Received: 2 February 2021 | Accepted: 6 February 2021

DOI: 10.1002/mana.202100068

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Framed motivic Donaldson–Thomas invariants of small crepant resolutions

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Funding information
Scuola Internazionale Superiore di Studi Avanzati

Abstract

For an arbitrary integer $r \geq 1$, we compute r-framed motivic DT and PT invariants of small crepant resolutions of toric Calabi–Yau 3-folds, establishing a "higher rank" version of the motivic DT/PT wall-crossing formula. This generalises the work of Morrison and Nagao. Our formulae, in particular their relationship with the $r = 1$ theory, fit nicely in the current development of higher rank refined DT invariants.

KEYWORDS

motivic Donaldson–Thomas invariants, motivic hall algebra, quiver representations, wall-crossing

MSC (2020)

14C05, 14N35

1 | INTRODUCTION

Let Y be a smooth Calabi–Yau 3-fold. Donaldson–Thomas (DT in short) theory in rank 1 is an enumerative theory virtually enumerating curves embedded in Y. The moduli space being "enumerated" is the Hilbert scheme of 1-dimensional subschemes of Y. On the other hand, Pandharipande–Thomas (PT in short) theory has as its main character the moduli space of (rank 1) stable pairs on Y, which are pairs (F, s) where $F \in \operatorname{Coh} Y$ is a purely 1-dimensional sheaf and $s : \mathscr{O}_Y \to F$ is a section with 0-dimensional cokernel. Both enumerative theories admit motivic refinements; in general it is very hard to produce explicit formulae for the generating functions of the motivic DT and PT invariants, but when the moduli spaces in question admit a description in terms of stable representations of the Jacobi algebra of a quiver with potential (Q, ω), the problem might become more tractable. For instance, Morrison and Nagao computed in [15] motivic DT and PT invariants of small crepant resolutions Y_σ of the affine toric Calabi–Yau 3-fold

$$X = \operatorname{Spec} \mathbb{C}[x, y, z, w]/\left(xy - z^{N_0} w^{N_1}\right) \subset \mathbb{A}^4,$$

generalising previous results on the resolved conifold [16], corresponding to the case $N_0 = N_1 = 1$. Such resolutions $Y_\sigma \to X$ are indexed by *partitions* σ of a polygon Γ_{N_0,N_1} naturally attached to X (more details in § 3). Each partition σ defines a quiver with potential $\left(Q_\sigma, \omega_\sigma\right)$ with $N = N_0 + N_1$ vertices (see Figure 3 for an example of such a Q_σ), and for any

$r \geq 1$ one can consider the r-*framed* quiver (Definition 2.2) with potential $(\widetilde{Q}_\sigma, \omega_\sigma)$. We denote by $\widetilde{J}_\sigma$ the corresponding *Jacobi algebra*. A generic choice of stability parameters ζ_{PT} and ζ_{DT}, respectively in the PT and DT regions of the space of all stability parameters of Q_σ, gives rise to generating functions

$$\mathsf{PT}_r(Y_\sigma; s, T) \quad \text{and} \quad \mathsf{DT}_r(Y_\sigma; s, T)$$

of motivic invariants, where (at least in the $r = 1$ case) s represents the point class and T is a vector of curve classes. The definition of the series PT_r and DT_r is as follows. One first sets, for a generic stability parameter ζ,

$$\mathsf{Z}_\zeta(y_0, y_1, \dots, y_{N-1}) = \sum_{\alpha \in \mathbb{N}^{(Q_\sigma)_0}} \left[\mathfrak{M}_\zeta(\widetilde{J}_\sigma, \alpha)\right]_{\mathrm{vir}} \cdot y^\alpha$$

where the *virtual motive* $[\,\cdot\,]_{\mathrm{vir}}$ of the moduli stack $\mathfrak{M}_\zeta(\widetilde{J}_\sigma, \alpha)$ of ζ-stable $\widetilde{J}_\sigma$-modules with dimension vector $(\alpha, 1)$ is introduced in Definition 2.11. One then defines

$$\begin{aligned} \mathsf{PT}_r(Y_\sigma; s, T) &= \mathsf{Z}_{\zeta_{\mathsf{PT}}}(s, T_1, \dots, T_{N-1}), \\ \mathsf{DT}_r(Y_\sigma; s, T) &= \mathsf{Z}_{\zeta_{\mathsf{DT}}}(s, T_1, \dots, T_{N-1}) \end{aligned} \tag{1.1}$$

where $s = y_0 y_1 \cdots y_{N-1}$, $T_i = y_i^{-1}$ and $T = (T_1, \dots, T_{N-1})$.

The generating functions (1.1) are computed explicitly for $r = 1$ in [15, Cor. 0.3]. The result, recalled in § 5.2, is the following: one has

$$\mathsf{PT}_1(Y_\sigma; s, T) = \prod_{1 \leq a \leq b \leq N-1} Z_{[a,b]}(s, T_a \cdots T_b),$$

where, letting $\{C_i \mid 1 \leq i \leq N-1\}$ be the set of components of the exceptional curve and $c(a, b)$ the number of $(-1, -1)$-curves in $\{C_i \mid a \leq i \leq b\}$, one sets

$$Z_{[a,b]}(s, T_a \cdots T_b) = \begin{cases} \displaystyle\prod_{m \geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{j + \frac{1}{2} - \frac{m}{2}} (-s)^m T_a \cdots T_b\right) & \text{if } c(a, b) \text{ is odd,} \\ \displaystyle\prod_{m \geq 1} \prod_{j=0}^{m-1} \left(1 - \mathbb{L}^{j + 1 - \frac{m}{2}} (-s)^m T_a \cdots T_b\right)^{-1} & \text{if } c(a, b) \text{ is even.} \end{cases}$$

As for the DT series in rank 1, one has the DT/PT correspondence

$$\mathsf{DT}_1(Y_\sigma; s, T) = \mathsf{DT}_1^{\text{points}}(Y_\sigma, s) \cdot \mathsf{PT}_1(Y_\sigma; s, T),$$

where $\mathsf{DT}_1^{\text{points}}(Y_\sigma, s)$ is the Behrend–Bryan–Szendrői generating function [2], that we recall in (4.2).

The goal of this paper is to compute the generating functions $\mathsf{PT}_r(Y_\sigma; s, T)$ and $\mathsf{DT}_r(Y_\sigma; s, T)$ for arbitrary r. The result, as we will show, is a full factorisation of the above series as r-fold (twisted) products of the $r = 1$ generating functions. Moreover, we establish an r-framed version of the motivic DT/PT correspondence for Y_σ.

Our main result, proved in § 5.2, is the following.

Theorem 1.1. *Let Y_σ be the crepant resolution of X corresponding to a partition σ. There are factorisations*

$$\begin{aligned} \mathsf{PT}_r(Y_\sigma; s, T) &= \prod_{i=1}^{r} \mathsf{PT}_1\left(Y_\sigma; (-1)^{r+1} \mathbb{L}^{\frac{-r-1}{2} + i} s, T\right), \\ \mathsf{DT}_r(Y_\sigma; s, T) &= \prod_{i=1}^{r} \mathsf{DT}_1\left(Y_\sigma; (-1)^{r+1} \mathbb{L}^{\frac{-r-1}{2} + i} s, T\right). \end{aligned} \tag{1.2}$$

Furthermore, the r-framed motivic DT/PT correspondence holds: there is an identity

$$\mathsf{DT}_r\big(Y_\sigma; s, T\big) \;=\; \mathsf{DT}_r^{\mathrm{points}}\big(Y_\sigma, s\big)\cdot \mathsf{PT}_r\big(Y_\sigma; s, T\big),$$

where $\mathsf{DT}_r^{\mathrm{points}}\big(Y_\sigma, s\big)$ *is the virtual motivic partition function of the Quot scheme of points on* Y_σ*.*

The series $\mathsf{DT}_r^{\mathrm{points}}(\mathbb{A}^3, s) = \sum_{n\geq 0}\big[\mathrm{Quot}_{\mathbb{A}^3}\big(\mathscr{O}^{\oplus r}, n\big)\big]_{\mathrm{vir}}\cdot s^n$, originating from the critical locus structure on $\mathrm{Quot}_{\mathbb{A}^3}\big(\mathscr{O}^{\oplus r}, n\big)$, is studied in detail in [5, 6, 22]. The series $\mathsf{DT}_r^{\mathrm{points}}(Y, s)$ was introduced and computed for all 3-folds Y in [26, § 4], generalising the $r = 1$ case corresponding to $\mathrm{Hilb}^n Y$ [2]. See § 4 for more details — for instance, an explicit formula for $\mathsf{DT}_r^{\mathrm{points}}\big(Y_\sigma, s\big)$ will be given in Equation (4.3).

A first instance of Formulae (1.2) was computed in [5, Chap. 3] for the case of the resolved conifold and the resolution of a line of A_2 singularities.

The same factorisation of generating functions of "rank r objects" into r copies of generating functions of rank 1 objects, shifted precisely as in Formulae (1.2), has recently been observed in the context of higher rank K-theoretic DT invariants [10] and in string theory [20].

Even though the geometric meaning of the moduli spaces of quiver representations giving rise to the r-framed invariants (1.2), for arbitrary r, is not as clear as in the $r = 1$ case, we do believe that such moduli spaces have a sensible geometric interpretation as suitable "higher rank" analogues of the Hilbert scheme of curves in Y_σ (DT side) and the moduli space of stable pairs on Y_σ (PT side). We come back to this in Remark 5.9, where we discuss a geometric interpretation of the framed moduli spaces in the PT chamber for the case of the conifold and $\widetilde{A}_2$ quivers.

2 | BACKGROUND MATERIAL

2.1 | Rings of motives

In this subsection we recall the definitions of various rings where the motivic invariants we want to study live.

As in [15, 16], we let $\mathcal{M}_{\mathbb{C}}$ be the Grothendieck ring of the category of effective Chow motives over $\mathbb{C}$ with rational coefficients [14], extended with $\mathbb{L}^{-1/2}$. A lambda-ring structure on $\mathcal{M}_{\mathbb{C}}$ is obtained by setting $\sigma_n([X]) = \big[\mathrm{Sym}^n X\big]$ and $\sigma_n\big(\mathbb{L}^{1/2}\big) = \mathbb{L}^{n/2}$ to define the lambda operations. In particular, there is a well defined notion of power structure and plethystic exponential on $\mathcal{M}_{\mathbb{C}}$ (see e.g. [2, § 2.5] or [8, § 1.5.1] for their formal properties). We consider the dimensional completion [3]

$$\widetilde{\mathcal{M}}_{\mathbb{C}} = \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}]\!],$$

which is also a lambda-ring, and in which the motives $\big[\,\mathrm{GL}_k\,\big]$ of all general linear groups are invertible.

2.1.1 | The virtual motive of a critical locus

Let U be a smooth d-dimensional $\mathbb{C}$-scheme, let $f : U \to \mathbb{A}^1$ be a regular function. The *virtual motive* of the critical locus $\mathrm{crit}(f) = Z(\mathrm{d}f) \subset U$, depending on the pair (U, f), is defined in [15, 16] as the motivic class

$$\big[\mathrm{crit}(f)\big]_{\mathrm{vir}} = -\Big(-\mathbb{L}^{\frac{1}{2}}\Big)^{-d}\cdot\big[\phi_f\big] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}},$$

where $\big[\phi_f\big] \in K_0^{\hat{\mu}}\big(\,\mathrm{Var}_{\mathbb{C}}\,\big)$ is the (absolute) motivic vanishing cycle class defined by Denef and Loeser [9] and the "$\hat{\mu}$" decoration refers to $\hat{\mu}$-equivariant motives, where $\hat{\mu}$ is the group of all roots of unity. However, all the motivic invariants studied here will live in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of classes carrying the trivial $\hat{\mu}$-action, so we will not be concerned with the subtle structure of this larger ring.

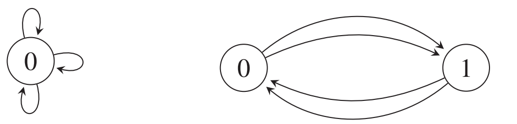

FIGURE 1 The 3-loop quiver L_3 and the conifold quiver Q_{con}

As an example, consider the function $f = 0 \in \Gamma(U)$. Then $\mathrm{crit}(f) = U$ and $[\phi_f] = -[U]$, so $[U]_{\mathrm{vir}} = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{-\dim U} \cdot [U]$. For instance,

$$\left[\mathrm{GL}_k\right]_{\mathrm{vir}} = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{-k^2} \cdot \left[\mathrm{GL}_k\right]. \tag{2.1}$$

Remark 2.1. Our definition of $[\mathrm{crit}(f)]_{\mathrm{vir}}$ differs from the original one [2, § 2.8], which is also the one used in [6, 8]. We decided to adopt the conventions in [15, 16] to keep close to the original formulae. In practice, the difference amounts to the substitution $\mathbb{L}^{1/2} \leftrightarrow -\mathbb{L}^{1/2}$. In particular, the Euler number specialisation with our conventions is $\mathbb{L}^{1/2} \to 1$, instead of $\mathbb{L}^{1/2} \to -1$.

2.2 | Quivers: framings, and motivic quantum torus

A quiver Q is a finite directed graph, determined by its sets Q_0 and Q_1 of vertices and edges, respectively, along with the maps $h, t : Q_1 \to Q_0$ specifying where an edge starts or ends. We use the notation

$$t(a) \bullet \xrightarrow{\quad a \quad} \bullet\ h(a)$$

to denote the *tail* and the *head* of an edge $a \in Q_1$.

All quivers in this paper will be assumed connected. The *path algebra* $\mathbb{C}Q$ of a quiver Q is defined, as a $\mathbb{C}$-vector space, by using as a $\mathbb{C}$-basis the set of all paths in the quiver, including a trivial path e_i for each $i \in Q_0$. The product is defined by concatenation of paths whenever the operation is possible, and 0 otherwise. The identity element is $\sum_{i \in Q_0} e_i \in \mathbb{C}Q$.

On a quiver Q one can define the *Euler–Ringel form* $\chi(-,-) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ by

$$\chi(\alpha, \beta) = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)},$$

as well as the skew-symmetric form

$$\langle \alpha, \beta \rangle = \chi(\alpha, \beta) - \chi(\beta, \alpha).$$

The following construction will be central in our paper.

Definition 2.2 (r-framing). Let Q be a quiver with a distinguished vertex $0 \in Q_0$, and let r be a positive integer. We define the quiver $\widetilde{Q}$ by adding one vertex, labelled ∞, to the original vertices in Q_0, and r edges $\infty \to 0$. We refer to $\widetilde{Q}$ as the *r-framed* quiver obtained out of $(Q, 0)$.

The r-framing construction was applied to the 3-loop quiver (on the left in Figure 1) in [1, 5, 6, 22], following the $r = 1$ case studied by Behrend–Bryan–Szendrői [2], and to the conifold quiver (on the right in Figure 1) in [5]. In this paper, it will be applied more generally to the quivers arising in the work of Morrison–Nagao [15], which we briefly discuss in § 3. The case $r = 1$ was covered in [15, 16].

Let Q be a quiver. Define its *motivic quantum torus* (or *twisted motivic algebra*) as

$$\mathcal{T}_Q = \prod_{\alpha \in \mathbb{N}^{Q_0}} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y^{\alpha}$$

with product rule

$$y^\alpha \cdot y^\beta = \left(-\mathbb{L}^{\frac{1}{2}}\right)^{\langle\alpha,\beta\rangle} y^{\alpha+\beta}. \tag{2.2}$$

If $\widetilde{Q}$ is the r-framed quiver associated to $(Q,0)$ via Definition 2.2, one has that $\mathcal{T}_Q$ sits inside $\mathcal{T}_{\widetilde{Q}}$ as a $\widetilde{\mathcal{M}}_{\mathbb{C}}$-subalgebra, and there is a $\mathbb{Z}$-module decomposition

$$\mathcal{T}_{\widetilde{Q}} = \mathcal{T}_Q \oplus \prod_{d\geq 0} \widetilde{\mathcal{M}}_{\mathbb{C}} \cdot y_\infty^d,$$

where we have set $y_\infty = y^{(\mathbf{0},1)}$. Similarly, a generator $y^\alpha \in \mathcal{T}_Q$ will be identified with its image $y^{(\alpha,0)} \in \mathcal{T}_{\widetilde{Q}}$.

2.3 | Quiver representations and their stability

Let Q be a quiver. A *representation* ρ of Q is the datum of a finite dimensional $\mathbb{C}$-vector space ρ_i for every vertex $i \in Q_0$, and a linear map $\rho(a) : \rho_i \to \rho_j$ for every edge $a : i \to j$ in Q_1. The *dimension vector* of ρ is the vector $\underline{\dim}\,\rho = \left(\dim_{\mathbb{C}} \rho_i\right)_i \in \mathbb{N}^{Q_0}$, where $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Convention 1. Let Q be a quiver, let $\widetilde{Q}$ be the associated r-framed quiver. The dimension vector of a representation $\widetilde{\rho}$ of $\widetilde{Q}$ will be denoted (α, d), where $\alpha \in \mathbb{N}^{Q_0}$ and $\dim_{\mathbb{C}} \widetilde{\rho}_\infty = d \in \mathbb{N}$.

Representations of a quiver Q form an abelian category, which is equivalent to the category of left modules over the path algebra $\mathbb{C}Q$ of the quiver. The space of all representations of Q, with a fixed dimension vector $\alpha \in \mathbb{N}^{Q_0}$, is the affine space

$$\mathrm{R}(Q,\alpha) = \prod_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{\alpha_{t(a)}}, \mathbb{C}^{\alpha_{h(a)}}\right).$$

The gauge group $\mathrm{GL}_\alpha = \prod_{i\in Q_0} \mathrm{GL}_{\alpha_i}$ acts on $\mathrm{R}(Q,\alpha)$ by $\left(g_i\right)_i \cdot (\rho(a))_{a\in Q_1} = \left(g_{h(a)} \circ \rho(a) \circ g_{t(a)}^{-1}\right)_{a\in Q_1}$. The quotient stack

$$\mathfrak{M}(Q,\alpha) = \left[\mathrm{R}(Q,\alpha)/\,\mathrm{GL}_\alpha\right]$$

parametrises isomorphism classes of representations of Q with dimension vector α.

Following [15, 16], we recall the notion of (semi)stability of a representation.

Definition 2.3. A *central charge* is a group homomorphism $\mathrm{Z} : \mathbb{Z}^{Q_0} \to \mathbb{C}$ such that the image of $\mathbb{N}^{Q_0} \setminus 0$ lies inside $\mathbb{H}_+ = \left\{ te^{\sqrt{-1}\pi\varphi} \mid t > 0,\ 0 < \varphi \leq 1 \right\}$. For every $\alpha \in \mathbb{N}^{Q_0} \setminus 0$, we denote by $\varphi(\alpha)$ the real number φ such that $\mathrm{Z}(\alpha) = te^{\sqrt{-1}\pi\varphi}$. It is called the *phase* of α with respect to Z.

Note that every vector $\zeta \in \mathbb{R}^{Q_0}$ induces a central charge Z_ζ if we set $\mathrm{Z}_\zeta(\alpha) = -\zeta \cdot \alpha + |\alpha|\sqrt{-1}$, where $|\alpha| = \sum_{i\in Q_0} \alpha_i$. We denote by φ_ζ the induced phase function, and we set $\varphi_\zeta(\rho) = \varphi_\zeta(\underline{\dim}\,\rho)$ for every representation ρ of Q. The *slope function* attached to Z_ζ assigns to $\alpha \in \mathbb{N}^{Q_0} \setminus 0$ the real number $\mu_\zeta(\alpha) = \zeta \cdot \alpha / |\alpha|$. Note that $\varphi_\zeta(\alpha) < \varphi(\beta)$ if and only if $\mu_\zeta(\alpha) < \mu_\zeta(\beta)$ (cf. [15, Rem. 3.5]).

Definition 2.4. Fix $\zeta \in \mathbb{R}^{Q_0}$. A representation ρ of Q is called ζ-*semistable* if

$$\varphi_\zeta(\rho') \leq \varphi_\zeta(\rho)$$

for every nonzero proper subrepresentation $0 \neq \rho' \subsetneq \rho$. When strict inequality holds, we say that ρ is ζ-*stable*. Vectors $\zeta \in \mathbb{R}^{Q_0}$ are referred to as *stability parameters*.

For a fixed ζ, every representation ρ admits a unique filtration

$$\mathrm{HN}_\zeta(\rho) \quad : \qquad 0 = \rho_0 \subset \rho_1 \subset \cdots \subset \rho_s = \rho,$$

called the *Harder–Narasimhan filtration*, such that ρ_i/ρ_{i-1} is ζ-semistable for $1 \leq i \leq s$, and there are strict inequalities $\varphi_\zeta(\rho_1/\rho_0) > \varphi_\zeta(\rho_2/\rho_1) > \cdots > \varphi_\zeta(\rho/\rho_{s-1})$.

Remark 2.5. The existence, uniqueness and functoriality of the Harder–Narasimhan filtration yields a stratification of the moduli stack of all Q-representations into locally closed substacks, indexed by Harder–Narasimhan type (this is a direct consequence of [21, Prop. 3.4]); this stratification induces relations in the motivic quantum torus, which are implicitly used in Lemma 5.4.

Definition 2.6 ([16, § 1.3]). Let $\alpha \in \mathbb{N}^{Q_0}$ be a dimension vector. A stability parameter ζ is called α-*generic* if for any $0 < \beta < \alpha$ one has $\varphi_\zeta(\beta) \neq \varphi_\zeta(\alpha)$.

The sets of ζ-stable and ζ-semistable representations with given dimension vector α form a chain of open subsets

$$\mathrm{R}^{\zeta\text{-st}}(Q,\alpha) \subset \mathrm{R}^{\zeta\text{-ss}}(Q,\alpha) \subset \mathrm{R}(Q,\alpha).$$

If ζ is α-generic, one has $\mathrm{R}^{\zeta\text{-st}}(Q,\alpha) = \mathrm{R}^{\zeta\text{-ss}}(Q,\alpha)$.

2.4 | Quivers with potential

Let Q be a quiver. Consider the quotient $\mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$ of the path algebra by the vector space spanned by commutators. An element $W \in \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$, which can be represented by a finite linear combination, is called a *potential*. Given a cyclic path w and an arrow $a \in Q_1$, one defines the noncommutative derivative

$$\frac{\partial w}{\partial a} = \sum_{\substack{w=cac' \\ c,c' \text{ paths in } Q}} c'c \ \in \mathbb{C}Q.$$

This rule extends to an operator $\partial/\partial a : \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q] \to \mathbb{C}Q$ acting on every potential. Thus every potential W gives rise to a (two-sided) ideal $I_W \subset \mathbb{C}Q$ generated by the paths $\partial W/\partial a$ for all $a \in Q_1$. The quotient $J = J(Q,W) = \mathbb{C}Q/I_W$ is called the *Jacobi algebra* of the quiver with potential (Q,W). For every $\alpha \in \mathbb{N}^{Q_0}$, a potential $W = \sum_c a_c c$ determines a regular function

$$f_\alpha : \mathrm{R}(Q,\alpha) \to \mathbb{A}^1, \quad \rho \mapsto \sum_{c \text{ cycle in } Q} a_c \cdot \mathrm{Tr}(\rho(c)).$$

The points in the critical locus $\mathrm{crit}(f_\alpha) \subset \mathrm{R}(Q,\alpha)$ correspond to J-*modules* with dimension vector α. Fix an α-generic stability parameter $\zeta \in \mathbb{R}^{Q_0}$. If $f_{\zeta,\alpha} : \mathrm{R}^{\zeta\text{-st}}(Q,\alpha) \to \mathbb{A}^1$ is the restriction of f_α, then

$$\mathfrak{M}(J,\alpha) = [\mathrm{crit}(f_\alpha)/G_\alpha], \quad \mathfrak{M}_\zeta(J,\alpha) = [\mathrm{crit}(f_{\zeta,\alpha})/\,\mathrm{GL}_\alpha]$$

are, by definition, the stacks of α-dimensional J-modules and ζ-stable J-modules.

Definition 2.7. A quiver with potential (Q,W) admits a *cut* if there is a subset $I \subset Q_1$ such that every cyclic monomial appearing in W contains exactly one edge in I.

From now on we assume (Q,W) admits a cut. This condition ensures that the motive $[\mathfrak{M}(J,\alpha)]_{\mathrm{vir}}$ introduced in the next definition is monodromy-free, i.e. it lives in $\widetilde{\mathcal{M}}_{\mathbb{C}}$. See [16, § 1.4] for more details. All quivers considered in this paper admit a cut [15, § 4].

Definition 2.8 ([16]). We define motivic Donaldson–Thomas invariants

$$\begin{aligned} \left[\mathfrak{M}(J,\alpha)\right]_{\mathrm{vir}} &= \frac{\left[\mathrm{crit}(f_\alpha)\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}}, \\ \left[\mathfrak{M}_\zeta(J,\alpha)\right]_{\mathrm{vir}} &= \left(-\mathbb{L}^{\frac{1}{2}}\right)^{\chi(\alpha,\alpha)} \frac{\left[f_{\zeta,\alpha}^{-1}(0)\right]-\left[f_{\zeta,\alpha}^{-1}(1)\right]}{[\mathrm{GL}_\alpha]}, \end{aligned} \tag{2.3}$$

in $\widetilde{\mathcal{M}}_{\mathbb{C}}$, where $\left[\,\mathrm{GL}_\alpha\,\right]_{\mathrm{vir}}$ is as in Equation (2.1). The generating function

$$A_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} \left[\mathfrak{M}(J,\alpha)\right]_{\mathrm{vir}} \cdot y^\alpha \in \mathcal{T}_Q \tag{2.4}$$

is called the *universal series* attached to (Q,W).

Definition 2.9 ([16, § 2.4]). A stability parameter $\zeta\in\mathbb{R}^{Q_0}$ is called *generic* if $\zeta\cdot\underline{\dim}\,\rho\neq 0$ for every nontrivial ζ-stable J-module ρ.

2.5 | Framed motivic DT invariants

Let $r\geq 1$ be an integer, let Q be a quiver, and $\widetilde{Q}$ its r-framing with respect to a vertex $0\in Q_0$ (Definition 2.2). A representation $\widetilde{\rho}$ of $\widetilde{Q}$ can be uniquely written as a pair (ρ,u), where ρ is a representation of Q and $u=(u_1,\dots,u_r)$ is an r-tuple of linear maps $u_i : \widetilde{\rho}_\infty\to\rho_0$.

From now on, we assume all r-framed representations to satisfy $\dim_{\mathbb{C}}\widetilde{\rho}_\infty=1$, so that by Convention 1 one has $\underline{\dim}\,\widetilde{\rho}=(\underline{\dim}\,\rho,1)$.

Definition 2.10 ([19] and [16, Def. 3.1]). Let $\zeta\in\mathbb{R}^{Q_0}$ be a stability parameter. A representation (ρ,u) of $\widetilde{Q}$ (or a $\widetilde{J}$-module) with $\dim_{\mathbb{C}}\widetilde{\rho}_\infty=1$ is said to be ζ*-(semi)stable* if it is (ζ,ζ_∞)-(semi)stable in the sense of Definition 2.4, where $\zeta_\infty=-\zeta\cdot\underline{\dim}\,\rho$.

Now fix a potential W on Q. We define motivic DT invariants for moduli stacks of r-framed J-modules on Q. Let $\widetilde{J}$ be the Jacobi algebra $J_{\widetilde{Q},W}$, where W is viewed as a potential on $\widetilde{Q}$ in the obvious way. For a generic stability parameter $\zeta\in\mathbb{R}^{Q_0}$, and a dimension vector $\alpha\in\mathbb{N}^{Q_0}$, set

$$\zeta_\infty=-\zeta\cdot\alpha,\quad \widetilde{\zeta}=(\zeta,\zeta_\infty),\quad \widetilde{\alpha}=(\alpha,1).$$

As in § 2.4, consider the functions

$$\begin{array}{ccc} \mathrm{R}^{\widetilde{\zeta}\text{-st}}\big(\widetilde{Q},\widetilde{\alpha}\big) & \hookrightarrow & \mathrm{R}\big(\widetilde{Q},\widetilde{\alpha}\big) \\ & \searrow^{f_{\widetilde{\zeta},\widetilde{\alpha}}} & \downarrow f_{\widetilde{\alpha}} \\ & & \mathbb{A}^1 \end{array}$$

associated to the potential W. Define the moduli stacks

$$\mathfrak{M}\big(\widetilde{J},\alpha\big)=\left[\mathrm{crit}\big(f_{\widetilde{\alpha}}\big)\,/\,\mathrm{GL}_\alpha\right],\quad \mathfrak{M}_\zeta\big(\widetilde{J},\alpha\big)=\left[\mathrm{crit}\big(f_{\widetilde{\zeta},\widetilde{\alpha}}\big)\,/\,\mathrm{GL}_\alpha\right].$$

Definition 2.11. We define r-framed motivic Donaldson–Thomas invariants

$$\begin{aligned} \left[\mathfrak{M}\big(\widetilde{J},\alpha\big)\right]_{\mathrm{vir}} &= \frac{\left[\mathrm{crit}\big(f_{\widetilde{\alpha}}\big)\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}}, \\ \left[\mathfrak{M}_\zeta\big(\widetilde{J},\alpha\big)\right]_{\mathrm{vir}} &= \frac{\left[\mathrm{crit}\big(f_{\widetilde{\zeta},\widetilde{\alpha}}\big)\right]_{\mathrm{vir}}}{[\mathrm{GL}_\alpha]_{\mathrm{vir}}} \end{aligned}$$

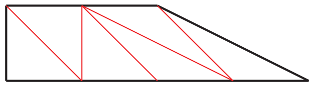

FIGURE 2 A partition Γ_σ of $\Gamma_{4,2}$

in $\widetilde{\mathcal{M}}_{\mathbb{C}}$, and the associated motivic generating functions

$$\widetilde{A}_U = \sum_{\alpha\in\mathbb{N}^{Q_0}} \left[\mathfrak{M}(\widetilde{J},\alpha)\right]_{\mathrm{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}},$$

$$\widetilde{A}_\zeta = \sum_{\alpha\in\mathbb{N}^{Q_0}} \left[\mathfrak{M}_\zeta(\widetilde{J},\alpha)\right]_{\mathrm{vir}} \cdot y^{\widetilde{\alpha}} \in \mathcal{T}_{\widetilde{Q}},$$

$$\mathsf{Z}_\zeta = \sum_{\alpha\in\mathbb{N}^{Q_0}} \left[\mathfrak{M}_\zeta(\widetilde{J},\alpha)\right]_{\mathrm{vir}} \cdot y^{\alpha} \in \mathcal{T}_{Q}.$$

The fact that the r-framed invariants live in $\widetilde{\mathcal{M}}_{\mathbb{C}}$ (i.e., have no monodromy) follows from [16, Lemma 1.10]. The reason is that the dimension vector $\widetilde{\alpha} = (\alpha, 1)$ contains "1" as a component.

Our main goal is to give a formula for Z_ζ, where ζ is chosen in a PT (resp. DT) chamber.

3 | NONCOMMUTATIVE CREPANT RESOLUTIONS

Fix integers $N_0 > 0$ and $0 \leq N_1 \leq N_0$, and set $N = N_0 + N_1$. The cone realising the singular Calabi–Yau 3-fold $X = \operatorname{Spec}\mathbb{C}[x,y,z,w]/\left(xy - z^{N_0}w^{N_1}\right)$ as a toric variety is the cone over the quadrilateral Γ_{N_0,N_1} with vertices (0,0), $(N_0, 0)$, $(N_1, 1)$ and (0,1), which becomes a triangle when $N_1 = 0$.

A *partition* σ of Γ_{N_0,N_1} is, roughly speaking, a subdivision of the polygon Γ_{N_0,N_1} into N triangles $\{\sigma_i\}_{0\leq i\leq N-1}$ of area 1/2. We refer the reader to [18, § 1.1] for the precise definition. We denote by Γ_σ the resulting object — see Figure 2 for an example with $N_0 = 4$, $N_1 = 2$. Each internal edge $\sigma_{i,i+1}$ corresponds to a component C_i of the exceptional curve in the resolution Y_σ attached to Γ_σ, and C_i is a $(-1,-1)$-curve (resp. a $(-2,0)$-curve) if $\sigma_i \cup \sigma_{i+1}$ is a quadrilateral (resp. a triangle).

As explained in [15, 18], any partition σ gives rise to a small crepant resolution $Y_\sigma \to X$ by taking the fan of Γ_σ, and any two such resolutions are related by a sequence of mutations. On the other hand, Nagao [18] explains how to associate to σ a bipartite tiling of the plane. The general construction in [13] then produces a quiver with potential $(Q_\sigma, \omega_\sigma)$. Its Jacobi algebra J_σ is derived equivalent to Y_σ [18, § 1].

The quiver Q_σ has vertex set $\widehat{I} = \{0, 1, \dots, N-1\}$, which we identify with the cyclic group $\mathbb{Z}/N\mathbb{Z}$. This in turn yields an identification

$$\mathbb{Z}^{\widehat{I}} = \mathbb{Z}^{(Q_\sigma)_0}. \tag{3.1}$$

Each vertex of Q_σ has an edge in and out of the next vertex. The partition prescribes which vertices carry a loop, as we now explain using the specific example of Figure 2. In that case, the partition $\sigma = \{\sigma_i\}_{0\leq i\leq 5}$ can be identified with the ordered set of half-points

$$\sigma = \left\{ \left(\frac{1}{2},0\right), \left(\frac{1}{2},1\right), \left(\frac{3}{2},0\right), \left(\frac{5}{2},0\right), \left(\frac{3}{2},1\right), \left(\frac{7}{2},0\right) \right\}, \tag{3.2}$$

where the ith element corresponds to the mid-point of the base of the ith triangle σ_i. A vertex $k \in \widehat{I}$ will carry a loop if and only if σ_{k-1} and σ_k have the same y-coordinate. Thus, by cyclicity, in our case we get two vertices ($k = 0, 3$) carrying a loop. The resulting quiver is drawn in Figure 3.

For the definition of the potential ω_σ, we refer the reader to [18, § 1.2] or [15, § 2.A]. It is proved in [15, § 4] that $(Q_\sigma, \omega_\sigma)$ has a cut for all σ.

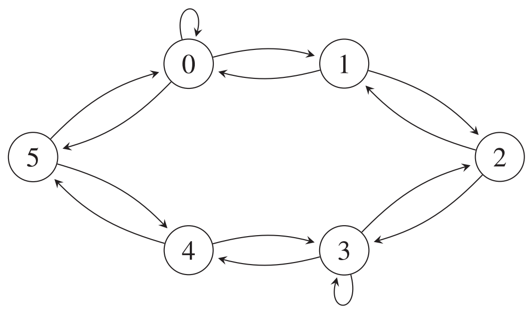

FIGURE 3 The quiver Q_σ associated to the partition (3.2)

Remark 3.1. The quiver Q_σ is *symmetric*. This implies that its motivic quantum torus $\mathcal{T}_{Q_\sigma}$ is commutative.

We fix $\epsilon_0, \dots, \epsilon_{N-1}$ to be the basis of $\mathbb{Z}^{(Q_\sigma)_0}$ corresponding to the canonical basis of $\mathbb{Z}^{\widehat{I}}$ under (3.1). We call ϵ_i a *simple root*, and $\delta = \epsilon_0 + \epsilon_1 + \cdots + \epsilon_{N-1}$ the positive minimal imaginary root. Following the notation in [15], we set $\epsilon_{[a,b]} = \sum_{a\le i\le b} \epsilon_i$ for all $1 \le a \le b \le N-1$, and

$$\begin{aligned}
\Delta_+^{\mathrm{re},+} &= \left\{ \epsilon_{[a,b]} + n\cdot\delta \mid 1 \le a \le b \le N-1,\ n \in \mathbb{Z}_{\ge 0} \right\},\\
\Delta_+^{\mathrm{re},-} &= \left\{ -\epsilon_{[a,b]} + n\cdot\delta \mid 1 \le a \le b \le N-1,\ n \in \mathbb{Z}_{> 0} \right\},\\
\Delta_+^{\mathrm{im}} &= \{ n\cdot\delta \mid n \in \mathbb{Z}_{>0} \}.
\end{aligned} \tag{3.3}$$

From the above sets we form the larger sets

$$\Delta_+^{\mathrm{re}} = \Delta_+^{\mathrm{re},+} \amalg \Delta_+^{\mathrm{re},-}, \quad \Delta_+ = \Delta_+^{\mathrm{re}} \amalg \Delta_+^{\mathrm{im}}.$$

Remark 3.2. The above sets depend on σ, but we omit this dependence to ease notation; in the language of [15], we have $\Delta_+ = \Delta_{\sigma,+}$, $\Delta_+^{\mathrm{re}} = \Delta_{\sigma,+}^{\mathrm{re}}$ and $\Delta_+^{\mathrm{im}} = \Delta_{\sigma,+}^{\mathrm{im}}$.

4 | HIGHER RANK MOTIVIC DT THEORY OF POINTS

The rank 1 DT theory of points on a 3-fold Y is entirely solved, see e.g. [4] for the case of $\mathrm{Hilb}^n Y$ and [11] for the *reduced* DT theory of points on an abelian 3-fold. In higher rank, to define the theory we fix a locally free sheaf F of rank r on Y. Building on the case of $Y = \mathbb{A}^3$, fully explored in [5–7, 22], a virtual motive for the Quot scheme $\mathrm{Quot}_Y(F,n)$ was defined in [26, Def. 4.10] via power structures, along the same lines of the rank 1 case [2, § 4.1].

The generating function

$$\mathsf{DT}_r^{\mathrm{points}}\big(Y, (-1)^r s\big) = \sum_{n\ge 0} \big[\mathrm{Quot}_Y(F,n)\big]_{\mathrm{vir}} \cdot ((-1)^r s)^n$$

was computed in [26, Thm. 4.11] as a plethystic exponential. Just as in the case of the naive motives [25], the generating function does not depend on F but only on r and on the motive of Y.

Consider the singular affine toric Calabi–Yau 3-fold $X = \operatorname{Spec} \mathbb{C}[x,y,z,w]/\big(xy - z^{N_0}w^{N_1}\big) \subset \mathbb{A}^4$, and fix a partition σ associated to the polygon Γ_{N_0,N_1}.

Lemma 4.1. *Let Y_σ be the crepant resolution of X corresponding to σ. Then*

$$\big[Y_\sigma\big] = \mathbb{L}^3 + (N-1)\mathbb{L}^2 \in K_0\big(\mathrm{Var}_{\mathbb{C}}\big).$$

Proof. The toric polygon of Y_σ consists of $N = N_0 + N_1$ triangles $\{\sigma_i\}$ intersecting pairwise along the edges $\{\sigma_{i,i+1}\}$. The toric resolution Y_σ is constructed by gluing the toric charts U_{σ_i} along the open affine subvarieties $U_{\sigma_{i,i+1}}$. Thus, the class $[Y_\sigma]$ can be computed using the cut-and-paste relations, after noticing that $U_{\sigma_i} \simeq \mathbb{A}^3$ and $U_{\sigma_{i,i+1}} \simeq \mathbb{A}^2 \times \mathbb{C}^*$. The result is

$$[Y_\sigma] = \sum_{i=1}^{N} \mathbb{L}^3 - \sum_{i=1}^{N-1} \mathbb{L}^2(\mathbb{L}-1) = \mathbb{L}^3 + (N-1)\mathbb{L}^2.$$

□

By [6, Thm. A] (but see also [5, 22] for different proofs), after rephrasing the result using the conventions adopted in this paper (cf. Remark 2.1), one has

$$\mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, (-1)^r s\right) = \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{k+2-\frac{rm}{2}}s^m\right)^{-1} = \prod_{i=1}^{r}\mathsf{DT}_1^{\text{points}}\left(\mathbb{A}^3, -\mathbb{L}^{\frac{-r-1}{2}+i}s\right).$$

An easy power structure argument shows that the same decomposition into r rank 1 pieces holds for every smooth 3-fold Y. In a little more detail (we refer the reader to [12] or to [2, 8] for the formal properties of the power structure on $\mathcal{M}_{\mathbb{C}}$), we have

$$\begin{aligned}\mathsf{DT}_r^{\text{points}}\left(Y, (-1)^r s\right) &= \mathsf{DT}_r^{\text{points}}\left(\mathbb{A}^3, (-1)^r s\right)^{\mathbb{L}^{-3}[Y]}\\ &= \prod_{i=1}^{r}\mathsf{DT}_1^{\text{points}}\left(\mathbb{A}^3, -\mathbb{L}^{\frac{-r-1}{2}+i}s\right)^{\mathbb{L}^{-3}[Y]}\\ &= \prod_{i=1}^{r}\mathsf{DT}_1^{\text{points}}\left(Y, -\mathbb{L}^{\frac{-r-1}{2}+i}s\right).\end{aligned}$$

Therefore, for any smooth 3-fold Y, we can write

$$\mathsf{DT}_r^{\text{points}}(Y,s) = \prod_{i=1}^{r}\mathsf{DT}_1^{\text{points}}\left(Y, (-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i}s\right). \tag{4.1}$$

By Lemma 4.1, the motivic partition of the Hilbert scheme of points on Y_σ is

$$\mathsf{DT}_1^{\text{points}}(Y_\sigma, s) = \prod_{m\geq 1}\prod_{k=0}^{m-1}\left(1-\mathbb{L}^{k+1-\frac{m}{2}}(-s)^m\right)^{1-N}\left(1-\mathbb{L}^{k+2-\frac{m}{2}}(-s)^m\right)^{-1} \tag{4.2}$$

and this determines $\mathsf{DT}_r^{\text{points}}(Y_\sigma, s)$ via Equation (4.1). The result is

$$\mathsf{DT}_r^{\text{points}}(Y_\sigma, s) = \prod_{m\geq 1}\prod_{k=0}^{rm-1}\left(1-\mathbb{L}^{k+1-\frac{rm}{2}}\left((-1)^r s\right)^m\right)^{1-N}\left(1-\mathbb{L}^{k+2-\frac{rm}{2}}\left((-1)^r s\right)^m\right)^{-1}. \tag{4.3}$$

5 | MOTIVIC INVARIANTS OF NONCOMMUTATIVE CREPANT RESOLUTIONS

5.1 | Relations among motivic partition functions

Fix integers $N_0 > 0$ and $0 \leq N_1 \leq N_0$, and set $N = N_0 + N_1$. We consider the affine singular toric Calabi–Yau 3-fold

$$X_{N_0,N_1} = \operatorname{Spec}\mathbb{C}[x,y,z,w]/\left(xy - z^{N_0}w^{N_1}\right) \subset \mathbb{A}^4.$$

Fix a partition σ of the polygon Γ_{N_0,N_1}, and set $(Q,W,J)=(Q_\sigma,\omega_\sigma,J_\sigma)$ to ease notation, where J_σ is the Jacobi algebra of the quiver with potential (Q_σ,ω_σ) whose construction we sketched in § 3. The universal series

$$A_U^\sigma(y)=A_U^\sigma(y_0,\dots,y_{N-1})=\sum_{\alpha\in\mathbb{N}^{Q_0}}\left[\mathfrak{M}(J_\sigma,\alpha)\right]_{\mathrm{vir}}\cdot y^\alpha\in\mathcal{T}_Q,$$

defined in Equation (2.4), is the main object of study in the work of Morrison and Nagao [15].

Fix a generic stability parameter ζ (cf. Definition 2.9) on the unframed quiver Q. Consider the stacks $\mathfrak{M}_\zeta^\pm(J,\alpha)$ of J-modules all of whose Harder–Narasimhan factors have positive (resp. negative) slope with respect to ζ. These stacks are defined as follows. Restrict the function $f_\alpha:\mathrm{R}(Q,\alpha)\to\mathbb{A}^1$, defined by taking the trace of ω_σ, to the open subschemes $\mathrm{R}_\zeta^\pm(Q,\alpha)\subset\mathrm{R}(Q,\alpha)$ of representations satisfying the above properties. This yields two regular functions $f_\zeta^\pm:\mathrm{R}_\zeta^\pm(Q,\alpha)\to\mathbb{A}^1$, and we set $\mathfrak{M}_\zeta^\pm(J,\alpha)=\left[\mathrm{crit}(f_\zeta^\pm)/\,\mathrm{GL}_\alpha\right]$. We define the virtual motives $\left[\mathfrak{M}_\zeta^\pm(J,\alpha)\right]_{\mathrm{vir}}$ as in the second identity in Equation (2.3), and the associated motivic generating functions (depending on σ via $J=J_\sigma$)

$$A_\zeta^\pm=\sum_{\alpha\in\mathbb{N}^{Q_0}}\left[\mathfrak{M}_\zeta^\pm(J,\alpha)\right]_{\mathrm{vir}}\cdot y^\alpha\in\mathcal{T}_Q.$$

The vertices of Q are labeled from 0 up to $N-1$. Let $\widetilde{Q}$ be the r-framed quiver associated to $(Q,0)$ (Definition 2.2). We let $\widetilde{J}=J_{\widetilde{Q},W}$ be the Jacobi algebra of $(\widetilde{Q},W)=(\widetilde{Q}_\sigma,\omega_\sigma)$. Now recall the motivic generating functions

$$\widetilde{A}_U,\quad \widetilde{A}_\zeta,\quad \mathsf{Z}_\zeta$$

introduced in Definition 2.11. We have to extend the relations between framed and unframed generating functions (in the same spirit of Mozgovoy's work [17]) to general r. By the following lemma, the arguments are going to be essentially formal.

Lemma 5.1. *In $\mathcal{T}_{\widetilde{Q}}$ there are identities*

$$y_\infty\cdot y^{(\alpha,0)}=\left(-\mathbb{L}^{\frac12}\right)^{-r\alpha_0}\cdot y^{\widetilde{\alpha}},\quad y^{(\alpha,0)}\cdot y_\infty=\left(-\mathbb{L}^{\frac12}\right)^{r\alpha_0}\cdot y^{\widetilde{\alpha}}.$$

Proof. Since $\infty\in\widetilde{Q}_0$ has edges only reaching 0, and no vertex of Q reaches ∞, we have $\chi((\alpha,0),(\mathbf{0},1))=0$, and $\chi((\mathbf{0},1),(\alpha,0))=-r\alpha_0$. The result follows by the product rule (2.2). □

Corollary 5.2. *In $\mathcal{T}_{\widetilde{Q}}$, there are identities*

$$\widetilde{A}_\zeta=y_\infty\cdot\mathsf{Z}_\zeta\left(\left(-\mathbb{L}^{\frac12}\right)^r y_0,y_1,\dots,y_{N-1}\right),\tag{5.1}$$

$$A_\zeta^-\cdot y_\infty=y_\infty\cdot A_\zeta^-\left(\mathbb{L}^r y_0,y_1,\dots,y_{N-1}\right).\tag{5.2}$$

Proof. We have

$$\begin{aligned}y_\infty\cdot\mathsf{Z}_\zeta\left(\left(-\mathbb{L}^{\frac12}\right)^r y_0,y_1,\dots,y_{N-1}\right)&=\sum_{\alpha\in\mathbb{N}^{Q_0}}\left[\mathfrak{M}_\zeta(\widetilde{J},\alpha)\right]_{\mathrm{vir}}\cdot y_\infty\cdot\left(\left(-\mathbb{L}^{\frac12}\right)^r y_0\right)^{\alpha_0}\cdot y_1^{\alpha_1}\cdots y_{N-1}^{\alpha_{N-1}}\\&=\sum_{\alpha\in\mathbb{N}^{Q_0}}\left[\mathfrak{M}_\zeta(\widetilde{J},\alpha)\right]_{\mathrm{vir}}\left(-\mathbb{L}^{\frac12}\right)^{r\alpha_0}\cdot\left(y_\infty\cdot y^\alpha\right)\\&=\sum_{\alpha\in\mathbb{N}^{Q_0}}\left[\mathfrak{M}_\zeta(\widetilde{J},\alpha)\right]_{\mathrm{vir}}\cdot y^{\widetilde{\alpha}}\\&=\widetilde{A}_\zeta,\end{aligned}$$

where we have applied Lemma 5.1 in the last step. The identity (5.2) follows by an identical argument. □

Lemma 5.3 ([16, Proposition 3.5]). *Let Q be a quiver, $\zeta \in \mathbb{R}^{Q_0}$ a generic stability parameter, $\widetilde{\rho}$ a representation (resp. $\widetilde{J}$-module) of the r-framed quiver $\widetilde{Q}$ with $\dim_{\mathbb{C}} \widetilde{\rho}_\infty = 1$. Then there is a unique filtration $0 = \widetilde{\rho}^0 \subset \widetilde{\rho}^1 \subset \widetilde{\rho}^2 \subset \widetilde{\rho}^3 = \widetilde{\rho}$ such that the quotients $\widetilde{\pi}^i = \widetilde{\rho}^i/\widetilde{\rho}^{i-1}$ satisfy:*

1. $\widetilde{\pi}^1_\infty = 0$, *and* $\widetilde{\pi}^1 \in \mathrm{R}^+_\zeta(Q, \underline{\dim}\, \widetilde{\pi}^1)$,
2. $\dim_{\mathbb{C}} \widetilde{\pi}^2_\infty = 1$ *and* $\widetilde{\pi}^2$ *is* ζ*-stable,*
3. $\widetilde{\pi}^3_\infty = 0$, *and* $\widetilde{\pi}^3 \in \mathrm{R}^-_\zeta(Q, \underline{\dim}\, \widetilde{\pi}^3)$.

Lemma 5.4. *Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_{\widetilde{Q}}$, there are factorisations*

$$\widetilde{A}_U = A^+_\zeta \cdot \widetilde{A}_\zeta \cdot A^-_\zeta, \tag{5.3}$$

$$\widetilde{A}_U = A^\sigma_U \cdot y_\infty. \tag{5.4}$$

Proof. Equation (5.3) is a direct consequence of the existence of the filtration of Lemma 5.3. Equation (5.4) follows directly from the following observation: given a framed representation $\widetilde{\rho} = (\rho, u)$ with $\dim_{\mathbb{C}} \widetilde{\rho}_\infty = 1$, one can view ρ as a submodule $\rho \subset \widetilde{\rho}$ of dimension $(\underline{\dim}\,\rho, 0)$, and the quotient $\widetilde{\rho}/\rho$ is the unique simple module of dimension (0,1), based at the framing vertex. □

Following [15, § 0], we define, for $\alpha \in \Delta_+$, the infinite products

$$A_\alpha(y) = \begin{cases} \prod_{j\geq 0} \left(1 - \mathbb{L}^{-j-\frac{1}{2}} y^\alpha\right) & \text{if } \alpha \in \Delta^{\mathrm{re}}_+ \text{ and } \sum_{k \notin \widehat{I}_\ell} \alpha_k \text{ is odd,} \\ \prod_{j\geq 0} \left(1 - \mathbb{L}^{-j} y^\alpha\right)^{-1} & \text{if } \alpha \in \Delta^{\mathrm{re}}_+ \text{ and } \sum_{k \notin \widehat{I}_\ell} \alpha_k \text{ is even,} \\ \prod_{j\geq 0} \left(1 - \mathbb{L}^{-j} y^\alpha\right)^{1-N} \left(1 - \mathbb{L}^{-j+1} y^\alpha\right)^{-1} & \text{if } \alpha \in \Delta^{\mathrm{im}}_+, \end{cases} \tag{5.5}$$

where $\widehat{I}_\ell \subset \widehat{I} = (Q_\sigma)_0$ denotes[1] the set of vertices carrying a loop, and $\alpha_k \in \mathbb{N}$ is the component of α corresponding to a vertex k.

Lemma 5.5 ([16, Lemma 2.6]). *Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there are identities*

$$A^\pm_\zeta(y) = \prod_{\substack{\alpha \in \Delta_+ \\ \pm\zeta\cdot\alpha > 0}} A_\alpha(y). \tag{5.6}$$

Lemma 5.6. *Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there is an identity*

$$A^\sigma_U = A^+_\zeta \cdot A^-_\zeta. \tag{5.7}$$

Proof. By [15, Thm. 0.1] there is a factorisation

$$A^\sigma_U(y) = \prod_{\alpha \in \Delta_+} A_\alpha(y).$$

Since ζ is generic, $\zeta \cdot \alpha \neq 0$ for all $\alpha \in \Delta_+$. The result then follows by combining this factorisation with Equation (5.6). □

Theorem 5.7. *Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there is an identity*

$$\mathsf{Z}_\zeta(y) = \frac{A_\zeta^-\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right)}{A_\zeta^-\left(\left(-\mathbb{L}^{-\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right)}. \tag{5.8}$$

Proof. Since $Q = Q_\sigma$ is symmetric (Remark 3.1), the algebra $\mathcal{T}_Q$ is commutative, therefore a power series $F \in \mathcal{T}_Q$ starting with the invertible element $1 \in \widetilde{\mathcal{M}}_\mathbb{C}$ will be invertible. For instance A_ζ^+ and A_ζ^- are invertible. Therefore we can write

$$\begin{aligned}
y_\infty \cdot \mathsf{Z}_\zeta\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right) &= \widetilde{A}_\zeta && \text{by (5.1)}\\
&= \left(A_\zeta^+\right)^{-1} \cdot \widetilde{A}_U \cdot \left(A_\zeta^-\right)^{-1} && \text{by (5.3)}\\
&= \left(A_\zeta^+\right)^{-1} \cdot \left(A_U^\sigma \cdot y_\infty\right) \cdot \left(A_\zeta^-\right)^{-1} && \text{by (5.4)}\\
&= \left(A_\zeta^+\right)^{-1} \cdot \left(A_\zeta^+ \cdot A_\zeta^- \cdot y_\infty\right) \cdot \left(A_\zeta^-\right)^{-1} && \text{by (5.7)}\\
&= y_\infty \cdot A_\zeta^-\left(\mathbb{L}^r y_0, y_1, \dots, y_{N-1}\right) \cdot \left(A_\zeta^-\right)^{-1} && \text{by (5.2)}
\end{aligned}$$

from which it follows that

$$\mathsf{Z}_\zeta\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right) = \frac{A_\zeta^-\left(\mathbb{L}^r y_0, y_1, \dots, y_{N-1}\right)}{A_\zeta^-\left(y_0, y_1, \dots, y_{N-1}\right)}.$$

Thus the change of variable $y_0 \to \left(-\mathbb{L}^{-\frac{1}{2}}\right)^r y_0$ yields the result. □

5.2 | Computing invariants in the DT and PT chambers

In this subsection we prove Theorem 1.1.

Define, for $\alpha \in \Delta_+$, the fraction

$$Z_\alpha^{(r)}(y_0, y_1, \dots, y_{N-1}) = \frac{A_\alpha\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right)}{A_\alpha\left(\left(-\mathbb{L}^{-\frac{1}{2}}\right)^r y_0, y_1, \dots, y_{N-1}\right)}, \tag{5.9}$$

where A_α is defined case by case in (5.5). Then one deduces the following explicit formulae:

$$Z_\alpha^{(r)}((-1)^r y_0, y_1, \dots, y_{N-1}) = \begin{cases} \displaystyle\prod_{k=0}^{r\alpha_0-1}\left(1-\mathbb{L}^{k+\frac{1}{2}-\frac{r\alpha_0}{2}} y^\alpha\right) & \text{if } \alpha \in \Delta_+^{\mathrm{re}} \text{ and } \displaystyle\sum_{k \notin \widehat{I}_\ell} \alpha_k \text{ is odd,}\\ \displaystyle\prod_{k=0}^{r\alpha_0-1}\left(1-\mathbb{L}^{k+1-\frac{r\alpha_0}{2}} y^\alpha\right)^{-1} & \text{if } \alpha \in \Delta_+^{\mathrm{re}} \text{ and } \displaystyle\sum_{k \notin \widehat{I}_\ell} \alpha_k \text{ is even,}\\ \displaystyle\prod_{k=0}^{r\alpha_0-1}\left(1-\mathbb{L}^{k+1-\frac{r\alpha_0}{2}} y^\alpha\right)^{1-N}\left(1-\mathbb{L}^{k+2-\frac{rm}{2}} y^\alpha\right)^{-1} & \text{if } \alpha \in \Delta_+^{\mathrm{im}}. \end{cases}$$

These identities can be easily rewritten uniformly in terms of the 'rank 1' generating functions:

$$Z_\alpha^{(r)}\left((-1)^r y_0, y_1, \dots, y_{N-1}\right) = \prod_{i=1}^{r} Z_\alpha^{(1)}\left(-\mathbb{L}^{\frac{-r-1}{2}+i} y_0, y_1, \dots, y_{N-1}\right). \tag{5.10}$$

Let us set

$$s = y_0 y_1 \cdots y_{N-1}, \quad T_i = y_i^{-1}, \quad T = (T_1, \dots, T_{N-1}).$$

For $1 \le a \le b \le N-1$, we let $T_{[a,b]} = T_a \cdots T_b$ be the monomial corresponding to the homology class

$$C_{[a,b]} = [C_a] + \cdots + [C_b] \in H_2(Y_\sigma, \mathbb{Z}),$$

where $C_i \subset Y_\sigma$ is a component of the exceptional curve. Let $c(a,b)$ be the number of $(-1,-1)$-curves in $\{C_i \mid a \le i \le b\}$. Then we set

$$Z_{[a,b]}(s, T_{[a,b]}) = \begin{cases} \prod_{m\ge 1}\prod_{j=0}^{m-1}\left(1 - \mathbb{L}^{j+\frac{1}{2}-\frac{m}{2}}(-s)^m T_{[a,b]}\right) & \text{if } c(a,b) \text{ is odd,} \\ \prod_{m\ge 1}\prod_{j=0}^{m-1}\left(1 - \mathbb{L}^{j+1-\frac{m}{2}}(-s)^m T_{[a,b]}\right)^{-1} & \text{if } c(a,b) \text{ is even} \end{cases}$$

and

$$Z_{\mathrm{im}}(s) = \prod_{m\ge 1}\prod_{j=0}^{m-1}\left(1 - \mathbb{L}^{j+1-\frac{m}{2}}(-s)^m\right)^{1-N}\left(1 - \mathbb{L}^{j+2-\frac{m}{2}}(-s)^m\right)^{-1}.$$

Fix, as in [15, § 6.C], stability parameters

$$\zeta_{\mathsf{PT}} = (1-N+\varepsilon, 1, \dots, 1), \quad \zeta_{\mathsf{DT}} = (1-N-\varepsilon, 1, \dots, 1),$$

with $0 < \varepsilon \ll 1$ chosen so that they are generic. We want to compute

$$\mathsf{PT}_r(Y_\sigma; s, T) = \mathsf{Z}_{\zeta_{\mathsf{PT}}}(s, T_1, \dots, T_{N-1}), \quad \mathsf{DT}_r(Y_\sigma; s, T) = \mathsf{Z}_{\zeta_{\mathsf{DT}}}(s, T_1, \dots, T_{N-1}).$$

For $r = 1$, these are the generating functions computed in [15, Cor. 0.3]. We know by Equation (4.2) (see also [15, Cor. 0.3 (2)]) that

$$Z_{\mathrm{im}}(s) = \mathsf{DT}_1^{\mathrm{points}}(Y_\sigma, s), \tag{5.11}$$

and Morrison–Nagao proved that

$$\begin{aligned} \mathsf{PT}_1(Y_\sigma; s, T) &= \prod_{1\le a\le b\le N-1} Z_{[a,b]}(s, T_{[a,b]}), \\ \mathsf{DT}_1(Y_\sigma; s, T) &= Z_{\mathrm{im}}(s)\cdot \mathsf{PT}_1(Y_\sigma; s, T). \end{aligned} \tag{5.12}$$

We have

$$\begin{aligned} \{\, \alpha \in \Delta_+ \mid \zeta_{\mathsf{PT}}\cdot\alpha < 0 \,\} &= \Delta_+^{\mathrm{re},-}, \\ \{\, \alpha \in \Delta_+ \mid \zeta_{\mathsf{DT}}\cdot\alpha < 0 \,\} &= \Delta_+^{\mathrm{re},-} \amalg \Delta_+^{\mathrm{im}}, \end{aligned} \tag{5.13}$$

where the definition of the sets in the right hand sides was given in Equation (3.3). For the PT stability condition, we thus obtain

$$\begin{aligned}
\mathsf{PT}_r(Y_\sigma; s, T) &= \frac{A^-_{\zeta_{\mathrm{PT}}}\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)}{A^-_{\zeta_{\mathrm{PT}}}\left(\left(-\mathbb{L}^{-\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)} && \text{by (5.8)}\\
&= \prod_{\alpha\in\Delta_+^{\mathrm{re},-}} \frac{A_\alpha\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)}{A_\alpha\left(\left(-\mathbb{L}^{-\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)} && \text{by (5.6) and (5.13)}\\
&= \prod_{\alpha\in\Delta_+^{\mathrm{re},-}} Z_\alpha^{(r)}\left(s, T_1, \dots, T_{N-1}\right) && \text{by (5.9)}\\
&= \prod_{i=1}^r \prod_{\alpha\in\Delta_+^{\mathrm{re},-}} Z_\alpha^{(1)}\left((-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i} s, T_1, \dots, T_{N-1}\right) && \text{by (5.10)}\\
&= \prod_{i=1}^r \prod_{1\le a\le b\le N-1} Z_{[a,b]}\left((-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i} s, T_{[a,b]}\right) && \text{by (3.3)}\\
&= \prod_{i=1}^r \mathsf{PT}_1\left(Y_\sigma; (-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i} s, T\right), && \text{by (5.12)}
\end{aligned}$$

which proves the first identity in Theorem 1.1.

Similarly,

$$\begin{aligned}
\prod_{\alpha\in\Delta_+^{\mathrm{im}}} \frac{A_\alpha\left(\left(-\mathbb{L}^{\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)}{A_\alpha\left(\left(-\mathbb{L}^{-\frac{1}{2}}\right)^r s, T_1, \dots, T_{N-1}\right)} &= \prod_{\alpha\in\Delta_+^{\mathrm{im}}} Z_\alpha^{(r)}\left(s, T_1, \dots, T_{N-1}\right) && \text{by (5.9)}\\
&= \prod_{i=1}^r Z_{\mathrm{im}}\left((-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i} s\right) && \text{by (5.10)}\\
&= \prod_{i=1}^r \mathsf{DT}_1^{\mathrm{points}}\left(Y_\sigma, (-1)^{r+1}\mathbb{L}^{\frac{-r-1}{2}+i} s\right) && \text{by (5.11)}\\
&= \mathsf{DT}_r^{\mathrm{points}}\left(Y_\sigma, s\right). && \text{by (4.1)}
\end{aligned}$$

In particular, thanks to (5.13), the motivic DT/PT correspondence

$$\mathsf{DT}_r\left(Y_\sigma; s, T\right) = \mathsf{DT}_r^{\mathrm{points}}\left(Y_\sigma, s\right) \cdot \mathsf{PT}_r\left(Y_\sigma; s, T\right)$$

holds. Note that, thanks to Equation (4.3), the right hand side is entirely explicit. Finally, the relation

$$\mathsf{DT}_r(Y_\sigma; s, T) = \prod_{i=1}^r \mathsf{DT}_1\left(Y_\sigma; (-1)^{r+1} s \mathbb{L}^{\frac{-r-1}{2}+i}, T\right)$$

follows from the factorisations of PT_r and $\mathsf{DT}_r^{\text{points}}$ as products of (equally shifted) $r = 1$ pieces, combined with the rank 1 DT/PT correspondence (5.12). The proof of Theorem 1.1 is complete.

Remark 5.8. A motivic DT/PT correspondence was obtained in [8] in the rank 1 case for the motivic contribution of a smooth curve in a 3-fold, refining the corresponding enumerative calculations [23, 24].

Remark 5.9. In the case when Y_σ is the crepant resolution of the conifold singularity, corresponding to $N_0 = N_1 = 1$, the moduli space of framed quiver representation has a clear geometric interpretation for a choice of PT stability condition. Consider the moduli space $\mathcal{P}_\alpha^r(Y_\sigma)$ parametrising Shesmani's highly frozen stable triples [27], whose geometric points consist of framed multi-sections $\mathcal{O}_{Y_\sigma}^{\oplus r} \to F$ with 0-dimensional cokernel, where F is a pure 1-dimensional sheaf F satisfying $\mathrm{ch}_2(F) = (\alpha_0 - \alpha_1)[\mathbb{P}^1]$ and $\chi(F) = \alpha_0$. In [5, Chap. 3] a scheme theoretic isomorphism $\mathfrak{M}_{\zeta_{\mathrm{PT}}}(\widetilde{J}_\sigma, \alpha) \simeq \mathcal{P}_\alpha^r(Y_\sigma)$ is constructed, and it is used to compute a first instance of Formula (1.2). A completely analogous result holds when Y_σ is the resolution of a line of A_2 singularities, corresponding to the case $N_0 = 2, N_1 = 0$ [5, Appendix 3.A]. We leave to future work a full geometric interpretation of the more general moduli spaces of framed quiver representations that we studied in this paper.

ACKNOWLEDGMENTS

We thank Balázs Szendrői for reading a first draft of this paper and for providing us with helpful comments and suggestions. We thank the anonymous referee for suggesting various improvements and corrections. A. C. thanks CNR-IOM for support and the excellent working conditions. A. R. thanks Dipartimenti di Eccellenza for support and SISSA the excellent working conditions.

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ENDNOTE

[1] The set $\widehat{I}_\ell$ is denoted $\widehat{I}_r$ in [15]. We changed the notation to avoid conflict with the number r of framings.

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How to cite this article: Cazzaniga A, Ricolfi AT. Framed motivic Donaldson–Thomas invariants of small crepant resolutions. *Mathematische Nachrichten*. 2022;**295:**1096–1112. https://doi.org/10.1002/mana.202100068