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(Article begins on next page)

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Continuity of some non-local functionals with respect to a convergence of the underlying measures

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Abstract

We study some non-local functionals on the Sobolev space $W^{1,p}_0(\Omega)$ involving a double integral on $\Omega\times\Omega$ with respect to a measure μ. We introduce a suitable notion of convergence of measures on product spaces which implies a stability property in the sense of Γ-convergence of the corresponding functionals.

Keywords: non-local functionals, Γ-convergence, Mosco convergence, graphons, cut norm

AMS Class: 49J45, 46E35, 28A33, 28A35

1 Introduction

In this paper we give a contribution to the study of Γ-limits of non-local integral functional, for which only few results are available in the literature (see for instance [13]). We consider sequences of integrals of the type

$$\int_{\Omega\times\Omega} f(u(x),u(y))d\mu_k(x,y)+\int_\Omega g_k(x,\nabla u(x))\,dx, \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$, with $d\ge 1$. These functionals have a non-local term

$$F_k(u):=\int_{\Omega\times\Omega} f(u(x),u(y))d\mu_k(x,y)$$

depending on a fixed function $f\colon\mathbb{R}\times\mathbb{R}\to[0,+\infty)$ and varying positive bounded measures μ_k on $\Omega\times\Omega$, while the local term

$$G_k(u):=\int_\Omega g_k(x,\nabla u(x))\,dx$$

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depends on a function $g_k\colon \Omega \times \mathbb{R}^d \to [0, +\infty)$. These functionals are defined for u in the Sobolev space $W_0^{1,p}(\Omega)$.

We assume that the functions g_k satisfy usual growth conditions and that the integral functionals G_k Γ-converge in the weak topology in $W_0^{1,p}(\Omega)$ to a functional G of the same form, with integrand g.

We address the question of the stability for functionals in (1.1); more precisely, we focus on a notion of convergence on μ_k such that the functionals $F_k + G_k$ Γ-convergence with respect to the weak topology in $W_0^{1,p}(\Omega)$ to a functional of the form

$$\int_{\Omega\times\Omega} f(u(x), u(y))d\mu(x, y) + \int_\Omega g(x, \nabla u(x))\, dx \tag{1.2}$$

for a limit measure μ. Under some additional assumptions, we also obtain the convergence of $F_k + G_k$ in the sense of Mosco convergence in $W_0^{1,p}(\Omega)$.

If $p = 2$, $g_k(x, \cdot)$ are quadratic forms, and $f(s, t) = |t - s|^2$, then the study of such functionals can be framed within the theory of Dirichlet Forms [8], where the Beurling-Deny formula ensures, under suitable assumptions, that the Γ-limit of $F_k + G_k$ can be represented analogously (see [13]). The extension of that theory is not immediate in a non-quadratic setting or when f is an arbitrary continuous function.

Note that, under suitable growth conditions on f, stability is easily proved under the strong assumption of convergence of μ_k to μ in the space $W^{-1,q}(\Omega\times\Omega)$ dual to $W_0^{1,p}(\Omega\times\Omega)$. However, this result is not satisfactory, since such a space may fail to contain relevant measures μ, depending on the value of p, such as Dirac deltas if $p < 2d$.

We introduce a wider space of measures on $\Omega \times \Omega$, together with a new notion of norm, inspired by a convergence that is used in the theory of *graphons* [5, 11]. The latter can be be seen as limits of Dirichlet forms on dense graphs (for an interpretation in terms of Γ-convergence we refer to [2]).

We prove that, if μ_k are non-negative measures that converge to μ with respect to that 'graphon' norm and $\mu_k(\Omega\times\Omega) \to \mu(\Omega\times\Omega)$, then the functionals defined by (1.1) Γ-converge under the only assumption that f be continuous and a very mild technical assumption (see (5.2)).

We now describe more in detail the content of the paper. In Section 2 we recall some preliminaries on Sobolev functions and introduce the quasicontinuous representative $\widetilde{u}$ of a function $u \in W_0^{1,p}(\Omega)$, which is needed in the precise definition of the functionals (1.1) and (1.2), when μ_k or μ are not absolutely continuous with respect to the Lebesgue measure.

In Section 3 we introduce the space $\mathcal{M}^{1,p}(\Omega \times \Omega)$ of Radon measures on $\Omega \times \Omega$ with finite 'Sobolev cut norm', defined as

$$\|\mu\|_\square := \sup\Big\{\Big|\int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x, y)\Big| : \varphi, \psi \in C_c^\infty(\Omega), \|\varphi\|_{1,p}\,, \|\psi\|_{1,p} \le 1\Big\},$$

where $\|\cdot\|_{1,p}$ is the norm in $W_0^{1,p}(\Omega)$. We prove that, if $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $u, v \in$

$W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω, we have

$$\Big| \int_{\Omega\times\Omega} \widetilde{u}(x)\widetilde{v}(y) d\mu(x,y) \Big| \le \|\mu\|_\square \|u\|_{1,p} \|v\|_{1,p},$$

and the integral does not depend on the choice of the quasicontinuous representatives (see Theorem 3.4).

In Section 4 we prove some continuity results for double integrals. We consider $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega\times\Omega)$ with $\mu_k(\Omega\times\Omega) < +\infty$, $\mu(\Omega\times\Omega) < +\infty$, and

$$\|\mu_k - \mu\|_\square \to 0 \quad \text{and} \quad \mu_k(\Omega\times\Omega) \to \mu(\Omega\times\Omega).$$

If u_k, v_k are sequences in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$ and with $\sup_k(\|u_k\|_\infty + \|v_k\|_\infty) < +\infty$, then we have

$$\lim_{k\to+\infty} \int_{\Omega\times\Omega} f(\widetilde{u}_k(x), \widetilde{v}_k(y)) d\mu_k(x,y) = \int_{\Omega\times\Omega} f(\widetilde{u}(x), \widetilde{v}(y)) d\mu(x,y)$$

for all continuous functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ (see Corollary 4.5). This result is obtained by considering first the special case $f(s,t) = st$ (see Corollary 4.2), from which the result can be obtained for a general polynomial (see Proposition 4.4), and eventually for an arbitrary continuous function by approximation.

Finally, in Section 5 we consider non-negative continuous functions f, and prove the above-mentioned Γ-convergence results (see Theorem 5.1). Note that no convexity assumption on f is needed.

2 Preliminaries on fine properties of Sobolev functions

Throughout the paper we fix $1 < p < +\infty$, $d \ge 1$, and a bounded open subset Ω of $\mathbb{R}^d$. We consider the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{1,p} = \Big(\int_\Omega |\nabla u|^p dx \Big)^{1/p}.$$

Its dual is denoted by $W^{-1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and this norm is denoted by $\|\cdot\|_{-1,q}$.

For all subset $A \subset \Omega$ the *capacity* of in Ω is defined as

$$C_{1,p}(A) := \inf\Big\{ \int_\Omega |\nabla u|^p \, dx : u \in W_0^{1,p}(\Omega),\ u \ge 1 \text{ a.e. in a neighbourhood of } A \Big\}.$$

A function $f\colon \Omega \to \mathbb{R}$ is *quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $C_{1,p}(U) < \varepsilon$ such that the restriction of to $\Omega \setminus U$ is continuous. It is known (see e.g. [10, 7]) that each function $u \in W_0^{1,p}(\Omega)$ has a *Borel quasicontinuous representative*,

which we denote by $\widetilde{u}$, in the sense that $\widetilde{u}$ is Borel measurable and quasicontinuous and $\widetilde{u} = u$ almost everywhere in Ω. Such a representative is unique up to sets of zero capacity.

Let $\mathcal{M}(\Omega)$ denote the space of signed Radon measures on Ω, which can be identified with the dual of $C_c^0(\Omega)$. We say that $\mu \in \mathcal{M}(\Omega)$ belongs to $W^{-1,q}(\Omega)$ if there existe $C \geq 0$ such that

$$\left| \int_\Omega \varphi \, d\mu \right| \leq C \|\varphi\|_{1,p} \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In this case there exists a unique $T_\mu \in W^{-1,q}(\Omega)$ such that

$$\langle T_\mu, \varphi \rangle = \int_\Omega \varphi d\mu \qquad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. In other words μ and T_μ coincide as distributions on Ω. For notational convenience, we shall sometimes directly write μ in the place of T_ν when the distinction between the two is not relevant.

In the following theorem we recall a property of sets of zero $C_{1,p}$-capacity (see [9]) and an integral representation of T_μ that can be deduced from a result of Brezis and Browder [3].

In order to simplify the notation, we introduce the space

$$W_c^{1,p}(\Omega) := \{u \in W_0^{1,p}(\Omega) : u \text{ has compact support in } \Omega\}.$$

Theorem 2.1. *Let $\mu \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$. Then for every $A \subset \Omega$ with $C_{1,p}(A) = 0$-capacity A is $|\mu|$-measurable and*

$$|\mu|(A) = 0 \tag{2.2}$$

If $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$, then

$$\langle T_\mu, u \rangle = \int_\Omega \widetilde{u} d\mu. \tag{2.3}$$

If, in addition, $\mu \geq 0$, then, for all $u \in W_0^{1,p}(\Omega)$, we have $\widetilde{u} \in L^1(\Omega; \mu)$ and (2.3) holds.

3 Sobolev graphons

Graphons are functions ρ defined on $(0,1) \times (0,1)$ introduced to study functionals of the form

$$\int_{(0,1)\times(0,1)} f(u(x), u(y)) \rho(x,y) \, dx \, dy,$$

defined for $u \in L^\infty(0,1)$, especially when $f(u,v) = |u - v|^2$. The functionals above are introduced as a generalization, and in some sense a limit, of energies on dense graphs [5, 11]. To that end, the space of graphons is equipped with the so-called *cut norm*

$$\|\rho\|_\square := \sup_{\varphi,\psi\colon (0,1)\to[0,1]} \left| \int_{(0,1)\times(0,1)} \varphi(x)\psi(y)\rho(x,y) \, dx \, dy \right|. \tag{3.1}$$

We extend the definition of cut norm to arbitrary measures defined on $\Omega \times \Omega$, with Ω in the place of $(0,1)$. For our purpose it is convenient to take test functions in the Sobolev space $W_0^{1,p}(\Omega)$.

Definition 3.1. *The space $\mathcal{M}^{1,p}(\Omega \times \Omega)$ is defined as*

$$\mathcal{M}^{1,p}(\Omega \times \Omega) := \big\{\mu \in \mathcal{M}(\Omega \times \Omega) : \|\mu\|_\square < +\infty\big\}, \tag{3.2}$$

where the Sobolev cut norm *of μ is defined as*

$$\|\mu\|_\square := \sup\Big\{\Big|\int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x,y)\Big| : \varphi,\psi \in C_c^\infty(\Omega), \|\varphi\|_{1,p}, \|\psi\|_{1,p} \le 1\Big\}. \tag{3.3}$$

We let $\mathcal{M}_+^{1,p}(\Omega \times \Omega)$ denote the cone of the positive measures in $\mathcal{M}^{1,p}(\Omega \times \Omega)$.

Remark 3.2. We make some observations on the convergence in the space $\mathcal{M}^{1,p}(\Omega \times \Omega)$, and in particular we compare it with weak* convergence in $\mathcal{M}(\Omega \times \Omega)$ and with the convergence $W^{-1,q}(\Omega\times\Omega)$.

(i) $\|\cdot\|_\square$ defines a norm on $\mathcal{M}^{1,p}(\Omega \times \Omega)$;

(ii) if μ belongs to $W^{-1,q}(\Omega \times \Omega)$, then $\|\mu\|_\square \le \|\mu\|_{-1,q}$. Indeed in such a case the function $\Phi(x,y) = \varphi(x)\psi(y)$ belongs to $W_0^{1,p}(\Omega \times \Omega)$ and

$$\begin{aligned}\int_{\Omega\times\Omega} |\nabla\Phi|^p dx\,dy &= \int_{\Omega\times\Omega} |\psi(y)\nabla\varphi(x) + \varphi(x)\nabla\psi(y)|^p dx\,dy \\ &\le C\int_{\Omega\times\Omega} |\psi(y)|^p|\nabla\varphi(x)|^p + |\varphi(x)|^p|\nabla\psi(y)|^p dx\,dy \le C\end{aligned}$$

so that

$$\Big|\int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x,y)\Big| \le C\|\mu\|_{-1,q};$$

(iii) if $\mu_k, \mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $\|\mu_k - \mu\|_\square \to 0$, then

$$\int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu_k(x,y) \to \int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x,y) \tag{3.4}$$

for all $\varphi,\psi \in C_c^\infty(\Omega)$. If in addition $\sup_k |\mu_k|(\Omega\times\Omega) < +\infty$, then (3.4) and a density argument imply that μ_k converges to μ weakly* in $\mathcal{M}(\Omega \times \Omega)$;

(iv) if μ_k^1 and μ_k^2 are such that μ_k^j converge to μ^j in $W^{-1,q}(\Omega)$, then the measures $\mu_k = \mu_k^1 \otimes \mu_k^2$ converge to $\mu = \mu^1 \otimes \mu^2$;

(v) if $\sup_k \|\mu_k\|_\square < +\infty$ and μ_k converges to some μ weakly* in $\mathcal{M}(\Omega \times \Omega)$, then $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $\|\mu\|_\square \le \liminf_k \|\mu_k\|_\square$;

Note that $\mathcal{M}^{1,p}(\Omega\times\Omega)$ is strictly larger than $\mathcal{M}(\Omega\times\Omega) \cap W^{-1,q}(\Omega\times\Omega)$, and in particular its convergence is weaker than convergence in $W^{-1,q}(\Omega\times\Omega)$. Some examples of this inclusion are given below.

Example 3.3. (i) The first example is simply a Dirac delta $\mu = \delta(x_0, y_0)$ for $x_0, y_0 \in \Omega$. Indeed if $d < p < 2d$, then μ does not belong to $W^{-1,q}(\Omega \times \Omega)$, while

$$\Big|\int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x,y)\Big| = \Big|\varphi(x_0)\psi(y_0)\Big| \le C\|\varphi\|_{1,p}\|\psi\|_{1,p}$$

where the inequality follows from the embedding of $W_0^{1,p}(\Omega)$ into $L^\infty(\Omega)$.

(ii) We can also exhibit an example where μ is absolutely continuous with respect to $\mathcal{L}^d$ with density of the form $m(x)m(y)$. We choose $d = 1$, $p = 2$, $\Omega = (-1, 1)$, and

$$m(x) = \frac{1}{|x|\,\log|x|\,\log|\log x|\,\log^2|\log|\log x||}.$$

Note that $m \in L^1(0, 1)$ so that $\mu \in \mathcal{M}^{1,2}(\Omega \times \Omega)$. If we take

$$w(x,y) = \begin{cases} \Big|\log\big|\log\sqrt{x^2+y^2}\big|\Big| & \text{if } \sqrt{x^2+y^2} < 1/e, \\ 0 & \text{otherwise,} \end{cases}$$

then $w \in W_0^{1,2}((-1, 1)^2)$ but $\int_{(-1,1)^2} w\, d\mu = +\infty$. By Theorem 2.1, this shows that $\mu \notin W^{-1,2}((-1, 1)^2)$.

Theorem 3.4. *Let $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$. Then for every $A \subset \Omega$ with $C_{1,p}(A) = 0$ the sets $A \times \Omega$ and $\Omega \times A$ are $|\mu|$-measurable and*

$$|\mu|(A \times \Omega) = |\mu|(\Omega \times A) = 0. \tag{3.5}$$

Moreover, for all $u, v \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have

$$\Big|\int_{\Omega\times\Omega} \widetilde{u}(x)\widetilde{v}(y)d\mu(x,y)\Big| \le \|\mu\|_\square\|u\|_{1,p}\|v\|_{1,p}\,. \tag{3.6}$$

Proof. Fix $\psi \in C_c^\infty(\Omega)$, and let $\mu_\psi \in \mathcal{M}(\Omega)$ be defined by

$$\mu_\psi(B) = \int_{B\times\Omega} \psi(y)d\mu(x,y) \text{ for all Borel sets } B \subset\subset \Omega. \tag{3.7}$$

Since

$$\int_\Omega \varphi(x)d\mu_\psi(x) = \int_{\Omega\times\Omega} \varphi(x)\psi(y)d\mu(x,y) \text{ for all } \varphi \in C_c^\infty(\Omega),$$

by the definition of $\|\mu\|_\square$ we have

$$\Big|\int_\Omega \varphi(x)d\mu_\psi(x)\Big| \le \|\mu\|_\square\|\psi\|_{1,p}\|\varphi\|_{1,p} \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{3.8}$$

Hence, $\mu_\psi \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$.

Let $B \subset \Omega$ be a Borel set with $C_{1,p}(B) = 0$. Let Ω' be an open set with $\Omega' \subset\subset \Omega$, and let ψ_k be a non-decreasing sequence of non-negative functions in $C_c^\infty(\Omega')$ converging to the constant 1 in Ω. By the definition of μ_{ψ_k} and Theorem 2.1, we have

$$\int_{B\times\Omega} \psi_k(y) d\mu(x,y) = \mu_{\psi_k}(B) = 0.$$

Letting $k \to +\infty$ we deduce $\mu(B \times \Omega') = 0$. Since this holds for all Borel subsets of B we deduce that $|\mu|(B \times \Omega') = 0$. By the arbitrariness of $\Omega' \subset\subset \Omega$ we finally obtain $|\mu|(B \times \Omega) = 0$. For a general $A \subset \Omega$ with $C_{1,p}(A) = 0$ it is sufficient to observe that there exists a Borel set B such that $A \subset B \subset \Omega$ and $C_{1,p}(B) = 0$. This implies that $A \times \Omega$ and is $|\mu|$-measurable and $|\mu|(A \times \Omega) = 0$. A similar argument proves the same result for $\Omega \times A$.

Let $T_{\mu_\psi} \in W^{-1,q}(\Omega)$ be defined as in (2.1) with μ replaced by μ_ψ. Fix $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, by Theorem 2.1 and the definition of μ_ψ, we have

$$\langle T_{\mu_\psi}, u\rangle = \int_\Omega \widetilde{u}(x) d\mu_\psi(x) = \int_{\Omega\times\Omega} \widetilde{u}(x)\psi(y) d\mu(x,y).$$

By (3.8) we have $\|T_{\mu_\psi}\|_{-1,q} \le \|\mu\|_\square \|\psi\|_{1,p}$, so that the previous equality gives

$$\Big|\int_{\Omega\times\Omega} \widetilde{u}(x)\psi(y) d\mu(x,y)\Big| \le \|\mu\|_\square \|\psi\|_{1,p}\|u\|_{1,p} \tag{3.9}$$

for all $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ and every $\psi \in C_c^\infty(\Omega)$.

Fix $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $\mu^u \in \mathcal{M}(\Omega)$ be defined by

$$\mu^u(B) := \int_{\Omega\times B} \widetilde{u}(x) d\mu(x,y) \text{ for all Borel sets } B \subset\subset \Omega. \tag{3.10}$$

Since

$$\int_\Omega \psi(y) d\mu^u(y) = \int_{\Omega\times\Omega} \widetilde{u}(x)\psi(y) d\mu(x,y) \text{ for all } \psi \in C_c^\infty(\Omega),$$

thanks to (3.9) we then have

$$\Big|\int_\Omega \psi(y) d\mu^u(y)\Big| \le \|\mu\|_\square \|\psi\|_{1,p}\|u\|_{1,p} \text{ for all } \psi \in C_c^\infty(\Omega).$$

Hence, $\mu^u \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$, and, using the notation above, we obtain

$$\|T_{\mu^u}\|_{-1,q} \le \|\mu\|_\square \|u\|_{1,p}\,.$$

By Theorem 2.1 and the definition of μ^u, we then have

$$\langle T_{\mu^u}, v\rangle = \int_\Omega \widetilde{v}(y) d\mu^u(y) = \int_{\Omega\times\Omega} \widetilde{u}(x)\widetilde{v}(y) d\mu(x,y).$$

Together with the previous inequality this gives

$$\Big|\int_{\Omega\times\Omega}\widetilde{u}(x)\widetilde{v}(y)d\mu(x,y)\Big| \le \|\mu\|_{\square}\|u\|_{1,p}\|v\|_{1,p}\,,$$

which concludes the proof of (3.6). □

4 Continuity properties of some double integrals

In this section we find conditions on f, u_k, v_k, u, v, μ_k, and μ which imply the convergence

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega} f(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega\times\Omega} f(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y).$$

We begin with the case $f(s,t) = st$. Subsequently, we consider the case when f is a polynomial, and finally an arbitrary continuous function by approximation.

Lemma 4.1. *Let $\mu \in \mathcal{M}^{1,p}_+(\Omega\times\Omega)$ and let u_k, v_k be sequences in $W^{1,p}_c(\Omega)\cap L^\infty(\Omega)$ converging to u, v weakly in $W^{1,p}(\Omega)$. Assume that there exist a compact set $K\subset\Omega$ and a constant M such that*

$$\operatorname{supp}(u_k)\cup\operatorname{supp}(v_k)\subset K \text{ and } \|u_k\|_\infty + \|v_k\|_\infty \le M, \tag{4.1}$$

where $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$ Then

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega}\widetilde{u}_k(x)\widetilde{v}_k(y)d\mu(x,y) = \int_{\Omega\times\Omega}\widetilde{u}(x)\widetilde{v}(y)d\mu(x,y). \tag{4.2}$$

Proof. We write

$$\begin{aligned}&\int_{\Omega\times\Omega}\widetilde{u}_k(x)\widetilde{v}_k(y)d\mu(x,y) - \int_{\Omega\times\Omega}\widetilde{u}(x)\widetilde{v}(y)d\mu(x,y)\\ =\ &\int_{\Omega\times\Omega}(\widetilde{u}_k(x)-\widetilde{u}(x))\widetilde{v}_k(y)d\mu(x,y) + \int_{\Omega\times\Omega}\widetilde{u}(x)(\widetilde{v}_k(y)-\widetilde{v}(y))d\mu(x,y).\end{aligned} \tag{4.3}$$

The first integral in (4.3) is estimated by

$$\begin{aligned}\Big|\int_{\Omega\times\Omega}(\widetilde{u}_k(x)-\widetilde{u}(x))\widetilde{v}_k(y)d\mu(x,y)\Big| &\le M\int_{\Omega\times\Omega}|\widetilde{u}_k(x)-\widetilde{u}(x)|\psi(y)d\mu(x,y)\\ &= M\langle T_{\mu_\psi}, |u_k-u|\rangle = o(1)\end{aligned} \tag{4.4}$$

as $k\to+\infty$, where ψ is any function in $C^\infty_c(\Omega)$ with $0\le\psi\le 1$ in Ω and $\psi = 1$ on K, and μ_ψ is defined in (3.7), and the convergence to 0 follows from the fact that $|u_k-u| \rightharpoonup 0$ weakly in $W^{1,p}_0(\Omega)$. Moreover, if μ^u is defined in (3.10), we have

$$\int_{\Omega\times\Omega}\widetilde{u}(x)(\widetilde{v}_k(y)-\widetilde{v}(y))d\mu(x,y) = \langle T_{\mu^u}, v_k - v\rangle = o(1) \tag{4.5}$$

as $k\to+\infty$. The convergence to 0 follows from the fact that $v_k - v \rightharpoonup 0$ weakly in $W^{1,p}_0(\Omega)$. □

Corollary 4.2. *Under the same hypotheses of Lemma* 4.1*, let* $\mu_k \in \mathcal{M}^{1,p}(\Omega\times\Omega)$ *with* $\|\mu_k-\mu\|_\square \to 0$*. Then*

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega}\widetilde{u}_k(x)\widetilde{v}_k(y)d\mu_k(x,y)=\int_{\Omega\times\Omega}\widetilde{u}(x)\widetilde{v}(y)d\mu(x,y). \tag{4.6}$$

Proof. It suffices to remark that

$$\Big|\int_{\Omega\times\Omega}\widetilde{u}_k(x)\widetilde{v}_k(y)d\mu_k(x,y)-\int_{\Omega\times\Omega}\widetilde{u}_k(x)\widetilde{v}_k(y)d\mu(x,y)\Big|\le\|\mu_k-\mu\|_\square\|u_k\|_{1,p}\|v_k\|_{1,p}$$

and that the right-hand side tends to 0. The conclusion then follows from Lemma 4.1. □

Proposition 4.3. *Let* $P\colon\mathbb{R}^2\to\mathbb{R}$ *be a polynomial function and let* $\varphi,\psi\in C_c^\infty(\Omega)$*. Let* $\mu_k,\mu\in\mathcal{M}^{1,p}_+(\Omega\times\Omega)$ *with* $\|\mu_k-\mu\|_\square\to 0$*, let* u_k,v_k *be sequences in* $W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$ *converging to* u,v *weakly in* $W^{1,p}_0(\Omega)$ *and equibounded in* $L^\infty(\Omega)$*. Then*

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega}\varphi(x)\psi(y)P(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y)=\int_{\Omega\times\Omega}\varphi(x)\psi(y)P(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y). \tag{4.7}$$

Proof. It is sufficient to consider $P(s,t)=s^mt^n$ for some non-negative integers m,n. We define $w_k(x)=\varphi(x)u_k(x)^m$ and $z_k(y)=\psi(y)v_k(y)^n$. Observe that w_k,z_k satisfy the hypotheses of Lemma 4.1, weakly converging in $W^{1,p}_0(\Omega)$ to w,z given by $w(x)=\varphi(x)u(x)^m$ and $z(y)=\psi(y)v(y)^n$. Since

$$\varphi(x)\psi(y)P(\widetilde{u}_k(x),\widetilde{v}_k(y))=\widetilde{w}_k(x)\,\widetilde{z}_k(y),\qquad\varphi(x)\psi(y)P(\widetilde{u}(x),\widetilde{v}(y))=\widetilde{w}(x)\,\widetilde{z}(y),$$

the claim then follows by Corollary 4.2. □

In the next proposition we consider a stronger condition on the convergence of μ_k to μ which allows to avoid the multiplication by the cut-off functions in the previous proposition. Note that the second condition in (4.8) is necessary for the validity of (4.9) when P is a constant.

Proposition 4.4. *Let* $P\colon\mathbb{R}^2\to\mathbb{R}$ *be a polynomial function. Let* $\mu_k,\mu\in\mathcal{M}^{1,p}_+(\Omega\times\Omega)$ *with* $\mu_k(\Omega\times\Omega)<+\infty$ *and* $\mu(\Omega\times\Omega)<+\infty$*. Suppose that*

$$\|\mu_k-\mu\|_\square\to 0\quad\text{ and }\quad\mu_k(\Omega\times\Omega)\to\mu(\Omega\times\Omega). \tag{4.8}$$

Let u_k,v_k *be sequences in* $W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$ *converging to* u,v *weakly in* $W^{1,p}_0(\Omega)$ *and with* $\sup_k(\|u_k\|_\infty+\|v_k\|_\infty)=M<+\infty$*. Then*

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega}P(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y)=\int_{\Omega\times\Omega}P(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y). \tag{4.9}$$

Proof. By Remark 3.2(iii) the first condition in (4.8) implies that $\mu_k \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega\times\Omega)$. This, together with the second condition in (4.8), gives $\mu_k(B) \to \mu(B)$ for all Borel sets in $\Omega \times \Omega$ such that $\mu(\partial B \cap (\Omega \times \Omega)) = 0$. Then, for every $\varepsilon > 0$ there exists a compact set K_ε of Ω such that

$$\mu((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \quad \text{and} \quad \mu_k((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \text{ for every } k. \tag{4.10}$$

Let $\varphi_\varepsilon \in C_c^\infty(\Omega)$ with $0 \le \varphi_\varepsilon \le 1$ in Ω and $\varphi_\varepsilon = 1$ on K_ε, and let

$$C_M = \max\{P(s,t) : s,t \in [-M,M]\}.$$

With this choice

$$\Big|\int_{\Omega\times\Omega} P(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y) - \int_{\Omega\times\Omega} \varphi_\varepsilon(x)\varphi_\varepsilon(y)P(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y)\Big| \le C_M\varepsilon$$

and

$$\Big|\int_{\Omega\times\Omega} P(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y) - \int_{\Omega\times\Omega} \varphi_\varepsilon(x)\varphi_\varepsilon(y)P(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y)\Big| \le C_M\varepsilon.$$

By (4.7) with $\varphi = \psi = \varphi_\varepsilon$ we then deduce

$$\limsup_{k\to+\infty}\Big|\int_{\Omega\times\Omega} P(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y) - \int_{\Omega\times\Omega} P(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y)\Big| \le 2C_M\varepsilon,$$

from which we obtain (4.9) by the arbitrariness of ε. □

We are now ready to prove the result for an arbitrary continuous function f.

Corollary 4.5. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Under the assumptions of Proposition* 4.4 *we have*

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega} f(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega\times\Omega} f(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y). \tag{4.11}$$

Proof. It suffices to approximate uniformly f by polynomials on $[-M,M]^2$ and apply the previous proposition. □

Finally, if f is bounded we can remove the hypothesis that u_k and v_k are bounded in L^∞.

Theorem 4.6. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded continuous function. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega\times\Omega)$ with $\mu_k(\Omega\times\Omega) < +\infty$ and $\mu(\Omega\times\Omega) < +\infty$. Suppose that*

$$\|\mu_k - \mu\|_\square \to 0 \quad \text{and} \quad \mu_k(\Omega\times\Omega) \to \mu(\Omega\times\Omega). \tag{4.12}$$

Let u_k, v_k be sequences in $W_0^{1,p}(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$, then

$$\lim_{k\to+\infty}\int_{\Omega\times\Omega} f(\widetilde{u}_k(x),\widetilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega\times\Omega} f(\widetilde{u}(x),\widetilde{v}(y))d\mu(x,y). \tag{4.13}$$

Proof. For all $\lambda > 0$ we define the truncation operator

$$\tau^\lambda(s) := (s \vee (-\lambda)) \wedge \lambda. \tag{4.14}$$

With fixed $\lambda > 1$ we set $u_k^\lambda(x) := \tau^\lambda(u_k(x))$ and $v_k^\lambda(y) := \tau^\lambda(v_k(y))$, and correspondingly, $u^\lambda(x) := \tau^\lambda(u(x))$ and $v^\lambda(y) := \tau^\lambda(v(y))$. By the uniqueness of the quasicontinuous representatives, we deduce from (3.5) that $\widetilde{u}_k^\lambda(x) = \tau^\lambda(\widetilde{u}_k(x))$ and $\widetilde{v}_k^\lambda(x) = \tau^\lambda(\widetilde{v}_k(x))$ for μ_k-almost every $(x,y) \in \Omega\times\Omega$. Similarly, we have $\widetilde{u}^\lambda(x) = \tau^\lambda(\widetilde{u}(x))$ and $\widetilde{v}^\lambda(x) = \tau^\lambda(\widetilde{v}(x))$ for μ-almost every $(x,y) \in \Omega\times\Omega$.

Since $u_k^\lambda \rharpoonup u^\lambda$ and $v_k^\lambda \rharpoonup v^\lambda$ weakly in $W_0^{1,p}(\Omega)$, we have

$$\lim_{k\to+\infty} \int_{\Omega\times\Omega} f(\tau^\lambda(\widetilde{u}_k(x)), \tau^\lambda(\widetilde{v}_k(y)))d\mu_k(x,y) = \int_{\Omega\times\Omega} f(\tau^\lambda(\widetilde{u}(x)), \tau^\lambda(\widetilde{v}(y)))d\mu(x,y) \tag{4.15}$$

by Corollary 4.5. To conclude the proof of the result is suffices to estimate

$$A_k^\lambda := \left| \int_{\Omega\times\Omega} f(\tau^\lambda(\widetilde{u}_k(x)), \tau^\lambda(\widetilde{v}_k(y)))d\mu_k(x,y) - \int_{\Omega\times\Omega} f(\widetilde{u}_k(x), \widetilde{v}_k(y))d\mu_k(x,y) \right|, \tag{4.16}$$

$$A^\lambda := \left| \int_{\Omega\times\Omega} f(\tau^\lambda(\widetilde{u}(x)), \tau^\lambda(\widetilde{v}(y)))d\mu(x,y) - \int_{\Omega\times\Omega} f(\widetilde{u}(x), \widetilde{v}(y))d\mu(x,y) \right|. \tag{4.17}$$

Let $M_0 = \sup|f|$. Since

$$A_k^\lambda \le 2M_0\, \mu_k(\{(x,y) \in \Omega\times\Omega : (\widetilde{u}_k(x), \widetilde{v}_k(y)) \notin [-\lambda,\lambda]^2\}),$$

it is enough to separately estimate

$$\mu_k(\{(x,y) \in \Omega\times\Omega : \widetilde{u}_k(x) > \lambda\}), \qquad \mu_k(\{(x,y) \in \Omega\times\Omega : \widetilde{u}_k(x) < -\lambda\}), \tag{4.18}$$

$$\mu_k(\{(x,y) \in \Omega\times\Omega : \widetilde{v}_k(y) > \lambda\}), \qquad \mu_k(\{(x,y) \in \Omega\times\Omega : \widetilde{v}_k(y) < -\lambda\}). \tag{4.19}$$

For fixed $\varepsilon > 0$ let K_ε be the compact sets introduced at the beginning of the proof of Proposition 4.4. By (4.10) we have

$$\mu_k(\{(x,y) \in \Omega\times\Omega : \widetilde{u}_k(x) > \lambda\}) \le \mu_k(\{(x,y) \in K_\varepsilon\times K_\varepsilon : \widetilde{u}_k(x) > \lambda\}) + \varepsilon. \tag{4.20}$$

Let $\psi_\varepsilon \in C_c^\infty(\Omega)$ with $0 \le \psi_\varepsilon \le 1$ and $\psi_\varepsilon = 1$ on K_ε. Let T_ε be the element in $W^{-1,q}(\Omega)$ defined by

$$\langle T_\varepsilon, v\rangle := \int_{\Omega\times\Omega} \widetilde{v}(x)\psi_\varepsilon(y)d\mu(x,y) \text{ for every } v \in W_0^{1,p}(\Omega).$$

Since u_k are equibounded in $W_0^{1,p}(\Omega)$, by (3.6) there exists $C_\varepsilon > 0$ and $k_\varepsilon \in \mathbb{N}$ such that

$$\mu_k(\{(x,y) \in K_\varepsilon\times K_\varepsilon : \widetilde{u}_k(x) > \lambda\}) \le \int_{\Omega\times\Omega} [\widetilde{u}_k(x) - \lambda + 1]^+\psi_\varepsilon(x)\psi_\varepsilon(y)d\mu_k(x,y)$$

$$\le \left| \int_{\Omega\times\Omega} [\widetilde{u}_k(x) - \lambda + 1]^+\psi_\varepsilon(x)\psi_\varepsilon(y)d(\mu_k - \mu)(x,y) \right| + \int_{\Omega\times\Omega} [\widetilde{u}_k(x) - \lambda + 1]^+\psi_\varepsilon(y)d\mu(x,y)$$

$$\le C_\varepsilon\|\mu_k - \mu\|_\square + \langle T_{\psi_\varepsilon}, [u_k - \lambda + 1]^+\rangle = \varepsilon + \langle T_{\psi_\varepsilon}, [u - \lambda + 1]^+]\rangle$$

for all $k \geq k_\varepsilon$. Now, since $\lim_{\lambda\to+\infty} \langle T_{\psi_\varepsilon}, [u-\lambda+1]^+]\rangle = 0$, we can choose $\lambda_\varepsilon > 0$ such that

$$\mu_k(\{(x,y) \in K_\varepsilon \times K_\varepsilon : \widetilde{u}_k(x) > \lambda\}) \leq 2\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. By (4.20) this in turn gives

$$\mu_k(\{(x,y) \in \Omega \times \Omega : \widetilde{u}_k(x) > \lambda\}) \leq 3\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$.

In the same way, we can prove analogue estimates for the other measures in (4.18) and (4.19), which we may assume to hold for the same K_ε and λ_ε, and conclude that $A_k^\lambda \leq 24M_0\varepsilon$ for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. Similarly we can prove that $A^\lambda \leq 24M_0\varepsilon$ for $\lambda \geq \lambda_\varepsilon$. From these estimates, by (4.15)–(4.17) the claim follows by the arbitrariness of ε. □

We finally prove a lower bound for limits of double integrals.

Corollary 4.7. *Let $f\colon \mathbb{R}^2 \to [0,+\infty)$ be a continuous function. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega\times\Omega)$ with $\mu_k(\Omega\times\Omega) < +\infty$, $\mu(\Omega\times\Omega) < +\infty$, satisfying (4.12). Let u_k, v_k be sequences in $W_0^{1,p}(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$. Then*

$$\liminf_{k\to+\infty} \int_{\Omega\times\Omega} f(\widetilde{u}_k(x), \widetilde{v}_k(y))d\mu_k(x,y) \geq \int_{\Omega\times\Omega} f(\widetilde{u}(x), \widetilde{v}(y))d\mu(x,y). \tag{4.21}$$

Proof. For every $\lambda > 0$ let $f^\lambda := \tau^\lambda(f)$, where τ^λ is the truncation operator as in (4.14). By Theorem 4.6 we then have

$$\begin{aligned}\liminf_{k\to+\infty} \int_{\Omega\times\Omega} f(\widetilde{u}_k(x), \widetilde{v}_k(y))d\mu_k(x,y) &\geq \lim_{k\to+\infty} \int_{\Omega\times\Omega} f^\lambda(\widetilde{u}_k(x), \widetilde{v}_k(y))d\mu_k(x,y)\\ &= \int_{\Omega\times\Omega} f^\lambda(\widetilde{u}(x), \widetilde{v}(y))d\mu(x,y).\end{aligned}$$

We then conclude by letting $\lambda \to +\infty$. □

5 Γ-convergence

In this final section we shall prove the Γ-convergence and the Mosco convergence of sequences of functionals as in (1.1).

Let $f\colon \mathbb{R}^2 \to [0,+\infty)$ be a continuous function, and let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega\times\Omega)$. We define $F_k, F\colon W_0^{1,p}(\Omega) \to [0,+\infty]$ by

$$F_k(u) := \int_{\Omega\times\Omega} f(\widetilde{u}(x), \widetilde{u}(y))d\mu_k(x,y) \quad \text{and} \quad F(u) := \int_{\Omega\times\Omega} f(\widetilde{u}(x), \widetilde{u}(y))d\mu(x,y). \tag{5.1}$$

We assume that f satisfies the following condition: there exists an unbounded set $\Lambda \subset [0,+\infty)$ and two constants $a, b \geq 0$ such that

$$f(\tau^\lambda(s), \tau^\lambda(t)) \leq a\, f(s,t) + b \quad \text{for all } s,t \in \mathbb{R} \text{ and for all } \lambda \in \Lambda, \tag{5.2}$$

where τ^λ is the truncation operator defined in (4.14). Note that this condition is valid if f is bounded (with $a = 0$) or when f is decreasing by truncations (with $a = 1$ and $b = 0$).

Let $g_k, g\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be Carathéodory functions satisfying the growth conditions

$$c_0|\xi|^p \leq g_k(x,\xi) \leq c_1|\xi|^p + a(x), \quad c_0|\xi|^p \leq g(x,\xi) \leq c_1|\xi|^p + a(x) \ \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^d, \tag{5.3}$$

for some constants $c_0, c_1 > 0$ and some function $a \in L^1(\Omega)$. Let $G_k, G\colon W^{1,p}_0(\Omega) \to [0,+\infty)$ be defined by

$$G_k(u) := \int_\Omega g_k(x, \nabla u)dx \quad \text{and} \quad G(u) := \int_\Omega g(x, \nabla u)dx. \tag{5.4}$$

Theorem 5.1. *Let $\mu_k, \mu \in \mathcal{M}^{1,p}_+(\Omega \times \Omega)$ with $\mu_k(\Omega\times\Omega) < +\infty$ and $\mu(\Omega\times\Omega) < +\infty$. Let F_k, F be defined as in (5.1) with $f\colon \mathbb{R}^2 \to [0,+\infty)$ a continuous function satisfying (5.2). Let G_k, G be defined as in (5.4) with $g_k, g\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ Carathéodory functions satisfying (5.3). Suppose that*

$$\|\mu_k - \mu\|_\square \to 0 \quad \text{and} \quad \mu_k(\Omega\times\Omega) \to \mu(\Omega\times\Omega), \tag{5.5}$$

$$G = \Gamma\text{-}\lim_{k\to+\infty} G_k \text{ with respect to the weak convergence in } W^{1,p}_0(\Omega). \tag{5.6}$$

Then $F + G = \Gamma\text{-}\lim_{k\to+\infty}(F_k + G_k)$ with respect to the weak convergence in $W^{1,p}_0(\Omega)$.

Proof. By [6, Proposition 8.10] we have to prove that for all $u \in W^{1,p}_0(\Omega)$ the following properties hold:

(i) for all u_k converging to u weakly in $W^{1,p}_0(\Omega)$ we have

$$F(u) + G(u) \leq \liminf_{k\to+\infty}(F_k(u_k) + G_k(u_k));$$

(ii) there exist u_k converging to u weakly in $W^{1,p}_0(\Omega)$ such that

$$F(u) + G(u) = \lim_{k\to+\infty}(F_k(u_k) + G_k(u_k)).$$

Claim (i) follows from Corollary 4.7 with $v_k = u_k$ and from the liminf inequality for G_k, which follows from (5.6).

If $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, we deduce from (5.6) that there exists a sequence u_k converging to u weakly in $W^{1,p}_0(\Omega)$ such that $G(u_k) \to G(u)$ and $\|u_k\|_\infty \leq M$ for some constant M and

for all k (see for instance [4, Proposition 2.5]). Setting $\lambda = \max\{f(s,t) : |s| \le M, |t| \le M\}$, we may apply Theorem 4.6 with $v_k = u_k$ and $\tau^\lambda(f)$ in the place of f, obtaining (ii).

Let now $u \in W_0^{1,p}(\Omega)$. If $F(u) = +\infty$, then claim (ii) follows from claim (i). Suppose then that $F(u) < +\infty$. By the validity of claim (ii) for $\tau^\lambda(u)$ we obtain

$$F(\tau^\lambda(u)) + G(\tau^\lambda(u)) = \Big(\Gamma\text{-}\limsup_{k\to+\infty}(F_k + G_k)\Big)(\tau^\lambda(u)). \tag{5.7}$$

By (5.3) we have $G(\tau^\lambda(u)) \to G(u)$ as $\lambda \to +\infty$. Since $F(u) < +\infty$, by (5.2) and the Dominated Convergence Theorem we have $F(\tau^\lambda(u)) \to F(u)$ as $\lambda \to +\infty$ with $\lambda \in \Lambda$. By (5.7), using the lower semicontinuity of the Γ-limsup we get

$$F(u) + G(u) \ge \Big(\Gamma\text{-}\limsup_{k\to+\infty}(F_k + G_k)\Big)(u).$$

By [6, Proposition 8.10] we then obtain an inequality in claim (ii). The proof is completed by using claim (i). □

Remark 5.2 (Mosco convergence)**.** If the functionals G_k converge to G in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$; that is, for all $u \in W_0^{1,p}(\Omega)$ we have

(i) for all u_k converging to u weakly in $W_0^{1,p}(\Omega)$ we have

$$G(u) \le \liminf_{k\to+\infty} G_k(u_k);$$

(ii) there exist u_k converging to u strongly in $W_0^{1,p}(\Omega)$ such that

$$G(u) = \lim_{k\to+\infty} G_k(u_k),$$

then also $F_k + G_k$ converges in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$.

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