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Continuity of some non-local functionals with respect to a convergence of the underlying measures

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Abstract

We study some non-local functionals on the Sobolev space $W_0^{1,p}(\Omega)$ involving a double integral on $\Omega \times \Omega$ with respect to a measure μ . We introduce a suitable notion of convergence of measures on product spaces which implies a stability property in the sense of Γ -convergence of the corresponding functionals.

Keywords: non-local functionals, Γ -convergence, Mosco convergence, graphons, cut norm

AMS Class: 49J45, 46E35, 28A33, 28A35

1 Introduction

In this paper we give a contribution to the study of Γ -limits of non-local integral functional, for which only few results are available in the literature (see for instance [13]). We consider sequences of integrals of the type

$$\int_{\Omega \times \Omega} f(u(x), u(y)) d\mu_k(x, y) + \int_{\Omega} g_k(x, \nabla u(x)) dx, \quad (1.1)$$

where Ω is a bounded open subset of \mathbb{R}^d , with $d \geq 1$. These functionals have a non-local term

$$F_k(u) := \int_{\Omega \times \Omega} f(u(x), u(y)) d\mu_k(x, y)$$

depending on a fixed function $f: \mathbb{R} \times \mathbb{R} \rightarrow [0, +\infty)$ and varying positive bounded measures μ_k on $\Omega \times \Omega$, while the local term

$$G_k(u) := \int_{\Omega} g_k(x, \nabla u(x)) dx$$

depends on a function $g_k: \Omega \times \mathbb{R}^d \rightarrow [0, +\infty)$. These functionals are defined for u in the Sobolev space $W_0^{1,p}(\Omega)$.

We assume that the functions g_k satisfy usual growth conditions and that the integral functionals G_k Γ -converge in the weak topology in $W_0^{1,p}(\Omega)$ to a functional G of the same form, with integrand g .

We address the question of the stability for functionals in (1.1); more precisely, we focus on a notion of convergence on μ_k such that the functionals $F_k + G_k$ Γ -converge with respect to the weak topology in $W_0^{1,p}(\Omega)$ to a functional of the form

$$\int_{\Omega \times \Omega} f(u(x), u(y)) d\mu(x, y) + \int_{\Omega} g(x, \nabla u(x)) dx \quad (1.2)$$

for a limit measure μ . Under some additional assumptions, we also obtain the convergence of $F_k + G_k$ in the sense of Mosco convergence in $W_0^{1,p}(\Omega)$.

If $p = 2$, $g_k(x, \cdot)$ are quadratic forms, and $f(s, t) = |t - s|^2$, then the study of such functionals can be framed within the theory of Dirichlet Forms [8], where the Beurling-Deny formula ensures, under suitable assumptions, that the Γ -limit of $F_k + G_k$ can be represented analogously (see [13]). The extension of that theory is not immediate in a non-quadratic setting or when f is an arbitrary continuous function.

Note that, under suitable growth conditions on f , stability is easily proved under the strong assumption of convergence of μ_k to μ in the space $W^{-1,q}(\Omega \times \Omega)$ dual to $W_0^{1,p}(\Omega \times \Omega)$. However, this result is not satisfactory, since such a space may fail to contain relevant measures μ , depending on the value of p , such as Dirac deltas if $p < 2d$.

We introduce a wider space of measures on $\Omega \times \Omega$, together with a new notion of norm, inspired by a convergence that is used in the theory of *graphons* [5, 11]. The latter can be seen as limits of Dirichlet forms on dense graphs (for an interpretation in terms of Γ -convergence we refer to [2]).

We prove that, if μ_k are non-negative measures that converge to μ with respect to that ‘graphon’ norm and $\mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega)$, then the functionals defined by (1.1) Γ -converge under the only assumption that f be continuous and a very mild technical assumption (see (5.2)).

We now describe more in detail the content of the paper. In Section 2 we recall some preliminaries on Sobolev functions and introduce the quasicontinuous representative \tilde{u} of a function $u \in W_0^{1,p}(\Omega)$, which is needed in the precise definition of the functionals (1.1) and (1.2), when μ_k or μ are not absolutely continuous with respect to the Lebesgue measure.

In Section 3 we introduce the space $\mathcal{M}^{1,p}(\Omega \times \Omega)$ of Radon measures on $\Omega \times \Omega$ with finite ‘Sobolev cut norm’, defined as

$$\|\mu\|_{\square} := \sup \left\{ \left| \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \right| : \varphi, \psi \in C_c^\infty(\Omega), \|\varphi\|_{1,p}, \|\psi\|_{1,p} \leq 1 \right\},$$

where $\|\cdot\|_{1,p}$ is the norm in $W_0^{1,p}(\Omega)$. We prove that, if $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $u, v \in$

$W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω , we have

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y) \right| \leq \|\mu\|_{\square} \|u\|_{1,p} \|v\|_{1,p},$$

and the integral does not depend on the choice of the quasicontinuous representatives (see Theorem 3.4).

In Section 4 we prove some continuity results for double integrals. We consider $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$, $\mu(\Omega \times \Omega) < +\infty$, and

$$\|\mu_k - \mu\|_{\square} \rightarrow 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega).$$

If u_k, v_k are sequences in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$ and with $\sup_k (\|u_k\|_\infty + \|v_k\|_\infty) < +\infty$, then we have

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y)$$

for all continuous functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ (see Corollary 4.5). This result is obtained by considering first the special case $f(s, t) = st$ (see Corollary 4.2), from which the result can be obtained for a general polynomial (see Proposition 4.4), and eventually for an arbitrary continuous function by approximation.

Finally, in Section 5 we consider non-negative continuous functions f , and prove the above-mentioned Γ -convergence results (see Theorem 5.1). Note that no convexity assumption on f is needed.

2 Preliminaries on fine properties of Sobolev functions

Throughout the paper we fix $1 < p < +\infty$, $d \geq 1$, and a bounded open subset Ω of \mathbb{R}^d . We consider the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{1,p} = \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}.$$

Its dual is denoted by $W^{-1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and this norm is denoted by $\|\cdot\|_{-1,q}$.

For all subset $A \subset \Omega$ the *capacity* of in Ω is defined as

$$C_{1,p}(A) := \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ a.e. in a neighbourhood of } A \right\}.$$

A function $f: \Omega \rightarrow \mathbb{R}$ is *quasicontinuous* if for every $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $C_{1,p}(U) < \varepsilon$ such that the restriction of f to $\Omega \setminus U$ is continuous. It is known (see e.g. [10, 7]) that each function $u \in W_0^{1,p}(\Omega)$ has a *Borel quasicontinuous representative*,

which we denote by \tilde{u} , in the sense that \tilde{u} is Borel measurable and quasicontinuous and $\tilde{u} = u$ almost everywhere in Ω . Such a representative is unique up to sets of zero capacity.

Let $\mathcal{M}(\Omega)$ denote the space of signed Radon measures on Ω , which can be identified with the dual of $C_c^0(\Omega)$. We say that $\mu \in \mathcal{M}(\Omega)$ belongs to $W^{-1,q}(\Omega)$ if there exists $C \geq 0$ such that

$$\left| \int_{\Omega} \varphi d\mu \right| \leq C \|\varphi\|_{1,p} \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In this case there exists a unique $T_\mu \in W^{-1,q}(\Omega)$ such that

$$\langle T_\mu, \varphi \rangle = \int_{\Omega} \varphi d\mu \quad \text{for all } \varphi \in C_c^\infty(\Omega), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. In other words μ and T_μ coincide as distributions on Ω . For notational convenience, we shall sometimes directly write μ in the place of T_μ when the distinction between the two is not relevant.

In the following theorem we recall a property of sets of zero $C_{1,p}$ -capacity (see [9]) and an integral representation of T_μ that can be deduced from a result of Brezis and Browder [3].

In order to simplify the notation, we introduce the space

$$W_c^{1,p}(\Omega) := \{u \in W_0^{1,p}(\Omega) : u \text{ has compact support in } \Omega\}.$$

Theorem 2.1. *Let $\mu \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$. Then for every $A \subset \Omega$ with $C_{1,p}(A) = 0$ -capacity A is $|\mu|$ -measurable and*

$$|\mu|(A) = 0 \quad (2.2)$$

If $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$, then

$$\langle T_\mu, u \rangle = \int_{\Omega} \tilde{u} d\mu. \quad (2.3)$$

If, in addition, $\mu \geq 0$, then, for all $u \in W_0^{1,p}(\Omega)$, we have $\tilde{u} \in L^1(\Omega; \mu)$ and (2.3) holds.

3 Sobolev graphons

Graphons are functions ρ defined on $(0,1) \times (0,1)$ introduced to study functionals of the form

$$\int_{(0,1) \times (0,1)} f(u(x), u(y)) \rho(x, y) dx dy,$$

defined for $u \in L^\infty(0,1)$, especially when $f(u, v) = |u - v|^2$. The functionals above are introduced as a generalization, and in some sense a limit, of energies on dense graphs [5, 11]. To that end, the space of graphons is equipped with the so-called *cut norm*

$$\|\rho\|_{\square} := \sup_{\varphi, \psi: (0,1) \rightarrow [0,1]} \left| \int_{(0,1) \times (0,1)} \varphi(x) \psi(y) \rho(x, y) dx dy \right|. \quad (3.1)$$

We extend the definition of cut norm to arbitrary measures defined on $\Omega \times \Omega$, with Ω in the place of $(0, 1)$. For our purpose it is convenient to take test functions in the Sobolev space $W_0^{1,p}(\Omega)$.

Definition 3.1. *The space $\mathcal{M}^{1,p}(\Omega \times \Omega)$ is defined as*

$$\mathcal{M}^{1,p}(\Omega \times \Omega) := \{\mu \in \mathcal{M}(\Omega \times \Omega) : \|\mu\|_{\square} < +\infty\}, \quad (3.2)$$

where the Sobolev cut norm of μ is defined as

$$\|\mu\|_{\square} := \sup \left\{ \left| \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \right| : \varphi, \psi \in C_c^{\infty}(\Omega), \|\varphi\|_{1,p}, \|\psi\|_{1,p} \leq 1 \right\}. \quad (3.3)$$

We let $\mathcal{M}_+^{1,p}(\Omega \times \Omega)$ denote the cone of the positive measures in $\mathcal{M}^{1,p}(\Omega \times \Omega)$.

Remark 3.2. We make some observations on the convergence in the space $\mathcal{M}^{1,p}(\Omega \times \Omega)$, and in particular we compare it with weak* convergence in $\mathcal{M}(\Omega \times \Omega)$ and with the convergence $W^{-1,q}(\Omega \times \Omega)$.

- (i) $\|\cdot\|_{\square}$ defines a norm on $\mathcal{M}^{1,p}(\Omega \times \Omega)$;
- (ii) if μ belongs to $W^{-1,q}(\Omega \times \Omega)$, then $\|\mu\|_{\square} \leq \|\mu\|_{-1,q}$. Indeed in such a case the function $\Phi(x, y) = \varphi(x)\psi(y)$ belongs to $W_0^{1,p}(\Omega \times \Omega)$ and

$$\begin{aligned} \int_{\Omega \times \Omega} |\nabla \Phi|^p dx dy &= \int_{\Omega \times \Omega} |\psi(y) \nabla \varphi(x) + \varphi(x) \nabla \psi(y)|^p dx dy \\ &\leq C \int_{\Omega \times \Omega} |\psi(y)|^p |\nabla \varphi(x)|^p + |\varphi(x)|^p |\nabla \psi(y)|^p dx dy \leq C \end{aligned}$$

so that

$$\left| \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \right| \leq C \|\mu\|_{-1,q};$$

- (iii) if $\mu_k, \mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $\|\mu_k - \mu\|_{\square} \rightarrow 0$, then

$$\int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu_k(x, y) \rightarrow \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \quad (3.4)$$

for all $\varphi, \psi \in C_c^{\infty}(\Omega)$. If in addition $\sup_k |\mu_k|(\Omega \times \Omega) < +\infty$, then (3.4) and a density argument imply that μ_k converges to μ weakly* in $\mathcal{M}(\Omega \times \Omega)$;

- (iv) if μ_k^1 and μ_k^2 are such that μ_k^j converge to μ^j in $W^{-1,q}(\Omega)$, then the measures $\mu_k = \mu_k^1 \otimes \mu_k^2$ converge to $\mu = \mu^1 \otimes \mu^2$;

- (v) if $\sup_k \|\mu_k\|_{\square} < +\infty$ and μ_k converges to some μ weakly* in $\mathcal{M}(\Omega \times \Omega)$, then $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $\|\mu\|_{\square} \leq \liminf_k \|\mu_k\|_{\square}$;

Note that $\mathcal{M}^{1,p}(\Omega \times \Omega)$ is strictly larger than $\mathcal{M}(\Omega \times \Omega) \cap W^{-1,q}(\Omega \times \Omega)$, and in particular its convergence is weaker than convergence in $W^{-1,q}(\Omega \times \Omega)$. Some examples of this inclusion are given below.

Example 3.3. (i) The first example is simply a Dirac delta $\mu = \delta(x_0, y_0)$ for $x_0, y_0 \in \Omega$. Indeed if $d < p < 2d$, then μ does not belong to $W^{-1,q}(\Omega \times \Omega)$, while

$$\left| \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \right| = \left| \varphi(x_0) \psi(y_0) \right| \leq C \|\varphi\|_{1,p} \|\psi\|_{1,p}$$

where the inequality follows from the embedding of $W_0^{1,p}(\Omega)$ into $L^\infty(\Omega)$.

(ii) We can also exhibit an example where μ is absolutely continuous with respect to \mathcal{L}^d with density of the form $m(x)m(y)$. We choose $d = 1$, $p = 2$, $\Omega = (-1, 1)$, and

$$m(x) = \frac{1}{|x| \log |x| \log |\log x| \log^2 |\log |\log x||}.$$

Note that $m \in L^1(0, 1)$ so that $\mu \in \mathcal{M}^{1,2}(\Omega \times \Omega)$. If we take

$$w(x, y) = \begin{cases} \left| \log |\log \sqrt{x^2 + y^2}| \right| & \text{if } \sqrt{x^2 + y^2} < 1/e, \\ 0 & \text{otherwise,} \end{cases}$$

then $w \in W_0^{1,2}((-1, 1)^2)$ but $\int_{(-1, 1)^2} w d\mu = +\infty$. By Theorem 2.1, this shows that $\mu \notin W^{-1,2}((-1, 1)^2)$.

Theorem 3.4. *Let $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$. Then for every $A \subset \Omega$ with $C_{1,p}(A) = 0$ the sets $A \times \Omega$ and $\Omega \times A$ are $|\mu|$ -measurable and*

$$|\mu|(A \times \Omega) = |\mu|(\Omega \times A) = 0. \quad (3.5)$$

Moreover, for all $u, v \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y) \right| \leq \|\mu\|_{\square} \|u\|_{1,p} \|v\|_{1,p}. \quad (3.6)$$

Proof. Fix $\psi \in C_c^\infty(\Omega)$, and let $\mu_\psi \in \mathcal{M}(\Omega)$ be defined by

$$\mu_\psi(B) = \int_{B \times \Omega} \psi(y) d\mu(x, y) \text{ for all Borel sets } B \subset \subset \Omega. \quad (3.7)$$

Since

$$\int_{\Omega} \varphi(x) d\mu_\psi(x) = \int_{\Omega \times \Omega} \varphi(x) \psi(y) d\mu(x, y) \text{ for all } \varphi \in C_c^\infty(\Omega),$$

by the definition of $\|\mu\|_{\square}$ we have

$$\left| \int_{\Omega} \varphi(x) d\mu_\psi(x) \right| \leq \|\mu\|_{\square} \|\psi\|_{1,p} \|\varphi\|_{1,p} \text{ for all } \varphi \in C_c^\infty(\Omega). \quad (3.8)$$

Hence, $\mu_\psi \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$.

Let $B \subset \Omega$ be a Borel set with $C_{1,p}(B) = 0$. Let Ω' be an open set with $\Omega' \subset \subset \Omega$, and let ψ_k be a non-decreasing sequence of non-negative functions in $C_c^\infty(\Omega')$ converging to the constant 1 in Ω . By the definition of μ_{ψ_k} and Theorem 2.1, we have

$$\int_{B \times \Omega} \psi_k(y) d\mu(x, y) = \mu_{\psi_k}(B) = 0.$$

Letting $k \rightarrow +\infty$ we deduce $\mu(B \times \Omega') = 0$. Since this holds for all Borel subsets of B we deduce that $|\mu|(B \times \Omega') = 0$. By the arbitrariness of $\Omega' \subset \subset \Omega$ we finally obtain $|\mu|(B \times \Omega) = 0$. For a general $A \subset \Omega$ with $C_{1,p}(A) = 0$ it is sufficient to observe that there exists a Borel set B such that $A \subset B \subset \Omega$ and $C_{1,p}(B) = 0$. This implies that $A \times \Omega$ and is $|\mu|$ -measurable and $|\mu|(A \times \Omega) = 0$. A similar argument proves the same result for $\Omega \times A$.

Let $T_{\mu_\psi} \in W^{-1,q}(\Omega)$ be defined as in (2.1) with μ replaced by μ_ψ . Fix $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, by Theorem 2.1 and the definition of μ_ψ , we have

$$\langle T_{\mu_\psi}, u \rangle = \int_{\Omega} \tilde{u}(x) d\mu_\psi(x) = \int_{\Omega \times \Omega} \tilde{u}(x) \psi(y) d\mu(x, y).$$

By (3.8) we have $\|T_{\mu_\psi}\|_{-1,q} \leq \|\mu\|_{\square} \|\psi\|_{1,p}$, so that the previous equality gives

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x) \psi(y) d\mu(x, y) \right| \leq \|\mu\|_{\square} \|\psi\|_{1,p} \|u\|_{1,p} \quad (3.9)$$

for all $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ and every $\psi \in C_c^\infty(\Omega)$.

Fix $u \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $\mu^u \in \mathcal{M}(\Omega)$ be defined by

$$\mu^u(B) := \int_{\Omega \times B} \tilde{u}(x) d\mu(x, y) \text{ for all Borel sets } B \subset \subset \Omega. \quad (3.10)$$

Since

$$\int_{\Omega} \psi(y) d\mu^u(y) = \int_{\Omega \times \Omega} \tilde{u}(x) \psi(y) d\mu(x, y) \text{ for all } \psi \in C_c^\infty(\Omega),$$

thanks to (3.9) we then have

$$\left| \int_{\Omega} \psi(y) d\mu^u(y) \right| \leq \|\mu\|_{\square} \|\psi\|_{1,p} \|u\|_{1,p} \text{ for all } \psi \in C_c^\infty(\Omega).$$

Hence, $\mu^u \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$, and, using the notation above, we obtain

$$\|T_{\mu^u}\|_{-1,q} \leq \|\mu\|_{\square} \|u\|_{1,p}.$$

By Theorem 2.1 and the definition of μ^u , we then have

$$\langle T_{\mu^u}, v \rangle = \int_{\Omega} \tilde{v}(y) d\mu^u(y) = \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y).$$

Together with the previous inequality this gives

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y) \right| \leq \|\mu\|_{\square} \|u\|_{1,p} \|v\|_{1,p},$$

which concludes the proof of (3.6). \square

4 Continuity properties of some double integrals

In this section we find conditions on f, u_k, v_k, u, v, μ_k , and μ which imply the convergence

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y).$$

We begin with the case $f(s, t) = st$. Subsequently, we consider the case when f is a polynomial, and finally an arbitrary continuous function by approximation.

Lemma 4.1. *Let $\mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ and let u_k, v_k be sequences in $W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to u, v weakly in $W^{1,p}(\Omega)$. Assume that there exist a compact set $K \subset \Omega$ and a constant M such that*

$$\text{supp}(u_k) \cup \text{supp}(v_k) \subset K \text{ and } \|u_k\|_\infty + \|v_k\|_\infty \leq M, \quad (4.1)$$

where $\|\cdot\|_\infty$ denotes the norm in $L^\infty(\Omega)$. Then

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} \tilde{u}_k(x) \tilde{v}_k(y) d\mu(x, y) = \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y). \quad (4.2)$$

Proof. We write

$$\begin{aligned} & \int_{\Omega \times \Omega} \tilde{u}_k(x) \tilde{v}_k(y) d\mu(x, y) - \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y) \\ &= \int_{\Omega \times \Omega} (\tilde{u}_k(x) - \tilde{u}(x)) \tilde{v}_k(y) d\mu(x, y) + \int_{\Omega \times \Omega} \tilde{u}(x) (\tilde{v}_k(y) - \tilde{v}(y)) d\mu(x, y). \end{aligned} \quad (4.3)$$

The first integral in (4.3) is estimated by

$$\begin{aligned} \left| \int_{\Omega \times \Omega} (\tilde{u}_k(x) - \tilde{u}(x)) \tilde{v}_k(y) d\mu(x, y) \right| &\leq M \int_{\Omega \times \Omega} |\tilde{u}_k(x) - \tilde{u}(x)| \psi(y) d\mu(x, y) \\ &= M \langle T_{\mu_\psi}, |u_k - u| \rangle = o(1) \end{aligned} \quad (4.4)$$

as $k \rightarrow +\infty$, where ψ is any function in $C_c^\infty(\Omega)$ with $0 \leq \psi \leq 1$ in Ω and $\psi = 1$ on K , and μ_ψ is defined in (3.7), and the convergence to 0 follows from the fact that $|u_k - u| \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$. Moreover, if μ^u is defined in (3.10), we have

$$\int_{\Omega \times \Omega} \tilde{u}(x) (\tilde{v}_k(y) - \tilde{v}(y)) d\mu(x, y) = \langle T_{\mu^u}, v_k - v \rangle = o(1) \quad (4.5)$$

as $k \rightarrow +\infty$. The convergence to 0 follows from the fact that $v_k - v \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$. \square

Corollary 4.2. *Under the same hypotheses of Lemma 4.1, let $\mu_k \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ with $\|\mu_k - \mu\|_{\square} \rightarrow 0$. Then*

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} \tilde{u}_k(x) \tilde{v}_k(y) d\mu_k(x, y) = \int_{\Omega \times \Omega} \tilde{u}(x) \tilde{v}(y) d\mu(x, y). \quad (4.6)$$

Proof. It suffices to remark that

$$\left| \int_{\Omega \times \Omega} \tilde{u}_k(x) \tilde{v}_k(y) d\mu_k(x, y) - \int_{\Omega \times \Omega} \tilde{u}_k(x) \tilde{v}_k(y) d\mu(x, y) \right| \leq \|\mu_k - \mu\|_{\square} \|u_k\|_{1,p} \|v_k\|_{1,p}$$

and that the right-hand side tends to 0. The conclusion then follows from Lemma 4.1. \square

Proposition 4.3. *Let $P: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a polynomial function and let $\varphi, \psi \in C_c^\infty(\Omega)$. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\|\mu_k - \mu\|_{\square} \rightarrow 0$, let u_k, v_k be sequences in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$ and equibounded in $L^\infty(\Omega)$. Then*

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} \varphi(x) \psi(y) P(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} \varphi(x) \psi(y) P(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \quad (4.7)$$

Proof. It is sufficient to consider $P(s, t) = s^m t^n$ for some non-negative integers m, n . We define $w_k(x) = \varphi(x) u_k(x)^m$ and $z_k(y) = \psi(y) v_k(y)^n$. Observe that w_k, z_k satisfy the hypotheses of Lemma 4.1, weakly converging in $W_0^{1,p}(\Omega)$ to w, z given by $w(x) = \varphi(x) u(x)^m$ and $z(y) = \psi(y) v(y)^n$. Since

$$\varphi(x) \psi(y) P(\tilde{u}_k(x), \tilde{v}_k(y)) = \tilde{w}_k(x) \tilde{z}_k(y), \quad \varphi(x) \psi(y) P(\tilde{u}(x), \tilde{v}(y)) = \tilde{w}(x) \tilde{z}(y),$$

the claim then follows by Corollary 4.2. \square

In the next proposition we consider a stronger condition on the convergence of μ_k to μ which allows to avoid the multiplication by the cut-off functions in the previous proposition. Note that the second condition in (4.8) is necessary for the validity of (4.9) when P is a constant.

Proposition 4.4. *Let $P: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a polynomial function. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$ and $\mu(\Omega \times \Omega) < +\infty$. Suppose that*

$$\|\mu_k - \mu\|_{\square} \rightarrow 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega). \quad (4.8)$$

Let u_k, v_k be sequences in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$ and with $\sup_k (\|u_k\|_\infty + \|v_k\|_\infty) = M < +\infty$. Then

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} P(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} P(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \quad (4.9)$$

Proof. By Remark 3.2(iii) the first condition in (4.8) implies that $\mu_k \rightharpoonup \mu$ weakly* in $\mathcal{M}(\Omega \times \Omega)$. This, together with the second condition in (4.8), gives $\mu_k(B) \rightarrow \mu(B)$ for all Borel sets in $\Omega \times \Omega$ such that $\mu(\partial B \cap (\Omega \times \Omega)) = 0$. Then, for every $\varepsilon > 0$ there exists a compact set K_ε of Ω such that

$$\mu((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \quad \text{and} \quad \mu_k((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \quad \text{for every } k. \quad (4.10)$$

Let $\varphi_\varepsilon \in C_c^\infty(\Omega)$ with $0 \leq \varphi_\varepsilon \leq 1$ in Ω and $\varphi_\varepsilon = 1$ on K_ε , and let

$$C_M = \max\{P(s, t) : s, t \in [-M, M]\}.$$

With this choice

$$\left| \int_{\Omega \times \Omega} P(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) - \int_{\Omega \times \Omega} \varphi_\varepsilon(x) \varphi_\varepsilon(y) P(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) \right| \leq C_M \varepsilon$$

and

$$\left| \int_{\Omega \times \Omega} P(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y) - \int_{\Omega \times \Omega} \varphi_\varepsilon(x) \varphi_\varepsilon(y) P(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y) \right| \leq C_M \varepsilon.$$

By (4.7) with $\varphi = \psi = \varphi_\varepsilon$ we then deduce

$$\limsup_{k \rightarrow +\infty} \left| \int_{\Omega \times \Omega} P(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) - \int_{\Omega \times \Omega} P(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y) \right| \leq 2C_M \varepsilon,$$

from which we obtain (4.9) by the arbitrariness of ε . \square

We are now ready to prove the result for an arbitrary continuous function f .

Corollary 4.5. *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function. Under the assumptions of Proposition 4.4 we have*

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \quad (4.11)$$

Proof. It suffices to approximate uniformly f by polynomials on $[-M, M]^2$ and apply the previous proposition. \square

Finally, if f is bounded we can remove the hypothesis that u_k and v_k are bounded in L^∞ .

Theorem 4.6. *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded continuous function. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$ and $\mu(\Omega \times \Omega) < +\infty$. Suppose that*

$$\|\mu_k - \mu\|_\square \rightarrow 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega). \quad (4.12)$$

Let u_k, v_k be sequences in $W_0^{1,p}(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$, then

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \quad (4.13)$$

Proof. For all $\lambda > 0$ we define the truncation operator

$$\tau^\lambda(s) := (s \vee (-\lambda)) \wedge \lambda. \quad (4.14)$$

With fixed $\lambda > 1$ we set $u_k^\lambda(x) := \tau^\lambda(u_k(x))$ and $v_k^\lambda(y) := \tau^\lambda(v_k(y))$, and correspondingly, $u^\lambda(x) := \tau^\lambda(u(x))$ and $v^\lambda(y) := \tau^\lambda(v(y))$. By the uniqueness of the quasicontinuous representatives, we deduce from (3.5) that $\tilde{u}_k^\lambda(x) = \tau^\lambda(\tilde{u}_k(x))$ and $\tilde{v}_k^\lambda(x) = \tau^\lambda(\tilde{v}_k(x))$ for μ_k -almost every $(x, y) \in \Omega \times \Omega$. Similarly, we have $\tilde{u}^\lambda(x) = \tau^\lambda(\tilde{u}(x))$ and $\tilde{v}^\lambda(x) = \tau^\lambda(\tilde{v}(x))$ for μ -almost every $(x, y) \in \Omega \times \Omega$.

Since $u_k^\lambda \rightharpoonup u^\lambda$ and $v_k^\lambda \rightharpoonup v^\lambda$ weakly in $W_0^{1,p}(\Omega)$, we have

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}_k(x)), \tau^\lambda(\tilde{v}_k(y))) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}(x)), \tau^\lambda(\tilde{v}(y))) d\mu(x, y) \quad (4.15)$$

by Corollary 4.5. To conclude the proof of the result it suffices to estimate

$$A_k^\lambda := \left| \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}_k(x)), \tau^\lambda(\tilde{v}_k(y))) d\mu_k(x, y) - \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) \right|, \quad (4.16)$$

$$A^\lambda := \left| \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}(x)), \tau^\lambda(\tilde{v}(y))) d\mu(x, y) - \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y) \right|. \quad (4.17)$$

Let $M_0 = \sup |f|$. Since

$$A_k^\lambda \leq 2M_0 \mu_k(\{(x, y) \in \Omega \times \Omega : (\tilde{u}_k(x), \tilde{v}_k(y)) \notin [-\lambda, \lambda]^2\}),$$

it is enough to separately estimate

$$\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) > \lambda\}), \quad \mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) < -\lambda\}), \quad (4.18)$$

$$\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{v}_k(y) > \lambda\}), \quad \mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{v}_k(y) < -\lambda\}). \quad (4.19)$$

For fixed $\varepsilon > 0$ let K_ε be the compact sets introduced at the beginning of the proof of Proposition 4.4. By (4.10) we have

$$\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) > \lambda\}) \leq \mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \tilde{u}_k(x) > \lambda\}) + \varepsilon. \quad (4.20)$$

Let $\psi_\varepsilon \in C_c^\infty(\Omega)$ with $0 \leq \psi_\varepsilon \leq 1$ and $\psi_\varepsilon = 1$ on K_ε . Let T_ε be the element in $W^{-1,q}(\Omega)$ defined by

$$\langle T_\varepsilon, v \rangle := \int_{\Omega \times \Omega} \tilde{v}(x) \psi_\varepsilon(y) d\mu(x, y) \text{ for every } v \in W_0^{1,p}(\Omega).$$

Since u_k are equibounded in $W_0^{1,p}(\Omega)$, by (3.6) there exists $C_\varepsilon > 0$ and $k_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned} \mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \tilde{u}_k(x) > \lambda\}) &\leq \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi_\varepsilon(x) \psi_\varepsilon(y) d\mu_k(x, y) \\ &\leq \left| \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi_\varepsilon(x) \psi_\varepsilon(y) d(\mu_k - \mu)(x, y) \right| + \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi_\varepsilon(y) d\mu(x, y) \\ &\leq C_\varepsilon \|\mu_k - \mu\|_\square + \langle T_{\psi_\varepsilon}, [\tilde{u}_k - \lambda + 1]^+ \rangle = \varepsilon + \langle T_{\psi_\varepsilon}, [u - \lambda + 1]^+ \rangle \end{aligned}$$

for all $k \geq k_\varepsilon$. Now, since $\lim_{\lambda \rightarrow +\infty} \langle T_{\psi_\varepsilon}, [u - \lambda + 1]^+ \rangle = 0$, we can choose $\lambda_\varepsilon > 0$ such that

$$\mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \tilde{u}_k(x) > \lambda\}) \leq 2\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. By (4.20) this in turn gives

$$\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) > \lambda\}) \leq 3\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$.

In the same way, we can prove analogue estimates for the other measures in (4.18) and (4.19), which we may assume to hold for the same K_ε and λ_ε , and conclude that $A_k^\lambda \leq 24M_0\varepsilon$ for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. Similarly we can prove that $A^\lambda \leq 24M_0\varepsilon$ for $\lambda \geq \lambda_\varepsilon$. From these estimates, by (4.15)–(4.17) the claim follows by the arbitrariness of ε . \square

We finally prove a lower bound for limits of double integrals.

Corollary 4.7. *Let $f: \mathbb{R}^2 \rightarrow [0, +\infty)$ be a continuous function. Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$, $\mu(\Omega \times \Omega) < +\infty$, satisfying (4.12). Let u_k, v_k be sequences in $W_0^{1,p}(\Omega)$ converging to u, v weakly in $W_0^{1,p}(\Omega)$. Then*

$$\liminf_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) \geq \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \quad (4.21)$$

Proof. For every $\lambda > 0$ let $f^\lambda := \tau^\lambda(f)$, where τ^λ is the truncation operator as in (4.14). By Theorem 4.6 we then have

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) &\geq \lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} f^\lambda(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) \\ &= \int_{\Omega \times \Omega} f^\lambda(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y). \end{aligned}$$

We then conclude by letting $\lambda \rightarrow +\infty$. \square

5 Γ -convergence

In this final section we shall prove the Γ -convergence and the Mosco convergence of sequences of functionals as in (1.1).

Let $f: \mathbb{R}^2 \rightarrow [0, +\infty)$ be a continuous function, and let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$. We define $F_k, F: W_0^{1,p}(\Omega) \rightarrow [0, +\infty]$ by

$$F_k(u) := \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{u}(y)) d\mu_k(x, y) \quad \text{and} \quad F(u) := \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{u}(y)) d\mu(x, y). \quad (5.1)$$

We assume that f satisfies the following condition: there exists an unbounded set $\Lambda \subset [0, +\infty)$ and two constants $a, b \geq 0$ such that

$$f(\tau^\lambda(s), \tau^\lambda(t)) \leq a f(s, t) + b \quad \text{for all } s, t \in \mathbb{R} \text{ and for all } \lambda \in \Lambda, \quad (5.2)$$

where τ^λ is the truncation operator defined in (4.14). Note that this condition is valid if f is bounded (with $a = 0$) or when f is decreasing by truncations (with $a = 1$ and $b = 0$).

Let $g_k, g: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ be Carathéodory functions satisfying the growth conditions

$$c_0 |\xi|^p \leq g_k(x, \xi) \leq c_1 |\xi|^p + a(x), \quad c_0 |\xi|^p \leq g(x, \xi) \leq c_1 |\xi|^p + a(x) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^d, \quad (5.3)$$

for some constants $c_0, c_1 > 0$ and some function $a \in L^1(\Omega)$. Let $G_k, G: W_0^{1,p}(\Omega) \rightarrow [0, +\infty)$ be defined by

$$G_k(u) := \int_{\Omega} g_k(x, \nabla u) dx \quad \text{and} \quad G(u) := \int_{\Omega} g(x, \nabla u) dx. \quad (5.4)$$

Theorem 5.1. *Let $\mu_k, \mu \in \mathcal{M}_+^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$ and $\mu(\Omega \times \Omega) < +\infty$. Let F_k, F be defined as in (5.1) with $f: \mathbb{R}^2 \rightarrow [0, +\infty)$ a continuous function satisfying (5.2). Let G_k, G be defined as in (5.4) with $g_k, g: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ Carathéodory functions satisfying (5.3). Suppose that*

$$\|\mu_k - \mu\|_{\square} \rightarrow 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega), \quad (5.5)$$

$$G = \Gamma\text{-}\lim_{k \rightarrow +\infty} G_k \text{ with respect to the weak convergence in } W_0^{1,p}(\Omega). \quad (5.6)$$

Then $F + G = \Gamma\text{-}\lim_{k \rightarrow +\infty} (F_k + G_k)$ with respect to the weak convergence in $W_0^{1,p}(\Omega)$.

Proof. By [6, Proposition 8.10] we have to prove that for all $u \in W_0^{1,p}(\Omega)$ the following properties hold:

(i) for all u_k converging to u weakly in $W_0^{1,p}(\Omega)$ we have

$$F(u) + G(u) \leq \liminf_{k \rightarrow +\infty} (F_k(u_k) + G_k(u_k));$$

(ii) there exist u_k converging to u weakly in $W_0^{1,p}(\Omega)$ such that

$$F(u) + G(u) = \lim_{k \rightarrow +\infty} (F_k(u_k) + G_k(u_k)).$$

Claim (i) follows from Corollary 4.7 with $v_k = u_k$ and from the liminf inequality for G_k , which follows from (5.6).

If $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we deduce from (5.6) that there exists a sequence u_k converging to u weakly in $W_0^{1,p}(\Omega)$ such that $G(u_k) \rightarrow G(u)$ and $\|u_k\|_\infty \leq M$ for some constant M and

for all k (see for instance [4, Proposition 2.5]). Setting $\lambda = \max\{f(s, t) : |s| \leq M, |t| \leq M\}$, we may apply Theorem 4.6 with $v_k = u_k$ and $\tau^\lambda(f)$ in the place of f , obtaining (ii).

Let now $u \in W_0^{1,p}(\Omega)$. If $F(u) = +\infty$, then claim (ii) follows from claim (i). Suppose then that $F(u) < +\infty$. By the validity of claim (ii) for $\tau^\lambda(u)$ we obtain

$$F(\tau^\lambda(u)) + G(\tau^\lambda(u)) = \left(\Gamma\text{-}\limsup_{k \rightarrow +\infty} (F_k + G_k) \right)(\tau^\lambda(u)). \quad (5.7)$$

By (5.3) we have $G(\tau^\lambda(u)) \rightarrow G(u)$ as $\lambda \rightarrow +\infty$. Since $F(u) < +\infty$, by (5.2) and the Dominated Convergence Theorem we have $F(\tau^\lambda(u)) \rightarrow F(u)$ as $\lambda \rightarrow +\infty$ with $\lambda \in \Lambda$. By (5.7), using the lower semicontinuity of the Γ -limsup we get

$$F(u) + G(u) \geq \left(\Gamma\text{-}\limsup_{k \rightarrow +\infty} (F_k + G_k) \right)(u).$$

By [6, Proposition 8.10] we then obtain an inequality in claim (ii). The proof is completed by using claim (i). \square

Remark 5.2 (Mosco convergence). If the functionals G_k converge to G in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$; that is, for all $u \in W_0^{1,p}(\Omega)$ we have

(i) for all u_k converging to u weakly in $W_0^{1,p}(\Omega)$ we have

$$G(u) \leq \liminf_{k \rightarrow +\infty} G_k(u_k);$$

(ii) there exist u_k converging to u strongly in $W_0^{1,p}(\Omega)$ such that

$$G(u) = \lim_{k \rightarrow +\infty} G_k(u_k),$$

then also $F_k + G_k$ converges in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$.

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