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On the spectrum of critical almost Mathieu operators in the rational case

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Abstract. We derive a new Chambers-type formula and prove sharper upper bounds on the measure of the spectrum of critical almost Mathieu operators with rational frequencies.

Dedicated to the memory of M. A. Shubin

1. Introduction

The Harper operator, also know as "discrete magnetic Laplacian," (the name "discrete magnetic Laplacian" was first introduced by M. Shubin in [19]) is a tight-binding model of an electron confined to a 2D square lattice in a uniform magnetic field orthogonal to the lattice plane and with flux $2\pi\alpha$ through an elementary cell. It acts on $\ell^2(\mathbb{Z}^2)$ and is usually given in the Landau gauge representation

$$(H(\alpha)\psi)_{m,n} = \psi_{m,n-1} + \psi_{m,n+1} + e^{-i2\pi\alpha n}\psi_{m-1,n} + e^{i2\pi\alpha n}\psi_{m+1,n}, \quad (1)$$

first considered by Peierls [18], who noticed that it makes the Hamiltonian separable and turns it into the direct integral in θ of operators on $\ell^2(\mathbb{Z})$ given by

$$(H_{\alpha,\theta}\varphi)(n) = \varphi(n-1) + \varphi(n+1) + 2\cos 2\pi(\alpha n + \theta)\varphi(n), \quad \alpha, \theta \in [0,1). \quad (2)$$

In physics literature, it also appears as "Harper's model" or "Azbel–Hofstadter model," with both names used also for the discrete magnetic Laplacian $H(\alpha)$. In mathematics, it is universally called "critical almost Mathieu operator." (This name was originally introduced by Barry Simon [20].) In addition to its importance in physics, this model is of special interest, being at the boundary of two reasonably well understood regimes – (almost) localization and (almost) reducibility – and not being amenable to methods of either side. Recently, there has been some progress in the study of the fine structure of its spectrum [7–10, 14, 16].

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Denote the spectrum of an operator H, as a set, by $\sigma(H)$. An important object is the union of $\sigma(H_{\alpha,\theta})$ over θ, which coincides with the spectrum of $H(\alpha)$. We denote it by

$$S(\alpha) := \sigma(H(\alpha)) = \bigcup_{\theta\in[0,1)} \sigma(H_{\alpha,\theta}).$$

Note that, by the general theory of ergodic operators, if α is irrational, then $\sigma(H_{\alpha,\theta})$ is independent of θ. We denote the Lebesgue measure of a set A by $|A|$.

For irrational α, the Lebesgue measure $|S(\alpha)| = 0$, see [2, 9, 15]. For rational $\alpha = \frac{p_0}{q_0}$, where p_0, q_0 are coprime positive integers, Last obtained the bounds [15, Lemma 1]

$$\frac{2(\sqrt{5}+1)}{q_0} < \left|S\left(\frac{p_0}{q_0}\right)\right| < \frac{8e}{q_0}, \tag{3}$$

where $e = \exp(1) = 2.71\ldots$. The measure of the spectrum is subject to a conjecture of Thouless [21,22]: that, in the limit $p_n/q_n \to \alpha$, we have $q_n|S(p_n/q_n)| \to c$, where $c = 32C_c/\pi = 9.32\ldots$, $C_c = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}$ being the Catalan constant. Thouless provided a partly heuristic argument in the case $p_n = 1, q_n \to \infty$. A rigorous proof for $\alpha = 0$ and $p_n = 1$ or $p_n = 2$, q_n odd was given in [6].

The purpose of this note is to present a sharper upper bound, for all $\alpha \in \mathbb{Q}$:

Theorem 1. *For all positive coprime integers p_0 and q_0,*

$$\left|S\left(\frac{p_0}{q_0}\right)\right| \le \frac{4\pi}{q_0}.$$

Thus, the upper bound is reduced from $8e = 21.74\ldots$ to $4\pi = 12.56\ldots$. The way we prove Theorem 1 is very different from that of [15]; we use the chiral gauge representation [9] and Lidskii's inequalities. The chiral gauge representation of the almost Mathieu operator also leads to a new type of Chambers' relation (equations (14) and (15) below).

2. Proof of Theorem 1

Consider the operator on $\ell^2(\mathbb{Z})$

$$(\widetilde{H}_{\alpha,\theta}\varphi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\varphi(n-1) + 2\sin 2\pi(\alpha n+\theta)\varphi(n+1), \tag{4}$$

with $\alpha, \theta \in [0, 1)$, and define

$$\widetilde{S}(\alpha) := \bigcup_{\theta\in[0,1)} \sigma(\widetilde{H}_{\alpha,\theta}).$$

It was shown in [9, Theorem 3.1] that the two copies of the operator

$$M_{2\alpha} := \bigoplus_{\theta\in[0,1)} H_{2\alpha,\theta}$$

and the operator

$$\widetilde{M}_\alpha := \bigoplus_{\theta\in[0,1)} \widetilde{H}_{\alpha,\theta}$$

are unitarily equivalent, so that $S(\alpha) = \widetilde{S}(\alpha/2)$. (Note that $\sigma(H_{2\alpha,\theta}) \neq \sigma(\widetilde{H}_{\alpha,\theta})$, in general.) See also related partly non-rigorous considerations in [11–13,17,23], and an application of the rational case in [14]. Operator (4) corresponds to the chiral gauge representation of the Harper operator.

From now on, we always consider the case of rational α. Furthermore, the analysis below for $q_0 = 1$, $q_0 = 2$ becomes especially elementary, and gives $|S(1)| = 8$, $|S(1/2)| = 4\sqrt{2}$, so that Theorem 1 obviously holds in these cases. From now on, we assume $q_0 \geq 3$.

If p_0 is even, define $p := \frac{p_0}{2}$ and $q := q_0$ (note that q is necessarily odd in this case). This corresponds to Case I below. If p_0 is odd, define $p := p_0$ and $q := 2q_0$. This corresponds to Case II below. We note that in either case p and q are coprime and $S(p_0/q_0) = \widetilde{S}(p/q)$.

Let $b(x) := 2\sin(2\pi x)$, and further identify $b_n(\theta) := b((p/q)n + \theta)$. For the operator $\widetilde{H}_{\frac{p}{q},\theta}$, Floquet theory states that $E \in \sigma(\widetilde{H}_{\frac{p}{q},\theta})$ if and only if the equation $(\widetilde{H}_{\frac{p}{q},\theta}\varphi)(n) = E\varphi(n)$ has a solution $\{\varphi(n)\}_{n\in\mathbb{Z}}$ satisfying $\varphi(n+q) = e^{ikq}\varphi(n)$ for all n and for some real k. Therefore, for a fixed k, there exist q values of E satisfying the eigenvalue equation

$$B_{\theta,k,\ell}\begin{pmatrix} \varphi(\ell) \\ \vdots \\ \varphi(\ell+q-1) \end{pmatrix} = E\begin{pmatrix} \varphi(\ell) \\ \vdots \\ \varphi(\ell+q-1) \end{pmatrix} \tag{5}$$

for any ℓ, where

$$B_{\theta,k,\ell} := \begin{pmatrix} 0 & b_\ell & 0 & 0 & \cdots & 0 & 0 & e^{-ikq}b_{\ell+q-1} \\ b_\ell & 0 & b_{\ell+1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_{\ell+1} & 0 & b_{\ell+2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{\ell+q-3} & 0 & b_{\ell+q-2} \\ e^{ikq}b_{\ell+q-1} & 0 & 0 & 0 & \cdots & 0 & b_{\ell+q-2} & 0 \end{pmatrix}. \tag{6}$$

Thus, the eigenvalues of $B_{\theta,k,\ell}$ are independent of ℓ.

2.1. Chambers-type formula

The celebrated Chambers' formula presents the dependence of the determinant of the almost Mathieu operator with $\alpha = p_0/q_0$ restricted to the period q_0 with Floquet boundary conditions, on the phase θ and quasimomentum k. In the critical, case it is given by (see, e.g., [15])

$$\det(A_{\theta,k,\ell} - E) = \Delta(E) - 2(-1)^{q_0}\big(\cos(2\pi q_0\theta) + \cos(kq_0)\big), \tag{7}$$

where

$$A_{\theta,k,\ell} := \begin{pmatrix} a_\ell & 1 & 0 & 0 & \cdots & 0 & 0 & e^{-ikq} \\ 1 & a_{\ell+1} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & a_{\ell+2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{\ell+q-2} & 1 \\ e^{ikq} & 0 & 0 & 0 & \cdots & 0 & 1 & a_{\ell+q-1} \end{pmatrix}, \quad \ell \in \mathbb{Z}, \tag{8}$$

$$a(x) := 2\cos(2\pi x), \quad a_n(\theta) := a((p_0/q_0)n + \theta), \tag{9}$$

and Δ, the discriminant,[1] is independent of θ and k. An immediate corollary of this formula is that $S(\frac{p_0}{q_0}) = \Delta^{-1}([-4,4])$, see, e.g., [15].

Here we obtain a formula of this type for $\det(B_{\theta,k,\ell} - E)$. Indeed, as usual, separating the terms containing k in the determinant, for the characteristic polynomial $D_{\theta,k}(E) := \det(B_{\theta,k,\ell} - E)$, we obtain

$$D_{\theta,k}(E) = D_\theta^{(0)}(E) - (-1)^q b_0 \cdots b_{q-1} \cdot 2\cos(kq), \tag{10}$$

where $D_\theta^{(0)}(E)$ is independent of k and equal therefore to $D_{\theta,k=\frac{\pi}{2q}}(E)$.

Lemma 1. *For the product of b_j's we have*

$$\begin{aligned} b_0 \cdots b_{q-1} &= \prod_{j=0}^{q-1} 2\sin 2\pi\left(\frac{p}{q} j + \theta\right) \\ &= 4\sin(\pi q\theta)\sin(\pi q(\theta + 1/2)) \\ &= 2\big(\cos(\pi q/2) - \cos(\pi q(2\theta + 1/2))\big). \end{aligned} \tag{11}$$

Proof. To evaluate the product of b_j's, we expand the sine in the first product in terms of exponentials and use the formula $1 - z^{-q} = \prod_{j=0}^{q-1}(1 - z^{-1}e^{2\pi i \frac{p}{q} j})$. An alternative derivation can go along the lines of the proof of [1, Lemma 9.6]. ■

Substituting (11) into (10), we have

$$D_{\theta,k}(E) = D_\theta^{(0)}(E) - 8(-1)^q \sin(\pi q\theta)\sin\pi q(\theta + 1/2)\cos(kq). \tag{12}$$

We can further obtain the dependence of $D_\theta^{(0)}(E)$ on θ:

Lemma 2. *We have*

$$D_\theta^{(0)}(E) = \widetilde{\Delta}(E) + \begin{cases} 0 & \text{if } q \text{ is odd}, \\ 4(\cos(2\pi q\theta) - 1) & \text{if } q \text{ is even}, \end{cases}$$

where the discriminant $\widetilde{\Delta}(E) := D_{\theta=0}^{(0)}(E)$ is independent of θ.

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[1] In [15], the discriminant differs from $\Delta(E)$ by the factor $(-1)^{q_0}$.

Proof. Since $D_{\theta,k}(E)$ is independent of ℓ, it is $1/q$ periodic in θ, i.e., $D_{\theta,k}(E) = D_{\theta+1/q,k}(E)$, and by (10) so is $D_\theta^{(0)}(E)$. Therefore, since, clearly,

$$D_\theta^{(0)}(E) = \sum_{n=-q}^{q} c_n(E) e^{2\pi i \theta n},$$

the terms c_k other than $k = mq$ vanish, and $D_\theta^{(0)}(E)$ has the following Fourier expansion:

$$D_\theta^{(0)}(E) = c_0(E) + c_q e^{2\pi i q\theta} + c_{-q} e^{-2\pi i q\theta}.$$

It is easily seen that the c_q and c_{-q} can be obtained from the expansion of the determinant and that, moreover, they do not depend on E. Expanding $D_\theta^{(0)}(E)$ with $E = 0$ in rows and columns (cf. [14]), we obtain

$$\begin{aligned} D_\theta^{(0)}(0) &= D_{\theta,k=\frac{\pi}{2q}}(0) \\ &= \begin{cases} 0 & \text{if } q \text{ is odd,} \\ (-1)^{q/2}(b_0^2 b_2^2 \cdots b_{q-2}^2 + b_1^2 b_3^2 \cdots b_{q-1}^2) & \text{if } q \text{ is even.} \end{cases} \end{aligned} \tag{13}$$

This gives $c_q = c_{-q} = 0$ for q odd, and

$$c_q = \prod_{j=0}^{\frac{q-2}{2}} e^{8\pi i \frac{p}{q} j} + \prod_{j=0}^{\frac{q-2}{2}} e^{4\pi i \frac{p}{q}(2j+1)} = 2 = c_{-q}$$

for q even. It remains to denote $\widetilde{\Delta}(E) = c_0(E)$ for q odd, and $\widetilde{\Delta}(E) = c_0(E) + 4$ for q even, and the proof is complete. ■

We therefore have, by (12) and Lemma 2:

Lemma 3 (Chambers-type formula). *If q is odd,*

$$D_{\theta,k}(E) = \widetilde{\Delta}(E) + 4(-1)^{(q-1)/2} \sin(2\pi q\theta)\cos(kq); \tag{14}$$

if q is even,

$$D_{\theta,k}(E) = \widetilde{\Delta}(E) - 4(1 - \cos(2\pi q\theta))\big(1 + (-1)^{q/2}\cos(kq)\big). \tag{15}$$

Note that $\widetilde{\Delta}(E)$ is a polynomial of degree q independent of $k \in \mathbb{R}$ and $\theta \in [0, 1)$. By Floquet theory, the spectrum $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the union of the eigenvalues of $B_{\theta,k,\ell}$ over k, a collection of q intervals.

We make the following observations.

Case I: q is odd. By (14),

$$D_{\theta,k}(E) \equiv \det(B_{\theta,k,\ell} - E) = 0 \iff \widetilde{\Delta}(E) = 4(-1)^{(q+1)/2} \sin(2\pi q\theta)\cos(kq).$$

Thus, $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the preimage of $[-4|\sin(2\pi q\theta)|, 4|\sin(2\pi q\theta)|]$ under the mapping $\widetilde{\Delta}(E)$. If $\theta = m/(2q)$, $m \in \mathbb{Z}$, then $\sigma(\widetilde{H}_{\frac{p}{q},\frac{m}{2q}})$ is a collection of q points where $\widetilde{\Delta}(E) = 0$. (In this case, $b_0(m/(2q)) = 0$, so that $\widetilde{H}$ splits into the direct sum of an infinite number of copies of a q-dimensional matrix.) We note that the spectra $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ for different θ are nested in one another as θ grows from 0 to $1/(4q)$; in particular, for each $\theta \in [0, 1)$,

$$\sigma(\widetilde{H}_{\frac{p}{q},\theta}) = \widetilde{\Delta}^{-1}([-4|\sin(2\pi q\theta)|, 4|\sin(2\pi q\theta)|]) \subseteq \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}}) = \widetilde{\Delta}^{-1}([-4, 4]). \tag{16}$$

This implies that all the maxima of $\widetilde{\Delta}(E)$ are no less than 4, and all the minima are no greater than -4. Moreover, taking the union over all $\theta \in [0, 1)$ gives:

$$\widetilde{S}\left(\frac{p}{q}\right) = \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}}) = \widetilde{\Delta}^{-1}([-4, 4]). \tag{17}$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(4q)]$.

Case II: q is even. This case is similar to Case I, so we omit some details for brevity. By (15), $D_{\theta,k}(E) = 0$ if and only if $\widetilde{\Delta}(E) = 4(1 - \cos(2\pi q\theta))(1 + (-1)^{q/2}\cos(kq))$. Considering the cases $k = 0, \frac{\pi}{q}$, it is easy to see that $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the preimage of $[0, 8 - 8\cos(2\pi q\theta)]$ under the mapping $\widetilde{\Delta}(E)$. If $\theta = m/q$, $m \in \mathbb{Z}$, then $\sigma(\widetilde{H}_{\frac{p}{q},\frac{m}{q}})$ is a collection of q points where $\widetilde{\Delta}(E) = 0$. We note that the spectra $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ for different θ are nested in one another as θ grows from 0 to $1/(2q)$; in particular, for each $\theta \in [0, 1)$,

$$\sigma(\widetilde{H}_{\frac{p}{q},\theta}) = \widetilde{\Delta}^{-1}([0, 8 - 8\cos(2\pi q\theta)]) \subseteq \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}}) = \widetilde{\Delta}^{-1}([0, 16]). \tag{18}$$

This implies that all the maxima of $\widetilde{\Delta}(E)$ are no less than 16, and all the minima are no greater than 0. Moreover, taking the union over all $\theta \in [0, 1)$ gives

$$\widetilde{S}\left(\frac{p}{q}\right) = \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}}) = \widetilde{\Delta}^{-1}([0, 16]). \tag{19}$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(2q)]$.

In this case of even q, we can say more about the form of $\widetilde{\Delta}(E)$. Note that $b_0(0) = b_{q/2}(0) = 0$ and $b_k(0) = b_{-k}(0)$. Recall that, by Floquet theory, $D_{\theta,k}(E) = \det(B_{\theta,k,\ell} - E)$ is independent of the choice of ℓ. For convenience, choose $\ell = -q/2 + 1$. It is easily seen that $B_{\theta=0,k,\ell=-q/2+1}$ decomposes into a direct sum, and moreover $\widetilde{\Delta}(E) = D_{\theta=0,k}(E) = (-1)^{q/2} P_{q/2}(-E) P_{q/2}(E)$, where $P_{q/2}(E)$ is a polynomial of degree $q/2$, odd if $q/2$ is odd, and even if $q/2$ is even (as it is a characteristic polynomial of a tridiagonal matrix with zero main diagonal). Thus $\widetilde{\Delta}(E) = P_{q/2}(E)^2$ is a square.

The discriminants $\widetilde{\Delta}(E) \equiv \widetilde{\Delta}_{p/q}(E)$ and $\Delta(E) \equiv \Delta_{p_0/q_0}(E)$ are related in the following way:

Lemma 4. *For q odd,*

$$\widetilde{\Delta}_{p/q}(E) = \Delta_{p_0/q_0}(E), \quad p_0 = 2p, \quad q_0 = q; \tag{20}$$

for q even,

$$\widetilde{\Delta}_{p/q}(E) = \Delta^2_{p_0/q_0}(E), \quad p_0 = p, \quad q_0 = q/2. \tag{21}$$

Proof. Case I: *q is odd.* Here, by our definitions at the start of the section, $p_0 = 2p$ and $q_0 = q$. Both $\widetilde{\Delta}_{p/q}(E)$ and $\Delta_{p_0/q_0}(E)$ are polynomials in E of degree q with the same coefficient -1 of E^q. Since $\widetilde{\Delta}(E) = \Delta(E) = \pm 4$ at the $2q \geq q + 1$ distinct edges of the bands (cf. [5, Section 3.3]), these polynomials coincide: $\widetilde{\Delta}(E) = \Delta(E)$ for each E.

Case II: *q is even.* Here, $p_0 = p$ and $q_0 = q/2$. We observe that $\widetilde{S}(\frac{p}{q}) = S(\frac{p_0}{q_0})$ is the preimage of [0, 16] under $\widetilde{\Delta}_{p/q}$ and of $[-4, 4]$ under Δ_{p_0/q_0}, hence also of [0, 16] under $\Delta^2_{p_0/q_0}$. On the other hand, we have seen above that $\widetilde{\Delta}(E) = P^2_{q/2}(E)$ for some polynomial $P_{q/2}(E)$ of degree $q/2 = q_0$. Thus, $P^2_{q/2}(E)$ and $\Delta^2(E)$ coincide at the $2q_0 \geq q_0 + 1$ (for q_0 odd) and $2q_0 - 1 \geq q_0 + 1$ (for q_0 even) distinct edges of the bands (cf. [5, Section 3.3]; the central bands merge for q_0 even), so these polynomials of degree q are equal: $\widetilde{\Delta}(E) = \Delta^2(E)$ for each E. ∎

2.2. Measure of the spectrum

The rest of the proof follows the argument of [3], namely it uses Lidskii's inequalities to bound $|\widetilde{S}(\frac{p}{q})|$. The key observation is that, choosing ℓ appropriately, we can make the corner elements of the matrix $B_{\theta,k,\ell}$ very small, of order $1/q$ when q is large. This is not possible to do in the standard representation for the almost Mathieu operator. Here are the details.

Case I: q is odd. Assume, without loss of generality, that one has $(-1)^{(q+1)/2} > 0$, for $\theta \in (0, 1/(4q)]$. (If $(-1)^{(q+1)/2} < 0$, the analysis is similar.) Then the eigenvalues $\{\lambda_i(\theta)\}_{i=1}^q$ of $B_{\theta,k=0,\ell}$ labelled in decreasing order are the edges of the spectral bands where $\widetilde{\Delta}(E)$ reaches its maximum $4\sin(2\pi q\theta)$ on the band; and the eigenvalues $\{\hat{\lambda}_i(\theta)\}_{i=1}^q$ of $B_{\theta,k=\pi/q,\ell}$ labelled in decreasing order are the edges of the spectral bands where $\widetilde{\Delta}(E)$ reaches its minimum $-4\sin(2\pi q\theta)$ on the band. Then,

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| = \sum_{j=1}^{q}(-1)^{q-j}(\hat{\lambda}_j(\theta)-\lambda_j(\theta))$$
$$= \sum_{j=1}^{(q+1)/2}(\hat{\lambda}_{2j-1}(\theta)-\lambda_{2j-1}(\theta)) + \sum_{j=1}^{(q-1)/2}(\lambda_{2j}(\theta)-\hat{\lambda}_{2j}(\theta)); \tag{22a}$$

$$\hat{\lambda}_j(\theta)-\lambda_j(\theta) > 0 \quad \text{if } j \text{ is odd;} \qquad \hat{\lambda}_j(\theta)-\lambda_j(\theta) < 0 \quad \text{if } j \text{ is even.} \tag{22b}$$

Now we view $B_{\theta,k=\pi/q,\ell}$ as $B_{\theta,k=0,\ell}$ with the added perturbation

$$B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell} = \begin{pmatrix} & -2b_{\ell+q-1} \\ -2b_{\ell+q-1} & \end{pmatrix},$$

which has the eigenvalues $\{E_i(\theta)\}_{i=1}^{q}$ given by

$$E_q(\theta) = -2|b_{\ell+q-1}(\theta)| < 0 = E_{q-1}(\theta) = \cdots = E_2(\theta) = 0 < 2|b_{\ell+q-1}(\theta)| = E_1(\theta).$$

Theorem 2 (Lidskii inequalities; e.g., [4]). *For any $q \times q$ self-adjoint matrix M, we denote its eigenvalues by*

$$E_1(M) \geq E_2(M) \geq \cdots \geq E_q(M).$$

For $q \times q$ self-adjoint matrices A and B, we have

$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B)$$
$$\leq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_1(B) + \cdots + E_m(B);$$

$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B)$$
$$\geq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_{q-m+1}(B) + \cdots + E_q(B),$$

for any $1 \leq i_1 < \cdots < i_m \leq q$.

Applying these inequalities with $A = B_{\theta,k=0,\ell}$, $B = B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell}$ gives

$$\sum_{j=1}^{(q+1)/2}(\hat{\lambda}_{2j-1}(\theta)-\lambda_{2j-1}(\theta)) \leq \sum_{j=1}^{(q+1)/2} E_j(\theta) = E_1(\theta);$$
$$\sum_{j=1}^{(q-1)/2}(\lambda_{2j}(\theta)-\hat{\lambda}_{2j}(\theta)) \leq -\sum_{j=(q-1)/2}^{q} E_j(\theta) = -E_q(\theta).$$

Substituting these into (22), we obtain

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \leq E_1(\theta) - E_q(\theta) = 4|b_{\ell+q-1}(\theta)|. \tag{23}$$

Moreover, by the invariance of $D_{\theta,k}(E)$ under the mapping $b_n \mapsto b_{n+m}$, for $n = 0, 1, \ldots, q-1$ and any m, we can choose any ℓ in (23), so that

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \le 4 \min_{\ell} |b_{\ell+q-1}(\theta)|. \tag{24}$$

In particular,

$$\left|\widetilde{S}\Big(\frac{p}{q}\Big)\right| = |\sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}})| \le 4 \min_{\ell} \left|b_{\ell+q-1}\Big(\frac{1}{4q}\Big)\right| = 4 \cdot 2 \left|\sin 2\pi \Big(\frac{1}{4q}\Big)\right| \le \frac{4\pi}{q}. \tag{25}$$

Therefore, $|S(\frac{p_0}{q_0})| = |\widetilde{S}(\frac{p}{q})| \le \frac{4\pi}{q} = \frac{4\pi}{q_0}$, as required.

Case II: q is even. This case is similar to Case I, so we omit some details for brevity. This time, the Lidskii equations of Theorem 2 show that $|\widetilde{S}(\frac{p}{q})| \le \frac{8\pi}{q}$. Indeed, as in (24), we have (note the doubling of the eigenvalues for $\widetilde{\Delta}(E) = 0$)

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \le 4 \min_{\ell} |b_{\ell+q-1}(\theta)|. \tag{26}$$

In particular,

$$\left|\widetilde{S}\Big(\frac{p}{q}\Big)\right| = |\sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}})| \le 4 \min_{\ell} \left|b_{\ell+q-1}\Big(\frac{1}{2q}\Big)\right| = 4 \cdot 2 \left|\sin 2\pi \Big(\frac{1}{2q}\Big)\right| \le \frac{8\pi}{q}. \tag{27}$$

Therefore, $|S(\frac{p_0}{q_0})| = |\widetilde{S}(\frac{p}{q})| \le \frac{8\pi}{q} = \frac{4\pi}{q_0}$, as required.

This completes the proof of Theorem 1.

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