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A study of BPS and near-BPS black holes via AdS/CFT

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Abstract

In the settings of various AdS/CFT dual pairs, we use results from supersymmetric localization to gain insights into the physics of asymptotically-AdS, BPS black holes in 5 dimensions, and near-BPS black holes in 4 dimensions.

We first begin with BPS black holes embedded in the known examples of $\mathrm{AdS}_5/\mathrm{CFT}_4$ dualities. Using the Bethe Ansatz formulation, we compute the superconformal index at large N with arbitrary chemical potentials for all charges and angular momenta, for general $\mathcal{N}=1$ four-dimensional conformal theories with a holographic dual. We conjecture and bring some evidence that a particular universal contribution to the sum over Bethe vacua dominates the index at large N. For $\mathcal{N}=4$ SYM, this contribution correctly leads to the entropy of BPS Kerr-Newman black holes in $\mathrm{AdS}_5 \times S^5$ for arbitrary values of the conserved charges, thus completing the microscopic derivation of their microstates. We also consider theories dual to $\mathrm{AdS}_5 \times \mathrm{SE}_5$, where SE_5 is a Sasaki-Einstein manifold. We first check our results against the so-called universal black hole. We then explicitly construct the near-horizon geometry of BPS Kerr-Newman black holes in $\mathrm{AdS}_5 \times T^{1,1}$, charged under the baryonic symmetry of the conifold theory and with equal angular momenta. We compute the entropy of these black holes using the attractor mechanism and find complete agreement with field theory predictions.

Next, we consider the 3d Chern-Simons matter theory that is holographically dual to massive Type IIA string theory on $\mathrm{AdS}_4 \times S^6$. By Kaluza-Klein reducing on S^2 with a background that is dual to the asymptotics of static dyonic BPS black holes in AdS_4, we construct a $\mathcal{N}=2$ supersymmetric gauged quantum mechanics whose ground-state degeneracy reproduces the entropy of BPS black holes. We expect its low-lying spectrum to contain information about near-extremal horizons. Interestingly, the model has a large number of statistically-distributed couplings, reminiscent of SYK models.

Preface

This thesis is based on the following publications and preprints:

- F. Benini, E. Colombo, S. Soltani, A. Zaffaroni, and Z. Zhang, "Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes," *Class. Quant. Grav.* **37** no. 21, (2020) 215021, `arXiv:2005.12308 [hep-th]`.
- F. Benini, S. Soltani, and Z. Zhang, "A quantum mechanics for magnetic horizons," `arXiv:2212.00672 [hep-th]`.

Contents

Introduction

The central objects which are studied in this thesis are black holes. Black holes are interesting because they are where our ignorance about gravity is sharply focused into various puzzles and paradoxes. The most famous of these is the black hole information paradox [1], which exposes an apparent incompatibility between gravity and quantum mechanics. In the following, we will instead focus on two other puzzles brought forward by black holes: the statistical mechanical interpretation of black hole entropy, and the thermodynamics of near-extremal black holes.

Microstate counting for supersymmetric black holes

Within the framework of classical gravity, the laws of black hole thermodynamics discovered by Bekenstein, Hawking and others [2,3] are rather mysterious. They are exactly analogous to the usual laws of thermodynamics obeyed by a system in thermal equilibrium, but the corresponding microscopic degrees of freedom are obscure, especially in light of "no-hair theorems" [4,5] stating that black hole solutions in classical gravity are completely specified by just a handful of quantum numbers. Given these limitations, it is natural to suppose that a microscopic description of black holes which gives a statistical mechanical interpretation for black hole thermodynamics must come from a quantum theory of gravity.

In particular, we know due to Bekenstein and Hawking [6,7] that black holes carry entropy proportional to the area of their horizons

$$S = \frac{\text{Area}}{4} \tag{1}$$

in natural units. A theory of quantum gravity must account for this entropy as $S = \log W$ where W is the number of microstates of the black hole. This has become a standard yardstick to measure the success of any candidate theory of quantum gravity.

String theory has been incredibly successful in this regard. For supersymmetric (BPS) asymptotically-flat black holes in string theory, their study was initiated by the seminal work by Strominger and Vafa [8], which managed to account for their entropy at large charges through the asymptotic degeneracy of D-brane bound states. Beyond the large charge limit and the area formula (1), the microscopic counting was found to agree with the macroscopic entropy up to great precision after incorporating higher-derivative corrections in supergravity [9–15]. The relevant technology is reviewed in [16] and [17], with more references therein.

For asymptotically anti-de-Sitter (AdS) BPS black holes, string theory also provides a natural set-up to account for their entropy, which is the AdS/CFT correspondence [18, 19]. This will be our main focus. In its strongest form, the conjecture states that various string theories (or M theory) on $\mathrm{AdS}_d \times M$, where M is some compact manifold, are defined non-perturbatively by a superconformal field theory (SCFT) living on the boundary of AdS_d. In this set-up, the BPS black hole is described by BPS states of the SCFT, whose degeneracy can be checked against the Bekenstein-Hawking formula. In the best-understood example of $\mathrm{AdS}_3/\mathrm{CFT}_2$, this check was performed for BTZ black holes [20, 21] in a generic quantum gravity theory on AdS_3 [22, 23].

In the developments described thus far, for the purpose of reproducing the area formula (1), the microscopic counting almost always boils down to computing the spectrum of a 2-dimensional CFT at large charges. For that, it is sufficient to know the central charge of the CFT and use Cardy's formula [24]. In higher-dimensional strongly-coupled CFTs, it is much more difficult to access the asymptotics of the spectrum, and this was limiting progress for some time. More recently, as supersymmetric localization [25–27] made it possible to compute various supersymmetric indices of higher dimensional SCFTs, the program of microstate counting was extended to higher dimensional BPS black holes. Further advances began with the analysis of static dyonic BPS black holes in AdS_4, whose states are counted by the topologically twisted index [28] of the dual field theory. Firstly, the entropy of static dyonic BPS black holes in M-theory on $\mathrm{AdS}_4 \times S^7$ was reproduced using the topologically twisted index of the dual ABJM theory [29, 30]. This was followed by a similar computation for static dyonic BPS black holes in massive Type IIA string theory on $\mathrm{AdS}_4 \times S^6$ [31–33], and generalized to include rotation in [34][1]. We shall return to this example later in the next section when we introduce the work [39].

For black holes in AdS_5, there was a puzzle which has stood for some time. The holographic description of electrically-charged and rotating (Kerr-Newman) BPS black holes in Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is in terms of 1/16 BPS states of the dual four-dimensional $\mathcal{N} = 4$ super-Yang-Mills (SYM) boundary theory on S^3. These states are counted (with sign) by the superconformal index [40–42], and one would expect that the contribution from black holes saturates the index at large N. However, the large N computation of the superconformal index performed in [41] gave a result of order one, while the entropy for the black holes is of order N^2, suggesting a large cancellation between bosonic and fermionic BPS states. Recently, this puzzle was resolved in [43–45] by realizing that the computation of [41] is only valid for real fugacities, and that allowing the fugacities to

[1]The microstate counting for rotating non-magnetically charged black holes in AdS_4 was performed in [35–37]. See the review [38] for a more complete list of references.

be complex obstructs cancellations between states due to the same $\mathcal{I}$-extremization principle found in [29, 30]. Using two different technical approaches, the works [44, 45] found that the superconformal index with complex fugacities does indeed capture the entropy of Kerr-Newman BPS black holes in $\mathrm{AdS}_5 \times S^5$.

Our work [46] is an extension of these results to Kerr-Newman black holes in Type IIB on $\mathrm{AdS}_5 \times SE_5$ without using the Cardy limit as in [44] or specializing to equal angular momenta, as in [45, 47]. This work will be presented in Chapter 1.

A quantum mechanics for near-BPS magnetic horizons

Ever since the beginning of AdS/CFT, there has always been an interest to find $\mathrm{AdS}_2/\mathrm{CFT}_1$ dualities [18, 48] because many black holes have near-horizon geometries which contain an AdS_2 factor, and therefore such dualities would provide access to the quantum properties of black holes, including their spectrum. Resolving the spectrum of near-BPS black holes is important because thermodynamics breaks down for very-near-BPS black holes according to their classical spectrum, where emitting a single quantum of Hawking radiation can change the temperature of the black hole by a substantial amount [49]. Therefore quantum gravity effects must play an important role.

However, it quickly became apparent in [48] that backreaction from any excitation would destroy an AdS_2 geometry, and thus the dual CFT_1 can only describe the ground states of the extremal (BPS) black hole. Following some years of inactivity, there has been a breakthrough recently when it was realized that a useful way to think about the near-horizon limit of black holes is to account for a leading order backreaction, leading to JT gravity [50–53]. The study of JT gravity has led to significant progress on the information paradox due to the model's solvability at the quantum level [54, 55]. At the same time, a dual quantum mechanical description was found in the SYK model [56–58], which was of independent interest to the condensed matter and quantum computing communities. This is sometimes referred to as a near-AdS_2/near-CFT_1 duality.

Similar methods were then applied to the JT gravity theories obtained via the dimensional reductions of specific asymptotically-flat and asymptotically-AdS_5 black holes in [59–62], where it was possible to compute the density of states of near-extremal and near-BPS black holes. Although there have been attempts such as [63], open questions remain about what is the SYK-like quantum mechanical dual that can reproduce the low energy dynamics and spectra of these works [59–61]. The same dimensional reduction can be performed around static magnetically-charged BPS black holes in massive Type IIA string theory on $\mathrm{AdS}_4 \times S^6$ like in [64], to obtain an effective theory of 2d gravity. Although a universal JT sector was found, it is unclear whether the near-extremal spectrum will be captured by the JT sector due to the presence of relevant deformations coming from the matter sector. It is also unclear whether the spectrum of operators found in [64] can be reproduced by a dual quantum mechanics. The relevant deformations suggest that the dual quantum mechanics might be significantly different from SYK.

In our work [39], we propose such a candidate quantum mechanics. The static dyonic black holes in massive Type IIA that we are interested in [65–67] interpolate between the AdS_4 vacuum and an $\mathrm{AdS}_2 \times S^2$ near-horizon geometry. This suggests a natural holographic interpretation for the solution as a RG flow across dimensions. To be precise, we have the dual 3d $\mathcal{N}=2$ $\mathrm{SU}(N)_k$ Chern-Simons-matter theory [68][2] placed on S^2 with a topological twist, flowing to a superconformal quantum mechanics. The topological twist is present due to the magnetic charges of the black hole. To obtain the superconformal quantum mechanics at the endpoint of the RG flow, we reduce the dual 3d field theory on S^2 in the presence of the topological twist [3]. The resulting quantum mechanics has a Witten index that reproduces the Bekenstein-Hawking entropy of the BPS black holes at large N, and we expect that its near-BPS spectrum should coincide with the spectrum of near-BPS black holes. This work [39] will be presented in Chapter 2.

[2]The theory has three adjoint chiral multiplets and a superpotential. It is essentially the 4d $\mathcal{N}=4$ $\mathrm{SU}(N)$ super-Yang-Mills theory reduced to 3d and deformed by an $\mathcal{N}=2$ Chern-Simons term.

[3]The background is dual to the black-hole chemical potentials, or charges, depending on the ensemble.

Chapter 1

Superconformal indices at large N and the entropy of $\mathrm{AdS}_5 \times \mathrm{SE}_5$ black holes

The family of $\mathrm{AdS}_5 \times S^5$ supersymmetric black holes found in [69–73] depends on three charges Q_a associated with the Cartan subgroup of the internal isometry SO(6), and two angular momenta J_i in AdS_5, subject to a non-linear constraint.[1] The entropy can be written as the value at the critical point (*i.e.*, as a Legendre transform) of the function [76]

$$\mathcal{S}(X_a,\tau,\sigma) = -i\pi N^2 \, \frac{X_1 X_2 X_3}{\tau\,\sigma} - 2\pi i \left(\sum_{a=1}^{3} X_a Q_a + \tau J_1 + \sigma J_2 \right) \tag{1.0.1}$$

with the constraint $X_1 + X_2 + X_3 - \tau - \sigma = \pm 1$, where N is the number of colors of the dual 4d $\mathcal{N}=4$ SU(N) SYM theory. The same entropy function can also be obtained by computing the zero-temperature limit of the on-shell action of a class of supersymmetric but non-extremal complexified Euclidean black holes [43, 77]. The two constraints with $\pm$ sign lead to the same value for the entropy, which is real precisely when the non-linear constraint on the black hole charges is imposed. The parameters X_a, τ and σ are chemical potentials for the conserved charges Q_a and J_i and can also be identified with the parameters the superconformal index depends on. With this identification, we expect that the entropy $S(Q_a, J_1, J_2)$ is just the constrained Legendre transform of $\log \mathcal{I}(X_a,\tau,\sigma)$, where $\mathcal{I}(X_a,\tau,\sigma)$ is the superconformal index.

Up to the work we present in this thesis, the entropy of $\mathrm{AdS}_5 \times S^5$ Kerr-Newman black holes has been derived from the superconformal index and shown to be in agreement with (1.0.1) only in particular limits. In [44], the entropy was derived for large black holes (whose size is much larger than the AdS radius) using a Cardy limit of the superconformal index where $\mathbb{I}\mathrm{m}(X_a), \tau, \sigma \ll 1$. In [45], the entropy was instead derived in the large N limit in the case of black holes with equal angular momenta, $J_1 = J_2$.[2] The large N limit has been evaluated by writing the index as a sum over Bethe vacua [79], an approach that has been successful for AdS black holes in many other contexts.

[1] Supersymmetric hairy black holes depending on all charges have been recently found in [74, 75], but their entropy seems to be parametrically smaller in the range of parameters where our considerations apply.

[2] The same result has been later reproduced with a different approach in [78].

One of the purposes of our work is to extend the derivation of [45] to the case of unequal angular momenta, thus providing a large N microscopic counting of the microstates of BPS Kerr-Newman black holes in $\mathrm{AdS}_5 \times S^5$ for arbitrary values of the conserved charges. We will make use of the Bethe Ansatz formulation of the superconformal index derived for $\tau = \sigma$ in [80] and generalized to unequal angular chemical potentials in [79]. This formulation allows us to write the index as a sum over the solutions to a set of Bethe Ansatz Equations (BAEs) — whose explicit form and solutions have been studied in [45, 81–85] — and over some auxiliary integer parameters m_i. We expect that, in the large N limit, one particular solution dominates the sum.[3] We will show that the "basic solution" to the BAEs, already used in [45], correctly reproduces the entropy of black holes in the form (1.0.1) for a choice of integers m_i. We stress that our result comes from a single contribution to the index, which is an infinite sum. Such a contribution might not be the dominant one — and so our estimate of the index might be incorrect — in some regions of the space of chemical potentials. It is known from the analysis in [45] that when the charges become smaller than a given threshold, new solutions take over and dominate the asymptotic behavior of the index. This suggests the existence of a rich structure where other black holes might also contribute. However, we conjecture and we will bring some evidence that the contribution of the basic solution is the dominant one in the region of the space of chemical potentials corresponding to sufficiently large charges.

We will also extend the large N computation of the index to a general class of superconformal theories dual to $\mathrm{AdS}_5 \times \mathrm{SE}_5$, where SE_5 is a five-dimensional Sasaki-Einstein manifold. The analysis for $J_1 = J_2$ was already performed in [83]. For toric holographic quiver gauge theories, we find a prediction for the entropy of black holes in $\mathrm{AdS}_5 \times \mathrm{SE}_5$ in the form of the entropy function

$$\mathcal{S}(X_a, \tau, \sigma) = -\frac{i\pi N^2}{6} \sum_{a,b,c}^{D} C_{abc} \frac{X_a X_b X_c}{\tau\, \sigma} - 2\pi i \left(\sum_{a=1}^{D} X_a Q_a + \tau J_1 + \sigma J_2 \right) , \tag{1.0.2}$$

with the constraint $\sum_{a=1}^{D} X_a - \tau - \sigma = \pm 1$, in terms of chemical potentials X_a for a basis of independent R-symmetries R_a. The coefficients $C_{abc}\, N^2 = \frac{1}{4} \operatorname{Tr} R_a R_b R_c$ are the 't Hooft anomaly coefficients for this basis of R-symmetries. The form of the entropy function (1.0.2) was conjectured in [86] and reproduced for various toric models in the special case $\tau = \sigma$ in [83]. We will give a general derivation, valid for all toric quivers and even more. We will also show that both constraints in (1.0.2), which lead to the same value for the entropy, naturally arise from the index in different regions of the space of chemical potentials. The function (1.0.2) was also derived in the Cardy limit in [87].

[3] It is argued in [84] that there exist families of continuous solutions. This does not affect our argument provided the corresponding contribution to the index is subleading.

In the last subsection of the chapter we will provide some evidence that (1.0.2) correctly reproduces the entropy of black holes in $\mathrm{AdS}_5 \times \mathrm{SE}_5$. We first check that our formula correctly reproduce the entropy of the *universal* black hole that arises as a solution in five-dimensional minimal gauged supergravity, and, as such, can be embedded in any $\mathrm{AdS}_5 \times \mathrm{SE}_5$ compactification. It corresponds to a black hole with electric charges aligned with the exact R-symmetry of the dual superconformal field theory and with arbitrary angular momenta J_1 and J_2. Since the solution is universal, the computation can be reduced to that of $\mathcal{N} = 4$ SYM and it is almost trivial. More interesting are black holes with general electric charges. Unfortunately, to the best of our knowledge, there are no available such black hole solutions in compactifications based on Sasaki-Einstein manifolds SE_5 other than S^5. To overcome this obstacle, we will explicitly construct the near-horizon geometry of supersymmetric black holes in $\mathrm{AdS}_5 \times T^{1,1}$ with equal angular momenta and charged under the baryonic symmetry of the dual Klebanov-Witten theory [88]. Luckily, the background $\mathrm{AdS}_5 \times T^{1,1}$ admits a consistent truncation to a five-dimensional gauged supergravity containing the massless gauge field associated to the baryonic symmetry [89–91]. We then use the strategy suggested in [76]: a rotating black hole in five dimensions with $J_1 = J_2$ can be dimensionally reduced along the Hopf fiber of the horizon three-sphere to a static solution of four-dimensional $\mathcal{N} = 2$ gauged supergravity. We will explicitly solve the BPS equations [92–94] for the horizon of static black holes with the appropriate electric and magnetic charges in $\mathcal{N} = 2$ gauge supergravity in four dimensions. The main complication is the presence of hypermultiplets. By solving the hyperino equations at the horizon, we will be able to recast all other supersymmetric conditions as a set of attractor equations, and we will show that these are equivalent to the extremization of (1.0.2) for the Klebanov-Witten theory with $\tau = \sigma$. This provides a highly non-trivial check of our result, and the conjecture that the basic solution to the BAEs dominates the index.

This chapter is organized as follows. In Section 1.1 we review the setting introduced in [45] and we evaluate the large N contribution of the "basic solution" to the BAEs to the index for generic angular fugacities. We show that it correctly captures the semiclassical Bekenstein-Hawking entropy of BPS black holes in $\mathrm{AdS}_5 \times S^5$. In Section 1.2 we discuss the generalization of this result to general toric quiver theories and find agreement with the entropy function prediction (1.0.2) in certain corners of the space of chemical potentials. In Section 1.3 we discuss the particular case of the universal black hole, which can be embedded in all string and M-theory supersymmetric compactifications with an AdS_5 factor. In Section 1.4 we match formula (1.0.2) with the entropy of a supersymmetric black hole in $\mathrm{AdS}_5 \times T^{1,1}$, whose near-horizon geometry we explicitly construct. Technical computations as well as some review material can be found in Appendices A to E.

1.1 The index of $\mathcal{N} = 4$ SYM at large N

We are interested in evaluating the large N limit of the superconformal index of 4d $\mathcal{N} = 1$ holographic theories. We will consider in this section the simplest example, namely $\mathcal{N} = 4$ SU(N) SYM. The superconformal index counts (with sign) the 1/16 BPS states of the theory on $\mathbb{R} \times S^3$ that preserve one complex supercharge $\mathcal{Q}$. These states are characterized by two angular momenta $J_{1,2}$ on S^3 and three R-charges for $\mathrm{U}(1)^3 \subset \mathrm{SO}(6)_R$. We write $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ notation in terms of a vector multiplet and three chiral multiplets Φ_I and introduce a symmetric basis of R-symmetry generators $R_{1,2,3}$ such that R_I assigns R-charge 2 to Φ_I and zero to Φ_J with $J \neq I$. The index is defined by the trace [40, 41]

$$\mathcal{I}(p, q, v_1, v_2) = \mathrm{Tr}\,(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{r}{2}}\, q^{J_2+\frac{r}{2}}\, v_1^{q_1}\, v_2^{q_2} \;, \tag{1.1.1}$$

in terms of two flavor generators $q_{1,2} = \frac{1}{2}(R_{1,2} - R_3)$ commuting with $\mathcal{Q}$, and the R-charge $r = \frac{1}{3}(R_1 + R_2 + R_3)$. Notice that $(-1)^F = e^{2\pi i J_{1,2}} = e^{i\pi R_{1,2,3}}$. Here p, q, v_I with $I = 1, 2$ are complex fugacities associated to the various quantum numbers, while the corresponding chemical potentials τ, σ, ξ_I are defined by

$$p = e^{2\pi i\tau} \;, \qquad q = e^{2\pi i\sigma} \;, \qquad v_I = e^{2\pi i\xi_I} \;. \tag{1.1.2}$$

The index is well-defined for $|p|, |q| < 1$.

It is convenient to redefine the flavor chemical potentials in terms of

$$\Delta_I = \xi_I + \frac{\tau + \sigma}{3} \qquad \text{for} \qquad I = 1, 2 \;. \tag{1.1.3}$$

It is also convenient to introduce an auxiliary chemical potential Δ_3 such that

$$\tau + \sigma - \Delta_1 - \Delta_2 - \Delta_3 \in 2\mathbb{Z} + 1 \;, \tag{1.1.4}$$

and use the corresponding fugacities

$$y_I = e^{2\pi i\Delta_I} \;. \tag{1.1.5}$$

The index then takes the more transparent form

$$\mathcal{I} = \mathrm{Tr}_{\mathrm{BPS}}\, p^{J_1}\, q^{J_2}\, y_1^{R_1/2}\, y_2^{R_2/2}\, y_3^{R_3/2} \;. \tag{1.1.6}$$

It shows that the constrained fugacities p, q, y_I with $I = 1, 2, 3$ are associated to the angular momenta $J_{1,2}$ and the charges $Q_I \equiv \frac{1}{2}R_I$.

Our starting point is the so-called Bethe Ansatz formulation of the superconformal index [79, 80]. The special case that the two angular chemical potentials are equal, $\tau = \sigma$, was already studied in [45] (see also [84]). Here we take them unequal. The formula of [79] can be applied when the ratio between the two angular chemical potentials is a rational number.[4] We thus set

$$\tau = a\omega \;, \qquad \sigma = b\omega \qquad \text{with} \qquad \mathbb{I}\mathrm{m}\,\omega > 0 \tag{1.1.7}$$

and with $a, b \in \mathbb{N}$ coprime positive integers. We call $\mathbb{H} = \{\omega \,|\, \mathbb{I}\mathrm{m}\,\omega > 0\}$ the upper half-plane. We then have the fugacities

$$h = e^{2\pi i\omega} \;, \qquad p = h^a = e^{2\pi i\tau} \;, \qquad q = h^b = e^{2\pi i\sigma} \qquad \text{with} \qquad |h|, |p|, |q| < 1 \;. \tag{1.1.8}$$

The formula in [79] allows us to write the superconformal index as a sum over the solutions to a set of Bethe Ansatz Equations (BAEs). Explicitly, the index reads

$$\mathcal{I} = \kappa_N \sum_{\hat{u} \,\in\, \mathrm{BAE}} \mathcal{Z}_{\mathrm{tot}}\, H^{-1}\Big|_{\hat{u}} \;. \tag{1.1.9}$$

The expressions of κ_N, H and $\mathcal{Z}_{\mathrm{tot}}$ for a generic $\mathcal{N} = 1$ theory are given in [79]. Here, we specialize them to $\mathcal{N} = 4$ SU(N) SYM. The quantity

$$\kappa_N = \frac{1}{N!}\left(\frac{(p;p)_\infty\,(q;q)_\infty\,\widetilde{\Gamma}(\Delta_1;\tau,\sigma)\,\widetilde{\Gamma}(\Delta_2;\tau,\sigma)}{\widetilde{\Gamma}(\Delta_1+\Delta_2;\tau,\sigma)}\right)^{N-1} \tag{1.1.10}$$

is a prefactor written in terms of the elliptic gamma function $\widetilde{\Gamma}$ and the Pochhammer symbol:

$$\widetilde{\Gamma}(u;\tau,\sigma) \equiv \Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1}q^{n+1}/z}{1 - p^m q^n z} \;, \qquad (z;q)_\infty = \prod_{n=0}^{\infty}(1 - zq^n) \;, \tag{1.1.11}$$

where $z = e^{2\pi i u}$. The sum in (1.1.9) is over the solution set to the following BAEs:[5]

$$1 = Q_i(u;\Delta,\omega) \equiv e^{2\pi i\left(\lambda + 3\sum_j u_{ij}\right)} \prod_{j=1}^{N} \frac{\theta_0\big(u_{ji}+\Delta_1;\omega\big)\,\theta_0\big(u_{ji}+\Delta_2;\omega\big)\,\theta_0\big(u_{ji}-\Delta_1-\Delta_2;\omega\big)}{\theta_0\big(u_{ij}+\Delta_1;\omega\big)\,\theta_0\big(u_{ij}+\Delta_2;\omega\big)\,\theta_0\big(u_{ij}-\Delta_1-\Delta_2;\omega\big)} \tag{1.1.12}$$

written in terms of $u_{ij} = u_i - u_j$ with $i, j = 1, \ldots, N$ and the theta function

$$\theta_0(u;\omega) = (z;h)_\infty (h/z;h)_\infty \;. \tag{1.1.13}$$

The unknowns are the "complexified SU(N) holonomies", which are expressed here in terms of U(N) holonomies u_i further constrained by

$$\sum_{i=1}^{N} u_i = 0 \pmod{\mathbb{Z}} \;, \tag{1.1.14}$$

[4] This might sound like a strong limitation. However, the index (1.1.6) is invariant under integer shifts of τ and σ compatible with (1.1.4). As proven in [79], the set of complex number pairs $\{\tau, \sigma\} \in \mathbb{H}^2$ (two copies of the upper half-plane) whose ratio becomes a (real) rational number after some integer shifts of τ and σ, is dense in $\mathbb{H}^2$. Thus, by continuity, the formula of [79] fixes the large N limit of the superconformal index for generic complex chemical potentials.

[5] The Bethe operators Q_i should not be confused with the charges Q_I introduced before.

as well as a "Lagrange multiplier" λ. The SU(N) holonomies are to be identified with the first $N-1$ variables $u_{i=1,\dots,N-1}$. As unknowns in the BAEs, they are subject to the identifications

$$u_i \sim u_i + 1 \sim u_i + \omega \,, \tag{1.1.15}$$

meaning that each one of them naturally lives on a torus of modular parameter ω. Instead, the last holonomy u_N is determined by the constraint (1.1.14). The relation between SU(N) and U(N) holonomies will be further clarified in Appendix A.2. The prescription in (1.1.9) is to sum over all the inequivalent solutions on the torus [79]. The function H is the Jacobian

$$H = \det\left[\frac{1}{2\pi i}\,\frac{\partial(Q_1,\dots,Q_N)}{\partial(u_1,\dots,u_{N-1},\lambda)}\right] \,. \tag{1.1.16}$$

Finally, the function $\mathcal{Z}_{\text{tot}}$ is the following sum over a set of integers $m_i = 1,\dots,ab$:

$$\mathcal{Z}_{\text{tot}} = \sum_{\{m_i\}=1}^{ab} \mathcal{Z}\big(u - m\omega;\tau,\sigma\big) \,, \tag{1.1.17}$$

where $\mathcal{Z}$, for $\mathcal{N}=4$ SU(N) SYM, reads

$$\mathcal{Z} = \prod_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\widetilde{\Gamma}(u_{ij}+\Delta_1;\tau,\sigma)\,\widetilde{\Gamma}(u_{ij}+\Delta_2;\tau,\sigma)}{\widetilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\tau,\sigma)\,\widetilde{\Gamma}(u_{ij};\tau,\sigma)} \,. \tag{1.1.18}$$

The sum in (1.1.17) freely varies over the first $N-1$ integers $m_{i=1,\dots,N-1}$ as indicated, while m_N is determined by the constraint

$$\sum_{i=1}^{N} m_i = 0 \,. \tag{1.1.19}$$

More details can be found in [45, 79]. In the following, when a double sum starts from 1 we will leave it implicit.

1.1.1 The building block

We will show that one particular contribution to the sums in (1.1.9) and (1.1.17) alone reproduces the entropy function of [76] and therefore it captures the Bekenstein-Hawking entropy of BPS black holes in $\mathrm{AdS}_5\times S^5$. To that aim, we are interested in the contribution from the so-called "basic solution" to the BAEs [45, 81, 82], namely

$$u_i = \frac{N-i}{N}\,\omega + \overline{u} \,, \qquad u_{ij} \equiv u_i - u_j = \frac{j-i}{N}\,\omega \,, \qquad \lambda = \frac{N-1}{2} \,. \tag{1.1.20}$$

Here $\overline{u}$ is fixed by enforcing the constraint (1.1.14). We also consider the contribution from a particular choice for the integers $\{m_j\}$:

$$m_j \in \{1,\dots,ab\} \qquad \text{such that} \qquad m_j = j \mod ab \,. \tag{1.1.21}$$

Note that this choice for $\{m_j\}$ does not satisfy the constraint (1.1.19). Nevertheless, we show in Appendix A.2 that this does not affect the contribution to leading order in N, in the sense that changing the single entry m_N has a subleading effect.

Now, the crucial technical point is to evaluate the following basic building block:

$$\Psi = \sum_{i\neq j}^{N} \log\widetilde{\Gamma}\left(\Delta + \omega\,\frac{j-i}{N} + \omega\big(m_j - m_i\big); a\omega, b\omega\right) \tag{1.1.22}$$

for $N\to\infty$. Here Δ plays the role of an electric chemical potential. In order to simplify the discussion, we assume that N is a multiple of ab, *i.e.*, we take $N = ab\widetilde{N}$. As we show in Appendix A.3 this assumption can be removed without affecting the leading behavior at large N.

We make use of the following identity [95]:

$$\widetilde{\Gamma}(u;\tau,\sigma) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\widetilde{\Gamma}\Big(u + \big(r\tau + s\sigma\big); a\tau, b\sigma\Big) \tag{1.1.23}$$

for any $\tau,\sigma\in\mathbb{H}$ and any $a,b\in\mathbb{N}$. This is immediate to prove exploiting the infinite product expression of $\widetilde{\Gamma}$. Now, exchanging $a\leftrightarrow b$ and $r\leftrightarrow s$ in the formula, and then setting $\tau\to a\omega$, $\sigma\to b\omega$, we obtain the formula of [96]:

$$\widetilde{\Gamma}(u;a\omega,b\omega) = \prod_{r=0}^{a-1}\prod_{s=0}^{b-1}\widetilde{\Gamma}\Big(u + \big(as + br\big)\omega; ab\omega, ab\omega\Big)\,. \tag{1.1.24}$$

Going back to Ψ, we can thus write

$$\Psi = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{i\neq j}^{N}\log\widetilde{\Gamma}\left(\Delta + \omega\,\frac{j-i}{N} + \omega\big(m_j - m_i + as + br\big); ab\omega, ab\omega\right)\,. \tag{1.1.25}$$

Let us set $i = \gamma ab + c$, $j = \delta ab + d$ with $\gamma,\delta = 0,\dots,\widetilde{N}-1$ and $c,d = 1,\dots,ab$. Then

$$\Psi = \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\underbrace{\sum_{\gamma,\delta=0}^{\widetilde{N}-1}\sum_{c,d=1}^{ab}}_{\text{s.t. } i\neq j}\log\widetilde{\Gamma}\left(\Delta + \omega\frac{\delta-\gamma}{\widetilde{N}} + \omega\frac{d-c}{N} + \omega\big(d - c + as + br\big); ab\omega, ab\omega\right)\,. \tag{1.1.26}$$

We will now perform two simplifications, and prove in Appendix A.1 that their effect is of subleading order at large N. More precisely, Ψ is of order N^2 while the two simplifications modify it at most at order N if $\mathbb{Im}\big(\Delta/\omega\big)\notin\mathbb{Z}\times\mathbb{Im}\big(1/\omega\big)$, or at most at order $N\log N$ if $\Delta = 0$. First, we substitute the condition $i\neq j$ with the condition $\gamma\neq\delta$ in the summation. Second and more importantly, we drop the term $\omega(d-c)/N$ in the argument. We then redefine $c \to ab - c$, $d \to d+1$, $\gamma \to \gamma - 1$, $\delta \to \delta - 1$ and obtain

$$\Psi \simeq \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{\gamma \neq \delta}^{\widetilde{N}} \sum_{c,d=0}^{ab-1} \log \widetilde{\Gamma} \left(\Delta + \omega \frac{\delta - \gamma}{\widetilde{N}} + \omega \big(d + c + 1 - ab + as + br \big); ab\omega, ab\omega \right) \tag{1.1.27}$$

where $\simeq$ means equality at leading order in N. At this point we can resum over c, d using (1.1.23) (with $\tau, \sigma \to \omega$ and $a, b \to ab$):

$$\Psi \simeq \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{\gamma \neq \delta}^{\widetilde{N}} \log \widetilde{\Gamma} \left(\Delta + \omega \frac{\delta - \gamma}{\widetilde{N}} + \omega \big(1 - ab + as + br \big); \omega, \omega \right) . \tag{1.1.28}$$

We recall the large N limit computed in [45]:

$$\sum_{i \neq j}^{N} \log \widetilde{\Gamma} \left(\Delta + \omega \, \frac{j-i}{N}; \omega, \omega \right) = -\pi i N^2 \, \frac{B_3\big([\Delta]'_\omega - \omega \big)}{3\omega^2} + \mathcal{O}(N) \tag{1.1.29}$$

valid for $\mathbb{Im}\big(\Delta/\omega\big) \notin \mathbb{Z} \times \mathbb{Im}\big(1/\omega\big)$. Here $B_3(x)$ is a Bernoulli polynomial:

$$B_3(x) = x \left(x - \tfrac{1}{2} \right) \left(x - 1 \right) . \tag{1.1.30}$$

It has the property that $B_3(1-x) = -B_3(x)$. The function $[\Delta]'_\omega$ was defined in [45] in the following way:

$$[\Delta]'_\omega = \left\{ z \;\middle|\; z = \Delta \bmod 1 \,, \quad 0 > \mathbb{Im}\left(\frac{z}{\omega} \right) > \mathbb{Im}\Big(\frac{1}{\omega} \Big) \right\} . \tag{1.1.31}$$

This function is only defined for $\mathbb{Im}\big(\Delta/\omega\big) \notin \mathbb{Z} \times \mathbb{Im}\big(1/\omega\big)$, it is continuous in each open connected domain, and it is periodic by construction under $\Delta \to \Delta + 1$. In the following we will also use the function $[\Delta]_\omega = [\Delta]'_\omega - 1$,

$$[\Delta]_\omega = \left\{ z \;\middle|\; z = \Delta \bmod 1 \,, \quad \mathbb{Im}\Big(-\frac{1}{\omega} \Big) > \mathbb{Im}\left(\frac{z}{\omega} \right) > 0 \right\} . \tag{1.1.32}$$

The functions $[\Delta]_\omega$ and $[\Delta]'_\omega$ are the mod 1 reductions of Δ to the fundamental strips shown in Figure 1.1. Then we use the following formula:

$$\begin{aligned} \frac{1}{ab} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} B_3\big(x + \omega(as + br - ab) \big) &= \\ = B_3 \left(x - \frac{a+b}{2} \omega \right) &+ \frac{2a^2b^2 - a^2 - b^2}{4} \, \omega^2 \, B_1 \left(x - \frac{a+b}{2} \omega \right) , \end{aligned} \tag{1.1.33}$$

where $B_1(x) = x - \frac{1}{2}$ is another Bernoulli polynomial — and $B_1(1-x) = -B_1(x)$. Thus

$$\Psi = -\pi i N^2 \, \frac{B_3\Big([\Delta]'_\omega - \frac{\tau+\sigma}{2} \Big)}{3\tau\sigma} - \frac{\pi i N^2}{12} \Big(2ab - \frac{a}{b} - \frac{b}{a} \Big) \, B_1\Big([\Delta]'_\omega - \tfrac{\tau+\sigma}{2} \Big) + \mathcal{O}(N) \tag{1.1.34}$$

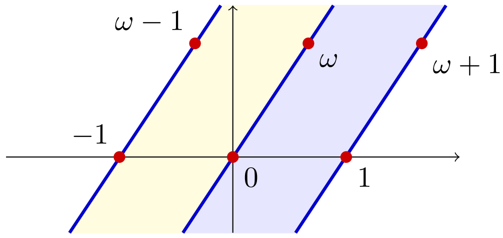

Figure 1.1: Fundamental strips for $[\Delta]_\omega$ and $[\Delta]'_\omega$. The function $[\Delta]_\omega$ is the restriction of Δ mod 1 to the region $\mathbb{Im}(-1/\omega) > \mathbb{Im}(\Delta/\omega) > 0$ (in yellow, on the left), while $[\Delta]'_\omega$ is the restriction of Δ mod 1 to the region $0 > \mathbb{Im}(\Delta/\omega) > \mathbb{Im}(1/\omega)$ (in blue, on the right).

for $\mathbb{Im}\big(\Delta/\omega\big) \notin \mathbb{Z} \times \mathbb{Im}\big(1/\omega\big)$. As a check, notice that $\big[\tau+\sigma-\Delta\big]'_\omega = \tau+\sigma+1-[\Delta]'_\omega$. From the properties of $B_{1,3}(x)$ noticed above, it follows

$$\Psi(\tau+\sigma-\Delta) = -\Psi(\Delta) \tag{1.1.35}$$

at leading order in N. This is in accordance with the inversion formula of the elliptic gamma function:

$$\widetilde{\Gamma}(u;\tau,\sigma) = 1/\widetilde{\Gamma}(\tau+\sigma-u;\tau,\sigma)\ . \tag{1.1.36}$$

The case $\Delta = 0$ requires some care, because $[0]_\omega$ is undefined. Taking the limit of Ψ as $\Delta \to 0$ from the left or the right, one obtains two values that differ by an imaginary quantity. The limit from the right corresponds to taking $[\Delta]'_\omega \to 0$ in (1.1.34), while the limit from the left corresponds to $[\Delta]_\omega \to 0$ (*i.e.*, $[\Delta]'_\omega \to 1$). The difference is

$$\Psi\Big|_{[\Delta]'_\omega\to 0} - \Psi\Big|_{[\Delta]_\omega\to 0} = \frac{i\pi N^2}{6}\left(3+ab+\frac{a}{b}+\frac{b}{a}\right)\ . \tag{1.1.37}$$

Since Ψ is in any case ambiguous by shifts of $2\pi i$ because it is a logarithm, only the remainder modulo $2\pi i$ is meaningful but this is an order 1 quantity which can be neglected. In fact it turns out that, with $N = ab\widetilde{N}$, the quantity on the right-hand-side of (1.1.37) is always an integer multiple of $i\pi\widetilde{N}$, and so its exponential is a sign. We should also notice that, for $\Delta = 0$, our approximation gets corrections at order $N\log N$.

1.1.2 The index and the entropy function

We are now ready to put all the ingredients together. Our working assumption is that, in the large N limit, the index (1.1.9) is dominated by the basic solution (1.1.20) and the choice of integers (1.1.21). Some evidence that the basic solution dominates the index for $\tau = \sigma$ has been given in [45] (see also [84]).

The leading contribution to (1.1.9) originates from $\mathcal{Z}_{\text{tot}}$ that can be evaluated using (1.1.34). Indeed, the term κ_N is manifestly sub-leading. That the contribution of H is also subleading follows from the analysis in [45] for $\tau=\sigma$, since H only depends on the solutions to the BAEs and not explicitly on τ and σ. The large N limit of the index at leading order is then

$$\log\mathcal{I}=\Psi(\Delta_1)+\Psi(\Delta_2)-\Psi(\Delta_1+\Delta_2)-\Psi(0)\ , \tag{1.1.38}$$

where the definition of the last term has an ambiguity of order 1.

Recall that in (1.1.4) we introduced the auxiliary chemical potential Δ_3. Notice in particular that the chemical potentials are defined modulo 1. Using the basic properties

$$[\Delta+1]_\omega=[\Delta]_\omega\ ,\qquad [\Delta+\omega]_\omega=[\Delta]_\omega+\omega\ ,\qquad [-\Delta]_\omega=-[\Delta]_\omega-1\ , \tag{1.1.39}$$

we find

$$[\Delta_3]_\omega=\tau+\sigma-1-[\Delta_1+\Delta_2]_\omega\ . \tag{1.1.40}$$

It follows from the definition of the function $[\Delta]_\omega$ that $[\Delta_1+\Delta_2]_\omega=[\Delta_1]_\omega+[\Delta_2]_\omega+n$ where $n=0$ or $n=1$. The result then breaks into two cases.

If $[\Delta_1+\Delta_2]_\omega=[\Delta_1]_\omega+[\Delta_2]_\omega$ then

$$[\Delta_1]_\omega+[\Delta_2]_\omega+[\Delta_3]_\omega-\tau-\sigma=-1\ , \tag{1.1.41}$$

and, using (1.1.38) and (1.1.34),

$$\begin{aligned}\log\mathcal{I}&=-\pi i N^2\,\frac{[\Delta_1]_\omega\,[\Delta_2]_\omega\left(\tau+\sigma-1-[\Delta_1]_\omega-[\Delta_2]_\omega\right)}{\tau\,\sigma}\\&=-i\pi N^2\,\frac{[\Delta_1]_\omega\,[\Delta_2]_\omega\,[\Delta_3]_\omega}{\tau\,\sigma}\ .\end{aligned} \tag{1.1.42}$$

To obtain this formula we used $\Psi(0)=\Psi\big|_{[\Delta]_\omega\to 0}$. Notice that the contributions from B_1 cancel out. As we will see in Section 1.2, this is a consequence of the relation $a=c$ among the two four-dimensional central charges in the large N limit.

If $[\Delta_1+\Delta_2]_\omega=[\Delta_1]_\omega+[\Delta_2]_\omega+1$, namely $[\Delta_1+\Delta_2]'_\omega=[\Delta_1]'_\omega+[\Delta_2]'_\omega$, then

$$[\Delta_1]'_\omega+[\Delta_2]'_\omega+[\Delta_3]'_\omega-\tau-\sigma=1\ , \tag{1.1.43}$$

and

$$\begin{aligned}\log\mathcal{I}&=-\pi i N^2\,\frac{[\Delta_1]'_\omega\,[\Delta_2]'_\omega\left(\tau+\sigma+1-[\Delta_1]'_\omega-[\Delta_2]'_\omega\right)}{\tau\,\sigma}\\&=-i\pi N^2\,\frac{[\Delta_1]'_\omega\,[\Delta_2]'_\omega\,[\Delta_3]'_\omega}{\tau\,\sigma}\ .\end{aligned} \tag{1.1.44}$$

This time we used $\Psi(0)=\Psi\big|_{[\Delta]'_\omega\to 0}$.

As in [45], we can extract the entropy of the dual black holes by taking the Legendre transform of the logarithm of the index. The precise identification of the charges associated with the chemical potentials follows from (1.1.6). The prediction for the entropy can then be combined into two constrained entropy functions

$$\mathcal{S}_{\pm}(X_I,\tau,\sigma,\Lambda) = -i\pi N^2\,\frac{X_1X_2X_3}{\tau\,\sigma} - 2\pi i\left(\sum_{I=1}^{3} X_I Q_I + \tau J_1 + \sigma J_2\right) \\ - 2\pi i\Lambda\Big(X_1+X_2+X_3-\tau-\sigma\pm 1\Big)\,, \quad (1.1.45)$$

where we used a neutral variable X_I to denote either $[\Delta_I]_\omega$ or $[\Delta_I]'_\omega$, we introduced a Lagrange multiplier Λ to enforce the constraint, and we recall that $Q_I = \frac{1}{2}R_I$. This completes our derivation of the entropy of supersymmetric black holes in $\mathrm{AdS}_5\times S^5$ for general angular momenta and electric charges. The expression (1.1.45) represents indeed the two entropy functions derived in [76], where it was shown that the (constrained) extremization of (1.1.45) reproduces the entropy of a black hole of angular momenta J_1 and J_2 and charges Q_I. The two results correspond to the two entropy functions that reproduce the same black hole entropy, and are associated to two Euclidean complex solutions that regularize the black hole horizon [43].

1.2 The index of quiver theories with a holographic dual

We want to generalize the large N computation of the superconformal index to theories dual to $\mathrm{AdS}_5\times\mathrm{SE}_5$ compactifications, where SE_5 is a five-dimensional Sasaki-Einstein manifold. We can write general formulæ with very few assumptions. We consider 4d $\mathcal{N}=1$ theories with $\mathrm{SU}(N)$ gauge groups as well as adjoint and bi-fundamental chiral multiplet fields. To cancel gauge anomalies, the total number of fields transforming in the fundamental representation of a group must be the same as the number of anti-fundamentals. We also require equality of the conformal central charges $c=a$ in the large N limit, as dictated by holography. Our analysis extends the results found in [83] for equal angular momenta.

We then assume that in the large N limit, as for $\mathcal{N}=4$ SYM, the leading contribution to the superconformal index comes from the basic solution and the choice of integers $\{m_i\}$ discussed in (1.1.21). As already shown in [83,85], the basic solution to the BAEs for $\mathcal{N}=4$ SYM [45,81,82] can easily be extended to quiver gauge theories by setting

$$u_{ij}^{\alpha\beta} \equiv u_i^\alpha - u_j^\beta = \frac{j-i}{N}\,\omega \qquad\qquad \alpha,\beta = 1,\dots,G\,, \quad (1.2.1)$$

where α,β run over the various gauge groups in the theory and G is the number of gauge groups. Similarly, we choose the integers

$$m_j^\alpha \in \{1,\dots,ab\} \qquad \text{such that} \qquad m_j^\alpha = j \mod ab\,. \quad (1.2.2)$$

Notice in particular that neither $u_{ij}^{\alpha\beta}$ nor m_j^α depend on α, β. As for $\mathcal{N} = 4$ SYM, the contribution of the determinant H to the Bethe Ansatz expansion (1.1.9) is subleading [83].

Using the general expressions given in [79] and following the logic of Section 1.1, it is easy to write the large N limit of the leading contribution to the superconformal index of a holographic theory, with adjoint and bi-fundamental chiral fields. We find

$$\log \mathcal{I} = \sum_{i\neq j}^{N} \left[\sum_{I_{\alpha\beta}} \log \widetilde{\Gamma}\Big(u_{ij}^{\alpha\beta} - \omega\big(m_i^\alpha - m_j^\beta\big) + \Delta_{I_{\alpha\beta}}; \tau, \sigma \Big) - \sum_{\alpha=1}^{G} \log \widetilde{\Gamma}\Big(u_{ij}^{\alpha\alpha} - \omega\big(m_i^\alpha - m_j^\beta\big); \tau, \sigma \Big) \right] \tag{1.2.3}$$

where $z_i^\alpha = e^{2\pi i u_i^\alpha}$ are the gauge fugacities, u_i^α represent the basic solution (1.2.1) and m_i^α are given in (1.2.2). The sum over $I_{\alpha\beta}$ is over all adjoint (if $\alpha = \beta$) and bi-fundamental (if $\alpha \neq \beta$) chiral multiplets in the theory. The second sum is the contribution of vector multiplets. When no confusion is possible, we will keep the gauge group indices implicit and just write $\Delta_{I_{\alpha\beta}} \equiv \Delta_I$. In the previous formula,

$$\Delta_I = \xi_I + r_I \, \frac{\tau + \sigma}{2} \,, \tag{1.2.4}$$

where r_I is the exact R-charge of the field and ξ_I are the flavor chemical potentials. The R-charges satisfy

$$\sum_{I \in W} r_I = 2 \tag{1.2.5}$$

for each superpotential term W in the Lagrangian. In this notation, the index W runs over the monomials in the superpotential, while $I \in W$ indicates all chiral fields appearing in a given monomial. Using that each superpotential term must be invariant under the flavor symmetries, but chemical potentials are only defined up to integers, we also require

$$\sum_{I \in W} \xi_I = n_W \qquad \text{for some} \qquad n_W \in \mathbb{Z} \,. \tag{1.2.6}$$

The values $n_W \equiv n_0 = \pm 1$ have been used in [97, 98] to study the Cardy limit. As a consequence of the previous formulæ, for each superpotential term we have

$$\sum_{I \in W} \Delta_I = \tau + \sigma + n_W \,. \tag{1.2.7}$$

Hence, we stress that the chemical potentials Δ_I are *not* independent. Notice that the expression (1.2.3) correctly reduces to the one for $\mathcal{N} = 4$ SYM, Eqn. (1.1.18), once we use the definition (1.1.4) as well as the inversion formula for the elliptic gamma function (1.1.36). We also need to use the exact R-charges $r_I = 2/3$ of the chiral fields Φ_I.

Applying (1.1.34), we can evaluate the large N limit of (1.2.3) and obtain

$$
\begin{aligned}
\log \mathcal{I} \;\simeq\; & -\frac{\pi i N^2}{3\tau\sigma} \sum_I \Big[B_3\Big([\Delta_I]_\omega + 1 - \tfrac{\tau+\sigma}{2} \Big) + \frac{\tau\sigma}{4}\Big(2ab - \frac{a}{b} - \frac{b}{a}\Big) B_1\Big([\Delta_I]_\omega + 1 - \tfrac{\tau+\sigma}{2} \Big) \Big] \\
& + \frac{\pi i G N^2}{3\tau\sigma} \Big[B_3\Big(1 - \tfrac{\tau+\sigma}{2} \Big) + \frac{\tau\sigma}{4}\Big(2ab - \frac{a}{b} - \frac{b}{a}\Big) B_1\Big(1 - \tfrac{\tau+\sigma}{2} \Big) \Big] \,.
\end{aligned} \tag{1.2.8}
$$

The corrections are of order $N \log N$ or smaller. The formula is obtained by summing (1.1.34) for each chiral multiplet, as well as (1.1.34) with $[\Delta]_\omega \to 0$ (and opposite sign) for each vector multiplet. We stress that (1.2.8) comes from a *single* contribution — in the Bethe Ansatz expansion — to the index. Such a contribution might not be the dominant one, and so our estimate of the index might be incorrect, in some regions of the space of chemical potentials. However, we conjecture and we will bring some evidence that this contribution always captures the semiclassical Bekenstein-Hawking entropy of BPS black holes.

Due to the presence of the brackets $[\Delta_I]_\omega$, the expression (1.2.8) assumes different analytic forms in different regions of the space of chemical potentials Δ_I. There are two regions where the expression greatly simplifies. They correspond to the natural generalization of the two regions for $\mathcal{N} = 4$ SYM discussed in Section 1.1.2 and are expected to lead to the correct black hole entropy. In particular, they smoothly reduce to the results obtained in the Cardy limit [87, 97, 98] and match the previous analysis done for equal angular momenta [83]. The first region corresponds to chemical potentials Δ_I satisfying

$$
\sum_{I \in W} [\Delta_I]_\omega = \tau + \sigma - 1 \,. \tag{1.2.9}
$$

As we will discuss later, many models — in particular all toric ones — exhibit a corner in the space of chemical potentials where this constraint is satisfied. We can define the rescaled variables

$$
\widehat{\Delta}_I = 2\, \frac{[\Delta_I]_\omega}{\tau + \sigma - 1} \tag{1.2.10}
$$

which, under the assumption (1.2.9), satisfy

$$
\sum_{I \in W} \widehat{\Delta}_I = 2 \tag{1.2.11}
$$

and can be interpreted as an assignment of R-charges to the chiral fields in the theory. In terms of $\widehat{\Delta}_I$ the contributions in (1.2.8) combine into

$$
\begin{aligned}
\log \mathcal{I} \;\simeq\; & -\frac{\pi i N^2}{24} \frac{(\tau+\sigma-1)^3}{\tau\sigma} \Big[\sum\nolimits_I \big(\widehat{\Delta}_I - 1\big)^3 + G \Big] \\
& + \frac{\pi i N^2}{24} \frac{(\tau+\sigma-1)}{\tau\sigma} \Big(1 - \tau\sigma\Big(2ab - \frac{a}{b} - \frac{b}{a}\Big)\Big) \Big[\sum\nolimits_I \big(\widehat{\Delta}_I - 1\big) + G \Big] \,.
\end{aligned} \tag{1.2.12}
$$

Introducing the charge operator $R(\widehat{\Delta})$ of R-charges parametrized by $\widehat{\Delta}_I$ and indicating with Tr the sum over all fermions in the theory, we can also write

$$\log \mathcal{I} \;\simeq\; -\frac{\pi i}{24}\left[\frac{(\tau+\sigma-1)^3}{\tau\sigma}\,\mathrm{Tr}\,R(\widehat{\Delta})^3 - \frac{(\tau+\sigma-1)}{\tau\sigma}\left(1-\tau\sigma\Big(2ab-\frac{a}{b}-\frac{b}{a}\Big)\right)\mathrm{Tr}\,R(\widehat{\Delta})\right], \tag{1.2.13}$$

valid at leading order in N.

In the large N limit, theories with a holographic dual satisfy $c = a$. Using standard formulæ for the central charges a and c in terms of the fermion R-charges [99], one finds $\mathrm{Tr}\,R = \mathcal{O}(1)$ and $a = \frac{9}{32}\,\mathrm{Tr}\,R^3 + \mathcal{O}(1)$ from which we obtain the final expression

$$\log \mathcal{I} \;\simeq\; -\frac{4\pi i}{27}\,\frac{(\tau+\sigma-1)^3}{\tau\sigma}\,a(\widehat{\Delta})\;, \tag{1.2.14}$$

where

$$a = \frac{9}{32}\,N^2\left(\sum\nolimits_I\big(\widehat{\Delta}_I-1\big)^3+G\right) \tag{1.2.15}$$

at leading order in N. The result (1.2.14) was conjectured in [86] — see Eqn. (A.7). It is also compatible with the Cardy limit performed in [97, 98].

We can find an analogous result in a second region of chemical potentials where

$$\sum_{I\in W}[\Delta_I]'_\omega = \tau+\sigma+1\;, \tag{1.2.16}$$

written in terms of the primed bracket $[\Delta]'_\omega = [\Delta]_\omega + 1$. As discussed at the end of Section 1.1, the contribution of vector multiplets can be written, up to subleading terms, as minus the contribution of a chiral multiplet with $[\Delta_I]'_\omega \to 0$. After defining another set of normalised R-charges,

$$\widehat{\Delta}'_I = 2\,\frac{[\Delta_I]'_\omega}{\tau+\sigma+1} \tag{1.2.17}$$

which satisfy

$$\sum_{I\in W}\widehat{\Delta}'_I = 2 \tag{1.2.18}$$

under the assumption (1.2.16), we can rewrite the index as

$$\log \mathcal{I} \;\simeq\; -\frac{\pi i}{24}\left[\frac{(\tau+\sigma+1)^3}{\tau\sigma}\,\mathrm{Tr}\,R(\widehat{\Delta}')^3 - \frac{(\tau+\sigma+1)}{\tau\sigma}\left(1-\tau\sigma\Big(2ab-\frac{a}{b}-\frac{b}{a}\Big)\right)\mathrm{Tr}\,R(\widehat{\Delta}')\right] \tag{1.2.19}$$

at leading order in N. This reduces to the simple expression

$$\log \mathcal{I} \;\simeq\; -\frac{4\pi i}{27}\,\frac{(\tau+\sigma+1)^3}{\tau\sigma}\,a(\widehat{\Delta}') \tag{1.2.20}$$

for holographic theories.

Field	r	Q_{F_1}	Q_{F_2}	Q_B	R_1	R_2	R_3	R_4
A_1	$\frac{1}{2}$	1	0	1	2	0	0	0
A_2	$\frac{1}{2}$	-1	0	1	0	2	0	0
B_1	$\frac{1}{2}$	0	1	-1	0	0	2	0
B_2	$\frac{1}{2}$	0	-1	-1	0	0	0	2

Table 1.1: Charges of chiral multiplets in the Klebanov-Witten theory, under the maximal torus of the global symmetry $U(1)_R \times SU(2)_{F_1} \times SU(2)_{F_2} \times U(1)_B$. In the table we indicate two useful basis. Notice that r and R_I are R-charges, while $Q_{F_{1,2}}$ and Q_B are flavor charges.

In the remainder of this section we will interpret the general results (1.2.14) and (1.2.20) and provide examples. In particular, we will show that both regions (1.2.9) and (1.2.16) in the space of chemical potentials always exist in toric quiver gauge theories. We will also see that the two expressions (1.2.14) and (1.2.20) lead to the very same result for the semiclassical entropy of dual black holes, generalizing what happens for $\mathcal{N} = 4$ SYM.

1.2.1 Example: the conifold

We start with the example of the Klebanov-Witten theory dual to $\mathrm{AdS}_5 \times T^{1,1}$, the near-horizon limit of a set of N D3-branes sitting at a conifold singularity [88]. This example was already studied for equal angular momenta in [83] and our results are consistent with those found there when we set $\tau = \sigma$.

The theory has gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$, bi-fundamental chiral multiplets A_1, A_2 transforming in the representation $(N, \overline{N})$ and B_1, B_2 transforming in the representation $(\overline{N}, N)$, and a superpotential

$$W = \mathrm{Tr}\big(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\big) \,. \tag{1.2.21}$$

The global symmetry of the theory is $U(1)_R \times SU(2)_{F_1} \times SU(2)_{F_2} \times U(1)_B$, where the first factor is the superconformal R-symmetry with charge r, while the other three factors are flavor symmetries. The charge assignments of chiral multiplets under the maximal torus are in Table 1.1. The index is defined as

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F \, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} \, p^{J_1 + r/2} \, q^{J_2 + r/2} \, v_{F_1}^{Q_{F_1}} \, v_{F_2}^{Q_{F_2}} \, v_B^{Q_B} \,. \tag{1.2.22}$$

It is convenient to introduce an alternative basis of R-charges R_I with $I = 1, 2, 3, 4$, such that each of them assigns R-charge 2 to one of the chiral multiplets and zero to the other ones. Correspondingly, we associate a variable Δ_I to each chiral multiplet. Notice that

$(-1)^F = e^{2\pi i J_{1,2}} = e^{\pi i R_{1,2,3,4}}$. According to (1.2.4) and up to integer ambiguities, the variables Δ_I are related to the chemical potentials for the charges in Table 1.1 by

$$\begin{aligned} \Delta_1 &= \xi_{F_1} + \xi_B + \frac{\tau+\sigma}{4} \,, & \Delta_3 &= \xi_{F_2} - \xi_B + \frac{\tau+\sigma}{4} \,, \\ \Delta_2 &= -\xi_{F_1} + \xi_B + \frac{\tau+\sigma}{4} \,, & \Delta_4 &= -\xi_{F_2} - \xi_B + \frac{\tau+\sigma}{4} + (2\mathbb{Z}+1) \,. \end{aligned} \tag{1.2.23}$$

Then, the constraint (1.2.7) reads

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = \tau + \sigma + n_W \tag{1.2.24}$$

and the index takes the more transparent form

$$\mathcal{I} = \mathrm{Tr}_{\mathrm{BPS}}\, p^{J_1}\, q^{J_2}\, y_1^{R_1/2}\, y_2^{R_2/2}\, y_3^{R_3/2}\, y_4^{R_4/2} \,. \tag{1.2.25}$$

This shows that Δ_I are the chemical potentials associated to the charges $Q_I \equiv R_I/2$.

We select three independent variables, say Δ_1, Δ_2 and Δ_3. Then, using (1.1.39) we find that

$$[\Delta_4]_\omega = \tau + \sigma - 1 - [\Delta_1 + \Delta_2 + \Delta_3]_\omega \,. \tag{1.2.26}$$

In general there are three possible cases:

$$[\Delta_1 + \Delta_2 + \Delta_3]_\omega = [\Delta_1]_\omega + [\Delta_2]_\omega + [\Delta_3]_\omega + n \qquad \text{with} \qquad n = 0, 1, 2 \tag{1.2.27}$$

that we call Case I, II and III, respectively.[6]

Case I corresponds to the corner of moduli space (1.2.9) where

$$[\Delta_1]_\omega + [\Delta_2]_\omega + [\Delta_3]_\omega + [\Delta_4]_\omega = \tau + \sigma - 1 \,. \tag{1.2.28}$$

In this corner, we can use (1.2.14). One can explicitly compute, at leading order in N,

$$\mathrm{Tr}\, R(\widehat{\Delta})^3 = N^2\Big(2 + \sum_{I=1}^4 \big(\widehat{\Delta}_I - 1\big)^3\Big) = 3N^2\Big(\widehat{\Delta}_1\widehat{\Delta}_2\widehat{\Delta}_3 + \widehat{\Delta}_1\widehat{\Delta}_2\widehat{\Delta}_4 + \widehat{\Delta}_1\widehat{\Delta}_3\widehat{\Delta}_4 + \widehat{\Delta}_2\widehat{\Delta}_3\widehat{\Delta}_4\Big) \tag{1.2.29}$$

imposing $\sum_{I=1}^4 \widehat{\Delta}_I = 2$. Using (1.2.10), we can write the index (1.2.14) as

$$\log \mathcal{I} \;\simeq\; -\frac{\pi i N^2}{\tau\sigma}\Big([\Delta_1]_\omega[\Delta_2]_\omega[\Delta_3]_\omega + [\Delta_1]_\omega[\Delta_2]_\omega[\Delta_4]_\omega + [\Delta_1]_\omega[\Delta_3]_\omega[\Delta_4]_\omega + [\Delta_2]_\omega[\Delta_3]_\omega[\Delta_4]_\omega\Big) \tag{1.2.30}$$

with the constraint (1.2.28).[7]

[6] For the sake of comparison, the notation is the same as in [83].

[7] For toric models, discussed in detail in Section 1.2.2, we can compute the index using formula (1.2.52). The 't Hooft coefficients are expressed in terms of toric data as $C_{abc} = \big|\det\{v_a, v_b, v_c\}\big|$, where v_a are the integer vectors defining the toric fan [100]. For the conifold: $v_1 = (1,0,0)$, $v_2 = (1,1,0)$, $v_3 = (1,1,1)\,, v_4 = (1,0,1)$ and thus $C_{123} = C_{124} = C_{134} = C_{234} = 1$ (and symmetrizations), recovering the expression above.

Case III corresponds to the corner of moduli space (1.2.16). Indeed

$$[\Delta_1]'_\omega + [\Delta_2]'_\omega + [\Delta_3]'_\omega + [\Delta_4]'_\omega = \tau + \sigma + 1 \,. \tag{1.2.31}$$

In this corner, we can use (1.2.20) and (1.2.17) and find

$$\log \mathcal{I} \;\simeq\; -\frac{\pi i N^2}{\tau\sigma}\Big([\Delta_1]'_\omega[\Delta_2]'_\omega[\Delta_3]'_\omega + [\Delta_1]'_\omega[\Delta_2]'_\omega[\Delta_4]'_\omega + [\Delta_1]'_\omega[\Delta_3]'_\omega[\Delta_4]'_\omega + [\Delta_2]'_\omega[\Delta_3]'_\omega[\Delta_4]'_\omega\Big) \tag{1.2.32}$$

with the constraint (1.2.31).

The entropy, which is the logarithm of the number of states, is given by the Legendre transform of the index, *i.e.*, by the critical value of the entropy function

$$\begin{aligned} \mathcal{S} = &-\frac{\pi i N^2}{\tau\sigma}\Big(X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4\Big) \\ &- 2\pi i\Big(\tau J_1 + \sigma J_2 + \sum_{I=1}^{4} X_I Q_I\Big) - 2\pi i\Lambda\Big(\sum_{I=1}^{4} X_I - \tau - \sigma \pm 1\Big) \,. \end{aligned} \tag{1.2.33}$$

Here the variables X_I stand for $[\Delta_I]_\omega$ or $[\Delta_I]'_\omega$ depending on whether we are in case I or III, respectively, and the $\pm$ sign is chosen accordingly. One can check that the two signs lead to the same entropy. We will give a general argument in Section 1.2.3.

In Section 1.4 we will compare the field theory result (1.2.33) with the entropy of black holes in $\mathrm{AdS}_5 \times T^{1,1}$, in the special case that $J_1 = J_2 \equiv J$ and the $SU(2)_{F_1} \times SU(2)_{F_2}$ symmetry is unbroken. To that purpose, let us specialize the index to the case that $\tau = \sigma$ and $\xi_{F_1} = \xi_{F_2} = 0$, which corresponds to $X_1 = X_2$ and $X_3 = X_4$. It is then useful to define the new variables

$$X_R = X_1 + X_3 \,, \qquad\qquad X_B = \frac{X_1 - X_3}{2} \,, \tag{1.2.34}$$

associated to R-symmetry and baryonic symmetry, respectively. The entropy function takes the simplified form

$$\mathcal{S} = -\frac{\pi i N^2}{2\tau^2}\, X_R\big(X_R^2 - 4X_B^2\big) - 2\pi i\Big(2\tau J + X_R r + X_B Q_B\Big) - 2\pi i\Lambda\Big(2X_R - 2\tau \pm 1\Big) \,. \tag{1.2.35}$$

1.2.2 Example: toric models

In this section we consider the gauge theory dual to an $\mathrm{AdS}_5 \times \mathrm{SE}_5$ geometry, where SE_5 is a toric Sasaki-Einstein manifold. The theory lives on a stack of N D3-branes sitting at the toric Calabi-Yau singularity $C(\mathrm{SE}_5)$ obtained by taking the cone over SE_5 [88, 101]. There is a general construction to extract gauge theory data from the geometry of the Calabi-Yau singularity [102–105]. The main complication compared to the $\mathbb{C}^3$ and the conifold cases is that there is no one-to-one correspondence between bi-fundamental fields Φ_I (and associated variables Δ_I) and R-symmetries R_a. However, we will argue in general that there always exist two corners of the space of chemical potentials where (1.2.9) and (1.2.16) are satisfied and the results (1.2.14) and (1.2.20) are valid. There are also other corners that should be analyzed separately for every specific model. Our findings are consistent with the case-by-case analysis performed in [83] for equal angular momenta.

We first need to understand how to write the trial central charges $a(\widehat{\Delta})$ and $a(\widehat{\Delta}')$ that enter in the expressions (1.2.14) and (1.2.20). Since the quantities $\widehat{\Delta}_I$ and $\widehat{\Delta}'_I$ satisfy the constraints (1.2.11), they can be interpreted as a set of trial R-charges for the chiral fields in the quiver. In the toric case, we can find an efficient parametrization of the trial R-charges of fields using the data of the toric diagram. Let us review how this is done.

A toric Calabi-Yau threefold singularity can be specified by a fan, *i.e.*, a convex cone in $\mathbb{R}^3$ defined by D integer vectors $v_a = (1, \vec{v}_a)$ lying on a plane. The restrictions $\vec{v}_a$ of those vectors to the plane define a regular convex polygon with integer vertices called the toric diagram. In the list $\{v_a\}$ we should include all integer vectors such that $\vec{v}_a$ is along the perimeter of the polygon, *i.e.*, we should include all integer points along the edges of the toric diagram. Moreover, we take the points $\vec{v}_a$ to be ordered in a counterclockwise fashion. The number of vectors in the fan is associated with the total rank of the global symmetry of the dual field theory [104]: for a toric model with D vectors in the fan (including integer points along the edges of the toric diagram) there is a flavor symmetry of rank $D-1$, besides the R-symmetry $U(1)_R$.[8] This allows us to parametrize flavor and R-symmetries in terms of variables associated with the vertices of (and integer points along) the toric diagram. In particular, the possible R-charges of fields in a toric theory can be parametrized using D variables δ_a satisfying the constraint

$$\sum_{a=1}^{D} \delta_a = 2 \, , \tag{1.2.36}$$

and the corresponding R-charge can be written as

$$R(\delta) = \sum_{a=1}^{D} \frac{\delta_a}{2} R_a \tag{1.2.37}$$

in terms of a basis $\{R_a\}$. This is done as follows [106]. In a minimal toric phase,[9] the theory contains a number G of gauge group factors SU(N) equal to twice the area of the

[8] The distinction between R- and flavor symmetries changes in the case of extended supersymmetry.

[9] There are many different quiver theories that describe the same IR SCFT. They are called "phases", and are related by Seiberg dualities. The toric phases are the quiver theories where all gauge groups are $SU(N)$ with the same rank N. It turns out that all toric phases have the same number G of gauge groups, but have different matter content. The "minimal" phases correspond to the quivers with the smallest number of chiral fields. There could be one or more minimal toric phases, for a given IR SCFT.

toric diagram. Moreover, defining the vectors $\vec{w}_a = \vec{v}_{a+1} - \vec{v}_a$ lying in the plane (we identify indices modulo D, so that, for example, $\vec{v}_{D+1} \equiv \vec{v}_1$), for each pair (a, b) such that $\vec{w}_a$ can be rotated counterclockwise into $\vec{w}_b$ in the plane with an angle smaller than π, there are precisely[10] $\det\{\vec{w}_a, \vec{w}_b\}$ bi-fundamental chiral fields Φ_{ab} with R-charge

$$R[\Phi_{ab}] = \delta_{a+1} + \delta_{a+2} + \ldots + \delta_b \, . \tag{1.2.38}$$

Interestingly, for all toric models the trial central charge $a(\delta)$ is a homogeneous function of degree three at large N:

$$a(\delta) = \frac{9}{32} \operatorname{Tr} R(\delta)^3 = \frac{9N^2}{64} \sum_{a,b,c=1}^{D} C_{abc} \, \delta_a \, \delta_b \, \delta_c \, . \tag{1.2.39}$$

Here $N^2 C_{abc} = \frac{1}{4} \operatorname{Tr} R_a R_b R_c$ are the 't Hooft anomaly coefficients, which can be read from the toric data through $C_{abc} = \big|\det\{v_a, v_b, v_c\}\big|$ [100]. Another important property of toric models that we will use in the following is that the constraints

$$\sum_{I \in W} R[\Phi_I] = 2 \, , \tag{1.2.40}$$

that must be satisfied for each monomial term W in the superpotential, always reduce to (1.2.36). Indeed, it follows from tiling techniques [102–106] that the R-charges $R[\Phi_I]$, $I \in W$, of the chiral fields entering in a superpotential monomial W correspond to a partition of the D elementary R-charges $\{\delta_1, \ldots, \delta_D\}$ into sums of the form (1.2.38), with each δ_a entering in just one $R[\Phi_I]$.

We can similarly parametrize the chemical potentials $\Delta[\Phi]$ entering the superconformal index in terms of D basic quantities Δ_a, $a = 1, \ldots, D$. For the chiral fields Φ_{ab} we have

$$\Delta[\Phi_{ab}] = \Delta_{a+1} + \Delta_{a+2} + \ldots + \Delta_b \, . \tag{1.2.41}$$

The conditions

$$\sum_{I \in W} \Delta[\Phi_I] = \tau + \sigma + n_W \, , \tag{1.2.42}$$

to be imposed for each monomial term W in the superpotential (and where n_W is the same for all monomial terms), are then equivalent to

$$\sum_{a=1}^{D} \Delta_a = \tau + \sigma + n_W \, . \tag{1.2.43}$$

Independently of the value of n_W, we have

$$[\Delta_D]_\omega = \tau + \sigma - 1 - \Big[\sum\nolimits_{a=1}^{D-1} \Delta_a\Big]_\omega \, . \tag{1.2.44}$$

[10] The condition on the angle guarantees that the formula for the number of fields gives a non-negative integer.

In general

$$\left[\sum_{a=1}^{D-1} \Delta_a\right]_\omega = \sum_{a=1}^{D-1} [\Delta_a]_\omega + n \tag{1.2.45}$$

where $n = 0, \ldots, D-2$, thus dividing the space of parameters into $D-1$ regions.

Two regions are particularly important for our analysis. The region $n = 0$ corresponds to

$$\sum_{a=1}^{D} [\Delta_a]_\omega = \tau + \sigma - 1 \,, \tag{1.2.46}$$

while $n = D-2$ corresponds to

$$\sum_{a=1}^{D} [\Delta_a]'_\omega = \tau + \sigma + 1 \,. \tag{1.2.47}$$

We can argue that the two regions (1.2.46) and (1.2.47) are always realized somewhere in the space of parameters. For example, we can choose one elementary variable, say Δ_1, to live in the fundamental strip $\mathbb{Im}(-1/\omega) > \mathbb{Im}\big(\Delta_1/\omega\big) > 0$ (see Fig. 1.1) and slightly on the right of the vertical line passing through $\tau + \sigma - 1$, while all the other Δ_a to live in the fundamental strip and slightly on the left of the vertical line passing through zero. One easily verifies that they can be arranged to satisfy (1.2.46). A similar construction gives parameters satisfying (1.2.47). We now argue that (1.2.46) and (1.2.47) imply (1.2.9) and (1.2.16), respectively. We start noticing that

$$\sum_{a=1}^{D} [\Delta_a]_\omega = \tau + \sigma - 1 \qquad \Rightarrow \qquad \mathbb{Im}\left(\frac{1}{\omega} \sum\nolimits_{a=1}^{D} [\Delta_a]_\omega\right) = \mathbb{Im}\left(-\frac{1}{\omega}\right) \,. \tag{1.2.48}$$

Since each of the $[\Delta_a]_\omega$ lives in the fundamental strip $\mathbb{Im}(-1/\omega) > \mathbb{Im}\big([\Delta_a]_\omega/\omega\big) > 0$, the previous equation implies that $\mathbb{Im}(-1/\omega) > \mathbb{Im}\big(\sum_{a\in S}[\Delta_a]_\omega/\omega\big) > 0$ for any proper subset S of the indices $\{1, \ldots, D\}$. Thus (1.2.46) implies that

$$\left[\sum\nolimits_{a\in S} \Delta_a\right]_\omega = \sum\nolimits_{a\in S} [\Delta_a]_\omega \tag{1.2.49}$$

for any proper subset $S \subsetneq \{1, \ldots, D\}$. This implies that all charges in (1.2.41) *split*, in the sense that $\big[\Delta_{a+1} + \ldots + \Delta_b\big]_\omega = [\Delta_{a+1}]_\omega + \ldots + [\Delta_b]_\omega$. At this point, since all $\big[\Delta[\Phi_I]\big]_\omega$ split and each Δ_a enters precisely once in every superpotential constraint, the condition (1.2.9) is a consequence of (1.2.46).[11] A similar argument shows that (1.2.47) implies (1.2.16). Notice

[11] There is an alternative algorithm that produces potentials Δ_I satisfying (1.2.9). Choose a perfect matching p_α of the dimer model of the theory [104]. It divides the chiral fields into two groups: those Φ_{P} appearing in the perfect matching, and those Φ_{NP} not doing so. Choose the potentials Δ_{NP} to be in the fundamental strip and slightly on the left of the origin. Each superpotential term W contains one and only one of the fields Φ_{P} (by definition of perfect matching): choose the corresponding Δ_{P} to be in the fundamental strip and slightly on the right of the point $\tau + \sigma - 1$, in such a way that (1.2.9) for that particular W is satisfied. The drawback of this construction is that it does not tell us what the independent variables Δ_a are.

that the region specified by (1.2.9) can be larger than (1.2.46) and, similarly, the region specified by (1.2.16) can be larger than (1.2.47). This, in particular, happens for Calabi-Yau cones with codimension-one orbifold singularities. This is the case of the models SPP and dP_4 discussed in [83].[12] For all the cones without orbifold singularities that we checked, the two regions (1.2.9) and (1.2.46) coincide. It would be interesting to see if this is a general result.

We are now ready to evaluate the index. Consider region (1.2.9) first. Since the chemical potentials $[\Delta_I]_\omega$ split, the rescaled quantities

$$\widehat{\Delta}_a = 2\,\frac{[\Delta_a]_\omega}{\tau+\sigma-1} \qquad \text{with} \qquad \sum_{a=1}^{D} \widehat{\Delta}_a = 2 \tag{1.2.50}$$

provide a parametrization of the R-charges of chiral fields in the quiver in the sense discussed above. Using the general formula (1.2.39) we can then write

$$a(\widehat{\Delta}) = \frac{9N^2}{64} \sum_{a,b,c=1}^{D} C_{abc}\, \widehat{\Delta}_a\, \widehat{\Delta}_b\, \widehat{\Delta}_c \,. \tag{1.2.51}$$

Plugging it into (1.2.14) and re-expressing the result in terms of the chemical potentials $[\Delta_a]_\omega$, we find the large N limit of the superconformal index in region (1.2.9):

$$\log \mathcal{I} \;\simeq\; -\pi i N^2 \sum_{a,b,c=1}^{D} \frac{C_{abc}}{6}\, \frac{[\Delta_a]_\omega [\Delta_b]_\omega [\Delta_c]_\omega}{\tau\sigma} \,, \qquad\qquad \sum_{a=1}^{D} [\Delta_a]_\omega = \tau+\sigma-1 \,. \tag{1.2.52}$$

A similar argument shows that, in region (1.2.16),

$$\log \mathcal{I} \;\simeq\; -\pi i N^2 \sum_{a,b,c=1}^{D} \frac{C_{abc}}{6}\, \frac{[\Delta_a]'_\omega [\Delta_b]'_\omega [\Delta_c]'_\omega}{\tau\sigma} \,, \qquad\qquad \sum_{a=1}^{D} [\Delta_a]'_\omega = \tau+\sigma+1 \,. \tag{1.2.53}$$

We will show in the next section that both (1.2.52) and (1.2.53) lead to the same entropy.

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[12] Models with codimension-one orbifold singularities are characterized by toric diagrams where at least one vector $\vec{v}_a$ lies in the interior of an edge. The parameters δ_a associated with integer points lying in the interior of an edge of the polygon enter in the parametrization (1.2.38) of the R-charges of chiral fields, but no elementary field carries precisely charge δ_a. In order to recover the region (1.2.9), we can require the following. Construct a set M by grouping the points $\{1,\dots,D\}$ along the toric diagram in the following way: Break each edge in two pieces at a non-integer point, and then for each vertex form a group (that will be an element of M) that contains the vertex itself and all other integer points (if any) along the two pieces of edges on the two sides. (In the absence of orbifold singularity, M necessarily coincides with $\{1,\dots,D\}$.) Then require that the sums split over the groups in M for every proper subgroup $S' \subsetneq M$, and for every possible choice of M. This region is typically larger than (1.2.46).

1.2.3 The entropy function

For toric holographic quivers, we have found two different expressions, (1.2.52) and (1.2.53), for the large N limit of the superconformal index that are valid in two different regions in the space of chemical potentials. The two expressions differ only for the constraint and give rise to the very same entropy. This generalizes an observation made in [43] for $\mathcal{N}=4$ SYM and holds for general quivers.

To show that, we define two entropy functions

$$S_\pm = -\pi i N^2 \sum_{a,b,c=1}^{D} \frac{C_{abc}}{6} \frac{X_a X_b X_c}{\tau\sigma} - 2\pi i\left(\tau J_1 + \sigma J_2 + \sum_{a=1}^{D} X_a Q_a\right) \\ - 2\pi i \Lambda\left(\sum_{a=1}^{D} X_a - \tau - \sigma \pm 1\right) , \quad (1.2.54)$$

where Λ is a Lagrange multiplier and we used neutral variables X_a to denote either $[\Delta_a]_\omega$ or $[\Delta_a]'_\omega$. Each of the electric charges $Q_a \equiv R_a/2$ is defined in terms of an R-charge R_a that assigns charge 2 to all chiral multiplets Φ_{ab} such that δ_a appears in the decomposition (1.2.38), and zero to all the other ones. The 't Hooft anomaly coefficients are defined by

$$C_{abc} N^2 = \frac{1}{4} \operatorname{Tr} R_a R_b R_c . \quad (1.2.55)$$

Above, S_+ is the prediction for the entropy of the dual black hole based on the superconformal index in the region of parameters (1.2.9) while S_- in the region (1.2.16). The form of the entropy function (1.2.54) was first conjectured in [86].

Observe that, since $S_\pm \pm 2\pi i\Lambda$ are homogeneous functions of degree one in (X_a, τ, σ), the values of the functions $S_\pm(X_a, \tau, \sigma, \Lambda)$ at the critical point are related to the Lagrange multiplier by

$$S_\pm\big|_{\text{crit}} = \mp 2\pi i \Lambda . \quad (1.2.56)$$

Observe also that, if Q_a, J_i are real (as charges should be), then the two functions are related by $\overline{S_+(X_a, \tau, \sigma, \Lambda)} = S_-\big(-\overline{X}_a, -\overline{\tau}, -\overline{\sigma}, \overline{\Lambda}\big)$. Hence, if $(X_a, \tau, \sigma, \Lambda)$ is a critical point of S_+, then $\big(-\overline{X}_a, -\overline{\tau}, -\overline{\sigma}, \overline{\Lambda}\big)$ is a critical point of S_- with critical value

$$S_-\big|_{\text{crit}} = \overline{S}_+\big|_{\text{crit}} . \quad (1.2.57)$$

For arbitrary and general real charges Q_a and J_i, the critical value of S_+ is not real. For $\mathcal{N}=4$ SYM, however, it becomes real and equal to the entropy when imposing the non-linear constraint on conserved charges that characterizes supersymmetric black holes [43,76]. The same phenomenon was already observed in AdS$_4$ in [30]. We expect the same to be true for general black holes in Sasaki-Einstein compactifications. Even if this were wrong and S_+

were not real, it would still makes sense to identify the entropy with $\mathbb{R}\mathrm{e}\, S_+$. In all cases, we see from (1.2.57) that both constraints in (1.2.54) lead to the very same result for the entropy.

The entropy functions (1.2.54) give our general result for the entropy of black holes in $\mathrm{AdS}_5 \times \mathrm{SE}_5$. We derived it for toric quiver gauge theories, but the very same argument can be extended to a class of more general non-toric quivers. In particular, the expression (1.2.54) only depends on the 't Hooft anomaly coefficients C_{abc} for a basis of R-symmetries and, as such, we expect that it is the correct result for generic holographic quiver theories.

1.3 The universal rotating black hole

In this section we discuss the case of the *universal rotating black hole* which has electric charge aligned with the exact R-symmetry of the theory. The black hole arises as a solution of minimal gauged supergravity in five dimensions and, as such, it can be embedded in any $\mathrm{AdS}_5 \times \mathrm{SE}_5$ compactification of type IIB and, more generally, in any AdS_5 solution of type II or M theory.[13] Due to its universal character, most of the analysis is identical to the one for $\mathrm{AdS}_5 \times S^5$. It is however interesting to see how the details work.

The universal black hole in AdS_5 was found in [72] in minimal gauged supergravity in five dimensions. It has charge Q under the graviphoton and angular momenta J_1 and J_2 in AdS_5.[14] The entropy can be compactly written as [107]

$$S(Q, J) = 2\pi\sqrt{3Q^2 - 2a(J_1 + J_2)} \tag{1.3.1}$$

where we introduced the quantity

$$a = \frac{\pi \ell_5^3}{8G_{\mathrm{N}}^{(5)}}\,, \tag{1.3.2}$$

where $G_{\mathrm{N}}^{(5)}$ is the five-dimensional Newton constant and ℓ_5 is the radius of AdS_5. The conserved charges must satisfy the nonlinear constraint

$$8Q^3 + 6aQ^2 - 6a(J_1 + J_2)Q - 2aJ_1J_2 - 4a^2(J_1 + J_2) = 0 \tag{1.3.3}$$

for the BPS black hole to have a smooth horizon.

Consider now the uplift of the universal black hole to $\mathrm{AdS}_5 \times \mathrm{SE}_5$, where SE_5 is a Sasaki-Einstein manifold. In such an embedding, the standard holographic dictionary identifies a

[13]It is believed and checked in many cases that the effective theory for all such compactifications can be consistently truncated to minimal gauged supergravity.

[14]To compare with the notations of [72]: $Q_{\text{there}} = -\sqrt{3}gQ_{\text{here}}$ and $G_{\mathrm{N}}^{(5)} = 1$, $\ell_5 = 1/g$.

with the central charge of the dual CFT_4. The black hole carries angular momenta J_1 and J_2 and an electric charge aligned with the exact R-symmetry of the dual CFT_4. We need to check that its entropy is reproduced by our result (1.2.14) (the same result can be similarly obtained using (1.2.20) instead). It is convenient to parametrize the chemical potentials as

$$\Delta_a = \frac{\tau+\sigma-1}{2}\left(\widehat{\Delta}_a^{(0)} + \widehat{\delta}_a\right) , \tag{1.3.4}$$

where $\widehat{\Delta}_a^{(0)}$ is the *exact* superconformal R-symmetry of the dual CFT_4 while $\widehat{\delta}_a$ parametrize a basis of flavor symmetries. These quantities satisfy

$$\sum_{a=1}^{D} \widehat{\Delta}_a^{(0)} = 2 , \qquad \sum_{a=1}^{D} \widehat{\delta}_a = 0 . \tag{1.3.5}$$

The entropy of the universal black hole is given by the Legendre transform of (1.2.14). Using (1.2.46) we can write the entropy function as

$$\mathcal{S} = -\frac{4\pi i}{27}\frac{(\tau+\sigma-1)^3}{\tau\sigma}\, a\Big(\widehat{\Delta}^{(0)} + \widehat{\delta}\Big) - 2\pi i\Big((\tau+\sigma-1)Q + \tau J_1 + \sigma J_2\Big) , \tag{1.3.6}$$

where we introduced a charge $Q = \frac{1}{2}\sum_{a=1}^{D}\widehat{\Delta}_a^{(0)}\, Q_a$ in the direction of the exact R-symmetry, and set all other charges to zero. We need to extremize the function $\mathcal{S}$ with respect to τ, σ and $\widehat{\delta}_a$ subject to the constraint (1.3.5). By a-maximization, since $\widehat{\Delta}_a^{(0)}$ is the exact R-symmetry, the function is extremized at $\widehat{\delta}_a = 0$. We can then restrict the entropy function to

$$\mathcal{S} = -\frac{4\pi i a}{27}\frac{(\tau+\sigma-1)^3}{\tau\sigma} - 2\pi i\Big((\tau+\sigma-1)Q + \tau J_1 + \sigma J_2\Big) , \tag{1.3.7}$$

where $a \equiv a\big(\widehat{\Delta}^{(0)}\big)$ is the central charge of the CFT_4, or, introducing a Lagrange multiplier Λ,

$$\mathcal{S} = -4\pi i a\frac{\Delta^3}{\tau\sigma} - 2\pi i\Big(3\Delta Q + \tau J_1 + \sigma J_2\Big) - 2\pi i\Lambda\big(3\Delta - \tau - \sigma + 1\big) . \tag{1.3.8}$$

If we set $a = a_{\mathcal{N}=4} = \frac{1}{4}N^2$, the function (1.3.8) becomes identical to the entropy function of $\mathcal{N}=4$ SYM for equal charges $Q_1 = Q_2 = Q_3 \equiv Q$, which is known to correctly reproduce (1.3.1) [76]. An analytic derivation of (1.3.1) and (1.3.3) for $\mathcal{N}=4$ SYM is explicitly discussed in [43] and for equal angular momenta in [45]. The charge constraint (1.3.3) is obtained as the requirement that the extremum of $\mathcal{S}$ be real.

At this point, the result for the universal black hole simply follows from the homogeneity properties of (1.3.8):

$$S(Q,J_1,J_2) = \frac{a}{a_{\mathcal{N}=4}}\, S_{\mathcal{N}=4}\left(\frac{a_{\mathcal{N}=4}}{a}\,Q,\ \frac{a_{\mathcal{N}=4}}{a}\,J_1,\ \frac{a_{\mathcal{N}=4}}{a}\,J_2\right) . \tag{1.3.9}$$

It is then immediate to derive the relations (1.3.1) and (1.3.3), thus completing our derivation.

1.4 AdS_5 Kerr-Newman black holes in $T^{1,1}$

We would like to compare the entropy function we obtained in Section 1.2 from the large N limit of the superconformal index of generic (toric) quiver gauge theories, with the Bekenstein-Hawking entropy of BPS black holes in the corresponding 5d gauged supergravities. In particular, the setup we would like to analyze is that of type IIB supergravity on asymptotically $\mathrm{AdS}_5 \times \mathrm{SE}_5$ spacetimes, where SE_5 is a toric Sasaki-Einstein manifold,[15] reduced and truncated to a 5d $\mathcal{N} = 2$ gauged supergravity on AdS_5. Unfortunately, with the exception of the case of S^5 truncated to the so-called 5d STU model, and the case of any SE_5 truncated to minimal $\mathcal{N} = 2$ gauged supergravity (that we analyzed in Section 1.3), all other known consistent truncations are to gauged supergravities with hypermultiplets (besides vector multiplets), and no supersymmetric black hole solutions have been constructed in such theories to date.

The strategy we propose to perform a test of our field theory results is as in [76]. We assume that a 5d BPS rotating black hole solution exists. Such a solution has the topology of a fibration of AdS_2 over S^3 (the three-sphere being the topology of the event horizon), and thus we can reduce it along the Hopf fiber of S^3. This gives a (putative) 4d BPS rotating black hole solution, with the same entropy.[16] The reduction generates an extra vector field A^0, corresponding to the isometry along the Hopf fiber. The 4d black hole has one unit of magnetic charge under A^0, corresponding to the first Chern class of the Hopf fibration. Calling J_1 and J_2 the 5d angular momenta along two orthogonal planes, the quantity $J_1 + J_2$ appears in 4d as the electric charge under A^0, while $J_1 - J_2$ becomes the angular momentum of the 4d black hole. Constructing such a 4d rotating black hole solution is still a difficult task, and an attractor mechanism is not known in general.[17] However, if we restrict to 5d black holes with two equal angular momenta $J_1 = J_2$ (so that the isometry of the squashed S^3 is enhanced from $U(1)^2$ to $U(1) \times SU(2)$), then the 4d black hole is static: in this case we can determine its entropy by exploiting the attractor mechanism in the near-horizon geometry [92–94], without actually constructing the whole solution.

The simplest non-trivial example is when SE_5 is $T^{1,1}$, the base of the conifold Calabi-Yau threefold, whose holographic dual is the Klebanov-Witten gauge theory [88]. We already presented the field theory analysis in Section 1.2.1. On the other hand, starting from 10d type IIB supergravity on $T^{1,1}$, we can exploit a consistent truncation that preserves $SU(2)^2 \times U(1)$ isometry, down to a 5d $\mathcal{N} = 2$ gauged supergravity with the graviton multiplet, two vector

[15] More precisely, the cone over SE_5 is a toric Calabi-Yau threefold.

[16] The 4d solution has an exotic asymptotic behavior, that follows from the reduction of AdS_5 [108]. Nonetheless, it has a regular extremal horizon, whose area determines the entropy.

[17] There are however some general results for theories with vector multiplets [109, 110].

multiplets and two hypermultiplets. This is the second truncation presented in Section 7 of [89] (see also [90,91]). On the AdS$_5$ vacuum, one vector multiplet (sometimes called "Betti multiplet") is massless and is associated to the baryonic symmetry, while the other vector multiplet is massive.

Hence, with the simplification that $J_1 = J_2$ and only the R-symmetry and baryonic symmetry charges are turned on (while the $SU(2)^2$ isometry of $T^{1,1}$ is unbroken), we will be able to match the Legendre transform of the superconformal index at large N with the extremization problem that comes from the attractor mechanism in supergravity. It follows that the bulk and boundary computations of the entropy exactly match.

1.4.1 Reduction from 5d to 4d and the attractor mechanism

A 5d $\mathcal{N} = 2$ Abelian gauged supergravity with n_V vector multiplets and n_H hypermultiplets — whose main building blocks we summarize in Appendix B — is specified by the following data [111–113]:

1. A very special real manifold $\mathcal{SM}$ of real dimension n_V, specified by a symmetric tensor of Chern-Simons couplings C_{IJK} with $I, J, K = 1, \ldots, n_V + 1$. The coordinates are Φ^I with the cubic constraint
$$\mathcal{V}(\Phi) \equiv \frac{1}{6} C_{IJK} \Phi^I \Phi^J \Phi^K = 1 \,. \tag{1.4.1}$$
2. A quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$ with coordinates q^u.
3. A set of $n_V + 1$ Killing vectors k_I^u (that could be linearly dependent, or vanish) on $\mathcal{QM}$, compatible with the quaternionic-Kähler structure, representing the isometries to be gauged by the vector fields A^I. Each Killing vector comes equipped with a triplet of moment maps $\vec{P}_I$.[18]

On the other hand, a 4d $\mathcal{N} = 2$ Abelian gauged supergravity with $n_V + 1$ vector multiplets and n_H hypermultiplets — that we summarize in Appendix C — is specified by the following data (see for instance [114, 115]):

1. A special Kähler manifold $\mathcal{KM}$ of complex dimension $n_V + 1$, with coordinates z^I and $I = 1, \ldots, n_V + 1$. We will work in a duality frame in which the geometry is specified by holomorphic sections $X^\Lambda(z)$, with $\Lambda = 0, \ldots, n_V + 1$, and a holomorphic prepotential $F(X)$, homogeneous of degree two.

[18] If $n_H = 0$, instead, one has to specify n_V Fayet-Iliopoulos parameters ζ^I, not all vanishing.

2. A quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$ with coordinates q^u.

3. In duality frames in which all gaugings are purely electric, a set of $n_V + 2$ Killing vectors k^u_Λ (that could be linearly dependent, or vanish) on $\mathcal{QM}$, compatible with the quaternionic-Kähler structure, representing the isometries to be electrically gauged by the vector fields A^Λ. Each Killing vector comes equipped with a triplet of moment maps $\vec{P}_\Lambda$ (see footnote 18).

We reduce the 5d theory on a circle, that will eventually be the Hopf fiber of S^3. Following [76, 116–120] we use the ansatz

$$\begin{aligned} ds^2_{(5)} &= e^{2\widetilde{\phi}} ds^2_{(4)} + e^{-4\widetilde{\phi}} \big(dy - A^0_{(4)} \big)^2 \\ \Phi^I &= -e^{2\widetilde{\phi}}\, \mathbb{I}\mathrm{m}\, z^I \;. \end{aligned} \tag{1.4.2}$$

Here y is the direction of the circular fiber, that we take with range $4\pi/g$ in terms of the coupling $g = \ell_5^{-1}$ inversely proportional to the AdS$_5$ radius ℓ_5, therefore the size of the circle is $e^{-2\widetilde{\phi}}\, 4\pi/g$. Because of the constraint $\mathcal{V}(\Phi) = 1$ in (1.4.1), the field $\widetilde{\phi}$ is redundant and can be eliminated with $e^{-6\widetilde{\phi}} = -\mathcal{V}(\mathbb{I}\mathrm{m}\, z^I)$. On the other hand, $A^0_{(4)}$ is the Kaluza-Klein vector. As noted in [76, 121], a Scherk-Schwarz twist for the gravitino as in [119] is necessary to satisfy the BPS conditions in 4d. We prefer to work in a gauge in which all bosonic fields are periodic around the circle, but there are flat gauge connections ξ^I turned on along y. This corresponds to the ansatz

$$A^I_{(5)} = A^I_{(4)} + \mathbb{R}\mathrm{e}\, z^I \big(dy - A^0_{(4)} \big) + \xi^I dy \;, \tag{1.4.3}$$

together with no y-dependence for any field. Notice that this ansatz is invariant under the redefinitions

$$z^I \to z^I + \delta\xi^I \;, \qquad A^I_{(4)} \to A^I_{(4)} + \delta\xi^I A^0_{(4)} \;, \qquad \xi^I \to \xi^I - \delta\xi^I \tag{1.4.4}$$

where $\delta\xi^I$ are real parameters. We will fix this redundancy below. The reduction of the 5d theory can be found in Appendix D. The resulting 4d data in terms of 5d ones are as follows.

1. The special Kähler manifold in 4d is described by the prepotential

$$F(X) = \frac{1}{6}\, C_{IJK}\, \frac{\check{X}^I \check{X}^J \check{X}^K}{X^0} \qquad \text{with} \qquad \check{X}^I = X^I + \xi^I X^0 \;. \tag{1.4.5}$$

The holomorphic sections X^Λ can be used as homogeneous coordinates, and the physical scalars are identified with the special coordinates $z^I = X^I/X^0$.

2. The quaternionic-Kähler manifold in 4d is the same as in 5d.

3. The 4d Killing vectors k_I^u are inherited from 5d, while the additional Killing vector is

$$k_0^u = \xi^I k_I^u \qquad \Rightarrow \qquad \vec{P}_0 = \xi^I \vec{P}_I \,, \tag{1.4.6}$$

and is gauged by the Kaluza-Klein vector field $A^0_{(4)}$.

Next, we study the attractor equations for the near-horizon limit of 4d BPS static black hole solutions [92–94]. Our goal is to use the BPS equations to fix the VEVs in massive vector multiplets and hypermultiplets, and be left with an extremization principle for the scalars in massless vector multiplets, similarly to [31, 32]. We consider the near-horizon geometry $\mathrm{AdS}_2 \times S^2$:

$$ds^2_{\text{near-horizon}} = -\frac{r^2}{L_{\mathrm{A}}^2}\, dt^2 + \frac{L_{\mathrm{A}}^2}{r^2}\, dr^2 + L_{\mathrm{S}}^2\, ds^2_{S^2} \,, \tag{1.4.7}$$

where L_{A} and L_{S} are the radii of AdS_2 and S^2, respectively. Electric and magnetic charges are defined as appropriate integrals over S^2 in the near-horizon region, respectively:

$$q_\Lambda = \frac{g}{4\pi} \int_{S^2} 16\pi G_{\mathrm{N}}^{(4)} \frac{\delta S_{4\mathrm{d}}}{\delta F^\Lambda} \,, \qquad\qquad p^\Lambda = \frac{g}{4\pi} \int_{S^2} F^\Lambda \,. \tag{1.4.8}$$

Here $G_{\mathrm{N}}^{(4)}$ is the 4d Newton constant, related to the 5d one by

$$\frac{4\pi}{G_{\mathrm{N}}^{(5)} g} = \frac{1}{G_{\mathrm{N}}^{(4)}} \,, \tag{1.4.9}$$

while $S_{4\mathrm{d}}$ is the 4d supergravity action. The 4d black holes we are interested in have both electric and magnetic charges. The magnetic charge $p^0 = 1$ is equal to the first Chern class of the Hopf fibration. On the other hand, we fix the redundancy (1.4.4) by setting the remaining magnetic charges to zero. In Appendix E we compute the relation of the 5d charges Q_I and angular momentum J measured at infinity, with the 4d charges measured at the horizon. We should be careful that only massless vector fields are associated to conserved charges. We indicate as $\mathbb{B}^I{}_J$ the matrix of linear redefinitions such that $\mathbb{B}^I{}_J A^J_\mu$ are the 5d mass eigenstates in the AdS_5 vacuum, and we take the index $\mathfrak{I}$ to run only over the massless vectors $\mathbb{B}^{\mathfrak{I}}{}_J A^J_\mu$. The corresponding conserved charges are $Q_{\mathfrak{I}} \equiv Q_J (\mathbb{B}^{-1})^J{}_{\mathfrak{I}}$. We find

$$\begin{aligned} p^0 &= 1 \,, \qquad\qquad & q_0 &= 4G_{\mathrm{N}}^{(4)} g^2 J + \frac{1}{3} C_{IJK} \xi^I \xi^J \xi^K \,, \\ p^I &= 0 \,, & q_{\mathfrak{I}} &= 4G_{\mathrm{N}}^{(4)} g^2 Q_{\mathfrak{I}} + \frac{1}{2} C_{\mathfrak{I}JK} \xi^J \xi^K \,, \end{aligned} \tag{1.4.10}$$

where $J_1 = J_2 \equiv J$, while the "non-conserved charges" $q_{J \neq \mathfrak{I}}$ will be fixed by the equations of motion. Notice that the conserved charges $Q_{\mathfrak{I}}$ are the same, but possibly in a different basis, as the charges Q_a introduced in Sections 1.2.2 and 1.2.3.[19]

[19] Similarly, the restriction of C_{IJK} to $C_{\mathfrak{I}\mathfrak{J}\mathfrak{K}}$ with curly indices is the same, but possibly in a different basis, as the 't Hooft anomaly coefficients C_{abc} previously defined.

Using a symplectic covariant notation, electric and magnetic charges form a symplectic vector

$$\mathcal{Q} = (p^\Lambda, q_\Lambda) \ . \tag{1.4.11}$$

One also defines

$$\vec{\mathcal{P}} = (0, \vec{P}_\Lambda) \ , \qquad\qquad \vec{\mathcal{Q}} = \langle \vec{\mathcal{P}}, \mathcal{Q} \rangle \ , \tag{1.4.12}$$

where vectors are triplets and $\langle V, W \rangle = V_\Lambda W^\Lambda - V^\Lambda W_\Lambda$ is the symplectic-invariant antisymmetric form.

To find covariantly-constant spinors, we impose the following twisting ansatz:

$$\epsilon_i = -\vec{\mathcal{Q}} \cdot \vec{\sigma}_i^{\ j} \, \Gamma^{\hat{t}\hat{r}} \epsilon_j \ , \tag{1.4.13}$$

whose square gives $\vec{\mathcal{Q}} \cdot \vec{\mathcal{Q}} = 1$. Here $\Gamma^{\hat{t}\hat{r}}$ is the antisymmetric product of two gamma matrices with flat indices $\hat{t}$ and $\hat{r}$. We choose a gauge in which $\mathcal{Q}^1 = \mathcal{Q}^2 = 0$ and

$$\mathcal{Q}^3 = -1 \tag{1.4.14}$$

at the horizon, as in [31].

The remaining BPS conditions are in general complicated, but they simplify at the horizon [92–94]. First, Maxwell's equations give

$$\mathcal{K}^u h_{uv} \langle \mathcal{K}^v, \mathcal{Q} \rangle = 0 \ , \tag{1.4.15}$$

where we defined

$$\mathcal{K}^u = (0, k^u_\Lambda) \tag{1.4.16}$$

because we work in a duality frame with purely electric gaugings. In fact, (1.4.15) in this case is equivalent to

$$p^\Lambda k^u_\Lambda = 0 \tag{1.4.17}$$

that must hold in the full solution simply because of spherical symmetry (see Appendix E). Second, vanishing of the hyperino variation implies

$$\langle \mathcal{K}^u, \mathcal{V} \rangle = 0 \ , \tag{1.4.18}$$

where $\mathcal{V}(z, \bar{z}) = e^{\mathcal{K}/2}(X^\Lambda, F_\Lambda)$ is the covariantly-holomorphic section defined in (C.0.3) and $F_\Lambda = \partial_\Lambda F(X)$. Third, we have the attractor equations[20]

$$\frac{\partial}{\partial z^I} \left(\frac{\mathcal{Z}}{\mathcal{L}} \right) = 0 \ , \qquad\qquad \frac{\mathcal{Z}}{\mathcal{L}} = 2ig^2 L_{\mathrm{S}}^2 \ , \tag{1.4.19}$$

where the derivatives are with respect to the physical scalars z^I and we defined

$$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle \ , \qquad\qquad \mathcal{L} = \langle \mathcal{P}^3, \mathcal{V} \rangle \ . \tag{1.4.20}$$

The equation on the right in (1.4.19) determines L_{S}, and thus the horizon area.

[20]There is an extra factor of 2 in front of L_{S}^2 compared to [30, 31, 114] due to the different normalization of kinetic terms in the Lagrangian (C.0.2): this is noticed footnote 4 of [119] and in footnote 10 of [76].

1.4.2 Example: the conifold

We apply the general strategy to the case of the conifold. We start with the 5d $\mathcal{N}=2$ gauged supergravity with $n_V=2$ vector multiplets and $n_H=2$ hypermultiplets constructed in Section 7 of [89] (called the "second model" in that paper), obtained from a consistent reduction of 10d type IIB supergravity on $T^{1,1}$ that preserves the $SU(2)^2 \times U(1)$ isometry. In Appendix B.1 we have recast its action as in the general formalism, and in Appendix D.1 we have reduced it down to 4d $\mathcal{N}=2$ gauged supergravity. We are now ready to look for BPS near-horizon black hole solutions.

Using (B.1.6) and (B.1.7), the conditions (1.4.14) and (1.4.17) take the form:

$$\begin{cases} P_0^3 = -1 \\ k_0^u = 0 \end{cases} \qquad \Rightarrow \qquad b_{1,2}^{\Omega} = c_{1,2}^{\Omega} = 0\,, \qquad \xi^1 = -\xi^2 = -\frac{1}{3}\,, \tag{1.4.21}$$

where $b_{1,2}^{\Omega}, c_{1,2}^{\Omega}, a, \phi, C_0, u$ are the scalar fields in hypermultiplets. In fact, since (1.4.17) must hold in the whole solution, so (1.4.21) does. Using the form (B.1.7) of the moment maps, this is consistent with $\mathcal{Q}^1 = \mathcal{Q}^2 = 0$. The hyperino condition (1.4.18) then gives

$$X^1 + X^2 = 0 \tag{1.4.22}$$

at the horizon, where X^Λ are the holomorphic sections. The fields C_0 and ϕ are not fixed by the equations of motion. However, together they form the axiodilaton of type IIB supergravity and are thus fixed by the boundary conditions that set them in terms of the complexified gauge coupling of the boundary theory. As apparent from the expression of k_2^u in (B.1.6), a is a Stückelberg field that breaks an Abelian gauge symmetry and is eaten up as the corresponding gauge field becomes massive via Higgs mechanism.

The remaining BPS conditions are the attractor equations (1.4.19). Given C_{IJK} in (B.1.2), the prepotential is

$$F(X) = \frac{\check{X}^1\big((\check{X}^2)^2 - (\check{X}^3)^2\big)}{X^0} \qquad \text{where} \qquad \check{X}^I = X^I + \xi^I X^0\,. \tag{1.4.23}$$

Using special coordinates $z^I = X^I/X^0$ as well as homogeneity of the prepotential $F(X)$, one can easily show that the two equations in (1.4.19) are equivalent to

$$\partial_\Lambda \Big[e^{-\mathcal{K}/2} \Big(\mathcal{Z}(X) - 2ig^2 L_{\mathrm{S}}^2\, \mathcal{L}(X) \Big) \Big] = 0\,, \tag{1.4.24}$$

where the derivatives are with respect to *independent* sections X^Λ. In these equations L_{S} should be regarded as one of the unknowns. Notice that (1.4.19) or (1.4.24) give, in general, isolated solutions in terms of (z^I, L_{S}), however the sections X^Λ are only fixed up to the "gauge" redundancy (related to Kähler transformations on $\mathcal{KM}$) $X^\Lambda \to e^f X^\Lambda$. In order to remove the redundancy, we choose to fix $\mathcal{L}(X)$ to a constant, which can elegantly be imposed by taking a derivative of the square bracket in (1.4.24) with respect to $L_{\rm S}^2$ as well. More precisely, expanding $\mathcal{Z}$ and $\mathcal{L}$ using (B.1.7), we consider the following set of equations:

$$\begin{aligned}
\partial_\Lambda \left[\frac{X^1\big((X^2)^2-(X^3)^2\big)}{(X^0)^2} + \widehat{q}_\Lambda X^\Lambda - 2ig^2 L_{\rm S}^2 \Big(3X^1 - X^0 - 2e^{-4u}(X^1+X^2) - \alpha \Big) \right] &= 0 \\
\frac{\partial}{\partial L_{\rm S}^2} \left[\frac{X^1\big((X^2)^2-(X^3)^2\big)}{(X^0)^2} + \widehat{q}_\Lambda X^\Lambda - 2ig^2 L_{\rm S}^2 \Big(3X^1 - X^0 - 2e^{-4u}(X^1+X^2) - \alpha \Big) \right] &= 0
\end{aligned} \tag{1.4.25}$$

where

$$\widehat{q}_I = q_I - \frac{1}{2} C_{IJK}\xi^J\xi^K \,, \qquad \widehat{q}_0 = q_0 - \frac{1}{3} C_{IJK}\xi^I\xi^J\xi^K \,. \tag{1.4.26}$$

The first line is the same as (1.4.24), except for the addition of the constant α that does not affect the equations. The second line fixes the gauge $\mathcal{L} = \alpha$. Notice that (1.4.22) should be imposed after solving (1.4.25).

From the point of view of AdS/CFT, only massless vector fields correspond to symmetries of the boundary theory and only their charges are conserved and fixed by the boundary conditions. On the contrary, the "charges" under massive vector fields are not conserved, and their radial profile should be determined by the equations of motion. The spectrum of the 5d supergravity under consideration around its supersymmetric AdS$_5$ vacuum was computed in [89] and we report it in our conventions in (B.1.9). In the basis

$$\begin{aligned}
A^R &\equiv A^1 - 2A^2 \,, \qquad & A^3 \,, \qquad & A^W \equiv A^1 + A^2 \,, \\
k_R &\equiv \tfrac{1}{3}(k_1 - k_2) \,, \qquad & k_3 \,, \qquad & k_W \equiv \tfrac{1}{3}(2k_1 + k_2) \,,
\end{aligned} \tag{1.4.27}$$

it turns out that A^R (corresponding to the R-symmetry) and A^3 are massless, while A^W is massive because of Higgs mechanism eating up the Stückelberg field a. In (1.4.27) we have indicated also the Killing vectors of the corresponding gauged isometries. On the black hole background the mass eigenstates may change (because the gauge kinetic functions have a non-trivial radial profile), however the fact that

$$k_R = k_3 = 0 \tag{1.4.28}$$

everywhere — which follows from (1.4.21) — guarantees that there is no hypermultiplet source in the 5d Maxwell equations (E.0.3) and thus the Page charges Q_R and Q_3 are conserved (while Q_W is not).

Indeed, the variation in (1.4.25) with respect to X^2 gives the complex equation

$$2\,\frac{X^1X^2}{(X^0)^2} + \widehat{q}_2 + 4ig^2 L_{\rm S}^2\, e^{-4u} = 0 \tag{1.4.29}$$

that fixes u and the "non-conserved charge" q_2 in terms of the sections and $L_{\rm S}$. We can then use the hyperino condition (1.4.22) to eliminate X^2 as well. Notice that the second condition in (1.4.21) implies that in 5d we cannot turn on a "flat connection" for A^W along the circle.

We are left with the unknowns $X^0, X^1, X^3, L_{\rm S}^2$. One can check that, when (1.4.22) and (1.4.29) are in place, the remaining equations in (1.4.25) are equivalent to the conditions of extremization of the function

$$\mathcal{S} = \beta\left[\frac{X^1\big((X^1)^2-(X^3)^2\big)}{(X^0)^2} + \widehat{q}_0X^0 + 3\widehat{q}_RX^1 + \widehat{q}_3X^3 - 2ig^2L_{\rm S}^2\big(3X^1 - X^0 - \alpha\big)\right] \tag{1.4.30}$$

with respect to the variables $X^0, X^1, X^3, L_{\rm S}^2$. Here β is a constant included for later convenience, while $\widehat{q}_R$ is the charge with respect to the massless vector $A_{\rm R}$:

$$\widehat{q}_R = \frac{\widehat{q}_1-\widehat{q}_2}{3} = \frac{g}{4\pi}\int_{S^2}16\pi G_{\rm N}^{(4)}\frac{\delta S_{\rm 4d}}{\delta F^R} - \frac{1}{6}\big(C_{1JK}-C_{2JK}\big)\xi^J\xi^K = 4g^2G_{\rm N}^{(4)}Q_R\,. \tag{1.4.31}$$

It is encouraging that we find an extremization problem in which only conserved charges appear. Since $\mathcal{S}$ is homogeneous in X^Λ of degree 1 except for the term involving α, it follows that $\mathcal{S}\big|_{\rm crit} = 2i\alpha\beta g^2L_{\rm S}^2$ at the critical point. With the choice

$$\alpha\beta = \frac{\pi}{2iG_{\rm N}^{(4)}g^2} \tag{1.4.32}$$

we obtain that $\mathcal{S}\big|_{\rm crit}$ is the black hole entropy:

$$\mathcal{S}\Big|_{\rm crit} = \frac{4\pi L_{\rm S}^2}{4G_{\rm N}^{(4)}} = S_{\rm BH}\,, \tag{1.4.33}$$

and therefore $\mathcal{S}$ is the entropy function. Using (1.4.10) and (1.4.26) we can express the 4d charges $\widehat{q}_0$, $\widehat{q}_{\mathfrak{T}}$ computed at the horizon in terms of the 5d black hole charges J, $Q_{\mathfrak{T}}$ computed at infinity:

$$\begin{aligned}\mathcal{S} = \frac{1}{\alpha}\Bigg[&\frac{\pi}{2iG_{\rm N}^{(4)}g^2}\,\frac{(X^1)^3-X^1(X^3)^2}{(X^0)^2} - 2\pi i\Big(JX^0 + 3Q_RX^1 + Q_3X^3\Big)\\ &- 2\pi i\Lambda\Big(3X^1 - X^0 - \alpha\Big)\Bigg]\,,\end{aligned} \tag{1.4.34}$$

where we redefined the Lagrange multiplier $L_{\rm S}^2 = 2iG_{\rm N}^{(4)}\Lambda$ for convenience.

It remains to spell out the AdS/CFT dictionary between gravity and field theory charges. First, the gauge group ranks in field theory are determined by (see Appendix E.1)

$$N^2 = \frac{8\pi}{27\,G_{\rm N}^{(5)}g^3} = \frac{2}{27\,G_{\rm N}^{(4)}g^2}\,. \tag{1.4.35}$$

This is in agreement with (1.3.2) using $a = \frac{27}{64}N^2$ for the Klebanov-Witten theory. Second, the angular momentum J is the same in gravity and in field theory. Third, the electric charges are identified as

$$r = 2Q_R \,, \qquad\qquad Q_B = \frac{4}{3}Q_3 \,. \tag{1.4.36}$$

This is determined as follows. From (B.0.36) we infer that the gravitino components have charge $Q_R = \pm\frac{1}{2}$. In the boundary field theory, the corresponding operators are of the schematic form $\mathrm{Tr}(F_{\mu\nu}\Gamma^\nu\lambda)$ (where F is a field strength and λ a gaugino) and have charge $r = \pm 1$ under $U(1)_R$. We deduce the first relation in (1.4.36). Obtaining the second relation is more subtle because no supergravity field is charged under A^3: what is charged are massive particles obtained from D3-branes wrapped on the 3-cycle of $T^{1,1}$, corresponding to dibaryon operators $A^N_{1,2}$ or $B^N_{1,2}$ in field theory. The 5d supergravity gauge field A^3 comes from the reduction of the Ramond-Ramond field strength F_5^{RR} of 10d type IIB supergravity on $T^{1,1}$. Therefore, from the 10d flux quantization condition we can deduce the 5d charge quantization condition $4Q_3/3N \in \mathbb{Z}$ (see the details in Appendix E.1). In field theory the dibaryon operators have charge $Q_B = \pm N$, implying the second relation in (1.4.36). Alternatively, we could compare the Chern-Simons terms restricted to massless vector fields in the 5d Lagrangian with the 't Hooft anomalies of the boundary theory. Taking into account the 't Hooft anomalies $\mathrm{Tr}(r^3) = \frac{3}{2}N^2$ and $\mathrm{Tr}(rQ_B^2) = -2N^2$ at leading order in N, the restriction of the 5d Chern-Simons action in (B.0.2) to $A^W \to 0$ matches the general expression

$$S_{\mathrm{CS}} = \frac{g^3}{24\pi^2}\int \mathrm{Tr}(Q_aQ_bQ_c)\, F^a\wedge F^b\wedge A^c \tag{1.4.37}$$

after setting $A^R \to 2A_r$ and $A^3 \to \frac{4}{3}A_B$. These correspond to (1.4.36).

Rewriting the entropy function (1.4.34) in terms of field theory charges, we find

$$\begin{aligned}\mathcal{S} = \frac{1}{\alpha}\bigg[-\frac{27\pi i N^2}{4}\,\frac{(X^1)^3 - X^1(X^3)^2}{(X^0)^2} - 2\pi i\Big(JX^0 + \frac{3}{2}rX^1 + \frac{3}{4}Q_BX^3\Big)\\ - 2\pi i\Lambda\Big(3X^1 - X^0 - \alpha\Big)\bigg]\,.\end{aligned} \tag{1.4.38}$$

This exactly matches the entropy function (1.2.35) we found in field theory from the large N limit of the superconformal index of the Klebanov-Witten theory, after the change of coordinates $X^0 \to 2\alpha\tau$, $X^1 \to 2\alpha X_R/3$, $X^3 \to 4\alpha X_B/3$.

Chapter 2

A quantum mechanics for magnetic horizons

We are interested in static dyonic near-BPS black holes embedded in massive Type IIA string theory on S^6, which is dual to a 3d $\mathcal{N} = 2$ $\mathrm{SU}(N)_k$ Chern-Simons-matter theory [68]. We find an effective quantum mechanics by reducing the dual 3d field theory on S^2, with a specific background that corresponds to the black-hole asymptotics.

More specifically, the entropy of static magnetically-charged BPS black holes in AdS_4 is captured by the topologically twisted (TT) index [28, 122] of the dual 3d boundary theory [29, 30, 33, 123–126], see in particular [31, 32] for the specific example in massive Type IIA studied here. In the Lagrangian formulation, the TT index is the Euclidean partition function of the theory on $S^2 \times S^1$, in the presence of a supersymmetric background that holographically reflects the asymptotics of the BPS black hole. The background can be thought of as a topological twist on S^2 that preserves two supercharges, or equivalently as an external magnetic flux for the R-symmetry. One observes that the TT index takes the form of the Witten index of a quantum mechanics, obtained by reducing the 3d theory on S^2 with the twisted background. Up to exponentially small corrections at large N, the index is the grand-canonical partition function for the BPS ground states of that quantum mechanics. In other words, the ground states of that quantum mechanics are the microstates of a BPS black hole with given charges, and one expects the excited states to describe near-extremal black holes. The goal of this work is to construct such a quantum mechanics.

The procedure we outlined has a technical complication: the formula for the TT index — schematically in (2.1.1) — has an infinite sum over gauge fluxes on S^2. For each term in the sum, one obtains a different quantum mechanics upon reduction. Thus it appears that, even at finite N, one has to deal with a quantum mechanical model with an infinite number of sectors, on which we do not have good control.[1] Nevertheless, in the large N limit we expect one sector to dominate the entropy[2] and thus to contribute the majority of

[1] This is partially due to the fact that the reduction is in the grand-canonical ensemble for the electric charges (though it is microcanonical for the magnetic charges), with fixed chemical potentials. Therefore, the states of all BPS and near-BPS black holes are mixed up together.

[2] We are grateful to Juan M. Maldacena for suggesting this possibility to us years ago.

the states. We determine such a sector by performing a saddle-point evaluation of the index in the sum over fluxes. This gives us an $\mathcal{N}=2$ supersymmetric gauged quantum mechanics with a finite number of fields (at finite N).

The resulting $\mathcal{N}=2$ supersymmetric QM, that we exhibit in Section 2.3, has some interesting features. The gauge group is $\mathrm{U}(1)^N$, while the number of fields scales as $N^{\frac{7}{3}}$. There is an SU(2) global symmetry, dual to the isometry of the S^2 black-hole horizon. More importantly, there is a large number of couplings among the fields, expressed in terms of Clebsh-Gordan coefficients (they arise in the reduction from the overlap of Landau-level wave-functions on S^2). Therefore, although the quantum mechanics is specific and well defined, at large N the couplings can be approximated by random variables following a statistical distribution. This makes us hopeful that the IR dynamics might have some traits in common with supersymmetric SYK models [127]. The idea of obtaining a supersymmetric QM with fixed, but statistically distributed, couplings in order to describe near-extremal horizons already appeared in [128] in the context of asymptotically-flat black holes in string theory.

In the large N saddle-point evaluation of the TT index, we noticed that there is actually a series of saddle points — one of which dominates the large N expansion. These saddle points are labelled by shifts of the chemical potentials by 2π, and likely correspond to a series of complex supergravity solutions with the very same boundary conditions, as in [129, 130].

The chapter is organized as follows. In Section 2.1 we re-examine the large N limit of the TT index by performing a saddle-point approximation both in the integration variables as well as in the sum over fluxes. This analysis already appeared recently in [34]. Section 2.2, which is the most technical one, is devoted to the dimensional reduction of the 3d theory on S^2 in the presence of gauge magnetic fluxes. This reduction involves a judicious choice of gauge fixing. In Section 2.3 we exhibit the effective $\mathcal{N}=2$ supersymmetric gauged quantum mechanics; the hurried reader who is only interested in the final result can directly jump there. Finally, in Section 2.4 we comment on which type of classical and quantum corrections to our analysis one might expect. Many technical details are collected in Appendices F to J.

2.1 Saddle-point approach to the TT index

We begin by re-examining the evaluation of the TT index of 3d $\mathcal{N}=2$ gauge theories at large N. The localization formula for the index found in [28] involves a sum over gauge fluxes $\mathfrak{m}$ on S^2, as well as a contour integral in the space of complexified gauge connections u on S^1. At large N, we apply a saddle-point approximation both to the integral over u as well as to the sum over fluxes, treated as a continuous variable $\mathfrak{m}$. The idea to compute a supersymmetric index in this way was put forward, for instance, in [35, 131] (see also [34, 132, 133]).[3] The upshot is to identify a specific gauge flux sector that dominates the index and, via holography, the BPS black-hole entropy. In Section 2.2 we will use that flux sector to perform a reduction of the 3d theory on S^2 down to a quantum mechanics.

The analysis in this and the following sections is performed in a specific (and simple) model, presented in Section 2.1.2. This choice is made for the sake of concreteness, but other theories (for instance ABJM [134]) could be studied in a similar way.

2.1.1 The basic idea

We are interested in the topologically twisted index [28] of the theory, because this quantity is known to reproduce the entropy of a class of BPS AdS$_4$ dyonic black holes [31, 32]. Using the notation of [31], this index can be written schematically as

$$\mathcal{I}_{S^2\times S^1} = \frac{1}{|\mathsf{W}|} \sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{du^i}{2\pi}\, e^{\mathfrak{m}V'(u)+\Omega(u)} \,. \tag{2.1.1}$$

Here $|\mathsf{W}|$ is the order of the Weyl group, $\Gamma_{\mathfrak{h}}$ is the co-root lattice, N is the rank of the gauge group, and $\mathcal{C}$ is an appropriate integration contour for the complexified Cartan-subalgebra-valued holonomies $\{u^i\} \in \mathfrak{h}_{\mathbb{C}}/2\pi\Gamma_{\mathfrak{h}}$. Let us outline three different approaches to this expression at large N.

1. The approach developed in [28] was to resum over $\mathfrak{m}$, schematically

$$\mathcal{I}_{S^2\times S^1} = \frac{1}{|\mathsf{W}|} \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{du^i}{2\pi}\, \frac{e^{\Omega(u)}}{1-e^{V'(u)}} \,, \tag{2.1.2}$$

then determine the positions $\bar{u}$ of the poles by solving the "Bethe Ansatz Equations" (BAEs)

$$e^{V'(\bar{u})} = 1 \,, \tag{2.1.3}$$

and finally take the residues

$$\mathcal{I}^{\mathrm{BAE}}_{S^2\times S^1} = \frac{1}{|W|} \sum_{\bar{u}\in\mathrm{BAE}} \frac{e^{\Omega(\bar{u})}}{i^N\, V''(\bar{u})} \,. \tag{2.1.4}$$

2. Alternatively, we can evaluate both the sum over $\mathfrak{m}$ and the integral over u in (2.1.1) in the saddle-point approximation, treating $\mathfrak{m}$ as a continuous variable. The simultaneous

[3] In particular, the evaluation of the (refined) TT index of the specific model studied here, through a saddle-point approximation of the sum over fluxes has recently already appeared in [34].

saddle-point equations for $\mathfrak{m}$ and u are, schematically:

$$\begin{cases} 0 = V'(\bar{u}) \\ 0 = \bar{\mathfrak{m}} V''(\bar{u}) + \Omega'(\bar{u}) \; . \end{cases} \tag{2.1.5}$$

Taking into account that $V'(u)$ in (2.1.1) is defined up to integer shifts by $2\pi i$, the first set of equations is exactly the set of BAEs, while the second set of equations uniquely fixes $\bar{\mathfrak{m}}$ in terms of $\bar{u}$. The Jacobian at the saddle point is

$$J^{\text{3d}}(\mathfrak{m}, u) = \det \begin{pmatrix} 0 & V''(u) \\ V''(u) & \mathfrak{m} V'''(u) + \Omega''(u) \end{pmatrix} = -\big(V''(u)\big)^2 \; . \tag{2.1.6}$$

Therefore, in the saddle-point approximation:

$$\mathcal{I}^{\text{saddle}}_{S^2\times S^1} \simeq \frac{1}{|\mathsf{W}|} \sum_{\bar{u}\in\text{saddles}} \frac{e^{\Omega(\bar{u})}}{\sqrt{J^{\text{3d}}}} = \frac{1}{|\mathsf{W}|} \sum_{\bar{u}\in\text{BAEs}} \frac{e^{\Omega(\bar{u})}}{i^N \, V''(\bar{u})} \; . \tag{2.1.7}$$

This method gives exactly the same answer as the previous method.

3. A more rough approximation is to fix $\mathfrak{m}$ in (2.1.1) to the value determined by the equations (2.1.5),

$$\mathcal{I}^{\text{fix } \bar{\mathfrak{m}}}_{S^2\times S^1} \simeq \mathcal{I}_{S^1} \equiv \frac{1}{|\mathsf{W}|} \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{du^i}{2\pi} \, e^{\bar{\mathfrak{m}} V'(u) + \Omega(u)} \; , \tag{2.1.8}$$

and then solve the integral in u in the saddle-point approximation. The saddle-point equations are $\bar{\mathfrak{m}} V''(u) + \Omega'(u) = 0$, therefore all solutions $\bar{u}$ of (2.1.5) are also saddle points of (2.1.8). Assuming that there are no other solutions, we find

$$\mathcal{I}_{S^1} \simeq \frac{1}{|\mathsf{W}|} \sum_{\bar{u}\in\text{BAEs}} \frac{e^{\Omega(\bar{u})}}{\sqrt{J^{\text{1d}}}} \; . \tag{2.1.9}$$

The Jacobian in this case is $J^{\text{1d}} = \bar{\mathfrak{m}} V'''(\bar{u}) + \Omega''(\bar{u}) = V''\big(\frac{\Omega'}{V''}\big)'(\bar{u})$ and is different from before, however as long as the Jacobian is subleading with respect to the exponential contribution, this approach captures the leading behavior.

In our setup we will find a series of saddle points $(\bar{u}, \bar{\mathfrak{m}})$, and the expression $\mathcal{I}_{S^1}$ in (2.1.8) evaluated on the dominant one will turn out to be the Witten index of an effective quantum mechanics we will construct. In order to do that, we will first have to find the saddle-point flux $\bar{\mathfrak{m}}$, and then reduce the 3d theory on S^2 in the presence of such a flux.

2.1.2 The model

We consider the AdS/CFT pair discovered in [68], that was used in [31,32] to study certain magnetic black holes in massive type IIA on $\mathrm{AdS}_4 \times S^6$ [65–67]. The field theory is a 3d $\mathcal{N}=2$ Chern-Simons-matter theory with gauge group $\mathrm{SU}(N)_k$, coupled to three chiral multiplets $\Phi_{a=1,2,3}$ in the adjoint representation. We can simplify the computation by considering a $\mathrm{U}(N)_k$ gauge theory, with no sources for the new topological symmetry. No field is charged under $\mathrm{U}(1) \subset U(N)$ and thus the only effect of this is to introduce a decoupled sector, whose Hilbert space on $\Sigma_\mathfrak{g}$ consists of $k^\mathfrak{g}$ states. This is just one state in the case of S^2. The theory has a superpotential

$$W = \lambda_{3\mathrm{d}} \operatorname{Tr} \Phi_1 \left[\Phi_2, \Phi_3\right] . \tag{2.1.10}$$

The global symmetry is $\mathrm{SU}(3) \times \mathrm{U}(1)_R$. We parametrize its Cartan subalgebra with three R-charges R_a, characterized by the charge assignment $R_a(\Phi_b) = 2\delta_{ab}$. We choose the Cartan generators of the flavor symmetry to be $q_{1,2} = (R_{1,2} - R_3)/2$. In this basis, all fields have integer global charges. Notice that $e^{i\pi R_a} = (-1)^F$ for $a = 1,2,3$.

To study AdS_4 BPS dyonic black holes, we place the theory on[4] $S^2 \times \mathbb{R}$ using a topological twist on S^2, so that one complex supercharge is preserved [135]. This is precisely the background of the topologically twisted index in [28]. In other words, there is a background gauge field A_R corresponding to an R-symmetry that is equal and opposite to the spin connection when acting on the top component of the supersymmetry parameter ϵ:

$$\frac{1}{2\pi} \int_{S^2} dA_R = -1 \,. \tag{2.1.11}$$

The R-symmetry used for the twist must have integer charge assignments, and a generic such R-charge can be written as $q_R = R_3 - \mathfrak{n}_1 q_1 - \mathfrak{n}_2 q_2$ for $\mathfrak{n}_{1,2} \in \mathbb{Z}$. Note that $\sum_a (q_R)_a = 2$ and the superpotential correctly has R-charge 2. Under these inequivalent twists, the scalar component of Φ_a experiences a flux $\mathfrak{n}_a = (q_R)_a \int_{S^2} \frac{dA_R}{2\pi} = -(R_3)_a + \mathfrak{n}_1 (q_1)_a + \mathfrak{n}_2 (q_2)_a$. This formula provides a definition of $\mathfrak{n}_3 \equiv -2 - \mathfrak{n}_1 - \mathfrak{n}_2$. Thus, twisting by a generic R-symmetry with integer charge assignments is the same as twisting with respect to R_3 and simultaneously turning on background gauge fields $A_{1,2}$ coupled to the flavor charges $q_{1,2}$ with

$$\frac{1}{2\pi} \int_{S^2} dA_{1,2} = \mathfrak{n}_{1,2} \,. \tag{2.1.12}$$

The theory that we are considering has a UV Lagrangian consisting of various building blocks which are individually supersymmetric off-shell. The vector multiplet V (in Wess-Zumino gauge) contains the adjoint-valued fields $(\sigma, \lambda, \overline{\lambda}, A_\mu, D)$, where σ is a dynamical real scalar field and D a real auxiliary field. We consider a supersymmetrized Chern-Simons

[4] One could also study the theory on a Riemann surface $\Sigma_\mathfrak{g}$ [122,124], but here we will focus on the sphere.

Lagrangian for it, but we also add the super-Yang-Mills Lagrangian as a regulator. The chiral multiplets Φ_a contain the adjoint-valued fields (Φ_a, Ψ_a, F_a), for which we consider the kinetic Lagrangian and the superpotential term. These Lagrangians, in Lorentzian signature and Wess-Zumino gauge, are:

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= \frac{1}{2e_{\text{3d}}^2} \operatorname{Tr}\Big[-\frac{1}{2} F_{\mu\nu}F^{\mu\nu} - D_\mu\sigma D^\mu\sigma + D^2 - i\bar{\lambda}\big(\not{D} - \sigma\big)\lambda \Big] , \qquad (2.1.13)\\
\mathcal{L}_{\text{CS}} &= \frac{k}{4\pi} \operatorname{Tr}\Big[-\epsilon^{\mu\nu\rho}\Big(A_\mu\partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \Big) - i\bar{\lambda}\lambda - 2D\sigma \Big] ,\\
\mathcal{L}_{\text{chiral}} &= -D_\mu\Phi_a^\dagger D^\mu\Phi_a - \Phi_a^\dagger\big(\sigma^2 + D\big)\Phi_a + F_a^\dagger F_a - i\overline{\Psi}_a\big(\not{D} + \sigma\big)\Psi_a + i\overline{\Psi}_a\lambda\Phi_a + i\Phi_a^\dagger\bar{\lambda}\Psi_a ,\\
\mathcal{L}_{\text{W}} &= \frac{\partial W}{\partial\Phi_a} F_a + \frac{1}{2} \frac{\partial^2 W}{\partial\Phi_a\partial\Phi_b} \overline{\Psi_b^c}\, \Psi_a + \text{c.c.} ,
\end{aligned}$$

where we used the convention $\Psi^c \equiv i\sigma_1\Psi^*$ for the conjugated spinor. The superpotential must be a gauge-invariant holomorphic function of R-charge 2. The supersymmetry variations preserved by these Lagrangians are in Appendix G.

In order to obtain a microscopic description of the black-hole entropy, one counts the ground states of this theory. It is convenient to work in the grand-canonical ensemble, in which one introduces a set of chemical potentials Δ_a, $a = 1, 2$ for each flavor Cartan generator. Like the fluxes, it is useful to introduce a third chemical potential Δ_3, constrained because of supersymmetry [30], such that

$$\Delta_1 + \Delta_2 + \Delta_3 \,\in\, 2\pi\mathbb{Z} \,. \qquad (2.1.14)$$

All chemical potentials are only defined modulo 2π. Computing the thermal partition function is hard because the theory is strongly coupled in the IR, therefore one can start from a quantity protected by supersymmetry: the topologically twisted index

$$\mathcal{I}_{\text{3d}}(\mathfrak{n}, \Delta) = \operatorname{Tr}\,(-1)^F\, e^{-\beta H}\, e^{iq_a\Delta_a} \;, \qquad (2.1.15)$$

where F is the Fermion number, H the Hamiltonian on the sphere S^2 in the presence of the magnetic fluxes (2.1.11)-(2.1.12), and the trace is over the Hilbert space of states. This quantity only gets contributions from the ground states of the theory. It was argued in [29], exploiting the $\mathfrak{su}(1,1|1)$ superconformal symmetry algebra expected to emerge from the $\text{AdS}_2 \times S^2$ near-horizon region in gravity, that the BPS states of a pure single-center black hole have constant statistics $(-1)^F$ in each charge sector, meaning that the index gets non-interfering contributions (at least at leading order in N) and can account for the black-hole entropy.[5]

[5]This expectation was confirmed for rotating black holes in AdS_5 in [61].

The TT index (2.1.15) can be computed exactly using supersymmetric-localization techniques [28, 122], and for the model considered here one obtains [31, 32]:

$$\mathcal{I}_{3\mathrm{d}}(\mathfrak{n},\Delta) = \frac{(-1)^N}{N!}\prod_{a=1}^{3}\frac{y_a^{N^2(\mathfrak{n}_a+1)/2}}{(1-y_a)^{N(\mathfrak{n}_a+1)}}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}\oint_{\mathrm{JK}}\prod_{i=1}^{N}\frac{dz_i}{2\pi i z_i}\,z_i^{k\mathfrak{m}_i}\times$$
$$\times\prod_{i\neq j}^{N}\left(1-\frac{z_i}{z_j}\right)\prod_{a=1}^{3}\prod_{i\neq j}^{N}\left(\frac{z_i-y_a z_j}{z_j-y_a z_i}\right)^{\mathfrak{m}_i}\left(1-y_a\frac{z_i}{z_j}\right)^{-\mathfrak{n}_a-1}\,. \quad (2.1.16)$$

Here $z_i \equiv e^{iu_i}$ and $y_a \equiv e^{i\Delta_a}$. This expression can be conveniently compiled into the same form as (2.1.1):

$$\mathcal{I}_{3\mathrm{d}}(\mathfrak{n},\Delta) = \frac{1}{N!}\sum_{\mathfrak{m}\in\Gamma_{\mathfrak{h}}}\oint_{\mathrm{JK}}\left(\prod_{i=1}^{N}\frac{du_i}{2\pi}\right)e^{\sum_i \mathfrak{m}_i V_i'(u,\Delta)+\Omega(u,\mathfrak{n},\Delta)}\,. \quad (2.1.17)$$

The two functions appearing in the exponent are

$$\sum_{i=1}^{N}\mathfrak{m}_i V_i'(u,\Delta) = \sum_{i=1}^{N}\mathfrak{m}_i\left\{iku_i+\sum_{j=1}^{N}\sum_{a=1}^{3}\left[\mathrm{Li}_1\left(e^{i(u_{ji}-\Delta_a)}\right)-\mathrm{Li}_1\left(e^{i(u_{ji}+\Delta_a)}\right)\right]+i\pi\left(N-2n_i\right)\right\}, \quad (2.1.18)$$

and

$$\Omega(u,\mathfrak{n},\Delta) = \sum_{a=1}^{3}(1+\mathfrak{n}_a)\sum_{i,j}^{N}\mathrm{Li}_1\left(e^{i(u_{ij}+\Delta_a)}\right)-\sum_{i\neq j}^{N}\mathrm{Li}_1\left(e^{iu_{ij}}\right)$$
$$+i\frac{N^2}{2}\sum_{a=1}^{3}(1+\mathfrak{n}_a)\Delta_a+\pi i(2M+N)\,, \quad (2.1.19)$$

where $u_{ji} = u_j - u_i$ whilst n_i and M are integer ambiguities. The JK integration contour is the so-called Jeffrey-Kirwan residue [136]. We used the polylogarithm function

$$\mathrm{Li}_1(z) = -\log(1-z)\,, \quad (2.1.20)$$

while more properties are in Appendix F.2.

2.1.3 The large N limit

To obtain the saddle-point equations, we first formulate (2.1.17) in a large N continuum description as in [29, 31, 137], and subsequently take functional derivatives. The Weyl symmetry permuting the discrete Cartan-subalgebra index i can be used to order the holonomies u_i such that $\mathbb{I}\mathrm{m}\, u_i$ increases with i. The discrete index i is then substituted with a continuous variable $t \in [t_-, t_+]$, after which u and the flux $\mathfrak{m}$ become functions of t. The reparametrization symmetry in t is fixed by identifying, up to normalization, t with $\mathbb{I}\mathrm{m}\, u(t)$:

$$u(t) = N^\alpha \big(it + v(t)\big) \ . \tag{2.1.21}$$

This introduces the density

$$\rho(t) \equiv \frac{1}{N}\,\frac{di}{dt} \ , \tag{2.1.22}$$

in terms of which any sum will be replaced by an integral: $\sum_i \to N \int dt\, \rho(t)$. The density ρ must be real, positive, and integrate to 1 in the defining range.

We perform the large N computation in Appendix F. In (F.0.11) and (F.0.12) we find:

$$\begin{aligned} \int dt\, \mathfrak{m}\, V' = ikN \int dt\, \rho\, \mathfrak{m}\, u + iN^{2-2\alpha}\, G(\Delta) \int dt\, \frac{\dot{\mathfrak{m}}\, \rho^2}{(1-i\dot v)^2} + \mathcal{O}\big(\mathfrak{m} N^{2-3\alpha}\big) \ , \\ \Omega = -N^{2-\alpha}\, f_+(\mathfrak{n}, \Delta) \int dt\, \frac{\rho^2}{1-i\dot v} + \mathcal{O}\big(N^{2-2\alpha}\big) \ , \end{aligned} \tag{2.1.23}$$

where we introduced the functions

$$G(\Delta) = \sum_{a=1}^{3} g_+(\Delta_a) \ , \qquad\qquad f_+(\mathfrak{n}, \Delta) = -\sum_{a=1}^{3} (1+\mathfrak{n}_a)\Big(g'_+(\Delta_a) - g'_+(0)\Big) \ , \tag{2.1.24}$$

and

$$g_+(\Delta) = \frac{1}{6}\Delta^3 - \frac{\pi}{2}\Delta^2 + \frac{\pi^2}{3}\Delta \ . \tag{2.1.25}$$

The entire exponent in the integrand of (2.1.17) is the functional:

$$\begin{aligned} \mathcal{V} = ikN^{1+\alpha} \int dt\, \rho\, \mathfrak{m}\, (it+v) + iN^{2-2\alpha}\, G(\Delta) \int dt\, \frac{\dot{\mathfrak{m}}\, \rho^2}{(1-i\dot v)^2} + \\ - N^{2-\alpha}\, f_+(\mathfrak{n}, \Delta) \int dt\, \frac{\rho^2}{1-i\dot v} + N^{2-\alpha}\, \mu \bigg(\int dt\, \rho - 1 \bigg) \ , \end{aligned} \tag{2.1.26}$$

where we added a Lagrange multiplier μ to enforce the normalization of ρ. For the terms in $\mathcal{V}$ to compete, we need $\alpha = \frac{1}{3}$ and $\mathfrak{m} \propto N^{\frac{1}{3}}$.

To find the saddle-point configurations at large N, we extremize $\mathcal{V}$ with respect to ρ, v, $\mathfrak{m}$ and μ. After some massaging, the saddle-point equations are:

$$0 = \frac{d}{dt}\bigg[2G\, \frac{\mathfrak{m}\, \rho}{1-i\dot v} - N^{\frac{1}{3}} \mu\, (it+v) \bigg] + 2iN^{\frac{1}{3}} f_+\, \rho \ , \tag{2.1.27}$$

$$0 = \rho\, \mathfrak{m} - \frac{2iG}{k}\, \frac{d}{dt}\bigg[\frac{\dot{\mathfrak{m}}\, \rho^2}{(1-i\dot v)^3} \bigg] + \frac{f_+}{G}\, \rho\, u \ , \tag{2.1.28}$$

$$0 = \frac{d}{dt}\bigg[k\, (it+v)^2 - 4iG\, \frac{\rho}{1-i\dot v} \bigg] \ , \tag{2.1.29}$$

together with $\int dt\, \rho = 1$. One can check that the functional $\mathcal{V}$ is invariant under reparametrizations of t that preserve the scaling ansatz (2.1.21) for the holonomies. Such reparametrizations act as:

$$\begin{aligned} & t = t(t') \;, && v(t) = i\big[t' - t(t')\big] + v'(t') \;, \\ & \rho(t) = \left(\frac{dt(t')}{dt'}\right)^{-1} \rho'(t') \;, && \mathfrak{m}(t) = \mathfrak{m}'(t') \;. \end{aligned} \tag{2.1.30}$$

Notice in particular that v' becomes complex after the transformation.

As we review in Appendix F.1, the equations (2.1.27)–(2.1.29) can be solved, yielding:

$$u(t) = \left(\frac{3NG}{k}\right)^{\frac{1}{3}} t \;, \quad \mathfrak{m}(t) = \left(\frac{N}{9kG^2}\right)^{\frac{1}{3}} f_+\, t \;, \quad \rho(t) = \frac{3}{4}\big(1 - t^2\big) \;, \quad t \in [-1, 1] \;. \tag{2.1.31}$$

This solution is obtained after making use of the reparametrization symmetry, so in particular $v(t)$ is complex. The value of the functional $\mathcal{V}$ at the saddle point for ρ, v and $\mathfrak{m}$ — which reproduces the logarithm of the index at leading order — is

$$\mathcal{V} = -\frac{iN^{\frac{5}{3}}}{5} \left(\frac{9k}{G(\Delta)}\right)^{\frac{1}{3}} f_+(\mathfrak{n}, \Delta) \;. \tag{2.1.32}$$

If $\sum_a \Delta_a = 2\pi$, the two functions G and f_+ take the particularly simple form

$$G(\Delta) = \frac{1}{2}\, \Delta_1\Delta_2\Delta_3 \;, \qquad\qquad f_+(\mathfrak{n}, \Delta) = -\frac{1}{2}\, \Delta_1\Delta_2\Delta_3 \sum_{a=1}^{3} \frac{\mathfrak{n}_a}{\Delta_a} \;. \tag{2.1.33}$$

In this case, the saddle-point value of the logarithm of the index is

$$\mathcal{V} = \frac{iN^{\frac{5}{3}}}{5} \left(\frac{9k}{4}\right)^{\frac{1}{3}} \big(\Delta_1\Delta_2\Delta_3\big)^{\frac{2}{3}} \sum_{a=1}^{3} \frac{\mathfrak{n}_a}{\Delta_a} \;. \tag{2.1.34}$$

When the Δ_a's are real this expression matches the result of [31], which reproduces the black-hole entropy upon performing a Legendre transform.

As mentioned above, the chemical potentials Δ_a are defined modulo 2π. The expression for $\mathcal{V}$ in (2.1.32), however, is not periodic under $\Delta_a \to \Delta_a + 2\pi$. This means that we have actually found an infinite number of saddle points, parametrized by the shifts.[6] This suggests that — as in AdS_3 [129] and AdS_5 [130] — there might be an infinite number of complex BPS black-hole-like supergravity solutions dual to the semiclassical expansion of the TT index. This issue deserves more study.

[6] In general, only a subset of the complex saddle points contribute to the contour integral: which ones do (depending on the contour) should be determined with steepest descent.

2.2 KK reduction on a flux background

The next step is to perform a Kaluza-Klein (KK) reduction of the 3d $\mathcal{N}=2$ gauge theory on the sphere S^2, in the presence of the flux background $\mathfrak{m}$ (2.1.31), determined as the saddle point of the TT index. By keeping only the light modes, we will obtain a 1d quantum mechanical model which we expect to be dual to the horizon degrees of freedom of the dyonic AdS_4 black holes we are interested in. This section is rather technical, and the reader only interested in the final result can directly jump to Section 2.3.

Here we will first show how the full twisted theory can be seen as a gauged $\mathcal{N}=2$ quantum mechanics. Afterwards, we will introduce the background of the reduction and review the standard procedure to fix the 3d gauge group down to the 1d gauge group. We will then explain why complications arise when computing the KK spectrum of the vector multiplet, and how they can be resolved by a further modification of the gauge-fixing Lagrangian. Lastly, we will exhibit the KK spectra of the vector and chiral multiplets.

2.2.1 Decomposing 3d multiplets into 1d multiplets

After the topological twist, the theory exactly fits into the framework of a gauged $\mathcal{N}=2$ quantum mechanics, and we perform various changes of variables in this section to make it explicit. A similar discussion can be found in [138]. We give a brief review of 1d $\mathcal{N}=2$ supersymmetry in Appendix I. Although it is adapated from [139], it also presents in Appendix I.5 and I.6 new supersymmetric Lagrangians peculiar to our 3d theory.

We shall write the supersymmetry transformations in terms of anticommuting generators Q and $\overline{Q}$, with the understanding that generators should be multiplied by a complex anticommuting parameter to produce a generic supersymmetry transformation. With $\epsilon=(1,0)^{\mathsf{T}}$, Q is obtained from $\widetilde{Q}_{3\text{d}}$ while $\overline{Q}$ is obtained from $Q_{3\text{d}}$ in (G.0.1) and (G.0.2). Note that Q and $\overline{Q}$ are related by Hermitian conjugation, that is $\overline{(QX)}=(-1)^F\,\overline{Q}\,\overline{X}$. The supersymmetry algebra is

$$Q^2=\overline{Q}^2=0\ ,\qquad\qquad \{Q,\overline{Q}\}=i\big[\partial_t-\delta_{\text{gauge}}(A_t+\sigma)\big]\ ,\tag{2.2.1}$$

where $\delta_{\text{gauge}}(\alpha)$ is a gauge tranformation with parameter α. We will use frame fields e^1_μ, $e^{\bar{1}}_\mu$ on S^2, which we introduce in Appendix H, and write differential forms on S^2 with flat indices $1,\bar{1}$. From now on, $\overline{X}$ will denote the Hermitian conjugate of X (since Dirac conjugates are no longer present anyway). After this rewriting, the supersymmetry variations and supersymmetric Lagrangians are as described below.

Vector multiplet. In Wess-Zumino gauge, the 3d vector multiplet consists of the gauge field A_μ, a real scalar σ, a real auxiliary scalar D, and a Dirac spinor λ. The bosonic components are R-neutral while λ has R-charge -1. We decompose λ in components as

$$\lambda = \begin{pmatrix} -\overline{\Lambda}_t \\ \Lambda_{\bar{1}} \end{pmatrix} , \tag{2.2.2}$$

and redefine D with a shift

$$D' = D - 2iF_{1\bar{1}} . \tag{2.2.3}$$

Now, $\Lambda_{\bar{1}}$ has R-charge -1 whereas Λ_t has R-charge $+1$. These field redefinitions have trivial Jacobian. Under the supercharges preserved by the twist, the supersymmetry variations of the vector multiplet split into 2 sets of variations. The first set (Hermitian conjugate relations being implied) is:

$$\begin{aligned} QA_t &= -Q\sigma = -\frac{i}{2}\overline{\Lambda}_t , &\qquad Q\Lambda_t &= -D_t\sigma - iD , \\ QD &= -\frac{1}{2}(D_t - i\sigma)\overline{\Lambda}_t , &\qquad \overline{Q}\Lambda_t &= 0 . \end{aligned} \tag{2.2.4}$$

These coincide with the supersymmetry variations (I.4.1) of a 1d $\mathrm{U}(N)$ vector multiplet in Wess-Zumino gauge. Note that here the fields and gauge transformations are also functions on S^2. The second set is:

$$QA_{\bar{1}} = \frac{1}{2}\Lambda_{\bar{1}} , \qquad \overline{Q}A_{\bar{1}} = 0 , \qquad Q\Lambda_{\bar{1}} = 0 , \qquad \overline{Q}\Lambda_{\bar{1}} = 2i\big(\partial_t A_{\bar{1}} - D_{\bar{1}}(A_t + \sigma)\big) . \tag{2.2.5}$$

These coincide with the supersymmetry variations (I.4.3) of a chiral multiplet $\big(A_{\bar{1}}, \tfrac{1}{2}\Lambda_{\bar{1}}\big)$ in Wess-Zumino gauge, provided that the corresponding superfields

$$\Xi_{\bar{1},h} = A_{\bar{1}} + \frac{\theta}{2}\Lambda_{\bar{1}} - \frac{i}{2}\theta\bar{\theta}\,\partial_t A_{\bar{1}} , \qquad\qquad \Xi_{1,\bar{h}} \equiv \overline{\Xi_{\bar{1},h}} = A_1 - \frac{\bar{\theta}}{2}\overline{\Lambda}_1 + \frac{i}{2}\theta\bar{\theta}\,\partial_t A_1 \tag{2.2.6}$$

satisfying $\overline{D}\,\Xi_{\bar{1},h} = D\,\Xi_{1,\bar{h}} = 0$, transform as connections under super-gauge transformations:

$$\Xi_{\bar{1},h} \to h\left(\Xi_{\bar{1},h} + i\partial_{\bar{1}}\right)h^{-1} , \qquad \Xi_{1,\bar{h}} \to \overline{h}^{\,-1}\big(\Xi_{1,\bar{h}} + i\partial_1\big)\,\overline{h} , \tag{2.2.7}$$

with $h = e^\chi$ and $\overline{D}\chi = 0$. We indicated as $\overline{\Lambda}_1$ the complex conjugate to $\Lambda_{\bar{1}}$.

The Yang-Mills Lagrangian is composed of two pieces, independently supersymmetric:

$$\begin{aligned} 2e_{3\mathrm{d}}^2\,\mathcal{L}_{\mathrm{YM}} = \;& \mathrm{Tr}\Big[4|F_{t\bar{1}}|^2 + 4iDF_{1\bar{1}} - 4|D_{\bar{1}}\sigma|^2 + i\overline{\Lambda}_1(D_t + i\sigma)\Lambda_{\bar{1}} + 2\Lambda_t D_1\Lambda_{\bar{1}} - 2\overline{\Lambda}_1 D_{\bar{1}}\overline{\Lambda}_t\Big] \\ & + \mathrm{Tr}\Big[(D_t\sigma)^2 + D^2 + i\overline{\Lambda}_t(D_t - i\sigma)\Lambda_t\Big] . \end{aligned} \tag{2.2.8}$$

Note that $2e_{3\mathrm{d}}^2\,\mathcal{L}_{\mathrm{YM}} = Q\overline{Q}\,\mathrm{Tr}\big[-4iA_1\partial_t A_{\bar{1}} + 4i(A_t - \sigma)F_{1\bar{1}}\big] + Q\overline{Q}\,\mathrm{Tr}\big[-\overline{\Lambda}_t\Lambda_t\big]$, so both terms are exact. The first piece is the appropriate kinetic term for a chiral transforming as a connection and its superspace expression is in (I.6.1). The second piece is the standard 1d gauge-kinetic term (I.5.6). Likewise, the Chern-Simons Lagrangian splits into two pieces which are separately supersymmetric:

$$\frac{4\pi}{k}\,\mathcal{L}_{\rm CS} = \mathrm{Tr}\Big[4iA_1\partial_t A_{\bar 1} - 4i(A_t+\sigma)F_{1\bar 1} + \overline{\Lambda}_1\,\Lambda_{\bar 1}\Big] \;+\; \mathrm{Tr}\Big[\overline{\Lambda}_t\Lambda_t - 2D\sigma\Big]\;. \tag{2.2.9}$$

The superspace expression of the first piece is given in (I.6.9), whereas the second piece matches (I.5.9).

Chiral multiplet. A 3d chiral multiplet consists of a complex scalar ϕ, a Dirac spinor Ψ and a complex auxiliary field f. We split Ψ into components as

$$\Psi = -i\begin{pmatrix}\psi\\ \eta\end{pmatrix}\;. \tag{2.2.10}$$

Their R-charges are $R(\psi) = R(\eta) = R(\phi) - 1$. Under the supercharges preserved by the twist, the supersymmetry variations of the 3d chiral multiplet can also be organized into two sets. The first set (Hermitian conjugate relations are again implicit) is:

$$Q\phi = \psi\;, \qquad \overline{Q}\phi = 0\;, \qquad Q\psi = 0\;, \qquad \overline{Q}\psi = i(D_t - i\sigma)\phi\;. \tag{2.2.11}$$

They coincide with the supersymmetry variations (I.4.3) of a 1d chiral multiplet (ϕ,ψ) in Wess-Zumino gauge , with corresponding superfield $\Phi_h = \phi + \theta\psi - \frac{i}{2}\theta\bar\theta\,\partial_t\phi$. The second is:

$$Q\eta = -f\;, \qquad \overline{Q}\eta = -2D_{\bar 1}\phi\;, \qquad Qf = 0\;, \qquad \overline{Q}f = -i(D_t - i\sigma)\eta - 2D_{\bar 1}\psi + i\Lambda_{\bar 1}\phi\;. \tag{2.2.12}$$

They match the variations (I.4.5) of a 1d Fermi multiplet (η, f) in Wess-Zumino gauge, whose corresponding superfield

$$\mathcal{Y}_h = \eta - \theta f + 2\bar\theta D_{\bar 1}\phi + \theta\bar\theta\left(-\frac{i}{2}\,\partial_t\eta - 2D_{\bar 1}\psi + i\Lambda_{\bar 1}\phi\right) \tag{2.2.13}$$

satisfies

$$\overline{D}\,\mathcal{Y}_h = E\big(\Phi_h, \Xi_{\bar 1,h}\big) = -2\big(\partial_{\bar 1} - i\Xi_{\bar 1,h}\big)\Phi_h\;. \tag{2.2.14}$$

Here $\partial_{\bar 1}$ contains the background $\mathrm{U}(1)_R$ connection. In the language of 1d supersymmetry, there is an E-term superpotential for $\mathcal{Y}_h$. After the shift (2.2.3), the kinetic term of a 3d chiral multiplet also splits into two separately supersymmetric pieces, *i.e.*, the kinetic terms of the 1d chiral (I.5.10) and of the 1d Fermi (I.5.13):

$$\begin{aligned}\mathcal{L}_{\rm chiral} = &\;\Big[|D_t\phi|^2 - |\sigma\phi|^2 - \overline{\phi}D\phi + i\overline{\psi}(D_t + i\sigma)\psi - i\overline{\psi}\,\overline{\Lambda}_t\phi + i\overline{\phi}\Lambda_t\psi\Big] \\ &+ \Big[i\overline{\eta}(D_t - i\sigma)\eta + \overline{f}f - |2D_{\bar 1}\phi|^2 - 2\overline{\psi}D_1\eta + 2\overline{\eta}D_{\bar 1}\psi - i\overline{\eta}\Lambda_{\bar 1}\phi + i\overline{\phi}\,\overline{\Lambda}_1\eta\Big]\;.\end{aligned} \tag{2.2.15}$$

Note that $\mathcal{L}_{\text{chiral}} = Q\overline{Q}\big(-i\overline{\phi}(D_t + i\sigma)\phi\big) + Q\overline{Q}\big(-\overline{\eta}\eta\big)$, so both terms are exact.

The superpotential terms can be written as $\mathcal{L}_{\text{W}} = -Q\big(\eta_a \frac{\partial W}{\partial \phi_a}\big) + \overline{Q}\big(\overline{\eta}_a \frac{\partial \overline{W}}{\partial \overline{\phi}_a}\big)$, which in the language of 1d supersymmetry are J terms for the Fermi multiplets η_a with $J_a = -\frac{\partial W}{\partial \phi_a}$. Supersymmetry of the first term under Q, and of the second term under $\overline{Q}$, are obvious. When $\overline{Q}$ acts on the first term we get, up to a total time derivative,

$$Q\overline{Q}\left(\eta_a \frac{\partial W}{\partial \phi_a}\right) = -2Q\left(D_{\bar{1}}\phi_a \frac{\partial W}{\partial \phi_a}\right) = -2Q(\partial_{\bar{1}} W) = -2\partial_{\bar{1}} Q W \ , \tag{2.2.16}$$

which is another total derivative. Thus the superpotential terms are $(Q + \overline{Q})$-exact. The supersymmetric Chern-Simons Lagrangian is the only piece that is not exact under any supercharge.

2.2.2 Reduction background

As mentioned at the beginning of this section, we want to reduce the theory in the presence of background fluxes for the global symmetries. In particular, we turn on a (negative) unit flux for the R-symmetry q_R. Since it is a background for a non-dynamical field, it can be off-shell without any consequences. The presence of this background, under which the chiral multiplets are differently charged, generically breaks the SU(3) flavor symmetry down to its diagonal subgroup $\mathrm{U}(1)^2_F$. We also single out a configuration of fluxes for the dynamical gauge fields:

$$F_{1\bar{1}} = \frac{i\,\mathfrak{m}}{4R^2} \ , \qquad \text{where } \mathfrak{m} \text{ is a constant in the Cartan subalgebra.} \tag{2.2.17}$$

The choice of $\mathfrak{m}$ will eventually be the one dictated by the saddle-point approximation to the topologically twisted index, discussed in Section 2.1. Since $F_{1\bar{1}}$ couples to the auxiliary field D in (2.2.8) like a FI parameter, the D-term equation for supersymmetric vacua is:

$$\frac{2i}{e^2_{\text{3d}}}\, F_{1\bar{1}} + \sum_a\, [\overline{\phi}_a, \phi_a] - \frac{k}{2\pi}\sigma = 0 \ . \tag{2.2.18}$$

The background should satisfy the D-term equation in order to be supersymmetric, and it is simplest to turn on a background for σ to cancel the background flux. This falls into the class of "topological" vacua discussed in [140]. Moreover, since $A_t + \sigma$ appears in the algebra (2.2.1), we also find it appropriate to turn on a background for A_t, opposite to that of σ, so that the background of $A_t + \sigma$ is zero. This ensures that BPS states have zero energy even before projecting onto gauge singlets. Thus, the background we use for the reduction is:

$$F_{1\bar{1}} = \frac{i\mathfrak{m}}{4R^2} \ , \qquad \sigma = -\frac{\mathfrak{m}}{2m_k R^2} \ , \qquad A_t = \frac{\mathfrak{m}}{2m_k R^2} \ , \qquad \text{where} \quad m_k \equiv \frac{k\, e^2_{\text{3d}}}{2\pi} \ . \tag{2.2.19}$$

One can check that all the equations of motion are satisfied on this background, except for that of $A_t + \sigma$, unless $\mathfrak{m} = 0$. Consequently, when expanding the action, there will be a Lagrangian term linear in $A_t + \sigma$, that is

$$\mathrm{Tr}\left(\frac{k\mathfrak{m}}{4\pi R^2}\,(A_t+\sigma)\right) . \tag{2.2.20}$$

In other words, background fluxes produce a background electric charge in the presence of Chern-Simons terms. As we will discuss later, the presence of this linear term is crucial and it is the main source of complications when computing the vector multiplet spectrum.

We parametrize the Lie algebra $\mathfrak{su}(N)$ by $N\times N$ matrices E_{ij} $(i,j=1,\dots,N)$ which have a single nonzero entry 1 in row i and column j: $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Elements with $i=j$ are a basis for the Cartan subalgebra, while those with $i\neq j$ correspond to roots with root vector $(\alpha_{ij})_k = \delta_{ki} - \delta_{kj}$. The commutation relations in this basis are

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \ . \tag{2.2.21}$$

Note also that $\overline{E_{ij}} = E_{ji}$ and

$$\mathrm{Tr}\, E_{ij}E_{kl} = \delta_{jk}\delta_{il} \ , \qquad \mathrm{Tr}\, E_{ij}[E_{kl}, E_{mn}] = \delta_{jk}\delta_{lm}\delta_{ni} - \delta_{il}\delta_{jm}\delta_{kn} \ . \tag{2.2.22}$$

We write the expansion of adjoint fields in this basis as $X = X^{ij}E_{ij}$. Note that $\overline{X}^{ij} = \overline{X^{ji}}$. The Cartan components will sometimes be written as $X^i \equiv X^{ii}$ for simplicity.

In the presence of global and gauge fluxes, the Lie algebra components of various fields in the vector multiplet and chiral multiplets are $\mathrm{U}(1)_{\text{spin}}$ sections with different monopole charges q (see Appendix H for details). A field $\chi_q(t,\theta,\varphi)$ with monopole charge q can then be expanded in a complete set of monopole harmonics $Y_{q,l,m}(\theta,\varphi)$, and the time-dependent expansion coefficients $\chi_{q,l,m}(t)$ are the 1d fields after the reduction:

$$\chi_q(t,\theta,\varphi) = \sum_{l\geq|q|}\ \sum_{|m|\leq l} \chi_{q,l,m}(t)\ Y_{q,l,m}(\theta,\varphi) \ . \tag{2.2.23}$$

Defining the quantities

$$q_{ij} \equiv \frac{\mathfrak{m}_i - \mathfrak{m}_j}{2} \ , \qquad\qquad q^a_{ij} \equiv \frac{\mathfrak{m}_i - \mathfrak{m}_j + \mathfrak{n}_a}{2} \ , \tag{2.2.24}$$

the monopole charges of the fields and their charges under the global symmetries of the theory are summarized in Table 2.1.

We assume that $\mathfrak{m}_i \neq \mathfrak{m}_j$, $\forall\ i\neq j$, since this is true for the saddle-point flux, and thus $q_{ij}\neq 0$ for $i\neq j$. Given a Hermitian adjoint field $X = X^{ij}E_{ij} = \overline{X}$ in a vector multiplet (*i.e.*, A_t, σ, D), its components satisfy $X^{ji} = \overline{X^{ij}}$. We parametrize the off-diagonal components in terms of complex fields X^{ij} with ij such that $q_{ij} > 0$. For complex adjoint fields $Y = Y^{ij}E_{ij}$ in

VM	σ^{ij}, A_t^{ij}, D^{ij}	Λ_t^{ij}	$A_{\bar{1}}^{ij}$	A_1^{ij}	$\Lambda_{\bar{1}}^{ij}$	$\overline{\Lambda}_1^{ij}$
q	q_{ij}	q_{ij}	$q_{ij}+1$	$q_{ij}-1$	$q_{ij}+1$	$q_{ij}-1$
q_R	0	1	0	0	-1	1
q_1	0	0	0	0	0	0
q_2	0	0	0	0	0	0

CM	ϕ_a^{ij}	ψ_a^{ij}	η_a^{ij}	f_a^{ij}
q	q_{ij}^a	q_{ij}^a	q_{ij}^a+1	q_{ij}^a+1
q_R	$-\mathfrak{n}_a$	$-\mathfrak{n}_a-1$	$-\mathfrak{n}_a-1$	$-\mathfrak{n}_a-2$
q_1	$\delta_{1a}-\delta_{3a}$	$\delta_{1a}-\delta_{3a}$	$\delta_{1a}-\delta_{3a}$	$\delta_{1a}-\delta_{3a}$
q_2	$\delta_{2a}-\delta_{3a}$	$\delta_{2a}-\delta_{3a}$	$\delta_{2a}-\delta_{3a}$	$\delta_{2a}-\delta_{3a}$

Table 2.1: Monopole and global charges of all fields. The R-charge is q_R, while $q_{1,2}$ are flavor charges. Above: modes from 3d vector multiplets. The modes are defined for pairs i, j such that $q_{ij} > 0$. Below: modes from 3d chiral multiplets, defined for any ij. In both cases, the modes are in SU(2) representations with $l \geq |q|$ and $l = q \bmod 1$.

vector multiplets (*i.e.*, $A_{\bar{1}}$, A_1, $\Lambda_{\bar{1}}$, $\overline{\Lambda}_1$), we initially parametrize the off-diagonal components in terms of complex fields Y^{ij}, $\overline{Y}^{ij}$ with ij such that $q_{ij} > 0$. For complex adjoint fields in chiral multiplets, instead, we simply use all components Y^{ij}.

The flux breaks the gauge group $\mathrm{U}(N)$ to its maximal torus $\mathrm{U}(1)^N$, and the 1d gauge group will consequently be $\mathrm{U}(1)^N$. Indeed, the generators of 1d gauge transformations have to be constant on S^2. However the components ϵ^{ij} of the gauge-transformation parameter have monopole charges q_{ij}, and since $l \geq |q_{ij}|$, only those in the Cartan subalgebra have an $l = 0$ mode which is constant on S^2.

2.2.3 Partial gauge fixing

In order to reduce to a gauged quantum mechanics, we need to fix the 3d gauge group to the unbroken 1d gauge group, consisting of time-dependent transformations that are constant on S^2. A systematic procedure to achieve that is presented in Appendix J and we refer the reader to [141] for more details. We choose the Coulomb gauge with gauge-fixing function

$$G_{\mathrm{gf}} = \frac{2}{\sqrt{\xi}} \left(D_1^B A_{\bar{1}} + D_{\bar{1}}^B A_1 \right) . \tag{2.2.25}$$

One can check that it leaves the 1d gauge group unfixed. The covariant derivatives above only contain the spin connection and monopole background. In general, for any G_{gf}, the gauge-fixing procedure adds the following terms to the Lagrangian:

$$\frac{1}{e_{3\mathrm{d}}^2}\,\mathrm{Tr}\left[\frac{b^2}{2}+b\left(G_{\mathrm{gf}}-\{\widetilde{c},c\}\right)+i\,\widetilde{c}\,\delta_{\mathrm{gauge}}(c)\,G_{\mathrm{gf}}+\frac{1}{2}\,\{\widetilde{c},c\}^2\right]. \tag{2.2.26}$$

Here c and $\widetilde{c}$ are independent Grassmann scalars, while b is a bosonic auxiliary field. Importantly, all of them are valued in the part of the gauge algebra that is broken by G_{gf}, and do not contain modes in the residual gauge algebra. In the following, a subscript $\mathfrak{r}$ will indicate a restriction to the residual gauge algebra, and a subscript $\mathfrak{f}$ a restriction to the complement containing fixed (or broken) gauge generators.[7] We define a BRST supercharge s as:

$$sX=\delta_{\mathrm{gauge}}(c)\,X\;,\quad sc=\frac{i}{2}\{c,c\}_{\mathfrak{f}}\;,\quad s\widetilde{c}=ib\;,\quad sb=\delta_{\mathrm{gauge}}(R)\,\widetilde{c}\;,\quad R\equiv-\frac{1}{2}\{c,c\}_{\mathfrak{r}}\;. \tag{2.2.27}$$

One can check that

$$s^2=i\,\delta_{\mathrm{gauge}}(R)\;,\qquad\qquad sR=0\;. \tag{2.2.28}$$

This allows us to define an s-cohomology on invariants of the residual gauge group. The terms produced by gauge fixing can then be written in a BRST-exact form:

$$(2.2.26)=\frac{1}{e_{3\mathrm{d}}^2}\,s\,\mathrm{Tr}\,\widetilde{c}\left(-i\,G_{\mathrm{gf}}-\frac{i}{2}\,b+\frac{i}{2}\,\{\widetilde{c},c\}\right)\equiv s\Psi_{\mathrm{gf}}\;. \tag{2.2.29}$$

We defined Ψ_{gf} as the function in parentheses. We note that there is still complete freedom in specifying the inner product in the ghost sector, *i.e.*, the Hermiticity properties of c and $\widetilde{c}$. In order for the theory to be unitary and have a consistent Hamiltonian formulation [142], one needs that c and $\widetilde{c}$ are Hermitian, so that s is a real supercharge and (2.2.26) is real. With this choice, (2.2.26) is invariant under a ghost-number symmetry valued in $\mathbb{R}^*$, which acts as:

$$c\;\mapsto\;e^{\alpha}\,c\;,\qquad\widetilde{c}\;\mapsto\;e^{-\alpha}\,\widetilde{c}\;,\qquad s\;\mapsto\;e^{\alpha}\,s\;, \tag{2.2.30}$$

with $\alpha\in\mathbb{R}$. We say that c has ghost number $n_g=1$ and $\widetilde{c}$ has $n_g=-1$. Physical observables are identified with the s-cohomology at $n_g=0$, since external states must be gauge-invariant and cannot contain ghosts. Since c, $\widetilde{c}$, and b are Hermitian, they are neutral under $\mathrm{U}(1)_R$, and (2.2.26) is invariant under $\mathrm{U}(1)_R$, since G_{gf} is R-neutral.

2.2.4 Supersymmetrized gauge fixing

As anticipated, the linear term (2.2.20) causes complications in the computation of the KK spectrum of the vector multiplet, and the following discussion aims to explain why. The standard Faddeev-Popov gauge-fixing procedure we just reviewed generically breaks

[7] In the Coulomb gauge (2.2.25), $\mathfrak{r}$ contains diagonal transformations with $l=0$, while $\mathfrak{f}$ contains diagonal transformations with $l>0$ as well as all off-diagonal transformations.

the supersymmetries that were defined on the original action because of the presence of the BRST-exact term $s\Psi_{\text{gf}}$, which might not be supersymmetric. Considering a supercharge Q, and assuming that it does not act on the fields in the gauge-fixing complex, the transformation of $s\Psi_{\text{gf}}$ is $-sQ\Psi_{\text{gf}}$. When computing s-closed (*i.e.*, gauge-invariant) quantities, this is harmless because the potentially violating term is s-exact, and it does not affect the result. For example, supersymmetric Ward identities can be derived for any observable in the theory, since their correlators do not depend on s-exact terms.

However, the spectrum of the Chern-Simons-matter theory around a monopole background is not gauge-invariant, because the quadratic action is not invariant under linearized BRST transformations.[8] This can be seen from the presence of the linear term (2.2.20). Its BRST variation is $\frac{1}{4\pi R^2}\operatorname{Tr}\big(ik\mathfrak{m}\,[c, A_t+\sigma]\big)$, and it must cancel with the linearized BRST variation of the quadratic action, which is then nonzero. Consequently, there is no guarantee that the spectrum will be supersymmetric, because it is computed from a quadratic action that is not s-closed, and therefore s-exact terms violating supersymmetry cannot be neglected.

A way to resolve this issue takes inspiration from [26]. In addition to adding $s\Psi_{\text{gf}}$ to gauge-fix our path-integral, we can further add $\mathcal{Q}\Psi_{\text{gf}}$. The real supercharge $\mathcal{Q}$ acts as $\mathcal{Q} = Q + \overline{Q}$ on physical fields, and we choose its action on the gauge-fixing complex such that $\delta \equiv (s+\mathcal{Q})$ closes on symmetries and unfixed gauge transformations. We will show that the further addition of $\mathcal{Q}\Psi_{\text{gf}}$ does not change the expectation value of any (possibly non-supersymmetric) operator $\mathcal{O}$ with ghost number $n_g \leq 0$. In particular, physical observables with $n_g = 0$ are not affected. At this point, we have added $\delta\Psi_{\text{gf}}$ to the original action. The real supercharge δ is explicitly preserved because our choice that δ^2 contains symmetries and unfixed gauge transformations implies $\delta^2\Psi_{\text{gf}} = 0$. With this procedure, the number of preserved supercharges has not changed; while the gauge-fixed action with $s\Psi_{\text{gf}}$ is invariant under s, the gauge-fixed action with $\delta\Psi_{\text{gf}}$ is invariant under δ. Its usefulness for computing the spectrum lies in the fact that $A_t + \sigma$ can be redefined by shifting with a quadratic combination of ghosts such that $\delta(A_t' + \sigma') = 0$, making the linear term (2.2.20) δ-closed. By extension, the quadratic action which is modified by the shift is also δ-closed, and its spectrum is supersymmetric.

In order for $\delta\Psi_{\text{gf}} = (s+\mathcal{Q})\Psi_{\text{gf}}$ to be invariant under δ, δ^2 should only contain residual gauge transformations and possibly other symmetries of Ψ_{gf}. This condition constrains how $\mathcal{Q}$ can act on fields in the gauge-fixing complex. The supersymmetry transformations of the physical fields X under $\mathcal{Q}$ are given in (2.2.4)-(2.2.5) and (2.2.11)-(2.2.12). Without

[8] Although the BRST transformations are non-linear in the fields, to have a gauge-invariant spectrum, it would be enough that the quadratic action be invariant under the *linearized* transformations.

specifying how $\mathcal{Q}$ acts on the fields Y in the gauge-fixing complex, we find:

$$\begin{aligned} \mathcal{Q}^2 X &= \{Q, \overline{Q}\} X = i\big[\partial_t - \delta_{\text{gauge}}(A_t + \sigma)\big]\, X \;, \qquad \{\mathcal{Q}, s\} X = \delta_{\text{gauge}}\big(\mathcal{Q}c\big)\, X \;, \\ \delta^2 X &= i\Big[\partial_t - \delta_{\text{gauge}}\big(A_t + \sigma + i\mathcal{Q}c - R\big)\Big]\, X \;. \end{aligned} \tag{2.2.31}$$

If we want δ to close on time translations and residual gauge transformations, the only possibility is to set $\mathcal{Q}c = i(A_t + \sigma)_{\mathfrak{f}}$. Hence, physical fields satisfy the algebra:

$$\delta^2 X = i\Big[\partial_t - \delta_{\text{gauge}}\big(A_{t,\mathfrak{r}} + \sigma_{\mathfrak{r}} - R\big)\Big]\, X \;. \tag{2.2.32}$$

Having fixed $\mathcal{Q}c$, we find that c also satisfies (2.2.32) and specifically

$$\mathcal{Q}^2 c = 0 \;, \qquad\qquad \{\mathcal{Q}, s\} c = i\big[\partial_t - \delta_{\text{gauge}}(A_{t,\mathfrak{r}} + \sigma_{\mathfrak{r}})\big]\, c \;, \tag{2.2.33}$$

which imply (2.2.32). For uniformity, we demand that (2.2.33) is satisfied on all fields Y in the gauge-fixing complex. Setting $\mathcal{Q}\widetilde{c} = 0$ for simplicity, we find that this fixes $\mathcal{Q}b$ and, altogether, $\mathcal{Q}$ acts on the fields in the gauge-fixing complex as:

$$\mathcal{Q}\, c = i(A_t + \sigma)_{\mathfrak{f}} \;, \qquad \mathcal{Q}\, \widetilde{c} = 0 \;, \qquad \mathcal{Q}\, b = \big[\partial_t - \delta_{\text{gauge}}(A_{t,\mathfrak{r}} + \sigma_{\mathfrak{r}})\big]\, \widetilde{c} \;. \tag{2.2.34}$$

Given Ψ_{gf} that we defined in (2.2.29), we can now determine

$$\mathcal{Q}\, \Psi_{\text{gf}} = \frac{1}{e_{3\text{d}}^2} \operatorname{Tr}\Big[i\, \widetilde{c}\, \mathcal{Q} G_{\text{gf}} + \frac{i}{2}\, \widetilde{c}(D_t - i\sigma)\widetilde{c} \Big] \;, \tag{2.2.35}$$

where σ acts in the adjoint representation (namely, $\sigma\widetilde{c}$ stands for $[\sigma, \widetilde{c}]$ in matrix notation). Hence, collecting the contributions from (2.2.26) and (2.2.35), the supersymmetrized gauge-fixing procedure requires us to add the following terms to the original Lagrangian:

$$\delta\Psi_{\text{gf}} = \frac{1}{e_{3\text{d}}^2} \operatorname{Tr}\bigg[\frac{b^2}{2} + b\,\Big(G_{\text{gf}} - \{\widetilde{c}, c\}\Big) + i\, \widetilde{c}\,\Big(\delta_{\text{gauge}}(c) + \mathcal{Q}\Big)\, G_{\text{gf}} + \frac{1}{2}\{\widetilde{c}, c\}^2 + \frac{i}{2}\, \widetilde{c}\,\big(D_t - i\sigma\big)\, \widetilde{c} \bigg] . \tag{2.2.36}$$

With the choice that c and $\widetilde{c}$ are Hermitian, $\delta\Psi_{\text{gf}}$ is real.

It is important to note (following [26]) that adding $\mathcal{Q}\Psi_{\text{gf}}$ to $s\Psi_{\text{gf}}$ does not change the expectation value of any operators with $n_g \leq 0$, even if they are *not* invariant under $\mathcal{Q}$. In particular, it does not change physical observables. This can be shown explicitly for the thermal partition function. We first integrate in an adjoint-valued auxiliary field a to rewrite the quartic ghost interactions, after which the gauge-fixing action becomes:

$$\delta\Psi_{\text{gf}} = \frac{1}{e_{3\text{d}}^2} \operatorname{Tr}\bigg[\frac{b^2 - a^2}{2} + b\, G_{\text{gf}} + \widetilde{c}\, \big[a + b, c\big] + i\widetilde{c}\Big(\delta_{\text{gauge}}(c) + \mathcal{Q}\Big) G_{\text{gf}} + \frac{i}{2}\, \widetilde{c}\big(D_t - i\sigma\big)\widetilde{c} \bigg] . \tag{2.2.37}$$

Note that a has both gauge-fixed and residual components. Since the full action is quadratic in the Grassmann fields $\{F_{\text{phys}}, c, \widetilde{c}\}$, where F_{phys} is the set of physical fermions, we can formally perform the path integral over them, obtaining:

$$\det\begin{pmatrix} S_0|_{F,F} & 0 & \mathcal{Q}\Psi_{\rm gf}|_{F,\widetilde{c}} \\ 0 & 0 & s\Psi_{\rm gf}|_{c,\widetilde{c}} \\ \mathcal{Q}\Psi_{\rm gf}|_{\widetilde{c},F} & s\Psi_{\rm gf}|_{\widetilde{c},c} & \mathcal{Q}\Psi_{\rm gf}|_{\widetilde{c},\widetilde{c}} \end{pmatrix} \sim \det\Big(s\Psi_{\rm gf}|_{c,\widetilde{c}}\Big)\,\det\Big(S_0|_{F,F}\Big)\,. \tag{2.2.38}$$

All entries of the matrix on the l.h.s. are (possibly differential) operators involving the bosons. This proves that the thermal partition function does not depend on the term $\mathcal{Q}\Psi_{\rm gf}$.

More generally, we prove that the expectation value of any operator $\mathcal{O}$ with ghost number $n_g \leq 0$ is unchanged by the addition of $\mathcal{Q}\Psi_{\rm gf}$ to the Lagrangian. The key property is that $\mathcal{Q}\Psi_{\rm gh}$ is the sum of two terms, of ghost number -1 and -2, respectively. Let $\langle\cdot\rangle_s$ be the path integral with $s\Psi_{\rm gf}$ as gauge fixing, and let $\langle\cdot\rangle_\delta$ be the path integral with $\delta\Psi_{\rm gf}$ as gauge fixing. We have

$$\langle\mathcal{O}\rangle_\delta = \left\langle\mathcal{O}\,e^{i\mathcal{Q}\Psi_{\rm gf}}\right\rangle_s = \langle\mathcal{O}\rangle_s + \sum_{n=1}^{\infty}\frac{(i)^n}{n!}\left\langle\mathcal{O}\,(\mathcal{Q}\Psi_{\rm gf})^n\right\rangle_s = \langle\mathcal{O}\rangle_s\,. \tag{2.2.39}$$

The last equality holds because ghost number is a symmetry of $\langle\cdot\rangle_s$, implying null expectation value for any correlator that has $n_g \neq 0$. Since $\mathcal{O}\,(\mathcal{Q}\Psi_{\rm gf})^n$ has $n_g < 0$, one concludes that $\langle\mathcal{O}\,(\mathcal{Q}\Psi_{\rm gf})^n\rangle_s = 0$ for every n. For the restricted set of operators $\mathcal{O}$ with $n_g \leq 0$, one can constrain $\langle\cdot\rangle_\delta$ using the symmetries of $\langle\cdot\rangle_s$. In particular, although both supersymmetry and $\mathrm{U}(1)_R$ are not symmetries of $\langle\cdot\rangle_\delta$ because $\mathcal{Q}\Psi_{\rm gf}$ breaks them, their Ward identities can still be used to constrain the correlators $\langle\mathcal{O}\rangle_\delta$. This result will play a crucial role in Section 2.4.

We can now show how the linear Lagrangian term containing $A_t+\sigma$ can be made δ-invariant using a field redefinition. This is crucial in order to have a reliable and supersymmetric spectrum. The linear term (2.2.20) only contains modes $(A_t+\sigma)_\mathfrak{r}$ which are constant on S^2, due to the integral over S^2. Since $A_{t,\mathfrak{r}}+\sigma_\mathfrak{r}-R$ appears in (2.2.32) as a central charge, $\delta(A_{t,\mathfrak{r}}+\sigma_\mathfrak{r}-R)=0$. Therefore, by redefining

$$A'_{t,\mathfrak{r}}+\sigma'_\mathfrak{r} = A_{t,\mathfrak{r}}+\sigma_\mathfrak{r}+\frac{1}{2}\{c,c\}_\mathfrak{r}\,, \tag{2.2.40}$$

the linear term (2.2.20) becomes (dropping the $'$ on $A'_{t,\mathfrak{r}}+\sigma'_\mathfrak{r}$):

$$\frac{k}{4\pi R^2}\,\mathrm{Tr}\Big(\mathfrak{m}\,(A_t+\sigma)\Big)+\frac{m_k}{4R^2e_{\rm 3d}^2}\,\mathrm{Tr}\Big(c\,[\mathfrak{m},c]\Big)\,, \tag{2.2.41}$$

where $\mathfrak{m}$ is diagonal and m_k was defined in (2.2.19). The first term is invariant under δ, therefore after adding the second term to the quadratic action, the latter becomes invariant under δ as well, and the spectrum has to be supersymmetric (*i.e.*, δ-symmetric). Notice that the newly shifted field $A_{t,\mathfrak{r}}+\sigma_\mathfrak{r}$ is still Hermitian because c is Hermitian.

2.2.5 Vector multiplet spectrum

We are now ready to compute the spectrum of the (gauge-fixed) vector multiplet action. We start by considering the off-diagonal components. The Yang-Mills, Chern-Simons, and gauge-fixing terms are expanded to quadratic order in fluctuations around (2.2.19). After integrating out the auxiliary fields D and b, the independent components consist of 4 complex bosons $\left(A_1^{ij}, A_t^{ij}, \sigma^{ij}, A_{\bar{1}}^{ij}\right)$ and 6 complex fermions $\left(\overline{\Lambda}_1^{ij}, \Lambda_t^{ij}, \overline{\Lambda}_t^{ij}, c^{ij}, \widetilde{c}^{ij}, \Lambda_{\bar{1}}^{ij}\right)$ for every $i \neq j$ such that $q_{ij} > 0$.[9] All components are then rescaled by a factor of $e_{3\mathrm{d}}/R$. Moreover A_1^{ij}, $A_{\bar{1}}^{ij}$ get an extra factor of $1/\sqrt{2}$, while $\Lambda_{\bar{1}}^{ij}$, $\overline{\Lambda}_1^{ij}$, Λ_t^{ij}, $\overline{\Lambda}_t^{ij}$ get an extra factor of $\sqrt{2}$. This is to ensure that the standard 1d kinetic terms are canonically normalized. After expanding in monopole harmonics according to Table 2.1 and integrating over S^2, the quadratic action for off-diagonal components in momentum space becomes:

$$\int \frac{dp}{2\pi} \sum_{i,j\,|\,q_{ij}>0} \; \sum_{l,\,|m|\leq l} \left(\overline{B_{l,m}^{ij}(p)}\, M_B\, B_{l,m}^{ij}(p) + \overline{F_{l,m}^{ij}(p)}\, M_F\, F_{l,m}^{ij}(p) \right) \tag{2.2.42}$$

where the vectors of bosonic and fermionic fields are, respectively,

$$\begin{aligned} B_{l,m}^{ij} &= \left(A_{1,l,m}^{ij}\,,\; A_{t,l,m}^{ij}\,,\; \sigma_{l,m}^{ij}\,,\; A_{\bar{1},l,m}^{ij}\right)^{\mathsf{T}}\,, \\ F_{l,m}^{ij} &= \left(\overline{\Lambda}_{1,l,m}^{ij}\,,\; \Lambda_{t,l,m}^{ij}\,,\; \overline{\Lambda}_{t,l,m}^{ij}\,,\; c_{l,m}^{ij}\,,\; \widetilde{c}_{l,m}^{ij}\,,\; \Lambda_{\bar{1},l,m}^{ij}\right)^{\mathsf{T}}\,. \end{aligned} \tag{2.2.43}$$

The operators acting on the bosonic and fermionic fields are:

$$M_B = \begin{pmatrix} p(p+m_k+2\sigma_0) - \frac{\xi+1}{\xi}\frac{s_-^2}{2R^2} & -\frac{is_-(p+m_k+\sigma_0)}{\sqrt{2}R} & -\frac{i\sigma_0 s_-}{\sqrt{2}R} & \frac{1-\xi}{\xi}\frac{s_+s_-}{2R^2} \\ \frac{is_-(p+m_k+\sigma_0)}{\sqrt{2}R} & \frac{s_0^2}{R^2}+\sigma_0^2 & \sigma_0(p+\sigma_0) & -\frac{is_+(p-m_k+\sigma_0)}{\sqrt{2}R} \\ \frac{i\sigma_0 s_-}{\sqrt{2}R} & \sigma_0(p+\sigma_0) & (p+\sigma_0)^2 - m_k^2 - \frac{s_0^2}{R^2} & -\frac{i\sigma_0 s_+}{\sqrt{2}R} \\ \frac{1-\xi}{\xi}\frac{s_+s_-}{2R^2} & \frac{is_+(p-m_k+\sigma_0)}{\sqrt{2}R} & \frac{i\sigma_0 s_+}{\sqrt{2}R} & p(p-m_k+2\sigma_0) - \frac{\xi+1}{\xi}\frac{s_+^2}{2R^2} \end{pmatrix} \tag{2.2.44}$$

with

$$\sigma_0 = -\frac{q_{ij}}{m_k R^2}\,, \quad s_0 = \sqrt{l(l+1) - q_{ij}^2}\,, \quad s_\pm = \sqrt{l(l+1) - q_{ij}(q_{ij} \pm 1)} = \sqrt{s_0^2 \mp q_{ij}} \tag{2.2.45}$$

[9] We have chosen to write $A_1^{ij} = \overline{A_{\bar{1}}^{ji}}$, $\overline{\Lambda}_1^{ij} = \overline{\Lambda_{\bar{1}}^{ji}}$ and $\overline{\Lambda}_t^{ij} = \overline{\Lambda_t^{ji}}$.

(notice that σ_0, s_0, and $s_\pm$ depend on ij) and

$$M_F = \begin{pmatrix} -p-m_k-2\sigma_0 & -\frac{s_-}{R} & 0 & 0 & -\frac{is_-}{\sqrt{2\xi}R} & 0 \\ -\frac{s_-}{R} & -p+m_k & 0 & 0 & 0 & 0 \\ 0 & 0 & -p-m_k & 0 & 0 & -\frac{s_+}{R} \\ 0 & 0 & 0 & \frac{m_k q_{ij}}{R^2} & \frac{is_0^2}{\sqrt{\xi}R^2} & 0 \\ \frac{is_-}{\sqrt{2\xi}R} & 0 & 0 & -\frac{is_0^2}{\sqrt{\xi}R^2} & -p & -\frac{is_+}{\sqrt{2\xi}R} \\ 0 & 0 & -\frac{s_+}{R} & 0 & \frac{is_+}{\sqrt{2\xi}R} & -p+m_k-2\sigma_0 \end{pmatrix}. \tag{2.2.46}$$

For $l \geq q_{ij}+1$, all modes exist and are massive. Moreover, the masses of the modes[10] from bosons and fermions are paired thanks to the δ-invariance of the action, and the ratio of fermionic to bosonic determinants is 1. For $l = q_{ij}$, the modes of $A_{\bar{1}}^{ij}$ and $\Lambda_{\bar{1}}^{ij}$ do not exist (see Table 2.1), so the rightmost column and the bottom row of the matrices M_B, M_F should be removed. In this case, there is a massless fermionic mode while the other massive modes are paired between bosons and fermions. The ratio of determinants is $-p$. For $l = q_{ij}-1$ (this case takes place only if $q_{ij} \geq 1$), modes only exist in A_1^{ij} and $\overline{\Lambda}_1^{ij}$. The bosonic field A_1^{ij} has a massless pole, and a massive pole that cancels with that of $\overline{\Lambda}_1^{ij}$.

The effective degrees of freedom at energies much smaller than m_k and $\frac{1}{R}$ are the massless fermionic modes with $l = q_{ij}$ and the massless modes in A_1^{ij} with $l = q_{ij}-1$ (if $q_{ij} \geq 1$). The identity of the massless fermionic modes is not immediately clear due to the off-diagonal entries in (2.2.46). We can first rescale the fields $c_{l,m}^{ij} \to R\, c_{l,m}^{ij}$, so that they have the same mass dimension as the other fermions. Defining the dimensionless ratio $\alpha = 1/(m_k R)$ for convenience, the fermionic kinetic operator above becomes:

$$M_F\big|_{l=q_{ij}} = \begin{pmatrix} -p-(1-2q_{ij}\alpha^2)\,m_k & -\sqrt{2q_{ij}}\,\alpha\, m_k & 0 & 0 & -i\sqrt{\frac{q_{ij}}{\xi}}\,\alpha\, m_k \\ -\sqrt{2q_{ij}}\,\alpha\, m_k & -p+m_k & 0 & 0 & 0 \\ 0 & 0 & -p-m_k & 0 & 0 \\ 0 & 0 & 0 & q_{ij}\, m_k & i\frac{q_{ij}}{\sqrt{\xi}}\,\alpha\, m_k \\ i\sqrt{\frac{q_{ij}}{\xi}}\,\alpha\, m_k & 0 & 0 & -i\frac{q_{ij}}{\sqrt{\xi}}\,\alpha\, m_k & -p \end{pmatrix}. \tag{2.2.47}$$

By introducing a kinetic term $i\varepsilon\, \overline{c^{ij}}\, \partial_t c^{ij}$ by hand for the fermion c^{ij}, the problem of finding mass eigenstates is reduced to the usual problem of diagonalizing a mass matrix. Taking $\varepsilon \to 0$ at the end of the computation, we obtain the desired $\mathrm{SL}(5,\mathbb{C})$ transformation that

[10]The counting of modes works as follows. A complex field with 2-derivative kinetic term gives two modes, with only 1-derivative kinetic term gives one mode, whereas with no kinetic term gives no modes.

diagonalizes (2.2.47):

$$S=\begin{pmatrix}
-\frac{A_-}{\sqrt{8q_{ij}^2\alpha^4\xi+A_-^2+B_-^2}} & -\frac{A_+}{\sqrt{8q_{ij}^2\alpha^4\xi+A_+^2+B_+^2}} & 0 & 0 & \frac{\alpha}{\sqrt{\xi+q_{ij}\alpha^2+2q_{ij}^2\alpha^4}}\\
\frac{B_-}{\sqrt{8q_{ij}^2\alpha^4\xi+A_-^2+B_-^2}} & \frac{B_+}{\sqrt{8q_{ij}^2\alpha^4\xi+A_+^2+B_+^2}} & 0 & 0 & \frac{\sqrt{2}\alpha^2}{\sqrt{\xi+q_{ij}\alpha^2+2q_{ij}^2\alpha^4}}\\
0 & 0 & 1 & 0 & 0\\
-\frac{2\sqrt{2\xi}q_{ij}\alpha^3}{\sqrt{8q_{ij}^2\alpha^4\xi+A_-^2+B_-^2}} & -\frac{2\sqrt{2\xi}q_{ij}\alpha^3}{\sqrt{8q_{ij}^2\alpha^4\xi+A_+^2+B_+^2}} & 0 & -i\sqrt{\frac{\xi}{q_{ij}}} & \frac{\sqrt{\xi}\alpha}{\sqrt{\xi+q_{ij}\alpha^2+2q_{ij}^2\alpha^4}}\\
-\frac{i2\sqrt{2}q_{ij}\alpha^2}{\sqrt{8q_{ij}^2\alpha^4\xi+A_-^2+B_-^2}} & -\frac{i2\sqrt{2}q_{ij}\alpha^2}{\sqrt{8q_{ij}^2\alpha^4\xi+A_+^2+B_+^2}} & 0 & 0 & \frac{i}{\sqrt{\xi+q_{ij}\alpha^2+2q_{ij}^2\alpha^4}}
\end{pmatrix}, \tag{2.2.48}$$

where we have defined

$$\begin{aligned}
A_\pm &= \sqrt{2q_{ij}}\alpha\left(q_{ij}\alpha^2\left(1+2\xi\right)\pm\sqrt{q_{ij}^2\alpha^4\left(1+2\xi\right)^2+4\xi(q_{ij}\alpha^2+\xi)}\right)\\
B_\pm &= 2\xi+q_{ij}\alpha^2\left(1+2\xi\right)\pm\sqrt{q_{ij}^2\alpha^4\left(1+2\xi\right)^2+4\xi(q_{ij}\alpha^2+\xi)}\,.
\end{aligned} \tag{2.2.49}$$

The resulting fermionic kinetic operator is

$$S^\dagger\, M_F\big|_{l=q_{ij}}\, S=\begin{pmatrix}
-p-\lambda_+m_k & 0 & 0 & 0 & 0\\
0 & -p-\lambda_-m_k & 0 & 0 & 0\\
0 & 0 & -p-m_k & 0 & 0\\
0 & 0 & 0 & m_k & 0\\
0 & 0 & 0 & 0 & -p
\end{pmatrix} \tag{2.2.50}$$

with

$$\lambda_\pm=\frac{q_{ij}\,\alpha^2\left(1-2\xi\right)\pm\sqrt{q_{ij}^2\,\alpha^4(1+2\xi)^2+4\xi\big(q_{ij}\,\alpha^2+\xi\big)}}{2\xi}\,. \tag{2.2.51}$$

Each row of the matrix S expresses an original fermion in terms of the mass eigenstates. The linear combinations are generically complicated, but they simplify in the physical regime of interest. Since we want to reduce a Chern-Simons-matter theory on S^2, and the Yang-Mills term was only introduced to make propagating gauge degrees of freedom massive, we are motivated to take $m_k\gg\frac{1}{R}$ or $\alpha\to 0$. In this limit, the massless fermion at $l=q_{ij}$ is $-i\sqrt{\xi}\,\widetilde{c}$ (last row of S), and $\lambda_\pm\to\pm 1$.

The spectrum of the diagonal components can be analyzed in the same way and we will be brief. One finds that every mode is massive for $l>0$. After integrating out the $l=0$ mode of the auxiliary fields D^i, the quadratic Lagrangian (including the linear terms) for the remaining diagonal $l=0$ modes is:

$$\sum_i\left\{k\mathfrak{m}_i\left(A^i_{t,0,0}+\sigma^i_{0,0}\right)+\frac{4\pi R^2}{e_{3\mathrm{d}}^2}\left[\frac{1}{2}\big(\partial_t\sigma^i_{0,0}\big)^2-\frac{1}{2}m_k^2\big(\sigma^i_{0,0}\big)^2+\frac{1}{2}\,\overline{\Lambda}^i_{t,0,0}\big(i\partial_t+m_k\big)\Lambda^i_{t,0,0}\right]\right\}. \tag{2.2.52}$$

We observe that $\sigma^i_{0,0}$ and $\Lambda^i_{t,0,0}$ have mass m_k and should be integrated out at low energies $p \ll m_k$. Only the combination $\left(A^i_{t,0,0} + \sigma^i_{0,0}\right)$ remains, which is a 1d gauge field for the gauge group $\mathrm{U}(1)^N$.[11]

To summarize, we write the quadratic Lagrangian for the modes from the vector multiplet that contain massless poles, including fermionic partners which are necessary for supersymmetry. After having rescaled $A_{\bar{1}}$ and $\Lambda_{\bar{1}}$ by $m_k^{-1/2}$ we have:

$$k \sum_i \mathfrak{m}_i \left(A^i_t + \sigma^i\right) + \sum_{i \neq j} \Bigg\{ \Theta(q_{ij} - 1) \sum_{|m| \leq q_{ij}-1} \Bigg[\overline{A^{ji}_{\bar{1},q_{ij}-1,m}}\, i\partial_t\, A^{ji}_{\bar{1},q_{ij}-1,m} + \overline{\Lambda^{ji}_{\bar{1},q_{ij}-1,m}}\, \Lambda^{ji}_{\bar{1},q_{ij}-1,m} + \frac{1}{m_k}\left(\left| \partial_t A^{ji}_{\bar{1},q_{ij}-1,m} \right|^2 + \overline{\Lambda^{ji}_{\bar{1},q_{ij}-1,m}}\, i\partial_t\, \Lambda^{ji}_{\bar{1},q_{ij}-1,m} \right) \Bigg] + \Theta(q_{ij}) \sum_{|m| \leq q_{ij}} \left(\overline{\widetilde{c}^{ij}_{q_{ij},m}}\, i\partial_t\, \widetilde{c}^{ij}_{q_{ij},m} \right) \Bigg\} \tag{2.2.53}$$

where $\Theta(n) = 1$ for $n \geq 0$ and it vanishes otherwise. Here we have changed notation, and used the fields $\left(A^{ji}_{\bar{1}}, \Lambda^{ji}_{\bar{1}}\right)$ in place of A^{ij}_1, $\overline{\Lambda}^{ij}_1$ because the former live in a chiral multiplet, see (2.2.5), while the latter in an anti-chiral multiplet. Besides, notice that there are matching degrees of freedom in $A^{ji}_{\bar{1}}$ and $\Lambda^{ji}_{\bar{1}}$ with mass m_k, which should not be included in the effective theory at energies $p \ll m_k$. These modes are encoded in the term proportional to $1/m_k$ and can be integrated out by neglecting that kinetic term. The workings are explained in [143]. The quadratic Lagrangian for the massless modes is then:[12]

$$k \sum_i \mathfrak{m}_i \left(A^i_t + \sigma^i\right) + \sum_{ij} \Bigg\{ \Theta(q_{ij} - 1) \sum_{|m| \leq q_{ij}-1} \Bigg(\overline{A^{ji}_{\bar{1},q_{ij}-1,m}}\, i\partial_t\, A^{ji}_{\bar{1},q_{ij}-1,m} + \\ + \overline{\Lambda^{ji}_{\bar{1},q_{ij}-1,m}}\, \Lambda^{ji}_{\bar{1},q_{ij}-1,m} \Bigg) + \Theta(q_{ij} - \tfrac{1}{2}) \sum_{|m| \leq q_{ij}} \overline{\widetilde{c}^{ij}_{q_{ij},m}}\, i\partial_t\, \widetilde{c}^{ij}_{q_{ij},m} \Bigg\} . \tag{2.2.54}$$

The bosons $A^{ji}_{\bar{1}}$ and the fermions $\widetilde{c}^{\,ij}$ have a 1-derivative action, while the fermions $\Lambda^{ji}_{\bar{1}}$ are auxiliary.

2.2.6 Matter spectrum

To find the spectrum of modes coming from the 3d chiral multiplets, we expand the chiral multiplet Lagrangian (2.2.15) to quadratic order in fluctuations around (2.2.19). All fields in the chiral multiplet are rescaled by $\frac{1}{R}$. After expanding in monopole harmonics according

[11] In other words, in the language of Appendic I, we find that the superfield V^- is massive, while Ω stays light and enforces gauge-invariance.

[12] Using the assumption that $q_{ij} \neq 0$ for $i \neq j$, we have substituted $\Theta(q_{ij}) \to \Theta(q_{ij} - \frac{1}{2})$ in (2.2.53), and consequently we have substituted $\sum_{i \neq j} \to \sum_{ij}$.

to Table 2.1 and integrating over S^2, the quadratic action in momentum space is:

$$\int \frac{dp}{2\pi} \sum_a \sum_{i,j} \sum_{l,\,|m|\leq l} \left\{ \left[p(p+2\sigma_0) - \frac{s_{+,a}^2}{R^2} \right] \left|\phi^{ij}_{a,l,m}(p)\right|^2 + \left|f^{ij}_{a,l,m}(p)\right|^2 + \right.$$

$$\left. + \left(\overline{\psi^{ij}_{a,l,m}(p)}\,,\ \overline{\eta^{ij}_{a,l,m}(p)} \right) \begin{pmatrix} -p-2\sigma_0 & \frac{s_{+,a}}{R} \\ \frac{s_{+,a}}{R} & -p \end{pmatrix} \begin{pmatrix} \psi^{ij}_{a,l,m}(p) \\ \eta^{ij}_{a,l,m}(p) \end{pmatrix} \right\} \tag{2.2.55}$$

where

$$\sigma_0 = -q_{ij}\alpha^2 m_k \equiv -\frac{m_\sigma}{2}\,, \qquad s_{\pm,a} \equiv \sqrt{l(l+1) - q^a_{ij}(q^a_{ij} \pm 1)}\,. \tag{2.2.56}$$

For $l \geq |q^a_{ij}| + 1$, all modes exist (see Table 2.1) and are massive. Moreover, the masses of bosons and fermions are paired and the ratio of determinants is 1. The modes with $l = |q^a_{ij}|$ exist in all fields if $q^a_{ij} \leq -\frac{1}{2}$, whereas they only exist in ϕ^{ij}_a and ψ^{ij}_a if $q^a_{ij} \geq 0$. In the former case, all modes are massive. In the latter case, the field ϕ^{ij}_a has a massless pole, and a massive pole that cancels with that of ψ^{ij}_a. Provided that $q^a_{ij} \leq -1$, there exist modes with $l = |q^a_{ij}| - 1 = -q^{ij}_a - 1$ in η^{ij}_a and f^{ij}_a, such that η^{ij}_a is massless while f^{ij}_a is auxiliary.

To summarize, the quadratic Lagrangian for modes which contain massless poles, and that of their supersymmetry partners is

$$\sum_{ij,\,a} \left\{ \Theta(q^a_{ij}) \sum_{|m|\leq q^a_{ij}} \left[m^{ij}_\sigma \left(\overline{\phi^{ij}_{a,q^a_{ij},m}}\, i\partial_t\, \phi^{ij}_{a,q^a_{ij},m} + \overline{\psi^{ij}_{a,q^a_{ij},m}}\, \psi^{ij}_{a,q^a_{ij},m} \right) + \left|\partial_t \phi^{ij}_{a,q^a_{ij},m}\right|^2 + \right. \right. \tag{2.2.57}$$

$$\left. \left. + \overline{\psi^{ij}_{a,q^a_{ij},m}}\, i\partial_t\, \psi^{ij}_{a,q^a_{ij},m} \right] + \Theta(-q^a_{ij} - 1) \sum_{|m|\leq -q^a_{ij}-1} \left(\overline{\eta^{ij}_{a,-q^a_{ij}-1,m}}\, i\partial_t\, \eta^{ij}_{a,-q^a_{ij}-1,m} + \left|f^{ij}_{a,-q^a_{ij}-1,m}\right|^2 \right) \right\},$$

where the i, j dependence of m_σ was made explicit. At low energies $p \ll m^{ij}_\sigma$, the quadratic kinetic term of $\phi^{ij}_{a,q^a_{ij},m}$ and the kinetic term of $\psi^{ij}_{a,q^a_{ij},m}$ can again be neglected. Note that $q^a_{ij} \geq 0$ does not exclude the possibility that $i = j$, in which case $m^{ij}_\sigma = 0$. We might also have $m^{ij}_\sigma \to 0$ as $\alpha \to 0$.[13] In either case, all of $\phi^{ij}_{a,q^a_{ij},m}$ and $\psi^{ij}_{a,q^a_{ij},m}$ would be classically massless. However, quantum effects would still generically generate supersymmetric mass terms like

$$m^{ij}_{\sigma(\mathrm{q})} \left(\overline{\phi^{ij}_{a,q^a_{ij},m}}\, i\partial_t\, \phi^{ij}_{a,q^a_{ij},m} + \overline{\psi^{ij}_{a,q^a_{ij},m}}\, \psi^{ij}_{a,q^a_{ij},m} \right)\,, \tag{2.2.58}$$

whose superspace expression is (I.5.12). At scales $p \ll m^{ij}_{\sigma(\mathrm{q})}$, the quadratic kinetic term of $\phi^{ij}_{a,q^a_{ij},m}$ and the kinetic term of $\psi^{ij}_{a,q^a_{ij},m}$ would still be negligible. Therefore, rescaling $\phi^{ij}_{a,q^a_{ij},m}$ and $\psi^{ij}_{a,q^a_{ij},m}$ by $1/(m^{ij}_\sigma)^{1/2}$ (including quantum corretions), the resulting quadratic effective

[13] Indeed $m_\sigma \sim \alpha^2 m_k \sim \alpha/R$, therefore its scaling is not fixed by the choices we already made.

Lagrangian is:

$$\sum_{ij,\,a}\Bigg[\Theta(q^a_{ij})\sum_{|m|\le q^a_{ij}}\Big(\overline{\phi^{ij}_{a,q^a_{ij},m}}\,i\partial_t\,\phi^{ij}_{a,q^a_{ij},m}+\overline{\psi^{ij}_{a,q^a_{ij},m}}\,\psi^{ij}_{a,q^a_{ij},m}\Big)+ \\ +\Theta(-q^a_{ij}-1)\sum_{|m|\le -q^a_{ij}-1}\Big(\overline{\eta^{ij}_{a,-q^a_{ij}-1,m}}\,i\partial_t\,\eta^{ij}_{a,-q^a_{ij}-1,m}+\big|f^{ij}_{a,-q^a_{ij}-1,m}\big|^2\Big)\Bigg]\,. \tag{2.2.59}$$

2.3 The effective Quantum Mechanics

In this section we present the proposed low-energy quantum mechanical model, which is the result of setting to zero all massive modes in the gauge-fixed 3d Lagrangian while only keeping the light modes.

The gauge group is $\mathrm{U}(1)^N$ and the vector multiplet only contains the gauge fields $A^i_t+\sigma^i$, with $i=1,\ldots,N$.[14] Their role is to impose Gauss's law. Because of the Wilson line of charges $k\mathfrak{m}_i$, coming from the 3d Chern-Simons term, Gauss's law projects onto a sector of non-vanishing gauge charges.

The matter content consists of various chiral and Fermi multiplets X^{ij} with charges $+1$ under $\mathrm{U}(1)_i\subset\mathrm{U}(1)^N$ and -1 under $\mathrm{U}(1)_j$. They interact with the gauge fields via the covariant derivative

$$D^+_t X^{ij}=\Big(\partial_t-i\big(A^i_t+\sigma^i-A^j_t-\sigma^j\big)\Big)X^{ij}\,. \tag{2.3.1}$$

The matter content depends on the fluxes $\mathfrak{m}_i$ and $\mathfrak{n}_a$ through the combinations q_{ij} and q^a_{ij} defined in (2.2.24). For every pair of indices ij, from the 3d vector multiplet we get the following matter multiplets. If $q_{ij}\le -1$, there are 1d chiral multiplets $\Xi^{ij}_{\bar{1},m}=\big(A^{ij}_{\bar{1},m},\Lambda^{ij}_{\bar{1},m}\big)$ in the SU(2) representation of highest weight $l=-q_{ij}-1$. Otherwise, if $q_{ij}\ge\frac{1}{2}$, there are 1d Fermi multiplets $C^{ij}_m=\big(\widetilde{c}^{ij}_m,g^{ij}_m\big)$ with $l=q_{ij}$. Here we introduce the auxiliary fields g^{ij}_m, even though they are not present in the 3d theory, in order to make off-shell supersymmetry manifest. From the 3d chiral multiplet with flavor index a, we get 1d chiral multiplets $\Phi^{ij}_{a,m}=\big(\phi^{ij}_{a,m},\psi^{ij}_{a,m}\big)$ with $l=q^a_{ij}$ if $q^a_{ij}\ge 0$, and otherwise 1d Fermi multiplets $\mathcal{Y}^{ij}_{a,m}=\big(\eta^{ij}_{a,m},f^{ij}_{a,m}\big)$ with $l=-q^a_{ij}-1$ if $q^a_{ij}\le -1$. We summarize this content in Table 2.2, where we also list the representations and charges of each multiplet under the global symmetries SU(2), $\mathrm{U}(1)^2_F$ and $\mathrm{U}(1)_R$.

In addition to gauge-interactions, other interactions are specified by E and J superpotentials. We have as many E and J functions as there are Fermi multiplets. For a given Fermi multiplet η, E is in the same gauge and flavour representation as η, and its R-charge

[14] In Wess-Zumino gauge, the only non-vanishing component of the superfield V (or equivalently of Ω) is $A_t+\sigma$. See Appendix I.3.

	$A^{ij}_{\bar{1},m}$ chiral	$\widetilde{c}^{ij}_{m}$ Fermi	$\phi^{ij}_{a,m}$ chiral	$\eta^{ij}_{a,m}$ Fermi
existence:	$q_{ij} \leq -1$	$q_{ij} \geq \frac{1}{2}$	$q^a_{ij} \geq 0$	$q^a_{ij} \leq -1$
l	$\lvert q_{ij}\rvert - 1$	q_{ij}	q^a_{ij}	$\lvert q^a_{ij}\rvert - 1$
R_3	0	0	$2\delta_{3a}$	$2\delta_{3a} - 1$
q_1	0	0	$\delta_{1a} - \delta_{3a}$	$\delta_{1a} - \delta_{3a}$
q_2	0	0	$\delta_{2a} - \delta_{3a}$	$\delta_{2a} - \delta_{3a}$

Table 2.2: Matter multiplets (we indicate the bottom components) for indices ij and their representations under the global symmetries. We label the SU(2) representation by the highest weight $l \in \mathbb{Z}/2$. The charges of the lowest components in each multiplet are indicated, while their superpartners have R-charges R_3 which are shifted by -1.

is $R(\eta)+1$. On the contrary, J is in the dual gauge and flavor representation with respect to η, and its R-charge is $-R(\eta)+1$. We find that the E and J functions are zero for the Fermi multiplets $\widetilde{c}^{ij}_m$. For the Fermi multiplets $\eta^{ij}_{a,m}$, the E and J superpotentials are:

$$
\begin{aligned}
E^{ij}_{a,m} = i \sum_k \Bigg[\quad & \Theta(q^a_{kj}) \sum_{|m'|\leq q^a_{kj}} e^{kj}_{\mathrm{1d}} \sqrt{2q^a_{kj}+1}\; C\Big(\begin{smallmatrix} |q_{ik}|-1 & q^a_{kj} & |q^a_{ij}|-1 \\ m-m' & m' & m \end{smallmatrix}\Big)\, A^{ik}_{\bar{1},m-m'}\, \phi^{kj}_{a,m'} \\
& - \Theta(q^a_{ik}) \sum_{|m'|\leq q^a_{ik}} e^{ik}_{\mathrm{1d}} \sqrt{2q^a_{ik}+1}\; C\Big(\begin{smallmatrix} |q_{kj}|-1 & q^a_{ik} & |q^a_{ij}|-1 \\ m-m' & m' & m \end{smallmatrix}\Big)\, \phi^{ik}_{a,m'}\, A^{kj}_{\bar{1},m-m'} \quad \Bigg] ,
\end{aligned}
\tag{2.3.2}
$$

$$
\begin{aligned}
J^{ji}_{a,-m} = & -\sum_{b,c,k} \epsilon_{abc}\, \Theta(q^b_{jk})\, \Theta(q^c_{ki}) \times \\
& \times \sum_{\substack{|m'|\leq q^b_{jk} \\ |m+m'|\leq q^c_{ki}}} \lambda^{jki}_{\mathrm{1d}} \left[\frac{(2q^b_{jk}+1)(2q^c_{ki}+1)}{2|q^a_{ij}|-1}\right]^{\frac{1}{2}} (-1)^{-q^a_{ij}-1-m}\, C\Big(\begin{smallmatrix} q^b_{jk} & q^c_{ki} & |q^a_{ij}|-1 \\ m' & -m-m' & -m \end{smallmatrix}\Big)\, \phi^{jk}_{b,m'}\, \phi^{ki}_{c,-m-m'} ,
\end{aligned}
\tag{2.3.3}
$$

where $C\big(\begin{smallmatrix} l & l' & l'' \\ m & m' & m'' \end{smallmatrix}\big)$ are the Clebsch-Gordan coefficients given in (H.0.20) and we defined

$$
e^{ij}_{\mathrm{1d}} = \frac{1}{R\sqrt{k\, m^{ij}_\sigma}} , \qquad \lambda^{ijk}_{\mathrm{1d}} = \frac{\lambda_{\mathrm{3d}}}{R\sqrt{4\pi\, m^{ij}_\sigma\, m^{jk}_\sigma}} . \tag{2.3.4}
$$

The sign $(-1)^{-q^a_{ij}-1-m}$ in the J-term is necessary for SU(2) invariance. Given that the term E^{ij}_a in (2.3.2) exists for $q^a_{ij} \leq -1$, the condition $q^a_{kj} \geq 0$ in the first line guarantees that $A^{ij}_{\bar{1}}$ and ϕ^{kj}_a both exist, and the condition $q^a_{ik} \geq 0$ in the second line guarantees that ϕ^{ik}_a and $A^{kj}_{\bar{1}}$ both exist. Also, the term J^{ji}_a in (2.3.3) exists for $q^a_{ij} \leq -1$, which is guaranteed by the two conditions $q^b_{jk} \geq 0$, $q^c_{ki} \geq 0$ on the r.h.s.. The E-term comes from the reduction of (2.2.14) whereas the J-term from the reduction of the 3d superpotential (2.1.10). One can check, by substituting (H.0.22) and relabeling indices, that

$$\sum_{ij,\,a} \Theta\big(-q_{ij}^a - 1\big) \sum_{|m|\le -q_{ij}^a - 1} E^{ij}_{a,m}\, J^{ji}_{a,-m} = 0 \;, \tag{2.3.5}$$

which is required for supersymmetry. The couplings $e_{\rm 1d}$ and $\lambda_{\rm 1d}$ are obtained by tree-level matching.

The complete Lagrangian in terms of the E and J given above is:

$$\begin{aligned}
\mathcal{L}_{\rm QM} = k \sum_i \mathfrak{m}_i \big(A_t^i + \sigma^i\big) + \sum_{ij} \bigg\{ \Theta(q_{ij}-1) \sum_{|m|\le q_{ij}-1} \Big(\overline{A^{ji}_{\bar 1,m}}\, iD_t^+ A^{ji}_{\bar 1,m} + \overline{\Lambda^{ji}_{\bar 1,m}}\, \Lambda^{ji}_{\bar 1,m} \Big) \qquad (2.3.6) \\
+ \Theta(q_{ij}-\tfrac12) \sum_{|m|\le q_{ij}} \Big(\overline{\widetilde c^{\,ij}_m}\, iD_t^+ \widetilde c^{\,ij}_m + \big|g^{ij}_m\big|^2 \Big) + \sum_{ij,\,a} \bigg\{ \Theta(q_{ij}^a) \sum_{|m|\le q_{ij}^a} \Big(\overline{\phi^{ij}_{a,m}}\, iD_t^+ \phi^{ij}_{a,m} + \overline{\psi^{ij}_{a,m}}\, \psi^{ij}_{a,m} \Big) \\
+ \Theta(-q_{ij}^a - 1) \sum_{|m|\le -q_{ij}^a - 1} \bigg(\overline{\eta^{ij}_{a,m}}\, iD_t^+ \eta^{ij}_{a,m} + \big|f^{ij}_{a,m}\big|^2 - \big|E^{ij}_{a,m}\big|^2 - \overline{\eta^{ij}_{a,m}}\, Q E^{ij}_{a,m} - \overline{Q}\, \overline{E^{ij}_{a,m}}\, \eta^{ij}_{a,m} \\
- f^{ij}_{a,m} J^{ji}_{a,-m} - \overline{J^{ji}_{a,-m}}\; \overline{f^{ij}_{a,m}} - \eta^{ij}_{a,m}\, Q J^{ji}_{a,-m} - \overline{Q}\, \overline{J^{ji}_{a,-m}}\; \overline{\eta^{ij}_{a,m}} \bigg) \bigg\} \;,
\end{aligned}$$

where $i,j = 1,\dots,N$, while $a = 1,2,3$. Note that both bosons and fermions have 1-derivative kinetic terms. The Lagrangian can be more compactly written in superspace:

$$\begin{aligned}
\mathcal{L}_{\rm QM} = \int d\theta d\bar\theta\, \bigg\{ k \sum_i \mathfrak{m}_i V^i + \sum_{ij} \bigg[\Theta(q_{ij}-1) \sum_{|m|\le q_{ij}-1} \overline{\Xi^{ji}_{\bar 1,m}}\, \Xi^{ji}_{\bar 1,m} + \Theta(q_{ij}-\tfrac12) \sum_{|m|\le q_{ij}} \overline{C^{ij}_m}\, C^{ij}_m \bigg] \\
+ \sum_{ij,\,a} \bigg[\Theta(q_{ij}^a) \sum_{|m|\le q_{ij}^a} \overline{\Phi^{ij}_{a,m}}\, \Phi^{ij}_{a,m} + \Theta(-q_{ij}^a - 1) \sum_{|m|\le -q_{ij}^a - 1} \overline{\mathcal{Y}^{ij}_{a,m}}\, \mathcal{Y}^{ij}_{a,m} \bigg] \bigg\} \\
+ \sum_{ij,\,a} \Theta(-q_{ij}^a - 1) \sum_{|m|\le q_{ij}^a} \bigg\{ \int d\theta\, \mathcal{Y}^{ij}_{a,m}\, J^{ji}_{a,-m}(\Phi) + \int d\bar\theta\, \overline{\mathcal{Y}^{ij}_{a,m}}\; \overline{J^{ji}_{a,-m}}(\overline\Phi) \bigg\} \,.
\end{aligned} \tag{2.3.7}$$

Here we promoted the scalar fields in J to be chiral superfields.

After gauge fixing by $s\Psi_{\rm gf}$, the observables of the 3d theory are the BRST-closed operators which are invariant under the residual gauge symmetry, and have ghost number $n_g = 0$. The further addition of $\mathcal{Q}\Psi_{\rm gf}$ to the Lagrangian does not modify their correlators, see (2.2.39). When we go to the effective 1d description (2.3.6), the ghost field c is completely integrated out. Any operator containing $\widetilde c^{\,ij}_m$ should not be regarded as a physical observable, because it will have $n_g < 0$. For instance, one might have noticed that the Lagrangian (2.3.6) has a large number of additional global U(1) symmetries that rotate each $\widetilde c^{\,ij}_m$ independently. However, their currents are not physical observables (because they are constructed with $\widetilde c^{\,ij}_m$), and indeed the symmetries act trivially on the sector of physical observables.[15] They should not

[15] In view of holographic applications of the low-energy quantum mechanics, one should not expect the extra symmetries to appear as gauge fields in AdS$_2$.

be regarded as emergent symmetries of the physical theory. On the other hand, all operators constructed from the fields of the low-energy 1d description other than $\widetilde{c}_m^{ij}$ and invariant under $\mathrm{U}(1)^N$, are physical observables. This is because the BRST transformations of the physical fields X are $sX = \delta_{\text{gauge}}(c)X$, but c is massive and set to zero in the low-energy description.

2.3.1 1-loop determinants and the Witten index

A simple check that we can perform of the proposed 1d quantum mechanics (2.3.7) is that its Witten index matches the TT index of the 3d theory, at leading order at large N. This ensures that its ground-state degeneracy reproduces the entropy of BPS black holes.

The Witten index of an $\mathcal{N}=2$ supersymmetric quantum mechanics is defined in exactly the same way as the TT index in (2.1.15). In the Lagrangian formulation, the chemical potentials Δ_a are introduced as twisted boundary conditions on the fields. For a class of these models, the Witten index has already been computed in [139] (see also [144,145]), and it takes the form of a Jeffrey-Kirwan contour integral as in (2.1.16). We want to make sure that the quantum mechanics (2.3.7) reproduces the integrand in (2.1.16) for the value of $\mathfrak{m}_i$ singled out by the saddle-point approximation.

After fixing the 1d gauge $\partial_t\big(A_t^i + \sigma^i\big) = 0$, the Wilson line gives a classical contribution $\exp\big(i\sum_i k\mathfrak{m}_i u_i\big)$, where u is the constant mode of the Wick-rotated $A_t + \sigma$. The chirals $\Xi_{\bar{1}}$ and Fermi's C coming from the 3d vector multiplet contribute to the 1-loop determinant as

$$\mathcal{Z}_{\Xi_{\bar{1}}} = \prod_{i\neq j} \left(\frac{e^{i\,u_{ij}/2}}{1-e^{iu_{ij}}} \right)^{\Theta(-q_{ij}-1)\,(-2q_{ij}-1)} , \qquad \mathcal{Z}_C = \prod_{i\neq j} \left(\frac{e^{iu_{ij}}-1}{e^{i\,u_{ij}/2}} \right)^{\Theta(q_{ij})\,(2q_{ij}+1)} , \tag{2.3.8}$$

where $u_{ij} = u_i - u_j$. The exponents come from the $2l+1$ degeneracy in each SU(2) representation of highest weight l, and the Θ functions ensure that nontrivial contributions only enter when the multiplets exist. Recalling that $q_{ij} \neq 0$ for $i \neq j$, their product simplifies:

$$\mathcal{Z}_{\Xi_{\bar{1}}}\, \mathcal{Z}_C = (-1)^{\frac{N(N-1)}{2}} \prod_{i\neq j} \left(1 - \frac{z_i}{z_j} \right) , \tag{2.3.9}$$

where $z_i = e^{iu_i}$. The result above matches (up to an inconsequential sign) the 1-loop determinant of a 3d vector multiplet given in [28] and appearing in (2.1.16).[16] As opposed

[16]The 1-loop determinant of a Fermi multiplet has a sign ambiguity coming from the assignment of fermion number to states in the fermionic Fock space. We have fixed this ambiguity in a specific way to get (2.3.9). Additional signs might appear if a different convention is chosen. Notice, for example, the different choice with respect to (2.3.10).

to the indirect Higgsing argument which was used in [28], the result here provides an explicit derivation based on a careful gauge-fixing procedure. This computation shows that the ghost multiplet C^{ij} appearing in the quantum mechanics is needed to reproduce the correct degeneracy of BPS states. Lastly, the chirals Φ_a and Fermis $\mathcal{Y}_a$ coming from the 3d chiral multiplets contribute to the 1-loop determinant as

$$\mathcal{Z}_{\Phi_a} = \prod_{i,j} \left(\frac{e^{i(u_{ij}+\Delta_a)/2}}{1-e^{i(u_{ij}+\Delta_a)}} \right)^{\Theta(q^a_{ij})\,(2q^a_{ij}+1)} , \qquad \mathcal{Z}_{\mathcal{Y}_a} = \prod_{i,j} \left(\frac{1-e^{i(u_{ij}+\Delta_a)}}{e^{i(u_{ij}+\Delta_a)/2}} \right)^{\Theta(-q^a_{ij}-1)\,(-2q^a_{ij}-1)} . \tag{2.3.10}$$

Their product is

$$\mathcal{Z}_{\Phi_a}\mathcal{Z}_{\mathcal{Y}_a} = \prod_{i,j} \left(\frac{e^{i(u_{ij}+\Delta_a)/2}}{1-e^{i(u_{ij}+\Delta_a)}} \right)^{2q^a_{ij}+1} = \frac{y_a^{N^2(\mathfrak{n}_a+1)/2}}{(1-y_a)^{N(\mathfrak{n}_a+1)}} \prod_{i\neq j} \left(\frac{z_i - y_a z_j}{z_j - y_a z_i} \right)^{\mathfrak{m}_i} \left(1 - y_a \frac{z_i}{z_j} \right)^{-\mathfrak{n}_a-1} . \tag{2.3.11}$$

The complete integrand is thus

$$\mathcal{Z}_{\text{tot}} = e^{ik\sum_i \mathfrak{m}_i u_i}\ \mathcal{Z}_\Xi\, \mathcal{Z}_C \prod_a \mathcal{Z}_{\Phi_a}\, \mathcal{Z}_{\mathcal{Y}_a} , \tag{2.3.12}$$

matching the integrand in (2.1.16). This guarantees that a large N saddle-point computation of the 3d TT index matches a saddle-point computation of the 1d Witten index.

2.4 Stability under quantum corrections

The gauge-fixing action $\delta\Psi_{\text{gf}}$ preserves the real supercharge δ, $\mathrm{U}(1)^2_F$, and $\mathrm{SU}(2)$. We first use the δ invariance of the full action to show that the fermion $\widetilde{c}^{ij}_m$ only has gauge-interactions. This allows us to focus on fields other than $\widetilde{c}^{ij}_m$. Although the gauge fixing breaks Q, $\overline{Q}$ and $\mathrm{U}(1)_R$, we will then give arguments for why they should be preserved in the effective action. The key observation will be (2.2.39). Finally, we will use all the symmetries Q, $\overline{Q}$, $\mathrm{U}(1)^2_F$, $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)$ to argue for the absence of various interactions.

2.4.1 Interactions involving $\widetilde{c}$

Using the fermionic symmetry δ, we can argue that the part of the Lagrangian involving the fermions $\widetilde{c}^{ij}_m$ cannot be anything other than (2.3.6) at low energies. Let $\langle\cdot\rangle_\delta$ denote the gauge-fixed path integral, as in (2.2.39). For i,j such that $q_{ij}>0$, we consider the quantity

$$\left\langle \overline{\widetilde{c}^{ij}_m}(t)\, D^+_t\, \widetilde{c}^{ij}_m(t') \right\rangle_\delta = \left\langle \overline{\widetilde{c}^{ij}_m}(t)\, \delta b^{ij}_m(t') \right\rangle_\delta - \left\langle \overline{\widetilde{c}^{ij}_m}(t)\, \delta_{\text{gauge}}(R)\, \widetilde{c}^{ij}_m(t') \right\rangle_\delta \approx \left\langle \overline{\widetilde{c}^{ij}_m}(t)\, \delta b^{ij}_m(t') \right\rangle_\delta . \tag{2.4.1}$$

Here b^{ij}_m is the $l = q_{ij}$ mode of the auxiliary field b in the gauge-fixing complex. In the first equality we used (2.2.27) and (2.2.34). The approximate equality $\approx$ only holds in the IR limit because the term that was discarded is a correlation function involving massive ghosts c in $R = -\frac{1}{2}\{c, c\}_{\mathfrak{r}}$, which is exponentially suppressed at large $t - t'$. We continue using the Leibniz rule on δ and the fact that δ-exact correlators vanish, to write

$$\big\langle \overline{\widetilde{c}^{ij}_m}(t)\, \delta b^{ij}_m(t') \big\rangle_\delta = -\big\langle \delta\overline{\widetilde{c}^{ij}_m}(t)\, b^{ij}_m(t') \big\rangle_\delta = i\big\langle \overline{b^{ij}_m}(t)\, b^{ij}_m(t') \big\rangle_\delta \,. \tag{2.4.2}$$

The path integral over b^{ij}_m is quadratic and can be done exactly, yielding

$$\big\langle \overline{\widetilde{c}^{ij}_m}(t)\, D^+_t\, \widetilde{c}^{ij}_m(t') \big\rangle_\delta \approx i\big\langle \overline{b^{ij}_m}(t)\, b^{ij}_m(t') \big\rangle_\delta = -\delta(t - t') + i\big\langle \overline{\mathcal{O}_H}(t)\, \mathcal{O}_H(t') \big\rangle_\delta \approx -\delta(t - t') \,, \tag{2.4.3}$$

where

$$\mathcal{O}_H = \sqrt{\frac{q_{ij}}{\xi R^2}}\, A^{ij}_{1,q_{ij},m} - \frac{e_{3\mathrm{d}}}{R}\, \{\widetilde{c}, c\}^{ij}_{l=q_{ij},m} \,. \tag{2.4.4}$$

The expression $\{\widetilde{c}, c\}^{ij}_{l=q_{ij},m}$ stands for the $\big(l = q_{ij}, m\big)$ mode of $\{\widetilde{c}, c\}^{ij}$. Both terms inside $\mathcal{O}_H$ contain massive fields only, therefore $\big\langle \overline{\mathcal{O}_H}(t)\, \mathcal{O}_H(t') \big\rangle_\delta$ is exponentially suppressed at large distances and the approximation holds to increasing accuracy in the IR. Using only symmetry arguments for δ, we have shown that $\widetilde{c}^{ij}_m$ must satisfy the Schwinger-Dyson equation derived from (2.3.6) in the IR limit. Any modification of (2.3.6) containing $\widetilde{c}^{ij}_m$ would change the Schwinger-Dyson equation, and can thus be excluded.

2.4.2 Presence of $\mathcal{N} = 2$ supersymmetry and R-symmetry

Having taken care of $\widetilde{c}^{ij}_m$, we want to constrain the effective Lagrangian for the remaining fields. Here we show that in the IR it must preserve 1d $\mathcal{N} = 2$ supersymmetry and $\mathrm{U}(1)_R$, even though these symmetries are broken by the gauge-fixing term $\delta\Psi_{\mathrm{gf}}$.

First, we show that the Ward identities for the supercharges Q and $\overline{Q}$ are satisfied on correlators $\mathcal{O}$ constructed from 1d fields excluding $\widetilde{c}^{ij}_m$, which are modes of physical fields in 3d. More precisely, we show that $\langle Q\mathcal{O}\rangle_\delta \approx 0$ (and analogously for $\overline{Q}$). As before, approximate equalities $\approx$ hold in the IR limit. Firstly, since $\mathcal{O}$ is constructed from modes of physical fields, it has $n_g = 0$, and the same goes for $Q\mathcal{O}$. Then (2.2.39) tells us that $\langle Q\mathcal{O}\rangle_\delta = \langle Q\mathcal{O}\rangle_s$. It remains to show that $\langle Q\mathcal{O}\rangle_s \approx 0$.

We then follow the standard procedure to derive a Ward identity. In the path integral $\langle\mathcal{O}\rangle_s$ we perform a field redefinition $X' = X + \epsilon\, QX$ on physical fields X in the form of a supersymmetry transformation, while keeping the fields Y in the gauge-fixing complex unchanged. Let S_{ph} be the original action before gauge fixing. At first order in ϵ we get

$$\begin{aligned} \langle\mathcal{O}\rangle_s &= \int \mathcal{D}\phi\, \mathcal{O}\, e^{i\left(S_{\mathrm{ph}} + s\Psi_{\mathrm{gf}}\right)} = \int \mathcal{D}\phi\, \big(\mathcal{O} + \epsilon\, Q\mathcal{O}\big)\, e^{i\left(S_{\mathrm{ph}} + s\Psi_{\mathrm{gf}}\right) - i\epsilon\, s\, Q\Psi_{\mathrm{gf}}} \\ &= \langle\mathcal{O}\rangle_s + \epsilon\Big(\langle Q\mathcal{O}\rangle_s - i\langle \mathcal{O}\, s\, Q\Psi_{\mathrm{gf}}\rangle_s\Big) + \dots \end{aligned} \tag{2.4.5}$$

Suppose that $\mathcal{O}$ is fermionic so that $\langle Q\mathcal{O}\rangle_s \approx 0$ is a non-trivial statement. At order ϵ, that equality implies

$$\langle Q\mathcal{O}\rangle_s = i\langle \mathcal{O}\, s\, Q\Psi_{\text{gf}}\rangle_s = i\Big\langle (s\mathcal{O})\,(Q\Psi_{\text{gf}})\Big\rangle_s = i\Big\langle \Big(\delta_{\text{gauge}}(c)\mathcal{O}\Big)\,(Q\Psi_{\text{gf}})\Big\rangle_s \approx 0\,. \tag{2.4.6}$$

We used that $\big\langle s(\mathcal{O}\,Q\Psi_{\text{gf}})\big\rangle_s = 0$ because the action $S_{\text{ph}} + s\Psi_{\text{gf}}$ is s-closed. In the last step, c is massive and therefore its correlators vanish in the IR. We can now use (2.2.39) to conclude that $\langle Q\mathcal{O}\rangle_\delta = \langle Q\mathcal{O}\rangle_s \approx 0$.

The Ward identity for $\mathrm{U}(1)_R$ can be derived with much less work. Any $\mathcal{O}$ built out of 1d fields excluding $\widetilde{c}^{\,ij}_m$ has $n_g = 0$, and $\langle \mathcal{O}\rangle_\delta = \langle \mathcal{O}\rangle_s$ by (2.2.39). Since $s\Psi_{\text{gf}}$ is $\mathrm{U}(1)_R$ invariant, $\langle \mathcal{O}\rangle_s = 0$ if $\mathcal{O}$ has nonzero R-charge. Therefore $\langle \mathcal{O}\rangle_\delta = 0$ if $\mathcal{O}$ has nonzero R-charge.

Given the above Ward identities, any effective action in the IR should have 1d $\mathcal{N} = 2$ supersymmetry and $\mathrm{U}(1)_R$ symmetry. For $\mathrm{U}(1)_R$, we can see this in the following way (the argument for supersymmetry is analogous). Formally, the exact effective action for the fields in the quantum mechanics is given by

$$e^{i\left(S_0 + \sum_{r\neq 0} S_r\right)} = \int \mathcal{D}\phi_H\; e^{i\left(S_{\text{ph}} + \delta\Psi_{\text{gf}}\right)}\,, \tag{2.4.7}$$

where S_r, $r \in \mathbb{Z}$ are pieces of the effective action with R-charge r, and ϕ_H are the massive fields which are integrated out. Note that the $\mathrm{U}(1)_R$ violating pieces $S_{r\neq 0}$ can in principle be generated because $\delta\Psi_{\text{gf}}$ breaks $\mathrm{U}(1)_R$. However, the presence of any $S_{r\neq 0}$ would generically violate the $\mathrm{U}(1)_R$ Ward identity. Indeed, suppose S_r is present for some $r \neq 0$ and consider an operator $\mathcal{O}$ with R-charge $-r$ which is constructed out of the fields ϕ_L in the quantum mechanics excluding $\widetilde{c}^{\,ij}_m$. The Ward identity tells us that $\langle \mathcal{O}\rangle_\delta = 0$. However, computing $\langle \mathcal{O}\rangle_\delta$ directly gives:

$$\langle \mathcal{O}\rangle_\delta = \int \mathcal{D}\phi_L\; \mathcal{O}\, e^{i(S_0 + S_r)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathcal{D}\phi_L\; \mathcal{O}\, S_r^n\, e^{iS_0} = i\int \mathcal{D}\phi_L\; \mathcal{O}\, S_r\, e^{iS_0} \neq 0\,. \tag{2.4.8}$$

We used that S_0 is $\mathrm{U}(1)_R$ invariant, while $\mathcal{O}$ and $\mathcal{O}S_r^{n\geq 2}$ have nonzero R-charge. The operator $\mathcal{O}S_r$ has zero R-charge and its expectation value is generically nonzero.

2.4.3 Symmetry constraints

We can use $\mathrm{U}(1)_R$, Q, and $\overline{Q}$, together with the other symmetries, to constrain the interactions that could appear in the effective action. We work within the framework of [139] (see also [146]), where the interactions in an $\mathcal{N} = 2$ supersymmetric quantum mechanics are specified by E and J functions, *i.e.*, holomorphic functions of chiral superfields satisfying

(2.3.5). The argument in Section 2.4.1 tells us that the E and J functions corresponding to C must vanish in the IR:

$$E^{ij}_{C,m} = 0 \ , \qquad J^{ji}_{C,-m} = 0 \ . \tag{2.4.9}$$

Besides, C cannot appear in the E- and J-terms of the other Fermi multiplets $\mathcal{Y}_a$. Since it is already true classically that $\overline{D}\mathcal{Y}_a \neq 0$ for every $\mathcal{Y}_a$, one expects that $\mathcal{Y}_a$'s cannot appear in E or J functions, because quantum corrections would need to be finely tuned to make them chiral. Therefore, E and J functions can only be holomorphic functions of Φ_a and $\Xi_{\bar{1}}$.

Let us neglect gauge charges and SU(2) invariance momentarily, and suppress the corresponding indices. To have the same $\mathrm{U}(1)^2_F$ charges as $\mathcal{Y}_a$ and R-charge $R(\mathcal{Y}_a)+1$, the E function corresponding to $\mathcal{Y}_a$ must have the simple form

$$E_a \ \sim \ \Phi_a \, h_E(\Xi_{\bar{1}}) \ , \tag{2.4.10}$$

where h_E is a holomorphic function. Fleshing out the gauge and SU(2) indices, we enforce that $E^{ij}_{a,m}$ have the same gauge charges and be in the same SU(2) representation as $\mathcal{Y}^{ij}_{a,m}$. Imposing those conditions on the constant term in h_E, we get $E^{ij}_{a,m} \sim \Phi^{ij}_{a,m}$. However, such a term is impossible because $\mathcal{Y}^{ij}_{a,m}$ (and therefore $E^{ij}_{a,m}$) exists when $q^a_{ij} \leq -1$, while $\Phi^{ij}_{a,m}$ exists when $q^a_{ij} \geq 0$. The two conditions are mutually exclusive.[17] We remain with terms in h_E which are at least linear in $\Xi_{\bar{1}}$. Writing the first term explicitly, we find:

$$\begin{aligned} E^{ij}_{a,m} = \ & \sum_k e^{ij}_{a,k} \, \Theta(q^a_{kj}) \sum_{|m'| \leq q^a_{kj}} C\Big(\begin{smallmatrix} |q_{ik}|-1 & q^a_{kj} & |q^a_{ij}|-1 \\ m-m' & m' & m \end{smallmatrix} \Big) \, \Xi^{ik}_{\bar{1},m-m'} \, \Phi^{kj}_{a,m'} \\ & + \sum_k \widetilde{e}^{ij}_{a,k} \, \Theta(q^a_{ik}) \sum_{|m'| \leq q^a_{ik}} C\Big(\begin{smallmatrix} |q_{kj}|-1 & q^a_{ik} & |q^a_{ij}|-1 \\ m-m' & m' & m \end{smallmatrix} \Big) \, \Phi^{ik}_{a,m'} \, \Xi^{kj}_{\bar{1},m-m'} + \dots \end{aligned} \tag{2.4.11}$$

The Θ functions are necessary to ensure that the fields Φ_a and $\Xi_{\bar{1}}$ exist with their corresponding gauge charges. The Clebsch-Gordan coefficients project the product of $\Xi_{\bar{1}}$ and Φ_a to the same SU(2) representation carried by $E^{ij}_{a,m}$, *i.e.*, $l = |q^a_{ij}| - 1$. Finally, $e^{ij}_{a,k}$ and $\widetilde{e}^{ij}_{a,k}$ are free coefficients. Analogously, terms of the form $\Phi_a(\Xi_{\bar{1}})^{n\geq 2}$ should contain a product of n Clebsch-Gordan coefficients and balanced gauge indices.

When constraining the functions J_a corresponding to $\mathcal{Y}_a$, we again start with $\mathrm{U}(1)^2_F$ and $\mathrm{U}(1)_R$. Now, J_a must have the opposite $\mathrm{U}(1)^2_F$ charges to $\mathcal{Y}_a$, and R-charge $-R(\mathcal{Y}_a)+1$. Thus J_a must have the form

$$J_a \ \sim \ \Phi_b \, \Phi_c \, h_J(\Xi_{\bar{1}}) \ , \tag{2.4.12}$$

where b and c are different flavor indices complementary to a. Again, h_J is a holomorphic function. We should impose gauge and SU(2) invariance. Expanding h_J as a polynomial in

[17] Because of this, the chirals and Fermi's in the quantum mechanics cannot gap each other out through a dynamically generated E-term.

$\Xi_{\bar{1}}$ and writing the first (constant) term explicitly, we have

$$J^{ji}_{a,-m} = \sum_k \Bigg[\frac{\lambda^{ji}_{a,k}}{\sqrt{2|q^a_{ij}|-1}} \,\Theta(q^b_{jk})\,\Theta(q^c_{ki}) \sum_{\substack{|m'|\leq q^b_{jk}\\ |m+m'|\leq q^c_{ki}}} (-1)^{-q^a_{ij}-1-m}\, C\begin{pmatrix} q^b_{jk} & q^c_{ki} & |q^a_{ij}|-1 \\ m' & -m-m' & -m \end{pmatrix} \Phi^{jk}_{b,m'}\,\Phi^{ki}_{c,-m-m'}$$
$$+ \frac{\widetilde{\lambda}^{ji}_{a,k}}{\sqrt{2|q^a_{ij}|-1}} \,\Theta(q^c_{jk})\,\Theta(q^b_{ki}) \sum_{\substack{|m'|\leq q^c_{jk}\\ |m+m'|\leq q^b_{ki}}} (-1)^{-q^a_{ij}-1-m}\, C\begin{pmatrix} q^c_{jk} & q^b_{ki} & |q^a_{ij}|-1 \\ m' & -m-m' & -m \end{pmatrix} \Phi^{jk}_{c,m'}\,\Phi^{ki}_{b,-m-m'} \Bigg] + \ldots \tag{2.4.13}$$

The indices b and c above are chosen such that $\epsilon^{abc} = 1$, and the factor $1/\sqrt{2|q^{ij}_a|-1}$ was added for later convenience. Similarly to the E function, there are two unfixed coefficients $\lambda^{ji}_{a,k}$ and $\widetilde{\lambda}^{ji}_{a,k}$. Terms of the form $\Phi_b\Phi_c(\Xi_{\bar{1}})^{n\geq 1}$ should contain a product of $n+1$ Clebsch-Gordan coefficients and gauge indices should be balanced.

Lastly, supersymmetry requires (2.3.5). If we restrict $E^{ij}_{a,m}$ and $J^{ji}_{a,-m}$ to the terms written explicitly in (2.4.11) and (2.4.13), this condition implies

$$\begin{aligned} &e^{ij}_{a,k}\,\lambda^{ji}_{a,l} + \widetilde{e}^{lk}_{c,i}\,\lambda^{kl}_{c,j} = 0 \qquad \text{if} \quad \epsilon^{abc} = 1 \quad \text{and} \quad \Theta(q^a_{kj})\,\Theta(q^b_{jl})\,\Theta(q^c_{li}) = 1 \\ &e^{ij}_{a,k}\,\widetilde{\lambda}^{ji}_{a,l} + \widetilde{e}^{lk}_{b,i}\,\widetilde{\lambda}^{kl}_{b,j} = 0 \qquad \text{if} \quad \epsilon^{abc} = 1 \quad \text{and} \quad \Theta(q^a_{kj})\,\Theta(q^c_{jl})\,\Theta(q^b_{li}) = 1\,. \end{aligned} \tag{2.4.14}$$

Note that none of the indices above are summed over. The coefficients in (2.3.2) and (2.3.3) that we found from the reduction satisfy these equations, but they might not be the unique choice. The constraint (2.3.5) would also have to be enforced on terms with higher powers of $\Xi_{\bar{1}}$, strongly constraining their coefficients.

From classical scaling arguments, we are not able to rule out the presence in (2.4.11) and (2.4.13) of terms which have higher powers of $\Xi_{\bar{1}}$. They could be generated both at tree and at loop level. It would be consistent to neglect those terms if $\Xi_{\bar{1}}$, which is classically dimensionless, gained a positive anomalous dimension. This is indeed the case for classically dimensionless fermions in SYK models such as [127], but it remains to be checked in the theory discussed here.

Chapter 3

Future directions

In our work on black hole microstate counting, a crucial ingredient was the Bethe ansatz formula for the superconformal index [79, 80], which expresses the index as a sum over solutions to a set of "Bethe ansatz equations" (BAEs). The BAEs were so named because they bear a striking resemblance to the Bethe equations of an elliptic integrable system. This is not a coincidence, since we expect that when 4d $\mathcal{N} = 1$ theories are compactified on T^2 to 2d $(2, 2)$ theories, their vacuum equations should match the Bethe ansatz equations of various integrable systems [147]. This is an instance of the so-called Bethe/gauge correspondence. The general expectation is that such a correspondence works for any 4d $\mathcal{N} = 1$ theory. However, given $\mathcal{N} = 4$ SYM or any 4d holographic quiver gauge theory, the corresponding elliptic integrable systems are unknown. Moreover, no-go theorems recently found in [148] make it tricky to construct the integrable systems corresponding to chiral toric quivers, as defined in [148]. It would be interesting to examine whether it is indeed possible to find the corresponding integrable systems for holographic quiver gauge theories. Perhaps the framework of integrable systems will give a new perspective on some properties of black hole microstates.

When it came to checking the large N limit of the 4d superconformal index against the Bekenstein-Hawking area formula, a limitation was the lack of known black hole solutions in type IIB supergravity which have $\mathrm{AdS}_5 \times \mathrm{SE}_5$ asymptotics. In order to perform more thorough and comprehensive tests of AdS/CFT, it would desirable to construct such solutions.

On the other hand, continuing from [39], we would like to compute the thermal partition function of the quantum mechanics obtained via dimensional reduction. This would allow us to extract the density of states for near-BPS black holes, which can be compared with expectations from supergravity such as [64]. We expect that such a computation is possible even though the theory is strongly coupled, due to simplifications at large N. In particular, preliminary explorations have suggested that the coupling constants of the theory can be replaced by gaussian random variables at leading order in N, in a way that is reminiscent of SYK and similar toy models such as [128].

Appendix A

Subleading effect of simplifications

A.1 Simplifications of the building block

We want to show that the terms neglected in passing from (1.1.26) to (1.1.27) are subleading at large N. We will first analyze the effect of dropping the term $\omega(d-c)/N$ from the arguments of the gamma functions, in all those terms with $\gamma \neq \delta$. We will later estimate the contribution from the terms with $\gamma = \delta$ that were discarded from the sum.

Defining

$$f(z) = \sum_{\gamma \neq \delta}^{\widetilde{N}} \log \widetilde{\Gamma}\left(z + \omega \frac{\delta - \gamma}{\widetilde{N}}; ab\omega, ab\omega\right) , \tag{A.1.1}$$

we want to show that

$$\left| f\big(z + C\omega/\widetilde{N}\big) - f(z) \right| \leq \mathcal{O}(N \log N) , \tag{A.1.2}$$

where $C = (d-c)/ab$, $z = \Delta + \omega\big(d - c + as + br\big)$ and $c, d = 1, \dots, ab$, $r = 0, \dots, a-1$, $s = 0, \dots, b-1$. Without loss of generality we can assume $C > 0$, because the case $C < 0$ is analogous while $C = 0$ is trivial. As in [45], we discard the Stokes lines $\Delta \in \mathbb{Z} + \mathbb{R}\,\omega$ except the point $\Delta = 0$, because the limit we compute would be singular along those lines anyway. If Δ is not on a Stokes line, then the restriction of f to a straight line in the complex plane passing through the points z and $z+\omega$ is a C^∞ complex function. In the case $\Delta = 0$, instead, we consider the restriction of f to a straight closed segment from z to $z + C\omega/\widetilde{N}$ and one can check that f is C^∞ along that segment, because for $\delta \neq \gamma$ the segment, suitably shifted, does not hit zeros nor poles of any of the gamma functions in (A.1.1) (in both cases, f is a holomorphic function in a neighbourhood of the restricted domain). A complex analogue of the Mean Value Theorem (MVT) then states that

$$\begin{aligned} \mathbb{R}\mathrm{e}\, \frac{f\big(z + C\omega/\widetilde{N}\big) - f(z)}{\omega} &= \frac{C}{\widetilde{N}}\, \mathbb{R}\mathrm{e}\, f'\big(z + \bar{c}_1\omega/\widetilde{N}\big) \\ \mathbb{I}\mathrm{m}\, \frac{f\big(z + C\omega/\widetilde{N}\big) - f(z)}{\omega} &= \frac{C}{\widetilde{N}}\, \mathbb{I}\mathrm{m}\, f'\big(z + \bar{c}_2\omega/\widetilde{N}\big) \end{aligned} \tag{A.1.3}$$

with $\bar{c}_1, \bar{c}_2 \in (0, C)$. Summing the absolute values, it follows the bound

$$\left| \frac{f\big(z + C\omega/\widetilde{N}\big) - f(z)}{\omega} \right| \leq \frac{1}{\widetilde{N}} \left(\left| f'\big(z + \bar{c}_1\omega/\widetilde{N}\big) \right| + \left| f'\big(z + \bar{c}_2\omega/\widetilde{N}\big) \right| \right) \tag{A.1.4}$$

where we used $|C| \leq 1 - \frac{1}{ab} < 1$. It is therefore sufficient to show that

$$\frac{1}{\widetilde{N}} \left| f'\big(z + \bar{c}\omega/\widetilde{N}\big) \right| \leq \mathcal{O}(N \log N) \tag{A.1.5}$$

for any $\bar{c} \in (0, C)$. Notice that $0 < \bar{c} < 1 - \frac{1}{ab}$.

We reason as follows. For $\Delta \notin \mathbb{Z} + \mathbb{R}\,\omega$, the arguments of the elliptic gamma functions in (A.1.1) remain at an $\widetilde{N}$-independent distance from the zeros and poles, that in our case are placed at the points

$$u_{0,i} = (1+i)ab\,\omega\ , \qquad u_{\infty,j} = (1-j)ab\,\omega \qquad \text{for} \qquad i,j \in \mathbb{Z}_{\geq 1}\ . \tag{A.1.6}$$

The orders of the zeros and poles are i and j respectively. The ratio $\big|\widetilde{\Gamma}'/\widetilde{\Gamma}\big|$ is bounded on the range of possible arguments, therefore

$$\frac{1}{\widetilde{N}} \left| f'\big(z + \bar{c}\omega/\widetilde{N}\big) \right| \leq \widetilde{N} \max_{t \in [-ab,\, 3ab-a-b]} \left| \frac{\widetilde{\Gamma}'\big(\Delta + t\omega; ab\omega, ab\omega\big)}{\widetilde{\Gamma}\big(\Delta + t\omega; ab\omega, ab\omega\big)} \right| = \mathcal{O}(\widetilde{N})\ . \tag{A.1.7}$$

The case $\Delta = 0$ is more subtle since, as $\widetilde{N}$ grows, the arguments of some of the gamma functions can get increasingly close to zeros or poles instead of staying at an $\widetilde{N}$-independent distance, and the $\widetilde{N}$-independent bound above does not apply. This happens when

$$z = \bar{u}_{0,i} \in \{u_{0,i}, u_{0,i} \pm \omega\} \qquad \text{or} \qquad z = \bar{u}_{\infty,j} \in \{u_{\infty,j}, u_{\infty,j} \pm \omega\}\ . \tag{A.1.8}$$

One can easily see that for $\Delta = 0$, z can range from $(1-ab)\omega$ to $(3ab - a - b - 1)\omega$, so that the problematic points we may approach are the simple zero at $u_{0,1}$, the simple pole at $u_{\infty,1}$, and the double pole at $u_{\infty,2}$.

We now introduce a few results for later use. For a meromorphic function g whose zeros include $\{z_i\}$ of order $\{m_i\}$ and whose poles include $\{p_j\}$ of order $\{n_j\}$, one can write

$$g(z) = \frac{\prod_i (z - z_i)^{m_i}}{\prod_j (z - p_j)^{n_j}} s(z), \tag{A.1.9}$$

where $s(z)$ is meromorphic with zeros and poles at the remaining zeros and poles of g that were not included in $\{z_i\}$ and $\{p_j\}$. Taking the derivative of this expression and computing g'/g, one finds

$$\frac{g'(z)}{g(z)} = \sum_i \frac{m_i}{z - z_i} - \sum_j \frac{n_j}{z - p_j} + h(z), \tag{A.1.10}$$

where $h(z) = s'(z)/s(z)$ is meromorphic with simple poles at the remaining zeros and poles of g that were not included in $\{z_i\}$ and $\{p_j\}$. Therefore we can apply (A.1.10) to the meromorphic function $\widetilde{\Gamma}$ and say that

$$\begin{aligned}
\frac{1}{\widetilde{N}}\left|f'\big(z+\bar{c}\omega/\widetilde{N}\big)\right| &\leq \frac{1}{\widetilde{N}}\sum_{\gamma\neq\delta}^{\widetilde{N}}\left|\frac{\widetilde{\Gamma}'\big(z+u^{\bar{c}}_{\gamma,\delta};ab\omega,ab\omega\big)}{\widetilde{\Gamma}\big(z+u^{\bar{c}}_{\gamma,\delta};ab\omega,ab\omega\big)}\right| \\
&\leq \frac{1}{\widetilde{N}}\sum_{\gamma\neq\delta}^{\widetilde{N}}\left(\frac{1}{|z+u^{\bar{c}}_{\gamma,\delta}-2ab\omega|}+\frac{1}{|z+u^{\bar{c}}_{\gamma,\delta}|}+\frac{2}{|z+u^{\bar{c}}_{\gamma,\delta}+ab\omega|}\right)+(\widetilde{N}-1)K \qquad \text{(A.1.11)}
\end{aligned}$$

where we defined

$$u^{\bar{c}}_{\gamma,\delta} = \omega\,\frac{\delta-\gamma+\bar{c}}{\widetilde{N}}\,, \qquad\qquad K = \max_{t\in[-ab,\,3ab-a-b]}\left|h_{\widetilde{\Gamma}}(t\omega)\right| \qquad \text{(A.1.12)}$$

and $h_{\widetilde{\Gamma}}$ is the meromorphic function associated to $\widetilde{\Gamma}$ in (A.1.10). We can bound its value with an $\widetilde{N}$-independent quantity because it is holomorphic on the range of possible arguments. If $z \neq \bar{u}_{0,1}, \bar{u}_{\infty,1}, \bar{u}_{\infty,2}$, the outlying sums in (A.1.11) will be of order $\mathcal{O}(\widetilde{N})$ since $z + u^{\bar{c}}_{\gamma,\delta}$ will be at least at a distance $|\omega|$ away from the zeros and poles. To complete our proof when $z = \bar{u}_{0,1}, \bar{u}_{\infty,1}, \bar{u}_{\infty,2}$, we now need to bound the quantities

$$R_x = \frac{1}{N}\sum_{\gamma\neq\delta}^{N}\frac{1}{\left|x+\frac{\delta-\gamma+\bar{c}}{N}\right|} \qquad \text{with} \quad x = 0, \pm 1\,, \qquad \text{(A.1.13)}$$

where we wrote N in place of $\widetilde{N}$ in order not to clutter the formulae. We recall that $0 < \bar{c} < 1 - 1/ab$. Considering $x = 0$ first, we reparametrize the sum in terms of $\delta - \gamma$ so that, after some manipulations, it becomes

$$R_0 = \sum_{M=1}^{N-1}\left(\frac{N-M}{M+\bar{c}}+\frac{N-M}{M-\bar{c}}\right) < 2\sum_{M=1}^{N-1}\frac{N-M}{M-\bar{c}}\,. \qquad \text{(A.1.14)}$$

The summand on the right is a positive decreasing function of M, therefore it can be bound by its integral:

$$R_0 < \frac{2(N-1)}{1-\bar{c}} + 2\int_1^{N-1}\frac{N-x}{x-\bar{c}}\,dx = \mathcal{O}(N\log N)\,. \qquad \text{(A.1.15)}$$

To ensure convergence of sums and integrals it is crucial to recall that $1 - \bar{c} > (ab)^{-1}$. In a similar way, for $x = +1$ we can write

$$R_1 = \sum_{M=1}^{N-1}\left(\frac{N-M}{N+M+\bar{c}}+\frac{N-M}{N-M+\bar{c}}\right) < \sum_{M=1}^{N-1}2 = \mathcal{O}(N)\,, \qquad \text{(A.1.16)}$$

while for $x = -1$ we can write

$$R_{-1} = \sum_{M=1}^{N-1}\left(\frac{N-M}{N-M-\bar{c}}+\frac{N-M}{N+M-\bar{c}}\right) < 2\sum_{M=1}^{N-1}\frac{N-M}{N-M-\bar{c}} < \frac{2(N-1)}{1-\bar{c}} = \mathcal{O}(N)\,. \qquad \text{(A.1.17)}$$

It remains to show that the terms we discarded from (1.1.26) when substituting the condition $i \neq j$ with the condition $\gamma \neq \delta$ give a subleading contribution. These are the terms in (1.1.26) with $\gamma = \delta$, whose total contribution is

$$\Phi = \widetilde{N} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{c \neq d}^{ab} \log \widetilde{\Gamma} \left(\Delta + \omega \frac{d-c}{N} + \omega\big(d - c + as + br\big); ab\omega, ab\omega \right) . \tag{A.1.18}$$

We need to show that this is subleading in the large N limit. We will bound the absolute value of the summand for all possible $c \neq d$, r, s and drop the sums since they give an overall order $\mathcal{O}(1)$ factor. After choosing a branch of the logarithm, the phases of $\widetilde{\Gamma}$ can clearly only give an order $\widetilde{N}$ contribution to (the imaginary part of) Φ.

For what concerns the absolute value of $\widetilde{\Gamma}$, reasoning in a very similar way to the $\gamma \neq \delta$ case discussed above, we see that if Δ is not on a Stokes line then $\big|\log|\widetilde{\Gamma}|\big|$ is bounded above by an N-independent quantity and thus Φ is of order $\mathcal{O}(N)$. When $\Delta = 0$, the argument of $\widetilde{\Gamma}$ can only approach zeros or poles if $z = \omega\big(d - c + as + br\big) \in \{u_{0,1}, u_{\infty,1}, u_{\infty,2}\}$. Using (A.1.9), we can write

$$\log \left| \widetilde{\Gamma}\Big(z + \omega \frac{d-c}{N}; ab\omega, ab\omega\Big) \right| = \log \left| \frac{\big(z + \omega \frac{d-c}{N} - u_{0,1}\big)\, s_{\widetilde{\Gamma}}\Big(z + \omega \frac{d-c}{N}\Big)}{\big(z + \omega \frac{d-c}{N} - u_{\infty,1}\big)\, \big(z + \omega \frac{d-c}{N} - u_{\infty,2}\big)^2} \right| \tag{A.1.19}$$

where $s_{\widetilde{\Gamma}}$ is a function which is regular at $u_{\infty,1}, u_{\infty,2}$ and non-zero at $u_{0,1}$. We can therefore bound $\big|\log|s_{\widetilde{\Gamma}}|\big|$ over its possible arguments with an N-independent constant, so that it contributes to Φ at order $\mathcal{O}(N)$. When $z = u_{0,1}, u_{\infty,1}, u_{\infty,2}$, only one of the factors multiplying $s_{\widetilde{\Gamma}}$ is of order $\mathcal{O}(\log N)$ while the other two do not approach zero and can be bounded by an N-independent constant. Explicitly,

$$\widetilde{N} \left| \log \left| \widetilde{\Gamma}\Big(z + \omega \frac{d-c}{N}; ab\omega, ab\omega\Big) \right| \right| \leq 2\widetilde{N} \left| \log \left| \omega \frac{d-c}{N} \right| \right| + \mathcal{O}(N) = \mathcal{O}(N \log N) . \tag{A.1.20}$$

A.2 SU(N) *vs.* U(N) holonomies

In what follows, as it is done in Section 1.1 and Section 1.2, in order to parametrize the SU(N) holonomies u^{SU} we introduce U(N) holonomies u^{U}, constrained by

$$\sum_{i=1}^{N} u_i^{\mathrm{U}} = 0 . \tag{A.2.1}$$

With the choice of bases for the Cartan subalgebras of SU(N) and U(N) required to write the BA operators as in (1.1.12), the relation between the two sets of holonomies when expressing a generic element of the Cartan subalgebra of SU(N) is

$$u_i^{\mathrm{U}} = u_i^{\mathrm{SU}} \quad \text{for } i \neq N \, , \qquad\qquad u_N^{\mathrm{U}} = -\sum_{j=1}^{N-1} u_j^{\mathrm{SU}} \, . \tag{A.2.2}$$

Note that the holonomies are only defined modulo $\mathbb{Z}$.

The SU(N) superconformal index defined by (1.1.9) contains a sum over $\{m_i^{\mathrm{SU}}\}$ that picks up (representatives of) solutions to the BAEs whose residue can contribute to the index, as explained in [79] and made explicit in (1.1.17). Under a shift $\{m_i^{\mathrm{SU}}\}$ of the SU(N) holonomies, the U(N) holonomies shift by corresponding amounts given by

$$m_i^{\mathrm{U}} = m_i^{\mathrm{SU}} \, , \qquad\qquad m_N^{\mathrm{U}} = -\sum_{j=1}^{N-1} m_j^{\mathrm{SU}} \, . \tag{A.2.3}$$

Given these identifications for the holonomies and shifts, the SU(N) quantities are always equal to the first $N-1$ U(N) quantities, so that in the following we will drop the superscripts SU and U, remembering that $u_{1,\ldots,N-1}$ and $m_{1,\ldots,N-1}$ are independent while u_N and m_N are determined by (A.2.2) and (A.2.3), respectively.

One might then worry that the choice of $\{m_j\}$ given in (1.1.21) is not allowed, since the last integer m_N there does not satisfy (A.2.3). Specifically, let us choose

$$m_j \in \{1, \ldots, ab\} \qquad \text{such that} \qquad m_j = j \mod ab \, , \qquad \text{for } j = 1, \ldots, N-1 \, , \tag{A.2.4}$$

so that m_N is fixed by (A.2.3) to be a negative integer of $\mathcal{O}(N)$. To match with the choice in (1.1.21), we want to replace this with $m_N = N \mod ab$ and in $\{1, \ldots, ab\}$. We will show that this replacement does not affect the value of $\mathcal{Z}$ to leading order in N. This will be done in two steps. We will first show that the function $\mathcal{Z}$ evaluated on a configuration $\{u_1, \ldots, u_N\}$ which is obtained from the basic solution by shifting one or more variables u_i by multiples of $2ab\omega$, is the same as $\mathcal{Z}$ evaluated on the basic solution. Using this property, $\mathcal{Z}$ is unaltered if evaluated on the following shifted value of m_N:

$$\widetilde{m}_N \in \{1, \ldots, 2ab\} \qquad \text{such that} \qquad \widetilde{m}_N = \left(-\sum_{i=1}^{N-1} m_i \right) \mod 2ab \, . \tag{A.2.5}$$

We will then show that the contribution to $\mathcal{Z}$ of the single holonomy u_N is subleading, provided $\widetilde{m}_N \in \{1, \ldots, 2ab\}$. Therefore, choosing instead $m_N = N \mod ab$ and in $\{1, \ldots, ab\}$ as we did in (1.1.21) does not change $\mathcal{Z}$ at leading order in N. This completes the proof.

As shown in [79], when evaluated on solutions to the BAEs, the function $\mathcal{Z}$ for a general semi-simple gauge group is invariant under independent shifts of any gauge holonomy by $ab\omega$. This is proven assuming that gauge and global symmetries are non-anomalous. In our case, this result only allows us to shift the u_i's while preserving the SU(N) constraint. This property does not allow us to independently shift the last holonomy u_N, since it is always fixed by the SU(N) constraint. We now show that an *independent* shift of u_N by a multiple of $ab\omega$ of u_N is also an invariance of $\mathcal{Z}$ for $\mathcal{N}=4$ SU(N) SYM, when this function is evaluated on the basic solution. In order to prove this, one has to use the property

$$\begin{aligned}&\widetilde{\Gamma}\big(u+mab\omega, a\omega, b\omega\big) \\ &\quad= \left(-e^{2\pi i u}\right)^{-\frac{ab}{2}m^2+\frac{a+b-1}{2}m}\left(e^{2\pi i\omega}\right)^{-\frac{ab}{6}m^3+\frac{ab(a+b)}{4}m^2-\frac{a^2+b^2+3ab-1}{12}m}\theta_0(u,\omega)^m\,\widetilde{\Gamma}(u,a\omega,b\omega)\end{aligned} \tag{A.2.6}$$

that was proven in [79], the fact that the U(N) BA operators are periodic modulo ω in the u_i's, and the explicit form of the basic solution (1.1.20). Applying (A.2.6), we first have that

$$\begin{aligned}&\prod_{i\neq j}\widetilde{\Gamma}\Big(u_{ij}+\Delta+mab\omega(\delta_{iN}-\delta_{jN}); a\omega, b\omega\Big) \\ &\quad= e^{-\pi i ab m^2(1+2\Delta)+2\pi i(a+b-1)m\sum_i u_{iN}+\pi i ab(a+b)m^2\omega}\prod_i\frac{\theta_0(u_{Ni}+\Delta,\omega)^m}{\theta_0(u_{iN}+\Delta;\omega)^m}\prod_{i\neq j}\widetilde{\Gamma}(u_{ij}+\Delta;a\omega,b\omega)\ ,\end{aligned} \tag{A.2.7}$$

and so from (1.1.12), (1.1.18) and (1.1.20) one obtains

$$\begin{aligned}&\mathcal{Z}(u_i+mab\omega\delta_{iN};a\omega,b\omega,\Delta) \\ &= \prod_i\left(\frac{\theta_0(u_{Ni}+\Delta_1,\omega)\,\theta_0(u_{Ni}+\Delta_2,\omega)\,\theta_0(u_{iN},\omega)\,\theta_0(u_{iN}+\Delta_1+\Delta_2,\omega)}{\theta_0(u_{iN}+\Delta_1,\omega)\,\theta_0(u_{iN}+\Delta_2,\omega)\,\theta_0(u_{Ni},\omega)\,\theta_0(u_{Ni}+\Delta_1+\Delta_2,\omega)}\right)^m\mathcal{Z}(u_i;a\omega,b\omega,\Delta) \\ &= (-1)^{m(N-1)}\,e^{2\pi i m\lambda}\,Q_N^{-m}(u_i;\omega,\Delta)\,\mathcal{Z}(u_i;a\omega,b\omega,\Delta) \\ &= \mathcal{Z}(u_i;a\omega,b\omega,\Delta)\ .\end{aligned} \tag{A.2.8}$$

In the steps above we also used the theta function reflection property

$$\theta_0(u;\omega) = -e^{2\pi i u}\,\theta_0(-u;\omega)\ . \tag{A.2.9}$$

More generally, we can show that this shift invariance is true for quiver gauge theories, when $\mathcal{Z}$ is evaluated on the basic solution and the chemical potentials u_N^α are shifted by a multiple of $2ab\omega$ simultaneously for all gauge groups SU(N)$_\alpha$. The steps are the same as in (A.2.8). We should notice that the expression for any particular Lagrange multiplier λ_α is more complicated than for $\mathcal{N}=4$ SYM, but the sum of all Lagrange multipliers is simple:

$$e^{2\pi i\sum_{\alpha=1}^G\lambda_\alpha} = (-1)^{n_\chi(N-1)}\ , \tag{A.2.10}$$

where α runs over the G SU(N) gauge group factors and n_χ is the number of chiral multiplets in the theory. Performing these steps one obtains

$$\begin{aligned}&\mathcal{Z}\big(u_i^\alpha+2mab\omega\delta_{iN};a\omega,b\omega,\Delta\big) \\ &\quad= \frac{e^{2\pi i 2m\sum_{\alpha=1}^G\lambda_\alpha}\,(-1)^{-2mG(N-1)+2mab(N-1)(n_\chi-G)}}{\big(\prod_\alpha Q_N^\alpha(u_i^\alpha;\omega,\Delta)\big)^{2m}}\,\mathcal{Z}(u_i^\alpha;a\omega,b\omega,\Delta) \\ &\quad= (-1)^{2m(G-n_\chi)(ab+1)(N-1)}\,\mathcal{Z}(u_i^a;a\omega,b\omega,\Delta) = \mathcal{Z}(u_i^a;a\omega,b\omega,\Delta)\ .\end{aligned} \tag{A.2.11}$$

We now proceed to show that the contribution to $\mathcal{Z}$ of a single holonomy u_i is subleading, provided that $m_{i<N} \in \{1, \dots, ab\}$ and $m_N \in \{1, \dots, 2ab\}$. In the building block Ψ defined in (1.1.22), the contribution of a single holonomy u_i consists of the two terms

$$\Phi_i^\pm \equiv \sum_{j(\neq i)}^{N} \log \widetilde{\Gamma} \left(z_\pm \pm \omega \, \frac{j-i}{N}; ab\omega, ab\omega \right) \,, \tag{A.2.12}$$

where we have defined

$$z_\pm \equiv \Delta \pm \omega \left(m_j - m_i \right) + \omega \left(as + br \right) \,. \tag{A.2.13}$$

In particular, for the case $i = N$ we will use the shift property just proven to substitute m_N with $\widetilde{m}_N$ defined in (A.2.5).

We will now show that $\Phi_i^\pm$ is subleading. In the case $i = N$ this will allow us to choose $\widetilde{m}_N$ as in (1.1.21). In order to do this we want to bound the absolute value of the summand $\log \widetilde{\Gamma}$ in $\Phi_i^\pm$. What follows will be completely analogous to the argument used to show that (A.1.18) is subleading. After choosing a branch of the logarithm, the phases of $\widetilde{\Gamma}$ can only contribute at order $\mathcal{O}(N)$ to $\Phi_i^\pm$. As before, we exclude Stokes lines and note that for $\Delta \neq 0$ we can bound $\left|\log |\widetilde{\Gamma}|\right|$ with an N-independent constant so that $|\Phi_i^\pm| = \mathcal{O}(N)$. For $\Delta = 0$, $z_\pm$ have the range

$$z_\pm \in \{-2ab + 1, \dots, 4ab - a - b - 1\}\omega \,, \tag{A.2.14}$$

and the argument of $\widetilde{\Gamma}$ may approach zeros or poles when $z_\pm = \bar{u}_{0,1}, \bar{u}_{0,2}, \bar{u}_{\infty,1}, \bar{u}_{\infty,2}, \bar{u}_{\infty,3}$, which are defined in (A.1.8). If this is the case, further inspection is required. Using again (A.1.9), we can write

$$\log \widetilde{\Gamma} \left(z_\pm \pm \omega \frac{j-i}{N}; ab\omega, ab\omega \right) = \log \left[\frac{\prod_{m=1}^{2} \left(z_\pm \pm \omega \frac{j-i}{N} - u_{0,m} \right)^m s_{\widetilde{\Gamma}} \left(z_\pm \pm \omega \frac{j-i}{N} \right)}{\prod_{n=1}^{3} \left(z_\pm \pm \omega \frac{j-i}{N} - u_{\infty,n} \right)^n} \right] \tag{A.2.15}$$

where $s_{\widetilde{\Gamma}}$ is a function that is regular at $u_{\infty,1}$, $u_{\infty,2}$, $u_{\infty,3}$, and non-zero at $u_{0,1}$, $u_{0,2}$. This allows us to bound $\left|\log |s_{\widetilde{\Gamma}}|\right|$ with an N-independent constant, and its contribution to $\Phi_i^\pm$ is of order $\mathcal{O}(N)$. When $z_\pm = \bar{u}_{0,1}$, $\bar{u}_{0,2}$, $\bar{u}_{\infty,1}$, $\bar{u}_{\infty,2}$, $\bar{u}_{\infty,3}$ the logarithms of the other factors are either bounded by an N-independent constant, or are of the form

$$\sum_{j\neq i}^{N} \left| \log \left| x \pm \frac{j-i}{N} \right| \right| \leq (N-1) \log N \,, \tag{A.2.16}$$

where $x = 0, \pm 1$. Notice that the use of the shift property previously proved plays a major role here. If we tried to apply this argument directly without first shifting m_N, we would have to consider a $\mathcal{O}(N)$ number of poles or zeros whose order is also $\mathcal{O}(N)$. This would lead to a $\mathcal{O}(N^3 \log N)$ bound, which is useless. What we did shows that a single $\Phi_i^\pm$ is of order

$\mathcal{O}(N \log N)$ for any choice of the corresponding m_i. In particular this allows us to choose $\widetilde{m}_N = N \bmod ab \in \{1, \ldots, ab\}$ as we do in (1.1.21), without affecting the leading behavior of the building block Ψ.

A.3 Generic N

Here we generalize the computation done in Section 1.1.1 and consider a generic N which is not necessarily a multiple of ab. We will exploit many of the arguments in Section A.1. Let $N = ab\widetilde{N} + q$, where $q \in \{0, \ldots, ab-1\}$. We need to examine the leading order contribution of the building block

$$\Psi = \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{i \neq j}^{N} \log \widetilde{\Gamma}\left(\Delta + \omega \frac{j-i}{N} + \omega\big(m_j - m_i + as + br\big); ab\omega, ab\omega\right) . \tag{A.3.1}$$

As shown in the final part of Section A.2, the contribution to the building block of a single holonomy u_i is subleading. Therefore the contribution of the last q holonomies $u_{ab\widetilde{N}+1}, \ldots, u_N$ is also subleading and can be discarded. Now, the sum over $i \neq j$ only goes up to $ab\widetilde{N}$, and we can decompose the indices as in (1.1.26). Neglecting the $\gamma = \delta$ terms using the same argument below (A.1.18), we get

$$\Psi \simeq \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{\gamma \neq \delta = 0}^{\widetilde{N}-1} \sum_{c,d=0}^{ab-1} \log \widetilde{\Gamma}\left(\Delta + \omega \frac{\delta - \gamma}{\widetilde{N} + \frac{q}{ab}} + \omega \frac{d-c}{N} + \omega\big(d - c + as + br\big); ab\omega, ab\omega\right) . \tag{A.3.2}$$

As in Section A.1, we want to drop $\omega(d-c)/N$ in the argument of the elliptic gamma function, and we can use the same reasoning given there, with the minor change that (A.1.13) takes the form

$$\widetilde{R}_x = \frac{1}{N + \frac{q}{ab}} \sum_{\gamma \neq \delta}^{N} \frac{1}{\left| x + \frac{\delta - \gamma + \bar{c}}{N + q/ab} \right|} , \qquad x = 0, \pm 1 . \tag{A.3.3}$$

The same bounds as for R_x can be used here, since one can show that

$$\widetilde{R}_0 = R_0 , \qquad \widetilde{R}_{\pm 1} \leq R_{\pm 1} . \tag{A.3.4}$$

We can then use (1.1.23), as we did in Section 1.1.1, to change the moduli of the elliptic gamma function from $(ab\omega, ab\omega)$ to (ω, ω):

$$
\begin{aligned}
\Psi &\simeq \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma\neq\delta=0}^{\widetilde{N}-1}\sum_{c,d=0}^{ab-1} \log\widetilde{\Gamma}\left(\Delta+\omega\,\frac{\delta-\gamma}{\widetilde{N}+\frac{q}{ab}}+\omega\big(d-c+as+br\big);ab\omega,ab\omega\right) \\
&= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma\neq\delta=0}^{\widetilde{N}-1} \log\widetilde{\Gamma}\left(\Delta+\omega\,\frac{\delta-\gamma}{\widetilde{N}+\frac{q}{ab}}+\omega\big(1-ab+as+br\big);\omega,\omega\right) \\
&= \frac{1}{(ab)^2}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{\gamma\neq\delta=0}^{\widetilde{N}-1}\sum_{c,d=0}^{ab-1} \log\widetilde{\Gamma}\left(\Delta+\omega\,\frac{\delta-\gamma}{\widetilde{N}+\frac{q}{ab}}+\omega\big(1-ab+as+br\big);\omega,\omega\right)\,.
\end{aligned}
\tag{A.3.5}
$$

In the last equality, to make future steps clearer, we added a sum over c, d even though nothing depends on c and d.

Now in order to get the desired result we trace our steps backwards. First, we will reintroduce the term $\omega(d-c)/N$ into the argument of the elliptic gamma functions. Then we will add to the sum in (A.3.5) the $\gamma = \delta$ terms to form the sum over $i \neq j$ up to $ab\widetilde{N}$. Finally we will add terms containing the last q holonomies $u_{ab\widetilde{N}+1}, \ldots, u_N$ in order to build the complete sum up to N. These are the exact same steps we just performed to express Ψ as in (A.3.5) up to subleading terms, with the only difference being that the moduli of $\widetilde{\Gamma}$ are now (ω, ω) rather than $(ab\omega, ab\omega)$. Therefore the same arguments can be used, with only slight modifications involving the number and order of zeros and poles, but since these are parametrized here by r and s that are N-independent, this is of no consequence. At this point, Ψ at leading order is

$$
\Psi \simeq \frac{1}{(ab)^2}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{i\neq j}^{N} \log\widetilde{\Gamma}\left(\Delta+\omega\,\frac{j-i}{N}+\omega\big(1-ab+as+br\big);\omega,\omega\right)\,,
\tag{A.3.6}
$$

and using the result of [45] (that is our equation (1.1.29)) we obtain

$$
\Psi \simeq -\frac{\pi i N^2}{3(a\omega)(b\omega)}\,\frac{1}{ab}\sum_{r=0}^{a-1}\sum_{s=0}^{b-1} B_3\Big([\Delta]'_\omega+\omega\big(as+br-ab\big)\Big)\,.
\tag{A.3.7}
$$

Then, using the property of Bernoulli polynomials (1.1.33), we finally get (1.1.34).

Appendix B

5d $\mathcal{N} = 2$ Abelian gauged supergravity

We report here the general form of 5d $\mathcal{N} = 2$ Abelian gauged supergravity with n_V vector multiplets and n_H hypermultiplets [111–113].[1] The graviton multiplet contains a graviton, a gravitino and a vector; each vector multiplet contains a vector, a gaugino and a real scalar; each hypermultiplet contains four real scalars and a hyperino. All fermions are Dirac, but can conveniently be doubled with a symplectic Majorana condition. We follow the notation of [150]. We use indices

$$I, J, K = 1, \ldots, n_V + 1 \ , \qquad i, j = 1, \ldots, n_V \ , \qquad u, v = 1, \ldots, 4n_H \tag{B.0.1}$$

for the gauge fields A_μ^I, for the scalars ϕ^i in vector multiplets, and for the scalars q^u in hypermultiplets, respectively. The data that define the theory are:

1. A very special real manifold $\mathcal{SM}$ of real dimension n_V.

2. A quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$.

3. A set of $n_V + 1$ Killing vectors on $\mathcal{QM}$ compatible with the quaternionic-Kähler structure (if $n_H = 0$, $n_V + 1$ FI parameters not all vanishing).

The Killing vectors could be linearly dependent or vanish.

The bosonic Lagrangian is given by

$$\begin{aligned} 8\pi G_{\mathrm{N}}^{(5)} e^{-1} \mathscr{L}_{5\mathrm{d}} = \frac{R_s}{2} - \frac{1}{2} \mathcal{G}_{ij}(\phi)\, \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{2} h_{uv}(q)\, \mathcal{D}_\mu q^u \mathcal{D}^\mu q^v - \frac{1}{4} G_{IJ}(\phi)\, F_{\mu\nu}^I F^{J\mu\nu} \\ + \frac{e^{-1}}{48} C_{IJK}\, \epsilon^{\mu\nu\rho\sigma\lambda}\, F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K - g^2 V(\phi, q) \ . \end{aligned} \tag{B.0.2}$$

Here $G_{\mathrm{N}}^{(5)}$ is the 5d Newton constant, $e\, d^5x$ is the spacetime volume form, R_s is the scalar curvature, $F_{\mu\nu}^I$ is the field strength of A_μ^I, g is a coupling constant, and V is the scalar potential. Let us explain the other terms.

[1] A more complete discussion was developed in [149].

Very special geometry. The scalars ϕ^i are real coordinates on the very special real manifold $\mathcal{SM}$ [151]. The latter is specified by the totally symmetric tensor C_{IJK} (which, controlling also the Chern-Simons couplings, should be suitably quantized) as the submanifold

$$\mathcal{SM} = \Big\{ \mathcal{V}(\Phi) \equiv \frac{1}{6}\, C_{IJK}\, \Phi^I\Phi^J\Phi^K = 1 \Big\} \subset \mathbb{R}^{n_V+1} \,. \tag{B.0.3}$$

Here Φ^I are coordinates on $\mathbb{R}^{n_V+1}$, and give rise to "sections" $\Phi^I(\phi^i)$ on $\mathcal{SM}$. The metrics G_{IJ} and $\mathcal{G}_{ij}$ for vector fields and vector multiplet scalar fields are

$$G_{IJ}(\phi) = -\frac{1}{2}\,\frac{\partial}{\partial\Phi^I}\frac{\partial}{\partial\Phi^J}\log\mathcal{V}\Big|_{\mathcal{V}=1} \,, \qquad\qquad \mathcal{G}_{ij}(\phi) = \partial_i\Phi^I\,\partial_j\Phi^J\,G_{IJ}\Big|_{\mathcal{V}=1} \tag{B.0.4}$$

where $\partial_i \equiv \partial/\partial\phi^i$. We recognize that $\mathcal{G}$ is the pull-back of G from $\mathbb{R}^{n_V+1}$ to $\mathcal{SM}$. From (B.0.3) it immediately follows

$$C_{IJK}\,\Phi^I\Phi^J\partial_i\Phi^K\,\big|_{\mathcal{V}=1} = 0 \,. \tag{B.0.5}$$

With a little bit of algebra one then obtains a more explicit expression for G:

$$G_{IJ} = -\frac{1}{2}C_{IJK}\Phi^K + \frac{1}{8}C_{IKL}C_{JMN}\Phi^K\Phi^L\Phi^M\Phi^N\Big|_{\mathcal{V}=1} \,. \tag{B.0.6}$$

It follows that the kinetic term for vector multiplet scalars can also be written as

$$-\frac{1}{2}\,\mathcal{G}_{ij}\,\partial_\mu\phi^i\partial^\mu\phi^j = \frac{1}{4}\,C_{IJK}\,\Phi^I\partial_\mu\Phi^J\partial^\mu\Phi^K\Big|_{\mathcal{V}=1} \,. \tag{B.0.7}$$

One can define on $\mathcal{SM}$ the sections with lower indices:

$$\Phi_I \equiv \frac{2}{3}G_{IJ}\Phi^J\Big|_{\mathcal{V}=1} = \frac{1}{6}C_{IJK}\Phi^J\Phi^K\Big|_{\mathcal{V}=1} = \frac{1}{3}\frac{\partial\mathcal{V}}{\partial\Phi^I}\Big|_{\mathcal{V}=1} \,. \tag{B.0.8}$$

With simple algebra one can show the following identities:

$$\begin{aligned} &\Phi_I\Phi^I = 1 \,, && G_{IJ} = \frac{9}{2}\Phi_I\Phi_J - \frac{1}{2}C_{IJK}\Phi^K \,, \\ &\partial_i\Phi_I = -\frac{2}{3}G_{IJ}\,\partial_i\Phi^J \,, && \Phi_I\,\partial_i\Phi^I = \Phi^I\partial_i\Phi_I = 0 \,. \end{aligned} \tag{B.0.9}$$

In particular, $\partial_i\Phi^I$ for $i = 1,\ldots,n_V$ are the tangent vectors to $\mathcal{SM}$ in $\mathbb{R}^{n_V+1}$ while Φ_I is a 1-form orthogonal to $\mathcal{SM}$. Another identity (and similar ones obtained by lowering one or both of the indices I, J with the metric G) is

$$\mathcal{G}^{ij}\,\partial_i\Phi^I\partial_j\Phi^J = G^{IJ} - \frac{2}{3}\Phi^I\Phi^J \,, \tag{B.0.10}$$

where G^{IJ} is the inverse of G_{IJ}. To prove it, one observes that the tensor on the LHS is the projector on $\mathcal{SM}$, and then verifies that the expression on the RHS has the same property.

When the manifold $\mathcal{SM}$ is a locally symmetric space, one can find a constant symmetric tensor C^{IJK} with upper indices such that [111]

$$C^{IPQ}\,C_{P(JK}\,C_{LM)Q} = \frac{4}{3}\,\delta^I_{(J}\,C_{KLM)}\;. \tag{B.0.11}$$

With some algebra, it follows that

$$\Phi^I = \frac{3}{2}\,G^{IJ}\Phi_J = \frac{9}{2}\,C^{IJK}\Phi_J\Phi_K\;, \qquad\qquad G^{IJ} = 2\Phi^I\Phi^J - 6C^{IJK}\Phi_K\;, \tag{B.0.12}$$

as well as

$$C^{IJK} = \frac{1}{8}\,G^{IL}\,G^{JM}\,G^{KN}\,C_{LMN}\;. \tag{B.0.13}$$

Quaternionic-Kähler geometry. The scalars q^u are real coordinates on the quaternionic-Kähler manifold $\mathcal{QM}$ with metric $h_{uv}(q)$ [152]. For $n_H \geq 2$,[2] this is a $4n_H$-dimensional Riemannian manifold with holonomy $SU(2)\times Sp(n_H)/\mathbb{Z}_2$. To express this fact, it is convenient to introduce local "vielbeins" $f_u{}^{iA}$ with $i=1,2$ (not to be confused with the index i of very special geometry) in the fundamental of $SU(2)$ and $A=1,\ldots,2n_H$ in the fundamental of $Sp(n_H)$, such that

$$h_{uv} = f_u{}^{iA} f_v{}^{jB}\epsilon_{ij}\Omega_{AB}\;, \tag{B.0.14}$$

where ϵ_{ij} and Ω_{AB} are the invariant tensors of $SU(2)$ and $Sp(n_H)$, respectively. Regarding (iA) as a composite index, the inverse of the matrix $f_u{}^{iA}$ is $f_{iA}{}^u = h^{uv} f_v{}^{jB}\epsilon_{ji}\Omega_{BA}$. One can then construct a locally-defined triplet of almost complex structures

$$\vec{J}_u{}^v \equiv (J^x)_u{}^v = -i f_u{}^{iA} f_{jA}{}^v (\sigma^x)_i{}^j \tag{B.0.15}$$

where $x=1,2,3$ is in the adjoint of $SU(2)$ and $\vec{\sigma}$ are the Pauli matrices. The derived triplet of almost symplectic forms is $\vec{J}_{uv} = \vec{J}_u{}^t\,h_{tv}$. They are antisymmetric, using that $\vec{\sigma}_i{}^j\epsilon_{jk}$ is symmetric.[3] The almost complex structures automatically satisfy the quaternion relation

$$(J^x)_u{}^s (J^y)_s{}^t = -\delta^{xy}\delta^t_u + \epsilon^{xyz}(J^z)_u{}^t\;. \tag{B.0.17}$$

The Levi-Civita connection takes values in $\mathfrak{su}(2)\times\mathfrak{sp}(n_H)$. Calling $\omega_{uj}{}^i$ and $\rho_{uB}{}^A$ the two projections, respectively, they are determined by the requirement that $f_u{}^{iA}$ be covariantly

[2]The case $n_H = 1$ is special because $SU(2)^2/\mathbb{Z}_2 \cong SO(4)$ and so the holonomy condition does not impose any constraint on (orientable) Riemannian manifolds. However, supersymmetry requires (B.0.25) which we can take as the definition of a quaternionic-Kähler manifold of dimension 4. A 4-dimensional space satisfying (B.0.25) is Einstein with self-dual Weyl curvature.

[3]Using the fact that a 2×2 matrix can be expanded in the basis $\{\mathbb{1},\vec{\sigma}\}$, we also find

$$2f_u{}^{iA} f_{jA}{}^v = \delta^v_u\delta^i_j + i\,\vec{J}_u{}^v\cdot\vec{\sigma}_j{}^i\;. \tag{B.0.16}$$

constant with respect to the full connection:

$$0 = \nabla_v f_u{}^{iA} + f_u{}^{jA} \omega_{vj}{}^i + f_u{}^{iB} \rho_{vB}{}^A \, . \tag{B.0.18}$$

We can alternate between the vector and bispinor notations of $SU(2)$ with[4]

$$\vec{\omega}_u = -i\, \omega_{ui}{}^j\, \vec{\sigma}_j{}^i \, , \qquad\qquad \omega_{ui}{}^j = \frac{i}{2}\, \vec{\omega}_u \cdot \vec{\sigma}_i{}^j \, . \tag{B.0.20}$$

The two connections are extracted from (B.0.18) through: $\omega_{ui}{}^j\, \delta_A^B + \delta_i^j\, \rho_{uA}{}^B = -f_{iA}{}^w \nabla_u f_w{}^{jB}$. From (B.0.18) it immediately follows

$$\widetilde{\nabla}_w \vec{J}_u{}^v \equiv \nabla_w \vec{J}_u{}^v + \vec{\omega}_w \times \vec{J}_u{}^v = 0 \, . \tag{B.0.21}$$

In other words, $\vec{J}$ is covariantly constant with respect to its natural $SU(2)$ connection $\vec{\omega}$. From the integrability condition of (B.0.18) one also obtains (in bispinor and vector notation):

$$R_{uv}{}^s{}_t = \mathcal{R}_{uvi}{}^j\, f_{jA}{}^s f_t{}^{iA} + \mathcal{R}_{uvA}{}^B\, f_{jB}{}^s f_t{}^{jA} = -\frac{1}{2}\, \vec{\mathcal{R}}_{uv} \cdot \vec{J}_t{}^s + \mathcal{R}_{uvA}{}^B\, f_{jB}{}^s f_t{}^{jA} \, , \tag{B.0.22}$$

where $R_{uv}{}^s{}_t$ is the Riemann tensor of h_{uv} and we defined

$$\begin{aligned} \mathcal{R}_{uvi}{}^j &\equiv 2\partial_{[u} \omega_{v]i}{}^j - 2\omega_{[u|i}{}^k \omega_{v]k}{}^j \qquad\qquad \text{or} \qquad\qquad \vec{\mathcal{R}}_{uv} \equiv 2\partial_{[u} \vec{\omega}_{v]} + \vec{\omega}_u \times \vec{\omega}_v \\ \mathcal{R}_{uvA}{}^B &\equiv 2\partial_{[u} \rho_{v]A}{}^B - 2\rho_{[u|A}{}^C \rho_{v]C}{}^B \, . \end{aligned} \tag{B.0.23}$$

In particular

$$R_{uv}{}^s{}_t\, \vec{J}_s{}^t = 2 n_H \vec{\mathcal{R}}_{uv} \, , \tag{B.0.24}$$

i.e., the $SU(2)$ field strength $\vec{\mathcal{R}}_{uv}$ is the $\mathfrak{su}(2)$ projection of the Riemann curvature.

One can prove [153] (see also [152, 154]) that $SU(2) \times Sp(n_H)$ holonomy manifolds with $n_H \geq 2$ are automatically Einstein. In fact, they satisfy a stronger property: the Riemann curvature is the sum of the Riemann tensor of $\mathbb{HP}^{n_H}$ and of a Weyl part,

$$\begin{aligned} R_{uvst} = \frac{R}{8 n_H (n_H + 2)} \Big(h_{s[u} h_{v]t} + \vec{J}_{uv} \cdot \vec{J}_{st} - \vec{J}_{s[u} \cdot \vec{J}_{v]t} \Big) + \\ + \big(f_u{}^{iA} f_v{}^{jB} \epsilon_{ij} \big) \big(f_s{}^{kC} f_t{}^{\ell D} \epsilon_{k\ell} \big)\, \mathcal{W}_{ABCD} \, . \end{aligned} \tag{B.0.25}$$

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[4] The $SU(2)$ connection satisfies $\epsilon^{jm} \omega_{um}{}^n \epsilon_{ni} = \omega_{ui}{}^j$, in particular $\omega_{uj}{}^j = 0$, and a similar condition is satisfied by ρ. This follows from the properties of the Pauli matrices. In going between the vector and bispinor notation one can use the identities

$$\vec{\sigma}_n{}^m \cdot \vec{\sigma}_i{}^j = \delta_n^j \delta_i^m - \epsilon^{mj} \epsilon_{ni} \, , \qquad\qquad \vec{\sigma}_i{}^j \times \vec{\sigma}_\ell{}^m = i \big(\vec{\sigma}_i{}^m\, \delta_\ell^j - \delta_i^m\, \vec{\sigma}_\ell{}^j \big) \, . \tag{B.0.19}$$

The tensor $\mathcal{W}_{ABCD}$ is totally symmetric and controls the Weyl curvature, which is contained in $Sp(n_H)$: it gives rise to a traceless (and thus Ricci flat) contribution to the Riemann curvature. From that expression we obtain

$$R_{vt} = \frac{R}{4n_H}\, h_{vt} \;, \qquad\qquad \vec{\mathcal{R}}_{uv} = \frac{R}{4n_H(n_H+2)}\, \vec{J}_{uv} \;. \tag{B.0.26}$$

The first equation shows that the manifold is Einstein. The second equation shows that the $SU(2)$ part of the curvature is completely fixed in terms of the triplet of complex structures. The tensor $\mathcal{W}_{ABCD}$ expresses the freedom in the $Sp(n_H)$ part.

While quaternionic-Kähler manifolds can have any size, local supersymmetry requires[5]

$$\lambda \equiv \frac{R}{4n_H(n_H+2)} = -1 \;, \tag{B.0.27}$$

fixing the scalar curvature [152]. Hence the manifold of hypermultiplet scalars is a non-trivial quaternionic-Kähler manifold with negative scalar curvature.

Isometries and gauging. We consider gaugings of Abelian isometries of the quaternionic-Kähler manifold $\mathcal{QM}$ by the vectors A^I_μ. The isometries are generated by (possibly vanishing or linearly dependent) Killing vectors $k^u_I(q)$ that also satisfy a quaternionic version of the triholomorphic condition:

$$h_{w(u}\nabla_{v)}k^w_I = 0 \;, \qquad\qquad \vec{J}_u{}^w(\nabla_w k^v_I) - (\nabla_u k^w_I)\vec{J}_w{}^v = \lambda\, \vec{J}_u{}^v \times \vec{P}_I \;. \tag{B.0.28}$$

The second equation expresses the fact that the derivative of each Killing vector commutes with the triplet of complex structures, up to a rotation parametrized by the $SU(2)$ sections $\vec{P}_I$. Notice that the LHS can be written, after lowering v, as $2\widetilde{\nabla}_{[u}\big(\vec{J}_{v]s}k^s_I\big)$, therefore in the hyper-Kähler case that $\lambda = 0$ and the $SU(2)$ bundle is trivial, this reduces to the familiar condition that the three symplectic forms $\vec{J}_{uv}$ be preserved by the isometries. By taking the cross product of the second equation in (B.0.28) with $\vec{J}_v{}^u$ we obtain

$$2n_H\lambda\, \vec{P}_I = \vec{J}_u{}^v\, \nabla_v k^u_I \;. \tag{B.0.29}$$

This shows that on quaternionic Kähler manifolds, the sections $\vec{P}_I$ are completely fixed in terms of the Killing vectors. With a little bit of work[6] we obtain

$$\widetilde{\nabla}_u \vec{P}_I = \vec{J}_{uw}\, k^w_I \;. \tag{B.0.30}$$

[5] Had we chosen a canonical normalization for the action of hypermultiplet scalars, the scalar curvature would be fixed in terms of the Planck mass to $\lambda = -m_{\rm Pl}^{-2}$ [152]. This reproduces the fact that the manifold of hypermultiplet scalars is hyper-Kähler in rigid supersymmetry.

[6] We take the derivative $\widetilde{\nabla}$ of (B.0.29), recalling that $\vec{J}$ is covariantly constant. From the algebraic Bianchi identity we have $R_{uvst}\,\vec{J}^{us} = \frac{1}{2}R_{vt}{}^s{}_u\,\vec{J}_s{}^u = n_H\vec{\mathcal{R}}_{vt} = n_H\lambda\,\vec{J}_{vt}$. Then we use that the vectors are Killing, as well as the properties of quaternionic-Kähler manifolds.

This shows that $\vec{P}_I$ are a triplet of moment maps for the action of k_I^u. Taking a derivative and using that $2\widetilde{\nabla}_{[u}\widetilde{\nabla}_{v]}\vec{P}_I = \vec{\mathcal{R}}_{uv} \times \vec{P}_I$ we get back the second equation in (B.0.28), showing that the correction term on the RHS is unavoidable. The divergence of (B.0.30) gives

$$\widetilde{\nabla}^u\widetilde{\nabla}_u\vec{P}_I = -2n_H\lambda\vec{P}_I \ , \tag{B.0.31}$$

showing that the moment maps are eigenfunctions of the Laplacian.

Finally, let us consider for the moment the general case that the Killing vectors might form a non-Abelian group:

$$[k_I, k_J]^u = 2k^s_{[I}\nabla_s k^u_{J]} = f_{IJ}{}^K k^u_K \ , \tag{B.0.32}$$

where on the LHS is the Lie bracket and $f_{IJ}{}^K$ are the structure constants. Multiplying (B.0.28) by $\nabla_v k^u_J$ and using (B.0.29), and then exploiting the derivative ∇_w of (B.0.32), we obtain

$$k_I^u\,\vec{J}_{uv}\,k_J^v = f_{IJ}{}^K\vec{P}_K + \lambda\,\vec{P}_I \times \vec{P}_J \ . \tag{B.0.33}$$

This is called the equivariance relation. In the Abelian case we just set f to zero. In the special case $n_H = 0$ that there are no hypermultiplets, all Killing vectors vanish and the only remnant of the quaternionic-Kähler structure is the condition $\vec{P}_I \times \vec{P}_J = 0$. The solution, up to $SU(2)$ rotations, is $P^x_I = \delta^{x3}\zeta_I$ where ζ_I are the so-called Fayet-Iliopoulos (FI) parameters, which in this case are extra parameters one needs to specify.

We now have all the ingredients to write the covariant derivative

$$\mathcal{D}_\mu q^u = \partial_\mu q^u + g\,A^I_\mu k^u_I \ , \tag{B.0.34}$$

as well as the scalar potential

$$\begin{aligned} V &= P^x_I P^x_J \left(\frac{1}{2}\mathcal{G}^{ij}\partial_i\Phi^I\partial_j\Phi^J - \frac{2}{3}\Phi^I\Phi^J\right) + \frac{1}{2}h_{uv}\,k^u_I k^v_J\,\Phi^I\Phi^J \\ &= P^x_I P^x_J \left(\frac{1}{2}G^{IJ} - \Phi^I\Phi^J\right) + \frac{1}{2}h_{uv}\,k^u_I k^v_J\,\Phi^I\Phi^J \end{aligned} \tag{B.0.35}$$

that couples the scalars on $\mathcal{SM}$ and $\mathcal{QM}$. To go to the second line we used (B.0.10).

The covariant derivative of the supersymmetry parameter $\epsilon_i^{\rm SUSY}$ (subject to symplectic-Majorana condition, with $i = 1, 2$) is

$$D_\mu\epsilon_i^{\rm SUSY} = \left(\nabla_\mu\delta_i^j - \frac{i}{2}\,\vec{\mathcal{V}}_\mu\cdot\vec{\sigma}_i{}^j\right)\epsilon_j^{\rm SUSY} \tag{B.0.36}$$

with connection

$$\begin{aligned} \vec{\mathcal{V}}_\mu &= \mathcal{D}_\mu q^u\,\vec{\omega}_u - g\,A^I_\mu\vec{r}_I \qquad \text{and} \qquad \vec{r}_I = k^u_I\,\vec{\omega}_u - \lambda\,\vec{P}_I \ , \\ &= \partial_\mu q^u\,\vec{\omega}_u + g\lambda\,A^I_\mu\vec{P}_I \end{aligned} \tag{B.0.37}$$

where λ is the constant (B.0.27). Under gauge transformations[7]

$$\delta q^u = g\,\alpha^I k_I^u\,, \qquad\qquad \delta A_\mu^I = -\partial_\mu \alpha^I \tag{B.0.38}$$

with parameters α^I, using (B.0.26), (B.0.30) and (B.0.33) one can show that $\vec{\mathcal{V}}_\mu$ transforms as an $SU(2)$ connection:

$$\delta \vec{\mathcal{V}}_\mu = \partial_\mu \vec{\Lambda} + \vec{\mathcal{V}}_\mu \times \vec{\Lambda} \qquad \text{with} \qquad \vec{\Lambda} = g\,\alpha^I \vec{r}_I\,. \tag{B.0.39}$$

Therefore, $D_\mu \epsilon_i^{\rm SUSY}$ is covariant if $\epsilon_i^{\rm SUSY}$ transforms as

$$\delta\epsilon_i^{\rm SUSY} = \frac{i}{2}\,\vec{\Lambda}\cdot\vec{\sigma}_i{}^j\epsilon_j^{\rm SUSY}\,. \tag{B.0.40}$$

B.1 Conifold truncation in the general framework

Here we embed the consistent truncation of type IIB supergravity on $T^{1,1}$ to a 5d $\mathcal{N}=2$ gauged supergravity with a so-called "Betti multiplet", described in Section 7 of [89] (called the "second model" in that paper), in the general framework. The model has $n_V = 2$ and $n_H = 2$. We identify the fields

$$\phi^i = \begin{pmatrix} u+v \\ w \end{pmatrix}_{\rm CF}, \quad \Phi^I = \begin{pmatrix} e^{-4(u+v)/3} \\ -e^{2(u+v)/3}\cosh 2w \\ -e^{2(u+v)/3}\sinh 2w \end{pmatrix}_{\rm CF}, \quad A^I = \begin{pmatrix} A \\ a_1^J \\ a_1^\Phi \end{pmatrix}_{\rm CF}, \quad q^u = \begin{pmatrix} b_1^\Omega \\ b_2^\Omega \\ c_1^\Omega \\ c_2^\Omega \\ a \\ \phi \\ C_0 \\ u \end{pmatrix}_{\rm CF} \tag{B.1.1}$$

where "CF" indicates the notation of [89]. The scalar fields b^Ω, c^Ω are complex and we used $z_1 = \mathbb{R}\mathrm{e}(z)$, $z_2 = \mathbb{I}\mathrm{m}(z)$ to indicate their real and imaginary parts, while u, v, w, a, ϕ, C_0 are real. The hypermultiplet scalars C_0 and ϕ together form the type IIB axiodilaton $C_0 + ie^{-\phi}$. Then we identify the Chern-Simons couplings

$$C_{122} = -C_{133} = 2 \tag{B.1.2}$$

and symmetric permutations thereof, while all other components vanish, and the very special geometry of $SO(1,1)\times SO(1,1)$:

$$\mathcal{G}_{ij} = \begin{pmatrix} 4/3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad G_{IJ} = e^{-\frac{4}{3}(u+v)} \begin{pmatrix} \frac{1}{2}e^{4(u+v)} & 0 & 0 \\ 0 & \cosh(4w) & -\sinh(4w) \\ 0 & -\sinh(4w) & \cosh(4w) \end{pmatrix}. \tag{B.1.3}$$

[7] The covariant derivative transforms as $\delta\,\mathcal{D}_\mu q^u = g\,\alpha^I \mathcal{D}_\mu k_I^u$.

The tensor C^{IJK} has non-vanishing components $C^{122} = -C^{133} = 1/2$ and permutations.

The quaternionic-Kähler manifold is $\frac{SO(4,2)}{SO(4)\times SO(2)}$. Its metric is

$$\begin{aligned} h_{uv}dq^u dq^v = e^{-4u-\phi}db^\Omega d\overline{b^\Omega} + e^{-4u+\phi}\big(dc^\Omega - C_0 db^\Omega\big)\big(d\overline{c^\Omega} - C_0 d\overline{b^\Omega}\big) \\ + \frac{1}{2}e^{-8u}\Big(2da + \mathbb{R}\mathrm{e}\big(b^\Omega d\overline{c^\Omega} - c^\Omega d\overline{b^\Omega}\big)\Big)^2 + \frac{1}{2}d\phi^2 + \frac{1}{2}e^{2\phi}dC_0^2 + 8du^2 \,. \end{aligned} \tag{B.1.4}$$

In this normalization $R = -32$ and thus $\lambda = -1$. The $SU(2)$ connection is

$$\begin{aligned} \omega^1 - i\omega^2 &= e^{-2u-\phi/2}db^\Omega + i\, e^{-2u+\phi/2}\big(dc^\Omega - C_0 db^\Omega\big) \\ \omega^3 &= \frac{1}{2}e^{-4u}\Big(2da + \mathbb{R}\mathrm{e}\big(b^\Omega d\overline{c^\Omega} - c^\Omega d\overline{b^\Omega}\big)\Big) - \frac{1}{2}e^\phi dC_0 \,. \end{aligned} \tag{B.1.5}$$

Finally, we identify the Killing vectors

$$k_1 = 3\left(-b_2^\Omega\frac{\partial}{\partial b_1^\Omega} + b_1^\Omega\frac{\partial}{\partial b_2^\Omega} - c_2^\Omega\frac{\partial}{\partial c_1^\Omega} + c_1^\Omega\frac{\partial}{\partial c_2^\Omega}\right) + 2\frac{\partial}{\partial a}\,, \qquad k_2 = 2\frac{\partial}{\partial a}\,, \qquad k_3 = 0 \tag{B.1.6}$$

and the corresponding moment maps

$$P_1^x = \begin{pmatrix} 3e^{\phi/2-2u}(c_1^\Omega - C_0 b_1^\Omega + e^{-\phi}b_2^\Omega) \\ 3e^{\phi/2-2u}(C_0 b_2^\Omega - c_2^\Omega + e^{-\phi}b_1^\Omega) \\ 3 - e^{-4u}(2 + 3b_2^\Omega c_1^\Omega - 3b_1^\Omega c_2^\Omega) \end{pmatrix}\,, \qquad P_2^x = \begin{pmatrix} 0 \\ 0 \\ -2e^{-4u} \end{pmatrix}\,, \qquad P_3^x = 0\,. \tag{B.1.7}$$

The $SU(2)$ connection and the moment maps were given in [91] and can be translated into the notation of [89] (up to a conventional minus sign in the gauge fields) using the identifications

$$\phi^i = \begin{pmatrix} -3u_3 \\ u_2 \end{pmatrix}_{\rm HLS},\quad A^I = \begin{pmatrix} A_1 \\ \dfrac{k_{11}-k_{12}}{2} \\ \dfrac{k_{11}+k_{12}}{2} \end{pmatrix}_{\rm HLS},\quad q^u = \Big(2\,\mathbb{R}\mathrm{e}\, b_0^1, 2\,\mathbb{I}\mathrm{m}\, b_0^1, 2\,\mathbb{R}\mathrm{e}\, b_0^2, 2\,\mathbb{I}\mathrm{m}\, b_0^2, \frac{k}{2}, \phi, a, u_1\Big)^{\mathsf{T}}_{\rm HLS} \tag{B.1.8}$$

where "HLS" indicates the notation of [91].

The theory has a supersymmetric AdS$_5$ vacuum at $u = v = w = b^\Omega = c^\Omega = 0$ and any value of a, C_0, ϕ (in particular, the axiodilaton can take any value). The potential is $V\big|_{\rm AdS} = -6$ leading to AdS radius $\ell_5 = g^{-1}$. The spectrum therein was computed in [89] (see its Table 2). We are particularly interested in the spectrum of vector fields and the Killing vectors they couple to:

$$\begin{aligned} &A^R \equiv A^1 - 2A^2\,, \quad A^3 \;:\quad m^2 = 0\,, && A^W \equiv A^1 + A^2 \;:\quad m^2 = 24g^2\,. \\ &k_R = \tfrac{1}{3}(k_1 - k_2)\,, \quad k_3 && k_W = \tfrac{1}{3}(2k_1 + k_2) \end{aligned} \tag{B.1.9}$$

The vector A^W acquires a mass by Higgs mechanism, eating the Stückelberg scalar a. The mass eigenstates are

$$\mathbb{B}^I{}_J A^J_\mu \qquad \text{where} \qquad \mathbb{B} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{B.1.10}$$

is the matrix that diagonalizes them (see also Appendix E).

Appendix C

4d $\mathcal{N} = 2$ Abelian gauged supergravity

We summarize the salient features of 4d $\mathcal{N} = 2$ Abelian gauged supergravity with n_V vector multiplets and n_H hypermultiplets, following [114, 115, 150]. The graviton multiplet contains a graviton, two gravitini and a vector; each vector multiplet contains a vector, two gaugini and a complex scalar; each hypermultiplet contains four real scalars and two hyperini (all fermions can be taken Majorana). We use indices

$$\Lambda, \Sigma = 0, \ldots, n_V \, , \qquad i, j = 1, \ldots, n_V \, , \qquad u, v = 1, \ldots, 4n_H \tag{C.0.1}$$

for the gauge fields A_μ^Λ, for the complex scalars z^i in vector multiplets, and for the real scalars q^u in hypermultiplets, respectively. The data that define the theory are:

1. A special Kähler manifold $\mathcal{KM}$ of complex dimension n_V.
2. A quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$.
3. A set of $n_V + 1$ Killing vectors on $\mathcal{QM}$ compatible with the quaternionic-Kähler structure (if $n_H = 0$, $n_V + 1$ FI parameters not all vanishing).

The Killing vectors could be linearly dependent or vanish.

It is always possible to find a duality frame in which all gaugings are purely electric. In such frames the bosonic Lagrangian is

$$\begin{aligned} 8\pi G_{\mathrm{N}}^{(4)} e^{-1} \mathscr{L}_{4\mathrm{d}} &= \frac{R_s}{2} - g_{i\bar{j}}(z,\bar{z})\, \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - \frac{1}{2}\, h_{uv}(q)\, \mathcal{D}_\mu q^u \mathcal{D}^\mu q^v \\ &+ \frac{1}{8}\, \mathbb{I}\mathrm{m}\, \mathcal{N}_{\Lambda\Sigma}(z,\bar{z})\, F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} + \frac{e^{-1}}{16}\, \mathbb{R}\mathrm{e}\, \mathcal{N}_{\Lambda\Sigma}(z,\bar{z})\, F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma \epsilon^{\mu\nu\rho\sigma} - g^2 V(z,\bar{z},q) \, . \end{aligned} \tag{C.0.2}$$

The notation is mostly as in Appendix B. Let us explain the other terms.

Special Kähler geometry. The scalars z^i are complex coordinates on the special Kähler manifold $\mathcal{KM}$ [115]. This is a Kähler-Hodge manifold — *i.e.*, a Kähler manifold with Kähler potential $\mathcal{K}(z,\bar{z})$ and metric $g_{i\bar{j}}(z,\bar{z}) = \partial_i\partial_{\bar{j}}\mathcal{K}$ as well as a line bundle (*i.e.*, a holomorphic vector bundle of rank 1) $\mathcal{L}$ such that its first Chern class coincides (up to a constant) with the Kähler class $\omega = i\partial\bar{\partial}\mathcal{K}$ of the manifold[1] — further endowed with a flat $Sp(n_V+1,\mathbb{R})$ symplectic bundle. The manifold comes equipped with a covariantly-holomorphic section of the tensor product of the symplectic bundle with the $U(1)$-bundle $\mathcal{U}$ associated to $\mathcal{L}$,

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \qquad \text{such that} \qquad \begin{aligned} D_i\mathcal{V} &\equiv \partial_i\mathcal{V} + \tfrac{1}{2}(\partial_i\mathcal{K})\mathcal{V} \\ D_{\bar{\imath}}\mathcal{V} &\equiv \partial_{\bar{\imath}}\mathcal{V} - \tfrac{1}{2}(\partial_{\bar{\imath}}\mathcal{K})\mathcal{V} = 0\,, \end{aligned} \tag{C.0.3}$$

obeying the constraints

$$\langle \mathcal{V}, \overline{\mathcal{V}} \rangle \equiv M_\Lambda \overline{L}^\Lambda - L^\Lambda \overline{M}_\Lambda = -i \tag{C.0.4}$$

and

$$\langle \mathcal{V}, D_i\mathcal{V} \rangle = 0\,, \tag{C.0.5}$$

where we introduced the Sp-invariant antisymmetric form $i\langle\ ,\ \rangle$. Equivalently, there is a holomorphic section of the tensor product of the symplectic bundle with $\mathcal{L}$,[2]

$$v(z) = e^{-\mathcal{K}/2}\,\mathcal{V} \equiv \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \qquad \text{such that} \qquad \begin{aligned} D_i v &\equiv \partial_i v + (\partial_i\mathcal{K})\, v \\ D_{\bar{\imath}} v &\equiv \partial_{\bar{\imath}} v = 0\,, \end{aligned} \tag{C.0.6}$$

in terms of which the constraint (C.0.4) reads

$$\mathcal{K} = -\log\big(i\,\langle v, \bar{v}\rangle\big) = -\log\Big[2\,\mathbb{Im}\big(X^\Lambda \overline{F}_\Lambda\big)\Big]\,, \tag{C.0.7}$$

while the constraint (C.0.5) becomes $\langle v, D_i v\rangle = \langle v, \partial_i v\rangle = 0$. From (C.0.3)–(C.0.5) it is easy to prove the following properties (or equivalent ones written in terms of v):

$$\begin{aligned} &\langle D_i\mathcal{V}, \overline{\mathcal{V}}\rangle = 0\,, \qquad && D_{\bar{j}} D_i \mathcal{V} = g_{i\bar{j}}\,\mathcal{V}\,, \qquad && \langle D_i\mathcal{V}, D_{\bar{j}}\overline{\mathcal{V}}\rangle = i\,g_{i\bar{j}} \\ &\langle D_i\mathcal{V}, D_j\mathcal{V}\rangle = 0\,, && D_{[i}D_{j]}\mathcal{V} = 0 && \end{aligned} \tag{C.0.8}$$

from which the Kähler metric is extracted in a symplectic-invariant way.

The rescaling of X^Λ, F_Λ under Kähler transformations suggests to use X^Λ as homogeneous coordinates on $\mathcal{KM}$. It is always possible to find symplectic frames[3] in which the Jacobian matrix $e^\lambda{}_i(z) = \partial_i\big(X^\lambda/X^0\big)$ (with $\lambda = 1,\ldots,n_V$) is invertible. Notice that

$$\det\big(e^\lambda{}_i\big) = (X^0)^{n_V+1}\det\big(X^\Lambda, \partial_i X^\Lambda\big) = (X^0)^{n_V+1}\det\big(X^\Lambda, D_i X^\Lambda\big) \tag{C.0.9}$$

where the two square matrices on the right have size n_V+1, therefore the matrix $\big(X^\Lambda, \partial_i X^\Lambda\big)$ is invertible as well. Invertibility of the Jacobian ensures that we can use X^Λ as homogeneous

[1] Because fermions are sections of the square root of $\mathcal{L}$, the Kähler class of $\mathcal{KM}$ equal to the first Chern class of $\mathcal{L}$ is required to be an even integer cohomology class.

[2] In particular, $A = \partial\mathcal{K}$ is the Chern connection on $\mathcal{L}$. Moreover, $D_i\mathcal{V} = e^{\mathcal{K}/2}D_i v$ and $D_{\bar{\imath}}\mathcal{V} = e^{\mathcal{K}/2}D_{\bar{\imath}}v$.

[3] See [155] for examples of frames in which, instead, a prepotential does not exist.

coordinates, and regard $F_\Lambda(X)$ as homogeneous functions of degree 1, namely $X^\Sigma \partial_\Sigma F_\Lambda = F_\Lambda$. From (C.0.5) and (C.0.8), written as $\langle v, \partial_i v\rangle = \langle \partial_i v, \partial_j v\rangle = 0$, one obtains the equations

$$\left(X^\Lambda, \partial_i X^\Lambda\right) \partial_{[\Lambda} F_{\Sigma]} \left(X^\Sigma, \partial_j X^\Sigma\right) = 0 \; . \tag{C.0.10}$$

Invertibility of the matrix implies $\partial_{[\Lambda} F_{\Sigma]} = 0$. Hence, in these frames, the sections F_Λ are the derivatives of a holomorphic homogeneous function $F(X)$ of degree 2, called the *prepotential*, namely $F_\Lambda = \partial_\Lambda F$. In such frames, the Kähler potential and thus the geometry are completely specified by the prepotential. The coordinates $t^i \equiv X^i/X^0$ with $i = 1, \dots, n_V$ are called *special coordinates*.

The couplings of vector fields to the scalars z^i are determined by the $(n_V+1)\times(n_V+1)$ period matrix $\mathcal{N}$, which is uniquely defined by the relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma}\, L^\Sigma \; , \qquad\qquad D_{\bar\imath} \overline{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\, D_{\bar\imath} \overline{L}^\Sigma \; . \tag{C.0.11}$$

Explicitly, one needs to invert the matrix relation $\left(F_\Lambda, D_{\bar\imath}\overline{F}_\Lambda\right) = \mathcal{N}_{\Lambda\Sigma}\left(X^\Sigma, D_{\bar\imath}\overline{X}^\Sigma\right)$. The requirement that $g_{i\bar\jmath}$ be positive definite guarantees that the rightmost matrix is invertible [115]. Indeed, introducing the square matrix $\mathcal{L}^\Lambda{}_I = \left(L^\Lambda, D_{\bar\imath}\overline{L}^\Lambda\right)$ of size n_V+1, one can rewrite the scalar products in (C.0.4), (C.0.5) and (C.0.8) as

$$\mathcal{L}^\mathsf{T}\left(\mathcal{N} - \mathcal{N}^\mathsf{T}\right)\mathcal{L} = 0 \; , \qquad\qquad \mathcal{L}^\dagger\left(\mathcal{N} - \mathcal{N}^\dagger\right)\mathcal{L} = -i\,\mathrm{diag}\left(1, g_{i\bar\jmath}\right) \; . \tag{C.0.12}$$

The first equation shows that $\mathcal{N}_{\Lambda\Sigma}$ is a symmetric matrix, given the invertibility of $\mathcal{L}$. The second equation then, assuming that $g_{i\bar\jmath}$ is positive definite, proves that $\mathcal{L}$ is invertible and that $\mathbb{I}\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma}$ is negative definite. It also gives an expression for $\mathbb{I}\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma}$ that, after taking the inverse, reads

$$D_i L^\Lambda D_{\bar\jmath} \overline{L}^\Sigma g^{i\bar\jmath} + \overline{L}^\Lambda L^\Sigma = -\frac{1}{2}\Big(\left(\mathbb{I}\mathrm{m}\,\mathcal{N}\right)^{-1}\Big)^{\Lambda\Sigma} \; . \tag{C.0.13}$$

This relation, or the equivalent one in terms of the holomorphic section, will be used to rewrite the scalar potential below. When a prepotential exists, $\mathcal{N}$ is obtained from

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2i\, \frac{(\mathbb{I}\mathrm{m}\, F_{\Lambda\Gamma}) X^\Gamma\, (\mathbb{I}\mathrm{m}\, F_{\Sigma\Delta}) X^\Delta}{X^\Omega (\mathbb{I}\mathrm{m}\, F_{\Omega\Psi}) X^\Psi} \; , \tag{C.0.14}$$

where $F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F$. In this expression $\mathcal{N}$ is manifestly symmetric.

Finally, one can define the tensor

$$\widetilde{C}_{ijk} = \langle D_i D_j \mathcal{V}, D_k \mathcal{V}\rangle = \langle \mathcal{V}, D_k D_i D_j \mathcal{V}\rangle \; . \tag{C.0.15}$$

Using (C.0.3)–(C.0.8) and the fact that the metric is Kähler, one easily proves that $\widetilde{C}_{ijk}$ is totally symmetric and covariantly holomorphic, $D_{\bar\ell}\widetilde{C}_{ijk} = 0$ where $\widetilde{C}$ has twice the charge of $\mathcal{V}$. One can prove that $(\mathcal{V}, D_i\mathcal{V}, \overline{\mathcal{V}}, D_{\bar{\imath}}\overline{\mathcal{V}})$ pointwise form a basis for the symplectic bundle [115], hence

$$D_i D_j \mathcal{V} = i\,\widetilde{C}_{ijk} g^{k\bar{k}} D_{\bar{k}} \overline{\mathcal{V}} \tag{C.0.16}$$

follows by taking the product of the LHS with the basis. Among other things, $\widetilde{C}$ controls the curvature tensor: $R_{\bar{\imath}j\bar{k}\ell} = g_{j\bar{\imath}} g_{\ell\bar{k}} + g_{j\bar{k}} g_{\ell\bar{\imath}} - \widetilde{C}_{j\ell m} \widetilde{C}_{\bar{\imath}\bar{k}\bar{n}} g^{m\bar{n}}$. In special coordinates the tensor $\widetilde{C}$ takes the particularly simple form

$$\widetilde{C}_{ijk} = e^{\mathcal{K}}\, \partial_i \partial_j \partial_k \mathcal{F}(t) \qquad \text{with} \qquad \mathcal{F}(t) = (X^0)^{-2} F(X) \tag{C.0.17}$$

and $t^i = X^i/X^0$.

Hypermultiplets and gauging. The part of the action involving the hypermultiplets has the same features as in the 5d case, summarized in Appendix B: the hypermultiplet scalars q^u (with $u = 1, \ldots, 4n_H$) are coordinates on a quaternionic-Kähler manifold $\mathcal{QM}$ with metric $h_{uv}(q)$. As before, we consider gauging of Abelian isometries of $\mathcal{QM}$, generated by $n_V + 1$ (possibly vanishing or linearly dependent) Killing vectors $k^u_\Lambda(q)$ that must be compatible with the quaternionic-Kähler structure, with associated triplets of moment maps $\vec{P}_\Lambda(q)$. In full generality one could consider both electric and magnetic gaugings, described by Killing vectors k^u_Λ and $k^{u\Lambda}$, respectively, and transforming as a vector under $Sp(n_V + 1, \mathbb{R})$ duality transformations. It is always possible to find a duality frame in which all gaugings are purely electric, and we will work in such a frame. Notice that there is no guarantee that in this frame a prepotential exists.

The scalar potential is

$$\begin{aligned} V &= 2\, P^x_\Lambda P^x_\Sigma\, e^{\mathcal{K}} \Big(g^{i\bar{\jmath}} D_i X^\Lambda D_{\bar{\jmath}} \overline{X}^\Sigma - 3 X^\Lambda \overline{X}^\Sigma \Big) + 4\, e^{\mathcal{K}} h_{uv}\, k^u_\Lambda k^v_\Sigma X^\Lambda \overline{X}^\Sigma \\ &= -P^x_\Lambda P^x_\Sigma \Big(\big(\mathbb{I}\mathrm{m}\, \mathcal{N}\big)^{-1\,\Lambda\Sigma} + 8\, e^{\mathcal{K}} X^\Lambda \overline{X}^\Sigma \Big) + 4\, e^{\mathcal{K}} h_{uv}\, k^u_\Lambda k^v_\Sigma X^\Lambda \overline{X}^\Sigma \,. \end{aligned} \tag{C.0.18}$$

To go to the second line we used (C.0.13).

The covariant derivative of the supersymmetry parameter $\epsilon_i^{\mathrm{SUSY}}$ (subject to symplectic-Majorana condition, with $i = 1, 2$) is

$$D_\mu \epsilon_i^{\mathrm{SUSY}} = \left(\nabla_\mu \delta_i^j - \frac{i}{2}\, \mathcal{A}_\mu \delta_i^j - \frac{i}{2}\, \vec{\mathcal{V}}_\mu \cdot \vec{\sigma}_i{}^j \right) \epsilon_j^{\mathrm{SUSY}} \tag{C.0.19}$$

with connections

$$\begin{aligned} \vec{\mathcal{V}}_\mu &= \partial_\mu q^u\, \vec{\omega}_u + g\lambda\, A^I_\mu \vec{P}_I \\ \mathcal{A}_\mu &= \frac{i}{2} \lambda \Big[(\partial_\alpha \mathcal{K}) \partial_\mu z^\alpha - (\partial_{\bar{\alpha}} \mathcal{K}) \partial_\mu \bar{z}^{\bar{\alpha}} \Big] \,. \end{aligned} \tag{C.0.20}$$

Here $\vec{\mathcal{V}}_\mu$ is the $SU(2)$ connection that descends from the quaternionic-Kähler manifold $\mathcal{QM}$, as in the 5d case (B.0.37). Instead $\mathcal{A}_\mu$ descends from the connection on the $U(1)$-bundle $\mathcal{U}$ on the special Kähler manifold $\mathcal{KM}$.

Appendix D

Supergravity reduction with background gauge fields

Following [119] we will now reduce, piece by piece, the bosonic Lagrangian (B.0.2) of 5d $\mathcal{N} = 2$ gauged supergravity down to 4d. We start in 5d with n_V vector multiplets and n_H hypermultiplets. We use indices

$$I, J = 1, \dots, n_V + 1 \,, \qquad \Lambda, \Sigma = 0, \dots, n_V + 1 \,, \qquad u, v = 1, \dots, 4n_H \,. \tag{D.0.1}$$

We indicate the 5d vector fields as $\widehat{A}^I_M$ (where $M, N = 0, \dots, 4$ are spacetime indices) and parametrize the vector multiplet scalars in terms of sections Φ^I subject to the cubic constraint $\mathcal{V}(\Phi) = 1$ in (B.0.3). The hypermultiplet scalars are q^u. We employ the rather standard Kaluza-Klein reduction ansatz (1.4.2) and (1.4.3):

$$\begin{gathered}
\widehat{g}_{MN} = \begin{pmatrix} e^{2\widetilde{\phi}} g_{\mu\nu} + e^{-4\widetilde{\phi}} A^0_\mu A^0_\nu & -e^{-4\widetilde{\phi}} A^0_\mu \\ -e^{-4\widetilde{\phi}} A^0_\nu & e^{-4\widetilde{\phi}} \end{pmatrix} , \qquad \widehat{g}^{MN} = \begin{pmatrix} e^{-2\widetilde{\phi}} g^{\mu\nu} & e^{-2\widetilde{\phi}} A^{0\mu} \\ e^{-2\widetilde{\phi}} A^{0\nu} & e^{4\widetilde{\phi}} + e^{-2\widetilde{\phi}} A^0_\rho A^{0\rho} \end{pmatrix} , \\
e_{(5)} = e^{2\widetilde{\phi}}\, e_{(4)} \,, \qquad \Phi^I = -e^{2\widetilde{\phi}}\, z_2^I \,, \qquad \widehat{A}^I_M = \big(A^I_\mu - z_1^I A^0_\mu ,\, z_1^I + \xi^I \big) \,.
\end{gathered} \tag{D.0.2}$$

The last coordinate, that we call y and whose range Δy we leave generic for now, is compactified on a circle of length $e^{-2\widetilde{\phi}} \Delta y$, and no field depends on it. We indicated as $\widehat{g}_{MN}$ and $e_{(5)}$ the 5d metric and the square root of its determinant, and as $g_{\mu\nu}$ and $e_{(4)}$ (with $\mu, \nu = 0, \dots, 3$ spacetime indices) their 4d counterparts. In 4d we end up with $n_V + 1$ vector multiplets, and we indicate as A^Λ_μ the vector fields. The physical scalars in 4d vector multiplets are the complex fields z^i. With a useful abuse of notation, we utilize the very same index I for 5d vector fields and 4d physical scalars, z^I, because in 4d we have one more vector field than in 5d. We also use the notation

$$z_1^I \equiv \mathbb{R}\mathrm{e}\, z^I \,, \qquad z_2^I \equiv \mathbb{I}\mathrm{m}\, z^I \,. \tag{D.0.3}$$

Notice that the real scalar $\widetilde{\phi}$ can be eliminated with the 5d constraint,

$$e^{-6\widetilde{\phi}} = -\mathcal{V}(z_2) \,, \tag{D.0.4}$$

then the scalars z^I can be treated as independent. The real parameters ξ^I represent background gauge fields along the circle, therefore, up to a gauge transformation, this ansatz is equivalent to a Scherk-Schwarz reduction.

The reduction of the Einstein term gives

$$8\pi G_{\rm N}^{(4)}\mathscr{L}_1 = e_{(5)}\,\frac{\widehat{R}_s}{2} = e_{(4)}\left[\frac{R_s}{2} - 3\,\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi} - \frac{e^{-6\widetilde{\phi}}}{8}\,F^0_{\mu\nu}F^{0\mu\nu}\right] + \text{total derivatives}\,. \tag{D.0.5}$$

Here $\widehat{R}_s$ and R_s are the 5d and 4d Ricci scalars, respectively. The 4d and 5d Newton constants are related by

$$\frac{1}{G_{\rm N}^{(4)}} = \frac{\Delta y}{G_{\rm N}^{(5)}}\,. \tag{D.0.6}$$

In the following, for clarity, we will omit the factor $8\pi G_{\rm N}^{(4)}$. The reduction of the kinetic term of vector multiplet scalars gives

$$\mathscr{L}_2 = -e_{(5)}\frac{1}{2}\,G_{IJ}\,\widehat{g}^{MN}\partial_M\Phi^I\partial_N\Phi^J = e_{(4)}\left[-\frac{e^{4\widetilde{\phi}}}{2}\,G_{IJ}\partial_\mu z_2^I\partial^\mu z_2^J + 3\,\partial_\mu\widetilde{\phi}\,\partial^\mu\widetilde{\phi}\right]. \tag{D.0.7}$$

The last term exactly cancels the second term in $\mathscr{L}_1$, therefore

$$\mathscr{L}_1 + \mathscr{L}_2 = e_{(4)}\left[\frac{R_s}{2} - \frac{e^{4\widetilde{\phi}}}{2}\,G_{IJ}\,\partial_\mu z_2^I\,\partial^\mu z_2^J - \frac{e^{-6\widetilde{\phi}}}{8}\,F^0_{\mu\nu}F^{0\mu\nu}\right]. \tag{D.0.8}$$

The reduction of the kinetic term of hypermultiplet scalars gives

$$\begin{aligned}\mathscr{L}_3 &= -e_{(5)}\frac{1}{2}\,h_{uv}\,\widehat{g}^{MN}\widehat{\mathcal{D}}_Mq^u\widehat{\mathcal{D}}_Nq^v\\ &= e_{(4)}\left[-\frac{1}{2}\,h_{uv}\,\mathcal{D}_\mu q^u\mathcal{D}^\mu q^v - \frac{g^2e^{6\widetilde{\phi}}}{2}\,\big(k_0^u + z_1^Ik_I^u\big)h_{uv}\big(k_0^v + z_1^Jk_J^v\big)\right].\end{aligned} \tag{D.0.9}$$

Here $\widehat{\mathcal{D}}_Mq^u = \partial_Mq^u + g\,\widehat{A}^I_Mk_I^u$ is the 5d covariant derivative in (B.0.34), while

$$\mathcal{D}_\mu q^u = \partial_\mu q^u + g\,A^I_\mu k_I^u + g\,A^0_\mu\,\xi^Ik_I^u = \partial_\mu q^u + g\,A^\Lambda_\mu k_\Lambda^u \tag{D.0.10}$$

is the 4d covariant derivative, and we defined the new Killing vector

$$k_0^u \;\equiv\; \xi^Ik_I^u\,. \tag{D.0.11}$$

The reduction of the gauge kinetic term gives

$$\begin{aligned}\mathscr{L}_4 &= -e_{(5)}\frac{1}{4}\,G_{IJ}\,\widehat{F}^I_{MN}\widehat{F}^{JMN}\\ &= e_{(4)}\left[-\frac{e^{-2\widetilde{\phi}}}{4}\,G_{IJ}\big(F^I_\mu - z_1^IF^0_{\mu\nu}\big)\big(F^{J\mu\nu} - z_1^JF^{0\mu\nu}\big) - \frac{e^{4\widetilde{\phi}}}{2}\,G_{IJ}\,\partial_\mu z_1^I\partial^\mu z_1^J\right],\end{aligned} \tag{D.0.12}$$

where $\widehat{F}_{MN}$ and $F_{\mu\nu}$ are the 5d and 4d field strengths, respectively. We used $\widehat{F}^I_{\mu4} = \partial_\mu z_1^I$ and $\widehat{F}^I_{\mu\nu} = F^I_{\mu\nu} - z_1^IF^0_{\mu\nu} + 2A^0_{[\mu}\partial_{\nu]}z_1^I$.

In order to reduce the Chern-Simons term, we extend the geometry (1.4.2) to a 6d bulk whose boundary is the original 5d space. A convenient way to do that is to complete the circle parametrized by y into a unit disk with radius $\rho \in [0,1]$. We extend the 5d connections $\widehat{A}^I$ in (1.4.3) to 6d connections $\widetilde{A}^I$ as follows:

$$\widetilde{A}^I = A^I + \xi^I A^0 + \rho^2 (z_1^I + \xi^I)\big(dy - A^0\big) \ . \tag{D.0.13}$$

We then write the Chern-Simons action term as

$$\int_{5\text{d}} \mathscr{L}_5 = \int_{5\text{d}} \frac{1}{12}\, C_{IJK}\, \widehat{F}^I \wedge \widehat{F}^J \wedge \widehat{A}^K = \int_{6\text{d}} \frac{1}{12}\, C_{IJK}\, \widetilde{F}^I \wedge \widetilde{F}^J \wedge \widetilde{F}^K \ . \tag{D.0.14}$$

Substituting $\widetilde{F}^I = d\widetilde{A}^I$ and performing the integrals over $d\rho^2 \wedge (dy - A^0)$, we extract the 4d reduced Lagrangian

$$\mathscr{L}_5 = \frac{1}{16}\, C_{IJK} \epsilon^{\mu\nu\rho\sigma} \left[\big(z_1^I + \xi^I\big) F_{\mu\nu}^J F_{\rho\sigma}^K - \big(z_1^I z_1^J - \xi^I \xi^J\big) F_{\mu\nu}^K F_{\rho\sigma}^0 + \frac{z_1^I z_1^J z_1^K + \xi^I \xi^J \xi^K}{3} F_{\mu\nu}^0 F_{\rho\sigma}^0 \right] . \tag{D.0.15}$$

Notice that the contributions containing the ξ^I's are standard theta terms.

Finally, the reduction of the scalar potential gives

$$\mathscr{L}_6 = -e_{(5)} g^2 V = -e_{(4)} g^2 \left[P_I^x P_J^x \left(\frac{e^{2\widetilde{\phi}}}{2} \mathcal{G}^{ij} \partial_i \Phi^I \partial_j \Phi^J - \frac{2 e^{6\widetilde{\phi}}}{3} z_2^I z_2^J \right) + \frac{e^{6\widetilde{\phi}}}{2} h_{uv} k_I^u k_J^v z_2^I z_2^J \right] . \tag{D.0.16}$$

We proceed now with recasting the various pieces of the action in the general form (C.0.2) of 4d $\mathcal{N} = 2$ gauged supergravity with $n_V + 1$ vector multiplets and n_H hypermultiplets. The Einstein term receives its contribution from $\mathscr{L}_1$:

$$\mathscr{L}_1' = e_{(4)}\, \frac{R_s}{2} \ . \tag{D.0.17}$$

The kinetic term of vector multiplet scalars gets contributions from $\mathscr{L}_2$ and $\mathscr{L}_4$:

$$\mathscr{L}_2' = -e_{(4)} \frac{e^{4\widetilde{\phi}}}{2}\, G_{IJ} \Big(\partial_\mu z_2^I \partial^\mu z_2^J + \partial_\mu z_1^I \partial^\mu z_1^J \Big) = -e_{(4)}\, g_{I\bar{J}}\, \partial z^I \partial^\mu \bar{z}^{\bar{J}} \ , \tag{D.0.18}$$

where we defined the Hermitian metric

$$g_{I\bar{J}} = \frac{e^{4\widetilde{\phi}}}{2}\, G_{I\bar{J}} \ . \tag{D.0.19}$$

The kinetic term of hypermultiplet scalars gets its contribution from $\mathscr{L}_3$,

$$\mathscr{L}_3' = -e_{(4)}\, \frac{1}{2}\, h_{uv}\, \mathcal{D}_\mu q^u \mathcal{D}^\mu q^v \ , \tag{D.0.20}$$

with the covariant derivative $\mathcal{D}_\mu$ defined in (D.0.10)-(D.0.11). The gauge kinetic term receives contributions from $\mathscr{L}_1$ and $\mathscr{L}_4$:

$$\mathscr{L}_4' = -e_{(4)} \frac{e^{-6\widetilde{\phi}}}{8} \left[F_{\mu\nu}^0 F^{0\mu\nu} + 4 g_{IJ} \big(F_{\mu\nu}^I - z_1^I F_{\mu\nu}^0\big)\big(F^{J\mu\nu} - z_1^J F^{0\mu\nu}\big) \right] = e_{(4)}\, \frac{1}{8}\, \mathbb{I}\mathrm{m}\, \mathcal{N}_{\Lambda\Sigma}\, F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} \tag{D.0.21}$$

where we defined the field-dependent matrix of gauge couplings

$$\mathbb{I}\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma} = -e^{-6\widetilde{\phi}} \begin{pmatrix} 1+4g_{KL}z_1^K z_1^L & -4g_{KJ}z_1^K \\ -4g_{IK}z_1^K & 4g_{IJ} \end{pmatrix} \tag{D.0.22}$$

in which the indices Λ, Σ run over 0 and then the values of I, J. On the other hand, the field-dependent theta terms are contained in $\mathscr{L}_5$:

$$\mathscr{L}_5' = \mathscr{L}_5 = \frac{1}{16}\,\mathbb{R}\mathrm{e}\,\mathcal{N}_{\Lambda\Sigma}\,\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma} \tag{D.0.23}$$

where

$$\mathbb{R}\mathrm{e}\,\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3}C_{KLM}\big(z_1^K z_1^L z_1^M + \xi^K\xi^L\xi^M\big) & -\frac{1}{2}C_{JKL}\big(z_1^K z_1^L - \xi^K\xi^L\big) \\ -\frac{1}{2}C_{IKL}\big(z_1^K z_1^L - \xi^K\xi^L\big) & C_{IJK}\big(z_1^K + \xi^K\big) \end{pmatrix}. \tag{D.0.24}$$

It turns out that $g_{I\bar{J}}$ and $\mathcal{N}_{\Lambda\Sigma}$ descend from the following prepotential:

$$\begin{aligned} F(X) &= \frac{1}{6}\,C_{IJK}\,\frac{\check{X}^I\check{X}^J\check{X}^K}{X^0} \qquad\qquad \text{with } \check{X}^I \equiv X^I + \xi^I X^0 \\ &= \frac{1}{6}\,C_{IJK}\,\frac{X^IX^JX^K}{X^0} + \frac{1}{2}C_{IJK}\Big(\xi^IX^JX^K + \xi^I\xi^JX^KX^0 + \frac{1}{3}\xi^I\xi^J\xi^K(X^0)^2\Big)\,. \end{aligned} \tag{D.0.25}$$

The terms in parenthesis involving the ξ^I's only affect standard theta terms, which are topological and thus do not enter in the equations of motion. Indeed, using special coordinates $z^I = X^I/X^0$ and in the Kähler frame $|X^0|^2 = 1$, one derives the Kähler potential[1]

$$\mathcal{K} = -\log\Big(\frac{1}{6i}\,C_{IJK}\,\big(z^I-\bar{z}^I\big)\big(z^J-\bar{z}^J\big)\big(z^K-\bar{z}^K\big)\Big) = -\log\big(8\,e^{-6\widetilde{\phi}}\big) \tag{D.0.26}$$

from which the Kähler metric (D.0.19) with (B.0.6) follows. On the other hand

$$F_{\Lambda\Sigma} = \begin{pmatrix} \frac{1}{3}C_{KLM}\big(z^Kz^Lz^M + \xi^K\xi^L\xi^M\big) & -\frac{1}{2}C_{JKM}\big(z^Kz^M - \xi^K\xi^L\big) \\ -\frac{1}{2}C_{IKM}\big(z^Kz^M - \xi^K\xi^L\big) & C_{IJK}\big(z^K + \xi^K\big) \end{pmatrix} \tag{D.0.27}$$

from which the matrix $\mathcal{N}$ in (D.0.22) and (D.0.24) follows. It might be useful

$$\begin{aligned} (X^0)^{-2}X^\Lambda\big(\mathbb{I}\mathrm{m}\,F_{\Lambda\Sigma}\big)X^\Sigma &= 4\,C_{IJK}\Big(\tfrac{1}{3}\,\mathbb{I}\mathrm{m}(z^Iz^Jz^K) - \tfrac{1}{2}\,\mathbb{I}\mathrm{m}(z^Iz^J)\,\mathbb{R}\mathrm{e}(z^K)\Big) \\ &= -\frac{4}{3}\,C_{IJK}z_2^Iz_2^Jz_2^K = e^{-\mathcal{K}} = 8\,e^{-6\widetilde{\phi}}\,, \end{aligned} \tag{D.0.28}$$

as well as $\big(\mathbb{I}\mathrm{m}\,F_{I\Sigma}\big)X^\Sigma/X^0 = i\,C_{IKM}z_2^Kz_2^M$.

[1] The completely covariant expression for the Kähler potential is $e^{-\mathcal{K}} = 8\,|X^0|^2\,e^{-6\widetilde{\phi}}$.

Finally, the scalar potential gets contributions from $\mathscr{L}_3$ and $\mathscr{L}_6$:

$$
\begin{aligned}
\mathscr{L}_6' &= -e_{(4)} g^2 \Bigg[P_I^x P_J^x \bigg(\frac{e^{2\widetilde{\phi}}}{2} \mathcal{G}^{ij} \partial_i \Phi^I \partial_j \Phi^J - \frac{2e^{6\widetilde{\phi}}}{3} z_2^I z_2^J \bigg) + \\
&\qquad\qquad + \frac{e^{6\widetilde{\phi}}}{2} h_{uv} \Big(k_I^u k_J^v z_2^I z_2^J + (k_0^u + z_1^I k_I^u)(k_0^v + z_1^J k_J^v) \Big) \Bigg] \\
&= -e_{(4)} g^2 \Big[-P_\Lambda^x P_\Sigma^x \Big((\mathbb{I}\mathrm{m}\,\mathcal{N})^{-1\,\Lambda\Sigma} + 8\, e^{\mathcal{K}} X^\Lambda \overline{X}^\Sigma \Big) + 4\, e^{\mathcal{K}} h_{uv} k_\Lambda^u k_\Sigma^v X^\Lambda \overline{X}^\Sigma \Big] \,.
\end{aligned}
\tag{D.0.29}
$$

To manipulate the first line we used (B.0.10) as well as

$$
\Big((\mathbb{I}\mathrm{m}\,\mathcal{N})^{-1} \Big)^{\Lambda\Sigma} + 8\, e^{\mathcal{K}} X^{(\Lambda} \overline{X}^{\Sigma)} = -e^{6\widetilde{\phi}} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} g^{IJ} - z_2^I z_2^J \end{pmatrix} \,, \tag{D.0.30}
$$

which immediately follows from (D.0.22). Notice in particular that $\vec{P}_0$ drops out of the potential and cannot be extracted from it, but it is still determined as $\vec{P}_0 = \xi^I \vec{P}_I$ from (D.0.11). The action $\mathscr{L}_6'$ exactly reproduces the potential in (C.0.18).

Summarizing, the compactification gives the following map from 5d to 4d data:

$$
\begin{array}{ccc}
\text{5d} & & \text{4d} \\
n_V \text{ vector multiplets} & & n_V + 1 \text{ vector multiplets} \\
\mathcal{SM} \text{ with } C_{IJK} & \xrightarrow[\text{background fields}]{\text{reduction with } \xi^I} & \mathcal{KM} \text{ with } F = \dfrac{1}{6} C_{IJK} \dfrac{\check{X}^I \check{X}^J \check{X}^K}{X^0} \\
\mathcal{QM} \text{ with } h_{uv}(q) & & \mathcal{QM} \text{ with } h_{uv}(q) \\
\text{gauging of } k_I^u & & \text{gauging of } k_\Lambda^u = \big(\xi^J k_J^u, k_I^u \big)
\end{array}
\tag{D.0.31}
$$

where $\check{X}^I = X^I + \xi^I X^0$.

D.1 Reduction of the conifold truncation

The reduction of the 5d conifold truncation described in Appendix B.1 gives a 4d supergravity with the following data. The prepotential is

$$
F = \frac{\check{X}^1 \big((\check{X}^2)^2 - (\check{X}^3)^2 \big)}{X^0} \,. \tag{D.1.1}
$$

It induces the vector multiplet scalar metric

$$g_{I\bar{J}} = \frac{1}{2}\begin{pmatrix} \frac{1}{2(z_2^1)^2} & 0 & 0 \\ & \frac{(z_2^2)^2+(z_2^3)^2}{\left((z_2^2)^2-(z_2^3)^2\right)^2} & -\frac{2\,z_2^2\,z_2^3}{\left((z_2^2)^2-(z_2^3)^2\right)^2} \\ & \text{Symmetrized} & \frac{(z_2^2)^2+(z_2^3)^2}{\left((z_2^2)^2-(z_2^3)^2\right)^2} \end{pmatrix} \tag{D.1.2}$$

that depends on z_2^I, the theta terms (D.0.24) that depend on z_1^I and ξ^I, while the gauge coupling function $\mathbb{I}\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma}$ takes a lengthier expression that depends on z_1^I and z_2^I and can be easily derived from (D.0.22). Since in 5d $k_3 = 0$, the 4d extra Killing vector is $k_0 = \xi^1 k_1 + \xi^2 k_2$.

Appendix E

Black hole charges and their reduction

The electric black hole charges computed in [73] in our notation read

$$Q_{\mathfrak{T}} = -\frac{1}{8\pi G_{\rm N}^{(5)} g}\int_{S^3_\infty} G_{\mathfrak{T}J}\star_5 \widehat{F}^J \ , \tag{E.0.1}$$

where the integral is taken on the three-sphere at infinity, and are dimensionless. We recall that only a subspace $\widehat{A}^{\mathfrak{T}}_\mu$ of the vector fields are massless on the AdS$_5$ vacuum, and the index $\mathfrak{T}$ runs over them. The massless vectors are such that the hypermultiplet scalars sit at a fixed point of the gauged isometries, and are thus identified by the conditions

$$k^u_{\mathfrak{T}}(q) = 0 \ . \tag{E.0.2}$$

Indeed, let $\mathbb{B}^I{}_J$ be a matrix of linear redefinitions such that $\mathbb{B}^I{}_J\widehat{A}^J_\mu$ are mass eigenstates. Such a matrix is characterized by $\mathbb{B}^I{}_J G^{JN} k^u_N h_{uv} k^v_L = \lambda^I_N \mathbb{B}^N{}_L$ where λ is the diagonal matrix of squared masses (in units of g^2). The corresponding linear transformation of charges is $Q_I \to Q_J(\mathbb{B}^{-1})^J{}_I$, while the Killing vectors corresponding to the mass eigenstates are $k^u_J(\mathbb{B}^{-1})^J{}_I$. Now consider a massless vector and let the index $\mathfrak{T}$ be such that $\lambda^{\mathfrak{T}}_{\mathfrak{T}} = 0$ (not summed over $\mathfrak{T}$). Using non-degeneracy of the metrics G_{IJ} and h_{uv}, one easily proves that $k^u_J(\mathbb{B}^{-1})^J{}_{\mathfrak{T}} = 0$, which is (E.0.2).

Now, the equations of motion for the bosonic fields of 5d gauged supergravity that follow from (B.0.2) are

$$\begin{aligned} d\Big(G_{IJ}\star_5 \widehat{F}^J\Big) &= \frac{1}{4}C_{IJK}\widehat{F}^J\wedge\widehat{F}^K - g\, h_{uv}\, k^u_I \star_5 \widehat{\mathcal{D}}q^v \\ \widehat{R}_{MN} &= G_{IJ}\bigg(\widehat{F}^I_{MP}\widehat{F}^{JP}_N - \frac{1}{6}\,\widehat{g}_{MN}\widehat{F}^I_{PQ}\widehat{F}^{JPQ}\bigg) \\ &\qquad + \mathcal{G}_{ij}\,\partial_M\phi^i\partial_N\phi^j + h_{uv}\,\widehat{\mathcal{D}}_M q^u\widehat{\mathcal{D}}_N q^v + \frac{2}{3}\,\widehat{g}_{MN}\, g^2 V \ . \end{aligned} \tag{E.0.3}$$

Notice that (E.0.2) is just the condition not to have a source in the $\mathfrak{T}$-th component of Maxwell's equation from the hypermultiplets. We can express the charges $\mathcal{Q}_{\mathfrak{T}}$ in terms of integrals at the horizon [156] using the EOMs (E.0.3):

$$Q_{\mathfrak{T}} = -\frac{1}{8\pi G_{\rm N}^{(5)} g}\bigg[\int_{S^3_r} G_{\mathfrak{T}J}\star_5\widehat{F}^J + \int_{S^3_r\times I[r,\infty]}\bigg(\frac{1}{4}\,C_{\mathfrak{T}JK}\,\widehat{F}^J\wedge\widehat{F}^K - g\,h_{uv}\,k^u_{\mathfrak{T}}\star_5\widehat{\mathcal{D}}q^v\bigg)\bigg] \ . \tag{E.0.4}$$

The first term is an integral evaluated at radius r, that we will take to be the horizon location. The second term is a correction, integrated on a cylinder $S^3 \times I$ where I is the interval from r to ∞, that leads to a Page charge. Assuming that the condition $k^u_{\mathfrak{I}}(q) = 0$ remains true also on the black hole background,[1] the third term vanishes.

We can apply a similar manipulation to the angular momenta $J_{a=1,2}$. Given the spacetime Killing vectors $K_a \equiv K_a^M \partial_M$, the angular momenta are defined in [73] as

$$J_a = \frac{1}{16\pi G_{\rm N}^{(5)}} \int_{S^3_\infty} \star_5 \, dK_a \tag{E.0.5}$$

where we have indicated with the same symbol $K_a \equiv K_{aM} dx^M$ the 1-forms dual to the Killing vectors, and the integral is evaluated once again at infinity. One can show that the Killing equation implies

$$d \star_5 dK = 2\widehat{R}_{MN} K^M \star_5 dx^N \ . \tag{E.0.6}$$

We can then use the EOMs (E.0.3) to replace the Ricci scalar $\widehat{R}_{MN}$. We assume that S^3 is invariant under the isometries generated by K_a, therefore, indicating as i_K the interior product, the integral of $\widehat{g}_{MN} K^M \star_5 dx^N = \mathrm{i}_K(\star_5 1)$ vanishes. We also assume that $\mathrm{i}_K d\phi^i = 0$. We obtain

$$J_a = \frac{1}{16\pi G_{\rm N}^{(5)}} \left[\int_{S^3_r} \star_5 \, dK_a + 2 \int_{S^3 \times I} \Big(G_{IJ} \big(\mathrm{i}_{K_a} \widehat{F}^I \big) \wedge \star_5 \widehat{F}^J + h_{uv} \big(\mathrm{i}_{K_a} \widehat{\mathcal{D}} q^u \big) \star_5 \widehat{\mathcal{D}} q^v \right] \ . \tag{E.0.7}$$

Now let us proceed and reduce the charges to 4d imposing the ansatz (D.0.2), in particular

$$\begin{aligned} \widehat{A}^I &= A^I + \xi^I A^0 + (z_1^I + \xi^I)(dy - A^0) \\ \widehat{F}^I &= F^I - z_1^I F^0 + dz_1^I \wedge (dy - A^0) \ , \end{aligned} \tag{E.0.8}$$

and performing the integrals along the circle. Notice that because of (D.0.6) and since the horizon areas in 5d and 4d are related by $\text{Area}_{(5)} = \Delta y \, \text{Area}_{(4)}$, the black hole entropy is the same in 5d and 4d. We find

$$\begin{aligned} \int_{S^3} G_{IJ} \star_5 \widehat{F}^J &= \Delta y \int_{S^2} e^{-2\widetilde{\phi}} G_{IJ} \star_4 \left(F^J - z_1^J F^0 \right) \\ C_{IJK} \int_{S^3 \times I} \widehat{F}^J \wedge \widehat{F}^K &= -\Delta y \, C_{IJK} \int_{S^2_r} \Big(2 z_1^J F^K - z_1^J z_1^K F^0 \Big) \ . \end{aligned} \tag{E.0.9}$$

In the second equality we used that $z_1^I \to 0$ at infinity. The electric charges are thus

$$\mathcal{Q}_{\mathfrak{I}} = \frac{1}{g} \int_{S^2_r} \frac{\delta S_{\rm 4d}}{\delta F^{\mathfrak{I}}} - \frac{1}{8\pi G_{\rm N}^{(4)} g} C_{\mathfrak{I}JK} \int_{S^2_r} \left(\frac{1}{2} \xi^J F^K + \frac{1}{4} \xi^J \xi^K F^0 \right) , \tag{E.0.10}$$

[1] In the case of the conifold compactification discussed in Section 1.4.2, this assumption is true, see (1.4.28). We expect the assumption to be true in all cases.

where

$$\frac{\delta S_{4\mathrm{d}}}{\delta F^\Lambda} = \frac{1}{16\pi G_{\mathrm{N}}^{(4)}}\Big(\mathbb{I}\mathrm{m}\,\mathcal{N}_{\Lambda\Sigma}\star_4 F^\Sigma + \mathbb{R}\mathrm{e}\,\mathcal{N}_{\Lambda\Sigma}F^\Sigma\Big) \tag{E.0.11}$$

are the derivatives of the action obtained from (C.0.2) with (D.0.22) and (D.0.24).

We define the 4d dimensionless magnetic charges as

$$p^\Lambda = \frac{g}{4\pi}\int_{S^2} F^\Lambda\,, \tag{E.0.12}$$

where the integral can be done at any radius because of the Bianchi identities. On the other hand, the first Chern class of the circle fibration — that we take to be the Hopf fibration of S^3 — is $\frac{1}{\Delta y}\int dA^0 = 1$. Thus, we obtain a properly quantized $p^0 = 1$ if we set

$$\Delta y = \frac{4\pi}{g}\,. \tag{E.0.13}$$

We will use this normalization from now on.

Let us now reduce the angular momentum. We consider the case $J_1 = J_2$, with $J_{1,2}$ normalized such that they generate orbits of length 2π, and define $J = (J_1+J_2)/2$. The corresponding Killing vector and dual 1-form are

$$K^M\partial_M = \frac{\Delta y}{4\pi}\frac{\partial}{\partial y} = \frac{1}{g}\frac{\partial}{\partial y}\,, \qquad K_M dx^M = \frac{1}{g}\,e^{-4\widetilde{\phi}}(dy - A^0)\,. \tag{E.0.14}$$

The first term in (E.0.7) gives

$$\int_{S^3}\star_5\, dK = -\frac{\Delta y}{g}\int_{S^2} e^{-6\widetilde{\phi}}\star_4 F^0\,. \tag{E.0.15}$$

To reduce the second term we use $\mathrm{i}_K\widehat{F}^I = -\frac{1}{g}dz_1^I$, integrate by parts, and use the EOMs (E.0.3). To reduce the third term we use $\mathrm{i}_K\widehat{\mathcal{D}}q^u = \big(z_1^I+\xi^I\big)k_I^u$ and $\mathrm{i}_\omega(\star 1) = \star\,\omega$ for a 1-form ω. Eventually

$$\begin{aligned} J = \frac{1}{8\pi G_{\mathrm{N}}^{(4)} g}\bigg\{\int_{S^2_r}\bigg[&-\frac{1}{2}\,e^{-6\widetilde{\phi}}\star_4 F^0 + e^{-2\widetilde{\phi}}\,G_{IJ}\,z_1^I\star_4\big(F^J - z_1^J F^0\big) \\ &- C_{IJK}\bigg(\frac{1}{4}\,z_1^I z_1^J F^K - \frac{1}{6}\,z_1^I z_1^J z_1^K F^0\bigg)\bigg] + \int_{S^2\times I}\star_4\, g\, k_0^u\, h_{uv}\,\mathcal{D}q^v\bigg\} \\ = \frac{1}{g}\int_{S^2_r}\frac{\delta S_{4\mathrm{d}}}{\delta F^0} - \frac{1}{8\pi G_{\mathrm{N}}^{(4)} g}&\bigg[C_{IJK}\xi^I\xi^J\int_{S^2_r}\bigg(\frac{1}{4}\,F^K + \frac{1}{6}\,\xi^K F^0\bigg) + \int_{S^2\times I}\star_4\, g\, k_0^u\, h_{uv}\,\mathcal{D}q^v\bigg]\,. \end{aligned} \tag{E.0.16}$$

The four-dimensional angular momentum of the black hole solution is proportional to $J_1 - J_2$, which vanishes in the case under consideration. This implies that we can impose spherical symmetry on S^2. The section $\mathcal{D}q^v$ is charged under the Abelian vector fields A_μ^Λ, therefore the magnetic fluxes p^Λ give rise to an effective spin s on S^2. However, the spin spherical harmonics [157, 158] have total angular momentum $j \geq |s|$, which should vanish in order to have a spherically-symmetric configuration. Since the Abelian symmetries are realized non-linearly on $\mathcal{D}q^v$ as soon as $k_\Lambda^u \neq 0$, we obtain the condition

$$p^\Lambda k_\Lambda^u(q) = 0 \tag{E.0.17}$$

for spherically-symmetric black hole solutions. Without loss of generality, in Section 1.4 we have set $p^I = 0$ which implies $k_0^u = 0$. We then see that the last term in (E.0.16) vanishes.

The magnetic charges that appear in the attractor equations of [94], in our conventions, are (E.0.12) while the electric charges are

$$q_\Lambda = \frac{g}{4\pi}\int_{S_r^2} G_\Lambda \qquad \text{with} \qquad G_\Lambda = 16\pi G_{\mathrm{N}}^{(4)}\,\frac{\delta S_{4\mathrm{d}}}{\delta F^\Lambda}\,. \tag{E.0.18}$$

Setting $p^I = 0$, we obtain the following dictionary between 5d and 4d charges:

$$\begin{aligned} q_0 &= 4G_{\mathrm{N}}^{(4)} g^2\, J + \frac{1}{3}\, C_{IJK}\xi^I\xi^J\xi^K p^0 \\ q_{\mathfrak{I}} &= 4G_{\mathrm{N}}^{(4)} g^2\, Q_{\mathfrak{I}} + \frac{1}{2}\, C_{\mathfrak{I}JK}\frac{1}{2}\xi^J\xi^K p^0\,. \end{aligned} \tag{E.0.19}$$

E.1 Baryonic charge quantization in the conifold theory

In order to fix the exact relation between the supergravity charge Q_3 and the field theory baryonic charge Q_B, we deduce the Dirac quantization condition satisfied by A_μ^3 from the consistent reduction of [89].

The metric of $T^{1,1}$ is

$$ds^2 = \frac{1}{6}\sum_{i=1,2}\Big(d\theta_i^2 + \sin^2\theta_i\, d\varphi_i^2\Big) + \eta^2 \qquad \text{with} \qquad \eta = -\frac{1}{3}\Big(d\psi + \sum_{i=1,2}\cos\theta_i\, d\varphi_i\Big)\,. \tag{E.1.1}$$

We define the 2-forms[2]

$$\begin{aligned} J &= \frac{1}{6}\Big(\sin\theta_1\, d\theta_1\wedge d\varphi_1 + \sin\theta_2\, d\theta_2\wedge d\varphi_2\Big) = \frac{1}{2}d\eta \\ \Phi &= \frac{1}{6}\Big(\sin\theta_1\, d\theta_1\wedge d\varphi_1 - \sin\theta_2\, d\theta_2\wedge d\varphi_2\Big)\,. \end{aligned} \tag{E.1.2}$$

The expansion of the 10d RR field strength F_5^{RR} in [89] around the $\mathrm{AdS}_5\times T^{1,1}$ vacuum (where $u = v = w = b^\Omega = c^\Omega = 0$), keeping only the dependence on the gauge fields and the

[2] The 2-form J should not be confused with the angular momentum of the black hole.

Stückelberg scalar a, in our conventions reads

$$\begin{aligned} F_5^{\mathrm{RR}} = 4g \star_5 1 - 2g^{-1} (\star_5 Da) \wedge (\eta - g\widehat{A}^1) - g^{-2} (\star_5 d\widehat{A}^2) \wedge J + g^{-2} (\star_5 d\widehat{A}^3) \wedge \Phi \\ - g^{-3} d\widehat{A}^2 \wedge J \wedge (\eta - g\widehat{A}^1) - g^{-3} d\widehat{A}^3 \wedge \Phi \wedge (\eta - g\widehat{A}^1) \\ + g^{-4} J \wedge J \wedge \big(Da + 2(\eta - g\widehat{A}^1)\big) \,, \end{aligned} \tag{E.1.3}$$

where $\star_5$ is the Poincaré dual in AdS$_5$ while $Da = da + 2g(\widehat{A}^1 + \widehat{A}^2)$. Dirac's quantization condition reads

$$\frac{1}{2\sqrt{\pi}\,\kappa_{10}} \int_{\mathcal{C}_5} F_5^{\mathrm{RR}} \in \mathbb{Z} \tag{E.1.4}$$

for any closed 5-cycle $\mathcal{C}_5$. Here κ_{10} is the 10d gravitational coupling, related to the 5d Newton constant by

$$\frac{\mathrm{Vol}(T^{1,1})}{g^5 \kappa_{10}^2} = \frac{1}{8\pi G_{\mathrm{N}}^{(5)}} \tag{E.1.5}$$

where $\mathrm{Vol}(T^{1,1}) = 16\pi^3/27$. Applying (E.1.4) to $\mathcal{C}_5 = T^{1,1}$ and imposing that there are N units of 5-form flux, we recover (1.4.35). On the other hand, let us apply (E.1.4) to the 5-cycle $X_2 \times S^3$, where X_2 is the non-trivial 2-cycle of $T^{1,1}$ while S^3 is a spatial 3-sphere in AdS$_5$. Using $\int_{X_2} J = 0$ and $\int_{X_2} \Phi = 4\pi/3$ as well as (1.4.35), we obtain

$$\frac{1}{2\sqrt{\pi}\,\kappa_{10}} \int_{X_2 \times S^3} F_5^{\mathrm{RR}} = \frac{1}{6\pi G_{\mathrm{N}}^{(5)} g N} \int_{S^3} \Big(\star_5 \widehat{F}^3 + \widehat{F}^3 \wedge \widehat{A}^1 \Big) = -\frac{4}{3N}\, Q_3 \in \mathbb{Z} \,, \tag{E.1.6}$$

where $\widehat{F}^3 = d\widehat{A}^3$. According to (E.0.3) and using (B.1.2) and (B.1.3), the combination in parenthesis gives the Page charge Q_3, which is conserved and quantized. Taking the 3-sphere to spatial infinity, it coincides with the charge defined in (E.0.1).

Appendix F

Large N limit of TT index

Let us start by studying (2.1.18), and in particular the terms involving the Li_1 function, whose definition and properties can be found in Section F.2. We first perform the sum over j (that becomes an integral over t'), leaving the sum over i (that becomes an integral over t) untouched.

The integral in t' has to be broken in two parts, above and below $t_{\pm\Delta} \equiv t \pm N^{-\alpha}\,\mathbb{I}\mathrm{m}\,\Delta$. When $\mathbb{I}\mathrm{m}(u_{ji} \mp \Delta) > 0$ (for one of the two signs), we can use the series expansion (F.2.1). This allows us to treat the integral above $t_{\pm\Delta}$:

$$\begin{aligned}\sum_j \Theta\big(\mathbb{I}\mathrm{m}(u_{ji} \mp \Delta)\big)\,\mathrm{Li}_1\Big(e^{i(u_{ji}\mp\Delta)}\Big) \;&\to\; N\int_{t_{\pm\Delta}} dt'\rho(t')\sum_{\ell=1}^{\infty}\frac{1}{\ell}\,e^{i\ell\big(u(t')\,-\,u(t)\,\mp\,\Delta\big)} \\ &\equiv\; N\sum_{\ell=1}^{\infty}\frac{e^{\mp i\ell\Delta}}{\ell}\,I_{\mathrm{L},\ell}[\rho](t,\Delta)\;. \qquad \text{(F.0.1)}\end{aligned}$$

In Section F.3 we define and manipulate these integrals. Using (F.3.6), we write (F.0.1) as:

$$\begin{aligned}&N^{1-\alpha}\,\mathrm{Li}_2\Big(e^{\mp i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\,\frac{\rho}{1-i\dot{v}} \qquad \text{(F.0.2)}\\ &+N^{1-2\alpha}\Big[\mathrm{Li}_3\Big(e^{\mp i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big) \pm (\mathbb{I}\mathrm{m}\,\Delta)(1-i\dot{v})\,\mathrm{Li}_2\Big(e^{\mp i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\Big]\Big[\frac{\dot{\rho}}{(1-i\dot{v})^2}+\frac{i\rho\,\ddot{v}}{(1-i\dot{v})^3}\Big]\\ &+\frac{i}{2}N^{1-2\alpha}(\mathbb{I}\mathrm{m}\,\Delta)^2(1-i\dot{v})^2\,\mathrm{Li}_1\Big(e^{\mp i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\frac{\rho\,\ddot{v}}{(1-i\dot{v})^3}+\mathcal{O}(N^{1-3\alpha})\;.\end{aligned}$$

When $\mathbb{I}\mathrm{m}(u_{ji} \mp \Delta) < 0$, the steps above are not applicable because the series expansion for Li_1 does not converge, but we can use (F.2.5) so that

$$\mathrm{Li}_1\Big(e^{i(u_{ji}\mp\Delta)}\Big) = \mathrm{Li}_1\Big(e^{i(u_{ij}\pm\Delta)}\Big) - i\big(u_{ji}\mp\Delta-\pi\big)\;. \qquad \text{(F.0.3)}$$

Now the Li_1 terms on the RHS can be analyzed in the same way as before using (F.3.7):

$$\begin{aligned}&\sum_j \Theta\big(\mathbb{I}\mathrm{m}(u_{ij} \pm \Delta)\big)\,\mathrm{Li}_1\Big(e^{i(u_{ij}\pm\Delta)}\Big) \to N\int^{t_{\pm\Delta}} dt'\,\rho(t')\sum_{\ell=1}^{\infty}\frac{e^{i\ell\big(u(t)-u(t')\pm\Delta\big)}}{\ell} = N\sum_{\ell=1}^{\infty}\frac{e^{\pm i\ell\Delta}}{\ell}\,I_{\mathrm{U},\ell}[\rho]\\ &= N^{1-\alpha}\,\mathrm{Li}_2\Big(e^{\pm i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\,\frac{\rho}{1-i\dot{v}}\\ &-N^{1-2\alpha}\Big[\mathrm{Li}_3\Big(e^{\pm i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big) \mp (\mathbb{I}\mathrm{m}\,\Delta)(1-i\dot{v})\,\mathrm{Li}_2\Big(e^{\pm i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\Big]\Big[\frac{\dot{\rho}}{(1-i\dot{v})^2}+\frac{i\rho\,\ddot{v}}{(1-i\dot{v})^3}\Big]\\ &-\frac{i}{2}N^{1-2\alpha}(\mathbb{I}\mathrm{m}\,\Delta)^2(1-i\dot{v})^2\,\mathrm{Li}_1\Big(e^{\pm i(\mathbb{R}\mathrm{e}\,\Delta-\dot{v}\,\mathbb{I}\mathrm{m}\,\Delta)}\Big)\frac{\rho\,\ddot{v}}{(1-i\dot{v})^3}+\mathcal{O}(N^{-3\alpha})\;. \qquad \text{(F.0.4)}\end{aligned}$$

To obtain the full integral over t', the contributions (F.0.2) and (F.0.4) with upper sign must be summed with minus the ones with lower sign, and the result can be simplified using (F.2.5). As in (2.1.18), we then integrate over t together with $\mathfrak{m}(t)$, and sum over $a = 1, 2, 3$. We obtain:

$$
\begin{aligned}
& iN^{2-2\alpha} \int dt \, \frac{i\,\mathfrak{m}\,\rho^2\,\ddot{v}}{(1-i\dot{v})^3} \sum_{a=1}^{3} (\mathbb{I}\mathrm{m}\,\Delta_a)^2 (1-i\dot{v})^2\, g''_+\big(\mathbb{R}\mathrm{e}\,\Delta_a - \dot{v}\,\mathbb{I}\mathrm{m}\,\Delta_a\big) \\
& - iN^{2-2\alpha} \int dt \; \mathfrak{m} \, \frac{d}{dt}\left[\frac{\rho^2}{(1-i\dot{v})^2}\right] \sum_{a=1}^{3} \Big[g_+\big(\mathbb{R}\mathrm{e}\,\Delta_a - \dot{v}\,\mathbb{I}\mathrm{m}\,\Delta_a\big) \\
& \qquad\qquad + i\,(\mathbb{I}\mathrm{m}\,\Delta_a)\,(1-i\dot{v})\, g'_+\big(\mathbb{R}\mathrm{e}\,\Delta_a - \dot{v}\,\mathbb{I}\mathrm{m}\,\Delta_a\big) \Big] \,.
\end{aligned}
\tag{F.0.5}
$$

The function $g_+(u)$ is defined in (F.2.6). It remains to add the contribution from the second term on the RHS of (F.0.3). We choose the integer ambiguities n_i in (2.1.18) such that

$$
\pi(N - 2n_i) = -\sum_{a=1}^{3}\sum_{j=1}^{N}\Big[2\pi\Big(\Theta\big(\mathbb{I}\mathrm{m}(u_{ij}+\Delta_a)\big) - \Theta\big(\mathbb{I}\mathrm{m}\,u_{ij}\big)\Big) + 2\Delta_a\Theta(\mathbb{I}\mathrm{m}\,u_{ij})\Big] + \mathcal{O}(1) \,. \tag{F.0.6}
$$

The subleading $\mathcal{O}(1)$ term accounts for the possibility that N might be odd and we would not be able to cancel it completely. The contributions from the second term on the RHS of (F.0.3) and from (F.0.6) sum up to

$$
\begin{aligned}
& i\sum_{a,i,j} \mathfrak{m}_i \Big[\Big(\Theta\big(\mathbb{I}\mathrm{m}(u_{ij}+\Delta_a)\big) - \Theta\big(\mathbb{I}\mathrm{m}\,u_{ij}\big)\Big)\big(-u_{ji} + \Delta_a - \pi\big) + \\
& \qquad + \Big(\Theta\big(\mathbb{I}\mathrm{m}(u_{ij}-\Delta_a)\big) - \Theta\big(\mathbb{I}\mathrm{m}\,u_{ij}\big)\Big)\big(u_{ji} + \Delta_a - \pi\big)\Big] \\
& = iN^2 \sum_{a=1}^{3}\sum_{+,-} \int dt\, \mathfrak{m}(t)\,\rho(t) \int_t^{t\pm\Delta_a} dt'\,\rho(t')\Big[\pm N^\alpha\big(it - it' + v(t) - v(t')\big) + \Delta_a - \pi\Big] \,.
\end{aligned}
\tag{F.0.7}
$$

In each integral we perform the change of variables $t' = t \pm N^{-\alpha}(\mathbb{I}\mathrm{m}\,\Delta_a)\varepsilon$, obtaining:

$$
\begin{aligned}
(\text{F.0.7}) = {} & iN^{2-\alpha} \sum_{a=1}^{3}\sum_{+,-} \mathbb{I}\mathrm{m}\,\Delta_a \int dt\,\mathfrak{m}(t)\,\rho(t) \int_0^1 d\varepsilon \, \times \\
& \times \Big\{ \pm\rho\Big(t \pm N^{-\alpha}(\mathbb{I}\mathrm{m}\,\Delta_a)\varepsilon\Big)\Big[-i(\mathbb{I}\mathrm{m}\,\Delta_a)\varepsilon \mp N^\alpha\, v\Big(t \pm N^{-\alpha}(\mathbb{I}\mathrm{m}\,\Delta_a)\varepsilon\Big) \pm N^\alpha v(t) + \Delta_a - \pi\Big]\Big\} \,.
\end{aligned}
\tag{F.0.8}
$$

We expand ρ and v in Taylor series and keep only the terms at leading order. Then we integrate in ε and use that $g''_+(\Delta) = \Delta - \pi$. We obtain the expression:

$$
\begin{aligned}
(\text{F.0.7}) = {} & iN^{2-2\alpha} \sum_{a=1}^{3} (\mathbb{I}\mathrm{m}\,\Delta_a)^2 \int dt\,\mathfrak{m} \Big\{ \rho\,\dot{\rho}\, g''_+\big(\mathbb{R}\mathrm{e}\,\Delta_a - \dot{v}\,\mathbb{I}\mathrm{m}\,\Delta_a\big) + \\
& \qquad + i\,\frac{\mathbb{I}\mathrm{m}\,\Delta_a}{6}\,\frac{d}{dt}\left[\frac{\rho^2}{(1-i\dot{v})^2}\right](1-i\dot{v})^3 \Big\} + \mathcal{O}(\mathfrak{m}N^{2-3\alpha}) \,.
\end{aligned}
\tag{F.0.9}
$$

We sum (F.0.5) and (F.0.9). We notice that the various terms can be organized into the Taylor series of $g_+(\Delta_a)$ around the point $\mathbb{R}\mathrm{e}(\Delta_a) - \dot{v}\, \mathbb{I}\mathrm{m}(\Delta_a)$, which has four terms because g_+ is a cubic polynomial. We obtain the compact expression

$$(\text{F.0.5}) + (\text{F.0.9}) = -iN^{2-2\alpha}\, G(\Delta) \int dt\, \mathfrak{m}\, \frac{d}{dt}\left[\frac{\rho^2}{(1-i\dot{v})^2}\right] + \mathcal{O}\big(\mathfrak{m}N^{2-3\alpha}, 1\big) \;, \tag{F.0.10}$$

where $G(\Delta)$ is the function defined in (2.1.24). It remains to add the first term on the RHS of (2.1.18). We obtain the final expression:

$$\int dt\, \mathfrak{m}\, V' = ikN \int dt\, \rho\, \mathfrak{m}\, u + iN^{2-2\alpha}\, G(\Delta) \int dt\, \frac{\dot{\mathfrak{m}}\, \rho^2}{(1-i\dot{v})^2} + \mathcal{O}\big(\mathfrak{m}N^{2-3\alpha}\big) \;. \tag{F.0.11}$$

We apply the same steps to obtain the large N limit of (2.1.19). To avoid repetition, we only present the result. We set the integer ambiguity M to $N/2 + \mathcal{O}(1)$. We obtain:

$$\Omega = -N^{2-\alpha}\, f_+(\mathfrak{n}, \Delta) \int dt\, \frac{\rho^2}{1 - i\dot{v}} + \mathcal{O}\big(N^{2-2\alpha}\big) \;, \tag{F.0.12}$$

where the function $f_+(\mathfrak{n}, \Delta)$ is defined in (2.1.24).

F.1 Solutions to the saddle-point equations

In this Section we solve the saddle-point equations (2.1.27)–(2.1.29), in the original parametrization in which $v(t)$ is a real function. Let us first solve (2.1.29). After integrating to

$$k\,(it + v)^2 + \frac{4G\rho}{i + \dot{v}} = A \in \mathbb{C} \;, \tag{F.1.1}$$

its real and imaginary parts give

$$4\rho = -\big(1 + \dot{v}^2\big)\, \mathbb{I}\mathrm{m}\Big[G^{-1}\left(A - k\,(it+v)^2\right)\Big] \;, \qquad \dot{v} = -\frac{\mathbb{R}\mathrm{e}\big[G^{-1}\big(A - k(it+v)^2\big)\big]}{\mathbb{I}\mathrm{m}\big[G^{-1}\big(A - k(it+v)^2\big)\big]} \;. \tag{F.1.2}$$

We impose that ρ is integrable. This necessarily implies that $\rho \to 0$ as $t \to \pm\infty$, or that ρ is defined on compact intervals where ρ is zero at the endpoints. At infinity, or at an endpoint, $\rho = 0$ implies $A - k\,(it+v)^2 = 0$. By considering real and imaginary parts, we see that this equation cannot be satisfied as $t \to \pm\infty$, and ρ must have compact support. In order for ρ to have two endpoints $t_\pm$ and be defined on the interval $[t_-, t_+]$, A cannot be on the positive real axis. Let $A^{\frac{1}{2}}$ be the square root whose imaginary part is positive. The boundary conditions are

$$t_\pm = \pm\, k^{-\frac{1}{2}}\; \mathbb{I}\mathrm{m}(A^{\frac{1}{2}}) \;, \qquad\qquad v(t_\pm) = \pm\, k^{-\frac{1}{2}}\; \mathbb{R}\mathrm{e}(A^{\frac{1}{2}}) \;. \tag{F.1.3}$$

We then solve the equation for $\dot{v}$ in (F.1.2) using (F.1.3) as boundary conditions. The equation can be rewritten and integrated to

$$\mathbb{Im}\left[G^{-1}\,(it+v)\left(A-\frac{k}{3}\,(it+v)^2\right)\right]=D\;, \tag{F.1.4}$$

where $D\in\mathbb{R}$ is an integration constant. The boundary conditions (F.1.3) imply $D=0$ and $\mathbb{Im}\big(G^{-1}A^{\frac{3}{2}}\big)=0$. Using a real constant B to parametrize the real part of $G^{-1}A^{\frac{3}{2}}$, we write

$$A=k\,(BG)^{\frac{2}{3}}\;,\qquad B\in\mathbb{R}\;, \tag{F.1.5}$$

where k is included for convenience. It is important to keep in mind that there are 3 branches for $G^{\frac{1}{3}}$ and the same branch is to be used in every expression. There is a triplet of solutions at this point. The equation (F.1.4) can be written as

$$0=\mathbb{Im}\big(G^{-\frac{1}{3}}(it+v)\big)\left[3B^{\frac{2}{3}}+\Big(\mathbb{Im}\big(G^{-\frac{1}{3}}(it+v)\big)\Big)^2-3\Big(\mathbb{Re}\big(G^{-\frac{1}{3}}(it+v)\big)\Big)^2\right]. \tag{F.1.6}$$

The solutions obtained by setting to zero the square bracket lead to profiles for ρ with a single zero, and so they have to be discarded. We remain with

$$\mathbb{Im}\Big(G^{-\frac{1}{3}}\,(it+v)\Big)=0\qquad\Rightarrow\qquad v(t)=\frac{\mathbb{Re}\,G^{\frac{1}{3}}}{\mathbb{Im}\,G^{\frac{1}{3}}}\,t\;, \tag{F.1.7}$$

which through (F.1.2) gives the following profile for ρ:

$$\rho(t)=\frac{k}{4\big(\mathbb{Im}\,G^{\frac{1}{3}}\big)^3}\left[B^{\frac{2}{3}}\big(\mathbb{Im}\,G^{\frac{1}{3}}\big)^2-t^2\right]. \tag{F.1.8}$$

Requiring that $\rho>0$ within (t_-,t_+) imposes

$$\mathbb{Im}\,G^{\frac{1}{3}}>0\;, \tag{F.1.9}$$

which restricts the branches we can take for $G^{\frac{1}{3}}$. Requiring that $\int dt\,\rho=1$ fixes $B=3/k$ and the final result for u and ρ is:

$$u(t)=N^{\frac{1}{3}}\,\frac{G^{\frac{1}{3}}}{\mathbb{Im}\,G^{\frac{1}{3}}}\,t\;,\quad \rho(t)=\frac{(9k)^{\frac{1}{3}}}{4\,\mathbb{Im}\,G^{\frac{1}{3}}}-\frac{k}{4\big(\mathbb{Im}\,G^{\frac{1}{3}}\big)^3}\,t^2\;,\quad t_\pm=\pm\left(\frac{3}{k}\right)^{\frac{1}{3}}\mathbb{Im}\,G^{\frac{1}{3}}\;. \tag{F.1.10}$$

Notice that if Δ_a are real and $G>0$, (F.1.9) fixes the branch of the cube root such that $G^{\frac{1}{3}}$ has phase $e^{\frac{2\pi i}{3}}$, and the solutions for u, ρ reduce to those found in [31]. We can now solve for $\mathfrak{m}$ using (2.1.28). Inserting (F.1.10) for u and ρ, the former reduces to:

$$\big(t^2-t_+^2\big)\,\ddot{\mathfrak{m}}+4t\,\dot{\mathfrak{m}}+2\mathfrak{m}=\frac{d^2}{dt^2}\Big[\big(t^2-t_+^2\big)\,\mathfrak{m}\Big]=-2\,\frac{f_+}{G}\,u\;, \tag{F.1.11}$$

whose general solution is

$$\mathfrak{m}(t) = -\frac{1}{\left(t^2 - t_+^2\right)} \frac{N^{\frac{1}{3}} f_+}{3G} \frac{G^{\frac{1}{3}}}{\mathbb{Im}\, G^{\frac{1}{3}}} \left(t^3 + Ct + D\right) , \tag{F.1.12}$$

where C and D are integration constants. The requirement that $\mathfrak{m}$ has compact image, namely that it does not diverge at $t = t_\pm$, fixes $C = -t_+^2$ and $D = 0$. This leads to the simple solution

$$\mathfrak{m}(t) = -\frac{f_+}{3G}\, u(t) \, . \tag{F.1.13}$$

One can then verify that (2.1.27) is automatically solved, with the following value of the Lagrange multiplier:

$$\mu = i f_+ \left(\frac{k}{3G}\right)^{\frac{1}{3}} . \tag{F.1.14}$$

The solution can be expressed more neatly by making use of the reparametrization symmetry (2.1.30), performing the transformation $t = (3/k)^{1/3} (\mathbb{Im}\, G^{1/3})\, t'$. This brings the solution to the form (2.1.31), in which primes have been omitted.

F.2 Polylogarithms

Polylogarithms are defined through their Taylor series around $z = 0$:

$$\mathrm{Li}_k(z) = \sum\nolimits_{\ell=1}^{\infty} \frac{z^\ell}{\ell^k} \, , \tag{F.2.1}$$

which is absolutely convergent for $|z| < 1$. This definition can be analytically continued to the whole complex plane, with a branch cut on the real axis from $z = 1$ to $z = \infty$. In particular $\mathrm{Li}_1(z) = -\log(1 - z)$, where the principal sheet defined by (F.2.1) is such that $\mathbb{Im} \log \in (-\pi, \pi)$. The functions $\mathrm{Li}_{k\geq 2}$ have an absolutely convergent series (F.2.1) on the unit circle and are thus continuous at $z = 1$, while the functions $\mathrm{Li}_{k\leq 0}$ have a pole at $z = 1$ but no branch cut (in particular $\mathrm{Li}_0(z) = \frac{z}{1-z}$). One can define the single-valued analytic functions

$$F_k(u) = \mathrm{Li}_k\left(1 - e^{-iu}\right) \tag{F.2.2}$$

defined by (F.2.1) in the domain $\left|1 - e^{-iu}\right| < 1$ with $\mathbb{Re}\, u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (implying that $F_k(0) = 0$) and by analytic continuation elsewhere. For instance $F_0(u) = e^{iu} - 1$ whereas $F_1(u) = iu$.

Whenever the function is differentiable, we have

$$z\, \partial_z \mathrm{Li}_k(z) = \mathrm{Li}_{k-1}(z) \tag{F.2.3}$$

or alternatively

$$-\, i\, \partial_u \mathrm{Li}_k(e^{iu}) = \mathrm{Li}_{k-1}(e^{iu}) \qquad \text{or} \qquad \partial_u F_k(u) = \frac{i}{e^{iu} - 1}\, F_{k-1}(u) \, . \tag{F.2.4}$$

The last relation allows one to define $F_k(u) = \int_0^u \frac{i}{e^{iw}-1} F_{k-1}(w)$ which is single-valued because the integrand is analytic with no poles. Polylogarithms satisfy the following identities:

$$\begin{aligned}
\mathrm{Li}_0(e^{iu}) + \mathrm{Li}_0(e^{-iu}) &= -g_+'''(u) = -1 \\
\mathrm{Li}_1(e^{iu}) - \mathrm{Li}_1(e^{-iu}) &= -ig_+''(u) \\
\mathrm{Li}_2(e^{iu}) + \mathrm{Li}_2(e^{-iu}) &= g_+'(u) \\
\mathrm{Li}_3(e^{iu}) - \mathrm{Li}_3(e^{-iu}) &= ig_+(u) \,,
\end{aligned} \tag{F.2.5}$$

where

$$g_+(u) = \frac{1}{6}u^3 - \frac{\pi}{2}u^2 + \frac{\pi^2}{3}u \tag{F.2.6}$$

is the same function defined in (2.1.25). These relations are valid for $\mathbb{R}\mathrm{e}\, u \in (0, 2\pi)$ and the polylogarithms in their principal determination, and can then be extended to the whole complex plane by analytic continuation (notice that the functions on the RHS are polynomials with no branch cuts).

F.3 Large N integrals

Let us evaluate, at large N, the following integrals:

$$\begin{aligned}
I_{\mathrm{L},\ell}[\rho](t,\Delta) \;&\equiv\; \int_{t_{\pm\Delta}} dt'\, \rho(t')\, e^{i\ell\left(u(t')-u(t)\right)} \,, \\
I_{\mathrm{U},\ell}[\rho](t,\Delta) \;&\equiv\; \int^{t_{\pm\Delta}} dt'\, \rho(t')\, e^{i\ell\left(u(t)-u(t')\right)} \,,
\end{aligned} \tag{F.3.1}$$

where $u(t) = N^\alpha\big(it + v(t)\big)$ and $t_{\pm\Delta} \equiv t \pm N^{-\alpha}\, \mathbb{I}\mathrm{m}\, \Delta$ (the subscripts L and U stand for lower and upper, respectively). We Taylor expand part of the integrand around $t_{\pm\Delta}$:

$$I_{\mathrm{L},\ell}[\rho](t,\Delta) = e^{-i\ell u(t)} \sum_{m=0}^{\infty} \frac{1}{m!}\, \partial_x^m \Big[\rho(x)\, e^{i\ell N^\alpha v(x)}\Big]_{x=t_{\pm\Delta}} \int_{t_{\pm\Delta}} dt'\, e^{-\ell N^\alpha t'} \left(t' - t_{\pm\Delta}\right)^m \,. \tag{F.3.2}$$

The integral on the RHS can be evaluated integrating by parts:

$$\int_{t_{\pm\Delta}} dt'\, e^{-\ell N^\alpha t'} \left(t' - t_{\pm\Delta}\right)^m = -\sum_{k=0}^{m} \frac{m!\,(t_+ - t_{\pm\Delta})^k}{k!\,(N^\alpha \ell)^{m-k+1}}\, e^{-\ell N^\alpha t_+} + \frac{m!}{(N^\alpha \ell)^{m+1}}\, e^{-\ell N^\alpha t_{\pm\Delta}} \,, \tag{F.3.3}$$

where t_+ is the upper limit of integration. The boundary terms at t_+ can be neglected because of an overall factor $e^{-\ell N^\alpha (t_+ - t_{\pm\Delta})}$, which is exponentially suppressed, with respect to the last term. This gives

$$\int_{t_{\pm\Delta}} dt'\, e^{-\ell N^\alpha t'} \left(t' - t_{\pm\Delta}\right)^m \;\simeq\; \frac{m!}{(N^\alpha l)^{m+1}}\, e^{-\ell N^\alpha t_{\pm\Delta}} \,. \tag{F.3.4}$$

For the derivatives in (F.3.2), the terms up to NLO in the large N expansion are

$$\begin{aligned}
&\partial^m \Big[\rho\, e^{i\ell N^\alpha v}\Big]_{x=t_{\pm\Delta}} = \\
&\quad = e^{i\ell N^\alpha v}\,(i\ell N^\alpha)^{m-1}\left(i\ell N^\alpha\,\rho\,\dot v^m + m\,\dot\rho\,\dot v^{m-1} + \tfrac{m(m-1)}{2}\,\rho\,\dot v^{m-2}\,\ddot v + \dots\right)\Bigg|_{x=t_{\pm\Delta}} \\
&\quad = e^{i\ell\left(N^\alpha v \pm \mathbb{I}\mathrm{m}(\Delta)\dot v\right)}(i\ell N^\alpha)^{m-1}\Big[i\ell N^\alpha\,\rho\,\dot v^m + m\,\dot\rho\,\dot v^{m-1} + \tfrac{m(m-1)}{2}\,\rho\,\dot v^{m-2}\,\ddot v + \\
&\qquad\qquad \pm\, i\ell\,\mathbb{I}\mathrm{m}(\Delta)\Big(\dot\rho\,\dot v^m + m\,\rho\,\dot v^{m-1}\,\ddot v \pm \tfrac{1}{2}\,i\ell\,\mathbb{I}\mathrm{m}(\Delta)\,\rho\,\dot v^m\,\ddot v\Big) + \dots\Big]\,.
\end{aligned} \tag{F.3.5}$$

In the last expression ρ and v are functions of t. Other contributions are subleading by powers of $N^{-\alpha}$. Plugging this back in (F.3.2), we get

$$\begin{aligned}
I_{\mathrm{L},\ell}[\rho](t,\Delta) &= e^{\mp\ell\,\mathbb{I}\mathrm{m}(\Delta)\,(1-i\dot v)}\Bigg[\frac{1}{\ell N^\alpha}\,\frac{\rho}{1-i\dot v} + \\
&+ \frac{1}{\ell^2 N^{2\alpha}}\Big(1 \pm \ell\;\mathbb{I}\mathrm{m}(\Delta)\,(1-i\dot v)\Big)\left(\frac{\dot\rho}{(1-i\dot v)^2} + \frac{i\,\rho\,\ddot v}{(1-i\dot v)^3}\right) + \frac{1}{2N^{2\alpha}}(\mathbb{I}\mathrm{m}\,\Delta)^2\frac{i\,\rho\,\ddot v}{1-i\dot v}\Bigg]\,.
\end{aligned} \tag{F.3.6}$$

Repeating the same steps for the other integral we find

$$\begin{aligned}
I_{\mathrm{U},\ell}[\rho](t,\Delta) &= e^{\pm\ell\,\mathbb{I}\mathrm{m}(\Delta)\,(1-i\dot v)}\Bigg[\frac{1}{\ell N^\alpha}\,\frac{\rho}{1-i\dot v} \\
&- \frac{1}{\ell^2 N^{2\alpha}}\Big(1 \mp \ell\;\mathbb{I}\mathrm{m}(\Delta)\,(1-i\dot v)\Big)\left(\frac{\dot\rho}{(1-i\dot v)^2} + \frac{i\,\rho\,\ddot v}{(1-i\dot v)^3}\right) - \frac{1}{2N^{2\alpha}}(\mathbb{I}\mathrm{m}\,\Delta)^2\frac{i\,\rho\,\ddot v}{1-i\dot v}\Bigg]\,.
\end{aligned} \tag{F.3.7}$$

Appendix G

3d SUSY variations

In terms of a single Dirac spinor ϵ, the 3d supersymmetry transformations under which the Lagrangians in (2.1.13) are invariant, for chiral and vector multiplets, respectively, are:

$$
\begin{aligned}
Q\Phi &= 0 & Q\Psi &= \big(i\gamma^\mu D_\mu\Phi - i\sigma\Phi\big)\,\epsilon & \widetilde{Q}\Psi &= \epsilon^c F \\
\widetilde{Q}\Phi &= -\overline{\epsilon}\,\Psi & \widetilde{Q}\overline{\Psi} &= -\overline{\epsilon}\,\big(i\gamma^\mu D_\mu\Phi^\dagger + i\Phi^\dagger\sigma\big) & Q\overline{\Psi} &= -\overline{\epsilon^c}F^\dagger \\
Q\Phi^\dagger &= \overline{\Psi}\,\epsilon & QF &= -\overline{\epsilon^c}\big(i\gamma^\mu D_\mu\Psi + i\sigma\Psi - i\lambda\Phi\big) & \widetilde{Q}F &= 0 \\
\widetilde{Q}\Phi^\dagger &= 0 & \widetilde{Q}F^\dagger &= \big(iD_\mu\overline{\Psi}\gamma^\mu - i\overline{\Psi}\sigma + i\Phi^\dagger\overline{\lambda}\big)\,\epsilon^c & QF^\dagger &= 0
\end{aligned}
\tag{G.0.1}
$$

and

$$
\begin{aligned}
QA_\mu &= -\frac{i}{2}\,\overline{\lambda}\gamma_\mu\epsilon & Q\lambda &= \left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + iD - i\gamma^\mu D_\mu\sigma\right)\epsilon & \widetilde{Q}\lambda &= 0 \\
\widetilde{Q}A_\mu &= \frac{i}{2}\,\overline{\epsilon}\gamma_\mu\lambda & \widetilde{Q}\overline{\lambda} &= \overline{\epsilon}\left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + iD + i\gamma^\mu D_\mu\sigma\right) & Q\overline{\lambda} &= 0 \\
Q\sigma &= -\frac{1}{2}\,\overline{\lambda}\epsilon & QD &= -\frac{1}{2}\big(D_\mu\overline{\lambda}\gamma^\mu - \sigma\overline{\lambda}\big)\epsilon \\
\widetilde{Q}\sigma &= \frac{1}{2}\,\overline{\epsilon}\lambda & \widetilde{Q}D &= -\frac{1}{2}\overline{\epsilon}\big(\gamma^\mu D_\mu\lambda - \sigma\lambda\big)\,.
\end{aligned}
\tag{G.0.2}
$$

Appendix H

Monopole spherical harmonics on S^2

We use complex coordinates on S^2 to perform the reduction. We define stereographic coordinates

$$z = e^{i\varphi} \tan\frac{\theta}{2} \qquad \text{for } \theta < \pi \;, \qquad v = e^{-i\varphi} \cot\frac{\theta}{2} \qquad \text{for } \theta > 0 \;, \tag{H.0.1}$$

related by $v = 1/z$, which exhibit S^2 as $\mathbb{CP}^1$. The round metric with radius R is proportional to the Fubini-Study metric, and the Lorentzian metric on $S^2 \times \mathbb{R}$ is

$$ds^2 = \frac{4R^2}{(1+z\bar{z})^2}\, dz\, d\bar{z} - dt^2 \;\equiv\; g^{\frac{1}{2}} dz\, d\bar{z} - dt^2 \;= e^1 e^{\bar{1}} - (e^3)^2 \;, \tag{H.0.2}$$

where we defined the vielbein

$$e^3 = dt \;, \qquad e^1 = g^{\frac{1}{4}} dz \;, \qquad e^{\bar{1}} = g^{\frac{1}{4}} d\bar{z} \;. \tag{H.0.3}$$

Here e^1 and $e^{\bar{1}}$ are complex conjugates of each other and therefore any real p-form expressed in this basis has components satisfying the reality property $X^*_{1\dots} = X_{\bar{1}\dots}$. Flat indices are lowered and raised by the flat metric η_{ab} with $\eta_{1\bar{1}} = \eta_{\bar{1}1} = \frac{1}{2}$. The volume form has flat components $\epsilon_{01\bar{1}} = i/2$.

Let us now move to spinors. We choose the set of gamma matrices

$$\gamma_t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \;, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;, \qquad \gamma_{\bar{1}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;, \tag{H.0.4}$$

satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}$. The generators of the Dirac representation are $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$. On $S^2 \times \mathbb{R}$ the 3d Lorentz group $SO(2,1)$ is broken to the U(1) generated by $\gamma^{1\bar{1}}$, and fields are characterized by a spin that is the charge under this U(1). The spin connection, defined by $(\omega^a{}_b)_\mu = e^a{}_\nu \big(\partial_\mu e^\nu{}_b + \Gamma^\nu_{\mu\rho} e^\rho{}_b\big)$, has non-zero components

$$(\omega^1{}_1)_z = -(\omega^{\bar{1}}{}_{\bar{1}})_z = -\frac{\bar{z}}{1+z\bar{z}} \;, \qquad (\omega^1{}_1)_{\bar{z}} = -(\omega^{\bar{1}}{}_{\bar{1}})_{\bar{z}} = \frac{z}{1+z\bar{z}} \;. \tag{H.0.5}$$

The spinor covariant derivative (without gauge connections) $D_\mu \big({\psi_+ \atop \psi_-} \big) \equiv (D_\mu\psi_+, D_\mu\psi_-)^\mathsf{T}$ can be written as

$$D = d - is\omega \qquad \text{with} \qquad \omega = i\,\frac{\bar{z}\, dz - z\, d\bar{z}}{1+z\bar{z}} = (\cos\theta - 1)\, d\varphi \tag{H.0.6}$$

and $s = \pm\frac{1}{2}$ is the spin. Note that $\frac{1}{2\pi}\int_{S^2} d\omega = -2$. The components $\psi_\pm$ are sections of the U(1) bundles associated to the line bundles $\mathcal{K}^{\pm\frac{1}{2}} \cong \mathcal{O}(\mp 1)$, where $\mathcal{K}$ is the canonical bundle. A generic U(1) bundle is labelled by a half-integer monopole charge q, and has covariant derivative $D = d - iqa$. To conform with the conventions of [158] for the monopole harmonics, we write the connection as a half-integer multiple of $a = -\omega$.

Similarly, the Levi-Civita connection on 1-forms is a U(1) connection when projected onto the frame fields:

$$e^z_1 \nabla_\mu A_z = (\partial_\mu - i\omega_\mu) e^z_1 A_z \equiv D_\mu A_1 \;, \qquad e^{\bar{z}}_{\bar{1}} \nabla_\mu A_{\bar{z}} = (\partial_\mu + i\omega_\mu) e^{\bar{z}}_{\bar{1}} A_{\bar{z}} \equiv D_\mu A_{\bar{1}} \;. \tag{H.0.7}$$

Thus $A_1 = e^z_1 A_z$ and $A_{\bar{1}} = e^{\bar{z}}_{\bar{1}} A_{\bar{z}}$ are sections with $q = -1$ and $q = +1$, respectively. On the other hand, $D_\mu A_3 = \partial_\mu A_3$ and thus A_3 is a section of the trivial bundle, like a scalar. Defining $D_a = e^\mu_a D_\mu$, one finds $(dA)_{ab} = e^\mu_a e^\nu_b (\nabla_\mu A_\nu - \nabla_\nu A_\mu) = D_a A_b - D_b A_a$. If, in addition, the fields are in the adjoint representation of the gauge group and there is a background gauge field with fluxes,

$$A = \frac{1}{2}\mathfrak{m}_i H^i\, a \qquad \Rightarrow \qquad \frac{1}{2\pi}\int_{S^2} dA = \mathfrak{m}_i H^i \;, \tag{H.0.8}$$

then including this background in the covariant derivatives D_μ shifts the spin $s \to s - \frac{\alpha(\mathfrak{m})}{2}$, or equivalently $q \to q + \frac{\alpha(\mathfrak{m})}{2}$, where α are the roots.

The derivatives D_1 and $D_{\bar{1}}$ raise and lower the spin by 1, respectively. This is opposite in terms of the charge q. Their explicit expressions are

$$D^{(q)}_1 = \frac{1}{2R}\Big((1 + z\bar{z})\, \partial_z - q\bar{z} \Big) \;, \qquad D^{(q)}_{\bar{1}} = \frac{1}{2R}\Big((1 + z\bar{z})\, \partial_{\bar{z}} + qz \Big) \;, \tag{H.0.9}$$

where the superscript indicates the charge of the section they act on, whereas under complex conjugation $\overline{D^{(q)}_1} = D^{(-q)}_{\bar{1}}$ and $\overline{D^{(q)}_{\bar{1}}} = D^{(-q)}_1$. We define the operators

$$L_+ = z^2\partial_z + \partial_{\bar{z}} - qz \;, \qquad L_- = -\bar{z}^2\partial_{\bar{z}} - \partial_z - q\bar{z} \;, \qquad L_z = z\partial_z - \bar{z}\partial_{\bar{z}} - q \;, \tag{H.0.10}$$

satisfying the $\mathfrak{su}(2)$ algebra $[L_z, L_\pm] = \pm L_\pm$ and $[L_+, L_-] = 2L_z$. The covariant Laplacian is

$$\begin{aligned} -D^2 &\equiv L^2 - q^2 = \frac{1}{2}\{L_+, L_-\} + L_z^2 - q^2 = -\big(1 + z\bar{z}\big)^2 \partial_z \partial_{\bar{z}} - q(1 + z\bar{z}) L_z - q^2 \\ &= -\frac{1}{\sin\theta}\partial_\theta \left(\sin\theta\, \partial_\theta\right) + \frac{1}{\sin^2\theta}\big(-i\partial_\varphi - q + q\cos\theta\big)^2 \;, \end{aligned} \tag{H.0.11}$$

which can be diagonalized simultaneously with L^2 and L_z. Its eigenfunctions are the monopole spherical harmonics $Y_{q,l,m}$ with $|m| \le l$, that we choose to be orthonormal on a S^2 of radius 1:

$$\int_{S^2} \sqrt{g}\; \overline{Y_{q,l,m}}\, Y_{q,l',m'} = \delta_{l,l'}\, \delta_{m,m'} \;. \tag{H.0.12}$$

The highest harmonic with $m = l$, annihilated by L_+, is

$$Y_{q,l,l}(z,\bar{z}) \propto \frac{z^{l+q}}{(1+z\bar{z})^l} \; . \tag{H.0.13}$$

Regularity at the poles implies $l + q \in \mathbb{Z}_{\geq 0}$ and $l \geq |q|$.

The Laplacian can be written in terms of the derivatives as

$$-D^2 = -4R^2 D_1 D_{\bar{1}} + q = -4R^2 D_{\bar{1}} D_1 - q = -2R^2 \{D_1, D_{\bar{1}}\} \; . \tag{H.0.14}$$

Besides, one can verify that

$$[D_1, L_z] = [D_1, L_\pm] = [D_{\bar{1}}, L_z] = [D_{\bar{1}}, L_\pm] = 0 \; . \tag{H.0.15}$$

Therefore the derivatives acts as bundle-changing operators mapping $Y_{q,m,l}$ to $Y_{q\pm1,m,l}$. The exact relations can be derived integrating by parts the orthonormality conditions. For a suitable choice of phases one finds [158, 159]:

$$\begin{aligned}
D_1^{(q)}\, Y_{q,l,m} &= -\frac{s_-(q,l)}{2R}\, Y_{q-1,l,m} \qquad && \text{with} \quad s_-(q,l) = \big[l(l+1) - q(q-1)\big]^{\frac{1}{2}} \; , \\
D_{\bar{1}}^{(q)}\, Y_{q,l,m} &= \frac{s_+(q,l)}{2R}\, Y_{q+1,l,m} \qquad && \text{with} \quad s_+(q,l) = \big[l(l+1) - q(q+1)\big]^{\frac{1}{2}} \; .
\end{aligned} \tag{H.0.16}$$

Following the same conventions as in [159], the monopole harmonics satisfy

$$\overline{Y_{q,l,m}} = (-1)^{q+m}\, Y_{-q,l,-m} \tag{H.0.17}$$

under complex conjugation.

Finally, the triple overlap of harmonics is given in terms of Wigner $3j$-symbols:

$$\begin{aligned}
&\int d\Omega\; Y_{q,l,m} Y_{q',l',m'} Y_{q'',l'',m''} = \\
&\qquad = (-1)^{l+l'+l''} \left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{\frac{1}{2}} \begin{pmatrix} l & l' & l'' \\ q & q' & q'' \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} \; ,
\end{aligned} \tag{H.0.18}$$

or equivalently

$$\begin{aligned}
&Y_{q,l,m}\, Y_{q',l',m'} = \\
&\sum_{l''} (-1)^{l+l'+l''+q''+m''} \left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{\frac{1}{2}} \begin{pmatrix} l & l' & l'' \\ q & q' & q'' \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} Y_{-q'',l'',-m''}
\end{aligned} \tag{H.0.19}$$

The $3j$-symbols are directly related to Clebsch-Gordan coefficients that decompose the angular momentum state $|l''\, m''\rangle$ in terms of $|l\, m\, l'\, m'\rangle = |l\, m\rangle \otimes |l'\, m'\rangle$:

$$C\left(\begin{smallmatrix} l & l' & l'' \\ m & m' & m'' \end{smallmatrix}\right) \equiv \langle l\, m\, l'\, m' | \, l''\, m'' \rangle = (-1)^{l-l'+m''} \sqrt{2l''+1} \begin{pmatrix} l & l' & l'' \\ m & m' & -m'' \end{pmatrix} \; . \tag{H.0.20}$$

In particular, the Clebsh-Gordan coefficients are zero unless $m+m'=m''$, $\left|m^{(i)}\right| \leq l^{(i)}$ with $m^{(i)} = l^{(i)}$ mod 1, and $l^{(i)} \leq l^{(j)} + l^{(k)}$. The $3j$-symbol is symmetric under even permutations of its columns, and gains a sign $(-1)^{l+l'+l''}$ under odd permutations. It also gains a sign $(-1)^{l+l'+l''}$ when one changes sign to m, m' and m''. This implies the following relations among Clebsch-Gordan coefficients:

$$
\begin{aligned}
C\left(\begin{smallmatrix} l' & l'' & l \\ m' & -m'' & -m \end{smallmatrix}\right) &= (-1)^{l-l''+m'} \left[\frac{2l+1}{2l''+1}\right]^{1/2} C\left(\begin{smallmatrix} l & l' & l'' \\ m & m' & m'' \end{smallmatrix}\right) , \\
C\left(\begin{smallmatrix} l'' & l & l' \\ -m'' & m & -m' \end{smallmatrix}\right) &= (-1)^{l''-l'+m} \left[\frac{2l'+1}{2l''+1}\right]^{1/2} C\left(\begin{smallmatrix} l & l' & l'' \\ m & m' & m'' \end{smallmatrix}\right) , \\
C\left(\begin{smallmatrix} l' & l & l'' \\ m' & m & m'' \end{smallmatrix}\right) &= (-1)^{l+l'-l''} C\left(\begin{smallmatrix} l & l' & l'' \\ m & m' & m'' \end{smallmatrix}\right) .
\end{aligned}
\tag{H.0.21}
$$

In the special case that $l'' = l + l' \equiv L$ (and $m + m' = -m'' \equiv M$ as in the general case):

$$
\begin{aligned}
\begin{pmatrix} l & l' & L \\ m & m' & -M \end{pmatrix} &= (-1)^{l-l'+M} \left[\frac{1}{2L+1} \binom{2L}{L+M}^{-1} \binom{2l}{l+m} \binom{2l'}{l'+m'}\right]^{\frac{1}{2}} , \\
C\left(\begin{smallmatrix} l & l' & L \\ m & m' & M \end{smallmatrix}\right) &= \left[\binom{2L}{L+M}^{-1} \binom{2l}{l+m} \binom{2l'}{l'+m'}\right]^{\frac{1}{2}} .
\end{aligned}
\tag{H.0.22}
$$

Appendix I

1d $\mathcal{N} = 2$ superspace

We review here the 1d $\mathcal{N} = 2$ superspace formalism, drawing from Appendix A of [139]. The $\mathcal{N} = 2$ superspace in quantum mechanics, which we denote as $\mathbb{R}^{1|2}$, has coordinates $(t, \theta, \bar{\theta})$, where θ is a complex fermionic coordinate. A supersymmetry transformation is $\delta = -\epsilon\, Q + \bar{\epsilon}\,\overline{Q}$, where ϵ, $\bar{\epsilon}$ are anticommuting parameters, and Q, $\overline{Q}$ are anticommuting generators so that δ is commuting. Here Q and $\overline{Q}$ are defined as differential operators acting on superfields:

$$Q \equiv \partial_\theta + \frac{i}{2}\,\bar{\theta}\,\partial_t\,, \qquad \overline{Q} \equiv -\partial_{\bar{\theta}} - \frac{i}{2}\,\theta\,\partial_t\,. \tag{I.0.1}$$

They satisfy the algebra $Q^2 = \overline{Q}^2 = 0$ and $\{Q, \overline{Q}\} = -i\partial_t$. Moreover, Q and $\overline{Q}$ anticommute with another set of differential operators

$$D \equiv \partial_\theta - \frac{i}{2}\,\bar{\theta}\,\partial_t\,, \qquad \overline{D} \equiv -\partial_{\bar{\theta}} + \frac{i}{2}\,\theta\,\partial_t\,, \tag{I.0.2}$$

which satisfy the algebra $D^2 = \overline{D}^2 = 0$ and $\{D, \overline{D}\} = i\partial_t$. One has $\overline{(DX)} = (-1)^F\,\overline{D}\,\overline{X}$ and $\overline{(\overline{D}X)} = (-1)^F D\overline{X}$.

I.1 Matter multiplets

A chiral superfield Φ_h is defined by $\overline{D}\Phi_h = 0$. Gauge transformations act as

$$\Phi_h \to h\,\Phi_h\,, \qquad h = e^{\chi}\,, \qquad \chi : \mathbb{R}^{1|2} \to \mathbb{C} \otimes r\,, \qquad \overline{D}\chi = 0\,, \tag{I.1.1}$$

where r is some representation of the gauge group. $\overline{D}\Phi_h = 0$ implies that Φ_h and its complex conjugate anti-chiral superfield $\overline{\Phi}_h$ have expansions:

$$\Phi_h = \phi + \theta\psi - \frac{i}{2}\theta\bar{\theta}\,\partial_t\phi\,, \qquad \overline{\Phi}_h = \overline{\phi} - \bar{\theta}\,\overline{\psi} + \frac{i}{2}\theta\bar{\theta}\,\partial_t\overline{\phi}\,. \tag{I.1.2}$$

Acting with (I.0.1) on Φ_h and $\overline{\Phi}_h$, we find the following supersymmetry variations:

$$Q\phi = \psi\,, \qquad Q\psi = 0\,, \qquad \overline{Q}\phi = 0\,, \qquad \overline{Q}\psi = i\partial_t\phi\,. \tag{I.1.3}$$

Suppose that $\Phi_{a,h}$ are a collection of bosonic chiral superfields. We can also have fermionic Fermi superfields $\mathcal{Y}_h$, satisfying $\overline{D}\mathcal{Y}_h = E(\Phi_h)$ for some holomorphic function $E(\Phi_h)$, and transforming as $\mathcal{Y}_h \to h\mathcal{Y}_h$ under some representation of the gauge group. $\overline{D}\mathcal{Y}_h = E(\Phi_h)$ implies that $\mathcal{Y}_h$ and its conjugate $\overline{\mathcal{Y}}_h$ have expansions:

$$\begin{aligned} \mathcal{Y}_h &= \eta - \theta f - \bar\theta E(\phi) + \theta\bar\theta\Big(\partial_a E(\phi)\psi_a - \tfrac{i}{2}\partial_t\eta\Big) = \eta - \theta f - \bar\theta E(\Phi) - \tfrac{i}{2}\theta\bar\theta\partial_t\eta \ , \\ \overline{\mathcal{Y}}_h &= \overline{\eta} - \bar\theta\, \overline{f} - \theta\overline{E}(\overline{\phi}) + \theta\bar\theta\Big(\overline{\psi}_a\overline{\partial}_a\overline{E}(\overline{\phi}) + \tfrac{i}{2}\partial_t\overline{\eta}\Big) = \overline{\eta} - \bar\theta\,\overline{f} - \theta\overline{E}(\overline{\Phi}) + \tfrac{i}{2}\theta\bar\theta\partial_t\overline{\eta} \ . \end{aligned} \tag{I.1.4}$$

Acting with (I.0.1) gives the supersymmetry variations:

$$Q\eta = -f \ , \qquad Qf = 0 \ , \qquad \overline{Q}\eta = E(\phi) \ , \qquad \overline{Q}f = -i\partial_t\eta + \partial_a E(\phi)\,\psi_a \ . \tag{I.1.5}$$

I.2 Vector multiplet

We assume that the gauge group G is semi-simple (inclusion of U(1) factors is trivial) with Lie algebra $\mathfrak{g}$. Denote the complexified algebra as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g}$, with Killing form given by the trace operation Tr. It admits a root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus_{\alpha\in\Phi} L_\alpha$, where $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra and Φ is the set of all roots. We can use the Chevalley basis $\mathfrak{g}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{H^{i=1,\dots,\mathrm{rk}\,G}, E^\alpha \mid \alpha \in \Phi\}$, where i indexes a set of simple roots α^i and H^i is defined in the following way:

$$\exists!\ H^i \in \mathfrak{h}_{\mathbb{C}} \ \Big|\ \alpha^i(h) = \mathrm{Tr}(H^i h)\ , \ \forall\, h \in \mathfrak{h}_{\mathbb{C}} \ . \tag{I.2.1}$$

The element E^α is also normalized so that $\mathrm{Tr}\, E^\alpha E^{-\alpha} = 1$. The compact real form is

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\big\{iH^i, E^\alpha - E^{-\alpha}, i(E^\alpha + E^{-\alpha}) \ \big|\ \alpha \in \Phi^+\big\} \ , \tag{I.2.2}$$

where Φ^+ is the set of positive roots. Using the fact that Tr splits between each summand in $\mathfrak{h}_{\mathbb{C}} \oplus_{\alpha\in\Phi^+} (L_\alpha \oplus L_{-\alpha})$, and that Tr is positive definite on H^i, it quickly follows that Tr is negative (positive) definite on $\mathfrak{g}$ ($i\mathfrak{g}$). Any $\Lambda \in i\mathfrak{g}$ can be expressed with Λ^i, Λ_1^α, $\Lambda_2^\alpha \in \mathbb{R}$ as

$$\begin{aligned} \Lambda &= \sum\nolimits_i \Lambda^i H^i + \sum\nolimits_{\alpha\in\Phi^+}\Big[\Lambda_1^\alpha(E^\alpha + E^{-\alpha}) + \Lambda_2^\alpha i(E^\alpha - E^{-\alpha})\Big] \\ &= \sum\nolimits_i \Lambda^i H^i + \sum\nolimits_{\alpha\in\Phi^+}\big(\Lambda^\alpha E^\alpha + \overline{\Lambda^\alpha}E^{-\alpha}\big) \ , \qquad \Lambda^\alpha \equiv \Lambda_1^\alpha + i\Lambda_2^\alpha \ . \end{aligned} \tag{I.2.3}$$

Therefore, defining a formal Hermitian conjugation on elements of $\mathfrak{g}_{\mathbb{C}}$ as $\overline{H^i} \equiv H^i$, $\overline{E^\alpha} \equiv E^{-\alpha}$, we can alternatively define $i\mathfrak{g}$ as $i\mathfrak{g} = \big\{\Lambda \in \mathfrak{g}_{\mathbb{C}} \,\big|\, \overline{\Lambda} = \Lambda\big\}$. A generic group element $k = e^{i\Lambda}$ then satisfies $\overline{k} = e^{-i\Lambda} = k^{-1}$. If $G = U(N)$, this formal Hermitian conjugation becomes the actual conjugate transpose on $N \times N$ matrices.

To build gauge interactions, we introduce the independent superfields Ω and V^-. Ω is valued in $\mathfrak{g}_{\mathbb{C}}$, while V^- is valued in $i\mathfrak{g}$, *i.e.*, $\overline{V^-} = V^-$. One can either use Ω alone, or include *both* Ω and V^- in the theory. The crucial role played by Ω is to allow for gauge-covariant chiral and Fermi conditions. Under gauge transformations, they transform as:

$$\begin{aligned} &e^{\Omega} \to k\, e^{\Omega}\, h^{-1} \,, && V^- \to kV^-k^{-1} + ik(\partial_t k^{-1}) \,, && \\ &h = e^{\chi} \,, && \chi : \mathbb{R}^{1|2} \to \mathfrak{g}_{\mathbb{C}} \,, && \overline{D}\chi = 0 \,, \\ &k = e^{i\Lambda} \,, && \Lambda : \mathbb{R}^{1|2} \to i\mathfrak{g} \,, && \overline{\Lambda} = \Lambda \,. \end{aligned} \tag{I.2.4}$$

Without loss of generality, V^- can be expanded as

$$V^- = A_t - \sigma - i\theta\overline{\lambda} - i\bar{\theta}\lambda + \theta\bar{\theta}D \,, \tag{I.2.5}$$

where $(A_t - \sigma,\ D)$ are valued in $i\mathfrak{g}$ and λ is valued in $\mathfrak{g}_{\mathbb{C}}$. We now define the various ingredients used to construct supersymmetric actions. The gauge-covariant superspace derivatives are defined as

$$\mathcal{D} \,\equiv\, e^{-\overline{\Omega}}\, D\, e^{\overline{\Omega}} \,, \qquad \overline{\mathcal{D}} \,\equiv\, e^{\Omega}\, \overline{D}\, e^{-\Omega} \,, \qquad \mathcal{D}_t^- \,\equiv\, \partial_t - iV^- \,, \tag{I.2.6}$$

which, according to (I.2.4) and using $\overline{D}h = D\overline{h} = 0$, transform as

$$\mathcal{D} \,\to\, k\mathcal{D}k^{-1} \,, \qquad \overline{\mathcal{D}} \,\to\, k\overline{\mathcal{D}}k^{-1} \,, \qquad \mathcal{D}_t^- \,\to\, k\mathcal{D}_t^- k^{-1} \,. \tag{I.2.7}$$

They satisfy the algebra

$$\mathcal{D}^2 = \overline{\mathcal{D}}^2 = 0 \,, \qquad \{\mathcal{D}, \overline{\mathcal{D}}\} = i(\partial_t - iV^+) \,\equiv\, i\mathcal{D}_t^+ \,, \tag{I.2.8}$$

where V^+ is an $i\mathfrak{g}$-valued superfield constructed out of Ω only:

$$V^+ \equiv D\big[e^{\Omega}\big(\overline{D}e^{-\Omega}\big)\big] + \overline{D}\big[e^{-\overline{\Omega}}\big(De^{\overline{\Omega}}\big)\big] + \big\{e^{\Omega}\big(\overline{D}e^{-\Omega}\big), e^{-\overline{\Omega}}\big(De^{\overline{\Omega}}\big)\big\} \,. \tag{I.2.9}$$

If the gauge group is Abelian this simplifies to $V^+ = -[D, \overline{D}]\,\Omega$. As it was for D and $\overline{D}$, one has $\overline{(\mathcal{D}X)} = (-1)^F\, \overline{\mathcal{D}}\,\overline{X}$ and $\overline{(\overline{\mathcal{D}}X)} = (-1)^F\, \mathcal{D}\overline{X}$. One can check that the gauge transformation of V^+ is identical to that of V^-:

$$V^+ \,\to\, kV^+k^{-1} + ik(\partial_t k^{-1}) \,, \tag{I.2.10}$$

which is consistent with (I.2.7) and (I.2.8). We will also have occasion to use the field strength superfield

$$\Upsilon \equiv [\overline{\mathcal{D}}, \mathcal{D}_t^-] = -i\overline{D}V^- - \partial_t\big[e^{\Omega}\big(\overline{D}e^{-\Omega}\big)\big] - i\big[e^{\Omega}\big(\overline{D}e^{-\Omega}\big), V^-\big] \,, \tag{I.2.11}$$

which also transforms covariantly as $\Upsilon \to k\Upsilon k^{-1}$. From the definition, it follows directly that $\overline{\mathcal{D}}\Upsilon = 0$.

Instead of Ω and V^-, we can equivalently use two other superfields V and V_h^- defined as

$$e^V \equiv e^{\overline{\Omega}} e^{\Omega} \,, \qquad V_h^- \,\equiv\, e^{\overline{\Omega}}\, V^- e^{\Omega} + \frac{i}{2} e^{\overline{\Omega}} \partial_t e^{\Omega} - \frac{i}{2}\big(\partial_t e^{\overline{\Omega}}\big) e^{\Omega} \,, \qquad \overline{V_h^-} = V_h^- \,, \tag{I.2.12}$$

which only transform under the complexified gauge transformations as:

$$e^V \to \overline{h}^{-1} e^V h^{-1} , \qquad V_h^- \to \overline{h}^{-1} V_h^- h^{-1} + \frac{i}{2}\overline{h}^{-1} e^V \partial_t h^{-1} - \frac{i}{2}\big(\partial_t \overline{h}^{-1}\big) e^V h^{-1} . \tag{I.2.13}$$

Note that V is constructed solely out of Ω, while V_h^- is built out of both V^- and Ω. In this formulation, the theory might contain V only, or both V_h^- and V. Analogously to the above, out of V and V_h^- we can construct

$$\begin{aligned} V_h^+ &\equiv \frac{1}{2} e^V \overline{D}\big(e^{-V} D e^V\big) + \frac{1}{2} D\big(e^V \overline{D} e^{-V}\big) e^V = e^{\overline{\Omega}} V^+ e^{\Omega} + \frac{i}{2} e^{\overline{\Omega}} \partial_t e^{\Omega} - \frac{i}{2}\big(\partial_t e^{\overline{\Omega}}\big) e^{\Omega} , \\ \Upsilon_h &\equiv -i\, e^V \overline{D}\Big[e^{-V}\Big(V_h^- + \frac{i}{2}\partial_t e^V\Big)\Big] = e^{\overline{\Omega}} \Upsilon e^{\Omega} . \end{aligned} \tag{I.2.14}$$

One can check that V_h^+ transforms in the same way as V_h^-, and Υ_h transforms in the same way as e^V. In an Abelian theory,

$$V_h^+ = \frac{1}{2} e^V \big(\overline{D} D - D \overline{D}\big) V . \tag{I.2.15}$$

When writing matter Lagrangians in terms of Φ_h and $\mathcal{Y}_h$ which transform with chiral gauge transformations h, it will be convenient to use V and V_h^-.

Given any chiral or Fermi superfield, one can define covariantly-chiral counterparts

$$\Phi_k \equiv e^{\Omega}\, \Phi_h , \qquad \mathcal{Y}_k \equiv e^{\Omega} \mathcal{Y}_h , \qquad \overline{\mathcal{D}} \Phi_k = 0 , \qquad \overline{\mathcal{D}} \mathcal{Y}_k = E(\Phi_k) , \tag{I.2.16}$$

which transform under the gauge group as $\Phi_k \to k\, \Phi_k$ and $\mathcal{Y}_k \to k\, \mathcal{Y}_k$. These fields are useful when one is using Ω and V^- to describe the vector multiplet.

I.3 Wess-Zumino gauge

We can expand Ω and the gauge-transformation parameters χ, Λ as:

$$\Omega = \Omega_0 + \theta \Omega_\theta + \bar\theta \Omega_{\bar\theta} + \theta\bar\theta \Omega_{\theta\bar\theta} , \qquad \chi = \chi_0 + \theta \chi_\theta - \frac{i}{2}\theta\bar\theta \partial_t \chi_0 , \qquad \Lambda = \Lambda_0 + \theta \Lambda_\theta - \bar\theta\, \overline{\Lambda}_\theta + \theta\bar\theta \Lambda_{\theta\bar\theta} . \tag{I.3.1}$$

We show that, using gauge transformations, every component of Ω can be canceled except for $\Omega_{\theta\bar\theta}$, and we can further set $\overline{\Omega}_{\theta\bar\theta} = \Omega_{\theta\bar\theta}$, *i.e.*, $\Omega_{\theta\bar\theta}$ is valued in $i\mathfrak{g}$. We shall call this component $-\frac{1}{2}(A_t + \sigma)$, where both A_t and σ are valued in $i\mathfrak{g}$. Due to the relative sign, this is independent from $(A_t - \sigma)$ in V^-. In other words, we can bring Ω to the form

$$\Omega = -\frac{1}{2}\, \theta\bar\theta\, (A_t + \sigma) , \tag{I.3.2}$$

that we dub the Wess-Zumino gauge. First, we use the transformation $\chi = \Omega_0 - \frac{i}{2}\theta\bar\theta\partial_t\Omega_0$, $\Lambda = 0$ to set $\Omega_0 \to 0$, after which only transformations with $\chi_0 = i\Lambda_0$ preserve $\Omega_0 = 0$

and are allowed. Next, performing the transformation $\chi = \theta(\Omega_\theta + \overline{\Omega}_{\bar\theta})$, $\Lambda = i\theta\overline{\Omega}_{\bar\theta} + i\bar\theta\Omega_{\bar\theta}$ sets Ω_θ, $\Omega_{\bar\theta} \to 0$. Further transformation parameters cannot have θ or $\bar\theta$ components since otherwise a nonzero $\Omega_{\bar\theta}$ would be generated. Lastly, we perform $\chi = 0$, $\Lambda = \frac{i}{2}\theta\bar\theta(\Omega_{\theta\bar\theta} - \overline{\Omega}_{\theta\bar\theta})$, after which $\Omega_{\theta\bar\theta} \to \frac{1}{2}(\Omega_{\theta\bar\theta} + \overline{\Omega}_{\theta\bar\theta})$ is valued in $i\mathfrak{g}$. The residual gauge transformations are $\chi = i\Lambda_0 + \frac{1}{2}\theta\bar\theta\partial_t\Lambda_0$, $\Lambda = \Lambda_0$, under which

$$A_t + \sigma \;\to\; e^{i\Lambda_0}(A_t + \sigma)e^{-i\Lambda_0} + i\, e^{i\Lambda_0}\partial_t e^{-i\Lambda_0} \; . \tag{I.3.3}$$

These are purely time-dependent gauge transformations, as expected. In this gauge, the gauge-covariant superspace derivatives simplify to

$$\mathcal{D}_t^+ = D_t^+ = \partial_t - i(A_t + \sigma) \; , \qquad \mathcal{D} = \partial_\theta - \frac{i}{2}\bar\theta D_t^+ \; , \qquad \overline{\mathcal{D}} = -\partial_{\bar\theta} + \frac{i}{2}\theta D_t^+ \; , \tag{I.3.4}$$

and

$$V^+ = A_t + \sigma \; , \qquad \Upsilon = \lambda - \theta\big(D_t\sigma + iD\big) - \frac{i}{2}\theta\bar\theta D_t^+\lambda \; . \tag{I.3.5}$$

The action of supersymmetry on Ω, using (I.0.1), is $\delta\Omega = \frac{1}{2}\epsilon\bar\theta(A_t + \sigma) - \frac{1}{2}\bar\epsilon\theta(A_t + \sigma)$ and the Wess-Zumino gauge is not preserved. This can be compensated by an infinitesimal gauge transformation with parameters

$$\Lambda = \frac{i}{2}\,\epsilon\bar\theta(A_t + \sigma) + \frac{i}{2}\bar\epsilon\theta(A_t + \sigma) + \mathcal{O}(\epsilon^2) \; , \qquad \chi = -\bar\epsilon\theta(A_t + \sigma) + \mathcal{O}(\epsilon^2) \; . \tag{I.3.6}$$

The supersymmetry transformations that preserve Wess-Zumino gauge are computed using δ with the addition of the compensating gauge transformation above. For Ω, its variation under the combined supersymmetry and gauge transformation is $\delta\Omega + i\Lambda - \chi = 0$ by construction. The superfields Φ_k, $\mathcal{Y}_k$ are only sensitive to the gauge transformations generated by Λ, and not to those generated by χ. The addition of the Λ-transformation (I.3.6) to δ can be directly absorbed into the supercharges:

$$Q_{\rm WZ} \equiv \partial_\theta + \frac{i}{2}\,\bar\theta\big[\partial_t - \delta_{\rm gauge}(A_t + \sigma)\big] \; , \qquad \overline{Q}_{\rm WZ} \equiv -\partial_{\bar\theta} - \frac{i}{2}\,\theta\big[\partial_t - \delta_{\rm gauge}(A_t + \sigma)\big] \; . \tag{I.3.7}$$

Note that $\delta_{\rm gauge}(\Lambda)$ acts according to the gauge representation of each superfield, except for $V^\pm$, on which $\delta_{\rm gauge}(\Lambda)\, V^\pm = \partial_t\Lambda - i[V^\pm, \Lambda]$. The modified supercharges satisfy the algebra

$$Q_{\rm WZ}^2 = \overline{Q}_{\rm WZ}^2 = 0 \; , \qquad \{Q_{\rm WZ}, \overline{Q}_{\rm WZ}\} = -i\big[\partial_t - \delta_{\rm gauge}(A_t + \sigma)\big] \; . \tag{I.3.8}$$

I.4 Transformations in Wess-Zumino gauge

Acting with (I.3.7) on $V^\pm$ and reading off the variations of each component, we find the following supersymmetry variations (and their complex conjugate) for the vector multiplet:

$$\begin{aligned} Q_{\rm WZ}\, A_t = -Q_{\rm WZ}\, \sigma &= -\frac{i}{2}\,\overline{\lambda} \; , \qquad & Q_{\rm WZ}\, \lambda &= -D_t\sigma - iD \; , \\ Q_{\rm WZ}\, D &= -\frac{1}{2} D_t^+ \overline{\lambda} \; , & \overline{Q}_{\rm WZ}\, \lambda &= 0 \; . \end{aligned} \tag{I.4.1}$$

Note that $Q_{\rm WZ}(A_t+\sigma) = \overline{Q}_{\rm WZ}(A_t+\sigma) = 0$, consistently with (I.3.8). In Wess-Zumino gauge, Φ_k and its conjugate $\overline{\Phi}_k$ have expansion:

$$\Phi_k = \phi + \theta\psi - \frac{i}{2}\theta\bar\theta D_t^+\phi\ , \qquad\qquad \overline{\Phi}_k = \overline{\phi} - \bar\theta\,\overline{\psi} + \frac{i}{2}\theta\bar\theta D_t^+\overline{\phi}\ . \tag{I.4.2}$$

Acting with (I.3.7) on Φ_k we find the following supersymmetry variations:

$$Q_{\rm WZ}\,\phi = \psi\ , \qquad Q_{\rm WZ}\,\psi = 0\ , \qquad \overline{Q}_{\rm WZ}\,\phi = 0\ , \qquad \overline{Q}_{\rm WZ}\,\psi = iD_t^+\phi\ . \tag{I.4.3}$$

Alternatively, we can obtain the same variations by acting with $\delta+\chi = -\epsilon\, Q_{\rm WZ} + \bar\epsilon\,\overline{Q}_{\rm WZ}$ on Φ_h, with χ given in (I.3.6). Analogously, $\mathcal{Y}_k$ and its conjugate $\overline{\mathcal{Y}}_k$ have the expansions

$$\begin{aligned}
\mathcal{Y}_k &= \eta - \theta f - \bar\theta E(\phi) + \theta\bar\theta\Big(\partial_a E(\phi)\psi_a - \tfrac{i}{2}D_t^+\eta\Big) = \eta - \theta f - \bar\theta E(\Phi) - \tfrac{i}{2}\theta\bar\theta D_t^+\eta \\
\overline{\mathcal{Y}}_k &= \overline{\eta} - \bar\theta\,\overline{f} - \theta\overline{E}(\overline{\phi}) + \theta\bar\theta\Big(\overline{\psi}_a\overline{\partial}_a\overline{E}(\overline{\phi}) + \tfrac{i}{2}D_t^+\overline{\eta}\Big) = \overline{\eta} - \bar\theta\,\overline{f} - \theta\overline{E}(\overline{\Phi}) + \tfrac{i}{2}\theta\bar\theta D_t^+\overline{\eta}\ ,
\end{aligned} \tag{I.4.4}$$

and acting with (I.3.7) gives the supersymmetry variations:

$$Q_{\rm WZ}\,\eta = -f\ , \qquad Q_{\rm WZ}\,f = 0\ , \qquad \overline{Q}_{\rm WZ}\,\eta = E(\phi)\ , \qquad \overline{Q}_{\rm WZ}\,f = -iD_t^+\eta + \partial_a E(\phi)\,\psi_a\ . \tag{I.4.5}$$

Again, we can obtain the same variations by acting with $\delta+\chi$ on $\mathcal{Y}_h$.

I.5 Supersymmetric Lagrangians

As with the prototypical 4d $\mathcal{N}=1$ supersymmetry, there are two broad classes of supersymmetric terms: D-terms and F-terms. Let X be a bosonic, gauge-invariant, real-valued superfield with expansion

$$X = X_0 + \theta X_\theta - \bar\theta\,\overline{X_\theta} + \theta\bar\theta X_{\theta\bar\theta}\ . \tag{I.5.1}$$

Acting with Q and $\overline{Q}$, we find that $QX_{\theta\bar\theta} = -\frac{i}{2}\partial_t X_\theta$ and $\overline{Q}X_{\theta\bar\theta} = \frac{i}{2}\partial_t\overline{X_\theta}$ are total derivatives. Moreover, $Q\overline{Q}X_0 = X_{\theta\bar\theta}$ up to a total derivative. Therefore,

$$\int d\theta d\bar\theta\, X = -X_{\theta\bar\theta} = Q\overline{Q}\,(-X_0) \tag{I.5.2}$$

is supersymmetric, and we call such terms D-terms. They are always Q and $\overline{Q}$ exact. Conversely, suppose there is a term in the Lagrangian of the form $Q\overline{Q}(-X_0)$ where X_0 is real and gauge-invariant. If there is a real-valued superfield X with bottom component X_0, it must have the same expansion (I.5.1). Therefore (I.5.2) holds and this term can be written as a D-term in superspace.

Let Y be a fermionic, gauge-invariant, complex-valued chiral superfield, $\overline{\mathcal{D}}Y = \overline{D}Y = 0$. Its complex conjugate $\overline{Y}$ is anti-chiral and satisfies $D\overline{Y} = 0$. They have expansion:

$$Y = Y_0 + \theta Y_\theta - \frac{i}{2}\,\theta\bar{\theta}\,\partial_t Y_0\ , \qquad\qquad \overline{Y} = \overline{Y_0} + \bar{\theta}\,\overline{Y_\theta} + \frac{i}{2}\,\theta\bar{\theta}\,\partial_t \overline{Y_0}\ . \tag{I.5.3}$$

Acting with Q and $\overline{Q}$ on Y and $\overline{Y}$, one finds that Y_θ and $\overline{Y_\theta}$ are separately supersymmetric up to total derivatives. Moreover, $Y_\theta = QY_0$ and $\overline{Y_\theta} = -\overline{Q}\,\overline{Y_0}$. Therefore:

$$\int d\theta\, Y + \int d\bar{\theta}\,\overline{Y} = Y_\theta + \overline{Y_\theta} = Q\,Y_0 - \overline{Q}\,\overline{Y_0} = (Q+\overline{Q})(Y_0 - \overline{Y_0}) \tag{I.5.4}$$

is supersymmetric, and we call such terms F-terms. They are always $(Q+\overline{Q})$ exact.

We can now write the following supersymmetric Lagrangians, with component expressions in Wess-Zumino gauge. In the gauge sector, if the theory only contains Ω or equivalently V, the only term we can think of is a Wilson line in $A_t + \sigma$. For a U(1) gauge group, the supersymmetric Wilson loop of charge q can be written as

$$\exp\!\left(iq \oint dt \int d\theta d\bar{\theta}\; V\right) \overset{\text{WZ}}{=} \exp\!\left(iq \oint dt\,(A_t + \sigma)\right)\ . \tag{I.5.5}$$

If both V^- and Ω are present, we can write the following terms. The conventional gauge kinetic term is

$$\frac{1}{2e_{\text{1d}}^2}\int d\theta d\bar{\theta}\ \operatorname{Tr}\overline{\Upsilon}\Upsilon = \frac{1}{2e_{\text{1d}}^2}\int d\theta d\bar{\theta}\ \operatorname{Tr}\overline{\Upsilon_h}e^{-V}\Upsilon_h e^{-V} \overset{\text{WZ}}{=} \frac{1}{2e_{\text{1d}}^2}\operatorname{Tr}\Big[(D_t\sigma)^2 + D^2 + i\bar{\lambda}D_t^+\lambda\Big]\ . \tag{I.5.6}$$

Note that the superfield $V^- - V^+$ transforms covariantly, $V^- - V^+ \to k\,(V^- - V^+)\,k^{-1}$, under gauge transformations. For an adjoint-invariant form $\zeta : i\mathfrak{g} \to \mathbb{R}$, the Fayet-Iliopoulos term is:

$$\int d\theta d\bar{\theta}\ \zeta\big(V^- - V^+\big) = \int d\theta d\bar{\theta}\ \zeta\Big(\big(V_h^- - V_h^+\big)\,e^{-V}\Big) \overset{\text{WZ}}{=} -\zeta(D)\ . \tag{I.5.7}$$

If the gauge group is Abelian, $V_h^+ e^{-V} = \frac{1}{2}(\overline{D}D - D\overline{D})V$ becomes a total derivative under the superspace integral. Therefore, FI terms for Abelian gauge groups can be written as

$$\int d\theta d\bar{\theta}\ \zeta\big(V_h^- e^{-V}\big)\ . \tag{I.5.8}$$

We can also write a mass term that gaps V^- (or equivalently the gaugino and σ):

$$-\frac{1}{2}\int d\theta d\bar{\theta}\ \operatorname{Tr}\big(V^- - V^+\big)^2 = -\frac{1}{2}\int d\theta d\bar{\theta}\ \operatorname{Tr}\Big(\big(V_h^- - V_h^+\big)\,e^{-V}\Big)^2 \overset{\text{WZ}}{=} \operatorname{Tr}\big(\bar{\lambda}\lambda - 2\sigma D\big)\ . \tag{I.5.9}$$

Moving on to the matter sector, the conventional kinetic term for a chiral multiplet is:

$$\begin{aligned} i\int d\theta d\bar{\theta}\ \overline{\Phi}_k \mathcal{D}_t^- \Phi_k &= \int d\theta d\bar{\theta}\left(\frac{i}{2}\,\overline{\Phi_h}\,e^V\partial_t\Phi_h - \frac{i}{2}\,\partial_t\overline{\Phi_h}\,e^V\Phi_h + \overline{\Phi_h}\,V_h^-\,\Phi_h\right. \\ &\overset{\text{WZ}}{=} -\bar{\phi}\big(D_t^2 + \sigma^2 + D\big)\phi + i\bar{\psi}D_t^-\psi + i\bar{\phi}\lambda\psi - i\bar{\psi}\,\bar{\lambda}\phi\ , \end{aligned} \tag{I.5.10}$$

where $D_t^- \equiv \partial_t - i(A_t - \sigma)$. It requires the presence of both V^- and Ω. Alternatively, we can write a kinetic term that couples to V^+ in place of V^-, in which case only Ω (or V) is required:

$$\begin{aligned} i\int d\theta d\bar{\theta}\, \overline{\Phi}_k \mathcal{D}_t^+ \Phi_k &= \int d\theta d\bar{\theta} \left(\frac{i}{2}\, \overline{\Phi_h}\, e^V \partial_t \Phi_h - \frac{i}{2}\, \partial_t \overline{\Phi_h}\, e^V \Phi_h + \overline{\Phi_h}\, V_h^+\, \Phi_h \right) \\ &\stackrel{\text{WZ}}{=} D_t^+ \overline{\phi}\, D_t^+ \phi + i\overline{\psi} D_t^+ \psi \;. \end{aligned} \tag{I.5.11}$$

We can also write a term with a first order action for ϕ, and it only requires Ω:

$$\int d\theta d\bar{\theta}\, \overline{\Phi}_k \Phi_k = \int d\theta d\bar{\theta}\, \overline{\Phi_h}\, e^V \Phi_h \stackrel{\text{WZ}}{=} i\,\overline{\phi} D_t^+ \phi + \overline{\psi}\psi \;. \tag{I.5.12}$$

The conventional kinetic term for a Fermi multiplet is

$$\int d\theta d\bar{\theta}\, \overline{\mathcal{Y}}_k \mathcal{Y}_k = \int d\theta d\bar{\theta}\, \overline{\mathcal{Y}_h} e^V \mathcal{Y}_h \stackrel{\text{WZ}}{=} i\overline{\eta} D_t^+ \eta + \overline{f} f - \big|E(\phi)\big|^2 - \overline{\eta}\, \partial_a E(\phi)\, \psi_a - \overline{\psi}_a\, \overline{\partial}_a \overline{E}(\overline{\phi})\, \eta \,, \tag{I.5.13}$$

and it only requires Ω. If present, terms in $E(\Phi)$ that are linear in the chiral superfields Φ_a gives rise to mass terms which gap out the chiral and Fermi multiplets together. Quadratic or higher-order terms in $E(\Phi)$ produce cubic or higher-order interactions. We shall call them E-interactions. Suppose now that we have a collection of Fermi superfields $\mathcal{Y}_i$ with $\overline{D}\mathcal{Y}_i = E_i(\Phi)$. In addition to E_i, we associate another holomorphic function $J_i(\Phi)$ of the chiral superfields to each Fermi such that $E_i J_i$ (with repeated indices summed) is gauge-invariant and $E_i J_i = 0$. Then $\mathcal{Y}_i\, J_i(\Phi)$ is a gauge-invariant fermionic chiral superfield. We can therefore write the F-terms:

$$\int d\theta\; \mathcal{Y}_i\, J_i(\Phi) + \int d\bar{\theta}\; \overline{\mathcal{Y}_i}\, \overline{J}_i(\overline{\Phi}) = -f_i J_i(\phi) - \eta_i\, \partial_a J_i(\phi)\, \psi_a - \overline{f}_i\, \overline{J}_i(\overline{\phi}) - \overline{\psi}_a\, \overline{\partial}_a \overline{J}_i(\overline{\phi})\, \overline{\eta}_i \;. \tag{I.5.14}$$

Note that because $\mathcal{Y}_i J_i$ is gauge-invariant, $\mathcal{Y}_{i,h} J_i(\Phi_h) = \mathcal{Y}_{i,k} J_i(\Phi_k)$. We will call interactions that are constructed in this way J-interactions.

I.6 Twisted 3d Yang-Mills and Chern-Simons terms

In this subsection, we show how the parts of the topologically twisted 3d Yang-Mills and Chern-Simons Lagrangians containing $\Xi_{\bar{1}}$ can be written in 1d superspace. The terms lie slightly beyond the scope of the exposition above, because $\Xi_{\bar{1}}$ transforms as a connection on S^2 under gauge transformations, as reported in (2.2.7).

Yang-Mills. The first line in (2.2.8) can be written in superspace as:

$$\begin{aligned} &\mathrm{Tr}\Big[4|F_{t\bar{1}}|^2 + 4iDF_{1\bar{1}} - 4|D_{\bar{1}}\sigma|^2 + i\overline{\Lambda}_1(D_t + i\sigma\Lambda_{\bar{1}} + 2\Lambda_t\, D_1 \Lambda_{\bar{1}} - 2\overline{\Lambda}_1\, D_{\bar{1}} \overline{\Lambda}_t \Big] \\ &\stackrel{\text{WZ}}{=} 4i \int d\theta d\bar{\theta}\; \mathrm{Tr}\Big(\Xi_{1,k}\, \partial_t \Xi_{\bar{1},k} - \mathcal{F}_{1\bar{1},k}\, V^- \Big) \,, \end{aligned} \tag{I.6.1}$$

where we defined the superfield

$$\mathcal{F}_{1\bar{1},k} \equiv \partial_1 \Xi_{\bar{1},k} - \partial_{\bar{1}} \Xi_{1,k} - i\big[\Xi_{1,k}, \Xi_{\bar{1},k}\big] \ . \tag{I.6.2}$$

Here $\mathcal{F}_{1\bar{1},k}$ transforms covariantly under super-gauge transformations as $\mathcal{F}_{1\bar{1},k} \mapsto k\mathcal{F}_{1\bar{1},k}k^{-1}$. Note that the superspace expression has the same form as a Chern-Simons term for superfields, with V^- playing the role of the connection along t. Therefore, under finite gauge transformations:

$$\begin{aligned}\delta_{\text{gauge}}\, 4i\int d\theta d\bar{\theta}\ \mathrm{Tr}\Big(\Xi_{1,k}\,\partial_t\Xi_{\bar{1},k} - \mathcal{F}_{1\bar{1},k}\,V^-\Big) &= 2i\int d\theta d\bar{\theta}\ \mathrm{Tr}\, k^{-1}\partial_t k\,\big[k^{-1}\partial_1 k,\ k^{-1}\partial_{\bar{1}}k\big]\ ,\\ &= 2i\,\mathrm{Tr}\,\partial_t\partial_\theta\Big(k^{-1}\partial_{\bar{\theta}}k\big[k^{-1}\partial_1 k,\ k^{-1}\partial_{\bar{1}}k\big]\Big) + \text{cyclic}\ .\end{aligned} \tag{I.6.3}$$

The omitted terms contain cyclic permutations of $(t, 1, \bar{1})$. This gauge variation looks like a winding number for super-gauge transformations. Since we are taking derivatives of the winding number density (albeit with respect to fermionic variables), a total derivative is expected because the winding number is homotopy invariant.

Alternatively, we can use superfields which are only sensitive to complexified gauge transformations. The superspace expression in (I.6.1) can then be written as

$$\text{(I.6.1)} = 4i\int d\theta d\bar{\theta}\ \mathrm{Tr}\Big(\Xi_{1,h}\,\partial_t\Xi_{\bar{1},h} - \mathcal{F}_{1\bar{1},h}\,e^{-V}V_h^-\Big)\ , \tag{I.6.4}$$

where total derivatives of the kind (I.6.3) have been neglected. One can check that (I.6.4) is real and gauge-invariant up to total derivatives.

Chern-Simons. We now want to write the first piece of (2.2.9) in superspace. To do this, we follow a similar procedure as in [160]. First, the fields X are extended to be functions $\widehat{X}$ of an auxiliary coordinate $y \in (0,1)$ in an arbitrary way, except that they must fulfill boundary conditions

$$\widehat{X}(\theta,\varphi,t,y=0) = 0\ , \qquad \widehat{X}(\theta,\varphi,t,y=1) = X(\theta,\varphi,t)\ . \tag{I.6.5}$$

Extended quantities will be denoted with a hat. Given (I.6.5), we have:

$$\mathcal{L}_{\text{CS},\Xi}\Big|_{\text{WZ}} = \widehat{\mathcal{L}}_{\text{CS},\Xi}(y=1)\Big|_{\text{WZ}} = \int_0^1 dy\ \partial_y\widehat{\mathcal{L}}_{\text{CS},\Xi}\Big|_{\text{WZ}}\ . \tag{I.6.6}$$

Now, $\partial_y\widehat{\mathcal{L}}_{\text{CS},\Xi}$ can be written in superspace as:

$$\begin{aligned}\partial_y\widehat{\mathcal{L}}_{\text{CS},\Xi}\Big|_{\text{WZ}} &= \mathrm{Tr}\Big[-4i\partial_y(\hat{A}_t+\hat{\sigma})\hat{F}_{1\bar{1}} + 4\partial_y\hat{A}_1\big(i\partial_t\hat{A}_{\bar{1}} - i\hat{D}_{\bar{1}}(\hat{A}_t+\hat{\sigma})\big) + \partial_y\hat{\bar{\Lambda}}_1\hat{\Lambda}_{\bar{1}}\\ &\qquad + 4\partial_y\hat{A}_{\bar{1}}\big(-i\partial_t\hat{A}_1 + i\hat{D}_1(\hat{A}_t+\hat{\sigma})\big) - \partial_y\hat{\Lambda}_{\bar{1}}\hat{\bar{\Lambda}}_1\Big]\\ &= 4\partial_y\int d\theta d\bar{\theta}\ \mathrm{Tr}\Big[\hat{\Xi}_{1,\bar{h}}\,\hat{\Xi}_{\bar{1},h} - i\hat{V}\Big(\partial_1\hat{\Xi}_{\bar{1},h} - \partial_{\bar{1}}\hat{\Xi}_{1,\bar{h}} - i\big[\hat{\Xi}_{1,\bar{h}},\,\hat{\Xi}_{\bar{1},h}\big]\Big)\Big]\ .\end{aligned} \tag{I.6.7}$$

This superspace expression is only valid in Wess-Zumino gauge where $V = -\theta\bar{\theta}(A_t + \sigma)$, and it is not invariant under super-gauge transformations. Even so, we can take it as a starting point for constructing the gauge-invariant completion. A gauge-invariant expression that reduces to the above in Wess-Zumino gauge is

$$\partial_y \widehat{\mathcal{L}}_{\mathrm{CS},\Xi} = 4 \int d\theta d\bar{\theta}\ \mathrm{Tr}\Big[- i\, e^{-\hat{V}} \partial_y \big(e^{\hat{V}}\big) \hat{\mathcal{F}}_{1\bar{1},h} + \hat{\Xi}_{1,h}\, \partial_y \hat{\Xi}_{\bar{1},h} + \partial_y \hat{\Xi}_{1,\bar{h}}\, \hat{\Xi}_{\bar{1},\bar{h}} \Big] \,. \tag{I.6.8}$$

One can check that the first term is Hermitian, while the second and third terms are Hermitian conjugates of each other. Therefore

$$\begin{aligned} \mathcal{L}_{\mathrm{CS},\Xi} \ &= \ \mathrm{Tr}\Big[4iA_1\partial_t A_{\bar{1}} - 4i(A_t+\sigma)F_{1\bar{1}} + \overline{\Lambda}_1 \Lambda_{\bar{1}}\Big] \\ &\overset{\mathrm{WZ}}{=} 4 \int_0^1 dy\, d\theta d\bar{\theta}\ \mathrm{Tr}\Big[-i\, e^{-\hat{V}} \partial_y \big(e^{\hat{V}}\big) \hat{\mathcal{F}}_{1\bar{1},h} + \hat{\Xi}_{1,h}\, \partial_y \hat{\Xi}_{\bar{1},h} + \partial_y \hat{\Xi}_{1,\bar{h}}\, \hat{\Xi}_{\bar{1},\bar{h}} \Big] \,. \end{aligned} \tag{I.6.9}$$

If the gauge group is Abelian, (I.6.8) is a total derivative in y and the auxiliary coordinate y can be eliminated to give

$$\mathcal{L}_{\mathrm{CS},\Xi} = 4 \int d\theta d\bar{\theta} \left[\Xi_{1,\bar{h}}\, \Xi_{\bar{1},h} - iV\big(\partial_1 \Xi_{\bar{1},h} - \partial_{\bar{1}} \Xi_{1,\bar{h}}\big) + \frac{1}{2} \partial_1 V\, \partial_{\bar{1}} V \right] . \tag{I.6.10}$$

For a non-Abelian gauge groups there is no compact expression for the integral in y, but we can expand in powers of V. Choosing

$$\hat{\Xi}_{\bar{1},h} = y\, \Xi_{\bar{1},h} \ , \qquad\qquad \hat{V} = y\, V \ , \tag{I.6.11}$$

one obtains the following expression up to quadratic terms in V:

$$\begin{aligned} \mathcal{L}_{\mathrm{CS},\Xi} = 4 \int d\theta d\bar{\theta}\ \mathrm{Tr}\Big[&\Xi_{1,\bar{h}}\, \Xi_{\bar{1},h} - iV\Big(\partial_1 \Xi_{\bar{1},h} - \partial_{\bar{1}} \Xi_{1,\bar{h}} - i\big[\Xi_{1,\bar{h}},\, \Xi_{\bar{1},h}\big]\Big) \\ &+ \frac{1}{2}\Big(\partial_1 V - i\big[\Xi_{1,\bar{h}}, V\big]\Big)\Big(\partial_{\bar{1}} V - i\big[\Xi_{\bar{1},h}, V\big]\Big) + \mathcal{O}(V^3) \Big] \,. \end{aligned} \tag{I.6.12}$$

Appendix J

Partial gauge fixing

In this appendix we follow [141] and review the general procedure for partial gauge fixing. Let $\mathcal{G}$ be the infinite-dimensional group of gauge transformations, and $\{e_A\}$ a Hermitian basis for its algebra $\mathfrak{g}$. Denote the structure constants of $\mathfrak{g}$ as $[e_A, e_B] = i f_{ABC}\, e_C$. The basis $\{e_A\}$ is also chosen such that it is orthonormal under the inner product

$$\int \operatorname{Tr}\left(e_A\, e_B\right) = \delta_{AB} \ . \tag{J.0.1}$$

Let $\mathcal{R} \subset \mathcal{G}$ be a subgroup, which will be the group of residual gauge transformations after partial gauge fixing. We call its algebra $\mathfrak{r} \subset \mathfrak{g}$ ($\mathfrak{r}$ stands for residual). We split the basis as $\{e_A\} = \{e_i, e_a\}$, where $\{e_i\}$ is a basis for $\mathfrak{r}$ whereas $\{e_a\}$ is a basis for $\mathfrak{f} \cong \mathfrak{g}/\mathfrak{r}$ ($\mathfrak{f}$ stands for gauge-fixed). Since $\mathcal{R}$ is a subgroup, $\mathfrak{r}$ is a subalgebra and $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}$, or $f_{ija} = 0$. By anti-symmetry of the structure constants this implies $f_{iaj} = 0$, or $[\mathfrak{r}, \mathfrak{f}] \subset \mathfrak{f}$. In summary, the algebra of $\mathfrak{g}$ decomposes as

$$[e_i, e_j] = i f_{ijk}\, e_k \ , \qquad [e_i, e_a] = i f_{iab}\, e_b \ , \qquad [e_a, e_b] = i f_{abi}\, e_i + i f_{abc}\, e_c \ . \tag{J.0.2}$$

In particular, this implies that the e_a's transform under the adjoint action in a real orthogonal representation of $\mathcal{R}$, which we call R_f.

In order to fix $\mathcal{G}$ to $\mathcal{R}$, we need to choose as many gauge-fixing conditions as there are generators in $\mathfrak{f}$. In other words we need to choose gauge-fixing functions $G^a_{\text{gf}}(X)$, where X collectively denotes physical fields in the chiral and vector multiplets. Notice that $G^a_{\text{gf}}(X)$ should transform in R_f under $\mathcal{R}$. This is true for all the gauge-fixing functions we can think of. The first step in the gauge-fixing procedure is to integrate in an adjoint scalar $\Lambda \in \mathfrak{g}$, and add $\int \frac{1}{2} \operatorname{Tr} \Lambda^2$ to the action. Notice that Λ will have mass dimension $[\Lambda] = 3/2$. Since Λ is completely decoupled from everything else, introducing it does not change the path integral. We then insert 1 in the path integral, written as

$$1 = \Delta(X, \Lambda) \int_{\mathcal{G}} \mathcal{D}g \prod_a \delta\big(G^a_{\text{gf}}(X^g) - (\Lambda^g)^a\big) \ , \tag{J.0.3}$$

where superscripts $(\cdot)^g$ denote a finite gauge transformation by g. Suppose that $g_{X,\Lambda} \in \mathcal{G}$ satisfies $G^a_{\text{gf}}(X^{g_{X,\Lambda}}) - (\Lambda^{g_{X,\Lambda}})^a = 0$, then so does $r g_{X,\Lambda}$ for any $r \in \mathcal{R}$, due to the covariant transformations of G^a_{gf} and Λ^a under $\mathcal{R}$. Therefore, $\mathcal{R}$ remains as the residual gauge group.

Notice that it is necessary for Λ to transform under gauge transformations. This is different from the standard Faddeev-Popov procedure, in which Λ is only integrated over at the very last step. That would have been sufficient if the gauge were completely fixed ($\mathcal{R} = 0$). The slightly different procedure described here will produce extra interaction terms in the ghost action. Now, as usual, the invariance of $\mathcal{D}g$ ensures that the determinant Δ is gauge-invariant, and

$$\Delta(X,\Lambda)^{-1} = \Delta(X^{g_{X,\Lambda}},\Lambda^{g_{X,\Lambda}})^{-1} = \int_{\mathcal{G}} \mathcal{D}g \prod_a \delta\Big(G^a_{\rm gf}(X^{g\cdot g_{X,\Lambda}}) - (\Lambda^{g\cdot g_{X,\Lambda}})^a\Big)\ . \tag{J.0.4}$$

Assuming no Gribov copies and writing $g = 1 + \epsilon^A e_A$, $\delta_A \equiv \delta_{\rm gauge}(e_A)$, one can expand the argument of the delta function to linear order in ϵ^A and obtain $\epsilon^b\,\delta_b\big[G_{\rm gf}(X^{g_{X,\Lambda}}) - \Lambda^{g_{X,\Lambda}}\big]^a$. The fact that the terms with ϵ^i disappear ensures that $\mathrm{Vol}(\mathcal{R})$ is factorized as an overall factor in the Faddeev-Popov determinant:

$$\Delta(X,\Lambda) = \det \delta_b\Big[G^a_{\rm gf}(X^{g_{X,\Lambda}}) - (\Lambda^{g_{X,\Lambda}})^a\Big] / \,\mathrm{Vol}(\mathcal{R})\ . \tag{J.0.5}$$

The determinant can be shown to be well-defined on the coset $\mathcal{R}g_{X,\Lambda}$. Having determined $\Delta(X,\Lambda)$, inserting 1 in the path integral gives

$$\int \mathcal{D}X\,\mathcal{D}\Lambda\,\mathcal{D}g\; e^{iS(X)-\frac{i}{2}\int \mathrm{Tr}\,\Lambda^2}\;\Delta(X,\Lambda)\prod_a \delta\big(G^a_{\rm gf}(X^g) - (\Lambda^g)^a\big)\ . \tag{J.0.6}$$

Undoing the gauge transformation in the delta function, the integral over the gauge group factorizes and one gets

$$\int \mathcal{D}X\,\mathcal{D}\Lambda\; e^{iS(X)-\frac{i}{2}\int \mathrm{Tr}\,\Lambda^2}\det\big(\delta_b G^a_{\rm gf}(X) - \delta_b\Lambda^a\big)\prod_a \delta\big(G^a_{\rm gf}(X) - \Lambda^a\big)\ . \tag{J.0.7}$$

By means of $\delta_b\Lambda^a = i\Lambda^A[e_b,e_A]^a = -\Lambda^A f_{bAa} = -f_{abi}\Lambda^i - f_{abc}\Lambda^c$ we can explicitly write:

$$\det\Big(\delta_b G^a_{\rm gf}(X) - \delta_b\Lambda^a\Big) = \int\Big(\prod_a \mathcal{D}\widetilde{c}^a\,\mathcal{D}c^a\Big)\,\exp\!\Big[-\widetilde{c}^a\Big(\delta_b G^a_{\rm gf}(X) + f_{abi}\Lambda^i + f_{abc}\Lambda^c\Big)c^b\Big]\ , \tag{J.0.8}$$

where we have introduced the Grassmann scalars c^a, $\widetilde{c}^a$. Note that they are valued in $\mathfrak{f}$ and not in $\mathfrak{g}$: modes corresponding to residual gauge transformations are not present. Also note that by dimensional analysis, $[\widetilde{c}]+[c] = [G_{\rm gf}] = 3/2$. Without loss of generality, we can take $[c] = 0$, $[\widetilde{c}] = 3/2$. Integrating out Λ^i and imposing the delta functions for Λ^a, one gets the action:

$$S(X) + \int \mathrm{Tr}\Big[-\frac{G_{\rm gf}^2}{2} + G_{\rm gf}\{\widetilde{c},c\} + i\,\widetilde{c}\,\delta_{\rm gauge}(c)\,G_{\rm gf} + \frac{1}{2}\{\widetilde{c},c\}_{\mathfrak{r}}\{\widetilde{c},c\}_{\mathfrak{r}}\Big]\ . \tag{J.0.9}$$

This is equivalent to the following action with extra scalars b^a integrated in:

$$S(X) + \int \mathrm{Tr}\Big[\frac{b^2}{2} + b\Big(G_{\rm gf} - \{\widetilde{c},c\}\Big) + i\,\widetilde{c}\,\delta_{\rm gauge}(c)\,G_{\rm gf} + \frac{1}{2}\{\widetilde{c},c\}^2\Big]\ . \tag{J.0.10}$$

Notice that b^a have dimension $[b] = 3/2$. One should keep in mind that $c, \widetilde{c}, b$ only contain modes in $\mathfrak{f}$. We will now rescale

$$G_{\rm gf} \to e_{\rm 3d}^{-1}\, G_{\rm gf} \qquad b \to e_{\rm 3d}^{-1}\, b\,, \qquad c \to e_{\rm 3d}^{-1}\, c\,, \tag{J.0.11}$$

after which $[G_{\rm gf}] = 2$, $[c] = \frac{1}{2}$, and $[b] = 2$. The gauge-fixing action gains an overall factor of $1/e_{\rm 3d}^2$. This is useful because the background Coulomb gauge $G_{\rm gf} = D_i^B A^i/\sqrt{\xi}$ (with ξ a positive dimensionless parameter) that we choose in the main text has dimension $[G_{\rm gf}] = 2$. This is true for many other standard gauge-fixing functions, such as the Lorenz gauge $\partial_\mu A^\mu/\sqrt{\xi}$ and the background Lorenz gauge $D_\mu^B A^\mu/\sqrt{\xi}$.

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