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Geometric analysis and measure theory in general ambient spaces

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Contents

Chapter 1

Introduction

This thesis is divided in four main chapters: in the first and present one we shall describe the content of the thesis and introduce some general preliminaries.

The second chapter is based on the two works [BGLL23] and [GG23] and deals with measure theory in the non-smooth setting: the first part consists on a proof of a conjecture raised by David H. Fremlin, concerning the non-triviality of the n-dimensional Hausdorff measure of $\mathbb{R}^n$, endowed with a general distance inducing the Euclidean topology. In the second part we shall instead study a type of differentiable structure for metric measure spaces, proving that some well-known properties holding on Riemannian manifolds remain valid in this setting.

The third chapter is based on [CG23] and deals with the asymptotics of the s-fractional perimeter on general (possibly weighted) Riemannian manifolds and RCD spaces when $s \to 0^+$, proving at the same time general results of independent interest concerning the heat kernel.

The fourth and last chapter is based on [GGZZ24] and deals with the regularity of harmonic maps with domain an $\mathrm{RCD}(K, N)$ space and target which is a $\mathrm{CAT}(\kappa)$ space, extending to the non-smooth setting various result holding in the smooth framework of Riemannian manifolds.

During my PhD I also studied the Fourier transform of functions of bounded variation, leading to the preprint [BG24a]. One of the results of the paper consists in an equivalent characterisation of sets of finite perimeter in terms of the Fourier transform of their characteristic function.

1.1 Contents of Chapter 2

In the first chapter we shall discuss about measure theory in non-smooth spaces. The first result we are going to present is an answer to a question of David H. Fremlin: it is a standard result that the Lebesgue measure in $\mathbb{R}^n$ is, when normalized in the correct way, equivalent to the n-dimensional Hausdorff measure computed with respect to the Euclidean distance. One of the consequences of this equivalence (such consequence can be inferred in many different ways) is that $\mathcal{H}^n_{\mathrm{d}_{\mathrm{eucl.}}}(U) > 0$ for every $U \subset \mathbb{R}^n$ open set. The behaviour of the Hausdorff measure should in principle strongly depend on what is the distance we use to define it, however in [BGLL23] we are able to show that if the distance induces the Euclidean topology, the property of giving positive measure to open sets is preserved, no matter which distance function is chosen. We can summarize the results obtained in [BGLL23] in the following

Theorem 1.1.1. *Let ρ be a distance on $\mathbb{R}^n$ inducing the Euclidean topology, then for every open set U we have $\mathcal{H}^n_\rho(U) > 0$.*

The strategy to prove the latter is to exploit topological techniques such as degree theory (in this thesis we shall present a proof via Brouwer fixed point theorem, instead of introducing the topological degree). After the completion of the thesis, Sylvester Eriksson-Bique has pointed out that the question of Fremlin had an answer long before it was posed. Indeed the first solution of such problems appears in [Szp37] and it is for example repeated in [HW41] and, with a more general statement and using the concept of topological dimension, in [Hei01, Theorem 8.15]. However our proof differs from the classical ones, which use Coarea inequality.

The second part of the first chapter is instead devoted to the study of the reference measure on the so-called metric measure spaces. To introduce the problem let us first discuss what happens in the smooth setting: consider an n-dimensional Riemannian manifold (M, g) endowed with its volume measure μ. It is a standard result that $\mu = \mathcal{H}^n_{\mathrm{d}_g}$, where d_g is the distance function induced by the Riemannian metric. The latter distance is defined as follows

$$\mathrm{d}_g(x,y) = \inf\left\{ \int_0^1 \sqrt{g_{\gamma_t}(\dot\gamma_t, \dot\gamma_t)}\,\mathrm{d}t :\ \gamma : [0,1] \to M,\ \gamma(0) = x,\ \gamma(1) = y,\ \gamma \in C^1([0,1]; M)\right\},$$

for all $x, y \in M$.

Now consider a chart $\varphi : U \subseteq M \to \mathbb{R}^n$ and the measure $\nu := \varphi_\sharp(\mu_{|U})$: we claim that $\nu \ll \mathscr{L}^n$. This is essentially due to the fact that φ^{-1} is a Lipschitz map from $\varphi(U)$ to U (say that U is bounded for simplicity) and the fact that for a Lipschitz map ψ between metric spaces $(\mathsf{X}, \mathrm{d}_\mathsf{X})$ and $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ there holds

$$\mathcal{H}^n_{\mathrm{d}_\mathsf{Y}}(\psi(A)) \le \mathrm{Lip}^n(\varphi)\mathcal{H}^n_{\mathrm{d}_\mathsf{X}}(A).$$

for every $A \subseteq \mathsf{X}$ Borel. In these types of arguments the fact that one deals with a chart which is invertible and that the measure on the Riemannian manifold is the Hausdorff measure is somewhat crucial. However this kind of assumption is not necessary and such a phenomenon occurs even in non-smooth structures as the one of metric measure spaces.

In the recent, very interesting, paper [EBS21] the authors provided a general construction of charts on metric measure spaces, key features of their notion being: the compatibility with Sobolev calculus (and thus in particular with the differential calculus as developed by Cheeger in [Che99] and Gigli [Gig15]), a very general existence result, notable consequences in terms of the structure of the Sobolev spaces (see also [EBRS22a] and [EBRS22b]). An example in this latter direction is the proof that the space $W^{1,p}(\mathsf{X})$, $p \in (1, \infty)$ (and actually the cotangent module $L^p(T^*\mathsf{X})$ as well), is reflexive as soon as the space X can be covered by a countable number of sets with finite Hausdorff measure (the 'previous best' result appeared in [ACDM14] and required the metric to be locally doubling).

A crucial step in [EBS21] is the proof that if $\varphi : E \subset \mathsf{X} \to \mathbb{R}^n$ is a 'p-independent weak chart', then n is bounded from above by the Hausdorff dimension of E: more precisely the authors prove the following:

Proposition 1.1.2. *Suppose $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathbb{R}^n)$ is p-independent on U. Then $n \le \dim_\mathcal{H}(U)$.*

For the precise meaning of 'p-independent weak chart' we refer to 2.2.8; for the purpose of this introduction we shall limit ourselves to point out that in the smooth setting this would be equivalent to requiring the image of the differential of φ at every point to span the whole tangent space of $\mathbb{R}^d$. Starting from this result, existence of actual charts is obtained via a suitable maximality argument.

Interestingly, this upper bound is proved via means that have, in principle, little to do with analysis in non-smooth setting: key ingredients are indeed the elliptic regularity result in [DPR] and the study of the structure of the set of non-differentiability points of Lipschitz functions in [AM16].

This sort of procedure has a recent analogue in the theory of RCD spaces. Let us recall indeed that in [MN14] it has been proved that finite dimensional RCD spaces admit bi-Lipschitz charts covering almost all the space. In [MN14] no information about the behaviour of the reference measure w.r.t. these charts has been provided: this topic has been later studied in [KM18], [DPMR], [GP21] where, relying in a way or another on [DPR] and [AM16], it has been proved that $\varphi_*(\mathfrak{m}_{|E}) \ll \mathscr{L}^n$ for a Mondino-Naber chart $\varphi : E \to \mathbb{R}^n$.

Of particular interest for the discussion here is the fact that in [GP21] only the results in [DPR] have been used, while in [KM18] also those in [AM16] were necessary. Comparing this with the results in [EBS21] it is natural to wonder whether the use of [AM16] is really crucial or can be avoided: this is the question motivating the present note. Of course, there is nothing wrong in using a well-established result in doing research, our study is simply motivated by the desire of better understanding the interesting construction done in [EBS21]. The result of our investigation is that [AM16] is not really needed and the line of thought presented here simplifies not only some of the steps done in [EBS21], but also some of those in [GP21]: see Section 2.3.

Another remark that we make, consequence of the studies in [EBS21], is that the dimension of the (co)tangent module (in the sense of [Gig15]) on a subset $E \subset \mathsf{X}$ is bounded from above from the Hausdorff dimension of E, see Remark 2.3.11.

1.2 Contents of Chapter 3

This chapter deals with the asymptotic behaviour of the s-fractional perimeter as s goes to zero and shows how harmonic functions are very much related to this problem.

The study of this kind of asymptotic was initiated in [DFPV13], where the authors focus on $\mathbb{R}^n$. In the Euclidean setting one can define the s-perimeter as follows (up to dimensional constants)

$$P_s(E) = s \int_{E^c} \int_E \frac{1}{|x-y|^{n+s}} \, \mathrm{d}x \, \mathrm{d}y.$$

It is possible to show that the previous quantity is equal (again up to dimensional constants) to

$$P_s(E) = \int_{\mathbb{R}^n} |\xi|^s |\mathcal{F}\{\mathbb{1}_E\}|^2(\xi) \, \mathrm{d}\xi.$$

If one assumes $|E| < \infty$ and $P_{s_0}(E) < \infty$ for some $s_0 \in (0,1)$ it is easy to infer from a dominated convergence argument and Plancherel theorem that

$$\lim_{s \to 0^+} P_s(E) = |E|.$$

Indeed the Fourier transform of $\mathbb{1}_E$ is continuous and bounded by $|E|$, hence we can pass to the limit inside the integral thanks to dominated convergence; one can take as domination the function $g(\xi) = |E|\mathbb{1}_{B_1} + |\xi|^{s_0} |\mathcal{F}\{\mathbb{1}_E\}|^2(\xi)\mathbb{1}_{B_1^c}$. This technique is difficult to adapt if one wants to find the convergence for the relative s-perimeter and even more difficult if one considers general Riemannian manifolds. From a dimensional point of view however one should expect an analogous behaviour for what concerns the asymptotic as $s \to 0^+$.

To approach this problem, on a general Riemannian manifold, given a Borel set $E \subset M$, we shall introduce and study the following quantity

$$\theta_E(p) := \lim_{s\to 0^+} \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x)\,, \tag{1.2.1}$$

where

$$\mathcal{K}_s(x,y) := \frac{1}{|\Gamma(-s/2)|}\int_0^\infty H_M(x,y,t)\,\frac{dt}{t^{1+s/2}} \tag{1.2.2}$$

and $H_M(x,y,t) : M \times M \times (0,\infty)$ is the heat kernel of M, that is the minimal[1], positive fundamental solution to the heat equation $\partial_t u - \Delta_g u = 0$ on M with $u(t,\cdot) \to \delta_{\{y\}}$ in the sense of distributions as $t \to 0^+$. The quantity analogous to (1.2.1) on $\mathbb{R}^n$ was previously studied in [DFPV13], where the authors deal with the study of the fractional s-perimeter as $s \to 0^+$. In the case of $M = \mathbb{R}^n$, the limit in (1.2.1) does not depend on p (whenever it exists); hence, θ_E is a constant function.

One of the main observations of this work is that θ_E is always an harmonic function on M, with values in $[0,1]$, and in general can be non-constant if M does not satisfy the $L^\infty-$Liouville property (see Definition 3.2.4). Moreover, for $E \equiv M$, the function θ_M encodes the asymptotics of the fractional Laplacian as $s \to 0^+$ on every complete (M,g).

The following are the main results of this chapter.

Theorem 1.2.1. *Let (M,g) be a complete Riemannian manifold with $\mu(M) = +\infty$, and let $E \subset M$ be a measurable set. Then*

(i) If for some $R > 0$ and every $p \in M$, the following limit exists

$$\theta_E(p) := \lim_{s\to 0^+} \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x) \in [0,1]\,, \tag{1.2.3}$$

then it is independent of the choice of R, and $\theta_E : M \to [0,1]$ is a bounded harmonic function on M.

(ii) For $R > 0$ and $p \in M$ the limit

$$\theta_M(p) := \lim_{s\to 0^+} \int_{M\setminus B_R(p)} \mathcal{K}_s(x,p)d\mu(x) \in [0,1] \tag{1.2.4}$$

always exists, does not depend on the choice of R, and equals

$$\theta_M(p) = \lim_{t\to\infty} \int_M H_M(p,x,t)\,d\mu(x)\,. \tag{1.2.5}$$

Moreover, $\theta_M : M \to [0,1]$ is a bounded harmonic function on M.

[1] Here, minimal means the following: if $v : (0,\infty) \times M \to \mathbb{R}$ is another function with $\partial_t v - \Delta_g v = 0$ on M and $v(t,\cdot) \to \delta_{\{y\}}$ as $t \to 0^+$, then $H_M(\cdot,y,\cdot) \le v$. See Section 9.1 in [Gri09] for details on this property.

Unless otherwise stated, when we will say *"assume θ_E exists"* we intend that the limit in (1.2.3) exists for some $R > 0$ and every $p \in M$. We shall also briefly discuss on the existence/nonexistence of this limit for different points p.

Next is the asymptotics of the fractional Laplacian. Note that, on well-behaved ambient spaces, one would expect (as it happens on $\mathbb{R}^n$) that the fractional $(s/2)$-Laplacian tends to the identity as $s \to 0^+$. With the following result, we show that this is not true on general Riemannian manifolds and that the harmonic function θ_M defined in (1.2.4) encodes how this limit differs from the identity.

Theorem 1.2.2. *Let (M, g) be a complete Riemannian manifold with $\mu(M) = +\infty$, and let θ_M be given by (1.2.4). Let also $s_\circ \in (0, 2)$ and $u \in H^{s_\circ/2}(M) \cap L^\infty(M)$ (see Definition 3.1.1) with bounded support. Then, as $s \to 0^+$ there holds*

$$(-\Delta)^{s/2}_{\mathrm{Si}} u \to \theta_M u \;\; a.e. \; on \; M, \tag{1.2.6}$$

where $(-\Delta)^{s/2}_{\mathrm{Si}}$ is the singular integral fractional Laplacian (3.1.4).

With this result, we also make an interesting observation regarding a Riemannian manifold constructed by Pinchover in [Pin95]. This Riemannian manifold satisfies the $L^\infty - \mathrm{Liouville}$ property (see Definition 3.2.4), but it is not stochastically complete, and we show that it satisfies $\theta_M \equiv 0$. We describe the construction of this manifold in Example 3.5.2. Consequently, there exist complete Riemannian manifolds where the mass of the heat kernel escapes so rapidly that the asymptotics of the fractional Laplacian not only differs from the identity but becomes identically zero, even for regular functions.

In the following result, we address the equivalence (actually, equality) of different definitions of the fractional Laplacian on stochastically complete manifolds. Moreover, we also find the asymptotics of the fractional Laplacian on manifolds with finite volume.

Theorem 1.2.3. *Let (M, g) be a stochastically complete Riemannian manifold. Let also $s_\circ \in (0, 2)$ and $u \in H^{s_\circ/2}(M)$ (see Definition 3.1.1). Then, for all $s < s_\circ$ the three definitions of the fractional Laplacian (3.1.4), (3.1.2), and (3.6.13) coincide a.e., that is*

$$(-\Delta)^{s/2}_{\mathrm{Si}} u = (-\Delta)^{s/2}_{\mathrm{B}} u = (-\Delta)^{s/2}_{\mathrm{Spec}} u \,.$$

Moreover, as $s \to 0^+$

$$(-\Delta)^{s/2} u \xrightarrow{L^2} u - \frac{1}{\mu(M)} \int_M u \, d\mu \quad \text{if} \;\; \mu(M) < +\infty \,, \tag{1.2.7}$$

and

$$(-\Delta)^{s/2} u \xrightarrow{L^2} u \quad \text{if} \;\; \mu(M) = +\infty \,, \tag{1.2.8}$$

where $(-\Delta)^{s/2}$ is any of the equivalent fractional Laplacians.

In proving the previous theorems, we also provide an equivalent characterization of being stochastically complete (see Definition 3.2.1) in the case of infinite volume.

Proposition 1.2.4. *Let (M, g) be a complete (possibly weighted) Riemannian manifold with $\mu(M) = +\infty$, and let $\theta_M(p)$ be given by (1.2.4). If M is stochastically complete, then*

$$\theta_M = \lim_{s \to 0^+} \int_{M \setminus B_1(p)} \mathcal{K}_s(x, p) d\mu(x) = 1 \quad \forall p \in M \,. \tag{1.2.9}$$

Conversely, if there exists $p \in M$ such that

$$\theta_M(p) = \lim_{s\to 0^+} \int_{M\setminus B_1(p)} \mathcal{K}_s(x,p)d\mu(x) = 1\,, \tag{1.2.10}$$

then M is stochastically complete.

We will prove this result at the beginning of section 3.4.

As a corollary of the results above we are able to obtain the asymptotics of the fractional perimeter as $s \to 0^+$ in an extremely general setting, generalizing both the existing results [DFPV13] for $\mathbb{R}^n$ and [CCLMP22] for the Gaussian space. Although these outcomes currently stem from broader results obtained in our investigation, we emphasize that the initial motivation behind this research was to explore the asymptotic properties of the fractional perimeter on general Riemannian manifolds.

In particular, with Theorem 1.2.5 and 1.2.7, we show that these two known behaviors of the asymptotics, the one of $\mathbb{R}^n$ and the one of the Gaussian space, are essentially the only two possible also in this general setting.

Theorem 1.2.5 (Infinite volume asymptotics). *Let (M,g) be a complete, stochastically complete Riemannian manifold with $\mu(M) = +\infty$ and such that the $L^\infty - \text{Liouville}$ property holds (see Definition 3.2.4). Let $\Omega \subset M$ be an open, bounded, connected set with Lipschitz boundary. Let also $E \subset M$ be a measurable set with[2] $P_{s_\circ}(E,\Omega) < +\infty$, for some $s_\circ \in (0,1)$, and such that θ_E exists (see (1.2.3)). Then*

(i) The limit $\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega)$ exists and

$$\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega) = \theta_{M\setminus E}\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)\,. \tag{1.2.11}$$

(ii) Conversely, if $\mu(\Omega\cap E) \neq \mu(\Omega\setminus E)$ and the limit $\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega)$ exists, then the limit in (1.2.3) exists and there holds

$$\theta_E = \frac{\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega) - \mu(E\cap\Omega)}{\mu(\Omega\setminus E) - \mu(E\cap\Omega)}\,.$$

(iii) If $\mu(\Omega\cap E) = \mu(\Omega\setminus E)$ then the limit $\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega)$ always exists and

$$\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega) = \mu(\Omega\cap E) = \mu(\Omega\setminus E)\,.$$

Remark 1.2.6. Without the assumption of stochastic completeness of M the situation can be different. We will describe in Example 3.5.2 a complete Riemannian manifold N, with the $L^\infty - \text{Liouville}$ property but not stochastically complete such that $\lim_{s\to 0^+} P_s(E) = 0$ for every subset $E \subset N$. Moreover observe that the r.h.s. of (1.2.11) can be rewritten as $(1-\theta_E)\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)$ since $\theta_{M\setminus E} = 1-\theta_E$ in this case.

Theorem 1.2.7 (Finite volume asymptotics). *Let (M,g) be a complete Riemannian manifold with $\mu(M) < +\infty$, and let $\Omega \subset M$ be an open and connected set with Lipschitz boundary. If for some set $E \subset M$ there exists $s_\circ \in (0,1)$ such that $P_{s_\circ}(E,\Omega) < +\infty$, then the limit $\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega)$ exists and*

$$\lim_{s\to 0^+} \frac{1}{2}P_s(E,\Omega) = \frac{1}{\mu(M)}\Big(\mu(E)\mu(E^c\cap\Omega) + \mu(E\cap\Omega)\mu(E^c\cap\Omega^c)\Big).$$

[2] We refer to Definition 3.1.3 for the definition of the fractional perimeter $P_s(\cdot,\Omega)$.

Finally in 3.6 we shall extend the results concerning the asymptotic of the fractional perimeter to RCD spaces, where there is enough technology to repeat the proofs in a similar fashion.

1.3 Contents of Chapter 4

In the last 50 years the study of harmonic maps has been blooming and gained a lot of interest from the mathematical community. One of the main questions is the one of existence of such mappings and parallel to that there is the issue of their regularity.

When $u : \Omega \subseteq M^n \to N^k$ is an harmonic map between Riemannian manifolds (M^n, g_M) and (N^k, g_N) the picture nowadays is quite clear: the existence of such mappings has been established via the study of parabolic problems by Hamilton (see [Ham75] for a discussion on the topic) and then by looking at the problem in a variational way. The latter approach can be tailored to the non-smooth setting as well, indeed in the recent [Sak23] the author has been able to prove the existence of harmonic maps between an RCD space and a $\mathrm{CAT}(\kappa)$ space if the image of u is contained in a sufficiently small ball.

Back to the case of an harmonic map between smooth Riemannian manifolds, the Bochner-Eells-Sampson formula states that

$$\Delta\left(\frac{|\,\mathrm{d}u|^2_{\mathsf{HS}}}{2}\right) = |\nabla\,\mathrm{d}u|^2_{\mathsf{HS}} + \mathrm{Ric}_{g_M}(\nabla u, \nabla u) - \sum_{i,j\leq n}\langle u_*\mathcal{R}^N(e_i, e_j)e_i, e_j\rangle,$$

where Ric_{g_M} is the Ricci tensor of the source space, $u_*\mathcal{R}^N$ is the pullback of the curvature tensor of the target space via the map u and $\mathrm{e}_{\alpha}{}_{\alpha=1}^{n}$ is an orthonormal frame for the tangent bundle TM. If we assume that $\mathrm{Ric}_{g_M} \geq -K$ (lower bound on the Ricci tensor) and $\mathrm{R_N} \leq \kappa$ (upper bound on the sectional curvatures), the previous identity can be turned into the following inequality

$$\Delta\left(\frac{|\,\mathrm{d}u|^2_{\mathsf{HS}}}{2}\right) \geq |\nabla\,\mathrm{d}u|^2_{\mathsf{HS}} + K|\,\mathrm{d}u|^2_{\mathsf{HS}} - \kappa|\,\mathrm{d}u|^4_{\mathsf{HS}}. \tag{1.3.1}$$

From this inequality, at least if $\kappa = 0$ it is possible to quickly deduce that harmonic maps are locally Lipschitz, as in this case we have

$$\Delta\left(\frac{|\,\mathrm{d}u|^2_{\mathsf{HS}}}{2}\right) \geq K|\,\mathrm{d}u|^2_{\mathsf{HS}} \tag{1.3.2}$$

and thus a De Giorgi-Nash-Moser argument shows that the function $f := |\,\mathrm{d}u|^2_{\mathsf{HS}}$ is locally bounded. The case $\kappa > 0$, say $\kappa = 1$, is more delicate and is known to require the additional assumption that the range of u is contained in a ball $B_r(p) \subset N^k$ of radius $r < \frac{\pi}{2}$ (otherwise there are known counterexamples to regularity [Riv95]). On top of this, the term $|\,\mathrm{d}u|^4$ is a priori not in L^1, making it hard to extract informations from (1.3.1). To overcome these difficulties, Serbinowski argued as follows: the function $f(x) := \mathsf{d}_N(u(x), p)$ satisfies $-\Delta\cos(f) \geq |\,\mathrm{d}u|^2_{\mathsf{HS}}\cos(f)$ (as a consequence of the fact that u is harmonic and of the curvature assumption on N), and quite trivially we have $|\,\mathrm{d}|\,\mathrm{d}u|_{\mathsf{HS}}|^2 \leq |\nabla\,\mathrm{d}u|^2_{\mathsf{HS}}$. These consideration and little algebraic manipulation show that (1.3.1) implies

$$\tfrac{|\,\mathrm{d}u|_{\mathsf{HS}}}{\cos(f)}\mathrm{div}\left(\cos^2(f)\nabla\left(\tfrac{|\,\mathrm{d}u|_{\mathsf{HS}}}{\cos(f)}\right)\right) \geq K|\,\mathrm{d}u|^2_{\mathsf{HS}}, \tag{1.3.3}$$

and since $\cos(f)$ is far from zero, a Moser iteration argument can be called into play to prove that $\frac{|\,\mathrm{d}u|_{\mathsf{HS}}}{\cos(f)}$, and thus $|\,\mathrm{d}u|_{\mathsf{HS}}$, is locally bounded, as desired.

This type of reasoning allows to conjecture that, in the non-smooth setting, one should impose a lower bound on the Ricci curvature of the source and an upper bound on the sectional curvature on the target to get that an harmonic maps is (locally) Lipschitz.

Many contributions in this direction have appeared in the recent years: for an account of the story we refer to the extensive introductions in [ZZ18], [ZZZ19], [MSa] and [Gig23b]. Here we just recall that in [ZZ18] the authors proved the Lipschitz regularity of harmonic maps between Alexandrov spaces and a weak Bochner-Eells-Sampson inequality. Building on this, in the more recent [Gig23b] and [MSa] the authors where able to establish such regularity when the source space is an $\mathrm{RCD}(K, N)$ space, namely a space with a synthetic notion of Ricci curvature bounded below by K and dimension bounded above by N, and the target is a $\mathrm{CAT}(0)$ space, namely a space with a synthetic notion of sectional curvature bounded above by 0.

Very roughly said, the basic argument to get a sort of (1.3.2) and local Lipschitz regularity of harmonic maps is to build two families $(g_t), (h_t)$ of functions (via a kind of Hopf-Lax formula for metric-valued maps) converging to $|\,\mathrm{d}u|^2$ in L^1 as $t \downarrow 0$ satisfying

$$\tfrac{1}{2}\Delta g_t \geq K h_t \qquad \forall t > 0. \tag{1.3.4}$$

Quite clearly, from this it is possible to pass to the limit and obtain that

$$\Delta\left(\frac{|\,\mathrm{d}u|^2}{2}\right) \geq K|\,\mathrm{d}u|^2. \tag{1.3.5}$$

Notice that in this the quantity $|\,\mathrm{d}u|$ is the operator norm of $\mathrm{d}u$, not its Hilbert-Schmidt norm as in (1.3.2), thus (1.3.5) is not the same as (1.3.2), but the effect is the same: a Moser iteration argument shows that $|\,\mathrm{d}u|$ must be locally bounded and thus that u is locally Lipschitz.

When dealing with the case $\kappa > 0$ this strategy encounters a problem, as the approximation procedure does not work well in conjunction with Serbinowski's technique: shortly said, at the approximated level the right hand side of (1.3.1) still contains a term that does not go to 0 in L^1 as $t \downarrow 0$.

Because of these difficulties, we do not achieve Lipschitz regularity of harmonic maps in the setting, our main results are rather:

1) the proof of Hölder continuity, see Theorem 4.2.12. Here we follow the strategy in [Jos97].

2) the higher integrability of the energy density, see Theorem 4.2.15, by using a Caccioppoli inequality and the Gehring lemma in [Maa08, AHT17].

3) Under the a priori assumption that the harmonic function is Lipschitz, possibly with a sub-optimal control on the Lipschitz constant, we prove a version of inequality (1.3.1) and thus get sharper Lipschitz estimates; see Theorem 4.2.26. To achieve this we suitable combine ideas from [ZZZ19], [MSa] and [Gig23b]. Once we have this, following the arguments in [ZZZ19] one can obtain a sharp estimate on the Lipschitz constant and, as a consequence, a Liouville-type of result, Theorem 4.2.27 and Corollary 4.2.28 for the precise statements.

4) the boundary regularity, see Theorem 4.2.31.

1.4 Preliminaries

In this section we shall focus on the necessary preliminaries for the forthcoming chapters.

1.4.1 Measure theory and Sobolev calculus

As references for what follows we would like to point out [GP20a], the monographs [Bog07b] and [Fed69]. Let (X, d) be a complete and separable metric space and $\mathfrak{m}$ a measure on (X, d)

Definition 1.4.1 (regular Borel measure)**.** We say that $\mathfrak{m}$ is a regular Borel measure on X if it is defined on the Borel sigma algebra $\mathscr{B}(\mathsf{X})$ and for all $E \in \mathscr{B}(\mathsf{X})$ the two following conditions hold

$$\begin{aligned}\mathfrak{m}(E) &= \inf\{\mathfrak{m}(U): \quad E \subseteq U,\ U \text{ open}\}\\ \mathfrak{m}(E) &= \inf\{\mathfrak{m}(C): \quad C \subseteq E,\ C \text{ closed}\}.\end{aligned}$$

Definition 1.4.2 (Radon measure)**.** We say that a regular Borel measure is Radon if

$$\mathfrak{m}(E) = \inf\{\mathfrak{m}(K): \quad K \subseteq E,\ K \text{ compact}\}.$$

for all $E \in \mathscr{B}(\mathsf{X})$. We denote the set of Radon measures as $\mathcal{M}(\mathsf{X})$, while with $\mathcal{M}_+\mathsf{X}$ we denote the subset of $\mathcal{M}(\mathsf{X})$ made of positive ones.

We shall write $\mu \in \mathcal{P}(\mathsf{X})$ if $\mu \in \mathcal{M}_+(\mathsf{X})$ and $\mu(\mathsf{X}) = 1$. This last definition allows us to introduce the main class of spaces we shall deal with in this thesis, which are the so-called *metric measure spaces*.

Definition 1.4.3 (Metric measure space)**.** We say that $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ is a metric measure space if (X, d) is a complete and separable metric space and $\mathfrak{m} \in \mathcal{M}(\mathsf{X})$ is finite on balls.

Remark 1.4.4. The requirement of finiteness over balls is crucial when one deals with spaces which are infinite dimensional (balls are not compact with respect to the strong topology).

We shall now introduce an important class of measures which one can naturally associate to a metric space, namely the so-called Hausdorff measures (in the following $\mathcal{P}(\mathsf{X})$ denotes the power set of X).

Definition 1.4.5 (Hausdorff measure)**.** Let $s \in (0, \infty)$ be a real number. We define the s-dimensional Hausdorff outer measure of $A \in \mathcal{P}(\mathsf{X})$ as

$$\mathcal{H}^s_{\mathrm{d}}(A) := \sup_{\delta>0} \mathcal{H}^s_{\delta,\mathrm{d}}(A), \qquad \text{with} \tag{1.4.1}$$

$$\mathcal{H}^s_{\delta,\mathrm{d}}(A) := \inf\Big\{\sum_{i\in I} \mathrm{diam}(A_i)^s \ : \ A \subseteq \cup_{i\in I} A_i,\ \mathrm{diam}(A_i) \le \delta\Big\}, \tag{1.4.2}$$

where $\mathrm{diam}(U) = \sup_{x,y\in U} \mathrm{d}(x, y)$ and I is an at most countable collection of indices.

Remark 1.4.6. The usual definition of Hausdorff measure is given scaling the result by a dimensional constant that, for instance, in the Euclidean case is equal to $2^{-n}\omega_n$, where ω_n is the volume of the unit n-ball. We opted to overlook the constant in order to simplify the notation.

Definition 1.4.7 (Pushforward measure)**.** Let μ be a Borel measure on X and let $T : \mathsf{X} \to \mathsf{Y}$ be a Borel map, where Y is a topological space. We define the *pushforward* of μ via T as the Borel measure $T_\sharp\mu$ such that

$$T_\sharp\mu(E) = \mu\big(T^{-1}(E)\big)$$

for all $E \in \mathscr{B}(\mathsf{X})$.

The following property of the pushforward of a measure holds

Proposition 1.4.8. *Let μ be a Borel measure on X and $T : \mathsf{X} \to \mathsf{Y}$ be a Borel map, where Y is a topological space. Then for all $\varphi : \mathsf{Y} \to [0, \infty]$ Borel we have the following* change of variables *formula*

$$\int_{\mathsf{Y}} \varphi \,\mathrm{d}T_\sharp\mu = \int_{\mathsf{X}} \varphi \circ T \,\mathrm{d}\mu.$$

We shall now recall the classical Disintegration theorem. The statement below is taken from [AGS08, Theorem 5.3.1], see also [Fre06, Chapter 452] and [Bog07a, Chapter 10.6].

Theorem 1.4.9 (Disintegration)**.** *Let X, Y be complete and separable metric spaces, $\mu \in \mathcal{P}(\mathsf{X})$, let $\pi : \mathsf{X} \to \mathsf{Y}$ be a Borel map and let $\nu = \pi_\sharp\mu \in \mathcal{P}(\mathsf{Y})$. Then there exists a ν-a.e. uniquely determined Borel family of probability measures $\{\mu_y\}_{y\in\mathsf{Y}} \subseteq \mathcal{P}(\mathsf{X})$ such that $\mu_x(\mathsf{X}\backslash\pi^{-1}(\{y\})) = 0$ for ν-a.e. $y \in \mathsf{X}$ and*

$$\int_{\mathsf{X}} f \,\mathrm{d}\mu = \int_{\mathsf{Y}} \bigg(\int_{\pi^{-1}(\{y\})} f \,\mathrm{d}\mu_y\bigg) \,\mathrm{d}\nu(y) \tag{1.4.3}$$

for every Borel map $f : \mathsf{X} \to [0, +\infty]$.

Remark 1.4.10. Two remarks are in order here: the first one is that the above theorem in [AGS08] is stated for Radon separable metric space but in our setting it suffices to state it for complete and separable ones (which in particular are Radon), the second is that the result easily extends to any $f : \mathsf{X} \to \mathbb{R}$ Borel provided for example that $f \in L^1(\mu)$.

We shall now develop calculus tools on metric measure spaces: the material comes from the seminal work [AGS14a] [Gig18] (see also [GP20a]).

A function $f : \mathsf{X} \to \mathbb{R}$ is said to be Lipschitz (continuous) if there exists $L \in [0, +\infty)$ which we call $\mathrm{Lip}(f)$ such that $|f(y) - f(x)| \leq L\mathsf{d}(x, y)$ holds for all $x, y \in \mathsf{X}$. We denote with $\mathrm{Lip}(\mathsf{X})$, $\mathrm{Lip_b}(\mathsf{X})$ and $\mathrm{Lip_{bs}}(\mathsf{X})$ the set of Lipschitz functions, the set of bounded Lipschitz functions and the set of Lipschitz functions with bounded support respectively. For a generic function $f : \mathsf{X} \to \mathbb{R}$ we set

$$\mathrm{lip} f(x) := \begin{cases} \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}_{\mathsf{X}}(x,y)} & \text{if } x \text{ is not isolated} \\ 0 & \text{if } x \text{ is isolated} \end{cases}$$

and we call it *local Lipschitz constant* of f. If $f \in \mathrm{Lip}(\mathsf{X})$ obviously one has $\mathrm{lip}(f)(x) \leq \mathrm{Lip}(f)$ for all $x \in \mathsf{X}$.

With these definitions it is possible to introduce the Sobolev space $W^{1,p}(\mathsf{X})$ following the approach of [Che99]. To do so we define the so-called pre-Cheeger energy

$$\mathrm{prCh}_p(f) := \begin{cases} \frac{1}{p}\int_{\mathsf{X}} \mathrm{lip}^p(f) \,\mathrm{d}\mathfrak{m} & \text{if } \ f \in \mathrm{Lip_{bs}}(\mathsf{X}) \\ +\infty & \text{otherwise,} \end{cases}$$

mimicking the Dirichlet energy. We then define the relaxation of the latter to obtain an $L^p(\mathfrak{m})$ ($p \in (1,\infty)$) lower semicontinuous functional, defining the following *Cheeger energy*

$$\mathrm{Ch}_p(f) := \inf\left\{ \liminf_{k\to+\infty} \mathrm{prCh}_p(f_k) : \ (f_k) \subseteq \mathrm{Lip_{bs}}(\mathsf{X}) \text{ s.t. } f_k \to f \text{ in } L^p(\mathfrak{m}) \right\}. \tag{1.4.4}$$

Definition 1.4.11 (Sobolev spaces)**.** We say that a function u belongs to $W^{1,p}(\mathsf{X})$ if $u \in L^p(\mathfrak{m})$ and $\mathrm{Ch}_p(f) < +\infty$.

It is possible to prove that for every $f \in W^{1,p}(\mathsf{X})$ there exists a minimal $L^p(\mathfrak{m})$ function, which we denote with $|Df|_p$, such that $\mathrm{Ch}_p(f) := 1/p \int_{\mathsf{X}} |Df_p|^p \,\mathrm{d}\mathfrak{m}$. The latter function has the meaning of modulus of the differential of a function and it is not only minimal in the L^p sense but also in the $\mathfrak{m}$-a.e. sense. It turns out that $W^{1,p}(\mathsf{X})$ equipped with the norm $\|u\|_{W^{1,p}(\mathsf{X})} := \big(\|u\|_{L^p(\mathfrak{m})} + p\mathrm{Ch}(f)\big)^{\frac{1}{p}}$ is a Banach space.

We shall continue the investigation of Sobolev spaces via the approach introduced in [AGS14a], which is there proven to be equivalent to the one in [Che99]. We shall denote with $AC^p([0,1];\mathsf{X})$,$p \in [1,\infty)$, the subset of the continuous curves ($C([0,1];\mathsf{X})$) $\gamma : \mathsf{X} \to [0,1]$ which are such that there exists $g \in L^p([0,1])$ for which

$$\mathrm{d}(\gamma_t,\gamma_s) \le \int_s^t g(r)\,\mathrm{d}r \quad \forall s < t \in [0,1]. \tag{1.4.5}$$

It can be proven that the following limit

$$\lim_{h\to 0^+} \frac{\mathrm{d}(\gamma_{t+h},\gamma_t)}{h} =: |\dot\gamma_t| \tag{1.4.6}$$

exists for a.e. $t \in (0,1)$ (with respect to the Lebesgue measure). We call $|\dot\gamma_t|$ the *metric speed* of the curve γ. Moreover one can show that $|\dot\gamma_t|$ is the minimal (in the a.e. sense) L^p function that can be chosen in (1.4.5) as function g. Let $p,q \in [1,\infty]$ be such that $\frac{1}{p}+\frac{1}{q} = 1$ (where we agree that $\frac{1}{\infty} = 0$), we have the following.

Definition 1.4.12 (Test plan)**.** We say that a probability measure π on $C([0,1];\mathsf{X})$ is a *q-test plan* if it is concentrated on $AC([0,1];\mathsf{X})$ and the following two conditions are met:

1. $\exists\, C = C(\pi) > 0$ such that $e_{t\sharp}\pi \le C\mathfrak{m}$, where $\mathfrak{m}$ is the reference measure on X and $e_t : C([0,1];\mathsf{X}) \to \mathsf{X}$ is the evaluation map $e_t(\gamma) = \gamma_t$.
2. The following quantity, called *kinetic energy*, is finite
$$\mathrm{K.E.}(\pi) = \int\int_0^1 |\dot\gamma_t|^q \,\mathrm{d}t\,\mathrm{d}\pi(\gamma).$$

We are now ready to give the second definition of Sobolev space: due to its equivalence with Definition 1.4.11 proved in [AGS14a] we will refer to this spaces again as $W^{1,p}(\mathsf{X})$.

Definition 1.4.13. We say that a function $f : \mathsf{X} \to \mathbb{R}$ is in $S^p(\mathsf{X})$ if $f \in L^0(\mathfrak{m})$ and if there exists $G \in L^p(\mathfrak{m})$ such that

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\pi(\gamma) \le \int_0^1 \int G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t \tag{1.4.7}$$

holds for all $\pi \in \mathcal{P}(C([0,1];\mathsf{X}))$ q-test plan.

Remark 1.4.14. Again as for the case of Definition 1.4.11 there exists minimal function G (both in the L^p norm and in the $\mathfrak{m}$-a.e. sense), which we call *minimal p-weak upper gradient* and keep denoting by $|Df|_p$ (since it is $\mathfrak{m}$-a.e. equal to the one minimizing the Cheeger energy $\mathrm{Ch}_p(f)$), satisfying (1.4.7). These two objects are actually equivalent but the proof is not immediate; see [AGS13] for the proof.

A fundamental notion to develop calculus tools on this non-smooth setting is the one of *infinitesimal Hilbertianity*, introduced in [Gig15].

Definition 1.4.15 (Infinitesimal Hilbertianity)**.** We say that a metric measure space $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ is infinitesimally Hilbertian if Ch_2 is a quadratic form, or equivalently if $W^{1,2}(\mathsf{X})$ is an Hilbert space.

1.4.2 L^p-normed L^∞-modules

We now switch our attention to the theory of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules developed in [Gig18]: the following material can be found there, unless otherwise stated.

Definition 1.4.16 ($L^p(\mathfrak{m})$-normed module)**.** We say that a Banach space $(\mathcal{M}, \|\cdot\|_\mathcal{M})$ is an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module if there exists a bilinear continuous map $\cdot : L^\infty(\mathfrak{m}) \times \mathcal{M} \to \mathcal{M}$ which makes $\mathcal{M}$ a module with unity over the ring of $L^\infty(\mathfrak{m})$ functions and another map $|\cdot| : \mathcal{M} \longrightarrow L^p(\mathfrak{m})$ with nonnegative values such that

$$\||v|\|_{L^p(\mathfrak{m})} = \|v\|_\mathcal{M}, \tag{1.4.8}$$

$$|f \cdot v| = |f||v| \qquad \mathfrak{m} - \text{a.e.} \tag{1.4.9}$$

for all $v \in \mathcal{M}$, $f \in L^\infty(\mathfrak{m})$. We call $\cdot$ the multiplication and $|\cdot|$ the *pointwise norm*.

Remark 1.4.17. Note that the pointwise norm is continuous thanks to the triangular inequality, in fact

$$\||v| - |w|\|_{L^p(\mathfrak{m})} \leq \||v - w|\|_{L^p(\mathfrak{m})} = \|v - w\|_\mathcal{M}.$$

Moreover with a little bit of abuse of notation we will write fv instead of $f \cdot v$ and write $L^p(\mathfrak{m})$-normed module instead of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

A related interesting concept is the one of *localization* of a module, indeed it is easy to see that the following object

$$\mathcal{M}_{|E} := \{\chi_E v : \ v \in \mathcal{M}\}$$

is a submodule of $\mathcal{M}$ and it clearly inherits the normed structure from $\mathcal{M}$.

Definition 1.4.18 (Local independence)**.** Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed module and $A \in \mathscr{B}(X)$ with $\mathfrak{m}(A) > 0$, we say that a family $v_1, ..., v_n \in \mathcal{M}$ is independent on A if for every $f_1, ..., f_n \in \mathrm{L}^\infty(\mathfrak{m})$ we have

$$\sum_{i=1}^n f_i v_i = 0 \quad \mathfrak{m} - \text{a.e. on } A \implies f_i = 0 \quad \mathfrak{m} - \text{a.e. on } A \quad \forall i = 1, ..., n. \tag{1.4.10}$$

In the spirit of linear algebra we shall also define what is the *span* of a set of vectors

Definition 1.4.19 (Span)**.** Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed module, $V \subset \mathcal{M}$ a subset and $A \in \mathscr{B}(X)$. We denote with $\mathrm{Span}_A(V)$ the closure in $\mathcal{M}$ of the $L^\infty(\mathfrak{m})$-linear combinations of elements of V. Moreover we say that $\mathrm{Span}_A(V)$ is the space generated by V on A.

After this definition, the one of basis and of dimension for an $L^p(\mathfrak{m})$-normed module $\mathcal{M}$ arise naturally:

Definition 1.4.20. We say that a finite family $v_1, ..., v_n \in \mathcal{M}$ is a basis on $A \in \mathscr{B}(X)$ if it is independent on A and $\mathrm{Span}_A\{v_1, ..., v_n\} = \mathcal{M}_{|A}$. If the above happens we say that the *local dimension* of $\mathcal{M}$ on A is n and in case $\mathcal{M}$ has not dimension k for any $k \in \mathbb{N}$ we say that it has infinite dimension.

It can be proved that the notion of dimension is well-posed, namely if we have $v_1, ..., v_n$ generating $\mathcal{M}$ on a set A and $w_1, ..., w_m$ are independent on A, then $n \geq m$. Ultimately this means that two different basis must have the same cardinality.

Building over these tools we have the following proposition:

Proposition 1.4.21. *Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed module. Then there is a unique partition $\{E_i\}_{i \in \mathbb{N} \cup \{\infty\}}$ of X, up to $\mathfrak{m}$-a.e. equality, such that:*

1. *for every $i \in \mathbb{N}$ such that $\mathfrak{m}(E_i) > 0$, $\mathcal{M}$ has dimension i on E_i,*

2. *for every $E \subset E_\infty$ with $\mathfrak{m}(E) > 0$, $\mathcal{M}$ has infinite dimension on E.*

We shall now introduce the notion of dual module, in analogy with that of dual of a Banach space.

Definition 1.4.22 (Dual module)**.** 2.2. We say that the space of $L^\infty(\mathfrak{m})$-linear and continuous maps $L : \mathcal{M} \to L^1(\mathfrak{m})$ is the dual module of the module $\mathcal{M}$ and we shall denote this space by $\mathcal{M}^*$.

Remark 1.4.23. Being $\mathcal{M}$ $L^p(\mathfrak{m})$-normed, we can endow $\mathcal{M}^*$ with a natural structure of $L^q(\mathfrak{m})$-normed module.

When dealing with a module the reader can observe that there are two ways to consider "functionals" acting on it. The first way is to view the module $\mathcal{M}$ as a Banach space and consider its dual $\mathcal{M}'$, the second is by considering its dual in the sense of modules, $\mathcal{M}^*$. To pass from one description to the other one we can introduce the map $\mathrm{Int}_{\mathcal{M}} : \mathcal{M}^* \to \mathcal{M}'$ such that

$$\mathrm{Int}_{\mathcal{M}}(T)(\cdot) := \int_{\mathsf{X}} T(\cdot)\, \mathrm{d}\mathfrak{m}.$$

It is easy to verify that such map is an isometry from $\mathcal{M}^*$ to $\mathcal{M}'$; with a little bit of work it is also possible to show that for $L^p(\mathfrak{m})$-normed modules such map is also onto.

Finally we can introduce the following map $\mathcal{J}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^{**}$, which to every $v \in \mathcal{M}$ associates the map $\mathcal{J}_{\mathcal{M}} : \mathcal{M}^* \to L^1(\mathfrak{m})$ given by

$$J_{\mathcal{M}}(v)(L) := L(v) \qquad \forall L \in \mathcal{M}^*.$$

Exploiting the fact that the map $\mathrm{Int}_{\mathcal{M}}$ is an isometric isomorphism, it is possible to prove that $J_{\mathcal{M}}$ is an isometry which may however fail to be surjective. This allows us to give the following definition

Definition 1.4.24 (Reflexive module)**.** We say that an $L^p(\mathfrak{m})$-normed module is reflexive if the map $J_{\mathcal{M}}$ is onto.

As one may notice, this definition resembles a lot the one given for Banach spaces and the similarity is not purely formal. Indeed we have

Proposition 1.4.25. *Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed module with $p \in (1, \infty)$. Then $\mathcal{M}$ is reflexive as a module if and only if it is reflexive as a Banach space.*

With the following we Theorem we shall build differentials of functions and subsequently link Sobolev calculus with $L^p(\mathfrak{m})$-normed modules. The idea is to construct an $L^p(\mathfrak{m})$-normed module which plays the role of the cotangent bundle on a Riemannian manifold. By duality we shall then construct the *tangent* module and we will finally be able to speak about derivations, vector fields and gradients.

We have the following:

Theorem 1.4.26 (Existence of the cotangent module)**.** *There exists a unique couple $(L^p(T^*\mathsf{X}), \mathrm{d})$, where $L^p(T^*\mathsf{X})$ is an $L^p(\mathfrak{m})$-normed module and $\mathrm{d}_p : S^p(\mathsf{X}) \to L^p(\mathfrak{m})$ is a linear operator such that*

- $|\,\mathrm{d}_p f| = |Df|_p$ *holds* $\mathfrak{m}$*-a.e. for every* $f \in S^p(\mathfrak{m})$.
- $L^p(T^*\mathsf{X})$ *is generated in the sense of modules by* $\{\,\mathrm{d}f :\ f \in S^p(\mathsf{X})\}$.

Remark 1.4.27. Uniqueness in the previous theorem is meant up to unique isomorphism: if there is another couple $(\mathcal{M}, T)$ satisfying the properties of the theorem, then there is a unique module isomorphism $\Phi : L^p(T^*\mathsf{X}) \to \mathcal{M}$ such that $\Phi \circ \mathrm{d}_p = T$.

We therefore call $L^p(T^*\mathsf{X})$ the *p-cotangent module* and d_p the *differential*: from an euristic point of view one can think to the elements of $L^p(T^*\mathsf{X})$ as *differential forms*.

Definition 1.4.28 (Tangent module)**.** We define the *tangent module* as the dual module of $L^p(T^*\mathsf{X})$ and we denote it by $L^q(T\mathsf{X})$, where q is the conjugate exponent of p.

In analogy with the smooth case one can think to the elements of $L^q(T\mathsf{X})$ as *vector fields*, *derivations* or *gradients*.

Remark 1.4.29. It is possible to prove that $L^q(T\mathsf{X})$ is an $L^q(\mathfrak{m})$-normed module whose pointwise norm is given by

$$|b_*| := \operatorname{ess\,sup}\{|b(\omega)| :\ \omega \in L^p(T^*\mathsf{X}), |\omega| \leq 1\}.$$

Finally we can introduce the *gradient* of a Sobolev function as the following

$$\mathrm{Grad}_q(f) := \{v \in L^q(T\mathsf{X}) :\ v(\,\mathrm{d}_p f) = |\,\mathrm{d}_p f|^p = |v|^q\ \mathfrak{m} - \text{a.e.}\}. \tag{1.4.11}$$

As one can observe $\mathrm{Grad}_q(f)$ is a set, rather than a single element of a module. Via the Hahn-Banach theorem it is possible to prove that this setter is never empty, however it may not be a singleton. Whenever it will be a singleton we shall denote its single element by $\nabla_q f$ (or simply ∇f when clear from the context).

Since we are now able to speak about gradients and vector fields, by duality we can introduce the concept of divergence.

Definition 1.4.30 (Divergence)**.** We say that a vector field $v \in L^q(T\mathsf{X})$ is in $\mathrm{Div}^q(\mathsf{X})$ if there exists a function $g \in L^q(\mathfrak{m})$ such that for all $f \in W^{1,p}(\mathsf{X})$ one has

$$\int_{\mathsf{X}} v(\,\mathrm{d}_p f)\,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} g f\,\mathrm{d}\mathfrak{m}.$$

We call g the divergence of v and we denote it by $\mathrm{div}(v)$.

Now that we are able to speak about vector fields it would be interesting to speak about *derivations* as well. We shall do this thanks to the ideas developed in [Mar14] and the Sobolev calculus.

Definition 1.4.31 (Derivations). We say that a linear map $b : S^p(\mathsf{X}) \to L^1(\mathfrak{m})$ is a derivation and we write $b \in \mathrm{Der}^q(\mathsf{X})$ if there exists $\ell \in L^q(\mathfrak{m})$ such that

$$|b(\mathrm{d}_p f)| \leq \ell |Df|_p \qquad \mathfrak{m} - \text{a.e.} \tag{1.4.12}$$

holds for every $f \in S^p(\mathfrak{m})$.

We immediately state (and prove, for the reader convenience) a result connecting the notion of derivation with the one of vector field, which is [Gig18, Theorem 2.3.3]

Theorem 1.4.32 (Derivations and vector fields). *For any vector field $X \in L^q(T\mathsf{X})$ the map $X \circ \mathrm{d}_p : S^p(\mathsf{X}) \to L^1(\mathfrak{m})$ is a derivation. Conversely, given a derivation $b \in \mathrm{Der}^q(\mathsf{X})$ there exists a unique vector field $X \in L^q(T\mathsf{X})$ such that the diagram*

$$\begin{array}{ccc} S^p(\mathsf{X}) & \xrightarrow{\ \mathrm{d}\ } & L^q(T^*\mathsf{X}) \\ & {\scriptstyle b}\searrow & \downarrow {\scriptstyle X} \\ & & L^1(\mathfrak{m}) \end{array}$$

commutes.

Proof. The map $X \circ \mathrm{d}_p$ is linear and satisfies

$$|(X \circ \mathrm{d}_p)(f)| = |\,\mathrm{d}_p f(X)| \leq |X|\,|\,\mathrm{d}_p f|_* = |X|\,|Df|_p, \qquad \mathfrak{m} - \text{a.e.} \qquad \forall f \in S^p(\mathsf{X}).$$

Since $|X| \in L^q(\mathfrak{m})$, the first claim is proved thanks to Hölder inequality.

For the second implication consider $b \in \mathrm{Der}^q(\mathsf{X})$ and define a linear map from $D := \{\,\mathrm{d}_p f : f \in S^p(\mathsf{X})\}$ to $L^1(\mathsf{X})$ as follows

$$\mathrm{d}_p f \qquad \mapsto \qquad X(\,\mathrm{d}_p f) := b(f).$$

Thanks to (1.4.12) (and the identity $|\,\mathrm{d}_p f|_* = |Df|_p$) the definition is well-posed and we have

$$|X(\,\mathrm{d}_p f)| \leq \ell |\,\mathrm{d}_p f|_*.$$

Now thanks to the continuity (boundedness) of the latter map and the fact that D generates $L^p(T^*\mathsf{X})$ (Theorem 1.4.26) we can extend X to an $L^\infty(\mathfrak{m})$-linear and continuous functional from $L^p(T^*\mathsf{X})$ to $L^1(\mathfrak{m})$, that is we built a vector field. This concludes the proof. □

First we recall Theorem 1.21 and Proposition 1.23 from [Pas22], which do not require any assumption on the Sobolev space $W^{1,p}(\mathsf{X})$. Before stating the results a few comments are in order: first of all we are going to consider the map $e : C([0,1]; \mathsf{X}) \times [0,1] \to \mathsf{X}$ satysfing $e(\gamma, t) = \gamma_t$. Recall that it is possible to endow the space $C([0,1] : \mathsf{X})$ with the distance d_∞ which is such that

$$\mathrm{d}_\infty(\gamma, \eta) := \sup_{t \in [0,1]} \mathrm{d}(\gamma_t, \eta_t).$$

Such distance makes the space $C([0,1];\mathsf{X})$ a complete and separable metric space. Now let us endow $H := C([0,1];\mathsf{X}) \times [0,1]$ with the pythagorean distance

$$\mathrm{d}_H := \sqrt{\mathrm{d}_\infty^2 + |\cdot|^2},$$

and the measure $\pi \otimes \mathcal{L}^1$. It is easy to verify that $(H, \mathrm{d}_H, \pi \otimes \mathcal{L}^1)$ is a metric measure space.

Theorem 1.4.33 (Velocity of a test plan)**.** *Let $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ be a metric measure space and fix exponents $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let π be a given q-test plan on $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$. Then there exists a unique element $\pi' \in \big(e^* L^p(T^*\mathsf{X})\big)^*$ such that, for any function $f \in W^{1,p}(\mathsf{X})$ and $\mathscr{L}^1$-a.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ e_t := \lim_{h \to 0} \frac{f \circ e_{t+h} - f \circ e_t}{h} = \pi'(e^* \,\mathrm{d}_p f)(\cdot, t), \tag{1.4.13}$$

where the derivative is taken with respect to the strong topology of $L^1(\pi)$. Moreover,

$$|\pi'|(\gamma, t) = |\dot\gamma_t| \qquad \text{for } \ \pi \otimes \mathscr{L}^1 - a.e. \ \ (\gamma, t) \in AC^q([0,1];\mathsf{X}) \times [0,1].$$

We call π' the p-velocity of the test plan π.

Proposition 1.4.34. *Let $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $p, q \in (1, +\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let π be a q-test plan on X. Then for every function $f \in W^{1,p}(\mathsf{X})$ the mapping $t \mapsto f \circ e_t$ belongs to $AC^q([0,1]; L^1(\pi))$ and satisfies*

$$f \circ e_t - f \circ e_s = \int_s^t \pi'(e^* \,\mathrm{d}_p f)(\cdot, r) dr \qquad \forall s, t \in [0,1] \ \ \textit{with} \ \ s < t. \tag{1.4.14}$$

Remark 1.4.35. From 1.4.14 we deduce that for π-a.e. $\gamma \in AC^q([0,1];\mathsf{X})$ we have

$$\int_s^t (f \circ \gamma)'_r \,\mathrm{d}r = \int_s^t \pi'(e^* \,\mathrm{d}_p f)(\cdot, r) dr \qquad \forall s, t \in [0,1] \ \ \text{with} \ \ s < t.$$

By Lebesgue differentiation and Fubini this means that for $\pi \otimes \mathscr{L}^1$-a.e. $(\gamma, t) \in \mathrm{AC}^q([0,1];\mathsf{X})$ we have

$$(f \circ \gamma)'_t = \pi'(e^* \,\mathrm{d} f)(\gamma, t). \tag{1.4.15}$$

Following [DM14] we first need to disintegrate the measure $\pi \otimes \mathscr{L}^1_{|[0,1]}$ with the evaluation map $e : \mathrm{AC}^q([0,1];\mathsf{X}) \times [0,1] \to \mathsf{X}$ so that we obtain a family of probability measures on curves $\{\pi_x\}_{x \in \mathsf{X}}$ satisfying, thanks to 1.4.3,

$$\int_0^1 \int f(\gamma, t) \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) = \int_{\mathsf{X}} \int_0^1 \int f(\gamma, t) \,\mathrm{d}\pi_x(\gamma, t) \,\mathrm{d}e_\sharp \pi(x)$$

for every $f \in L^1(\tilde\pi)$, with π_x concentrated on the set $\{(\gamma, t) \in \mathrm{AC}^q([0,1];\mathsf{X}) \times [0,1] : \ \gamma_t = x\}$ for $\mathfrak{m}$-a.e. $x \in \mathsf{X}$.

Note moreover that due to the compression bound $e_{t\sharp}\pi \leq C\mathfrak{m}$ we easily deduce a bound on the product measure $\tilde\pi = \pi \otimes \mathscr{L}^1_{|[0,1]}$, indeed $e_\sharp \tilde\pi \leq C\mathfrak{m}$, so that there exists $\rho \in L^1(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$ such that $e_\sharp \tilde\pi = \rho\mathfrak{m}$. We can therefore write

$$\int_0^1 \int f(\gamma, t) \,\mathrm{d}t \,\mathrm{d}\pi(\gamma) = \int_{\mathsf{X}} \int_0^1 \int f(\gamma, t) \,\mathrm{d}\pi_x(\gamma, t) \rho(x) \,\mathrm{d}\mathfrak{m}$$

for every $f \in L^1(\tilde\pi)$.

Remark 1.4.36. It is a standard result that if (X, d) is a complete and separable metric space then so is $C([0,1];\mathsf{X})$ with respect to the uniform topology and a fortiori so are $AC^p([0,1];\mathsf{X})$ and $AC^p([0,1];\mathsf{X}) \times [0,1]$.

Let us now define $V := \Big\{ \sum_{i=1}^n \chi_{E_i} \, \mathrm{d} f_i \ : \ (E_i) \subseteq \mathscr{B}(\mathsf{X}) \text{ disjoint}, \ f_i \in \mathsf{W}^{1,p}(\mathsf{X}) \ \forall i = 1, ..., n \Big\}$, which is a dense subset of $L^p(\mathrm{T}^*\mathsf{X})$, then we have the following:

Proposition 1.4.37. *Consider the map* $\underline{b}_\pi : V \longrightarrow L^1(\mathfrak{m})$ *defined by*

$$\underline{b}_\pi(\mathrm{d}f)(x) := \int_0^1 \int \pi'(e^* \, \mathrm{d}f)(\gamma, t) \rho(x) \, \mathrm{d}\pi_x(\gamma, t), \tag{1.4.16}$$

with π_x *being the disintegrations of the measure* $\hat{\pi} = \pi \otimes \mathrm{d}t$, π *q-test plan, with respect to the evaluation map* $e : C([0,1];\mathsf{X}) \times [0,1] \longrightarrow \mathsf{X}$ *such that* $e(\gamma, t) = \gamma_t$, *while* $\rho \in L^1(\mathfrak{m}) \cap L^\infty(\mathsf{X})$ *is such that* $e_\sharp \hat{\pi} = \rho \mathfrak{m}$.

Then $\underline{b}_\pi$ *is continuous and uniquely extends to a continuous map from* $L^p(T^*\mathsf{X})$ *to* $L^1(\mathfrak{m})$.

Proof. Let us prove that there exists a function $G_\pi \in L^q(\mathfrak{m})$ with nonnegative values such that

$$|\underline{b}_\pi(\mathrm{d}f)| \leq G_\pi |Df|_p \quad \mathfrak{m}-\text{a.e.}.$$

Note that the regularity of ρ follows from the fact that $\forall A \in \mathscr{B}(\mathsf{X})$ we have

$$e_\sharp \hat{\pi}(A) = \int_0^1 \pi(\{\gamma : \gamma_t \in A\}) \, \mathrm{d}t \leq C(\pi) \mathfrak{m}(A).$$

We have

$$|\underline{b}_\pi(\mathrm{d}f)|(x) \leq \rho(x) \int_0^1 \int |Df|_p(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}\pi_x(\gamma, t) = |Df|_p(x) \rho(x) \int_0^1 \int |\dot{\gamma}_t| \, \mathrm{d}\pi_x(\gamma, t),$$

since π_x is concentrated on $\{(\gamma, t) \ : \ \gamma_t = x\}$ for $\mathfrak{m}$-a.e. $x \in \mathsf{X}$ and defining $G_\pi(x) := \rho(x) \int_0^1 \int |\dot{\gamma}_t| \, \mathrm{d}\pi_x(\gamma, t)$ we have $G_\pi \in L^q(\mathfrak{m})$ since

$$\int_{\mathsf{X}} |G_\pi(x)|^q \, \mathrm{d}\mathfrak{m}(x) \leq \int_0^1 \int_{\mathsf{X}} \int |\dot{\gamma}_t|^q \rho(x)^q \, \mathrm{d}\pi_x(\gamma, t) \, \mathrm{d}\mathfrak{m}(x) = C(\pi)^{q-1} \text{K.E.}(\pi),$$

where we used Jensen inequality and the properties of the disintegration. Note moreover that the latter map is $L^\infty(\mathfrak{m})$-linear, indeed $\forall g \in L^\infty(\mathfrak{m})$ we have

$$\underline{b}_\pi(g \, \mathrm{d}f) = \int_0^1 \int \pi'(e^*(g \, \mathrm{d}f))(\gamma, t) \rho(x) \, \mathrm{d}\pi_x(\gamma, t) = \int_0^1 \int \pi'(e^*(\mathrm{d}f))(\gamma, t) \rho(x) g(x) \, \mathrm{d}\pi_x(\gamma, t),$$

as in the last equality we used the fact that $e^*(g\omega) = g \circ e e^*(\omega)$, the $L^\infty(\hat{\pi})$-linearity of π' and again the properties of the disintegration.

Thanks to the previous estimate and the locality property we have that $\underline{b}_\pi$ extends to an $L^\infty(\mathfrak{m})$-linear map from $L^p(\mathrm{T}^*\mathsf{X})$ to $L^1(\mathfrak{m})$, proving the result. □

Remark 1.4.38. The maps we built in 1.4.16 enjoy additional regularity properties with respect to standard integrable vector fields, since we can compute their divergence:

$$\int_{\mathsf{X}} \underline{b}_\pi(\mathrm{d}f) \, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \int_0^1 \int \pi'(e^* \, \mathrm{d}f)(\gamma, t) \rho(x) \, \mathrm{d}\pi_x(\gamma, t) \, \mathrm{d}\mathfrak{m}(x) = \int_0^1 \int \pi'(e^* \, \mathrm{d}f)(\gamma, t) \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t \tag{1.4.17}$$

$$= \int \int_0^1 \frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}(t) \, \mathrm{d}\pi(\gamma) \, \mathrm{d}t = \int f(\gamma_1) - f(\gamma_0) \, \mathrm{d}\pi(\gamma) = \int_{\mathsf{X}} f(x) \big(\rho_1(x) - \rho_0(x)\big) \, \mathrm{d}\mathfrak{m}(x), \tag{1.4.18}$$

so that we have $\operatorname{div}(\underline{\mathrm{b}}_\pi) = \rho_0(x) - \rho_1(x) \in L^1(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$, where ρ_t is such that $e_{t\sharp}\pi = \rho_t\mathfrak{m}$ for every $t \in [0,1]$.

Thanks to the previous Proposition we are able to establish a density result concerning vector fields in general metric measure spaces, namely that we can approximate (in the sense of weak star topology) integrable vector fields with vector fields with bounded divergence.

Proposition 1.4.39. *The set* $\mathrm{Div}^q(\mathsf{X})$ *is weakly* dense in* $L^q(T\mathsf{X})$*.*

Proof. Note that to prove the Proposition we just need to show that

$$\int_X \underline{\mathrm{b}}_\pi(\omega)\,\mathrm{d}\mathfrak{m} = 0 \quad \forall \pi \text{ test plan} \implies \quad \omega = 0$$

for every $\omega = \sum_{n=1}^N \chi_{E_i}\,\mathrm{d}_p f_i$ with $f_i \in W^{1,p}(\mathsf{X})$ for every $i = 1, ..., N$ and $(E_i)_{i=1}^N$ partition of X.

Therefore let us assume

$$\int_X \underline{\mathrm{b}}_\pi\Big(\chi_E\,\mathrm{d}f\Big)\,\mathrm{d}\mathfrak{m} = 0 \quad \forall \pi\ q\text{-test plan}$$

for some $f \in W^{1,p}(\mathsf{X})$, $E \in \mathscr{B}(\mathsf{X})$ (the general case follows by $L^\infty(m)$-linearity of vector fields and the linearity of the integral): we get

$$\int_0^1 \int \chi_E(\gamma_r)(f\circ\gamma)'_r\,\mathrm{d}\pi(\gamma)\,\mathrm{d}r = 0. \quad \forall \pi\ q\text{-test plan} \tag{1.4.19}$$

Now fix a test plan π and define the following sets $\Gamma_t^+ := \{\gamma \in \mathrm{AC}^q([0,1];\mathsf{X}) : \ (f\circ\gamma)'_t > 0\}$, $\Gamma_t^- := \{\gamma \in \mathrm{AC}^q([0,1];\mathsf{X}) : \ (f\circ\gamma)'_t < 0\}$ and the following q-test plans $\pi_t^+ := \frac{\pi_{|\Gamma_t^+}}{\pi(\Gamma_t^+)}$, $\pi_t^- := \frac{\pi_{|\Gamma_t^-}}{\pi(\Gamma_t^-)}$ for every $t \in [0,1]$.

By assumption we have

$$\int_0^1 \int \chi_E(\gamma_r)(f\circ\gamma)'_r\,\mathrm{d}\pi_s^+(\gamma)\,\mathrm{d}r = 0. \tag{1.4.20}$$

Now define $\pi^{ts} := \mathrm{Restr}_{s\sharp}^t\pi_s^+$ with $t > s$, which is again test plan, so that we have

$$\int \int_s^t \chi_E(\gamma_r)(f\circ\gamma)'_r\,\mathrm{d}\pi_s^+(\gamma)\,\mathrm{d}r = 0 \tag{1.4.21}$$

and after dividing by $(t-s)$ and taking the limit as $t \to s$ we deduce

$$\int \chi_E(\gamma_s)(f\circ\gamma)'_s\,\mathrm{d}\pi_s^+(\gamma) = 0 \quad \mathscr{L}^1 - \text{a.e.} \tag{1.4.22}$$

Applying the same reasoning with the test plans π_s^- we finally get

$$\int_0^1 \int \chi_E(\gamma_s)|(f\circ\gamma)'_s|\,\mathrm{d}\pi(\gamma)\,\mathrm{d}t = 0$$

for every q-test plan π, which amounts to say that $|\chi_E\,\mathrm{d}f| = 0$ $\mathfrak{m}$-a.e. since $\forall\pi$ it holds $|(f\circ\gamma)'_t| = \big(\chi_E(\gamma_t) + \chi_{E^c}(\gamma_t)\big)|(f\circ\gamma)'_t| \le \chi_{E^c}|Df|_p(\gamma_t)|\dot\gamma_t|$ for π-a.e. γ, $\mathscr{L}^1$-a.e., therefore $\chi_{E^c}|Df|_p$ is a p-weak upper gradient and by the minimality property this suffices to prove that $|\chi_E\,\mathrm{d}f| = 0$, therefore proving the result. □

Thanks to the previous theorem we can also give a different proof of the Coarea formula in general metric measure spaces (for the original proof see [Mir03]) assuming the cotangent module to be separable, but first we shall recall a technical result whose proof can be found in [GP20b, Theorem 2.1.18]

Proposition 1.4.40. *Let $u : \mathsf{X} \to \mathbb{R}$ be a Lipschitz function (actually u Borel with $u \in S^2(\mathsf{X})$ would be enough), then $|Du| = 0$ $\mathfrak{m}$-a.e. on $f^{-1}(N)$ for every N with $\mathscr{L}^1(N) = 0$.*

We can now state and prove Coarea formula.

Theorem 1.4.41 (Coarea formula)**.** *Assume $L^2(T^*\mathsf{X})$ separable and let $u : \mathsf{X} \to \mathbb{R}$ be a Lipschitz function and $f : \mathsf{X} \to [0, \infty]$ another Borel function, then*

$$\int_{\mathsf{X}} f|Du| \, \mathrm{d}\mathfrak{m} = \int_{\mathbb{R}} \int f \, \mathrm{dPer}(\{u < r\}, \cdot) \, \mathrm{d}r.$$

Proof. We begin by noticing that, thanks to Proposition 1.4.40, $|Du| = 0$ $\mathfrak{m}$-a.e. on $u^{-1}(B)$ for every B such that $\mathscr{L}^1(B) = 0$: therefore we get that $u_\sharp(|Du|\mathfrak{m}) << \mathscr{L}^1$ and so $u_\sharp(|Du|\mathfrak{m}) = \rho\mathscr{L}^1$ with $\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$. Now let us apply Theorem 1.4.9 and disintegrate the measure $\mu = |Du|\mathfrak{m}$ with respect to the map u so that for every $f : \mathsf{X} \to [0, \infty]$ Borel we get

$$\int_{\mathsf{X}} f|Du| \, \mathrm{d}\mathfrak{m} = \int_{\mathbb{R}} \int f \, \mathrm{d}\mathfrak{m}_t \rho(t) \, \mathrm{d}t.$$

where the map $t \mapsto \mathfrak{m}_t$ is weakly Borel and $\mathfrak{m}_t$ is unique in the sense of Theorem 1.4.9: finally let us define $\mu_t = \mathfrak{m}_t\rho(t)$. Thanks to [BG24b, Proposition 2.1], for any open set U we have

$$\mathrm{Per}(E, U) = \sup\left\{ \int_{\mathsf{X}} \chi_E \mathrm{div}(b) \, \mathrm{d}\mathfrak{m} : b \in D(\mathrm{div}), |b| \leq 1, \mathrm{spt(b)} \Subset \mathrm{U} \right\}.$$

Now let us consider such an open set U, $s, t \in \mathbb{R}$ with $s < t$ and a vector field b as in the previous realization of the perimeter. We then have

$$\int_s^t \int_{\mathsf{X}} \chi_{\{u<r\}}(x) \mathrm{div}(b) \, \mathrm{d}\mathfrak{m} \, \mathrm{d}r = \int_{\mathsf{X}} (s \wedge u \vee t) \mathrm{div}(b) \, \mathrm{d}\mathfrak{m} = -\int_{\{s<u<t\}} b\left(\frac{\mathrm{d}u}{|\nabla u|}\right) \mathrm{d}\mathfrak{m},$$

where we used Fubini, intergation by parts and the locality of the differential. We then have

$$\int_{\{s<u<t\}} b(\,\mathrm{d}u) \, \mathrm{d}\mathfrak{m} = \int_s^t \int_{\mathsf{X}} b\left(\frac{\mathrm{d}u}{|\nabla u|}\right) \mathrm{d}\mu_t \, \mathrm{d}t.$$

Therefore for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ we have

$$\int_{\mathsf{X}} \chi_{\{u<t\}} \mathrm{div}(b) \, \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} b\left(\frac{\mathrm{d}u}{|\nabla u|}\right) \mathrm{d}\mu_t. \tag{1.4.23}$$

Now since $L^2(\mathrm{T}^*\mathsf{X})$ is separable the weak-star topology of $L^2(\mathrm{T}\mathsf{X})$ is metrizable and by Banach-Alaoglu the unit ball in $L^2(\mathrm{T}\mathsf{X})$ is compact, therefore $L^2(\mathrm{T}\mathsf{X})$ is separable. Moreover by Proposition 1.4.39 and the previous observation there exists a countable set $\mathcal{D} \subseteq D(\mathrm{div})$ which is weakly-star dense in $D(\mathrm{div})$. We then define a further countable set

$$\tilde{\mathcal{D}} := \left\{ \sum_{i=1}^N q_i b_i : \ (q_i)_{i=1}^N \subseteq \mathbb{Q}_+, \ (b_i)_{i=1}^N \subseteq \mathcal{D} \right\}$$

so that $\forall b \in D(\mathrm{div})$ there exists $(b_n)_n \subseteq \tilde{\mathcal{D}}$ such that

$$\int_{\mathsf{X}} g b_n(\omega)\,\mathrm{d}\mathfrak{m} \to \int_{\mathsf{X}} g b(\omega)\,\mathrm{d}\mathfrak{m} \quad \forall g \in L^\infty(\mathfrak{m})$$

and thanks to the density of $W^{1,2}(\mathsf{X})$ into $L^2(\mathfrak{m})$ we also get $\mathrm{div}(b_n) \rightharpoonup \mathrm{div}(b)$ in $L^2(\mathfrak{m})$. Moreover thanks to Mazur lemma and the definition of $\tilde{\mathcal{D}}$ we can choose the previous sequence in such a way that

$$\lim_{n\to+\infty} \int_{\mathsf{X}} \left|(b_n - b)\left(\frac{\mathrm{d}u}{|\nabla u|}\right)\right| \mathrm{d}\mathfrak{m} = 0.$$

Up to throwing away a set of measure zero we have

$$\int_{\mathsf{X}} \chi_{\{u<t\}} \mathrm{div}(b)\,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} b\left(\frac{\mathrm{d}u}{|\nabla u|}\right) \mathrm{d}\mu_t.$$

for every $b \in \tilde{\mathcal{D}}$ with $\mathrm{spt}(b_n) \Subset U$. Now we can just take the supremum on both sides and note that for a sequence f_n converging to f in $L^1(\mathfrak{m})$ we get

$$\int_{\mathsf{X}} f_n\,\mathrm{d}\mu_t \to \int_{\mathsf{X}} f\,\mathrm{d}\mu_t \quad \mathscr{L}^1 - a.e.$$

so that up to throwing away a further set of $\mathscr{L}^1$ measure zero (call N the union of these sets) we can take the supremum on $n \in \mathbb{N}$ in 1.4.23 to get

$$\mathrm{Per}(\{u < t\}, U) = \mu_t(U) \quad \forall t \in \mathbb{R} \setminus N.$$

Finally taking a countable base of open sets for the Borel sigma-algebra gives that the previous identity holds for $\mathscr{L}^1$-almost everywhere for every Borel set thanks to Borel regularity, meaning that the measure $\mathrm{Per}(\{u < t\}, \cdot)$ coincides with μ_t for almost every $t \in \mathbb{R}$ and thus proving the statement. □

Chapter 2

Measure theory in non-smooth spaces

2.1 Introduction and Notation

The following question was asked by professor David H. Fremlin: consider a distance ρ on $\mathbb{R}^2$ inducing the Euclidean topology, is it possible that $\mathcal{H}^2_\rho(\mathbb{R}^2) = 0$? In [BGLL23] we are able to answer this question negatively in any dimension.

By $\mathcal{H}^n_\rho$ we denote the n-dimensional Hausdorff measure according to Definition 2.1.1 below. We give an answer to this problem in full generality, since our proof is valid in $\mathbb{R}^n$, $\forall n \geq 1$, showing that such a behaviour cannot happen. On the other hand, we will show in Remark 2.3.4 that, when the metric does not induce the usual Euclidean topology, counterexamples can be found.
Before stating our main theorem in Section 2.3, we recall in this introductory section some classical tools for convenience of the reader (see [Fed69] for further details).

Definition 2.1.1 (Hausdorff measure)**.** Let (X, d) be a metric space. We define the n-dimensional Hausdorff outer measure of $A \in \mathcal{P}(X)$ as

$$\mathcal{H}^n_{\mathrm{d}}(A) := \sup\nolimits_{\delta>0} \mathcal{H}^n_{\delta,\mathrm{d}}(A), \qquad \text{with} \tag{2.1.1}$$

$$\mathcal{H}^n_{\delta,\mathrm{d}}(A) := \inf\Big\{\sum_{i\in I} \mathrm{diam}(A_i)^n \ : \ A \subseteq \cup_{i\in I} A_i,\ \mathrm{diam}(A_i) \leq \delta\Big\}, \tag{2.1.2}$$

where $\mathrm{diam}(U) = \sup_{x,y\in U} \mathrm{d}(x, y)$ and I is an at most countable collection of indices.

Remark 2.1.2. The usual definition of Hausdorff measure is given scaling the result by a dimensional constant that, for instance, in the Euclidean case is equal to $2^{-n}\omega_n$, where ω_n is the volume of the unit n-ball. We opted to overlook the constant in order to simplify the notation. Clearly Theorem 2.3.2 is not affected by this choice.

To prove our result we will exploit the following well-known theorem.

Lemma 2.1.3 (Dini)**.** *Let (K, d) be a compact metric space. Let $f_n : K \to \mathbb{R}$ be continuous functions such that*

$$f_n \leq f_{n+1} \qquad \forall n \in \mathbb{N} \tag{2.1.3}$$

and assume that

$$f(x) = \lim_{n\to+\infty} f_n(x) \qquad \forall x \in K, \tag{2.1.4}$$

exists and the function $f : K \to \mathbb{R}$ is also continuous. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on K.

We shall now recall Brouwer fixed point theorem: we shall indeed present a slightly different proof of Theorem 2.3.2, however exploiting the same ideas contained in [BGLL23]. We have (see [Pat19, Theorem 10.1]) the following classical result.

Theorem 2.1.4 (Schauder-Tychonoff Fixed Point). *Let X be a locally convex space, let $K \subset X$ be nonempty and convex (not necessarily closed), and let $K_0 \subset K$ be a compact set. Given a continuous map $f : K \to K_0$, there exists $\bar{x} \in K_0$ such that $f(\bar{x}) = \bar{x}$.*

Remark 2.1.5. The same result holds if f is defined only from K_0 instead of K.

2.2 More on normed modules

Definition 2.2.1 (Maps of bounded compression/deformation). Let $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ and $(\mathsf{Y}, \mathrm{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y})$ be two metric measure spaces. We say that a map $\varphi : \mathsf{X} \to \mathsf{Y}$ is a map of bounded compression if it is a Borel map such that $\varphi_\sharp \mathfrak{m}_\mathsf{X} \le C\mathfrak{m}_\mathsf{Y}$. If the map φ is Lipschitz continuous then we say that φ is a map of bounded deformation.

We now introduce the notion of *pullback module* which, roughly speaking, is nothing but a module over a space X obtained by pulling back a module on another space Y via a map of bounded compression.

Definition 2.2.2 (Pullback). Let $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ and $(\mathsf{Y}, \mathrm{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y})$ be metric measure spaces, $\varphi : \mathsf{X} \longrightarrow \mathsf{Y}$ a map of bounded compression and $\mathcal{M}$ and $L^p(\mathfrak{m}_\mathsf{Y})$-normed module. Then there exists a unique, up to unique isomorphism, couple $(\varphi^*\mathcal{M}, \varphi^*)$ with $\varphi^*\mathcal{M}$ being an $L^p(\mathfrak{m}_\mathsf{X})$-normed module and $\varphi^* : \mathcal{M} \longrightarrow \varphi^*\mathcal{M}$ being a linear and continuous operator such that:

1. $|\varphi^* v| = |v| \circ \varphi$ holds $\mathfrak{m}_\mathsf{X}$-a.e., for every $v \in \mathcal{M}$,

2. the set $\{\varphi^* v \ : \ v \in \mathcal{M}\}$ generates $\varphi^*\mathcal{M}$ as a module.

At this point one can try to understand what is the relation between the dimension of a module and the one of its pullback via the map φ and in order to do so we need to introduce a sort of *left inverse* of the pullback operator φ^*. To do so let us assume $\varphi_\sharp \mathfrak{m}_\mathsf{X} = \mathfrak{m}_Y$ to simplify the exposition.

For $f \in L^p(\mathfrak{m}_\mathsf{X})$ nonnegative we put

$$\mathrm{Pr}_\varphi(f) := \frac{\mathrm{d}\varphi_\sharp(f\mathfrak{m}_\mathsf{X})}{\mathrm{d}\mathfrak{m}_\mathsf{Y}} \tag{2.2.1}$$

and in a natural way we set $\mathrm{Pr}_\varphi(f) := \mathrm{Pr}_\varphi(f^+) - \mathrm{Pr}_\varphi(f^-)$ for general $f \in L^p(\mathfrak{m}_\mathsf{X})$.

We now recall some properties of the map Pr_φ.

Proposition 2.2.3. *The operator $\mathrm{Pr}_\varphi : L^p(\mathfrak{m}_\mathsf{X}) \longrightarrow L^p(\mathfrak{m}_Y)$ is linear, continuous and*

$$\mathrm{Pr}_\varphi(f)(y) = \int_\mathsf{X} f(x)\,\mathrm{d}\mathfrak{m}_y(x) \quad \mathfrak{m}_\mathsf{Y} - a.e., \quad \forall f \in L^p(\mathfrak{m}_\mathsf{X}), \tag{2.2.2}$$

where $y \mapsto m_y$ denotes the disintegration of $\mathfrak{m}_\mathsf{X}$ with respect to the map φ. Finally it holds

$$|\mathrm{Pr}_\varphi(f)| \le \mathrm{Pr}_\varphi(|f|) \quad \mathfrak{m}_\mathsf{Y} - a.e. \tag{2.2.3}$$

Proof. Linearity is a consequence of the linearity of the integral. Formula (2.2.3) is also trivial while for (2.2.2) we have for any $A \in \mathscr{B}(\mathsf{Y})$

$$\int_A \mathrm{Pr}_\varphi(f)(y)\,\mathrm{d}\mathfrak{m}_\mathsf{Y} = \int_A \mathrm{d}\varphi_\sharp(f\,\mathrm{d}\mathfrak{m}_\mathsf{X}) = \int_{\varphi^{-1}(A)} f(x)\,\mathrm{d}\mathfrak{m}_\mathsf{X},$$

and by the properties of the disintegration we have

$$\int_{\varphi^{-1}(A)} f(x)\,\mathrm{d}\mathfrak{m}_\mathsf{X} = \int_\mathsf{Y}\int_{\varphi^{-1}(A)} f(x)\,\mathrm{d}\mathfrak{m}_y(x)\,\mathrm{d}\mathfrak{m}_\mathsf{Y}(y) = \int_A\int_\mathsf{X} f(x)\,\mathrm{d}\mathfrak{m}_y(x)\,\mathrm{d}\mathfrak{m}_\mathsf{Y}(y),$$

therefore proving (2.2.2).

To prove continuity note that the case $p=\infty$ is due to formula (2.2.3) while continuity in $L^p(\mathfrak{m})$ for every $p \in [1,+\infty)$ follows from the following

$$\int_\mathsf{Y} |\mathrm{Pr}_\varphi|^p\,\mathrm{d}\mathfrak{m}_\mathsf{Y} = \int_\mathsf{Y}\left|\int_\mathsf{X} f(x)\,\mathrm{d}\mathfrak{m}_y(x)\right|^p \mathrm{d}\mathfrak{m}_\mathsf{Y}(y) \le \int_\mathsf{Y}\int_\mathsf{X} |f(x)|^p\,\mathrm{d}\mathfrak{m}_y(x)\,\mathrm{d}\mathfrak{m}_\mathsf{Y}(y) = \|f\|^p_{L^p(\mathfrak{m})},$$

where we used Jensen's inequality and the properties of the disintegration. □

In the case of a general $L^p(\mathfrak{m}_\mathsf{X})$-normed module the continuous operator $\mathrm{Pr}_\varphi : \varphi^*\mathcal{M} :\longrightarrow \mathcal{M}$ can be characterized by the following properties:

$$g\mathrm{Pr}_\varphi(v) = \mathrm{Pr}_\varphi(g\circ\varphi v), \quad \forall v \in \mathcal{M} \quad \forall g \in \mathrm{L}^\infty(\mathfrak{m}_\mathsf{X}) \tag{2.2.4}$$

$$\mathrm{Pr}_\varphi(g\varphi^* v) = \mathrm{Pr}_\varphi(g)v \quad \forall v \in \mathcal{M} \quad \forall g \in \mathrm{L}^\infty(\mathfrak{m}_\mathsf{X}), \tag{2.2.5}$$

with the bound $|\mathrm{Pr}_\varphi(V)| \le \mathrm{Pr}_\varphi(|V|)$ still holding $\mathfrak{m}_\mathsf{Y}$-a.e. for every $V \in \varphi^*\mathcal{M}$.

With these objects we are now able to describe the structure of the pullback module, in particular (as one can expect by reasoning via pre-composition) the pullback of an n-dimensional module $\mathcal{M}$ over E is an n-dimensional module over $\varphi^{-1}(E)$ (see also [Pas18]).

Proposition 2.2.4. *Let $\mathcal{M}$ be an $L^p(\mathfrak{m}_\mathsf{Y})$-normed module over the m.m.s. $(\mathsf{Y}, d_\mathsf{Y}, \mu)$ and let $E \in \mathcal{B}(Y)$ be a Borel set where $\mathcal{M}$ has dimension n, with $\{v_1, ..., v_n\}$ being a basis. Let $(\mathsf{X}, d_\mathsf{X}, \mathfrak{m})$ be another m.m.s. and $\varphi : \mathsf{X} \to \mathsf{Y}$ be a Borel map such that $\varphi_\sharp\mathfrak{m}_\mathsf{X} = \mathfrak{m}_\mathsf{Y}$, then $\{\varphi^*v_1, ..., \varphi^*v_n\}$ is a basis of $\varphi^*\mathcal{M}$ over $\varphi^{-1}(E)$.*

Proof. We first prove that $\{\varphi^*v_1, ..., \varphi^*v_n\}$ generate $\varphi^*\mathcal{M}$ over $\varphi^{-1}(E)$.

First recall that $\varphi^*\mathcal{M}$ is generated (as module) by $\{\varphi^*v : v \in \mathcal{M}\} =: V$. Let us show that $V \subseteq \mathrm{Span}_{\varphi^{-1}(E)}\{\varphi^*v_1, ..., \varphi^*v_n\}$: pick $w \in V$, then there exists $v \in \mathcal{M}$ such that $w = \varphi^*v$ so that there exists $(A_j)_j \subseteq \mathscr{B}(\mathsf{X})$ partition of E and $(g_i^j)_{j\in\mathbb{N}} \subset \mathrm{L}^\infty(\mathfrak{m}_\mathsf{Y})$ $\forall i = 1, ..., n$ such that

$$\chi_{A_j} v = \sum_{i=1}^n g_i^j v_i \qquad \forall j \in \mathbb{N}$$

Using the linearity of the pullback map and the fact that $\varphi^*(gv) = g\circ\varphi\varphi^*v$ for all $v \in \mathcal{M}$, $g \in \mathrm{L}^\infty(\mathfrak{m}_\mathsf{Y})$ we get

$$\chi_{\varphi^{-1}(A_j)} w = \sum_{i=1}^n g_i^j \circ \varphi\varphi^* v_i.$$

Finally, since the pullback module has a natural structure of $L^p(\mathfrak{m})$-normed $\mathrm{L}^\infty(\mathfrak{m})$-module, we get that $\mathrm{Span}_{\varphi^{-1}(E)}\{\varphi^*v_1, ..., \varphi^*v_n\}$ is closed, proving the first result.

We now turn to local independence: assume by contradiction $\{\varphi^* v_1, ..., \varphi^* v_n\}$ are not independent on $\varphi^{-1}(E)$ then there exist $f_1, ..., f_n \in \mathrm{L}^\infty(\mathfrak{m}_\mathsf{X})$ such that $\sum_{i=1}^n f_i \varphi^* v_i = 0$ $\mathfrak{m}$-a.e. with (upon relabeling indexes) $|f_1| > 0$ $\mathfrak{m}$-a.e. on some subset $\tilde{E}$ of positive measure. Without loss of generality, possibly considering a smaller set, we shall assume $f_1 > 0$ $\mathfrak{m}$-a.e. so that

$$\sum_{i=1}^n f_i \varphi^* v_i = 0 \quad \mathfrak{m} - \text{a.e. on } \tilde{E} \implies \sum_{i=1}^n \mathrm{Pr}_\varphi(f_i) v_i = 0 \quad \mathfrak{m} - \text{a.e. on } \tilde{E}.$$

However note that $\mathrm{Pr}_\varphi(f_1) > 0$ on some set of positive $\mathfrak{m}_\mathsf{Y}$ measure, contradicting the independence of the v_is.

□

Besides the differential of a Sobolev function introduced in Theorem 1.4.26, one can give another definition which exploits the fact that the map is Lipschitz and such that $\varphi_\sharp \mathfrak{m}_\mathsf{X} \le C \mathfrak{m}_\mathsf{Y}$ for some $C > 0$ (namely a map of *bounded compression*): this class of maps is that of *bounded deformation*. In this direction we need to recall the notion of *pullback of forms*: in order to distinguish it from the pullback of a module we shall proceed denoting with $\omega \mapsto [\varphi^* \omega]$ the pullback map and with φ^* the pullback of 1-forms which is the following:

Definition 2.2.5. Let $\varphi : \mathsf{X} \to \mathsf{Y}$ be a map of bounded deformation, then we define $\varphi^* : L^p(\mathrm{T}^*\mathsf{Y}) \to L^p(\mathrm{T}^*\mathsf{X})$ to be the linear map such that $\varphi^*(\,\mathrm{d} f) = \mathrm{d}(f \circ \varphi)$ for all $f \in W^{1,p}(\mathsf{Y})$ and $\varphi^*(g\omega) = g \circ \varphi \varphi^* \omega$ for all $g \in \mathrm{L}^\infty(\mathsf{Y})$ and $\omega \in L^p(\mathrm{T}^*\mathsf{Y})$.

Remark 2.2.6. It is easy to see that, thanks to the regularity properties of φ, the pullback of 1-forms φ^* is well defined.

Definition 2.2.7. Given $\varphi : \mathsf{X} \longrightarrow \mathsf{Y}$ of bounded deformation we define for all $p \ge 1$ its p-differential as an operator $\underline{\mathrm{d}_p \varphi} : L^q(\mathrm{T}\mathsf{X}) \longrightarrow \varphi^* \big(L^p(\mathrm{T}^*\mathsf{Y})\big)^*$ such that

$$[\varphi^* \omega](\underline{\mathrm{d}_p \varphi}(v)) = \varphi^* \omega(v) \quad \forall v \in L^q(\mathrm{T}\mathsf{X}), \ \ \forall \omega \in L^p(\mathrm{T}^*\mathsf{Y}). \tag{2.2.6}$$

In the recent work [EBS21] the authors provide some "charts" over Borel sets $(E_i)_{i \in \mathbb{N}}$ partitioning the metric measure space $\mathfrak{m}$-a.e.: we will briefly recall here the definition

Definition 2.2.8. We say $\varphi : \mathsf{X} \to \mathbb{R}^N$ is an EBS chart over the Borel set E if it is a Lipschitz map with the following properties

1. (p-independence) $\operatorname{ess\,inf}_{v \in \mathbb{S}^{N-1}} |D(v \cdot \varphi)|_p > 0$ $\mathfrak{m}$-a.e on E.

2. (maximality) There is no other Lipschitz map $\varphi : \mathsf{X} \to \mathbb{R}^M$ with $M > N$ which is p-independent on a subset of E of positive measure.

The authors proved that the condition of p-independence over a set E is equivalent to the fact that the $L^p(\mathrm{T}^*\mathsf{X})$ module over E is generated by the differentials of the components of the chart: in other words $\{\,\mathrm{d}_p \varphi^1, ..., \,\mathrm{d}_p \varphi^N\}$ is a basis for $L^p(\mathrm{T}^*\mathsf{X})_{|E}$ (see Lemma 6.3 in [EBS21]) and as a consequence of Theorem 1.4.7 in [Gig18] we are able to deduce that $L^q(\mathrm{T}\mathsf{X})_{|E}$ is also an N-dimensional normed module.

2.3 Main results

Here, following the previous section, we shall first present the main result of [BGLL23] and then the one of [GG23].

Before proving our main theorems we present a further key result; denoting with $\mathbb{B}(0,r)$ the closed Euclidean ball (in $\mathbb{R}^n$) of radius r centered at 0, we have the following:

Proposition 2.3.1. *Let* $\mathrm{id} : (\mathbb{B}(0,1),\rho) \longrightarrow (\mathbb{B}(0,1),d_{eucl})$ *be the identity map, where* ρ *is a distance inducing the Euclidean topology. Let* $(F_k)_k : (\mathbb{B}(0,1),\rho) \longrightarrow (\mathbb{B}(0,1),d_{eucl})$ *be a sequence of continuous maps converging uniformly to* id*. Then there exists* $k_0 \in \mathbb{N}$ *such that* $\mathrm{int}(F_k(\mathbb{B}(0,1)))$ *is non-empty for all* $k \geq k_0$*.*

Proof. Fix $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\|F_k - \mathrm{id}\|_\infty \leq \varepsilon/n$. To prove the proposition it is enough to show that for all $z \in \mathbb{B}(0,1-\varepsilon/n)$ there exists $x \in \mathbb{B}(0,1)$ such that $F_\varepsilon(z) = x$. To this aim, let $z \in \mathbb{B}(0,1-\varepsilon/n)$ and define the map $T : \mathbb{B}(0,1) \to \mathbb{R}^n$ as $T(w) := z + w - F_\varepsilon(w)$. For all $w \in \mathbb{B}(0,1)$ we have

$$\|T(w)\| \leq \|z\| + \|w - F_\varepsilon(w)\| \leq 1 - \varepsilon + n\|w - F_\varepsilon(w)\|_\infty \leq 1,$$

where in the last line we used that $\|v\| \leq n\|v\|_\infty$. This means that the image of $\mathbb{B}(0,1)$ under T is contained in $\mathbb{B}(0,1)$. Being T continuous by assumption, we can apply Theorem 2.1.4 and deduce the existence of $x \in \mathbb{B}(0,1)$ such that $T(x) = x$, meaning that $F_\varepsilon(x) = z$ and concluding the proof. □

We are now in the position to state our main theorem.

Theorem 2.3.2. *Let* $(\mathbb{R}^n,\rho)$ *be a metric space with* ρ *inducing the Euclidean topology, then* $\mathcal{H}^n_\rho(\mathbb{R}^n) > 0$*.*

Proof. Assume by contradiction that there exists a distance ρ in $\mathbb{R}^n$ such that $\mathcal{H}^n_\rho(\mathbb{R}^n) = 0$. We denote by $\mathbb{B}(0,1)$ the closed unit ball with respect to Euclidean metric and we consider the identity map

$$\mathrm{id} : (\mathbb{B}(0,1),\rho) \longrightarrow (\mathbb{B}(0,1),\mathrm{d}_{\mathrm{eucl}}). \tag{2.3.1}$$

Such a map is an homeomorphism by assumption, but it carries no metric information a priori. Let us write

$$\mathrm{id}(x) = \big(\pi_1(x),\ldots,\pi_n(x)\big) \tag{2.3.2}$$

and define

$$\pi_i^\varepsilon(x) := \min_{z\in\mathbb{B}(0,1)} \left[\pi_i(z) + \frac{1}{\varepsilon}\rho(x,z)\right] \qquad \forall i = 1,...,n \quad \forall x \in \mathbb{B}(0,1), \tag{2.3.3}$$

where we are using that $\mathbb{B}(0,1)$ is compact also for the metric ρ. The latter functions are Lipschitz, since they are the infimum of a family of equi-Lipschitz functions, more precisely

$$|\pi_i^\varepsilon(x) - \pi_i^\varepsilon(y)| \leq \frac{1}{\varepsilon}\rho(x,y) \quad \forall x,y \in \mathbb{B}(0,1). \tag{2.3.4}$$

We say that the functions π_i^ε converge uniformly in the compact ball $\mathbb{B}(0,1)$ to the components of the identity as $\varepsilon \to 0$. In order to prove that, for every $\varepsilon_m \to 0$ consider a sequence $(z_{\varepsilon_m})_m \subseteq \mathbb{B}(0,1)$ such that

$$\pi_i^{\varepsilon_m}(x) = \pi_i(z_{\varepsilon_m}) + \frac{1}{\varepsilon_m}\rho(x,z_{\varepsilon_m}). \tag{2.3.5}$$

Since $(z_{\varepsilon_m})_m$ is bounded, by compactness there exists a convergent subsequence. Due to equation (2.3.5) and the bound

$$1 \geq \pi_i \geq \pi_i^{\varepsilon_m} \geq -1, \tag{2.3.6}$$

it follows that $\lim_{m\to+\infty} \rho(z_{\varepsilon_m}, x) = 0$, which means that $(z_{\varepsilon_m})_m$ converges to x, leading to the pointwise convergence. Now, since we have $\pi_i^{\varepsilon}(x) \geq \pi_i^{\varepsilon+\gamma}(x)$ for every $\gamma, \varepsilon > 0$ and for every $x \in \mathbb{B}(0,1)$, by Dini's theorem $\pi_i^{\varepsilon_m}$ converges uniformly to π_i on $\mathbb{B}(0,1)$ for every $i = 1, ..., n$. Summing up we have obtained a sequence

$$F^{\varepsilon} = (\pi_1^{\varepsilon}, ..., \pi_n^{\varepsilon}) : (\mathbb{B}(0,1), \rho) \longrightarrow (\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}) \tag{2.3.7}$$

such that

$$\mathrm{d}_{\mathrm{eucl}}\big(F^{\varepsilon}(x), F^{\varepsilon}(y)\big) \leq C_{\varepsilon}\rho(x,y) \quad \forall x, y \in \mathbb{B}(0,1) \tag{2.3.8}$$

with $C_{\varepsilon} > 0$ and such that it converges uniformly to the identity in $\mathbb{B}(0,1)$. We can now choose a countable sequence $\varepsilon_k \to 0$ and apply Proposition 2.3.1 to deduce the existence of $k_0 \in \mathbb{N}$ such that $F^{\varepsilon_k}(\mathbb{B}(0,1))$ has non-empty interior for every $k \geq k_0$. For simplicity set $F^{\hat{\varepsilon}} := F^{\varepsilon_{k_0}}$.
Since $F^{\hat{\varepsilon}}(\mathbb{B}(0,1))$ contains a non-empty open set and $F^{\hat{\varepsilon}}$ is Lipschitz, we get

$$\mathcal{H}^n_{\mathrm{d}_{\mathrm{eucl}}}\big(F^{\hat{\varepsilon}}(\mathbb{B}(0,1))\big) \leq C_{\varepsilon}^n \mathcal{H}^n_{\rho}(\mathbb{B}(0,1)) = 0, \tag{2.3.9}$$

which is a contradiction since the n-dimensional Hausdorff measure on $\mathbb{R}^n$ with the Euclidean distance gives positive measure to not empty open sets. □

Remark 2.3.3. The same proof of Theorem 2.3.2 can be adapted to prove that any nonempty open set A is such that $\mathcal{H}^n_{\rho}(A) > 0$.

Remark 2.3.4. Removing the assumption that ρ induces the Euclidean topology, counterexamples show that $\mathcal{H}^n_{\rho}(\mathbb{R}^n)$ might vanish. Consider, for instance, the metric space $(\mathcal{C}, \mathrm{d})$, where $\mathcal{C} \subset \mathbb{R}$ is the Cantor set and d denotes the usual one-dimensional Euclidean distance. Having $\mathcal{C}$ the cardinality of the continuum, there exist bijections $g_n : \mathcal{C} \to \mathbb{R}^n$. Then, define on $\mathbb{R}^n$ the metric $\rho(x,y) = \mathrm{d}(g_n^{-1}(x), g_n^{-1}(y))$.
Given any collection $(A_i)_{i\in\mathbb{N}}$ that covers $\mathcal{C}$, follows that $(g_n(A_i))_{i\in\mathbb{N}}$ covers $\mathbb{R}^n$ and $\mathrm{diam}(A_i) = \mathrm{diam}(g_n(A_i))\,\forall i \in \mathbb{N}$. Clearly, also the opposite direction applies. Therefore, we have

$$\mathcal{H}^n_{\rho}(\mathbb{R}^n) = \mathcal{H}^n_d(\mathcal{C}) = 0 \tag{2.3.10}$$

that shows a counterexample.

Remark 2.3.5. Note that, under previous assumptions on ρ, it is not true in general that $\dim_H^{\rho}(\mathbb{R}^n) = n$. In fact, choosing $\rho(x,y) = \mathrm{d}_{\mathrm{eucl}}(x,y)^{1/2}$, the distance ρ induces the Euclidean topology, but in this case

$$\mathcal{H}^s_{\mathrm{d}_{\mathrm{eucl}}}(A) = \mathcal{H}^{2s}_{\rho}(A)$$

for all $A \subseteq \mathbb{R}^n,\ s \geq 0$. For this reason we get that $\dim_H^{\rho}(\mathbb{R}^n) = 2n$.

Now we shall focus on the main result of [GG23], giving an alternative proof to Proposition 4.13 in [EBS21]. First we remark that with $\mathrm{d}_p\varphi$ we will denote the differential of a map of bounded deformation in the sense of definition 2.2.7, while with $\mathrm{d}_p f$ we denote the differential in the sense of Proposition Theorem 1.4.26. Lastly let us assume that $\mathfrak{m}$ is a finite measure: we can do so because of the inner regularity of the measure $\mathfrak{m}$. Indeed if for a Borel map $\psi : \mathsf{X} \to \mathbb{R}^n$ we have $\psi_\sharp(\mathfrak{m}_{|E_k}) << \mathscr{L}^n$ for every $k \in \mathbb{N}$ with $(E_k)_k$ compact, such that $E_k \subseteq E_{k+1}$ and $\mathfrak{m}(E \setminus \cup_k E_k) = 0$, then $\psi_\sharp(\mathfrak{m}_{|E}) << \mathscr{L}^n$.

We begin with the following simple lemma which follows standard arguments in linear algebra:

Lemma 2.3.6. *Let $\mathcal{M}$ be an $L^p(\mathfrak{m})$-normed module and $\mathcal{M}^*$ be its dual module. Assume that $\mathcal{M}$ has dimension n over E: then $\{v_1, ..., v_n\}$ and $\{\omega_1, ..., \omega_n\}$ are basis of $\mathcal{M}^*$ and $\mathcal{M}$ (respectively) over E if and only if $\det[\omega_i(v_j)]_{ij} > 0$ $\mathfrak{m}$-a.e. on E.*

Proof. Define $A_{ij} := [\omega_i(v_j)]_{ij}$ and let us assume first that $\det A > 0$ $\mathfrak{m}$-a.e.. It is clearly sufficient to prove the independence: assume by contradiction that $\sum_{i=1}^n g_i v_i = 0$ $\mathfrak{m}$-a.e. on some subset B of positive measure, for some $g_1, ..., g_n$ which are not all zero on B (in the measure theoretic sense). Then consider $\underline{\mathrm{g}} := (g_1, ..., g_n)$ and note that $A\underline{\mathrm{g}} \neq 0$ $\mathfrak{m}$-a.e. on B because of the condition on the determinant. However $(A\underline{\mathrm{g}})_i = \sum_{j=1}^n g_j v_j(\omega_i)$=0 $\mathfrak{m}$-a.e. on B for every $i = 1, ..., n$, which is clearly a contradiction. This argument trivially applies for $\{\omega_1, ..., \omega_n\}$ as well by considering the transpose of A.

Assume now that $\{\omega_1, ..., \omega_n\}$ and $\{v_1, ..., v_n\}$ are basis over E of $\mathcal{M}$ and $\mathcal{M}^*$ respectively and by contradiction let $\det A = 0$ $\mathfrak{m}$-a.e. on a Borel subset C of positive measure. Then there exists a further measurable subset (which we won't relabel) C of positive measure and $\underline{\mathrm{g}} \in \mathrm{L}^\infty(\mathfrak{m})^n$ for which $A\underline{\mathrm{g}} = 0$ and $\underline{\mathrm{g}} \neq 0$ $\mathfrak{m}$-a.e. on C. The latter system of equations means that we have

$$v_i\left(\sum_{j=1}^n g_j \omega_j\right) = 0 \quad \mathfrak{m} - \text{a.e. on } C,\ \forall i = 1, ..., n. \tag{2.3.11}$$

Set $\tilde{\omega} = \sum_{j=1}^n g_j \omega_j$ and suppose that $|\tilde{\omega}| \neq 0$ $\mathfrak{m}$-a.e. on C, then there exists a non-zero continuous functional $\ell \in \mathcal{M}'$ (which is the Banach dual) such that $\ell(\chi_C \tilde{\omega}) = ||\chi_C \tilde{\omega}||_{\mathcal{M}}$ and there exists $L \in \mathcal{M}^*$ (see Proposition 1.2.13 in [Gig18]) such that

$$\ell(\omega) = \int_{\mathrm{X}} L(\omega)\,\mathrm{d}\mathfrak{m} \quad \forall \omega \in \mathcal{M}.$$

In our case this means that $||\chi_C \tilde{\omega}||_{\mathcal{M}} = \int_C L(\tilde{\omega})\,\mathrm{d}\mathfrak{m} > 0$, so that there must be a Borel set of positive measure where $\chi_C L(\tilde{\omega}) > 0$, which contradicts (2.3.11) since there exists $D \subset C$ with $\mathfrak{m}(D) > 0$ such that $\chi_D L = \sum_{i=1}^n f_i v_i$ for some $f_1, ..., f_n \in \mathrm{L}^\infty(\mathfrak{m})$. □

Lemma 2.3.7. *Let φ be an EBS chart over the Borel set E and $\{v_1, ..., v_n\} \in L^p(T\mathrm{X})$ be independent over E, then $\{\underline{\mathrm{d}_p\varphi}(v_1), ..., \underline{\mathrm{d}_p\varphi}(v_n)\} \in \varphi^* L^p_\mu(T\mathbb{R}^n)$ are independent over the same set, where $\mu = \varphi_\sharp(\mathfrak{m}_{|E})$ and $L^p_\mu(T\mathbb{R}^n)$ is the tangent module built over $(\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}, \mu)$.*

Proof. Consider $f_1, ..., f_n \in \mathrm{L}^\infty(\mathfrak{m})$ such that

$$\sum_{i=1}^n f_i\, \underline{\mathrm{d}_p\varphi}(v_i) = 0 \quad \mathfrak{m} - \text{a.e. on } E,$$

then set $v := \sum_{i=1}^n f_i v_i$. Note that the maps $\Pi^j : \mathbb{R}^n \longrightarrow \mathbb{R}$ being the projection on the j-th component are all 1-Lipschitz with respect to the Euclidean distance and for this reason they belong to $W^{1,p}(\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}, \mu)$: following equation (2.2.6) we have that, for every $j = 1, ..., n$ and choosing $\omega = \mathrm{d}_p \Pi_j$,

$$0 = \mathrm{d}_p\varphi^j(v) = \sum_{i=1}^n f_i\, \mathrm{d}_p\varphi^j(v_i) \quad \mathfrak{m} - \text{a.e. on } E,$$

where φ^j is the j-th component of the map φ.

Being the matrix $A = (A_{ij})_{ij} = \langle \mathrm{d}_p\varphi^j, v_i \rangle$ such that $\det A > 0$ $\mathfrak{m}$-a.e., the equations above can be rewritten as $A\underline{\mathrm{f}} = 0$ $\mathfrak{m}-$a.e. on E with $\underline{\mathrm{f}} = (f_1, ..., f_n)$, meaning $\underline{\mathrm{f}} = 0$ thanks to Lemma 2.3.6. □

The following result is borrowed from [LPR21] (Proposition 4.5) where only the metric measure space $(\mathbb{R}^n, \mathrm{d}_{\mathrm{eucl}}, \mu)$ is considered.

Proposition 2.3.8. *Assume that there exists a Borel set E such that $\dim L^p_\mu(\mathrm{T}^*\mathbb{R}^n)_{|E} = n$ for some $p \in (1, +\infty)$, then $\mu_{|E} << \mathscr{L}^n$.*

Remark 2.3.9. It is in the proof of the latter proposition that the results contained in [DPR] are used.

Now we are in place to apply Proposition 2.3.8 to prove the following:

Theorem 2.3.10. *Let $\varphi : \mathsf{X} \to \mathbb{R}^N$ be a p-independent weak chart over a Borel set E of positive measure and with $p \geq 1$, then $\mu = \varphi_\sharp(\mathfrak{m}_{|E}) << \mathscr{L}^N$ and $N \leq dim_H(E)$.*

Proof. For the moment assume $p \in (1, +\infty)$ and without loss of generality assume E to be compact. Thanks to Lemma 2.3.7 we deduce that $\varphi^* L^p_\mu(\mathrm{T}^*\mathbb{R}^N)$ has dimension N over the set E, meaning that $L^p_\mu(\mathrm{T}^*\mathbb{R}^N)$ has dimension N over the set $\varphi(E)$. Being the latter module top dimensional, by Proposition 2.3.8 we have that $\mu << \mathscr{L}^N$ which is the first part of the statement. The second part is immediate since if we had $N > \dim_H(E)$ we would get $\mathcal{H}^N(E) = 0$ and since the map φ is Lipschitz this implies $\mathcal{H}^N(\varphi(E)) = \mathscr{L}^N(\varphi(E)) \leq C \cdot 0 = 0$, so that by absolute continuity $\mu(\varphi(E)) = \mathfrak{m}(E) = 0$, which is clearly a contradiction.

For the case $p = 1$ note that, since the measure $\mathfrak{m}$ is finite, we have $|D(v \cdot \varphi)|_1 \leq |D(v \cdot \varphi)|_p$ $\mathfrak{m}$-a.e. and for every $v \in \mathbb{S}^{N-1}$, meaning that φ is also p-independent and the same argument applies. □

Remark 2.3.11. By virtue of the latter theorem one can see that a control on the Hausdorff dimension l of a subset E of a metric measure space grants that the dimension of $L^p(\mathrm{T}^*\mathsf{X})_{|E}$ is bounded by l, hence the cotangent module is finite dimensional there. Moreover the proof presented here simplifies the one in [GP21] since there the authors needed to build independent vector fields in $L^2(\mathrm{T}\mathsf{X})$ with $L^2(\mathfrak{m})$-integrable divergence and push them to $\mathbb{R}^n$ keeping them independent and regular: to do so they had to use additional properties of the map Pr_φ and the bi-Lipschitz regularity of their chart φ was essential. Here instead we mainly exploit the properties of $\mathbb{R}^n$.

Chapter 3

On the asymptotics of the s-fractional perimeter

3.1 Introduction and Notation

3.1.1 The fractional perimeter on Riemannian manifolds

It was recently pointed out in [CFSS23] a canonical definition of the fractional s-perimeter on every closed Riemannian manifold (M, g): this boils down to giving a canonical definition of the fractional Sobolev seminorm $H^{s/2}(M)$ for $s \in (0, 1)$. Consider a closed (even though we will deal with general complete ones), connected Riemannian manifold (M, g) with $n \geq 2$. In [CFSS23] the authors show that a canonical definition of the fractional Sobolev seminorm $H^{s/2}(M)$ can be given in at least four equivalent (up to absolute constants) ways:

(i) By the *singular integral*

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(x) - u(y))^2 \mathcal{K}_s(x, y) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \,, \tag{3.1.1}$$

where $\mathcal{K}_s(x, y)$ is given by (1.2.2).

(ii) Following the *Bochner definition* of the fractional Laplacian

$$(-\Delta)^{s/2}_{\mathrm{B}} u = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta}u - u) \frac{\mathrm{d}t}{t^{1+s/2}} \,, \tag{3.1.2}$$

via

$$[u]^2_{H^{s/2}(M)} = 2 \int_M u(-\Delta)^{s/2}_{\mathrm{B}} u \, \mathrm{d}\mu \,.$$

(iii) By *spectral theory*, one can set

$$[u]^2_{H^{s/2}(M)} = \sum_{k\geq 1} \lambda_k^{s/2} \langle u, \phi_k\rangle^2_{L^2(M)} \tag{3.1.3}$$

where $\{\phi_k\}_k$ is an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator $(-\Delta_g)$ and $\{\lambda_k\}_k$ are the corresponding eigenvalues. Note that for $s = 2$ this gives the usual $[u]^2_{H^1(M)}$ seminorm.

(iv) Considering a *Caffarelli-Silvestre type extension (cf. [CS07, BGS15, CG11])*, namely, a degenerate-harmonic extension problem in one extra dimension. One can set

$$[u]^2_{H^{s/2}(M)} = \inf\left\{\int_{M\times[0,\infty)} z^{1-s}|\widetilde{\nabla} U(x,z)|^2 \,\mathrm{d}\mu(x)dz \;\text{ s.t. }\; U(x,0)=u(x)\right\}.$$

Here $\widetilde{\nabla}$ denotes the Riemannian gradient of the manifold $\widetilde{M} = M\times[0,\infty)$, with respect to natural product metric, and the infimum is taken over all the extensions $U \in X$, where $X = H^1(\widetilde{M}\,; z^{1-s}\,\mathrm{d}\mu dz)$ is the classical weighted Sobolev space of the functions $U \in L^2(\widetilde{M}\,; \mathrm{d}\mu)$ with respect to the measure $\mathrm{d}\mu = z^{1-s}\,\mathrm{d}\mu dz$ that admit a weak gradient $\widetilde{\nabla} U \in L^2(\widetilde{M}\,; \mathrm{d}\mu)$.

The spectral definition (iii) can be extended to manifolds that are not closed, where the spectrum of the Laplacian is not discrete. Nevertheless, the equivalence between (i) and (iv) also holds on many (but not every) complete Riemannian manifolds, which are not necessarily compact. For example, a lower Ricci curvature bound is sufficient. See [BGS15] for general conditions for which the equivalence of $(i) \iff (iv)$ holds. Moreover, under suitable assumptions on u, the equivalence between (i) and (ii) holds if and only if M is stochastically complete; we will treat this equivalence in subsection 3.6.3.

Since in the present work, we aim to study the asymptotics of the fractional s-perimeter on complete Riemannian manifolds (not necessarily closed or with curvature bounded below), we work with the singular integral definition (3.1.1) since it extends naturally to the case of general manifolds and weighted manifolds. Then, the fractional s-perimeter on a Riemannian manifold is naturally defined by means of the fractional Sobolev seminorm.

Here and in the rest of the work, (M,g) will denote a general complete, connected Riemannian manifold, and hence also geodesically complete. We denote by $\mathrm{d}\mu$ its Riemannian volume form and by $H_M(x,y,t)$ the heat kernel of (M,g). To see how to build the heat kernel on a general (weighted) manifold, see the classical reference [Gri09]. Moreover, we denote by $B_R(p) \subset M$ the geodesics ball on M and by $\mathcal{B}_R(0) \subset \mathbb{R}^n$ the one on $\mathbb{R}^n$.

Definition 3.1.1. Let (M,g) be a complete Riemannian manifold and $s \in (0,2)$. Then, we set

$$H^{s/2}(M) := \left\{u \in L^2(M) : [u]^2_{H^{s/2}(M)} < \infty\right\},$$

where

$$[u]^2_{H^{s/2}(M)} := \iint_{M\times M} (u(x)-u(y))^2 \mathcal{K}_s(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

and $\mathcal{K}_s$ is defined as in (1.2.2).

Moreover, we will use the singular integral

$$(-\Delta)^{s/2}_{\mathrm{Si}} u(x) := \mathrm{P.V.}\,\frac{1}{|\Gamma(-s/2)|}\int_M (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) \tag{3.1.4}$$

as our main definition of *"the fractional Laplacian"* on M. We stress that in the general setting of complete Riemannian manifolds, this integro-differential operator cannot be regarded as a fractional power of the Laplacian in any reasonable sense. In particular:

- If M is not stochastically complete (see Definition 3.2.1), then (i) and (ii) do not coincide. In this case, since $e^{t\Delta}1 \neq 1$, the Bochner fractional Laplacian (ii) of a constant does not equal zero. In particular, defining the fractional Sobolev seminorm with (ii) would imply that the s-perimeter is not invariant under complementation $P_s(E) \neq P_s(E^c)$. Nevertheless, with our definition via the singular integral (i), one has that the seminorm of a constant is always zero, and hence, in this work, the fractional perimeter is always invariant under complementation.
- The semigroup property $(-\Delta)^{\alpha+\beta} = (-\Delta)^{\alpha} \circ (-\Delta)^{\beta}$ also fails in general for our definition (3.1.4). Indeed, one can see that the equivalence $(i) \iff (iv)$ above is sufficient for the semigroup property to hold. For example, a Ricci curvature lower bound would be sufficient. See [BGS15] for many sufficient conditions for the equivalence $(i) \iff (iv)$.

Definition 3.1.2. For a measurable set $E \subset M$, we define the fractional s-perimeter of E on (M, g) as

$$P_s(E) := [\chi_E]^2_{H^{s/2}(M)} = 2 \iint_{E \times E^c} \mathcal{K}_s(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\,,$$

where $[\,\cdot\,]^2_{H^{s/2}(M)}$ is defined by (3.1.1) and χ_E is the characteristic function of E.

Apart from the above definition of the fractional perimeter of a set E on the entire M, we will also consider its localized version. For $A, B \subset M$ disjoint and measurable sets, let

$$\mathcal{J}_s(A, B) := \iint_{A \times B} \mathcal{K}_s(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)$$

be the s-interaction functional between the sets A and B.

Definition 3.1.3. Let (M, g) be a complete Riemannian manifold, and let $\Omega \subset M$ be an open and connected set with Lipschitz boundary. We define the s-perimeter of E in Ω as

$$\begin{aligned}\frac{1}{2} P_s(E, \Omega) &:= \iint_{(M\times M)\setminus(\Omega^c \times \Omega^c)} (\chi_E(x) - \chi_E(y))^2 \mathcal{K}_s(x, y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \\ &= \mathcal{J}_s(E \cap \Omega, E^c \cap \Omega) + \mathcal{J}_s(E \cap \Omega, E^c \cap \Omega^c) + \mathcal{J}_s(E \cap \Omega^c, E^c \cap \Omega)\,.\end{aligned}$$

For any measurable $E \subset M$, it is clear by the definition above that $P_s(E, \Omega) = P_s(E^c, \Omega)$, that $P_s(E, M) = P_s(E) = [\chi_E]^2_{H^{s/2}(M)}$ and also that $P_s(E, \Omega) = P_s(E)$ if $E \subset \Omega$ or $E^c \subset \Omega$.

Remark 3.1.4. The hypothesis $P_{s_\circ}(E, \Omega) < +\infty$ for some $s_\circ \in (0, 1)$ cannot be removed in neither of these results. Indeed, in [DFPV13, Example 2.10] the authors exhibit a bounded set $E \subset \mathbb{R}$ such that $P_s(E) = +\infty$ for all $s \in (0, 1)$.

Remark 3.1.5. Note that, taking $M = \mathbb{R}^n$ with its standard metric in Theorem 1.2.5 gives $\mathcal{K}_s(x, p) = \frac{\beta_{n,s}}{|x-p|^{n+s}}$, where

$$\beta_{n,s} = \frac{s 2^{s-1} \Gamma\left(\frac{n+s}{2}\right)}{\pi^{n/2} \Gamma(1 - s/2)}\,.$$

Hence

$$\theta_{\mathbb{R}^n} = \lim_{s \to 0^+} \int_{\mathbb{R}^n \cap B_R^c(p)} \frac{\beta_{n,s}}{|x - p|^{n+s}} dx = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \lim_{s \to 0^+} s \int_{\mathbb{R}^n \cap B_1^c(0)} \frac{1}{|x|^{n+s}} dx = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}} \alpha_{n-1} = 1\,,$$

where α_{n-1} is the volume of the unit sphere $\mathbb{S}^{n-1}$. Moreover, analogously for $E \subset \mathbb{R}^n$ (if the limit exists)

$$\theta_E = \lim_{s \to 0^+} \int_{E \cap B_1^c(0)} \frac{\beta_{n,s}}{|x|^{n+s}} dx = \frac{1}{\alpha_{n-1}} \lim_{s \to 0^+} s \int_{E \cap B_1^c(0)} \frac{1}{|x|^{n+s}} dx \in [0,1] \,,$$

which is (up to the absolute multiplicative constant α_{n-1}^{-1}) what is denoted by $\alpha(E)$ in [DFPV13]. Hence, we see that in the case of the Euclidean space our result Theorem 1.2.5 recovers the one in [DFPV13].

Remark 3.1.6. Note that, as $s \to 0^+$, the constant in (1.2.2) satisfies

$$\frac{1}{|\Gamma(-s/2)|} = \frac{s/2}{\Gamma(1-s/2)} \sim \frac{s}{2} \,.$$

We will use this fact many times in the computations of the asymptotics.

The paper is divided as follows. In section 3.2 we recall some facts and definitions that we will need regarding the heat kernel and harmonic functions on general complete manifolds. In section 3.3 we prove the all the main results stated at the beginning of the introduction. Then, building on our main results, in section 3.4 and section 3.5 we prove Theorem 1.2.5 and Theorem 1.2.7 regarding the asymptotics of the fractional perimeter in infinite volume and finite volume respectively.

Lastly, in section 3.6 we explain why our results hold in a much more general setting than the one of Riemannian manifolds, namely RCD spaces. We could have proved our theorem directly in this generality, but we believe that a presentation for Riemannian manifolds is easier to follow and already captures all the possible (two) behaviors of the limit of the asymptotics: this also allows us to present different proofs. For these reasons, we have moved everything regarding non-smooth spaces to section 3.6.

3.2 The heat kernel on Riemannian manifolds

Let us start by recalling a few classical definitions and results.

Definition 3.2.1 (Stochastical completeness)**.** We call a Riemannian manifold (M,g) stochastically complete if, for every $t > 0$ and for every $p \in M$

$$\int_M H_M(x,p,t) \, \mathrm{d}\mu(x) = 1 \,. \tag{3.2.1}$$

For equivalent definitions of stochastical completeness, one can refer to the manuscript [Gri09] or to the more recent [GIM20] and [GIMP23].

Lemma 3.2.2. *Let (M,g) be a complete Riemannian manifold, then for every $p \in M$*

$$\mathcal{M}(t,p) = \int_M H_M(x,p,t) \, \mathrm{d}\mu(x) \ \text{ is nonincreasing in } t.$$

Proof. The proof is an easy consequence of the semigroup property. Indeed, for $t > s$ we can write

$$H_M(z,p,t) = \int_M H_M(z,x,t-s) H_M(x,p,s) \, \mathrm{d}\mu(x).$$

Integrating in $\mathrm{d}\mu(z)$, using Fubini's theorem and the fact that $\int_M H_M(z,x,t-s)\,\mathrm{d}\mu(x) \le 1$ we get

$$\int_M H_M(z,p,t)\,\mathrm{d}\mu(z) \le \int_M H_M(x,p,s)\,\mathrm{d}\mu(x),$$

which is the thesis. □

Note that, because of Lemma 3.2.2, being stochastically complete is equivalent to the fact that (3.2.1) holds for one single time $t = t_\circ > 0$.

Theorem 3.2.3 (Yau). *Let (M,g) be a complete Riemannian manifold. Then every $L^2(M)$ harmonic function is constant.*

Proof. Let $u \in L^2(M)$ be harmonic. It is a standard result by Yau (see for example [Li12, Lemma 7.1]) that, on every complete Riemannian manifold M, the Caccioppoli-type inequality

$$\int_{B_R(p)} |\nabla u|^2 \,\mathrm{d}\mu \le \frac{4}{R^2} \int_{B_{2R}(p)} |u|^2 \,\mathrm{d}\mu \tag{3.2.2}$$

holds. Since $u \in L^2(M)$, letting $R \to \infty$ gives that u is constant. □

Definition 3.2.4 ($L^\infty - \mathrm{Liouville}$ property)**.** We say that a Riemannian manifold (M,g) has the $L^\infty - \mathrm{Liouville}$ property if every bounded harmonic function on M is constant.

Since the validity of the $L^\infty - \mathrm{Liouville}$ property will be a key feature in our result for infinite volume, we shall recall few conditions that imply this property. See [Gri99] for more general conditions under which the $L^\infty - \mathrm{Liouville}$ property holds.

Proposition 3.2.5. *Let (M,g) be a complete Riemannian manifold. Then, each of the following properties implies the $L^\infty - \mathrm{Liouville}$ property for M:*

(i) $\mathrm{Ric}_M \ge 0$.

(ii) $\mu(B_R(p))/R^2 \to 0$ *as* $R \to \infty$ *for some (and hence any)* $p \in M$.

(iii) There exists a metric $\widetilde{g}$ on M and $K \subset M$ compact such that $\widetilde{g} = g$ in $M \setminus K$ and $(M,\widetilde{g})$ has the $L^\infty - \mathrm{Liouville}$ property.

Proof. To show (i) we just need to apply the $L^\infty - \mathrm{Lip}$ regularization of (3.6.2), that we state in general for RCD spaces in section 3.6 and we give a simple proof at the end of the Appendix. Indeed let $u \in L^\infty(M)$ be such a function: we can clearly assume $\|u\|_{L^\infty} = 1$ so that we have $\|\nabla e^{t\Delta} u\|_{L^\infty} \le C/\sqrt{t}$. The previous estimate tells us that $\|\nabla e^{t\Delta} u\|_{L^\infty} \to 0$ as $t \to \infty$ so that $e^{t\Delta} u \to \mathrm{const}$ weakly star in $L^\infty(M)$. However, we also know that $e^{t\Delta} u = u$ for every $t \in (0,\infty)$ because of the uniqueness of the solution of the heat equation (due to stochastical completeness which holds in the presence of a lower Ricci curvature bound) and this means that u has to be constant.

Part (ii) follows from Yau's estimate (3.2.2) letting $R \to \infty$. Lastly, the proof of part (iii) is contained in [Gri99, Proposition 4.2] and [Gri99, Theorem 5.1]. □

Notice that $\mathrm{Ric}_M \ge -K$ for some $K > 0$ is not sufficient for the $L^\infty - \mathrm{Liouville}$ property to hold since there exist non-constant bounded harmonic functions on the hyperbolic space $\mathbb{H}^n$. Since $\mathbb{H}^n$ is stochastically complete, this means that stochastical completeness does not imply the $L^\infty - \mathrm{Liouville}$ property. Moreover, quite surprisingly, stochastical completeness of M is not implied by the $L^\infty - \mathrm{Liouville}$ property. The first example of such a manifold was constructed by Pinchover in [Pin95], we briefly explain this construction in Example 3.5.2.

In the next lemma, we give the proof of a result that, perhaps, is well-known to the experts, but we could not find an appropriate reference. The case $\mu(M) < +\infty$ is stated in [Gri09] as Exercise 11.21.

We stress that these results easily extend to the context of weighted Riemannian manifolds.

Lemma 3.2.6. *Let (M, g) be a complete, connected Riemannian manifold. Then*

(i) If $\mu(M) < +\infty$, then for all $x, y \in M$

$$\lim_{t\to+\infty} H_M(x, y, t) = \frac{1}{\mu(M)}\,,$$

and the convergence is uniform in every bounded $\Omega \subset M$, that is

$$\lim_{t\to+\infty} \sup_{x,y\in\Omega} \left| H_M(x, y, t) - \frac{1}{\mu(M)} \right| = 0\,.$$

(ii) If $\mu(M) = +\infty$, then for all $x, y \in M$

$$\lim_{t\to+\infty} H_M(x, y, t) = 0\,,$$

and the convergence is uniform in every bounded $\Omega \subset M$, that is

$$\lim_{t\to+\infty} \sup_{x,y\in\Omega} H_M(x, y, t) = 0\,.$$

Moreover, for every fixed $p \in M$ there holds also

$$\lim_{t\to+\infty} \sup_{x\in M} H_M(x, p, t) = 0\,. \tag{3.2.3}$$

Proof. To prove the result we use standard spectral theory. Let us first do the case $\mu(M) = +\infty$. The spectrum of the Laplacian $\sigma(-\Delta)$ is contained in $[0, \infty)$ and by Theorem 3.2.3 we know that the eigenspace of $\lambda = 0$ contains no constant function except for the function identically 0.

Let $\{E_\lambda\}_{\lambda\geq 0}$ be the spectral resolution of the Laplacian, then for every $f \in L^2(M)$ (here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the $L^2(M)$ scalar product)

$$\langle e^{t\Delta} f, f\rangle = \int_0^\infty e^{-t\lambda} d\langle E_\lambda f, f\rangle.$$

Since $\lim_{t\to\infty} e^{-\lambda t} = \chi_{\{0\}}(\lambda)$ we can apply dominated convergence to deduce that

$$\lim_{t\to\infty} \langle e^{t\Delta} f, f\rangle = \langle E_0 f, f\rangle\,,$$

and since E_0 projects onto the eigenspace of $\lambda = 0$, made only by the constant function identically zero, we get

$$\lim_{t\to\infty} \langle e^{t\Delta} f, f\rangle = 0\,. \tag{3.2.4}$$

Now note that for all $f, g \in L^2(M)$ we have $|\langle e^{t\Delta} f, g\rangle| = |\langle e^{t/2\Delta} f, e^{t/2\Delta} g\rangle|$. Thus by Cauchy-Schwartz

$$\begin{aligned}\langle e^{t\Delta} f, g\rangle = \langle e^{t/2\Delta} f, e^{t/2\Delta} g\rangle &\leq \|e^{t/2\Delta} f\|_{L^2} \|e^{t/2\Delta} g\|_{L^2} \\ &= \langle e^{t\Delta} f, f\rangle \langle e^{t\Delta} g, g\rangle\,.\end{aligned}$$

Taking the supremum over $g \in L^2(M)$ with $\|g\|_{L^2} \leq 1$ and sending $t \to \infty$ gives that $e^{t\Delta} f \to 0$ strongly in $L^2(M)$. Since this holds for all $f \in L^2(M)$, this implies $H_M(\cdot, y, t) \to 0$ in $L^2(M)$ as $t \to \infty$.

Now, by a local parabolic Harnack inequality, we are able to turn this convergence into pointwise convergence that is actually locally uniform. Indeed for $p \in M$, $R \ll 1$ to be chosen depending on p, and $t \geq 10$, taking $f = \chi_{B_R(p)}$ above gives

$$\langle e^{t\Delta} \chi_{B_R(p)}, \chi_{B_R(p)}\rangle = \int_{B_R(p)} \int_{B_R(p)} H_M(x, y, t)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \geq \mu(B_R(p))^2 \inf_{x,y \in B_R(p)} H_M(x, y, t)\,.$$

By the parabolic Harnack inequality (see Remark 3.2.8 after this proof) applied two times

$$\begin{aligned}\inf_{x,y \in B_R(p)} H_M(x, y, t) &\geq C^{-1} \inf_{x \in B_R(p)} \sup_{y \in B_R(p)} H_M(x, y, t - 1/2) \\ &\geq C^{-1} \sup_{x \in B_R(p)} \inf_{y \in B_R(p)} H_M(x, y, t - 1/2) \\ &\geq C^{-2} \sup_{x,y \in B_R(p)} H_M(x, y, t - 1)\,,\end{aligned}$$

for some $C > 0$ depending on $B_R(p) \subset M$ but independent of t. Hence

$$\sup_{x,y \in B_R(p)} H_M(x, y, t) \leq C(B_R(p)) \langle e^{(t+1)\Delta} \chi_{B_R(p)}, \chi_{B_R(p)}\rangle \to 0\,,$$

as $t \to \infty$. Covering any bounded set with small balls allows us to infer the desired local uniform convergence.

We are left to prove (3.2.3). By the properties of the heat kernel, we have

$$H_M(p, p, t) = \int_M H_M^2(p, z, t/2)\, \mathrm{d}\mu(z) = \|H_M(p, \cdot, t/2)\|_{L^2(M)}^2.$$

Moreover

$$H_M(x, p, t) = \int_M H_M(x, z, t/2) H_M(p, z, t/2)\, \mathrm{d}\mu(z) \leq \sqrt{H_M(p, p, t)} \|H_M(x, \cdot, t/2)\|_{L^2(M)},$$

which concludes the proof if we are able to show that $\sup_{x \in M} \|H_M(x, \cdot, t/2)\|_{L^2(M)}$ is bounded as $t \to \infty$.

However since $H_M(x, y, t) = e^{(t-1)\Delta}(H_M(x, \cdot, 1))(y)$ and we have the contraction estimate $\|e^{s\Delta}(f)\|_{L^2(M)} \leq \|f\|_{L^2(M)}$ for every $s \in (0, \infty)$ and for every $f \in L^2(M)$ we can write

$$\|H_M(x, \cdot, t)\|_{L^2(M)} = \|e^{(t-1)\Delta}(H_M(x, \cdot, 1))\|_{L^2} \leq \|H_M(x, \cdot, 1)\|_{L^2} \quad \forall t > 1.$$

Therefore, we reach the sought conclusion. This concludes the proof of (ii).

Now assume $\mu(M) < +\infty$. Since the proof in this case is almost identical to the one for infinite volume, we just sketch the argument, highlighting the differences. The only essential difference is that in the case $\mu(M) < +\infty$, the eigenspace relative to $\lambda = 0$ is made only by the constant function $\mu(M)^{-1/2}$. Hence

$$E_0 f = \frac{1}{\mu(M)} \int_M f \, \mathrm{d}\mu =: \fint_M f \,,$$

and in place of (3.2.4) we get

$$\lim_{t\to\infty} \left\langle e^{t\Delta} f - \fint_M f, f \right\rangle \to 0 \,.$$

From here, the proof proceeds exactly the same showing that $H_M(\cdot, y, t) - 1/\mu(M) \to 0$ strongly in $L^2(M)$. Then, one can turn the convergence into pointwise and locally uniform by a similar argument with the parabolic Harnack inequality.

Indeed, the function $v := (H_M(\cdot, y, t) - 1/\mu(M))_+$ (where $(f)_+$ denotes the positive part of f) is a nonnegative subsolution to the heat equation. Then, by the parabolic version of the Moser-Harnack inequality (see, for example, [SC95, Theorem 5.1]) we have (here C depends on R and the geometry of M in $B_{2R}(p)$)

$$\sup_{[t+R^2/2, t+R^2]} v^2 \leq C \int_t^{t+R^2} \int_{B_R(p)} |v|^2 \, \mathrm{d}\mu \to 0 \,. \tag{3.2.5}$$

Hence $\limsup_{t\to\infty} H_M(\cdot, y, t) \leq 1/\mu(M)$. Argiung similarly with the negative part gives also the $\liminf$ inequality, and hence the pointwise convergence. The fact that the convergence is uniform follows from (3.2.5).

□

Remark 3.2.7. Since $\|H_M(x, \cdot, t)\|_{L^1(M)} \leq 1$ and $\|H_M(x, \cdot, t)\|_{L^\infty(M)} \to 0$ as $t \to \infty$, we conclude that also $\|H_M(x, \cdot, t)\|_{L^p(M)} \to 0$ for any $p \in (1, \infty]$. The convergence to zero in $L^1(M)$ is clearly prevented if M is stochastically complete.

Remark 3.2.8. We emphasize that we have used only a local (non-uniform) Harnack inequality in $B_R(p) \subset M$, that is where the constant is allowed to depend on the point p and radius R. This is clear since, for fixed $p \in M$ one can take $R \ll 1$ such that, in normal coordinates at p, the metric coefficients satisfy $\|g_{ij} - \delta_{ij}\|_{C^2(B_R(p))} \leq 1/100$. Then, any solution $u : B_R(p) \to \mathbb{R}$ to the heat equation on M satisfies (in coordinates)

$$u_t - Lu = 0 \,, \quad \textit{in } \mathcal{B}_R(0) \times (0, +\infty) \,,$$

where $-L$ is a uniformly elliptic operator with uniformly bounded coefficients. Hence, by the standard Harnack inequality on $\mathbb{R}^n$ one can conclude the local estimate.

On the other hand, for general Riemannian manifolds, a uniform Harnack inequality (that is, with the constant independent of R and the point p) fails, and strong assumptions are required for it to hold. Actually, the validity of a volume doubling property and a uniform Poincarè inequality is equivalent to the uniform Harnack inequality, this was first proved in [SC92].

Remark 3.2.9. One can turn the previous local uniform convergence in (3.2.3) into the convergence of solutions of the heat equation. Indeed, in the case $\mu(M) = +\infty$, since $H_M(\cdot, p, t)$ converges uniformly to zero we get (by dominated convergence)

$$e^{t\Delta} f(y) = \int_M H_M(x, y, t) f(x) \, \mathrm{d}\mu(x) \to 0 \quad \text{as } t \to \infty \,,$$

for every $y \in M$ and $f \in L^1(M)$.

3.3 Proof of the main results

First, we shall briefly comment on the following quantity

$$\alpha(E) = \lim_{s\to 0^+} s \int_{E\setminus B_1(0)} \frac{1}{|y|^{n+s}}\, dy,$$

introduced by Dipierro, Figalli, Palatucci, and Valdinoci in [DFPV13] as a measure of the behavior of the set E near infinity, and which is (up to a dimensional constant) the limit in (1.2.4) in the case $M = \mathbb{R}^n$ with its standard metric. This quantity is invariant by rescaling of E and, at first, can be thought as a measure of "how conical" is E near infinity. Indeed, if the blow-down E/λ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a regular cone E_∞ as $\lambda \to \infty$, then $\alpha(E) = \mathcal{H}^{n-1}(E_\infty \cap \mathbb{S}^{n-1})$. Nevertheless, the fact that this limit exists in not equivalent to having a conical blow-down. Indeed, one can easily construct examples where the limit in $\alpha(E)$ exists but the blow-downs of E converge to two different cones along two different subsequences.

Finally, the authors in [DFPV13] refer to $\alpha(E)$ as the weighted volume towards infinity of the set E; however in light of our results and description, it would be more appropriate to call this quantity *heat density over E*. Indeed, $\alpha(E)$ represents the fraction of heat kernel that flows through the set towards infinity (this explains why $\theta_M \equiv 1$ on stochastically complete manifolds).

Because of this intuitive reason, the limit in the definition of $\alpha(E)$ needs not to exist in general if E, for example, oscillates between two cones near infinity. See [DFPV13, Example 2.8] for the construction of such an example.

On a Riemannian manifold, a similar quantity is needed but, since no canonical origin (as in $\mathbb{R}^n$) is present, the singular kernel $1/|y|^{n+s}$ has to be replaced with $\mathcal{K}_s(y,p)$ and it has to be proved if and when the limit (1.2.3) becomes independent of $p \in M$. On Riemannian manifolds, this property of the limit being independent of the base point p turns out to be quite delicate and, as a consequence of Theorem 1.2.1, we will see that is implied by the $L^\infty - \mathrm{Liouville}$ property of Definition 3.2.4.

Definition 3.3.1 (Heat density of a set)**.** Let $E \subset M$ be a measurable set with $P_{s_\circ}(E,\Omega) < +\infty$ for some $s_\circ \in (0,1)$. We define, for every $p \in M$ and $R > 0$, the *heat density of E* as the following limit

$$\theta_E(p,R) := \lim_{s\to 0} \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)\, \mathrm{d}\mu(x)\,,$$

when it exists. At this level, this may depend on p and R.

Note that, at this point, it is not even clear whether the limit (1.2.4) of the heat density θ_M of the whole M exists or is different from zero. For example, as a consequence of the proof of Theorem 1.2.5, if there were complete Riemannian manifolds with $\mu(M) = +\infty$ and $\theta_M \neq 1$, then we would see the asymptotic

$$\lim_{s\to 0^+} \frac{1}{2} P_s(E,\Omega) = (\theta_M - \theta_E)\mu(E\cap\Omega) + \theta_E \mu(E^c \cap \Omega)$$

holding (even when $\theta_M \neq 1$), and if $\theta_M = 0$ this would mean that there are Riemannian manifolds where the asymptotic of the fractional s-perimeter of any set E is zero. These type of Riemannian manifolds exist; since $\theta_M \neq 1$ in this case, they are not stochastically complete. We will describe such a manifold in Example 3.5.2.

Now, we show that this does not happen if M is stochastically complete: the limit (1.2.4) always exists, and it is equal to one. Actually, more is true: if there is a point $p \in M$ for which the limit is 1, then the manifold is stochastically complete. Indeed, this is the statement of Proposition 1.2.4 that we now prove.

Proof of Proposition 1.2.4. Note that since $\mu(M) = +\infty$ we have $\mu(M \setminus B_1(p)) > 0$. We want to compute the following

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x).$$

Claim 1. There holds

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^1 H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = 0.$$

Indeed, this directly follows by writing

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_0^1 H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = \lim_{s\to 0^+} \frac{s}{2} \int_0^1 e^{t\Delta}(\chi_{M\setminus B_r(p)})(p) \frac{\mathrm{d}t}{t^{1+s/2}}$$

and exploiting the estimate of Lemma 3.6.18.

Claim 2. There holds

$$\lim_{s\to 0^+} \frac{s}{2} \int_{B_1(p)} \int_1^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = 0. \tag{3.3.1}$$

By the uniform convergence of the heat kernel to zero (in particular, by the result contained in Remark 3.2.9) we get that $e^{t\Delta}(\chi_{B_1(p)})(p) \to 0$ as $t \to \infty$. Therefore, for all $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that $e^{t\Delta}(\chi_{B_1(p)})(p) \leq \varepsilon$ for all $t \geq T$, whence

$$\limsup_{s\to 0^+} \frac{s}{2} \int_1^\infty e^{t\Delta}(\chi_{B_1(p)})(p) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) \leq \lim_{s\to 0} \frac{s}{2} \int_1^T \frac{\mathrm{d}t}{t^{1+s/2}} + \varepsilon \limsup_{s\to 0} \frac{s}{2} \int_T^\infty \frac{\mathrm{d}t}{t^{1+s/2}} \leq \varepsilon\,,$$

for all $\varepsilon > 0$, proving the second claim.

Now, thanks to the first claim we can reduce ourselves to computing

$$\lim_{s\to 0^+} \frac{s}{2} \int_{M\setminus B_1(p)} \int_1^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x).$$

Then we can then add (3.3.1) to the previous limit, which gives zero contribution, and we end up with

$$\lim_{s\to 0^+} \frac{s}{2} \int_M \int_1^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x).$$

Using Fubini and the stochastical completeness of M we get

$$\lim_{s\to 0^+} \frac{s}{2} \int_M \int_1^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = \lim_{s\to 0^+} \frac{s}{2} \int_1^\infty \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = 1\,,$$

and this concludes the proof.

Conversely assume that (1.2.9) holds, then since both the previous claims hold on any connected and geodesically complete Riemannian manifold we have

$$\lim_{s\to 0^+} \frac{s}{2} \int_M \int_1^\infty H_M(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \, \mathrm{d}\mu(x) = 1.$$

Setting $\mathcal{M}(t,p) = \int_M H_M(x,p,t)\, \mathrm{d}\mu(x) \leq 1$ we can infer that, for every $T > 0$

$$1 = \lim_{s\to 0} \frac{s}{2} \int_T^\infty \frac{\mathcal{M}(t,p)}{t^{1+s/2}} \, \mathrm{d}t \leq \lim_{s\to 0} \frac{s}{2} \int_T^\infty \frac{1}{t^{1+s/2}} \, \mathrm{d}t = 1\,.$$

Now, assume by contradiction that M is not stochastically complete. Then since $\mathcal{M}(t,p)$ is nonincreasing in time and nonnegative, there holds $\lim_{t\to\infty} \mathcal{M}(t,p) \leq 1-\delta$ for some $\delta > 0$, and we would have $\mathcal{M}(t,p) \leq 1-\delta/2$ for every $t \geq T = T(\delta)$. This gives

$$1 = \lim_{s\to 0} \frac{s}{2} \int_T^\infty \frac{\mathcal{M}(t,p)}{t^{1+s/2}} \, \mathrm{d}t \leq \lim_{s\to 0} \frac{s}{2} \int_T^\infty \frac{1-\delta/2}{t^{1+s/2}} \, \mathrm{d}t = 1-\delta/2\,,$$

reaching a contradiction, hence $\lim_{t\to\infty} \mathcal{M}(t,p) = 1$ and thanks to Lemma (3.2.2) we conclude. □

Remark 3.3.2. Following the proof of Proposition 1.2.4, one can see a clear picture of what happens to the limit in $\theta_M(p)$ even when M is not stochastically complete. Indeed, for every Riemannian manifold (not necessarily stochastically complete) and $p \in M$, the limit $\lim_{t\to\infty} \mathcal{M}(t,p)$ exists. This follows from the fact that $\mathcal{M}(\cdot,p)$ is nonincreasing and nonnegative; see Lemma 3.2.2. Since

$$\mathcal{M}(t,p) = \int_M H_M(p,x,t) \; \mathrm{d}\mu(x) = e^{t\Delta}1$$

is a solution to the heat equation starting from the function equal to one; it follows from the proof above and from standard parabolic estimates that $\mathcal{M}(t,\cdot) \to \theta_M$ in $C^2_{\mathrm{loc}}(M)$ as $t \to \infty$, where $\theta_M : M \to \mathbb{R}$ is a bounded, nonnegative harmonic function on M. Therefore:

(i) If M is stochastically complete, we have $\theta_M \equiv 1$ (in particular, the value of θ_M does not depend on the point), and the proof above shows $\theta_M = 1$.

(ii) If M is not stochastically complete but satisfies the $L^\infty - \mathrm{Liouville}$ property (see Definition 3.2.4) we know that $\theta_M \equiv \theta_\circ \in [0,1)$ and, following the proof of the proposition, one finds that the limit in the definition of θ_M exists, does not depend on the point p and there holds $\theta_M = \theta_\circ$. Note that such Riemannian manifolds exist and were first constructed in [Pin95]. In Example 3.5.2, we describe one with $\theta_\circ = 0$.

(iii) If M is not stochastically complete and does not satisfy the $L^\infty - \mathrm{Liouville}$ property, then in general θ_M is a nonconstant harmonic function on M, and the value of $\theta_M(p)$ can depend on the point p.

Now we are in the position to prove our first main result.

Proof of Theorem 1.2.1. With no loss of generality assume $r < R$. First, we show that the limit does not depend on the radius, that is

$$\theta_E(p,R) = \theta_E(p,r)\,.$$

We have

$$\begin{aligned}\left|\int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x) - \int_{E\setminus B_r(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x)\right| &\leq \int_{B_R(p)\setminus B_r(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x)\\ &\leq Cs\int_{B_R(p)\setminus B_r(p)}\int_0^1 H_M(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x)\\ &+ Cs\int_{B_R(p)\setminus B_r(p)}\int_1^\infty H_M(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x) =: I_1+I_2\,.\end{aligned}$$

For the first integral, by Lemma 3.6.18 as $s\to 0^+$

$$I_1 \leq Cs\int_0^1 e^{t\Delta}(\chi_{M\setminus B_r(p)})(p)\frac{\mathrm{d}t}{t^{1+s/2}} \leq Cs\int_0^1 \frac{e^{-c/t}}{t^{1+s}}\,\mathrm{d}t \to 0\,.$$

Regarding the second integral, for all $\varepsilon>0$ by Lemma 3.2.6 there is $T=T(\varepsilon)>0$ such that $|H_M(x,p,t)|\leq\varepsilon$ for all $x\in B_R(p)$ and $t\geq T$, hence

$$\begin{aligned}I_2 &\leq Cs\int_1^T\int_{B_R(p)} H_M(x,p,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t^{1+s/2}} + Cs\int_T^\infty\int_{B_R(p)} H_M(x,p,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t^{1+s/2}}\\ &\leq Cs\int_1^T\frac{\mathrm{d}t}{t^{1+s/2}} + Cs\varepsilon\mu(B_R(p))\int_T^\infty\frac{\mathrm{d}t}{t^{1+s/2}}\\ &= C(1-T^{-s/2}) + C\varepsilon\mu(B_R(p))T^{-s/2}\,,\end{aligned}$$

letting $s\to 0^+$ (and then $\varepsilon\to 0$) gives $I_2\to 0$. Hence, taking $s\to 0^+$ shows $\theta_E(p,R)=\theta_E(p,r)$, showing that the limit never depends on the radius. Note that what we have just proved already implies that if E is bounded then the limit exists and $\theta_E=0$, since one can just take $R\gg 1$ so that $E\setminus B_R(p)=\varnothing$.

Now fix $q\in M$. For every $p\in B_{1/2}(q)$ we can write

$$\theta_E(p) = \lim_{s\to 0^+}\int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x).$$

This is possible because we always have independence on the radius. Indeed

$$\left|\int_{E\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x) - \int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x)\right| \leq \int_{B_1(q)\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)\;\mathrm{d}\mu(x)\,,$$

hence

$$\limsup_{s\to 0^+}\left|\int_{E\setminus B_{1/2}(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x) - \int_{E\setminus B_1(q)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x)\right| \leq \theta_{B_1(q)} = 0\,.$$

Now set

$$\Theta_{E,s}(p) := \frac{s}{2}\int_0^\infty e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{\mathrm{d}t}{t^{1+s/2}}\,, \tag{3.3.2}$$

so that $\theta_E(p)=\lim_{s\to 0^+}\Theta_{E,s}(p)$. By Lemma 3.6.18 we have that that $0\leq\Theta_{E,s}(p)\leq C$, for some constant $C>0$ depending only on M. Now fix $\varphi\in C_c^\infty(B_{1/2}(q))$, by dominated convergence

$$\int_M \theta_E(\Delta\varphi)\;\mathrm{d}\mu = \lim_{s\to 0^+}\int_M \Theta_{E,s}(\Delta\varphi)\;\mathrm{d}\mu = \lim_{s\to 0^+}\int_M (\Delta\Theta_{E,s})\varphi\;\mathrm{d}\mu\,. \tag{3.3.3}$$

Note that, for fixed s and $p \in B_{1/2}(q)$, we can write

$$\Delta\Theta_{E,s}(p) = \frac{s}{2}\int_0^\infty \Delta e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{\mathrm{d}t}{t^{1+s/2}} = \frac{s}{2}\int_0^\infty \partial_t e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{\mathrm{d}t}{t^{1+s/2}},$$

which, after integration by parts, becomes (note that the boundary term at $t = 0^+$ is zero due to Lemma 3.6.18) equal to

$$\frac{s}{2}\int_0^\infty e^{t\Delta}(\chi_{E\setminus B_1(q)})(p)\frac{(1+s/2)}{t^{2+s/2}}\,\mathrm{d}t\,.$$

The latter quantity goes to 0 as $s \to 0^+$, and is uniformly bounded for $s \in (0,1)$, for every $p \in B_{1/2}(q)$. Hence, going back to (3.3.3) we get

$$\int_M \theta_E(\Delta\varphi)\,\mathrm{d}\mu = 0\,.$$

That is, $\theta_E \in L^1_{\mathrm{loc}}(M)$ is a very weak solution of $\Delta\theta_E = 0$. We're left to prove that θ_E is smooth and is a classical solution of $\Delta\theta_E = 0$.

In a small chart, in coordinates, one can see that u is (locally) a very weak solution of $\partial_i\big(\sqrt{|\det(g)|}g^{ij}\partial_j\theta_E\big) = 0$. Choosing the chart sufficiently small, we get that the coefficients $\sqrt{|\det(g)|}g^{ij}$ are smooth and uniformly elliptic. Then, for example by [ZB12, Theorem 1.3], we get that $\theta_E \in W^{2,2}_{\mathrm{loc}}(M)$ and bootstrapping classical elliptic regularity gives that θ_E is smooth and harmonic.

Lastly, (1.2.5) follows from the last part of the proof of Proposition 1.2.4, and the fact that $p \mapsto \theta_M(p)$ is harmonic is verbatim the proof we did for $E \subset M$ above. □

Note that, according to Theorem 1.2.1, if M possesses the L^∞ − Liouville property, then θ_E is constant for every set E for which it exists. A natural question to ask would be whether some type of converse is true, however we are not able to provide an answer.

Now we turn to the proof of Theorem 1.2.2. To prove this result, we will need Lemma 3.3.3, which essentially says that for manifolds with $\mu(M) = +\infty$, the singular kernel $\mathcal{K}_s$ locally behaves like that of $\mathbb{R}^n$ as $s \to 0^+$. This is not the case for finite volume manifolds[1]. Recall the notation of Remark 3.1.5, where we denote by $\frac{\beta_{n,s}}{|x-y|^{n+s}}$ the singular kernel of $\mathbb{R}^n$ with its standard metric. Note also that $cs(2-s) \le \beta_{n,s} \le Cs(2-s)$ for some dimensional $c, C > 0$.

The following lemma is a sharpening of [CFSS23, Lemma 2.19] for manifolds with infinite volume. Indeed, in [CFSS23], the authors are not interested in characterizing the sharp dependence from s of $\mathcal{K}_s$ as $s \to 0^+$. Moreover, in [CFSS23], the authors estimate $\mathcal{K}_s$ locally on every complete Riemannian manifold M (both with finite and infinite volume), but the result stated in Lemma 3.3.3 is not true on manifolds with finite volume.

Lemma 3.3.3. *Let (M,g) be a complete n-dimensional Riemannian manifold with $\mu(M) = +\infty$, and let $p \in M$. Assume that in normal coordinates at p there holds $\frac{99}{100}|v|^2 \le g_{ij}(q)v^iv^j \le \frac{101}{100}|v|^2$ and $|\nabla g_{ij}(q)| \le 1/100$ for all $v \in \mathbb{R}^n$ and $q \in B_1(p)$. Then there exists $\mathcal{K}'_s : B_1(p)\times B_1(p) \to [0,\infty)$ such that*

$$\lim_{s\to 0^+}\sup_{x,y\in B_{1/8}(p)}|\mathcal{K}_s(x,y) - \mathcal{K}'_s(x,y)| = 0\,,$$

[1] Indeed, for finite volume manifolds, the same conclusion (3.3.4) holds with constants depending on s, but as $s \to 0^+$ the constants do not behave like the ones of $\mathbb{R}^n$.

and for all $x, y \in B_{1/8}(p)$

$$c\frac{\beta_{n,s}}{d(x,y)^{n+s}} \le \mathcal{K}'_s(x,y) \le C\frac{\beta_{n,s}}{d(x,y)^{n+s}}\,, \tag{3.3.4}$$

for some dimensional constants $c, C > 0$.

We postpone the proof of Lemma 3.3.3 to subsection 3.6.2 in the Appendix.

Proof of Theorem 1.2.2. As we can assume $s < s_\circ/2$, it follows from the proof of Proposition 3.6.21 that the integral in $(-\Delta)^{s/2}_{\mathrm{Si}}u$ is absolutely convergent[2] for a.e. $x \in M$, and the principal value is not needed. Moreover, since $u \in H^{s_\circ/2}(M)$ we have

$$\int_M (u(x)-u(y))^2\mathcal{K}_{s_\circ}(x,y)\,\mathrm{d}\mu(y) < +\infty$$

for a.e. $x \in M$. Fix $x \in M$ in the intersection of these two sets of full measure, and take R such that $\mathrm{supp}(u) \subset B_R(x)$. Then

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{Si}}u(x) &= \int_M (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\\ &= \int_{B_R(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) + u(x)\int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\,. \end{aligned} \tag{3.3.5}$$

Note that being $\mu(M) = +\infty$ we have

$$\int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) \neq 0\,.$$

Claim. As $s \to 0^+$ there holds

$$\lim_{s\to 0^+}\int_{B_R(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) = 0\,.$$

Indeed, let $\rho \ll 1$ small that will be chosen later. We denote here by C a constant which does not depend on s. Then

$$\begin{aligned}&\left|\int_{B_R(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\right|\\ &\qquad= \left|\int_{B_\rho(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) + \int_{B_R(x)\setminus B_\rho(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\right|\\ &\qquad\le \int_{B_\rho(x)} |u(x)-u(y)|\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) + 2\|u\|_{L^\infty}\int_{B_R(x)\setminus B_\rho(x)} \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\,.\end{aligned}$$

We estimate these two integrals separately. Let $\mathcal{K}'_s$ be the singular kernel given by Lemma 3.3.3, applied with ρ sufficiently small and suitably rescaled. For the first integral, Lemma 3.3.3 gives

$$\limsup_{s\to 0^+}\int_{B_\rho(x)} |u(x)-u(y)|\big(\mathcal{K}_s(x,y)-\mathcal{K}'_s(x,y)\big)\,\mathrm{d}\mu(y) = 0\,. \tag{3.3.6}$$

[2]Here we are not assuming M being stochastically complete, but in Proposition 3.6.21 stochastical completeness is only used to have that $(-\Delta)^{s/2}_{\mathrm{B}}u = (-\Delta)^{s/2}_{\mathrm{Si}}u$ a.e., not to show the absolute convergence of the integrals.

Moreover, by the bounds of Lemma 3.3.3 and since $u \in H^{s_\circ/2}(M)$, for a.e. $x \in M$

$$\int_{B_\rho(x)} \frac{(u(x)-u(y))^2}{d(x,y)^{n+s_\circ}}\,dy \le C(s_\circ)\int_{B_\rho(x)} (u(x)-u(y))^2 \mathcal{K}_{s_\circ}(x,y)\,dy < +\infty.$$

Hence, by Lemma 3.3.3 again and Holder's inequality

$$\begin{aligned}\int_{B_\rho(x)} |u(x)-u(y)|\mathcal{K}_s'(x,y)\,\mathrm{d}\mu(y) &\le Cs\int_{B_\rho(x)} \frac{|u(x)-u(y)|}{d(x,y)^{n+s}}\,\mathrm{d}\mu(y)\\ &\le Cs\left(\int_{B_\rho(x)} \frac{(u(x)-u(y))^2}{d(x,y)^{n+s_\circ}}\,dy\right)^{1/2}\left(\int_{B_\rho(x)} \frac{1}{d(x,y)^{n+2s-s_\circ}dy}\right)^{1/2}\\ &\le Cs\left(\frac{\rho^{s_\circ-2s}}{s_\circ-2s}\right)^{1/2} \to 0\,,\end{aligned}$$

as $s \to 0^+$, where in the second-last inequality we have used polar coordinates for ρ sufficiently small (possibly depending on x). Thus, with (3.3.6) we have that the first integral tends to zero.

Regarding the second integral, one can note that we have proved in part (i) of Theorem 1.2.1 that, for every $x \in M$ and $r, R > 0$

$$\lim_{s\to 0^+} \int_{B_R(x)\setminus B_r(x)} \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) = 0\,,$$

since $B_R(x)$ is a bounded set, and this concludes the proof of the claim.

Moreover, by the very definition of θ_M we have

$$\lim_{s\to 0^+} \int_{M\setminus B_R(x)} \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y) = \theta_M(x)\,, \tag{3.3.7}$$

hence letting $s \to 0^+$ in (3.3.5) gives

$$\lim_{s\to 0^+} (-\Delta)^{s/2}_{\mathrm{Si}} u(x) = \theta_M(x)u(x)\,,$$

for a.e. $x \in M$, and this concludes the proof. □

To prove our result Theorem 1.2.7 on the asymptotics for infinite volume, one needs also to know the asymptotics as $s \to 0^+$ of the fractional s-perimeter on the entire M, that is when $\Omega \equiv M$. This is addressed by Theorem 3.3.4 below on the asymptotics of the fractional Sobolev seminorms. This result is the counterpart of Theorem 3.4.1 in the case of infinite volume.

Theorem 3.3.4. *Let (M,g) be a complete Riemannian manifold with $\mu(M) = +\infty$, and let $s_\circ \in (0,1)$. Then, for every $u \in H^{s_\circ/2}(M) \cap L^\infty(M)$ with bounded support there holds*

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_M u^2\theta_M\,\mathrm{d}\mu\,.$$

Proof. Formally, one would like to infer that

$$\begin{aligned}\frac{1}{2}[u]^2_{H^{s/2}(M)} &:= \frac{1}{2}\iint_{M\times M} (u(x)-u(y))^2\mathcal{K}_s(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &= \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u\,\mathrm{d}\mu \xrightarrow{s\to 0^+} \int_M u^2\theta_M\,\mathrm{d}\mu\,,\end{aligned}$$

where the first equality is the very definition of the seminorm. The second inequality is nontrivial since the integrals one would write in the few lines of a proof are not absolutely convergent in general. Moreover, for the last step of taking the limit as $s \to 0^+$ one needs to show that the a.e. convergence $(-\Delta)^{s/2}_{\mathrm{Si}} u \to \theta_M u$ of Theorem 1.2.2 can be upgraded to weak convergence in $L^2(M)$. Now we shall justify both steps.

Step 1. We have

$$\frac{1}{2}\iint_{M\times M} (u(x)-u(y))^2 \mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) = \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u\, \mathrm{d}\mu\,. \tag{3.3.8}$$

Fix $\varepsilon > 0$ and let

$$(-\Delta)^{s/2}_{\varepsilon} u(x) := \int_{M\setminus B_\varepsilon(x)} (u(x)-u(y))\mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,.$$

Let also $D := \{(z,z) \,:\, z\in M\}$ denote the diagonal of $M\times M$ and D_δ a δ-neighborhood of D. We have

$$\begin{aligned}
&\iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} (u(x)-u(y))^2 \mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad = \iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(x)(u(x)-u(y))\mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad\quad - \iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(y)(u(x)-u(y))\mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad = 2\iint_{M\times M\setminus D_{\varepsilon/\sqrt{2}}} u(x)(u(x)-u(y))\mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\qquad = 2\int_M\int_{M\setminus B_\varepsilon(x)} u(x)(u(x)-u(y))\mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\qquad = 2\int_M u(-\Delta)^{s/2}_{\varepsilon} u\, \mathrm{d}\mu\,,
\end{aligned}$$

where splitting the integral and Fubini are justified since the integrals are absolutely convergent. Indeed

$$\begin{aligned}
&\int_M\int_{M\setminus B_\varepsilon(x)} |u(x)(u(x)-u(y))|\mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le \int_M |u(x)|^2 \int_{M\setminus B_\varepsilon(x)} \mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,\mathrm{d}\mu(x) + \int_M |u(x)| \int_{M\setminus B_\varepsilon(x)} |u(y)|\mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\,,
\end{aligned}$$

but by Lemma 3.6.18

$$\begin{aligned}
\int_{M\setminus B_\varepsilon(x)} \mathcal{K}_s(x,y)\, \mathrm{d}\mu(y) &= C\int_0^\infty \left(\int_{M\setminus B_\varepsilon(x)} H_M(x,y,t)\, \mathrm{d}\mu(y)\right)\frac{\mathrm{d}t}{t^{1+s/2}}\\
&\le C\int_0^\infty \frac{e^{-c/t}}{t^{1+s/2}}\,\mathrm{d}t \le C\,,
\end{aligned}$$

for some C depending on s and ε. Hence

$$\int_M\int_{M\setminus B_\varepsilon(x)} |u(x)(u(x)-u(y))|\mathcal{K}_s(x,y)\, \mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \le C(\|u\|_{L^\infty}, \mu(\mathrm{supp}(u)), \varepsilon, s) < +\infty\,,$$

and this shows the absolute convergence.

Moreover, by Proposition 3.6.21 for a.e. $x \in M$ the integral in $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is absolutely convergent, then

$$\int_M \left|(-\Delta)^{s/2}_{\mathrm{Si}} u - (-\Delta)^{s/2}_{\varepsilon} u\right|^2 \mathrm{d}\mu \le \int_M \left| \int_{B_\varepsilon(x)} |u(x) - u(y)| \mathcal{K}_s(x,y)\, \mathrm{d}\mu(y) \right|^2 \mathrm{d}\mu(x)\,,$$

and the right hand side ternds to 0 as $\varepsilon \to 0$. Indeed, as $\varepsilon \to 0$, by the very same argument at the end of the proof of Theorem 1.2.2 there holds

$$\int_{B_\varepsilon(x)} |u(x) - u(y)| \mathcal{K}_s(x,y)\, \mathrm{d}\mu(y) \to 0\,,$$

for a.e. $x \in M$, and for x fixed the convergence is monotone (decreasing) since the integrand is positive. Hence we have proved $(-\Delta)^{s/2}_{\varepsilon} u \to (-\Delta)^{s/2}_{\mathrm{Si}} u$ in $L^2(M)$ as $\varepsilon \to 0$. Now, letting $\varepsilon \to 0$ in

$$\frac{1}{2} \iint_{M \times M \setminus D_{\varepsilon/\sqrt{2}}} (u(x) - u(y))^2 \mathcal{K}_s(x,y)\ \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) = \int_M u (-\Delta)^{s/2}_{\varepsilon} u\ \mathrm{d}\mu\,,$$

together with the monotone convergence theorem on the left-hand side, we get the equality of the seminorms and this completes the proof of Step 1.

Step 2. There holds

$$(-\Delta)^{s/2}_{\mathrm{Si}} u \rightharpoonup \theta_M u \ \text{ weakly in } L^2(M)\,.$$

The convergence a.e. is given by Theorem 1.2.2. To prove that the convergence holds weakly in $L^2(M)$, we show that $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is equibounded in $L^2(M)$. By (3.6.12) there is C depending only on $s_\circ$ such that

$$\|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} \le C \|u\|^2_{L^2(M)} + C s^2 \|u\|^2_{H^{s_\circ}(M)}\,,$$

and hence

$$\limsup_{s \to 0^+} \|(-\Delta)^{s/2}_{\mathrm{Si}} u\|^2_{L^2(M)} \le C \|u\|^2_{L^2(M)} < +\infty\,.$$

This concludes Step 2 and, sending $s \to 0^+$ in (3.3.8) concludes the proof. □

Remark 3.3.5. Note that the equivalence of the seminorms (3.3.8) always holds for characteristic functions, without any assumption. Indeed for every measurable $E \subset M$

$$\begin{aligned} 2 \int_M \chi_E \cdot (-\Delta)^{s/2}_{\mathrm{Si}} \chi_E \, dx &= 2 \int_E \left(\lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(x)} (1 - \chi_E(y)) \mathcal{K}_s(x,y) dy \right) dx \\ &= 2 \int_E \left(\lim_{\varepsilon \to 0} \int_{(M \setminus B_\varepsilon(x)) \cap E^c} \mathcal{K}_s(x,y) dy \right) dx \\ &= 2 \int_E \int_{E^c} \mathcal{K}_s(x,y) dy = [\chi_E]^2_{H^{s/2}(M)}\,, \end{aligned}$$

where the second-last equality follows by the monotone convergence theorem.

Proof of Theorem 1.2.3. Since M is stochastically complete, by Proposition 3.6.23 we have $H^{s_\circ/2}(M) \subset \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$. The equality a.e. of the fractional Laplacian, then follows by Proposition 3.6.21 and Proposition 3.6.24.

To prove (1.2.7), (1.2.8) one can argue similarly to the proof of Theorem 3.4.1. Indeed, as $s \to 0^+$ for every $v \in L^2(M)$ we have

$$\langle (-\Delta)^{s/2}_{\mathrm{Spec}} u, v\rangle = \int_{\sigma(-\Delta)} \lambda^{s/2} d\langle E_\lambda u, v\rangle \to \int_{\sigma(-\Delta)\setminus\{0\}} d\langle E_\lambda u, v\rangle = \langle u, v\rangle - \langle E_0 u, v\rangle \,,$$

where E_0 is the projector onto the eigenspace of $-\Delta$ relative to the eigenvalue $\lambda = 0$. By Theorem 3.2.3 every $L^2(M)$ harmonic function is constant, hence we have two cases:

(i) If $\mu(M) < +\infty$ then the eigenspace of $\lambda = 0$ is the span of the eigenfunction $\mu(M)^{-1/2}$, then $E_0 u = \frac{1}{\mu(M)} \int_M u \,\mathrm{d}\mu$ and this gives (1.2.7).

(ii) If $\mu(M) = +\infty$ then $E_0 u = 0$ and we have (1.2.8).

This concludes the proof. □

Remark 3.3.6. When M is stochastically complete with $\mu(M) = +\infty$ the convergence in (1.2.8) also follows by Theorem 1.2.2, since $\theta_M \equiv 1$ in this case. Nevertheless, the argument carried on in Theorem 1.2.2 is much more general and shows what happens in the limit on any manifold with $\mu(M) = +\infty$, even when M is not stochastically complete (i.e. when $(-\Delta)^{s/2}_{\mathrm{Si}}$ and $(-\Delta)^{s/2}_{\mathrm{Spec}}$ do not coincide).

3.4 Asymptotics: finite volume manifolds

3.4.1 Global asymptotics

We first give a simple proof of Theorem 1.2.7 in the case $\Omega = M$, using our results from subsection 3.6.3 on the equivalence of the spectral fractional Laplacian and ours defined by the singular integral (3.1.4).

Theorem 3.4.1. *Let (M, g) be a complete Riemannian manifold with $\mu(M) < +\infty$ and let $s_\circ \in (0, 1)$. Then, for every $u \in H^{s_\circ/2}(M)$ there holds*

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(M)} = \|u\|^2_{L^2(M)} - \frac{1}{\mu(M)}\left(\int_M u \,\mathrm{d}\mu\right)^2 .$$

Proof. Let $\{E_\lambda\}_{\lambda\geq 0}$ be the spectral resolution of the Laplacian $-\Delta$ on $L^2(M)$, and let $\sigma(-\Delta) \subset [0, \infty)$ be the spectrum of $-\Delta$. In particular, for every $u \in L^2(M)$, $d\langle E_\lambda u, u\rangle$ is a regular Borel (real valued) measure on $[0, \infty)$ concentrated on $\sigma(-\Delta)$, and with

$$\|u\|^2_{L^2(M)} = \int_{\sigma(-\Delta)} d\langle E_\lambda u, u\rangle \,.$$

We refer to [Gri09, Appendix A.5] for an introduction and properties of the spectral resolution. Since $\mu(M) < +\infty$, we have that $0 \in \sigma(-\Delta)$ lies in the point spectrum with eigenfunction $\phi_0 = \mu(M)^{-1/2}$. Then

$$-\Delta = \int_{\sigma(-\Delta)} \lambda dE_\lambda \,, \quad \text{and} \quad (-\Delta)^{s/2}_{\mathrm{Spec}} = \int_{\sigma(-\Delta)} \lambda^{s/2} dE_\lambda \,,$$

on $\mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}}) := \left\{u \in L^2(M) \,:\, \int_{\sigma(-\Delta)} \lambda^s \, d\langle E_\lambda u, u\rangle < +\infty\right\}$.

Hence, for all $s < s_\circ$ by Corollary 3.6.25

$$\frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u \,\mathrm{d}\mu = \int_{\sigma(-\Delta)\setminus\{0\}} \lambda^{s/2} d\langle E_\lambda u, u\rangle \,.$$

Taking the limit as $s \to 0^+$ gives

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_{\sigma(-\Delta)\setminus\{0\}} d\langle E_\lambda u, u\rangle = \|u\|^2_{L^2(M)} - \langle E_0 u, u\rangle = \|u\|^2_{L^2(M)} - \frac{1}{\mu(M)}\left(\int_M u \,\mathrm{d}\mu\right)^2 ,$$

where in the last line we have used that E_0 is the projector onto the eigenspace of $-\Delta$ relative to the eigenvalue $\lambda = 0$, but by a result of Yau (see Theorem 3.2.3) on a complete manifold every $L^2(M)$ harmonic function is constant and then $\langle E_0 u, u\rangle = \langle \phi_0, u\rangle^2_{L^2(M)} = \frac{1}{\mu(M)}\left(\int_M u \,\mathrm{d}\mu\right)^2$. □

Remark 3.4.2. This result allows us to prove our main theorem in the case $\Omega = M$. Indeed, if $E \subset M$ is such that $P_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$, then taking $u = \chi_E$ in Theorem 3.4.1 gives

$$\lim_{s\to 0^+} \frac{1}{2} P_s(E) = \mu(E) - \frac{1}{\mu(M)}\mu(E)^2 = \frac{\mu(E)\mu(E^c)}{\mu(M)} \,.$$

3.4.2 Localized asymptotics and proof of Theorem 1.2.7

Now we turn to the proof of the main result on the asymptotics for finite volume Theorem 1.2.7

Lemma 3.4.3. *Let (M,g) be a complete Riemannian manifold, and let $A, B \subset M$ two disjoint measurable sets with (say) $\mu(A) < +\infty$. If $\mathcal{J}_{s_\circ}(A,B) < +\infty$ for some $s_\circ \in (0,1)$ then*

$$\lim_{s\to 0^+} \left| \mathcal{J}_s(A,B) - \frac{1}{|\Gamma(-s/2)|} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \right| = 0 \,.$$

Proof. Since $\int_M H_M(x,y,t)\,\mathrm{d}\mu(x) \le 1$ for all $y \in M$ and $t \in (0,\infty)$ we have

$$\begin{aligned}
&\left| \mathcal{J}_s(A,B) - \frac{1}{|\Gamma(-s/2)|} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \right| \\
&= \iint_{A\times B} \left(\mathcal{K}_s(x,y) - \frac{1}{|\Gamma(-s/2)|} \int_{1/s}^{\infty} H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}} \right) \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \frac{1}{|\Gamma(-s/2)|} \iint_{A\times B} \left(\int_0^1 H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}} + \int_1^{1/s} H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}} \right) \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \frac{1}{|\Gamma(-s/2)|} \left(\iint_{A\times B} \int_0^1 H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) + \int_A \int_1^{1/s} \left(\int_B H_M(x,y,t)\,\mathrm{d}\mu(x) \right) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(y) \right) \\
&\le Cs \iint_{A\times B} \int_0^{\infty} H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s_\circ/2}} + Cs\mu(A) \int_1^{1/s} \frac{\mathrm{d}t}{t^{1+s/2}} \\
&= Cs\mathcal{J}_{s_\circ}(A,B) + C\mu(A)(1 - s^{s/2}) \,,
\end{aligned}$$

and taking $s \to 0^+$ concludes the proof. □

Proof of Theorem 1.2.7. First, we claim that

$$\lim_{s\to 0^+}\frac{1}{|\Gamma(-s/2)|}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)=\frac{\mu(A)\mu(B)}{\mu(M)}\,. \tag{3.4.1}$$

Indeed

$$s\int_{1/s}^{\infty} H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}}=s^{1+s/2}\int_1^{\infty} H(x,y,r/s)\frac{dr}{r^{1+s/2}}\,,$$

and since by Lemma 3.2.6 as $t\to+\infty$ the heat kernel $H_M(x,y,t)$ converges to $1/\mu(M)$ for all $x,y\in M$, we get

$$\begin{aligned}\lim_{s\to 0^+}\frac{s}{2}\iint_{A\times B}\int_{1/s}^{\infty} H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)&=\frac{\mu(A)\mu(B)}{\mu(M)}\lim_{s\to 0^+}(s/2)\,s^{s/2}\int_1^{\infty}\frac{dr}{r^{1+s/2}}\\&=\frac{\mu(A)\mu(B)}{\mu(M)}\,.\end{aligned}$$

Then, putting together Lemma 3.4.3 and (3.4.1) readily implies

$$\lim_{s\to 0^+}\mathcal{J}_s(A,B)=\frac{\mu(A)\mu(B)}{\mu(M)}\,.$$

Lastly, since $P_{s_\circ}(E,\Omega)<+\infty$ and

$$\frac{1}{2}P_s(E,\Omega)=\mathcal{J}_s(E\cap\Omega,E^c\cap\Omega)+\mathcal{J}_s(E\cap\Omega,E^c\cap\Omega^c)+\mathcal{J}_s(E\cap\Omega^c,E^c\cap\Omega)\,,$$

the theorem follows by letting $s\to 0^+$.

□

In [CCLMP22] the authors prove the following result regarding the s-perimeter of the Gaussian space. Since the total mass of the Gaussian space is one, we see that this is formally identical to our Theorem 1.2.7 for finite volume.

Theorem 3.4.4 (Main Theorem in [CCLMP22])**.** *Let $\Omega\subset\mathbb{R}^n$ be an open and connected set with Lipschitz boundary. Then, for any $E\subset\mathbb{R}^n$ measurable set such that $P^\gamma_{s_\circ}(E,\Omega)<+\infty$ for some $s_\circ\in(0,1)$ there holds*

$$\lim_{s\to 0^+}\frac{s}{2}P^\gamma_s(E;\Omega)=\gamma(E)\gamma(E^c\cap\Omega)+\gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)\,,$$

where $P^\gamma_s(E,\Omega)$ is the fractional Gaussian perimeter

$$\begin{aligned}&P^\gamma_s(E,\Omega)\\&=\iint_{E\cap\Omega\times E^c\cap\Omega}\mathcal{K}_s(x,y)\,d\gamma_x d\gamma_y+\iint_{E\cap\Omega\times E^c\cap\Omega}\mathcal{K}_s(x,y)\,d\gamma_x d\gamma_y+\iint_{E\cap\Omega\times E^c\cap\Omega}\mathcal{K}_s(x,y)\,d\gamma_x d\gamma_y\,,\end{aligned}$$

and $\mathcal{K}_s(x,y)$ is defined as in (1.2.2) with on the right-hand side the heat kernel H_γ of the Gaussian space $(\mathbb{R}^n,\gamma)$, where $d\gamma(x)=\frac{1}{(2\pi)^{n/2}}e^{-|x|^2/2}\mathcal{L}^n(dx)$.

The proof in [CCLMP22] follows the same lines as our proof of Theorem 1.2.7, but the authors heavily use the fact that they know the explicit form of the heat kernel H_γ for the Gaussian space. In the next subsection, we briefly explain how our method implies their result when applied to weighted manifolds.

3.4.3 Weighted manifolds

Our result for finite volume manifolds extends, with proofs mutatis mutandis, to the case of weighted manifolds with finite volume, implying the one in [CCLMP22].

A weighted manifold is a Riemannian manifold (M, g) endowed with a measure μ that has a smooth positive density with respect to the Riemannian volume form dV_g. The space (M, g, μ) features the so-called weighted Laplace operator $-\Delta_\mu$, generalizing the Laplace-Beltrami operator, which is symmetric with respect to measure μ. It is possible to extend $-\Delta_\mu$ to a self-adjoint operator in $L^2(M, \mu)$, which allows one to define the heat semigroup $e^{t\Delta_\mu}$ as one would on a classical Riemannian manifold. The heat semigroup has the integral kernel $H_\mu(x, y, t)$, which is called the heat kernel of (M, g, μ), and has completely analogous properties as the classical one. For every detail regarding the heat kernel on weighted manifolds, we refer to the survey [Gri06].

In this case, we see that our proof applies since Lemma 3.2.6 also holds (with the same proof) on geodesically complete weighted manifolds, and also Theorem 3.4.1 holds with the same proof, since our results from subsection 3.6.3 are valid for weighted manifolds too.

Moreover, our method works also for manifolds with boundary and finite volume. Indeed, if (M, g) is a complete manifold with (possibly empty) boundary and finite volume, and one defines $\mathcal{K}_s(x, y)$ by (1.2.2) with the heat kernel with Neumann boundary conditions on the right-hand side, then the same proof applies.

3.5 Asymptotics: infinite volume manifolds

3.5.1 Global asymptotics

Corollary 3.5.1. *Let (M, g) be stochastically complete and with $\mu(M) = +\infty$. Let $E \subset M$ be bounded and such that $P_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0, 1)$. Then*

$$\lim_{s\to 0^+} \frac{1}{2} P_s(E) = \mu(E)\,.$$

Proof. Since M is stochastically complete, by Proposition 1.2.4 we have $\theta_M \equiv 1$. Then the result follows taking $u = \chi_E$ in Theorem 3.3.4. □

One can note that stochastical completeness is not really needed in Corollary 3.5.1. Even when M is not stochastically complete, by Theorem 1.2.1, we know that θ_M is a (possibly non-constant) bounded harmonic function with values in $[0, 1]$. Then, by Theorem 3.3.4 with $u = \chi_E$ again

$$\lim_{s\to 0^+} \frac{1}{2} P_s(E) = \int_E \theta_M \; \mathrm{d}\mu$$

Consequently, if in particular $\theta_M \equiv \theta_\circ \in [0, 1]$ we have

$$\lim_{s\to 0^+} \frac{1}{2} P_s(E) = \theta_\circ \mu(E)\,, \tag{3.5.1}$$

for every E bounded with $P_{s_\circ}(E) < +\infty$. This feature led us to note the following example, which shows that, interestingly enough, Riemannian manifolds with $\theta_M \equiv \theta_\circ = 0$ exists.

Example 3.5.2. There exists a complete Riemannian manifold N where the asymptotics of the fractional s-perimeter as $s \to 0^+$ is zero for every set, that is: for every bounded E with $P_{s_\circ}(E) < +\infty$ for some $s_\circ \in (0,1)$ there holds

$$\lim_{s \to 0^+} P_s(E) = 0 \,.$$

By (3.5.1) above we see that it is enough to provide an example of a Riemannian manifold N with $\theta_N(p) \equiv \theta_\circ = 0$, meaning that the limit does not depend on the point p and is always zero. Moreover, by part (ii) of Remark 3.3.2 this is satisfied if N has the $L^\infty - \text{Liouville}$ property, is not stochastically complete and

$$\mathcal{N}(t,p) = \int_N H_N(x,p,t)\, \mathrm{d}\mu(x) \to 0 \,, \quad \text{as } t \to \infty \,.$$

A complete Riemannian manifold N with these properties actually exists, and we now sketch how it is constructed. We want N such that

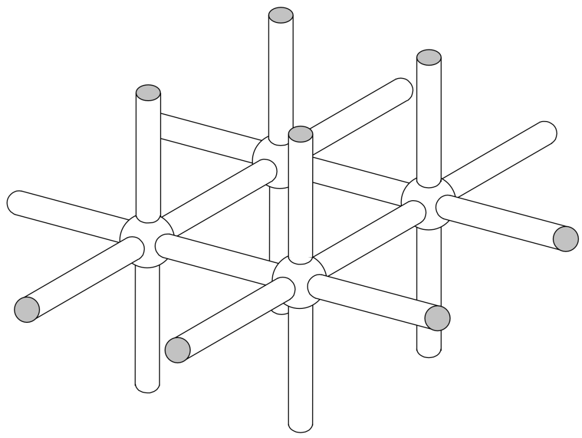
(i) N has the $L^\infty - \text{Liouville}$ property.

(ii) N is not stochastically complete.

(iii) For every $p \in N$ we have $\mathcal{N}(t,p) = \int_N H_N(x,p,t)\, \mathrm{d}\mu(x) \to 0$.

The construction of N that satisfies $(i), (ii)$ is taken from [Gri99, Section 13.5], which in turn builds on the first such example found by Pinchover in [Pin95]. Here, we note that it satisfies also (iii).

Figure 3.1: The two dimensional jungle-gym in $\mathbb{R}^3$. Picture taken from [Gri99].



Start from the two-dimensional jungle-gym JG^2 in $\mathbb{R}^3$ as in Figure 3.1. This is done by smoothly connecting the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ with necks. Let g be the standard metric on JG^2 induced by the embedding in $\mathbb{R}^3$. Fix $o \in JG^2$ and let $r := d(o,x)$. One can show that JG^2 has the $L^\infty - \text{Liouville}$ property. Moreover, there holds $\mu(B_R(o)) \leq CR^3$, and the Green function grows at most as $G(o,x) \leq C/r$ for large r. Let $\rho : JG^2 \to [0,+\infty)$ be a smooth positive function with $\rho = 1$ in $[0,1]$ and $\rho(r) \sim \frac{1}{r\log(r)}$ for large r, and consider the conformal metric $\widehat{g} := \rho^2(r) g$ on JG^2. We claim that $N := (JG^2, \widehat{g})$ has the desired properties. Since

$$\int_1^\infty \rho(r) dr = \infty \,,$$

then N is geodesically complete and hence complete. Moreover, as the Laplacian is conformally invariant in dimension two, JG^2 with its standard metric and N have the same harmonic functions, and thus N also has the L^∞ − Liouville property and satisfies (i). Denote by $\widehat{G}$ the Green's function of N. Then, by the choice of ρ, for R big

$$\int_{N\setminus B_R(o)} \widehat{G}(o,x)\, d\widehat{\mu}(x) = \int_{JG^2\setminus B_R(o)} G(o,x)\rho^2(r)\ \mathrm{d}\mu(x) < +\infty\,,$$

and by [Gri99, Corollary 6.7] this implies (ii). Consequently, note that also

$$\begin{aligned}\int_0^\infty \mathcal{N}(p,t)\ \mathrm{d}t &= \int_0^\infty \int_N H_N(x,p,t)\, d\widehat{\mu}(x)\ \mathrm{d}t = \int_N \left(\int_0^\infty H_N(x,p,t)\ \mathrm{d}t\right) d\widehat{\mu}(x)\\ &= \int_N \widehat{G}(o,x)\, d\widehat{\mu}(x) = \int_{N\setminus B_R(o)} \widehat{G}(o,x)\, d\widehat{\mu}(x) + \int_{B_R(o)} \widehat{G}(o,x)\, d\widehat{\mu}(x) < +\infty\,,\end{aligned}$$

and since the function $\mathcal{N}(p,\cdot)$ is also nonincreasing this implies that N also satisfies (iii).

3.5.2 Localized asymptotics and proof of Theorem 1.2.5

We now show (among other things) that (1.2.3) is well-posed as in $\mathbb{R}^n$ for manifolds with the L^∞ − Liouville property, in the sense that it does not even depend on the choice of p.

Lemma 3.5.3. *Let (M,g) be a complete Riemannian manifold with $\mu(M) = +\infty$ and $E \subset M$ be a set for which the limit (1.2.3) exists for some $p \in M$. If M has the L^∞ − Liouville property, then $\theta_E(p) \equiv \theta_E$ is constant, meaning that the limit in $\theta_E(q)$ exists for all $q \neq p$ and equals $\theta_E(p)$.*

Proof. We adopt the notation in the proof of Theorem 1.2.1. In particular, let $q \mapsto \Theta_{E,s}(q)$ be defined in (3.3.2). Arguing exactly as in the proof of Theorem 1.2.1, every subsequential limit (say, in $C^{2,\alpha}_{\mathrm{loc}}(M)$) of $\Theta_{E,s}$ as $s \to 0^+$ is a bounded harmonic function on M.

Since M has the L^∞ − Liouville property, every such subsequential limit is constant. Then, since the limit $\lim_{s\to 0^+} \Theta_{E,s}(p) = \theta_E(p)$ exists by hypothesis, all the subsequential limits must coincide with $\theta_E(p)$ everywhere. □

Let us note that the conclusion of Lemma 3.5.3 is not completely trivial in general and is particular of Riemannian manifolds that have the L^∞ − Liouville property. Indeed, we believe that on a general complete Riemannian manifold, it can happen that the limit in $\theta_E(\cdot)$ exists for some $p \in M$ but does not exist for some other $q \in M$ with $q \neq p$. See subsection 3.6.1 for a brief discussion on this feature.

Lemma 3.5.4. *In the hypothesis of Lemma 3.5.3, for every bounded $F \subset M$ and $R > 0$ with $F \subset B_{R/2}(p)$ there holds*

$$\mu(F)\theta_E = \lim_{s\to 0^+} \mathcal{J}_s(F, E\setminus B_R(p)) = \lim_{s\to 0^+} \int_F \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\ \mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Proof. Now since $F \subset B_{R/2}(p)$, we have that $B_{R/10}(y) \subset B_R(p) \subset B_{10R}(y)$ for every $y \in F$. Since the kernel $\mathcal{K}_s$ is nonnegative we get

$$\int_{E\setminus B_{10R}(y)} \mathcal{K}_s(x,y)\ \mathrm{d}\mu(x) \le \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\ \mathrm{d}\mu(x) \le \int_{E\setminus B_{R/10}(y)} \mathcal{K}_s(x,y)\ \mathrm{d}\mu(x)\,.$$

By the very definition of θ_E (1.2.3) and the fact that the limit does not depend on the radius whenever it exists (see part (i) of Theorem 1.2.1) both the left-hand side and right-hand side of the last inequality converge to $\theta_E(y) = \theta_E$, since θ_E is constant by Lemma 3.5.3, as $s \to 0^+$. Hence, integrating in $y \in F$ and letting $s \to 0^+$, by dominated convergence

$$\lim_{s\to 0^+} \int_F \int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) = \int_F \theta_E\, \mathrm{d}\mu(y) = \mu(F)\theta_E\,,$$

which is what we wanted to prove. □

Lemma 3.5.5. *Let (M, g) be complete with $\mu(M) = +\infty$, and let $A, B \subset M$ be two disjoint measurable sets with $\mu(A), \mu(B) < +\infty$ and with $\mathcal{J}_{s_\circ}(A, B) < +\infty$, for some $s_\circ \in (0, 1)$. Then*

$$\lim_{s\to 0^+} \mathcal{J}_s(A, B) = 0\,.$$

Proof. First, by Lemma 3.4.3 we have

$$\limsup_{s\to 0^+} \mathcal{J}_s(A,B) \leq \limsup_{s\to 0^+} \frac{s}{2} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y)\,.$$

Then

$$\begin{aligned} \frac{s}{2} \iint_{A\times B} \int_{1/s}^{\infty} H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) &= C s^{1+s/2} \int_A \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}}\, \mathrm{d}\mu(x) \\ &\leq C \int_A \left(s \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}} \right) \mathrm{d}\mu(x)\,. \end{aligned}$$

Since $\chi_B \in L^1(M)$, for every $x \in A$ (see Remark 3.2.9) there holds by dominated convergence

$$s \int_1^{\infty} e^{(\xi/s)\Delta}(\chi_B)(x) \frac{d\xi}{\xi^{1+s/2}} \to 0\,,$$

as $s \to 0^+$. From here, the result follows by dominated convergence using that $\mu(A) < +\infty$. □

The results above directly imply the following.

Corollary 3.5.6. *Let (M, g) be complete with $\mu(M) = +\infty$ and with the $L^\infty-$Liouville property, and let $\Omega \subset M$ be bounded. Then, for every $F \subset \Omega$ with $P_{s_\circ}(F, \Omega) < +\infty$, for some $s_\circ \in (0, 1)$, there holds*

$$\lim_{s\to 0^+} \mathcal{J}_s(F, E \cap \Omega^c) = \mu(F)\theta_E\,.$$

Proof. Let $p \in M$ and $R \gg 1$ be such that $\Omega \subset B_R(p)$, then

$$\begin{aligned} \mathcal{J}_s(F, E\cap\Omega^c) &= \mathcal{J}_s(F, E\cap\Omega^c \cap B_R(p)) + \mathcal{J}_s(F, E\cap\Omega^c\cap B_R^c(p)) \\ &= \mathcal{J}_s(F, E\cap\Omega^c \cap B_R(p)) + \mathcal{J}_s(F, E\cap B_R^c(p))\,. \end{aligned}$$

From here, since $\Omega^c \cap B_R(p)$ and F are disjoint and both with finite volume, the first term tends to zero as

$$\mathcal{J}_s(F, E\cap\Omega^c\cap B_R(p)) \leq \mathcal{J}_s(F, \Omega^c\cap B_R(p)) \to 0\,,$$

as $s \to 0^+$. Moreover, the second term tends to $\mu(F)\theta_E$ by Lemma 3.5.4. □

The proof of our main theorem in the infinite volume case is just a simple application of all the results we have derived above.

Proof of Theorem 1.2.5. Write

$$\frac{1}{2}P_s(E,\Omega) = \mathcal{J}_s(E\cap\Omega, E^c\cap\Omega) + \mathcal{J}_s(E\cap\Omega, E^c\cap\Omega^c) + \mathcal{J}_s(E\cap\Omega^c, E^c\cap\Omega)$$
$$= \frac{1}{2}P_s(E\cap\Omega) - \mathcal{J}_s(E\cap\Omega, E\cap\Omega^c) + \mathcal{J}_s(E^c\cap\Omega, E\cap\Omega^c)\,.$$

By Corollary 3.5.1 applied to the first term, and by Corollary 3.5.6 applied with $F = E\cap\Omega$ and $F = E^c\cap\Omega$ respectively on the second and third term, taking the limit as $s\to 0^+$ we get

$$\lim_{s\to 0^+}\frac{1}{2}P_s(E,\Omega) = \mu(E\cap\Omega) - \theta_E\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)$$
$$= (1-\theta_E)\mu(E\cap\Omega) + \theta_E\mu(E^c\cap\Omega)\,,$$

and this shows (i).

To prove (ii) and (iii) we follow closely the proof of in [DFPV13, Theorem 2.7], which deals with the analogous property in the case of the Euclidean space $\mathbb{R}^n$. We just sketch the argument since in the reference [DFPV13], the proof is carried on in full detail, and in our case, it is analogous. Let us denote

$$\Theta_{E,s} := \int_{E\setminus B_R(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x)\,, \tag{3.5.2}$$

and fix $R>0$ such that $\Omega\subset B_{R/2}(p)$. Note that

$$\int_{\Omega\setminus E}\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\;\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) - \int_{\Omega\cap E}\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\;\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$
$$= \frac{1}{2}P_s(E,\Omega) - \frac{1}{2}P_s(E\cap\Omega,\Omega) - \mathcal{J}_s(\Omega\setminus E, (E\setminus\Omega)\cap B_R(p)) + \mathcal{J}_s(\Omega\cap E, (E\setminus\Omega)\cap B_R(p))\,.$$

Now, arguing exactly as in the proof of Lemma 3.5.4 we have that for every $F\subset\Omega$ there holds

$$\lim_{s\to 0^+}\left|\mu(F)\Theta_{E,s} - \int_F\int_{E\setminus B_R(p)} \mathcal{K}_s(x,y)\;\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right| = 0\,. \tag{3.5.3}$$

Since $\Omega\setminus E$ and $(E\setminus\Omega)\cap B_R(p)$ are disjoint and both with finite volume (since they are bounded), by Lemma 3.5.5 we have

$$\lim_{s\to 0^+}\mathcal{J}_s(\Omega\setminus E, (E\setminus\Omega)\cap B_R(p)) = 0\,,$$

and similarly

$$\lim_{s\to 0^+}\mathcal{J}_s(\Omega\cap E, (E\setminus\Omega)\cap B_R(p)) = 0\,.$$

Hence, taking the limit as $s\to 0^+$ above using (3.5.3) for the left-hand side with $F=\Omega\setminus E$ and $F=\Omega\cap E$ respectively gives

$$\lim_{s\to 0^+}\Theta_{E,s}\big(\mu(\Omega\setminus E) - \mu(\Omega\cap E)\big) = \lim_{s\to 0^+}\frac{1}{2}\big(P_s(E,\Omega) - P_s(E\cap\Omega,\Omega)\big)\,.$$

Since $E\cap\Omega\subset\Omega$ is bounded, by Corollary 3.5.1 we have

$$\lim_{s\to 0^+}\frac{1}{2}P_s(E\cap\Omega,\Omega) = \lim_{s\to 0^+}\frac{1}{2}P_s(E\cap\Omega) = \mu(E\cap\Omega)\,,$$

thus

$$\lim_{s\to 0^+} \Theta_{E,s}\big(\mu(\Omega\setminus E)-\mu(\Omega\cap E)\big) = \left(\lim_{s\to 0^+}\frac{1}{2}P_s(E,\Omega)\right) - \mu(E\cap\Omega)\,.$$

From here, the conclusion of the theorem easily follows. Indeed, if $\mu(\Omega\setminus E)=\mu(\Omega\cap E)$ then the limit $\lim_{s\to 0^+}\frac{1}{2}P_s(E,\Omega)$ always exists and is equal to $\mu(E\cap\Omega)$. On the other hand, if the limit $\lim_{s\to 0^+}\frac{1}{2}P_s(E,\Omega)$ exists then from above the limit in θ_E also exists and there holds

$$\theta_E = \frac{\big(\lim_{s\to 0^+}\frac{1}{2}P_s(E,\Omega)\big)-\mu(E\cap\Omega)}{\mu(\Omega\setminus E)-\mu(E\cap\Omega)}\,,$$

and this concludes the proof. □

3.6 Extension to RCD spaces

In this section, we briefly explain how our results extend to the case of $\mathrm{RCD}(K,N)$ spaces, which are a generalization of Riemannian manifolds with an upper bound on the dimension N and Ricci curvature bounded from below by the real number K (and they include weighted manifolds). For the definition of RCD spaces we refer the reader to Chapter 4. We stress that we won't reprove every result of the smooth case, but only the ones presenting significant changes needed to perform the asymptotic analysis.

First of all, on any $\mathrm{RCD}(K,N)$ space with $K\in\mathbb{R}$ and $N\in\mathbb{N}\cup\{\infty\}$ it is possible to define a heat kernel and to do so we shall exploit the theory of gradient flows.

We call the *heat flow* $(e^{t\Delta})_{t>0}$ the gradient flow (in the sense of Komura-Brezis theory) of the Cheeger energy, which displays the following properties: for an L^2 function f the curve $t\in(0,\infty)\to e^{t\Delta}f\in L^2$ is locally absolutely continuous, it is such that $e^{t\Delta}f\in\mathrm{D}(\Delta)$, $\lim_{t\to 0}e^{t\Delta}f=f$ in L^2 and satisfies the *heat equation*

$$\frac{de^{t\Delta}}{\mathrm{d}t} = \Delta e^{t\Delta}f \quad \forall t>0.$$

We will now collect some other properties of the heat flow holding on infinitesimally Hilbertian metric measure spaces which we will exploit (see [GP20b] for a reference):

Proposition 3.6.1. *Let (X,d,μ) be an infinitesimally Hilbertian metric measure space, then we have*

(i) (Weak maximum principle)*: Given any $f\in L^2(\mu)$ such that $f\le C$ μ-almost everywhere we have*

$$e^{t\Delta}f\le C \quad \mu-\text{a.e.}$$

(ii) ($e^{t\Delta}$ is self-adjoint): *For all $f,g\in L^2(\mu)$ we have*

$$\int_X e^{t\Delta}fg\,\mathrm{d}\mu = \int_X e^{t\Delta}gf\,\mathrm{d}\mu \quad \forall t>0.$$

(iii) (Δ and $e^{t\Delta}$ commute): *For all $f\in\mathrm{D}(\Delta)$ we have*

$$\Delta e^{t\Delta}f = e^{t\Delta}\Delta f \quad \mu-\text{a.e.},\ \forall t>0.$$

Moreover if (X,d,μ) is an $\mathrm{RCD}(K,\infty)$ space we have the following additional properties:

(iv) (Bakry-Émery estimate): *For all* $f \in W^{1,2}(X)$ *and* $t > 0$ *we have*

$$|\nabla e^{t\Delta} f|^2 \leq e^{-2Kt} e^{t\Delta}\big(|\nabla f|^2\big) \quad \mu - \text{a.e.} \tag{3.6.1}$$

(v) ($L^\infty - \mathrm{Lip}$ regularization): *For all* $f \in L^\infty(\mu)$ *and* $t > 0$ *we have*

$$\|\nabla(e^{t\Delta} f)\|_{L^\infty(\mu)} \leq \frac{e^{-2Kt}}{\sqrt{t}} \|f\|_{L^\infty(\mu)}. \tag{3.6.2}$$

It is then possible to define the heat flow for all probability measures with finite second moment as the EVI_K (again, we assume the reader to be familiar with the terminology) gradient flow of the entropy functional. More precisely for every $\mu \in \mathcal{P}_2(X)$, $e^{t\Delta}\mu$ (with a little abuse of notation here) is the unique measure such that

$$\int_X \varphi d e^{t\Delta}\mu = \int_X e^{t\Delta}\varphi \, \mathrm{d}\mu \quad \forall \varphi \in \mathrm{Lip}_{bs}(X),$$

where $\mathrm{Lip}_{bs}(X)$ is the set of Lipschitz functions with bounded support and $e^{t\Delta}\varphi$ is the Lipschitz continuous representative of its equivalence class (which is well-posed thanks to the $L^\infty - \mathrm{Lip}$ regularization property).

On $\mathrm{RCD}(K, \infty)$ it is possible to define the heat kernel $H_X(x, \cdot, t) := \frac{d e^{t\Delta}\delta_x}{\mathrm{d}\mu}$ and we have the following (see [JLZ16] for a reference):

Proposition 3.6.2. *Let* (X, d, μ) *be an* $\mathrm{RCD}(K, N)$ *space with* $N \in \mathbb{N}$*, then for all* $\varepsilon > 0$*, for some* C_1, C_2, C_3, C_4 *nonnegative constants (possibly depending on* ε *and* N*) we have*

$$\frac{1}{C_1 \mu(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x,y)}{(4-\varepsilon)t} - C_2 t\right) \leq H_X(x,y,t) \leq \frac{C_1}{\mu(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t} + C_2 t\right) \tag{3.6.3}$$

for all $x, y \in X$*,* $t > 0$ *and*

$$|\nabla H_X(x, \cdot, t)|(y) \leq \frac{C_3}{\sqrt{t}\mu(B_{\sqrt{t}}(y))} \exp\left(\frac{-d^2(x,y)}{(4+\varepsilon)t} - C_4 t\right) \tag{3.6.4}$$

$\mu \times \mu$*-a.e.* $(x, y) \in X \times X$*, for all* $t > 0$.

Moreover, if $K = 0$ *then estimate* (3.6.3) *holds with* $C_2 = C_4 = 0$.

On any $\mathrm{RCD}(K, \infty)$ space we have

$$\int_X H_X(x, y, t) \, \mathrm{d}\mu(x) = 1$$

for all $y \in X$, $t > 0$. That is, X is stochastically complete.

In the setting of $\mathrm{RCD}(K, N)$ (actually infinitesimal hilbertianity is not required) we also have Bishop-Gromov's comparison theorem, holding both for the perimeter measure and the volume measure (see [Stu06]). Finally, it is possible to prove that the following version of the Harnack inequality holds (see [Li16] for the proof)

Proposition 3.6.3 (Harnack inequality)**.** *Let* (X, d, μ) *be an* $\mathrm{RCD}(K, \infty)$ *space,* $p \in (1, \infty)$ *and* $f \in L^1(\mu) + L^\infty(\mu)$*, then*

$$|(e^{t\Delta} f)(x)|^p \leq (e^{t\Delta}|f|^p)(y) \exp\left(\frac{pKd^2(x,y)}{2(p-1)(e^{2Kt}-1)}\right)$$

for all $x, y \in X \times X$ *and* $t > 0$.

From the previous Harnack inequality, it is possible to prove the following Gaussian bound (see [Tam19, Theorem 4.1]) for $\mathrm{RCD}(K,\infty)$ spaces (compare with (3.6.3) above for $\mathrm{RCD}(K,N)$ spaces).

Proposition 3.6.4. *Let (X,d,μ) be an $\mathrm{RCD}(K,\infty)$ space, then there exists $C_K>0$ and for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that*

$$H_X(x,y,t)\le \frac{1}{\sqrt{\mu(B_{\sqrt{t}}(x))}\sqrt{\mu(B_{\sqrt{t}}(y))}}\exp\Big(C_\varepsilon(1+C_K)t-\frac{d^2(x,y)}{(4+\varepsilon)t}\Big). \tag{3.6.5}$$

If $K\ge 0$ one can take $C_K=0$.

The second ingredient we need is a generalization to $\mathrm{RCD}(K,\infty)$ spaces of the L^2- Liouville property of Yau (our Theorem 3.2.3).

Proposition 3.6.5. *Let (X,d,μ) be an $\mathrm{RCD}(K,\infty)$ space. Then, any $L^2(\mu)$ harmonic function is constant.*

Proof. Denote $w(t,x):=e^{t\Delta}u(x)$. Assume $u\in L^2(\mu)$ is harmonic, then by applying the heat flow to $\Delta u=0$ and using item (iii) of Proposition 3.6.1 we have

$$\Delta w=0.$$

By gradient flow theory we have

$$\int_X|\nabla w|^2\,\mathrm{d}\mu\le\frac{1}{2t}\int_X|u|^2\,\mathrm{d}\mu,$$

whence

$$0=-\int_X w\Delta w\ \mathrm{d}\mu=\int_X|\nabla w|^2\,\mathrm{d}\mu.$$

This means $|\nabla w|=0$ μ-a.e. and by the Sobolev to Lipschitz property, this implies that w is constant. Therefore there exists $C=C(t)$ such that $w(t,\cdot)=C(t)$.

Now if $\mu(X)<+\infty$ we can infer (as $u\in L^2(\mu)$ implies $u\in L^1(\mu)$)

$$\int_X w\ \mathrm{d}\mu=\int_X u\ \mathrm{d}\mu=\mu(X)C(t),$$

hence C does not actually depend on t and by taking the limit as $t\to 0^+$ we infer that u is constant.

If $\mu(X)=+\infty$, then for every t, we have $w=0$ because the only constant in $L^2(\mu)$ is zero, and we conclude. □

Remark 3.6.6. The previous proposition actually does not require a curvature condition: working in a space in which having zero weak upper gradient implies being constant suffices (see [DLDPMT23, Proposition 3.3]).

We then have the following result, which is a non-smooth analogous of Proposition 3.2.6.

Proposition 3.6.7. *Let (X,d,μ) be an $\mathrm{RCD}(K,\infty)$ space, then we have the following dicotomy:*

(i) *If $\mu(X)<+\infty$ then*

$$H_X(t,x,y)\to\frac{1}{\mu(X)}\quad \text{as } t\to\infty\ \ \forall x,y\in X.$$

(ii) *If* $\mu(X) = +\infty$ *we have*

$$H_X(\cdot,\cdot,t) \to 0 \quad \text{as } t \to \infty \tag{3.6.6}$$

locally uniformly and $H_X(p,\cdot,t) \to 0$ *uniformly as* $t \to \infty$ *for every* $p \in M$.

Proof. The proof follows along the same lines of Proposition 3.2.6. If $\mu(X) < +\infty$ let $f = H_X(p,\cdot,1) - 1/\mu(X)$, otherwise let $f = H_X(p,\cdot,1)$, then $\max\{\|e^{t\Delta} f\|_{L^1}, \|e^{t\Delta} f\|_{L^\infty}\} \le C$ due to the properties of the heat flow. Moreover by the semigroup property of it is easy to see that weak convergence in $L^2(\mu)$ of $e^{t\Delta} f$ is equivalent to strong convergence and we again have the inequality

$$|(e^{t\Delta} f, g)| \le |(e^{t\Delta} f, f)| |(e^{t\Delta} g.g)| \le \|g\|^2_{L^2(\mu)} |(e^{t\Delta} f, f)|$$

for all $t \in (0,\infty)$ and for all $f, g \in L^2(\mu)$. Using the spectral measure representation and Proposition 3.6.5, we infer the desired L^2 convergence. This convergence can be upgraded to be locally uniform by the Harnack inequality (Proposition 3.6.3) with $p = 2$ and by the fact that $|f|^2 \le \|f\|_{L^\infty} |f|$, together with the maximum principle to get

$$|e^{t\Delta} f(x)|^2 \le \|f\|_{L^\infty} e^{t\Delta}(|f|)(y) \exp\left(\frac{2KR^2}{2(e^{2Kt} - 1)}\right)$$

for every $y \in B_R(x)$. Integrating over the latter set in $\mathrm{d}\mu(y)$ and taking the supremum allows to conclude. The global uniform convergence follows as in the smooth case. □

Remark 3.6.8. As in the smooth case if $\mu(X) = +\infty$ we have that for every $f \in L^1(\mu)$

$$\lim_{t\to\infty} e^{t\Delta} f(x) = 0$$

for every $x \in X$.

We refer to [BLS20] for an introduction to H^s spaces on very general ambient space, like RCD spaces and more. We have the analog of Theorem 3.4.1.

Theorem 3.6.9. *Let* (X, d, μ) *be an* $\mathrm{RCD}(K,\infty)$ *space with* $K \in \mathbb{R}$ *and* $\mu(X) < +\infty$. *Let* $u \in H^{s_\circ/2}(X)$ *for some* $s_\circ \in (0,1)$ *with bounded support. Then*

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(X)} = \|u\|^2_{L^2(X)} - \frac{1}{\mu(M)}\left(\int_X u \,\mathrm{d}\mu\right)^2.$$

Proof. The proof is exactly the same as in the smooth case exploiting the $L^2 - \mathrm{Liouville}$ property of Proposition 3.6.5. □

To prove the convergence result for the case of infinite volume we need a convergence result for the solution of the heat equation to the initial datum. We, therefore, recall the following (upper) Large Deviation Principle on proper $\mathrm{RCD}(K,\infty)$ spaces (see [GTT22, Theorem 5.3])

Theorem 3.6.10. *Let* (X, d, μ) *be a proper* $\mathrm{RCD}(K,\infty)$ *space, then for every* $x \in X$ *and closed set* $C \subseteq X$ *we have, setting* $\mu_t[x] = H_X(\cdot, x, t)\mu$,

$$\limsup_{t\to 0} t \log(\mu_t[x](C)) \le -\inf_{y\in C} \frac{d^2(x,y)}{4}. \tag{3.6.7}$$

Remark 3.6.11. In (3.6.7) we can choose $C = X \setminus B_r(p)$ and obtain the following estimate for small times (depending on $r > 0$ and $\varepsilon > 0$)

$$|e^{t\Delta}(\chi_{X\setminus B_r(p)})(p)| \le \exp\left(-\frac{r^2-\varepsilon}{4t}\right) \tag{3.6.8}$$

We are finally ready to prove the following proposition (analog of Proposition 1.2.4)

Proposition 3.6.12. *Let* (X, d, μ) *be a proper* $\mathrm{RCD}(K,\infty)$ *space with* $\mu(X) = +\infty$. *Then for every* $p \in X$

$$\theta_M(p) = \lim_{s\to 0} \int_{X\setminus B_1(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x) = 1.$$

Proof. As for the smooth case, we first show that

$$\lim_{s\to 0^+} \frac{s}{2} \int_{X\setminus B_1(p)} \int_0^1 H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x) = 0.$$

Indeed there exists $\delta > 0$ such that for all $t \le \delta$ (3.6.8) holds, so that the previous integral can be estimated with the following

$$\frac{s}{2}\int_0^\delta e^{-r^2/5t}\frac{\mathrm{d}t}{t^{1+s/2}} + \frac{s}{2}\int_{X\setminus B_1(p)}\int_\delta^1 H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}.$$

The first term clearly goes to zero as $s \to 0^+$ and to handle the second we use Fubini to deduce that (here stochastical completeness is not necessary but $\mathrm{RCD}(K,\infty)$ spaces enjoy this property so we write the equality sign)

$$\frac{s}{2}\int_{X\setminus B_1(p)}\int_\delta^1 H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}} = \frac{s}{2}\int_\delta^1 \frac{\mathrm{d}t}{t^{1+s/2}} - \frac{s}{2}\int_{B_1(p)}\int_\delta^1 H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x).$$

Again the first term trivially goes to zero while for the second we apply (3.6.5) and exploit properness of the space to infer that $H_X(\cdot,\cdot,\cdot)$ is equibounded in $B_1(p)\times[\delta,1]$ so that

$$\limsup_{s\to 0}\left|\frac{s}{2}\int_{B_1(p)}\int_\delta^1 H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x)\right| \le \limsup_{s\to 0^+} C\frac{s}{2}\int_\delta^1 \frac{\mathrm{d}t}{t^{1+s/2}} = 0\,.$$

We now claim that

$$\lim_{s\to 0^+}\frac{s}{2}\int_{B_1(p)}\int_1^\infty H_X(x,p,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x) = 0.$$

Indeed, thanks to the local uniform convergence proved in (3.6.6) and reasoning as in the previous step the latter result easily follows.

Finally, we can perform the same steps and write

$$\theta_M(p) = \lim_{s\to 0}\frac{s}{2}\int_X\int_1^\infty H_X(x,y,t)\frac{\mathrm{d}t}{t^{1+s/2}}\,\mathrm{d}\mu(x),$$

which equals 1 by using stochastical completeness. □

In the following proposition, we study the behavior of the singular kernel $\mathcal{K}_s(x,y)$.

Proposition 3.6.13. *Let (X, d, μ) be an $\mathrm{RCD}(K, N)$ space with $\mu(X) = +\infty$ and essential dimension equal to n. Then, for every $x \in X$ which is a regular point we have*

$$\frac{Cs}{r^{n+s}} \leq \mathcal{K}_s(x, y) \leq \frac{Cs}{r^{n+s}} + o_s(1) + \sup_{t \geq 1/s} H_M(x, y, t), \tag{3.6.9}$$

for every $y \in X$, where $r = d(x, y)$. In particular $\mathcal{K}_s(x, \cdot) \to 0$ as $s \to 0^+$ locally uniformly away from x.

Proof. Let us define

$$\mathcal{K}_s(x, y) = \frac{s}{2} \int_0^1 H_X(x, y, t) \frac{\mathrm{d}t}{t^{1+s/2}} + \frac{s}{2} \int_1^{1/s} H_X(x, y, t) \frac{\mathrm{d}t}{t^{1+s/2}} + \frac{s}{2} \int_{1/s}^{\infty} H_X(x, y, t) \frac{\mathrm{d}t}{t^{1+s/2}}$$
$$=: I_1 + I_2 + I_3.$$

By the Gaussian estimates (3.6.3) and using the fact that x is a regular point we have

$$I_1 \leq Cs \int_0^1 e^{-r^2/5t} \frac{\mathrm{d}t}{t^{1+s/2+n/2}} \leq \frac{Cs}{r^{n+s}}.$$

Moreover, since $\mu(X) = +\infty$ by (3.6.6) the heat kernel converges locally uniformly to zero, and we also get

$$I_2 \leq Cs \int_1^{1/s} \frac{\mathrm{d}t}{t^{1+s/2}} = C(1 - s^{s/2}) = o_s(1),$$

for some constant C which is bounded in a neighborhood of x. Finally, we have

$$I_3 \leq \frac{s}{2} \sup_{t \geq 1/s} H_M(x, y, t) \int_{1/s}^{\infty} \frac{\mathrm{d}t}{t^{1+s/2}},$$

thus proving the upper bound in (3.6.13). For the lower bound it is enough to neglect I_2 and I_3 and apply the Gaussian estimate from below to I_1.

Finally, the local uniform convergence $\mathcal{K}_s(x, \cdot) \to 0$ is apparent due to the local uniform convergence (3.6.6) of the heat kernel to zero and the other quantities involved. □

With the next proposition, we show that the heat density of a set, whenever it exists, is independent of the radius and also on the point if the $L^\infty - \mathrm{Liouville}$ property holds, analogously to the case of manifolds.

Proposition 3.6.14. *Let (X, d, μ) be an $\mathrm{RCD}(K, \infty)$ space with $\mu(X) = +\infty$, let $E \subset M$ be measurable and set*

$$\Theta_{E,s}(p, r) := \int_{X \setminus B_r(p)} \mathcal{K}_s(p, x) \, \mathrm{d}\mu(x).$$

Then for all $0 < r \leq R$ one has

$$\limsup_{s \to 0^+} \left| \Theta_{E,s}(p, R) - \Theta_{E,s}(p, r) \right| = 0.$$

meaning that if $\lim_{s \to 0^+} \Theta_{E,s}(p, r) = \theta_E(p)$ exists for some $p \in M$, then it does not depend on r. Moreover, if the $L^\infty - \mathrm{Liouville}$ property holds on X and $\theta_E(p)$ exists for all $p \in X$, then $\theta_E \equiv \theta_E(p)$ is constant.

Proof. We first show the independence on the radius; therefore we fix any two $0 < r < R$, and we show that

$$\limsup_{s\to 0^+} \frac{s}{2} \int_{\overline{B_R(p)}\setminus B_r(p)} \int_0^\infty H_X(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) = 0.$$

We split the integral over time in three pieces: one from 0 to ε, one from ε to T, and the last one from T to ∞. The first piece goes to zero since $\overline{B_R(p)} \setminus B_r(p)$ is a closed set and we can apply (3.6.8), the second piece goes to zero for every $T \gg 1$ thanks to the properness of the space, the Gaussian upper bound (3.6.5) and easy calculations, while the last piece is such that, for all $T \geq T_0(\varepsilon)$

$$\limsup_{s\to 0^+} \frac{s}{2} \int_T^\infty H_X(x,p,t) \frac{\mathrm{d}t}{t^{1+s/2}} \,\mathrm{d}\mu(x) \leq \varepsilon.$$

Since this holds for every ε we get the convergence to zero.

For what concerns the independence on the point, we first take r big that $q \in B_{r/10}(p)$ and wlog, we assume E to be closed. We have

$$\begin{aligned}\limsup_{s\to 0^+} &\left| \int_{E\setminus B_r(p)} \mathcal{K}_s(x,p)\,\mathrm{d}\mu(x) - \int_{E\setminus B_{2r}(q)} \mathcal{K}_s(x,q)\,\mathrm{d}\mu(x) \right| \\ &\leq \limsup_{s\to 0^+} \left| \int_{E\setminus B_r(p)} \mathcal{K}_s(x,q)\,\mathrm{d}\mu(x) - \int_{E\setminus B_{2r}(q)} \mathcal{K}_s(x,q)\,\mathrm{d}\mu(x) \right| \\ &+ \limsup_{s\to 0^+} \left| \int_{E\setminus B_r(p)} \mathcal{K}_s(x,p) - \mathcal{K}_s(x,q)\,\mathrm{d}\mu(x) \right| =: I_1 + I_2.\end{aligned}$$

The first integral is zero since

$$I_1 \leq \limsup_{s\to 0^+} \int_{B_{2r}(q)\setminus B_r(p)} \mathcal{K}_s(x,q)\,\mathrm{d}\mu(x) \leq \theta_{B_{2r}(q)}(q) = 0\,,$$

where we have used the independence on the radius.

While for I_2 we shall exploit the $L^\infty - \mathrm{Liouville}$ property of X. We can, as usual, expand the singular kernel and split the integral in time into three pieces in time, one going from 0 to 1, another from 1 to $T \gg 1$, and lastly, from T to ∞. The first two are handled thanks to the exponential convergence (3.6.8) and the boundedness of the heat kernel, while for the last one, we have

$$\limsup_{s\to 0^+} \left| \int_T^\infty e^{t\Delta}(\chi_{E\setminus B_r(p)})(p) - e^{t\Delta}(\chi_{E\setminus B_r(p)})(q) \frac{\mathrm{d}t}{t^{1+s/2}} \right| = 0\,,$$

thanks to the $L^\infty - \mathrm{Liouville}$ property.

Indeed $e^{t\Delta}(\chi_{E\setminus B_r(p)})$ converges up to subsequences to a constant harmonic function; hence its (of the limit function) value at the points p and q is the same so that, being this true for any subsequence, $e^{t\Delta}(\chi_{E\setminus B_r(p)})(p) - e^{t\Delta}(\chi_{E\setminus B_r(p)})(q) \to 0$ as $t \to \infty$. □

Remark 3.6.15. In the previous proposition, we only care about spaces satisfying the $L^\infty - \mathrm{Liouville}$ property. However, with a little work, it is possible to show that the function $p \mapsto \theta_E(p)$, whenever it exists, is a bounded harmonic function in a suitable weak sense.

Finally, we have the analog of Theorem 3.3.4.

Theorem 3.6.16. *Let (X, d, μ) be a proper $\mathrm{RCD}(K, N)$ space with $\mu(X) = +\infty$, $N < +\infty$, essential dimension equal to n and let $s_\circ \in (0, 1)$. Then for every $u \in H^{s_\circ/2}(X) \cap L^\infty(X)$ with bounded support there holds*

$$\lim_{s\to 0^+} \frac{1}{2}[u]^2_{H^{s/2}(X)} = \|u\|^2_{L^2(X)}.$$

Proof. The proof is similar to the smooth case; we just need to handle the computations more carefully. We advise the reader to first see the proof in the smooth case of Theorem 3.3.4.

By Proposition 3.6.21 (which also holds for RCD spaces, see Remark 3.6.22) for μ-a.e. $x \in X$ the integral in $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is absolutely convergent. Fix $x \in X$ in this full-measure set and $R > 0$ such that $\mathrm{supp}(u) \subseteq B_R(x)$. Now we prove that, as $s \to 0^+$, $(-\Delta)^{s/2}_{\mathrm{Si}} u \to u$ μ-a.e. with the same strategy of the smooth case. Take also $x \in X$ to be a regular point, we have

$$(-\Delta)^{s/2}_{\mathrm{Si}} u(x) = \int_{B_R(x)} (u(x) - u(y))\mathcal{K}_s(x, y)\,\mathrm{d}\mu(y) + u(x) \int_{X\setminus B_R(x)} \mathcal{K}_s(x, y)\,\mathrm{d}\mu(y)$$

and we are left to prove that the first term goes to zero as $s \to 0^+$, as the second one in the limit is precisely $u(x)$. Now fix $\rho \ll 1$ and let us split the first integral as follows

$$\begin{aligned}\left|\int_{B_R(x)} (u(x) - u(y))\mathcal{K}_s(x, y)\ \mathrm{d}\mu(y)\right| &= \int_{B_\rho(x)} |u(x) - u(y)|\mathcal{K}_s(x, y)\ \mathrm{d}\mu(y)\\ &\quad + \int_{B_R(x)\setminus B_\rho(x)} |u(x) - u(y)|\mathcal{K}_s(x, y)\ \mathrm{d}\mu(y).\end{aligned}$$

For the first integral, we can apply Proposition 3.6.13 to obtain

$$\int_{B_\rho(x)} |u(x) - u(y)|\mathcal{K}_s(x, y)\ \mathrm{d}\mu(y) \le Cs \int_{B_\rho(x)} \frac{|u(x) - u(y)|}{d(x, y)^{n+s}}\,\mathrm{d}\mu(y) + o_s(1). \tag{3.6.10}$$

Applying Hölder inequality as in the smooth case (take s small so that $2s < s_0$) we now get

$$\int_{B_\rho(x)} \frac{|u(x) - u(y)|}{d(x, y)^{n+s}}\,\mathrm{d}\mu(y) \le \left(\int_{B_\rho(x)} \frac{(u(x) - u(y))^2}{d(x, y)^{n+s_\circ}}\,\mathrm{d}\mu(y)\right)^{1/2} \left(\int_{B_\rho(x)} \frac{1}{d(x, y)^{n+2s-s_\circ}}\,\mathrm{d}\mu(y)\right)^{1/2}$$

and conclude in the same way that taking the limit as $s \to 0^+$ in (3.6.10) gives zero. For the second term, we just use the fact that $\mathcal{K}_s(x, \cdot)$ goes to zero locally uniformly away from x together with dominated convergence. Therefore we have proved that $(-\Delta)^{s/2}_{\mathrm{Si}} u \to u$ μ-a.e. as $s \to 0^+$. To establish the seminorms' convergence, we exploit Corollary 3.6.25, which also holds in this non-smooth setting with the same proof. To conclude we just need to prove that $(-\Delta)^{s/2}_{\mathrm{Si}} u \rightharpoonup u$ weakly in $L^2(\mu)$: this is however apparent because of the equiboundedness of $\|(-\Delta)^{s/2}_{\mathrm{Si}} u\|_{L^2(\mu)}$ given by the estimate (3.6.12). □

Thanks to the previous results we would be in the position of stating and proving (which we won't do since the proofs are exactly the same as in the smooth case) the theorems regarding the asymptotics of the fractional perimeter Theorem 1.2.7 and Theorem 1.2.5, also in this non-smooth setting.

3.6.1 On the existence/nonexistence of $\theta_E(\cdot)$ at different points.

Let (M,g) be a complete Riemannian manifold with infinite volume and $E \subset M$. As we have proved in Lemma 3.5.3 if M has the $L^\infty -$ Liouville property and $\theta_E(p)$ exists for some $p \in M$ then it exists for all $p \in M$ and the two values coincide. Let us stress that, on manifolds with $L^\infty -$ Liouville property, the limit does not need to exist, but if it does not exist at some point, then it does not exist everywhere. For example, even on $\mathbb{R}^n$ in [DFPV13, Example 2.8], the authors exhibit a set for which the limit $\theta_E(x)$ does not exist at every point $x \in \mathbb{R}^n$.

On the other hand, on a general M without the $L^\infty -$ Liouville property, we believe that $\theta_E(\cdot)$ could exist for some $p \in M$ and fail to exist for some $q \neq p$. Let $\Theta_{E,s}(\cdot)$ defined as in (3.3.2) so that $\theta_E(p) = \lim_{s\to 0^+} \Theta_{E,s}$, if the limit exists. It can be proved with the Li-Yau Harnack inequality [Li12, Corollary 12.3] that if the limit in $\theta_E(p)$ does not exist and

$$\limsup_{s\to 0^+} \Theta_{E,s}(p) - \liminf_{s\to 0^+} \Theta_{E,s}(p) = \delta > 0\,,$$

then the limit still does not exist for every $q \in B_{C\delta}(p)$, where $C > 0$ is a constant that depends on M. But in this estimate the lower bound for $\limsup_{s\to 0^+} \Theta_{E,s}(q) - \liminf_{s\to 0^+} \Theta_{E,s}(q)$ tends to 0 as q approaches $\partial B_{C\delta}(p)$. Without further information in M, we do not see any reason why the limit should not exist at some point outside $B_{C\delta}(p)$.

3.6.2 Heat kernel estimates and $H^s(M)$ spaces.

Here (M,g) denotes a complete, connected Riemannian manifold. First, we present a simple interpolation inequality for $H^{s/2}(M)$ spaces.

This inequality is known in the case of $M = \mathbb{R}^n$ or $M = \Omega \subset \mathbb{R}^n$ for fractional Sobolev spaces $W^{s,p}$, also when $p \neq 2$. Here, we carry on a structural proof using a few properties of the heat kernel, which gives the interpolation inequality on general ambient spaces.

Lemma 3.6.17. *Let $u \in H^\sigma(M)$ for some $\sigma \in (0,1)$, and let $0 < s < \sigma < 1$. Then $u \in H^s(M)$ and the following inequality holds*

$$[u]_{H^s(M)} \le C\|u\|_{L^2(M)}^{1-s/\sigma}[u]_{H^\sigma(M)}^{s/\sigma}.$$

for some absolute constant $C > 0$.

Proof. We have

$$\begin{aligned} |\Gamma(-s)|[u]^2_{H^s(M)} &= \iint_{M\times M} (u(x)-u(y))^2 \int_0^\infty H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\le \iint_{M\times M} (u(x)-u(y))^2 \int_0^\xi H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &+ \iint_{M\times M} (u(x)-u(y))^2 \int_\xi^\infty H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \end{aligned}$$

where $\xi \in (0,\infty)$ will be chosen at the end. Note that for all $t \in (0,\xi)$ we have $(\xi/t)^{1+s} \le (\xi/t)^{1+\sigma}$ so that we can estimate from above the first integral of the previous inequality with

$$\xi^{\sigma-s}\iint_{M\times M} (u(x)-u(y))^2 \int_0^\xi H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+\sigma}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \le \xi^{\sigma-s}|\Gamma(-\sigma)|[u]^2_{H^\sigma(M)}.$$

The symmetry of the heat kernel and the fact that $\mathcal{M}(t,y) \le 1$, for all $y \in M$, together imply that the second integral can be bounded by

$$\iint_{M\times M} (u(x)-u(y))^2 \int_\xi^\infty H_M(x,y,t)\frac{\mathrm{d}t}{t^{1+s}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \le \frac{4}{s\xi^s}\|u\|^2_{L^2(M)}.$$

These two inequalities lead to

$$|\Gamma(-s)|[u]^2_{H^s(M)} \le \xi^{\sigma-s}|\Gamma(-\sigma)|[u]^2_{H^\sigma(M)} + \frac{4}{s\xi^s}\|u\|^2_{L^2(M)}.$$

Optimizing the right-hand side in ξ gives that the optimal value is

$$\xi = \left(\frac{4\|u\|^2_{L^2(M)}}{(\sigma-s)|\Gamma(-\sigma)|[u]^2_{H^\sigma(M)}}\right)^{1/\sigma}.$$

Putting everything together gives

$$|\Gamma(-s)|[u]^2_{H^s(M)} \le \frac{C}{s}\|u\|^{2(1-s/\sigma)}_{L^2(M)}[u]^{2s/\sigma}_{H^\sigma(M)},$$

and this implies

$$[u]_{H^s(M)} \le C\|u\|^{1-s/\sigma}_{L^2(M)}[u]^{s/\sigma}_{H^\sigma(M)},$$

as desired. □

Lemma 3.6.18 ([CFSS23]). *Let (M^n,g) be a complete n-dimensional Riemannian manifold and let $B_R(p)\subset M$. Then*

$$e^{t\Delta}(\chi_{M\setminus B_R(p)})(p) = \int_{M\setminus B_R(p)} H_M(x,p,t)\,\mathrm{d}\mu(x) \le Ce^{-c/t},$$

for some $C,c>0$ depending on R and the geometry of M in $B_R(p)$.

Proof. This is essentially [CFSS23, Lemma 2.9]. Indeed, in [CFSS23, Lemma 2.9] the authors prove that if (M,g) is a complete Riemannian manifold and $B_r(p)\subset M$ is a ball diffeomorphic to $\mathcal{B}_r(0)\subset T_pM$ with metric coefficients g_{ij} (say, in normal coordinates) uniformly close to δ_{ij}, then

$$\int_{M\setminus B_r(p)} H_M(x,p,t)\,\mathrm{d}\mu(x) \le Ce^{-cr^2/t},$$

for some $C,c>0$ dimensional. Then, taking $r \ll 1$ very small and writing

$$\int_{M\setminus B_R(p)} H_M(x,p,t)\,\mathrm{d}\mu(x) \le \int_{M\setminus B_r(p)} H_M(x,p,t)\,\mathrm{d}\mu(x)$$

allows to bound the desired integral. □

Now we present the proof of Lemma 3.3.3, that we needed to prove the asymptotics of the full $H^{s/2}(M)$ seminorm of Theorem 3.3.4.

Proof of Lemma 3.3.3. Let $\varphi^{-1} : B_1(p) \to \mathbb{R}^n$ be the inverse of the exponential map at p. Take $\eta \in C_c^\infty(\mathcal{B}_{4/5}(0))$ with $\chi_{\mathcal{B}_{2/5}(0)} \le \eta \le \chi_{\mathcal{B}_{4/5}(0)}$ and let $g'_{ij} := g_{ij}\eta + (1-\eta)\delta_{ij}$. This is a metric on $\mathbb{R}^n$ with $g'_{ij} = g_{ij}$ in $\mathcal{B}_{2/5}(0)$. Denote by $\mathcal{K}_s, \mathcal{K}'_s$ the singular kernels of (M,g) and $M' := (\mathbb{R}^n, g')$ respectively. Let $\Lambda := \sup_{x\in B_{1/5}(p)} H_M(x,x,1)$ and $\Lambda' := \sup_{x\in \mathcal{B}_{1/5}(0)} H_{M'}(x,x,1)$. Then, by [CFSS23, Lemma 2.17] applied to the Riemannian manifolds (M,g) and $(\mathbb{R}^n, g')$ we have, for $x,y \in \mathcal{B}_{1/5}(0)$

$$\begin{aligned}
\left|\mathcal{K}_s(\varphi(x),\varphi(y)) - \mathcal{K}'_s(x,y)\right| &\le \frac{s/2}{\Gamma(1-s/2)} \int_0^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \\
&\le Cs(2-s)\int_0^1 \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \\
&+ Cs(2-s)\int_1^{1/s} \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \\
&+ Cs(2-s)\int_{1/s}^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \\
&:= Cs(2-s)\left[I_1 + I_2 + I_3\right].
\end{aligned}$$

By [CFSS23, Lemma 2.17] there holds

$$I_1 = \int_0^1 \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \le C\int_0^1 e^{-c/t}\frac{\mathrm{d}t}{t^{1+s/2}} \le C,$$

for some dimensional $C = C(n) > 0$. Regarding the second integral

$$I_2 \le \int_1^{1/s} (\Lambda + \Lambda')\frac{\mathrm{d}t}{t^{1+s/2}} = (\Lambda+\Lambda')\frac{1-s^{s/2}}{s/2},$$

and lastly

$$\begin{aligned}
I_3 &= \int_{1/s}^\infty \left|H_M(\varphi(x),\varphi(y),t) - H_{M'}(x,y,t)\right| \frac{\mathrm{d}t}{t^{1+s/2}} \\
&\le s^{s/2}\int_1^\infty \Big[H_M(\varphi(x),\varphi(y),\xi/s) + H_{M'}(x,y,\xi/s)\Big]\frac{d\xi}{\xi^{1+s/2}} = o_s(1) \to 0
\end{aligned}$$

as $s \to 0^+$, since both M and M' have infinite volume, and thus, their heat kernel tends to zero as $t \to +\infty$ (see Lemma 3.2.6). Hence as $s \to 0^+$

$$\left|\mathcal{K}_s(\varphi(x),\varphi(y)) - \mathcal{K}'_s(x,y)\right| \le Cs + C(\Lambda+\Lambda')(1-s^{s/2}) + o_s(1) = o_s(1),$$

and note that this estimate is uniform in $x,y \in \mathcal{B}_{1/5}(0)$. This follows, for example, from the parabolic Harnack inequality since one can locally estimate the supremum of H_M and $H_{M'}$ with the L^1 norm at later times; see the end of the proof of Lemma 3.2.6. Then

$$\lim_{s\to 0^+} \sup_{x,y\in B_{1/8}(p)} |\mathcal{K}_s(x,y) - \mathcal{K}'_s(x,y)| = 0.$$

Lastly, by [CFSS23, Lemma 2.5] there exists dimensional constants $c, C > 0$ such that

$$c\frac{\beta_{n,s}}{d(x,y)^{n+s}} \le \mathcal{K}'_s(x,y) \le C\frac{\beta_{n,s}}{d(x,y)^{n+s}},$$

and this concludes the proof. □

3.6.3 On the equivalence and well-posedness of different fractional Laplacians.

In this subsection we shall prove some results concerning the equivalence between different definitions of the fractional Laplacian, and the fractional Sobolev seminorms on (possibly weighted) Riemannian manifolds.

Next we want to show that the fractional laplacian defined with the heat semigroup $(-\Delta)^{s/2}_{\mathrm{B}}$ and the one defined via the singular integral $(-\Delta)^{s/2}_{\mathrm{Si}}$ coincide. Note that the two following propositions do not hold when M is not stochastically complete. Indeed, using definition (3.1.4) gives $(-\Delta)^{s/2}_{\mathrm{Si}}(1) \equiv 0$, while if M is not stochastically complete equation (3.1.2) gives $(-\Delta)^{s/2}_{\mathrm{B}}(1) \neq 0$.

Proposition 3.6.19. *Let (M, g) be a complete, stochastically complete Riemannian manifold, and let $u \in C^\infty_c(M)$. Then:*

(i) For $s < 1$ the integral in $(-\Delta)^{s/2}_{\mathrm{Si}} u$ is absolutely convergent and the principal value is not needed.

(ii) The singular integral $(-\Delta)^{s/2}_{\mathrm{Si}} u$ (defined in (3.1.4)) and the Bochner $(-\Delta)^{s/2}_{\mathrm{B}} u$ (defined in (3.1.2)) definition coincide.

Proof. For what concerns the absolute convergence for $s \in (0, 1)$, we have

$$\int_M (u(x) - u(y))\mathcal{K}_s(x, y)\,\mathrm{d}\mu(y) = \int_{B_r(x)} (\dots)\,\mathrm{d}\mu(y) + \int_{M\setminus B_r(x)} (\dots)\,\mathrm{d}\mu(y) =: I_1 + I_2.$$

For r small, arguing exactly as in the proof of Theorem 3.3.4

$$I_1 \leq C \int_{B_r(x)} \frac{1}{d(x, y)^{n+s-1}}\,\mathrm{d}\mu(y) \leq C \int_0^r \frac{1}{\rho^s} d\rho < +\infty\,.$$

On the other hand, for the second integral

$$I_2 \leq 2\|u\|_{L^\infty} \int_{M\setminus B_r(x)} \mathcal{K}_s(x, y)\,\mathrm{d}\mu(y)\,,$$

and thanks to Lemma 3.6.18 and Fubini

$$\begin{aligned}\int_{M\setminus B_r(x)} \mathcal{K}_s(x, y)\,\mathrm{d}\mu(y) &= \int_0^\infty \frac{1}{t^{1+s/2}} \int_{M\setminus B_r(x)} H_M(x, y, t)\,\mathrm{d}\mu(y)\,\mathrm{d}t\\ &\leq C \int_0^1 e^{-c/t} \frac{\mathrm{d}t}{t^{1+s/2}} + \int_1^\infty \frac{1}{t^{1+s/2}}\,\mathrm{d}t < +\infty.\end{aligned}$$

This concludes the proof of (i).

Now, let us define

$$\mathfrak{I}(t) := \frac{e^{t\Delta}u(x) - u(x)}{t^{1+s/2}} = \frac{1}{t^{1+s/2}} \int_M H_M(x, y, t)(u(y) - u(x))\,\mathrm{d}\mu(y),$$

where the second equality is due to the stochastical completeness. Note that $\mathfrak{I} \in L^1(0, +\infty)$ since $|e^{t\Delta}u(x) - u(x)| \leq Ct$, where the constant C depends on $\|\Delta u\|_{L^\infty}$. We can now define

$$\mathfrak{I}_k(t) := \frac{1}{t^{1+s/2}} \int_{M\setminus B_{1/k}(x)} H_M(x, y, t)(u(y) - u(x))\,\mathrm{d}\mu(y).$$

and observe that $\mathfrak{J}_k(t) \to \mathfrak{J}(t)$ for all $t \in (0,\infty)$. Now if $t \geq 1$ (estimating the mass of the heat kernel by 1) we get $\mathfrak{J}_k(t) \leq 2\|u\|_{L^\infty}/t^{1+s/2}$, while by [CFSS23, Lemma 2.11] we have

$$\begin{aligned}\left|\mathfrak{J}(t) - \mathfrak{J}_k(t)\right| &\leq \frac{1}{t^{1+s/2}} \int_{B_{1/k}(x)} H_M(x,y,t)|u(y)-u(x)|\,\mathrm{d}\mu(y) \\ &\leq \frac{C}{t^{1+s/2+n/2}} \int_{B_{1/k}(x)} e^{-d^2(x,y)/5t} d(x,y)\,\mathrm{d}\mu(y)\,.\end{aligned}$$

Applying Coarea formula and using the fact that $\mathrm{Per}(B_r(x)) \leq Cr^{n-1}$ if k is big we get

$$\left|\mathfrak{J}(t) - \mathfrak{J}_k(t)\right| \leq \frac{C}{t^{1+s/2+n/s}} \int_0^{1/k} e^{-r^2/5t} r^n dr = \frac{C}{t^{s/2}} \int_0^{1/(5tk^2)} e^{-z} z^{n/2-1} dz \leq \frac{C}{t^{s/2}}.$$

Therefore if $t \geq 1$ we have $\mathfrak{J}_k(t) \leq C/t^{1+s/2} \in L^1(1,+\infty)$ while if $t \leq 1$ we have $\mathfrak{J}_k(t) \leq C/t^{s/2} + \mathfrak{J}(t) \in L^1(0,1)$. Hence by dominated convergence we can write

$$(-\Delta)^{s/2}_{\mathrm{B}} u(x) = \int_0^\infty \mathfrak{J}(t)\ \mathrm{d}t = \lim_{k\to\infty} \int_0^\infty \int_{M\setminus B_{1/k}(x)} (u(y)-u(x)) H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}}\ \mathrm{d}\mu(y).$$

Now for any $k \in \mathbb{N}$ fixed, by Lemma (3.6.18) and the fact that u is bounded, we get

$$\int_0^\infty \int_{M\setminus B_{1/k}(x)} |u(y)-u(x)| H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}} \leq 2\|u\|_{L^\infty} \int_0^1 e^{-c/t} \frac{\mathrm{d}t}{t^{1+s/2}} + 2\|u\|_{L^\infty} \int_1^\infty \frac{\mathrm{d}t}{t^{1+s/2}} < +\infty.$$

Therefore we can apply Fubini and infer

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{B}} u(x) &= \lim_{k\to\infty} \int_{M\setminus B_{1/k}(x)} \int_0^\infty (u(y)-u(x)) H_M(x,y,t) \frac{\mathrm{d}t}{t^{1+s/2}}\ \mathrm{d}\mu(y) \\ &= \mathrm{P.V.} \int_M (u(y)-u(x)) \mathcal{K}_s(x,y)\,\mathrm{d}\mu(y).\end{aligned}$$

□

Remark 3.6.20. One can note that the proof above of the absolute convergence of $(-\Delta)^{s/2}_{\mathrm{Si}} u$ for $s \in (0,1)$ actually shows that the integral is absolutely convergent if $u \in C^\alpha_{\mathrm{loc}}(M) \cap L^\infty(M)$ for some $\alpha > s$.

Regarding the following two results, we couldn't find any proof in the case of an ambient Riemannian manifold (M,g), even though they appear to be well-known in the community in the case $M = \mathbb{R}^n$ or a domain $M = \Omega \subset \mathbb{R}^n$. For example, a proof that $\mathrm{Dom}((-\Delta_\Omega)^{s/2}_{\mathrm{Spec}}) = H^s(\Omega)$ for the Dirichlet Laplacian on $\Omega \subset \mathbb{R}^n$ can be found in [BSV15, Section 3.1.3], but it heavily uses the discreteness of the spectrum and interpolation theory.

Our results are not sharp, in particular, we believe that Proposition 3.6.21 and 3.6.23 hold also for $s = \sigma$ since this is the case for domains in $\mathbb{R}^n$. Here we focus on providing structural (and short) proofs that apply verbatim to the case of any weighted manifold, and we avoid using any local Euclidean-like structure of M.

Proposition 3.6.21. *Let (M,g) be a stochastically complete Riemannian manifold, $\sigma \in (0,1)$ and $u \in H^\sigma(M)$ (as defined in Definition 3.1.1). Then, for every $s < \sigma$ the singular integral $(-\Delta)^{s/2}_{\mathrm{Si}} u$ (defined in (3.1.4)) and the Bochner $(-\Delta)^{s/2}_{\mathrm{B}} u$ (defined in (3.1.2)) definition coincide a.e. Moreover $(-\Delta)^{s/2}_{\mathrm{B}} u = (-\Delta)^{s/2}_{\mathrm{Si}} u \in L^2(M)$.*

Proof. Let $u \in H^\sigma(M)$ and $x \in M$. Since M is stochastically complete, if we could exchange the order of integration we would have

$$\begin{aligned}(-\Delta)^{s/2}_{\mathrm{B}} u(x) &= \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta}u(x) - u(x)) \frac{\mathrm{d}t}{t^{1+s/2}} \\ &= \frac{1}{\Gamma(-s/2)} \int_0^\infty \left(\int_M H_M(x,y,t)(u(y) - u(x)) \,\mathrm{d}\mu(y) \right) \frac{\mathrm{d}t}{t^{1+s/2}} \\ &= \int_M (u(y) - u(x)) \mathcal{K}_s(x,y) \,\mathrm{d}\mu(y) = (-\Delta)^{s/2}_{\mathrm{Si}} u(x)\,.\end{aligned}$$

Now we shall justify the steps above, showing that the integral is absolutely convergent. Note that this will also justify the last equality, since we have defined $(-\Delta)^{s/2}_{\mathrm{Si}}$ with the Cauchy principal value. In particular, we show that

$$\int_M \left(\int_M |u(x) - u(y)| \mathcal{K}_s(x,y) \,\mathrm{d}\mu(y) \right)^2 \mathrm{d}\mu(x) < +\infty\,.$$

This will prove at the same time that the integral above is absolutely convergent for a.e. $x \in M$ and that $(-\Delta)^{s/2}_{\mathrm{Si}} u \in L^2(M)$. Let us call

$$\mathfrak{I}(t) := \int_M |u(x) - u(y)| H_M(x,y,t) \,\mathrm{d}\mu(y)\,,$$

and denote by C a constant that depends at most on σ.

Note that, by Jensen's inequality

$$\begin{aligned}\int_0^\infty \mathfrak{I}(t)^2 \frac{\mathrm{d}t}{t^{1+\sigma}} &= \int_0^\infty \left(\int_M |u(x) - u(y)| H_M(x,y,t) \,\mathrm{d}\mu(y) \right)^2 \frac{\mathrm{d}t}{t^{1+\sigma}} \\ &\le \int_0^\infty \int_M |u(x) - u(y)|^2 H_M(x,y,t) \,\mathrm{d}\mu(y) \frac{\mathrm{d}t}{t^{1+\sigma}} \\ &= C \int_M |u(x) - u(y)|^2 \mathcal{K}_{2\sigma}(x,y) \,\mathrm{d}\mu(y)\,. \end{aligned} \tag{3.6.11}$$

Write

$$\begin{aligned}&\int_M \left(\int_M |u(x) - u(y)| \mathcal{K}_s(x,y) \,\mathrm{d}\mu(y) \right)^2 \mathrm{d}\mu(x) \\ &\qquad = Cs^2 \int_M \left(\int_0^\infty \mathfrak{I}(t) \frac{\mathrm{d}t}{t^{1+s/2}} \right)^2 \mathrm{d}\mu \\ &\qquad \le Cs^2 \int_M \left(\int_0^1 \mathfrak{I}(t) \frac{\mathrm{d}t}{t^{1+s/2}} \right)^2 \mathrm{d}\mu + Cs^2 \int_M \left(\int_1^\infty \mathfrak{I}(t) \frac{\mathrm{d}t}{t^{1+s/2}} \right)^2 \mathrm{d}\mu\,.\end{aligned}$$

For the first integral, since $s < \sigma$, by Hölder's inequality and (3.6.11) we have

$$\int_M \left(\int_0^1 \mathfrak{I}(t) \frac{\mathrm{d}t}{t^{1+s/2}} \right)^2 \mathrm{d}\mu \le \int_M \left(\int_0^1 \mathfrak{I}(t)^2 \frac{\mathrm{d}t}{t^{1+\sigma}} \right) \left(\int_0^1 \frac{\mathrm{d}t}{t^{1-\sigma+s}} \right) \mathrm{d}\mu \le C[u]^2_{H^\sigma(M)} < +\infty.$$

For the second integral, let us first renormalize the measure $\nu := C\,\mathrm{d}t/t^{1+s/2}$ in a way that it becomes a probability measure on $[1,\infty)$. Then, by Jensen again (applied two times: to

$d\nu(t)$ and then $H_M(x,y,t)\,\mathrm{d}\mu(y)$)

$$\begin{aligned}\int_M \left(\int_1^\infty \mathfrak{I}(t)\frac{\mathrm{d}t}{t^{1+s/2}}\right)^2 \mathrm{d}\mu &\le \frac{C}{s^2}\iint_{M\times M}\int_1^\infty |u(x)-u(y)|^2 H_M(x,y,t)\,d\nu(t)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\le \frac{4C}{s^2}\iint_{M\times M}\int_1^\infty |u(x)|^2 H_M(x,y,t)\,d\nu(t)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\ &\le \frac{4C}{s^2}\|u\|^2_{L^2(M)} < +\infty\,.\end{aligned}$$

Hence, we have proved

$$\begin{aligned}\|(-\Delta)^{s/2}_{\mathrm{Si}}u\|^2_{L^2(M)} &\le \int_M \left(\int_M |u(x)-u(y)|\mathcal{K}_s(x,y)\,\mathrm{d}\mu(y)\right)^2 \mathrm{d}\mu(x)\\ &\le C\|u\|^2_{L^2(M)} + Cs^2\|u\|^2_{H^\sigma(M)},\end{aligned} \tag{3.6.12}$$

and this concludes the proof. □

Remark 3.6.22. Note that the proof of Proposition 3.6.21 applies verbatim to the case of $\mathrm{RCD}(K,N)$ spaces, since every $\mathrm{RCD}(K,N)$ space is stochastically complete. We will use this fact in the proof of Theorem 3.6.16.

Next, we address the equivalence of the spectral fractional Laplacian $(-\Delta)^{s/2}_{\mathrm{Spec}}$ with the other definitions. We refer to [Gri09] and [EBGK+22, Section 2.6] and the references therein for an introduction of the spectral theory of the fractional Laplacian on general spaces.

Let E_λ be the spectral resolvent of (minus) the Laplacian on (M,g). Then, for $s\in(0,2)$ in the classical sense of spectral theory

$$\mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}}) := \left\{u\in L^2(M)\ :\ \int_{\sigma(-\Delta)} \lambda^s\, d\langle E_\lambda u, u\rangle < +\infty\right\},$$

and for $u\in \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$

$$(-\Delta)^{s/2}_{\mathrm{Spec}}u := \int_{\sigma(-\Delta)} \lambda^{s/2} d\langle E_\lambda u,\cdot\rangle\,. \tag{3.6.13}$$

Proposition 3.6.23. *Let (M,g) be a stochastically complete Riemannian manifold, $\sigma\in(0,1)$ and $s<\sigma$. Then $H^\sigma(M)\subseteq \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$.*

Proof. Let $u\in H^\sigma(M)$, and let

$$\varphi(\lambda) := \lambda^{s/2} = \frac{1}{\Gamma(-s/2)}\int_0^\infty (e^{-\lambda t}-1)\frac{\mathrm{d}t}{t^{1+s/2}}\,.$$

Since $u\in L^2(M)$, by standard spectral theory (see [Gri09] for example)

$$\begin{aligned}\int_0^\infty \lambda^s d\langle E_\lambda u,u\rangle &= \int_0^\infty |\varphi(\lambda)|^2 d\langle E_\lambda u,u\rangle = \|\varphi(-\Delta)u\|^2_{L^2(M)}\\ &= \left\|\int_0^\infty (e^{t\Delta}u-u)\frac{\mathrm{d}t}{t^{1+s/2}}\right\|^2_{L^2(M)} = \|(-\Delta)^{s/2}_{\mathrm{B}}u\|^2_{L^2(M)} = \|(-\Delta)^{s/2}_{\mathrm{Si}}u\|^2_{L^2(M)} < +\infty\,,\end{aligned}$$

where we have used that by Proposition 3.6.21 $(-\Delta)^{s/2}_{\mathrm{B}}u = (-\Delta)^{s/2}_{\mathrm{Si}}u \in L^2(M)$. □

Proposition 3.6.24. *Let* $u \in \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$. *Then*

$$(-\Delta)^{s/2}_{\mathrm{B}} u := \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{t\Delta}u - u)\frac{\mathrm{d}t}{t^{1+s/2}} = \int_{\sigma(-\Delta)} \lambda^{s/2} d\langle E_\lambda u, \cdot\rangle =: (-\Delta)^{s/2}_{\mathrm{Spec}} u\,,$$

where the equality is in duality with $\mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$.

Proof. We follow [CS16, Lemma 2.2] which deals with the analogous proposition in the case of discrete spectrum in a domain $\Omega \subset \mathbb{R}^n$. Recall the numerical formula

$$\lambda^{s/2} = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{-\lambda t} - 1)\frac{\mathrm{d}t}{t^{1+s/2}}\,,$$

valid for $\lambda > 0$, $0 < s < 2$. Let $\psi \in \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$, and write $\psi = \int_{\sigma(-\Delta)} dE_\lambda\langle\psi, \cdot\rangle$. Then

$$\begin{aligned}\int_{\sigma(-\Delta)} \lambda^{s/2} d\langle E_\lambda u, \psi\rangle &= \frac{1}{\Gamma(-s/2)} \int_{\sigma(-\Delta)} \int_0^\infty (e^{-\lambda t} - 1)\frac{\mathrm{d}t}{t^{1+s/2}} d\langle E_\lambda u, \psi\rangle \\ &= \frac{1}{\Gamma(-s/2)} \int_0^\infty \left(\int_{\sigma(-\Delta)} (e^{-\lambda t} - 1) d\langle E_\lambda u, \psi\rangle\right)\frac{\mathrm{d}t}{t^{1+s/2}} \\ &= \frac{1}{\Gamma(-s/2)} \int_0^\infty \left(\langle e^{t\Delta}u, \psi\rangle - \langle u, \psi\rangle\right)\frac{\mathrm{d}t}{t^{1+s/2}}\,,\end{aligned}$$

where the second-last inequality follows by Fubini's theorem since $u, \psi \in \mathrm{Dom}((-\Delta)^{s/2}_{\mathrm{Spec}})$. □

Corollary 3.6.25. *Let* (M, g) *be a stochastically complete Riemannian manifold,* $\sigma \in (0, 1)$, $s < \sigma$ *and* $u \in H^\sigma(M)$. *Then*

$$\frac{1}{2}[u]^2_{H^{s/2}(M)} = \int_M u(-\Delta)^{s/2}_{\mathrm{Si}} u \,\mathrm{d}\mu = \int_0^\infty \lambda^{s/2} d\langle E_\lambda u, u\rangle\,.$$

Proof. The first equality is (3.3.8), and the second equality is a direct consequence of Proposition 3.6.21, Proposition 3.6.23 and Proposition 3.6.24. □

3.6.4 Manifolds with nonnegative Ricci curvature.

We recall a theorem of Yau which gives a lower bound on the growth of the volume of geodesic balls under the nonnegative Ricci curvature assumption. Note that the same holds with the same proof on $\mathrm{CD}(K, N)$ spaces.

Theorem 3.6.26. *Let* (M, g) *be a complete non-compact Riemannian manifold with* $\mathrm{Ric}_M \geq 0$. *Then, there exists a constant* $C = C(n) > 0$ *such that for every* $x \in M$ *and* $\lambda > 0$

$$V_x(r\lambda) \geq C r V_x(\lambda), \quad \forall\, r > 1\,.$$

Proof. By scaling invariance of the hypothesis $\mathrm{Ric}_M \geq 0$ one can assume $\lambda = 1$. Then, the result is [Li12, Theorem 2.5]. □

Next, we present here a result concerning the growth of the singular kernel $\mathcal{K}_s$ in the case of nonnegative Ricci curvature. We will not use this result anywhere but we believe it can be interesting per se. For example, it implies that on cylinders $M = \mathbb{S}^{n-k} \times \mathbb{R}^k$ (with their product metric) the singular kernel $\mathcal{K}_s(x, y)$ decays like $1/d(x, y)^{k+s}$ and not $1/d(x, y)^{n+s}$ for large distances.

Lemma 3.6.27. *Let (M,g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq 0$ and $s \in (0,2)$. Then, there exists dimensional constants $0 < c < C$ such that*

$$c\frac{s(2-s)}{r^s\mu(B_r(x))} \leq \mathcal{K}_s(x,y) \leq C\frac{s(2-s)}{r^s\mu(B_r(x))}$$

with $r = d(x,y)$ for all $x, y \in M$.

Proof. In the definition of the singular kernel $\mathcal{K}_s$ we first perform the change of variables $r^2t = k$ with $r = d(x,y)$ so that we obtain

$$\mathcal{K}_s(x,y) = \frac{r^{-s}}{|\Gamma(-s/2)|}\int_0^\infty H_M(x,y,r^2k)\frac{dk}{k^{1+s/2}}.$$

Now we employ the Gaussian estimates from above to get

$$\mathcal{K}_s(x,y) \leq \frac{Cs(2-s)}{r^s}\left[\int_0^1 \frac{1}{\mu(B_{r\sqrt{k}}(x))}e^{-1/5k}\frac{dk}{k^{1+s/2}} + \int_1^\infty \frac{1}{\mu(B_{r\sqrt{k}}(x))}e^{-1/5k}\frac{dk}{k^{1+s/2}}\right] =: I_1 + I_2.$$

Using Bishop-Gromov's inequality we get

$$I_1 \leq \frac{Cs(2-s)}{\mu(B_r(x))}\int_0^1 \frac{e^{-1/5k}}{k^{n/2+1+s/2}}dk \leq \frac{Cs(2-s)}{\mu(B_r(x))},$$

while for $k \in (1,\infty)$ we can use Theorem 3.6.26 to write

$$I_2 \leq \frac{Cs(2-s)}{\mu(B_r(x))}\int_1^\infty e^{-1/5k}\frac{dk}{k^{3/2+s/2}} \leq \frac{Cs(2-s)}{\mu(B_r(x))}$$

and this concludes the upper estimate. For the one from below we again use the Gaussian estimates to infer

$$\mathcal{K}_s(x,y) \geq \frac{cs(2-s)}{r^s}\left[\int_0^1 \frac{1}{\mu(B_{r\sqrt{k}}(x))}e^{-1/3k}\frac{dk}{k^{1+s/2}} + \int_1^\infty \frac{1}{\mu(B_{r\sqrt{k}}(x))}e^{-1/3k}\frac{dk}{k^{1+s/2}}\right] =: I_3 + I_4.$$

We now get

$$I_3 \geq \frac{cs(2-s)}{\mu(B_r(x))}\int_0^1 e^{-1/3k}\frac{dk}{k^{1+s/2}} = \frac{cs(2-s)}{\mu(B_r(x))}.$$

Since $I_4 \geq 0$ we infer the lower bound as well. □

Remark 3.6.28. If we assume $\mathrm{AVR}(M) = \lim_{r\to\infty}\frac{\mu(B_r(x))}{\omega_n r^n} = \theta > 0$ then we have the more Euclidean-like bounds

$$\frac{cs(2-s)}{\theta r^{n+s}} \leq \mathcal{K}_s(x,y) \leq \frac{Cs(2-s)}{\theta r^{n+s}}.$$

Note moreover that the same proof works in the singular setting of $\mathrm{RCD}(0,N)$ spaces.

The following is a well-known result concerning the regularization of bounded functions via the heat flow whose proof is based on Bakry-Emery inequality. Here we shall present a direct proof exploiting Gaussian estimates, Yau's inequality and the following fact

$$\frac{\mathrm{Per}(B_r(x))}{\mu(B_r(x))} \leq \frac{n}{r} \quad \forall r > 0\,, \tag{3.6.14}$$

which is an easy consequence of the Bishop-Gromov theorem. We stress that our proof is not fully general since it does not cover the case of $\mathrm{RCD}(0,\infty)$ spaces (our inequality is not dimension free).

Proposition 3.6.29 ($L^\infty - \mathrm{Lip}$ regularization). *Let (M,g) be a geodesically complete Riemannian manifold with $\mathrm{Ric}_M \geq 0$, then for every $u \in L^\infty(\mu)$ we have $\mathrm{Lip}(e^{t\Delta}u) \leq C/\sqrt{t}$ for some $C = C(n) > 0$, for all $t > 0$.*

Proof. Fix any $u \in L^\infty(\mu)$ and $p, q \in M$. Then we have

$$|H_M(x,p,t) - H_M(x,q,t)| \leq d(p,q) \int_0^1 |\nabla H_M(x,\gamma_s,t)| ds \quad \forall x \in M,\ \forall t > 0$$

where γ is a constant speed geodesic joining p and q. Now we have

$$|e^{t\Delta}u(p) - e^{t\Delta}u(q)| \leq d(p,q)\|u\|_{L^\infty} \int_0^1 \int_M |\nabla H_M(x,\gamma_s,t)|\, \mathrm{d}\mu(x)\, \mathrm{d}t$$

which exploiting the Gaussian estimates for the gradient (3.6.4) and the Coarea formula becomes

$$|e^{t\Delta}u(p) - e^{t\Delta}u(q)| \leq \frac{C}{\sqrt{t}} d(p,q)\|u\|_{L^\infty} \int_0^1 \int_0^\infty e^{-d^2(x,\gamma_s)/5t} \frac{\mathrm{Per}(B_r(\gamma_s))}{\mu(B_{\sqrt{t}}(\gamma_s))} drds.$$

Now we can multiply and divide the integrand by $\mu(B_r(\gamma_s))$, use Coarea formula and apply (3.6.14) to get

$$|e^{t\Delta}u(p) - e^{t\Delta}u(q)| \leq \frac{Cn}{\sqrt{t}} d(p,q)\|u\|_{L^\infty} \int_0^1 \int_0^\infty e^{-r^2/5t} \frac{\mu(B_r(\gamma_s))}{\mu(B_{\sqrt{t}}(\gamma_s))} \frac{1}{r} drds.$$

We now set $r^2/5t = z$ and we get, applying Theorem 3.6.26 and relabeling constants,

$$|e^{t\Delta}u(p) - e^{t\Delta}u(q)| \leq \frac{Cn}{\sqrt{t}} d(p,q)\|u\|_{L^\infty} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} dz,$$

that is the thesis. □

Finally let us observe that on a Riemannian manifold (M,g) (or actually $\mathrm{RCD}(K,N)$ space) with $\mathrm{Ric}_g \geq -K$ we have that bounded harmonic functions have uniformly bounded Lipschitz constant, indeed we have the following

Lemma 3.6.30. *Let (M,g) be a Riemannian manifold with $\mathrm{Ric} \geq -K$ with $K \geq 0$. Then any bounded harmonic function is* globally *Lipschitz continuous.*

Proof. Let $u : M \to \mathbb{R}$ be a bounded harmonic function, then we have $e^{t\Delta}u = u$. Therefore by the $L^\infty - \mathrm{Lip}$ regularization of the heat flow we get, for all $t \in (0,\infty)$

$$\|\nabla u\|_\infty = \|\nabla e^{t\Delta}u\|_\infty \leq \frac{e^{2Kt}}{\sqrt{t}} \|u\|_\infty, \tag{3.6.15}$$

meaning that u is Lipschitz continuous. Optimizing in t the inequality (3.6.15) and get $\|\nabla u\|_\infty = \mathrm{Lip}(u) \leq 2\sqrt{eK}\|u\|_\infty$, which is a *dimension free* inequality. □

Chapter 4

Regularity of harmonic maps

4.1 Introduction and Notation

4.1.1 The source: $\mathrm{RCD}(K, N)$ spaces

In this section we shall introduce the notion of RCD space. If we assume a metric measure space $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m})$ to be infinitesimally Hilbertian (in the sense of Definition 1.4.15), it is possible to give a meaning to the object

$$\int_\mathsf{X} \langle \nabla\varphi, \nabla f\rangle \,\mathrm{d}\mathfrak{m}_\mathsf{X}$$

by setting

$$\int_\mathsf{X} \langle \nabla\varphi, \nabla f\rangle \,\mathrm{d}\mathfrak{m}_\mathsf{X} := \mathrm{Ch}(f+\varphi) - \mathrm{Ch}(f) - \mathrm{Ch}(\varphi).$$

With the latter object we are able then to speak about 'Laplacian of a function': one of the ways to introduce such object is the following.

Definition 4.1.1 (L^2 Laplacian)**.** We say that $f : \mathsf{X} \to \mathbb{R}$ in $W^{1,2}(\mathsf{X})$ is such that $f \in D(\Delta) \subset L^2(\mathfrak{m}_\mathsf{X})$ if there exists $g \in L^2(\mathfrak{m}_\mathsf{X})$ such that

$$-\int_\mathsf{X} \langle \nabla\varphi, \nabla f\rangle \,\mathrm{d}\mathfrak{m}_\mathsf{X} = \int_\mathsf{X} g\varphi \,\mathrm{d}\mathfrak{m}_\mathsf{X}$$

for all $\varphi \in W^{1,2}(\mathsf{X})$. We shall set $\Delta f := g$.

Definition 4.1.2 (Measure-valued Laplacian)**.** We say that $f : \mathsf{X} \to \mathbb{R}$ in $W^{1,2}_{\mathrm{loc}}(\mathsf{X})$ has measure-valued Laplacian in Ω if there exists a Radon measure $\mu \in \mathcal{M}(\Omega)$ such that

$$-\int_\mathsf{X} \langle \nabla\varphi, \nabla f\rangle \,\mathrm{d}\mathfrak{m}_\mathsf{X} = \int_\mathsf{X} \varphi \,\mathrm{d}\mu$$

for all $\varphi \in \mathrm{Lip}_c(\Omega)$, the latter being the space of Lipschitz functions with compact support inside Ω.

Remark 4.1.3. With a little bit of abuse of notation we shall call $\Delta f = \mu$ the measure-valued Laplacian as well. We will do this since if $\mu \ll \mathfrak{m}_\mathsf{X}$ with density in L^2_{loc}, then $\mu = \Delta f \mathfrak{m}_\mathsf{X}$. Notice also that we are using the term *Radon measure* to denote what are more properly called Radon functionals (see [CM20]).

We are now ready to introduce the class of spaces which we will use as source space for the definition of our harmonic map u. We can introduce $\mathrm{RCD}(K,N)$ spaces building on the tools we have just presented. Following an Eulerian approach it is possible to characterize them via the Bochner inequality (see [GKO13], [AGS14b], [AGS15], [EKS14], [AMS15], [CM16]). For a more detailed discussion on such notions and for the interplay with optimal transport we refer to the recent [Gig23a] and [Amb18].

Definition 4.1.4 ($\mathrm{RCD}(K,N)$ space)**.** We say that a metric measure space $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ is an $\mathrm{RCD}(K,N)$ space if the following conditions are met:

1. There exists $c_1, c_2 \geq 0$ such that for some $x \in \mathsf{X}$ we have
$$\mathfrak{m}(B_r(x)) \leq C_1 e^{c_2 r^2}.$$
2. $W^{1,2}(\mathsf{X})$ is a Hilbert space.
3. If $f \in W^{1,2}(\mathsf{X})$ is such that $|\,\mathrm{d}f| \leq 1$ $\mathfrak{m}$-a.e., then f has a 1-Lipschitz representative.
4. For every $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(\mathsf{X})$ and $g \in L^\infty(\mathfrak{m}) \cap D(\Delta)$ the following *Bochner inequality* holds
$$\int_\mathsf{X} \frac{|\,\mathrm{d}f|^2}{2} \Delta g \,\mathrm{d}\mathfrak{m} \geq \int_\mathsf{X} g\Big(K|\,\mathrm{d}f|^2 + \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f\rangle\Big)\,\mathrm{d}\mathfrak{m}.$$

The final object we shall introduce is the heat semigroup $e^{t\Delta} : L^2(\mathfrak{m}_\mathsf{X}) \to L^2(\mathfrak{m}_\mathsf{X})$: it can be introduced as the gradient flow of the Cheeger energy. Therefore we shall call $(e^{t\Delta} f)_{t\geq 0}$ such a gradient flow starting from $f \in L^2(\mathfrak{m}_\mathsf{X})$ (as in Chapter 3). For an account of its properties the reader can consult [GP20a]. If the space X is an $\mathrm{RCD}(K,N)$ space then it is possible to consider the EVI_K gradient flow of the entropy functional on the space of probability measures. If we denote with $e^{t\Delta}\delta_x$ the gradient flow of the entropy starting from a Dirac mass centered at x we have $e^{t\Delta}\delta_x \ll \mathfrak{m}_\mathsf{X}$ and we shall call $H_X(x,y,t) := \frac{\mathrm{d}e^{t\Delta}\delta_x}{\mathrm{d}\mathfrak{m}_\mathsf{X}}(y)$. It can be proved that $e^{t\Delta} f := \int_\mathsf{X} H_X(x,\cdot,t) f(x)\,\mathrm{d}\mathfrak{m}_\mathsf{X}$ and that $H_X(\cdot,\cdot,t)$ is Hölder continuous and satisfies the following Gaussian estimates

$$\frac{c}{\mathfrak{m}_\mathsf{X}(B_{\sqrt{t}}(x))} e^{-\mathsf{d}^2_\mathsf{X}(x,y)/3t - C_1 t} \leq H_X(x,y,t) \leq \frac{C}{\mathfrak{m}_\mathsf{X}(B_{\sqrt{t}}(x))} C e^{-\mathsf{d}^2_\mathsf{X}(x,y)/5t + C_2 t}, \tag{4.1.1}$$

for all $x, y \in \mathsf{X}$, $t > 0$ and for some $c, C, C_1, C_2 > 0$. There is also a gradient bound thanks to the Li-Yau inequality but for the sake of exposition we shall limit ourselves to this presentation (see Proposition 3.6.2): the interested reader can consult [JLZ16], [Stu94], [Stu95] and [Stu96] for more information on Gaussian estimates.

Since we are interested in giving a meaning to "$\Delta f \geq \eta$" we shall rigorously introduce such a notion:

Definition 4.1.5 (Weak Laplacian bound)**.** Let $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m})$ be a metric measure space and $\Omega \subset \mathsf{X}$ an open and bounded set. Let $\eta : \Omega \to \mathbb{R}$ be continuous and bounded. We say that a function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is such that $\Delta f \leq \eta$ in the weak sense if for all $\varphi \in \mathrm{Lip}_c^+(\Omega)$ (being $\mathrm{Lip}_c^+(\Omega)$ the subset of $\mathrm{Lip}_c(\Omega)$ made of nonnegative functions) we have

$$-\int_\mathsf{X} \nabla f \cdot \nabla \varphi \,\mathrm{d}\mathfrak{m}_\mathsf{X} \leq \int_\mathsf{X} \varphi\eta \,\mathrm{d}\mathfrak{m}_\mathsf{X}.$$

To introduce another (weaker) notion of Laplacian bounds we need to introduce the following space

$$\mathrm{Test}_c^\infty(\mathsf{X}) := \Big\{\varphi \in D(\Delta) \cap L^\infty : |\nabla\varphi| \in L^\infty, \Delta\varphi \in L^\infty \cap W^{1,2}\Big\}.$$

We write $\mathrm{Test}_c^\infty(\Omega)$ if $\mathrm{supp}\varphi \subset\subset \Omega$.

Definition 4.1.6 (Heat flow Laplacian bound)**.** Let $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ be an infinitesimally Hilbertian metric measure space and $\Omega \subset \mathsf{X}$ be an open and bounded set. Let $f : \Omega \to \mathbb{R}$ be a bounded and lower semicontinuous function and let $\eta \in C_b(\Omega)$. We say that $\Delta f \leq \eta$ in the heat flow sense if

$$\limsup_{t\to 0} \frac{e^{t\Delta}\tilde{f}(x) - \tilde{f}(x)}{t} \leq \eta(x)$$

for all $x \in \Omega$, where $\tilde{f} : \mathsf{X} \to \mathbb{R}$ is the global extension of f which is set to zero outiside of Ω.

Finally we recall the classical Laplacian comparison for the distance function from a point, which in this non-smooth setting has been obtained in [Gig15, Corollary 5.15].

Theorem 4.1.7 (Laplacian comparison)**.** *Let $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$, $N \in \mathbb{N}$ and fix $x_0 \in \mathsf{X}$. Then the map $x \to \mathrm{d}^2_\mathsf{X}(x_0, x) = \mathrm{d}^2_{\mathsf{X},x_0}(x)$ has measure-valued Laplacian and*

$$\Delta \frac{\mathrm{d}^2_{\mathsf{X},x_0}}{2} \leq C(N, K, \mathrm{d}_{\mathsf{X},x_0}(\cdot))\mathfrak{m}_\mathsf{X}$$

in the weak sense. Moreover the same holds for the map $x \to \mathrm{d}_{\mathsf{X},x_0}(x)$, on $\mathsf{X} \setminus \{x_0\}$, namely

$$\Delta \mathrm{d}_{\mathsf{X},x_0|\mathsf{X}\setminus\{x_0\}} \leq \frac{C(N, K, \mathrm{d}_{\mathsf{X},x_0}(\cdot)) - 1}{\mathrm{d}_{\mathsf{X},x_0}(\cdot)}\mathfrak{m}_\mathsf{X}.$$

4.1.2 The target: $\mathrm{CAT}(\kappa)$ spaces

For what concerns the target space, for our harmonic map we will consider a complete $\mathrm{CAT}(\kappa)$ space, namely a metric space with sectional curvature bounded above by κ. Let M_κ be the *model space*, namely the 2-dimensional connected, simply-connected and complete Riemannian manifold with constant sectional curvature equal to κ. Let us further denote by d_κ the geodesic distance on such a space and with $\mathrm{D}_\kappa = \mathrm{diam}(\mathrm{M}_\kappa)$ its diameter, i.e.

$$\mathrm{D}_\kappa = \begin{cases} \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0 \\ +\infty & \text{if } \kappa \leq 0. \end{cases}$$

We also set $\mathrm{R}_\kappa := \mathrm{D}_\kappa/2$. We have the following:

Definition 4.1.8 ($\mathrm{CAT}(\kappa)$ space)**.** Let $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ be a complete metric space. We say that $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ is a $\mathrm{CAT}(\kappa)$ space if it is geodesic and for any triple of points $a, b, c \in \mathsf{Y}$ such that $\mathrm{d}_\mathsf{Y}(a, b) + \mathrm{d}_\mathsf{Y}(b, c) + \mathrm{d}_\mathsf{Y}(a, c) < 2\mathrm{D}_\kappa$ and any intermediate point d between b and c there exist comparison points $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathrm{M}_\kappa$ such that $\mathrm{d}_\mathsf{Y}(a, b) = \mathrm{d}_\kappa(\bar{a}, \bar{b})$, $\mathrm{d}_\mathsf{Y}(b, c) = \mathrm{d}_\kappa(\bar{b}, \bar{c})$, $\mathrm{d}_\mathsf{Y}(a, c) = \mathrm{d}_\kappa(\bar{a}, \bar{c})$ and

$$\mathrm{d}_\mathsf{Y}(a, d) \leq \mathrm{d}_\kappa(\bar{a}, \bar{d}).$$

We now have a key technical Lemma holding in general $\mathrm{CAT}(\kappa)$ spaces which is [ZZZ19, Lemma 2.3]: we shall discuss only the case $\kappa = 1$ for the sake of exposition.

Lemma 4.1.9. *Let* (Y, d) *be a* $\mathrm{CAT}(1)$ *space. Take any ordered sequence of points* $\{P, Q, R, S\} \subset \mathsf{Y}$ *with* $\mathrm{d}_{\mathsf{Y}}(P,Q) + \mathrm{d}_{\mathsf{Y}}(Q,R) + \mathrm{d}_{\mathsf{Y}}(R,S) + \mathrm{d}_{\mathsf{Y}}(S,P) \leq 2\pi$ *and let* Q_m *be the mid-point of the geodesic joining* Q *and* R *(which in this case is unique). Then for any* $\alpha \in [0,1]$ *and* $\beta > 0$ *we get*

$$\begin{aligned} &\frac{1-\alpha}{2}\Big(4\sin^2(\mathrm{d}_{QR}/2) - 4\sin^2(\mathrm{d}_{PS}/2)\Big) + 2\alpha\sin(\mathrm{d}_{QR}/2)\Big(2\sin(\mathrm{d}_{QR}/2) - 2\sin(\mathrm{d}_{PS}/2)\Big) \\ &\qquad \leq \left[1 - \frac{1-\alpha}{2}\Big(1-\frac{1}{\beta}\Big)\right] 4\sin^2(\mathrm{d}_{PQ}/2) + 2\cos(\mathrm{d}_{QR}/2)\Big(\cos(\mathrm{d}_{PQ_m}) - \cos(\mathrm{d}_{QQ_m})\Big) \\ &\qquad + \left[1 - \frac{1-\alpha}{2}\Big(1-\beta\Big)\right] 4\sin^2(\mathrm{d}_{RS}/2) + 2\cos(\mathrm{d}_{QR}/2)\Big(\cos(\mathrm{d}_{SQ_m}) - \cos(\mathrm{d}_{RQ_m})\Big). \end{aligned} \tag{4.1.2}$$

Following [GT21] (after the seminal work [KS93]) we shall now introduce the Korevaar-Schoen energy and its main properties, being the main tool we need to speak about harmonic functions.

Let $u \in L^2(\Omega, \mathsf{Y})$ with $\Omega \subseteq \mathsf{X}$ open set. We call the 2-energy density of u at scale r inside Ω the quantity $\mathbf{ks}_{2,r}[u,\Omega] : \mathsf{X} \to \mathbb{R}_+$, defined as

$$\mathbf{ks}_{2,r}[u](x) := \begin{cases} \left(⨍_{B_r(x)} \frac{\mathrm{d}^2_{\mathsf{Y}}(u(x),u(y))}{r^2}\,\mathrm{d}\mathfrak{m}(x) \right)^{\frac{1}{2}} & \text{if } B_r(x) \subset U \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.3}$$

Moreover we introduce the *total energy* of u in Ω as

$$\mathrm{E}_2[u,\Omega] := \liminf_{r\to 0} \int_\Omega \mathbf{ks}_{2,r}[u,\Omega]^2(x)\,\mathrm{d}\mathfrak{m}(x). \tag{4.1.4}$$

We can now define Sobolev spaces as follows

Definition 4.1.10 (Korevaar-Schoen space and harmonic maps)**.** We say that a function $u \in L^2(\Omega, \mathsf{Y})$ is in $\mathrm{KS}^2(\Omega, \mathsf{Y})$ if $\mathrm{E}_2[u] < +\infty$. We say that u is *harmonic* in Ω if $u = \arg\min_{v \in \mathrm{KS}^{1,2}(\Omega,\mathsf{Y})} \mathrm{E}_2[v,\Omega]$.

Existence of minimizers for $\mathrm{E}_2[\cdot,\Omega]$ has been established in the recent [Sak23] (see Theorem 1.2 therein) under the condition that the image of such maps is contained in a sufficiently small ball of the target space. For maps $u : \mathsf{X} \to \mathsf{Y}$ with source space which is a *strongly rectifiable* metric measure space (which include the class of $\mathrm{RCD}(K,N)$ spaces) and target which is a complete metric space it is possible to speak about differential $\mathbf{md}_x(u)$ in the sense introduced by Kirchheim in [Kir94] in the Euclidean setting and adapted by the authors in [GT21] to the metric setting. We shall assume the reader to be familiar with these concepts as we are going to recall only part of [GT21, Theorem 3.13], stating it for RCD spaces instead of the more general class of strongly rectifiable metric measure spaces.

Theorem 4.1.11. *Let* $(\mathsf{X}, \mathrm{d}_{\mathsf{X}})$ *be an* $\mathrm{RCD}(K,N)$ *space and* $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ *a complete metric space. Then for every* $u \in \mathrm{KS}^{1,2}(\mathsf{X}, \mathsf{Y})$ *there exists a function* $e_p[u] \in L^2(\mathsf{X})$*, called* p-energy density *of* u*, such that*

$$\mathbf{ks}_{2,r}[u] \to e_2[u] \quad \mathfrak{m}-\text{a.e. and in } L^2 \text{ as } r \to 0.$$

In particular the $\liminf$ *in* (4.1.4) *is actually a limit.*

We shall now present a representation formula of the energy density $e_2[u]$ in terms of the Hilbert-Schmidt norm of the differential $|\,\mathrm{d}u|_{\mathrm{HS}}$: we will not discuss the meaning of the object $\mathrm{d}u$, referring to [GPS20] for the details. What follows is [GT21, Proposition 6.7].

Theorem 4.1.12. *Let $(\mathsf{X}, \mathsf{d}_\mathsf{X}, \mathfrak{m}_\mathsf{X})$ be an $\mathrm{RCD}(K, N)$ space and $\Omega \subset \mathsf{X}$ an open set. Let $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a $\mathrm{CAT}(\kappa)$ space and $u \in \mathbf{KS}^{1,2}(\Omega, \mathsf{Y})$, then for its energy density we have the following representation formula*

$$e_2[u] = (d+2)^{-\frac{1}{2}}|\,\mathrm{d}u|_{\mathrm{HS}}. \tag{4.1.5}$$

Proof. Note that in [GT21] the theorem is stated for X which is a strongly rectifiable space and Y which is a $\mathrm{CAT}(0)$ space. On one hand the proof for the case of $\mathrm{CAT}(\kappa)$ target is the same of the one for $\mathrm{CAT}(0)$ spaces, exploting the universal infinitesimal Hilbertianity of such spaces (see [DMGPS21]), on the other hand we shall avoid speaking about strongly rectifiable metric measure spaces since our main results are only stated for $\mathrm{RCD}(K, N)$ spaces. □

Finally we have the following definition:

Definition 4.1.13 (λ-convexity)**.** Let $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a complete and geodesic metric space. We say that a function $E : \mathsf{Y} \to \mathbb{R}$ is λ-convex if for all $x, y \in \mathsf{Y}$ and for all geodesics γ connecting $x = \gamma_0$ and $y = \gamma_1$ we have

$$E(\gamma_t) \leq tE(\gamma_1) + (1-t)E(\gamma_0) - \frac{\lambda}{2}t(1-t)\mathsf{d}^2_\mathsf{Y}(\gamma_0, \gamma_1).$$

4.2 Main results

4.2.1 Hölder regularity of harmonic maps

In this section we will prove Hölder regularity of our harmonic map with values in a sufficiently small ball of a $\mathrm{CAT}(\kappa)$ space. Note that without this assumption there may be a "big" set of discontinuity (singular set), for examples and a detailed discussion one can consult [Riv95]. Since we can always renormalize the target space in such a way that it becomes a $\mathrm{CAT}(1)$ space, to ease the notation and the computations we shall assume $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ to be a $\mathrm{CAT}(1)$ space here and in the rest of the work.

In the following we shall prove the convexity of three functions, namely $1 - \cos(\mathsf{d}_{\mathsf{Y},o})$, $\mathsf{d}_{\mathsf{Y},o}$ and $\mathsf{d}^2_{\mathsf{Y},o}$. The proof of the λ convexity of the squared distance is contained [Oht07, Lemma3.1] and the convexity of the distance $\mathsf{d}_{\mathsf{Y},o}$ is well-known but we shall prove them here anyway because they are natural consequences of the convexity of $1 - \cos(\mathsf{d}_{\mathsf{Y},o})$.

Proposition 4.2.1. *Let $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ be a $\mathrm{CAT}(1)$ space and consider $B_\rho(o) \subset \mathsf{Y}$ with $\rho < \pi/2$. Then the distance function $\mathsf{d}_{\mathsf{Y},o} = \mathsf{d}_\mathsf{Y}(o, \cdot)$ is convex on $B_\rho(o)$, $\mathsf{d}^2_{\mathsf{Y},o}$ is λ-convex and the function $\cos(\mathsf{d}_\mathsf{Y}(o, \cdot))$ is λ'-concave, with*

$$\lambda = 2\cos\rho, \qquad \lambda' = \cos\rho.$$

Finally $\mathsf{d}_\mathsf{Y}(\cdot, \cdot)$ restricted to $B_{\rho/2}(o)$ is jointly convex.

Proof. We show that the distance from the north pole on $\mathbb{S}^2$ is convex on the upper hemisphere. Consider three points $N, p, q \in \mathbb{S}^2$. Denote with $\mathsf{d}_N(y) := \mathsf{d}_{\mathbb{S}^2}(N, y)$ the distance from the north pole for every $y \in \mathbb{S}^2$ and let γ be the geodesic connecting p and q. By the cosine law for the sphere we can consider the triangle whose vertex are p, q and N and write

$$\cos(f(t)) = \cos(t\mathrm{d}_{\mathbb{S}^2}(p,q))\cos(\mathrm{d}_N(p)) + \sin(t\mathrm{d}(p,q))\sin(\mathrm{d}_N(p))\sin(\theta),$$

where $f(t) = \mathrm{d}_N(\gamma(t))$ and θ is the angle between $\gamma'(0)$ and $\eta'(1)$ (η being the geodesic connecting the north pole and the point p). Note that we also used the fact that $\mathrm{d}_{\mathbb{S}^2}(p,\gamma(t)) = t\mathrm{d}_{\mathbb{S}^2}(p,q)$. Now differentiate twice the previous identity to get

$$(\cos(f(t)))'' = -\mathrm{d}^2_{\mathbb{S}^2}(p,q)\cos(f(t)) \le -\mathrm{d}^2_{\mathbb{S}^2}(\gamma_1,\gamma_0)\cos\rho,$$

whence $\cos(f(t))$ is a λ'-concave function with $\lambda' = \cos(\rho)$. Now write $f = \arccos\cos(f)$ and let us call $g(t) := \cos(f(t))$: we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f = \frac{(g')^2 g - g''(1-g^2)}{(1-g^2)^{\frac{3}{2}}} \ge 0, \tag{4.2.1}$$

meaning that f is a convex function (we have used that $\mathrm{Im}(g) \subseteq (0,1]$ and $g'' \le 0$)- this is fully justified if $g \neq 1$, i.e. $f \neq 0$, otherwise the argument is justified by slightly moving the north pole N combined with the stability properties of convexity.

For what concerns the squared distance f^2 just use the product rule for the derivative to get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f^2 = 2|f'|^2 + 2ff'' \ge 2ff''.$$

Now plug (4.2.1) into the previous expression to get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f^2 \ge 2f\left[\frac{(g')^2 g - g''(1-g^2)}{(1-g^2)^{\frac{3}{2}}}\right] \ge -2f\frac{g''(1-g^2)}{(1-g^2)^{\frac{3}{2}}} \ge 2\mathrm{d}^2_{\mathsf{Y}}(p,q)\cos\rho\frac{f}{\sin f} \ge 2\mathrm{d}^2_{\mathbb{S}^2}(p,q)\cos\rho,$$

which is the λ convexity with $\lambda = 2\cos\rho$.

Now consider three points $x, y \in B_\rho(o) \subseteq \mathsf{Y}$ and let p, q, N be three comparison points of x, y, o in $\mathbb{S}^2$: by the $\mathrm{CAT}(1)$ condition we have $\mathrm{d}_{\mathsf{Y}}(\tilde\gamma(t), o) \le \mathrm{d}_{\mathbb{S}^2}(\gamma(t), N)$ (with γ geodesic joining p and q and with $\tilde\gamma$ geodesic joining x and y and), meaning that

$$\cos(\mathrm{d}_{\mathsf{Y}}(\tilde\gamma(t), o)) \ge \cos(\mathrm{d}_{\mathbb{S}^2}(\gamma(t), N)).$$

The definition of comparison points together with the previous observation allows to write

$$\cos(\mathrm{d}_{\mathsf{Y}}(\tilde\gamma(t), o)) \ge t\cos(\mathrm{d}_{\mathsf{Y}}(q,o)) + (1-t)\cos(\mathrm{d}_{\mathsf{Y}}(p,o)) + \frac{t(1-t)}{2}\mathrm{d}^2_{\mathsf{Y}}(p,q)\cos\rho,$$

which is the sought λ'-concavity with $\lambda' = \cos\rho$. Analogous arguments apply for $\mathrm{d}_{\mathsf{Y}}(o,\cdot)$ and $\mathrm{d}^2_{\mathsf{Y}}(o,\cdot)$.

For the final part of the proof fix $x \in B_{\rho/2}(o)$ and notice that for all $y \in B_{\rho/2}(o)$ we must have $\mathrm{d}_{\mathsf{Y}}(x,y) < \rho$ by triangle inequality. Therefore we can use the fact that $B_\rho(x)$ is convex and conclude. □

We recall now some lemmas of gradient flow theory on locally $\mathrm{CAT}(\kappa)$-spaces which will be useful to prove some Laplacian bounds. Let us start with the following, which is part of [GN21, Theorem 3.3], to which we also refer for the relevant definitions:

Theorem 4.2.2. *Let Y be a locally $\mathrm{CAT}(\kappa)$-space, $E : \mathsf{Y} \to \mathbb{R} \cup \{+\infty\}$ a λ-convex and lower semicontinuous functional. Then, the following hold:*

- *Existence*

  For every $y \in D(E)$ there exists a gradient flow trajectory for E starting from y.

- *Uniqueness and λ-contraction*

  For any two gradient flow trajectories $(y_t), (z_t)$ we have

$$\mathrm{d}_{\mathsf{Y}}(y_t, z_t) \le e^{-\lambda(t-s)}\mathrm{d}_{\mathsf{Y}}(y_s, z_s) \quad \forall t \ge s \ge 0. \tag{4.2.2}$$

Then we have the following a priori estimates for the gradient flow trajectory which is [GN21, Lemma 3.4], following the ideas contained in [Pet07]:

Lemma 4.2.3. *Let Y be locally $\mathrm{CAT}(\kappa)$ and $E : \mathsf{Y} \to \mathbb{R} \cup \{+\infty\}$ be a λ-convex and lower semicontinuous functional, $\lambda \in \mathbb{R}$. Let $y, z \in \mathsf{Y}$ and consider the gradient flow trajectories $(y_t), (z_t)$ associated with E. Then, for any $t \ge s > 0$, it holds*

$$\begin{aligned}\mathrm{d}^2_{\mathsf{Y}}(y_t, z_s) \le& e^{-2\lambda s}\Big(\mathrm{d}^2_{\mathsf{Y}}(y, z) + 2(t-s)(E(z) - E(y)) \\ &+ 2|\partial^- E|^2(y)\int_0^{t-s}\theta_\lambda(r)\,\mathrm{d}r - \lambda\int_0^{t-s}\mathrm{d}^2_{\mathsf{Y}}(y_r, z)\,\mathrm{d}r\Big).\end{aligned} \tag{4.2.3}$$

where $\theta_\lambda(t) := \int_0^t e^{-2\lambda r}\,\mathrm{d}r$.

With the previous two lemmas at hand we can prove the analogue of [GN21, Lemma 4.17] for $\mathrm{CAT}(\kappa)$ spaces. Below we shall denote with $\mathrm{Lip_{bs}}(\mathsf{X})$ the space of Lipschitz functions with bounded support and with $\mathrm{Lip}^+_{\mathrm{bs}}(\mathsf{X})$ the subset of $\mathrm{Lip_{bs}}(\mathsf{X})$ made of non-negative functions.

Lemma 4.2.4. *Let $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, Y a locally $\mathrm{CAT}(\kappa)$-space and $\Omega \subset \mathsf{X}$ open and bounded. Also, let $f \in \mathrm{Lip}(\mathsf{Y})$ be λ-convex, $\lambda \in \mathbb{R}$, and $u \in \mathrm{KS}^{1,2}(\Omega, \mathsf{Y})$. For $g \in \mathrm{Lip_{bs}}(\mathsf{X})^+$ define the (equivalence class of the) variation map $u_t(x) := \mathrm{GF}^f_{tg(x)}(u(x))$ $\forall t > 0, x \in \Omega$. Then, $u_t \in \mathrm{KS}^{1,2}(\Omega, \mathsf{Y})$ for every $t > 0$ and there is a constant $C > 0$ depending on f, g such that*

$$|\,\mathrm{d}u_t|^2_{\mathrm{HS}} \le e^{-2\lambda t g}\Big(|\,\mathrm{d}u|^2_{\mathrm{HS}} - 2t\langle\,\mathrm{d}g,\,\mathrm{d}(f\circ u)\rangle + Ct^2\Big)\ \mathfrak{m} - \text{q.o. in}\,\Omega \tag{4.2.4}$$

holds for every $t \in [0, 1]$. In particular

$$\limsup_{t\to 0}\frac{E^{\mathrm{KS}}(u_t) - E^{\mathrm{KS}}(u)}{t} \le -\frac{1}{d+2}\int_\Omega\Big(\lambda g|\,\mathrm{d}u|^2_{\mathrm{HS}} + \langle\,\mathrm{d}g,\,\mathrm{d}(f\circ u)\rangle\Big)\,\mathrm{d}\mathfrak{m}. \tag{4.2.5}$$

Proof. The fact that $u_t \in L^2(\Omega, \mathsf{Y})$ easily follows from the following inequalities and the fact that the support of g is bounded:

$$\begin{aligned}\mathrm{d}^2_{\mathsf{Y}}(u_t(x), o) &\le 2\mathrm{d}^2_{\mathsf{Y}}(u_t(x), u(x)) + 2\mathrm{d}^2_{\mathsf{Y}}(u(x), o) \\ &\le 2\mathrm{d}^2_{\mathsf{Y}}(u(x), o) + 2te^{2|\lambda|t}\,\mathrm{Lip}^2(f)g(x),\end{aligned}$$

where for the second inequality we applied the a priori estimates (4.2.3) and exploited the fact that $|\partial^- f|(y) \le \mathrm{Lip}(f)$ for all $y \in \mathsf{Y}$. Now thanks to (4.2.2) we have (w.l.o.g. assume $g(y) \ge g(x)$)

$$\mathrm{d}^2_{\mathsf{Y}}(u_t(x), u_t(y)) \le e^{2|\lambda||g(x)-g(y)|}\mathrm{d}^2_{\mathsf{Y}}\big(u(x), \mathrm{GF}^f_{t|g(y)-g(x)|}(u(y))\big).$$

Now we can use the *sharp dissipation rate* of the gradient flow (see [GN21, point (ii) of Theorem 3.2]) to establish the Lipschitzianity of the map $t \to \mathrm{GF}_t^f(u(x))$ and get

$$\begin{aligned}
\mathrm{d}_{\mathsf{Y}}^2\big(u(x), \mathrm{GF}^f_{t|g(y)-g(x)|}(u(y))\big) &\leq 2\mathrm{d}_{\mathsf{Y}}^2\big(u(x), \mathrm{GF}^f_{t|g(y)-g(x)|}(u(x))\big) \\
&\quad + 2\mathrm{d}_{\mathsf{Y}}^2\big(\mathrm{GF}^f_{t|g(y)-g(x)|}(u(x)), \mathrm{GF}^f_{t|g(y)-g(x)|}(u(y))\big) \\
&\leq C_1 t^2 |g(x)-g(y)|^2 + 2e^{2|\lambda||g(y)-g(x)|}\mathrm{d}_{\mathsf{Y}}^2(u(y), u(x)) \\
&\leq C_1 t^2 \mathrm{d}^2(x,y) + C_2 \mathrm{d}_{\mathsf{Y}}^2(u(y), u(x)).
\end{aligned}$$

Dividing by $r^2 := \mathrm{d}^2(x,y)$ and $\mathfrak{m}(B_r(x))$ and integrating over $B_r(x) \subseteq \Omega$ we get

$$\mathrm{ks}^2_{2,r}[u_t, \Omega](x) \leq C_1 t^2 + C_2 \mathrm{ks}^2_{2,r}[u, \Omega](x).$$

The fact that $\mathfrak{m}(\Omega) < +\infty$ allows to conclude $u_t \in \mathrm{KS}^2(\Omega, \mathsf{Y})$.

For what concerns estimate (4.2.4) the proof is verbatim the one in [GN21, Lemma 4.17].

Finally for the last point we just need to subtract from both sides of (4.2.4) the quantity $|\,\mathrm{d}u|^2_{\mathrm{HS}}$ and then integrate over Ω and divide by $2t(d+2)$. Taking the $\limsup$ as $t \to 0^+$ and exploiting a dominated convergence argument allows to conclude with (4.2.5). □

The following is a generalization to $\mathrm{CAT}(\kappa)$ spaces of well-known inequalities holding for functions in $\mathrm{CAT}(0)$ spaces. We begin with the following:

Proposition 4.2.5. *Let $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space and $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ be a locally $\mathrm{CAT}(\kappa)$ space. Let $\Omega \subset \mathsf{X}$ be open and bounded and let $u : \Omega \to \mathsf{Y}$ be an harmonic map and $f : \mathsf{Y} \to \mathbb{R}$ be a Lipschitz and λ-convex map, then $f \circ u \in W^{1,2}(\Omega)$ and*

$$\Delta(f \circ u) \geq \lambda |\mathrm{d}u|^2_{\mathrm{HS}} \mathfrak{m} \tag{4.2.6}$$

in the weak sense. In particular $\Delta(f \circ u)$ is a signed Radon measure.

Proof. The fact that $f \circ u \in W^{1,2}(\Omega)$ is well-known (see [GT21]). To prove (4.2.6) first observe that being u harmonic implies

$$\limsup_{t \to 0} \frac{E^{KS}(u_t) - E^{KS}(u)}{t} \geq 0,$$

so that (4.2.5) gives

$$\lambda \int_\Omega g |\,\mathrm{d}u|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \leq -\int_\Omega \langle\,\mathrm{d}g,\, \mathrm{d}(f \circ u)\rangle\,\mathrm{d}\mathfrak{m} = \int_\Omega \Delta(f \circ u) g\,\mathrm{d}\mathfrak{m}$$

for all $g \in \mathrm{Lip}^+_{bs}(\mathsf{X})$, whence (4.2.6) follows. □

Lemma 4.2.6. *Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space and $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ a $\mathrm{CAT}(1)$ space. Let $u : \Omega \subset \mathsf{X} \to \mathsf{Y}$ be an harmonic mapping such that $u(\Omega) \subset B_\rho(o)$ for some $\rho < \pi/2$, then consider the function $f_o : \mathsf{X} \to [0,1]$ given by $f_o(x) := \cos(\mathrm{d}_{\mathsf{Y}}(u(x), o))$. We have $f_o \in W^{1,2}(\Omega)$ and*

$$\Delta f_o \leq -\cos\rho |\,\mathrm{d}u|^2_{\mathrm{HS}} \tag{4.2.7}$$

in the weak sense in Ω.

Proof. This is indeed a consequence of Proposition 4.2.1 in combination with Proposition 4.2.5. Indeed one just needs to apply those results with the space $(\overline{B_\rho(o)}, \mathrm{d}_{\mathsf{Y}})$, which is a $\mathrm{CAT}(1)$ space. □

We now have the following result which holds in a more general setting than the present one (see [BM95, Theorem 5.4]) but we shall present it in the setting of RCD spaces to avoid further technicalities.

Theorem 4.2.7 (Elliptic Harnack inequality). *Let* $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and* $u : \mathsf{X} \to \mathbb{R}$ *be a weakly subharmonic function in* $B_{4r}(x_0)$*, i.e.* $u \in W^{1,2}(B_{4r}(x_0))$ *and*

$$\Delta u \geq 0$$

in the weak sense in $B_{4r}(x_0)$*. Then the following estimate holds*

$$\sup_{z \in B_{r/2}(x_0)} \max\{u, 0\}(z) \leq C(K^- r^2, N) \left(\frac{1}{\mathfrak{m}(B_r(x_0))} \int_{B_r(x_0)} u^2 \,\mathrm{d}\mathfrak{m} \right)^{1/2}, \tag{4.2.8}$$

where C *is equibounded as* $r \to 0^+$.

Remark 4.2.8. As a consequence of (4.2.8) we get that any weakly subharmonic function is locally bounded from above.

We shall now introduce the following notation: for a function $v : \mathsf{X} \to \mathbb{R}$ we set

$$v_R := \fint_{B_R(x_0)} v \,\mathrm{d}\mathfrak{m},$$

where $x_0 \in \mathsf{X}$ is a point which will be clear from the context. We further set

$$v_{+,R} := \sup_{x \in B_R(x_0)} \max\{v, 0\}(x)$$

The following is a combination of [Jos97, Corollary 1] and [Jos97, Lemma 7]:

Corollary 4.2.9. *Let* $u : \mathsf{X} \to \mathbb{R}$ *be as in the previous Theorem and nonnegative, then there exists* $\delta_0 > 0$ *independent of* R *such that*

$$\sup_{B_R(x_0)} u \leq (1 - \delta_0) u_{+,4R} + \delta_0 u_R.$$

Moreover if $\varepsilon \in (0, 1/4)$ *there exists* $m \in \mathbb{N}$ *(independent of* u *and* ε*) such that*

$$u_{+,\varepsilon^m R} \leq \varepsilon^2 u_{+,R} + (1 - \varepsilon^2) u_{R'} \tag{4.2.9}$$

where R' *(possibly depending on* ε *and* u*) is such that* $\varepsilon^m R \leq R' \leq R/4$.

We proceed recalling another useful lemma which again extends to the context of $\mathrm{CAT}(\kappa)$ spaces without modifications:

Lemma 4.2.10. *Let* $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}}, o)$ *be a pointed complete metric space, then for every* $u \in \mathrm{KS}^{1,2}(\mathsf{X}, \mathsf{Y})$ *there exists* $C = C(\mathrm{diam}(\Omega), K, N) \geq 1$ *such that for every* $r > 0$ *and* $p \in \Omega$ *for which* $B_{rC}(p) \subseteq \Omega$ *we have*

$$\fint_{B_r(p)} \int_{B_r(p)} \mathrm{d}^2_{\mathsf{Y}}(u(x), u(y)) \,\mathrm{d}\mathfrak{m}(x) \,\mathrm{d}\mathfrak{m}(y) \leq C r^2 \int_{B_{rC}(p)} e_2^2[u] \,\mathrm{d}\mathfrak{m}. \tag{4.2.10}$$

Proof. The proof can be found in [Guo, Lemma 4.9]. □

The next Lemma is basically [Jos97, Lemma 8] adapted to $\mathrm{CAT}(\kappa)$ setting.

Lemma 4.2.11. *Let* $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space,* $\Omega \in \mathsf{X}$ *an open set, and* $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ *be a* $\mathrm{CAT}(1)$ *space. Let* $u : \Omega \to \mathsf{Y}$ *be an harmonic map with values in* $B_\rho(o)$ *with* $\rho < \pi/2$ *and let* $B_{4R}(x_0) \subset\subset \Omega$*, then*

$$R^2 \fint_{B_R(x_0)} |\,\mathrm{d}u|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \le C(v_{+,4R} - v_{+,R}),$$

where $v(x) = \mathrm{d}^2_{\mathsf{Y}}(u(x), o)$ *and* $C = C(\mathrm{diam}(\Omega), K, N)$*.*

Proof. To begin with let us consider a mollified version of the Green function (whose existence can be proved for instance via Lax-Milgram theorem) which solves in the weak sense the following

$$\begin{cases} -\Delta G_p = \frac{\chi_{B_R(p)}}{\mathfrak{m}(B_R(p))} & \text{on } B_{2R}(p) \\ G_p = 0 & \text{on } B^c_{2R}(p). \end{cases}$$

We have (we shall omit the point p center of the ball)

$$\int_{B_{2R}} \langle\, \mathrm{d}\varphi,\, \mathrm{d}G_p\rangle\,\mathrm{d}\mathfrak{m} = \fint_{B_R} \varphi\,\mathrm{d}\mathfrak{m} \tag{4.2.11}$$

for all $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X})$ with $\mathrm{supp}\varphi \subset\subset B_{2R}(p)$. Now following [BM95, Section 6] we define for convenience a rescaled version of G, namely we set

$$G_{p,R} := \frac{\mathfrak{m}(B_R(p))}{R^2} G_p,$$

which satisfies

$$\int_{B_{2R}} \langle\, \mathrm{d}\varphi,\, \mathrm{d}G_{p,R}\rangle\,\mathrm{d}\mathfrak{m} = \frac{1}{R^2}\int_{B_R} \varphi\,\mathrm{d}\mathfrak{m}$$

and the following estimates (again we refer to [BM95, Theorem 6.1], which deals with more general metric spaces which include the class of $\mathrm{RCD}(K,N)$ spaces)

$$0 < C_1 \le G_{p,R} \quad \text{on } B_R,$$
$$0 \le G_{p,R} \le C_2 \quad \text{on } B_{2R},$$

where C_1, C_2 only depend on K,N and $\mathrm{diam}(\Omega)$. Now we can define $z := v - v_{+,4R}$ and write, exploiting (4.2.6) for $f(\cdot)$ equal to $\mathrm{d}^2_{\mathsf{Y}}(\cdot, o)$ with $\lambda = 2\cos\rho$ by Proposition 4.2.1,

$$\lambda \int_{B_{2R}} |\,\mathrm{d}u|^2_{\mathrm{HS}} G^2_{p,R}\,\mathrm{d}\mathfrak{m} \le \int_{B_{2R}} (\Delta z) G^2_{p,R}\,\mathrm{d}\mathfrak{m} = -2\int_{B_{2R}} \langle\, \mathrm{d}z,\, \mathrm{d}G_{p,R}\rangle G_{p,R}\,\mathrm{d}\mathfrak{m}.$$

Now we can use the Leibniz rule for the differential $\mathrm{d}(G_{p,R}v) = G_{p,R}\,\mathrm{d}z + z\,\mathrm{d}G_{p,R}$ and write

$$\lambda \int_{B_{2R}} |\,\mathrm{d}u|^2_{\mathrm{HS}} G^2_{p,R}\,\mathrm{d}\mathfrak{m} \le -2\int_{B_{2R}} \langle\, \mathrm{d}G_{p,R},\, \mathrm{d}(G_{p,R}z)\rangle\,\mathrm{d}\mathfrak{m} + 2\int_{B_{2R}} \langle\, \mathrm{d}G_{p,R},\, \mathrm{d}G_{p,R}\rangle z\,\mathrm{d}\mathfrak{m}.$$

Being $z \le 0$ we can neglect the second term and obtain

$$\begin{aligned}\lambda \int_{B_{2R}} |\,\mathrm{d}u|^2_{\mathrm{HS}} G^2_{p,R}\,\mathrm{d}\mathfrak{m} &\le -2\int_{B_{2R}} \langle\, \mathrm{d}G_{p,R},\, \mathrm{d}(G_{p,R}z)\rangle\,\mathrm{d}\mathfrak{m} = -\frac{1}{R^2}\int_{B_R} G_{p,R}z\,\mathrm{d}\mathfrak{m} \\ &\le -\frac{C_1\mathfrak{m}(B_R)}{R^2}(v_R - v_{+,4R}) = \frac{C_1\mathfrak{m}(B_R)}{R^2}(v_{+,4R} - v_R)\end{aligned}$$

where we used the definition of the mollified Green function. Finally, applying Corollary 4.2.9, we get the thesis. □

We are now in position to prove the desired Hölder continuity of harmonic maps.

Theorem 4.2.12. *Let $u : \Omega \subseteq \mathsf{X} \to \mathsf{Y}$ be an harmonic map such that $\mathrm{Im}(u) \subseteq B_\rho(o)$ with $\rho < \pi/2$ and with $(\mathsf{X}, \mathrm{d}, \mathfrak{m})$ which is an $\mathrm{RCD}(K, N)$ space and Y which is a $\mathrm{CAT}(1)$ space. Then u is locally Hölder continuous in Ω.*

Proof. The proof closely follows [Jos97, Theorem]. Let us fix $x_0 \in \Omega$ in such a way that $B_{4R}(x_0) \subset\subset \Omega$. Let us define the mean of u on a ball centered at x_0 with radius r, denoted by $\bar{u}_r$, as one of the minimums of

$$\mathsf{Y} \ni q \mapsto \fint_{B_r(x_0)} \mathrm{d}_{\mathsf{Y}}(u(x), q)\, \mathrm{d}\mathfrak{m}(x).$$

Finally set $v_p(x) := \mathrm{d}^2_{\mathsf{Y}}(u(x), p)$ where $p \in \mathsf{Y}$ will be chosen later and $w(x) := \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4})$. We want to exploit the result in Corollary 4.2.9: let us therefore fix $\varepsilon \le 1/10$ so that $\varepsilon^m R \le R' \le R/4$ and estimate as follows

$$w^m_{R'} = \frac{1}{\mathfrak{m}(B_{R'}(x_0))} \int_{B_{R'}(x_0))} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4})\, \mathrm{d}\mathfrak{m}(x) \le \frac{C}{\mathfrak{m}(B_{R/4}(x_0)} \int_{B_{R/4}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4})\, \mathrm{d}\mathfrak{m}(x)$$

where C is independent of R, exploiting the (uniformly) doubling property of the measure $\mathfrak{m}$ on Ω. Now applying Poincaré inequality to the previous expression we get

$$\frac{C}{\mathfrak{m}(B_{R/4}(x_0))} \int_{B_{R/4}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4})\, \mathrm{d}\mathfrak{m}(x) \le C_1 \frac{R^2}{\mathfrak{m}(B_R(x_0))} \int_{B_{R/(4\lambda)}(x_0)} |\,\mathrm{d}u|^2_{\mathrm{HS}}\, \mathrm{d}\mathfrak{m},$$

for some $\lambda \in (0, 1)$. Now we shall apply Lemma 4.2.11 and the doubling inequality again to obtain

$$w^m_{R'} \le C\big(v_{p,+,R/\lambda} - v_{p,+,R/4\lambda}\big). \tag{4.2.12}$$

Choose now $p \in \mathrm{conv}\big(u(B_{\varepsilon^m R}(x_0))\big)$ so that we have

$$\sup_{x \in B_{\varepsilon^m R}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p) \le 2 \sup_{x \in B_{\varepsilon^m R}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4}) + 2\mathrm{d}^2_{\mathsf{Y}}(\bar{u}_{R/4}, p) \le 4 \sup_{x \in B_{\varepsilon^m R}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4})$$

and at the same time

$$\sup_{x \in B_R(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4}) \le 4 \sup_{x \in B_R(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p)$$

Combining estimate (4.2.12) and the result of Corollary 4.2.9 we get

$$\begin{aligned} \sup_{x \in B_{\varepsilon^m R}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4}) &\le 4\varepsilon^2 \sup_{B_R(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), \bar{u}_{R/4}) + C\big(v_{p,+,R/\lambda} - v_{p,+,R/4\lambda}\big) \\ &\le 16\varepsilon^2 \sup_{x \in B_R(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p) + C\big(v_{p,+,R/\lambda} - v_{p,+,\varepsilon^m R}\big), \end{aligned}$$

where in the last line we also used that $\varepsilon^m \le (1/8)^m \le 1/4 \le 1/4\lambda$. In the end we obtain

$$\sup_{x \in B_{\varepsilon^m R}(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p) \le 64\varepsilon^2 \sup_{x \in B_R(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p) + C\big(v_{p,+,R/\lambda} - v_{p,+,\varepsilon^m R}\big).$$

Setting $\omega(r) := \sup_{x \in B_r(x_0)} \mathrm{d}^2_{\mathsf{Y}}(u(x), p)$ we can rewrite the previous inequality as

$$(1 + C)\omega(\varepsilon^m R) \le 64\varepsilon^2 \omega(R) + C\omega(R/\lambda) \le (64/100 + C)\omega(R/\lambda),$$

which means

$$\omega(\varepsilon^m R) \leq c\omega(R/\lambda),$$

where ε and λ are fixed and $c < 1$. By an iteration of the latter estimate (holding for every $R \leq R_0$ for which $B_{R_0}(x_0) \subset\subset \Omega$) we get

$$\frac{\omega(r)}{r^\alpha} \leq C\frac{\omega(R_0)}{R_0^\alpha},$$

where $\alpha \in (0,1)$, $C > 0$ and $r \leq R_0$. Choosing $p = \bar{u}_r$ we get

$$\sqrt{\omega(r)} \leq \operatorname{osc}(u, B_r(x_0)) \leq 2\sqrt{\omega(r)}$$

and this proves the (local) Hölder continuity of u. □

4.2.2 Higher integrability of energy densities

Let $\Omega \subset X$ be an open bounded set in an $\mathrm{RCD}(K,N)$ space with $X \setminus \Omega \neq \emptyset$, $K \in \mathbb{R}$ and $N \in [1,\infty)$. Let $(Y, \mathrm{d_Y})$ be a $\mathrm{CAT}(1\kappa)$ space with $\kappa > 0$ and suppose that $u \in \mathrm{KS}(\Omega; Y)$ is an harmonic map with values in a ball $B_\rho(o) \subset Y$ with $\rho \in (0, \frac{\pi}{2\sqrt{\kappa}})$. We shall always fix a Hölder continuous representative of u.

Let us recall some notations in [AHT17].

Definition 4.2.13. For any $q > 1$, a nonnegative $\mathfrak{m}$-measurable function w on Ω belongs to the *weak q-Reverse Hölder class RH_q^{weak}* if there exists a constant C_q such that

$$\left(\fint_B w^q \,\mathrm{d}\mathfrak{m}\right)^{1/q} \leqslant C_q \int_{2B} w \,\mathrm{d}\mathfrak{m}$$

for all ball $B := B_r(y)$ with $2B := B_{2r}(y) \subset \Omega$.

We need the following Gehring lemma, see [AHT17, Propostion 6.2] and [Maa08, Theorem 3.1].

Lemma 4.2.14. *If $1 < q < \infty$ and $w \in RH_q^{weak}$, then there exists $\varepsilon > 0$ such that $w \in RH_{q+\varepsilon}^{weak}$.*

Now we will prove the higher integrability of energy density.

Theorem 4.2.15. *Let Ω, Y and u be as above. Then there exists an $\varepsilon = \varepsilon(N, K, \mathrm{diam}(\Omega), \rho) > 0$ such that $|\,\mathrm{d}u|_{\mathrm{HS}} \in W^{1,2+\varepsilon}_{\mathrm{loc}}(\Omega)$ and*

$$\left(\fint_B |\,\mathrm{d}u|_{\mathrm{HS}}^{2+\varepsilon} \,\mathrm{d}\mathfrak{m}\right)^{\frac{2}{2+\varepsilon}} \leqslant C_\varepsilon \fint_B |\,\mathrm{d}u|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \tag{4.2.13}$$

for any ball B with $2B \subset \Omega$, where the constant $C_\varepsilon > 0$ depends only on ε.

Proof. Fix any ball B with $2B \subset \Omega$, then by Lemma 4.2.6, we have

$$\Delta(f_o - a) \leqslant -\cos\rho |\,\mathrm{d}u|_{\mathrm{HS}}^2, \quad \forall a \in \mathbb{R},$$

where $f_o(x) = \cos(\mathrm{d_Y}(u(x), o))$. Let $\phi : \Omega \to [0,1]$ be a cut-off function with $\phi = 1$ on B, $\phi = 0$ out of $\frac{3}{2}B$, and

$$|\nabla\phi| \leqslant C_1 r^{-1}, \quad |\Delta\phi| \leqslant C_2 r^{-2},$$

where the constants C_1, C_2 depend only on K, N and $\mathrm{diam}(\Omega)$. Then we get

$$\int_B |\,\mathrm{d}u|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \leqslant \int_{\frac{3}{2}B} |\,\mathrm{d}u|_{\mathrm{HS}}^2 \phi \,\mathrm{d}\mathfrak{m} \leqslant \frac{C_3}{r^2} \int_{\frac{3}{2}B} |f_o - a| \,\mathrm{d}\mathfrak{m},$$

for all $a \in \mathbb{R}$, where $C_3 = C_2/\cos\rho$. It is well-known that a weak $(1,2)$-Poincaré inequality holds on $\mathrm{RCD}(K,N)$ spaces and since the weak $(1,s)$-Poincaré inequality is an open ended condition (see [KZ08, Theorem 1.0.1]), there exists a number $s_0 \in (1,2)$ such that the weak $(1,s_0)$-Poincaré inequality holds on $\mathrm{RCD}(K,N)$ spaces. Therefore, we have

$$\inf_{a\in\mathbb{R}} \fint_{\frac{3}{2}B} |f_o - a| \,\mathrm{d}\mathfrak{m} \leqslant C_{K,N,\mathrm{diam}(\Omega),s_0} \cdot r \left(\fint_{2B} |\nabla f_o|^{s_0} \,\mathrm{d}\mathfrak{m} \right)^{1/s_0}.$$

Combining the above two inequalities, we conclude that

$$\left(\fint_B |\,\mathrm{d}u|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \right)^{1/2} \leqslant C_4 \left(\fint_{2B} |\nabla f_o|^{s_0} \,\mathrm{d}\mathfrak{m} \right)^{1/s_0} \leqslant C_4 \left(\fint_{2B} |\,\mathrm{d}u|_{\mathrm{HS}}^{s_0} \,\mathrm{d}\mathfrak{m} \right)^{1/s_0},$$

where we have used $|\nabla f_o| \leqslant |\sin \mathrm{d}_Y(o,u)| \cdot |\nabla \mathrm{d}_Y(o,u)| \leqslant |\,\mathrm{d}u|_{\mathrm{HS}}$. Now, applying 4.2.14 to $|\,\mathrm{d}u|_{\mathrm{HS}}^{s_0}$, we obtain $|\,\mathrm{d}u|_{\mathrm{HS}}$ is in $W^{2+\varepsilon}_{\mathrm{loc}}(\Omega)$, and moreover

$$\left(\fint_B |\,\mathrm{d}u|_{\mathrm{HS}}^{2+\varepsilon} \,\mathrm{d}\mathfrak{m} \right)^{1/(2+\varepsilon)} \leqslant C_\varepsilon \left(\fint_{2B} |\,\mathrm{d}u|_{\mathrm{HS}}^{s_0} \,\mathrm{d}\mathfrak{m} \right)^{1/s_0} \leqslant C_\varepsilon \left(\fint_{2B} |\,\mathrm{d}u|_{\mathrm{HS}}^{2} \,\mathrm{d}\mathfrak{m} \right)^{1/2},$$

since $s_0 < 2$ and Hölder inequality. □

4.2.3 Auxiliary results

In this section we shall work under the following assumptions:

1. $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space with essential dimension $d \in \mathbb{N}$.
2. $\Omega \subset \mathsf{X}$ is an open bounded set with $\mathfrak{m}(\mathsf{X} \setminus \Omega) > 0$.
3. $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ is a $\mathrm{CAT}(1)$ space: the results obtained for general $\mathrm{CAT}(\kappa)$ spaces will be obtained by a rescaling of the distance function.
4. $u \in \mathrm{KS}(\Omega;\mathsf{Y})$ is harmonic with values in a ball $B_\rho(o) \subset \mathsf{Y}$ with $\rho < \frac{\pi}{2}$. Finally we shall fix Borel representatives of u (the Hölder continuous one) and of $e_2[u]$ (and of $|\,\mathrm{d}u|_{\mathrm{HS}}$).
5. Let $\Omega' \subset\subset \Omega$ be open and consider $r > 0$ and $\hat{x} \in \Omega'$ such that $B_{4r}(\hat{x}) \subset \Omega'$ and $\|u\|_{C^\alpha(\Omega')} r^\alpha < \pi/10$. Finally call $B = B_r(\hat{x})$ $2B := B_{2r}(\hat{x})$ and $B' = B_{3r/2}(\hat{x})$.

Let us first define $F : \mathbb{R} \to \mathbb{R}$ as the following

$$F(t) := 2\sin\left(\frac{t}{2}\right) + 4\sin^2\left(\frac{t}{2}\right)$$

and observe that F is such that $F', F'' \geq 0$ on $[0, \pi/2]$. With a little abuse of notation let us also set

$$F(z,w) := 2\sin\left(\frac{\mathrm{d}_\mathsf{Y}(z,w)}{2}\right) + 4\sin^2\left(\frac{\mathrm{d}_\mathsf{Y}(z,w)}{2}\right)$$

for any $z, w \in \mathsf{Y}$.

We introduce the following quantities, in order to produce an Hopf-Lax formula for the function u.

$$f(x,y) = \begin{cases} -F(u(x),u(y)) & \text{if } x,y \in B' \\ -6 & \text{otherwise.} \end{cases}$$

Notice that f is lower semiconinuous since F is bounded between 0 and 6. We call f_t the p-Hopf-Lax semigroup applied to the function f, namely we set

$$f_t(x) := \inf_{y \in \mathsf{X}} \left[\frac{\mathrm{d}^p(x,y)}{pt^{p-1}} + f(x,y) \right], \tag{4.2.14}$$

where we avoid to include p in the definition of f_t to lighten the notation. Notice that $0 \geq f_t(x) \geq -6$ for every $x \in \mathsf{X}$. Moreover the infimum in (4.2.14) is actually a minimum (this follows by Weierstrass theorem exploiting the semicontinuity of the function we are minimizing). We also have a quantitative estimate for where to find a minimum, indeed denoting with $y_{t,x}$ a minimizer for $f_t(x)$, choosing x as a competitor, we get

$$f_t(x) \leq \frac{\mathrm{d}^2_{\mathsf{X}}(x, y_{t,x})}{2t} + f(x, y_{t,x}) \leq 0.$$

This means $\mathrm{d}_{\mathsf{X}}(x, y_{t,x}) \leq 12\sqrt{t}$ so that there exists $t_* = t_*(p) > 0$ such that we have

$$f_t(x) := \inf_{y \in B_{12\sqrt{t}}(x)} \left[\frac{\mathrm{d}^p_{\mathsf{X}}(x,y)}{pt^{p-1}} - F(u(x),u(y)) \right] \qquad \forall x \in B$$

for $t \in (0, t_*)$.

Now set

$$S_t(x) := \left\{ y \in \mathsf{X} : \; f_t(x) = \frac{\mathrm{d}^p_{\mathsf{X}}(x,y)}{pt^{p-1}} - F(u(x),u(y)) \right\}$$

and observe that the latter set is non-empty if $t < t_*$. Finally set

$$L_t(x) := \min_{y \in S_t(x)} \mathrm{d}_{\mathsf{X}}(x,y) \qquad \text{and} \qquad D_t(x) := \frac{L_t^p(x)}{pt^{p-1}} - f_t(x).$$

We now present a slight modification of [ZZZ19, Lemma 4.1] since we still don't know that the map u is Lipschitz continuous but we have Hölder regularity instead: if the map is assumed to be Lipschitz the proof works in the same way replacing α with 1.

Lemma 4.2.16. *With the above notation and assumptions:*

1. *f_t is Hölder continuous on B.*

2. *L_t and D_t are lower semicontinuous.*

3. *There exists a constant $C = C(p, \|u\|_{C^\alpha}, k) > 0$ such that*

$$L_t \leq Ct^{\beta}, \qquad D_t \leq \tilde{C}t^{\beta'}, \qquad -f_t \leq \tilde{C}t^{\beta'} \quad \text{on } B \tag{4.2.15}$$

for some $\beta, \beta' < 1$ and constant C which depends on p, the Hölder norm of u $\|u\|_{C^\alpha}$. If u is Lipschitz we have $\alpha = \beta = \beta' = 1$.

Proof. The proof of (1) is immediate since the infimum of equi-Hölder functions is Hölder.

The proof of (2) is contained in [ZZZ19, Lemma 4.1].

For the proof of (3) consider $y_t(x) \in S_t(x)$ such that $L_t(x) = \mathrm{d}_{\mathsf{X}}(x, y_t)$. We get, using $\sin\theta \le \theta$ for $\theta > 0$ and that $\mathrm{d}_{\mathsf{Y}}(u(x), u(y_t)) \le \pi$,

$$D_t(x) = \frac{L_t^p(x)}{pt^{p-1}} - f_t(x) = F(x, y_t) \le \mathrm{d}_{\mathsf{Y}}(u(x), u(y_t)) + \mathrm{d}_{\mathsf{Y}}^2(u(x), u(y_t))$$
$$\le (1+\pi)\mathrm{d}_{\mathsf{Y}}(u(x), u(y_t)) \le (1+\pi)\|u\|_{C^\alpha} L_t^\alpha(x) = CL_t^\alpha(x).$$

At the same time, being $f_t \le 0$ we have

$$\frac{L_t^p(x)}{pt^{p-1}} \le D_t(x) \le CL_t^\alpha(x)$$

so that we get $L_t \le Ct^\beta$, with $\beta = (p-1)/(p-\alpha)$. For D_t we have instead

$$D_t(x) \le CL_t^\alpha(x) \le \tilde{C}t^{\beta'}$$

with $\beta' = \alpha\beta$. Finally for f_t we have, since $-f_t \le D_t$,

$$-f_t \le \tilde{C}t^{\beta'}.$$

□

We now recall [ZZZ19, Lemma 4.4]:

Lemma 4.2.17. *Let q be such that $1/q + 1/p = 1$. For all $x \in B$ we have*

$$\liminf_{t\to 0} \frac{f_t(x)}{t} \ge -\frac{1}{q}\mathrm{lip}^q u(x). \tag{4.2.16}$$

Moreover for $\mathfrak{m}$-a.e. $x \in B$ (namely any point in B where u is metrically differentiable) we have

$$\lim_{t\to 0^+} \frac{f_t(x)}{t} = -\frac{\mathrm{lip}^q u(x)}{q} \tag{4.2.17}$$

and

$$\lim_{t\to 0^+} \frac{L_t(x)}{t} = \mathrm{lip}^{q/p} u(x), \qquad \lim_{t\to 0^+} \frac{D_t(x)}{t} = \mathrm{lip}^q u(x). \tag{4.2.18}$$

Proof. The proof follows as in [MSa, Proposition 7.5] combined with [ZZZ19, Lemma 4.4]. □

To establish the key variational inequality we shall exploit the following simple but useful lemma

Lemma 4.2.18. *With the above assumptions we have*

$$\Delta f(\cdot, y) \ge 0 \qquad \text{on } B$$

in the weak sense, for all $y \in B$.

Proof. Thanks to the assumptions it is sufficient to compute $\Delta F(u(\cdot), u(y))$ in the weak sense. By the chain rule we get

$$\Delta F(u(\cdot), u(y)) = F''|\nabla \mathrm{d}_{\mathsf{Y}}(u(\cdot), u(y))|^2 + F'\Delta \mathrm{d}_{\mathsf{Y}}(u(\cdot), u(y)),$$

whence the claim follows by the nonnegativity of the factors on the right hand side (recall that the Laplacian of $x \mapsto \mathrm{d}_{\mathsf{Y}}(u(x), u(y))$ is nonnegative thanks to Proposition 4.2.5). □

We now have a lemma on the heat flow Laplacian of the Hopf-Lax semigroup (the idea is from [MSb], see also [Gig23b] and [MSa])

Lemma 4.2.19. *Let $f : \mathsf{X} \to \mathbb{R}$ be a bounded Borel function. Assume that for some $x, y \in \mathsf{X}$ we have*

$$Q_t^p f(x) = f(y) + \frac{\mathrm{d}_{\mathsf{X}}^p(x,y)}{pt^{p-1}}. \tag{4.2.19}$$

Then

$$\Delta Q_t^p f(x) \leq \Delta f(y) - K\frac{\mathrm{d}_{\mathsf{X}}^p(x,y)}{t^{p-1}}. \tag{4.2.20}$$

holds in the heat flow sense.

Proof. First of all let $\pi_s \in \mathcal{P}(\mathsf{X}\times\mathsf{X})$ be an optimal transport plan between $e^{s\Delta}\delta_x \in \mathcal{P}(\mathsf{X})$ and $e^{s\Delta}\delta_y \in \mathcal{P}(\mathsf{X})$ for the cost $\mathrm{d}_{\mathsf{X}}^p$. Moreover we have the following estimate, which is the Wasserstein contractivity of the heat flow (holding in general $\mathrm{RCD}(K,\infty)$ spaces, see [AGS15]),

$$W_p^p(e^{s\Delta}\delta_x, e^{s\Delta}\delta_y) \leq e^{-pKs}\mathrm{d}_{\mathsf{X}}^p(x,y). \tag{4.2.21}$$

We can now estimate as follows

$$\begin{aligned} e^{s\Delta}Q_t f(x) &= \int_{\mathsf{X}} Q_t f(z)\,\mathrm{d}e^{s\Delta}\delta_x(z) = \int_{\mathsf{X}\times\mathsf{X}} Q_t f(z)\,\mathrm{d}\pi_s(z,z') \\ &\leq \int_{\mathsf{X}\times\mathsf{X}} \left[f(z') + \frac{\mathrm{d}^p(z,z')}{pt^{p-1}}\right]\mathrm{d}\pi_s(z,z') \\ \text{(by optimality of } \pi_s) \qquad &= \int_{\mathsf{X}} f(z')\,\mathrm{d}e^{s\Delta}\delta_y(z) + \frac{1}{pt^{p-1}}W_p^p(e^{s\Delta}\delta_x, e^{s\Delta}\delta_y) \\ &= e^{s\Delta}f(y) + \frac{1}{pt^{p-1}}W_p^p(e^{s\Delta}\delta_x, e^{s\Delta}\delta_y). \end{aligned}$$

Finally applying (4.2.21) to the previous inequality we get (note that the following would hold for any $w \in \mathsf{X}$ in place of y)

$$e^{s\Delta}Q_t f(x) \leq e^{s\Delta}f(y) + \frac{e^{-pKs}}{pt^{p-1}}\mathrm{d}_{\mathsf{X}}^p(x,y). \tag{4.2.22}$$

Subtracting (4.2.19) from (4.2.22), dividing by $s > 0$ and taking the $\limsup$ as $s \to 0$ finally gives (4.2.20). □

We now proceed with a refinement of (4.2.7), following [Ser95, Proposition 1.17], which will be crucial for obtaining an elliptic inequality involving the function f_t.

Lemma 4.2.20. *Let $u : \Omega \to \mathsf{Y}$ be an harmonic map with $\Omega \subset \mathsf{X}$ open set, $(\mathsf{Y}, \mathrm{d}_{\mathsf{Y}})$ which is a $\mathrm{CAT}(1)$ space and $\mathrm{Im}(u) \subseteq B_\rho(o)$ with $o \in \mathsf{Y}$, $\rho < \pi/2$. Let further $f_o(x) := \cos(\mathrm{d}_{\mathsf{Y}}(u(x), o))$, then we have $f_o \in W^{1,2}(\Omega)$ and*

$$\Delta f_o \leq -f_o|\,\mathrm{d}u|_{\mathrm{HS}}^2 = -f_o(n+2)e_2^2[u] \qquad \text{in } \Omega \tag{4.2.23}$$

in the weak sense.

Proof. Let us first set $R(x) := \mathrm{d}_{\mathsf{Y}}(u(x), o)$, denote with $x \to G_t^{u(x),o}$ the map which associates to each $x \in \Omega$ the point at time t lying in the geodesic (recall that geodesics are unique in our case) connecting o and $u(x)$. Finally set $u_\eta := G^{u,o}_\eta$ where $\eta \in W^{1,2}(\Omega) \cap C_c(\Omega)$ is such that $0 \le \eta \le 1$: then by [Sak23, Lemma 3.8] we have

$$e_2^2[u_{\eta t}] \le \frac{\sin^2\big[(1-\eta t)R\big]}{\sin^2 R}(e_2^2[u] - e_2^2[R]) + e_2^2[(1-\eta t)R] \tag{4.2.24}$$

$\mathfrak{m}$-a.e. in Ω, where t is a positive parameter that we will eventually send to zero. Now we shall use the duplication formula for the sinus to get

$$|\,\mathrm{d}u_{\eta t}|^2_{\mathrm{HS}} \le \left[\cos^2(t\eta R) + \frac{\sin^2(t\eta R)\cos^2 R}{\sin^2 R} - \frac{\cos R \sin(2t\eta R)}{\sin R}\right](|\,\mathrm{d}u|^2_{\mathrm{HS}} - |\,\mathrm{d}R|^2_{\mathrm{HS}}) + |\,\mathrm{d}R - t\eta\,\mathrm{d}R|^2_{\mathrm{HS}}.$$

Note that we have simultaneously used that $|\,\mathrm{d}u|^2_{\mathrm{HS}} = (n+2)e_2^2[u]$ (recall that if $f : \mathsf{X} \to \mathbb{R}$ then $|\,\mathrm{d}f| = |\,\mathrm{d}f|_{\mathrm{HS}}$). We proceed integrating over Ω, we divide by t and exploit the fact that $E_2(u_{t\eta}) - E_2(u) \ge 0$ (as u is harmonic) together with the asymptotics of the involved functions to get

$$0 \le \int_\Omega \left[-\eta R \frac{\cos R}{\sin R}|\,\mathrm{d}u|^2_{\mathrm{HS}} + \eta R\frac{\cos R}{\sin R}|\,\mathrm{d}R|^2 - \langle\,\mathrm{d}R,\,\mathrm{d}(\eta R)\rangle\right]\mathrm{d}\mathfrak{m}.$$

We can now use the following identity

$$\Big\langle \nabla\Big(\eta\frac{R}{\sin R}\Big), \nabla\cos R\Big\rangle = \eta R\frac{\cos R}{\sin R}|\,\mathrm{d}R|^2 - \langle\,\mathrm{d}R,\,\mathrm{d}(\eta R)\rangle$$

to get

$$0 \le \int_\Omega -\eta R\frac{\cos R}{\sin R}|\,\mathrm{d}u|^2_{\mathrm{HS}} + \Big\langle\nabla\Big(\eta\frac{R}{\sin R}\Big), \nabla\cos R\Big\rangle\,\mathrm{d}\mathfrak{m}.$$

Note that now we can choose the magnitude of η to be whatever we want since the inequality doesn't change if we divide everything by a positive constant. Now pick $\varphi \in \mathrm{Lip}_c(\Omega)$ nonnegative and set $\eta := \varphi R/\sin R$: it is clear that $\eta \in W^{1,2}(\Omega) \cap C_c(\Omega)$ because it is the product of a bounded $W^{1,2}(\Omega)$ and continuous function and a Lipschitz function with compact support. Finally this means that for all $\varphi \in \mathrm{Lip}_c(\Omega)$ nonnegative we have

$$\int_\Omega \varphi\cos R|\,\mathrm{d}u|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \le \int_\Omega \langle\nabla\varphi, \nabla\cos R\rangle\,\mathrm{d}\mathfrak{m}.$$

The latter is the conclusion. □

Finally define some parametric functions depending on the distance of the target space d_Y and deduce some Laplacian bounds on them that we shall exploit later in the proof of the "good" distributional bound.

Lemma 4.2.21. *Let $u : \Omega \subset \mathsf{X} \to \mathsf{Y}$ be an harmonic map with $\mathrm{Im}(u) \subset B_\rho(o)$ and $\rho < \pi/2$. Consider for any $z \in \Omega$ and $y \in \mathsf{Y}$ the function*

$$w_{a,b,y,z}(x) = a\mathsf{d}^2_\mathsf{Y}(u(x), u(z)) + b\cos(\mathsf{d}_\mathsf{Y}(u(x), y)).$$

For $\mathfrak{m}$−a.e. $x_0 \in \Omega$ we have

$$\begin{aligned}\Delta w_{a,b,o,x_0}(x_0) &\le \big(2a - b\cos(\mathsf{d}_\mathsf{Y}(u(x_0), o))\big)(n+2)e_2^2[u](x_0)\\ &= \big(2a - b\cos(\mathsf{d}_\mathsf{Y}(u(x_0), o))\big)|\,\mathrm{d}u|^2_{\mathrm{HS}}(x_0)\end{aligned}$$

in the heat flow sense.

Proof. First of all we shall notice that [MSa, Proposition 3.3] holds also in this setting with the same proof since by Lemma 4.2.12 we have the (Hölder) continuity of u. Therefore we have

$$e^{t\Delta}(\mathrm{d}^2_{\mathsf{Y}}(u(\cdot),u(x_o))(x_0) = 2|\,\mathrm{d}u|^2_{\mathrm{HS}}(x_0)t + o(t) \qquad \text{as } t \to 0^+. \tag{4.2.25}$$

for $\mathfrak{m}$-a.e. x_0. Secondly by the results contained in [GMS23] and Lemma 4.2.20 we have

$$\limsup_{t\to 0} \frac{e^{t\Delta}\cos(\mathrm{d}_{\mathsf{Y}}(u(\cdot),o))(x) - \cos(\mathrm{d}_{\mathsf{Y}}(u(x),o))}{t} \leq -\cos(\mathrm{d}_Y(u(x),o))|\,\mathrm{d}u|^2_{\mathrm{HS}}. \tag{4.2.26}$$

Combining (4.2.25) with (4.2.26) we finally get the thesis. □

4.2.4 A variant of the Bochner-Eells-Sampson inequality

The authors in [ZZ18] are able to prove the Lipschitz continuity of harmonic maps between Alexandrov spaces exploiting the properties of the Hopf-Lax semigroup. Moreover in [ZZZ19], given the Lipschitz continuity of the harmonic map proved in [Ser95], they are able to prove a weak version of the Bochner-Eells-Sampson inequality for maps from a Riemannian domain to a $\mathrm{CAT}(1)$ space. Here we shall exploit the ideas contained in [Gig23b] and fuel them with the ideas of [ZZZ19] (see also [MSa] for the non-smooth counterpart, as in our case) to obtain a variational inequality (the "good" distributional bound) which in the limit will be the desired inequality.

We now recall [Gig23b, Lemma 6.13].

Lemma 4.2.22. *There exists $T > 0$ such that, given a Borel set $E \subset B'$ such that $\mathfrak{m}(B' \setminus E) = 0$, we have: for all $0 < t < T$ there exists $z_t \in B$ such that for $\mathfrak{m}$-a.e. $x \in E \cap B_{4r/3}(\bar{x}) =: E \cap B''$ and every $n \in \mathbb{N}$ the function*

$$y \mapsto g_t(x,y,z_t) := \frac{\mathrm{d}^2(x,y)}{2t} + f(x,y) + \frac{\mathrm{d}^2_{\mathsf{X}}(y,z_t)}{2n}$$

admits a minimizer $T_t(x)$ and such minimizer belongs to the set $E \cap B''$.

Proof. The difference with respect to [Gig23b, Lemma 6.13] lies in the different definition of f, however since the proof follows with minor modifications we decided to omit it. Note moreover that from the proof in [Gig23b] we can infer that Sobolev regularity is not necessary for the function f. It would be sufficient to ask for f to be continuous and with a Laplacian bound $\Delta f \leq L\mathfrak{m}$ in the weak sense. □

We further define

$$f_{t,n}(x) := \inf_{y\in\mathsf{X}} \left[\frac{\mathrm{d}^2(x,y)}{2t} + f(x,y) + \frac{\mathrm{d}^2_{\mathsf{X}}(y,z_t)}{2n}\right]. \tag{4.2.27}$$

We now have the following distributional bound for the function $f_{t,n}$.

Lemma 4.2.23 ("Bad" distributional bound)**.** *Possibly choosing a smaller t_* the following holds. Let $f_{t,n}$ be defined as in (4.2.27): we have*

$$\Delta f_{t,n} \leq C(K,N)\left(\frac{1}{t} + \frac{1}{n}\right)\mathfrak{m} \qquad \text{on } B \tag{4.2.28}$$

in the weak sense, for all $t < t_*$, for all $n \in \mathbb{N}$.

Proof. Fix $y \in B$: combining Theorem 4.1.7 with Lemma 4.2.18 and [Gig23b, Lemma 4.7] we infer the result. □

To obtain the "good" distributional bound we need the following lemma for the function F to be able to let the heat flow and the Hopf-Lax semigroup combine in an efficient way.

Lemma 4.2.24 (Key technical Lemma). *Consider 4 points P, Q, R, S inside $u(B_r(x))$ in such a way that $P := u(x)$, $Q := u(\bar{x})$, $R := u(\bar{y})$, $S := u(y)$. Let us further set $l_0 := 2\sin\frac{\mathrm{d}_{\mathsf{Y}}(Q,R)}{2}$, $l_1 := 2\cos\frac{\mathrm{d}_{\mathsf{Y}}(Q,R)}{2}$, $\alpha := 1/(1+2l_0)$ and finally let $\beta > 0$. We have*

$$F(u(\bar{x}), u(\bar{y})) - F(u(x), u(y)) \leq \frac{\left[w_{a_1,b,Q_m,\bar{x}}(x) - w_{a_1,b,Q_m,\bar{x}}(\bar{x})\right] + \left[w_{a_2,b,Q_m,\bar{y}}(y) - w_{a_2,b,Q_m,\bar{y}}(\bar{y})\right]}{\alpha l_0}, \tag{4.2.29}$$

where Q_m is the middle point of the geodesic joining Q and R,

$$a_1 := 1 - \frac{1-\alpha}{2}\left(1 - \frac{1}{\beta}\right), \quad b := l_1, \quad a_2 := 1 - \frac{1-\alpha}{2}\left(1 - \beta\right)$$

and the function w is defined in Lemma 4.2.21.

Proof. We can apply (4.1.2) to get

$$\begin{aligned}
\alpha l_0\big(F(Q,R) - F(P,S)\big) &= \alpha l_0\left(4\sin^2\frac{\mathrm{d}_{QR}}{2} - 4\sin^2\frac{\mathrm{d}_{PS}}{2}\right) + \alpha l_0\left(2\sin\frac{\mathrm{d}_{QR}}{2} - 2\sin\frac{\mathrm{d}_{PS}}{2}\right)\\
&\leq \left[1 - \frac{1-\alpha}{2}\left(1-\frac{1}{\beta}\right)\right]4\sin^2\frac{\mathrm{d}_{PQ}}{2} + l_1\left(\cos\mathrm{d}_{PQ_m} - \cos\mathrm{d}_{QQ_m}\right)\\
&+ \left[1 - \frac{1-\alpha}{2}\left(1-\beta\right)\right]4\sin^2\frac{\mathrm{d}_{RS}}{2} + l_1\left(\cos\mathrm{d}_{SQ_m} - \cos\mathrm{d}_{RQ_m}\right)\\
&\leq \left[w_{a_1,b,Q_m,\bar{x}}(x) - w_{a_1,b,Q_m,\bar{x}}(\bar{x})\right] + \left[w_{a_2,b,Q_m,\bar{y}}(y) - w_{a_2,b,Q_m,\bar{y}}(\bar{y})\right],
\end{aligned}$$

which concludes the proof. □

The second tool we need is an improvement of the earlier distributional bound: this is the aim of the following proposition.

Proposition 4.2.25 ("Good" distributional bound). *We have*

$$\Delta f_t \leq -K\frac{L_t^p}{t^{p-1}} + \Big(1 + o_t(1)\Big)D_t|\,\mathrm{d}u|^2_{\mathrm{HS}} \quad \text{on } B \tag{4.2.30}$$

in the weak sense, for all $t < t^*$.

Proof. First of all let us recall the definition of $f_{t,n}$

$$f_{t,n}(x) := \inf_{y\in\mathsf{X}}\left[\frac{\mathrm{d}^2_{\mathsf{X}}(x,y)}{2t} + f(x,y) + \frac{\mathrm{d}^2_{\mathsf{X}}(y,z_t)}{2n}\right].$$

Thanks to the Lemma 4.2.22 we can find z_t in such a way that a minimizer of $g_t(x,y,z_t)$, i.e. a point $T_t(x)$ for which $g_t(x, T_t(x), z_t) = f_{t,n}(x)$, lies inside $E \cap B''$ for $\mathfrak{m}$-a.e. $x \in E \cap B''$ and we can choose E to be the set of regular points of the space intersected with the set of Lebesgue points of $|\,\mathrm{d}u|_{\mathrm{HS}}$ (which is clearly of full measure). Now let us fix $\bar{x} \in E \cap B''$ and call $\bar{y}$ the "good" minimiser of $f_{t,n}(x)$. Clearly for such points we have

$$f_{t,n}(\bar{x}) = f(\bar{x},\bar{y}) + \frac{\mathrm{d}^2_{\mathsf{X}}(\bar{x},\bar{y})}{2t} + \frac{\mathrm{d}^2_{\mathsf{X}}(\bar{y},z_t)}{2n} = -F(u(\bar{x}),u(\bar{y})) + \frac{\mathrm{d}^2_{\mathsf{X}}(\bar{x},\bar{y})}{2t} + \frac{\mathrm{d}^2_{\mathsf{X}}(\bar{y},z_t)}{2n}.$$

Now fix any other two points $x, y \in \Omega$. Setting $P := u(x)$, $Q := u(\bar{x})$, $R := u(\bar{y})$, $S := u(y)$. Using the inequality (4.2.29) of the key techcnical lemma (and its notation) we get

$$\begin{aligned} f_{t,n}(x) &= \inf_{y\in\mathsf{X}}\left[\frac{\mathrm{d}^p_{\mathsf{X}}(x,y)}{pt^{p-1}} + f(x,y) + \frac{\mathrm{d}^2_{\mathsf{X}}(y,z_t)}{2n}\right] \\ &= -F(u(\bar{x}),u(\bar{y})) + \inf_{y\in\mathsf{X}}\left[\frac{\mathrm{d}^p_{\mathsf{X}}(x,y)}{pt^{p-1}} + F(u(\bar{x}),u(\bar{y})) - F(u(x),u(y)) + \frac{\mathrm{d}^2_{\mathsf{X}}(y,z_t)}{2n}\right] \\ &\underbrace{\leq}_{(4.2.29)} -F(u(\bar{x}),u(\bar{y})) + \frac{w_{a_1,b,Q_m,\bar{x}}(x) - w_{a_1,b,Q_m,\bar{x}}(\bar{x})}{\alpha l_0} \\ &+ Q_t\left[\frac{w_{a_2,b,Q_m,\bar{y}}(\cdot) - w_{a_2,b,Q_m,\bar{y}}(\bar{y})}{\alpha l_0} + \frac{\mathrm{d}^2_{\mathsf{X}}(\cdot,z_t)}{2n}\right](x), \end{aligned}$$

with equality if $x = \bar{x}$. We now proceed to obtain a bound on the Laplacian of $f_{t,n}$ in the heat flow sense at the point $\bar{x}$, therefore we shall estimate

$$\limsup_{s\to 0^+} \frac{e^{s\Delta}(f_{t,n})(\bar{x}) - f_{t,n}(\bar{x})}{s} = \Delta f_{t,n}(\bar{x}).$$

Exploiting the previous inequalities and the monotonicity of the heat flow ($e^{t\Delta}f \leq e^{t\Delta}g$ if $f \leq g$) we get

$$\Delta f_{t,n}(\bar{x}) \leq \frac{\Delta w_{a_2,b,Q_m,\bar{y}}(\bar{x})}{\alpha l_0} + \Delta Q_t\left[\frac{w_{a_2,b,Q_m,\bar{y}}(\cdot)}{\alpha l_0} + \frac{\mathrm{d}^2_{\mathsf{X}}(\cdot,z_t)}{2n}\right](\bar{x}).$$

Moreover thanks to the properties of the Hopf-Lax semigroup (namely (4.2.20)) we get

$$\Delta Q_t\left[\frac{w_{a_2,b,Q_m,\bar{y}}(\cdot)}{\alpha l_0} + \frac{\mathrm{d}^2_{\mathsf{X}}(\cdot,z_t)}{2n}\right](\bar{x}) \leq \frac{\Delta w_{a_2,b,Q_m,\bar{y}}(\bar{y})}{\alpha l_0} + \frac{1}{n}\Delta \mathrm{d}^2_{\mathsf{X}}(\cdot,z_t)(\bar{y}) - K\frac{L^p_t(\bar{x})}{t^{p-1}}.$$

Now we can apply Lemma 4.2.21 and the Laplacian comparison to obtain

$$\begin{aligned} \Delta f_{t,n}(\bar{x}) \leq \frac{C(K,N,r)}{n} - K\frac{L^p_t(\bar{x})}{t^{p-1}} + \frac{2a_1 - b\cos(\mathrm{d}_{\mathsf{Y}}(u(\bar{x}),Q_m))}{\alpha l_0}|\,\mathrm{d}u|^2_{\mathrm{HS}}(\bar{x}) \\ + \frac{2a_2 - b\cos(\mathrm{d}_{\mathsf{Y}}(u(\bar{y}),Q_m))}{\alpha l_0}|\,\mathrm{d}u|^2_{\mathrm{HS}}(\bar{y}). \end{aligned}$$

Since $\cos(\mathrm{d}_{\mathsf{Y}}(Q_m),\bar{y}) = \cos(\mathrm{d}_{\mathsf{Y}}(Q_m),\bar{x}) = l_1/2$ and $1 - l_1^2/4 = l_0^2/4$ we can choose β such that $a_2 = l_1^2/4$, so that $2a_2 - b\cos(\mathrm{d}_{\mathsf{Y}}(u(\bar{y}),Q_m)) = 0$. This is achieved with

$$\beta = 1 - \frac{l_0(1+2l_0)}{4}.$$

Via standard computations we get

$$\frac{2a_1 - b\cos(\mathrm{d}_{\mathsf{Y}}(Q_m),\bar{x})}{\alpha l_0} = 2l_0(1+2l_0)\left(\frac{1}{4} + \frac{1}{4 - l_0(1+2l_0)}\right).$$

Therefore we get

$$\begin{aligned}\Delta f_{t,n}(\bar{x}) &\le \frac{C(K,N,r)}{n} - K\frac{L_t^p(\bar{x})}{t^{p-1}} + 2l_0(1+2l_0)\left(\frac{1}{4} + \frac{1}{4-l_0(1+2l_0)}\right)|\,\mathrm{d}u|^2_{\mathrm{HS}}(\bar{x}) \\ &\le \frac{C(K,N,r)}{n} - K\frac{L_t^p(\bar{x})}{t^{p-1}} + \left(1+o_t(1)\right)D_t(\bar{x})|\,\mathrm{d}u|^2_{\mathrm{HS}}(\bar{x}).\end{aligned}$$

where we also used that $D_t(\bar{x}) = l_0 + l_0^2$ and that u is Hölder continuous to estimate the remainder in $o_t(1)$ (observe also that $\hat{x}$ does not depend on $n \in \mathbb{N}$). Combining the latter with Lemma 4.2.28 and [Gig23b, Lemma 4.8] (recalling that u is continuous on Ω) we end up with

$$\Delta f_{t,n} \le \frac{C(K,N,r)}{n} - K\frac{L_t^p(\cdot)}{t^{p-1}} + \left(1+o_t(1)\right)D_t(\cdot)|\,\mathrm{d}u|^2_{\mathrm{HS}}(\cdot) \quad \text{on } B$$

in the weak sense, for all $n \in \mathbb{N}$ and for all $t < t_*$.

Now since $f_{t,n}$ converges to f_t uniformly as $n \to \infty$, thanks to the regularity of f_t and the stability of the Laplacian bounds we infer (4.2.30).

□

Theorem 4.2.26 (A variant of the BES inequality). *Let u be as above and assume that it is locally Lipschitz in Ω, then the inequality*

$$\Delta\left(\frac{\mathrm{lip}^2 u}{2}\right) \ge |\nabla \mathrm{lip} u|^2 - K\mathrm{lip}^2(u) - e_2^2[u]\mathrm{lip}^2 u \tag{4.2.31}$$

holds in the very weak sense in Ω.

Proof. By the chain rule it is easy to infer that (4.2.31) is equivalent to

$$\Delta \mathrm{lip} u \ge -K\mathrm{lip} u - e_2^2[u]\mathrm{lip} u. \tag{4.2.32}$$

We shall now verify that there exists a neighborhood $B_R(\bar{x})$ with $B_{2R}(\bar{x}) \subset \Omega$ such that $\mathrm{lip}(u) \in W^{1,2}(B_R(\bar{x}))$ and (4.2.32) holds in the sense of distributions in $B_R(\bar{x})$.

Due to the continuity of u there exists $R > 0$ such that $u(B_{2R}(\bar{x})) \subset B_{\pi/4}(u(\bar{x}))$, so that $\mathrm{diam}(u(B_{2R}(\bar{x}))) < \pi/2$ and $R < r/2$. By (4.2.30) and (4.2.15) we have $\Delta f_t/t \le C(\mathrm{Lip}\, u)$ on B_{2R} for all $t \in (0, t_*)$. Combining the elliptic inequality (4.2.30) with Lemma 4.2.15 and a Caccioppoli inequality we get $f_t/t \in W^{1,2}(B_{3R/2}(x))$ with $\|f_t/t\|_{W^{1,2}(B_{3R/2}(\bar{x}))} \le C$ and C depending only on the Lipschitz norm of u in Ω'. Therefore, exploiting Lemma 4.2.17, up to a subsequence we have that $-f_t/t$ converges weakly in $W^{1,2}$ to $\mathrm{lip}^q(u)/q$ and we get

$$\Delta(\mathrm{lip}^q u/q) \ge K\mathrm{lip}^q u - e_2^2[u] \cdot \mathrm{lip}^q u \tag{4.2.33}$$

in $B_{3R/2}(\bar{x})$ in the weak sense. Exploiting the Lipschitz continuity of u we get

$$\Delta(\mathrm{lip}^q u/q) \ge K(\mathrm{lip} u)^q - (\mathrm{lip} u)^{q+2} \ge -C$$

where the constant is uniform in q. Now again by Caccioppoli inequality we get

$$\|\mathrm{lip}^q u/q\|_{W^{1,2}(B_R)} \le C \quad \text{as } q \to 1.$$

This means that $\mathrm{lip}^q(u)/q$ converges to $\mathrm{lip}(u)$ in $W^{1,2}(B_R(\bar{x}))$ and we can pass to the limit in (4.2.33) and get (4.2.32), whence we also deduce (4.2.31). □

Finally we shall mention that the theorems in [ZZZ19, Section 5] hold also in the present setting: we refer to [ZZZ19] for the proofs which work mutatis mutandis in our context.

Theorem 4.2.27. *Let u be as above but with values in $B_\rho(o) \subset \mathsf{Y}$, where $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ is a $\mathrm{CAT}(\kappa)$ space and $\rho < \pi/2\sqrt{\kappa}$. Then letting $R > 0$ be such that $B_{2R}(x_0) \subset \Omega$ we have*

$$\sup_{x \in B_{R/2}(x_0)} \mathrm{lip}(u)(x) \leq \frac{C_{d,\sqrt{K}R,\pi/(2\sqrt{\kappa}-\rho)}}{R}, \tag{4.2.34}$$

where the constant C only depends on the parameters listed at its subscript.

As a consequence we obtain a Liouville type theorem for harmonic maps, which follows by estimate (4.2.34).

Corollary 4.2.28. *Let $(\mathsf{X}, \mathrm{d}_\mathsf{X}, \mathfrak{m}_X)$ be an $\mathrm{RCD}(0, N)$ space and $(\mathsf{Y}, \mathrm{d}_\mathsf{Y})$ be a $\mathrm{CAT}(\kappa)$ space. Consider an harmonic map $u : \mathsf{X} \to \mathsf{Y}$ such that $u(\mathsf{X}) \subset B_\rho(o)$ for some $o \in \mathsf{Y}$ and $\rho < \pi/(2\sqrt{\kappa})$ with sublinear growth, i.e.*

$$\liminf_{R \to \infty} \frac{\sup_{y \in B_R(x_0)} \mathrm{d}_\mathsf{Y}(u(y), o)}{R} = 0$$

for some $o \in \mathsf{Y}$. Then u must be a constant map.

4.2.5 Boundary regularity for harmonic maps

In this section, we continue to assume that $\Omega \subset X$ is an open bounded set in an $\mathrm{RCD}(K, N)$ space with $X \setminus \Omega \neq \emptyset$, $K \in \mathbb{R}$ and $N \in [1, \infty)$. Moreover we let $(Y, \mathrm{d}_\mathsf{Y})$ be a $\mathrm{CAT}(\kappa)$ space with $\kappa > 0$.

To study the boundary regularity of harmonic maps, we shall also impose some regularity conditions on the boundary of Ω.

Definition 4.2.29. Let $\Omega \subset X$ be a domain. We say that Ω satisfies an *exterior density condition* if there exist two numbers $\lambda \in (0, 1)$ and $R_0 > 0$ such that

$$\mathfrak{m}(\Omega \setminus B_r(x)) \geqslant \lambda \cdot \mathfrak{m}(B_r(x)) \quad \forall\, x \in \partial\Omega, \ \ \forall r \in (0, R_0). \tag{4.2.35}$$

Additionally we say that Ω satisfies a *uniform exterior sphere condition* if there exists a number $R_0 > 0$ such that for each $x_0 \in \partial\Omega$ there exists a ball $B_{R_0}(y_0)$ satisfying

$$\Omega \cap B_{R_0}(y_0) = \emptyset \quad \text{and} \quad x_0 \in \partial B_{R_0}(y_0). \tag{4.2.36}$$

Remark 4.2.30. It is easy to see that if the space satisfies a volume doubling condition (which is the case of $\mathrm{RCD}(K, N)$ spaces, thanks to Bishop-Gromov inequality), then the exterior density condition is implies by the exterior sphere condition.

The main result of this section is the following.

Theorem 4.2.31. *Let Ω and Y be as above. Suppose that $\Omega \subset X$ satisfies a uniform exterior sphere condition with constant R_0 and let $w \in \mathrm{Lip}(\overline{\Omega}, Y)$. Let $u \in \mathrm{KS}^{1,2}(\Omega, Y)$ be an harmonic map with boundary data w such that $\mathrm{Im}(u) \subset B_{\pi/4-\rho}(o)$ for some $o \in Y$ and $\rho > 0$. Then for any $\varepsilon \in (0, 1)$ it holds*

$$\mathrm{d}_\mathsf{Y}\big(u(x), w(x_0)\big) \leqslant C_\varepsilon L_w \mathrm{d}_\mathsf{X}^{1-\varepsilon}(x, x_0) \tag{4.2.37}$$

for all $x_0 \in \partial\Omega$ *and* $x \in \Omega$ *with* $\mathrm{d}_X(x, x_0) < R_\varepsilon$*, where both* R_ε *and* C_ε *depend only on* ε, N, K *and* $\mathrm{diam}(\Omega)$*, and*

$$L_w := \sup_{x,y\in\overline{\Omega}} \frac{\mathrm{d}_{\mathsf{Y}}\big(w(x), w(y)\big)}{\mathrm{d}_X(x, y)}.$$

In particular, u *is continuous at* x_0 *and* $u(x_0) = w(x_0)$*.*

To prove this result, we need the following two lemmas.

Lemma 4.2.32. *Let* $\Omega \subset X$ *be a bounded domain satisfying a uniformly exterior condition with constant* R_0*. Suppose that* $f \in W^{1,2}(\Omega)$ *is a harmonic function on* Ω *with boundary data* $g \in \mathrm{Lip}(\overline{\Omega})$*. Suppose* $g(z_0) = 0$ *for some* $z_0 \in \overline{\Omega}$*. Then for any* $\varepsilon \in (0, 1)$*, there exists a number* $R_\varepsilon \in (0, \min\{1, R_0/2\})$ *(depending only on* ε, N, K *and* $\mathrm{diam}(\Omega)$*) such that for any ball* $B_r(x_0)$ *with* $x_0 \in \partial\Omega$ *and* $r \in (0, R_\varepsilon)$ *it holds*

$$\sup_{B_r(x_0)\cap\Omega} |f(x) - f(x_0)| \leqslant C_\varepsilon L \cdot r^{1-\varepsilon}, \tag{4.2.38}$$

where the constant $C_\varepsilon > 0$ *depending only on* ε, N, K*, and the constant* L *is a Lipschitz constant of* g*.*

Proof. This is Theorem 4.3 in [ZZ24]. □

Lemma 4.2.33. *Let* Ω, Y *be as above. Suppose that* $u : \Omega \to Y$ *is an harmonic map. Then for any* $P \in Y$ *such that* $\mathrm{Im}(u) \in B_{\pi/2-\rho}(P)$ *it holds*

$$\Delta \mathrm{d}_{\mathsf{Y}}\big(u(x), P\big) \geqslant 0 \tag{4.2.39}$$

in the sense of distributions.

Proof. Since the function $\mathrm{d}_{\mathsf{Y}}(P, \cdot)$ is convex in $B_{\pi/2}(P) \subset Y$, the assertion follows directly from Proposition 4.2.5. □

We are now in the position to prove Theorem 4.2.31, whose proof is a modification of the one in [ZZ24, Theorem 4.6].

Proof of Theorem 4.2.31. Fix any a point $x_0 \in \partial\Omega$, and set $P = w(x_0)$. Then, by the triangle inequality and the fact that $\mathrm{Im}(u) \subset B_{\pi/4-\rho}(o)$, we have $\mathrm{d}_{\mathsf{Y}}(P, u(x)) \leqslant \pi/2 - 2\rho$ for any $x \in \Omega$. Moreover by Lemma 4.2.33, we observe that $\mathrm{d}_{\mathsf{Y}}(P, u(x))$ is sub-harmonic on Ω.

We can now solve the Dirichlet problem

$$\Delta f(x) = 0 \quad \text{on } \Omega \qquad \text{and} \qquad f(x) - \mathrm{d}_{\mathsf{Y}}(w(x_0), w(x)) \in W_0^{1,2}(\Omega).$$

Notice that, by the triangle inequality, the function $g_{x_0}(x) := \mathrm{d}_{\mathsf{Y}}(w(x_0), w(x))$ is Lipschitz continuous on $\overline{\Omega}$ with a Lipschitz constant

$$L_{g_{x_0}} \leqslant L_w \quad \text{and} \quad g_{x_0}(x_0) = 0.$$

According to Lemma 4.2.32, we have

$$\sup_{B_r(x_0)\cap\Omega} |f(x) - f(x_0)| \leqslant C_\varepsilon L_w r^{1-\varepsilon}, \tag{4.2.40}$$

for any ball $B_r(x_0)$ with $x_0 \in \partial\Omega$ and $r \in (0, R'_\varepsilon)$.

At last, since $\mathrm{d}_{\mathsf{Y}}(u(x), w(x_0)) - f(x)$ is sub-harmonic on Ω, and

$$[\mathrm{d}_{\mathsf{Y}}(u(x), w(x_0)) - f(x)]^+ \in W_0^{1,2}(\Omega),$$

the maximum principle yields

$$\mathrm{d}_{\mathsf{Y}}(u(x), w(x_0)) \leqslant f(x), \quad \text{a.e. in } \Omega.$$

Noticing that $u \in C(\Omega)$ (by Theorem 4.2.12) and $f \in C(\Omega)$, we get

$$\mathrm{d}_{\mathsf{Y}}(u(x), w(x_0)) \leqslant f(x), \quad \forall x \in \Omega.$$

The combination of the latter with (4.2.40) implies the desired result, concluding the proof. □

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