

MATHEMATICS AREA – PHD COURSE IN
GEOMETRY AND MATHEMATICAL PHYSICS

REFLECTION POSITIVITY,
MAGNETIC LATTICE LAPLACIANS AND
EMERGENT PHASES IN LATTICE
FERMIONIC SYSTEMS

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To those who helped me become
a better physicist,
a better mathematician
and, hopefully, a better man.

*Non puoi dare di più di quello che hai perché è impossibile.
Non puoi dare di meno perché è inaccettabile.*

— Kaos

Abstract

Emergent phenomena in interacting fermionic systems often arise as collective many-body effects whose formation can be understood by studying the interplay between geometry and symmetry. Understanding such phenomena from first principles remains one of the central challenges of Mathematical Physics, particularly in regimes where perturbative methods fail and non-trivial phases emerge through strongly correlated dynamics.

This thesis investigates a class of lattice fermionic systems exhibiting *Spontaneous Symmetry Breaking*, *Flux ordering* and *Topological Order*. Although the models considered belong to different physical frameworks, including Euclidean lattice field theories formulated in terms of Grassmann variables and Quantum Lattice Systems described by operators acting on Hilbert spaces, they share a common mathematical structure. In each case, geometric features encoded by lattice Dirac operators, magnetic fluxes, and discrete gauge fields interact with Reflection Positivity properties that allow for a rigorous non-perturbative analysis.

A central role throughout the thesis is indeed played by *Reflection Positivity* and one of its most striking consequences: the *Chessboard Estimates*. These methods provide powerful tools for converting geometric information into quantitative bounds on energies, correlation functions, and phase stability.

The first part of the thesis, based on [FG26], is devoted to lattice Gross–Neveu models in two dimensional Euclidean Spacetime. After a Hubbard–Stratonovich transformation, the fermionic theories are mapped to effective bosonic systems for which Reflection Positivity can be established. Combining Reflection Positivity methods with Peierls’ Argument, we prove the existence of Chiral Long-Range Order for a class of lattice regularizations and obtain quantitative bounds relating the order parameter to the minimizers of the effective potential.

The second part, based on [GP26], focuses on fermions coupled to dynamical $\mathbb{Z}_2$ -gauge fields. We prove the stability of the π -flux phase under gauging by showing that monopole excitations possess a strictly positive energy cost. These results imply the emergence of effective Dirac fermions at low energies and provide a rigorous characterization of the semimetallic phase of the model.

Finally, we investigate the topological phase obtained by introducing a fermionic mass gap in the π -flux phase, following [BGP26]. We prove the existence of an almost four-fold degenerate low-energy sector on the torus, separated from the rest of the spectrum by a uniform gap, and construct quasi-local loop and string operators whose algebra exhibits the characteristic toric-code braiding phases. This provides a rigorous realization of topological order emerging from interacting fermions coupled to a dynamical gauge field.

Taken together, these results illustrate how Reflection Positivity methods and spectral properties of the Magnetic Laplacian can be combined to obtain rigorous information on the phase diagram of interacting fermionic systems. They reveal a common mechanism underlying symmetry breaking, flux stabilization, and topological order across a variety of lattice models.

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di tutti gli anni passati,
dei pochi risultati concreti e
di tutti i giorni buttati..."*
Ape – Giorni Matti

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faran sempre parte della mia storia e dei mie ricordi
incido il mio presente perché viva nel futuro..."*

Duplici – 21 Grammi

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Part I

Foundations

Introduction

This thesis addresses three non-perturbative problems in lattice fermionic systems. The first is the formation of chiral long-range order in Euclidean lattice Gross–Neveu models, studied in [Chapter 4](#). The second, discussed in [Chapter 6](#), is the selection and stability of the uniform π -flux phase for fermions coupled to a $\mathbb{Z}_2$ -gauge field and their response property to external magnetic fields. The third is the emergence of Topological order and Abelian anyons in a massive deformation of the same gauge–Matter system, developed in [Chapter 7](#). These problems represent three different forms of collective organization: Spontaneous Symmetry Breaking detected through long-distance correlations, Flux order selected by an energetic competition between gauge fields and fermionic Matter and Topological order encoded in the global structure of the low-energy space.

The three problems belong to two mathematically distinct settings, introduced systematically in [Chapter 2](#). The Gross–Neveu models are Euclidean lattice field theories whose observables form a Grassmann algebra and whose expectations are defined by Berezin integration. The two gauge–matter models are Hamiltonian quantum systems defined on a spatial lattice, with fermionic observables acting on Fock space and gauge degrees of freedom associated with the lattice edges. The Euclidean and Hamiltonian formulations are not identified with one another in the applications. Instead, they are developed in parallel so that their common geometric features can be isolated without obscuring the differences between their algebraic implementations.

The Hamiltonian coupling between fermionic Matter and lattice gauge fields is recalled in [Chapter 5](#). In the models studied in [Chapter 6](#) and [Chapter 7](#), the gauge field is dynamical in the statistical-mechanical sense: the link variables are local degrees of freedom of the model and the partition function contains a sum over their configurations, rather than being evaluated in a prescribed external background. Since the electric-field term is absent, the Hamiltonian is block diagonal in the gauge configurations basis. The analysis can therefore be performed sector by sector, after which the gauge configurations are summed and the Gauss-law constraint is imposed.

Reflection positivity A common methodological ingredient is reflection positivity, whose Euclidean and Hamiltonian formulations are reviewed in [Chapter 3](#). Its role, however, is different in the three applications. In [Chapter 4](#), Grassmann reflection positivity is transferred to the auxiliary-field measure and, through chessboard estimates, controls the probability of disseminated defects; a Peierls argument then yields chiral long-range order. In [Chapter 6](#), the fermionic reflection positivity transformation is applied within fixed gauge sectors to compare partition functions in different flux configurations, select the uniform π -flux background, and obtain an extensive energy gap for

magnetic defects. In [Chapter 7](#), the strategy developed in [Chapter 6](#) provides the starting point for the construction of the topologically ordered phase. The proof additionally requires the spectral analysis of the massive magnetic lattice Hamiltonian, a fermionic cluster expansion, the Gauss-law projection, Lieb–Robinson bounds, and quasi-adiabatic flow.

In all three applications, the effectiveness of the chessboard estimate relies on the same mechanism: repeated reflections replace an arbitrary local defect by a periodic configuration. The resulting one-particle operators can then be analyzed using Fourier Transforms. In [Chapter 4](#) these are lattice Dirac operators coupled to periodic multiplication potentials, whereas in [Chapter 6](#) and [Chapter 7](#) they are discrete magnetic Laplacians with periodic flux background, with an additional periodic mass potential in the latter case. The relevant energetic penalties are extracted by controlling the eigenvalues of the corresponding finite-dimensional Bloch matrices.

Organization of the Thesis. The thesis is organized accordingly. In [Chapter 2](#) we introduce the Euclidean and the Hamiltonian descriptions of lattice fermionic systems. A general and quick account on the method of reflection positivity and its main consequences are then presented in [Chapter 3](#). In [Chapter 4](#) we discuss lattice regularizations of the Gross-Neveu model, and the proof of chiral long-range order. The last part of the thesis begins with the introduction of the general gauge-matter formalism of [Chapter 5](#), which is the starting points for two applications: the stability of the π -flux phase in [Chapter 6](#) and the construction of the massive topologically ordered phase in [Chapter 7](#).

The following three sections present the original contributions in greater detail. For each work, we describe the physical and mathematical problem, define the model, state the principal results, outline the architecture of the proof, and discuss the main questions that remain open.

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1.1 Chiral Long-Range Order in Three Euclidean Lattice Gross-Neveu Models

This section summarizes the results obtained in joint work with Simone Fabbri [\[FG26\]](#). The problem is to establish, by non-perturbative methods, chiral long-range order in two-dimensional Euclidean lattice models of interacting fermions and to compare the resulting order parameter with its Large- N mean-field prediction.

The general problem. Relativistic quantum field theories with four-fermion interactions provide a natural framework for studying dynamical mass generation and spontaneous chiral Symmetry Breaking. A prototypical example is the Nambu–Jona-Lasinio model [\[NJ61a; NJ61b\]](#), originally introduced as an effective theory of interacting nucleons and subsequently recognized as a precursor

of modern mechanisms of chiral Symmetry Breaking. Such models have been investigated through Large- N expansions [Kle92; HK94; Ber+25], Functional Renormalization Group Methods [BTW02; Bra12], and Lattice Regularizations [BV93]. Rigorous results are much rarer; an important example is [SS91], where chiral Symmetry Breaking was established by reflection positivity methods.

The Gross–Neveu model [GN74] can be viewed as a paradigmatic example within this framework, retaining the essential features of Spontaneous Symmetry Breaking while offering a more tractable setting for the analysis of non-perturbative phenomena. It describes N flavours of Dirac fermions coupled through a local quartic interaction in the scalar bilinear $\bar{\psi}\psi$. In $1 + 1$ dimensions it is renormalizable and asymptotically free. Its formal Large- N solution predicts the dynamical generation of a mass scale, a non-trivial bound-state spectrum, and the spontaneous breaking of a discrete chiral symmetry [GN74; DHN75]. The corresponding order parameter is the scalar condensate $\langle\bar{\psi}\psi\rangle$.

The Gross–Neveu model has also played a central role in the development of constructive and renormalization group techniques for fermionic field theories. Its ultraviolet construction was carried out rigorously in [Fel+85; GK85; DR02]; see also [DY24; Duc24; Dim25]. Its infrared behaviour was analysed by Kopper, Magnen, and Rivasseau [KMR95] through a combination of cluster expansions and Large- N estimates for a suitably regularized continuum model. A related strategy was used for mass generation in non-linear sigma models in [Kop99]. Nevertheless, there is presently no rigorous proof of chiral Symmetry Breaking for the Gross–Neveu model after removal of the ultraviolet cutoff.

At fixed ultraviolet cutoff, our analysis of the infrared regime is based on reflection positivity and its associated chessboard estimates. Reflection Positivity is first verified separately for each lattice discretization and is then used to control the probability of spatially disseminated defects. Combined with a Peierls argument, these estimates yield quantitative bounds on the long-distance correlations of the order parameter. This approach is more direct than the cluster-expansion method of [KMR95], but it is also more sensitive to the microscopic form of the action, since reflection positivity is a very fragile property. The reflection positivity framework used in this work is reviewed in Chapter 3.

The result and the role of the three discretizations. We address the infrared problem at fixed ultraviolet cutoff. More precisely, we prove long-range order in three Euclidean lattice regularizations of the two-dimensional Gross–Neveu model, for sufficiently weak coupling and a sufficiently large, even number of fermion flavours. The models, originally considered in [CER81; CER83], are:

1. the naive-fermion model, denoted by NN;
2. the staggered-fermion model with on-site interaction, denoted by SN;
3. the Staggered-fermion model with plaquette interaction, denoted by SP.

They have different Dirac operators, internal degrees of freedom, and ultraviolet interpretations. The Naive model suffers from fermion doubling, the Staggered on-site model is particularly simple but does not reproduce the standard Gross–Neveu continuum action, while the Plaquette model has the appropriate formal continuum limit [CER83; HKK93]. Treating the three models simultaneously therefore tests whether the ordering mechanism depends on a special discretization or instead follows from a common structural property.

To make this structure transparent, we introduce the common notation before recalling the three actions. Their microscopic differences are encoded in the lattice Dirac operators and in the

realization of the internal index $J \in \mathcal{J}$ introduced in [Subsection 2.1.1](#). Despite the differences at the microscopic level, these lattice realizations of the Gross-Neveu model present a quartic interaction term admitting an Hubbard–Stratonovich representation. This separation also mirrors the proof: establishing reflection positivity is model dependent, since it requires a detailed analysis of the hopping terms and boundary signs, whereas the subsequent chessboard and Peierls arguments depend only on the common effective bosonic structure.

Common lattice formulation. Following the conventions of the Euclidean formalism, we set the lattice spacing equal to one and work on the two-dimensional discrete torus

$$\Gamma_L \doteq \mathbb{Z}^2 \cap \left(-\frac{L}{2}, \frac{L}{2} \right]^2, \quad L \in 4\mathbb{N},$$

For the plaquette model we further assume $L \in 8\mathbb{N}$ and introduce the coarse lattice

$$\tilde{\Gamma}_L \doteq (2\mathbb{Z})^2 \cap \left(-\frac{L}{2}, \frac{L}{2} \right]^2.$$

We use the notation of the Gross–Neveu chapter

$$\Gamma_L^\alpha \doteq \begin{cases} \Gamma_L, & \alpha = \text{NN}, \\ \Gamma_L, & \alpha = \text{SN}, \\ \tilde{\Gamma}_L, & \alpha = \text{SP}, \end{cases}$$

and $\mathcal{I} \doteq \{A, B, C, D\}$ for the four positions inside a 2×2 unit cell of the SP model. In terms of the general internal index $J \in \mathcal{J}$, the three realizations are

$$J = \begin{cases} (s, j), & \alpha = \text{NN} \quad j \in \{1, \dots, N\} \quad s \in \{1, 2\}, \\ j, & \alpha = \text{SN} \quad j \in \{1, \dots, N\} \\ (a, j), & \alpha = \text{SP} \quad j \in \{1, \dots, N\} \quad a \in \mathcal{I}. \end{cases} \quad (1)$$

Thus $\psi_{x,j}$ and $\bar{\psi}_{x,j}$ are two-component spinors in the NN model, scalar Grassmann variables in the SN model, and are written as $\psi_{x,a,j}$ and $\bar{\psi}_{x,a,j}$ in the SP model, with $x \in \tilde{\Gamma}_L$ and $a \in \mathcal{I}$.

The fermionic fields satisfy antiperiodic boundary conditions in both lattice directions. Equivalently, one may use periodic Grassmann variables and an antiperiodic extension $(\not{D}_\alpha)_{\Gamma_L}^-$ of the corresponding lattice Dirac operator $\not{D}_\alpha$. On the torus, antiperiodicity is represented by a $\mathbb{Z}_2$ -holonomy line. In the proof of reflection positivity, its position must be chosen compatibly with the reflection plane: a change of reflection plane is accompanied by a $\mathbb{Z}_2$ -gauge transformation moving the holonomy defect.

We indicate the scalar fermionic bilinear appearing in the interaction by:

$$(\bar{\psi}\psi)_x \doteq \begin{cases} \sum_{j=1}^N \sum_{s=1}^2 \bar{\psi}_{x,s,j} \psi_{x,s,j}, & \alpha = \text{NN}, \\ \sum_{j=1}^N \bar{\psi}_{x,j} \psi_{x,j}, & \alpha = \text{SN}, \\ \sum_{j=1}^N \sum_{a \in \mathcal{I}} \bar{\psi}_{x,a,j} \psi_{x,a,j}, & \alpha = \text{SP}. \end{cases} \quad (2)$$

The three actions are:

$$\begin{aligned} S_{\text{NN}}(\bar{\psi}, \psi) &= \sum_{x,y \in \Gamma_L} \sum_{j=1}^N \bar{\psi}_{x,j} ((\mathcal{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j} - \frac{\lambda}{2N} \sum_{x \in \Gamma_L} (\bar{\psi}\psi)_x^2, \\ S_{\text{SN}}(\bar{\psi}, \psi) &= \sum_{x,y \in \Gamma_L} \sum_{j=1}^N \bar{\psi}_{x,j} ((\mathcal{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j} - \frac{\lambda}{2N} \sum_{x \in \Gamma_L} (\bar{\psi}\psi)_x^2, \\ S_{\text{SP}}(\bar{\psi}, \psi) &= \sum_{x,y \in \tilde{\Gamma}_L} \sum_{a,b \in \mathcal{I}} \sum_{j=1}^N \bar{\psi}_{x,a,j} ((\mathcal{D}_{\text{SP}})_{\tilde{\Gamma}_L}^-)_{(x,a),(y,b)} \psi_{y,b,j} - \frac{\lambda}{2N} \sum_{x \in \tilde{\Gamma}_L} (\bar{\psi}\psi)_x^2. \end{aligned}$$

Each action is invariant under its corresponding discrete chiral transformation, under which $(\bar{\psi}\psi)_x$ changes sign. We denote by $\langle \cdot \rangle_\alpha$ the Grassmann expectation associated with S_α , according to the general definition of [Subsection 2.1.1](#).

Auxiliary-field representation and order parameter. The common interaction in [Equation \(2\)](#) allows us to apply at every site the Hubbard–Stratonovich identity

$$e^{\frac{\lambda}{2N} (\bar{\psi}\psi)_x^2} = \sqrt{\frac{N}{2\pi\lambda}} \int_{-\infty}^{+\infty} d\phi_x e^{-\frac{N}{2\lambda} \phi_x^2 + \phi_x (\bar{\psi}\psi)_x}. \quad (3)$$

After introducing the scalar field $\phi \in \mathbb{R}^{\Gamma_L^\alpha}$ and integrating out the Grassmann variables, the partition function is represented by the effective bosonic measure

$$d\mu_\alpha(\phi) = \frac{1}{Z_{\Gamma_L^\alpha}^\alpha} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x e^{-\frac{N}{2\lambda} \phi_x^2} \right) \det \left((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi \right)^N. \quad (4)$$

Here M_ϕ is one of the following multiplication operators:

$$\begin{aligned} (M_\phi)_{(x,s),(y,s')} &= \delta_{x,y} \delta_{s,s'} \phi_x, & \alpha &= \text{NN}, \\ (M_\phi)_{x,y} &= \delta_{x,y} \phi_x, & \alpha &= \text{SN}, \\ (M_\phi)_{(x,a),(y,b)} &= \delta_{x,y} \delta_{a,b} \phi_x, & \alpha &= \text{SP}. \end{aligned}$$

For every real ϕ , the determinant in [Equation \(4\)](#) is real. We assume throughout that N is even, so that its N -th power is non-negative and μ_α is a probability measure. The effective measure is reflection positive for every N , however, even though in various parts of the proof we heavily rely on the positivity of the measure. Whether this restriction can be removed would require different techniques and will not be investigated here.

The Hubbard–Stratonovich transformation also identifies ϕ as the bosonic representative of the chiral order parameter. Gaussian integration by parts gives the exact identity

$$\langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha = \left(\frac{N}{\lambda} \right)^2 \langle \phi_x \phi_y \rangle_{\mu_\alpha} - \frac{N}{\lambda} \delta_{x,y}. \quad (5)$$

Consequently, after spatial averaging, the contact term is negligible in the thermodynamic limit. Proving long-range order for the fermionic bilinear is therefore equivalent to proving that

$$\liminf_{L \rightarrow \infty} \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} \neq 0.$$

Thus, the physical problem and the proof strategy meet precisely in the auxiliary-field representation: the discretization is encoded in the determinant, while the order parameter and the geometric estimates are expressed through the same scalar field in all three models.

Main result. The common formulation allows the principal theorem to be stated simultaneously for the three discretizations.

Theorem 1 (Chiral long-range order). *There exists $\lambda_0 > 0$ such that, for every $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, every $\lambda \in (0, \lambda_0]$, and every $\varepsilon \in (0, 1)$, there exist $N_0(\lambda, \varepsilon) \geq 1$ and $L_0(\lambda, \varepsilon) \geq 1$ with the following property. If*

$$N \in [N_0(\lambda, \varepsilon), \infty) \cap 2\mathbb{N} \quad \text{and} \quad L \in [L_0(\lambda, \varepsilon), \infty) \cap 4(1 + \delta_{\alpha, \text{SP}})\mathbb{N},$$

then

$$(1 + \varepsilon) \left(\frac{N}{\lambda} \right)^2 (\varphi_\alpha^*(\lambda))^2 \geq \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x, y \in \Gamma_L^\alpha} \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \geq (1 - \varepsilon) \left(\frac{N}{\lambda} \right)^2 (\varphi_\alpha^*(\lambda))^2 > 0. \quad (6)$$

Here $\varphi_\alpha^*(\lambda)$ is the positive minimizer of the effective potential (see Figure 1.1)

$$v_\lambda^\alpha(\varphi) = \frac{\varphi^2}{2\lambda} - C_\alpha \iint_{[-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \log(\varphi^2 + \sin^2 p_0 + \sin^2 p_1), \quad (7)$$

where

$$C_{\text{NN}} = 1, \quad C_{\text{SN}} = \frac{1}{2}, \quad C_{\text{SP}} = 2.$$

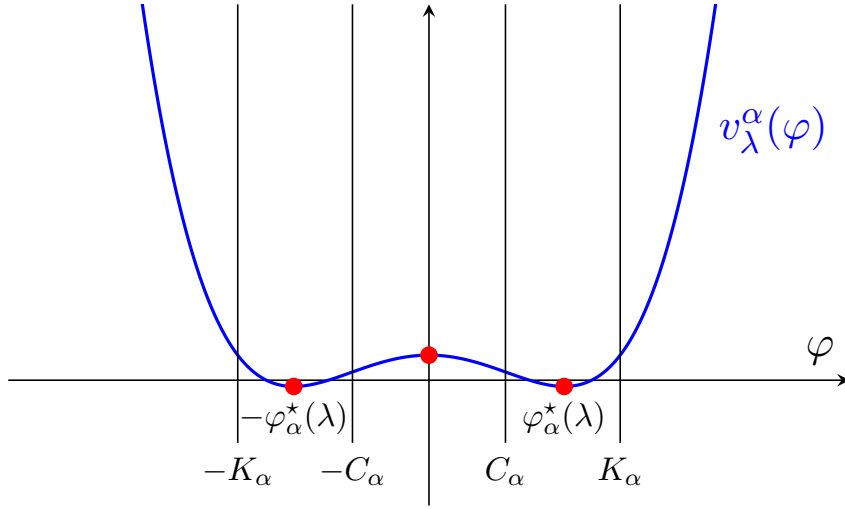


Figure 1.1: Sketch of the effective potential $v_\lambda^\alpha(\varphi)$

The theorem proves long-range order for the scalar fermionic bilinear and quantifies its relation to mean-field theory. In fact, the proof gives the stronger pointwise estimate

$$(1 - \varepsilon) \left(\frac{N}{\lambda} \right)^2 (\varphi_\alpha^*(\lambda))^2 \leq \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \leq (1 + \varepsilon) \left(\frac{N}{\lambda} \right)^2 (\varphi_\alpha^*(\lambda))^2$$

uniformly in $x, y \in \Gamma_L^\alpha$, once N and L satisfy the assumptions of the theorem. Thus the Large- N mean-field prediction is recovered at the level of individual two-point functions, without need for spatial averaging. The argument does not use a formal saddle-point or $1/N$ expansion: it allows to get matching upper and lower bounds for the finite-volume correlation function in terms of the minimizers of the same potential that appears in mean-field theory.

Structure of the proof. The proof separates naturally into a model-dependent and a model-independent part. For each discretization, the coupled Grassmann–boson functional obtained from Equation (3) is first shown to be reflection positive. The non-trivial step is the decomposition of the kinetic terms crossing the reflection plane. It depends on the specific Dirac operator, on the anti-multiplicative Grassmann reflection, and on the signs associated with the antiperiodic boundary conditions. Reflection positivity then passes to the bosonic effective measure μ_α .

From this point onward, the three models are treated by a common argument. Reflection positivity yields chessboard estimates for local events of the auxiliary field. The iterative application of the reflection bound turns the corresponding local constraints. For such configurations, Bloch–Floquet theory reduces $(\not{D}_\alpha)_{\Gamma_L}^- - M_\phi$, or equivalently the associated positive operator $((\not{D}_\alpha)_{\Gamma_L}^- - M_\phi)^* ((\not{D}_\alpha)_{\Gamma_L}^- - M_\phi)$, to a finite-dimensional matrix. Controlling its eigenvalues gives the required lower bounds on the fermionic determinant and on the cost of the disseminated configurations. The effective potential in Equation (7) has two symmetry-related minimizers $\pm\varphi_\alpha^*(\lambda)$. A Peierls-type construction shows that field configurations interpolating between neighbourhoods of the two minima carry a free-energy cost proportional to N and are therefore exponentially suppressed when N is large. The same estimates control Large-field configurations and configurations that stay away from either minimum. Combining these bounds gives both sides of Equation (6).

This approach resolves a difficulty specific to fermionic formulations of geometric contour arguments. A Peierls contour is not naturally defined directly in terms of Grassmann variables. The Hubbard–Stratonovich field provides a genuine real configuration space on which contours, local bad events, and chessboard patterns can be defined, while retaining exact information about the original fermionic order parameter through Equation (5).

Relation to previous results. The conclusion is conceptually close to the infrared results of [KMR95; Kop99], but our proof uses reflection positivity and a Peierls argument rather than cluster expansions. It should also be compared with the work of Goto and Koma on chiral Symmetry Breaking in Hamiltonian Nambu–Jona-Lasinio models [GK23]. Their analysis is formulated in the operator setting and uses infrared bounds, applying in spatial dimensions $d \geq 3$ at positive temperature and in $d = 2$ at zero temperature. The present models are instead defined directly by Grassmann fields on a two-dimensional Euclidean space–time lattice. Observe that, in our setting, the strategy of [GK23] would not apply, due to the logarithmic divergence of the integral entering the infrared bound. Instead of using infrared bounds, here we develop a Peierls’ type contour argument, which allows us to obtain a better control of the fluctuations around the ordered phases, and to establish the spontaneous breaking of chiral symmetry.

The three principal contributions may therefore be identified as follows:

1. reflection positivity is established for the three coupled Grassmann–boson lattice models and transferred to their effective bosonic measures;
2. Chessboard estimates and a Peierls argument are adapted to the unbounded auxiliary field arising from the Hubbard–Stratonovich representation;
3. Uniform pointwise upper and lower bounds are proved for the mass–mass two-point function, yielding chiral long-range order and a quantitative validation of the Large- N mean-field prediction.

Scope and open problems. The estimates are uniform in the volume but not in the lattice spacing. Accordingly, the theorem establishes chiral long-range order at fixed ultraviolet cutoff, or along a combined regime in which the lattice spacing tends to zero scaling N as $\sim \ell^{-\alpha}$ for some power $\alpha > 0$. A proof uniform in the lattice spacing would require combining the present infrared argument with a multiscale analysis of the ultraviolet regime and remains a major open problem.

There are two further natural directions. The first is the construction of distinct extremal infinite-volume Gibbs states associated with the two minimizers $\pm\varphi_\alpha^*(\lambda)$, together with a proof of clustering in the corresponding symmetry-broken phases. The second is the derivation of a dynamically generated mass gap from exponential decay of suitable truncated correlation functions. These questions go beyond long-range order itself and are not addressed by the finite-volume reflection positivity and Peierls estimates developed here.

The detailed construction of the three lattice models and the complete proof of [Theorem 1](#) are presented in [Chapter 4](#).

1.2 Stability of the π -Flux Phase for $\mathbb{Z}_2$ -Lattice gauge Theory Coupled to fermionic Matter

This section presents the problem and the main results of the joint work with Marcello Porta [\[GP26\]](#). The starting point is the flux-phase problem for lattice fermions: consider particles hopping on a planar bipartite lattice at half filling, with phases in the hopping amplitudes representing a static magnetic background. The flux-phase conjecture [\[AM88\]](#) states that the fermionic energy is minimized when the magnetic flux through each elementary plaquette is equal to π . Lieb proved this statement, at both zero and positive temperature and for a broad class of interacting models, by means of reflection positivity; a more general formulation was subsequently given by Macris and Nachtergaele [\[Lie94; MN96\]](#).

The question addressed here is what remains of this flux-selection mechanism when the $\mathbb{Z}_2$ -valued link variables are themselves included among the degrees of freedom of the system. This leads to a lattice gauge theory coupled to fermionic Matter. Numerical simulations [\[GRV17; AG16\]](#) indicate that, at half filling, the π -flux phase persists when the fermions are coupled to a dynamical $\mathbb{Z}_2$ gauge field, provided that the hopping strength is sufficiently large compared with the plaquette energy. This flux selection manifests itself in the fermionic sector through a semimetallic regime with Dirac-like low-energy excitations. Our goal is to put this picture on rigorous grounds in the case of vanishing electric-field strength. More precisely, for $\varepsilon = 0$ we prove that the π -flux phase is selected at half filling for sufficiently large t , uniformly in the volume, and we determine the corresponding low-momentum magnetic response.

Notation for the Lattice Let

$$\Lambda_L \doteq (\mathbb{Z}_L)^2, \quad L \in 4\mathbb{N},$$

be the square lattice with periodic boundary conditions. We will denote with $V(\Lambda_L)$, $E(\Lambda_L)$ and $F(\Lambda_L)$ respectively the set of vertices, edges and faces of the lattice Λ_L . Their elements will be written as $i \in V(\Lambda_L)$, $(i, j) \in E(\Lambda_L)$ (with the convention that j and i are respectively the starting

and arrival point according to the orientation shown in the picture) and $p = (i, j, k, l) \in F(\Lambda_L)$. Often, with a slight abuse of notation, we will identify the set of vertices $V(\Lambda_L)$ with the lattice Λ_L itself. The set of vertices, edges and faces of the dual lattice Γ_L^* of Λ_L (sketched with dotted lines) will be likewise denoted with $V(\Gamma_L^*)$, $E(\Gamma_L^*)$ and $F(\Gamma_L^*)$. Of course, there is a bijective correspondence between $F(\Lambda_L) \simeq V(\Gamma_L^*)$ and $V(\Lambda_L) \simeq F(\Gamma_L^*)$, which we denote with $*$ and we shall sometimes use without further mention. In particular, the four edges sharing one vertex can be thought of as the dual edges bounding the corresponding dual face.

The Lattice Gauge Model. The fermionic degrees of freedom are described by the Fock-space formalism reviewed in [Subsection 2.2.3](#), whereas the Hamiltonian formalism for lattice gauge fields and their coupling to fermionic Matter is recalled in [Chapter 5](#). We recall here only the ingredients needed to specify the present spinless model:

$$\mathcal{F}_L \doteq \bigoplus_{n=0}^{|\Lambda_L|} \bigwedge^n \ell^2(\Lambda_L).$$

For every $i \in V(\Lambda_L)$, we denote by a_i^+ and a_i^- the fermionic creation and annihilation operators on $\mathcal{F}_L$. The gauge-field Hilbert space and the total Hilbert space are

$$\mathcal{H}_L^g \doteq \bigotimes_{(i,j) \in E(\Lambda_L)} \mathbb{C}^2, \quad \mathcal{H}_L \doteq \mathcal{F}_L \otimes \mathcal{H}_L^g.$$

On the factor associated with (i, j) , the Pauli matrices $\hat{\sigma}_{ij}^z$ and $\hat{\sigma}_{ij}^x$ represent, respectively, the $\mathbb{Z}_2$ vector potential and electric field. The insertion of $\hat{\sigma}_{ij}^z$ in the hopping term is the $\mathbb{Z}_2$ -analogue of the Peierls substitution¹ briefly mentioned in [Example 12](#).

For $p \in P(\Lambda_L)$, define the plaquette-flux operator, represented graphically in [Figure 1.2](#), by

$$B_p \doteq \prod_{(i,j) \in \partial p} \hat{\sigma}_{ij}^z. \quad (8)$$

The eigenvalues $B_p = +1$ and $B_p = -1$ represent, respectively, zero flux and π flux through p .

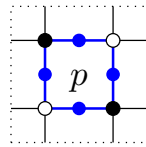


Figure 1.2: The magnetic-field operator B_p associated with the plaquette p .

The family of Hamiltonians is

$$H_\varepsilon \doteq - \sum_{p \in P(\Lambda_L)} B_p - t \sum_{(i,j) \in E(\Lambda_L)} \left(a_i^+ \hat{\sigma}_{ij}^z a_j^- + \text{h.c.} \right) + \varepsilon \sum_{(i,j) \in E(\Lambda_L)} \hat{\sigma}_{ij}^x. \quad (9)$$

Here $t > 0$ is the hopping strength and ε controls quantum fluctuations of the electric field. The model studied in [\[GP26\]](#) is $H = H_0$ at chemical potential $\mu = 0$, which is the half-filling point.

¹which is the lattice counterpart of the minimal coupling in gauge theories

The local gauge transformation acts jointly on the gauge and fermionic degrees of freedom. Introduce the star operator, represented graphically in Figure 1.3,

$$A_i \doteq \prod_{(k,l) \in \partial^* i} \hat{\sigma}_{kl}^x \quad (10)$$

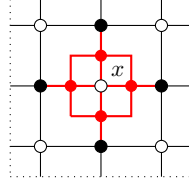


Figure 1.3: The operator A_i acts on the four edges incident on the lattice site $x \in \Lambda_L$.

and the gauge transformations generator

$$Q_i \doteq A_i (-1)^{n_i}, \quad n_i \doteq a_i^+ a_i^-.$$

Since $[H_\varepsilon, Q_i] = 0$ for every x , the Hamiltonian preserves the physical Hilbert space

$$\mathcal{H}_L^{\text{phys}} \doteq \{ \Psi \in \mathcal{H}_L : Q_i \Psi = \Psi \text{ for every } x \in \Lambda_L \}.$$

This is the exponentiated $\mathbb{Z}_2$ -Gauss' law: the electric flux emanating from a vertex is tied to the parity of the fermionic occupation at that vertex.

At $\varepsilon = 0$, all the operators $\hat{\sigma}_{ij}^z$ commute with H , and the Hamiltonian can therefore be decomposed into sectors labelled by classical link configurations $\sigma \doteq \{\sigma_{ij}\}$. This does not turn σ into a prescribed external background. The gauge variables remain part of the Hilbert space, the trace is restricted by the Gauss' law, and the physical partition function contains a sum over all gauge sectors. The problem is precisely to determine which sectors dominate this sum.

Competing Flux-Selection Mechanisms. The two terms in H favour different flux patterns. At $t = 0$, the magnetic term $-\sum_p B_p$ is minimized by $B_p = +1$ on every plaquette, that is, by the zero-flux sector. At large t , instead, the fermionic hopping energy dominates and Lieb's mechanism favours $B_p = -1$ on every plaquette. Thus the preferred flux sector must change between the two regimes. The result below concerns the large- t region; it does not locate or characterize the transition at intermediate coupling.

On the torus, specifying $B_p = -1$ for every plaquette does not determine a unique gauge-equivalence class. There remain two $\mathbb{Z}_2$ -holonomies $(a, b) \in \{\pm 1\}^2$, measured by Wilson loops along the two non-contractible cycles. Consequently, there are four inequivalent π -flux sectors, denoted by (π, a, b) .

Stability of the π -Flux Phase. At $\mu = 0$, define the physical partition function by

$$Z_{\beta,L} \doteq \text{Tr}_{\mathcal{H}_L} \left[\left(\prod_{i \in V(\Lambda_L)} \frac{1 + Q_i}{2} \right) e^{-\beta H} \right].$$

We write $Z_{\beta,L}(\Phi, a, b)$ for the contribution of the gauge-equivalence class with plaquette-flux configuration Φ and holonomies (a, b) . In particular, $Z_{\beta,L}(\pi, a, b)$ denotes a sector with uniform π flux. The full and restricted free-energy densities are

$$f_{\beta,L} \doteq -\frac{1}{\beta L^2} \log Z_{\beta,L} \quad f_{\beta,L}(\pi) \doteq -\frac{1}{\beta L^2} \log \left(\sum_{(a,b) \in \{\pm 1\}^2} Z_{\beta,L}(\pi, a, b) \right).$$

Here and below, a *magnetic gauge-invariant observable* is a gauge-invariant observable independent of the electric-field operators $\hat{\sigma}_{ij}^x$. For such an observable $\mathcal{O}$, let $\mathcal{O}(\pi, a, b)$ denote its restriction to the corresponding π -flux sector and set

$$\langle \mathcal{O} \rangle_{\beta,L}^\pi \doteq \frac{\sum_{(a,b) \in \{\pm 1\}^2} \text{Tr}_{\mathcal{F}_L} \left[e^{-\beta H_f(\pi, a, b)} \mathcal{O}(\pi, a, b) \left(\frac{1 + (-1)^N}{2} \right) \right]}{\sum_{(a,b) \in \{\pm 1\}^2} \text{Tr}_{\mathcal{F}_L} \left[e^{-\beta H_f(\pi, a, b)} \left(\frac{1 + (-1)^N}{2} \right) \right]}. \quad (11)$$

The factor $\frac{1+(-1)^N}{2}$ is the even-fermion-parity projection left by the Gauss-law constraint after the gauge degrees of freedom have been resolved.

Theorem 2 (Stability of the π -Flux Phase). *Let $L = 4\ell$, with $\ell \in \mathbb{N}$. There exists $t_0 > 0$, independent of β and L , such that the following statements hold for $t > t_0$.*

There is a quantity $\Delta_{\beta,L}$ satisfying, for large β and L ,

$$\Delta_{\beta,L} = \kappa t + o(1) \quad (12)$$

with a universal constant $\kappa > 0$, such that

$$0 \leq \beta L^2 (f_{\beta,L}(\pi) - f_{\beta,L}) \leq \log \left\{ 1 + 4 \left[\left(1 + e^{-\beta(\Delta_{\beta,L}-2)} \right)^{L^2} - 1 \right] \right\}. \quad (13)$$

Moreover, for every magnetic gauge-invariant observable $\mathcal{O}$ and for sufficiently large β and L ,

$$\left| \langle \mathcal{O} \rangle_{\beta,L} - \langle \mathcal{O} \rangle_{\beta,L}^\pi \right| \leq K_{\mathcal{O}} \left[\left(1 + e^{-\beta(\Delta_{\beta,L}-1)} \right)^{L^2} - 1 \right], \quad (14)$$

where $K_{\mathcal{O}}$ depends only on a uniform bound for the sector restrictions of $\mathcal{O}$.

Finally, let

$$E_{0;L}(\pi, a, b) \doteq -\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_{\beta,L}(\pi, a, b). \quad (15)$$

Then

$$E_{0;L}(\pi, a, b) \geq E_{0;L}(\pi, -1, -1), \quad (16)$$

and, for every $0 < \alpha < 1/3$,

$$|E_{0;L}(\pi, a, b) - E_{0;L}(\pi, -1, -1)| \leq \frac{C_\alpha}{L^{1-3\alpha}}. \quad (17)$$

The quantity $\Delta_{\beta,L}$ measures the fermionic free-energy penalty associated with replacing π -flux plaquettes by zero-flux plaquettes. The numerical value of the constant obtained in the proof is $\kappa \simeq 0.036$. Since the total estimate involves the entropy of possible defect locations, the comparison with the π -flux state is uniformly effective in the low-temperature regime $\beta \gtrsim \log L$. In particular, it determines the ground-state expectations of magnetic gauge-invariant observables. The last part of the theorem shows that the four torus sectors have the same thermodynamic ground-state energy, with a finite-volume splitting bounded by a power of L^{-1} .

Emergent Dirac Behaviour and Magnetic Susceptibility. Once the state has been reduced to the four π -flux sectors, the fermionic Hamiltonian can be diagonalized explicitly. In a convenient periodic gauge, its two bands have dispersion

$$e_{\pm}(k) = \pm 2t \sqrt{1 + \frac{1}{2} \cos(2k_1) + \frac{1}{2} \cos(2k_2)}, \quad (18)$$

which vanishes linearly at the Dirac points (see Figure 1.4). The corresponding Fermi velocity is $v = 2t$. This identifies the selected large- t regime as a gapless semimetal.

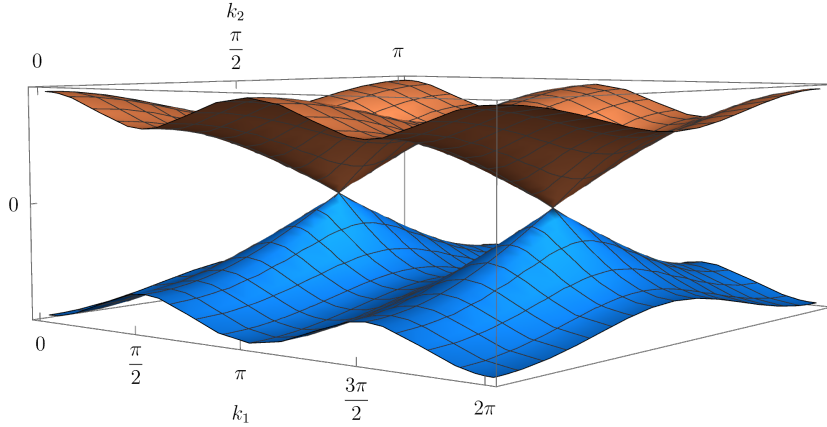


Figure 1.4: Energy bands associated with the π -flux phase.

The magnetic response is defined by coupling the fermions to an additional static $U(1)$ vector potential through the Peierls substitution. Let $\hat{J}_{\mu}(p)$ and $\hat{K}_{\mu,\nu}(p, q)$ be, respectively, the paramagnetic current and the diamagnetic term obtained from the first two derivatives of the perturbed Hamiltonian. For $p = (0, p_2)$, set

$$\chi(p_2) \doteq \frac{1}{p_2^2} \lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \left[\int_0^\beta ds \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} + \left\langle \hat{K}_{1,1}(p, -p) \right\rangle_{\beta, L} \right]. \quad (19)$$

Here the semicolon denotes the truncated imaginary-time correlation. The factor p_2^{-2} converts the response to the vector potential into the response to the magnetic field.

Proposition 3 (Ground-State Susceptibility). *Under the assumptions of Theorem 2, as $p_2 \rightarrow 0$,*

$$\chi(p_2) = -\frac{q^2 v}{8|p_2|} + o\left(\frac{1}{|p_2|}\right), \quad v = 2t. \quad (20)$$

This is the magnetic susceptibility of two-dimensional massless Dirac fermions and agrees with the value found numerically for the semimetallic regime of the $\mathbb{Z}_2$ -gauge theory [GRV17; KAA09].

Structure of the Proof. The proof has three main ingredients. First, at $\varepsilon = 0$, the Gauss-law projection can be resolved explicitly. For magnetic gauge-invariant observables, the physical trace becomes a sum over $\mathbb{Z}_2$ -link configurations of fermionic traces with the parity projector $\frac{1+(-1)^N}{2}$ where N is the total fermion number. This reduction keeps track of the gauge constraint while allowing each flux sector to be treated by fermionic methods.

Second, the Reflection-Positive transformation used by Lieb is applied to the fermionic Hamiltonian in a fixed gauge sector. Its reflected Cauchy–Schwarz inequality proves the optimality of uniform π -flux. To obtain stability rather than mere optimality, one must additionally control the cost of an arbitrary collection of zero-flux defects. Iterating the reflection inequality yields a chessboard estimate [Frö+78; Frö+80; Tas20]: every defect configuration is compared with a periodic pattern in which one quarter of the plaquettes carry zero flux. The crucial point is that the dissemination produced by the chessboard estimate replaces an arbitrary flux configuration by a periodic magnetic background. The corresponding one-particle Hamiltonian is therefore a periodic discrete magnetic Laplacian and can be reduced, by Bloch–Floquet theory, to a finite-dimensional matrix over the reduced Brillouin zone. Controlling its eigenvalues yields a strictly positive difference between the relevant band energies and hence a free-energy cost proportional to the number of defects. This is the origin of the constant $\Delta_{\beta,L}$ in Equation (12).

Third, after Theorem 2 reduces the ground-state expectations to the four π -flux sectors, the susceptibility is computed from the exact band structure. Lattice conservation laws enforce the cancellation between the paramagnetic and diamagnetic contributions, while the remaining non-analytic term is fixed by the Dirac dispersion, as in the analysis of graphene and related lattice semimetals [GMP12; Giu+16; GMP19].

The theorem addresses a restricted, but non-trivial, part of the phase diagram: $\varepsilon = 0$, half filling, and sufficiently large t . It neither proves stability under a non-zero electric-field term nor determines the location or the order of the change from the zero-flux regime. Moreover, the selected π -flux phase is gapless, and the four holonomy sectors are shown here to be only power-law close in energy. It also provides the flux-selection and defect-energy estimates on which the subsequent, massive analysis of the topologically ordered regime is built.

Open Problems. The results leave four immediate questions open. First, the comparison between the full Gibbs state and its restriction to the π -flux sectors is not uniform in the volume at fixed positive temperature. The present estimate becomes effective when $\beta \gtrsim \log L$, and therefore gives direct control of the ground-state regime, but not of the thermodynamic limit at fixed β .

Second, one would like to prove that the π -flux phase remains stable for sufficiently small $\varepsilon > 0$. The electric-field term then makes the gauge field genuinely dynamical: the operators $\hat{\sigma}_{ij}^z$ no longer commute with the Hamiltonian, and the decomposition into fixed gauge sectors is lost. More precisely, the individual plaquette-flux operators B_p are no longer conserved, and neither is the number of zero-flux defects. This makes the identification and control of the π -flux phase considerably more

subtle (see [KT94] for a discussion of the situation when an order parameter does not commute with the hamiltonian) and, in particular, the reflection positivity argument used here does not apply directly.

Third, although the zero-flux configuration is the exact minimizer at $t = 0$, the stability of the zero-flux phase for small but positive hopping strength has not been established uniformly in the volume. Proving both this small- t stability and the finite-temperature stability of the large- t π -flux phase would provide a natural starting point for a rigorous analysis of the transition between the two regimes.

A fourth question concerns the magnetic susceptibility in the presence of interactions. It would be interesting to determine whether the leading low-momentum singularity found in the non-interacting π -flux phase persists after adding a Reflection-Positive interaction, such as an on-site Hubbard term, and whether its coefficient satisfies a suitable non-renormalization or universality property. More precisely, one may ask whether the interacting susceptibility retains the same asymptotic form once the bare Fermi velocity is replaced by its renormalized value.

The general Hamiltonian formalism for lattice gauge fields and their coupling to fermionic Matter is reviewed in Chapter 5. The specific model considered here, together with the results stated above and their complete proofs, is presented in detail in Chapter 6.

1.3 Anyons in the π -Flux Phase of fermionic Matter Coupled to a $\mathbb{Z}_2$ -gauge Field

Historical Background and Motivation. In three or more spatial dimensions, the exchange of two identical quantum particles leads to the familiar boson–fermion dichotomy. In two dimensions, however, particle exchanges are described by the braid group rather than by the permutation group, and more general statistics become possible. The theoretical possibility of such particles, now called *Anyons*, was identified by Leinaas and Myrheim in a differential-geometric framework [LM77], by Goldin, Menikoff and Sharp from an algebraic perspective [GMS80], and through the magnetic flux–charge picture developed by Wilczek and collaborators [Wil82; WZ83]; related ideas were already implicit in [Frö76]. Wu subsequently formulated the connection between particle trajectories, the braid group, and the Feynman path integral [Wu84]. For abelian anyons a braid acts by a phase that is not restricted to ± 1 , whereas for non-abelian anyons it acts by a non-commuting unitary transformation on a degenerate state space.

The discovery of the Fractional Quantum Hall effect provided a physical setting in which these ideas became central. Laughlin’s wavefunction [Lau83], together with the subsequent description of fractionally charged excitations and their statistics [Hal84; ASW84], established the connection between anyons and phases of Matter that cannot be characterized by a local order parameter. Such phases are now understood as paradigmatic examples of *Topological order*. Relativistic quantum-field-theoretic descriptions of anyons were developed in [Frö21; FM88] (see also [FST94] for a review and [FRS89] for the operator-algebraic formulation in terms of superselection sectors with braid statistics). On a finite torus, a characteristic manifestation of topological order is a ground-state sector whose dimension reflects the topology of the underlying space and whose degeneracy cannot be detected or split by local observables, up to corrections vanishing with the system size [WN90].

The possibility of using such a ground-state sector as a stable quantum memory, and braiding as a fault-tolerant implementation of quantum gates, renewed the interest in the subject [Fre+03]. The toric-code and honeycomb models of Kitaev [Kit03; Kit06], as well as the string-net models of Levin and Wen [LW05], provide exactly solvable lattice realizations of abelian and non-abelian anyons. Their exact solvability makes the topological structure transparent, but it also relies on rather special microscopic Hamiltonians. Deriving the same structure from a natural interacting model, without reducing the problem to commuting projectors or to an explicitly diagonalizable spin system, is in general a very difficult problem. Recent experiments succeeded in probing anyonic statistics in engineered quantum systems [Ie24], thus making the construction and the analysis of more general microscopic models with exotic statistics particularly relevant.

This section presents the problem and the main results of the joint work with Sven Bachmann and Marcello Porta [BGP26]. We study spinful complex fermions coupled to a $\mathbb{Z}_2$ -gauge field on the square lattice. Models of this kind have a rich phase diagram and have been investigated extensively by numerical methods, both for spinful [AG16; GRV17; Gaz+18] and spinless fermions [Bor+22]. In the massless theory, the half-filled system has a semimetallic regime governed by Dirac excitations [GP26; GRV17], while away from half filling one encounters pairing phenomena, including a BCS–BEC crossover and superconducting phases. Proposals based on cold atoms [Bar+19; Hom+23], together with quantum-computing and quantum-simulation studies of confinement [Mil+25; Keb+25], further motivate the analysis of these gauge–Matter systems from first principles.

Our purpose is to establish rigorous finite-volume signatures of Abelian topological order in a natural member of this family. For a large finite torus, we identify an almost four-fold degenerate ground-state sector, prove a uniform spectral gap above it, and establish local ground-state indistinguishability. We then construct quasi-local loop operators by adiabatic flux insertion and use them to identify localized excitations and their braiding phases. The model is not exactly solvable once the Hubbard interaction is present; nevertheless, all these conclusions are obtained without any uncontrolled approximation. In particular, it provides one of the comparatively few non-exactly-solvable lattice systems for which local topological order can be proved directly; see also [LMY24]. We do not construct all thermodynamic-limit superselection sectors or a complete braided tensor category; the anyonic interpretation below refers to the finite-volume loop and string algebra that is proved in the thesis.

The Massive Deformation of the π -Flux Model. The model studied in Chapter 7 is a massive, spinful deformation of the $\varepsilon = 0$ theory introduced in Section 1.2. We retain the lattice Λ_L , the gauge-field Hilbert space $\mathcal{H}_L^g$, the link operators $\hat{\sigma}_{ij}^z$ and $\hat{\sigma}_{ij}^x$, and the plaquette and star operators B_p and A_i defined in Equation (8) and Equation (10). The Matter sector now contains two fermionic species,

$$\mathcal{S} \doteq \{\uparrow, \downarrow\}. \quad (21)$$

In this section, we denote the corresponding spinful Fock space by

$$\mathcal{F}_L \doteq \bigoplus_{n=0}^{2|\Lambda_L|} \bigwedge^n (\ell^2(\Lambda_L) \otimes \mathbb{C}^2), \quad \mathcal{H}_L \doteq \mathcal{F}_L \otimes \mathcal{H}_L^g. \quad (22)$$

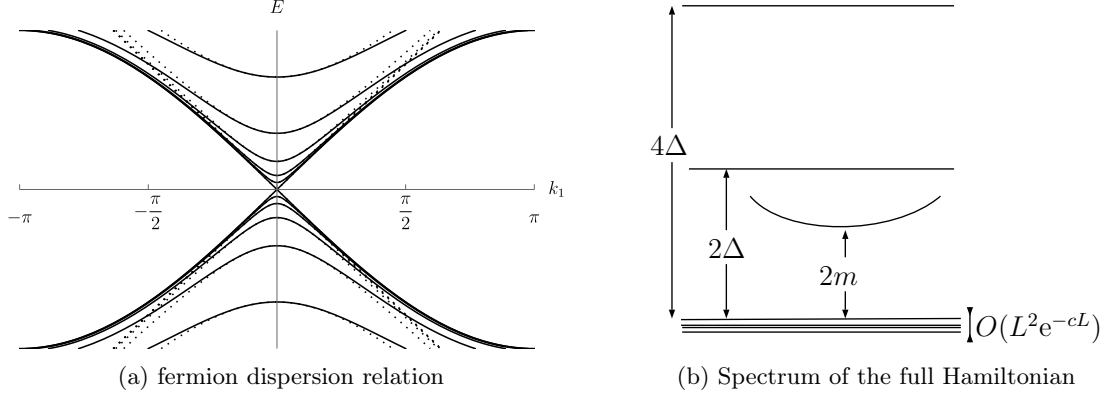


Figure 1.5: (a) Effect of mass term on π -flux dispersion relations: the gap opened by the perturbation is proportional to $\frac{m}{t}$. Solid and dotted lines represent respectively lattice and continuum dispersion relations. (b) The spectral structure of the full Hamiltonian: Δ is the energy necessary to create a monopole, while m is the minimal energy needed to create a fermion (for $U = 0$). The ground state splitting is exponentially small. The picture is stable for $|U|$ small enough.

For every $i \in V(\Lambda_L)$ and $\eta \in \mathcal{S}$, we denote by $a_{i,\eta}^+$ and $a_{i,\eta}^-$ the fermionic creation and annihilation operators associated with the spin state η , and set

$$n_{i,\eta} \doteq a_{i,\eta}^+ a_{i,\eta}^-, \quad n_i \doteq n_{i,\uparrow} + n_{i,\downarrow}. \quad (23)$$

For $t, m > 0$ and $U \in \mathbb{R}$, consider

$$\begin{aligned} H \doteq & -t \sum_{(i,j) \in E(\Lambda_L)} \sum_{\eta \in \mathcal{S}} \left(a_{i,\eta}^+ \hat{\sigma}_{ij}^z a_{j,\eta}^- + \text{h.c.} \right) \\ & + m \sum_{x \in \Lambda_L} \sum_{\eta \in \mathcal{S}} (-1)^{i_1+i_2} n_{i,\eta} + U \sum_{i \in V(\Lambda_L)} \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right). \end{aligned} \quad (24)$$

The first term is the nearest-neighbour hopping Hamiltonian coupled to the $\mathbb{Z}_2$ -vector potential through the lattice Peierls substitution. The second is a staggered mass, which opens a gap at the Dirac points of the uniform π -flux Hamiltonian. The final term is an on-site Hubbard interaction, repulsive for $U > 0$ and attractive for $U < 0$. Both the mass term and the Hubbard interaction are compatible with the Reflection-Positive transformation used in the flux-phase argument. The plaquette term appearing in Equation (9) may also be retained; the conclusions below continue to hold when the hopping strength is sufficiently large relative to its coefficient.

The gauge transformation has the same geometric action as in the spinless model, but the Matter parity at i now includes both species. Thus,

$$Q_i \doteq A_i (-1)^{n_i} = A_i (-1)^{n_{i,\uparrow} + n_{i,\downarrow}}. \quad (25)$$

The corresponding physical Hilbert space is

$$\mathcal{H}_L^{\text{phys}} \doteq \prod_{i \in V(\Lambda_L)} \left(\frac{1 + Q_i}{2} \right) \mathcal{H}_L. \quad (26)$$

An interesting class of its elements are obtained by projecting product states $|\psi\rangle \otimes |\sigma\rangle$:

$$\prod_{i \in V(\Lambda_L)} \left(\frac{1 + Q_i}{2} \right) (|\psi\rangle \otimes |\sigma\rangle) \quad (27)$$

This action creates a coherent superposition over gauge-equivalent configurations. This is the loop-gas structure characteristic of the physical states used below [Kit06; KP06; LW06].

As in the model described in Section 1.2, no electric-field term is present. Hence the Hamiltonian decomposes into fixed $\mathbb{Z}_2$ -gauge sectors, and its spectral data depend only on the plaquette fluxes and on the two holonomies of the torus.

From the Semimetal to a Fully Gapped Phase. At $m = U = 0$, the Matter Hamiltonian consists of two identical copies of the massless hopping Hamiltonian analyzed in the preceding Section 1.2. The results of Chapter 6 therefore provide the starting point of the present analysis: the uniform π -flux sector minimizes the energy, zero-flux defects have a strictly positive energy cost, and the fermionic spectrum contains gapless Dirac excitations. The corresponding four holonomy sectors have ground-state energies separated only by power-law finite-volume corrections.

The staggered mass changes the last two properties while preserving the flux-selection mechanism: it opens a fermionic gap and makes the dependence of the ground-state energy on the holonomies exponentially small in L . For sufficiently small $|U|$, a convergent fermionic cluster expansion shows that the Hubbard interaction closes neither the fermionic gap nor the monopole gap. The four holonomy sectors consequently form an almost degenerate spectral patch separated uniformly from the rest of the spectrum.

To state the result, let $H_f(\boldsymbol{\sigma}; e^{i\theta}, e^{i\phi})$ denote the fermionic Hamiltonian in the background $\boldsymbol{\sigma}$, with $U(1)$ twists $(e^{i\theta}, e^{i\phi})$ along two independent cycles of the torus, and let $E_{0;L}(\boldsymbol{\sigma}; e^{i\theta}, e^{i\phi})$ be its ground-state energy. We write π for a uniform π -flux background.

Theorem 4 (Spectral Structure). *Let $L \in 4\mathbb{N}$ and $t, m > 0$. There exists $U_0 = U_0(t, m) > 0$, independent of L , such that, for $|U| < U_0$, the following statements hold.*

1. *For every $\theta, \phi \in \mathbb{R}$, the ground state of*

$$H_f(\pi; e^{i\theta}, e^{i\phi})$$

is unique, lies at half filling, and is separated from the rest of the fermionic spectrum by a gap

$$\delta_L \geq m - C|U|^{1/3}. \quad (28)$$

2. *There exist $C, c > 0$, independent of L and of the holonomies, such that*

$$\left| E_{0;L}(\pi; e^{i\theta}, e^{i\phi}) - E_{0;L}(\pi; e^{i\theta'}, e^{i\phi'}) \right| \leq CL^2 e^{-cL}. \quad (29)$$

3. *Let $\boldsymbol{\sigma}$ contain $2k$ zero-flux plaquettes. Then*

$$E_{0;L}(\boldsymbol{\sigma}; e^{i\theta}, e^{i\phi}) \geq E_{0;L}(\pi; -1, -1) + 2k\Delta_L, \quad \Delta_L = \Delta_\infty + o_L(1), \quad (30)$$

where $\Delta_\infty > 0$.

The result proved in the cited work is slightly stronger than the ground-state formulation above: the monopole estimate is first established for the finite-temperature free energy and only then passed to the limit $\beta \rightarrow \infty$. The smallness condition on U is uniform in the volume, although it is not uniform as $m \downarrow 0$. This is natural: the exponential estimate in Equation (29) relies on the fermionic mass gap and degenerates when the Dirac points reappear.

The Topologically ordered Ground-State Sector. For $(a, b) \in \{\pm 1\}^2$, choose a uniform π -flux representative $\sigma_{\pi,a,b}$ with holonomies (a, b) , and let $|\psi_{\pi,a,b}\rangle$ be the unique ground state of the corresponding fermionic Hamiltonian. Define

$$|\Omega_{ab}\rangle \doteq \frac{1}{\mathcal{N}_{ab}} \Pi_{\text{phys}} (|\psi_{\pi,a,b}\rangle \otimes |\sigma_{\pi,a,b}\rangle), \quad (a, b) \in \{\pm 1\}^2, \quad (31)$$

where $\mathcal{N}_{ab}$ is a normalization factor, and let P be the orthogonal projection onto their span. The four labels are the two $\mathbb{Z}_2$ -holonomies of the torus; equivalently, they describe the four cohomology classes, or the four spin structures, relevant to the fermionic problem.

Theorem 5 (Ground-State Topological order). *Under the assumptions of Theorem 4, let $P^\perp$ denote the orthogonal complement of P inside $\mathcal{H}_L^{\text{phys}}$, and define*

$$g_L \doteq \inf \text{Spec}(P^\perp H P^\perp) - \sup \text{Spec}(P H P). \quad (32)$$

Then

$$g_L \geq \min\{2\delta_L, 2\Delta_L\} - CL^2 e^{-cL}. \quad (33)$$

Moreover, let $\mathcal{O}$ be a fixed observable which is local and gauge invariant. Then there exist $C_{\mathcal{O}}, c > 0$ such that

$$\left| \langle \Omega_{ab}, \mathcal{O} \Omega_{a'b'} \rangle - \delta_{aa'} \delta_{bb'} \frac{\text{Tr}(P \mathcal{O})}{\text{Tr}(P)} \right| \leq C_{\mathcal{O}} e^{-cL}. \quad (34)$$

Equation (33) identifies the two elementary energy scales above the ground-state patch. The quantity $2\delta_L$ is the cost of creating a pair of fermionic excitations, while $2\Delta_L$ is the cost of creating a pair of monopoles. Equation (34) is the local ground-state-indistinguishability condition: after restriction to the four-dimensional ground-state sector, every fixed local gauge-invariant observable is asymptotically proportional to the identity. Thus the degeneracy is not associated with a local order parameter and cannot be resolved by a local measurement.

Flux Threading and Loop Operators. The four ground states can be distinguished by Wilson loops. If $\mathcal{C}$ is a closed path in the direct lattice, set

$$\hat{Z}_{\mathcal{C}} \doteq \prod_{(ij) \in \mathcal{C}} \hat{\sigma}_{ij}^z. \quad (35)$$

For the two non-contractible cycles, the eigenvalues of these operators are precisely the holonomies (a, b) . Since local observables act trivially on the ground-state sector, an operator connecting distinct ground states must necessarily be non-local.

Such operators are constructed by threading a $U(1)$ flux through the torus. Let $\mathcal{C}^*$ be a closed path in the dual lattice and let $H_{\mathcal{C}^*}(\phi)$, $0 \leq \phi \leq \pi$, be obtained from H by multiplying every

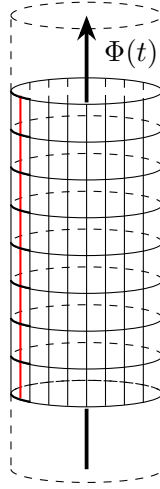


Figure 1.6: Picture of the flux threading procedure in a portion of the torus: the hoppings on the edges cut by the blue line acquire an extra $e^{\pm i\phi}$ (depending on the orientation).

hopping amplitude across $\mathcal{C}^*$ by $e^{\pm i\phi}$, with the sign fixed by the orientation (see Figure 1.6). The spectral gap stays open along this path by Theorem 4. Quasi-adiabatic continuation therefore gives a quasi-local unitary $V_{\mathcal{C}^*}(\phi)$ satisfying

$$P_{\mathcal{C}^*}(\phi) = V_{\mathcal{C}^*}(\phi)P_{\mathcal{C}^*}(0)V_{\mathcal{C}^*}(\phi)^*, \quad (36)$$

where $P_{\mathcal{C}^*}(\phi)$ is the ground-state projection of the twisted Hamiltonian.

At $\phi = \pi$, the $U(1)$ twist can be undone by the $\mathbb{Z}_2$ -Large gauge transformation

$$\hat{X}_{\mathcal{C}^*} \doteq \prod_{(i,j) \cap \mathcal{C}^* \neq \emptyset} \hat{\sigma}_{ij}^x. \quad (37)$$

This leads to the loop operator

$$W_{\mathcal{C}^*} \doteq \hat{X}_{\mathcal{C}^*} V_{\mathcal{C}^*}(\pi). \quad (38)$$

The $U(1)$ symmetry of the complex fermions is essential here: it provides the continuous flux-insertion path whose quasi-adiabatic transport dresses the bare $\mathbb{Z}_2$ -string.

We write $A \stackrel{L}{=} B$ when $\|A - B\| = O(L^{-n})$ for every $n \in \mathbb{N}$. If $\mathcal{I}(\mathcal{C}, \mathcal{C}^*) \in \mathbb{Z}_2$ denotes the mod-2 intersection number, the loop algebra and its physical interpretation may be summarized as follows.

Theorem 6 (Loop Algebra and Braiding). *Under the assumptions of Theorem 5, the following statements hold.*

1. The loop operators preserve the ground-state sector:

$$[W_{\mathcal{C}^*}, P] \stackrel{L}{=} 0. \quad (39)$$

2. Direct- and dual-lattice loops satisfy

$$\hat{Z}_{\mathcal{C}} W_{\mathcal{C}^*} = (-1)^{\mathcal{I}(\mathcal{C}, \mathcal{C}^*)} W_{\mathcal{C}^*} \hat{Z}_{\mathcal{C}}. \quad (40)$$

For two generators $\mathcal{C}_1^*, \mathcal{C}_2^*$ of the first cohomology of the torus, $W_{\mathcal{C}_1^*}$ and $W_{\mathcal{C}_2^*}$ permute the four states $|\Omega_{ab}\rangle$ by changing the corresponding holonomy.

3. The braiding of two dressed monopoles is trivial on the ground-state sector:

$$W_{C_1^*}^* W_{C_2^*}^* W_{C_1^*} W_{C_2^*} |\Omega_{ab}\rangle \stackrel{L}{=} |\Omega_{ab}\rangle. \quad (41)$$

4. Let $x, y \in \Lambda_L$ be separated by a distance of order L , and let $C_{x,y}$ be a path joining them. The normalized state

$$|\xi_{ab}^{x,y}\rangle \doteq \frac{1}{\mathcal{N}_{x,y}} a_{x,\eta}^+ \hat{Z}_{C_{x,y}} a_{y,\eta'}^+ |\Omega_{ab}\rangle \quad (42)$$

describes two gauge-invariant fermionic excitations localized near x and y . If C^* winds once around x but not around y , then

$$\frac{\langle \xi_{ab}^{x,y}, W_{C^*} \xi_{ab}^{x,y} \rangle}{\langle \Omega_{ab}, W_{C^*} \Omega_{ab} \rangle} \stackrel{L}{=} -1. \quad (43)$$

Opening a dressed dual loop creates a pair of monopoles at its endpoints, whereas the open Wilson line in Equation (42) creates a pair of fermions joined by a gauge string. Equation (43) is the universal Aharonov–Bohm phase acquired when a magnetic π -flux is braided around a unit electric charge. The corresponding localized excitations therefore exhibit the mutual-anyon braiding phase. The vanishing of the monopole–monopole phase in Equation (41) can be proved by relating it to the (vanishing) Hall conductance of the fermionic π -flux state [Bac+20; KS20].

At the level of this finite-volume loop and string algebra, it is convenient to denote the dressed monopole by $\mathbf{m}$ and the fermionic excitation by ϵ , and to summarize their $\mathbb{Z}_2$ composition laws by

$$\mathbf{m} \times \mathbf{m} = 1, \quad \epsilon \times \epsilon = 1, \quad \mathbf{e} \doteq \mathbf{m} \times \epsilon. \quad (44)$$

The corresponding loop relations assign trivial self-braiding to $\mathbf{m}$ and mutual phase -1 to $\mathbf{m}$ and ϵ , as in the toric code. Equation (44) is shorthand for these $\mathbb{Z}_2$ composition rules and is not a classification of superselection sectors. Thus the finite-volume algebra reproduces the characteristic toric-code fusion and braiding relations, even though the microscopic degrees of freedom and the construction of the excitations are different.

Comparison with Exactly Solvable Topological Phases. The comparison with Kitaev’s honeycomb model in its gapped phase is particularly instructive [Kit06]. The finite-volume loop algebra constructed here displays the same characteristic fusion and braiding relations as the toric-code sector of Kitaev’s model, but the microscopic degrees of freedom and the mechanisms producing the topological phase are different. This comparison concerns the relations established by the loop and string operators; it does not assert an equivalence of complete categories of superselection sectors. Kitaev’s model is formulated in terms of spin degrees of freedom and becomes exactly solvable after their fractionalization into Majorana fermions coupled to conserved $\mathbb{Z}_2$ gauge variables. By contrast, the present model is built from spinful complex fermions with a microscopic $U(1)$ particle-number symmetry coupled to a $\mathbb{Z}_2$ gauge field (see Section 7.1). This continuous symmetry is essential for defining the local flux-insertion unitaries and, through quasi-adiabatic continuation, the dressed loop operators used to create, transport, and braid monopoles. Following Lieb’s flux-phase argument [Lie94], Reflection Positivity allows us to identify rigorously the uniform π -flux sector as the ground-state sector, whereas the ground state of the honeycomb model belongs to the uniform zero-flux, or vortex-free, sector. More strongly, proceeding as in [GP26], the chessboard estimate proves both the positivity of the monopole mass and an energy lower bound proportional to the number

of monopoles. The corresponding extensivity statement for Kitaev’s honeycomb model remains conjectural and was investigated numerically in [Kit06, Appendix A]. Interestingly, the periodic chessboard configuration that arises naturally from our Reflection-Positive argument also appears in that numerical study as the flux configuration minimizing the energy density.

The microscopic distinction is also reflected in the construction of the excitations. In the present model, a monopole is a dressed version of a pure $\mathbb{Z}_2$ vortex: its creation and transport require the microscopic $U(1)$ symmetry together with quasi-adiabatic spectral flow. Fermionic excitations are simpler and are created in pairs by bare gauge-covariant string operators. The staggered mass term opens a gap for these charged fermionic excitations, while the chessboard estimate supplies the monopole gap; together, these two estimates yield a fully gapped many-body system. Unlike the static excitations of the exactly solvable toric code, the quasiparticles constructed here are dynamical: they are not exact eigenstates localized at prescribed lattice sites, but localized wave packets generated by quasi-local string operators. In this respect, the construction is related to the dynamical analysis of [BNV23], although that work is restricted to the Hilbert space spanned by two-quasiparticle states, whereas the present analysis is performed in the full many-body Hilbert space. Finally, the inclusion of a non-zero but sufficiently small Hubbard interaction shows that the spectral gap, the low-energy topological sector, and the finite-volume loop and braiding relations persist even when an explicit free-fermion diagonalization is no longer available. The model therefore provides a fermionic, Reflection-Positive realization of local topological order with the characteristic loop algebra of the toric-code phase [Sop25].

Structure of the Proof. The proof combines four ingredients. First, the staggered mass and the Hubbard interaction are shown to be compatible with the fermionic Reflection-Positive transformation used in Lieb’s flux-phase argument. reflection positivity and the chessboard estimate then select the uniform π -flux sector and give a positive, additive free-energy cost for monopole defects. A convergent fermionic cluster expansion controls the dependence of this cost on a sufficiently small Hubbard coupling. As in the massless model, the configurations produced by the chessboard estimate are periodic. Their quadratic fermionic Hamiltonians are discrete magnetic Laplacians supplemented by the periodic multiplication operator defined by the staggered mass. A Bloch–Floquet decomposition reduces them to finite-dimensional matrices, and the required monopole and fermionic gaps are obtained by controlling the resulting eigenvalues.

Second, the massive π -flux Hamiltonian is analyzed uniformly in the $U(1)$ holonomies. The single-particle gap, exponential decay estimates and the cluster expansion yield both the interacting fermionic gap and the exponentially small splitting in Equation (29). Combining this information with the monopole estimate produces the many-body gap in Equation (33).

Third, the Gauss-law projection is used to construct the four physical ground states. Locality and exponential clustering show that a local gauge-invariant observable cannot distinguish their holonomies, which gives Equation (34).

Finally, the uniform gap along the flux-insertion path permits the use of quasi-adiabatic continuation. Lieb–Robinson bounds make the resulting spectral flow quasi-local [HW05; Bac+11; Bac+19; NSY19]. Combining this locality with the Wilson-loop algebra proves the permutation of the ground states and the mutual braiding phase. The monopole self-braiding is related to the adiabatic curvature, equivalently to the Hall response, and is trivial in the present time-reversal-

symmetric π -flux phase.

The general Hamiltonian formalism for gauge fields coupled to fermionic Matter is developed in [Chapter 5](#). The flux-selection and monopole estimates used here are discussed in [Chapter 6](#). The massive model and the complete proofs of the results stated above are presented in [Chapter 7](#).

Open Problems. The most significant open question is whether one can find a gauge-invariant mass term for which the occupied fermionic bands have a non-zero Chern number C . For such a deformation, the spectral-flow argument would lead to the projective commutation relation

$$\check{W}_x \check{W}_y = e^{i\frac{\pi}{2}C} \check{W}_y \check{W}_x \quad (45)$$

between vortex-line operators associated with two transverse directions. This conclusion is conditional on proving that the spectral gap remains open along the entire flux-insertion path. Although this property holds for the time-reversal-symmetric model studied here, its analogue for a chiral deformation is presently missing since no known term with this property is reflection positive (and it is not even clear whether such term can actually exist). Establishing both the existence of such a mass term and the required uniform gap would provide a rigorous microscopic realization of a family of chiral Abelian topological phases².

A second direction is the extension from $\mathbb{Z}_2$ to other finite cyclic gauge groups. The natural candidates are the groups $\mathbb{Z}_{2n}$, which contain a distinguished element of order two representing π flux. Preliminary computations indicate that the reflection positivity argument can again select this flux and that the subsequent chessboard dissemination produces a periodic background with a fundamental cell containing 16 sites. The enlarged cell makes the Bloch-space analysis and the numerical verification of the resulting crossing estimates slightly more involved than in [\[GP26; BGP26\]](#).

²The relation with Kitaev's sixteen-fold way must be understood with some care. If the global $U(1)$ symmetry is forgotten and a complex Chern band is regarded as a system of Majorana fermions, its Majorana index is $\nu = 2C$. Thus only the even- ν sector is accessible, and the $\mathbb{Z}_2$ -vortices remain Abelian for every integer value of C , including odd C . The preserved $U(1)$ symmetry carries additional information, most notably the quantized Hall response, so that the resulting $U(1)$ -symmetric phases are not classified solely by C modulo 8. Moreover, the projective commutation relation above depends only on C modulo 4; a complete characterization would also require control of the vortex self-statistics and of the $U(1)$ response.

Fermionic Systems in Euclidean and Hamiltonian Formalisms

This chapter is expository and claims no new theorem. Its purpose is to review the basic formalism underlying lattice fermionic systems and to fix the notation used throughout the thesis. In particular, it provides a self-contained synthesis of the Euclidean and Hamiltonian frameworks needed in Chapters 3–7, including the finite-volume lattice conventions and boundary conditions, Grassmann orientations and signs, lattice Dirac operators, Fock space and second quantization, locality estimates, thermal correlation functions, and the passage from operators to a Euclidean functional integral.

The Euclidean and Hamiltonian descriptions emphasize complementary aspects of the same quantum system. The Hamiltonian formalism naturally describes real-time dynamics and spectral properties, whereas equilibrium and ground-state quantities can be expressed in terms of imaginary-time correlation functions. The latter representation connects quantum lattice systems with classical statistical mechanics and provides the natural setting for functional-integral methods. Since the Gross–Neveu models studied in [Chapter 4](#) are formulated directly in Euclidean space, we begin with finite-dimensional Grassmann calculus and Euclidean lattice fermions. We then introduce the one-particle and many-body Hamiltonian formalisms, review the dynamical tools used in the later chapters, and finally derive the Euclidean path integral from the operator formalism through fermionic coherent states.

The material reviewed here follows standard sources. The Euclidean lattice conventions, the finite-dimensional Grassmann calculus of [Subsection 2.1.1](#), the Gaussian identities, and the coherent-state construction of [Section 2.A](#) follow treatments such as [\[Weg16\]](#). The discussion of lattice Dirac operators collects the standard naive, Wilson, and staggered constructions [\[KS75; NN81\]](#), while the one-particle, many-body, and second-quantized descriptions, including the Fock-space and CAR conventions fixed in [Subsection 2.2.3](#), follow standard operator and lattice-fermion formulations. The dynamical material is likewise review: The Lieb–Robinson bounds reviewed in [Subsection 2.2.4](#) follows [\[Naa17; NSY19; NSY18\]](#), the Gaussian-filter proof of clustering in [Subsection 2.2.5](#) follows [\[NS06; Has04b; HK06b\]](#) and the quasi-adiabatic continuation reviewed in [Subsection 2.2.6](#) follows [\[HW05; Bac+11; Bac+19\]](#). The thermal and imaginary-time conventions used in the coherent-state derivation are introduced in [Subsection 2.2.7](#). The thermal and imaginary-time conventions subsequently used in the coherent-state derivation are introduced in [Subsection 2.2.7](#).

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2.1 Euclidean Formalism

In the Euclidean formalism, quantum systems are formulated on a finite $(d+1)$ -dimensional space–time lattice. The starting point is a discrete torus

$$\Gamma_{L,\ell,\beta,\tau} \doteq \Lambda_{L,\ell} \times (\tau\mathbb{Z}/\beta\mathbb{Z}) \doteq (\ell\mathbb{Z}^d/L\mathbb{Z}^d) \times (\tau\mathbb{Z}/\beta\mathbb{Z}), \quad L/\ell \in 2\mathbb{N}, \quad \beta/\tau \in 2\mathbb{N}.$$

The parameters ℓ and τ represent the spatial and temporal lattice spacings, whereas L and β determine the spatial and temporal extents of the system.¹ We use the following notations

$$\begin{aligned} \text{Vol}(\Gamma_{L,\ell,\beta,\tau}) &= L^d \beta & |\Gamma_{L,\ell,\beta,\tau}| &= \left(\frac{L}{\ell}\right)^d \left(\frac{\beta}{\tau}\right) \\ \text{Vol}(\Lambda_{L,\ell}) &= L^d & |\Lambda_{L,\ell}| &= \left(\frac{L}{\ell}\right)^d \end{aligned}$$

For finite-temperature quantum systems, the inverse temperature is identified with the circumference of the Euclidean time circle. The resulting Euclidean theory can be regarded as a classical statistical mechanics model on a $(d+1)$ -dimensional lattice, where the dynamical degrees of freedom are associated with suitable cells of the lattice and are weighted by the Gibbs factor e^{-S} , with S the Euclidean action. Scalar and Grassmann fields are naturally attached to the vertices of the lattice and gauge fields to its oriented edges, whereas the associated field strengths (or fluxes) are assigned to plaquettes. Higher-form lattice models are similarly described by fields living on higher-dimensional cells. We call $\Gamma_{L,\ell,\beta,\tau}$ and $\Lambda_{L,\ell}$ the *space–time lattice* and the *space lattice*, respectively.

When the Euclidean measure satisfies suitable hypotheses, including reflection positivity, time-translation invariance, and the regularity assumptions needed for reconstruction (see [OS73a; OS75] and discussion below in Section 3.1), one may reconstruct a quantum theory at finite inverse temperature β from its Euclidean correlation functions. This viewpoint underlies the Euclidean formulation of quantum lattice systems and Lattice gauge theories [Wil74; OS78], including lattice QCD, and

¹The spacings ℓ and τ act as ultraviolet cutoffs, whereas L and β act as infrared cutoffs.

is the standard framework used in numerical Monte Carlo simulations. Nevertheless, the Euclidean formalism is perfectly meaningful in its own right: one may equally regard the measure e^{-S} as defining a classical statistical mechanics model, independently of whether a corresponding quantum Hamiltonian exists.

Remark 1. In classical statistical mechanics, one often considers the isotropic case $\ell = \tau$ and $L = \beta$. The designation of one coordinate as “Euclidean time” then becomes conventional, and all directions are treated on an equal footing. Consequently, $\Gamma_{L,\ell,L,\ell} \equiv \Gamma_{L,\ell}$ is naturally regarded as a $(d+1)$ -dimensional Euclidean lattice rather than as a discretized space–time.

The finite lattice theory can be regarded as a regularized model. Different physical situations are obtained by taking suitable limits of the cutoff parameters, such as:

1. Classical statistical-mechanics regimes

- (a) *Continuum limit* $\ell = \tau \rightarrow 0$ (with $L = \beta$ fixed):

If a scaling limit exists, one expects to obtain a Euclidean field theory on the continuum torus $\mathbb{T}_L^{d+1} = (\mathbb{R}/L\mathbb{Z})^{d+1}$.

- (b) *Thermodynamic limit* $L = \beta \rightarrow \infty$ (with $\ell = \tau$ fixed).

This corresponds to an infinite-volume lattice system on $\ell\mathbb{Z}^{d+1}$.

2. Quantum statistical-mechanics regimes

- (a) *Continuous-time (or Hamiltonian) limit* $\tau \rightarrow 0$.

Under suitable assumptions, the Euclidean lattice model may admit a reconstruction as a quantum statistical mechanical system on the finite spatial lattice $(\ell\mathbb{Z}/L\mathbb{Z})^d$ at inverse temperature β .

- (b) *Quantum thermodynamic limit* $L \rightarrow \infty$:

Assuming the continuous-time limit has been performed, this yields a quantum system on the infinite lattice $\ell\mathbb{Z}^d$ at inverse temperature β .

- (c) *Zero-temperature limit* $\beta \rightarrow \infty$.

Assuming that the continuous-time limit has been performed, this corresponds formally to the ground-state regime of the quantum system.²

Remark 2. The continuous-time limit deserves particular attention. Establishing that a Euclidean lattice theory admits a corresponding quantum mechanical interpretation is, in general, a highly non-trivial problem. Reflection positivity provides a powerful sufficient criterion for such a reconstruction and lies at the heart of the Osterwalder–Schrader reconstruction theorem, which was developed precisely to establish a rigorous connection between Euclidean and quantum theories.

In most of this thesis, we will not be concerned with the continuum limit. Instead, we shall either keep the space–time lattice fixed (except for a brief discussion of the lack of uniformity in the lattice spacing $\ell = \tau$ for our Gross–Neveu results), or start directly from quantum lattice models in which Euclidean time is already continuous.

²The thermodynamic and zero-temperature limits can also be considered simultaneously. A common order of limits is first $L \rightarrow \infty$ and then $\beta \rightarrow \infty$. In the isotropic specialization $L = \beta$, sending the common extent to infinity combines the spatial thermodynamic limit with the zero-temperature limit of a reconstructed quantum system.

Accordingly, throughout the remainder of the Euclidean section we set

$$\tau = \ell = 1, \quad \beta = L,$$

and use the simplified notation

$$\Gamma_{L,1,L,1} \equiv \Gamma_L.$$

In the Euclidean representation, partition functions and correlation functions of quantum fermionic systems can be represented as finite-dimensional Grassmann integrals (for a complete treatment see [Weg16]). The Gaussian part of the Grassmann measure is determined by an appropriate lattice Dirac operator, while interactions are encoded through local Grassmann polynomials. Beyond its computational advantages, the Euclidean formalism plays a fundamental conceptual role. It provides the natural language for the study of thermodynamic limits, phase transitions, spontaneous Symmetry Breaking, and long-range order. Moreover, several powerful techniques of mathematical physics, such as reflection positivity, infrared bounds, renormalization group methods, and cluster expansions, are most naturally formulated within the Euclidean framework. For these reasons, Euclidean lattice field theories constitute the starting point of much of the modern rigorous analysis of interacting fermionic systems.

2.1.1 Grassmann Algebras and Grassmann Integrals

Grassmann algebras. To each lattice site $x \in \Gamma_L$, we associate a finite-dimensional complex vector space V describing the internal degrees of freedom. After choosing a basis $\{e_J\}_{J \in \mathcal{J}}$ of V , we associate a pair of Grassmann generators

$$\{\bar{\psi}_{x,J}, \psi_{x,J}\}_{J \in \mathcal{J}}.$$

The index $J \in \mathcal{J}$ labels the internal degrees of freedom, such as spin, flavour, colour, sublattice, or any combination thereof (see for example Equation (1)). In particular,

$$q_V \doteq \dim V = |\mathcal{J}|.$$

Remark 3. At this stage, V is an arbitrary finite-dimensional complex vector space; it need not carry a representation of the Euclidean Clifford algebra $\text{Cl}_{d+1,0}$. Such additional structure will be imposed only for specific lattice discretizations.

For notational convenience, we introduce an additional index $\epsilon \in \{+, -\}$ and write

$$(x, \epsilon, J) \longmapsto \psi_{x,J}^\epsilon \quad \text{with the convention:} \quad \psi_{x,J}^+ \equiv \bar{\psi}_{x,J}, \quad \psi_{x,J}^- \equiv \psi_{x,J}.$$

Whenever convenient, we shall also combine the lattice and internal indices into a single composite index

$$\alpha = (x, J) \in A \doteq \Gamma_L \times \mathcal{J},$$

so that the Grassmann generators may simply be denoted by ψ_α^ϵ .

Definition 7 (Grassmann Algebra). Given a (finite) index set A , the Grassmann algebra $\mathcal{G}_A$ over A is the unital $\mathbb{C}$ -algebra generated by the symbols $\{\psi_\alpha^\epsilon\}$, subject to the anticommutation relations

$$\psi_\alpha^\epsilon \psi_{\alpha'}^{\epsilon'} + \psi_{\alpha'}^{\epsilon'} \psi_\alpha^\epsilon = 0 \tag{1}$$

Remark 4.

1. In physical applications, one is usually interested in the following two relevant cases

$$A = \Gamma_L \times \mathcal{J}, \quad \text{and} \quad A = \Lambda_L \times \frac{2\pi}{\beta} \left(\mathbb{Z} + \frac{1}{2} \right) \times \mathcal{J}.$$

The first corresponds to the Euclidean space-time lattice together with the internal degrees of freedom labelled by $\mathcal{J}$. The second arises in the continuous-time path integral formulation of quantum lattice systems at finite temperature, where the imaginary-time direction is encoded by the fermionic Matsubara frequencies $\frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$. The second case must be dealt with a little bit of caution as A is not a finite index set in this case. Unless otherwise stated, all definitions and results will be formulated for finite index sets and subsequently extended to this setting whenever necessary.

2. By the anticommutation relations, a nonzero monomial contains each generator at most once. Consequently, $\mathcal{G}_A \simeq \Lambda(\mathbb{C}^{2|A|})$.
3. The Grassmann algebra is graded by total degree:

$$\mathcal{G}_A = \bigoplus_{k=0}^{2|A|} \mathcal{G}_A^{(k)} \quad \mathcal{G}_A^{(k)} \simeq \Lambda^k(\mathbb{C}^{2|A|})$$

4. It also has the $\mathbb{Z}_2$ -grading

$$\mathcal{G}_A = \mathcal{G}_A^{\text{even}} \oplus \mathcal{G}_A^{\text{odd}}, \quad \mathcal{G}_A^a \mathcal{G}_A^b \subseteq \mathcal{G}_A^{ab}, \quad a, b \in \{\text{even} = +, \text{odd} = -\},$$

where $\mathcal{G}_A^{\text{even}}$ and $\mathcal{G}_A^{\text{odd}}$ are the subspaces of even and odd polynomials, respectively.

Grassmann Integrals Given $(\alpha, \epsilon) \in A \times \{\pm\}$, define a functional:

$$\int d\psi_\alpha^\epsilon : \mathcal{G}_A \rightarrow \mathcal{G}_A$$

as the unique map determined by the following properties:

1. **Linearity:** for any $F(\psi), G(\psi)$ in $\mathcal{G}_A$ and complex numbers $a, b \in \mathbb{C}$:

$$\int d\psi_\alpha^\epsilon (aF(\psi) + bG(\psi)) = a \int d\psi_\alpha^\epsilon F(\psi) + b \int d\psi_\alpha^\epsilon G(\psi)$$

2. **Normalization:** for any $(\alpha, \epsilon) \in A \times \{\pm\}$ we have:

$$\int d\psi_\alpha^\epsilon 1 = 0 \quad \int d\psi_\alpha^\epsilon \psi_{\alpha'}^{\epsilon'} = \delta_{\alpha, \alpha'} \delta_{\epsilon, \epsilon'}$$

3. **Graded Leibniz identity:** for any monomials $F(\psi), G(\psi)$ in $\mathcal{G}_A$

$$\int d\psi_\alpha^\epsilon F(\psi) G(\psi) = \left[\int d\psi_\alpha^\epsilon F(\psi) \right] G(\psi) + (-1)^{\deg F} F(\psi) \left[\int d\psi_\alpha^\epsilon G(\psi) \right]$$

with $\deg F$ denoting the degree of F .

Notice that these are maps of degree -1 , *i.e.* they send degree k monomials to monomials of degree at most $k - 1$, for all $k = 1, \dots, 2|A|$. Also, one may check that any two such maps anti-commute, that is

$$\int d\psi_\alpha^\epsilon \circ \int d\psi_{\alpha'}^{\epsilon'} = - \int d\psi_{\alpha'}^{\epsilon'} \circ \int d\psi_\alpha^\epsilon \quad (2)$$

and we shall write their composition using the short-hand

$$\int d\psi_\alpha^\epsilon d\psi_{\alpha'}^{\epsilon'} \doteq \int d\psi_\alpha^\epsilon \circ \int d\psi_{\alpha'}^{\epsilon'}.$$

Definition 8 (Grassmann Integral). The *Grassmann Integral* is the linear functional

$$\int (d\bar{\psi} d\psi)_A : \mathcal{G}_A \rightarrow \mathbb{C}$$

that to each polynomial $F \in \mathcal{G}_A$ assigns

$$\int (d\bar{\psi} d\psi)_A F(\psi) \doteq \int \prod_{\alpha \in A} [d\bar{\psi}_\alpha d\psi_\alpha] F(\psi) \quad (3)$$

Remark 5.

1. In the following, we will refer to $(d\bar{\psi} d\psi)_A$, with an abuse of nomenclature, as the Grassmann Flat Measure.
2. The definition in Equation (3) is well posed as it does not depend on any choice of ordering of A . Indeed, exchanging two pairs of differentials is an even permutation and hence produces no sign.
3. The result is a complex number because Berezin integration has degree $-2|A|$. The integral vanishes on monomials of degree strictly less than $2|A|$, whereas

$$\int (d\bar{\psi} d\psi)_A \prod_{\alpha \in A} \psi_\alpha \bar{\psi}_\alpha = 1 \quad (4)$$

4. Using Equation (1), we have that any other monomial obtained from Equation (4) by an even/odd permutation is mapped to ± 1 respectively.

On a finite lattice, a Euclidean fermionic theory is specified by a choice of action

$$S(\bar{\psi}, \psi) = S_0(\bar{\psi}, \psi) + \lambda S_{\text{int}}(\bar{\psi}, \psi),$$

where S_0 is a quadratic form in the Grassmann variables, usually referred to as the *free action*, while S_{int} describes the interactions and $\lambda \in \mathbb{R}$ is the interaction strength. Expectation values of observables and, more generally, fermionic correlation functions are obtained by integrating with respect to the flat Grassmann measure in Equation (3) weighted by the Boltzmann factor $e^{-S(\bar{\psi}, \psi)}$. More precisely:

Definition 9 (Grassmann Expectation Values). Given an observable $F \in \mathcal{G}_A$ and assume $Z \neq 0$, we define the expectation value of F as:

$$\langle F \rangle_A \doteq \frac{1}{Z} \int (d\bar{\psi} d\psi)_A F(\bar{\psi}, \psi) e^{-S(\bar{\psi}, \psi)},$$

where

$$Z \doteq \int (d\bar{\psi} d\psi)_A e^{-S(\bar{\psi}, \psi)}$$

Equivalently, one may introduce the normalized Grassmann measure

$$d\mu_A(\bar{\psi}, \psi) \doteq \frac{1}{Z} (d\bar{\psi} d\psi)_A e^{-S(\bar{\psi}, \psi)},$$

so that expectation values take the compact form

$$\langle F \rangle_A = \int d\mu_A(\bar{\psi}, \psi) F(\bar{\psi}, \psi).$$

In particular, the fermionic n -point correlation functions are obtained by choosing F to be a suitable monomial in the Grassmann generators.

Remark 6. At first sight, one might worry that the partition function could vanish, making the normalized expectation values ill-defined. However, on a finite lattice, the exponential truncates and $Z(\lambda)$ is a polynomial in λ . If this polynomial is not identically zero, for example, if the free quadratic form is invertible (see [Proposition 11](#)), so that $Z(0) \neq 0$, it has only finitely many zeros. The normalized expectation is well-defined away from those zeros. Furthermore, in all Gross-Neveu models considered in this thesis, the partition functions admit an auxiliary-field representation from which it follows that

$$Z(\lambda) > 0 \quad \text{for } \lambda \geq 0.$$

The free action serves as the natural starting point for the study of interacting fermionic theories. Being exactly solvable, it provides the reference Gaussian theory around which perturbative expansions are developed and from which many non-perturbative constructions originate. In particular, the Gross-Neveu models studied in this thesis will be reformulated, via a Hubbard-Stratonovich transformation, in terms of Gaussian fermionic integrals coupled to an auxiliary bosonic field.

In the lattice setting, the free action is specified by a suitable quadratic operator acting on the fermionic fields. We now describe the class of operators that will be considered throughout this thesis.

2.1.2 Lattice Dirac Operators

The free fermionic action is determined by a lattice Dirac operator

$$\mathcal{D} : \ell^2(\Gamma_L) \otimes V \longrightarrow \ell^2(\Gamma_L) \otimes V,$$

designed so as to reproduce the Euclidean Dirac operator in the formal continuum limit $\ell \rightarrow 0$ while preserving as many of its symmetries as possible. Among these, the vector and chiral symmetries will play a central role in [Chapter 4](#).

The continuum Euclidean Dirac operator is constructed from a collection of matrices³ $\{\gamma_\mu\}_{\mu=0}^d \subset \text{End}(V)$ satisfying the Euclidean Clifford relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbf{1}. \tag{5}$$

³Equivalently, the matrices γ_μ arise from a complex representation of the Euclidean Clifford algebra $\text{Cl}_{d+1,0}$. More precisely, if $\rho : \text{Cl}_{d+1,0} \longrightarrow \text{End}(V)$ is such a representation and $\{\varepsilon_\mu\}_{\mu=0}^d$ denote the canonical generators of $\text{Cl}_{d+1,0}$ satisfying

$$\varepsilon_\mu \varepsilon_\nu + \varepsilon_\nu \varepsilon_\mu = 2\delta_{\mu\nu} \mathbf{1},$$

then $\{\gamma_\mu\}_{\mu=0}^d = \{\rho(\varepsilon_\mu)\}_{\mu=0}^d$ satisfy [Equation \(5\)](#).

The vector space V is called the *spinor space*. In the standard presentations of the Dirac Equation, one always works with the smallest irreducible complex representation of $\text{Cl}_{d+1,0}$, so that $\dim_{\mathbb{C}} V = 2^{\lfloor \frac{d+1}{2} \rfloor}$. Moreover, we always choose the representation to be Hermitian, namely

$$\gamma_{\mu}^{\dagger} = \gamma_{\mu}, \quad \mu = 0, \dots, d.$$

The Euclidean Dirac operator is then defined by

$$\not{\partial} + m = \sum_{\mu=0}^d \gamma_{\mu} \partial_{\mu} + m \mathbf{1},$$

and the corresponding Euclidean Lagrangian reads

$$\mathcal{L} = \bar{\Psi} (\not{\partial} + m) \Psi,$$

where $\Psi(x), \bar{\Psi}(x) \in V$ are two independent Spinor Grassmann fields.

For every space–time dimension, the Dirac Lagrangian is invariant under the *global* $U(1)$ *vector symmetry*

$$\Psi \mapsto e^{i\theta} \Psi \quad \bar{\Psi} \mapsto \bar{\Psi} e^{-i\theta},$$

which corresponds to conservation of the fermion number of particles. When the space–time dimension $d+1$ is even, one can further define the *chirality matrix*

$$\gamma_A \doteq (-i)^{\frac{d+1}{2}} \gamma_0 \gamma_1 \cdots \gamma_d,$$

which satisfies

$$\gamma_A^2 = \mathbf{1}, \quad \gamma_A^{\dagger} = \gamma_A, \quad \{\gamma_A, \gamma_{\mu}\} = 0.$$

For $m = 0$, the Dirac Lagrangian is moreover invariant under the continuous chiral transformation

$$\Psi \mapsto e^{i\phi \gamma_A} \Psi, \quad \bar{\Psi} \mapsto \bar{\Psi} e^{i\phi \gamma_A},$$

which defines the *chiral* $U(1)$ *symmetry*.

Different discretizations of the continuum Dirac operator lead to different lattice fermion formulations. The most common examples are described below. Any choice of discretization amounts to a choice of the internal vector space V , together with a discretization of the continuum Dirac operator $\not{\partial}$. Then one usually starts from an operator $\not{D}$ defined in the infinite lattice $\mathbb{Z}^{d+1}$ and then restricts it to the finite torus Γ_L by imposing suitable boundary conditions on the fermionic fields $\bar{\psi}_{\alpha}, \psi_{\alpha}$. The two most common choices are periodic/antiperiodic conditions in each direction. Given a vector $\underline{a} \in \{\pm 1\}^{d+1}$, then:

$$\bar{\psi}_{x+Le_{\mu},J} = a_{\mu} \bar{\psi}_{x,J}, \quad \psi_{x+Le_{\mu},J} = a_{\mu} \psi_{x,J}.$$

where here ψ_{α}^{ϵ} are defined on the full infinite lattice $\Gamma_{\infty} = \mathbb{Z}^{d+1}$. Since all lattice discretizations considered in this thesis are invariant under translations by Le_{μ} , the finite-volume operator with boundary conditions specified by $\underline{a} \in \{\pm 1\}^{d+1}$ may be obtained by the following signed periodization of the infinite-volume kernel⁴:

$$(\not{D}_{\Gamma_L}^{\underline{a}})_{xy} = \sum_{n \in \mathbb{Z}^{d+1}} \left(\prod_{\mu=0}^d a_{\mu}^{n_{\mu}} \right) \not{D}_{x+Ln,y}, \quad x, y \in \Gamma_L.$$

⁴Here we are implicitly assuming that our infinite volume operator is local meaning that it is finite range: $\not{D}_{xy} = 0$ whenever $|x - y| > R$. Here $|x - y|$ means the ℓ^1 -norm on $\mathbb{Z}^d$. In general one can allow for integrable decay of matrix elements.

In the rest of the discussion, we can then just specify the corresponding infinite-volume operators, as their finite-volume versions are obtained by applying the above periodization procedure.

Example 10.

1. **Naive fermions:** In this discretization, the internal vector space V is taken to be the smallest irreducible complex spinor representation of the Euclidean Clifford algebra $\text{Cl}_{d+1,0}$, and therefore

$$\dim_{\mathbb{C}} V = 2^{\lfloor \frac{d+1}{2} \rfloor}.$$

The corresponding naive lattice Dirac operator on the infinite lattice $\Gamma_{\infty} = \mathbb{Z}^{d+1}$ is

$$\mathcal{D}_{\text{naive}} = \frac{1}{2} \sum_{\mu=0}^d \gamma_{\mu} \otimes (T_{\mu} - T_{\mu}^{*}) + m \mathbf{1} \otimes \mathbf{1},$$

where T_{μ} denotes the right-translation operator in the μ -th direction acting on $\ell^2(\mathbb{Z}^{d+1})$. The corresponding formal lattice action is

$$S_{\text{naive}} = \frac{1}{2} \sum_{x \in \mathbb{Z}^{d+1}} \sum_{\mu=0}^d \sum_{s,s'=1}^{2^{\lfloor \frac{d+1}{2} \rfloor}} \bar{\psi}_{x,s} (\gamma_{\mu})_{ss'} (\psi_{x+\hat{\mu},s'} - \psi_{x-\hat{\mu},s'}) + m \sum_{x \in \mathbb{Z}^{d+1}} \sum_{s=1}^{2^{\lfloor \frac{d+1}{2} \rfloor}} \bar{\psi}_{x,s} \psi_{x,s}.$$

Passing to momentum space, the Fourier symbol of the naive Dirac operator is

$$\mathcal{D}_{\text{naive}}(p) = m + i \sum_{\mu=0}^d \gamma_{\mu} \sin p_{\mu}, \quad p \in [-\pi, \pi]^{d+1}.$$

For $m = 0$, the symbol vanishes at exactly 2^{d+1} points of the Brillouin zone. Consequently, in the continuum (or low-energy) limit, the naive discretization describes 2^{d+1} massless Dirac fermions instead of a single one. This is the well-known *fermion doubling problem* [KS75; NN81].

2. **Wilson fermions:** A natural strategy to remove the fermion doublers consists in adding to the naive Dirac operator the so-called *Wilson term*,

$$-\frac{r}{2} \sum_{\mu=0}^d (T_{\mu} + T_{\mu}^{*} - 2\mathbf{1}) \otimes \mathbf{1}.$$

The resulting Wilson operator is therefore

$$\mathcal{D}_{\text{Wilson}} = \mathcal{D}_{\text{naive}} - \frac{r}{2} \sum_{\mu=0}^d (T_{\mu} + T_{\mu}^{*} - 2\mathbf{1}) \otimes \mathbf{1}.$$

In momentum space,

$$\mathcal{D}_{\text{Wilson}}(p) = m + i \sum_{\mu=0}^d \gamma_{\mu} \sin p_{\mu} + r \sum_{\mu=0}^d (1 - \cos p_{\mu}).$$

The Wilson term vanishes at the physical Dirac point $p = 0$, while giving the doubler modes masses of order ℓ^{-1} . As a consequence, the unwanted species decouple in the continuum limit.

The drawback is that the Wilson term explicitly breaks the exact lattice chiral symmetry of the massless naive action,

$$\bar{\psi}_x \mapsto \bar{\psi}_x e^{i\phi\gamma_A}, \quad \psi_x \mapsto e^{i\phi\gamma_A} \psi_x.$$

This trade-off is one manifestation of the celebrated Nielsen–Ninomiya no-go theorem [NN81], which states that under suitable locality, regularity, translation-invariance, and symmetry hypotheses, a lattice Dirac operator with exact chiral symmetry cannot describe a single isolated relativistic fermion species.

3. **Staggered fermions:** A conceptually different strategy for reducing the fermion doubling problem was introduced by Kogut and Susskind [KS75]. Instead of employing spinor-valued Grassmann fields, one considers scalar Grassmann variables, namely

$$V \simeq \mathbb{C}.$$

The role of the gamma matrices is replaced by the staggered phase factors (also known as Kawamoto–Smit phases)

$$\Gamma_\mu(x) \doteq \begin{cases} 1, & \mu = 0, \\ (-1)^{x_0 + \dots + x_{\mu-1}}, & \mu = 1, \dots, d, \end{cases}$$

where $x = (x_0, \dots, x_d) \in \mathbb{Z}^{d+1}$. The staggered Dirac operator on the infinite lattice is

$$(\not{D}_{\text{stag}})_{xy} = \frac{1}{2} \sum_{\mu=0}^d \Gamma_\mu(x) (T_{e_\mu} - T_{e_\mu}^*)_{xy} + m \mathbf{1}_{xy},$$

and the corresponding lattice action is

$$S_{\text{stag}} = \frac{1}{2} \sum_{x \in \mathbb{Z}^{d+1}} \sum_{\mu=0}^d \Gamma_\mu(x) \bar{\psi}_x (\psi_{x+e_\mu} - \psi_{x-e_\mu}) + m \sum_{x \in \mathbb{Z}^{d+1}} \bar{\psi}_x \psi_x.$$

Unlike the naive discretization, the staggered Dirac operator is not invariant under ordinary lattice translations. Instead, translations are realized projectively: a unit translation changes the staggered phase factors, but this change can be compensated by a suitable local gauge transformation. Equivalently, the symmetry group is generated by *magnetic translations* M_μ , defined as:

$$(M_\nu f)_x = \begin{cases} (-1)^{x_{\nu+1} + \dots + x_d} f_{x+e_\nu}, & \nu = 0, \dots, d \\ f_{x+e_\nu} & \nu = d+1 \end{cases}.$$

From this perspective, the staggered operator may be regarded as a magnetic Laplacian in the presence of a background π -flux through every elementary plaquette. As a consequence, magnetic translations in different directions anticommute,

$$M_\mu M_\nu = -M_\nu M_\mu, \quad \mu \neq \nu,$$

and ordinary Bloch(/Fourier) theory is replaced by its magnetic analogue. Consequently, one works with an enlarged magnetic unit cell containing 2^d lattice sites. After Fourier transformation, the staggered Dirac operator becomes a $2^d \times 2^d$ matrix-valued function on the

reduced Brillouin zone. In the continuum limit, this construction reduces the number of fermion doublers from 2^{d+1} to $2^{\lceil \frac{d+1}{2} \rceil}$, which are commonly referred to as the *staggered tastes*. The discretization, in even space-time dimensions and $m = 0$, still possesses an exact $U(1)$ chiral symmetry, given by:

$$\bar{\psi}_x \rightarrow e^{i\alpha\epsilon(x)}\bar{\psi}_x \quad \psi_x \rightarrow e^{i\alpha\epsilon(x)}\psi_x \quad \epsilon(x) = (-1)^{x_0+\dots+x_d}.$$

4. **Modern chiral fermions** Besides the discretizations discussed above, there exist lattice formulations preserving an exact lattice realization of chiral symmetry while avoiding the fermion doubling problem. The two most prominent examples are *domain-wall fermions*, introduced by Kaplan [Kap92], and *overlap fermions*, developed by Narayanan and Neuberger [NN93; NN95; Neu98]. The former embed the theory into one extra dimension, where light chiral modes become localized on domain walls, while the latter provide an equivalent purely $(d+1)$ -dimensional formulation satisfying the Ginsparg–Wilson relation (in place of $\{\gamma_A, \not{D}(p)\} = 0$). Since neither of these discretizations will play a role in the remainder of this thesis, we do not discuss them further and refer the reader to the original literature.

Among the discretizations listed above, only naive and staggered fermions will play a direct role in this thesis. They constitute the lattice regularizations underlying the Gross–Neveu models discussed in Chapter 4. Wilson, domain-wall and overlap fermions are mentioned for completeness, as they provide alternative approaches to the fermion-doubling problem.

2.1.3 Fermionic Gaussian Integrals

The simplest fermionic theories are those whose action is quadratic in the Grassmann variables. Given a lattice Dirac operator $\not{D}$, the associated normalized Gaussian Grassmann measure is then

$$d\mu_{\not{D}}(\bar{\psi}, \psi) \doteq \frac{1}{Z_{\not{D}}} (d\bar{\psi} d\psi)_A e^{-\sum_{\alpha, \beta \in A} \bar{\psi}_\alpha \not{D}_{\alpha\beta} \psi_\beta}.$$

A fundamental feature of Grassmann integration is that Gaussian integrals can be evaluated exactly.

Proposition 11. *Assuming that the Dirac operator $\not{D} \in GL_{|A|}(\mathbb{C})$:*

1. *The partition function $Z_{\not{D}}$ is given by:*

$$Z_{\not{D}} = \det \not{D} \neq 0 \tag{6}$$

2. *The two point function is given by:*

$$\langle \psi_\beta \bar{\psi}_\alpha \rangle_{\not{D}} = (\not{D}^{-1})_{\beta\alpha}, \quad \langle \bar{\psi}_\alpha \psi_\beta \rangle_{\not{D}} = -(\not{D}^{-1})_{\beta\alpha}. \tag{7}$$

Remark 7. More generally, all higher-order correlation functions are completely determined by the two-point function through the fermionic version of Wick’s theorem.

Proof.

1. By expanding the exponential, the flat Grassmann Measure selects only degree $2|A|$ monomials:

$$\int (d\bar{\psi}d\psi)_A e^{-\sum_{\alpha,\beta \in A} \bar{\psi}_\alpha \not{D}_{\alpha\beta} \psi_\beta} = \frac{1}{|A|!} \sum_{\substack{\alpha_1, \dots, \alpha_{|A|} \\ \beta_1, \dots, \beta_{|A|}}} \int \prod_{\alpha \in A} d\bar{\psi}_\alpha d\psi_\alpha \prod_{i=1}^{|A|} \psi_{\beta_i} \not{D}_{\alpha_i \beta_i} \bar{\psi}_{\alpha_i} \quad (8)$$

Because Grassmann variables are anticommuting and nilpotent, any monomial containing two identical creation or annihilation variables vanishes. Consequently, the indices $\alpha_1, \dots, \alpha_{|A|}$ are pairwise distinct, and so are $\beta_1, \dots, \beta_{|A|}$. Therefore, there exist a permutation $\sigma \in S_{|A|}$ such that $\alpha_i = \beta_{\sigma(i)}$ for each $i \in \{1, \dots, |A|\}$ and thus:

$$\prod_{i=1}^{|A|} \psi_{\beta_i} \bar{\psi}_{\alpha_i} = \prod_{i=1}^{|A|} \psi_{\beta_i} \bar{\psi}_{\beta_{\sigma(i)}} = (-1)^{|\sigma|} \prod_{i=1}^{|A|} \psi_{\beta_i} \bar{\psi}_{\beta_i}$$

where $(-1)^{|\sigma|}$ indicates the parity of the permutation $\sigma \in S_{|A|}$. By plugging this expression in Equation (8), we get:

$$\begin{aligned} \int (d\bar{\psi}d\psi)_A e^{-\sum_{\alpha,\beta \in A} \bar{\psi}_\alpha \not{D}_{\alpha\beta} \psi_\beta} &= \frac{1}{|A|!} \sum_{\substack{\alpha_1, \dots, \alpha_{|A|} \\ \beta_1, \dots, \beta_{|A|}}} \int d\Psi \prod_{i=1}^{|A|} [\psi_{\beta_i} \not{D}_{\alpha_i \beta_i} \bar{\psi}_{\alpha_i}] \\ &= \frac{1}{|A|!} \sum_{\beta_1 \neq \dots \neq \beta_{|A|}} \sum_{\sigma \in S_{|A|}} (-1)^{|\sigma|} \int d\Psi \prod_{\beta \in A} \psi_\beta \not{D}_{\sigma(\beta)\beta} \bar{\psi}_\beta \\ &= \frac{|A|!}{|A|!} \sum_{\sigma \in S_{|A|}} (-1)^{|\sigma|} \prod_{\beta \in A} \not{D}_{\sigma(\beta)\beta} \\ &= \det \not{D} \end{aligned}$$

2. The connected correlation functions can be computed by differentiating the generating functional $W(\bar{J}, J) \doteq \log Z(\bar{J}, J)$, where

$$Z(\bar{J}, J) \doteq \int d\mu_{\not{D}} \exp \left(\sum_{\alpha \in A} (\bar{J}_\alpha \psi_\alpha + \bar{\psi}_\alpha J_\alpha) \right),$$

and $\{J_\alpha\}_{\alpha \in A}$ and $\{\bar{J}_\alpha\}_{\alpha \in A}$ are independent Grassmann sources (which anticommute among themselves and with the other Grassmann generators). In particular, it holds

$$-\frac{\partial^2 W(\bar{J}, J)}{\partial \bar{J}_\alpha \partial J_\beta} \Big|_{J=\bar{J}=0} = \langle \psi_\alpha \bar{\psi}_\beta \rangle - \langle \psi_\alpha \rangle \langle \bar{\psi}_\beta \rangle = \langle \psi_\alpha \bar{\psi}_\beta \rangle.$$

By completing the square and exploiting the translational invariance of the Grassmann measure, the generating functional can be evaluated exactly,

$$W(\bar{J}, J) = \sum_{\alpha, \beta \in A} \bar{J}_\alpha \not{D}_{\alpha\beta}^{-1} J_\beta.$$

Differentiating⁵ with respect to the Grassmann sources immediately gives the connected two-point function,

$$\langle \psi_\alpha \bar{\psi}_\beta \rangle = - \left. \frac{\partial^2 W(\bar{J}, J)}{\partial \bar{J}_\alpha \partial J_\beta} \right|_{J=\bar{J}=0} = \mathbb{D}_{\alpha\beta}^{-1}.$$

■

Remark 8. Notice, however, that the invertibility of $\mathbb{D}$ depends on the choice of boundary conditions. In particular, for the massless lattice Dirac operators considered in this thesis, periodic boundary conditions in every lattice direction typically give rise to exact zero modes, implying

$$\det \mathbb{D} = 0,$$

so that the normalized Gaussian measure is not well-defined. For this reason, it is customary to impose antiperiodic boundary conditions along one distinguished direction, conventionally interpreted as the Euclidean time direction. Besides removing the zero modes of the free theory, this choice has a direct physical interpretation: in the Euclidean formulation of finite-temperature quantum systems, fermionic fields are antiperiodic along the time circle of length β , in accordance with the Fermi–Dirac statistics. Moreover, under suitable positivity assumptions on the lattice action, this Euclidean theory admits a Hamiltonian reconstruction through the Osterwalder–Schrader theorem. The key ingredient underlying this reconstruction is the property of *reflection positivity*, which will play a central role throughout this thesis and will be discussed in detail in the [Chapter 3](#).

While the free theory can be solved exactly through the fermionic Gaussian formalism discussed above, the presence of interactions gives rise to genuinely non-trivial many-body phenomena including spontaneous Symmetry Breaking.

Several complementary approaches have been developed to study interacting fermionic systems. When the interaction is sufficiently weak, one may regard S_{int} as a perturbation of the Gaussian theory and construct the corresponding Schwinger functions through perturbative expansions. On the other hand, many physically relevant models admit exact reformulations in terms of auxiliary bosonic fields via Hubbard–Stratonovich transformations, reducing the fermionic sector to Gaussian integrals. After integrating out the fermionic degrees of freedom, one obtains an effective bosonic theory whose action typically contains the logarithm of a fermionic determinant. This latter point of view will be adopted in [Chapter 4](#), where all Gross–Neveu models considered will be analyzed through their auxiliary-field representation, where the fermions can be integrated out exactly, allowing the techniques of classical statistical mechanics to be combined with the algebraic structure of fermionic Gaussian integrals.

2.2 Hamiltonian Formalism

The Euclidean functional integral discussed in the previous section provides a convenient framework for the analysis of fermionic lattice models, particularly in connection with methods from classical statistical mechanics. Nevertheless, many physical systems are more naturally formulated in the

⁵Throughout this thesis we adopt the convention of left Grassmann derivatives. Grassmann differentiation satisfies the graded Leibniz rule and is algebraically equivalent to Grassmann integration:

$$\frac{\partial}{\partial \psi_\alpha^\epsilon} (F(\psi) G(\psi)) = \frac{\partial F(\psi)}{\partial \psi_\alpha^\epsilon} G(\psi) + (-1)^{\deg(F)} F(\psi) \frac{\partial G(\psi)}{\partial \psi_\alpha^\epsilon}$$

Hamiltonian setting, where the fundamental object is a self-adjoint operator H acting on a many-body Hilbert space associated to the spatial lattice $\Lambda_L \doteq \mathbb{Z}^d / L\mathbb{Z}^d$.

Unlike the Euclidean setting, the Hamiltonian formalism treats space and time on different footing: in particular, for the lattice systems under study, the space is a discrete lattice while the time is treated as a continuous evolution parameter. The purpose of this section is to briefly review this formulation and to explain how the Euclidean Grassmann path integral naturally arises from it.

Remark 9. It is worth stressing that even continuous systems can be described, within certain approximations, by lattice models. For example, one of the main physical situations described by lattice fermion models is that of electrons moving in a crystalline solid. In the presence of a periodic ionic potential, the electronic wave functions become localized around the atoms (or molecules) forming the crystal. Within the tight-binding approximation [Fra13], these localized orbitals are taken as a basis for the one-particle Hilbert space, and the dominant contribution to the kinetic energy arises from quantum tunnelling between neighbouring sites. This naturally leads to Hamiltonians describing fermions hopping on a lattice, supplemented by interaction terms accounting for the Coulomb repulsion and other many-body effects.

2.2.1 Single-Particle Quantum Mechanics

The mathematical description of quantum many-body systems begins with the quantum mechanics of a single particle. Let V be a finite-dimensional complex vector space describing the internal degrees of freedom of the particle (such as spin, flavor, or sublattice indices). A notable example for spin $s \in \frac{1}{2}\mathbb{N}$ particles is $V \simeq \mathbb{C}^{2s+1}$. The corresponding one-particle Hilbert space is

$$\mathcal{H}_L^1 = \ell^2(\Lambda_L) \otimes V.$$

A single-particle Hamiltonian is a *local* self-adjoint operator

$$h : \mathcal{H}_L^1 \longrightarrow \mathcal{H}_L^1, \tag{9}$$

whose spectral properties determine the quantum dynamics. Local here means that the matrix elements of h decay sufficiently fast. An example of such behavior is given by exponential decay:

$$|h(i, j)| \leq \epsilon e^{-\xi|i-j|} \quad i \neq j \in \Lambda_L \quad \epsilon, \xi > 0$$

Here the distance $|\cdot|$ denotes the standard distance on the discrete torus:

$$|i - j| \doteq \min_{n \in \mathbb{Z}^d} |i - j - nL|_1$$

Example 12. Many lattice Hamiltonians of physical interest are finite-range difference operators. To illustrate the basic formalism, we first consider particles without internal degrees of freedom (i.e. $V \simeq \mathbb{C}$). The extension to spinful cases and other internal quantum numbers is straightforward.

1. The simplest example is the discrete Schrödinger operator

$$(h\psi)(i) = -t \sum_{\substack{j \in \mathbb{V}(\Lambda_L): \\ (i,j) \in \mathbb{E}(\Lambda_L)}} \psi(j) + \mu(i)\psi(i),$$

where $t > 0$ denotes the hopping amplitude, $\mu : \Lambda_L \rightarrow \mathbb{R}$ is an external potential with $\|\mu\|_\infty < C$ with $C > 0$ independent of L .

2. The presence of a magnetic field is encoded through the so-called *Peierls substitution* [Pei33]. One assigns to each oriented edge (i, j) the (exponential of the) *Vector Potential* $A_{ij} \in [0, 2\pi)$

$$U_{ij} \in U(1), \quad U_{ji} = \overline{U_{ij}}, \quad U_{ij} = e^{iA_{ij}}$$

and defines the magnetic Schrödinger operator

$$(h_A \psi)(i) = -t \sum_{\substack{j \in V(\Lambda_L): \\ (i,j) \in E(\Lambda_L)}} U_{ij} \psi(j) + \mu(i) \psi(i).$$

Important examples that are encompassed by the hamiltonian above include the Harper [Har55], Hofstadter [Hof76] and π -flux Hamiltonians [AM88; Lie94], which play a central role in the theory of quantum Hall systems [Tho+82; DAZ85] and topological phases of Matter [HK06a; HFA06].

As it is well known from a standard course in electromagnetism, the vector potential A_{ij} is defined up to a gauge transformation

$$A_{ij} \rightarrow A'_{ij} = A_{ij} + \lambda_i - \lambda_j \quad \lambda_i, \lambda_j \in [0, 2\pi)$$

which leaves invariant the holonomy W_γ along any closed curve $\gamma \in Z_1(\Lambda_L, \mathbb{Z})$:

$$W_\gamma = \prod_{(x,y) \subset \gamma} U_{xy}.$$

In particular, the magnetic flux through an elementary plaquette p is

$$B_p = \prod_{(x,y) \subset \partial p} U_{xy},$$

where p is a 2-cell and $\partial p \in B_1(\Lambda_L; \mathbb{Z})$ is its oriented boundary. Gauge-equivalent link fields give unitarily equivalent operators, and the gauge-equivalence class is determined by its holonomies (see [LL93, Lemma 2.1] and the explicit proof of [Proposition 66](#)). In two dimensions, the plaquette fluxes obey one global constraint. Hence one may specify the fluxes through all but one plaquette, together with the holonomies around the two noncontractible cycles of the torus.

2.2.2 Many-Body Formalism

Although the one-particle theory already exhibits a rich mathematical structure, it is insufficient for describing systems containing an arbitrary number of interacting identical particles. The fundamental obstacle is that the Hilbert space $(\mathcal{H}_L^1)^{\otimes N}$ contains states with no definite exchange symmetry, whereas quantum mechanics requires identical particles to transform according to irreducible representations of the permutation group. Fermions satisfy the Pauli exclusion principle and therefore occupy the antisymmetric subspace⁶ $\bigwedge^N \mathcal{H}_L^1$ of $(\mathcal{H}_L^1)^{\otimes N}$, defined by the condition

$$\Omega_n(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) = (-1)^{|\sigma|} \Omega_n(\xi_1, \dots, \xi_N) \quad \forall \sigma \in S_N, \forall \xi_i \in \Lambda_L \times \mathcal{J}$$

⁶For a comprehensive account on the analogous construction of the Bosonic Fock space see [Lie+05].

Remark 10. Since $\dim \mathcal{H}_L^1 = |\Lambda_L| \dim V$, antisymmetry implies

$$\mathcal{H}_L^N \doteq \bigwedge^N \mathcal{H}_L^1 = \{0\} \quad \text{whenever } N > |\Lambda_L| \dim V.$$

For a fixed particle number N , the dynamics is generated by an N -body Hamiltonian acting on $\mathcal{H}_N$. A typical example is

$$H_N = \sum_{a=1}^N h^{(a)} + \sum_{1 \leq a < b \leq N} w^{(ab)},$$

where w is assumed to be a two-body self-adjoint operator invariant under exchange of its two one-particle factors. Under these hypotheses, H_N preserves $\bigwedge^N \mathcal{H}_L^1$. More precisely:

1. $h^{(a)}$ is a shorthand notation for

$$h^{(a)} \doteq \mathbf{1}^{\otimes a-1} \otimes h \otimes \mathbf{1}^{\otimes N-a}$$

denoting the one-particle Hamiltonian in [Equation \(9\)](#) (see examples 12) acting on the a -th particle.

2. $w^{(ij)}$ is a shorthand notation for the lifting of the two-particle interaction potential operator

$$w : \mathcal{H}_L^1 \otimes \mathcal{H}_L^1 \longrightarrow \mathcal{H}_L^1 \otimes \mathcal{H}_L^1 \quad (10)$$

to $(\mathcal{H}_L^1)^{\otimes N}$, acting on the i -th and j -th particles and as the identity on all the remaining ones.

Example 13.

1. For particles without internal degrees of freedom, a natural class of two-body interactions is given by multiplication operators. More precisely, let

$$\begin{aligned} v : \ell^2(\Lambda_L) \otimes \ell^2(\Lambda_L) &\longrightarrow \ell^2(\Lambda_L) \otimes \ell^2(\Lambda_L) \\ \psi(i, j) &\longmapsto v(i - j)\psi(i, j) \end{aligned}$$

where $v : \Lambda_L \rightarrow \mathbb{R}$ is a sufficiently fast decaying function. The corresponding interaction on the N -particle Hilbert space is

$$(v^{ab}\Omega_n)(x_1, \dots, x_N) = v(i_a - i_b)\Omega_n(i_1, \dots, i_N), \quad (11)$$

where v^{ab} acts as the two-body operator v on the a -th and b -th particles and as the identity on all the remaining factors. Notice that v is diagonal in the position basis of $(\mathcal{H}_L^1)_a \otimes (\mathcal{H}_L^1)_b$ and translational-invariant and therefore depends only on the difference of coordinates of the two particles.

2. When the particles carry spin $1/2$, so that the one-particle Hilbert space is

$$\mathcal{H}_L^1 = \ell^2(\Lambda_L) \otimes \mathbb{C}^2,$$

one may also consider spin-dependent interactions. A paradigmatic example is the Heisenberg exchange interaction $j_{ab} \mathbf{s}_a \cdot \mathbf{s}_b$, where $\mathbf{s}_a = \frac{1}{2}(\sigma_a^x, \sigma_a^y, \sigma_a^z)$ and J_{ab} acts on the spatial variables as

$$(j_{ab}\Omega_n)(i_1, \sigma_1, \dots, i_N, \sigma_N) = j(i_a - i_b)\Omega_n(i_1, \sigma_1, \dots, i_N, \sigma_N),$$

with $j : \Lambda_L \rightarrow \mathbb{R}$ a sufficiently fast decaying coupling. As before, j_{ab} acts on the a -th and b -th particles and as the identity on the remaining ones.

At finite temperature, equilibrium states are described by the canonical density matrix

$$\rho_{\beta, L, N} = \frac{e^{-\beta H_N}}{\text{Tr}_{\mathcal{H}_L^N} e^{-\beta H_N}},$$

which corresponds to a fixed number of particles. This is sometimes called *canonical ensemble formalism*⁷. Although this canonical formulation is mathematically natural, it is not well suited to many-body systems in which the particle number is allowed to fluctuate, nor is it convenient from a technical viewpoint, since one has to work separately in each sector $\mathcal{H}_L^N$.

2.2.3 Fock Space and Second Quantization

It is therefore preferable to adopt the *grand-canonical formalism*.⁸ In this formalism, all particle sectors are treated simultaneously by introducing the fermionic Fock space:

$$\mathcal{F}_L = \bigoplus_{N=0}^{\dim \mathcal{H}_L^1} \mathcal{H}_L^N,$$

whose vacuum sector, corresponding to the only state with 0 particles, is

$$\bigwedge^0 \mathcal{H}_L^1 = \mathbb{C}|0\rangle.$$

The decomposition in sectors with a fixed number of particles allows to introduce a particle number operator

$$N : \mathcal{F}_L \longrightarrow \mathcal{F}_L,$$

defined by

$$N|_{\mathcal{H}_L^N} = N \mathbf{1}.$$

The eigenspace associated with the eigenvalue N is precisely the N -particle sector $\mathcal{H}_L^N$. Consequently, operators commuting with N preserve the particle number and describe many-body systems which can be restricted in a fixed particle number sector.

The Fock space provides the natural setting for the *second quantization formalism*.

Definition 14 (Creation and annihilation operators). To every vector $f \in \mathcal{H}_L^1$ one associates creation and annihilation operators $a^\pm(f)$ acting on $\mathcal{F}_L$, satisfying:

⁷See [LS10; RS75] for further background on the first-quantized many-body formalism.

⁸In effective descriptions of superconductivity, particle number need not be conserved, although fermion parity is. Such dynamics cannot be restricted to a fixed particle-number sector.

1. $(a^-(f))^* = a^+(f)$
2. The canonical anticommutation relations (CAR)

$$\{a^-(f), a^-(g)\} = 0 \quad \{a^+(f), a^+(g)\} = 0 \quad \{a^-(f), a^+(g)\} = \langle f, g \rangle_{\mathcal{H}_L^1} \mathbf{1}.$$

Remark 11. Choosing an orthonormal basis $\{e_\alpha\}_{\alpha \in A}$ of $\mathcal{H}_L^1$, one writes $a_\alpha^\pm = a^\pm(e_\alpha)$, so that

$$\{a_\alpha^-, a_\beta^-\} = 0, \quad \{a_\alpha^+, a_\beta^+\} = 0, \quad \{a_\alpha^-, a_\beta^+\} = \delta_{\alpha\beta}. \quad (12)$$

The vacuum vector $|0\rangle$ is characterized by

$$a_\alpha^- |0\rangle = 0, \quad \forall \alpha \in A.$$

Taking linear combinations of the resulting vectors, the entire Fock space can be obtained via repeated applications of the creation operators over the vacuum state:

$$a_{\alpha_1}^+ \cdots a_{\alpha_N}^+ |0\rangle = |\alpha_1 \cdots \alpha_N\rangle,$$

Notice that in this way the antisymmetry of the wave functions in $\mathcal{H}_L^N$ follows automatically from the CAR.

Definition 15 (Algebra of Fermionic Observables). The (C^*) -algebra $\mathcal{A}_{\text{CAR}}$ of fermionic observables is the (C^*) -algebra generated by the creation and annihilation operators by taking any possible linear combination of monomials in the creation and annihilation operators.

Remark 12. $\mathcal{A}_{\text{CAR}}$ is normed by the operator norm:

$$\|\mathcal{O}\|_{\text{op}} = \sup_{\substack{|\psi\rangle \in \mathcal{F}_L: \\ \langle \psi | \psi \rangle = 1}} \sqrt{\langle \mathcal{O}^* \psi | \mathcal{O} \psi \rangle}$$

One can easily show that $\|a_\alpha^\pm\|_{\text{op}} \leq 1$ as:

For any observable $\mathcal{O} \in \mathcal{A}_{\text{CAR}}$, we define the parity automorphism as:

$$\mathcal{P}(\mathcal{O}) = (-1)^N \mathcal{O} (-1)^N. \quad (13)$$

Since $\mathcal{P}$ is an involution, its eigenvalues are ± 1 . We denote its even and odd eigenspaces by $\mathcal{A}_{\text{CAR}}^{\text{even}}$ and $\mathcal{A}_{\text{CAR}}^{\text{odd}}$, respectively. Thus $\mathcal{A}_{\text{CAR}} = \mathcal{A}_{\text{CAR}}^{\text{even}} \oplus \mathcal{A}_{\text{CAR}}^{\text{odd}}$ is a $\mathbb{Z}_2$ -graded algebra⁹ (cf. [Remark 4](#)). Physical observables will be taken in the even subalgebra $\mathcal{A}_{\text{CAR}}^{\text{even}}$, in accordance with fermion-parity superselection:

$$\mathcal{O} = \mathcal{P}(\mathcal{O}). \quad (14)$$

Even fermionic observables satisfy the following locality property. If $X, Y \subset \Lambda_L$ are disjoint and $\mathcal{O}_X, \mathcal{O}_Y$ are even observables localized¹⁰ in X and Y , respectively, then

$$[\mathcal{O}_X, \mathcal{O}_Y] = 0. \quad (15)$$

We will denote the algebra of even fermionic observables localized in a set $X \subset V(\Lambda_L)$ as $\mathcal{A}_{\text{CAR}}^{\text{even}}(X)$. [Equation \(15\)](#) says that:

⁹Note that $\mathcal{A}_{\text{CAR}}^{\text{odd}}$ is a vector subspace, not a subalgebra.

¹⁰The support of an operator is given by the union of the supports of the creation/annihilation operators that generates it. Alternatively, we can define the support of an operator as the set of the sites where it acts non trivially (i.e. as the identity).

$$[\mathcal{A}_{\text{CAR}}^{\text{even}}(X), \mathcal{A}_{\text{CAR}}^{\text{even}}(Y)] = \{0\} \quad X \cap Y = \emptyset.$$

One of the main advantages of the second quantization formalism is that any one-particle operator such those in Equation (9) admits a canonical lift to the fermionic Fock space. If

$$h_{\alpha\beta} = \langle e_\alpha, h e_\beta \rangle,$$

are the matrix elements of the one-particle Hamiltonian in an orthonormal basis $\{e_\alpha\}_{\alpha \in A}$ of $\mathcal{H}_L^1$, its second quantization is

$$H_0 = \sum_{\alpha, \beta \in A} h_{\alpha\beta} a_\alpha^+ a_\beta^-. \quad (16)$$

Similarly the second quantization of the number operator is

$$N = \sum_{\alpha \in A} a_\alpha^+ a_\alpha^-.$$

The second quantization formalism also accommodates interactions in a natural way. Let w be a two-particle operator as in Equation (10) with matrix elements

$$w_{\alpha\beta\gamma\delta} = \langle e_\alpha \otimes e_\beta, w(e_\gamma \otimes e_\delta) \rangle.$$

Its second quantization is defined by

$$W = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} w_{\alpha\beta\gamma\delta} : a_\alpha^+ a_\beta^+ a_\delta^- a_\gamma^- :, \quad (17)$$

where $:\dots:$ denotes normal ordering, namely all creation operators are placed to the left of all annihilation operators, taking into account the signs arising from the canonical anticommutation relations. Since the above monomial is already normally ordered, the colons may be omitted. More generally, one does not necessarily start from a fixed N -particle Hamiltonian. Instead, the many-body theory is usually formulated directly on the fermionic Fock space by considering self-adjoint operators built from creation and annihilation operators. Such Hamiltonians naturally decompose into a free and an interacting part,

$$H = H_0 + H_{\text{int}},$$

where H_0 is a quadratic operator as in Equation (16) and H_{int} contains normally ordered higher-order monomials, the leading contribution being typically quartic and corresponding to effective two-body interactions such as those in Equation (17). Although higher-order interaction terms may also be considered, Hamiltonians consisting of quadratic and quartic terms already encompass the vast majority of lattice models arising in Condensed Matter Physics and Lattice Quantum field Theory. The following examples illustrate this general framework.

Example 16.

1. The second quantization of the magnetic Laplacian introduced in Example 12-Item 2 is

$$h = -t\Delta_A + u \xrightarrow{\text{Second Quantization}} H = -t \sum_{\substack{i,j \in V(\Lambda_L): \\ (i,j) \in E(\Lambda_L)}} U_{ij} a_i^+ a_j^- + \sum_{i \in V(\Lambda_L)} \mu(i) n_i,$$

where $n_i \doteq a_i^+ a_i^-$ denotes the local number operator.

2. For the two-body interaction introduced in Equation (11), the second-quantized interaction reads

$$V = \frac{1}{2} \sum_{i,j \in V(\Lambda_L)} V(i-j) a_i^+ a_j^+ a_j^- a_i^-.$$

Similarly, the Heisenberg exchange interaction becomes

$$J = \frac{1}{4} \sum_{i,j \in V(\Lambda_L)} \sum_{\mu=1}^3 \sum_{a,b,c,d=\uparrow,\downarrow} J(i-j) a_{i,a}^+ \sigma_{ab}^\mu a_{i,b}^- a_{j,c}^+ \sigma_{cd}^\mu a_{j,d}^-.$$

3. A paradigmatic example combining a quadratic kinetic term with a local quartic interaction is provided by the Hubbard Hamiltonian,

$$H_{\text{Hub}} = -t \sum_{\substack{i,j \in V(\Lambda_L): \\ (i,j) \in E(\Lambda_L)}} \sum_{\sigma=\uparrow,\downarrow} a_{i,\sigma}^+ a_{j,\sigma}^- + \sum_{i \in V(\Lambda_L)} U(i) (n_{i,\uparrow} - \frac{1}{2})(n_{i,\downarrow} - \frac{1}{2}) \quad (18)$$

where $U : \Lambda_L \rightarrow \mathbb{R}$ is the on-site interaction strength with $\|U\|_\infty < C$ for some $C > 0$ independent of L . Despite its apparent simplicity, the Hubbard model captures the competition between electron delocalization and Coulomb repulsion and constitutes one of the fundamental models of condensed Matter physics.

From a mathematical perspective, second quantization is considerably more than a convenient notation: it provides a unified language for describing free and interacting many-body systems, automatically incorporating the Pauli exclusion principle, and allows one to formulate Hamiltonians independently of the number of particles in the system. Once a Hamiltonian on the fermionic Fock space has been specified, the next natural step is to describe its equilibrium statistical mechanics. This is achieved within the grand-canonical formalism by introducing the Gibbs density matrix, which encodes the thermal equilibrium state at fixed temperature and chemical potential but arbitrary number of particles. As we shall see in the following sections, the grand-canonical partition function and the corresponding correlation functions admit an equivalent representation in terms of Grassmann functional integrals on the Euclidean space-time manifold $\Lambda_L \times S_\beta^1$. In particular, the passage from Hamiltonian many-body systems to Grassmann functional integrals, and subsequently to their discretization on Euclidean space-time lattices, establishes the conceptual bridge between the Hamiltonian and Euclidean formulations.

2.2.4 Real-Time Evolution and Lieb-Robinson Bounds

Once a Hamiltonian H has been specified, one of the fundamental questions is to understand how local observables, and possibly their correlation functions, behave under the time evolution generated by H . In the Heisenberg picture, the dynamics of an observable $\mathcal{O} \in \mathcal{A}_{\text{CAR}}$ is given by

$$\tau_t(\mathcal{O}) \doteq e^{itH} \mathcal{O} e^{-itH}.$$

In the remainder of this section, we restrict ourselves to local even fermionic observables and to Hamiltonians generated by uniformly bounded finite-range even interactions. The presentation below follows primarily the treatment of Lieb–Robinson bounds given by Naaikens [Naa17] and the general quasi-locality framework of Nachtergaele, Sims and Young [NSY19]. We specialize their

discussion to even fermionic interactions and observables, for which ordinary locality is recovered on disjoint supports; for the general fermionic formulation, including odd observables and the corresponding graded locality structure, we refer to [NSY18].

More precisely, we consider finite range Hamiltonians, meaning that they can be written as:

$$H = \sum_{X \subset V(\Lambda_L)} \Phi(X) \quad \Phi(X) \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X)$$

where:

1. $\Phi(X)^* = \Phi(X)$.
2. There exists $R < \infty$, independent of L , such that

$$\Phi(X) = 0 \quad \text{if} \quad \ell(X) \doteq \sup_{x,y \in X} d(x,y) > R.$$

3. There exists $C_\Phi < \infty$, independent of L , such that

$$\|\Phi(X)\|_\infty \leq C_\Phi |X|.$$

where $|X|$ is the cardinality of $X \subset \Lambda_L$.

The constant R is referred to as the *range* of the interaction. Notice that the restriction to even interactions guarantees that the Hamiltonian preserves fermionic parity under the dynamics. The Hamiltonians introduced in [Example 16](#) - [Item 1](#) and [Item 3](#) are examples of finite-range interactions with $R = 1$. With the normalization adopted there, one can choose $C_\Phi = \max\{|\mu|, t\}$ for the former examples and $C_\Phi = \max\{|U|, 2t\}$ for the Hubbard Hamiltonian.

Remark 13. The finite-range assumption is convenient but by no means essential. Lieb–Robinson estimates continue to hold for interactions with sufficiently fast spatial decay, including suitable exponentially or polynomially decaying interactions. This allows one, for instance, to treat long-range interactions of the type introduced in [Item 2](#), provided that the corresponding functions V and J decay sufficiently rapidly. We shall not pursue this more general setting here, since finite-range interactions are sufficient for the applications considered below (see [NSY19; NSY18] for more general formulations).

Let us first observe that the above assumptions imply a uniform local bound on the strength of the interaction.

Lemma 17. *Under the above assumption on the interaction Φ , then:*

$$\|\Phi\|_{\text{loc}} \doteq \sup_{x \in \Lambda_L} \sum_{\substack{X \subset \Lambda_L \\ x \in X}} |X| \|\Phi(X)\|_{\text{op}}$$

is finite uniformly in L .

Proof. Indeed, if $x \in X$ and $\Phi(X) \neq 0$, the finite-range condition implies $X \subset B_R(x)$. On the d -dimensional lattice there exists a constant $K_d > 0$, depending only on the dimension, such that

$$|B_R(x)| \leq K_d R^d,$$

where, without loss of generality, we take $R \geq 1$. Moreover, a ball containing N sites has exactly 2^{N-1} subsets containing a prescribed site x . Since

$$|X| \|\Phi(X)\|_{\text{op}} \leq C_\Phi |X|^2 \leq C_\Phi |B_R(x)|^2,$$

we obtain the rough estimate

$$\|\Phi\|_{\text{loc}} \leq C_\Phi \tilde{K}_d R^{2d} 2^{K_d R^d - 1} \quad \text{for some } \tilde{K}_d > 0,$$

which is independent of the volume L . The precise value of this estimate will play no role below; its only purpose is to show that finite range together with the uniform bound on the interaction implies $\|\Phi\|_{\text{loc}} < \infty$. $\blacksquare$

Remark 14. The preceding estimate is deliberately crude, since it allows a non-vanishing interaction term $\Phi(X)$ for every subset of $B_R(x)$. It can be substantially improved when the interaction has bounded body order. If Φ is at most k -body, i.e.

$$\Phi(X) = 0 \quad \text{for} \quad |X| > 2k,$$

then, for fixed k ,

$$\#\left\{X \subset \Lambda_L | x \in X, X \subset B_R(x) \text{ and } |X| \leq k\right\} = O(R^{d(2k-1)}).$$

Consequently $\|\Phi\|_{\text{loc}} = O(C_\Phi R^{d(k-1)})$. In particular, for a uniformly bounded two-body interaction, one obtains the natural estimate

$$\|\Phi\|_{\text{loc}} = O(C_\Phi R^d).$$

For nearest-neighbour Hamiltonians, $R = 1$, so that $\|\Phi\|_{\text{loc}}$ is bounded by a constant depending only on the interaction strengths and on the spatial dimension.

We now turn to the dynamical consequences of locality. The natural question is not quite whether the support of an observable remains finite under time evolution but rather to what extent an initially local observable remains *approximately local*. More precisely, given $A \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X)$, one would like to understand how strongly $\tau_t(A)$ can affect observables supported far away from X .

At $t = 0$, ordinary locality is recovered in the even CAR algebra (see [Equation \(15\)](#)). Since an even Hamiltonian preserves fermionic parity, this observation allows us to use the ordinary commutator throughout the present discussion. The additional graded structure of the CAR algebra becomes relevant when one considers odd observables: this more general setting will not be needed here. Eventhough non-relativistic lattice Hamiltonians do not possess a strict causal structure, locality of the interaction nevertheless imposes an effective finite speed of propagation. This remarkable fact is quantified by the *Lieb–Robinson bounds*:

Theorem 18 (Lieb–Robinson Bounds). *Let*

$$A \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X), \quad B \in \mathcal{A}_{\text{CAR}}^{\text{even}}(Y),$$

be two observables supported in two disjoint regions X and Y , then there exist three L -independent constants $C, \mu, v_{\text{LR}} > 0$, such that for a finite range Hamiltonian, the following holds:

$$\|[\tau_t(A), B]\|_{\text{op}} \leq 2 \min\{|X|, |Y|\} \|A\|_{\text{op}} \|B\|_{\text{op}} e^{-\mu(d(X,Y) - v_{\text{LR}}|t|)} \quad d(X, Y) \doteq \inf_{x \in X, y \in Y} |x - y|.$$

The constant v_{LR} is called Lieb–Robinson velocity and is defined as:

$$v_{\text{LR}} = \frac{2e^{\mu R}}{R} \|\Phi\|_{\text{loc}}$$

Remark 15.

1. Although $\tau_t(A)$ is in general no longer strictly supported in X for $t \neq 0$, its non-local tails are exponentially small outside the effective cone

$$d(X, Y) \lesssim v_{\text{LR}}|t|.$$

In this sense, the Lieb–Robinson velocity v_{LR} plays the role of an emergent maximal propagation velocity for non-relativistic lattice systems. Equivalently, the dynamics maps local observables into *quasi-local* observables. More precisely, if $X^{(r)}$ denotes the r -neighbourhood of X , meaning the subset of Λ_L whose points have at most distance r from X , one can construct a strictly local observable

$$A^{(r)}(t) \in \mathcal{A}_{X^{(r)}}$$

such that, schematically,

$$\|\tau_t(A) - A^{(r)}(t)\|_{\text{op}} \leq C\|A\|_{\text{op}}e^{-\mu(r-v_{\text{LR}}|t|)}.$$

Thus, outside the Lieb–Robinson cone, the time-evolved observable can be replaced by a genuinely local one at the cost of an exponentially small error. We refer to [NSY19, Sect. 4.2] for a systematic discussion of such local approximations.

2. Lieb–Robinson bounds were first established by Lieb and Robinson [LR72] for finite range translationally invariant spin systems and provided the first quantitative formulation of this effective finite propagation speed. Such bounds give a useful non-relativistic substitute for the strict locality familiar from relativistic quantum field theory and have subsequently become a fundamental tool in the analysis of quantum many-body systems. An important conceptual development of these later formulations is that the mechanism behind Lieb–Robinson bounds does not fundamentally rely on translation invariance, Fourier analysis, or even on the existence of a regular underlying lattice. Instead, the estimates can be formulated for general quantum systems with finite dimensional local Hilbert spaces defined on rather general metric spaces, provided that the geometry and the decay of the interaction satisfy suitable summability conditions [NS06; NSY19]. This metric formulation makes particularly transparent that what drives the effective causal structure is the locality of the interaction itself.
3. An early application was given by Wreszinski, who used Lieb–Robinson estimates in the analysis of Goldstone-type results for non-relativistic quantum systems [Wre87]. Hastings later exploited this notion of approximate locality in his higher-dimensional extension of the Lieb–Schultz–Mattis theorem [LSM61; Has04a] and in establishing the connection between a non-vanishing spectral gap and exponential decay of correlations [Has04b]. The latter implication was subsequently developed in a general and rigorous form by Nachtergaele and Sims [NS06]. Lieb–Robinson bounds and the resulting notion of quasi-locality also play an important role in the operator-algebraic description of topological excitations in non-relativistic quantum lattice systems. Following the Doplicher–Haag–Roberts and Buchholz–Fredenhagen ideas from algebraic quantum field theory, localized and transportable endomorphisms of the quasi-local observable algebra can be used to describe superselection sectors corresponding to anyonic excitations. This program was developed for the toric code and abelian quantum double models in [Naa11; FN15], where the fusion and braiding properties of the excitations are encoded in the corresponding category of superselection sectors. In this context, Lieb–Robinson estimates

become particularly important when strict localization is replaced by approximate localization and one studies the stability of the sector structure under quasi-local deformations. In particular, Cha, Naaijken and Nachtergaele [CNN20] used Lieb–Robinson bounds adapted to cone-like regions to show that almost-localized endomorphisms, and hence the associated braided tensor structure, are preserved under suitable quasi-local dynamics. More recently, this operator-algebraic superselection framework has been extended to non-Abelian quantum double models, where the full category of anyonic sectors can be recovered within a DHR-type framework [Bol+26].

Proof. For a finite set $X \subset \Lambda_L$, define

$$C_B(X, t) \doteq \sup_{\substack{A \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X) \\ A \neq 0}} \frac{\|[\tau_t(A), B]\|}{\|A\|}.$$

We first derive an integral inequality for $C_B(X, t)$. Ordinary locality of the even CAR algebra (see Equation (15)) implies:

$$\frac{d}{dt} \tau_t(A) = i \sum_{\Omega: \Omega \cap X \neq \emptyset} [\tau_t([\Phi(\Omega), A]), B].$$

Using the Jacobi identity, we obtain

$$\frac{d}{dt} \tau_t(A) = i \sum_{\Omega: \Omega \cap X \neq \emptyset} [\tau_t(\Phi(\Omega)), f(t)] - i \sum_{\Omega: \Omega \cap X \neq \emptyset} [\tau_t(A), [\tau_t(\Phi(\Omega)), B]].$$

Using variation of the constants formula and the applying the norm on both sides, we get the following estimate:

$$\begin{aligned} \|[\tau_t(A), B]\|_{\text{op}} &\leq \| [A, B] \|_{\text{op}} + 2\|A\|_{\text{op}} \sum_{\Omega: \Omega \cap X \neq \emptyset} \int_0^{|t|} \|[\tau_s(\Phi(\Omega)), B]\|_{\text{op}} ds \\ &\leq \| [A, B] \|_{\text{op}} + 2\|A\|_{\text{op}} \sum_{\Omega: \Omega \cap X \neq \emptyset} \int_0^{|t|} \|\Phi(\Omega)\|_{\text{op}} C_B(\Omega, s) ds \\ &= 2\|A\|_{\text{op}} \sum_{\Omega: \Omega \cap X \neq \emptyset} \int_0^{|t|} \|\Phi(\Omega)\|_{\text{op}} C_B(\Omega, s) ds. \end{aligned}$$

Dividing by $\|A\|$ and taking the supremum over $A \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X)$ yields:

$$C_B(X, t) \leq 2 \sum_{\Omega: \Omega \cap X \neq \emptyset} \|\Phi(\Omega)\|_{\text{op}} \int_0^{|t|} C_B(\Omega, s) ds.$$

Iterating the recursive inequality therefore gives

$$C_B(X, t) \leq 2\|B\|_{\text{op}} \sum_{n=1}^{\infty} \frac{(2|t|)^n}{n!} a_n(X, Y),$$

where

$$a_n(X, Y) \doteq \sum_{\substack{\Omega_1, \dots, \Omega_n \\ \Omega_1 \cap X \neq \emptyset \\ \Omega_j \cap \Omega_{j-1} \neq \emptyset, j=2, \dots, n \\ \Omega_n \cap Y \neq \emptyset}} \prod_{j=1}^n \|\Phi(\Omega_j)\|_{\text{op}}.$$

Thus $a_n(X, Y)$ is the total weight of all chains of interaction terms starting from X , overlapping successively, and eventually reaching Y . We next estimate the total weight of such chains. Define

$$Q_n(X) \doteq \sum_{\substack{\Omega_1, \dots, \Omega_n \\ \Omega_1 \cap X \neq \emptyset \\ \Omega_j \cap \Omega_{j-1} \neq \emptyset, j=2, \dots, n}} \prod_{j=1}^n \|\Phi(\Omega_j)\|_{\text{op}}.$$

We claim that $Q_n(X) \leq |X| \|\Phi\|_{\text{loc}}^n$. For $n = 1$:

$$Q_1(X) = \sum_{\Omega: \Omega \cap X \neq \emptyset} \|\Phi(\Omega)\|_{\text{op}} \leq \sum_{x \in X} \sum_{\Omega: x \in \Omega} \|\Phi(\Omega)\|_{\text{op}} \leq \sum_{x \in X} \sum_{\substack{\Omega: \\ x \in \Omega}} |\Omega| \|\Phi(\Omega)\|_{\text{op}} \leq |X| \|\Phi\|_{\text{loc}}.$$

Suppose now, inductively, that $Q_{n-1}(\Omega) \leq |\Omega| \|\Phi\|_{\text{loc}}^{n-1}$, then:

$$\begin{aligned} Q_n(X) &\leq \sum_{\Omega: \Omega \cap X \neq \emptyset} \|\Phi(\Omega)\|_{\text{op}} Q_{n-1}(\Omega) \\ &\leq \|\Phi\|_{\text{loc}}^{n-1} \sum_{\Omega: \Omega \cap X \neq \emptyset} |\Omega| \|\Phi(\Omega)\|_{\text{op}} \\ &\leq \|\Phi\|_{\text{loc}}^{n-1} \sum_{x \in X} \sum_{\Omega \ni x} |\Omega| \|\Phi(\Omega)\|_{\text{op}} \\ &\leq |X| \|\Phi\|_{\text{loc}}^n. \end{aligned}$$

The claim follows by induction. In particular $a_n(X, Y) \leq Q_n(X) \leq |X| \|\Phi\|_{\text{loc}}^n$. It remains to use the finite-range assumption. If $\Omega_1, \dots, \Omega_n$ contributes to $a_n(X, Y)$, then

$$\Omega_1 \cap X \neq \emptyset, \quad \Omega_j \cap \Omega_{j-1} \neq \emptyset, \quad \Omega_n \cap Y \neq \emptyset.$$

Since $\text{diam}(\Omega_j) \leq R$ for every interaction term contributing to the chain, the triangle inequality gives

$$d(X, Y) \leq \sum_{j=1}^n \text{diam}(\Omega_j) \leq nR.$$

Consequently,

$$a_n(X, Y) = 0 \quad \text{whenever} \quad nR < d(X, Y).$$

Let now $\mu > 0$. Whenever a chain contributes, $nR \geq d(X, Y)$ and hence $1 \leq e^{-\mu d(X, Y)} e^{\mu nR}$. It follows that:

$$a_n(X, Y) \leq |X| \|\Phi\|_{\text{loc}}^n e^{-\mu d(X, Y)} e^{\mu nR}$$

so that, by substituting this estimate into the expansion for $C_B(X, t)$, we obtain:

$$\begin{aligned} C_B(X, t) &\leq 2\|B\| |X| e^{-\mu d(X, Y)} \sum_{n=1}^{\infty} \frac{(2\|\Phi\|_{\text{loc}} e^{\mu R} |t|)^n}{n!} \\ &= 2\|B\| |X| e^{-\mu d(X, Y)} \left(e^{2\|\Phi\|_{\text{loc}} e^{\mu R} |t|} - 1 \right) \\ &\leq 2\|B\| |X| e^{-\mu d(X, Y) + 2\|\Phi\|_{\text{loc}} e^{\mu R} |t|}. \end{aligned} \tag{19}$$

Multiplying by $\|A\|$, we conclude that

$$\|[\tau_t(A), B]\| \leq 2|X| \|A\| \|B\| e^{-\mu d(X, Y) + 2e^{\mu R} \|\Phi\|_{\text{loc}} |t|}.$$

Defining

$$v_{\text{LR}} \doteq \frac{2e^{\mu R} \|\Phi\|_{\text{loc}}}{\mu},$$

this can be rewritten as

$$\|[\tau_t(A), B]\| \leq 2|X| \|A\| \|B\| e^{-\mu(d(X,Y) - v_{\text{LR}}|t|)}.$$

Finally, since the dynamics preserves the operator norm, $\|[\tau_t(A), B]\| = \|[A, \tau_{-t}(B)]\|$, the same argument can be applied with X and Y interchanged. Therefore the prefactor $|X|$ can be replaced by $\min\{|X|, |Y|\}$. This concludes the proof. $\blacksquare$

2.2.5 Exponential clustering from a spectral gap

One of the most important consequences of the Lieb–Robinson bounds is the exponential clustering of ground-state correlations in gapped quantum systems. The basic mechanism is remarkably simple: the spectral gap suppresses contributions coming from sufficiently long times, whereas the Lieb–Robinson bound controls the propagation of local observables at shorter times. Combining the two estimates yields a finite correlation length.

The argument presented below follows the Gaussian-filter proof of Nachtergaele and Sims [NS06] (see also [Has04b; HK06b]). The latter reference treats both quantum-spin and fermionic systems. Since throughout this section we restrict ourselves to even fermionic observables, ordinary locality holds on disjoint supports and the proof can be formulated entirely in terms of ordinary commutators.

We shall use the slightly sharper form of the Lieb–Robinson estimate obtained in the proof of [Theorem 18](#), namely (see [Equation \(19\)](#)):

$$\|[\tau_t(A), B]\|_{\text{op}} \leq 2 \min\{|X|, |Y|\} \|A\|_{\text{op}} \|B\|_{\text{op}} e^{-\mu d(X,Y)} \left(e^{\mu v_{\text{LR}}|t|} - 1 \right) \quad (20)$$

The factor $e^{\mu v_{\text{LR}}|t|} - 1$, which is of order $|t|$ as $t \rightarrow 0$, will be useful below in compensating the singularity of the spectral filter at the origin.

Theorem 19 (Exponential clustering). *Let H be a finite-range fermionic Hamiltonian satisfying the assumptions of [Theorem 18](#). Assume that H has a unique normalized ground state $|\Omega_L\rangle$, and that the spectral gap above the ground state is bounded uniformly in the volume: there exists¹¹ $\gamma > 0$, independent of L , such that*

$$\text{spec}(H - E_0(L)) \subset \{0\} \cup [\gamma, \infty).$$

Then, for every $\mu > 0$, there exists $C_\mu, m_\mu < \infty$, independent of L , such that

$$|\langle \Omega_L | AB \Omega_L \rangle - \langle \Omega_L | A \Omega_L \rangle \langle \Omega_L | B \Omega_L \rangle| \leq C_\mu \min\{|X|, |Y|\} \|A\|_{\text{op}} \|B\|_{\text{op}} e^{-m_\mu d(X,Y)} \quad (21)$$

for all $A \in \mathcal{A}_{\text{CAR}}^{\text{even}}(X)$ and $B \in \mathcal{A}_{\text{CAR}}^{\text{even}}(Y)$ with disjoint support. The constant m_μ can be written as:

$$m_\mu = \frac{\mu\gamma}{2\mu v_{\text{LR}} + \gamma}$$

¹¹The gap γ can be defined as $\inf_L \gamma_L > 0$ where γ_L is the gap of the hamiltonian H_L on Λ_L . The statement is that finite volume gaps don't close as the volume is increased.

Proof. By subtracting the ground-state energy, we can always assume that:

$$H|\Omega_L\rangle = 0, \quad \text{spec}(H) \subset \{0\} \cup [\gamma, \infty). \quad (22)$$

By replacing B by $\tilde{B} \doteq B - \langle \Omega_L | B \Omega_L \rangle$, we have

$$\langle \Omega_L | A \tilde{B} \Omega_L \rangle = \langle \Omega_L | A B \Omega_L \rangle - \langle \Omega_L | A \Omega_L \rangle \langle \Omega_L | B \Omega_L \rangle.$$

Notice that $\tilde{B}$ can still be regarded as an even observable localized in Y since it acts non-trivially only in Y . The idea of the proof is to decompose $\tilde{B}$ approximately into its positive-frequency part. For $\alpha > 0$, define

$$\tilde{B}_\alpha^+ \doteq \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-\alpha t^2}}{t - i\epsilon} \tau_t(\tilde{B}) dt. \quad (23)$$

where the convergence is meant in operator norm. This is a Gaussian regularization of the projection onto positive frequencies. Indeed, if we consider the complete set of eigenvectors $\{|\Omega_n\rangle\}_{n \in \mathbb{N}}$ of H such that:

$$H|\Omega_n\rangle = E_n|\Omega_n\rangle \quad (|\Omega_0\rangle \doteq |\Omega_L\rangle),$$

then, using Sokhotski–Plemelj formula¹², we get:

$$\begin{aligned} \langle \Omega_m | \tilde{B}_\alpha^+ \Omega_n \rangle &= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dt \frac{e^{-\alpha t^2}}{t - i\epsilon} \langle \Omega_m | \tau_t(\tilde{B}) \Omega_n \rangle \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dt \frac{e^{-\alpha t^2}}{t - i\epsilon} e^{it(E_m - E_n)} \langle \Omega_m | \tilde{B} \Omega_n \rangle \\ &= \langle \Omega_m | \tilde{B} \Omega_n \rangle \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-\alpha t^2} \frac{\sin(E_m - E_n)t}{t} + \frac{1}{2} \right) \\ &\doteq \langle \Omega_m | \tilde{B} \Omega_n \rangle \left(\frac{1}{2\pi} G_\alpha(E_m - E_n) + \frac{1}{2} \right) \end{aligned}$$

The integral can be computed using Feynman's trick. Let us first compute the derivative of $G_\alpha(\omega)$:

$$G'_\alpha(\omega) = \int_{-\infty}^{+\infty} dt e^{-\alpha t^2} \cos(\omega t) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

so that:

$$G_\alpha(\omega) = G_\alpha(0) + \int_0^\omega \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega'^2}{4\alpha}} d\omega' = \pi \operatorname{erf} \left(\frac{\omega}{2\sqrt{\alpha}} \right).$$

Putting all together, we have:

$$\langle \Omega_m | \tilde{B}_\alpha^+ \Omega_n \rangle = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{E_m - E_n}{2\sqrt{\alpha}} \right) \right) \langle \Omega_m | \tilde{B} \Omega_n \rangle \doteq F_\alpha(E_m - E_n) \langle \Omega_m | \tilde{B} \Omega_n \rangle.$$

Let us notice a couple of properties of the function in the round brackets: there exists a numerical constant $C > 0$ such that, for every $\omega \geq \gamma$,

$$0 \leq 1 - F_\alpha(\omega) \leq C e^{-\gamma^2/(4\alpha)} \quad 0 \leq F_\alpha(-\omega) \leq C e^{-\gamma^2/(4\alpha)}. \quad (24)$$

¹² $\lim_{\epsilon \rightarrow 0^+} \frac{1}{t - i\epsilon} = \text{p.v.} \frac{1}{t} + i\pi \delta(t)$

Since:

$$\tilde{B}|\Omega_L\rangle = (1 - |\Omega_L\rangle\langle\Omega_L|)\tilde{B}|\Omega_L\rangle = \sum_{n \geq 1} |\Omega_n\rangle \langle\Omega_n|\tilde{B}\Omega_L\rangle,$$

Equation (22) and Equation (24) imply:

$$\begin{aligned} \|(\tilde{B} - \tilde{B}_\alpha^+)|\Omega_L\rangle\|^2 &= \left\| \sum_{n \geq 1} |\Omega_n\rangle \langle\Omega_n|\tilde{B}\Omega_L\rangle \left(1 - \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^{-\alpha t^2}}{t - i\epsilon} e^{itE_n}\right) \right\|^2 \\ &= \left\| \sum_{n \geq 1} |\Omega_n\rangle \langle\Omega_n|\tilde{B}\Omega_L\rangle (1 - F_\alpha(E_n)) \right\|^2 \\ &= \sum_{n \geq 1} |\langle\Omega_n|\tilde{B}\Omega_L\rangle|^2 (1 - F_\alpha(E_n))^2 \\ &\leq (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \left\| \sum_{n \geq 1} |\Omega_n\rangle \langle\Omega_n|\tilde{B}\Omega_L\rangle \right\|^2 \\ &= (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \left\| \sum_{n \geq 0} |\Omega_n\rangle \langle\Omega_n|\tilde{B}\Omega_L\rangle \right\|^2 \\ &= (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \|\tilde{B}|\Omega_L\rangle\|^2 \\ &\leq (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \|\tilde{B}\|_{\text{op}}^2 \end{aligned} \tag{25}$$

and similarly,

$$\begin{aligned} \| |\Omega_L\rangle\langle\Omega_L| \tilde{B}_\alpha^+ (1 - |\Omega_L\rangle\langle\Omega_L|) \|_{\text{op}}^2 &= \sup_{\substack{|\psi\rangle \in \mathcal{F}_L: \\ \langle\psi|\psi\rangle=1}} \sum_{n \geq 1} |\langle\Omega_n|\psi\rangle|^2 |\langle\Omega_L|\tilde{B}\Omega_n\rangle|^2 F_\alpha^2(-E_n) \\ &\leq (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \sup_{\substack{|\psi\rangle \in \mathcal{F}_L: \\ \langle\psi|\psi\rangle=1}} |\langle\Omega_L|\tilde{B}\psi\rangle|^2 \\ &\leq (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \sup_{\substack{|\psi\rangle \in \mathcal{F}_L: \\ \langle\psi|\psi\rangle=1}} \|\tilde{B}|\psi\rangle\|^2 \\ &= (Ce^{-\frac{\gamma^2}{4\alpha}})^2 \|\tilde{B}\|_{\text{op}}^2 \end{aligned} \tag{26}$$

Moreover,

$$|\Omega_L\rangle \langle\Omega_L| \tilde{B}_\alpha^+ \Omega_L \rangle \langle\Omega_L| = |\Omega_L\rangle \langle\Omega_L| \langle\Omega_L|\tilde{B}\Omega_L\rangle F_\alpha(0) = 0.$$

We can now write:

$$\begin{aligned} \langle\Omega_L|A\tilde{B}\Omega_L\rangle &= \langle\Omega_L|A\tilde{B}_\alpha^+\Omega_L\rangle + \langle\Omega_L|A(\tilde{B} - \tilde{B}_\alpha^+)\Omega_L\rangle \\ &= \langle\Omega_L|[A, \tilde{B}_\alpha^+]\Omega_L\rangle + \langle\Omega_L|\tilde{B}_\alpha^+A\Omega_L\rangle + \langle\Omega_L|A(\tilde{B} - \tilde{B}_\alpha^+)\Omega_L\rangle \end{aligned}$$

Using Equation (25) and Equation (26) together with $\|\tilde{B}\|_{\text{op}} \leq 2\|B\|_{\text{op}}$, we obtain:

$$\left| \langle\Omega_L|A\tilde{B}\Omega_L\rangle \right| \leq \| [A, \tilde{B}_\alpha^+] \|_{\text{op}} + 4C\|A\|_{\text{op}}\|B\|_{\text{op}}e^{-\gamma^2/(4\alpha)}. \tag{27}$$

It remains to estimate the commutator. From Equation (23), we get:

$$\begin{aligned}
\|[A, \tilde{B}_\alpha^+]\|_{\text{op}} &= \left\| \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-\alpha t^2}}{t - i\varepsilon} [A, \tau_t(B)] dt \right\|_{\text{op}} \\
&= \left\| \frac{1}{2} [A, B] + \frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{+\infty} dt \frac{e^{-\alpha t^2}}{t} [A, \tau_t(B)] \right\|_{\text{op}} \\
&= \left\| \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{e^{-\alpha t^2}}{t} [A, \tau_t(B)] \right\|_{\text{op}} \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{-\alpha t^2}}{|t|} \|[A, \tau_t(B)]\|_{\text{op}}
\end{aligned} \tag{28}$$

where we removed the principal value because the integrand was regular for small t due to Lieb-Robinson bounds.

In order to bound the integral in Equation (28), given some $T > 0$, we use Equation (20) for short times, $|t| < T$, whereas for large times, $|t| > T$ we use the trivial bound $\|[A, \tau_t(B)]\|_{\text{op}} \leq 2\|A\|_{\text{op}}\|B\|_{\text{op}}$. Let us split the integral accordingly:

$$\|[A, \tilde{B}_\alpha^+]\|_{\text{op}} \leq \frac{1}{2\pi} \int_{-T}^T dt \frac{e^{-\alpha t^2}}{t} \|[A, \tau_t(B)]\|_{\text{op}} + \frac{1}{2\pi} \int_{|t| \geq T} dt \frac{e^{-\alpha t^2}}{t} \|[A, \tau_t(B)]\|_{\text{op}} \tag{29}$$

The first term can be bounded as:

$$\begin{aligned}
\frac{1}{2\pi} \int_{-T}^T dt \frac{e^{-\alpha t^2}}{|t|} \|[A, \tau_t(B)]\|_{\text{op}} &\leq \frac{\|A\|_{\text{op}}\|B\|_{\text{op}} \min\{|X|, |Y|\}}{\pi} e^{-\mu d(X,Y)} \int_{-T}^T dt e^{-\alpha t^2} \frac{e^{\mu v_{\text{LR}}|t|} - 1}{|t|} \\
&\leq \frac{2\|A\|_{\text{op}}\|B\|_{\text{op}} \min\{|X|, |Y|\}}{\pi} e^{-\mu d(X,Y)} \int_0^T dt e^{-\alpha t^2} \frac{e^{\mu v_{\text{LR}}t} - 1}{t} \\
&\leq \frac{2\|A\|_{\text{op}}\|B\|_{\text{op}} \min\{|X|, |Y|\}}{\pi} e^{-\mu(d(X,Y) - v_{\text{LR}}T)}
\end{aligned} \tag{30}$$

where we have used $e^{-\alpha t^2} \leq 1$ and $\frac{e^{\xi x} - 1}{x} \leq \xi e^{\xi x}$. The second term can be bounded simply as:

$$\begin{aligned}
\frac{1}{2\pi} \int_{|t| \geq T} dt \frac{e^{-\alpha t^2}}{|t|} \|[A, \tau_t(B)]\|_{\text{op}} &\leq \frac{2\|A\|_{\text{op}}\|B\|_{\text{op}}}{\pi} \int_T^\infty dt \frac{e^{-\alpha t^2}}{t} \\
&\leq \frac{\|A\|_{\text{op}}\|B\|_{\text{op}}}{\pi} \int_{\alpha T^2}^\infty d\xi \frac{e^{-\xi}}{\xi} \\
&\leq \frac{\|A\|_{\text{op}}\|B\|_{\text{op}}}{\pi \alpha T^2} e^{-\alpha T^2}
\end{aligned} \tag{31}$$

Therefore, using the bounds Equation (30) and Equation (31), we obtain:

$$\text{Equation (28)} \leq \frac{2\|A\|_{\text{op}}\|B\|_{\text{op}} \min\{|X|, |Y|\}}{\pi} e^{-\mu(d(X,Y) - v_{\text{LR}}T)} + \frac{\|A\|_{\text{op}}\|B\|_{\text{op}}}{\pi \alpha T^2} e^{-\alpha T^2} \tag{32}$$

We thus get:

$$\text{Equation (27)} \leq \frac{2\|A\|_{\text{op}}\|B\|_{\text{op}} \min\{|X|, |Y|\}}{\pi} e^{-\mu(d(X,Y) - v_{\text{LR}}T)} + \frac{\|A\|_{\text{op}}\|B\|_{\text{op}}}{\pi\alpha T^2} e^{-\alpha T^2} + 4C\|A\|_{\text{op}}\|B\|_{\text{op}} e^{-\gamma^2/(4\alpha)}. \quad (33)$$

We now choose α and T so as to balance the three exponential contributions:

$$\alpha = \frac{\gamma}{2T} \quad T = \frac{\mu d(X, Y)}{\mu v_{\text{LR}} + \frac{\gamma}{2}}.$$

We therefore define

$$m_\mu \doteq \frac{\mu\gamma}{2\mu v_{\text{LR}} + \gamma}. \quad (34)$$

Since X and Y are disjoint subsets of the lattice, $d(X, Y) \geq 1$, and hence this prefactor is bounded by a constant depending only on μ , γ and the interaction. Absorbing all numerical constants into C_μ , we can conclude. $\blacksquare$

The physical content of [Theorem 19](#) can be summarized as follows. The inverse spectral gap γ^{-1} provides a characteristic time scale, while the Lieb–Robinson velocity bounds the distance over which information can propagate during such a time. The resulting correlation length is therefore naturally controlled by a quantity of order $\xi \sim \frac{v_{\text{LR}}}{\gamma}$. In this sense, a spectral gap together with an effective finite propagation velocity produces a finite correlation length.

Remark 16.

1. **Quasi-degenerate ground-state sectors with vanishing width.** The uniqueness assumption in [Theorem 19](#) is essential if one wants to conclude exponential decay of the ordinary connected correlation function. Indeed, a spectral gap above a degenerate ground-state sector does not exclude long-range correlations arising entirely from matrix elements within that sector.

More precisely, consider a sequence of finite-volume Hamiltonians H_L on Λ_L whose bottom of the spectrum consists of q eigenvalues $E_{0,1}^{(L)}, \dots, E_{0,q}^{(L)}$, where q is independent of L . Let P_L^0 denote the corresponding spectral projection. Assume that the width of this low-energy sector,

$$\delta_L \doteq \max_{1 \leq i, j \leq q} |E_{0,i}^{(L)} - E_{0,j}^{(L)}|,$$

vanishes in the thermodynamic limit $\delta_L \rightarrow 0$ as $L \rightarrow \infty$, while the sector remains uniformly separated from the rest of the spectrum: there exists $\gamma > 0$, independent of L , such that

$$\text{dist}\left(\{E_{0,1}^{(L)}, \dots, E_{0,q}^{(L)}\}, \text{spec}(H_L) \setminus \{E_{0,1}^{(L)}, \dots, E_{0,q}^{(L)}\}\right) \geq \gamma.$$

Thus, the low-energy eigenvalues are allowed to be only quasi-degenerate at finite volume, but their splitting has to disappear as $L \rightarrow \infty$. If we introduced the averaged correlator:

$$\langle \mathcal{O} \rangle_{0,L} \doteq \frac{1}{q} \sum_{j=1}^q \langle \Omega_L^j | \mathcal{O} \Omega_L^j \rangle = \frac{1}{q} \text{Tr}(\mathcal{O} P_L^0)$$

under the above assumptions, we the following exponential clustering inequality:

$$\left| \langle AB \rangle_{0,L} - \langle AP_L^0 B \rangle_{0,L} \right| \leq C \min\{|X|, |Y|\} \|A\|_{\text{op}} \|B\|_{\text{op}} e^{-md(X,Y)},$$

for suitable constants $C, m > 0$. The object on the right-hand side reduces to the ordinary connected correlation when $q = 1$, recovering the form appearing in [Theorem 19](#). See [\[HK06b\]](#).

2. **Low-energy spectral patches of non-vanishing width.** A more general finite-volume situation is considered in [\[Bac+21\]](#). Let $E_1^{(L)} \leq E_2^{(L)} \leq \dots$, be the eigenvalues of H_L , counted with multiplicity. Assume that there exist constants $p \in \mathbb{N}$, $\gamma, \Delta > 0$, all independent of L , such that

$$E_{p+1}^{(L)} - E_p^{(L)} \geq \gamma, \quad E_p^{(L)} - E_1^{(L)} \leq \Delta, \quad \gamma > 2\Delta.$$

Let P_L^p denote the rank- p spectral projection associated with the low-energy patch $\{E_1^{(L)}, \dots, E_p^{(L)}\}$. In contrast with the previous setting, the width of the spectral patch is not required to vanish as $L \rightarrow \infty$: the bound

$$E_p^{(L)} - E_1^{(L)} \leq \Delta$$

may hold with a fixed $\Delta > 0$. In this set-up, for any $|\Omega_L^j\rangle \in \text{ran}(P_L^p)$, one gets the following exponential clustering theorem:

$$\left| \langle \Omega_L^j | AB \Omega_L^j \rangle - \langle \Omega_L^j | AP_L^p B \Omega_L^j \rangle \right| \leq C \|A\|_{\text{op}} \|B\|_{\text{op}} \min\{|X|, |Y|\} O(d(X, Y)^{-\infty}),$$

uniformly in the volume. Here $A \leq BO(r^{-\infty})$ means that, for every $n \in \mathbb{N}$, there exists a constant $C_n > 0$, independent of L , such that:

$$A \leq \frac{C_n B}{(1+r)^n}$$

The weaker, superpolynomial spatial decay is the price paid for using a smooth spectral filter separating, uniformly in the distance of the supports of A and B , transitions within the order one low-energy patch from transitions connecting the patch to the rest of the spectrum. Such an exact smooth cutoff has a Fourier transform that decays faster than any power, rather than exponentially, and the corresponding quasi-locality estimate therefore yields $O(d(X, Y)^{-\infty})$ -decay.

Notice that the assumptions of this second result also include spectral patches whose actual width tends to zero. In that more restrictive situation, however, the superpolynomial bound above is not necessarily optimal; when the splitting collapses sufficiently fast and the hypotheses of the exponential clustering theorem apply, one can recover exponential decay in the appropriate projected correlation.

3. **Odd fermionic observables.** The restriction to the even CAR algebra allows us to use ordinary commutators throughout the proof. For odd observables localized in disjoint regions, the corresponding locality relation is formulated in terms of anticommutators. A version of the exponential clustering theorem covering this situation is proved in [\[HK06b\]](#).

2.2.6 Quasi-adiabatic continuation and spectral flow

The quasi-adiabatic continuation was introduced by Hastings and Wen as a tool to transport the low-energy sector along a gapped path of local Hamiltonians while preserving locality. The presentation below follows the finite-volume formulation used in [Bac+19], in particular its application to Thouless pumps, together with the original construction of [HW05]. As in the previous discussion of Lieb–Robinson bounds, all estimates are formulated uniformly in the volume.

Consider, at finite volume, a differentiable family of local Hamiltonians $H_L(s)$ with $s \in [0, 1]$ and assume that, for every s , the spectrum admits a decomposition

$$\text{spec}(H_L(s)) = \text{spec}_0(s) \cup \text{spec}_1(s) \quad \text{such that } \text{dist}(\text{spec}_0(s), \text{spec}_1(s)) \geq \gamma$$

for some $\gamma > 0$, independent of L and s . The spectral patch $\text{spec}_0(s)$ may consist of a single non-degenerate ground-state eigenvalue or, more generally, of several eigenvalues separated uniformly from the rest of the spectrum. In order to construct the flow, one defines an analogue of the gaussian filter in Equation (23) with a different rapidly decaying odd¹³ cut-off function $W_\gamma \in L^1(\mathbb{R})$, whose Fourier transform satisfies

$$\widehat{W}_\gamma(\omega) = -\frac{i}{\omega} \quad |\omega| \geq \gamma.$$

Notice that under such hypothesis W_γ can be chosen to decay faster than any power (see [Bac+11, Lemma 2.3] for an explicit choice of such filter function):

$$\sup_{|t| \geq 1} |t|^n |W_\gamma(t)| < \infty \quad \text{for every } n \in \mathbb{N}.$$

The quasi-adiabatic map is defined as:

$$\mathcal{I}_s(A) \doteq \int_{-\infty}^{+\infty} W_\gamma(t) \tau_t^{(s)}(A) dt = \int_{-\infty}^{+\infty} W_\gamma(t) e^{itH_L(s)} A e^{-itH_L(s)} dt$$

Proposition 20. *Let P_L^0 is the spectral projector on $\text{spec}_0(s)$, then for any observable of the form $P_L^0 A (\mathbf{1} - P_L^0)$ and $(\mathbf{1} - P_L^0) A P_L^0$, the operator $\mathcal{I}_s$ is an inverse of the Liouvillian $-i[H_L(s), (\cdot)]$:*

$$-i[H_L(s), \mathcal{I}_s(P_L^0 A (\mathbf{1} - P_L^0))] = P_L^0 A (\mathbf{1} - P_L^0) \quad -i[H_L(s), \mathcal{I}_s((\mathbf{1} - P_L^0) A P_L^0)] = (\mathbf{1} - P_L^0) A P_L^0$$

Proof. Let $\tilde{A} \doteq P_L^0 A (\mathbf{1} - P_L^0)$, then:

$$\begin{aligned} [H_L(s), \mathcal{I}_s(\tilde{A})] &= -i \sum_{n,m \geq 0} |\Omega_n(s)\rangle \langle \Omega_m(s)| \int_{-\infty}^{\infty} dt W_\gamma(t) \left\langle \Omega_n(s) \left| [H_L(s), e^{itH_L(s)} \tilde{A} e^{-itH_L(s)}] \Omega_m(s) \right\rangle \right. \\ &= -i \sum_{n,m \geq 0} |\Omega_n(s)\rangle \langle \Omega_m(s)| \int_{-\infty}^{\infty} dt W_\gamma(t) e^{it(E_n(s) - E_m(s))} (E_n(s) - E_m(s)) \\ &\quad \times \left\langle \Omega_n(s) \left| \tilde{A} \Omega_m(s) \right\rangle \right. \\ &= -i \sum_{\substack{n,m \geq 0: \\ E_n(s) \in \text{spec}_0(s) \\ E_m(s) \in \text{spec}_1(s)}} |\Omega_n(s)\rangle \langle \Omega_m(s)| \widehat{W}_\gamma(E_m(s) - E_n(s)) (E_n(s) - E_m(s)) \langle \Omega_n(s) | A \Omega_m(s) \rangle \\ &= \sum_{\substack{n,m \geq 0: \\ E_n(s) \in \text{spec}_0(s) \\ E_m(s) \in \text{spec}_1(s)}} |\Omega_n(s)\rangle \langle \Omega_m(s)| \langle \Omega_n(s) | A \Omega_m(s) \rangle \\ &= \tilde{A} \end{aligned}$$

¹³The Fourier transform of an odd real function is purely imaginary and odd in ω

The proof for $(\mathbf{1} - P_L^0)AP_L^0$ follows similarly. ■

As a consequence, the spectral projection satisfies the exact differential equation:

$$\dot{P}_L^0(s) = i [\mathcal{I}_s(\dot{H}_L(s)), P_L^0(s)] \quad (35)$$

Proof.

$$\begin{aligned} \dot{P}_L^0(s) &= \dot{P}_L^0(s)P_L^0(s) + P_L^0(s)\dot{P}_L^0(s) \xrightarrow{P_L^0(s)} \dot{P}_L^0(s)P_L^0(s) = 0 \\ \dot{P}_L^0(s) &= \dot{P}_L^0(s)(\mathbf{1} - P_L^0(s)) + (\mathbf{1} - P_L^0(s))\dot{P}_L^0(s) \xrightarrow{\mathbf{1} - P_L^0(s)} (\mathbf{1} - P_L^0(s))\dot{P}_L^0(s)(\mathbf{1} - P_L^0(s)) = 0 \end{aligned}$$

we have that:

$$\begin{aligned} \dot{P}_L^0(s) &= -i[H_L(s), \mathcal{I}_s(\dot{P}_L^0(s))] \\ &= -i \int dt W_\gamma(t) [H_L(s), e^{itH_L(s)} \dot{P}_L^0(s) e^{-itH_L(s)}] \\ &= i \int dt W_\gamma(t) e^{itH_L(s)} [H_L(s), \dot{P}_L^0(s)] e^{-itH_L(s)} \\ &= -i \int dt W_\gamma(t) e^{itH_L(s)} \left(\frac{d}{ds} [H_L(s), P_L^0(s)] - [\dot{H}_L(s), P_L^0(s)] \right) e^{-itH_L(s)} \\ &= i \int dt W_\gamma(t) [e^{itH_L(s)} \dot{H}_L(s) e^{-itH_L(s)}, P_L^0(s)] \\ &= i [\mathcal{I}_s(\dot{H}_L(s)), P_L^0(s)] \end{aligned}$$
■

Thus, unlike the physical adiabatic Schrödinger evolution, the quasi-adiabatic flow transports the instantaneous spectral subspace exactly: no adiabatic parameter is involved in this identity.

The crucial additional property is locality. If $H_L(s)$ and $\dot{H}_L(s)$ are generated by uniformly local interactions, the Lieb–Robinson bound implies (with a similar proof of [Remark 16](#)) that $\tau_t^{(s)}(\dot{H}_L(s))$ spreads only inside an effective light cone. Since $W_\gamma(t)$ decays faster than any power, contributions coming from times large enough to propagate information over a distance R are superpolynomially small in R . Consequently, $\mathcal{I}_s(\dot{H}_L(s))$ is itself generated by a quasi-local interaction.

In this sense, quasi-adiabatic continuation provides an exact spectral flow along the gapped path $H_L(s)$, while the Lieb–Robinson bound ensures that the corresponding generator remains quasi-local uniformly in the volume. In particular, for a cyclic path $H_L(1) = H_L(0)$, the resulting quasi-local unitary implements an adiabatic cycle on the low-energy sector. This is the framework used, for instance, in the description of Thouless pumps and in the definition of the many-body charge-transport index of [\[Bac+19\]](#), where the quasi-adiabatic flow allows one to relate the spectral evolution along the cycle to a quantized transport of charge across the system.

2.2.7 Thermal States and Imaginary-Time Correlation Functions

Having introduced Hamiltonians acting on the fermionic Fock space, we now turn to their equilibrium statistical mechanics. In the grand-canonical ensemble, both energy and particle number may fluctuate, while their mean values are controlled by the inverse temperature $\beta > 0$ and the chemical potential $\mu \in \mathbb{R}$ (see [Remark 17](#)). We will assume that $[H, N] = 0$, as is appropriate for

the standard grand-canonical ensemble. The corresponding equilibrium state is described by the grand-canonical Gibbs density matrix

$$\rho_{\beta,\mu,L} = \frac{1}{Z_{\beta,\mu,L}} e^{-\beta(H-\mu N)} \quad Z_{\beta,\mu,L} \doteq \text{Tr}_{\mathcal{F}_L}(e^{-\beta(H-\mu N)}),$$

and the expectation value of an observable $\mathcal{O}$ is given by

$$\langle \mathcal{O} \rangle_{\beta,\mu,L} = \frac{\text{Tr}(\mathcal{O} e^{-\beta(H-\mu N)})}{\text{Tr}(e^{-\beta(H-\mu N)})}. \quad (36)$$

Remark 17. At fixed temperature, the chemical potential controls the mean particle number:

$$\frac{1}{\beta} \partial_\mu \log Z_{\beta,\mu,L} = \langle N \rangle_{\beta,\mu,L}$$

Besides the real-time dynamics discussed in [Subsection 2.2.4](#), the Hamiltonian H induces another automorphism on the observable algebra, called imaginary-time evolution:

$$\gamma_s(\mathcal{O}) \doteq e^{s(H-\mu N)} \mathcal{O} e^{-s(H-\mu N)}.$$

The name comes from the fact that $\gamma_s(\mathcal{O})$ is obtained from $\tau_t(\mathcal{O})$ by formally substituting $t \rightarrow -is$. The imaginary-time evolution naturally gives rise to the so-called *Euclidean correlation functions*. Given two observables $A, B \in \mathcal{A}_{\text{CAR}}$ in a fixed parity sector, their two-point Euclidean correlation function is defined by

$$G_{A,B}(s, t) \doteq \langle \mathbf{T}_s \gamma_s(A) \gamma_t(B) \rangle_{\beta,\mu,L} \doteq \frac{\text{Tr}(e^{-\beta(H-\mu N)} \mathbf{T}_s \gamma_s(A) \gamma_t(B))}{\text{Tr}(e^{-\beta(H-\mu N)})},$$

where $\mathbf{T}_s$ denotes the imaginary-time ordering operator, defined by

$$\mathbf{T}_s \gamma_s(A) \gamma_t(B) \doteq \begin{cases} \gamma_s(A) \gamma_t(B) & s > t, \\ (-1)^{|A||B|} \gamma_t(B) \gamma_s(A) & t > s, \end{cases}$$

with $|A|, |B| \in \{0, 1\}$ denoting the parity of the observables, i.e. $|A| = 0, 1$ if $A \in \mathcal{A}^\pm$ respectively. In particular, if both observables are odd ($|A| = |B| = 1$), one recovers the familiar minus sign,

$$\mathbf{T}_s \gamma_s(A) \gamma_t(B) = \begin{cases} \gamma_s(A) \gamma_t(B), & s > t, \\ -\gamma_t(B) \gamma_s(A), & t > s. \end{cases}$$

Proposition 21 (KMS Identity). *For any $A, B \in \mathcal{A}$, it holds the Kubo-Martin-Schwinger Identity:*

$$\langle \gamma_s(A) \gamma_t(B) \rangle_{\beta,\mu,L} = \langle \gamma_{t+s}(B) \gamma_s(A) \rangle_{\beta,\mu,L} \quad \forall s, t \in \mathbb{R} \quad (37)$$

Proof.

$$\begin{aligned} \langle \gamma_s(A) \gamma_t(B) \rangle_{\beta,\mu,L} &= \frac{\text{Tr}(e^{s(H-\mu N)} A e^{-s(H-\mu N)} B(t) e^{-\beta(H-\mu N)})}{\text{Tr}(e^{-\beta(H-\mu N)})} \\ &= \frac{\text{Tr}(e^{-\beta(H-\mu N)} e^{\beta(H-\mu N)} B(t) e^{-\beta(H-\mu N)} e^{s(H-\mu N)} A e^{-s(H-\mu N)})}{\text{Tr}(e^{-\beta(H-\mu N)})} \\ &= \langle \gamma_{t+\beta}(B) \gamma_s(A) \rangle_{\beta,\mu,L} \end{aligned}$$

■

The KMS identity determines the continuation of the time-ordered function around the thermal circle. This statement must be applied separately in the two ordering sectors. For example, if $0 \leq t < s < \beta$, then

$$\begin{aligned} G_{A,B}(t + \beta, s) &= \langle \gamma_{t+\beta}(A) \gamma_s(B) \rangle_{\beta, \mu, L} \\ &= \langle \gamma_s(B) \gamma_t(A) \rangle_{\beta, \mu, L} \\ &= (-1)^{|A||B|} G_{A,B}(t, s). \end{aligned}$$

If $0 \leq s < t < \beta$, the equivalent relation is

$$G_{A,B}(t - \beta, s) = (-1)^{|A||B|} G_{A,B}(t, s).$$

Using the KMS identity in Equation (37), we have thus shown that the values of $G_{A,B}(t, s)$ are completely determined by their value for $t, s \in [0, \beta]$ as a periodic/antiperiodic extension. In particular, a two-point function of two odd observables is antiperiodic, while it is periodic if at least one observable is even.

Among the most relevant examples are the two-point creation–annihilation correlation functions, obtained by choosing $A = a_\alpha^-$ and $B = a_{\alpha'}^+$, which we denote by $G(t, \alpha|s, \alpha')$. This quantity, commonly referred to as the (imaginary-time) fermionic Green’s function, plays a central role in equilibrium many-body physics. Besides describing the propagation of single-particle excitations, it constitutes the fundamental building block of perturbation theory through Wick’s theorem, determines linear response functions via the Kubo formalism, and encodes spectral information after analytic continuation to real time.

Remark 18. At coincident times this identity must be understood in terms of one-sided limits, since time-ordered fermionic correlation functions are in general discontinuous across $(t = s)$. In particular, for $A = a_\alpha^-$ and $B = a_{\alpha'}^+$, the canonical anticommutation relations imply

$$G(s^+, \alpha|s, \alpha') - G(s^-, \alpha|s, \alpha') = \delta_{\alpha\alpha'}. \quad (38)$$

Accordingly, the antiperiodic boundary condition relates the corresponding one-sided limits,

$$G(\beta^-, \alpha|0, \alpha') = -G(0^-, \alpha|0, \alpha'), \quad G(-\beta^+, \alpha|0, \alpha') = -G(0^+, \alpha|0, \alpha').$$

Example 22. When $H = H_0$ is a quadratic hamiltonian as in Equation (16), the model is exactly solvable, which means that one can compute exactly correlation functions and the partition function¹⁴.

1. Partition function:

$$Z_{\beta, \mu, L} = \det \left(\mathbf{1} + e^{-\beta(h - \mu \mathbf{1})} \right)$$

Proof. In order to compute the partition function is convenient to use the basis of the Fock space $\mathcal{F}_L$ induced by the eigenstates of h . Indeed, let $\{\psi_\lambda\}_{\lambda \in \text{spec}(h)}$ be the complete set of eigenstates of h with eigenvalues $\{\epsilon_\lambda\}_{\lambda \in \text{spec}(h)}$ and $a_\lambda^\pm \doteq a^\pm(\psi_\lambda)$ the associated creation/annihilation operators. We denote with:

$$|\underline{n}_\lambda\rangle = \left(\prod_{\lambda \in \text{spec}(h)} (a_\lambda^+)^{n_\lambda} \right) |0\rangle \quad n_\lambda = 0, 1 \quad \forall \lambda \in \text{spec}(h)$$

¹⁴One should compare the following results with the analogous results in the case of quadratic Grassmann Integrals

the corresponding occupation basis. Thus:

$$\begin{aligned}
Z_{\beta,\mu,L} &= \text{Tr}_{\mathcal{F}_L}(e^{-\beta(H_0-\mu N)}) \\
&= \sum_{\{n_\lambda\}} \left\langle \underline{n}_\lambda \left| e^{-\beta(H_0-\mu N)} \right| \underline{n}_\lambda \right\rangle \\
&= \sum_{\{n_\lambda\}_{\lambda \in \text{spec}(H)}} \left\langle \underline{n}_\lambda \left| e^{-\beta(H_0-\mu N)} \right| \underline{n}_\lambda \right\rangle \\
&= \prod_{\lambda \in \text{spec}(H)} \sum_{n_\lambda=0,1} e^{-\beta(\epsilon_\lambda-\mu)n_\lambda} \\
&= \prod_{\lambda \in \text{spec}(H)} (1 + e^{-\beta(\epsilon_\lambda-\mu)}) \\
&= \det(\mathbf{1} + e^{-\beta(h-\mu\mathbf{1})})
\end{aligned}$$

■

From the Partition function one can easily compute the average number of particles as:

$$\langle N \rangle_{\beta,\mu,L} = \frac{1}{\beta} \sum_{\lambda \in \text{spec}(h)} \partial_\mu \log(1 + e^{-\beta(\epsilon_\lambda-\mu)}) = \sum_{\lambda \in \text{spec}(h)} \frac{1}{1 + e^{\beta(\epsilon_\lambda-\mu)}}$$

The object on the right-hand side is the trace of famous Fermi-Dirac Distribution:

$$n_{\text{FD}}(h) \doteq (\mathbf{1} + e^{\beta(h-\mu\mathbf{1})})^{-1}$$

2. Two Point function:

$$G(\alpha, t | \beta, s) = \begin{cases} \left(\frac{e^{-(t-s)(h-\mu\mathbf{1})}}{\mathbf{1} + e^{-\beta(h-\mu\mathbf{1})}} \right)_{\alpha\beta} & t > s \\ - \left(\frac{e^{-(\beta-(s-t)(h-\mu\mathbf{1})})}{\mathbf{1} + e^{-\beta(h-\mu\mathbf{1})}} \right)_{\alpha\beta} & t \leq s \end{cases}$$

As a consistency check, one can easily verify that G is discontinuous as it is prescribed by anticommutation relation in [Equation \(38\)](#):

$$G(\alpha, s^+ | \beta, s) - G(\alpha, s^- | \beta, s) = \left(\frac{\mathbf{1}}{\mathbf{1} + e^{-\beta(h-\mu\mathbf{1})}} \right)_{\alpha\beta} + \left(\frac{e^{-\beta(h-\mu\mathbf{1})}}{\mathbf{1} + e^{-\beta(h-\mu\mathbf{1})}} \right)_{\alpha\beta} = \delta_{\alpha\beta}$$

Proof. First let's observe that:

$$G(\alpha, t | \beta, s) = \langle \mathbf{T}_s \gamma_t(a_\alpha^-) \gamma_s(a_\beta^+) \rangle = \sum_{\lambda, \mu \in \text{spec}(h)} \bar{\psi}_\lambda(\alpha) \psi_\mu(\beta) \langle \mathbf{T}_s \gamma_t(a_\lambda^-) \gamma_s(a_\mu^+) \rangle$$

Thus, it is sufficient to evaluate the two-point Green's function $\langle \mathbf{T}_s \gamma_t(a_\lambda^-) \gamma_s(a_\mu^+) \rangle$. This can be done by considering separately the cases $t > s$ and $t < s$:

$$\begin{aligned}
\langle \gamma_t(a_\lambda^-) \gamma_s(a_\mu^+) \rangle &= \sum_{\{n_\nu\}} \prod_\nu e^{-(\beta-(t-s))(\epsilon_\nu-\mu)n_\nu} \left\langle a_\lambda^+ \underline{n}_\nu \left| e^{-(t-s)(H_0-\mu N)} \right| a_\mu^+ \underline{n}_\nu \right\rangle \\
&= \frac{e^{-(t-s)(\epsilon_\lambda-\mu)} \delta_{\lambda\mu}}{Z_{\beta,\mu,L}} \sum_{\{n_\nu\}} \prod_{\nu \neq \lambda} e^{-\beta(\epsilon_\nu-\mu)n_\nu} \\
&= \frac{e^{-(t-s)(\epsilon_\lambda-\mu)}}{1 + e^{-\beta(\epsilon_\lambda-\mu)}} \delta_{\lambda\mu} \\
\langle \gamma_s(a_\mu^+) \gamma_t(a_\lambda^-) \rangle &= \sum_{\{n_\nu\}} \prod_\nu e^{-(\beta-(s-t))(\epsilon_\nu-\mu)n_\nu} \left\langle a_\mu^- \underline{n}_\nu \left| e^{-(s-t)(H_0-\mu N)} \right| a_\lambda^- \underline{n}_\nu \right\rangle \\
&= \frac{e^{-(\beta-(s-t))(\epsilon_\lambda-\mu)} \delta_{\lambda\mu}}{Z_{\beta,\mu,L}} \sum_{\{n_\nu\}} \prod_{\nu \neq \lambda} e^{-\beta(\epsilon_\nu-\mu)n_\nu} \\
&= \frac{e^{-(\beta-(s-t))(\epsilon_\lambda-\mu)}}{1 + e^{-\beta(\epsilon_\lambda-\mu)}} \delta_{\lambda\mu}
\end{aligned}$$

■

2.A Fermionic Coherent States and the Euclidean Path Integral

The finite-dimensional construction below follows the standard Grassmann conventions and coherent-state trace formulas reviewed in [Weg16]. It is adapted to the generator ordering fixed in [Subsection 2.1.1](#) and to the Fock-space conventions of [Subsection 2.2.3](#), and provides the functional-integral representation of the thermal quantities introduced in [Subsection 2.2.7](#).

The imaginary-time formalism admits a functional-integral representation in terms of Grassmann variables. This representation provides the bridge between the Hamiltonian formulation of a fermionic system, in which observables act on the fermionic Fock space, and its Euclidean formulation, in which thermal correlation functions are expressed as Grassmann integrals over fields defined on the space-time cylinder $[0, \beta] \times \Lambda_L$. The construction is based on fermionic coherent states and on a Trotter decomposition of the Gibbs operator.

fermionic coherent state creation operators can be defined rigorously as certain operators living in an extension of the fermionic Observable Algebra $\mathcal{A}$ by a suitable Grassmann Algebra $\mathcal{G}_A$ associated to the index set A labeling one-particles degrees of freedom, i.e. spatial position in Λ_L and internal degrees of freedom labelled by a basis of V .

Fermionic coherent states. For the remainder of this appendix, fix a total ordering of the finite index set A . Products indexed by a subset $I \subseteq A$ are always taken in the induced increasing order. This convention is needed to define unambiguously ψ_I and $a_I^\pm$ and to track the associated signs.

For homogeneous elements, the multiplication on the graded tensor product is

$$(g \otimes A)(h \otimes B) = (-1)^{|A||h|} gh \otimes AB$$

where $|A|$ and $|h|$ denote the parities of $A \in \mathcal{A}_{\text{CAR}}$ and $h \in \mathcal{G}$, respectively. We denote the resulting graded tensor product $\mathcal{G} \hat{\otimes} \mathcal{A}_{\text{CAR}}$ by $\mathcal{A}_{\mathcal{G}}^{\text{CAR}}$.

Definition 23 (Coherent-state operators). For a family of Grassmann generators $\psi = \{\psi_\alpha\}_{\alpha \in A}$, we define the coherent state operators (which belong to $\mathcal{A}_{\mathcal{G}}$)

$$a_\psi^\pm = e^{\mp \sum_\alpha \psi_\alpha \otimes a_\alpha^\pm}$$

Remark 19.

1. Since all terms in the exponent commute:

$$(\psi_\alpha \otimes a_\alpha^\pm)(\psi_\beta \otimes a_\beta^\pm) = -(\psi_\alpha \psi_\beta \otimes a_\alpha^\pm a_\beta^\pm) = -(\psi_\beta \psi_\alpha \otimes a_\beta^\pm a_\alpha^\pm) = (\psi_\beta \otimes a_\beta^\pm)(\psi_\alpha \otimes a_\alpha^\pm)$$

the coherent state creation/annihilation operator can be written as:

$$a_\psi^\pm = \prod_\alpha e^{\mp \psi_\alpha \otimes a_\alpha^\pm}$$

2. Notice that due to the nilpotency of both Grassmann variables and creation operators, $a_\psi^\pm$ is actually a polynomial

$$a_\psi^\pm = \prod_\alpha (1 \otimes \mathbf{1} \mp \psi_\alpha \otimes a_\alpha^\pm)$$

and no norms have to be introduced to define the convergence of the power series.

Define the $\mathcal{G}$ -bimodules $\mathcal{F}_{L,\mathcal{G}}$ and $\check{\mathcal{F}}_{L,\mathcal{G}}$ by applying $\mathcal{A}_{\mathcal{G}}^{\text{CAR}}$ to $1 \otimes |0\rangle$ and $\langle 0| \otimes 1$, respectively. Here $1 \in \mathbb{C} \subset \mathcal{G}$, $|0\rangle \in \mathcal{F}_L$ is the vacuum, and $\langle 0| \in \mathcal{F}_L^*$ is its dual. The actions on homogeneous elements are:

$$g \otimes A(1 \otimes |0\rangle) = g \otimes (A|0\rangle) \quad (\langle 0| \otimes 1)g \otimes A = (-1)^{|g||A|}(\langle 0|A) \otimes g$$

The action on general states is the following:

$$(h \otimes B)(g \otimes A|0\rangle) = (-1)^{|B||g|}hg \otimes BA|0\rangle \quad (\langle 0|A \otimes g)(h \otimes B) = (-1)^{|B|(|h|+|g|)}(\langle 0|AB \otimes gh)$$

Remark 20.

1. The actions of $\mathcal{A}_{\mathcal{G}}^{\text{CAR}}$ on $\mathcal{F}_{L,\mathcal{G}}$ and $\check{\mathcal{F}}_{L,\mathcal{G}}$ are compatible with the graded product in $\mathcal{A}_{\mathcal{G}}^{\text{CAR}}$.

Indeed:

- (a) For any homogeneous vector $g \otimes A|0\rangle \in \mathcal{F}_{L,\mathcal{G}}$,

$$h \otimes B(g \otimes A|0\rangle) = (h \otimes B)(g \otimes A)1 \otimes |0\rangle = (-1)^{|B||g|}(hg \otimes BA)1 \otimes |0\rangle = (-1)^{|B||g|}hg \otimes BA|0\rangle.$$

- (b) For any homogeneous vector $\langle 0|A \otimes g \in \check{\mathcal{F}}_{L,\mathcal{G}}$, we have:

$$\begin{aligned} (\langle 0|A \otimes g)h \otimes B &= (-1)^{|A||g|}(\langle 0| \otimes 1)(g \otimes A)(h \otimes B) \\ &= (-1)^{|A|(|h|+|g|)}(\langle 0| \otimes 1)(gh \otimes AB) \\ &= (-1)^{|A|(|h|+|g|)}(-1)^{(|B|+|A|)(|g|+|h|)}(\langle 0|AB \otimes gh) \\ &= (-1)^{|B|(|g|+|h|)}(\langle 0|AB \otimes gh) \end{aligned}$$

2. Such spaces inherit the left/right actions of $\mathcal{G}$ from the Grassmann algebra itself as:

$$\begin{aligned} g \triangleright (h \otimes |v\rangle) &\doteq gh \otimes |v\rangle & (h \otimes |v\rangle) \triangleleft g &\doteq (-1)^{|h||v|}gh \otimes |v\rangle \\ (\langle v| \otimes h) \triangleleft g &\doteq \langle v| \otimes hg & g \triangleright (\langle v| \otimes h) &\doteq (-1)^{|g||v|}\langle v| \otimes gh \end{aligned}$$

Definition 24 (fermionic coherent states). The coherent ket and bra are the elements of $\mathcal{F}_{L,\mathcal{G}}$ and $\check{\mathcal{F}}_{L,\mathcal{G}}$ defined by

$$|\psi\rangle = a_\psi^+(1 \otimes |0\rangle) \quad \langle\psi| = (\langle 0| \otimes 1)a_\psi^-$$

Remark 21. The coherent states satisfy

$$1 \otimes a_\alpha^- |\psi\rangle = \psi_\alpha \triangleright |\psi\rangle \quad \langle\psi| 1 \otimes a_\alpha^+ = \langle\psi| \triangleleft \psi_\alpha$$

in analogy with bosonic coherent states.

Proof. Let us first observe that for any $\alpha \neq \beta$:

$$\begin{aligned} (1 \otimes a_\beta^\mp)(1 \otimes \mathbf{1} \mp \psi_\alpha \otimes a_\alpha^\pm) &= 1 \otimes a_\beta^\mp \pm \psi_\alpha \otimes a_\beta^\mp a_\alpha^\pm \\ &= 1 \otimes a_\beta^\mp \mp \psi_\alpha \otimes a_\alpha^\pm a_\beta^\mp \\ &= (1 \otimes \mathbf{1} \mp \psi_\alpha \otimes a_\alpha^\pm)(1 \otimes a_\beta^\mp) \end{aligned}$$

Therefore:

$$\begin{aligned} 1 \otimes a_\beta^- |\psi\rangle &= (1 \otimes a_\beta^-) \prod_\alpha (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) (1 \otimes |0\rangle) \\ &= \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (1 \otimes a_\beta^-) (1 \otimes \mathbf{1} - \psi_\beta \otimes a_\beta^+) (1 \otimes |0\rangle) \\ &= \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (\psi_\beta \otimes a_\beta^- a_\beta^+) (1 \otimes |0\rangle) \\ &= \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (\psi_\beta \otimes \{a_\beta^-, a_\beta^+\}) (1 \otimes |0\rangle) \\ &= \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (\psi_\beta \otimes \mathbf{1}) (1 \otimes |0\rangle) \\ &= (\psi_\beta \otimes \mathbf{1}) \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (1 \otimes |0\rangle) \\ &= (\psi_\beta \otimes \mathbf{1}) \left(\prod_\alpha (1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+) \right) (1 \otimes |0\rangle) \\ &= \psi_\beta \triangleright |\psi\rangle \end{aligned}$$

where we have used that we could add for free the term $(1 \otimes \mathbf{1} - \psi_\alpha \otimes a_\alpha^+)$ because:

$$(\psi_\beta \otimes \mathbf{1})(1 \otimes \mathbf{1} - \psi_\beta \otimes a_\beta^+) = (\psi_\beta \otimes \mathbf{1}).$$

Similarly:

$$\begin{aligned}
\langle \psi | (1 \otimes a_\beta^+) &= (\langle 0 | \otimes 1) \prod_{\alpha} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) (1 \otimes a_{\beta}^+) \\
&= (\langle 0 | \otimes 1) (1 \otimes \mathbf{1} + \psi_{\beta} \otimes a_{\beta}^{-}) (1 \otimes a_{\beta}^+) \prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \\
&= (\langle 0 | \otimes 1) (\psi_{\beta} \otimes a_{\beta}^{-} a_{\beta}^+) \prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \\
&= (\langle 0 | \otimes 1) (\psi_{\beta} \otimes \{a_{\beta}^{-}, a_{\beta}^+\}) \prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \\
&= (\langle 0 | \otimes 1) (\psi_{\beta} \otimes \mathbf{1}) \prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \\
&= (\langle 0 | \otimes 1) \left(\prod_{\alpha \neq \beta} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \right) (\psi_{\beta} \otimes \mathbf{1}) \\
&= (\langle 0 | \otimes 1) \left(\prod_{\alpha} (1 \otimes \mathbf{1} + \psi_{\alpha} \otimes a_{\alpha}^{-}) \right) (\psi_{\beta} \otimes \mathbf{1}) \\
&= \langle \psi | (\psi_{\beta} \otimes \mathbf{1}) \\
&= \langle \psi | \triangleleft \psi_{\beta}
\end{aligned}$$

■

We define a $\mathbb{C}$ -bilinear pairing between $\tilde{\mathcal{F}}_{L,G}$ and $\mathcal{F}_{L,G}$. On homogeneous pure tensors, set

$$(\langle v | \otimes g, h \otimes |w \rangle) \doteq (-1)^{|v|(|g|+|h|)} gh \langle v | w \rangle \quad (39)$$

and extend by bilinearity.

Remark 22.

1. Notice that we could have chosen the sign to be $(-1)^{|w|(|g|+|h|)}$ since $\langle v | w \rangle \neq 0$ if and only if $(-1)^{|v|} = (-1)^{|w|}$. Indeed:

$$(-1)^{|v|} \langle v | w \rangle = \langle (-1)^N v | w \rangle = \langle v | (-1)^N w \rangle = (-1)^{|w|} \langle v | w \rangle.$$

2. The definition is given in order for the following property to hold:

$$(\langle v | \otimes g, (l \otimes A)(h \otimes |w \rangle)) = ((\langle v | \otimes g)(l \otimes A), (h \otimes |w \rangle)) \quad (40)$$

Indeed:

$$\begin{aligned}
(\langle v | \otimes g, (l \otimes A)(h \otimes |w \rangle)) &= (-1)^{|A||h|} (\langle v | \otimes g, lh \otimes A |w \rangle) \\
&= (-1)^{|A||h|} (-1)^{|v|(|g|+|l|+|h|)} glh \langle v | A |w \rangle
\end{aligned}$$

and:

$$\begin{aligned}
((\langle v | \otimes g)(l \otimes A), (h \otimes |w \rangle)) &= (-1)^{|A|(|l|+|g|)} (\langle v | A \otimes gl, h \otimes |w \rangle) \\
&= (-1)^{|A||h|} (-1)^{|v|(|g|+|l|+|h|)} glh \langle v | A |w \rangle
\end{aligned}$$

3. Restricting the property Equation (40) to operators of the form $l \otimes \mathbf{1}$, we have:

$$((\langle v | \otimes g) \triangleleft l, (h \otimes |w \rangle)) = ((\langle v | \otimes g), l \triangleright (h \otimes |w \rangle))$$

If $(-1)^{|l|} = 1$, we have that:

$$l \triangleright ((\langle v| \otimes g), (h \otimes |w\rangle)) = ((\langle v| \otimes g), l \triangleright (h \otimes |w\rangle))$$

and similarly:

$$((\langle v| \otimes g), (h \otimes |w\rangle)) \triangleleft l = ((\langle v| \otimes g), l \triangleright (h \otimes |w\rangle))$$

Proposition 25. *The pairing of two coherent states is*

$$(\langle \xi|, |\psi\rangle) = e^{\sum_{\alpha} \xi_{\alpha} \psi_{\alpha}}$$

Proof. Using Equation (40), we can reduce the computation of the pairing to understanding:

$$\begin{aligned} a_{\xi}^{-} a_{\psi}^{+} &= \left(\prod_{\alpha} (1 \otimes \mathbf{1} + \xi_{\alpha} \otimes a_{\alpha}^{-}) \right) \left(\prod_{\alpha} (1 \otimes \mathbf{1} - \psi_{\alpha} \otimes a_{\alpha}^{+}) \right) \\ &= \prod_{\alpha} (1 \otimes \mathbf{1} + \xi_{\alpha} \otimes a_{\alpha}^{-}) (1 \otimes \mathbf{1} - \psi_{\alpha} \otimes a_{\alpha}^{+}) \\ &= \prod_{\alpha} (1 \otimes \mathbf{1} + \xi_{\alpha} \otimes a_{\alpha}^{-} - \psi_{\alpha} \otimes a_{\alpha}^{+} + \xi_{\alpha} \psi_{\alpha} \otimes a_{\alpha}^{-} a_{\alpha}^{+}) \end{aligned}$$

By applying such expression to $1 \otimes |0\rangle$, we get:

$$\begin{aligned} \prod_{\alpha} (1 \otimes \mathbf{1} + \xi_{\alpha} \otimes a_{\alpha}^{-} - \psi_{\alpha} \otimes a_{\alpha}^{+} + \xi_{\alpha} \psi_{\alpha} \otimes a_{\alpha}^{-} a_{\alpha}^{+}) (1 \otimes |0\rangle) &= \prod_{\alpha} (1 + \xi_{\alpha} \psi_{\alpha}) \otimes |0\rangle \\ &= e^{\sum_{\alpha} \xi_{\alpha} \psi_{\alpha}} \otimes |0\rangle \end{aligned}$$

It is enough now to use the expression Equation (39) to each monomial of $e^{\sum_{\alpha} \xi_{\alpha} \psi_{\alpha}}$ (no signs will be generated because all terms are even) to compute the pairing. $\blacksquare$

With an abuse of notation we will indicate:

$$(\langle \xi|, |\phi\rangle) = \langle \xi| \phi \rangle$$

Coherent states yield a resolution of the identity on Fock space and a convenient trace formula.

Proposition 26. *The following identities hold as operators on $\mathcal{F}_L$:*

$$\begin{aligned} \mathbf{1}(\cdot) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi}|, 1 \otimes \cdot) \triangleright |\psi\rangle \\ \mathbf{1}(\cdot) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft (\langle \bar{\psi}|, 1 \otimes \cdot) \end{aligned}$$

Proof. Let us apply the operators to the occupation basis of $\mathcal{F}_L$, given by $|I\rangle \doteq (\prod_{\alpha \in I} a_{\alpha}^{+})|0\rangle$:

$$\begin{aligned} &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi}|, 1 \otimes |I\rangle) \triangleright |\psi\rangle \\ &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft (\langle \bar{\psi}|, 1 \otimes |I\rangle) \end{aligned}$$

The pairing can be easily computed¹⁵ Thus

¹⁵The first sign comes from moving all annihilation operators past the Grassmann variables:

$$\begin{aligned}
(\langle \bar{\psi} |, 1 \otimes |I\rangle) &= \sum_{J \subseteq A} (\langle 0 | \otimes 1, (\prod_{\alpha \in J} \bar{\psi}_\alpha \otimes a_\alpha^-) 1 \otimes a_I^+ |0\rangle) \\
&= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (\langle 0 | \otimes 1, \bar{\psi}_J \otimes a_J^- a_I^+ |0\rangle) \\
&= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} \bar{\psi}_J \langle 0 | a_J^- a_I^+ |0\rangle \\
&= \bar{\psi}_I
\end{aligned}$$

Thus, we are left with:

$$\begin{aligned}
\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} (\langle \bar{\psi} |, 1 \otimes |I\rangle) \triangleright |\psi\rangle &= \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \bar{\psi}_I \triangleright |\psi\rangle \\
&= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|J|} \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \bar{\psi}_I \psi_J \otimes |J\rangle \\
&= (-1)^{\frac{|I|(|I|-1)}{2}} (-1)^{|I|} \left(\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \bar{\psi}_I \psi_I \right) |I\rangle \\
&= (-1)^{|I|} \left(\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \prod_{\beta \in I} \bar{\psi}_\beta \psi_\beta \right) |I\rangle \\
&= |I\rangle
\end{aligned}$$

and similarly:

$$\begin{aligned}
\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} |\psi\rangle \triangleleft (\langle \bar{\psi} |, 1 \otimes |I\rangle) &= \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} |\psi\rangle \triangleleft \bar{\psi}_I \\
&= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|J|} (-1)^{|I||J|} \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \psi_J \bar{\psi}_I \otimes |J\rangle \\
&= (-1)^{\frac{|I|(|I|-1)}{2}} (-1)^{|I|} \left(\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \bar{\psi}_I \psi_I \right) |I\rangle \\
&= (-1)^{|I|} \left(\int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha \bar{\psi}_\alpha \psi_\alpha} \prod_{\beta \in I} \bar{\psi}_\beta \psi_\beta \right) |I\rangle \\
&= |I\rangle
\end{aligned}$$

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$$\begin{aligned}
(\bar{\psi}_{\alpha_1} \otimes a_{\alpha_1}^-) \cdots (\bar{\psi}_{\alpha_{|I|}} \otimes a_{\alpha_{|I|}}^-) &= -(\bar{\psi}_{\alpha_1} \otimes a_{\alpha_1}^-) \cdots (\bar{\psi}_{\alpha_{|I|-1}} \bar{\psi}_{\alpha_{|I|}} \otimes a_{\alpha_{|I|-1}}^- a_{\alpha_{|I|}}^-) \\
&= -(\bar{\psi}_{\alpha_1} \otimes a_{\alpha_1}^-) \cdots (\bar{\psi}_{\alpha_{|I|-2}} \bar{\psi}_{\alpha_{|I|-1}} \bar{\psi}_{\alpha_{|I|}} \otimes a_{\alpha_{|I|-2}}^- a_{\alpha_{|I|-1}}^- a_{\alpha_{|I|}}^-) \\
&= (-1)^{\sum_{k=1}^{|I|-s} (|I|-k)} (\bar{\psi}_{\alpha_1} \otimes a_{\alpha_1}^-) \cdots (\bar{\psi}_{\alpha_{|I|-s}} \cdots \bar{\psi}_{\alpha_{|I|-2}} \bar{\psi}_{\alpha_{|I|-1}} \bar{\psi}_{\alpha_{|I|}} \otimes a_{\alpha_{|I|-s}}^- \cdots a_{\alpha_{|I|-2}}^- a_{\alpha_{|I|-1}}^- a_{\alpha_{|I|}}^-) \\
&= (-1)^{\sum_{k=1}^{|I|-1} k} \bar{\psi}_I \otimes a_I^- = (-1)^{\frac{|I|(|I|-1)}{2}} \bar{\psi}_I \otimes a_I^-
\end{aligned}$$

With the same ordering convention, $\langle 0 | a_J^- a_I^+ |0\rangle = (-1)^{\frac{|I|(|I|-1)}{2}} \delta_{IJ}$.

Proposition 27. *The following identities hold as operators on $\mathcal{F}_{L,g}$:*

$$\begin{aligned} 1 \otimes \mathbf{1}(\cdot) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi} |, \cdot) \triangleright |\psi\rangle \\ 1 \otimes \mathbf{1}(\cdot) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft (\langle \bar{\psi} |, \cdot) \end{aligned}$$

Proof. Apply the operator to a homogeneous vector $g \otimes |I\rangle$, where $|I\rangle \doteq (\prod_{\alpha \in I} a_{\alpha}^{+})|0\rangle$.

$$\begin{aligned} &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi} |, g \otimes |I\rangle) \triangleright |\psi\rangle, \\ &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft (\langle \bar{\psi} |, g \otimes |I\rangle). \end{aligned}$$

The pairing can be easily computed as:

$$\begin{aligned} (\langle \bar{\psi} |, g \otimes |I\rangle) &= \sum_{J \subseteq A} (\langle 0 | \otimes 1, (\prod_{\alpha \in J} \bar{\psi}_{\alpha} \otimes a_{\alpha}^{-}) g \otimes a_I^{+} |0\rangle) \\ &= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|g||J|} (\langle 0 | \otimes 1, \bar{\psi}_J g \otimes a_J^{-} a_I^{+} |0\rangle) \\ &= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|g||J|} \bar{\psi}_J g \langle 0 | a_J^{-} a_I^{+} |0\rangle \\ &= g \bar{\psi}_I \end{aligned}$$

Thus, we are left with:

$$\begin{aligned} &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi} |, g \otimes |I\rangle) \triangleright |\psi\rangle = \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} g \bar{\psi}_I \triangleright |\psi\rangle \\ &= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|J|} \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} g \bar{\psi}_I \psi_J \otimes |J\rangle \\ &= (-1)^{\frac{|I|(|I|-1)}{2}} (-1)^{|I|} g \triangleright \left(\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \bar{\psi}_I \psi_I \right) \triangleright |I\rangle \\ &= (-1)^{|I|} g \triangleright \left(\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \prod_{\beta \in I} \bar{\psi}_{\beta} \psi_{\beta} \right) \triangleright |I\rangle \\ &= g \otimes |I\rangle \end{aligned}$$

and similarly:

$$\begin{aligned} &\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft (\langle \bar{\psi} |, g \otimes |I\rangle) = \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} |\psi\rangle \triangleleft g \bar{\psi}_I \\ &= \sum_{J \subseteq A} (-1)^{\frac{|J|(|J|-1)}{2}} (-1)^{|J|} (-1)^{|I|(|J|+|g|)} \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \psi_J g \bar{\psi}_I \otimes |J\rangle \\ &= (-1)^{\frac{|I|(|I|-1)}{2}} (-1)^{|I|} g \triangleright \left(\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \bar{\psi}_I \psi_I \right) |I\rangle \\ &= (-1)^{|I|} g \triangleright \left(\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \prod_{\beta \in I} \bar{\psi}_{\beta} \psi_{\beta} \right) |I\rangle \\ &= g \otimes |I\rangle \end{aligned}$$

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Proposition 28. *Let $\mathcal{O} \in \mathcal{A}_{\text{CAR}}^{\text{even}}$. Then the following identities hold:*

$$\begin{aligned} \text{Tr}(\mathcal{O}) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle -\bar{\psi} |, (1 \otimes \mathcal{O}) | \psi \rangle) \\ \text{Tr}(\mathcal{O}^2) &= \int \prod_{\alpha} \prod_{j=0}^1 d\bar{\psi}_{\alpha,j} d\psi_{\alpha,j} e^{-\sum_{j=0}^1 \sum_{\alpha} \bar{\psi}_{\alpha,j} \psi_{\alpha,j}} (\langle -\bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_1 \rangle) (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) \\ \text{Tr}(\mathcal{O}^n) &= \int \prod_{\alpha} \prod_{j=0}^{n-1} d\bar{\psi}_{\alpha,j} d\psi_{\alpha,j} e^{-\sum_{j=0}^{n-1} \sum_{\alpha} \bar{\psi}_{\alpha,j} \psi_{\alpha,j}} (\langle -\bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_{n-1} \rangle) \cdots \\ &\quad \cdots (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) \end{aligned}$$

where, in the second and third identities, we have introduced an additional label $j \in \{0, \dots, n-1\}$ corresponding to the discretization of the imaginary-time direction (with $n \geq 2$). Equivalently, the Grassmann variables are indexed by the extended set $A \times \{0, \dots, n-1\}$, and the Grassmann algebra is naturally enlarged by requiring all generators to mutually anticommute. This construction will naturally give rise to the Euclidean space-time lattice appearing in the fermionic path integral representation.

Remark 23. The minus sign appearing in the bra $\langle -\bar{\psi} |$ can be interpreted as antiperiodic boundary conditions in the time direction for the fermions: it is the functional-integral manifestation of the fermionic KMS condition. Indeed, the minus sign in the coherent-state trace formula encodes the fact that an odd operator acquires a sign when transported once around the thermal circle. Accordingly, the Euclidean Grassmann fields naturally satisfy antiperiodic boundary conditions along the thermal circle, corresponding to its nontrivial spin structure. In this sense, the coherent-state construction translates the operator-algebraic thermal formalism into the Euclidean path-integral language.

Proof. We prove the three identities in order. The one-slice trace formula follows by evaluating both sides in the occupation basis. The formulas for $\text{Tr}(\mathcal{O}^2)$ and $\text{Tr}(\mathcal{O}^n)$ can then be obtained by inserting the resolution of the identity between successive factors. Since $\mathcal{O}$ is even, each coherent-state kernel appearing in the product is even and they may therefore be rearranged without additional graded signs. The single minus sign occurs only on the link that closes the trace, yielding the antiperiodic boundary condition. The explicit index computation below verifies these statements with the conventions used here.

1. The pairing can be computed explicitly:

$$(\langle -\bar{\psi} |, (1 \otimes \mathcal{O}) | \psi \rangle) = \sum_{I, J \subseteq A} c_I c_J (-1)^{|I|+|J|} (-1)^{|I||J|} \bar{\psi}_I \psi_J \langle 0 | a_I^- \mathcal{O} a_J^+ | 0 \rangle,$$

where $c_I \doteq (-1)^{\frac{|I|(|I|-1)}{2}}$. Furthermore,

$$\int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} \bar{\psi}_I \psi_J = \delta_{IJ} c_I (-1)^{|I|}.$$

Using the previous identity and setting $J = I$, we obtain

$$(-1)^{|I||I|} (-1)^{|I|} c_I \langle 0 | a_I^- \mathcal{O} a_I^+ | 0 \rangle = \langle I | \mathcal{O} | I \rangle,$$

which proves the first claim.

2. Set $\mathcal{O}_{IJ} \doteq \langle I | \mathcal{O} | J \rangle$. Then

$$\begin{aligned} (\langle -\bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_1 \rangle) (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) &= \sum_{I,J,K,L \subseteq A} c_J c_L (-1)^{|I|+|J|+|L|} \times \\ &\times (-1)^{|I||J|+|K||L|} (-1)^{|L|(|K|+|J|)} \bar{\psi}_{I,0} \psi_{L,0} \psi_{J,1} \bar{\psi}_{K,1} \mathcal{O}_{IJ} \mathcal{O}_{KL}. \end{aligned}$$

By applying the Grassmann integral we obtain

$$\int \prod_{\alpha} \prod_{j=0}^1 d\bar{\psi}_{\alpha,j} d\psi_{\alpha,j} e^{-\bar{\psi}_{\alpha,j} \psi_{\alpha,j}} \bar{\psi}_{I,0} \psi_{L,0} \psi_{J,1} \bar{\psi}_{K,1} = (-1)^{|I|} c_{IJ} \delta_{IL} \delta_{JK}.$$

Therefore,

$$\sum_{I,J \subseteq A} \mathcal{O}_{IJ} \mathcal{O}_{JI} (-1)^{|I|+|J|}.$$

To get rid of the signs, we observe that

$$(-1)^{|I|+|J|} \langle I | \mathcal{O} | J \rangle = \langle I | (-1)^N \mathcal{O} (-1)^N | J \rangle = (-1)^{|\mathcal{O}|} \langle I | \mathcal{O} | J \rangle = \langle I | \mathcal{O} | J \rangle,$$

where we used that $(-1)^N |I\rangle = (-1)^{|I|} |I\rangle$ and the fact that $\mathcal{O} \in \mathcal{A}_{\text{CAR}}^{\text{even}}$.

Hence:

$$\sum_{I,J \subseteq A} \mathcal{O}_{IJ} \mathcal{O}_{JI} (-1)^{|I|+|J|} = \sum_{I,J \subseteq A} \mathcal{O}_{IJ} \mathcal{O}_{JI} = \text{Tr}(\mathcal{O}^2).$$

3. Similarly,

$$\begin{aligned} (\langle -\bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_{n-1} \rangle) \cdots (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) &= \sum_{\substack{I_0, \dots, I_{n-1} \\ J_0, \dots, J_{n-1}}} c_{J_0} \cdots c_{J_{n-1}} \times \\ &\times (-1)^{|I_0| + \sum_{j=0}^{n-1} (|I_j|+1)|J_j|} \mathcal{O}_{I_0 J_0} \cdots \mathcal{O}_{I_{n-1} J_{n-1}} \bar{\psi}_{I_0,0} \psi_{J_0,n-1} \bar{\psi}_{I_1,n-1} \psi_{J_1,n-2} \cdots \bar{\psi}_{I_{n-1},1} \psi_{J_{n-1},0}. \end{aligned}$$

Moving $\psi_{J_{n-1},0}$ to the right of $\bar{\psi}_{I_0,0}$ produces the sign

$$(-1)^{J_{n-1}(|J_0|+|I_{n-1}|+\sum_{j=2}^{n-1} |I_{n-j}|+|J_{n-j}|)}.$$

The integration over the Grassmann variables gives

$$(-1)^{|I_0|} c_{I_0} \cdots c_{I_{n-1}} \delta_{I_0, J_{n-1}} \delta_{I_1, J_0} \delta_{I_2, J_1} \cdots \delta_{J_{n-2}, I_{n-1}}.$$

After imposing the constraints coming from the Grassmann integration, the only surviving contributions are those satisfying

$$J_{n-1} = I_0, \quad J_0 = I_1, \quad \dots, \quad J_{n-2} = I_{n-1}.$$

Therefore, the product of matrix elements reduces to

$$\mathcal{O}_{I_0 I_1} \mathcal{O}_{I_1 I_2} \cdots \mathcal{O}_{I_{n-1} I_0},$$

which is precisely the standard expression for $\text{Tr}(\mathcal{O}^n)$. This proves the third identity. ■

Euclidean path-integral representation of the thermal partition function. The coherent-state formalism gives a Grassmann representation of the thermal partition function and imaginary-time correlation functions. Writing $K = H - \mu N$, the exact time-slicing identity is

$$e^{-\beta K} = \left(e^{-\beta K/n} \right)^n.$$

which decomposes the imaginary-time evolution into infinitesimal steps. To obtain a local Grassmann action, we approximate each short-time factor by its first-order expansion.

Lemma 29. *There exists a constant $C = C(\beta, \|K\|)$ such that, for every $n \geq 1$,*

$$\left\| e^{-\beta K} - \left(\mathbf{1}_{\mathcal{F}_L} - \frac{\beta}{n} K \right)^n \right\| \leq \frac{C}{n}.$$

Proof. Set

$$X_n \doteq e^{-\beta K/n}, \quad R_n \doteq e^{-\beta K/n} - \left(\mathbf{1}_{\mathcal{F}_L} - \frac{\beta}{n} K \right).$$

The Taylor expansion of the exponential gives

$$\|R_n\| \leq \frac{\beta^2 \|K\|^2}{2n^2} e^{\beta \|K\|/n}.$$

Using the telescopic identity

$$X_n^n - (X_n - R_n)^n = \sum_{j=0}^{n-1} X_n^{n-1-j} R_n (X_n - R_n)^j,$$

we obtain

$$\begin{aligned} \left\| e^{-\beta K} - \left(\mathbf{1}_{\mathcal{F}_L} - \frac{\beta}{n} K \right)^n \right\| &\leq n \|R_n\| \max\{\|X_n\|, \|X_n - R_n\|\}^{n-1} \\ &\leq \frac{C(\beta, \|K\|)}{n}. \end{aligned}$$

■

Proposition 28 may now be applied to $(\mathbf{1} - \beta K/n)^n$. If H is normal ordered, its coherent-state matrix elements are obtained from its normal symbol (see [Remark 21](#)):

$$Z_{\beta, \mu, L} = \int \prod_{\alpha} \prod_{j=0}^{n-1} d\bar{\psi}_{\alpha, j} d\psi_{\alpha, j} e^{\sum_{j=0}^{n-1} \sum_{\alpha} (\bar{\psi}_{\alpha, j+1} - \bar{\psi}_{\alpha, j}) \psi_{\alpha, j}} \prod_{j=0}^{n-1} \left(1 - \frac{\beta}{n} (H - \mu N)(\bar{\psi}_{j+1}, \psi_j) \right) + O\left(\frac{1}{n}\right)$$

where $\bar{\psi}_{\alpha, n} = -\bar{\psi}_{\alpha, 0}$ and $\psi_{\alpha, n} = -\psi_{\alpha, 0}$. For the quadratic-plus-quartic Hamiltonian in [Equation \(16\)](#) and [Equation \(17\)](#), its normal symbol is

$$H(\bar{\psi}_{j+1}, \psi_j) = \sum_{\alpha, \beta} h_{\alpha\beta} \bar{\psi}_{\alpha, j+1} \psi_{\beta, j} + \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} w_{\alpha\beta\gamma\delta} \bar{\psi}_{\alpha, j+1} \bar{\psi}_{\beta, j+1} \psi_{\delta, j} \psi_{\gamma, j}.$$

Expanding the short-time coherent-state kernel shows that replacing each factor by its exponential changes one slice by $O(n^{-2})$ and hence the full finite-dimensional trace by $O(n^{-1})$:

$$1 - \frac{\beta}{n} K(\bar{\psi}_{j+1}, \psi_j) \longrightarrow \exp \left[-\frac{\beta}{n} K(\bar{\psi}_{j+1}, \psi_j) \right].$$

Thus one obtains a Euclidean action on $\Lambda_L \times (\frac{\beta}{n}\mathbb{Z}/\beta\mathbb{Z})$:

$$S_n(\bar{\psi}, \psi) = - \sum_{j=0}^{n-1} \sum_{\alpha} (\bar{\psi}_{\alpha,j+1} - \bar{\psi}_{\alpha,j}) \psi_{\alpha,j} + \frac{\beta}{n} \sum_{j=0}^{n-1} K(\bar{\psi}_{j+1}, \psi_j). \quad (41)$$

Before discussing the continuous time limit $n \rightarrow \infty$, let us consider the quadratic case

$$(H - \mu N) = \sum_{\alpha, \beta} (h_{\alpha\beta} - \mu \delta_{\alpha\beta}) a_{\alpha}^{\dagger} a_{\beta}^{-},$$

for which the action becomes Gaussian and can be diagonalized by means of a discrete Fourier transform in the imaginary-time direction. Setting

$$\varepsilon = \frac{\beta}{n}, \quad \tau_j = \frac{\beta j}{n} = \varepsilon j.$$

we define

$$\psi_{\alpha,j} \doteq \frac{1}{\sqrt{n}} \sum_{\omega \in \Omega_{\beta,n}^{-}} e^{-i\omega\tau_j} \psi_{\alpha,\omega}, \quad \bar{\psi}_{\alpha,j} \doteq \frac{1}{\sqrt{n}} \sum_{\omega \in \Omega_{\beta,n}^{-}} e^{i\omega\tau_j} \bar{\psi}_{\alpha,\omega},$$

where the antiperiodic boundary conditions imply that the frequencies belong to

$$\Omega_{\beta,n}^{-} = \left\{ \frac{2\pi}{\beta} \left(m + \frac{1}{2}\right) : m = -\frac{n}{2}, \dots, \frac{n}{2} - 1 \right\},$$

where we have assumed n to be even for simplicity. Plugging these expressions in the action [Equation \(41\)](#), we get:

$$S_n(\bar{\psi}, \psi) = \sum_{\omega \in \Omega_{\beta,n}^{-}} \sum_{\alpha, \beta} \bar{\psi}_{\alpha,\omega} [(1 - e^{i\omega\varepsilon})\mathbb{1} + \varepsilon e^{i\omega\varepsilon}(h - \mu\mathbb{1})]_{\alpha\beta} \psi_{\beta,\omega},$$

Since the action is quadratic, [Proposition 11](#) gives

$$Z_{\beta,\mu,L} = \prod_{\omega \in \Omega_{\beta,n}^{-}} \det_{\mathcal{H}_L^1} [\mathbb{1} - e^{i\omega\varepsilon}(\mathbb{1} - \varepsilon(h - \mu\mathbb{1}))] + O\left(\frac{1}{n}\right).$$

The numbers $z_{\omega} = e^{i\omega\varepsilon}$, with $\omega \in \Omega_{\beta,n}^{-}$, are precisely the n roots of -1 . Using

$$\prod_{z^n = -1} (1 - z\lambda) = 1 + \lambda^n, \quad \lambda \in \mathbb{C},$$

we get:

$$Z_{\beta,\mu,L} = \det_{\mathcal{H}_L^1} [\mathbb{1} + (\mathbb{1} - \varepsilon(h - \mu\mathbb{1}))^n] + O\left(\frac{1}{n}\right).$$

Since $\varepsilon = \beta/n$, taking the limit $n \rightarrow \infty$ yields:

$$(\mathbb{1} - \varepsilon(h - \mu\mathbb{1}))^n = \left(\mathbb{1} - \frac{\beta}{n}(h - \mu\mathbb{1}) \right)^n \longrightarrow e^{-\beta(h - \mu\mathbb{1})},$$

which reproduces the grand-canonical partition function previously obtained by operator methods.

Remark 24.

1. The previous construction extends to imaginary-time correlation functions. Indeed, inserting local observables at the appropriate imaginary-time slices before taking the limit $n \rightarrow \infty$ yields the Grassmann-integral representation of Euclidean Green functions. The time-slicing decomposition naturally endows the discretized imaginary-time direction with an ordering, so that observables inserted at different time slices are automatically arranged according to their Euclidean time. Consequently, the continuous-time limit produces the usual time-ordered correlation functions. In particular, when two observables are inserted at the same imaginary time, they necessarily belong to the same Trotter slice, and the coherent-state matrix elements reduce to their normal symbols. Thus this particular discretization selects the normal-ordered equal-time prescription. Other one-sided prescriptions differ by the contact terms dictated by the CAR, as in Equation (38). Time ordering and the chosen equal-time convention are therefore encoded by the underlying time slicing.
2. In the physics literature, one usually proceeds one step further by taking the continuous-time limit directly at the level of the Euclidean action. Formally, the Riemann sums appearing in the discretized action are replaced by integrals over the thermal circle, leading to:

$$S(\bar{\psi}, \psi) = \int_0^\beta d\tau \left[\sum_\alpha \bar{\psi}_\alpha(\tau) \partial_\tau \psi_\alpha(\tau) + (H - \mu N)(\bar{\psi}(\tau), \psi(\tau)) \right],$$

where the Grassmann fields satisfy the antiperiodic boundary conditions

$$\psi_\alpha(\beta) = -\psi_\alpha(0), \quad \bar{\psi}_\alpha(\beta) = -\bar{\psi}_\alpha(0).$$

For the normal-ordered Hamiltonian considered above, this action takes the explicit form

$$S(\bar{\psi}, \psi) = \int_0^\beta d\tau \left[\sum_\alpha \bar{\psi}_\alpha(\tau) \partial_\tau \psi_\alpha(\tau) + \sum_{\alpha, \beta} (h_{\alpha\beta} - \mu \delta_{\alpha\beta}) \bar{\psi}_\alpha(\tau) \psi_\beta(\tau) + \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} w_{\alpha\beta\gamma\delta} \bar{\psi}_\alpha(\tau) \bar{\psi}_\beta(\tau) \psi_\delta(\tau) \psi_\gamma(\tau) \right].$$

The antiperiodic boundary conditions naturally lead to the Fourier expansions

$$\psi_\alpha(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega \in \Omega_\beta^-} e^{-i\omega\tau} \psi_{\alpha, \omega}, \quad \bar{\psi}_\alpha(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega \in \Omega_\beta^-} e^{i\omega\tau} \bar{\psi}_{\alpha, \omega},$$

where

$$\Omega_\beta^- = \left\{ \omega_m = \frac{2\pi}{\beta} \left(m + \frac{1}{2} \right) : m \in \mathbb{Z} \right\}$$

is the full set of fermionic Matsubara frequencies. In this representation, the quadratic part of the action becomes

$$S_0(\bar{\psi}, \psi) = \sum_{\omega \in \Omega_\beta^-} \sum_{\alpha, \beta} \bar{\psi}_{\alpha, \omega} [-i\omega \delta_{\alpha\beta} + h_{\alpha\beta} - \mu \delta_{\alpha\beta}] \psi_{\beta, \omega}.$$

The interaction term is instead given by

$$S_{\text{int}}(\bar{\psi}, \psi) = \frac{1}{2\beta} \sum_{\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega_\beta^-} \sum_{\alpha, \beta, \gamma, \delta} w_{\alpha\beta\gamma\delta} \bar{\psi}_{\alpha, \omega_1} \bar{\psi}_{\beta, \omega_2} \psi_{\delta, \omega_4} \psi_{\gamma, \omega_3} \delta_{\omega_1 + \omega_2, \omega_3 + \omega_4},$$

where the Kronecker delta expresses conservation of the Matsubara frequency. The continuous-time Gaussian functional integral formally involves an infinite product over Matsubara frequencies and therefore requires a normalization prescription. Set

$$k \doteq h - \mu \mathbf{1}_{\mathcal{H}_L^1}, \quad d \doteq \dim \mathcal{H}_L^1,$$

and introduce the symmetric ultraviolet cutoff

$$\Omega_{\beta,-}^{(N)} \doteq \left\{ \omega_m = \frac{2\pi}{\beta} \left(m + \frac{1}{2} \right) : m = -N-1, \dots, N \right\}.$$

The corresponding cutoff Grassmann integral is

$$Z_{\beta,\mu,L}^{(N)} \doteq \prod_{\omega \in \Omega_{\beta,-}^{(N)}} \det_{\mathcal{H}_L^1} (-i\omega \mathbf{1}_{\mathcal{H}_L^1} + k).$$

Its high-frequency part is divergent. A natural symmetric normalization is

$$\begin{aligned} C_N^{\text{sym}} &\doteq \prod_{m=-N-1}^N \det_{\mathcal{H}_L^1} (-i\omega_m \mathbf{1}_{\mathcal{H}_L^1}) \\ &= \left[\prod_{m=0}^N \frac{4\pi^2}{\beta^2} \left(m + \frac{1}{2} \right)^2 \right]^d. \end{aligned}$$

Pairing the frequencies ω_m and $\omega_{-m-1} = -\omega_m$, one obtains

$$\lim_{N \rightarrow \infty} \frac{Z_{\beta,\mu,L}^{(N)}}{C_N^{\text{sym}}} = \det_{\mathcal{H}_L^1} \cosh \left(\frac{\beta k}{2} \right).$$

This symmetric expression is even in every eigenvalue of k and therefore does not by itself retain the normal-ordering convention fixed by the Hamiltonian trace. Indeed,

$$\mathbf{1}_{\mathcal{H}_L^1} + e^{-\beta k} = 2e^{-\beta k/2} \cosh \left(\frac{\beta k}{2} \right),$$

and hence

$$\det_{\mathcal{H}_L^1} (\mathbf{1}_{\mathcal{H}_L^1} + e^{-\beta k}) = 2^d e^{-\frac{\beta}{2} \text{Tr}_{\mathcal{H}_L^1} k} \det_{\mathcal{H}_L^1} \cosh \left(\frac{\beta k}{2} \right).$$

The symmetric determinant is naturally associated, up to the parameter-independent factor 2^{-d} , with the particle-hole-centered many-body Hamiltonian

$$K_{\text{sym}} \doteq d\Gamma(k) - \frac{1}{2} \text{Tr}_{\mathcal{H}_L^1}(k) \mathbf{1}_{\mathcal{F}_L}.$$

If $\{\varepsilon_\lambda\}_{\lambda=1}^d$ are the eigenvalues of k , then

$$K_{\text{sym}} = \sum_{\lambda=1}^d \varepsilon_\lambda \left(n_\lambda - \frac{1}{2} \right),$$

and therefore

$$\text{Tr}_{\mathcal{F}_L} e^{-\beta K_{\text{sym}}} = 2^d \prod_{\lambda=1}^d \cosh \left(\frac{\beta \varepsilon_\lambda}{2} \right).$$

Its ground-state energy is

$$E_{\text{GS}}^{\text{sym}} = -\frac{1}{2} \sum_{\lambda=1}^d |\varepsilon_{\lambda}|.$$

By contrast, for the normal-ordered many-body Hamiltonian

$$K = d\Gamma(k),$$

one has

$$E_{\text{GS}} = \sum_{\lambda: \varepsilon_{\lambda} < 0} \varepsilon_{\lambda}.$$

The normalization compatible with the operator trace is therefore

$$C_N^{\text{op}} \doteq 2^{-d} e^{\frac{\beta}{2} \text{Tr}_{\mathcal{H}_L^1} k} C_N^{\text{sym}}.$$

Combining the preceding identities gives

$$\lim_{N \rightarrow \infty} \frac{Z_{\beta, \mu, L}^{(N)}}{C_N^{\text{op}}} = \det_{\mathcal{H}_L^1} (\mathbf{1}_{\mathcal{H}_L^1} + e^{-\beta k}) = \text{Tr}_{\mathcal{F}_L} e^{-\beta d\Gamma(k)}.$$

Thus the continuous-time fermionic functional integral must be understood as an appropriately normalized limit of finite-dimensional Grassmann integrals. The discrete-time coherent-state construction selects the normalization compatible with the normal-ordered Hamiltonian and removes the ambiguity contained in the formal infinite product over Matsubara frequencies.

For $0 < t < \beta$, using the path-integral representation together with [Equation \(7\)](#), the two-point function can be written as:

$$\langle \mathbf{T}_s a_x^-(t) a_y^+(0) \rangle_{\beta, \mu, L} = \frac{1}{\beta} \lim_{N \rightarrow \infty} \sum_{n=-N-1}^N e^{-i\omega_n t} [-i\omega_n \mathbf{1} + h - \mu \mathbf{1}]_{xy}^{-1}. \quad (42)$$

The symmetric limit is understood away from the discontinuity. At $t = 0$ the value depends on the chosen one-sided prescription, consistently with [Equation \(38\)](#).

Reflection Positivity for Fermionic Lattice Systems

Reflection positivity is a positivity condition for a pairing between observables supported in two complementary halves of a system related by a geometric reflection. This chapter is expository: no independent new reflection-positivity theorem is claimed. Its contribution is a unified, thesis-specific formulation of the Euclidean Grassmann and Hamiltonian CAR settings, in which the different fermionic sign conventions and their roles in the reflected pairings are made explicit. The discussion uses the two complementary frameworks introduced in the preceding chapter and recalls only the geometric, algebraic, and functional-analytic structures needed for the applications developed below.

The first is the *Euclidean*, or Lagrangian, formulation. It is defined on the Euclidean space–time lattice Γ_L , and its algebra of observables is generated by fields supported on Γ_L . Ordinary classical fields generate a commutative algebra, whereas fermionic fields generate a Grassmann algebra whose homogeneous elements supercommute according to their parity. The corresponding functionals are defined by integration against a Gibbs measure in the former case and by a Grassmann integral in the latter (see [Subsection 2.1.1](#)). When the resulting model is studied as a problem in classical statistical mechanics or Euclidean field theory, no coordinate of Γ_L need be distinguished as the time direction.

The second is the *Hamiltonian*, or operator-algebraic, formulation. Its underlying geometry is the spatial lattice Λ_L , and its observables belong to a local operator algebra. In quantum-spin systems, local algebras associated with disjoint subsets of Λ_L commute. In fermionic systems, locality is instead graded: homogeneous elements of disjoint CAR subalgebras supercommute, so that two odd elements anticommute, whereas an even element commutes with every observable having disjoint support (see [Subsection 2.2.3](#)). Expectations are given by states or trace functionals on the operator algebra, as in [Equation \(36\)](#). Quantum-spin systems will be mentioned only as the ungraded operator-algebraic counterpart of CAR systems, since the Hamiltonian part of this chapter is concerned specifically with fermionic operator algebras. For background on reflection positivity in quantum-spin systems, see [\[DLS78; Tas20; BU22\]](#).

In both formulations, a reflection exchanges two half-lattices and induces a map between the corresponding spaces of local observables. Reflection positivity requires the resulting reflected pairing to define a positive semidefinite sesquilinear form and therefore implies a Cauchy–Schwarz inequality. In commutative classical systems and quantum-spin systems, the two half-algebras commute, and the reflected pairing is constructed using the ordinary product. Fermionic systems require additional algebraic care. For Grassmann fields, the relevant signs are incorporated into the anti-

multiplicative Euclidean conjugation, the ordering of the generators, and the orientation of the Grassmann integral. For CAR algebras, disjoint odd observables anticommute, and the reflected pairing must consequently be formulated either on the even half-algebra or by means of an explicitly twisted product.

The distinction between the Euclidean and Hamiltonian formulations should not be confused with the distinction between two different uses of reflection positivity. Historically, the concept arose in the Osterwalder–Schrader axiomatization of Euclidean quantum field theory [OS73a; OS73b; FO74; OS75; OS78]. In that setting, one Euclidean coordinate is interpreted as imaginary time, and reflection positivity with respect to this distinguished direction is one of the ingredients used to reconstruct a Hilbert space, a non-negative Hamiltonian, and the corresponding real-time dynamics. This construction provides the conceptual converse to the coherent-state derivation of the preceding chapter, which associates a Euclidean Grassmann functional with a given quantum Hamiltonian. We recall this connection only briefly, since the reconstruction theorem itself is not needed in this thesis.

Here reflection positivity is used primarily as a statistical-mechanical tool. In the Euclidean formulation, the functional defines a model on the full lattice Γ_L , and reflections may be used in every coordinate direction for which the action is reflection positive. In the Hamiltonian formulation, reflections are defined directly on the spatial lattice Λ_L and its operator algebra and may likewise be applied in every admissible spatial direction. These constructions are not obtained from one another through Osterwalder–Schrader reconstruction; rather, they are distinct algebraic implementations of the same reflection-and-inequality strategy. The statistical-mechanical use of reflection positivity was pioneered by Fröhlich, Simon, and Spencer in their infrared-bound proof of phase transitions and continuous-symmetry breaking [FSS76]. Fröhlich, Israel, Lieb, and Simon subsequently systematized reflection positivity, infrared bounds, and chessboard estimates for lattice systems [Frö+78; Frö+80]. We shall not discuss Gaussian domination or the associated infrared bounds, which form another major application of reflection positivity and provide a standard route to proving spontaneous symmetry breaking of continuous symmetries. Recent applications and extensions of infrared-bound methods to fermionic systems can be found in [GK23; GK25].

Accordingly, after briefly recalling the Osterwalder–Schrader origin of the method, we develop the two fermionic formulations separately and then discuss reflection inequalities and their principal statistical-mechanical consequence used in this thesis, namely chessboard estimates. The Osterwalder–Schrader discussion and its lattice transfer-matrix version in Section 3.1 and Subsection 3.1.2 follow [OS73a; OS73b; OS75; FO74]. The abstract reflected pairing, Cauchy–Schwarz inequality, and dissemination mechanism developed in Section 3.2 synthesize the statistical-mechanical framework of [FSS76; Frö+78; Frö+80; FV17]. The Euclidean Grassmann reflection and action criterion in Subsections 3.3.1 and 3.3.2 adapt the framework of [OS78]. The Hamiltonian charged-CAR formulation, reflection inequality, and chessboard argument in Section 3.4, Subsection 3.4.1 and Subsection 3.4.2 follow [Lie94; MN96; Tas20]¹. Finally, the flux theorem and its comparison with the Majorana formalism are reviewed in Subsection 3.4.3 and Subsection 3.4.4, with the latter based on [JP15; JJ16].

¹[Frö+80] sketches a fermionic application, but the particular transformation used there does not put the hopping Hamiltonian in reflection-positive form. As pointed out explicitly in [MN96], the hopping terms on the right half acquire the opposite sign from those on the left. The Hamiltonian treatment developed here instead uses the framework of Lieb [Lie94] together with the systematic Jordan–Wigner-type, particle–hole, and gauge transformation of Macris–Nachtergaele [MN96]; none of the subsequent arguments relies on the problematic step in [Frö+80].

Within this general framework, the Euclidean action criterion is presented in the anti-multiplicative Grassmann convention used in the joint work [FG26], while the Hamiltonian reflection inequality and chessboard symmetrization are stated in precisely the form required in the joint works [GP26; BGP26]. These formulations are adaptations and technical preparations for the original applications developed in Chapter 4, Chapter 6 and Chapter 7.

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3.1 Osterwalder–Schrader positivity and transfer matrices

3.1.1 Continuum formulation

Let $x = (x_0, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ be a point of Euclidean space and $\vartheta_{\mathbb{E}}^0$ the reflection of the Euclidean-time coordinate

$$\vartheta_{\mathbb{E}}^0(x_0, \mathbf{x}) = (-x_0, \mathbf{x}).$$

Let $\mathcal{E}_+$ be the algebra generated by fields smeared with test functions supported in the positive-time half-space $x_0 > 0$. The geometric reflection, combined with complex conjugation, induces an antilinear map $\Theta_{\mathbb{E}}^0$ from $\mathcal{E}_+$ to the algebra associated with the negative-time half-space.

The Euclidean n -point correlation functions, conventionally called *Schwinger functions*, are the moments of a Euclidean Schwinger functional $\mathcal{S}_{\mathbb{E}}$. For a scalar field, for instance, they are given schematically by

$$S_n(f_1, \dots, f_n) = \mathcal{S}_{\mathbb{E}}(\Phi(f_1) \cdots \Phi(f_n)).$$

When the Euclidean theory admits a quantum interpretation, its Schwinger functions coincide with the imaginary-time-ordered correlation functions of the corresponding quantum theory discussed eg. in Subsection 2.2.7. Intrinsically, however, they are Euclidean correlation functions and can be defined independently of whether a quantum reconstruction is available.

The Schwinger functional satisfies Osterwalder–Schrader positivity if

$$\mathcal{S}_{\mathbb{E}}(\Theta_{\mathbb{E}}^0(F)F) \geq 0, \quad F \in \mathcal{E}_+. \quad (1)$$

In the fermionic case, the notation in Equation (1) is understood with the appropriate Euclidean conjugation and graded ordering which were already identified in the original formulation of Euclidean Fermi fields [OS73b; FO74].

Osterwalder–Schrader positivity does not require the Schwinger functions, or their integral kernels, to be non-negative pointwise. In general, such a requirement would not even be well posed, since Schwinger functions may be distributions or may carry internal and spinorial indices. Rather, Equation (1) is a quadratic positivity condition obtained after reflecting and smearing the field insertions. Equivalently, for every finite family $F_1, \dots, F_m \in \mathcal{E}_+$, the matrix

$$(\mathcal{S}_E(\Theta_E^0(F_i)F_j))_{i,j=1}^m \quad (2)$$

is positive semidefinite. Indeed, applying Equation (1) to $F = \sum_{i=1}^m c_i F_i$, gives

$$\sum_{i,j=1}^m \bar{c}_i c_j, \mathcal{S}_E(\Theta_E^0(F_i)F_j) \geq 0.$$

Individual off-diagonal entries of Equation (2) need not be non-negative; positivity concerns the associated quadratic form. Consequently,

$$(F, G)_{\text{OS}} \doteq \mathcal{S}_E(\Theta_E^0(F)G)$$

defines a positive semidefinite sesquilinear form on $\mathcal{E}_+$. Quotienting $\mathcal{E}_+$ by the corresponding null space and completing the quotient yields the Osterwalder–Schrader Hilbert space $\mathcal{H}_{\text{OS}}$. Under the remaining Osterwalder–Schrader assumptions, positive Euclidean-time translations induce a strongly continuous contraction semigroup $e^{-tH_{\text{OS}}}$ for $t \geq 0$, whose generator H_{OS} is non-negative. Reflection positivity therefore converts reflection in Euclidean time into Hilbert-space positivity and provides one of the fundamental ingredients in the reconstruction of the corresponding quantum theory [OS73a; OS75]. We shall not need the full reconstruction theorem; only this structural relation between the reflected Euclidean pairing and the quantum inner product will be relevant.

3.1.2 Lattice formulation and the transfer matrix

The same mechanism has a direct lattice counterpart. Write the temporal coordinate in lattice units as

$$x_0 \in \tau\mathbb{Z}/\beta\mathbb{Z}, \quad \beta = N_\tau\tau,$$

where N_τ is the number of temporal slices and τ is the temporal lattice spacing. Reflection through the hyperplane lying between the slices $x_0 = 0$ and $x_0 = 1$ is given by

$$\vartheta_E^0(x_0, \mathbf{x}) = ((\tau - x_0) \bmod \beta, \mathbf{x}). \quad (3)$$

For even N_τ , this reflection exchanges two disjoint temporal halves of the Euclidean lattice.

Suppose that the action is local in the temporal direction and admits a transfer-matrix representation meaning that the Euclidean partition function can then be written as

$$Z_E = \text{Tr}_{\mathcal{H}_{\text{tr}}} \mathcal{T}^{N_\tau}, \quad (4)$$

where $\mathcal{T}$ is some operator that evolves the system by one temporal lattice spacing. Under the standard transfer-matrix hypotheses, reflection invariance gives self-adjointness of $\mathcal{T}$, while reflection

positivity across the intervening hyperplane implies $\mathcal{T} \geq 0$. This is the operator-theoretic content of reflection positivity in the temporal direction. If $\mathcal{T}$ has trivial kernel, one may define

$$H_{\text{tr}} = -\frac{1}{\tau} \log \mathcal{T}.$$

The operator H_{tr} is self-adjoint and bounded from below, and Equation (4) becomes

$$Z_{\text{E}} = \text{Tr}_{\mathcal{H}_{\text{tr}}} e^{-\beta H_{\text{tr}}}. \quad (5)$$

Thus, the Euclidean partition function is identified with the thermal partition function of the reconstructed Hamiltonian system. Multiplying $\mathcal{T}$ by a positive scalar amounts to adding a constant to H_{tr} . One may therefore shift the zero of the energy so that $H_{\text{tr}} \geq 0$; this changes the partition function only by an overall factor and leaves normalized correlation functions unchanged.

The reflected pairing also has a direct transfer-matrix interpretation. An observable F supported in one temporal half determines an operator B_F on $\mathcal{H}_{\text{tr}}$, obtained by composing its field insertions with the appropriate powers of $\mathcal{T}$. Temporal reflection reverses the order of the insertions and replaces them by their adjoints. After absorbing the transfer factors into B_F , one obtains schematically

$$\langle \Theta_{\text{E}}^0(F)F \rangle_{\text{E}} = \frac{1}{Z_{\text{E}}} \text{Tr}_{\mathcal{H}_{\text{tr}}} (B_F^* B_F) \geq 0. \quad (6)$$

The lattice Osterwalder–Schrader condition is therefore the Euclidean representation of an ordinary positive operator pairing. In the fermionic case, the Euclidean conjugation and the graded signs are precisely what is needed for the reflected Grassmann expression to reproduce the adjoint B_F^* .

The converse construction was discussed in the preceding chapter. Starting from a self-adjoint grand-canonical generator $H - \mu N$, one has the positive one-step operator

$$\mathcal{T}_{\tau} = e^{-\tau(H - \mu N)},$$

and hence

$$Z_{\beta, \mu, L} = \text{Tr}_{\mathcal{F}} \mathcal{T}_{\tau}^{N_{\tau}} = \text{Tr}_{\mathcal{F}} e^{-\beta K}. \quad (7)$$

As shown in the preceding chapter, inserting fermionic coherent-state resolutions of the identity between the factors of $\mathcal{T}_{\tau}$ converts this trace into a finite-dimensional Grassmann integral with the discretized action given in Equation (41). The antiperiodic temporal boundary condition originates from the trace in Equation (7), while the orientation of the Grassmann integral and the ordering of the Grassmann variables encode the corresponding fermionic signs. In particular, the temporal Grassmann reflection cannot in general be obtained by simply copying the spatial CAR reflection: its precise form depends on the coherent-state conventions and on the lattice Dirac operator.

This discussion explains the Osterwalder–Schrader origin of reflection positivity and its relation to the Euclidean representation of a quantum partition function. From the next section onward, reflection positivity will instead be treated directly as a positivity property of lattice functionals and used to derive statistical-mechanical inequalities.

3.2 General Structure of Reflection Positivity

We now turn to the statistical-mechanical formulation of reflection positivity. The settings considered below are organized by two independent distinctions: whether the observables are Euclidean

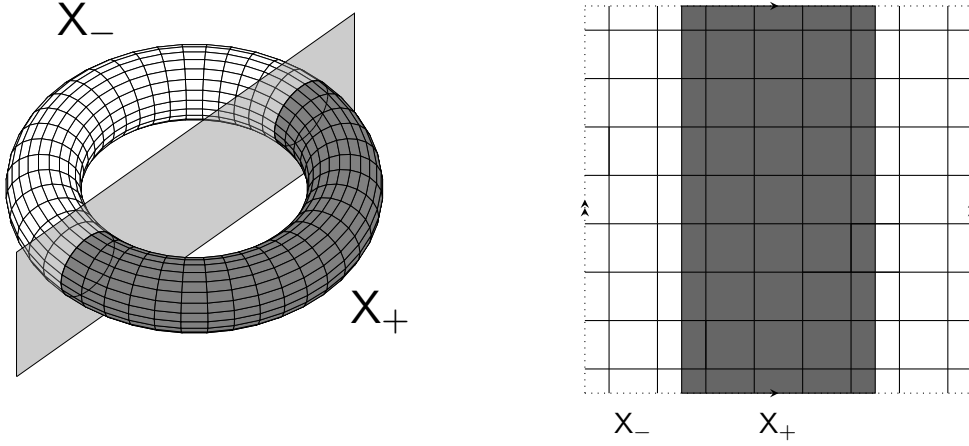


Figure 3.1: Graphical representation of the cut torus (© Davide Morgante)

fields or Hamiltonian operators, and whether their local algebra is ungraded or $\mathbb{Z}_2$ -graded. This gives the four realizations:

	Euclidean/Lagrangian	Hamiltonian/operator-algebraic
ungraded	classical Gibbs Reflection Positivity	quantum-spin Reflection Positivity
$\mathbb{Z}_2$ -graded	Grassmann Reflection Positivity	CAR Reflection Positivity.

The two columns correspond to different realizations of locality, while the two rows distinguish ordinary locality from graded locality. Despite these algebraic differences, all four cases implement the same geometric reflection and inequality mechanism.

Let $\mathbf{X}$ denote either the Euclidean lattice Γ_L or the spatial lattice Λ_L , and let $\vartheta : \mathbf{X} \rightarrow \mathbf{X}$ be a fixed-point-free reflection (see Figure 3.1) such as the reflection between vertices of $\mathbf{X}$. Thus

$$\mathbf{X} = \mathbf{X}_+ \sqcup \mathbf{X}_-, \quad \vartheta(\mathbf{X}_+) = \mathbf{X}_-, \quad \vartheta^2 = \text{id}. \quad (8)$$

On a torus the reflection has two interfaces, therefore the boundary conditions and all terms crossing both interfaces must therefore be included when reflection positivity is checked.

Denote by $\mathcal{B}_\pm$ the spaces of observables supported in $\mathbf{X}_\pm$. In all the applications below, geometric reflection induces an antilinear bijective involution

$$\Theta : \mathcal{B}_\pm \longrightarrow \mathcal{B}_\mp. \quad (9)$$

So far we have not required any compatibility between the reflection operation Θ on the algebras and their multiplicative structure: this point is essential. For classical observables and for the operator algebras used for quantum spins and CAR fermions, the reflections usually chosen are multiplicative. The Euclidean conjugation used for Grassmann fields is instead an anti-homomorphism².

²Notice, however, that a bijective (actually surjectivity is sufficient) antimorphism preserves the identity of the algebra. Indeed, we have that $\Theta(\mathbf{1})$ is both a left and a right identity on the image of Θ :

$$\Theta(A) = \Theta(\mathbf{1}A) = \Theta(A)\Theta(\mathbf{1}) \quad \Theta(A) = \Theta(A\mathbf{1}) = \Theta(\mathbf{1})\Theta(A) \quad \forall A \in \mathcal{B}_+$$

Therefore if Θ is surjective, by unicity of the identity in an algebra we can conclude.

Consequently, there is no model-independent algebraic expression that can be denoted by “the reflected square” before these conventions have been specified.

For each of these realizations, the reflection and the product between opposite half-algebras are chosen (when possible) to define a normalized sesquilinear pairing

$$\langle A, B \rangle_{\Theta, \omega} \doteq \omega(\Theta(A) \circ B), \quad A, B \in \mathcal{B}_+, \quad (10)$$

where ω is the relevant normalized expectation functional and $\circ$ denotes the reflected product appropriate to the setting under consideration.

Remark 25. If the reflected product is bilinear, antilinearity of Θ and the linearity of ω imply sesquilinearity:

$$\langle A + \lambda B, C \rangle_{\Theta, \omega} = \langle A, C \rangle_{\Theta, \omega} + \bar{\lambda} \langle B, C \rangle_{\Theta, \omega} \quad (11)$$

$$\langle C, A + \lambda B \rangle_{\Theta, \omega} = \langle C, A \rangle_{\Theta, \omega} + \lambda \langle C, B \rangle_{\Theta, \omega}. \quad (12)$$

Furthermore for normalized functionals $\omega(\mathbf{1}) = 1$, since $\Theta(\mathbf{1}_{X_+}) = \mathbf{1}_{X_-}$:

$$\langle \mathbf{1}_+, \mathbf{1}_+ \rangle_{\Theta, \omega} = 1.$$

In commutative and quantum-spin systems one simply has $\Theta(A) \circ B = \Theta(A)B$. We shall use the same notation in the Grassmann setting: there the fermionic signs are encoded in the anti-multiplicative Euclidean conjugation, in the ordering of the generators, and in the orientation of the Grassmann integral, rather than in an additional phase in the displayed product. For CAR algebras, by contrast, the reflection is multiplicative and the product $\circ$ must explicitly take the grading into account (see [JP15]). However, it is worth stressing that the graded product $\circ$ reduces to the ordinary product for even observables.

Definition 30 (Reflection Positivity). The functional ω is Reflection Positive, with respect to the specified reflection and reflected product, if it is

1. **Hermitian:**

$$\langle A, B \rangle_{\Theta, \omega} = \overline{\langle B, A \rangle_{\Theta, \omega}} \quad \text{for every } A, B \in \mathcal{B}_+ \quad (13)$$

2. **Positive Semi-Definite:**

$$\langle A, A \rangle_{\Theta, \omega} \geq 0 \quad \text{for every } A \in \mathcal{B}_+. \quad (14)$$

Remark 26. The hermiticity property in Equation (13) actually follows from sesquilinearity and Equation (14). Since:

$$\begin{aligned} \langle A + B, A + B \rangle_{\Theta, \omega} &= \langle A, A \rangle_{\Theta, \omega} + \langle A, B \rangle_{\Theta, \omega} + \langle B, A \rangle_{\Theta, \omega} + \langle B, B \rangle_{\Theta, \omega} \geq 0 \\ \langle A + iB, A + iB \rangle_{\Theta, \omega} &= \langle A, A \rangle_{\Theta, \omega} + i\langle A, B \rangle_{\Theta, \omega} - i\langle B, A \rangle_{\Theta, \omega} + \langle B, B \rangle_{\Theta, \omega} \geq 0 \end{aligned}$$

it holds that both:

$$\langle A, B \rangle_{\Theta, \omega} + \langle B, A \rangle_{\Theta, \omega} \in \mathbb{R} \quad \langle A, B \rangle_{\Theta, \omega} - \langle B, A \rangle_{\Theta, \omega} \in \mathbb{R}$$

Therefore, by setting $\langle A, B \rangle_{\Theta, \omega} = x + iy$ and $\langle B, A \rangle_{\Theta, \omega} = a + ib$, we have:

$$x + iy + a + ib \in \mathbb{R} \implies y + b = 0 \quad ix - y - ia + b \in \mathbb{R} \implies x - a = 0$$

so that:

$$\langle A, B \rangle_{\Theta, \omega} = x + iy \quad \text{and} \quad \langle B, A \rangle_{\Theta, \omega} = x - iy$$

Once the two conditions hold, the following consequences are independent of how the pairing was constructed.

Lemma 31 (Reflected Cauchy–Schwarz inequality). *For $A, B \in \mathcal{B}_+$,*

$$|\langle A, B \rangle_{\Theta, \omega}|^2 \leq \langle A, A \rangle_{\Theta, \omega} \langle B, B \rangle_{\Theta, \omega}. \quad (15)$$

Remark 27. From Equation (15), it follows that if $\langle A, A \rangle_{\Theta, \omega} = 0$, then $\langle A, B \rangle_{\Theta, \omega} = 0$ for every $B \in \mathcal{B}_+$. Therefore, the set

$$\mathcal{N}_{\Theta, \omega} \doteq \{A \in \mathcal{B}_+ : \langle A, A \rangle_{\Theta, \omega} = 0\}$$

is a linear subspace. The quotient $\mathcal{B}_+ / \mathcal{N}_{\Theta, \omega}$ carries a positive definite inner product and its completion $\mathcal{H}_{\Theta, \omega}$ is an Hilbert Space.

The reflected Cauchy–Schwarz inequality in Equation (15) is a consequence of Reflection Positivity with respect to a single reflection. Chessboard estimates require instead a compatible family of reflections and a precise procedure for transporting local observables across the lattice. We briefly describe the standard block formulation; see, for example, [FV17].

Suppose we can divide our lattice X in blocks Q_B of size B such that $2B$ divides the length of the lattice L as in Figure 3.2.

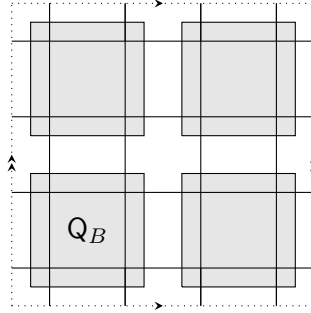


Figure 3.2: Division of the lattice X with blocks Q_B with $B = 2$ and $L = 4$

Introduce the reference block $\mathsf{Q}_B \doteq \{0, \dots, B-1\}^d$ and the block-index torus $\mathbb{T}_{L/B}^d \doteq \mathbb{Z}_{L/B}^d$. The torus then decomposes into the disjoint union

$$\mathsf{X} = \bigsqcup_{t \in \mathbb{T}_{L/B}^d} \mathsf{Q}_B^t, \quad \mathsf{Q}_B^t \doteq \mathsf{Q}_B + Bt.$$

The total number of blocks is therefore

$$N_B \doteq |\mathbb{T}_{L/B}^d| = \left(\frac{L}{B}\right)^d.$$

	λ	f	λ	f	λ	f	λ
	f	λ	f	λ	f	λ	f
	λ	f	λ	f	λ	f	λ
	f	λ	f	λ	f	λ	f

Figure 3.3: Successive reflections transport the local function f from the block $\mathbf{Q}_B$ to the block indexed by t . The shaded cells indicate one possible path of reflections. From [FV17, Figure 10.3]

An observable f is called $\mathbf{Q}_B$ -local if it is supported in the reference block $\mathbf{Q}_B$. For each $t \in \mathbb{T}_{L/B}^d$, its reflected copy in $\mathbf{Q}_B^t$ is defined as follows. Choose a nearest-neighbor path

$$0 = t_0, t_1, \dots, t_k = t$$

in the block-index torus, and let Θ_i denote the reflection through the bond hyperplane separating the adjacent blocks $\mathbf{Q}_B^{t_{i-1}}$ and $\mathbf{Q}_B^{t_i}$. We set

$$f^{[t]} \doteq \Theta_k \circ \dots \circ \Theta_1(f).$$

where $\circ$ denotes the composition of functions. For the standard coordinate reflections, the assumption that L/B is even ensures that this definition is independent of the chosen admissible path. More generally, path independence must be included among the compatibility requirements imposed on the reflection maps. Notice that $f^{[t]}$ is supported in $\mathbf{Q}_B^t$, but it need not be a simple translate of f : depending on t , the orientation of the observable inside the block may be reversed (see Figure 3.3).

The Gibbs functional is called B -periodic if it is invariant under translations by one block,

$$\langle \tau_{Be_j}(A) \rangle = \langle A \rangle, \quad j = 1, \dots, d.$$

Assume that it is B -periodic and Reflection Positive with respect to every bond reflection separating two neighboring layers of blocks. Then, for any family of $\mathbf{Q}_B$ -local observables $\{f_t\}_{t \in \mathbb{T}_{L/B}^d}$, which are either all bounded or all non-negative, the chessboard estimate states that

$$\left| \left\langle \prod_{t \in \mathbb{T}_{L/B}^d} f_t^{[t]} \right\rangle \right| \leq \prod_{t \in \mathbb{T}_{L/B}^d} \left\langle \prod_{s \in \mathbb{T}_{L/B}^d} f_t^{[s]} \right\rangle^{1/N_B}. \quad (16)$$

Each factor on the right-hand side is non-negative. It is obtained by placing the same block observable f_t , with the appropriate reflected orientation, in every block of the torus. This operation is usually called the *dissemination* of f_t .

The proof consists of repeated applications of the reflected Cauchy–Schwarz inequality in Equation (15). At each step, the observables in one half of the torus are reflected into the other half. Iterating over the relevant bond hyperplanes and coordinate directions eventually replaces every local insertion by a configuration in which that insertion occupies all blocks. Consequently, the

chessboard estimate is not a consequence of Reflection Positivity with respect to one fixed plane: it uses Reflection Positivity for the entire family of block-reflection planes, block periodicity, and the consistency of the transport maps.

Remark 28. In a graded system, Equation (16) applies in this form to even block observables, since reflected copies with disjoint supports can then be reordered without producing signs. For odd insertions, the ordinary product in the preceding formulas is not intrinsically defined independently of its ordering. One must instead prescribe a graded ordering, track the signs produced by every reflection, or formulate the estimate using an appropriate twisted product. None of these structures will be needed in this thesis.

In the remainder of this chapter, the Grassmann and CAR realizations of these structures will be developed systematically, while commutative classical systems and quantum-spin systems will serve as ungraded reference cases.

3.3 Euclidean Reflection Positivity

3.3.1 Euclidean Reflections and Reversal of Products

Throughout this section, ϑ_ν denotes a geometric reflection of the Euclidean lattice and Θ_ν its antilinear action on fields and observables. The subscripts E and H will be restored only when the Euclidean and Hamiltonian formulations are compared explicitly.

Let Γ_L be a Euclidean lattice and let ϑ_ν be a reflection through a bond hyperplane orthogonal to the coordinate direction ν which gives the following decomposition:

$$\Gamma_L = \Gamma_L^+ \sqcup \Gamma_L^-, \quad \vartheta_\nu(\Gamma_L^+) = \Gamma_L^-. \quad (17)$$

Definition 32. Let $(G_A)_\pm$ be the Grassmann algebras generated by $\psi_{x,J}$ and $\bar{\psi}_{x,J}$ with $x \in \Gamma_L^\pm$ respectively. The reflection (anti-)morphism is an involutive antilinear map:

$$\Theta_\nu : (G_A)_\pm \rightarrow (G_A)_\mp$$

defined on the generators as:

$$\begin{aligned} \Theta_\nu(\psi_{x,J}) &= \sum_{J'} (A_\nu(x))_{JJ'} \bar{\psi}_{\vartheta_\nu(x),J'}, \\ \Theta_\nu(\bar{\psi}_{x,J}) &= \sum_{J'} \psi_{\vartheta_\nu(x),J'} (B_\nu(x))_{J'J}, \end{aligned} \quad (18)$$

for some $|\mathcal{J}| \times |\mathcal{J}|$ complex matrices A_ν and B_ν chosen so that $\Theta_\nu^2 = \text{id}^3$, and then extended as an anti-homomorphism to the whole algebra:

$$\Theta_\nu(FG) = \Theta_\nu(G)\Theta_\nu(F). \quad (19)$$

Remark 29.

³Such condition can be easily recasted as two matrix equations:

$$\overline{A_\nu(x)} B_\nu(\vartheta_\nu(x)) = \mathbf{1} \quad \overline{B_\nu(x)} A_\nu(\vartheta_\nu(x)) = \mathbf{1}$$

1. The action of reflection on the generators is model dependent.
2. Θ_ν preserves Grassmann parity:

$$\Theta_\nu((\mathcal{G}_A)_\pm^{\text{even/odd}}) = (\mathcal{G}_A)_\mp^{\text{even/odd}}$$

where we have denoted with $(\mathcal{G}_A)_\pm^{\text{even/odd}}$ respectively denote the even and odd subspaces of the right/left half-space algebra.

3. For a monomial of m odd generators,

$$\Theta_\nu(g_1 \cdots g_m) = \Theta_\nu(g_m) \cdots \Theta_\nu(g_1). \quad (20)$$

Reordering the right-hand side back to the original order gives a sign $(-1)^{m(m-1)/2}$. In the convention of [Equation \(19\)](#), this order reversal is precisely what makes the action criterion below work without introducing the phase used in the twisted CAR product.

4. Generally one requires that $\Theta_\nu(\bar{\psi}_{x,J} \psi_{x,J}) = \bar{\psi}_{\vartheta_\nu(x),J} \psi_{\vartheta_\nu(x),J}$, which imposes that $A_\nu = B_\nu^{-1}$. In the following, we will always deal with reflections that preserves this property.

3.3.2 The Action Criterion for a Reflection Positive Grassmann Measure

A Grassmann integral is not a positive functional in the sense that it maps positive elements of the algebra into positive numbers⁴. Reflection positivity is instead a positivity property of the complex-valued linear functional obtained by integrating reflected products of the form $\Theta(F)F$. We can factor the flat Grassmann measure $(d\bar{\psi} d\psi)_A$ (defined in [Subsection 2.1.1](#)) into two half-space measures $d\nu^\pm$, as:

$$d\nu_\pm = \prod_{x \in \Gamma_L^\pm} \prod_{J \in \mathcal{J}} d\bar{\psi}_{x,J} d\psi_{x,J} \quad \text{such that} \quad (d\bar{\psi} d\psi)_A = d\nu^+ d\nu^-$$

We define the reflected measure $\Theta(d\nu_\pm)$ as a "push-forward":

$$\forall G_\pm \in \mathcal{G}_{A,\pm} \quad \int \Theta(d\nu_\pm) G_\mp = \overline{\int d\nu_\pm \Theta(G_\mp)}. \quad (21)$$

Proposition 33. *The following hold true:*

1. $\Theta(d\nu_\pm) = d\nu_\mp$
2. *The quadratic form:*

$$\langle A, B \rangle \doteq \int (d\bar{\psi} d\psi)_A \Theta(A) B \quad \forall A, B \in \mathcal{G}_{A,+}$$

is reflection positive

Proof.

⁴There is no non-trivial intrinsic notion of positivity on a Grassmann algebra analogous to the positive cone of a C^* -algebra. Indeed, with the natural involution one has

$$\bar{\psi}\psi = \psi^* \psi, \quad -\bar{\psi}\psi = \psi \bar{\psi} = (\bar{\psi})^* \bar{\psi}.$$

Thus a non-zero even element and its negative are both algebraic sums of squares. Consequently, the corresponding cone is not proper, and every ordinary positive functional must annihilate the nilpotent Grassmann ideal.

1. First, let us notice that we can restrict Equation (21) to observables of the form:

$$G_{\pm} = \prod_{x \in \Gamma_L^{\pm}} \prod_J \bar{\psi}_{x,J} \psi_{x,J}.$$

If G is not top degree, Equation (21) is trivially true as both sides of the equation are vanishing. Recalling Item 4 of Remark 29, we have:

$$\Theta(G_{\pm}) = G_{\mp},$$

therefore:

$$\int \Theta(d\nu_{\pm}) G_{\mp} = \int d\nu_{\pm} \Theta(G_{\mp}) = \int d\nu_{\pm} G_{\pm} = 1$$

2. The quadratic form is clearly sesquilinear (recall Remark 25) and positive semi-definite:

$$\int (d\bar{\psi} d\psi)_A \Theta(A) A = \int d\nu_- d\nu_+ \Theta(A) A = \left| \int d\nu_+ A \right|^2 \quad A \in \mathcal{G}_{A,+}$$

■

The following proposition gives the sufficient condition on the exponent that will be used in the Euclidean fermionic applications.

Proposition 34 (Euclidean Grassmann action criterion). *Suppose that the even element $-S \in (G_A)_+$ admits the decomposition*

$$-S = B + \Theta(B) + \sum_{\omega \in \Omega} C_{\omega} \Theta(C_{\omega}), \quad (22)$$

where Ω is finite and

$$B \in (G_A)_+^{\text{even}}, \quad C_{\omega} \in (G_A)_+^{\text{odd}}, \quad \omega \in \Omega. \quad (23)$$

Then, for every $F \in (G_A)_+^{\text{even}}$,

$$\int (d\bar{\psi} d\psi)_A \Theta(F) F e^{-S} \geq 0. \quad (24)$$

Remark 30. If⁵

$$Z \doteq \int (d\bar{\psi} d\psi)_A e^{-S} > 0, \quad (25)$$

then the normalized expectation

$$\langle F \rangle \doteq \frac{1}{Z} \int (d\bar{\psi} d\psi)_A F e^{-S} \quad (26)$$

is Reflection Positive on $(\mathcal{G}_A)_+^{\text{even}}$, namely

$$\langle \Theta(F) F \rangle \geq 0, \quad F \in (\mathcal{G}_A)_+^{\text{even}}. \quad (27)$$

⁵Note that, if we assume that the unnormalized expectation value is Reflection Positive, $Z \geq 0$.

Proof. Since:

$$e^{\Theta(B)} = \sum_{n \geq 0} \frac{\Theta(B)^n}{n!} = \sum_{n \geq 0} \frac{\Theta(B^n)}{n!} = \Theta\left(\sum_{n \geq 0} \frac{B^n}{n!}\right) = \Theta(e^B),$$

we are left with proving that:

$$\int d\nu_+ d\nu_- e^{B+\Theta(B)+\sum_{\omega} C_{\omega}\Theta(C_{\omega})} \Theta(F)F = \int d\nu_+ e^B \int d\nu_- \Theta(e^B) e^{\sum_{\omega} C_{\omega}\Theta(C_{\omega})} \Theta(F)F \geq 0.$$

Expanding the mixed term, we find that

$$\begin{aligned} e^{\sum_{\omega} C_{\omega}\Theta(C_{\omega})} &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\omega_1 \dots \omega_n} C^{\omega_1} \Theta(C^{\omega_1}) \dots C^{\omega_n} \Theta(C^{\omega_n}) \\ &= \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \sum_{\omega_1 \dots \omega_n} C^{\omega_1} \dots C^{\omega_n} \Theta(C^{\omega_1}) \dots \Theta(C^{\omega_n}) \\ &= \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \sum_{\omega_1 \dots \omega_n} C^{\omega_1} \dots C^{\omega_n} \Theta(C^{\omega_n} \dots C^{\omega_1}) \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\omega_1 \dots \omega_n} C^{\omega_1} \dots C^{\omega_n} \Theta(C^{\omega_1} \dots C^{\omega_n}). \end{aligned}$$

Thus:

$$\begin{aligned} \int d\nu_+ d\nu_- e^{B+\Theta(B)+\sum_{\omega} C_{\omega}\Theta(C_{\omega})} \Theta(F)F &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\omega_1 \dots \omega_n} \int d\nu_+ e^B C^{\omega_1} \dots C^{\omega_n} F \times \\ &\quad \times \int d\nu_- \Theta(e^B C^{\omega_1} \dots C^{\omega_n} F). \end{aligned}$$

Due to [Equation \(21\)](#), we have

$$\int d\nu_+ d\nu_- e^{B+\Theta(B)+\sum_{\omega} C_{\omega}\Theta(C_{\omega})} \Theta(F)F = \left| \int d\nu_+ e^B C^{\omega_1} \dots C^{\omega_n} F \right|^2 \geq 0.$$

■

Usually even terms of the action S supported entirely in the two separate half-lattices are included in $B + \Theta(B)$. The substantive condition concerns the terms crossing the reflection plane: they must admit a decomposition as a sum of $C_{\omega}\Theta(C_{\omega})$, with each C_{ω} odd and positive coefficient. The same argument applies when the Grassmann variables are coupled to commuting fields (see [Section 4.2](#) for details), provided that the commuting integration measure and all local even weights factorize over the each half-spacetime lattice and are exchanged by the reflection. This is the form used for the auxiliary-field representation of the Euclidean Gross–Neveu models. For each $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, the corresponding fermionic exponent is decomposed as

$$K_{\alpha} = B_{\alpha} + \Theta(B_{\alpha}) + \sum_{\omega \in \Omega_{\alpha}} C_{\alpha}^{\omega} \Theta(C_{\alpha}^{\omega}), \quad \text{with } B_{\alpha} \in (G_A)_+^{\text{even}}, \quad C_{\alpha}^{\omega} \in (G_A)_+^{\text{odd}}. \quad (28)$$

The verification of this decomposition depends on the chosen Dirac operator and crucially on the choice of antiperiodic boundary conditions. These model-dependent checks, together with the proof of reflection positivity for the resulting bosonic measures, are carried out in detail in [Chapter 4](#).

3.4 Hamiltonian Reflection Positivity for Fermions

We now turn to the Hamiltonian formulation of reflection positivity for fermionic systems. The observables now belong to the CAR algebra over the spatial lattice Λ_L . The fermionic signs arising under reflection are organized in different way compared to the Euclidean case (see [Remark 32](#)). Since the applications in this thesis use the formulation introduced by Lieb in the proof of the flux-phase theorem [[Lie94](#)], we take that formulation as our main framework. A more general treatment in terms of Majorana generators will be mentioned briefly at the end of the section.

3.4.1 Lieb's Reflection Inequality

Let Λ_L be a finite spatial lattice and let ϑ be a reflection through a bond hyperplane that decomposes it into two disjoint halves,

$$\Lambda_L = \Lambda_L^- \sqcup \Lambda_L^+ \quad \vartheta(\Lambda_L^+) = \Lambda_L^-$$

Let $\mathcal{A}_+$ and $\mathcal{A}_-$ be the CAR subalgebras generated by the fermionic creation and annihilation operators supported in the two halves.

Definition 35 (Reflection Operator). Let $\mathcal{R}$ be a unitary implementation of the geometric reflection and let τ implement particle-hole conjugation:

$$\mathcal{R}^* a_x^\pm \mathcal{R} = a_{\vartheta(x)}^\pm, \quad \tau^* a_x^\pm \tau = a_x^\mp.$$

Lieb's reflection is the antilinear involution defined on generators by:

$$\begin{aligned} \Theta : \mathcal{A}_\pm &\rightarrow \mathcal{A}_\mp \\ a_x^\pm &\mapsto \mathcal{R}^* \tau^* a_x^\pm \tau \mathcal{R} = a_{\vartheta(x)}^\mp, \end{aligned} \tag{29}$$

and then extended as an antilinear morphism on the whole algebra:

$$\Theta(AB) = \Theta(A)\Theta(B). \tag{30}$$

Remark 31.

1. By construction Θ preserves each parity sector, i.e. $\Theta(\mathcal{A}_\pm^{\text{even/odd}}) = \mathcal{A}_\mp^{\text{even/odd}}$.
2. The operator order is preserved by the action of Θ . This is different from the Euclidean Grassmann convention used in [Subsection 3.3.1](#), where the reflected conjugation reverses the order of products. Notice that because of this convention the local number operator $n_x = a_x^+ a_x^-$ transforms as

$$\Theta(n_x) = \Theta(a_x^+) \Theta(a_x^-) = a_{\vartheta(x)}^- a_{\vartheta(x)}^+ = 1 - n_{\vartheta(x)}.$$

3. The action on an intra-half hopping term contains an additional sign. For distinct $x, y \in \Lambda_L^+$,

$$\Theta(a_x^+ a_y^-) = a_{\vartheta(x)}^- a_{\vartheta(y)}^+ = -a_{\vartheta(y)}^+ a_{\vartheta(x)}^-. \tag{31}$$

This minus sign is the mechanism that produces the canonical flux under successive spatial reflections.

Consider the free Hamiltonian at half filling at $u(x) = 0^6$ introduced in [Example 16-Item 1](#) for $d \geq 2$. Then it can be written in the form:

$$H = H_+ + H_- + H_{\text{int}}, \quad (32)$$

where $H_- \in \mathcal{A}_-$ and $H_+ \in \mathcal{A}_+$ are even and self-adjoint. After a Gauge transformation, the hopping amplitudes on the bonds crossing P can be set to 1

$$H_{\text{int}} = - \sum_{\substack{\langle x\vartheta(x) \rangle: \\ x \in \Lambda_L^+ \\ \vartheta(x) \in \Lambda_L^-}} \left(a_x^+ a_{\vartheta(x)}^- + a_{\vartheta(x)}^+ a_x^- \right) = - \sum_{\substack{\langle x\vartheta(x) \rangle: \\ x \in \Lambda_L^+ \\ \vartheta(x) \in \Lambda_L^-}} \left(a_x^+ a_{\vartheta(x)}^- - a_x^- a_{\vartheta(x)}^+ \right), \quad (33)$$

The two summands transfer one particle across the reflection plane in opposite directions, and both occur with a non-negative coefficient once the explicit minus sign in the second monomial is retained. Certain classes of interaction can be taken into account as long as they respect reflection positivity requirements. The simplest case is the on-site Hubbard-Interaction introduced in [Equation \(18\)](#) and the staggered mass term studied in [Chapter 7](#).

The notation

$$Z_{\beta,L}^\pm(H_-, H_+) \doteq \text{Tr}_{\mathcal{F}_L} \left((\pm 1)^N e^{-\beta(H_- + H_+ + H_{\text{int}})} \right),$$

will be used for the ordinary and parity-twisted traces, respectively. They should not be confused with the partition functions projected onto the even and odd parity sectors, which are instead

$$Z_{\text{even}} = \frac{1}{2} \left(Z_{\beta,L}^+ + Z_{\beta,L}^- \right), \quad Z_{\text{odd}} = \frac{1}{2} \left(Z_{\beta,L}^+ - Z_{\beta,L}^- \right).$$

Lemma 36 (Lieb's Reflection Inequality). *Assume that H_- and H_+ are even, self-adjoint, and number-conserving, and that H_{int} has the form in [Equation \(33\)](#). Then, for every $\beta \geq 0$,*

$$\left[Z_{\beta,L}^\pm(H_-, H_+) \right]^2 \leq Z_{\beta,L}^\pm(H_-, \Theta(H_-)) Z_{\beta,L}^\pm(\Theta(H_+), H_+). \quad (34)$$

Proof. We recall the argument because it shows precisely how the signs of the charged fermions are handled. Introduce the Trotter approximation

$$\mathcal{T}_M \doteq \left[\left(1 - \frac{\beta}{M} H_{\text{int}} \right) e^{-\frac{\beta}{M} H_-} e^{-\frac{\beta}{M} H_+} \right]^M. \quad (35)$$

Since the system is finite dimensional, $\mathcal{T}_M \rightarrow e^{-\beta H}$ in norm as $M \rightarrow \infty$. At fixed M , expansion of the crossing factors gives a finite sum of operator monomials.

The factors $e^{-\beta H_-/M}$ and $e^{-\beta H_+/M}$ are even. All left-half operators can therefore be moved to the left and all right-half operators to the right without signs coming from the half-Hamiltonians. A term has a non-zero trace only if the number of particle transfers from left to right equals the number of transfers from right to left. Denote this common number by m . The term then contains $2m$ crossing monomials. Separating their left and right odd factors requires

$$0 + 1 + \cdots + (2m - 1) = m(2m - 1)$$

⁶One can easily prove that such class of hamiltonians are unitarily conjugated to minus themselves, therefore they have a spectrum of the form $(-\lambda, \lambda)$. Setting the chemical potential to zero, enforces the half-filling condition at zero temperature.

fermionic transpositions and hence produces $(-1)^{m(2m-1)} = (-1)^m$. The m monomials describing transfer in the opposite direction carry the explicit minus sign displayed in Equation (33). Their product contributes another factor $(-1)^m$. The two signs cancel:

$$(-1)^{m(2m-1)}(-1)^m = (-1)^{2m^2} = 1.$$

Every non-vanishing term can therefore be written with a non-negative coefficient as $X_\alpha^- X_\alpha^+$, where both half-space factors are even.

The trace consequently factorizes over the two half Fock spaces, and the Trotterized partition function takes the form

$$Z_{\beta, L; M}^\pm(H_-, H_+) = \sum_\alpha c_\alpha A_\alpha^\pm(H_-) B_\alpha^\pm(H_+), \quad c_\alpha \geq 0.$$

The antilinearity of Θ identifies the complex conjugate of a left-half amplitude with the corresponding amplitude of the reflected right half. For the parity-twisted trace, the same statement follows from:

$$\Theta((-1)^{N_-}) = (-1)^{|\Lambda_{L,+}| - N_+} = (-1)^{|\Lambda_{L,+}|} (-1)^{N_+} = (-1)^{N_+},$$

where the last equality holds since L is even. Applying the Cauchy–Schwarz inequality to the sum over α gives

$$\left| Z_{\beta, L; M}^\pm(H_-, H_+) \right|^2 \leq \left(\sum_\alpha c_\alpha |A_\alpha^\pm(H_-)|^2 \right) \left(\sum_\alpha c_\alpha |B_\alpha^\pm(H_+)|^2 \right).$$

Reversing the preceding factorization identifies the two factors on the right with the Trotter approximations of $Z_{\beta, L}^\pm(H_-, \Theta(H_-))$ and $Z_{\beta, L}^\pm(\Theta(H_+), H_+)$. Taking $M \rightarrow \infty$ proves Equation (34). ■

Remark 32.

1. The centered density $q_x = n_x - \frac{1}{2}$ is odd under the reflection $\Theta(q_x) = -q_{\vartheta(x)}$, therefore the partition function of any particle-hole symmetric Hamiltonian with non-vanishing chemical potential μ is not Reflection Positive. In particular there is an open conjecture which states that in a $1/p$ -filled electron band, the optimal flux is $2\pi/p$, which cannot be tackled for example with reflection positivity methods.
2. The difference between the Euclidean and Hamiltonian fermionic arguments is not exhausted by the anti-multiplicativity or multiplicativity of the reflection. In the Euclidean models, the two oriented terms of the anti-self-adjoint Dirac operator acquire the same sign after the Grassmann variables are reordered and can therefore be written as reflected squares. On the torus, however, the second interface carries the opposite sign: the antiperiodic $\mathbb{Z}_2$ -holonomy compensates this mismatch (see Section 4.2). The anti-multiplicativity of the Euclidean conjugation then guarantees that arbitrary products of crossing terms factorize without residual Koszul signs.

In Lieb’s Hamiltonian argument, the two oppositely oriented hopping operators arise instead as the two adjoint components of a self-adjoint Hamiltonian. Once both terms are written with the left-half operator first, they carry opposite signs. Particle-number conservation implies that every non-vanishing contribution to the trace contains the two hopping directions with equal multiplicity. The resulting explicit sign cancels exactly the CAR sign produced by separating the left and right factors. Thus Lieb’s multiplicative reflection relies on self-adjointness, charge conservation, and trace factorization; it is not a general untwisted Reflection-Positivity criterion for CAR Hamiltonians.

3.4.2 The Abstract Chessboard Argument

The Reflection Inequality compares an arbitrary configuration with the two configurations obtained by replacing one half by the reflected copy of the other. Iterating this operation disseminates a local pattern throughout the lattice. The combinatorial content of the iteration is isolated by the following lemma, adapted from [Tas20, Lemma 4.5].

Lemma 37 (Abstract chessboard symmetrization). *Let $\mathcal{X}$ be a set and let $F : \mathcal{X}^{2n} \rightarrow \mathbb{R}$ satisfy cyclicity,*

$$F(f_1, \dots, f_{2n}) = F(f_{2n}, f_1, \dots, f_{2n-1}), \quad (36)$$

and the reflection bound

$$F(f_1, \dots, f_n, f_{n+1}, \dots, f_{2n}) \geq \frac{1}{2}F(f_1, \dots, f_n, f_n, \dots, f_1) + \frac{1}{2}F(f_{2n}, \dots, f_{n+1}, f_{n+1}, \dots, f_{2n}).$$

Then

$$F(f_1, \dots, f_{2n}) \geq \frac{1}{2n} \sum_{j=1}^{2n} F(f_j, \dots, f_j). \quad (37)$$

Proof. Set

$$G(f_1, \dots, f_{2n}) \doteq F(f_1, \dots, f_{2n}) - \frac{1}{2n} \sum_{j=1}^{2n} F(f_j, \dots, f_j). \quad (38)$$

The function G satisfies the same cyclicity and reflection bound. Fix $f_1, \dots, f_{2n}$ and minimize G over the finite set of all strings whose entries are chosen from these $2n$ elements. If $(g_1, \dots, g_{2n})$ is a minimizing string, both reflected strings on the right-hand side of Equation (36) must also minimize G . By cyclicity, one of them can be chosen with its first two entries equal to g_1 . Repeating the argument makes successively four, eight, and eventually all entries equal to g_1 . Since $G(g_1, \dots, g_1) = 0$, the minimum is zero. Hence $G \geq 0$, which is Equation (37). ■

To apply the lemma to partition functions, incorporate the action of Θ into the definition of an oriented layer datum and set

$$F(f_1, \dots, f_{2n}) \doteq -\log Z(f_1, \dots, f_{2n}), \quad (39)$$

whenever $Z > 0$. Lieb's inequality gives Equation (36), while translation invariance gives cyclicity. Exponentiating Equation (37) yields

$$Z(f_1, \dots, f_{2n}) \leq \prod_{j=1}^{2n} Z(f_j^\#)^{1/(2n)}, \quad (40)$$

where $f_j^\#$ denotes the pattern obtained by disseminating f_j by successive reflections. Applying Equation (40) successively in every coordinate direction gives the usual multidimensional chessboard estimate. In a fermionic Gauge problem the copies appearing in $f_j^\#$ are reflected copies as it will be discussed in Lemma 70.

3.4.3 The π -Flux Theorem

Let us now discuss the key consequence of the bound in Equation (34), discovered in [Lie94]. Let:

$$Z^\pm(\sigma) \doteq Z_{\beta,L}^\pm(H(\sigma)) . \quad (41)$$

As discussed in Example 12-Item 2, the partition function only depends on the fluxes enclosed by the plaquettes, and by the non-contractible loops. In particular, without loss of generality we can assume that Equation (33) holds. Let us denote by:

$$Z_*^\pm \doteq \max_{\text{fluxes}} Z^\pm(\sigma) , \quad (42)$$

where the maximization is over the fluxes over all plaquettes and over the non-contractible cycles. The next result proves the optimality of the π -flux phase.

Theorem 38 (Optimality of the π -flux phase). *Under the same assumptions of Lemma 36, the following is true:*

$$Z_*^\pm = Z^\pm(\sigma_\pi) , \quad (43)$$

where σ_π is a gauge field configuration realizing the π -flux phase.

Proof. It easily follows from Lemma 36. Let us denote by $\sigma_*^\pm$ a spin configuration that attains the maximum Equation (42), and let $H_*^\pm \doteq H(\sigma_*^\pm)$. Using the bound Equation (34) and the maximality of $Z_*^\pm$, we have that:

$$Z_*^\pm = Z_{\beta,L}^\pm((H_*^\pm)_-, \Theta((H_*^\pm)_-)) , \quad Z_*^\pm = Z_{\beta,L}^\pm(\Theta((H_*^\pm)_+), (H_*^\pm)_+) . \quad (44)$$

The resulting spin configuration $\tilde{\sigma}_*^\pm$ has fluxes π in every square cut by the plane used for the reflection due to Equation (31). Iterating this procedure one finds a minimizing pattern of spins with fluxes equal to -1 in all plaquettes. ■

Remark 33.

1. If $L = 2n$, the minimizing π -flux phase has flux $(-1)^{n-1}$ along non-contractible cycles. To see this, observe that, after a reflection, the flux along non-contractible cycles orthogonal to the cut is $\prod_{i=1}^{n-1} (\sigma_i \cdot -\sigma_i) = (-1)^{n-1}$.
2. In the limit $\beta \rightarrow 0$, the optimality of the π -flux phase for the odd partition function $Z_{\beta,L}^-(\sigma)$ proved in Theorem 38 implies that $|\det H(\sigma)| \leq |\det H(\sigma_\pi)|$, with σ_π the π -flux gauge field configuration produced in the proof of the theorem. This recovers a result of Lieb and Loss [LL93], which was obtained without using reflection positivity.
3. One has to take a little bit of care for the odd partition function Z^- has in principle it does not need to be positive. However, for particle-hole symmetric hamiltonians on Λ_L , one can explicitly check it always holds that $Z_{\beta,L}^- \geq 0$. Therefore the same procedure can be applied also whenever $Z^- > 0$ and holds trivially for $Z^- = 0$.

3.4.4 Relation with the Majorana Formalism

Lieb's argument is especially efficient for the charged, number-conserving systems considered in this thesis. Charge conservation is used in the proof of Lemma 36 to match the two directions of particle transfer and cancel the fermionic signs. It is therefore not, by itself, a model-independent criterion for arbitrary CAR Hamiltonians.

A systematic treatment of reflection positivity for general CAR systems is most naturally obtained by passing to Majorana generators. For each complex fermionic mode, define two self-adjoint operators

$$\eta_i = a_i^+ + a_i^-, \quad \xi_i = i(a_i^+ - a_i^-).$$

They satisfy

$$\{\eta_i, \eta_j\} = \{\xi_i, \xi_j\} = 2\delta_{ij} \quad \{\eta_i, \xi_j\} = 0$$

Jaffe and Pedrocchi developed a general Reflection-Positivity criterion for Hamiltonians written in these variables [JP15]. In that formulation the reflection is multiplicative on Majorana monomials, while the signs associated with odd monomials are encoded by a fixed parity-dependent phase, or equivalently by a twisted reflected product. A positive-semidefinite condition on the matrix of couplings across the reflection plane then provides a general sufficient criterion, without assuming $U(1)$ charge conservation. The corresponding abstract characterization was further developed in [JJ16].

The Majorana and Lieb formulations are compatible, but they do not use the same reflection map on the charged generators and they do not place the fermionic phases in the same part of the construction. In the Majorana formalism the phase is included locally in the reflected product; in Lieb's proof no factor i appears, because the relevant signs cancel globally in the number-conserving trace expansion. Since the flux-phase results proved later in this thesis use the transformations in Equation (29), we retain Lieb's formulation in those applications and use the Jaffe–Pedrocchi construction only to indicate how the theory extends beyond the number-conserving setting.

Part II

Chiral Symmetry Breaking in Euclidean Field Theory

Chiral LRO in Euclidean Lattice Gross–Neveu Models

The three Euclidean Gross–Neveu models considered in this chapter and the main long-range-order result were introduced in [Section 1.1](#). This chapter, based on joint work with Simone Fabbri [FG26], provides their precise lattice formulation and the complete proof of that result. The three models and the theorem are formulated in [Section 4.1](#) in the notation used throughout the proof. We adopt the Euclidean Grassmann and Gaussian conventions introduced in [Subsection 2.1.1](#) and [Proposition 11](#), while the model-specific reflection-positive decompositions apply the general criterion of [Subsection 3.3.2](#) extended to fermion-boson models.

The original results of the joint work are the finite-volume, pointwise two-point bounds and the resulting chiral long-range order, established uniformly for all three discretizations in the stated regime of weak coupling and a sufficiently large, even number of fermionic flavors. Hubbard–Stratonovich transformations, reflection positivity, chessboard estimates, Peierls arguments, and Bloch–Floquet theory have standard antecedents and their use here is adapted to the particular algebraic and spectral structures of the three Gross–Neveu discretizations. The formal analysis in [Section 4.C](#) identifies the continuum Gross–Neveu theories formally associated with the three lattice discretizations, but does not establish a rigorous continuum limit. The weak-coupling analysis in [Section 4.B](#) and [Subsection 4.B.3](#), instead, determines the structure of the effective potentials used in the finite-lattice proof.

The proof separates into two main parts. First, reflection positivity is verified separately for each lattice Dirac operator. This step is model-dependent and requires careful control of the Grassmann signs and of the antiperiodic boundary conditions. Once reflection positivity has been established, the three models are treated by a common chessboard-estimate and Peierls strategy. Dissemination replaces local field constraints by periodic configurations, which can be analyzed through Bloch–Floquet theory. The resulting spectral estimates provide the energetic suppression of large fields, fields close to the local maximum of the effective potential, and configurations interpolating between its two minima.

More precisely, the model-dependent reflection-positivity arguments are given in [Section 4.2](#), while the common two-point decomposition and the classification according to field size are developed in [Section 4.3](#) and [Subsection 4.3.1](#). The quantitative suppression bounds are then proved in [Section 4.4](#). The main technical ingredients specific to the joint work are the proof of reflection positivity in the fermion-boson model, the control of large fields and fields near the local maximum

in Subsections 4.4.1 and 4.4.2, the Peierls estimate and the spectral cost of alternating field configurations in Subsection 4.4.3 and Subsection 4.4.4 and the final quantitative estimates assembled in Subsection 4.3.2 and Section 4.D.

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4.1 The Euclidean Gross–Neveu Lattice Models

4.1.1 Naive Fermions with Naive Interaction (NN)

The first model we consider is the lattice version of the Gross–Neveu model originally introduced in [CER81], formulated in terms of naive lattice fermions (see Example 10 - Item 1) and therefore subject to the usual fermion–doubling phenomenon. This formulation provides the simplest discretization of the fermionic action and serves as a natural starting point for our analysis.

Spinors. At each lattice site $x \in \Gamma_L$, we introduce $2N$ independent Grassmann-valued spinor fields $\psi_{x,j}$ and $\bar{\psi}_{x,j}$ ($j = 1, \dots, N$) with 2 entries, where $\psi_{x,j}$ is regarded as a column vector and $\bar{\psi}_{x,j}$ as a row vector:

$$\psi_{x,j} = \begin{pmatrix} \psi_{x,1,j} \\ \psi_{x,2,j} \end{pmatrix}, \quad \bar{\psi}_{x,j} = \begin{pmatrix} \bar{\psi}_{x,1,j} & \bar{\psi}_{x,2,j} \end{pmatrix}.$$

Gamma matrices. We choose two self-adjoint matrices $\{\gamma_\mu\}_{\mu=0,1} \subset M_2(\mathbb{C})$ as a possible irreducible representation of the 2-dimensional Euclidean Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbb{1}_2$. The Axial matrix γ_A is defined as $\gamma_A \doteq i\gamma_0\gamma_1$ and satisfies:

$$\gamma_A^2 = \mathbb{1}_2, \quad \{\gamma_A, \gamma_\mu\} = 0, \quad \gamma_A^\dagger = \gamma_A. \quad (\text{GA})$$

We adopt the following choice for the γ -matrices:

$$\gamma_0 = \sigma^y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = \sigma^x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_A = \sigma^z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac operator. The Lattice Dirac operator is chosen to correspond with its naive discretization, acting on $\ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ according to:

$$\not{D}_{\text{NN}} = \frac{1}{2} \sum_{\mu=0}^1 \gamma_\mu \otimes (T_\mu - T_\mu^*),$$

where T_μ is the right translation operator in the μ -th direction, namely:

$$(T_\mu \psi)_x = \psi_{x+\hat{e}_\mu} \quad \forall \psi \in \ell^2(\mathbb{Z}^2).$$

The Fourier representation for the Dirac operator is

$$(\not{D}_{\text{NN}})_{x,y} = -i \sum_{\mu=0,1} \iint_{(-\pi,\pi]^2} \frac{d^2 p}{(2\pi)^2} e^{-ip \cdot (x-y)} \gamma_\mu \sin(p_\mu). \quad (1)$$

The fermionic fields are taken to satisfy anti-periodic boundary conditions in both directions. This is implemented by using the anti-periodic extension of the lattice Dirac operator¹, denoted by $(\not{D}_{\text{NN}})_{\Gamma_L}^-$, as the kernel of the kinetic term:

$$((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{xy} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} (\not{D}_{\text{NN}})_{x+Ln,y}.$$

Lemma 39 (Properties of the NN Dirac operator). *The operator $(\not{D}_{\text{NN}})_{\Gamma_L}^-$ has the following properties, for every $\phi \in \mathbb{R}^{\Gamma_L}$.*

1. *Anti-self-adjointness:*

$$((\not{D}_{\text{NN}})_{\Gamma_L}^-)^* = -(\not{D}_{\text{NN}})_{\Gamma_L}^-. \quad (\text{PND1})$$

¹There are in general two equivalent ways to implement anti-periodic boundary conditions: either one considers anti-periodic Grassmann variables and a periodic kernel (both restricted to the finite lattice Γ_L) or periodic Grassmann variables and an anti-periodic kernel. In this thesis we adopt the latter formulation.

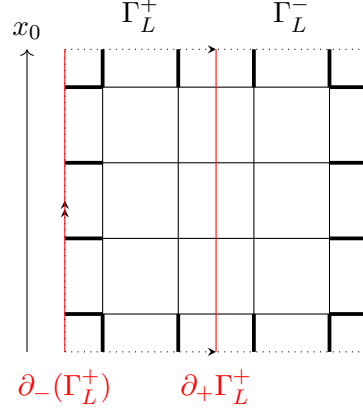


Figure 4.1: Distribution of Phases of the Anti-symmetrized Forward Dirac Operator $\gamma_\mu \otimes T_\mu$ in $L = 2$ Lattice. Solid Links represent a -1 sign. Notice that the set of edges intersecting $\partial_-(\Gamma_L^+)$ are opposite of those intersecting $\partial_+(\Gamma_L^+)$ because we have chosen an anti-symmetrization convention where the $\mathbb{Z}_2$ -twisting is localized on $\partial_-(\Gamma_L^+)$. By means of any $\mathbb{Z}_2$ -Gauge Transformation, we can move such line wherever we desire, without changing the physical properties of the system.

2. Chirality under adjoint:

$$((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi)^* = (\gamma_A \otimes 1)((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi)(\gamma_A \otimes 1), \quad (\text{PND2})$$

where $\gamma_A \otimes 1 : \mathbb{C}^2 \otimes \ell^2(\Gamma_L) \rightarrow \mathbb{C}^2 \otimes \ell^2(\Gamma_L)$ acts as multiplication by γ_A on the spinor indices.

3. Spectrum of the adjoint:

$$\text{spec} \left[((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi)^* \right] = \text{spec} \left[(\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi \right]. \quad (\text{PND3})$$

4. Reality of the determinant:

$$\det ((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi) \in \mathbb{R}. \quad (\text{PND4})$$

The proof of [Lemma 39](#) is discussed in [Subsection 4.A.1](#).

Lattice action. The Dirac fermions are coupled through a local four-fermion interaction of Gross–Neveu type:

$$S_{\text{NN}}(\bar{\psi}, \psi) = \sum_{x,y \in \Gamma_L} \sum_{j=1}^N \bar{\psi}_{x,j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j} - \frac{\lambda}{2N} \sum_{x \in \Gamma_L} \left(\sum_{j=1}^N \bar{\psi}_{x,j} \psi_{x,j} \right)^2, \quad \lambda > 0, \quad (2)$$

where all spinor products have to be understood as row times column products, such as:

$$\bar{\psi}_{x,j} \psi_{x,j} = \sum_{s=1}^2 \bar{\psi}_{x,s,j} \psi_{x,s,j}.$$

Chiral symmetry. The action is invariant under the discrete $\mathbb{Z}_2$ -Chiral symmetry defined by:

$$\begin{cases} \psi_{x,j} \rightarrow \gamma_A \psi_{x,j}, \\ \bar{\psi}_{x,j} \rightarrow -\bar{\psi}_{x,j} \gamma_A, \end{cases} \quad (3)$$

while the fermion bilinear $\bar{\psi}_{x,j} \psi_{x,j}$ obviously changes sign as one can easily infer from the first two of (GA).

Remark 34. Alternatively to the transformation shown in Equation (3), one can consider the following transformation:

$$\begin{cases} \psi_{x,j} \rightarrow e^{i\frac{\pi}{2}\gamma_A} \psi_{x,j} = i\gamma_A \psi_{x,j} \\ \bar{\psi}_{x,j} \rightarrow \bar{\psi}_{x,j} e^{i\frac{\pi}{2}\gamma_A} = i\bar{\psi}_{x,j} \gamma_A. \end{cases}$$

Such transformation seems apparently a $\mathbb{Z}_4$ -symmetry. However, when restricted to even polynomials (which are the actual observables) it reduces to a $\mathbb{Z}_2$ -symmetry.

Effective bosonic measure. By applying the general Hubbard–Stratonovich construction (see Equation (3)) to the action in Equation (2), and integrating out the fermionic fields (recall Equation (6)), we obtain the effective bosonic measure on $\mathbb{R}^{\Gamma_L}$:

$$d\mu_{\text{NN}}(\phi) = \frac{1}{Z_{\Gamma_L}^{\text{NN}}} \prod_{x \in \Gamma_L} d\phi_x e^{-\frac{N}{2\lambda} \phi_x^2} \det \left((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi \right)^N.$$

Issues, strong coupling and scaling limit. As anticipated above, the naive lattice formulation of the Gross–Neveu model suffers from the well-known fermion–doubling problem [KS75; NN81], implying that the number of low-energy (long-wavelength) Dirac fermions described by the free action is four, rather than one as suggested by the continuum theory. It has been pointed out that such a proliferation of fermionic degrees of freedom in free lattice theories may lead to significant difficulties once interactions among the long-wavelength modes are introduced [Sus77]. These issues were analyzed in detail in [CER83, Sect. 3.1], where it was argued that the interaction spoils Lorentz invariance at the quantum level. Nevertheless, at least heuristically, this model is still expected to exhibit asymptotic freedom and infrared properties analogous to those of the continuum Gross–Neveu model, up to an overall factor of four arising from the additional poles in the free propagator [GK80; CER83].

Motivated by these difficulties, alternative lattice formulations based on staggered fermions [KS75] were introduced in [CER83]. These models exhibit different behaviour in the strong-coupling regime. The SP model, introduced below as the third lattice realization studied in this chapter, has the appropriate formal continuum limit. Before turning to it, we introduce the simpler SN model, which does not reproduce the standard Gross–Neveu continuum action but retains the infrared ordering mechanism (see [HKK93, Sect. 3] and [Jol86]) relevant to the present analysis.

4.1.2 Staggered fermions with Naive Interaction (SN)

The second model we consider is formulated in terms of staggered fermions (See [KS75] and Example 10-Item 3). Compared with the naive discretization, spin degrees of freedom are encoded in site-dependent phases, while the interaction remains on-site and of Gross–Neveu type.

Staggered Fermions. At each lattice site $x \in \Gamma_L$, we introduce $2N$ independent Grassmann variables $\psi_{x,j}$ and $\bar{\psi}_{x,j}$, with $j = 1, \dots, N$.

Gamma matrices. Gamma Matrices are represented by phases:

$$\Gamma_\mu(x) = \begin{cases} 1, & \mu = 0 \\ (-1)^{x_0}, & \mu = 1. \end{cases}$$

The role of the Axial matrix γ_A is played by the multiplication operator:

$$\epsilon(x) = (-1)^{x_0+x_1}.$$

Dirac operator. The Lattice Dirac operator in the Staggered Fermions formalism is an operator acting on $\ell^2(\mathbb{Z}^2, \mathbb{C})$ according to:

$$\not{D}_{\text{SN}} = \frac{1}{2} \sum_{\mu=0}^1 \Gamma_{\mu} (T_{\mu} - T_{\mu}^*).$$

By labeling the sites of Γ_L by (x, a) , with $x \in \Gamma_L$ such that x_0 is even and $a \in \{1, 2\}$, so that every $y \in \Gamma_L$ can be written as $(x_0 + \frac{1}{2}(a-1), x_1)$, we recover translation invariance of the Dirac operator, which can be therefore expressed by Fourier transform:

$$(\not{D}_{\text{SN}})_{(x,a),(y,b)} = \iint_{(-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi]} \frac{d^2 p}{(2\pi)^2} e^{-ip(x-y)} \begin{pmatrix} i \sin p_1 & \frac{1}{2}(e^{2ip_0} - 1) \\ -\frac{1}{2}(e^{-2ip_0} - 1) & -i \sin p_1 \end{pmatrix}_{a,b}. \quad (4)$$

Again, we impose anti-periodic boundary conditions by considering an anti-periodic extension of the above Dirac Operator:

$$((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{xy} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} (\not{D}_{\text{SN}})_{x+Ln, y}.$$

Lemma 40 (Properties of the SN Dirac operator). *The operator $(\not{D}_{\text{SN}})_{\Gamma_L}^-$ has the following properties, for every $\phi \in \mathbb{R}^{\Gamma_L}$.*

1. **Anti-self-adjointness:**

$$((\not{D}_{\text{SN}})_{\Gamma_L}^-)^* = -(\not{D}_{\text{SN}})_{\Gamma_L}^-. \quad (\text{PSD1})$$

2. **Chirality under adjoint:**

$$((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi})^* = \epsilon((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi})\epsilon, \quad (\text{PSD2})$$

where $((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi})_{x,y} = ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} - \phi_x \delta_{x,y}$ and ϵ is the multiplication operator by $\epsilon(x) = (-1)^{x_0+x_1}$.

3. **Spectrum of the adjoint:**

$$\text{spec} \left[((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi})^* \right] = \text{spec} \left[(\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi} \right]. \quad (\text{PSD3})$$

4. **Reality of the determinant:**

$$\det((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_{\phi}) \in \mathbb{R}. \quad (\text{PSD4})$$

The proof of [Lemma 40](#) is discussed in [Appendix 4.A.2](#).

Lattice action. The Dirac fermions are coupled through a local four-fermion interaction of Gross-Neveu type:

$$S_{\text{SN}}(\bar{\psi}, \psi) = \sum_{x,y \in \Gamma_L} \sum_{j=1}^N \bar{\psi}_{x,j} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j} - \frac{\lambda}{2N} \sum_{x \in \Gamma_L} \left(\sum_{j=1}^N \bar{\psi}_{x,j} \psi_{x,j} \right)^2, \quad \lambda > 0. \quad (5)$$

Chiral symmetry. The action is invariant under the discrete $\mathbb{Z}_2$ -Chiral transformation defined by:

$$\begin{cases} \psi_{x,j} \rightarrow e^{i\frac{\pi}{2}\epsilon(x)}\psi_{x,j} \\ \bar{\psi}_{x,j} \rightarrow e^{i\frac{\pi}{2}\epsilon(x)}\bar{\psi}_{x,j}. \end{cases}$$

Indeed, under this transformation, we have that

$$\bar{\psi}_{x,j}((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y}\psi_{y,j} \rightarrow e^{i\frac{\pi}{2}(\epsilon(x)+\epsilon(y))}\bar{\psi}_{x,j}((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y}\psi_{y,j} = \bar{\psi}_{x,j}((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y}\psi_{y,j}$$

where we used the fact that $((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y}$ is non-zero only if $|x - y| = 1$, in which case $\epsilon(x) = -\epsilon(y)$. Conversely, under this transformation the bilinear $(\bar{\psi}\psi)_x$ changes sign:

$$(\bar{\psi}\psi)_x \rightarrow e^{i\pi\epsilon(x)}(\bar{\psi}\psi)_x = -(\bar{\psi}\psi)_x.$$

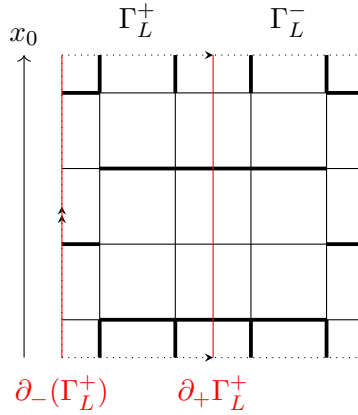


Figure 4.2: Distribution of Phases of the Anti-symmetrized Forward Dirac Operator $\Gamma_\mu(x)T_\mu$ in $L = 2$ Lattice. Solid Links represent a -1 sign. Notice that the set of edges intersecting $\partial_-(\Gamma_L^+)$ are opposite of those intersecting $\partial_+\Gamma_L^+$ because we have chosen an anti-symmetrization convention where the $\mathbb{Z}_2$ -twisting is localized on $\partial_-(\Gamma_L^+)$. By means of any $\mathbb{Z}_2$ Gauge Transformation, we can move such line wherever we desire, without changing the physical properties of the system.

Effective bosonic measure. By applying the general Hubbard–Stratonovich construction to the action in Equation (5), and integrating out the fermionic fields, we obtain the effective bosonic measure on $\mathbb{R}^{\Gamma_L}$:

$$d\mu_{\text{SN}}(\phi) = \frac{1}{Z_{\Gamma_L}^{\text{SN}}} \prod_{x \in \Gamma_L} d\phi_x e^{-\frac{N}{2\lambda}\phi_x^2} \det((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_\phi)^N.$$

4.1.3 Staggered Fermions with Plaquette Interaction (SP)

The third model, originally introduced in [CER83], is widely considered as the proper lattice discretization of the Gross-Neveu model, as after performing the formal continuum limit one gets exactly the usual Gross-Neveu action [CER83, Sect. 3.2], see also Appendix 4.C.

The model is most naturally formulated by partitioning the original lattice Γ_L into four-site unit cells, as illustrated in Figure 4.3. This construction gives rise to the reduced lattice

$$\tilde{\Gamma}_L = 2\mathbb{Z}^2 \cap \left(-\frac{L}{2}, \frac{L}{2}\right]^2,$$

so that $\Gamma_L = \{A, B, C, D\} \times \tilde{\Gamma}_L$. Here, $a \in A, B, C, D \doteq \mathcal{I}$ labels the position of a site within each unit cell. The model is constructed along the same lines as the SN model described above, but with a different interaction structure. We therefore retain the same notation, suitably adapted to the present labeling.

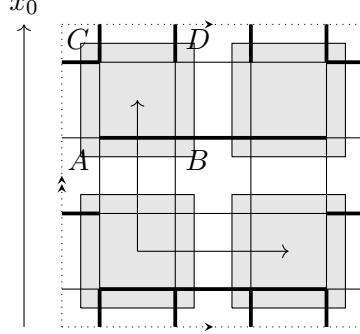


Figure 4.3: Blocking of the lattice Γ_L ($L = 4$ in this example) in terms of the unit cell with points indexed by $\mathcal{I}$. Lighter (resp. heavier) lines are used for denoting hoppings with a phase $+1$ (resp. -1).

Staggered fermions. At each lattice site $(x, a) \in \tilde{\Gamma}_L \times \mathcal{I}$, we introduce $2N$ independent Grassmann variables $\psi_{x,a,j}$ and $\bar{\psi}_{x,a,j}$.

Gamma matrices. The staggered phases are now encoded in

$$\Gamma_\mu(a) \doteq \begin{cases} +1 & \mu = 0, \\ -1 & \mu = 1, a = A, B, \\ +1 & \mu = 1, a = C, D. \end{cases}$$

The analogue of the axial matrix is the multiplication operator

$$\epsilon(a) = \begin{cases} +1 & a = A, D, \\ -1 & a = B, C. \end{cases}$$

Dirac operator. The lattice Dirac operator acts on $\ell^2(\tilde{\Gamma}_L \times \mathcal{I}, \mathbb{C})$ as

$$\not{D}_{\text{SP}} = \frac{1}{2} \sum_{\mu=0}^1 \Gamma_\mu(a) (T_\mu - T_\mu^*).$$

Note that it is translationally invariant on the reduced lattice $\tilde{\Gamma}_L$:

$$(\not{D}_{\text{SP}})_{xy}^{ab} = \iint_{(-\frac{\pi}{2}, \frac{\pi}{2}]^2} \frac{d^2 p}{2(2\pi)^2} e^{-ip \cdot (x-y)} \begin{pmatrix} 0 & -1 + e^{2ip_1} & 1 - e^{2ip_0} & 0 \\ 1 - e^{-2ip_1} & 0 & 0 & 1 - e^{2ip_0} \\ -1 + e^{-2ip_0} & 0 & 0 & 1 - e^{2ip_1} \\ 0 & -1 + e^{-2ip_0} & -1 + e^{-2ip_1} & 0 \end{pmatrix}_{a,b},$$

for every $x, y \in \tilde{\Gamma}_L$ and $a, b \in \mathcal{I}$ (the matrix row and columns are indexed in the order A, B, C, D). Anti-periodic boundary conditions are imposed as in the previous models by defining the anti-periodic extension

$$((\not{D}_{\text{SP}})_{\tilde{\Gamma}_L}^-)_{(x,a),(y,b)} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0 + n_1} (\not{D}_{\text{SP}})_{(x+Ln,a),(y,b)}, \quad x, y \in \tilde{\Gamma}_L, \quad a, b \in \mathcal{I}. \quad (6)$$

Lemma 41 (Properties of the SP Dirac operator). *The operator $(\not{D}_{\text{SP}})_{\Gamma_L}^-$ has the following properties, for every $\phi \in \mathbb{R}^{\tilde{\Gamma}_L}$.*

1. *Anti-self-adjointness:*

$$((\not{D}_{\text{SP}})_{\Gamma_L}^-)^* = -(\not{D}_{\text{SP}})_{\Gamma_L}^-. \quad (\text{PSDP1})$$

2. *Chirality under adjoint:*

$$((\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi)^* = \epsilon((\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi)\epsilon, \quad (\text{PSDP2})$$

where ϵ is the multiplication operator by $\epsilon(a)$.

3. *Spectrum of the Adjoint:*

$$\text{spec}\left[((\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi)^*\right] = \text{spec}\left[(\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi\right]. \quad (\text{PSDP3})$$

4. *Reality of the determinant:*

$$\det((\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi) \in \mathbb{R}. \quad (\text{PSDP4})$$

Remark 35. The properties stated in [Lemma 41](#) follow from their analogue related to the SN Dirac operator, [Lemma 40](#). This is straightforward since every field configuration ϕ associated with the plaquette interaction, can be regarded as a field configuration of the SN model (considered in [Lemma 40](#)) which is constant within each unit cell.

Lattice action. The third model is similar to the previous one but differs for the interaction, which couples degrees of freedom within the same unit cell:

$$S_{\text{SP}}(\bar{\psi}, \psi) = \sum_{x,y \in \tilde{\Gamma}_L} \sum_{a,b \in \mathcal{I}} \sum_{j=1}^N \bar{\psi}_{x,a,j} ((\not{D}_{\text{SP}})_{\Gamma_L}^-)_{(x,a),(y,b)} \psi_{y,b,j} - \frac{\lambda}{2N} \sum_{x \in \tilde{\Gamma}_L} \left(\sum_{j=1}^N \sum_{a \in \mathcal{I}} \bar{\psi}_{x,a,j} \psi_{x,a,j} \right)^2, \quad (7)$$

where again $\lambda > 0$.

Effective bosonic measure. Applying the general Hubbard–Stratonovich construction to the action in [Equation \(7\)](#) and integrating out the fermionic fields, we obtain the effective bosonic measure on $\mathbb{R}^{\tilde{\Gamma}_L}$:

$$d\mu_{\text{SP}}(\phi) = \frac{1}{Z_{\Gamma_L}^{\text{SP}}} \prod_{x \in \tilde{\Gamma}_L} d\phi_x e^{-\frac{N}{2\lambda} \phi_x^2} \det((\not{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi)^N.$$

This formulation makes it natural to view $\not{D}$ as an operator on the reduced lattice $\tilde{\Gamma}_L$, endowed with four internal degrees of freedom per site. Equivalently, the SP model may be regarded as a restriction of the SN model to Hubbard–Stratonovich fields that are constant within each unit cell. This restriction, however, has significant consequences for the (formal) continuum limit (see [\[CER83; HKK93\]](#)) and [Appendix 4.C](#).

4.1.4 The Chiral Long-Range-Order Theorem

Having defined the three models precisely, we now restate the main result, [Theorem 1](#), previously introduced in [Section 1.1](#).

Theorem (Chiral Long-Range Order). *There exists $\lambda_0 > 0$ such that for each $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, the following holds. For every $\lambda \in (0, \lambda_0]$ and $\varepsilon \in (0, 1)$, there exist $N_0(\lambda, \varepsilon) \geq 1$ and $L_0(\lambda, \varepsilon) \geq 1$ with the property that, for all $N \in [N_0(\lambda, \varepsilon), \infty) \cap 2\mathbb{N}$ and $L \in [L_0(\lambda, \varepsilon), \infty) \cap 4(1 + \delta_{\alpha, \text{SP}})\mathbb{N}$,*

$$(1 + \varepsilon) \left(\frac{N}{\lambda}\right)^2 \varphi_\alpha^*(\lambda)^2 \geq \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x, y \in \Gamma_L^\alpha} \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \geq (1 - \varepsilon) \left(\frac{N}{\lambda}\right)^2 \varphi_\alpha^*(\lambda)^2 > 0. \quad (8)$$

Here $\varphi_\alpha^*(\lambda)$ denotes the (strictly) positive minimizer of the effective potential:

$$v_\lambda^\alpha(\varphi) = \frac{\varphi^2}{2\lambda} - C_\alpha \iint_{[-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1)), \quad (9)$$

where $C_\alpha = 1, \frac{1}{2}, 2$ for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively.

Remark 36.

1. **Interpretation of the Result.** [Theorem 1](#) establishes Long-Range Order for the scalar fermionic bilinear $(\bar{\psi}\psi)_x$ in the regime of sufficiently weak coupling and sufficiently large flavor number. Actually, the proof yields a stronger pointwise statement than the spatially averaged formulation of the theorem: for every $\varepsilon \in (0, 1)$ there exists $N_0 = N_0(\lambda, \varepsilon)$ such that, for all $N \geq N_0$,

$$(1 - \varepsilon) \left(\frac{N}{\lambda}\right)^2 \varphi_\alpha^*(\lambda)^2 \leq \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \leq (1 + \varepsilon) \left(\frac{N}{\lambda}\right)^2 \varphi_\alpha^*(\lambda)^2 \quad \forall x, y \in \Gamma_L^\alpha.$$

Thus the mean-field prediction for the fermion condensate is recovered uniformly at the level of individual two-point functions of the fermion bilinear, and not only after spatial averaging. Remarkably, this conclusion is obtained without relying on a large- N expansion or on a saddle-point approximation. Instead, our rigorous upper and lower bounds are formulated directly in terms of the minimizers of the effective potential, namely the same quantities that arise in the heuristic mean-field analysis (similarly to what is done in [\[GN74\]](#)). As a consequence, the fermion bilinear two-point function is uniformly trapped, up to controlled errors, between its corresponding mean-field predictions.

2. **Current Limitations and Future Directions.** A notable feature of our analysis is that it is carried out entirely at finite volume, with estimates that are uniform in the lattice size. Consequently, any thermodynamic-limit point of the finite-volume Gibbs states considered in this work would automatically inherit the Long-Range Order established in [Theorem 1](#). The existence and characterization of such infinite-volume Gibbs states will be investigated elsewhere.

A natural next step is the rigorous construction of extremal infinite-volume Gibbs states $\langle \cdot \rangle_\alpha^{\infty, \pm}$, corresponding to the two symmetry-related minima of the effective potential. One would then like to prove that these states satisfy the clustering property and, more strongly, that the *truncated correlations* decay exponentially, namely:

$$\langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha^{\infty, \pm} - \left(\frac{N}{\lambda} \varphi_\alpha^*(\lambda)\right)^2 = \mathcal{O}(e^{-\mathfrak{k}|x-y|}), \quad \mathfrak{k} > 0,$$

up to the accuracy already controlled by the finite-volume estimates. From this perspective, the main remaining challenge is not the identification of the order parameter itself, but

rather the rigorous construction and characterization of the corresponding pure infinite-volume phases.

3. **Asymptotic Expansion for $\varphi_\alpha^*(\lambda)$.** The positive minimizer of the effective potential $v_\lambda^\alpha(\varphi)$ can be implicitly written [GK80, Sect. 5] as the (non-trivial) solution of the gap equation $\frac{d}{d\varphi}v_\lambda^\alpha(\varphi_\alpha^*) = 0$, which in turn can be rewritten as

$$\frac{\pi}{4\lambda} = \frac{C_\alpha}{1 + (\varphi_\alpha^*)^2} K\left(\sqrt{\frac{1}{1 + (\varphi_\alpha^*)^2}}\right), \quad (10)$$

where K is the Complete Elliptic Integral of the first kind² [AS64, Sect. 17.3]. Using [AS64, Eq. 17.3.26], it is possible to obtain the precise asymptotic of $\varphi_\alpha^*(\lambda)$ as $\lambda \rightarrow 0^+$. In fact:

$$\frac{C_\alpha}{1 + (\varphi_\alpha^*)^2} K\left(\sqrt{\frac{1}{1 + (\varphi_\alpha^*)^2}}\right) = C_\alpha \log\left(\frac{2\sqrt{2}}{\varphi_\alpha^*}\right) (1 + o(1)),$$

(with $o(1)$ denoting quantities which vanish as $\varphi_\alpha^* \rightarrow 0^+$) which leads to

$$\varphi_\alpha^*(\lambda) = 2\sqrt{2}e^{-\frac{\pi}{4C_\alpha\lambda}} (1 + o(1)) \quad (11)$$

(here instead $o(1)$ denotes quantities which vanish as $\lambda \rightarrow 0^+$). It is worth observing that this expression is closely related to the dynamically generated mass predicted by the continuum Gross–Neveu gap equation. The only difference lies in the numerical coefficients appearing in the exponent, which reflect the larger number of effective fermionic degrees of freedom arising from lattice fermion doubling.

4. **Extension to Lattices with Arbitrary Mesh Size.** Theorem 1 extends immediately to lattices $\Gamma_{\ell,L} \doteq \ell\mathbb{Z}^2 \cap \left(-\frac{L}{2}, \frac{L}{2}\right]^2$, with arbitrary mesh size ℓ such that $L/\ell \in 4\mathbb{N}$. Indeed, the dependence on the lattice spacing can be completely absorbed into a rescaling of the fermionic fields. We illustrate this observation for the NN model on $\Gamma_{\ell,L}$ described by the Gibbs state $\langle \cdot \rangle_{\text{NN}}^{\ell,L}$ given by

$$\langle \mathcal{O} \rangle_{\text{NN}}^{\ell,L} \doteq \frac{\int d\bar{\psi} d\psi \mathcal{O}(\bar{\psi}, \psi) e^{-S_{\text{NN}}^{\ell,L}(\bar{\psi}, \psi)}}{\int d\bar{\psi} d\psi e^{-S_{\text{NN}}^{\ell,L}(\bar{\psi}, \psi)}},$$

where³

$$S_{\text{NN}}^{\ell,L}(\bar{\psi}, \psi) = \ell^2 \sum_{x,y \in \Gamma_{\ell,L}} \sum_{j=1}^N \bar{\psi}_{x,j} ((\mathcal{D}_{\text{NN}})_{\Gamma_{\ell,L}}^-)_{x,y} \psi_{y,j} - \frac{\lambda}{2N} \ell^2 \sum_{x \in \Gamma_{\ell,L}} \left(\sum_{j=1}^N \bar{\psi}_{x,j} \psi_{x,j} \right)^2,$$

and

$$((\mathcal{D}_{\text{NN}})_{\Gamma_{\ell,L}}^-)_{x,y} = \frac{1}{2\ell} \sum_{\mu=0}^1 \gamma_\mu (\delta_{x+\ell e_\mu, y} - \delta_{x-\ell e_\mu, y}).$$

²Notice that [AS64] and [GK80] adopt different conventions for the complete elliptic integral of the first kind. Throughout [AS64] the argument is the parameter m , whereas [GK80] uses the modulus k . The two conventions are related by $m = k^2$.

³The extra normalization factor ℓ^2 is chosen in order for the action to scale as the volume of the system L^2 . Alternatively, one can think that such factor allows to reconstruct a Riemann Integral in the limit $\ell \rightarrow 0^+$.

Introducing the rescaled spinors

$$\xi_{x,s,j} \doteq \ell^{1/2} \psi_{\ell x,s,j}, \quad \bar{\xi}_{x,s,j} \doteq \ell^{1/2} \bar{\psi}_{\ell x,s,j}, \quad x \in \Gamma_{1,L/\ell}, \quad s \in \{1, 2\},$$

a direct computation shows that

$$S_{\text{NN}}^{\ell,L}(\bar{\psi}, \psi) = S_{\text{NN}}^{1,L/\ell}(\bar{\xi}, \xi),$$

and thus⁴ we obtain

$$\left\langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \right\rangle_{\text{NN}}^{\ell,L} = \ell^{-2} \left\langle (\bar{\xi}\xi)_{x/\ell} (\bar{\xi}\xi)_{y/\ell} \right\rangle_{\text{NN}}^{1,L/\ell}.$$

The same rescaling argument applies in the same way to the SN and SP models. Since the rescaled theories coincide exactly with the unit-lattice-spacing models studied throughout this chapter, [Theorem 1](#) applies verbatim for all three cases under study. Hence, for $\lambda \leq \lambda_0$, $L/\ell \geq L_0(\lambda, \varepsilon)$ and $N \geq N_0(\lambda, \varepsilon)$,

$$(1 + \varepsilon) \left(\frac{N}{\ell\lambda} \right)^2 \varphi_\alpha^*(\lambda)^2 \geq \frac{1}{|\Gamma_{\ell,L}^\alpha|^2} \sum_{x,y \in \Gamma_{\ell,L}^\alpha} \left\langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \right\rangle_\alpha^{\ell,L} \geq (1 - \varepsilon) \left(\frac{N}{\ell\lambda} \right)^2 \varphi_\alpha^*(\lambda)^2,$$

where recall that $|\Gamma_{\ell,L}^\alpha|$ denotes the cardinality of the lattice. It is worth stressing that the sole effect of the mesh size is the dimensional prefactor ℓ^{-1} in the condensate and ℓ^{-2} in its two-point function. A noteworthy feature is that, for fixed bare coupling λ , the dynamically generated mass scale grows like ℓ^{-1} as the lattice spacing is sent to zero. More precisely, if all bare parameters are kept fixed, the order parameter diverges proportionally to ℓ^{-1} in the limit $\ell \rightarrow 0^+$. This behavior can be compensated by allowing the bare coupling to run with the cutoff according to

$$\lambda(\ell) = \frac{\lambda(\ell_0)}{1 + \frac{4C_\alpha}{\pi} \lambda(\ell_0) \log\left(\frac{\ell_0}{\ell}\right)}, \quad (12)$$

where ℓ_0 is a fixed reference lattice spacing and $\lambda(\ell_0)$ is the corresponding physical coupling. By substituting this expression into [Equation \(11\)](#), one obtains

$$\begin{aligned} \frac{1}{|\Gamma_{\ell,L}^\alpha|^2} \sum_{x,y \in \Gamma_{\ell,L}^\alpha} \left\langle \phi_x \phi_y \right\rangle_\alpha^{\ell,L} &\leq (1 \pm \varepsilon) \left(\frac{1}{\ell} \right)^2 \varphi_\alpha^*(\lambda(\ell))^2 \\ &= (1 \pm \varepsilon) \left(\frac{1}{\ell_0} \right)^2 \varphi_\alpha^*(\lambda(\ell_0))^2 (1 + o(1)), \end{aligned}$$

where $o(1)$ denotes quantities tending to zero as $\lambda(\ell_0) \rightarrow 0^+$, showing that the order parameter expectation value can be kept fixed as the ultraviolet cutoff is removed. In this sense, the scaling of the bare coupling compensates exactly for the divergence induced by the shrinking lattice spacing and yields a finite continuum value for the order parameter.

Nevertheless, our results should not be interpreted as a genuine ultraviolet construction of the model. Indeed, the assumptions of [Theorem 1](#) are imposed directly at the ultraviolet scale and the corresponding estimates are not controlled uniformly in the lattice spacing ℓ , as the choice of bare coupling in [Equation \(12\)](#) forces the number of flavors N to diverge as $\ell \rightarrow 0^+$. This

⁴Since the Jacobian of the change of Grassmann variables cancels between numerator and denominator in the definition of the Gibbs state

limitation is not surprising: our proof establishes Symmetry Breaking directly at the cutoff scale, without performing a Renormalization Group analysis. A true continuum limit would require a multiscale argument capable of tracking the relevant estimates uniformly across scales and proving that the symmetry-breaking mechanism persists along the Renormalization Group flow.

5. Continuum Limit and Relation with the Gross–Neveu beta function.

The scaling relation in Equation (12) also suggests a natural Renormalization Group interpretation. Indeed, after upgrading ℓ to a continuum variable, the unique solution to

$$\ell \frac{d\lambda}{d\ell} = \frac{4C_\alpha}{\pi} \lambda^2. \quad (13)$$

is given by Equation (12). This turns out to be exactly the one-loop beta-function equation for the standard Gross–Neveu model, with a model-dependent coefficient reflecting the number of effective low-energy fermionic species. Let us stress, however, that this interpretation is most transparent for the SP model (see Appendix 4.C). In this case the low-energy degrees of freedom are expected to reproduce the Gross–Neveu interaction without extra spurious quartic terms due to the fermion doubling (similar e.g. to the Umklapp–Scattering effects). Such interactions (which are not of the standard Gross–Neveu form) are instead generated at intermediate scales within the NN and SN models, yielding some extra difficulties, as it is not a priori clear whether they are compatible with Lorentz invariance or whether they can be renormalized without introducing additional counterterms (even though it is claimed to be the case, at least formally: see [CER83, Sect. 3.1] for the NN model and [HKK93, Sect. 3], where the reader is referred to [Jol86], for the SN model).

Thus, while the one-loop running coupling constant, extracted from the scaling of the order parameter, has the expected Gross–Neveu form, a complete continuum Renormalization Group analysis for the NN and SN discretizations would require controlling these additional effective interactions and their Reflection Positivity properties.

Even for the SP model, despite being a natural candidate for reproducing the correct Gross–Neveu model (see Appendix 4.C), a detailed multiscale analysis would be required in order to constructively prove its renormalizability and to establish the rigorous connection with the flow equation (see Equation (13)). All in all, the above comparison with the continuum Gross–Neveu beta function should be, at least for this moment, regarded as a heuristic indication, rather than as a solid mathematical argument.

4.2 Reflection Positivity of the Effective Measures

We aim to show that the three bosonic measures $\mu_\alpha(\phi)$, $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, constructed in the previous section, are reflection positive [Frö+78] with respect to reflections across bond planes of the underlying lattice Γ_L^α , where:

$$\Gamma_L^{\text{NN}} = \Gamma_L^{\text{SN}} = \Gamma_L \quad \Gamma_L^{\text{SP}} = \tilde{\Gamma}_L.$$

As a very first step towards the notion of Reflection Positivity, we need the precise definition of reflections on Γ_L^α , regarded as a torus, with respect to some fixed *cut plane*.

Definition 42 (Cut Plane). For $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, a “cut plane” on Γ_L^α is a couple (ν, r) , with $\nu \in \{0, 1\}$ and

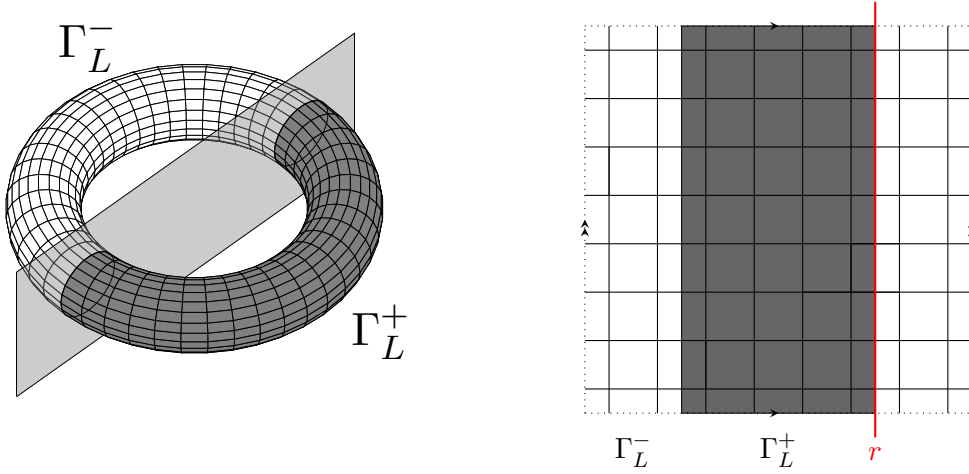


Figure 4.4: Graphical representation of the cut torus, with the cut plane highlighted in red (Credits: Davide Morgante)

- $r \in \left(-\frac{L-1}{2}, -\frac{L-3}{2}, \dots, \frac{L+1}{2}\right)$, if $\alpha = \text{NN}, \text{SN}$,
- $r \in \left(-\frac{L}{2} + 1, -\frac{L}{2} + 3, \dots, \frac{L}{2} + 1\right)$, if $\alpha = \text{SP}$,

which, from a geometric viewpoint, identifies the axis $\{x_\nu = r\}$. We call “canonical” the cut plane with $r = \frac{1}{2}$, if $\alpha = \text{NN}, \text{SN}$ or $r = 1$, if $\alpha = \text{SP}$. Notice that the canonical cut plane is the only axis that preserves the fundamental domain presentation of each lattice⁵.

Definition 43 (Reflection on the Discrete Torus). For $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, to any cut plane (ν, r) on Γ_L^α , we associate a “full-plane reflection” $\vartheta_\nu : \Gamma_\infty^\alpha \mapsto \Gamma_\infty^\alpha$, defined as:

$$\vartheta_\nu(x) \doteq \begin{cases} (2r - x_0, x_1), & \nu = 0, \\ (x_0, 2r - x_1), & \nu = 1. \end{cases}$$

Consequently, we define the “torus reflection”, $\vartheta_\nu^L : \Gamma_L^\alpha \mapsto \Gamma_L^\alpha$, as:

$$\vartheta_\nu^L \doteq \Pi_{\Gamma_L^\alpha} \circ \vartheta_\nu,$$

with $\Pi_{\Gamma_L^\alpha} : \Gamma_\infty^\alpha \mapsto \Gamma_L^\alpha$ the unique map such that $\Pi_{\Gamma_L^\alpha}(x) - x \in L\mathbb{Z}^2$ for every $x \in \Gamma_\infty^\alpha$.

Any cut plane (r, ν) induces the splitting $\Gamma_L^\alpha = (\Gamma_{L,r}^\alpha) \sqcup (\Gamma_L^{\alpha-})$, where (see Figure 4.4):

$$(\Gamma_L^\alpha)^+ \doteq \left\{ x \in \Gamma_L^\alpha : (x_\nu - r) \bmod L \in \left(-\frac{L}{2}, 0\right) \right\}$$

and $\Gamma_L^{\alpha-} \doteq \Gamma_L^\alpha \setminus (\Gamma_L^\alpha)^+$. It is straightforward to check that $\vartheta_\nu^L((\Gamma_L^\alpha)_\pm) = (\Gamma_L^\alpha)_\mp$.

The reflection ϑ_ν^L induces an action on field configurations $\phi \in \mathbb{R}^{\Gamma_L^\alpha}$ given by:

$$(\Theta\phi)_x \doteq \phi_{\vartheta_\nu^L(x)}.$$

In the following, in order to avoid cluttering of the notation, we will denote ϑ_ν^L simply by ϑ_ν as reflections on the plane will not play any role in the main text. In Appendix 4.A we will restore the

⁵In other words it preserves the open square lattice

double notation as we will need both reflections. Besides, we will possibly drop the index $\nu \in \{0, 1\}$ for the reflection, whenever its role will not be essential.

Let $\mathcal{A}_L^\alpha$ be the algebra of complex polynomials on $\mathbb{R}^{\Gamma_L^\alpha}$. We define the Osterwalder–Schrader reflection $\Theta : \mathcal{A}_L^\alpha \rightarrow \mathcal{A}_L^\alpha$ by

$$(\Theta F)(\phi) \doteq \overline{F(\Theta \phi)}.$$

With this definition, Θ satisfies the following properties discussed in the following proposition.

Proposition 44 (Properties of Θ). *Θ is an anti-linear algebra homomorphism of $\mathcal{A}_L^\alpha$:*

1. **Additivity:** $\Theta(F + G) = \Theta(F) + \Theta(G)$;
2. **Anti-homogeneity:** $\Theta(\lambda F) = \bar{\lambda}\Theta(F)$;
3. **Involutivity:** $\Theta(\Theta(F)) = F$;
4. **Multiplicativity:** $\Theta(FG) = \Theta(F)\Theta(G)$.

Let $(\mathcal{A}_L^\alpha)_\pm \subset \mathcal{A}_L^\alpha$ denote the subalgebra of observables depending only on $\{\phi_x\}_{x \in (\Gamma_L^\alpha)_\pm}$. We define the pairing:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mu_\alpha} : (\mathcal{A}_L^\alpha)_+ \times (\mathcal{A}_L^\alpha)_+ &\rightarrow \mathbb{C}, \\ (A, B) &\mapsto \int d\mu_\alpha(\phi) (\Theta A)(\phi) B(\phi). \end{aligned} \tag{14}$$

Theorem 45 (Reflection Positivity of the bosonic measures). *For each $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, the pairing in Equation (14) defines a positive semidefinite sesquilinear form on $\mathcal{A}_L^\alpha$. In particular, for all $A, B \in (\mathcal{A}_L^\alpha)_+$,*

1. $\langle A, B \rangle_{\mu_\alpha} = \overline{\langle B, A \rangle_{\mu_\alpha}}$,
2. $\langle A, A \rangle_{\mu_\alpha} \geq 0$.

Remark 37. Without loss of generality, we can always assume the cut plane to be the *canonical one*.

The canonical cut plane has two important properties:

- $\vartheta_\nu(\Gamma_L^\alpha) = \Gamma_L^\alpha$;
- the holonomy line, where the $\mathbb{Z}_2$ twisting for anti-periodic boundary conditions is imposed, corresponds to the boundary $\partial_-(\Gamma_L^\alpha)^+$ (see Figure 4.1).

Indeed, if one wishes to implement reflection with respect to an arbitrary reflection plane, one may first perform a $\mathbb{Z}_2$ -gauge transformation⁶ so that the corresponding holonomy defect coincides with the boundary $\partial_-(\Gamma_L^\alpha)^+$. After a suitable relabeling of the lattice coordinates, the problem is reduced to the canonical reflection plane considered above. In other words, the change of the reflection plane can always be complemented with a gauge transformation which moves the holonomy line over the boundary corresponding to $\partial_-(\Gamma_L^\alpha)^+$ (see Figure 4.5). For this reason, the proof of Theorem 45 will be carried out for reflections with respect to the canonical cut planes only.

Proof. The proof follows the same general strategy in all three cases $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$. Therefore, we first outline the common argument in an abstract setting and then proceed to the specialization to each model.

⁶Such a transformation leaves the multiplication operator M_ϕ invariant, since it acts by conjugation with a diagonal multiplication operator which clearly commutes with M_ϕ (see [LL93]).

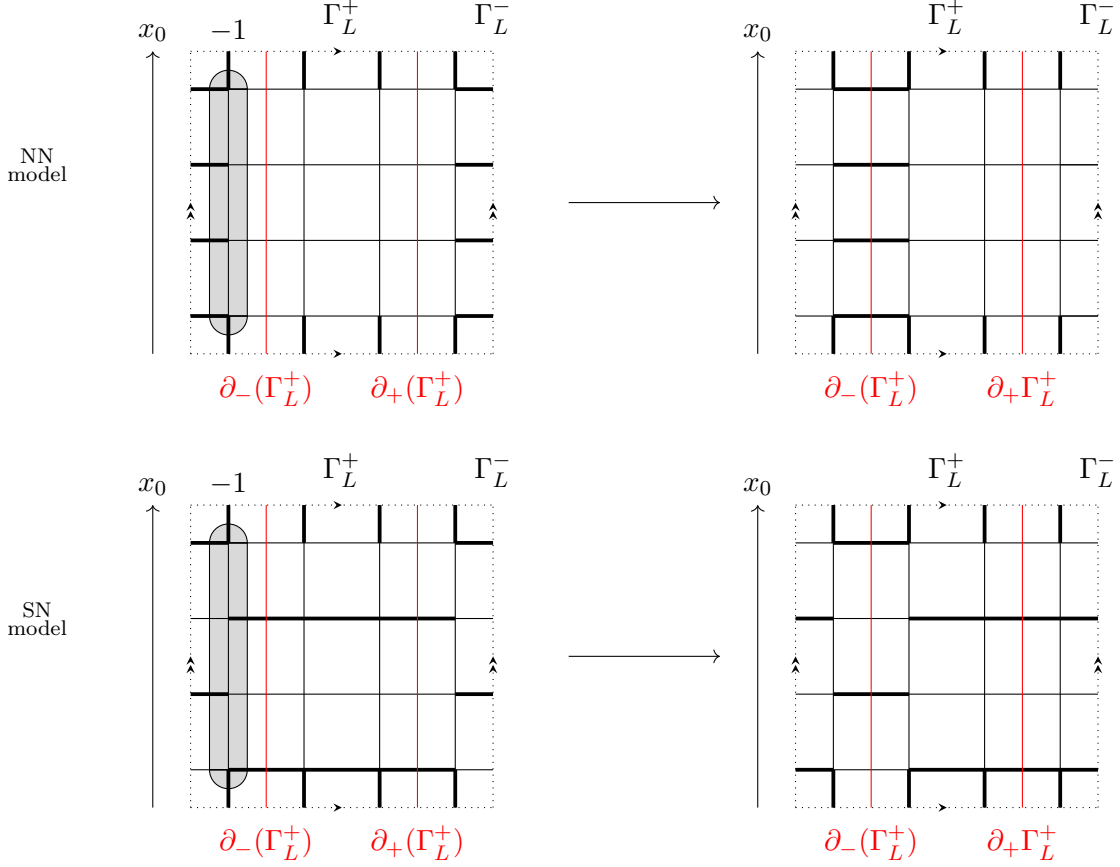


Figure 4.5: By means of a $\mathbb{Z}_2$ -Gauge Transformation supported on one side of the holonomy line, we can translate such line to the left making it overlap with the left boundary of Γ_L^+ . An example of this construction for the NN and SN models is shown in the upper and lower panels, respectively.

1. Let $A, B \in (\mathcal{A}_L^\alpha)_+$. Then:

$$\langle A, B \rangle_{\mu_\alpha} = \int d\mu_\alpha(\phi) (\Theta A)(\phi) B(\phi) = \int d\mu_\alpha(\phi) \overline{A(\Theta\phi)} B(\phi).$$

Performing the change of variables $\phi \mapsto \Theta\phi$ and using the reflection invariance of the measure:

$$\mu_\alpha \circ \Theta^{-1} = \mu_\alpha,$$

we obtain:

$$\langle A, B \rangle_{\mu_\alpha} = \int d\mu_\alpha(\phi) \overline{A(\phi)} B(\Theta\phi) = \overline{\int d\mu_\alpha(\phi) \overline{B(\Theta\phi)} A(\phi)} = \overline{\langle B, A \rangle_{\mu_\alpha}}.$$

The reflection invariance of the measure holds in each model as a consequence of the following identity (see [Lemma 54](#) and [Lemma 55](#)):

$$\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\Theta(\phi)}) = \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi). \quad (\text{H1})$$

2. Let $A \in (\mathcal{A}_L^\alpha)_+$, then:

$$\langle A, A \rangle_{\mu_\alpha} = \int d\mu_\alpha(\phi) (\Theta A)(\phi) A(\phi).$$

The key idea is to represent the determinant appearing in the bosonic measure $\mu_\alpha(\phi)$ by reintroducing auxiliary fermionic fields, thereby recovering a local fermionic action. One then

exploits the reflection covariance properties of the Dirac operator to decompose the action into a sum of terms supported on the positive and negative halves of the lattice, together with mixed boundary contributions. Using the Osterwalder–Schrader reflection and the specific structure of the cross terms, one shows that the pairing $\langle A, A \rangle_{\mu_\alpha}$ can be written as a sum of absolute squares and is therefore non-negative.

Step 1: Reintroduction of the fermionic variables. Using the Grassmann representation of the determinant, we rewrite the bosonic measure as the marginal of a local measure on $(\phi, \bar{\psi}, \psi)$ and thus it factorizes over $(\Gamma_L^\alpha)^\pm$. Writing $\Gamma_L^\alpha = (\Gamma_L^\alpha)^+ \sqcup (\Gamma_L^\alpha)^-$ and splitting the bosonic and on-site fermionic factors accordingly, we obtain:

$$d\mu_\alpha(\phi) = \int d\nu_+^\alpha(\phi, \bar{\psi}, \psi) d\nu_-^\alpha(\phi, \bar{\psi}, \psi) \exp \left(- \sum_{J, J'} \sum_{x, y \in \Gamma_L^\alpha} \bar{\psi}_{x, J} ((\not{D}_\alpha)_L^-)_{xy}^{J, J'} \psi_{y, J'} \right) \quad (\text{H2})$$

where:

$$d\nu_\pm^\alpha(\phi, \bar{\psi}, \psi) = \prod_{x \in \Gamma_L^\alpha} \prod_J d\phi_x d\bar{\psi}_{x, J} d\psi_{x, J} e^{-\frac{N}{2\lambda} \phi_x^2} e^{\phi_x \sum_J \bar{\psi}_{x, J} \psi_{x, J}}$$

and the integral is meant to be taken only on the fermionic part. Notice that the term $e^{\phi_x \sum_J \bar{\psi}_{x, J} \psi_{x, J}}$ is actually a polynomial in the bosonic field due to the nilpotency of the Grassmann Variables.

Step 2: RP Decomposition of the fermion kinetic term. Using the notation introduced in [Section 4.1](#), we write the fermionic kinetic term as

$$K_\alpha \doteq - \sum_{J, J'} \sum_{x, y \in \Gamma_L^\alpha} \bar{\psi}_{x, J} ((\not{D}_\alpha)_L^-)_{(x, J), (y, J')} \psi_{y, J'}.$$

The reflection Θ_ν , across the canonical axis orthogonal to e_ν , acts on fermions depending on the specific formulation⁷:

$$\begin{cases} \Theta_\nu(\bar{\psi}_{x, s, j}) = \sum_{s'} (\gamma_\nu)_{s, s'} \bar{\psi}_{\vartheta_\nu(x), s', j} \\ \Theta_\nu(\psi_{x, s, j}) = \sum_{s'} \bar{\psi}_{\vartheta_\nu(x), s', j} (\gamma_\nu)_{s', s} \end{cases} \quad (\text{RNF})$$

$$\begin{cases} \Theta_\nu(\bar{\psi}_{x, j}) = \Gamma_\nu(x) \bar{\psi}_{\vartheta_\nu(x), j} \\ \Theta_\nu(\psi_{x, j}) = \bar{\psi}_{\vartheta_\nu(x), j} \Gamma_\nu(x) \end{cases} \quad (\text{RSF})$$

⁷Notice that these definitions are the correct ones only for reflections across the *canonical cut plane* (see [Definition 42](#)). For reflections across generic cut planes, [Equation \(RNF\)](#), [Equation \(RSF\)](#) and [Equation \(RSFP\)](#) must be accompanied with the multiplication of the right-hand side by an extra sign:

$$\sigma_\nu^L(x) \doteq (-1)^{\sum_{\mu=0,1} \frac{1}{L} (\vartheta_\nu(x) - \vartheta_\nu^L(x))_\mu}.$$

See also [Appendix 4.A](#) for a discussion of reflections across generic cut planes.

$$\begin{cases} \Theta_\nu(\bar{\psi}_{x,a,j}) = \Gamma_\nu(a) \psi_{\vartheta_\nu(x),\vartheta_\nu(a),j} \\ \Theta_\nu(\psi_{x,a,j}) = \bar{\psi}_{\vartheta_\nu(x),\vartheta_\nu(a),j} \Gamma_\nu(a), \end{cases} \quad (\text{RSFP})$$

for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively. In the third case ϑ_ν also acts on the internal index $J = (j, a)$ because a describes a position within unit cells⁸. We will show that each such transformation is an involution. In the following, in order to lighten the notation, we will omit the index ν wherever is not needed (i.e. whenever no explicit computation is performed).

The Reflection operator is then extended to the full algebra of Polynomials in the Grassmann variables $\mathcal{G}_L^\alpha$ as an anti-linear anti-homomorphism, namely $\Theta(\lambda A) = \bar{\lambda}\Theta(A)$ and $\Theta(AB) = \Theta(B)\Theta(A)$ as discussed in [Definition 32](#).

Let us suppose that each fermionic kinetic term has the form:

$$K_\alpha = B_\alpha + \Theta(B_\alpha) + \sum_{\omega} C_\alpha^\omega \Theta(C_\alpha^\omega), \quad (\text{H3})$$

with ω an index running over a suitable finite set, $B_\alpha \in (\mathcal{G}_L^\alpha)_+^{\text{even}}$ and $C_\alpha^\omega \in (\mathcal{G}_L^\alpha)_+^{\text{odd}}$. Then, by [Proposition 34](#), such condition is sufficient to ensure $\langle A, A \rangle_{\mu_\alpha} \geq 0$.

The remaining part of the proof consists of checking [Equation \(H1\)](#), [Equation \(H2\)](#) and [Equation \(H3\)](#) for each lattice formulation. ■

4.2.1 Naive Fermions with Naive Interaction

Lemma 46. *Properties in [Equation \(H1\)](#), [Equation \(H2\)](#) and [Equation \(H3\)](#) hold for canonical reflections, within the NN model.*

Proof.

(H1) See [Lemma 54](#) in [Appendix 4.A.1](#).

(H2) By a simple computation.

(H3) Let us first show that the reflection in [Equation \(RNF\)](#) is an involution, as claimed right after its definition.

⁸Explicitly:

$$\theta_1(a) = \begin{cases} A & a = B \\ B & a = A \\ D & a = C \\ C & a = D \end{cases}, \quad \theta_0(a) = \begin{cases} A & a = C \\ C & a = A \\ D & a = B \\ B & a = D \end{cases}.$$

$$\begin{aligned}
\Theta_\nu(\Theta_\nu(\bar{\psi}_{x,s,j})) &= \Theta_\nu\left(\sum_{s'=1,2} (\gamma_\nu)_{s,s'} \psi_{\vartheta_\nu(x),s',j}\right) \\
&= \sum_{s'=1,2} \overline{(\gamma_\nu)_{s,s'}} \sum_{s''=1,2} \bar{\psi}_{x,s'',j} (\gamma_\nu)_{s'',s'} \\
&= \sum_{s',s''=1,2} (\gamma_\nu)_{s'',s'} (\gamma_\nu^*)_{s',s} \bar{\psi}_{x,s'',j} \\
&= \sum_{s',s''=1,2} (\gamma_\nu)_{s'',s'} (\gamma_\nu)_{s',s} \bar{\psi}_{x,s'',j} \\
&= \sum_{s''=1,2} (\gamma_\nu^2)_{s'',s} \bar{\psi}_{x,s'',j} \\
&= \bar{\psi}_{x,s,j}.
\end{aligned}$$

We need now to show that the kinetic term:

$$K = - \sum_{\substack{x,y \in \Gamma_L \\ j=1,\dots,N}} \bar{\psi}_{x,j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{xy} \psi_{y,j}$$

(with $K \equiv K_{\text{NN}}$ in this subsection) fits the decomposition in [Equation \(H3\)](#). This is not quite true. However, with a simple staggered phase transformation (which preserves the local bilinears and the fermionic measure):

$$\begin{cases} \bar{\psi}_{x,j} \rightarrow \mathbb{1}_2 \otimes (-1)^{x_0+x_1} \bar{\psi}_{x,j} \\ \psi_{x,j} \rightarrow \mathbb{1}_2 \otimes (-1)^{x_0+x_1} \psi_{x,j}. \end{cases}$$

we get:

$$K \rightarrow K' = \sum_{\substack{x,y \in \Gamma_L \\ j=1,\dots,N}} \bar{\psi}_{x,j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j}.$$

Let us split the sum in the following way:

$$K' = K'_{++} + K'_{--} + K'_{+-} + K'_{-+}, \quad (16)$$

where:

$$K'_{\epsilon\epsilon'} = \sum_{x \in \Gamma_L^\epsilon, y \in \Gamma_L^{\epsilon'}} \bar{\psi}_{x,j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j}, \quad \epsilon, \epsilon' \in \{\pm\}.$$

Let us analyze the four terms in the right-hand side of [Equation \(16\)](#).

- (a) Let us begin by discussing K'_{++} and K'_{--} . We want to show that $\Theta_\nu(K'_{++}) = K'_{--}$. By a direct computation:

$$\begin{aligned}
\Theta_\nu(K'_{++}) &= \sum_{\substack{x,y \in \Gamma_L^+ \\ j=1,\dots,N}} \Theta_\nu(\bar{\psi}_{x,j}((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j}) \\
&= \sum_{\substack{x,y \in \Gamma_L^+ \\ s_1,s_2,s_3,s_4=1,2 \\ j=1,\dots,N}} \bar{\psi}_{\vartheta_\nu(y),s_3,j}(\gamma_\nu)_{s_3,s_2} \overline{((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{(x,s_1),(y,s_2)}}(\gamma_\nu)_{s_1,s_4} \psi_{\vartheta(x),s_4,j} \\
&= \sum_{\substack{x,y \in \Gamma_L^+ \\ s_3,s_4=1,2 \\ j=1,\dots,N}} \bar{\psi}_{\vartheta(y),s_3,j} \left(\gamma_\nu ((\not{D}_{\text{NN}})_{\Gamma_L}^-)^*_{x,y} \gamma_\nu \right)_{s_3,s_4} \psi_{\vartheta(x),s_4,j} \\
&= \sum_{\substack{x,y \in \Gamma_L^+ \\ j=1,\dots,N}} \bar{\psi}_{\vartheta(y),j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{\vartheta(y),\vartheta(x)} \psi_{\vartheta(x),j} \\
&= \sum_{\substack{x,y \in \Gamma_L^- \\ j=1,\dots,N}} \bar{\psi}_{x,j} ((\not{D}_{\text{NN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j} \\
&= K_{--},
\end{aligned}$$

where we have used [Equation \(RND1\)](#) to pass from the fourth to the third line from the end.

- (b) The terms K'_{+-}, K'_{-+} are more tricky and show where the anti-symmetric boundary conditions are needed. We will show that:

$$K'_{+-} + K'_{-+} = \sum_{\substack{j=1,\dots,N \\ \epsilon=\pm 1 \\ s=1,2}} \left(\sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\mu) \in \partial_+(\Gamma_L^+)}} \psi_{x,s,j}^\epsilon \Theta(\psi_{x,s,j}^\epsilon) + \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+)}} \psi_{x,s,j}^\epsilon \Theta(\psi_{x,s,j}^\epsilon) \right), \quad (17)$$

with $\psi_{x,s,j}^- \equiv \psi_{x,s,j}$ and $\psi_{x,s,j}^+ \equiv \bar{\psi}_{x,s,j}$.

Let us consider the subset of cross terms localized around the right boundary $\partial_+ \Gamma_L^+$ (or in general on the boundary of the region Γ_L^+ where we have not inserted the $\mathbb{Z}_2$ -twisting):

$$\begin{aligned}
K'_{+-} &= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ s_1,s_2=1,2 \\ j=1,\dots,N}} \left(\bar{\psi}_{x,s_1,j}(\gamma_\nu)_{s_1,s_2} \psi_{x+e_\nu,s_2,j} - \bar{\psi}_{x+e_\nu,s_1,j}(\gamma_\nu)_{s_1,s_2} \psi_{x,s_2,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ s_1,s_2=1,2 \\ j=1,\dots,N}} \left(\bar{\psi}_{x,s_1,j}(\gamma_\nu)_{s_1,s_2} \psi_{x+e_\nu,s_2,j} + \psi_{x,s_2,j}(\gamma_\nu)_{s_1,s_2} \bar{\psi}_{x+e_\nu,s_1,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ s=1,2 \\ j=1,\dots,N}} \left(\bar{\psi}_{x,s,j} \Theta(\bar{\psi}_{x,s,j}) + \psi_{x,s,j} \Theta(\psi_{x,s,j}) \right),
\end{aligned}$$

where the Grassmann variables $\psi, \bar{\psi}$ are understood with *periodic* boundary conditions.

The terms on the other boundary of Γ_L^+ , $\partial_-(\Gamma_L^+)$, come with an extra minus sign which is compensated by the $\mathbb{Z}_2$ -holonomy (see Figure 4.1):

$$\begin{aligned}
K'_{-+} &= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ s_1, s_2=1,2 \\ j=1, \dots, N}} \left(-\bar{\psi}_{x-e_\nu, s_1, j}(\gamma_\nu)_{s_1, s_2} \psi_{x, s_2, j} + \bar{\psi}_{x, s_1, j}(\gamma_\nu)_{s_1, s_2} \psi_{x-e_\nu, s_2, j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ s_1, s_2=1,2 \\ j=1, \dots, N}} \left(\psi_{x, s_2, j}(\gamma_\nu)_{s_1, s_2} \bar{\psi}_{x-e_\nu, s_1, j} + \bar{\psi}_{x, s_1, j}(\gamma_\nu)_{s_1, s_2} \psi_{x-e_\nu, s_2, j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ s=1,2 \\ j=1, \dots, N}} \left(\psi_{x, s, j} \Theta(\psi_{x, s, j}) + \bar{\psi}_{x, s, j} \Theta(\bar{\psi}_{x, s, j}) \right).
\end{aligned}$$

■

4.2.2 Staggered Fermions with Naive Interaction

Lemma 47. *Properties in Equation (H1), Equation (H2) and Equation (H3) hold for canonical reflections, within the SN model.*

Proof.

(H1) See Lemma 55 in Subsection 4.A.2.

(H2) By a simple computation.

(H3) Let us first show that the reflection in Equation (RSF) is an involution:

$$\begin{aligned}
\Theta_\nu(\Theta_\nu(\bar{\psi}_{x, j})) &= \Theta_\nu(\Gamma_\nu(x) \psi_{\vartheta_\nu(x), j}) \\
&= \Gamma_\nu(x) \Gamma_\nu(\vartheta_\nu(x)) \bar{\psi}_{x, j} \\
&= \Gamma_\nu(x)^2 \bar{\psi}_{x, j} \\
&= \bar{\psi}_{x, j}, \\
\Theta_\nu(\Theta_\nu(\psi_{x, j})) &= \Theta_\nu(\Gamma_\nu(x) \bar{\psi}_{\vartheta_\nu(x), j}) \\
&= \Gamma_\nu(x) \Gamma_\nu(\vartheta_\nu(x)) \psi_{x, j} \\
&= \Gamma_\nu(x)^2 \psi_{x, j} \\
&= \psi_{x, j},
\end{aligned}$$

where we have used that

$$\Gamma_\nu(\vartheta_\nu(x)) = \begin{cases} 1 & \nu = 0 \\ (-1)^{\vartheta_1(x_0)} & \nu = 1 \end{cases} = \begin{cases} 1 & \nu = 0 \\ (-1)^{x_0} & \nu = 1 \end{cases} = \Gamma_\nu(x).$$

We need now to show that the kinetic term:

$$K = - \sum_{\substack{x, y \in \Gamma_L \\ j=1, \dots, N}} \bar{\psi}_{x, j} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{xy} \psi_{y, j}$$

(with $K \equiv K_{\text{SN}}$ in this subsection) fits the decomposition in Equation (H3). Again this is not strictly true; however it can be fixed by a simple staggered phase transformation (which preserves the local bilinears and the fermionic integration):

$$\begin{cases} \bar{\psi}_{x,j} \rightarrow (-1)^{x_0+x_1} \bar{\psi}_{x,j} \\ \psi_{x,j} \rightarrow (-1)^{x_0+x_1} \psi_{x,j}, \end{cases}$$

$$K \rightarrow K' = \sum_{\substack{x,y \in \Gamma_L \\ j=1,\dots,N}} \bar{\psi}_{x,j} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j}.$$

Let us split K' in the following way:

$$K' = K'_{++} + K'_{--} + K'_{+-} + K'_{-+}, \quad (18)$$

where:

$$K'_{\epsilon\epsilon'} = \sum_{x \in \Gamma_L^\epsilon, y \in \Gamma_L^{\epsilon'}} \bar{\psi}_{x,j} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} \psi_{y,j}, \quad \epsilon, \epsilon' \in \{\pm\}.$$

Let us analyze the terms in the right-hand side of Equation (18).

- (a) Let us begin by discussing K'_{++} and K'_{--} . We want to show that $\Theta(K'_{++}) = K'_{--}$. By a direct computation:

$$\begin{aligned} \Theta(K'_{++}) &= \sum_{x,y \in \Gamma_L^+} \Theta(\psi_y) \overline{((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y}} \Theta(\bar{\psi}_x) \\ &= \sum_{x,y \in \Gamma_L^+} \Gamma_\nu(x) \bar{\psi}_{\vartheta(y)} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)^*_{y,x} \Gamma_\nu(y) \psi_{\vartheta(x)} \\ &= \sum_{x,y \in \Gamma_L^+} \bar{\psi}_{\vartheta(y)} ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{\vartheta(y),\vartheta(x)} \psi_{\vartheta(x)} \\ &= \sum_{x,y \in \Gamma_L^-} \bar{\psi}_x ((\not{D}_{\text{SN}})_{\Gamma_L}^-)_{x,y} \psi_y \\ &= K'_{--}, \end{aligned}$$

where we have used Equation (RSD1) to pass from the second to the third line.

- (b) In analogy with the NN model, for the terms K'_{+-} and K'_{-+} we will show a decomposition of the form:

$$K'_{+-} + K'_{-+} = \sum_{j=1}^N \sum_{\epsilon=\pm 1} \left(\sum_{\substack{x \in \Gamma_L^+ \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+)}} \psi_{x,j}^\epsilon \Theta(\psi_{x,j}^\epsilon) + \sum_{\substack{x \in \Gamma_L^+ \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+)}} \psi_{x,j}^\epsilon \Theta(\psi_{x,j}^\epsilon) \right),$$

with $\psi_{x,j}^- \equiv \psi_{x,j}$ and $\psi_{x,j}^+ \equiv \bar{\psi}_{x,j}$.

Let us consider the subset of cross terms localized around the right boundary $\partial_+(\Gamma_L^+)$ (or in general on the boundary of the region Γ_L^+ where we have not inserted the $\mathbb{Z}_2$ -twisting):

$$\begin{aligned}
K'_{+-} &= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ j=1, \dots, N}} \left(\bar{\psi}_{x,j} \Gamma_\nu(x) \psi_{x+e_\nu,j} - \bar{\psi}_{x+e_\nu,j} \Gamma_\nu(x) \psi_{x,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ j=1, \dots, N}} \left(\bar{\psi}_{x,j} \Gamma_\nu(x) \psi_{x+e_\nu,j} + \psi_{x,j} \Gamma_\nu(x) \bar{\psi}_{x+e_\nu,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x, x+e_\nu) \in \partial_+(\Gamma_L^+) \\ j=1, \dots, N}} \left(\bar{\psi}_{x,j} \Theta(\bar{\psi}_{x,j}) + \psi_{x,j} \Theta(\psi_{x,j}) \right),
\end{aligned}$$

where again the Grassmann variables $\psi, \bar{\psi}$ are understood with *periodic* boundary conditions.

The terms on the other boundary of Γ_L^+ , $\partial_-(\Gamma_L^+)$, come with an extra minus sign which is compensated by the $\mathbb{Z}_2$ -holonomy (see [Figure 4.2](#)):

$$\begin{aligned}
K'_{-+} &= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ j=1, \dots, N}} \left(-\bar{\psi}_{x-e_\nu,j} \Gamma_\nu(x) \psi_{x,j} + \bar{\psi}_{x,j} \Gamma_\nu(x) \psi_{x-e_\nu,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ j=1, \dots, N}} \left(\psi_{x,j} \Gamma_\nu(x) \bar{\psi}_{x-e_\nu,j} + \bar{\psi}_{x,j} \Gamma_\nu(x) \psi_{x-e_\nu,j} \right) \\
&= \sum_{\substack{x \in \Gamma_L^+ : \\ (x-e_\nu, x) \in \partial_-(\Gamma_L^+) \\ j=1, \dots, N}} \left(\psi_{x,j} \Theta(\psi_{x,j}) + \bar{\psi}_{x,j} \Theta(\bar{\psi}_{x,j}) \right).
\end{aligned}$$

■

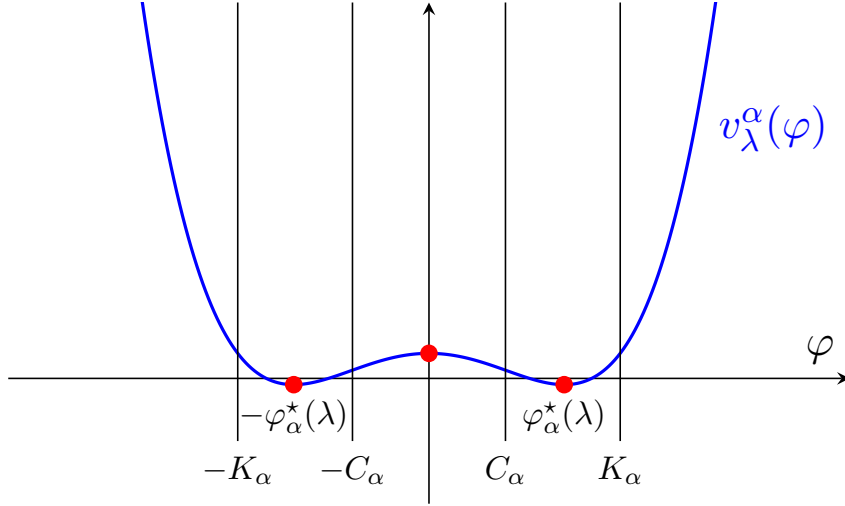
4.2.3 Staggered Fermions with Plaquette Interaction

Lemma 48. *Properties in [Equation \(H1\)](#), [Equation \(H2\)](#) and [Equation \(H3\)](#) hold for canonical reflections, within the SP model.*

Proof. The Lemma is a corollary of the related lemma for the SN model. Indeed, one can think of the SP model as a particular case of the SN model where the bosonic field is constant within each unit cell $\mathcal{I}$. Since reflections are performed between cells, the structure is preserved under reflections and every consideration made above can be easily translated here. ■

4.3 Decomposition of the Bosonic Two-Point Function

In this section we establish Long-Range Order for the bosonic field with respect to the effective measures $\mu_\alpha(\phi)$ and, by the Hubbard–Stratonovich identity in [Equation \(5\)](#), for the fermionic bilinear $\bar{\psi}\psi$.

Figure 4.6: Sketch of the Effective Potential $v_\lambda^\alpha(\varphi)$

In each model, the effective bosonic measure can be written in Gibbs form as:

$$d\mu_\alpha(\phi) = \prod_{x \in \Gamma_L^\alpha} d\phi_x e^{-NV_{L,\lambda}^\alpha(\phi)}$$

where the effective potential is given by

$$V_{L,\lambda}^\alpha(\phi) \doteq \frac{1}{2\lambda} \sum_{x \in \Gamma_L^\alpha} \phi_x^2 - \log |\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)|. \quad (\text{EP})$$

Remark 38. The determinant appearing in the definition of $V_{L,\lambda}^\alpha(\phi)$ may vanish for certain configurations $\phi \in \mathbb{R}^{\Gamma_L^\alpha}$. However, since it is a polynomial in the field variables, its zero set has zero Lebesgue measure (and hence also zero Gaussian measure). We adopt the convention that $V_{L,\lambda}^\alpha(\phi) = +\infty$ on such configurations. In particular, $V_{L,\lambda}^\alpha$ takes values in $\mathbb{R} \cup \{+\infty\}$.

The strategy of the proof is dictated by the structure of the effective potential. Although $V_{L,\lambda}^\alpha$ is highly non-local and does not admit a simple closed expression, Reflection Positivity implies that, among configurations constrained to lie in a given interval, its minimizers are spatially constant and can therefore be computed explicitly via Fourier transforms. The corresponding infinite-volume effective potential density:

$$v_\lambda^\alpha(\varphi) \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\phi) \Big|_{\phi = \varphi \Gamma_L^\alpha}, \quad (\text{TDEPD})$$

has a symmetric double-well structure, with minima at $\pm\varphi_\alpha^*(\lambda) \neq 0$ for any $\lambda > 0$ (see Figure 4.6). Long-Range Order is then proved by decomposing the configuration space into regions around the two minima $\varphi = \pm\varphi_\alpha^*(\lambda)$, the local maximum $\varphi = 0$, and the large-field regions, and by estimating the contribution from each region separately.

The following facts about the effective potential will be repeatedly used.

Proposition 49 (Structural properties of the effective potential). *For each $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$ and every interval $I \subseteq \mathbb{R}$, the following hold.*

1. The constrained infimum of $V_{L,\lambda}^\alpha$ over $I^{\Gamma_L^\alpha}$ is attained among constant configurations:

$$\inf_{\phi \in I^{\Gamma_L^\alpha}} V_{L,\lambda}^\alpha(\phi) = \inf_{\varphi \in I} V_{L,\lambda}^\alpha(\phi) \Big|_{\phi = \varphi \Gamma_L^\alpha} \quad (19)$$

2. There exists a constant $\mathfrak{c}_0 > 0$ such that:

$$\left| \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\varphi^{\Gamma_L^\alpha}) - v_\lambda^\alpha(\varphi) \right| \leq \frac{\mathfrak{c}_0}{L}, \quad (20)$$

uniformly with respect to $\varphi \in \mathbb{R}$, where the infinite-volume potential density v_λ^α admits the explicit representation:

$$v_\lambda^\alpha(\varphi) = \frac{\varphi^2}{2\Gamma} - C_\alpha \iint_{(-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1))$$

with $C_\alpha = 1, \frac{1}{2}, 2$ for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively.

3. The function v_λ^α has exactly two non-zero global minima at $\pm\varphi_\alpha^*(\lambda)$ and a local maximum at 0. φ_α^* is a C^∞ function of $\lambda \in (0, \infty)$ and satisfies, for $\lambda \leq 1$,

$$\mathfrak{a}_1 e^{-\frac{\mathfrak{b}_1}{\lambda}} \leq \varphi_\alpha^*(\lambda) \leq \mathfrak{a}_2 e^{-\frac{\mathfrak{b}_2}{\lambda}}, \quad (21)$$

for suitable constants $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 > 0$. Besides, v_λ^α is strictly decreasing over $[0, \varphi_\alpha^*(\lambda)]$ and strictly increasing over $[\varphi_\alpha^*(\lambda), \infty)$.

The proof of [Proposition 49](#) is discussed in [Appendix 4.B](#).

4.3.1 Field-size decomposition

By [Proposition 49](#) we have that the minima of the finite-volume energy density are asymptotically determined by those of v_λ^α . The latter exhibits a symmetric double-well structure (see [Figure 4.6](#)), with absolute minima at $\pm\varphi_\alpha^*(\lambda)$. This suggests partitioning the configuration space according to the position of the field relative to the minima, the local maximum at the origin, and the large-field regime. Configurations belonging to energetically unfavorable regions, namely large fields or fields near the local maximum, are exponentially suppressed in N . This suppression is quantified by comparing the energy of configurations constrained to a given region with that of optimal constant configurations, using chessboard estimates together with [Equation \(19\)](#). For completeness, let us recall the statement of chessboard estimates in [Equation \(16\)](#) adapted to the present case:

Theorem (Chessboard Estimates). *Let μ be a reflection-positive probability measure on $\mathbb{R}^{\Gamma_L}$ with respect to reflections between neighboring blocks $\mathbf{Q}_B + Bt$ with $t \in \mathbb{Z}_{L/B}^2$ (where both $L, L/B \in 2\mathbb{N}$). Then for any positive set of functions $f_1, \dots, f_n$ supported in cubes of size $\mathbf{Q}_B$ and for any set of vectors $t_1, \dots, t_n \in \mathbb{Z}_{L/B}^2$, the following inequality holds:*

$$\left\langle \prod_{j=1}^n \Theta_{t_j}(f_j) \right\rangle_\mu \leq \prod_{j=1}^n \left\langle \prod_{t \in \mathbb{Z}_{L/B}^2} \Theta_t(f_j) \right\rangle_\mu^{\frac{|\Gamma_B|}{|\Gamma_L|}} \quad (\text{CE})$$

where Θ_t is the composition of reflections that brings the block Γ_B in the block $\mathbf{Q}_B + Bt$

Proof. See [\[FV17, Pag. 444-449\]](#), [\[Bis+09, Pag. 43-46\]](#) or [\[Frö+78, Pag. 19-20\]](#) ■

The contributions which are not a priori suppressed arise from configurations where both fields lie near the minima. Configurations in which the field at different points lies around different minima can be shown to be again exponentially suppressed in N , via a Peierls-type argument combined with Reflection Positivity, in the spirit of [\[FL78\]](#) (see also [\[Bis+09; FV17\]](#)). While this strategy applies to all our three models, the explicit estimates depend on the specific lattice realization of

the Dirac operator and will be treated separately in the following subsections.

To implement this strategy, we introduce a quantitative partition of the configuration space at the level of single-site variables. Fix parameters $C_\alpha, K_\alpha > 0$ such that:

$$0 < C_\alpha < \varphi_\alpha^*(\lambda) < K_\alpha.$$

We then split the single-site field values into the following five regions:

1. **Large positive fields:** $\phi_x \in (K_\alpha, \infty)$;
2. **Large negative fields:** $\phi_x \in (-\infty, -K_\alpha)$;
3. **Neighborhood of the local maximum:** $\phi_x \in (-C_\alpha, C_\alpha)$;
4. **Neighborhood of the positive minimum:** $\phi_x \in [C_\alpha, K_\alpha]$;
5. **Neighborhood of the negative minimum:** $\phi_x \in [-K_\alpha, -C_\alpha]$.

Accordingly, introduce the following indicator functions:

$$I_x^{\pm K_\alpha}(\phi) \doteq \mathbb{1}_{\{\pm\phi_x > K_\alpha\}}, \quad I_x^{(-C_\alpha, C_\alpha)}(\phi) \doteq \mathbb{1}_{\{|\phi_x| < C_\alpha\}}, \quad I_x^{\star\pm}(\phi) \doteq \mathbb{1}_{\{C_\alpha \leq \pm\phi_x \leq K_\alpha\}}.$$

The lower bound on the two-point function is obtained by isolating the contribution in which both fields lie near the same minimum, and by estimating all the remaining contributions in terms of three errors: *large-field* events, *local-maximum* events, and *opposite-minimum* events. Following the decomposition carried out in [Appendix 4.D](#), we obtain:

$$\begin{aligned} \inf_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} &\geq C_\alpha^2 - \left(2(K_\alpha^2 + C_\alpha^2) \sup_{x,y \in \Lambda_L^\alpha} \langle I_x^{\star+} I_y^{\star-} \rangle_{\mu_\alpha} + 4C_\alpha(C_\alpha + K_\alpha) \sup_{x \in \Lambda_L^\alpha} \langle I_x^{(-C_\alpha, C_\alpha)} \rangle_{\mu_\alpha} \right. \\ &\quad \left. + 4\left(1 + \frac{C_\alpha^2}{K_\alpha^2}\right) \sup_{x \in \Lambda_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} + 4\left(C_\alpha + 2\frac{C_\alpha^2}{K_\alpha} + 2K_\alpha\right) \sup_{x \in \Lambda_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} \right). \end{aligned} \quad (22)$$

Similarly, see again [Appendix 4.D](#), for the upper bound we find:

$$\sup_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} \leq K_\alpha^2 + 2K_\alpha \sup_{x \in \Gamma_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} + 4 \sup_{x \in \Gamma_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}. \quad (23)$$

Thus, the proof reduces to estimating the following three expectation values:

$$\langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}, \quad \langle I_x^{(-C_\alpha, C_\alpha)} \rangle_{\mu_\alpha}, \quad \langle I_x^{\star+} I_y^{\star-} \rangle_{\mu_\alpha},$$

whose bounds are discussed in [Proposition 50](#) below. The first two correspond to *large-field* and *local-maximum* events and will be controlled directly by Chessboard Estimates together with [Proposition 49](#). The third one corresponds to configurations in which the field fluctuates near different minima at two distant sites and its control requires a Peierls-type argument combined with Reflection Positivity.

Proposition 50. *There exists $\lambda_0 > 0$ such that the following holds. For every $0 < \Gamma \leq \lambda_0$ and every choice of parameters*

$$0 < C_\alpha < \varphi_\alpha^*(\lambda) < K_\alpha,$$

with φ_α^* as in [Proposition 49](#), there exist $N_1(K_\alpha, C_\alpha, \lambda)$ and $L_1(K_\alpha, C_\alpha, \lambda)$ such that, for all

$$N \geq N_1(K_\alpha, C_\alpha, \lambda), \quad L \geq L_1(K_\alpha, C_\alpha, \lambda),$$

we have:

$$\sup_{x \in \Gamma_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} \leq e^{-\frac{N}{2} c_1^\alpha(\lambda)} \quad (24)$$

$$\sup_{x \in \Gamma_L^\alpha} \langle I_x^{(-C_\alpha, C_\alpha)} \rangle_{\mu_\alpha} \leq e^{-\frac{N}{2} c_2^\alpha(\lambda)} \quad (25)$$

$$\sup_{x, y \in \Gamma_L^\alpha} \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha} \leq e^{-\frac{N}{4} c_3^\alpha(\lambda)} \quad (26)$$

where:

1. $c_1^\alpha(\lambda)$ is the “large-field energy gap” associated with configurations where $\phi_x > K_\alpha$:

$$c_1^\alpha(\lambda) \doteq v_\lambda^\alpha(K_\alpha) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) > 0;$$

2. $c_2^\alpha(\lambda)$ is the “local-maximum energy gap” associated with configurations $-C_\alpha < \phi_x < C_\alpha$:

$$c_2^\alpha(\lambda) \doteq v_\lambda^\alpha(C_\alpha) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) > 0;$$

3. $c_3^\alpha(\lambda) > 0$ is the “Peierls gap” associated with alternating-sign configurations obtained by reflection of elementary blocks:

$$c_3^\alpha(\lambda) \doteq \min \left(v_\lambda^\alpha \left(\frac{1}{\sqrt{2}} \varphi_\alpha^*(\lambda) \right) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)), -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^\alpha \frac{\varphi_\alpha^*(\lambda)^{n_\alpha}}{1 + \varphi_\alpha^*(\lambda)^{n_\alpha}} \right) \right),$$

where $n_\alpha = 4, 2, 8$ for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively and $\mathfrak{C}_2^\alpha > 0$ are suitable constants.

Assuming [Proposition 50](#) for the moment, we first derive [Theorem 1](#). The proof of [Proposition 50](#) is given in [Section 4.4](#).

4.3.2 Derivation of Chiral Long-Range Order

We now show how [Proposition 50](#) implies the main theorem. By [Equation \(5\)](#), we have that

$$\frac{1}{|\Gamma_L^\alpha|^2} \sum_{x, y \in \Gamma_L^\alpha} \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha = \left(\frac{N}{\lambda} \right)^2 \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x, y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} - \frac{N}{\lambda} \frac{1}{|\Gamma_L^\alpha|}.$$

Therefore:

$$\left(\frac{N}{\lambda} \right)^2 \left(\min_{x, y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} - \frac{4\lambda}{NL^2} \right) \leq \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x, y \in \Gamma_L^\alpha} \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \leq \left(\frac{N}{\lambda} \right)^2 \max_{x, y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha}. \quad (27)$$

Given any $\varepsilon \in (0, 1)$, we apply [Proposition 50](#) with

$$C_\alpha = \sqrt{1 - \frac{\varepsilon}{2}} \varphi_\alpha^*(\lambda), \quad K_\alpha = \sqrt{1 + \frac{\varepsilon}{2}} \varphi_\alpha^*(\lambda).$$

Using the upper and lower bound for the bosonic correlator, [Equation \(23\)](#) and [Equation \(22\)](#) respectively, we find that

$$\begin{aligned} \sup_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} &\leq \left(1 + \frac{\varepsilon}{2}\right) \varphi_\alpha^*(\lambda)^2 + \mathcal{E}_1(N, \lambda, \varepsilon), \\ \inf_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} &\geq \left(1 - \frac{\varepsilon}{2}\right) \varphi_\alpha^*(\lambda)^2 - \mathcal{E}_2(N, \lambda, \varepsilon), \end{aligned}$$

for every $N \geq N_1(K_\alpha, C_\alpha, \lambda)$ and $L \geq L_1(K_\alpha, C_\alpha, \lambda)$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are suitable quantities, which are exponentially small in N for any $\lambda \in (0, \lambda_0]$ and $\varepsilon \in (0, 1)$ fixed. Going back to Equation (27), we have that

$$\begin{aligned} \left(\frac{N}{\lambda}\right)^2 \left[\left(1 - \frac{\varepsilon}{2}\right) \varphi_\alpha^*(\lambda)^2 - \mathcal{E}_2(N, \lambda, \varepsilon) - \frac{4\lambda}{NL^2} \right] &\leq \frac{1}{|\Gamma_L^\alpha|^2} \sum_{x,y \in \Gamma_L^\alpha} \langle (\bar{\psi}\psi)_x (\bar{\psi}\psi)_y \rangle_\alpha \\ &\leq \left(\frac{N}{\lambda}\right)^2 \left[\left(1 + \frac{\varepsilon}{2}\right) \varphi_\alpha^*(\lambda)^2 + \mathcal{E}_1(N, \lambda, \varepsilon) \right]. \end{aligned}$$

Finally we require N, L to be large enough (depending on λ and ε), in order to satisfy:

$$\mathcal{E}_1(N, \lambda, \varepsilon) \leq \frac{\varepsilon}{2} \varphi_\alpha^*(\lambda)^2, \quad \mathcal{E}_2(N, \lambda, \varepsilon) \leq \frac{\varepsilon}{4} \varphi_\alpha^*(\lambda)^2, \quad \frac{1}{NL^2} \leq \frac{\varepsilon}{8\lambda} \varphi_\alpha^*(\lambda)^2,$$

so that Equation (8) follows.

4.4 Suppression of Subdominant Field Configurations

This section is devoted to the analysis of the three main subdominant contributions to the two-point bosonic correlator $\langle \phi_x \phi_y \rangle_{\mu_\alpha}$, namely the *large-field*, the *local-maximum* and the *Peierls-like* term. These will be shown to satisfy the bounds in Equation (24), Equation (25) and Equation (26) respectively, which are the main building blocks for the proof of LRO, as discussed in Subsection 4.3.2. The first two estimates will be proved in Subsection 4.4.1 and Subsection 4.4.2. The third estimate, treated in Subsection 4.4.3, requires a separate analysis of explicit, alternating, periodic configurations, discussed in Subsection 4.4.4.

4.4.1 Large-field terms

We prove the large-field bound in Equation (24), namely we show that configurations with $\phi_x > K_\alpha$ are exponentially suppressed in N .

The proof combines Chessboard Estimates with the fact that, for $K_\alpha > \varphi_\alpha^*(\lambda)$, the mean-field potential in Equation (9) is strictly larger than its minimum. Applying the chessboard estimate (CE) to the single-site observable $I_x^{+K_\alpha}(\phi)$, we obtain:

$$\langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} \leq \left(\frac{\int_{[K_\alpha, \infty) \Gamma_L^\alpha} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \phi_x^2 \right) e^{-NV_{L,\lambda}^\alpha(\phi)}}{\int_{\mathbb{R} \Gamma_L^\alpha} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \right) e^{-NV_{L,\lambda}^\alpha(\phi)}} \right)^{\frac{1}{|\Gamma_L^\alpha|}}. \quad (28)$$

We will proceed by looking for an upper and a lower bound for the numerator and denominator of Equation (36) respectively.

Upper bound for the numerator of Equation (28). We proceed via the following intermediate steps.

1. We first rewrite the effective potential as:

$$V_{L,\lambda}^\alpha(\phi) = \frac{1}{2\lambda N} \sum_{x \in \Gamma_L^\alpha} \phi_x^2 + V_{L,\tilde{\lambda}_N}^\alpha(\phi), \quad \tilde{\lambda}_N \doteq \frac{\lambda}{1-\frac{1}{N}}.$$

2. As a second step, we extract the infimum of $V_{L,\tilde{\lambda}_N}^\alpha$ over the constrained set $\{\phi_x \geq K_\alpha \forall x \in \Gamma_L^\alpha\}$ and we integrate over ϕ :

$$\begin{aligned} \left(\int_{[K_\alpha, \infty)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \phi_x^2 \right) e^{-NV_{L,\lambda}^\alpha(\phi)} \right)^{\frac{1}{|\Gamma_L^\alpha|}} &= \left(\int_{[K_\alpha, \infty)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \phi_x^2 e^{-\frac{\phi_x^2}{2\lambda}} \right) e^{-NV_{L,\tilde{\lambda}_N}^\alpha(\phi)} \right)^{\frac{1}{|\Gamma_L^\alpha|}} \\ &\leq (\sqrt{2\pi}\lambda^{\frac{3}{2}}) \exp \left(-\frac{N}{|\Gamma_L^\alpha|} \inf_{\phi \in [K_\alpha, \infty)^{\Gamma_L^\alpha}} V_{L,\tilde{\lambda}_N}^\alpha(\phi) \right). \end{aligned}$$

3. By [Proposition 49-Item 1](#), the infimum of $V_{L,\lambda}^\alpha$ is attained for constant field configurations, whose large L asymptotics can be controlled, in force of [Proposition 49-Item 2](#), by the infinite-volume potential density v_λ^α :

$$\frac{1}{|\Gamma_L^\alpha|} \inf_{\phi \in [K_\alpha, \infty)^{\Gamma_L^\alpha}} V_{L,\tilde{\lambda}_N}^\alpha(\phi) \stackrel{(19)}{\geq} \frac{1}{|\Gamma_L^\alpha|} \inf_{\varphi \in [K_\alpha, \infty)} V_{L,\tilde{\lambda}_N}^\alpha(\varphi) \stackrel{(20)}{\geq} \inf_{\varphi \in [K_\alpha, \infty)} v_{\tilde{\lambda}_N}^\alpha(\varphi) - \frac{\mathfrak{c}_0}{L}. \quad (29)$$

4. Since $K_\alpha > \varphi_\alpha^*(\lambda)$ and the map $\lambda \mapsto \varphi_\alpha^*(\lambda)$ is continuous over $(0, \infty)$ (see [Proposition 49-Item 3](#)), for N sufficiently large one has $K_\alpha \geq \varphi_\alpha^*(\tilde{\lambda}_N)$. Hence, since $v_{\tilde{\lambda}_N}^\alpha$ is increasing over $[\varphi_\alpha^*(\tilde{\lambda}_N), \infty)$ (see [Proposition 49-Item 3](#)), the infimum over $[K_\alpha, \infty)$ is attained at $\varphi = K_\alpha$ and (29) yields:

$$\frac{1}{|\Gamma_L^\alpha|} \inf_{\phi \in [K_\alpha, \infty)^{\Gamma_L^\alpha}} V_{L,\tilde{\lambda}_N}^\alpha(\phi) \geq v_{\tilde{\lambda}_N}^\alpha(K_\alpha) - \frac{\mathfrak{c}_0}{L}.$$

All in all:

$$\left(\int_{[K_\alpha, \infty)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \phi_x^2 \right) e^{-NV_{L,\lambda}^\alpha(\phi)} \right)^{\frac{1}{|\Gamma_L^\alpha|}} \leq (\sqrt{2\pi}\lambda^{\frac{3}{2}}) \exp \left(-N(v_{\tilde{\lambda}_N}^\alpha(K_\alpha) - \frac{\mathfrak{c}_0}{L}) \right). \quad (30)$$

Upper bound for the denominator of (28). The strategy for bounding from below the denominator of (28) (a.k.a. the partition function) goes through the following steps.

1. First, we restrict the integral over ϕ to a neighborhood $Q_{\delta_N}^\alpha$ of the homogeneous minimizer $\varphi_\alpha^*(\lambda) > 0$, where

$$Q_{\delta_N}^\alpha \doteq [\varphi_\alpha^*(\lambda) - \frac{\delta_N}{2}, \varphi_\alpha^*(\lambda) + \frac{\delta_N}{2}]^{\Gamma_L^\alpha}$$

with $\delta_N > 0$ to be specified. Since the integrand is pointwise non-negative for N even, such restriction produces a lower-bound:

$$(Z_{\Gamma_L}^\alpha)^{\frac{1}{|\Lambda_L^\alpha|}} \geq \left[\int_{Q_{\delta_N}^\alpha} \prod_{x \in \Gamma_L^\alpha} d\phi_x e^{-\frac{N}{2\lambda} \phi_x^2} \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)^N \right]^{\frac{1}{|\Lambda_L^\alpha|}}. \quad (31)$$

2. In the second place, we perform a change of variables $\phi_x \mapsto \varphi_\alpha^*(\lambda) + \phi_x$ and we use the following bound (holding for $\phi \in Q_{\delta_N}^{\alpha,0} \doteq [-\frac{\delta_N}{2}, \frac{\delta_N}{2}]^{\Gamma_L^\alpha}$):

$$e^{-\frac{N}{2\lambda} \sum_x (\varphi_\alpha^*(\lambda) + \phi_x)^2} = e^{-\frac{N}{2\lambda} (\varphi_\alpha^*(\lambda)^2 |\Gamma_L^\alpha| + 2\varphi_\alpha^*(\lambda) \sum_x \phi_x + \sum_x \phi_x^2)} \geq e^{-\frac{N}{2\lambda} |\Gamma_L^\alpha| (\varphi_\alpha^*(\lambda)^2 + 2\varphi_\alpha^*(\lambda) \delta_N + \delta_N^2)}$$

to obtain:

$$\begin{aligned} (Z_{\Gamma_L}^\alpha)^{\frac{1}{|\Lambda_L^\alpha|}} &\geq e^{-\frac{N}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\varphi_\alpha^*(\lambda))} e^{-\frac{N}{2\lambda} (2\varphi_\alpha^*(\lambda) \delta_N + \delta_N^2)} \times \\ &\times \left[\int_{Q_{\delta_N}^{\alpha,0}} \prod_{x \in \Lambda_L^\alpha} d\phi_x \det \left(\mathbb{1} - ((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\varphi_\alpha^*(\lambda)})^{-1} M_\phi \right)^N \right]^{\frac{1}{|\Lambda_L^\alpha|}}. \end{aligned} \quad (32)$$

Remark 39. Observe that the Dirac operator is invertible since a non-vanishing, constant background field opens a gap in the dispersion relations (see [Appendix 4.B](#)). More precisely, introducing the operator norm:

$$\|\cdot\|_{\text{op}} \doteq \sup_{v \in \mathbb{C}^n, \|v\|=1} \|\cdot v\|_{\ell^2(X)} \text{ with } X = \begin{cases} \Gamma_L \times \{1, 2\} & \alpha = \text{NN} \\ \Gamma_L & \alpha = \text{SN} \\ \tilde{\Lambda}_L \times \{A, B, C, D\} & \alpha = \text{SP} \end{cases}$$

one has that

$$\left\| ((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\varphi_{\Gamma_L}^\alpha})^{-1} \right\|_{\text{op}} \leq \frac{1}{\varphi},$$

for any $\varphi \neq 0$. Indeed, it is readily checked (see [Appendix 4.B.2](#)) that the operator

$$(\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\varphi_{\Gamma_L}^\alpha} \equiv (\mathcal{D}_\alpha)_{\Gamma_L}^- - \varphi \mathbb{1}$$

is normal and has eigenvalues:

$$\pm i \sqrt{\sin^2 p_0 + \sin^2 p_1} - \varphi \Big|_{p \in (\Gamma_L^\alpha)^*_{--}} \quad \text{where } (\Gamma_L^\alpha)^*_{--} = \begin{cases} \frac{2\pi}{L} (\mathbb{Z} + \frac{1}{2})^2 \cap (-\pi, \pi]^2 & \alpha = \text{NN} \\ \frac{2\pi}{L} (\mathbb{Z} + \frac{1}{2})^2 \cap \left((-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi] \right) & \alpha = \text{SN} \\ \frac{2\pi}{L} (\mathbb{Z} + \frac{1}{2})^2 \cap \left((-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi] \right) & \alpha = \text{SP} \end{cases}$$

3. In order to find a lower bound for the determinant in the right-hand side of [Equation \(32\)](#), we use the following standard fact: letting A be a linear operator on a vector space $\mathcal{V}$ of dimension $n < \infty$, then

$$|\det(\mathbb{1}_{\mathcal{V}} + A)| \geq (1 - \|A\|_{\text{op}})^n. \quad (33)$$

In our case, setting $A = ((\mathcal{D}_\alpha)_{\Gamma_L}^- - \varphi_\alpha^*(\lambda))^{-1} M_\phi$, with $\phi \in Q_{\delta_N}^{\alpha,0}$, we see that, in force of [Remark 39](#),

$$\left\| ((\mathcal{D}_\alpha)_{\Gamma_L}^- - \varphi_\alpha^*(\lambda))^{-1} M_\phi \right\|_{\text{op}} \leq \left\| ((\mathcal{D}_\alpha)_{\Gamma_L}^- - \varphi_\alpha^*(\lambda))^{-1} \right\|_{\text{op}} \|M_\phi\|_{\text{op}} \leq \frac{\delta_N}{\varphi_\alpha^*(\lambda)}.$$

Therefore, for any choice $0 < \delta_N < \varphi_\alpha^*(\lambda)$, we have that

$$|\det \left(\mathbb{1} - ((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\varphi_\alpha^*}(\lambda))^{-1} M_\phi \right)| \geq \left(1 - \frac{\delta_N}{\varphi_\alpha^*(\lambda)} \right)^{2|\Gamma_L^\alpha|}.$$

Plugging the above bound in the right-hand side of Equation (32) and performing the integral over $Q_{\delta_N}^{\alpha,0}$, we find the following lower bound for the partition function:

$$(Z_{\Gamma_L}^\alpha)^{\frac{1}{|\Gamma_L^\alpha|}} \geq e^{N \left[-v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) - \frac{\delta_N}{\lambda} (2\varphi_\alpha^*(\lambda) + \delta_N) + \frac{1}{N} \log(2\delta_N) + 2 \log \left(1 - \frac{\delta_N}{\varphi_\alpha^*(\lambda)} \right) - \frac{\epsilon_0}{L} \right]}. \quad (34)$$

Proof of the bound (24). By plugging Equation (30) and Equation (34) into the right-hand side of Equation (28), we find:

$$\langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{|\Gamma_L^\alpha|}} \leq e^{-N \left[v_\lambda^\alpha(K_\alpha) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) - \mathcal{R}_{N,L}^{(1)}(\lambda) \right]} \doteq e^{-N \left[c_1^\alpha(\lambda) - \mathcal{R}_{N,L}^{(1)}(\lambda) \right]}, \quad (35)$$

where

$$\begin{aligned} \mathcal{R}_{N,L}^{(1)}(\lambda) &\doteq (v_\lambda^\alpha(K_\alpha) - v_{\lambda_N}^\alpha(K_\alpha)) + \frac{\delta_N}{\lambda} (2\varphi_\alpha^*(\lambda) + \delta_N) - \frac{1}{N} \log(2\delta_N) + \\ &\quad - 2 \log \left(1 - \frac{\delta_N}{\varphi_\alpha^*(\lambda)} \right) + \frac{1}{2N} (\log(2\pi) + 3 \log \lambda) + \frac{2\epsilon_0}{L}. \end{aligned}$$

The main observation is that

$$v_\lambda^\alpha(K_\alpha) - v_{\lambda_N}^\alpha(\varphi_\alpha^*(\lambda)) \doteq c_1^\alpha(\lambda) > 0,$$

for every choice of $K_\alpha > \varphi_\alpha^*(\lambda)$, since the potential v_λ^α is strictly-increasing in the interval $[\varphi_\alpha^*(\lambda), \infty)$. Furthermore, due to the smoothness of the map $\lambda \mapsto \varphi_\alpha^*(\lambda)$, for every $\varphi > 0$ fixed, we have that $v_\lambda^\alpha(K_\alpha) - v_{\lambda_N}^\alpha(K_\alpha) \rightarrow 0$ as $N \rightarrow \infty$. In this way, by also fixing $\delta_N \equiv \delta_N(\lambda) \doteq \frac{\varphi_\alpha^*(\lambda)}{N}$, we readily see that

$$\mathcal{R}_{N,L}^{(1)}(\lambda) \xrightarrow{N,L \rightarrow \infty} 0,$$

for every $\lambda > 0$ and $K_\alpha > \varphi_\alpha^*(\lambda)$ fixed. Therefore, for N, L sufficiently large (depending on λ and K_α), we can assume that $|\mathcal{R}_{N,L}^{(1)}(\lambda)| \leq \frac{1}{2} c_1^\alpha(\lambda)$, yielding the bound in Equation (24).

4.4.2 Local-maximum terms

In this subsection we prove the second inequality in Equation (25), namely

$$\sup_{x \in \Gamma_L^\alpha} \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \leq e^{-\frac{N}{2} c_2^\alpha(\lambda)}.$$

This estimate controls configurations in which the field lies in a neighborhood of a local maximum of the potential. Such configurations are energetically unfavorable and hence exponentially suppressed in N . The proof relies on Chessboard Estimates, combined with the fact that, as established by Proposition 49, the potential v_λ^α is strictly larger than its minimum whenever $\phi_x \in (-C_\alpha, C_\alpha) \subsetneq (-\varphi_\alpha^*, \varphi_\alpha^*)$. Applying the chessboard estimate (CE) to the single-site observable, we obtain:

$$\left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \leq \left(\frac{\int_{(-C_\alpha, C_\alpha)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \right) e^{-NV_{L,\lambda}^\alpha(\phi)}}{\int_{\mathbb{R}^{\Lambda_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \right) e^{-NV_{L,\lambda}^\alpha(\phi)}} \right)^{\frac{1}{|\Gamma_L^\alpha|}}. \quad (36)$$

Upper bound for the numerator of Equation (36). The bound of the numerator is a straightforward application of Proposition 49.

1. First, we extract the infimum of $V_{L,\lambda}^\alpha$ over the constrained set $-C_\alpha \leq \phi_x \leq C_\alpha$ and perform the integral over ϕ :

$$\left(\int_{(-C_\alpha, C_\alpha)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \right) e^{-NV_{L,\lambda}^\alpha(\phi)} \right)^{\frac{1}{|\Gamma_L^\alpha|}} \leq (2C_\alpha) \exp \left(-\frac{N}{|\Gamma_L^\alpha|} \inf_{\phi \in (-C_\alpha, C_\alpha)^{\Lambda_L^\alpha}} V_{L,\lambda}^\alpha(\phi) \right).$$

2. By Equation (19), the infimum of $V_{L,\lambda}^\alpha$ is attained over constant field configurations, whose large- L asymptotics can be controlled, due to Proposition 49-Item 2, by the infinite-volume potential density v_λ^α :

$$\frac{1}{|\Gamma_L^\alpha|} \inf_{\phi \in (-C_\alpha, C_\alpha)^{\Lambda_L^\alpha}} V_{L,\lambda}^\alpha(\phi) \stackrel{(19)}{\geq} \frac{1}{|\Gamma_L^\alpha|} \inf_{\varphi \in (-C_\alpha, C_\alpha)} V_{L,\lambda}^\alpha(\varphi) \stackrel{(20)}{\geq} \inf_{\varphi \in (-C_\alpha, C_\alpha)} v_\lambda^\alpha(\varphi) - \frac{c_0}{L}. \quad (37)$$

3. Since v_λ^α is strictly decreasing on $[0, \varphi_\alpha^*(\lambda)]$ (see Proposition 49-Item 3) and $C_\alpha < \varphi_\alpha^*(\lambda)$, the right-hand side of Equation (37) is larger than $v_\lambda^\alpha(C_\alpha) - \frac{c_0}{L}$.

Therefore, we find the following upper bound for the numerator of Equation (36):

$$\left(\int_{(-C_\alpha, C_\alpha)^{\Gamma_L^\alpha}} \left(\prod_{x \in \Gamma_L^\alpha} d\phi_x \right) e^{-NV_{L,\lambda}^\alpha(\phi)} \right)^{\frac{1}{|\Gamma_L^\alpha|}} \leq (2C_\alpha) e^{-N[v_\lambda^\alpha(C_\alpha) - \frac{c_0}{L}]}. \quad (38)$$

Proof of the bound Equation (25). Plugging equations Equation (30) and Equation (38) into the right-hand side of Equation (36), we find:

$$\left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \leq e^{-N(v_\lambda^\alpha(C_\alpha) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) - \mathcal{R}_{N,L}^{(2)}(\lambda))},$$

where

$$\mathcal{R}_{N,L}^{(2)}(\lambda) \doteq \frac{\delta_N}{\lambda} (2\varphi_\alpha^*(\lambda) + \delta_N) + \frac{1}{N} (-\log(2\delta_N) + \log(2C_\alpha)) - 2\log \left(1 - \frac{\delta_N}{\varphi_\alpha^*(\lambda)} \right) + \frac{2c_0}{L}.$$

We observe that

$$v_\lambda^\alpha(C_\alpha) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) \equiv c_2^\alpha(\lambda) > 0,$$

for every choice of $C_\alpha < \varphi_\alpha^*(\lambda)$, since the potential v_λ^α is strictly decreasing in the interval $[0, \varphi_\alpha^*(\lambda)]$ (see Proposition 49-Item 3). By fixing $\delta_N = \frac{\varphi_\alpha^*(\lambda)}{N}$, we see that for every $\lambda > 0$ and $0 < C_\alpha < \varphi_\alpha^*(\lambda)$,

$$\mathcal{R}_{N,L}^{(2)}(\lambda) \xrightarrow{N,L \rightarrow \infty} 0.$$

Therefore, for N, L sufficiently large (depending on λ and C_α), we can assume that $|\mathcal{R}_{N,L}^{(2)}(\lambda)| \leq \frac{1}{2}c_2^\alpha(\lambda)$, yielding the bound in Equation (25).

4.4.3 Peierls-like terms

In this subsection we prove the bound in Equation (26), namely

$$\sup_{x,y \in \Gamma_L^\alpha} \langle I_x^+ I_y^+ \rangle_{\mu_\alpha} \leq e^{-\frac{N}{4} c_3^\alpha(\lambda)}.$$

This estimate controls configurations in which the field takes values near opposite minimizers of the potential at two distinct sites. Such configurations necessarily contain an interface separating regions of opposite sign, and are therefore expected to be exponentially suppressed in N .

In contrast to the previous cases, this contribution cannot be handled directly by means of the chessboard estimates alone. Its analysis requires a more detailed control of the energy cost associated with configurations interpolating between the two minima. This is achieved via a Peierls-type argument (see e.g. [FV17, Sect. 3.7.3]), combined with reflection positivity, which allows us to control the weight of contours separating regions with opposite sign of ϕ by relating them to the energetic penalty of suitably constructed periodic configurations generated through reflections.

General Peierls' argument.

1. For any $x, y \in \Gamma_L^\alpha$, we have that

$$\langle I_x^+ I_y^+ \rangle_{\mu_\alpha} \leq \langle I_x^+ I_y^- \rangle_{\mu_\alpha},$$

where $I_x^\pm(\phi) \doteq \mathbb{1}_{\{\pm\phi_x > 0\}}$.

2. Since $I_x^+ + I_x^- = 1$ almost surely, we have that

$$\langle I_x^+ I_y^- \rangle_{\mu_\alpha} = \left\langle I_x^+ I_y^- \prod_{z \neq x, y} (I_z^+ + I_z^-) \right\rangle_{\mu_\alpha} = \sum_{\substack{\zeta \in \{\pm\}^{\Gamma_L^\alpha} \\ \zeta_x = +, \zeta_y = -}} \left\langle \prod_{z \in \Gamma_L^\alpha} I_z^{\zeta_z} \right\rangle_{\mu_\alpha}. \quad (39)$$

3. To each sign configuration $\zeta \in \{\pm\}^{\Gamma_L^\alpha}$, we associate its set of Peierls' contours, namely the closed curves in the dual lattice $(\Gamma_L^\alpha)^{\text{dual}} \doteq \Lambda_L^\alpha + (\frac{1}{2}, \frac{1}{2})$ crossing precisely those edges $\langle x', y' \rangle \subset \Gamma_L^\alpha$ for which $\zeta_{x'} \neq \zeta_{y'}$. As usual, crossings are resolved according to the deformation rule (see [FV17, pag. 111]) shown in Figure 4.7, so that the contour set is a collection of disjoint loops including the information of the values of the spins on both “sides”.

4. A standard Peierls argument (see [FV17, pag. 453-457]) then gives:

$$\langle I_x^+ I_y^+ \rangle_{\mu_\alpha} \leq 2 \sum_{\gamma} \left\langle \prod_{\langle x', y' \rangle \in E(\gamma)} I_{x'}^+ I_{y'}^+ \right\rangle_{\mu_\alpha}, \quad (40)$$

where the sum runs over all dual contours γ separating x and y , and $E(\gamma)$ denotes the set of crossed primal edges, ordered so that x' lies on the “left hand side” of γ and y' on the “right hand side” (with respect to some fixed orientation of the contour γ).

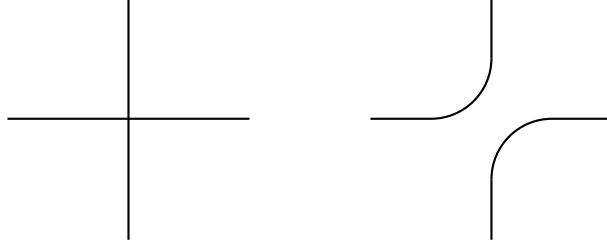


Figure 4.7: deformation rule adopted for associating a set of Peierls' contours to every sign configuration $\zeta \in \{\pm\}^{\Gamma_L^\alpha}$.

Reduction to contour estimates. The main bound for controlling the Peierls'-like contributions is provided by the following lemma.

Lemma 51. *For every contour γ , the following bound holds true:*

$$\left\langle \prod_{\langle x', y' \rangle \in E(\gamma)} I_{x'}^+ I_{y'}^- \right\rangle_{\mu_\alpha} \leq e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|}, \quad (41)$$

with $c_3^\alpha(\lambda) > 0$ defined in [Proposition 50](#).

In this way the estimate can be reduced to a simple contour-counting argument. It is readily checked that the sum over contours γ as in the right-hand side of [Equation \(40\)](#) can be bounded as

$$\sum_{\gamma \text{ around } x} + \sum_{\gamma \text{ around } y} + \sum_{\gamma: |\gamma| \geq L},$$

where “ γ around x ” means that the contour γ is contained in a square patch of the torus Γ_L^α with side L and centered around x . Therefore, assuming the bound [Equation \(41\)](#), we readily obtain that

$$\langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha} \leq 2 \sum_{\gamma \text{ around } x} e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|} + 2 \sum_{\gamma \text{ around } y} e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|} + 2 \sum_{\gamma: |\gamma| \geq L} e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|}. \quad (42)$$

Using the standard contour counting bounds:

$$\#\{\gamma \text{ around } x : |\gamma| = k\} \leq \frac{k}{2} 3^k, \quad \#\{\gamma : |\gamma| = k\} \leq L^2 3^k,$$

we easily find, for N and L sufficiently large:

$$\sum_{\gamma \text{ around } x} e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|} \leq \frac{1}{8} e^{-\frac{1}{4} N c_3^\alpha(\lambda)}, \quad \sum_{\gamma: |\gamma| \geq L} e^{-\frac{1}{4} N c_3^\alpha(\lambda) |\gamma|} \leq \frac{1}{4} e^{-\frac{1}{4} N c_3^\alpha(\lambda)}.$$

Substituting into the right-hand side of [Equation \(42\)](#), we obtain the bound written in [Equation \(26\)](#).

Proof of Lemma 51. We now prove (41) by combining Chessboard Estimates with an energetic analysis of alternating-sign periodic configurations.

1. For any contour γ , its crossed edges can be partitioned into four classes:

$$E(\gamma) = E_{h,e}(\gamma) \sqcup E_{h,o}(\gamma) \sqcup E_{v,e}(\gamma) \sqcup E_{v,o}(\gamma),$$

(here “h”/“v” stands for *horizontal*/*vertical* and “e”/“o” stands for *even*/*odd*) corresponding to the four disjoint tessellations (v,e) , (v,o) , (h,e) , (h,o) of the torus made of 2×1 rectangles

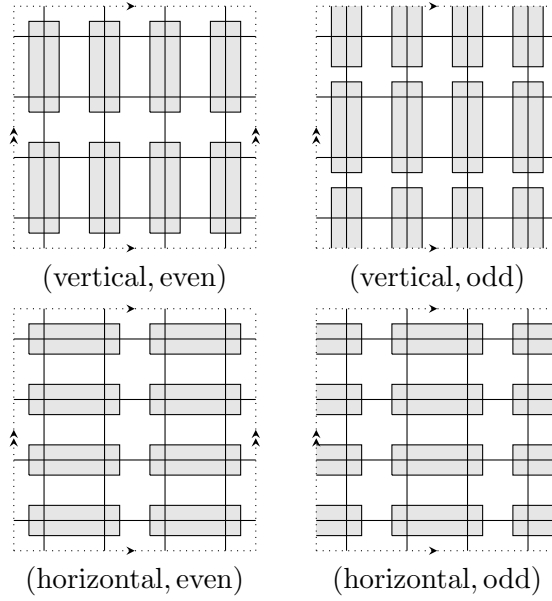


Figure 4.8: 4 Tessellation Patterns Used to Cover Lattices Γ_L^α : notice that using all such covering one is able to recover all edges crossed by the contour γ . (In the Lattice Γ_L^{SP} all spacings are doubled)

as in Figure 4.8.

By Cauchy–Schwartz’s inequality:

$$\left\langle \prod_{\langle x,y \rangle \in E(\gamma)} I_x^+ I_y^- \right\rangle_{\mu_\alpha} = \left\langle \prod_{\substack{a=h,v \\ b=e,o}} \prod_{\langle x,y \rangle \in E_{a,b}(\gamma)} I_x^+ I_y^- \right\rangle_{\mu_\alpha} \leq \prod_{\substack{a=h,v \\ b=e,o}} \left\langle \prod_{\langle x,y \rangle \in E_{a,b}(\gamma)} I_x^+ I_y^- \right\rangle_{\mu_\alpha}^{1/4}. \quad (43)$$

2. Applying the chessboard estimate (CE) to each factor produces, up to translations, one of the four periodic patterns shown in Figure 4.9. Therefore:

$$\left\langle \prod_{\langle x,y \rangle \in E(\gamma)} I_x^+ I_y^- \right\rangle_{\mu_\alpha} \leq \left\langle I^{(-+++)} \right\rangle_{\mu_\alpha}^{\frac{|E_{h,e}(\gamma)|}{2|\Gamma_L^\alpha|}} \left\langle I^{(--++)} \right\rangle_{\mu_\alpha}^{\frac{|E_{h,o}(\gamma)|}{2|\Gamma_L^\alpha|}} \left\langle I^{\left(\begin{smallmatrix} - \\ + \\ + \\ - \end{smallmatrix}\right)} \right\rangle_{\mu_\alpha}^{\frac{|E_{v,e}(\gamma)|}{2|\Gamma_L^\alpha|}} \left\langle I^{\left(\begin{smallmatrix} - \\ - \\ + \\ + \end{smallmatrix}\right)} \right\rangle_{\mu_\alpha}^{\frac{|E_{v,o}(\gamma)|}{2|\Gamma_L^\alpha|}} \quad (44)$$

where (see also Figure 4.9):

$$\begin{aligned} I^{(-+++)} &\doteq \prod_{x \in \Lambda_L^{\alpha,h}} I_x^- I_{x+e_0^\alpha}^+ I_{x+2e_0^\alpha}^+ I_{x+3e_0^\alpha}^-, & I^{(--++)} &\doteq \prod_{x \in \Lambda_L^{\alpha,h}} I_x^- I_{x+e_0^\alpha}^- I_{x+2e_0^\alpha}^+ I_{x+3e_0^\alpha}^+, \\ I^{\left(\begin{smallmatrix} - \\ + \\ + \\ - \end{smallmatrix}\right)} &\doteq \prod_{x \in \Lambda_L^{\alpha,v}} I_x^- I_{x+e_1^\alpha}^+ I_{x+2e_1^\alpha}^+ I_{x+3e_1^\alpha}^-, & I^{\left(\begin{smallmatrix} - \\ - \\ + \\ + \end{smallmatrix}\right)} &\doteq \prod_{x \in \Lambda_L^{\alpha,v}} I_x^- I_{x+e_1^\alpha}^- I_{x+2e_1^\alpha}^+ I_{x+3e_1^\alpha}^+, \end{aligned}$$

with $e_0^\alpha = 2^{\delta_{\alpha,\text{SP}}}(1,0)$, $e_1^\alpha = 2^{\delta_{\alpha,\text{SP}}}(0,1)$, and

$$\begin{aligned} \Lambda_L^{\alpha,h} &= \Gamma_L^\alpha \cap \begin{cases} (4\mathbb{Z}) \times \mathbb{Z}, & \alpha \in \{\text{NN}, \text{SN}\} \\ (8\mathbb{Z}) \times (2\mathbb{Z}), & \alpha = \text{SP} \end{cases} \\ \Lambda_L^{\alpha,v} &= \Gamma_L^\alpha \cap \begin{cases} \mathbb{Z} \times (4\mathbb{Z}), & \alpha \in \{\text{NN}, \text{SN}\} \\ (2\mathbb{Z}) \times (8\mathbb{Z}), & \alpha = \text{SP} \end{cases}. \end{aligned}$$

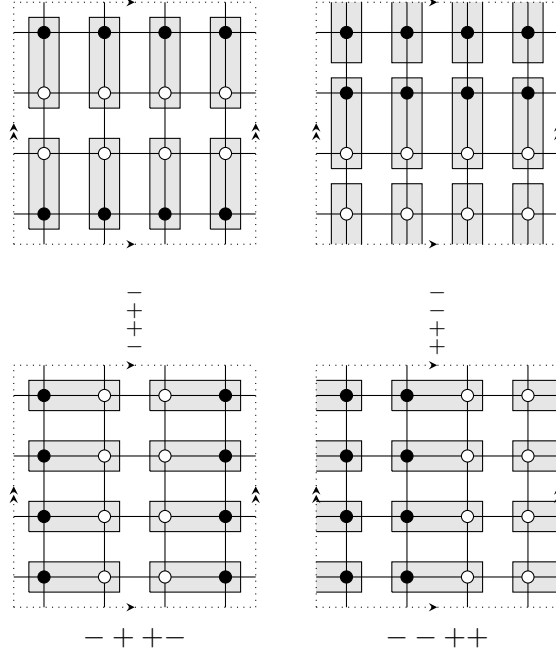


Figure 4.9: Periodic patterns generated by chessboard reflections of the gray boxes representing the covering configurations (h,e), (h,o), (v,e), and (v,o).

3. By translation and rotation invariance of μ_α , the four expectations coincide⁹ and thus, using also that $|\mathbb{E}_{h,e}(\gamma)| + |\mathbb{E}_{h,o}(\gamma)| + |\mathbb{E}_{v,e}(\gamma)| + |\mathbb{E}_{v,o}(\gamma)| = |\gamma|$,

$$\left\langle \prod_{\langle x,y \rangle \in \mathbb{E}(\gamma)} I_x^+ I_y^- \right\rangle_{\mu_\alpha} \leq \left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}}. \quad (45)$$

4. Following the strategy used to bound the numerator in the right-hand side of Equation (28), we obtain:

$$\left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}} \leq (2\pi\lambda)^{\frac{|\gamma|}{4}} \left[\frac{1}{Z_{\Gamma_L^\alpha}^\alpha} \exp \left\{ -N \inf_{\phi \in \text{supp } I^{(-++-)}} V_{L, \tilde{\lambda}_N}^\alpha(\phi) \right\} \right]^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}}, \quad (46)$$

with $\tilde{\lambda}_N = \lambda(1 - \frac{1}{N})^{-1}$.

5. A further RP argument, closely analogous to Proposition 49, shows that the infimum over $\text{supp } I^{(-++-)}$ is attained by periodic configurations of the form:

$$(\varphi_-, \varphi_+, \varphi_+, \varphi_-), \quad \varphi_+ \geq 0, \varphi_- \leq 0,$$

⁹ One can verify that the same conclusion holds for staggered fermions by observing the following facts. Any geometric transformation of the field ϕ is implemented by conjugation with a suitable unitary operator U . Since we are only interested in spectral properties, one may equivalently regard this transformation as acting on the Dirac operator, because

$$\det((\not{D}_\alpha)^- - U M_\phi U^* - \lambda \mathbf{1}) = \det(U(\not{D}_\alpha)^- U^* - M_\phi - \lambda \mathbf{1}).$$

Now, the $\mathbb{Z}_2$ -holonomy background is invariant under translations and rotations, since every plaquette carries flux -1 . Moreover, the spectrum of a magnetic Laplacian, of which the staggered Dirac operator is an example, depends only on the holonomies of the background Gauge field (see [LL93, Lemma 2.1]). In the proof of that result, the unitary conjugation relating two magnetic Laplacians with the same fluxes is constructed explicitly and shown to be a gauge transformation, namely multiplication by site-dependent phases. Such transformations commute with the multiplication operator M_ϕ , and therefore do not affect it.

depending only on *two real parameters* (φ_+, φ_-) . Accordingly:

$$\inf_{\phi \in \text{supp } I^{(-++-)}} V_{L, \tilde{\lambda}_N}^\alpha(\phi) = \inf_{\varphi_+ \geq 0, \varphi_- \leq 0} V_{L, \tilde{\lambda}_N}^\alpha(\varphi_+, \varphi_-) \quad (47)$$

where, by abuse of notation, $V_{L, \tilde{\lambda}_N}^\alpha(\varphi_+, \varphi_-)$ denotes the potential $V_{L, \tilde{\lambda}_N}^\alpha(\varphi_+, \varphi_-)$ evaluated at the periodic configuration corresponding to the repetition of the pattern $(\varphi_-, \varphi_+, \varphi_+, \varphi_-)$.

Moreover, for any $\lambda > 0$ and $\varphi_+, \varphi_- \in \mathbb{R}$, the finite-volume potential density $|\Gamma_L^\alpha|^{-1} V_{L, \lambda}^\alpha(\varphi_+, \varphi_-)$ is well approximated by its infinite-volume limit, to be denoted by $w_\lambda^\alpha(\varphi_+, \varphi_-)$ (in full analogy with Equation (20)):

$$\left| \frac{1}{|\Gamma_L^\alpha|} V_{L, \lambda}^\alpha(\varphi_+, \varphi_-) - w_\lambda^\alpha(\varphi_+, \varphi_-) \right| \leq \frac{\mathfrak{c}_1}{L}. \quad (48)$$

It is possible to write down an explicit expression for w_λ^α :

$$w_\lambda^\alpha(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - N_\alpha \iint_{(-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \log \Delta^\alpha(p; \varphi_+, \varphi_-), \quad (49)$$

where N_α is a multiplicity factor ($N_\alpha = \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$ for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively) and $\Delta^\alpha(p; \varphi_+, \varphi_-)$ is an explicit function (see Equation (51), Equation (54) and Equation (59)), related to the determinant of the Dirac operator in alternating background.

Therefore we have that

$$\left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}} \leq (2\pi\lambda)^{\frac{|\gamma|}{4}} \left[\frac{1}{|\Gamma_L^\alpha|} \exp \left(-N \left(\inf_{\varphi_+ \geq 0, \varphi_- \leq 0} w_{\tilde{\lambda}_N}^\alpha(\varphi_+, \varphi_-) - \frac{\mathfrak{c}_1}{L} \right) \right) \right]^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}}. \quad (50)$$

6. Plugging the lower-bound on the partition function Equation (34) in Equation (50), we get:

$$\left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}} \leq e^{-N \frac{|\gamma|}{2} (w_{\tilde{\lambda}_N}^\alpha(\varphi_+, \varphi_-) - v_{\tilde{\lambda}_N}^\alpha(\varphi_\alpha^*(\tilde{\lambda}_N)) - \mathcal{R}_{N,L}^{(3)}(\lambda))},$$

where

$$\begin{aligned} \mathcal{R}_{N,L}^{(3)}(\lambda) &\doteq (v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) - v_{\tilde{\lambda}_N}^\alpha(\varphi_\alpha^*(\tilde{\lambda}_N))) + \frac{\log(2\pi\lambda)}{2N} + \frac{\delta_N}{\lambda} (2\varphi_\alpha^*(\lambda) + \delta_N) \\ &\quad - \frac{1}{N} \log(2\delta_N) - 2 \log \left(1 - \frac{\delta_N}{\varphi_\alpha^*(\lambda)} \right) + \frac{\mathfrak{c}_0 + \mathfrak{c}_1}{L}. \end{aligned}$$

7. Since φ_α^* is a smooth function of $\lambda \in (0, \infty)$ (see Proposition 49 and $v_\lambda^\alpha(\varphi)$ is smooth with respect to λ and $\varphi \in (0, \infty)$), we easily get that

$$\lim_{N \rightarrow \infty} v_{\tilde{\lambda}_N}^\alpha(\varphi_\alpha^*(\tilde{\lambda}_N)) = v_\lambda^\alpha(\varphi_\alpha^*(\lambda)).$$

Using this information, we can follow a similar reasoning as before: by fixing $\delta_N = \frac{\varphi_\alpha^*(\lambda)}{N}$, we see that for every $\lambda > 0$:

$$\mathcal{R}_{N,L}^{(3)}(\lambda) \xrightarrow{N, L \rightarrow \infty} 0.$$

The crucial input, about the energetic penalty associated with the Peierls-like terms, is provided by the following lemma.

Lemma 52. *There exists $\lambda_0 > 0$ such that, for every $0 < \lambda \leq \lambda_0$,*

$$\inf_{\varphi_+ \geq 0, \varphi_- \leq 0} w_\lambda^\alpha(\varphi_+, \varphi_-) - v_\lambda^\alpha(\varphi_\alpha^*(\lambda)) \geq c_3^\alpha(\lambda) > 0,$$

where $c_3^\alpha(\lambda)$ as in [Proposition 50](#).

Remark 40. As clear from the incoming analysis, the assumption that the coupling constant λ is sufficiently small ensures that the effective potential $w_\lambda^\alpha(\varphi_+, \varphi_-)$ does not develop stationary points in the interior of the region $\{\varphi_+ \geq 0, \varphi_- \leq 0\}$. In fact, a direct inspection of the effective potential suggests that, for sufficiently large values of λ , this condition is violated and nontrivial stationary points appear (whose energy gap is not easily controlled via our method). This behavior is consistent with the expected phase structure of the model, where no spontaneous breaking of Chiral symmetry occurs at strong coupling due to the approximate product structure of the measure [[CER81](#); [CER83](#)].

8. If $\lambda \leq \lambda_0$, for N large enough, $\tilde{\lambda}_N \leq \lambda_0$ as well. Therefore, we can apply [Lemma 52](#) to obtain:

$$\left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}} \leq e^{-N \frac{|\gamma|}{2} (c_3^\alpha(\tilde{\lambda}_N) - \mathcal{R}_{N,L}^{(3)}(\lambda))}.$$

Also, for N, L large enough (depending on λ), we can assume that $|\mathcal{R}_{N,L}^{(3)}(\lambda)| \leq \frac{1}{4}c_3^\alpha(\lambda)$, leading to

$$\left\langle I^{(-++-)} \right\rangle_{\mu_\alpha}^{\frac{|\gamma|}{2|\Gamma_L^\alpha|}} \leq e^{-N \frac{|\gamma|}{2} (c_3^\alpha(\tilde{\lambda}_N) - \frac{1}{4}c_3^\alpha(\lambda))}.$$

9. Furthermore, by the very definition (cf. [Proposition 50](#)), c_3^α is a continuous function of $\lambda > 0$, so that $\lim_{N \rightarrow \infty} c_3^\alpha(\tilde{\lambda}_N) = c_3^\alpha(\lambda)$. Therefore, noting also that $c_3^\alpha(\lambda) > 0$, for N large enough we can assume that

$$|c_3^\alpha(\lambda) - c_3^\alpha(\tilde{\lambda}_N)| \leq \frac{1}{4}c_3^\alpha(\lambda),$$

which yields the bound [Equation \(26\)](#).

4.4.4 Spectral Cost of Alternating Field Configurations

We discuss the proof for the three models, $\alpha \in \{\text{NN}, \text{SN}, \text{SP}\}$, separately. We prefer to start by discussing the SN model, which involves slightly simpler computations than the other two models.

Proof for the SN model. Using the periodicity of the alternating field configurations ϕ , which depend only on the two independent variables φ_+ and φ_- (see [Figure 4.10](#)), one readily obtains the following expression for the potential $V_{L,\lambda}^{\text{SN}}(\varphi_+, \varphi_-)$ introduced after [Equation \(47\)](#):

$$V_{L,\lambda}^{\text{SN}}(\varphi_+, \varphi_-) = \frac{|\Gamma_L|}{2} \frac{\varphi_+^2 + \varphi_-^2}{2\lambda} - \sum_{p \in \Xi_L^*} \log \Delta^{\text{SN}}(p; \varphi_+, \varphi_-) \quad \Xi_L^* = \left(\frac{2\pi}{L} (\mathbb{Z} + \frac{1}{2})^2 \right) \cap \left(\left(-\frac{\pi}{4}, \frac{\pi}{4} \right] \times (-\pi, \pi] \right)$$

where:

$$\Delta^{\text{SN}}(p; \varphi_+, \varphi_-) = \det \begin{pmatrix} i \sin(p_1) - \varphi_- & -\frac{1}{2} & 0 & \frac{1}{2} e^{4ip_0} \\ \frac{1}{2} & -i \sin(p_1) - \varphi_+ & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & i \sin(p_1) - \varphi_+ & -\frac{1}{2} \\ -\frac{1}{2} e^{-4ip_0} & 0 & \frac{1}{2} & -i \sin(p_1) - \varphi_- \end{pmatrix}.$$

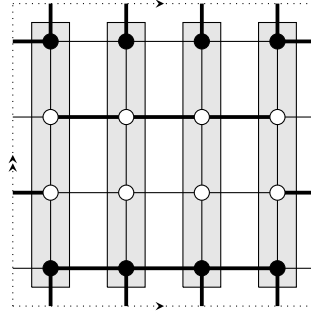


Figure 4.10: Graphical Representation of the Dirac Operator with periodic pattern produced by Chessboard's Estimates. In Grey we have drawn the unit cells used to perform Fourier Transforms

It follows that

$$w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) = \lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L|} V_{\lambda}^L(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\pi, \pi]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SN}}(p; \varphi_+, \varphi_-)$$

where:

$$\Delta^{\text{SN}}(p; \varphi_+, \varphi_-) = \frac{1}{4} \sin^2(2p_0) + \frac{1}{4}(\varphi_+ + \varphi_-)^2 + (\varphi_+ \varphi_-)^2 + \sin^2(p_1)(1 + \varphi_+^2 + \varphi_-^2 + \sin^2(p_1)). \quad (51)$$

Note that due to the periodicity of $\Delta^{\text{SN}}(\cdot; \varphi_+, \varphi_-)$ under translations of $\frac{\pi}{2}\mathbb{Z} \times \pi\mathbb{Z}$, we can rewrite w_{λ}^{SN} in Equation (49) as:

$$w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SN}}(p; \varphi_+, \varphi_-),$$

so that within the latter domain of integration, $\Delta^{\text{SN}}(\cdot; \varphi_+, \varphi_-)$ is singular only at $(p, \varphi_+, \varphi_-) = (0, 0, 0)$. We are interested in finding the minimizers of w_{λ}^{SN} and understanding their properties. By computing derivatives, we get:

$$\begin{aligned} \partial_{\varphi_+} w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) &= \frac{\varphi_+}{2\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{\frac{1}{2}(\varphi_+ + \varphi_-) + 2\varphi_+ \varphi_-^2 + 2 \sin^2(p_1) \varphi_+}{\Delta^{\text{SN}}(p; \varphi_+, \varphi_-)}, \\ \partial_{\varphi_-} w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) &= \frac{\varphi_-}{2\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{\frac{1}{2}(\varphi_+ + \varphi_-) + 2\varphi_- \varphi_+^2 + 2 \sin^2(p_1) \varphi_-}{\Delta^{\text{SN}}(p; \varphi_+, \varphi_-)}. \end{aligned}$$

Since we are looking for the minimizer over the constrained region $\{\varphi_+ \geq 0, \varphi_- \leq 0\}$, looking at the zeros of ∇w_{λ} is not enough. However, we observe the following:

$$(1, -1) \cdot \nabla w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) = (\varphi_+ - \varphi_-) \left(\frac{1}{2\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{-2\varphi_+ \varphi_- + 2 \sin^2(p_1)}{\Delta^{\text{SN}}(p; \varphi_+, \varphi_-)} \right). \quad (52)$$

The expression in brace can be shown to be positive for λ small enough, say $\lambda \leq \lambda_0$ with λ_0 a suitable constant (recall also the discussion in Remark 40). Indeed, since¹⁰

$$\Delta(p; \varphi_+, \varphi_-) \gtrsim m_{\text{SN}}(\varphi_+, \varphi_-)^2 + |p|^2 \quad \forall p \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

¹⁰The notation $a \gtrsim b$ means that $a \geq \mathfrak{C}b$, for some universal constant $\mathfrak{C} > 0$.

with $m_{\text{SN}}(\varphi_+, \varphi_-) = |\varphi_+ \varphi_-|$, it follows that

$$\begin{aligned} \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \left| \frac{-2\varphi_+ \varphi_- + 2\sin^2(p_1)}{\Delta^{\text{SN}}(p; \phi_+, \phi_-)} \right| &\lesssim \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{|p| + m_{\text{SN}}(\varphi_+, \varphi_-)}{|p|^2 + m_{\text{SN}}(\varphi_+, \varphi_-)^2} \\ &\lesssim \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{1}{\sqrt{|p|^2 + m_{\text{SN}}(\varphi_+, \varphi_-)^2}} \\ &\lesssim 1. \end{aligned}$$

As a consequence, for λ small enough, the right-hand side of Equation (52) is always non-negative on the domain of our interest $\{\varphi_+ \geq 0, \varphi_- \leq 0\}$, so that $w_\lambda(\varphi_+, \varphi_-)$ is minimized on the boundary of the region $\{\varphi_+ \geq 0, \varphi_- \leq 0\}$. Furthermore, by $\mathbb{Z}_2$ -invariance of $w_\lambda(\varphi_+, \varphi_-)$, it suffices to look for the minima on the set $\{\varphi_+ \geq 0, \varphi_- = 0\}$. In order to proceed, we note the following.

1. By construction:

$$w_\lambda^{\text{SN}}(\varphi, \varphi) = \lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L|} V_{L, \lambda}^{\text{SN}}(\phi) \Big|_{\phi = \varphi^{\Gamma_L^\alpha}} \equiv v_\lambda^{\text{SN}}(\varphi)$$

with v_λ^{SN} minimized by $\varphi = \varphi_{\text{SN}}^*(\lambda)$ (see Proposition 49).

2. Explicitly, we have that

$$\begin{aligned} \Delta^{\text{SN}}(p; \varphi, \varphi) &= \frac{1}{4} \sin^2(2p_0) + \varphi^2 + \varphi^4 + \sin^2(p_1)(1 + 2\varphi^2 + \sin^2(p_1)), \\ \Delta^{\text{SN}}(p; \varphi, 0) &= \frac{1}{4} \sin^2(2p_0) + \frac{1}{4} \varphi^2 + \sin^2(p_1)(1 + \varphi^2 + \sin^2(p_1)), \end{aligned}$$

from which we see that:

$$\Delta^{\text{SN}}(p; \varphi, 0) = \Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) - \frac{1}{4}\varphi^2 - \frac{1}{4}\varphi^4.$$

Therefore, we have that

$$\begin{aligned} w_\lambda^{\text{SN}}(\varphi, 0) &= \frac{\varphi^2}{4\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SN}}(p; \varphi, 0) \\ &= \frac{\varphi^2}{4\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(\Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) - \frac{1}{4}\varphi^2(1 + \varphi^2) \right) \\ &= \frac{\varphi^2}{4\lambda} - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) \\ &\quad - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \frac{\frac{1}{4}\varphi^2(1 + \varphi^2)}{\Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)} \right) \\ &= v_\lambda^{\text{SN}}(\frac{1}{\sqrt{2}}\varphi) - 2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \frac{\frac{1}{4}\varphi^2(1 + \varphi^2)}{\Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)} \right) \\ &\doteq v_\lambda^{\text{SN}}(\frac{1}{\sqrt{2}}\varphi) + \delta w^{\text{SN}}(\varphi). \end{aligned}$$

Observe that:

$$\Delta^{\text{SN}}(p; \varphi, \varphi) \lesssim 1 + \varphi^2 + \varphi^4 \leq (1 + \varphi^2)^2,$$

therefore:

$$\begin{aligned}
\delta w^{\text{SN}}(\varphi) &= -2 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \frac{\frac{1}{4}\varphi^2(1+\varphi^2)}{\Delta^{\text{SN}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)} \right) \\
&\geq -\frac{1}{4} \log \left(1 - \mathfrak{C}_2 \frac{\varphi^2(1+\varphi^2)}{(1+\varphi^2)^2} \right) \\
&= -\frac{1}{4} \log \left(1 - \mathfrak{C}_2 \frac{\varphi^2}{1+\varphi^2} \right),
\end{aligned} \tag{53}$$

for a suitable constant $\mathfrak{C}_2^{\text{SN}} > 0$. Note that the right-hand side of Equation (53) is positive and monotone increasing for $\varphi > 0$. We deduce the following:

- if $\varphi \in [0, \varphi_{\text{SN}}^*(\lambda)]$, we have that:

$$w_{\lambda}^{\text{SN}}(\varphi, 0) \geq v_{\lambda}^{\text{SN}}\left(\frac{1}{\sqrt{2}}\varphi\right) \geq v_{\lambda}^{\text{SN}}\left(\frac{1}{\sqrt{2}}\varphi_{\text{SN}}^*(\lambda)\right)$$

(recall Proposition 49 for the form of v_{λ}^{SN});

- if $\varphi \in [\varphi_{\text{SN}}^*(\lambda), \infty)$, then

$$w_{\lambda}^{\text{SN}}(\varphi, 0) \geq -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^{\text{SN}} \frac{\varphi^2}{1+\varphi^2} \right) + v_{\lambda}^{\text{SN}}\left(\frac{1}{\sqrt{2}}\varphi\right) \geq -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^{\text{SN}} \frac{(\varphi_{\text{SN}}^*(\lambda))^2}{1+(\varphi_{\text{SN}}^*(\lambda))^2} \right) + v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)).$$

Since:

$$\begin{aligned}
\min_{\varphi \geq 0} w_{\lambda}^{\text{SN}}(\varphi, 0) - v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)) &\geq \min \left(v_{\lambda}^{\text{SN}}\left(\frac{1}{\sqrt{2}}\varphi_{\text{SN}}^*(\lambda)\right) - v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)), -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^{\text{SN}} \frac{(\varphi_{\text{SN}}^*(\lambda))^2}{1+(\varphi_{\text{SN}}^*(\lambda))^2} \right) \right) \\
&\equiv c_3^{\text{SN}}(\lambda)
\end{aligned}$$

and

$$\min_{\varphi_+ \geq 0, \varphi_- \leq 0} w_{\lambda}^{\text{SN}}(\varphi_+, \varphi_-) - v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)) = \min_{\varphi \geq 0} w_{\lambda}^{\text{SN}}(\varphi, 0) - v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)),$$

we can conclude that:

$$\min_{\varphi \geq 0} w_{\lambda}^{\text{SN}}(\varphi, 0) - v_{\lambda}^{\text{SN}}(\varphi_{\text{SN}}^*(\lambda)) \geq c_3^{\text{SN}}(\lambda) > 0.$$

Proof for the NN model. Proceeding in analogy with the case of the SN model (see Figure 4.11 for a graphical understanding of the field configuration), we find that

$$w_{\lambda}^{\text{NN}}(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\pi, \pi]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{NN}}(p; \varphi_+, \varphi_-), \tag{54}$$

where:

$$\Delta^{\text{NN}}(p; \varphi_+, \varphi_-) = \frac{1}{64} [8 + 4\varphi_+\varphi_- + 6\varphi_+^2 + 6\varphi_-^2 + 8\varphi_+^2\varphi_-^2 - \cos(4p_0) - 4(2 + \varphi_+^2 + \varphi_-^2) \cos(2p_1) + \cos(4p_1)]^2.$$

Note that due to the periodicity of $\Delta^{\text{NN}}(\cdot; \varphi_+, \varphi_-)$ under translations, we can write:

$$w_{\lambda}^{\text{NN}}(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - 4 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \sqrt{\Delta^{\text{NN}}}(p; \varphi_+, \varphi_-),$$

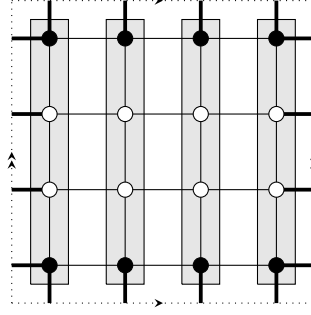


Figure 4.11: Graphical Representation of the Naive Dirac Operator with periodic pattern produced by Chessboard's Estimates. In Grey we have drawn the unit cells used to perform Fourier Transforms

so that, on the latter domain, the integrand has only one singularity at $p = 0$, if $\varphi_+ = \varphi_- = 0$. Some basic but cumbersome algebraic manipulations lead to the following rewriting:

$$\sqrt{\Delta^{\text{NN}}}(p; \varphi_+, \varphi_-) = \frac{1}{4}(\varphi_+ + \varphi_-)^2 + \varphi_+^2 \varphi_-^2 + \frac{1}{4} \sin^2(2p_0) + (1 + \varphi_+^2 + \varphi_-^2 + \sin^2(p_1)) \sin^2(p_1). \quad (55)$$

We are going to adapt the same arguments used in the proof for the SN model. As a first step, we shall show that $\partial_{\varphi_+} w_{\lambda}^{\text{NN}}(\varphi_+, \varphi_-) - \partial_{\varphi_-} w_{\lambda}^{\text{NN}}(\varphi_+, \varphi_-) \geq 0$, implying that the minimum of w_{λ}^{NN} is attained on the set $\{\varphi_+ = 0\} \cup \{\varphi_- = 0\}$. An explicit computation shows that:

$$(1, -1) \cdot \nabla w_{\lambda}^{\text{NN}} = (\varphi_+ - \varphi_-) \left(\frac{1}{2\lambda} - 8 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{-\varphi_+ \varphi_- + \sin^2 p_1}{\sqrt{\Delta^{\text{NN}}}(p; \varphi_+, \varphi_-)} \right). \quad (56)$$

Since in the integration region, we have that:

$$\sqrt{\Delta^{\text{NN}}}(p; \varphi_+, \varphi_-) \gtrsim |p|^2 + m_{\text{NN}}(\varphi_+, \varphi_-)^2 \text{ with } m_{\text{NN}}(\varphi_+, \varphi_-) \doteq \sqrt{(\varphi_+ + \varphi_-)^2 + \varphi_+^2 \varphi_-^2} \geq |\varphi_+ \varphi_-|,$$

it follows that the integral in the right-hand side of Equation (56) is uniformly bounded:

$$\begin{aligned} \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \left| \frac{-\varphi_+ \varphi_- + \sin^2 p_1}{\sqrt{\Delta^{\text{NN}}}(p; \varphi_+, \varphi_-)} \right| &\lesssim \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{m_{\text{NN}}(\varphi_+, \varphi_-) + |p|^2}{m_{\text{NN}}(\varphi_+, \varphi_-)^2 + |p|^2} \\ &\lesssim \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \left(\frac{1}{\sqrt{|p|^2 + m_{\text{NN}}(\varphi_+, \varphi_-)^2}} + 1 \right) \\ &\lesssim 1. \end{aligned}$$

With a similar reasoning as before, due to Equation (56) and symmetry considerations, we can conclude that for λ small enough:

$$\min_{\varphi_+ \geq 0, \varphi_- \leq 0} w_{\lambda}^{\text{NN}}(\varphi_+, \varphi_-) = \min_{\varphi \geq 0} w_{\lambda}^{\text{NN}}(\varphi, 0).$$

Following the strategy used for the SN model, we compare $w_{\lambda}^{\text{NN}}(\varphi, 0)$ with $w_{\lambda}^{\text{NN}}(\frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)$. From Equation (55), it is straightforward to check that

$$\sqrt{\Delta^{\text{NN}}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) = \sqrt{\Delta^{\text{NN}}}(p; \varphi, 0) + \frac{1}{4}(\varphi^2 + \varphi^4).$$

This allows us to rewrite:

$$\begin{aligned}
w_\lambda^{\text{NN}}(\varphi, 0) &= \frac{\varphi^2}{4\lambda} - 4 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \sqrt{\Delta^{\text{NN}}}(p; \varphi, 0) \\
&= w_\lambda^{\text{NN}}\left(\frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi\right) - 4 \iint \frac{d^2 p}{(2\pi)^2} \log \left(1 - \frac{\varphi^2 + \varphi^4}{4\sqrt{\Delta^{\text{NN}}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)}\right) \\
&\doteq v_\lambda\left(\frac{1}{\sqrt{2}}\varphi\right) + \delta w^{\text{NN}}(\varphi).
\end{aligned}$$

Noting that

$$\sqrt{\Delta^{\text{NN}}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) \lesssim 1 + \varphi^4,$$

it is easy to get the following lower bound for δw^{NN} :

$$\delta w^{\text{NN}}(\varphi) \geq -4 \iint_{(-\frac{\pi}{4}, \frac{\pi}{4}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \mathfrak{C}_2^{\text{NN}} \frac{\varphi^4}{1 + \varphi^4}\right) = -\frac{1}{2} \log \left(1 - \mathfrak{C}_2^{\text{NN}} \frac{\varphi^4}{1 + \varphi^4}\right), \quad (57)$$

for a suitable constant $\mathfrak{C}_2^{\text{NN}} > 0$. Note that the right-hand side of Equation (57) is positive and monotone increasing for $\varphi > 0$. All in all:

$$w_\lambda^{\text{NN}}(\varphi, 0) \geq v_\lambda^{\text{NN}}\left(\frac{1}{\sqrt{2}}\varphi\right) - \frac{1}{2} \log \left(1 - \mathfrak{C}_2^{\text{NN}} \frac{\varphi^4}{1 + \varphi^4}\right).$$

By replicating the same reasoning presented at the end of the previous paragraph, we find that

$$\min_{\varphi_+ \geq 0, \varphi_- \leq 0} w_\lambda^{\text{NN}}(\varphi_+, \varphi_-) - v_\lambda^{\text{NN}}(\varphi_{\text{NN}}^*(\lambda)) \geq c_3^{\text{NN}}(\lambda).$$

with:

$$c_3^{\text{NN}}(\lambda) \doteq \min \left(v_\lambda^{\text{NN}}\left(\frac{1}{\sqrt{2}}\varphi_{\text{NN}}^*(\lambda)\right) - v_\lambda^{\text{NN}}(\varphi_{\text{NN}}^*(\lambda)), -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^{\text{NN}} \frac{(\varphi_{\text{NN}}^*(\lambda))^4}{1 + (\varphi_{\text{NN}}^*(\lambda))^4}\right) \right)$$

Proof for the SP model. Again we follow the same ideas used for the SN model (see Figure 4.12 for a graphical understanding of the field configuration). An explicit computation leads to

$$w_\lambda^{\text{SP}}(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - 4 \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\pi, \pi]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SP}}(p; \varphi_+, \varphi_-), \quad (58)$$

where

$$\begin{aligned}
\Delta^{\text{SP}}(p; \varphi_+, \varphi_-) &= \\
&\frac{1}{256} \left\{ 16[\varphi_+^2 + \varphi_+^4 + 2\varphi_-(\varphi_+ + 2\varphi_+^3) + 4\varphi_-^3(\varphi_+ + 2\varphi_+^3) + \varphi_-^2(1 + 10\varphi_+^2 + 12\varphi_+^4) + \right. \\
&\quad \left. + \varphi_-^4(1 + 12\varphi_+^2 + 16\varphi_+^4)] + 4\sin^2(4p_0) + 64\sin^2 p_1(1 + \varphi_-^2 + \varphi_+^2 + \sin^2 p_1) \cdot \right. \\
&\quad \left. \cdot [1 + 2\varphi_-\varphi_+ + 3\varphi_+^2 + \varphi_-^2(3 + 8\varphi_+^2) + 4\sin^2 p_1(1 + \varphi_-^2 + \varphi_+^2 + \sin^2 p_1)] \right\}.
\end{aligned} \quad (59)$$

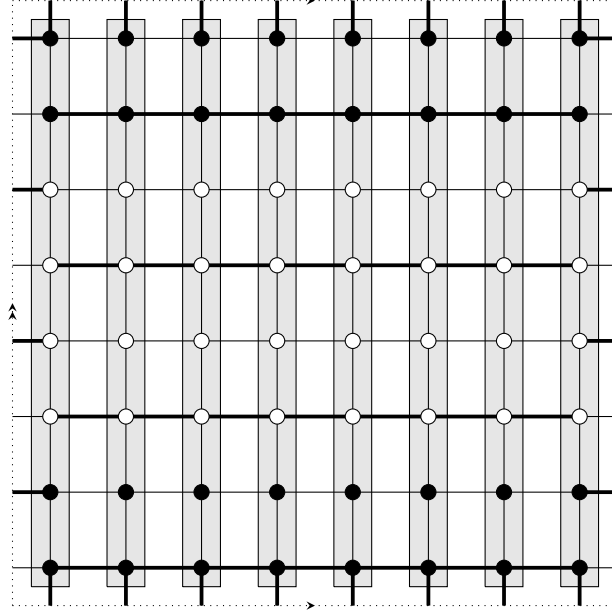


Figure 4.12: Graphical Representation of the Staggered Dirac Operator with Plaquette Interaction with periodic pattern produced by Chessboard Estimates. In Grey we have drawn the unit cells used to perform Fourier Transforms.

Note that due to the periodicity of $\Delta^{\text{SP}}(\cdot; \varphi_+, \varphi_-)$ under translations of $\frac{\pi}{4}\mathbb{Z} \times \pi\mathbb{Z}$, we can either rewrite w_λ^{SP} in the form of Equation (49), or alternatively

$$w_\lambda^{\text{SP}}(\varphi_+, \varphi_-) = \frac{\varphi_+^2 + \varphi_-^2}{4\lambda} - 8 \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \Delta^{\text{SP}}(p; \varphi_+, \varphi_-). \quad (60)$$

In this way, if $\varphi_+ = \varphi_- = 0$, the integrand has only one singularity at $p = 0$. Trivial though cumbersome algebraic manipulations lead to

$$\begin{aligned} \Delta^{\text{SP}}(p; \varphi_+, \varphi_-) = & \frac{1}{16} [16\varphi_+^4 \varphi_-^4 + 8\varphi_-^2 \varphi_+^2 (\varphi_-^2 + \varphi_+^2) + (\varphi_+^2 + \varphi_-^2)^2 + (1 + 2\varphi_- \varphi_+)^2 (\varphi_- + \varphi_+)^2] + \\ & + \frac{1}{64} \sin^2(4p_0) + \frac{1}{4} \sin^2 p_1 (1 + \varphi_-^2 + \varphi_+^2 + \sin^2 p_1) \cdot \\ & \cdot [1 + 2\varphi_- \varphi_+ + 3\varphi_+^2 + \varphi_-^2 (3 + 8\varphi_+^2) + 4 \sin^2 p_1 (1 + \varphi_-^2 + \varphi_+^2 + \sin^2 p_1)]. \end{aligned} \quad (61)$$

Notice that since $2\varphi_- \varphi_+ + 3\varphi_+^2 + \varphi_-^2 (3 + 8\varphi_+^2) \geq 2\varphi_- \varphi_+ + \varphi_+^2 + \varphi_-^2 \geq 0$, we have that

$$\Delta^{\text{SP}}(p; \varphi_+, \varphi_-) \gtrsim m_{\text{SP}}(\varphi_+, \varphi_-)^2 + |p|^2 \quad \forall p \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad (62)$$

where we have introduced

$$m_{\text{SP}}(\varphi_+, \varphi_-) \doteq \sqrt{\varphi_+^4 \varphi_-^4 + \varphi_-^2 \varphi_+^2 (\varphi_-^2 + \varphi_+^2) + (\varphi_+^2 + \varphi_-^2)^2}.$$

Once again our goal is to replicate the reasoning from the analysis of the SN model. An explicit computation shows that

$$(1, -1) \cdot \nabla w_\lambda^{\text{SP}}(\varphi_+, \varphi_-) = (\varphi_+ - \varphi_-) \left(\frac{1}{2\lambda} - 8 \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{\Gamma^{\text{SP}}(p; \varphi_+, \varphi_-)}{\Delta^{\text{SP}}(p; \varphi_+, \varphi_-)} \right), \quad (63)$$

where

$$\begin{aligned} \Gamma^{\text{SP}}(p; \varphi_+, \varphi_-) = & \frac{1}{2} \left(-\varphi_+ \varphi_- [1 + 3\varphi_+^2 + \varphi_-^2 (3 + 8\varphi_+^2)] + \right. \\ & + \sin^2(p_1) [3 - 8\varphi_-^3 \varphi_+ + 5\varphi_+^2 + \varphi_-^2 (5 + 8\varphi_+^2) - \varphi_+ \varphi_- (6 + 8\varphi_+^2)] \\ & \left. + 2\sin^2(p_1) [5 + 4(\varphi_-^2 + \varphi_+^2 - \varphi_- \varphi_+) + 4\sin^2(p_1)] \right). \end{aligned}$$

Observe that $|\Gamma^{\text{SP}}(p; \varphi_+, \varphi_-)| \lesssim m_{\text{SP}}(\varphi_+, \varphi_-) + m_{\text{SP}}(\varphi_+, \varphi_-)^2 + |p|^2$; therefore the integral in the right-hand side of Equation (63) is uniformly bounded:

$$\begin{aligned} \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \left| \frac{\Gamma^{\text{SP}}(p; \varphi_+, \varphi_-)}{\Delta^{\text{SP}}(p; \varphi_+, \varphi_-)} \right| & \lesssim \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \frac{m_{\text{SP}}(\varphi_+, \varphi_-) + m_{\text{SP}}(\varphi_+, \varphi_-)^2 + |p|^2}{|p|^2 + m_{\text{SP}}(\varphi_+, \varphi_-)^2} \\ & \lesssim 1. \end{aligned}$$

As an immediate consequence of Equation (63), we have that for λ small enough, $(1, -1) \cdot \nabla w_\lambda^{\text{SP}}(\varphi_+, \varphi_-) \geq 0$ on $\{\varphi_+ \geq 0, \varphi_- \leq 0\}$. Using also the symmetry $w_\lambda(\varphi_+, \varphi_-) = w_\lambda(-\varphi_-, -\varphi_+)$, it follows that

$$\min_{\varphi_+ \geq 0, \varphi_- \leq 0} w_\lambda^{\text{SP}}(\varphi_+, \varphi_-) = \min_{\varphi \geq 0} w_\lambda^{\text{SP}}(\varphi, 0). \quad (64)$$

Again we aim to compare $w_\lambda^{\text{SP}}(\varphi, 0)$ with $w_\lambda^{\text{SP}}(\frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)$, keeping in mind that $w_\lambda^{\text{SP}}(\varphi, \varphi) = v_\lambda^{\text{SP}}(\varphi)$. From Equation (61) we have that

$$\begin{aligned} \Delta^{\text{SP}}(p; \varphi, 0) = & \frac{1}{16} [\varphi^2 + \varphi^4] + \frac{1}{64} \sin^2(4p_0) + \frac{1}{4} \sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1) \\ & + \frac{1}{4} \sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1) [3\varphi^2 + 4\sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1)], \end{aligned}$$

while

$$\begin{aligned} \Delta^{\text{SP}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) = & \frac{1}{16} [2\varphi^2 + 5\varphi^4 + 4\varphi^6 + \varphi^8] + \frac{1}{64} \sin^2(4p_0) + \\ & + \frac{1}{4} \sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1) \cdot [1 + 4\varphi^2 + 2\varphi^4 + 4\sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1)]. \end{aligned}$$

Therefore we see that $\Delta^{\text{SP}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) = \Delta^{\text{SP}}(p; \varphi, 0) + \tilde{\Delta}^{\text{SP}}(p; \varphi)$, where

$$\tilde{\Delta}^{\text{SP}}(p; \varphi) = \frac{1}{16} (\varphi^2 + 4\varphi^4 + 4\varphi^6 + \varphi^8) + \frac{1}{4} \sin^2 p_1 (1 + \varphi^2 + \sin^2 p_1) (\varphi^2 + 2\varphi^4). \quad (65)$$

This allows us to rewrite

$$\begin{aligned} w_\lambda^{\text{SP}}(\varphi, 0) = & v_\lambda^{\text{SP}}(\frac{1}{\sqrt{2}}\varphi) - 8 \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \frac{\tilde{\Delta}^{\text{SP}}(p; \varphi)}{\Delta^{\text{SP}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi)} \right) \\ = & v_\lambda^{\text{SP}}(\frac{1}{\sqrt{2}}\varphi) + \delta w^{\text{SP}}(\varphi). \end{aligned}$$

In order to get a lower bound for $\delta w^{\text{SP}}(\varphi)$, we observe that:

$$\begin{aligned}\tilde{\Delta}^{\text{SP}}(p; \varphi) &\gtrsim \frac{1}{16}\varphi^8 \\ \Delta^{\text{SP}}(p; \frac{1}{\sqrt{2}}\varphi, \frac{1}{\sqrt{2}}\varphi) &\lesssim 1 + \varphi^8\end{aligned}$$

Therefore

$$\delta w^{\text{SP}}(\varphi) \geq -8 \iint_{(-\frac{\pi}{8}, \frac{\pi}{8}] \times (-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{d^2 p}{(2\pi)^2} \log \left(1 - \mathfrak{C}_2^{\text{SP}} \frac{\varphi^8}{1 + \varphi^8} \right) = -\frac{1}{2} \log \left(1 - \mathfrak{C}_2^{\text{SP}} \frac{\varphi^8}{1 + \varphi^8} \right),$$

for a suitable $\mathfrak{C}_2^{\text{SP}} > 0$. Note that the right-hand side is positive and monotone increasing for $\varphi > 0$. All in all:

$$w_{\lambda}^{\text{SP}}(\varphi, 0) \geq v_{\lambda}^{\text{SP}}(\frac{1}{\sqrt{2}}\varphi) - \frac{1}{2} \log \left(1 - \mathfrak{C}_2^{\text{SP}} \frac{\varphi^8}{1 + \varphi^8} \right).$$

Finally, the same argument applied to the SN and NN models yields

$$\min_{\varphi_+ \geq 0, \varphi_- \leq 0} w_{\lambda}^{\text{SP}}(\varphi_+, \varphi_-) - v_{\lambda}^{\text{SP}}(\varphi_{\text{SP}}^*(\lambda)) \geq c_3^{\text{SP}}(\lambda).$$

where we have set:

$$c_3^{\text{SP}}(\lambda) \doteq \min \left(v_{\lambda}^{\text{SP}}(\frac{1}{\sqrt{2}}\varphi_{\text{SP}}^*(\lambda)) - v_{\lambda}^{\text{SP}}(\varphi_{\text{SP}}^*(\lambda)), -\frac{1}{4} \log \left(1 - \mathfrak{C}_2^{\text{SP}} \frac{(\varphi_{\text{SP}}^*(\lambda))^8}{1 + (\varphi_{\text{SP}}^*(\lambda))^8} \right) \right)$$

4.A Algebraic Properties of the Lattice Dirac Operators

In this appendix, we collect and prove several technical results concerning the Dirac operators associated with the three lattice fermion models used in the main body of the chapter. These results are repeatedly employed in the proof of Reflection Positivity for the corresponding measures $d\mu_{\alpha}(\phi)$, but are of a purely operator-theoretic nature and can therefore be isolated from the main exposition.

For each model, we analyze the algebraic and spectral properties of the associated Dirac operators, with particular emphasis on adjointness relations, symmetry under reflections, and the reality and invariance of the fermionic determinant.

In order to avoid cluttering the notation, in [Subsection 4.A.1](#) and [Subsection 4.A.2](#) we will use the symbol $\mathbb{D}$ to denote $\mathbb{D}_{\text{NN}}$ and $\mathbb{D}_{\text{SN}}$, respectively. On the other hand, we will need to distinguish between reflections on the full plane and reflections on the torus (cf. [Definition 43](#)), which will be denoted by ϑ_{ν} and ϑ_{ν}^L , respectively.

4.A.1 Naive Dirac Operator

Proof of [Lemma 39](#).

(PND1): **Anti-self-adjointness.** Let us consider first the Infinite Plane Naive Dirac Operator $\mathbb{D}$:

$$\mathbb{D} = \frac{1}{2} \sum_{\mu=0}^1 \gamma_{\mu} \otimes (T_{\mu} - T_{\mu}^*), \quad \mathbb{D}^* = -\frac{1}{2} \sum_{\mu=0}^1 \gamma_{\mu}^{\dagger} \otimes (T_{\mu} - T_{\mu}^*) = -\mathbb{D};$$

thus:

$$\begin{aligned}
\overline{(\not{D}_{\Gamma_L}^-)_{(y,s'),(x,s)}} &= \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \overline{(\not{D})_{(y+nL,s'),(x,s)}} \\
&= \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \overline{(\not{D})_{(y,s'),(x-nL,s)}} \\
&= - \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} (\not{D})_{(x-nL,s),(y,s')} = -(\not{D}_{\Gamma_L}^-)_{(x,s),(y,s')}.
\end{aligned} \tag{66}$$

(PND2): **Chirality under adjoint.** Since $\gamma_A \gamma_\mu \gamma_A = -\gamma_\mu$ and $\gamma_A^2 = \mathbb{1}_2$, we have that

$$(\gamma_A \otimes 1) \not{D}_{\Gamma_L}^- (\gamma_A \otimes 1) = -\not{D}_{\Gamma_L}^-, \quad (\gamma_A \otimes 1) M_\phi (\gamma_A \otimes 1) = M_\phi,$$

so that

$$(\not{D}_{\Gamma_L}^- - M_\phi)^* = (\gamma_A \otimes 1) (\not{D}_{\Gamma_L}^- - M_\phi) (\gamma_A \otimes 1).$$

(PND3): **Spectrum of the adjoint.** Since $\gamma_A^2 = \mathbb{1}_2$ and $\gamma_A^\dagger = \gamma_A$, γ_A is a unitary matrix. Thus, Equation (PND2) implies that $(\not{D}_{\Gamma_L}^- - M_\phi)^*$ and $\not{D}_{\Gamma_L}^- - M_\phi$ are unitarily conjugated through $\gamma_A \otimes 1$ and thus have the same spectrum.

(PND4): **Reality of the determinant.** By applying the determinant to Equation (PND2), we get:

$$\det(\overline{\not{D}_{\Gamma_L}^- - M_\phi}) = \det((\not{D}_{\Gamma_L}^- - M_\phi)^*) \stackrel{(PND2)}{=} \det(\not{D}_{\Gamma_L}^- - M_\phi).$$

■

We now analyze the properties of the Dirac operator under reflections. To this purpose we will need the notion of the *Reflection Operator*.

Definition 53 (Reflection Operator). Given a cut plane (ν, r) on Γ_L , and letting ϑ_ν and ϑ_ν^L be its related reflections on the full plane and on the torus respectively (cf. Definition 42 and Definition 43), we define the linear operators $\hat{\vartheta}_\nu : \ell^2(\mathbb{Z}^2) \mapsto \ell^2(\mathbb{Z}^2)$ and $\hat{\vartheta}_\nu^L : \ell^2(\Gamma_L) \mapsto \ell^2(\Gamma_L)$ as:

$$\begin{aligned}
(\hat{\vartheta}_\nu f)_x &\doteq f_{\vartheta_\nu(x)}, & \forall f \in \ell^2(\mathbb{Z}^2), \\
(\hat{\vartheta}_\nu^L f)_x &\doteq \underbrace{(-1)^{\sum_{\mu=0,1} \frac{1}{L} (\vartheta_\nu(x) - \vartheta_\nu^L(x))_\mu}}_{\doteq \sigma_\nu^L(x)} f_{\vartheta_\nu^L(x)}, & \forall f \in \ell^2(\Gamma_L),
\end{aligned}$$

where note that $\vartheta_\nu(x) - \vartheta_\nu^L(x) \in L\mathbb{Z}^2$.

Lemma 54 (Properties under the Reflection Operator). *The following identities hold, for every reflection ϑ_ν on Γ_L .*

1. **Reflection Covariance of the Dirac Operator:**

$$(\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L) (\not{D}_{\Gamma_L}^-) (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L) = ((\gamma_\nu \otimes 1) (\not{D}_{\Gamma_L}^-) (\gamma_\nu \otimes 1))^*. \tag{RND1}$$

2. **Reflection Covariance :**

$$\not{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)} = (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L) ((\gamma_\nu \otimes 1) (\not{D}_{\Gamma_L}^- - M_\phi) (\gamma_\nu \otimes 1))^* (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L). \tag{RND2}$$

3. Determinant Invariance:

$$\det\left(\not{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)}\right) = \overline{\det\left(\not{D}_{\Gamma_L}^- - M_\phi\right)} = \det\left(\not{D}_{\Gamma_L}^- - M_\phi\right). \quad (\text{RND3})$$

Proof.

(RND1). Let us first analyze the action of the reflection on the infinite-plane Dirac operator.

$$\begin{aligned} ((\mathbb{1}_2 \otimes \hat{\vartheta}_\nu) \not{D} (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu))_{(x,s),(y,s')} &= \frac{1}{2} \sum_{\mu=0}^1 (\gamma_\mu)_{s,s'} (T_\mu - T_\mu^*)_{\vartheta_\nu(x), \vartheta_\nu(y)} \\ &= \frac{1}{2} \sum_{\mu=0}^1 (\gamma_\mu)_{s,s'} (\delta_{\vartheta_\nu(x), \vartheta_\nu(y) - e_\mu} - \delta_{\vartheta_\nu(x), \vartheta_\nu(y) + e_\mu}) \\ &= \frac{1}{2} \sum_{\mu=0}^1 (\gamma_\mu)_{s,s'} (\delta_{x,y - (-1)^{\delta_{\mu\nu}} e_\mu} - \delta_{x,y + (-1)^{\delta_{\mu\nu}} e_\mu}) \\ &= \frac{1}{2} \sum_{\mu=0}^1 (-1)^{\delta_{\mu\nu}} (\gamma_\mu)_{s,s'} (T_\mu - T_\mu^*)_{x,y}. \end{aligned}$$

With a simple manipulation with γ -matrices, we get:

$$\begin{aligned} \frac{1}{2} \sum_{\mu=0}^1 (-1)^{\delta_{\mu\nu}} (\gamma_\mu)_{s,s'} (T_\mu - T_\mu^*)_{x,y} &= -\frac{1}{2} \sum_{\mu=0}^1 (\gamma_\nu \gamma_\mu \gamma_\nu)_{s,s'} (T_\mu - T_\mu^*)_{x,y} \\ &= ((\gamma_\nu \otimes 1)(-\not{D})(\gamma_\nu \otimes 1))_{(x,s),(y,s')}. \end{aligned}$$

Therefore, we get that $(\mathbb{1}_2 \otimes \hat{\vartheta}_\nu) \not{D} (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu) = -(\gamma_\nu \otimes 1) \not{D} (\gamma_\nu \otimes 1)$. From this identity, first note that

$$\begin{aligned} ((\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L) (\not{D}_{\Gamma_L}^-) (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L))_{(x,s),(y,s')} &= \sigma_\nu^L(x) \sigma_\nu^L(y) (\not{D}_{\Gamma_L}^-)_{(\vartheta_\nu^L(x),s),(\vartheta_\nu^L(y),s')} \\ &= (\not{D}_{\Gamma_L}^-)_{(\vartheta_\nu(x),s),(\vartheta_\nu(y),s')}, \end{aligned}$$

essentially by the definition of the reflection operator and the anti-periodicity of $\not{D}_{\Gamma_L}^-$. Furthermore, notice that

$$(\not{D}_{\Gamma_L}^-)_{(\vartheta_\nu(x),s),(\vartheta_\nu(y),s')} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \not{D}_{(\vartheta_\nu(x)+nL,s),(\vartheta_\nu(y),s')} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \not{D}_{(\vartheta_\nu(x+nL),s),(\vartheta_\nu(y),s')}.$$

Therefore:

$$\begin{aligned} ((\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L) (\not{D}_{\Gamma_L}^-) (\mathbb{1}_2 \otimes \hat{\vartheta}_\nu^L))_{(x,s),(y,s')} &= \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} ((\mathbb{1}_2 \otimes \vartheta_\nu) (\not{D}) (\mathbb{1}_2 \otimes \vartheta_\nu))_{(x+nL,s),(y,s')} \\ &= - \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} ((\gamma_\nu \otimes 1) \not{D} (\gamma_\nu \otimes 1))_{(x+nL,s),(y,s')} \\ &= -((\gamma_\nu \otimes 1) (\not{D}_{\Gamma_L}^-) (\gamma_\nu \otimes 1))_{(x,s),(y,s')} \\ &= ((\gamma_\nu \otimes 1) (\not{D}_{\Gamma_L}^-) (\gamma_\nu \otimes 1))^*_{(x,s),(y,s')}, \end{aligned}$$

where we have also used that, by Equation (PND1), $(\not{D}_{\Gamma_L}^-)^* = -\not{D}_{\Gamma_L}^-$. By the arbitrariness of $x, y \in \Gamma_L$ and $s, s' \in \{1, 2\}$, it follows that

$$(\mathbb{1}_2 \otimes \hat{v}_\nu^L)(\not{D}_{\Gamma_L}^-)(\mathbb{1}_2 \otimes \hat{v}_\nu^L) = ((\gamma_\nu \otimes 1)(\not{D}_{\Gamma_L}^-)(\gamma_\nu \otimes 1))^*.$$

(RND2). Since $M_{\Theta_\nu(\phi)} = (\mathbb{1}_2 \otimes \hat{v}_\nu^L)M_\phi(\mathbb{1}_2 \otimes \hat{v}_\nu^L)$ and, as one can easily check, $(\hat{v}_\nu^L)^2 = 1$, we have:

$$\not{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)} = (\mathbb{1}_2 \otimes \hat{v}_\nu^L)((\mathbb{1}_2 \otimes \hat{v}_\nu^L)(\not{D}_{\Gamma_L}^-)(\mathbb{1}_2 \otimes \hat{v}_\nu^L) - M_\phi)(\mathbb{1}_2 \otimes \hat{v}_\nu^L), \quad (67)$$

which, using Equation (RND1), implies Equation (RND2).

(RND3) readily follows by taking the determinant at both sides of Equation (RND2), using the fact that $\det(\gamma_\nu \otimes 1)^2 = \det(\mathbb{1}_2 \otimes \hat{v}_\nu^L)^2 = 1$ and exploiting the reality of the determinant, established in Equation (PND4). ■

4.A.2 Staggered Dirac Operator

Proof of Lemma 40.

(PSD1): **Anti-self-adjointness.** The infinite-volume operator $\not{D} \equiv \frac{1}{2}\Gamma_\mu(T_\mu - T_\mu^*)$ is anti-self-adjoint by direct inspection. About $\not{D}_{\Gamma_L}^-$, we have that

$$\begin{aligned} ((\not{D}_{\Gamma_L}^-)^*)_{x,y} &= \overline{(\not{D}_{\Gamma_L}^-)_{y,x}} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \overline{\not{D}_{y+Ln,x}} \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \sum_{\mu=0}^1 \Gamma_\mu(y+Ln)(\delta_{y+Ln+e_\mu,x} - \delta_{y+Ln-e_\mu,x}) \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}^2} \sum_{\mu=0}^1 (-1)^{n_0+n_1} \Gamma_\mu(x)(\delta_{y,x-Ln-e_\mu} - \delta_{y,x-Ln+e_\mu}), \end{aligned}$$

where we used the fact that $\Gamma_\mu(x-Ln \pm e_\mu) = \Gamma_\mu(x \pm e_\mu) = \Gamma_\mu(x)$. From the previous formula we recognize that

$$\begin{aligned} ((\not{D}_{\Gamma_L}^-)^*)_{x,y} &= \frac{1}{2} \sum_{n \in \mathbb{Z}^2} \sum_{\mu=0}^1 (-1)^{n_0+n_1} \Gamma_\mu(x)(\delta_{x,y+Ln+e_\mu} - \delta_{x,y+Ln-e_\mu}) \\ &= -\frac{1}{2} \sum_{n \in \mathbb{Z}^2} \sum_{\mu=0}^1 (-1)^{n_0+n_1} \Gamma_\mu(x-nL)(T_\mu - T_\mu^*)_{x-nL,y} = -(\not{D}_{\Gamma_L}^-)_{x,y}. \end{aligned}$$

(PSD2): **Chirality under adjoint.** It is convenient to first check the properties for the Infinite Plane Dirac Operator $\not{D}$, whose only non-vanishing entries have different parity, because they are related by a single translation in the μ -th direction:

$$\begin{aligned} (\epsilon \not{D} \epsilon)_{x,y} &= (-1)^{x_1+x_2} \sum_{\mu=0,1} \Gamma_\mu(x)(\delta_{x+e_\mu,y} - \delta_{x-e_\mu,y})(-1)^{y_1+y_2} \\ &= - \sum_{\mu=0,1} \Gamma_\mu(x)(\delta_{x+e_\mu,y} - \delta_{x-e_\mu,y}) = -\not{D}_{x,y}. \end{aligned}$$

Thus, for the anti-periodic operator:

$$(\epsilon \not{D}_{\Gamma_L}^- \epsilon)_{x,y} = \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} (\epsilon \not{D} \epsilon)_{x+nL,y} = - \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \not{D}_{x+nL,y} = -(\not{D}_{\Gamma_L}^-)_{x,y}.$$

This shows, after Equation (PSD1), that $\epsilon \not{D}_{\Gamma_L}^- \epsilon = -\not{D}_{\Gamma_L}^- = (\not{D}_{\Gamma_L}^-)^*$. Besides, the fact that $\epsilon M_\phi \epsilon = M_\phi^* = M_\phi$ is obvious since $\epsilon^2 = 1$ and $\epsilon M_\phi = M_\phi \epsilon$.

(PSD3): **Spectrum of the adjoint.** Since $\epsilon^2 = 1$ and $\epsilon^* = \epsilon$, Equation (PSD2) shows that $(\not{D}_{\Gamma_L}^- - M_\phi)^*$ and $\not{D}_{\Gamma_L}^- - M_\phi$ are unitarily equivalent, so, in particular, they share the same spectrum.

(PSD4): **Reality of the determinant.** Taking the determinant at both sides of Equation (PSD2), we get:

$$\overline{\det(\not{D}_{\Gamma_L}^- - M_\phi)} = \det((\not{D}_{\Gamma_L}^- - M_\phi)^*) = \det(\epsilon(\not{D}_{\Gamma_L}^- - M_\phi)\epsilon) = \det(\epsilon)^2 \det(\not{D}_{\Gamma_L}^- - M_\phi).$$

■

Lemma 55 (Properties under the Reflection Operator). *The following identities hold, for every reflection ϑ_ν on Γ_L .*

1. **Reflection Covariance of the Dirac Operator:**

$$\hat{\vartheta}_\nu^L \not{D}_{\Gamma_L}^- \hat{\vartheta}_\nu^L = (\Gamma_\nu \not{D}_{\Gamma_L}^- \Gamma_\nu)^*, \quad (\text{RSD1})$$

where Γ_ν denotes the multiplication operator by $\Gamma_\nu(x)$.

2. **Reflection Covariance :**

$$\not{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)} = \hat{\vartheta}_\nu^L (\Gamma_\nu (\not{D}_{\Gamma_L}^- - M_\phi) \Gamma_\nu)^* \hat{\vartheta}_\nu^L. \quad (\text{RSD2})$$

3. **Determinant Invariance:**

$$\det(\not{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)}) = \overline{\det(\not{D}_{\Gamma_L}^- - M_\phi)} = \det(\not{D}_{\Gamma_L}^- - M_\phi). \quad (\text{RSD3})$$

Proof.

(RSD1). Let us first analyze the action of the reflection on the Infinite Plane Dirac operator.

$$\begin{aligned} (\hat{\vartheta}_\nu \not{D} \hat{\vartheta}_\nu)_{x,y} &= \not{D}_{\vartheta_\nu(x), \vartheta_\nu(y)} = \frac{1}{2} \sum_{\mu=0,1} \Gamma_\mu(\vartheta_\nu(x)) (\delta_{\vartheta_\nu(x)+e_\mu, \vartheta_\nu(y)} - \delta_{\vartheta_\nu(x)-e_\mu, \vartheta_\nu(y)}) \\ &= \frac{1}{2} \sum_{\mu=0,1} \Gamma_\mu(\vartheta_\nu(x)) (\delta_{x+(-1)^{\delta_{\mu,\nu}} e_\mu, y} - \delta_{x-(-1)^{\delta_{\mu,\nu}} e_\mu, y}). \end{aligned}$$

Using the fact that $\Gamma_\mu(\vartheta_\nu(x))$ equals $(-1)^{\delta_{\mu,\nu}+1} \Gamma_\mu(x)$, if $\nu = 0$, and $\Gamma_\mu(x)$, if $\nu = 1$, we find that

$$(\hat{\vartheta}_\nu \not{D} \hat{\vartheta}_\nu)_{x,y} = \frac{1}{2} \sum_{\mu} \Gamma_\mu(x) (\delta_{x+e_\mu, y} - \delta_{x-e_\mu, y}) \cdot \begin{cases} (-1), & (\nu = 0) \\ (-1)^{\delta_{\mu,\nu}}, & (\nu = 1) \end{cases} = -(\Gamma_\nu \not{D} \Gamma_\nu)_{x,y},$$

where we also used the fact that, for $y = x \pm e_\mu$, $\Gamma_0(x)\Gamma_0(y) = 1$ and $\Gamma_1(x)\Gamma_1(y) = (-1)^{\delta_{\mu,0}}$. Now, for the anti-periodic operator:

$$\begin{aligned}
(\hat{\vartheta}_\nu^L \mathcal{D}_{\Gamma_L}^- \hat{\vartheta}_\nu^L)_{x,y} &= \sigma_\nu^L(x) \sigma_\nu^L(y) (\mathcal{D}_{\Gamma_L}^-)_{\vartheta_\nu^L(x), \vartheta_\nu^L(y)} \\
&= (\mathcal{D}_{\Gamma_L}^-)_{\vartheta_\nu(x), \vartheta_\nu(y)} \\
&= \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \mathcal{D}_{\vartheta_\nu(x)+nL, \vartheta_\nu(y)} \\
&= \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} \mathcal{D}_{\vartheta_\nu(x+nL), \vartheta_\nu(y)} \\
&= - \sum_{n \in \mathbb{Z}^2} (-1)^{n_0+n_1} (\Gamma_\nu \mathcal{D}_{\Gamma_L}^-)_{x+nL, y} \\
&= -\Gamma_\nu(x) (\mathcal{D}_{\Gamma_L}^-)_{x, y} \Gamma_\nu(y) \\
&= -(\Gamma_\nu (\mathcal{D}_{\Gamma_L}^-) \Gamma_\nu).
\end{aligned} \tag{68}$$

Since $(\mathcal{D}_{\Gamma_L}^-)^* = -\mathcal{D}_{\Gamma_L}^-$ and the Γ_ν operators are self-adjoint, it follows that

$$\hat{\vartheta}_\nu^L \mathcal{D}_{\Gamma_L}^- \hat{\vartheta}_\nu^L = (\Gamma_\nu \mathcal{D}_{\Gamma_L}^- \Gamma_\nu)^*.$$

(RSD2). It is clear that

$$(\hat{\vartheta}_\nu^L M_\phi \hat{\vartheta}_\nu^L)_{x,y} = \phi_{\vartheta_\nu^L(x)} \delta_{\vartheta_\nu^L(x), \vartheta_\nu^L(y)} = (M_{\Theta_\nu(\phi)})_{x,y}$$

and also that $M_\phi = \Gamma_\nu M_\phi \Gamma_\nu$. Combining these two identities with the self-adjointness of M_ϕ , the fact that $(\hat{\vartheta}_\nu^L)^2 = 1$ and with [Equation \(RND1\)](#), we get:

$$\mathcal{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)} = \hat{\vartheta}_\nu^L (\hat{\vartheta}_\nu^L \mathcal{D}_{\Gamma_L}^- \hat{\vartheta}_\nu^L - M_\phi) \hat{\vartheta}_\nu^L = \hat{\vartheta}_\nu^L (\Gamma_\nu (\mathcal{D}_{\Gamma_L}^- - \phi) \Gamma_\nu)^* \hat{\vartheta}_\nu^L.$$

(RSD3). Taking the determinant at both sides of [Equation \(RSD2\)](#), and using that $\det(\hat{\vartheta}_\nu^L)^2 = \det(\Gamma_\nu^2) = 1$, we get:

$$\begin{aligned}
\det(\mathcal{D}_{\Gamma_L}^- - M_{\Theta_\nu(\phi)}) &= \det(\hat{\vartheta}_\nu^L (\Gamma_\nu (\mathcal{D}_{\Gamma_L}^- - \phi) \Gamma_\nu)^* \hat{\vartheta}_\nu^L) \\
&= \det(\mathcal{D}_{\Gamma_L}^- - M_\phi)^* \\
&= \overline{\det(\mathcal{D}_{\Gamma_L}^- - M_\phi)}.
\end{aligned}$$

Combining [Equation \(RSD3\)](#) with [Equation \(PSD4\)](#), [Equation \(RSD3\)](#) follows. ■

4.B Analysis of the Bosonic Potential

This appendix is dedicated to the proof of the main properties, stated in [Proposition 49](#), of the bosonic potential:

$$V_{L,\lambda}^\alpha(\phi) = \frac{1}{2\lambda} \sum_{x \in \Gamma_L^\alpha} \phi_x^2 - \log |\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)|$$

and its infinite volume limit: $v_\lambda^\alpha(\varphi) \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\phi) \Big|_{\phi=\varphi \Gamma_L^\alpha}$, with $\varphi \in \mathbb{R}$.

4.B.1 Minimization over Constant Field Configurations

Here we shall prove that the potential $V_{L,\lambda}^\alpha$ is minimized by constant field configurations, namely, given any interval $I \subseteq \mathbb{R}$,

$$\inf_{\phi \in I^{\Gamma_L^\alpha}} V_{L,\lambda}^\alpha(\phi) = \inf_{\varphi \in I} V_{L,\lambda}^\alpha(\phi) \Big|_{\phi=\varphi^{\Gamma_L^\alpha}}. \quad (69)$$

The key tool is again Reflection Positivity, which implies the following lemma.

Lemma 56. *Given any cut plane of Γ_L^α , with $\vartheta : (\Gamma_L^\alpha)^\pm \mapsto (\Gamma_L^\alpha)^\mp$ being the related reflection operator (see [Definition 42](#) and [Definition 43](#)), and given any $\phi \in \mathbb{R}^{\Gamma_L^\alpha}$, it holds true that*

$$(e^{-NV_{L,\lambda}^\alpha(\phi)})^2 \leq e^{-NV_{L,\lambda}^\alpha(\phi^+)} e^{-NV_{L,\lambda}^\alpha(\phi^-)}, \quad (70)$$

where $\phi^\pm$ is the configuration obtained from ϕ by reflecting, respectively, the left/right configuration (with respect to the cut plane) to the right/left:

$$\phi_x^\pm \doteq \begin{cases} \phi_x & x \in (\Gamma_L^\alpha)_\pm \\ \phi_{\vartheta(x)} & x \in (\Gamma_L^\alpha)_\mp. \end{cases}$$

Before proving [Lemma 56](#), let us show how it readily implies the validity of [Equation \(69\)](#). The idea is to prove that, for any given field configuration $\phi \in I^{\Gamma_L^\alpha}$, there exists a uniform configuration, $\phi = \varphi^{\Gamma_L^\alpha}$ for some $\varphi \in I$, with a smaller energy. First of all observe that, since $e^{-NV_{L,\lambda}^\alpha(\phi)} \geq 0$ and $e^{-NV_{L,\lambda}^\alpha(\phi)} \rightarrow 0^+$ as $\|\phi\|_{\ell^2(\Gamma_L^\alpha)} \rightarrow \infty$, we have that

$$\inf_{\phi \in I^{\Gamma_L^\alpha}} V_{L,\lambda}^\alpha(\phi) = -\frac{1}{N} \log \sup_{\phi \in I^{\Gamma_L^\alpha}} e^{-NV_{L,\lambda}^\alpha(\phi)} = -\frac{1}{N} \log \max_{\phi \in \bar{I}^{\Gamma_L^\alpha}} e^{-NV_{L,\lambda}^\alpha(\phi)},$$

where $\bar{I}$ denotes the closure of the interval. If we denote by $\tilde{\phi}$ the maximizer of $e^{-NV_{L,\lambda}^\alpha}$ over $\bar{I}^{\Gamma_L^\alpha}$, then by [Lemma 56](#) we have that

$$(e^{-NV_{L,\lambda}^\alpha(\tilde{\phi})})^2 \leq e^{-NV_{L,\lambda}^\alpha(\tilde{\phi}^+)} e^{-NV_{L,\lambda}^\alpha(\tilde{\phi}^-)} \leq e^{-NV_{L,\lambda}^\alpha(\tilde{\phi}^+)} e^{-NV_{L,\lambda}^\alpha(\tilde{\phi}^-)},$$

implying that $e^{-NV_{L,\lambda}^\alpha(\tilde{\phi})} = e^{-NV_{L,\lambda}^\alpha(\tilde{\phi}^+)}$. In other words, we have shown that a configuration obtained by reflections from the minimizing one has the same energy as the latter. We can repeat this argument first for all the horizontal and then for all the vertical cutting planes, so that we end up with a constant field configuration $\tilde{\varphi}^{\Gamma_L^\alpha}$, with $\tilde{\varphi} \in \bar{I}$, which has the same energy as $\tilde{\phi}$, thus proving [Equation \(69\)](#).

Proof of Lemma 56. We must employ Reflection Positivity for the fermionic integration. Recall that

$$\begin{aligned} e^{-NV_{L,\lambda}^\alpha(\phi)} &= \left(\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} \phi_x^2} \right) \det((\not{D}_\alpha)_{\Gamma_L}^- - M_\phi)^N \\ &= \left(\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} \phi_x^2} \right) \left(\int d\bar{\psi} d\psi e^{\sum_{x,y} \bar{\psi}_x ((\not{D}_\alpha)_{\Gamma_L}^- - M_\phi)_{xy} \psi_y} \right)^N. \end{aligned}$$

The idea is to think of the determinant $\det((\not{D}_\alpha)_{\Gamma_L}^- - M_\phi)$ as a bilinear product $\langle e^A, e^B \rangle_{\rho_\alpha}$, where, given two functions F, G over the Grassmann algebra generated by $\bar{\psi}, \psi$,

$$\langle F, G \rangle_{\rho_\alpha} \doteq \int d\rho_\alpha(\bar{\psi}, \psi) (F \Theta(G))(\bar{\psi}, \psi),$$

with the reflection operator Θ as in Equation (RNF), Equation (RSF), Equation (RSFP) and:

$$\begin{aligned} A(\bar{\psi}, \psi) &\doteq \sum_{x \in (\Gamma_L^\alpha)^+} \phi_x \bar{\psi}_x \psi_x, & B(\bar{\psi}, \psi) &\doteq \sum_{x \in (\Gamma_L^\alpha)^+} \phi_{\vartheta(x)} \bar{\psi}_x \psi_x, \\ d\rho_\alpha(\bar{\psi}, \psi) &\doteq d\bar{\psi} d\psi \exp \left(\sum_{x, y \in \Gamma_L^\alpha} \bar{\psi}_x ((\mathcal{D}_\alpha)_{\Gamma_L}^-)_{x, y} \psi_y \right). \end{aligned} \quad (71)$$

Notice that here ϕ has to be understood as a fixed static, external field (versus the dynamical field in the full measure $d\mu_\alpha$ as in Equation (4)). The Grassmann integration $d\rho_\alpha(\bar{\psi}, \psi)$ has already been (implicitly) shown to be reflection positive, in proving Theorem 45, so that the sesquilinear form $\langle \cdot, \cdot \rangle_{\rho_\alpha}$ is actually semi-positive definite. As a consequence, by the Cauchy-Schwartz inequality:

$$\begin{aligned} \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)^2 &= \langle e^A, e^B \rangle_{\rho_\alpha}^2 \\ &\leq \langle e^A, e^A \rangle_{\rho_\alpha} \langle e^B, e^B \rangle_{\rho_\alpha} \\ &= \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\phi^+}) \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\phi^-}), \end{aligned} \quad (72)$$

with A and B as in Equation (71). Observe that each factor in the right-hand side is non-negative, being of the form $\|e^A\|_{\rho_\alpha}^2 \geq 0$. Therefore, if either of the two determinants on the right-hand side vanishes, the left-hand side must vanish as well. Furthermore, since the left-hand side is manifestly non-negative, we also have, for any $N \in \mathbb{N}$,

$$\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)^{2N} \leq \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\phi^+})^N \det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_{\phi^-})^N.$$

Finally, by multiplying both sides by $\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} \phi_x^2}$ and noting that

$$\left(\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} \phi_x^2} \right)^2 = \left(\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} (\phi_x^+)^2} \right) \left(\prod_{x \in \Gamma_L^\alpha} e^{-\frac{N}{2\lambda} (\phi_x^-)^2} \right),$$

we find the validity of Equation (70), hence proving Lemma 56. $\blacksquare$

Remark 41. It is interesting to observe that inequality Equation (72) implies that $\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi)$ is positive whenever the configuration ϕ is symmetric under reflection around some cut plane, as it can be written as:

$$\det((\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi) = \langle e^A, e^A \rangle_{\rho_\alpha} \geq 0,$$

with A as in Equation (71) and Γ_L^+ the right half portion of Γ_L determined by the cut plane.

4.B.2 Uniform-Field Potentials and the Thermodynamic Limit

Computation of the uniform-field potential. Since ϕ is constant, namely $\phi = \varphi^{\Gamma_L^\alpha}$, the spectrum of the operator $(\mathcal{D}_\alpha)_{\Gamma_L}^- - M_\phi$ can be explicitly obtained using Fourier transforms. Such operation actually depends on the specific model under consideration, thus we will work out the three cases $\alpha = \text{NN}, \text{SN}, \text{SP}$ separately.

- **NN model.** The model has a full translational invariance by $\mathbb{Z}^2$, thus the space of allowed momenta with anti-periodic boundary conditions is

$$(\Gamma_L^{\text{NN}})_{--}^* \doteq \frac{2\pi}{L} \left(\mathbb{Z} + \frac{1}{2} \right)^2 \cap (-\pi, \pi]^2.$$

The determinant of the finite-volume Dirac operator is computed starting from [Equation \(1\)](#):

$$\begin{aligned} \det \left((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi \right) \Big|_{\phi=\varphi}^{\Gamma_L^\alpha} &= \prod_{p \in (\Gamma_L^{\text{NN}})^*_{--}} \det \left(-i\gamma_0 \sin(p_0) - i\gamma_1 \sin(p_1) - \mathbb{1}_2 \varphi \right) \\ &= \prod_{p \in (\Gamma_L^{\text{NN}})^*_{--}} (\varphi^2 + \sin^2(p_0) + \sin^2(p_1)), \end{aligned}$$

from which:

$$\frac{1}{|\Lambda_L^{\text{NN}}|} \log \left| \det \left((\not{D}_{\text{NN}})_{\Gamma_L}^- - M_\phi \right) \Big|_{\phi=\varphi}^{\Gamma_L^\alpha} \right| = \frac{1}{L^2} \sum_{p \in (\Gamma_L^{\text{NN}})^*_{--}} \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1)).$$

- **SN model.** The model has a reduced translational invariance by $(2\mathbb{Z}) \times \mathbb{Z}$. It is therefore convenient to consider an enlarged 2×1 fundamental cell. The resulting lattice is drawn in [Figure 4.13](#).

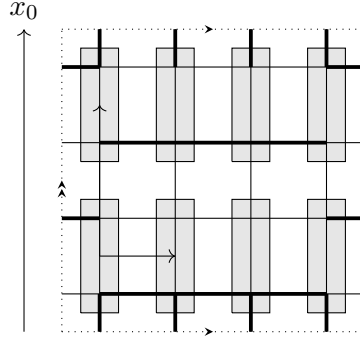


Figure 4.13: Fundamental Cells: In absence of twistings of the boundary conditions one gets a manifestly translation invariant system.

The set of allowed momenta is given by the dual of this reduced lattice:

$$(\Gamma_L^{\text{SN}})^*_{--} = \frac{2\pi}{L} \left(\mathbb{Z} + \frac{1}{2} \right)^2 \cap \left(\left(-\frac{\pi}{2}, \frac{\pi}{2} \right] \times (-\pi, \pi] \right).$$

Again the determinant can be computed starting from [Equation \(4\)](#):

$$\begin{aligned} \det \left((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_\phi \right) \Big|_{\phi=\varphi}^{\Gamma_L^\alpha} &= \prod_{p \in (\Gamma_L^{\text{SN}})^*_{--}} \det \begin{pmatrix} i \sin p_1 - \varphi & \frac{1}{2}(e^{2ip_0} - 1) \\ -\frac{1}{2}(e^{-2ip_0} - 1) & -i \sin p_1 - \varphi \end{pmatrix} \\ &= \prod_{p \in (\Gamma_L^{\text{SN}})^*_{--}} (\varphi^2 + \sin^2(p_0) + \sin^2(p_1)). \end{aligned}$$

Thus:

$$\frac{1}{|\Lambda_L^{\text{SN}}|} \log \left| \det \left((\not{D}_{\text{SN}})_{\Gamma_L}^- - M_\phi \right) \Big|_{\phi=\varphi}^{\Gamma_L^\alpha} \right| = \frac{1}{L^2} \sum_{p \in (\Gamma_L^{\text{SN}})^*_{--}} \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1)).$$

- **SP model.** The analysis is identical to the one for the model SN. Indeed, when the field ϕ is constant, the models SN and SP are actually the same, as well as the set of momenta: $(\Lambda_L^{\text{SP}})^*_{--} = (\Lambda_L^{\text{SN}})^*_{--}$. The only difference comes from the fact that $|\Lambda_L^{\text{SP}}| = \frac{1}{4} |\Lambda_L^{\text{SN}}|$, so that

$$\frac{1}{|\Lambda_L^{\text{SP}}|} \log \left| \det \left((\mathcal{D}_{\text{SP}})_{\Gamma_L}^- - M_\phi \right) \right|_{\phi=\varphi^{\Gamma_L^\alpha}} = \frac{4}{L^2} \sum_{p \in (\Gamma_L^{\text{SP}})^*_{--}} \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1)).$$

Notice that, due to the periodicity of $\hat{f}(p; \varphi) \doteq \log(\varphi^2 + \sin^2(p_0) + \sin^2(p_1))$ with respect to translations of $\pm\pi$ in both directions, it is possible to rewrite:

$$\begin{aligned} \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\varphi^{\Gamma_L^\alpha}) &= \frac{\varphi^2}{2\lambda} - \frac{1}{|\Gamma_L^\alpha|} \sum_{p \in (\Gamma_L^\alpha)^*} \hat{f}(p; \varphi) = \frac{\varphi^2}{2\lambda} - \frac{C_\alpha}{L^2} \sum_{p \in (\Lambda_L^{\text{NN}})^*} \hat{f}(p; \varphi) \\ &= \frac{\varphi^2}{2\lambda} - \frac{4C_\alpha}{L^2} \sum_{p \in \frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2})^2 \cap (-\frac{\pi}{2}, \frac{\pi}{2})^2} \hat{f}(p; \varphi), \end{aligned} \quad (73)$$

where $C_\alpha = 1, \frac{1}{2}, 2$ for $\alpha = \text{NN}, \text{SN}, \text{SP}$ respectively.

Convergence to the infinite-volume limit. By standard arguments (e.g. the Dominated Convergence Theorem), we have the convergence, in the limit $L \rightarrow \infty$, of the Riemann sum in [Equation \(73\)](#) to the respective integral $C_\alpha \iint_{(-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \hat{f}(p; \varphi)$, so that

$$\begin{aligned} v_\lambda^\alpha(\varphi) &\equiv \lim_{L \rightarrow \infty} \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\varphi^{\Gamma_L^\alpha}) = \frac{\varphi^2}{2\lambda} - C_\alpha \iint_{(-\pi, \pi]^2} \frac{d^2 p}{(2\pi)^2} \hat{f}(p; \varphi) \\ &= \frac{\varphi^2}{2\lambda} - 4C_\alpha \iint_{(-\frac{\pi}{2}, \frac{\pi}{2}]^2} \frac{d^2 p}{(2\pi)^2} \hat{f}(p; \varphi). \end{aligned} \quad (74)$$

The representation in the second line of [Equation \(74\)](#) is convenient because in the domain $(-\frac{\pi}{2}, \frac{\pi}{2}]^2$, the integrand $\hat{f}(p; \varphi)$ has only one singularity at $p = (0, 0)$, if $\varphi = 0$. Observe also that the Riemann sum in the right-hand side of [Equation \(73\)](#) can be rewritten as follows:

$$\frac{1}{L^2} \sum_{p \in \frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2})^2 \cap (-\frac{\pi}{2}, \frac{\pi}{2})^2} \hat{f}(p; \varphi) = \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2 p}{(2\pi)^2} \hat{f}(\pi_L(p); \varphi),$$

where $\pi_L : (-\frac{\pi}{2}, \frac{\pi}{2})^2 \mapsto (-\frac{\pi}{2}, \frac{\pi}{2})^2 \cap \frac{2\pi}{L}(\mathbb{Z} + \frac{1}{2})^2$ is the projection defined by the following condition (see [Figure 4.14](#)):

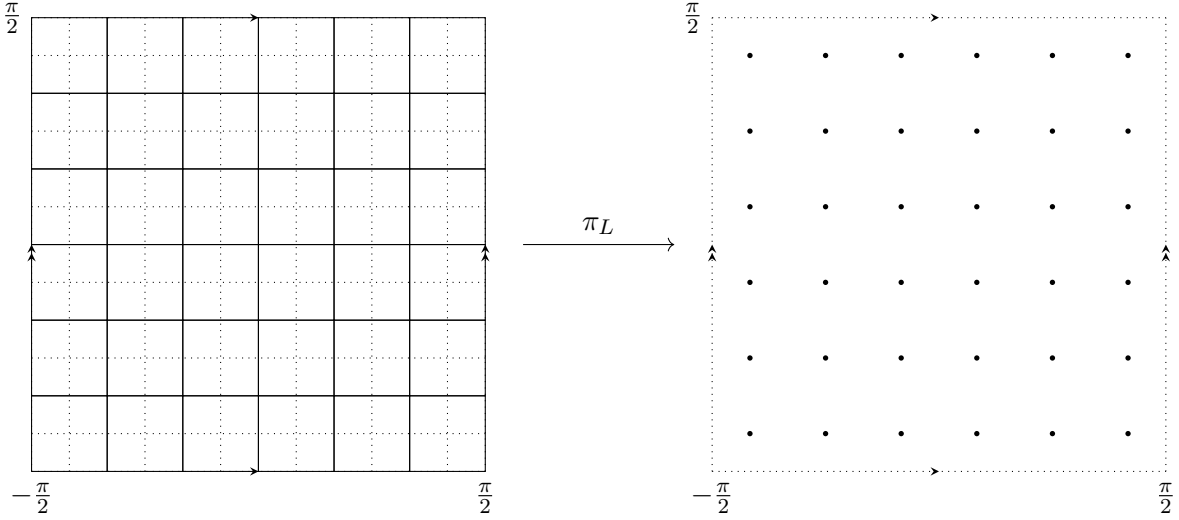
$$\text{if } p \in \frac{2\pi}{L}([n_0, n_0 + 1) \times [n_1, n_1 + 1)) \text{ then } \pi_L(p) = \frac{2\pi}{L}(n_0 + \frac{1}{2}, n_1 + \frac{1}{2})$$

.

In this way:

$$\left| \frac{1}{|\Gamma_L^\alpha|} V_{L,\lambda}^\alpha(\varphi^{\Gamma_L^\alpha}) - v_\lambda^\alpha(\varphi) \right| \leq 4C_\alpha \iint_{(-\frac{\pi}{2}, \frac{\pi}{2}]^2} \frac{d^2 p}{(2\pi)^2} \left| \hat{f}(\pi_L(p); \varphi) - \hat{f}(p; \varphi) \right|. \quad (75)$$

In order to proceed, observe that for any $p \in (-\frac{\pi}{2}, \frac{\pi}{2})^2$ and $\mu \in \{0, 1\}$, $|\pi_L(p)_\mu| \geq \frac{1}{2}|p_\mu|$. Note also that for any $p \in (-\frac{\pi}{2}, \frac{\pi}{2})^2$, we can write:

Figure 4.14: Picture of the action of the projector π_L : each cell is sent to its center

$$\begin{aligned}
\left| \hat{f}(\pi_L(p); \varphi) - \hat{f}(p; \varphi) \right| &\leq \sum_{\mu=0,1} |\pi_L(p)_\mu - p_\mu| \left| \int_0^1 ds \left[\frac{2 \sin(q_\mu) \cos(q_\mu)}{\varphi^2 + \sin^2(q_0) + \sin^2(q_1)} \right]_{q=s\pi_L(p) + (1-s)p} \right| \\
&\leq \sum_{\mu=0,1} |\pi_L(p)_\mu - p_\mu| \int_0^1 ds \left[\frac{2}{\sqrt{\sin^2(q_0) + \sin^2(q_1)}} \right]_{q=s\pi_L(p) + (1-s)p} \\
&\leq \frac{4\pi}{L} \frac{2}{\sqrt{\sin^2(\frac{1}{2}p_0) + \sin^2(\frac{1}{2}p_1)}} \leq \frac{8\pi^2}{L|p|},
\end{aligned}$$

where we also used that for $|t| \leq \frac{\pi}{2}$, $|\sin t| \geq \frac{2}{\pi}|t|$. Therefore from Equation (75) we find:

$$\left| \frac{1}{|\Gamma_L|} V_{L,\lambda}^\alpha(\varphi^{\Gamma_L^\alpha}) - v_\lambda^\alpha(\varphi) \right| \leq 8 \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2p}{(2\pi)^2} \frac{8\pi^2}{L|p|} \leq \frac{32\pi^2}{L},$$

concluding the proof of Item 2.

4.B.3 Non-Zero Minima and their Weak-Coupling Asymptotics

Existence of non-zero minima. It is clear by inspection that, for any $\lambda > 0$, the mean-field potential v_λ^α , as a function of φ , is even and C^1 over $\mathbb{R}$. In order to find its minima it is convenient to analyze its derivative:

$$\frac{dv_\lambda^\alpha}{d\varphi}(\varphi) = \varphi \left(\frac{1}{\lambda} - 4C_\alpha \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2p}{(2\pi)^2} \frac{1}{\varphi^2 + \sin^2 p_0 + \sin^2 p_1} \right) \doteq \varphi g_\lambda^\alpha(\varphi). \quad (76)$$

We see that v_λ^α has a local maximum at $\varphi = 0$. Moreover, as it is easy to check, for $\varphi > 0$ the function g_λ^α is monotone increasing and $\lim_{\varphi \rightarrow +\infty} g_\lambda^\alpha(\varphi) = 0^+$. Besides, $\lim_{\varphi \rightarrow 0^+} g_\lambda^\alpha(\varphi) = -\infty$, indeed:

$$g_\lambda^\alpha(\varphi) \leq \frac{1}{\lambda} - 4C_\alpha \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2p}{(2\pi)^2} \frac{1}{\varphi^2 + |p|^2} \leq \frac{1}{\lambda} - 2 \iint_{|p| \leq 1} \frac{d^2p}{(2\pi)^2} \frac{1}{\varphi^2 + |p|^2} = \frac{1}{\lambda} - \frac{1}{2\pi} \log \left(1 + \frac{1}{\varphi^2} \right). \quad (77)$$

Since g_λ^α is also obviously continuous over $(0, \infty)$, it follows that there exists a unique value $\varphi_\alpha^*(\lambda) \in (0, \infty)$ such that $g_\lambda^\alpha(\varphi_\alpha^*(\lambda)) = 0$. Such point is of course an absolute minimum for v_λ^α .

Regularity of the minima. As we just argued, the minimum $\varphi_\alpha^*(\lambda)$ can be interpreted as the unique positive solution to the *gap equation*:

$$g_\lambda^\alpha(\varphi) = 0, \quad \varphi \in (0, \infty). \quad (78)$$

The solution of such equation can be characterized by the Implicit Function Theorem. The map $(\lambda, \varphi) \mapsto g_\lambda^\alpha(\varphi)$ is clearly C^∞ over $(0, \infty)^2$. Besides, since the function

$$\frac{\partial g_\lambda^\alpha}{\partial \varphi}(\varphi) = -8C_\alpha \varphi \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2 p}{(2\pi)^2} \frac{1}{(\varphi^2 + \sin^2 p_0 + \sin^2 p_1)^2}$$

is never zero, we can apply the (global version of the) Implicit Function Theorem, which implies that the (unique) solution $\varphi_\alpha^*(\lambda)$ to Equation (78) is actually C^∞ with respect to $\lambda \in (0, \infty)$.

Exponential smallness of the minima. Finally we prove Equation (21). Using that $g_\lambda^\alpha(\varphi_\alpha^*(\lambda)) = 0$, from Equation (77) we have that $0 \leq \frac{1}{\lambda} - \frac{1}{2\pi} \log(1 + \varphi_\alpha^*(\lambda)^{-2})$, which implies that $\varphi_\alpha^*(\lambda) \geq e^{-\frac{\pi}{\lambda}}$. In order to provide an upper bound, we observe that

$$\begin{aligned} g_\lambda^\alpha(\varphi) &= \frac{1}{\lambda} - 4C_\alpha \iint_{(-\frac{\pi}{2}, \frac{\pi}{2})^2} \frac{d^2 p}{(2\pi)^2} \frac{1}{\varphi^2 + \sin^2 p_0 + \sin^2 p_1} \\ &\geq \frac{1}{\lambda} - 8 \iint_{|p| \leq \pi} \frac{d^2 p}{(2\pi)^2} \frac{1}{\varphi^2 + \frac{4}{\pi^2} |p|^2} = \frac{1}{\lambda} - \frac{\pi}{2} \log\left(1 + \frac{2}{\varphi^2}\right), \end{aligned}$$

where again we used the fact that $|\sin t| \geq \frac{2}{\pi}|t|$ for $|t| \leq \frac{\pi}{2}$. The condition $g_\lambda^\alpha(\varphi_\alpha^*(\lambda)) = 0$ then yields, for $\lambda \leq 1$,

$$\frac{2}{\varphi_\alpha^*(\lambda)^2} \geq e^{\frac{2}{\pi\lambda}} - 1 \geq e^{\frac{2}{\pi\lambda}} (1 - e^{-\frac{2}{\pi}}) \Rightarrow \varphi_\alpha^*(\lambda) \leq \sqrt{\frac{2}{1 - e^{-\frac{2}{\pi}}}} e^{-\frac{1}{\pi\lambda}}.$$

4.C Formal Continuum Limit of the Staggered Plaquette Model

In this appendix we derive the formal continuum limit of the Staggered Plaquette model and identify the corresponding continuum Gross–Neveu theory. We will show that:

1. after a suitable change of fermionic variables, the non-interacting lattice theory converges to one describing $N_{\text{eff}} \equiv 2N$ non-interacting Dirac fermions;
2. the plaquette interaction becomes the standard Gross–Neveu quartic interaction.

Lattice Action. We consider a two-dimensional lattice of linear size L and mesh $\ell > 0$, $\Gamma_{\ell,L} = \ell\mathbb{Z}^2 \cap (-\frac{L}{2}, \frac{L}{2}]^2$, with $L/\ell \in 4\mathbb{N}$. We partition $\Gamma_{\ell,L}$ into 2×2 plaquettes and introduce the coarse lattice: $\tilde{\Gamma}_{\ell,L} = 2\ell\mathbb{Z}^2 \cap (-\frac{L}{2}, \frac{L}{2}]^2$ and we consider the set of internal plaquette sites $\mathcal{I} = \{A, B, C, D\}$, so that we can identify $\Lambda_{\ell,L}$ with $\tilde{\Gamma}_{\ell,L} \times \mathcal{I}$ (see Figure 4.3). The action of the Staggered Plaquette model reads $S_{\text{SP}}^{\ell,L} = S_0^{\ell,L} + S_{\text{int}}^{\ell,L}$, where

$$S_0^{\ell,L} \doteq \ell^2 \sum_{x,y \in \tilde{\Lambda}_{\ell,L}} \sum_{j=1}^N \sum_{a,b \in \mathcal{I}} \bar{\psi}_{x,a,j} ((\not{D}_{\text{SP}})_{\Lambda_{\ell,L}}^-)_{(x,a),(y,b)} \psi_{y,b,j}, \quad (79)$$

$$S_{\text{int}}^{\ell,L} \doteq -\frac{\lambda(2\ell)^2}{8N} \sum_{x \in \tilde{\Gamma}_{\ell,L}} \left(\sum_{a \in \mathcal{I}} \sum_{j=1}^N \bar{\psi}_{x,a,j} \psi_{x,a,j} \right)^2 \quad (80)$$

and $((\not{D}_{\text{SP}})_{\Lambda_{\ell,L}}^-)_{x,y} \doteq \ell^{-1} ((\not{D}_{\text{SP}})_{\Lambda_{L/\ell}}^-)_{x/\ell,y/\ell}$, with the latter defined in Equation (6).

Fourier Representation. We define, for $p \in (\tilde{\Gamma}_{\ell,L})_{--}^* \doteq \frac{2\pi}{L} \left(\mathbb{Z} + \frac{1}{2} \right)^2 \cap \left(-\frac{\pi}{2\ell}, \frac{\pi}{2\ell} \right)^2$,

$$\hat{\psi}_{p,a,j}^- \doteq (2\ell)^2 \sum_{x \in \tilde{\Lambda}_{\ell,L}} e^{ipx} \psi_{x,a,j}, \quad \hat{\psi}_{p,a,j}^+ \doteq \ell^2 \sum_{x \in \tilde{\Lambda}_{\ell,L}} e^{-ipx} \bar{\psi}_{x,a,j},$$

so that

$$\psi_{x,a,j} = \frac{1}{L^2} \sum_{p \in (\tilde{\Gamma}_{\ell,L})_{--}^*} e^{-ipx} \hat{\psi}_{p,a,j}^-, \quad \bar{\psi}_{x,a,j} = \frac{1}{L^2} \sum_{p \in (\tilde{\Gamma}_{\ell,L})_{--}^*} e^{ipx} \hat{\psi}_{p,a,j}^+.$$

By introducing the four-component spinor:

$$\hat{\Psi}_{p,j}^- = \begin{pmatrix} \hat{\psi}_{p,A,j}^- \\ \hat{\psi}_{p,B,j}^- \\ \hat{\psi}_{p,C,j}^- \\ \hat{\psi}_{p,D,j}^- \end{pmatrix}, \quad \hat{\Psi}_{p,j}^+ = \begin{pmatrix} \hat{\psi}_{p,A,j}^+ & \hat{\psi}_{p,B,j}^+ & \hat{\psi}_{p,C,j}^+ & \hat{\psi}_{p,D,j}^+ \end{pmatrix},$$

the free action takes the form

$$S_0^{\ell,L} = \frac{1}{4L^2} \sum_{j=1}^N \sum_p \hat{\Psi}_{p,j}^+ \hat{D}(p) \hat{\Psi}_{p,j}^-,$$

where, letting $\sigma_\mu(p) \doteq \frac{1-e^{-2i\ell p_\mu}}{2i\ell}$,

$$\hat{D}(p) \doteq \begin{pmatrix} 0 & -i\bar{\sigma}_1(p) & i\bar{\sigma}_0(p) & 0 \\ -i\sigma_1(p) & 0 & 0 & i\bar{\sigma}_0(p) \\ i\sigma_0(p) & 0 & 0 & i\bar{\sigma}_1(p) \\ 0 & i\sigma_0(p) & i\sigma_1(p) & 0 \end{pmatrix}.$$

Spin–Taste Basis. To identify the proper fermionic degrees of freedom suited for the continuum limit, we perform a standard *spin–taste transformation* [GL10, pp. 243–248]. Namely we define:

$$\hat{\xi}_{p,\omega,s,j}^- \doteq \frac{1}{4} \sum_{a \in \mathcal{I}} \Gamma_{s,\omega}^{(a)} \hat{\psi}_{p,a,j}^-, \quad \hat{\xi}_{p,\omega,s,j}^+ \doteq \frac{1}{4} \sum_{a \in \mathcal{I}} \hat{\psi}_{p,a,j}^+ (\Gamma^{(a)})_{\omega,s}^\dagger,$$

with $\omega \in \{\pm\}$ the *taste index*, $s \in \{\uparrow, \downarrow\}$ the *spin index* and, letting $e_A = (0, 0)$, $e_B = (1, 0)$, $e_C = (0, 1)$, $e_D = (1, 1)$ be the normalized displacement within the unit cell (see Figure 4.3),

$$\Gamma^{(a)} \equiv \begin{pmatrix} \Gamma_{\uparrow,+}^{(a)} & \Gamma_{\uparrow,-}^{(a)} \\ \Gamma_{\downarrow,+}^{(a)} & \Gamma_{\downarrow,-}^{(a)} \end{pmatrix} \doteq \gamma_1^{e_a \cdot e_1} \gamma_0^{e_a \cdot e_0} = \begin{cases} \mathbb{1}_2, & a = A, \\ \sigma_x, & a = B, \\ \sigma_y, & a = C, \\ i\sigma_z, & a = D. \end{cases}$$

Alternatively, after introducing

$$\hat{\Xi}_{p,j}^- = \begin{pmatrix} \xi_{p,+,\uparrow,j}^- \\ \xi_{p,+,\downarrow,j}^- \\ \xi_{p,-,\uparrow,j}^- \\ \xi_{p,-,\downarrow,j}^- \end{pmatrix}, \quad \hat{\Xi}_{p,j}^+ = \begin{pmatrix} \xi_{p,+,\uparrow,j}^+ & \xi_{p,+,\downarrow,j}^+ & \xi_{p,-,\uparrow,j}^+ & \xi_{p,-,\downarrow,j}^+ \end{pmatrix},$$

the change of basis reads $\hat{\Psi}_{p,j}^- = U^\dagger \hat{\Xi}_{p,j}^-$ and $\hat{\Psi}_{p,j}^+ = \hat{\Xi}_{p,j}^+ U$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

is a unitary matrix. Hence:

$$S_0^{\ell,L} = \frac{1}{4L^2} \sum_{j=1}^N \sum_{p \in (\tilde{\Lambda}_{\ell,L})_{--}^*} \hat{\Xi}_{p,j}^+ \hat{S}(p) \hat{\Xi}_{p,j}^-,$$

with $\hat{S}(p) = U \hat{D}(p) U^\dagger$. A direct computation gives:

$$\hat{S}(p) = \mathbb{1}_{\text{taste}} \otimes \left(-ip_0 \gamma_0 - ip_1 \gamma_1 \right) + \mathcal{O}(\ell|p|^2). \quad (81)$$

Equation (81) shows that the taste sector becomes asymptotically diagonal and survives as an additional independent fermionic species. Note also that

$$\det \hat{S}(p) = \det \hat{D}(p) = \ell^{-2} (\sin^2(\ell p_0) + \sin^2(\ell p_1)),$$

showing that $\hat{S}(p)^{-1}$ has a pole only at $p = 0$.

Observe also that, in the spin-taste basis, the interaction term in Equation (80) becomes

$$S_{\text{int}}^{\ell,L} = -\frac{\lambda(2\ell)^2}{8N} \sum_{x \in \tilde{\Gamma}_{\ell,L}} \left(\Xi_x \Xi_x \right)^2 \equiv -\frac{\lambda(2\ell)^2}{8N} \sum_{x \in \tilde{\Gamma}_{\ell,L}} \left(\sum_{j=1}^N \sum_{\omega=\pm} \Xi_{x,\omega,j} \Xi_{x,\omega,j} \right)^2.$$

Continuum Gross–Neveu Theory. The formal limit $\ell \rightarrow 0^+$, $L \rightarrow \infty$ is obtained by replacing Riemann sums by the corresponding integrals:

$$(2\ell^2) \sum_{x \in \tilde{\Lambda}_{\ell,L}} \rightsquigarrow \iint_{\mathbb{R}^2} d^2x, \quad L^{-2} \sum_{p \in (\tilde{\Lambda}_{\ell,L})_{--}^*} \rightsquigarrow \iint_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}$$

and neglecting the remainders in the right-hand side of Equation (81). We obtain:

$$S_{\text{SP}}^{\ell,L} \rightsquigarrow S_{\text{SP}} = \iint d^2x \left[-\frac{1}{4} i \sum_{j=1}^N \sum_{\omega=\pm} \sum_{\mu=0,1} \Xi_{x,\omega,j} \gamma_\mu \partial_\mu \Xi_{x,\omega,j} - \frac{\lambda}{8N} \left(\sum_{j=1}^N \sum_{\omega=\pm} \Xi_{x,\omega,j} \Xi_{x,\omega,j} \right)^2 \right].$$

The kinetic term contains an overall factor $1/4$, which is easily removed by a trivial rescaling $\Xi \mapsto \Xi' = \frac{1}{2}\Xi$. In this way we get:

$$S_{\text{SP}} = \iint d^2x \left[-\frac{1}{4}i \sum_{j=1}^N \sum_{\omega=\pm} \sum_{\mu=0,1} \Xi'_{x,\omega,j} \gamma_\mu \partial_\mu \Xi'_{x,\omega,j} - \frac{\lambda_{\text{eff}}}{2N_{\text{eff}}} \left(\sum_{j=1}^N \sum_{\omega=\pm} \Xi'_{x,\omega,j} \Xi'_{x,\omega,j} \right)^2 \right], \quad (82)$$

with $N_{\text{eff}} = 2N$ and $\lambda_{\text{eff}} = 8\lambda$. We therefore conclude that the Staggered Plaquette model, after performing the formal continuum limit, reproduces the standard Gross-Neveu model with coupling λ_{eff} and N_{eff} flavors.

One-Loop Running Coupling. The Renormalization Group equation, for the running coupling constant $\lambda_{\text{eff}}(\ell)$ at energy scale ℓ^{-1} , reads:

$$\ell \frac{d\lambda_{\text{eff}}}{d\ell} = \beta(\lambda_{\text{eff}}),$$

with $\beta(\lambda_{\text{eff}})$ the Beta Function of the Gross-Neveu model, whose one-loop expression is [TV90]:

$$\beta(\lambda_{\text{eff}}) = \frac{1}{\pi} \left(1 - \frac{1}{N_{\text{eff}}} \right) \lambda_{\text{eff}}^2 + \mathcal{O}(\lambda_{\text{eff}}^3).$$

The integration of the Renormalization Group equation truncated at order λ_{eff}^2 , yields

$$\lambda_{\text{eff}}(\ell) = \frac{\lambda_{\text{eff}}(\ell_0)}{1 + \frac{1}{\pi} \left(1 - \frac{1}{N_{\text{eff}}} \right) \lambda_{\text{eff}}(\ell_0)^2 \log\left(\frac{\ell_0}{\ell}\right)}.$$

After writing $N_{\text{eff}} = 2N$ and $\lambda_{\text{eff}}(\ell) = 8\lambda(\ell)$, we obtain:

$$\lambda(\ell) = \frac{\lambda(\ell_0)}{1 + \frac{8}{\pi} \left(1 - \frac{1}{2N} \right) \lambda(\ell_0)^2 \log\left(\frac{\ell_0}{\ell}\right)}.$$

At the lowest order in $\frac{1}{N}$, this expression coincides with Equation (12) for $\alpha = \text{SP}$.

4.D Quantitative Bounds on the Two-Point Function

4.D.1 Lower Bound on the Two-Point Function

Since:

$$1 = I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)} + I_x^{\star+} + I_x^{\star-},$$

the two-point function can be decomposed as:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha} = \sum_{i,j \in \mathcal{R}} \langle \phi_x \phi_y I_x^{(i)} I_y^{(j)} \rangle_{\mu_\alpha}, \quad (83)$$

where the sum runs over the five regions $\mathcal{R}$ listed in Item 1–Item 5 in Subsection 4.3.1. These contributions can be naturally grouped into the three following classes.

- **DT (dominant term).** The two fields lie near the *same minimum*:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{DT}} \doteq \langle \phi_x \phi_y (I_x^{\star+} I_y^{\star+} + I_x^{\star-} I_y^{\star-}) \rangle_{\mu_\alpha}.$$

- **PT (Peierls' term).** The two fields lie around *different minima*:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{PT}} \doteq \langle \phi_x \phi_y (I_x^{*+} I_y^{*-} + I_x^{*-} I_y^{*+}) \rangle_{\mu_\alpha}.$$

- **EUT (energetic unfavorable term).** At least one of the two fields lies in an *energetically unfavorable region*:

$$\begin{aligned} \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{EUT}} &\doteq \left\langle \phi_x \phi_y (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} \\ &\quad + \left\langle \phi_x \phi_y (I_x^{*+} + I_x^{*-}) (I_y^{+K_\alpha} + I_y^{-K_\alpha} + I_y^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha}. \end{aligned}$$

First, it is convenient to perform a preliminary bound of the two-point function as:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha} \geq \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{DT}} - \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{PT}} \right| - \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{EUT}} \right|. \quad (84)$$

The bound in Equation (22) is then obtained by estimating separately the three terms in the right-hand side of Equation (84).

Dominant Term (DT). We shall derive the following lower bound:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{DT}} \geq C_\alpha^2 - \frac{4C_\alpha^2}{K_\alpha^2} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} - \frac{8C_\alpha^2}{K_\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} - 3C_\alpha^2 \langle I_x^{(-C_\alpha, C_\alpha)} \rangle_{\mu_\alpha} - 2C_\alpha^2 \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha}, \quad (85)$$

where notice that, due to the translation invariance of the model (see Footnote 9), averages of the form $\langle f(\phi_x) \rangle_{\mu_\alpha}$ are actually constant with respect to $x \in \Gamma_L^\alpha$.

Proof of Equation (85). Since $\phi_x \phi_y \geq C_\alpha^2$ on the support of $I_x^{\pm} I_y^{\pm}$, we obtain:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{DT}} = \langle \phi_x \phi_y (I_x^{*+} I_y^{*+} + I_x^{*-} I_y^{*-}) \rangle_{\mu_\alpha} \geq C_\alpha^2 \langle I_x^{*+} I_y^{*+} + I_x^{*-} I_y^{*-} \rangle_{\mu_\alpha}.$$

Moreover, using the identity:

$$I_x^{*+} I_y^{*+} + I_x^{*-} I_y^{*-} = (I_x^{*+} + I_x^{*-})(I_y^{*+} + I_y^{*-}) - (I_x^{*+} I_y^{*-} + I_x^{*-} I_y^{*+}),$$

we find that

$$\begin{aligned} \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{DT}} &\geq C_\alpha^2 \langle (I_x^{*+} I_y^{*+} + I_x^{*-} I_y^{*-}) \rangle_{\mu_\alpha} \\ &\geq C_\alpha^2 \langle (I_x^{*+} + I_x^{*-})(I_y^{*+} + I_y^{*-}) \rangle_{\mu_\alpha} - 2C_\alpha^2 \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha} \\ &= C_\alpha^2 \left\langle (1_x - I_x^{+K_\alpha} - I_x^{-K_\alpha} - I_x^{(-C_\alpha, C_\alpha)})(1_y - I_y^{+K_\alpha} - I_y^{-K_\alpha} - I_y^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} - 2C_\alpha^2 \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha} \\ &= C_\alpha^2 + C_\alpha^2 \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)})(I_y^{+K_\alpha} + I_y^{-K_\alpha} + I_y^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} \\ &\quad - 2C_\alpha^2 \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} - 2C_\alpha^2 \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha} \\ &\geq C_\alpha^2 - C_\alpha^2 \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)})(I_y^{+K_\alpha} + I_y^{-K_\alpha} + I_y^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} \\ &\quad - 2C_\alpha^2 \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} - 2C_\alpha^2 \langle I_x^{*+} I_y^{*-} \rangle_{\mu_\alpha}. \end{aligned} \quad (86)$$

This estimate yields a strictly positive lower bound on the dominant contribution, up to additional subdominant terms of the form PT and EUT, which can be controlled as follows, using the Cauchy-Schwartz inequality and the invariance of μ_α under the transformation $\phi \mapsto -\phi$.

$$\begin{aligned}
& \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)}) (I_y^{+K_\alpha} + I_y^{-K_\alpha} + I_y^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} \\
&= \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha}) (I_y^{+K_\alpha} + I_y^{-K_\alpha}) \right\rangle_{\mu_\alpha} + \left\langle I_x^{(-C_\alpha, C_\alpha)} I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} + \left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha}) I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \\
&+ \left\langle I_x^{(-C_\alpha, C_\alpha)} (I_y^{+K_\alpha} + I_y^{-K_\alpha}) \right\rangle_{\mu_\alpha} \\
&\leq 2 \left\langle I_x^{+K_\alpha} I_y^{+K_\alpha} \right\rangle_{\mu_\alpha} + 2 \left\langle I_x^{+K_\alpha} I_y^{-K_\alpha} \right\rangle_{\mu_\alpha} + 4 \left\langle I_x^{+K_\alpha} \right\rangle_{\mu_\alpha} + \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \\
&\leq \frac{2}{K_\alpha^2} \langle |\phi_x| |\phi_y| (I_x^{+K_\alpha} I_y^{+K_\alpha} + I_x^{+K_\alpha} I_y^{-K_\alpha}) \rangle_{\mu_\alpha} + \frac{4}{K_\alpha} \langle \phi_x I_x^{+K_\alpha} \rangle_{\mu_\alpha} + \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \\
&\leq \frac{4}{K_\alpha^2} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} + \frac{4}{K_\alpha} \langle \phi_x I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} + \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha},
\end{aligned}$$

and

$$\left\langle (I_x^{+K_\alpha} + I_x^{-K_\alpha} + I_x^{(-C_\alpha, C_\alpha)}) \right\rangle_{\mu_\alpha} \leq \frac{2}{K_\alpha} \langle \phi_x I_x^{+K_\alpha} \rangle_{\mu_\alpha} + \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \leq \frac{2}{K_\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} + \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}.$$

Collecting together all the contributions above, Equation (85) follows. ■

Energetically Unfavorable Terms (EUT). We shall prove the following upper bound:

$$\left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{EUT}} \right| \leq 4 \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha} + 4(C_\alpha + 2K_\alpha) \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}^{\frac{1}{2}} + (4K_\alpha C_\alpha + C_\alpha^2) \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}. \quad (87)$$

Proof of Equation (87). We can rewrite

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{EUT}} = \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K,K}} + \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K,C}} + \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K},\star} + \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{C},\star} + \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{C,C}},$$

where:

$$\begin{aligned}
\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K,K}} &\doteq \langle \phi_x \phi_y (I_x^{+K_\alpha} + I_x^{-K_\alpha}) (I_y^{+K_\alpha} + I_y^{-K_\alpha}) \rangle_{\mu_\alpha}, \\
\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K,C}} &\doteq \left\langle \phi_x \phi_y [(I_x^{+K_\alpha} + I_x^{-K_\alpha}) I_y^{(-C_\alpha, C_\alpha)} + I_x^{(-C_\alpha, C_\alpha)} (I_y^{+K_\alpha} + I_y^{-K_\alpha})] \right\rangle_{\mu_\alpha}, \\
\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K},\star} &\doteq \langle \phi_x \phi_y [(I_x^{+K_\alpha} + I_x^{-K_\alpha}) (I_y^{\star+} + I_y^{\star-}) + (I_x^{\star+} + I_x^{\star-}) (I_y^{+K_\alpha} + I_y^{-K_\alpha})] \rangle_{\mu_\alpha}, \\
\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{C},\star} &\doteq \left\langle \phi_x \phi_y [I_x^{(-C_\alpha, C_\alpha)} (I_y^{\star+} + I_y^{\star-}) + (I_x^{\star+} + I_x^{\star-}) I_y^{(-C_\alpha, C_\alpha)}] \right\rangle_{\mu_\alpha}, \\
\langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{C,C}} &\doteq \left\langle \phi_x \phi_y I_x^{(-C_\alpha, C_\alpha)} I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}.
\end{aligned} \quad (88)$$

We are going to bound the several terms above separately.

1. **Bound on K,K term.** Using the Cauchy-Schwartz inequality and the invariance of μ_α under the transformation $\phi \mapsto -\phi$, we get:

$$\begin{aligned}
\sup_{x,y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{K,K}} \right| &\leq 2 \sup_{x,y \in \Gamma_L^\alpha} \langle \phi_x \phi_y I_x^{+K_\alpha} I_y^{+K_\alpha} \rangle_{\mu_\alpha} + 2 \sup_{x,y \in \Gamma_L^\alpha} \langle |\phi_x \phi_y| I_x^{+K_\alpha} I_y^{-K_\alpha} \rangle_{\mu_\alpha} \\
&\leq 4 \sup_{x \in \Gamma_L^\alpha} \langle \phi_x^2 I_x^{+K_\alpha} \rangle_{\mu_\alpha}.
\end{aligned}$$

2. **Bound on K,C term.** Similarly to the previous item:

$$\begin{aligned} \sup_{x,y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{K,C} \right| &\leq 4 \sup_{x,y \in \Gamma_L^\alpha} \left\langle |\phi_x| |\phi_y| I_x^{+K_\alpha} I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \\ &\leq 4C_\alpha \sup_{x,y \in \Gamma_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \left\langle I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \leq 4C_\alpha \sup_{x \in \Gamma_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}}. \end{aligned}$$

3. **Bound on K,★ term:**

$$\begin{aligned} \sup_{x,y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{K,\star} \right| &\leq 4 \sup_{x,y \in \Gamma_L^\alpha} \left\langle \phi_x \phi_y I_x^{+K_\alpha} I_y^{\star+} \right\rangle_{\mu_\alpha} + 4 \sup_{x,y \in \Gamma_L^\alpha} \left\langle |\phi_x \phi_y| I_x^{+K_\alpha} I_y^{\star-} \right\rangle_{\mu_\alpha} \\ &\leq 8K_\alpha \sup_{x \in \Gamma_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}}. \end{aligned}$$

4. **Bound on C,★ term:**

$$\sup_{x,y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{C,\star} \right| \leq 4 \left\langle |\phi_x \phi_y| I_x^{(-C_\alpha, C_\alpha)} I_y^{\star+} \right\rangle_{\mu_\alpha} \leq 4K_\alpha C_\alpha \sup_{x \in \Gamma_L^\alpha} \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}.$$

5. **Bound on C,C term:**

$$\sup_{x,y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{C,C} \right| \leq \sup_{x,y \in \Gamma_L^\alpha} \left\langle |\phi_x \phi_y| I_x^{(-C_\alpha, C_\alpha)} I_y^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \leq C_\alpha^2 \sup_{x \in \Gamma_L^\alpha} \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha}.$$

Collecting all the contributions, we find the claimed inequality shown in Equation (87). ■

Peierls' terms (PT). Using again the invariance of μ_α under $\phi \mapsto -\phi$, we get that

$$\left| \langle \phi_x \phi_y \rangle_{\mu_\alpha}^{\text{PT}} \right| \leq 2 \left\langle |\phi_x \phi_y| I_x^{\star+} I_y^{\star-} \right\rangle_{\mu_\alpha} \leq 2K_\alpha^2 \left\langle I_x^{\star+} I_y^{\star-} \right\rangle_{\mu_\alpha}. \quad (89)$$

Plugging Equation (85), Equation (87) and Equation (89) into the right-hand side of Equation (84), we finally get:

$$\begin{aligned} \inf_{x,y \in \Lambda_L^\alpha} \left\langle \phi_x \phi_y \right\rangle_{\mu_\alpha} &\geq C_\alpha^2 - \left(2(K_\alpha^2 + C_\alpha^2) \sup_{x,y \in \Lambda_L^\alpha} \left\langle I_x^{\star+} I_y^{\star-} \right\rangle_{\mu_\alpha} + 4C_\alpha(C_\alpha + K_\alpha) \sup_{x \in \Lambda_L^\alpha} \left\langle I_x^{(-C_\alpha, C_\alpha)} \right\rangle_{\mu_\alpha} \right. \\ &\quad \left. + 4\left(1 + \frac{C_\alpha^2}{K_\alpha^2}\right) \sup_{x \in \Lambda_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha} + 4\left(C_\alpha + 2\frac{C_\alpha^2}{K_\alpha} + 2K_\alpha\right) \sup_{x \in \Lambda_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \right), \end{aligned}$$

which is nothing but Equation (22).

4.D.2 Upper Bound on the Two-Point Function

In order to perform an upper bound, we can decompose the two-point function as:

$$\langle \phi_x \phi_y \rangle_{\mu_\alpha} = \left\langle \phi_x \phi_y (I_x^{(-K_\alpha, K_\alpha)} + I_x^{(-K_\alpha, K_\alpha)^c}) (I_y^{(-K_\alpha, K_\alpha)} + I_y^{(-K_\alpha, K_\alpha)^c}) \right\rangle_{\mu_\alpha},$$

where $I_x^{(-K_\alpha, K_\alpha)} \doteq \mathbb{1}_{\{-L_\alpha < \phi_x < K_\alpha\}}$ and $I_x^{(-K_\alpha, K_\alpha)^c} \doteq 1 - I_x^{(-K_\alpha, K_\alpha)}$. Proceeding as in the previous subsection, we can decompose the correlator as:

$$\begin{aligned} \langle \phi_x \phi_y \rangle_{\mu_\alpha} &= \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)} \right\rangle_{\mu_\alpha} + 2 \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} + \\ &\quad + \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)^c} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha}. \end{aligned} \quad (90)$$

The three terms appearing in the right-hand side of Equation (90) can be upper-bounded as follows.

$$\begin{aligned} \left| \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)} \right\rangle_{\mu_\alpha} \right| &\leq \left\langle |\phi_x| |\phi_y| I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)} \right\rangle_{\mu_\alpha} \\ &\leq \left\langle I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)} \right\rangle_{\mu_\alpha} \\ &\leq K_\alpha^2; \\ \left| \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} \right| &\leq \left\langle |\phi_x| |\phi_y| I_x^{(-K_\alpha, K_\alpha)} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} \\ &\leq K_\alpha \left\langle |\phi_y| I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} \\ &= 2K_\alpha \left\langle \phi_x I_x^{+K_\alpha} \right\rangle_{\mu_\alpha} \\ &\leq 2K_\alpha \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}}; \\ \left| \left\langle \phi_x \phi_y I_x^{(-K_\alpha, K_\alpha)^c} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} \right| &\leq \left\langle |\phi_x| |\phi_y| I_x^{(-K_\alpha, K_\alpha)^c} I_y^{(-K_\alpha, K_\alpha)^c} \right\rangle_{\mu_\alpha} \\ &= 2 \left\langle \phi_x \phi_y I_x^{+K_\alpha} I_y^{+K_\alpha} \right\rangle_{\mu_\alpha} + 2 \left\langle |\phi_x \phi_y| I_x^{-K_\alpha} I_y^{+K_\alpha} \right\rangle_{\mu_\alpha} \\ &\leq 2 \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \left\langle \phi_y^2 I_y^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} + 2 \left\langle \phi_x^2 I_x^{-K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \left\langle \phi_y^2 I_y^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \\ &= 4 \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} \left\langle \phi_y^2 I_y^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}}. \end{aligned}$$

Therefore, going back to Equation (90):

$$\sup_{x, y \in \Gamma_L^\alpha} \langle \phi_x \phi_y \rangle_{\mu_\alpha} \leq \sup_{x, y \in \Gamma_L^\alpha} \left| \langle \phi_x \phi_y \rangle_{\mu_\alpha} \right| \leq K_\alpha^2 + 2K_\alpha \sup_{x \in \Gamma_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha}^{\frac{1}{2}} + 4 \sup_{x \in \Gamma_L^\alpha} \left\langle \phi_x^2 I_x^{+K_\alpha} \right\rangle_{\mu_\alpha},$$

which is nothing but Equation (23).

Part III

Flux Phases and gauge Fields

Fermions coupled to $\mathbb{Z}_2$ -Gauge Fields

This chapter introduces the common Hilbert-space formalism underlying the two quantum models studied in [Chapter 6](#) and [Chapter 7](#) and developed in the joint works [\[GP26; BGP26\]](#). In both models, fermionic matter on the vertices of a square lattice is coupled to $\mathbb{Z}_2$ -valued gauge degrees of freedom carried by its edges. The coupling is implemented by dressing each fermionic hopping term with the vector-potential operator on the corresponding edge, while the physical Hilbert space is selected by a local Gauss-law constraint.

The purpose of the chapter is to formulate these common ingredients once, independently of the particular matter Hamiltonian. We first describe the pure $\mathbb{Z}_2$ -gauge sector, including its Hilbert space, star and Wilson operators, gauge transformations, Gauss law, and decomposition into flux sectors. These are standard lattice-gauge constructions originating in works such as [\[Weg71; FS78\]](#) and reviewed, for example, in [\[Fra13\]](#) (see [Section 5.1](#)). We then couple the gauge field to fermionic matter by gauging fermion parity. The construction of [Section 5.2](#), including the physical observables and Gauss-law constraint defined in [Definition 64](#) and [Definition 65](#), uses standard ideas specialized to the finite square torus and to the models without an electric-field term considered in [Chapter 6](#) and [Chapter 7](#).

The final part of the chapter concerns the physical Gibbs state and its decomposition over fixed vector-potential backgrounds. The gauge variables are local degrees of freedom of the model, rather than prescribed external fields, and must therefore be included in the physical trace. This remains true when the vector-potential operators commute with the Hamiltonian because no electric-field term is present. In this case, the dynamics does not produce transitions between distinct backgrounds, but the partition function still sums over all of them subject to the Gauss-law constraint. This observation makes it possible to reduce the spectral and thermal analysis to fermionic Hamiltonians in fixed $\mathbb{Z}_2$ backgrounds.

The chapter does not claim a new general gauge-theory formalism. Its contribution is a reusable and internally consistent implementation of the fixed-background reduction in precisely the form required by the two subsequent applications. In particular, the isospectrality of gauge-equivalent backgrounds in [Proposition 66](#), the resolution of the Gauss-law projector in [Proposition 67](#), and the resulting parity-twisted trace formula are proved in the form used later. These technical propositions should therefore be understood as thesis-specific synthesis and infrastructure for the original results of [Chapter 6](#) and [Chapter 7](#), rather than as stand-alone foundational advances.

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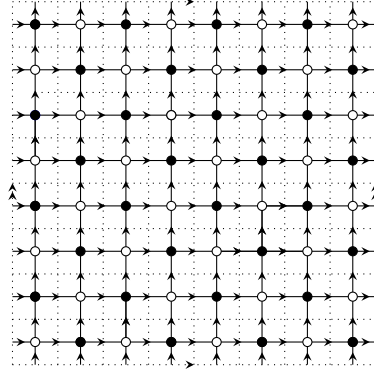


Figure 5.1: Oriented lattice Λ_L and its dual Λ_L^*

Notations. The lattice Λ_L is bipartite, meaning that it consists of two sublattices Λ_L^A and Λ_L^B which are represented with white and black dots respectively. We will denote with $V(\Lambda_L)$, $E(\Lambda_L)$ and $F(\Lambda_L)$ respectively the set of vertices, edges and faces of the lattice Λ_L . Their elements will be written as $i \in V(\Lambda_L)$, $(i, j) \in E(\Lambda_L)$ (with the convention that j and i are respectively the starting and arrival point according to the orientation shown in the picture) and $p = (i, j, k, l) \in F(\Lambda_L)$. Often, with a slight abuse of notation, we will identify the set of vertices $V(\Lambda_L)$ with the lattice Λ_L itself. The set of vertices, edges and faces of the dual lattice Λ_L^* of Λ_L (sketched with dotted lines) will be likewise denoted with $V(\Lambda_L^*)$, $E(\Lambda_L^*)$ and $F(\Lambda_L^*)$. Of course, there is a bijective correspondence between $F(\Lambda_L) \simeq V(\Lambda_L^*)$ and $V(\Lambda_L) \simeq F(\Lambda_L^*)$, which we shall sometimes use without further mention. In particular, the four edges sharing one vertex can be thought of as the dual edges bounding the corresponding dual face.

Homology of the torus: basic definitions and notations We recall that $\Lambda = \mathbb{Z}_L^2$ is the square lattice with periodic boundary conditions, and that $V(\Lambda)$, $E(\Lambda)$, $F(\Lambda)$ are the sets of all vertices, oriented edges and oriented faces of Λ . By construction $|V(\Lambda)| = |F(\Lambda)|$ and $|E(\Lambda)| = 2|V(\Lambda)|$. This structure is naturally understood as a CW-complex, and we briefly recall the basic constructions of cellular homology. Since orientation does not play a role in the analysis of this thesis, we do not insist on it here.

Definition 57 (Chains). For $i = 0, 1, 2$, the i -chain group $C_i(\Lambda)$ is a free abelian group with coefficients in $\mathbb{Z}_2$ generated respectively by $V(\Lambda)$, $E(\Lambda)$ and $F(\Lambda)$.

Geometrically, a 1-chain $\mathcal{C}$ is a string along the edges of the lattice, or a collection thereof. The fact that the coefficients are $\mathbb{Z}_2$ simply means (in the example of $i = 1$) that an edge may appear at most once.

The boundary maps $\partial_i : C_i(\Lambda) \rightarrow C_{i-1}(\Lambda)$ associate to a i -chain the $(i-1)$ -chain that forms its boundary. For example, if Λ is a 2-chain, namely a collection of plaquettes, then $\partial\Lambda$ is the 1-chain made of all the boundary edges. As usual, $\partial_i \circ \partial_{i+1} = 0$, namely $\text{Im}(\partial_{i+1}) \subset \text{Ker}(\partial_i)$.

Definition 58 (Cycles and Boundaries). The i -cycle group is $Z_i(\Lambda) = \text{Ker}(\partial_i)$. The i -boundary group is $B_i(\Lambda) = \text{Im}(\partial_{i+1})$.

In other words, a 1-cycle is geometrically a loop or a collection thereof. A 1-boundary is the geometric boundary of a collection of plaquettes. As pointed out above, $B_i(\Lambda) \subset Z_i(\Lambda)$: For the case $i = 1$, any boundary is indeed a collection of loops.

Definition 59 (homology groups). The homology groups are defined as:

$$H_i(\Lambda) = \frac{Z_i(\Lambda)}{B_i(\Lambda)}. \quad (1)$$

The group $H_1(\Lambda)$ is made up of equivalence classes of loops, where two loops are equivalent if they differ by a boundary (one could say: they can be ‘deformed’ into each other). On the torus, $H_1(\Lambda)$ has four elements: It is generated by the two types of loops that wind once around the torus in either direction. In this thesis, they are associated with the holonomies ([Chapter 4](#), [Chapter 6](#), and [Chapter 7](#)) or large gauge transformations ([Chapter 6](#) and [Chapter 7](#)).

Cohomology. The regular square lattice has a natural dual obtained by identifying each face with a dual vertex, each edge with a dual edge (which is geometrically perpendicular to it) and each vertex with a dual face. The construction above can be repeated with this dual complex, yielding cochain groups $C_i(\Lambda^*)$, cocycles $Z_i(\Lambda^*)$ and coboundaries $B_i(\Lambda^*)$, and cohomology groups. For example, a 1-cochain C^* is a string in the dual lattice or a collection thereof.

5.1 The Pure $\mathbb{Z}_2$ -gauge Sector

The gauge-field Hilbert space To every edge $(i, j) \in E(\Lambda_L)$ of the lattice, we associate the two-dimensional Hilbert space

$$\mathcal{H}_{ij}^g = \mathbb{C}[\mathbb{Z}_2] \simeq \mathbb{C}^2,$$

representing the local gauge-field degree of freedom. Its canonical basis is

$$|+1\rangle_{ij}, \quad |-1\rangle_{ij}.$$

These vectors are the eigenstates of the vector-potential operator $\hat{\sigma}_{ij}^z$:

$$\hat{\sigma}_{ij}^z |\pm 1\rangle_{ij} = \pm |\pm 1\rangle_{ij}.$$

The electric-field operator $\hat{\sigma}_{ij}^x$ exchanges the two vector-potential eigenstates. Thus, on each edge, the vector-potential and electric-field operators satisfy the $\mathbb{Z}_2$ -analogue of the Weyl relations:

$$\hat{\sigma}_{ij}^z \hat{\sigma}_{ij}^x = -\hat{\sigma}_{ij}^x \hat{\sigma}_{ij}^z, \quad (\hat{\sigma}_{ij}^x)^2 = (\hat{\sigma}_{ij}^z)^2 = \mathbf{1}_{ij}.$$

These relations provide the finite-dimensional counterpart of the canonical commutation relations between vector potential and electric field.

Definition 60 (The total Hilbert space of the gauge sector). The total Hilbert space of the gauge sector is

$$\mathcal{H}_L^g = \bigotimes_{(i,j) \in E(\Lambda_L)} \mathcal{H}_{ij}^g. \quad (2)$$

The algebra of operators acting on $\mathcal{H}_L^g$ is generated by the vector-potential and electric-field operators associated with all edges. Operators supported on distinct edges commute; in particular,

$$\hat{\sigma}_{ij}^z \hat{\sigma}_{kl}^x = \hat{\sigma}_{kl}^x \hat{\sigma}_{ij}^z, \quad \hat{\sigma}_{ij}^z \hat{\sigma}_{kl}^z = \hat{\sigma}_{kl}^z \hat{\sigma}_{ij}^z, \quad \hat{\sigma}_{ij}^x \hat{\sigma}_{kl}^x = \hat{\sigma}_{kl}^x \hat{\sigma}_{ij}^x,$$

whenever $(i,j) \neq (k,l)$. We denote this algebra by $\mathcal{A}_L^g$, in order to distinguish it from the CAR algebra describing fermionic matter.

Star operators, gauge transformations and gauge-invariant operators For every vertex $i \in V(\Lambda_L)$, introduce the star operator (see Figure 5.2)

$$A_i \doteq \prod_{(k,l) \in \partial * i} \hat{\sigma}_{kl}^x, \quad (3)$$

where $\partial * i$ denotes the set of edges incident on i .

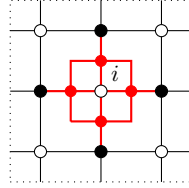


Figure 5.2: The operator A_i acts on the four edges incident on the lattice site $i \in \Lambda_L$.

The star operators are self-adjoint unitaries and mutually commute. Moreover, since every edge has two endpoints, each electric-field operator occurs exactly twice in the product of all star operators. Consequently,

$$\prod_{i \in V(\Lambda_L)} A_i = 1.$$

In particular, $\text{spec}(A_i) = \{\pm 1\}$. Conjugation by A_i implements the local gauge transformation at i :

$$A_i \hat{\sigma}_{kl}^z A_i = \begin{cases} -\hat{\sigma}_{kl}^z, & i \in \{k, l\}, \\ \hat{\sigma}_{kl}^z, & i \notin \{k, l\}, \end{cases} \quad A_i \hat{\sigma}_{kl}^x A_i = \hat{\sigma}_{kl}^x.$$

The star operators therefore allow us to identify the gauge-invariant operators. Let $\mathcal{A}$ denote the group generated by the A_i 's. The following definition expresses gauge invariance as commutation with every local gauge transformation.

Definition 61 (Pure-gauge physical observables). The algebra of gauge-invariant observables in the pure gauge sector is the commutant

$$\mathcal{C}_{\mathcal{A}_L^g}(\mathcal{A}) = \{\mathcal{O} \in \mathcal{A}_L^g : [\mathcal{O}, A_i] = 0 \text{ for every } i \in V(\Lambda_L)\}.$$

We next introduce string operators associated with chains of the direct and dual lattices:

$$\begin{aligned}\hat{Z} : C_1(\Lambda_L) &\longrightarrow \mathcal{A}_L^g, & \mathcal{C} &\longmapsto \hat{Z}_{\mathcal{C}} \doteq \prod_{(i,j) \in \mathcal{C}} \hat{\sigma}_{ij}^z, \\ \hat{X} : C_1(\Lambda_L^*) &\longrightarrow \mathcal{A}_L^g, & \mathcal{C}^* &\longmapsto \hat{X}_{\mathcal{C}^*} \doteq \prod_{\substack{(i,j) \in E(\Lambda_L) \\ (i,j) \cap \mathcal{C}^* \neq \emptyset}} \hat{\sigma}_{ij}^x.\end{aligned}$$

Here the chain groups are understood to have coefficients in $\mathbb{Z}_2$. Consequently, these maps are group homomorphisms:

$$\hat{Z}_{\mathcal{C}_1 + \mathcal{C}_2} = \hat{Z}_{\mathcal{C}_1} \hat{Z}_{\mathcal{C}_2}, \quad \hat{X}_{\mathcal{C}_1^* + \mathcal{C}_2^*} = \hat{X}_{\mathcal{C}_1^*} \hat{X}_{\mathcal{C}_2^*}. \quad (4)$$

For every plaquette $p \in P(\Lambda_L)$, define the magnetic-field, or plaquette-flux, operator

$$B_p \doteq \prod_{(j,l) \in \partial p} \hat{\sigma}_{jl}^z.$$

By construction,

$$\hat{Z}_{\partial p} = B_p.$$

More generally, if $\mathcal{C} = \partial \sum_{p \in \mathcal{S}} p$ is a boundary in Λ_L , then

$$\hat{Z}_{\mathcal{C}} = \prod_{p \in \mathcal{S}} B_p.$$

Likewise, if $\mathcal{C}^* = \partial \sum_{p^* \in \mathcal{S}^*} p^*$ is a boundary in the dual lattice, then

$$\hat{X}_{\mathcal{C}^*} = \prod_{\substack{i \in V(\Lambda_L) \\ i \leftrightarrow p^* \in \mathcal{S}^*}} A_i,$$

where the dual face p^* is identified with the corresponding vertex i of the direct lattice.

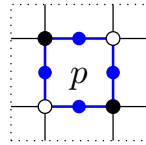


Figure 5.3: The magnetic-field operator B_p associated with a plaquette p .

On the torus, every edge belongs to the boundaries of exactly two plaquettes. Hence the plaquette-flux operators satisfy the global constraint

$$\prod_{p \in P(\Lambda_L)} B_p = \mathbf{1}. \quad (5)$$

The electric string $\hat{X}_{\mathcal{C}^*}$ is gauge-invariant for every dual chain $\mathcal{C}^* \in C_1(\Lambda_L^*)$, since it is constructed entirely from mutually commuting electric-field operators. A magnetic string is gauge-invariant precisely when every vertex belongs to an even number of edges of its support, namely when its support is a cycle. Thus,

$$\mathcal{C}^* \in C_1(\Lambda_L^*) \implies \hat{X}_{\mathcal{C}^*} \in \mathcal{C}_{\mathcal{A}_L^g}(\mathcal{A}), \quad (6)$$

$$\mathcal{C} \in Z_1(\Lambda_L) \implies \hat{Z}_{\mathcal{C}} \in \mathcal{C}_{\mathcal{A}_L^g}(\mathcal{A}). \quad (7)$$

Gauge orbits and holonomies A $\mathbb{Z}_2$ -vector-potential configuration is a collection

$$\sigma = \{\sigma_{ij}\}_{(i,j) \in E(\Lambda_L)}, \quad \sigma_{ij} \in \{\pm 1\}. \quad (8)$$

The corresponding vector-potential eigenstate is

$$|\sigma\rangle \doteq \bigotimes_{(i,j) \in E(\Lambda_L)} |\sigma_{ij}\rangle_{ij}.$$

Two configurations σ and σ' are gauge-equivalent if there exists a collection $\{\tau_i\}_{i \in V(\Lambda_L)}$, with $\tau_i \in \{\pm 1\}$, such that

$$\sigma'_{ij} = \tau_i \sigma_{ij} \tau_j \quad \text{for every } (i,j) \in E(\Lambda_L). \quad (9)$$

At the operator level, this transformation is implemented by

$$|\sigma'\rangle = \left(\prod_{i \in V(\Lambda_L)} A_i^{(1-\tau_i)/2} \right) |\sigma\rangle. \quad (10)$$

Since Λ_L has no boundary, the simultaneous choice $\tau_i = -1$ at every vertex is the only non-trivial choice of the τ_i 's that acts trivially on every edge. Consequently, the gauge orbit of any configuration contains

$$2^{|V(\Lambda_L)|-1}$$

distinct configurations. Since there are $2^{|E(\Lambda_L)|}$ vector-potential configurations altogether, the number of gauge-equivalence classes is

$$2^{|E(\Lambda_L)|-|V(\Lambda_L)|+1}.$$

For the square torus, $|E(\Lambda_L)| = 2|V(\Lambda_L)|$, and therefore this number becomes

$$2^{|V(\Lambda_L)|+1}.$$

The gauge-equivalence class of a configuration is completely determined by its $\mathbb{Z}_2$ -holonomies along closed circuits. A basis of the $\mathbb{Z}_2$ -cycle space may be chosen as the boundaries of all but one elementary plaquette, together with two non-contractible cycles $\mathcal{C}_x$ and $\mathcal{C}_y$ generating the homology of the torus. We denote the corresponding Wilson-loop operators by

$$\hat{Z}_x \doteq \hat{Z}_{\mathcal{C}_x}, \quad \hat{Z}_y \doteq \hat{Z}_{\mathcal{C}_y}.$$

Thus, a gauge-equivalence class is specified by the plaquette fluxes $\Phi_p \doteq B_p(\sigma)$, subject to the global constraint in Equation (5), together with the two global holonomies measured by $a \doteq \hat{Z}_x(\sigma)$ and $b \doteq \hat{Z}_y(\sigma)$.

The physical gauge-field Hilbert space Since the star operators mutually commute, the gauge-field Hilbert space decomposes into simultaneous eigenspaces:

$$\mathcal{H}_L^g = \bigoplus_{\substack{\bar{a} \in \{\pm 1\}^{V(\Lambda_L)} \\ \prod_i a_i = 1}} \mathcal{H}_L^{g,\bar{a}}, \quad (11)$$

where

$$\mathcal{H}_L^{g,\bar{a}} \doteq \{|\Psi\rangle \in \mathcal{H}_L^g : A_i|\Psi\rangle = a_i|\Psi\rangle \text{ for every } i \in V(\Lambda_L)\}.$$

The restriction $\prod_i a_i = 1$ follows from the identity $\prod_i A_i = \mathbf{1}$. The eigenvalues a_i may be interpreted as a distribution of fixed background $\mathbb{Z}_2$ -charges: $a_i = +1$ represents the absence of a charge at i , whereas $a_i = -1$ represents its presence.

Remark 42.

1. For every compatible collection $\bar{a}$, the orthogonal projector onto $\mathcal{H}_L^{g,\bar{a}}$ is

$$P_{\bar{a}} \doteq \prod_{i \in V(\Lambda_L)} \left(\frac{1 + a_i A_i}{2} \right), \quad \mathcal{H}_L^{g,\bar{a}} = P_{\bar{a}} \mathcal{H}_L^g.$$

2. By definition, every physical observable leaves each superselection sector invariant. Indeed, if $\mathcal{O} \in \mathcal{C}_{\mathcal{A}_L^g}(\mathcal{A})$, then, for every $|\Psi\rangle \in \mathcal{H}_L^{g,\bar{a}}$,

$$A_i \mathcal{O} |\Psi\rangle = \mathcal{O} A_i |\Psi\rangle = a_i \mathcal{O} |\Psi\rangle.$$

Thus, the algebra of physical observables is block diagonal with respect to the decomposition in Equation (11).

3. Since only $|V(\Lambda_L)| - 1$ of the conditions $A_i |\Psi\rangle = a_i |\Psi\rangle$ are independent,

$$\dim \mathcal{H}_L^{g,\bar{a}} = 2^{|E(\Lambda_L)| - |V(\Lambda_L)| + 1}.$$

For the square torus this becomes

$$\dim \mathcal{H}_L^{g,\bar{a}} = 2^{|V(\Lambda_L)| + 1}.$$

which are precisely the inequivalent classes of vector potentials.

We define the physical Hilbert space as the sector containing no background charges.

Definition 62 (Gauss' law). A state $|\Psi\rangle \in \mathcal{H}_L^g$ satisfies the vacuum Gauss law if

$$A_i |\Psi\rangle = |\Psi\rangle \quad \text{for every } i \in V(\Lambda_L).$$

The corresponding simultaneous eigenspace is denoted by

$$\mathcal{H}_L^{g,\text{phys}} \doteq \mathcal{H}_L^{g,\bar{1}}$$

and is called the physical pure-gauge Hilbert space.

The relation between the physical Hilbert space and the gauge orbits described above can be made explicit. For each gauge-equivalence class $[\sigma]$, define the normalized orbit state

$$|[\sigma]\rangle \doteq \frac{1}{2^{(|V(\Lambda_L)|-1)/2}} \sum_{\sigma' \sim \sigma} |\sigma'\rangle.$$

Every such state satisfies the vacuum Gauss law, and the collection of orbit states forms an orthonormal basis of $\mathcal{H}_L^{g,\text{phys}}$. The number of gauge orbits therefore agrees with the dimension of the physical Hilbert space.

A simple gauge-invariant Hamiltonian We finally introduce a simple Hamiltonian for the pure gauge sector of the $\mathbb{Z}_2$ -lattice gauge theory:

$$H_g \doteq - \sum_{p \in P(\Lambda_L)} B_p. \tag{12}$$

The Hamiltonian is gauge-invariant and preserves every superselection sector. On the physical Hilbert space, its ground states satisfy the flux-free condition

$$B_p|\Psi\rangle = |\Psi\rangle \quad \text{for every } p \in P(\Lambda_L).$$

Only $|P(\Lambda_L)| - 1$ of these conditions are independent, because of the global relation in Equation (5). The dimension of the ground-state space is therefore

$$\frac{2^{|\mathcal{E}(\Lambda_L)|}}{2^{|\mathcal{V}(\Lambda_L)|-1}2^{|P(\Lambda_L)|-1}} = 2^{-\chi(\Lambda_L)+2} = 4, \quad (13)$$

where

$$\chi(\Lambda_L) = |\mathcal{V}(\Lambda_L)| - |\mathcal{E}(\Lambda_L)| + |P(\Lambda_L)| = 0$$

is the Euler characteristic of the torus. The conditions $B_p = 1$ fix all contractible holonomies, while the holonomies around the two non-contractible cycles remain unconstrained. Consequently, the four ground states are uniquely labeled by the joint eigenvalues

$$(a, b) \in \{\pm 1\} \times \{\pm 1\}$$

of the Wilson-loop operators $\hat{Z}_x$ and $\hat{Z}_y$.

5.2 Gauging Fermion Parity

We now couple the pure gauge sector introduced above to fermionic matter by promoting the global fermion-parity symmetry to a $\mathbb{Z}_2$ -gauge symmetry. This construction provides the common framework for the two models studied in the following chapters.

The fermionic degrees of freedom are described by the Fock-space formalism introduced in Subsection 2.2.3. We only recall the notation needed here. Let $\mathcal{S}$ be the set of internal fermionic labels and let $\mathcal{F}_L$ be the Fock space over $\ell^2(\Lambda_L) \otimes \mathbb{C}^{\mathcal{S}}$. The spinless case corresponds to $|\mathcal{S}| = 1$, in which case the internal label will be suppressed, while for spin- $\frac{1}{2}$ fermions we take $\mathcal{S} = \{\uparrow, \downarrow\}$. We write

$$n_{i,\eta} \doteq a_{i,\eta}^+ a_{i,\eta}^-, \quad n_i \doteq \sum_{\eta \in \mathcal{S}} n_{i,\eta}, \quad N \doteq \sum_{i \in \mathcal{V}(\Lambda_L)} n_i,$$

and denote the local and global fermion-parity operators by

$$(-1)^{n_i} \quad \text{and} \quad (-1)^N = \prod_{i \in \mathcal{V}(\Lambda_L)} (-1)^{n_i},$$

respectively.

The Hilbert space of the coupled system is

$$\mathcal{H}_L \doteq \mathcal{H}_L^g \otimes \mathcal{F}_L. \quad (14)$$

The tensor product is the ordinary tensor product of Hilbert spaces. Accordingly, gauge-field and fermionic operators act on different factors and commute. We denote by $\mathcal{A}_L$ the algebra generated by $\mathcal{A}_L^{\text{gauge}}$ and by the fermionic creation and annihilation operators.

5.2.1 Local charges, physical observables and Gauss' law

The global fermion-parity transformation is promoted to a local $\mathbb{Z}_2$ transformation acting simultaneously on the fermionic degrees of freedom at a vertex and on the gauge fields carried by the incident edges.

Definition 63 ($\mathbb{Z}_2$ -charge operators). For every $i \in V(\Lambda_L)$, we define

$$Q_i \doteq A_i(-1)^{n_i}. \quad (15)$$

The charge operators are mutually commuting self-adjoint unitaries. Their action on the generators of $\mathcal{A}_L$ is

$$Q_i \hat{\sigma}_{kl}^z Q_i = \begin{cases} -\hat{\sigma}_{kl}^z, & i \in \{k, l\}, \\ \hat{\sigma}_{kl}^z, & i \notin \{k, l\}, \end{cases} \quad Q_i \hat{\sigma}_{kl}^x Q_i = \hat{\sigma}_{kl}^x \quad Q_i a_{j,\eta}^\pm Q_i = \begin{cases} -a_{j,\eta}^\pm, & j = i, \\ a_{j,\eta}^\pm, & j \neq i. \end{cases}$$

Thus, if $\tau_i \in \{\pm 1\}$, a local gauge transformation acts as

$$a_{i,\eta}^\pm \mapsto \tau_i a_{i,\eta}^\pm, \quad \hat{\sigma}_{ij}^z \mapsto \tau_i \hat{\sigma}_{ij}^z \tau_j.$$

Using $\prod_i A_i = \mathbf{1}$, we also obtain

$$\prod_{i \in V(\Lambda_L)} Q_i = (-1)^N.$$

Consequently, an operator commuting with all the local charges is automatically even under global fermion parity.

Definition 64 (Physical observables). Let $\mathcal{Q}$ be the group generated by the operators Q_i . The algebra of physical observables is the commutant

$$\mathcal{C}_{\mathcal{A}_L}(\mathcal{Q}) = \{\mathcal{O} \in \mathcal{A}_L : [\mathcal{O}, Q_i] = 0 \text{ for every } i \in V(\Lambda_L)\}.$$

A physical observable will be called magnetic if it belongs to the subalgebra generated by the vector-potential operators $\hat{\sigma}_{ij}^z$ and by the fermionic operators.

The pure-gauge string operators introduced in the preceding section remain physical observables of the coupled system. In addition, open Wilson lines can be used to dress fermionic bilinears. If $\mathcal{C}$ is a lattice path joining j to i , then

$$a_{i,\eta}^+ \hat{Z}_{\mathcal{C}} a_{j,\eta'}^-$$

is gauge-invariant: the signs generated at the endpoints by the fermionic operators are compensated by those generated by the open Wilson line.

Since the Q_i 's commute and have spectrum $\{\pm 1\}$, the total Hilbert space decomposes into charge sectors,

$$\mathcal{H}_L = \bigoplus_{\substack{\bar{q} \in \{\pm 1\}^{V(\Lambda_L)}: \\ \prod_i q_i = 1}} \mathcal{H}_L^{\bar{q}}, \quad \mathcal{H}_L^{\bar{q}} \doteq \{|\Psi\rangle \in \mathcal{H}_L : Q_i |\Psi\rangle = q_i |\Psi\rangle \text{ for every } i\}.$$

The corresponding orthogonal projector is

$$P_{\bar{q}} \doteq \prod_{i \in V(\Lambda_L)} \left(\frac{1 + q_i Q_i}{2} \right).$$

The relation $\prod_i Q_i = (-1)^N$ implies that the total fermion parity in the sector $\mathcal{H}_L^{\bar{q}}$ is

$$(-1)^N|_{\mathcal{H}_L^{\bar{q}}} = \prod_{i \in V(\Lambda_L)} q_i.$$

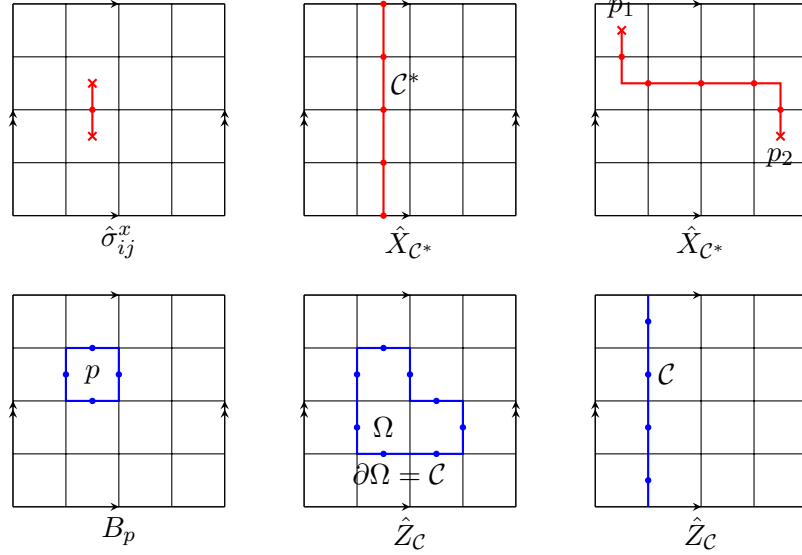


Figure 5.4: Graphical representation of some gauge-Invariant Operators

Definition 65 (Gauss' law). A state $|\Psi\rangle \in \mathcal{H}_L$ satisfies the vacuum Gauss law if

$$Q_i|\Psi\rangle = |\Psi\rangle \quad \text{for every } i \in V(\Lambda_L).$$

The physical Hilbert space of the coupled system is

$$\mathcal{H}_L^{\text{phys}} \doteq P_{\text{phys}} \mathcal{H}_L, \quad P_{\text{phys}} \doteq \prod_{i \in V(\Lambda_L)} \left(\frac{1 + Q_i}{2} \right). \quad (16)$$

Every physical state has even total fermion parity. Moreover, Gauss' law identifies the star operator with the local fermion parity:

$$A_i P_{\text{phys}} = (-1)^{n_i} P_{\text{phys}}. \quad (17)$$

For spinless fermions this becomes

$$A_i = 1 - 2n_i \quad \text{on } \mathcal{H}_L^{\text{phys}},$$

while for spin- $\frac{1}{2}$ fermions it gives

$$A_i = (1 - 2n_{i,\uparrow})(1 - 2n_{i,\downarrow}) = 4 \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right) \quad \text{on } \mathcal{H}_L^{\text{phys}}.$$

The latter identity explains why, after imposing Gauss' law, a Hubbard interaction can be interpreted as a star term.

5.2.2 Gauge-Invariant Hamiltonians and fixed backgrounds

The two models considered in the following chapters have no electric-field term. Their Hamiltonians belong to the common class

$$H = H_g(\hat{\sigma}^z) + H_f(\hat{\sigma}^z), \quad (18)$$

where H_g is given in Equation (12) and

$$H_f(\hat{\sigma}^z) = -t \sum_{(i,j) \in E(\Lambda_L)} \left(a_{i,\eta}^+ \hat{\sigma}_{ij}^z a_{j,\eta}^- + \text{h.c.} \right) + V.$$

Here V denotes a polynomial in the density operators encoding possible mass terms and self-interactions. Notice that different spins are possibly coupled only through the interaction term V . The spinless hopping model we study in [Chapter 6](#) is obtained by taking $|\mathcal{S}| = 1$ and $V = 0$, while the spinful model we will consider in [Chapter 7](#) staggered masses and Hubbard interactions.

The insertion of $\hat{\sigma}_{ij}^z$ in the hopping term is the lattice analogue of minimal coupling. A local gauge transformation at an endpoint of an edge changes the signs of both the fermionic operator and the vector potential, and the two signs cancel. Therefore,

$$[H, Q_i] = 0 \quad \text{for every } i \in V(\Lambda_L).$$

Since no electric-field operators occur in H , $[H, \hat{\sigma}_{ij}^z] = 0$ for every $(i, j) \in E(\Lambda_L)$. Thus the vector-potential configuration is conserved. In this sense the gauge background is frozen; it nevertheless remains a local degree of freedom that must be summed over in the partition function.

Let $\sigma = \{\sigma_{ij}\}_{(i,j) \in E(\Lambda_L)}$ be a joint eigenvalue configuration of the vector-potential operators. The total Hilbert space and Hamiltonian decompose as

$$\mathcal{H}_L = \bigoplus_{\sigma} (\mathbb{C}|\sigma\rangle \otimes \mathcal{F}_L), \quad H = \bigoplus_{\sigma} H(\sigma), \quad (19)$$

where $H(\sigma)$ acts on $\mathcal{F}_L$ and is obtained by replacing each $\hat{\sigma}_{ij}^z$ with its eigenvalue σ_{ij} .

5.2.3 Gauge-equivalent backgrounds and fluxes

The following result is the form of gauge covariance used in both applications. It shows that a fiber Hamiltonian does not depend spectrally on the choice of vector potential representing a fixed flux configuration.

Proposition 66 (Gauge-equivalent backgrounds are isospectral). *Suppose that σ and σ' have the same plaquette fluxes and the same holonomies around the two non-contractible cycles of Λ_L . Then there exists a unitary operator $U : \mathcal{F}_L \rightarrow \mathcal{F}_L$ such that*

$$UH(\sigma)U^* = H(\sigma'). \quad (20)$$

In particular, the spectrum of $H(\sigma)$ depends on the vector-potential configuration only through the plaquette fluxes and the two global holonomies.

Proof. For every edge (i, j) , define

$$w_{ij} \doteq \sigma_{ij} \sigma'_{ij}.$$

Since σ and σ' have the same plaquette fluxes and the same global holonomies, the product of the w_{ij} 's along every closed circuit is equal to 1. Indeed, every cycle on the torus is the sum, with $\mathbb{Z}_2$ coefficients, of plaquette boundaries and of the two chosen non-contractible cycles.

Fix a reference vertex i_0 . For every $i \in V(\Lambda_L)$, choose a path $\mathcal{C}_i$ from i_0 to i and set

$$\tau_i \doteq \prod_{(k,l) \in \mathcal{C}_i} w_{kl}.$$

The preceding observation implies that τ_i is independent of the chosen path. Applying it to the closed circuit obtained by following $\mathcal{C}_i$, then the edge (i, j) , and finally the reverse of $\mathcal{C}_j$, gives

$$\tau_i w_{ij} \tau_j = 1.$$

Since all factors take values in $\{\pm 1\}$, this is equivalent to $\sigma'_{ij} = \tau_i \sigma_{ij} \tau_j$. Thus the two backgrounds are gauge-equivalent.

Consider now the unitary operator

$$U \doteq \prod_{i \in V(\Lambda_L)} \tau_i^{n_i}.$$

It acts on every fermionic flavor at i as

$$U a_{i,\eta}^\pm U^* = \tau_i a_{i,\eta}^\pm.$$

Consequently,

$$U \left(a_{i,\eta}^+ \sigma_{ij} a_{j,\eta'}^- \right) U^* = \tau_i \tau_j a_{i,\eta}^+ \sigma_{ij} a_{j,\eta'}^- = a_{i,\eta}^+ \sigma'_{ij} a_{j,\eta'}^-.$$

The interaction V is gauge-invariant and is therefore left unchanged, while $H_g(\sigma) = H_g(\sigma')$ because the two backgrounds have the same plaquette fluxes. This proves Equation (20). $\blacksquare$

Remark 43. The same conclusion holds for the ordinary and parity-twisted thermal traces because U commutes with N and therefore with $(-1)^N$. Thus, both twisted/untwisted partition functions depend only on the plaquette fluxes and the two global holonomies.

5.2.4 Resolution of the gauge constraint

Let $\mathcal{O}$ be a magnetic physical observable and set

$$\mathcal{O}(\sigma) \doteq \langle \sigma | \mathcal{O} | \sigma \rangle.$$

Its Gibbs expectation on the physical Hilbert space is

$$\langle \mathcal{O} \rangle_{\beta,\mu,L}^{\text{phys}} \doteq \frac{\text{Tr}_{\mathcal{H}_L} [P_{\text{phys}} \mathcal{O} e^{-\beta(H-\mu N)}]}{\text{Tr}_{\mathcal{H}_L} [P_{\text{phys}} e^{-\beta(H-\mu N)}]}.$$

Proposition 67 (Resolution of the gauge constraint). *For every magnetic physical observable $\mathcal{O}$,*

$$\langle \mathcal{O} \rangle_{\beta,\mu,L}^{\text{phys}} = \frac{\sum_{\sigma} \text{Tr}_{\mathcal{F}_L} \left[(\mathbf{1} + (-1)^N) \mathcal{O}(\sigma) e^{-\beta(H(\sigma)-\mu N)} \right]}{\sum_{\sigma} \text{Tr}_{\mathcal{F}_L} \left[(\mathbf{1} + (-1)^N) e^{-\beta(H(\sigma)-\mu N)} \right]}. \quad (21)$$

In particular, the partition function on the physical Hilbert space is

$$Z_{\beta,\mu,L} = \frac{1}{2^{|V(\Lambda_L)|}} \sum_{\sigma} \text{Tr}_{\mathcal{F}_L} \left[(\mathbf{1} + (-1)^N) e^{-\beta(H(\sigma)-\mu N)} \right]. \quad (22)$$

Proof. We evaluate the trace in the occupation-number basis of $\mathcal{F}_L$ and in the joint eigenbasis of the vector-potential operators. For $\Omega \subseteq V(\Lambda_L)$, define

$$A_{\Omega} \doteq \prod_{i \in \Omega} A_i, \quad N_{\Omega} \doteq \sum_{i \in \Omega} n_i.$$

Expanding the projector in Equation (16) gives

$$P_{\text{phys}} = \frac{1}{2^{|V(\Lambda_L)|}} \sum_{\Omega \subseteq V(\Lambda_L)} A_{\Omega} (-1)^{N_{\Omega}}.$$

The operator A_Ω flips the vector potentials on the edges in the boundary of Ω . Since Λ_L is connected, the diagonal matrix element $\langle \sigma | A_\Omega | \sigma \rangle$ is non-zero only for $\Omega = \emptyset$ and $\Omega = V(\Lambda_L)$. The former contributes the identity, whereas in the latter case we get $(-1)^N$. Therefore,

$$\langle \sigma | P_{\text{phys}} | \sigma \rangle = \frac{1 + (-1)^N}{2^{|V(\Lambda_L)|}}$$

as an operator on $\mathcal{F}_L$. Since H and $\mathcal{O}$ contain no electric-field operators, they are diagonal in the vector-potential basis. Inserting the last identity in the trace gives Equation (22); the same computation with the insertion of $\mathcal{O}$, followed by cancellation of the common factor $2^{-|V(\Lambda_L)|}$, gives Equation (21). $\blacksquare$

Using Proposition 66, the sum in Equation (22) can be organized directly in terms of fluxes. Let $\Phi \doteq \{\Phi_p\}_p$ (with $\prod_p \Phi_p = 1$) be the set of magnetic fluxes thorough each plaquette and $a, b \in \{\pm 1\}$ the two global holonomies. By the counting performed in the preceding section, each choice of (Φ, a, b) has $2^{|V(\Lambda_L)|-1}$ vector-potential representatives.

Consequently,

$$Z_{\beta, \mu, L} = \frac{1}{2} \sum_{\substack{\Phi \in \{\pm 1\}^{P(\Lambda_L)} \\ \prod_p \Phi_p = 1}} \sum_{a, b = \pm 1} \text{Tr}_{\mathcal{F}_L} \left[(1 + (-1)^N) e^{-\beta(H(\Phi, a, b) - \mu N)} \right].$$

Thus the Gauss-law constraint produces the projection onto even total fermion parity, while the remaining sum runs over the independent local and global $\mathbb{Z}_2$ fluxes.

Remark 44 (Flux notation). In the following chapters, we identify the flux through a plaquette with the corresponding eigenvalue Φ_p . Thus $\Phi_p = 1$ denotes zero flux, whereas $\Phi_p = -1$ denotes π flux. The same convention is used for the global holonomies a and b .

Remark 45 (Observables containing electric fields). The restriction to magnetic observables in Proposition 67 does not lose the information carried by diagonal Gibbs expectations. Consider a monomial containing the electric-field factor

$$X_{\mathcal{B}} \doteq \prod_{(i,j) \in \mathcal{B}} \hat{\sigma}_{ij}^x.$$

After expanding P_{phys} , its diagonal matrix element in the vector-potential basis can be non-zero only if

$$X_{\mathcal{B}} = A_\Omega$$

for some $\Omega \subseteq V(\Lambda_L)$. If no such Ω exists, the expectation vanishes. If it does exist, Gauss' law gives

$$A_\Omega P_{\text{phys}} = (-1)^{N_\Omega} P_{\text{phys}},$$

so that the expectation can be rewritten as that of a magnetic observable.

Remark 46. Observe that the twisted partition function can be viewed as a partition function for a modified Gibbs state, with complex chemical potential $\mu + i\pi/\beta$. The presence of this imaginary shift in the chemical potential introduces an twisted version of the KMS identity (see Equation (37) in Proposition 21). Let:

$$\gamma_t(\mathcal{O}) \doteq e^{t(H - \mu N)} \mathcal{O} e^{-t(H - \mu N)}. \quad (23)$$

be the imaginary-time evolution of $\mathcal{O}$, for $t \in [0; \beta)$. Then:

$$\begin{aligned} \frac{\text{Tr}_{\mathcal{F}_L} (-1)^N \gamma_t(a_i^-) a_j^+ e^{-\beta(H(\boldsymbol{\sigma}) - \mu N)}}{Z_{\beta, \mu, L}^-} &= \frac{\text{Tr}_{\mathcal{F}_L} \gamma_t(a_i^-) (-1)^{N-1} a_j^+ e^{-\beta(H(\boldsymbol{\sigma}) - \mu N)}}{Z_{\beta, \mu, L}^-} \\ &= - \frac{\text{Tr}_{\mathcal{F}_L} (-1)^N a_j^+ \gamma_{t-\beta}(a_i^-) e^{-\beta(H(\boldsymbol{\sigma}) - \mu N)}}{Z_{\beta, \mu, L}^-}. \end{aligned} \quad (24)$$

This relation will be used, later on, to introduce periodic or antiperiodic time extension of the time-ordered correlation functions, for the modified Gibbs states.

There is an alternative way of understanding the emergence of periodic boundary conditions in the twisted partition function. Using the path integral representation introduced in [Section 2.A](#), one sees that the trace formulas in [Proposition 28](#) in presence of $(-1)^N$ are modified to:

$$\begin{aligned} \text{Tr}((-1)^N \mathcal{O}) &= \int \prod_{\alpha} d\bar{\psi}_{\alpha} d\psi_{\alpha} e^{-\sum_{\alpha} \bar{\psi}_{\alpha} \psi_{\alpha}} (\langle \bar{\psi} |, (1 \otimes \mathcal{O}) | \psi \rangle) \\ \text{Tr}((-1)^N \mathcal{O}^2) &= \int \prod_{\alpha} \prod_{j=0}^1 d\bar{\psi}_{\alpha, j} d\psi_{\alpha, j} e^{-\sum_{j=0}^1 \sum_{\alpha} \bar{\psi}_{\alpha, j} \psi_{\alpha, j}} (\langle \bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_1 \rangle) (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) \\ \text{Tr}((-1)^N \mathcal{O}^n) &= \int \prod_{\alpha} \prod_{j=0}^{n-1} d\bar{\psi}_{\alpha, j} d\psi_{\alpha, j} e^{-\sum_{j=0}^{n-1} \sum_{\alpha} \bar{\psi}_{\alpha, j} \psi_{\alpha, j}} (\langle \bar{\psi}_0 |, (1 \otimes \mathcal{O}) | \psi_{n-1} \rangle) \cdots \\ &\quad \cdots (\langle \bar{\psi}_1 |, (1 \otimes \mathcal{O}) | \psi_0 \rangle) \end{aligned}$$

Therefore, the path integral representation of the twisted partition function and the associated correlation functions involve periodic boundary conditions in the time direction. One can imagine the operator $(-1)^N$ as defect/monodromy for the fermionic trajectories in time.

Stability of π -Flux phase

The spinless $\mathbb{Z}_2$ -gauge model studied in this chapter and its main results were introduced in [Section 1.2](#). This chapter is based on joint work with Marcello Porta [\[GP26\]](#) and provides the precise formulation of the model together with the complete proofs of the stability and susceptibility results. We specialize the framework of [Chapter 5, Section 5.2](#) and the resolution of the Gauss-law constraint established in [Proposition 67](#). The resulting spinless model, its fixed-background decomposition, and its main statements are presented in [Section 6.1](#).

The model exhibits a direct competition between its gauge and matter energies. The plaquette term favours uniform zero flux, whereas the hopping energy of the half-filled fermions favours π -flux. The first main result proves that the fermionic mechanism dominates when the hopping strength is sufficiently large. More precisely, the uniform π -flux phase is quantitatively stable: zero-flux plaquette defects have a strictly positive energy cost, and the total penalty admits a lower bound proportional to the number of defects. This estimate suppresses all non- π -flux sectors in the low-temperature Gibbs state and reduces magnetic observables to the four uniform π -flux sectors distinguished by their torus holonomies. The ground-state energies of these four sectors become degenerate in the thermodynamic limit, with a quantitatively controlled finite-size splitting.

The second main result concerns the magnetic response of the model. Once stability has reduced the ground-state expectations to the uniform π -flux sectors, the translation-invariant fermionic Hamiltonian can be analyzed explicitly by Bloch–Floquet theory. Its spectrum has conical intersections and therefore produces the response characteristic of two-dimensional massless Dirac fermions. The resulting rigorous low-momentum asymptotics of the magnetic susceptibility justifies, for the present $\mathbb{Z}_2$ -lattice gauge theory, the numerical Dirac-fermion response observed in [\[GRV17\]](#).

The proof of stability separates into two main steps. First, the reflection-positive inequality is verified in every fixed $\mathbb{Z}_2$ -background, with careful control of the half-filling condition and the torus holonomies. Second, the chessboard estimate disseminates an arbitrary collection of flux defects into periodic magnetic configurations. Their fermionic energies can then be computed by Bloch–Floquet reduction, yielding the spectral cost that produces the extensive defect bound. The susceptibility is treated afterwards by using the stability theorem to restrict the zero-temperature state to the four π -flux sectors and evaluating the corresponding current response through the free-fermion representation and the lattice conservation laws.

The general ingredients of this argument are standard. The reflection inequality descends from Lieb’s half-filled flux-phase method [\[Lie94\]](#), reviewed in [Subsection 3.4.1](#) and [Subsection 3.4.3](#). The

dissemination step adapts the reflection-positivity and chessboard framework of [Frö+78; Frö+80; Tas20], reviewed in Subsection 3.4.2. The model-specific reflection and dissemination arguments are developed in Section 6.2 and Subsection 6.2.2, while the holonomy-resolved Bloch analysis is carried out in Subsection 6.2.1. The stability theorem is proved in Section 6.3, the susceptibility is computed in Section 6.4 and the finite-size expansion of the ground-state energies used in both arguments is collected in Section 6.A.

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6.1 The Spinless $\mathbb{Z}_2$ -Gauge Model and Its Main Results

We now specialize the general Gauge–Matter framework of Chapter 5 to the spinless model studied in this chapter. Throughout, $L \in 4\mathbb{N}$, the internal Fermionic label is suppressed, the chemical potential is fixed at $\mu = 0$, and the electric-field strength is set to zero. Thus, using the plaquette operators B_p introduced in the preceding chapter, the Hamiltonian is

$$H \doteq H_g + H_f = - \sum_{p \in \mathcal{P}(\Lambda_L)} B_p - t \sum_{(i,j) \in \mathcal{E}(\Lambda_L)} \left(a_i^\dagger \hat{\sigma}_{ij}^z a_j^- + \text{h.c.} \right), \quad t > 0. \quad (1)$$

This is the $\varepsilon = 0$ member of the family Equation (9). The first term favours zero flux, whereas the Fermionic hopping energy favours π flux at half filling. The purpose of the chapter is to prove that the latter mechanism dominates for sufficiently large t .

The total Hilbert space, the local charges Q_i , and the physical Hilbert space $\mathcal{H}_L^{\text{phys}}$ are those defined in Section 5.2, with $|\mathcal{S}| = 1$. Since H contains no electric-field operators, it decomposes into the fixed vector-potential fibers $H(\sigma)$ of Equation (19). Gauge-equivalent backgrounds give unitarily equivalent fiber Hamiltonians by Proposition 66. Their spectra therefore depend only on the plaquette fluxes

$$\Phi_p \doteq \prod_{(i,j) \in \partial p} \sigma_{ij}$$

and on the two holonomies $(a, b) \in \{\pm 1\}^2$ around the non-contractible cycles of the torus. We use $\Phi_p = 1$ for zero flux and $\Phi_p = -1$ for π flux, and write π for the uniform π -flux configuration. This convention will also be used for the holonomies a and b .

Remark 47 (Flux notation). The symbols Φ_p , a , and b denote Gauge-invariant flux data, not a particular choice of vector potential. Any representative may be used inside a fiber trace, because different representatives are related by unitary conjugation.

For a physical observable $\mathcal{O}$, its Gibbs expectation is

$$\langle \mathcal{O} \rangle_{\beta,L} \doteq \frac{\text{Tr}_{\mathcal{H}_L^{\text{phys}}}(\mathcal{O}e^{-\beta H})}{\text{Tr}_{\mathcal{H}_L^{\text{phys}}}e^{-\beta H}}, \quad Z_{\beta,L} \doteq \text{Tr}_{\mathcal{H}_L^{\text{phys}}}e^{-\beta H}. \quad (2)$$

For magnetic Gauge-invariant observables, the resolution of the Gauss-law constraint is exactly **Proposition 67**. We denote by $Z_{\beta,L}(\Phi, a, b)$ the contribution of a fixed flux sector in **Subsection 5.2.4**, and by $\mathcal{O}(\Phi, a, b)$ the corresponding restriction of $\mathcal{O}$ to the Fermionic fiber. The Gauge field is therefore frozen within each fiber but remains a local degree of freedom summed over in the physical partition function.

6.1.1 Stability of the π -Flux Phase and Magnetic Response

Stability of the π -Flux Phase We restate, without renumbering, the main **Theorem 2** announced in the Introduction.

Theorem (Stability of the π -flux phase). *Let $L = 4\ell$, $\ell \in \mathbb{N}$. There exists $t_0 > 0$ independent of β and L such that for $t > t_0$ the following is true. Let $f_{\beta,L}$ be the free energy of the gauge theory, and let $f_{\beta,L}(\pi)$ the free energy of the π -flux phase,*

$$f_{\beta,L} \doteq -\frac{1}{\beta L^2} \log Z_{\beta,L}, \quad f_{\beta,L}(\pi) \doteq -\frac{1}{\beta L^2} \log \left(\sum_{a,b} Z_{\beta,L}(\pi, a, b) \right). \quad (3)$$

Then:

$$0 \leq \beta L^2 (f_{\beta,L}(\pi) - f_{\beta,L}) \leq \log \left(1 + 4 \left((1 + e^{-\beta(\Delta_{\beta,L}-2)})^{L^2} - 1 \right) \right), \quad (4)$$

where, for $\beta, L \gg 1$, and for a suitable universal constant $\kappa > 0$:

$$\Delta_{\beta,L} = \Delta_{\infty} + o(1), \quad \Delta_{\infty} = \kappa t. \quad (5)$$

Furthermore, for any magnetic gauge invariant observable $\mathcal{O}$, and for β, L large enough:

$$\left| \langle \mathcal{O} \rangle_{\beta,L} - \langle \mathcal{O} \rangle_{\beta,L}^{\pi} \right| \leq K_{\mathcal{O}} ((1 + e^{-\beta(\Delta_{\beta,L}-1)})^{L^2} - 1) \quad (6)$$

where the constant $K_{\mathcal{O}}$ only depends on $\max_{\Phi,a,b} \|\mathcal{O}(\Phi, a, b)\|$ and:

$$\langle \mathcal{O} \rangle_{\beta,L}^{\pi} \doteq \frac{\sum_{(a,b) \in \mathbb{Z}_2^2} \text{Tr}_{\mathcal{F}_L} e^{-\beta H_{\text{f}}(\pi,a,b)} \mathcal{O}(\pi, a, b) (1 + (-1)^N)}{\sum_{(a,b) \in \mathbb{Z}_2^2} \text{Tr}_{\mathcal{F}_L} e^{-\beta H_{\text{f}}(\pi,a,b)} (1 + (-1)^N)}. \quad (7)$$

Also, let:

$$E_{0;L}(\pi, a, b) \doteq -\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_{\beta,L}(\pi, a, b) \quad (8)$$

be the ground state energy of $H_{\text{f}}(\pi, a, b)$. Then,

$$E_{0;L}(\pi, a, b) \geq E_{0;L}(\pi, -1, -1). \quad (9)$$

Furthermore, the ground state of the π -flux phase is almost degenerate, in the sense that, for any $0 < \alpha < 1/3$:

$$\left| E_{0;L}(\pi, a, b) - E_{0;L}(\pi, -1, -1) \right| \leq \frac{C_{\alpha}}{L^{1-3\alpha}}. \quad (10)$$

The proof of [Theorem 2](#) is based on reflection positivity methods. It relies on the key result of Lieb [[Lie94](#)], that proved that the optimum, energy minimizing magnetic flux for a half-filled band of electrons hopping on a planar, bipartite graph is π per square plaquette. A few remarks are in order.

Remark 48.

- (i) The quantity $\Delta_{\beta,L}$ can be thought as a lower bound on the monopole's mass: it quantifies the energetic penalty due to the removal of one monopole from the system. The number κ is given by the integral of a suitable trigonometric function, that will appear in the proof of the theorem. Its numerical evaluation with the software **Mathematica** gives $\kappa \simeq 0.036$.
- (ii) The error terms in [Equation \(6\)](#) are exponentially small for β large, however non uniformly in L . Thus, the result allows to compare the Gibbs states of the lattice gauge theory and of the π -flux phase for $L \gg 1$ and $\beta \gtrsim \log L$. In particular, the result allows to study the ground state of the gauge theory.
- (iii) The result of Lieb [[Lie94](#)] implies that the smallest energy of $H_f(\sigma)$ is attained in correspondence of configurations realizing the π -flux phase. The result of [[Lie94](#)] is actually much more general than this, since it does not require the background gauge fields to be $\mathbb{Z}_2$ valued, and it also applied to certain non-abelian gauge groups. See also [[MN96](#)] for a further generalization of [[Lie94](#)]. Here we prove that the ground state energy of Hamiltonians $H_f(\sigma)$ not associated with the π -flux phase are separated by the ground state energy of the π -flux phase by a positive gap. Furthermore, we provide a lower bound for this gap, that increases with the number of monopoles' removals, that is plaquettes with zero flux. The proof is based on reflection positivity, as [[Lie94](#)]. More precisely, to prove the bound for the gap, we rely on the chessboard estimate, see [[Tas20](#)] for a pedagogical review and The chessboard estimate allows to bound the gap from below, in terms of the ground state energy of a “chessboard configuration” of plaquettes with 0 and π fluxes, which we explicitly compute.

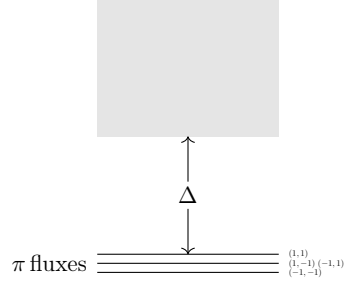


Figure 6.1: Schematic representation of the ground state energies of the fermionic Hamiltonians $H_f(\sigma)$, for $L = 4\ell$. The grey box contains the ground state energies of $H_f(\sigma)$ with at least one zero-flux plaquette. A numerical computation shows that $E_{0;L}(\pi, -1, -1) < E_{0;L}(\pi, 1, -1) < E_{0;L}(\pi, 1, 1)$ (and $E_{0;L}(\pi, 1, -1) = E_{0;L}(\pi, -1, 1)$). The difference of the π -flux ground state energies vanishes as $L \rightarrow \infty$.

- (iv) The fermionic ground state energy is minimized by the π -flux phase also for $L = 2(2\ell + 1)$. In this case, however, the optimal choice of a, b is $(a, b) = (1, 1)$. The restriction to $L = 4\ell$ is technical, and it comes from the argument used to prove a lower bound for the monopole's mass.

Remark 49 (On the π -flux phase). There are only four inequivalent π flux phases, labelled by the fluxes (a, b) corresponding to non-contractible loops of the torus. As discussed in [Subsection 6.2.1](#),

for the four π -flux phases it is possible to choose gauge field configurations σ that preserve translation invariance, paying the price of considering a fundamental cell formed by two lattice sites; we refer the reader to [Figure 6.3](#) and [Figure 6.4](#) for graphical representations. To be more precise, the π -flux phases can be described in terms of translation-invariant fermionic Hamiltonians, with periodic/antiperiodic boundary conditions, according to the value of (a, b) . The choice $a = 1$ corresponds to periodic boundary condition on the horizontal direction, while $a = -1$ correspond to antiperiodic boundary conditions; the same holds for b , for the vertical direction.

Remark 50 (Decay of fermionic correlations). As the proof of [Theorem 2](#) shows, all Euclidean (that is, imaginary time) ground state correlation functions of the gauge theory can be computed explicitly, via the fermionic Wick's rule. In particular, since the fermionic Hamiltonian with the π -flux background has no spectral gap, it turns out that Euclidean correlation functions decay algebraically, at an integrable rate in space-time. Let us denote by x the location of the fundamental cell of the π -flux phase, with elements labelled by $i = A, B$; see [Figure 6.3](#). Let us denote by $a_{x,i}^\pm$ the fermionic creation/annihilation operators, associated with the creation/annihilation of a particle in the fundamental cell labelled by x , and occupying the lattice site $i = A, B$ in the cell. Then, we have, for $|x - y| \neq 0$ and $t > 0$:

$$\lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \langle e^{tH} a_{x,i}^+ a_{x,i}^- e^{-tH} ; a_{y,j}^+ a_{y,j}^- \rangle_{\beta, L} = -\mathfrak{g}_{ij}((x, t); (y, 0)) \mathfrak{g}_{ji}((y, 0); (x, t)) , \quad (11)$$

where $\mathfrak{g}(\cdot; \cdot)$ is:

$$\mathfrak{g}((x, t); (y, s)) = \int_{[0, 2\pi] \times [0, \pi]} \frac{dk}{(2\pi)^2} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-s) + ik(x-y)}}{-i\omega + h(k)} , \quad (12)$$

with Bloch Hamiltonian:

$$h(k) = -t \begin{pmatrix} -e^{ik_1} - e^{-ik_1} & 1 + e^{2ik_2} \\ 1 + e^{-2ik_2} & e^{ik_1} + e^{-ik_1} \end{pmatrix} . \quad (13)$$

The left-hand side of [Equation \(11\)](#) is the Euclidean and time-ordered density-density correlation function, whose decay properties are ultimately determined by the spectrum of $h(k)$. The eigenvalues of $h(k)$, also called energy bands, are given by:

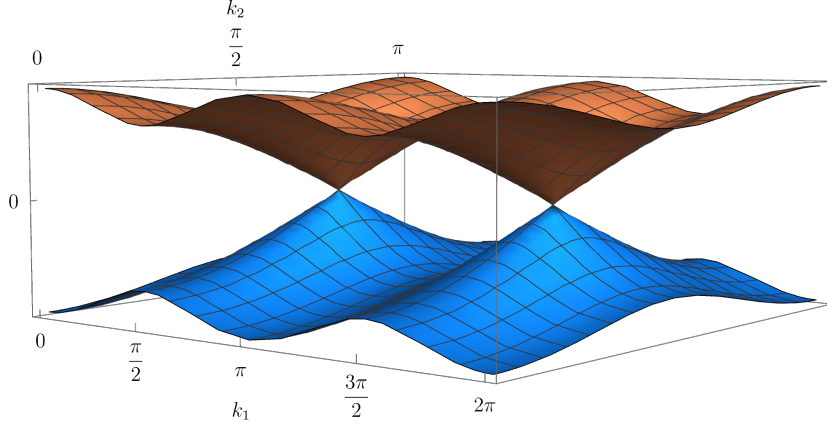
$$e_{\pm}(k) = \pm 2t \sqrt{1 + \frac{1}{2} \cos(2k_1) + \frac{1}{2} \cos(2k_2)} . \quad (14)$$

In particular, $e_{\pm}(k) = 0$ if and only if $k = (\pi/2, \pi/2) \doteq k_F^-$ or $k = (3\pi/2, \pi/2) \doteq k_F^+$. In correspondence with these points, the energy bands display conical intersections, see [Figure 6.2](#): for $\alpha = \pm$,

$$e_{\pm}(k_F^\alpha + q) = q \cdot \nabla_q e_{\pm}(k_F) + O(|q|^2) = \mp 2t|q| + O(|q|^2) , \quad \text{for } |q| \ll 1. \quad (15)$$

This implies that the function $\mathfrak{g}(\mathbf{x}; \mathbf{y})$, with $\mathbf{x} = (x, t)$ and $\mathbf{y} = (y, s)$, decays as $|\mathbf{x} - \mathbf{y}|^{-2}$, and hence that the ground-state density-density correlation function [Equation \(11\)](#) decays as $|\mathbf{x} - \mathbf{y}|^{-4}$, which is non-integrable in $2 + 1$ dimensions.

Susceptibility [Theorem 2](#) opens the way to explicitly compute observables for the lattice gauge theory. Here, we shall consider the magnetic susceptibility, following [\[GRV17\]](#). We anticipate that this quantity is defined as the response to an external $U(1)$ gauge field, rather than a $\mathbb{Z}_2$ field. This might look unnatural from a physics viewpoint. We stress however that the role of the magnetic susceptibility is computational: it is used in the literature, see e.g. [\[GRV17\]](#), as a tool to probe

Figure 6.2: Energy bands associated with the π -flux phase.

different regions of the phase diagram of the model. Our goal will be to rigorously justify the numerical findings of [GRV17].

We shall couple the system to an external, time-independent vector potential, which generates a static magnetic field. The vector potential only couples to the fermionic Hamiltonian, and it does so via the Peierls' substitution. Denoting by H_f the Fock space Hamiltonian associated with a single-particle, nearest-neighbour Schroedinger operator h , we define:

$$H_f(A) \doteq \sum_{x,y \in \Lambda_L} a_x^+ h(x; y) e^{iq \int_{x \rightarrow y} d\ell \cdot A(\ell)} a_y^- + \text{h.c.}, \quad (16)$$

with q the electric charge, and where the integral is over the bond connecting x to y . The external potential is defined in the continuum, over $\mathbb{R}^2/L\mathbb{R}^2$.

By [Theorem 2](#), we know that the ground state of the lattice gauge theory is represented by a π -flux phase; this corresponds to a periodic arrangement of spins on the lattice bonds, as anticipated in [Remark 48](#) and as discussed later in [Subsection 6.2.1](#). Thus, we find it convenient to rewrite the Hamiltonian $H_f(A)$ using the fundamental cell that accommodates for the periodicity of the π -flux configurations, see [Figure 6.3](#). Let Λ_L^{red} be the sublattice of Λ_L generated by the basis vectors e_1 and $2e_2$. We have:

$$H_f(A) = \sum_{x, x' \in \Lambda_L^{\text{red}}} \sum_{i, j \in I} h_{ij}(x, x') e^{iq \int_0^1 A((x' + r_j)(1-s) + s(x + r_i)) \cdot \xi ds} a_{x,i}^+ a_{x',j}^- \quad (17)$$

where $I = (A, B)$ collects the labels in the fundamental cell, and $h_{ij}(x, x')$ are the matrix elements of the single-particle Hamiltonian, labelled by the points x, x' of the reduced lattice Λ_L^{red} and by the points in the fundamental cell. We denote by $r_A = e_2$ and $r_B = 0$ the relative coordinates in the fundamental cell. The displacement vector ξ is directed along the bonds of the lattice, and it is $\xi = x + r_i - x' - r_j$ (and it can take values $\pm e_1, \pm e_2$). Next, we represent the vector potential in terms of its Fourier coefficients,

$$A_\mu(x + r_i) = \frac{1}{L^2} \sum_{p \in \frac{2\pi}{L}(\mathbb{Z}^2/L\mathbb{Z}^2)} \hat{A}_\mu(p) e^{ip \cdot (x + r_i)}, \quad \mu = 1, 2, \quad x \in \Lambda_L^{\text{red}}, \quad (18)$$

where here and below the sum runs over the independent momenta compatible with L -periodicity in space. The momentum-space current operator is then:

$$\hat{J}_\mu^{(A)}(p) \doteq -L^2 \frac{\partial H_f(A)}{\partial \hat{A}_\mu(-p)} ; \quad (19)$$

by expanding the Hamiltonian Equation (17) in A , we get:

$$H_f(A) = H_f - \frac{1}{L^2} \sum_p \sum_{\mu=1,2} \hat{J}_\mu(p) \hat{A}_\mu(-p) - \frac{1}{2} \frac{1}{L^4} \sum_{p,q} \sum_{\mu,\nu=1,2} \hat{A}_\mu(-p) \hat{K}_{\mu,\nu}(p,q) \hat{A}_\nu(-q) + O(A^3) , \quad (20)$$

and hence:

$$\hat{J}_\mu^{(A)}(p) = \hat{J}_\mu(p) + \frac{1}{L^2} \sum_q \sum_{\mu,\nu=1,2} \hat{K}_{\mu,\nu}(p,q) \hat{A}_\nu(-q) + O(A^2) , \quad (21)$$

where:

$$\begin{aligned} \hat{J}_\mu(p) &\doteq -\frac{iq}{2} \sum_{\substack{x,x' \in \Lambda_L^{\text{red}} \\ i,j \in I}} e^{-ip \cdot (x' + r_j)} \left(h_{ij}(x, x') a_{x,i}^+ a_{x',j}^- - h_{ji}(x', x) a_{x',j}^+ a_{x,i}^- \right) \eta_\xi(p) \xi_\mu \\ \hat{K}_{\mu,\nu}(p, q) &\doteq -\frac{q^2}{2} \sum_{\substack{x,x' \in \Lambda_L^{\text{red}} \\ i,j \in I}} e^{-i(p+q) \cdot (x' + r_j)} \left(h_{ij}(x, x') a_{x,i}^+ a_{x',j}^- + h_{ji}(x', x) a_{x',j}^+ a_{x,i}^- \right) \eta_\xi(p) \eta_\xi(q) \xi_\mu \xi_\nu \end{aligned} \quad (22)$$

with:

$$\eta_\xi(p) = \begin{cases} \frac{e^{-ip \cdot \xi} - 1}{-ip \cdot \xi} & p \cdot \xi \neq 0 \\ 1 & \text{otherwise.} \end{cases} \quad (23)$$

We are now ready to compute the average of the current operator in presence of the external gauge field, at first order in the perturbation. Let $H(A)$ be the total Hamiltonian of the gauge theory, coupled to the external gauge field. We have, by Duhamel formula:

$$\begin{aligned} \frac{\text{Tr}(e^{-\beta H(A)} \hat{J}_1^{(A)}(p))}{\text{Tr}(e^{-\beta H(A)})} - \frac{\text{Tr}(e^{-\beta H} \hat{J}_1(p))}{\text{Tr}(e^{-\beta H})} &= \frac{1}{L^2} \sum_{\substack{\nu=1,2 \\ q}} \left[\int_0^\beta ds \left\langle \hat{J}_\nu(q, s); \hat{J}_1(p) \right\rangle_{\beta, L} \hat{A}_\nu(q) + \right. \\ &\quad \left. + \left\langle \hat{K}_{1,\nu}(p, q) \right\rangle_{\beta, L} \hat{A}_\nu(q) \right] + O(A^2), \end{aligned}$$

where $\mathcal{O}(s) = \gamma_s(\mathcal{O})$ is the imaginary-time evolution of the operator $\mathcal{O}$. By Theorem 2, as $\beta \rightarrow \infty$ the Gibbs state average reduces to the linear combination of averages on the four π -flux phases. Since all π -flux configurations are translation-invariant, we can use momentum conservation to write:

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \sum_q \left[\int_0^\beta ds \left\langle \hat{J}_\nu(q, s); \hat{J}_1(p) \right\rangle_{\beta, L} + \left\langle \hat{K}_{1,\nu}(p, q) \right\rangle_{\beta, L} \right] \hat{A}_\nu(q) \\ = \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \sum_q \left[\int_0^\beta ds \left\langle \hat{J}_\nu(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} + \left\langle \hat{K}_{1,\nu}(p, -p) \right\rangle_{\beta, L} \right] \delta_{q,-p} \hat{A}_\nu(q) , \end{aligned} \quad (24)$$

with $\delta_{\cdot, \cdot}$ the periodic Kronecker symbol (of period π in both momentum components). Let us consider a vector potential $\hat{A}(p) = (\hat{A}_1(p), 0)$, with $\hat{A}_1(p)$ nonzero for $p = (0, p_2)$. By Maxwell's

laws, the associated magnetic field is $\hat{B}(p) = -ip_2\hat{A}_1(p) + ip_1\hat{A}_2(p)$. We define the magnetic susceptibility from the variation of Equation (24) at first order in the vector potential. That is:

$$\chi(p_2) \doteq \frac{1}{p_2^2} \lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \left[\int_0^\beta ds \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} + \left\langle \hat{K}_{1,1}(p, -p) \right\rangle_{\beta, L} \right]. \quad (25)$$

The factor $1/p_2^2$ takes into account the fact that we are interested in the variation with respect of the magnetic field $\hat{B}((0, p_2)) = -ip_2\hat{A}_1((0, p_2))$, and the fact that we are actually measuring the variation of the magnetization: for a (classical) conserved current $\hat{J}(p)$, we can write $\hat{J}_1(p) = ip_2\hat{M}(p)$ and $\hat{J}_2(p) = -ip_1\hat{M}(p)$, where $\hat{M}(p)$ is the Fourier transform of the local magnetic moment [GRV17; KAA09]. The next result provides the explicit form of the magnetic susceptibility, at small external momenta.

Proposition 68 (Ground state susceptibility). *Under the same assumption of Theorem 2, the following is true:*

$$\chi(p_2) = -\frac{q^2 v}{8|p_2|} + o\left(\frac{1}{|p_2|}\right), \quad (26)$$

where $v = 2t$ is the Fermi velocity of the fermionic sector in the π -flux phase.

Remark 51. Equation (26) is the same obtained for non-interacting $2d$ massless Dirac fermions and for graphene [KAA09]. Equation (26) puts on rigorous grounds the numerical evidence of [GRV17], for the $\mathbb{Z}_2$ -lattice gauge theory.

The proof of Proposition 68 follows from an explicit calculation of the transport coefficient on the π -flux phase, which correspond to a semimetallic system. The final expression for the susceptibility is completely determined by the Dirac-like excitation spectrum of the π -flux phase, and lattice conservation laws.

The rest of the chapter is devoted to the proof of Theorem 2 and of Proposition 68.

6.2 Reflection Positivity and Chessboard Estimates

The Hamiltonian formulation of Reflection Positivity, Lieb's Reflection Inequality, and the Abstract Chessboard Argument were developed in Chapter 3. We use those results here without repeating their proofs. The model-dependent work has two parts. First, one must verify that every fixed Gauge sector has the crossing structure required by Lieb's inequality. Second, after the chessboard estimate has replaced an arbitrary collection of 0-flux plaquettes by periodic flux patterns, the corresponding one-particle Hamiltonians must be diagonalized and their spectral gaps estimated explicitly. The latter step turns the abstract reflection bound into a volume-uniform energy cost per defect.

6.2.1 Analysis of the π -Flux Phases

The argument of the Subsection 3.4.3 shows that the twisted/untwisted partition functions are maximized by a gauge field configuration with flux π in all plaquettes, and with flux 0 or π along the non-contractible cycles, depending on whether $L/2 = \ell$ is odd or even, respectively. These configurations are of course far from being unique, due to the fact that all equivalent gauge field configurations produce the same partition function, recall Proposition 66.

One can produce different π -flux phases, choosing different fluxes on the non-contractible cycles: we can think of the π -flux phases as equivalence classes of gauge field configurations, parametrized by four integers, labelling the product of the spins over two inequivalent, non-contractible loops of the torus. The argument of Lieb shows that the partition function is maximized in correspondence with one particular π -flux phase (see [Remark 33 - Item 1](#)) it leaves open the possibility that the other π -flux phases might have the same energy. This is the question that we will address in the present section, via an explicit computation of the matter partition functions associated with the four inequivalent π -flux phases.

Let us introduce the ground state energies associated with the twisted/untwisted partition function as:

$$E_{0;L}^{\pm}(\sigma) \doteq - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log Z_{\beta,L}^{\pm}(\sigma) . \quad (27)$$

The reader might be worried about taking the log of $Z_{\beta,L}^{-}$, due to the lack of positivity of the argument of the trace. However, it is not difficult to see that the twisted partition function is non-negative at half-filling. Being the lattice bipartite, we can represent it as $\Lambda_L = \Lambda_L^A + \Lambda_L^B$, with Λ_L^B a translate by Λ_L^A of a basis vector. Let $h(\sigma)$ be the single-particle Hamiltonian, and let us denote by U the unitary operator that changes sign of a function f on Λ_L , in correspondence with one of the two sublattices and acts as the identity on the other sublattice. Then, it is easy to see that $U^*h(\sigma)U = -h(\sigma)$. This implies that the spectrum of $h(\sigma)$ is symmetric around zero, and that the multiplicities of the eigenvalues $\lambda, -\lambda$ are the same. Let $(\lambda_i)_{i=1,\dots,L^2}$ be the eigenvalues of $h(\sigma)$. The twisted partition function is:

$$Z_{\beta,L}^{-}(\sigma) = \prod_{i=1}^{L^2} (1 - e^{-\beta\lambda_i}) . \quad (28)$$

The existence of zero modes implies that the twisted partition function is zero, and in particular non-negative. Suppose now that there are no zero modes. Using the equality of the multiplicities of the eigenvalues of $h(\sigma)$, we can rewrite the twisted partition function as:

$$Z_{\beta,L}^{-}(\sigma) = \prod_{i:\lambda_i < 0} (1 - e^{-\beta\lambda_i})(1 - e^{\beta\lambda_i}) . \quad (29)$$

The function $(1 - e^{-\beta\lambda_i})(1 - e^{\beta\lambda_i})$ is non-positive. However, the right-hand side of [Equation \(29\)](#) is non-negative, since it involves the product of $L^2/2$ terms, which is even for L even. This shows that $Z_{\beta,L}^{-}(\sigma) > 0$, which implies that $E_{0;L}^{-}(\sigma)$ in [Equation \(27\)](#) is real.

Taking the zero-temperature limit in the Reflection-Positivity inequality above gives

$$E_{0;L}^{\pm}(\sigma) \geq E_{0;L}^{\pm}(\sigma_{\pi}) , \quad (30)$$

where σ_{π} is a configuration with flux equal to π in every plaquette. In this section we shall consider all such phases; these are parametrized by $(a, b) \in \mathbb{Z}_2^2$, which correspond to the fluxes along two non-contractible cycles of the torus (recall [Remark 47](#)). We shall denote by $E_{0;L}^{\pm}(a, b)$ the corresponding energies. The next proposition proves the energetic closeness of these phases. It proves, in particular, the statement [Equation \(10\)](#) in [Theorem 2](#).

Proposition 69 (Almost-degeneracy of the π -flux phases). *Let $L = 2\ell$. For $\ell = 2n$:*

$$Z_{\beta,L}^{-}(\pi; 1, 1) = 0 , \quad Z_{\beta,L}^{-}(\pi; a, b) > 0 \quad \text{for } (a, b) \neq (1, 1). \quad (31)$$

Furthermore, for $(\varepsilon, a, b) \neq (-1, 1, 1)$, and $0 < \alpha < 1/3$:

$$|E_{0;L}^\varepsilon(\pi; a, b) - E_{0;L}^\varepsilon(\pi; 1, 1)| \leq \frac{C_\alpha}{L^{1-3\alpha}}. \quad (32)$$

For $\ell = 2n + 1$:

$$Z_{\beta,L}^-(\pi; -1, -1) = 0, \quad Z_{\beta,L}^-(\pi; a, b) > 0 \quad \text{for } (a, b) \neq (-1, -1). \quad (33)$$

Furthermore, for $(\varepsilon, a, b) \neq (-1, -1, -1)$, and $0 < \alpha < 1/3$:

$$|E_{0;L}^\varepsilon(\pi; a, b) - E_{0;L}^\varepsilon(\pi; -1, -1)| \leq \frac{C_\alpha}{L^{1-3\alpha}}. \quad (34)$$

Remark 52.

- (i) We stress that bounds Equation (32) and Equation (34) estimate the difference of extensive quantities.
- (ii) From reflection positivity, we know that:

$$E_{0;L}^\varepsilon(\pi; -1, -1) \leq E_{0;L}^\varepsilon(\pi; a, b) \quad \text{for } \ell \text{ even}, \quad E_{0;L}^\varepsilon(\pi; 1, 1) \leq E_{0;L}^\varepsilon(\pi; a, b) \quad \text{for } \ell \text{ odd}, \quad (35)$$

as follows from the Reflection-Positivity argument above. The estimates Equation (32) and Equation (34) do not rule out possible degeneracies; however, they show that the energy variation due to the change of fluxes on non-contractible cycles is bounded by a vanishing quantity as $\ell \rightarrow \infty$.

Proof. We shall only discuss the case $L/2 = 2n$, as the case $L/2 = 2n + 1$ is completely analogous. The proof of Proposition 69 is based on the Bloch diagonalization of the Hamiltonian associated with the four inequivalent π -fluxes. Let us start from the π -flux phase parametrized by $(a, b) = (1, 1)$. A convenient gauge field configuration associated with this phase is represented in Figure 6.3. Clearly, the corresponding Hamiltonian is not translationally invariant with respect to all lattice translations; however, it becomes translation invariant after having introduced the appropriate fundamental cell. We introduce a fundamental cell formed by two lattice sites, labelled by A, B , as in Figure 6.3. The position of the cell is defined by the coordinate of the B -lattice site. We shall denote by Λ_L^{red} the two-dimensional lattice formed by the positions of the fundamental cells. The number of lattice sites contained in Λ_L^{red} is $L^2/2$. Observe that the lattice spacing between nearest-neighbours fundamental cells in Λ_L^{red} is 1 in the horizontal direction, and 2 in the vertical direction.

Using these coordinates, the Hamiltonian of the matter sector is:

$$H_f(\pi; 1, 1) = -t \sum_{x \in \Lambda_L^{\text{red}}} (a_{A,x}^+ a_{B,x+2e_2}^- - a_{A,x}^+ a_{A,x+e_1}^- + a_{B,x}^+ a_{A,x}^- + a_{B,x}^+ a_{B,x+e_1}^-) + \text{h.c.} \quad (36)$$

Let us introduce the Brillouin zone $B_L(1, 1)$ as:

$$B_L(1, 1) \doteq \left\{ k \in \frac{2\pi}{L}(n_1, n_2) \mid 0 \leq n_1 \leq L-1, 0 \leq n_2 \leq L/2-1 \right\}, \quad (37)$$

and let us define the momentum-space creation/annihilation operators as:

$$\hat{a}_{\alpha,k}^\pm = \sum_{x \in \Lambda_L^{\text{red}}} e^{\pm i k \cdot x} a_{\alpha,x}^\pm \iff a_{\alpha,x}^\pm = \frac{1}{|\Lambda_L^{\text{red}}|} \sum_{k \in B_L(1,1)} e^{\mp i k \cdot x} \hat{a}_{\alpha,k}^\pm, \quad (38)$$

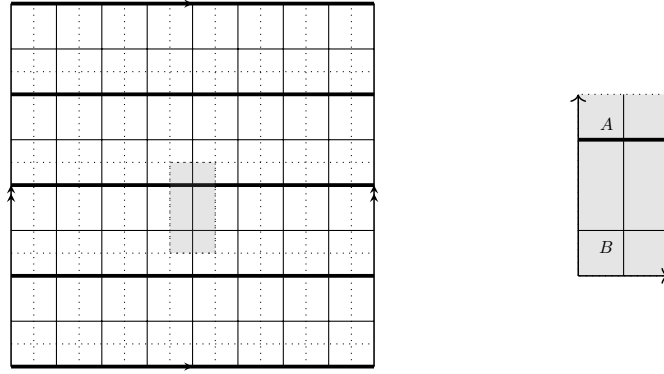


Figure 6.3: On the left, graphical representation of the spin configurations associated with the $(1,1)$ π flux phase. The thin bonds correspond to spin $+1$, while the solid bonds correspond to spin -1 . On the right, definition of the fundamental cell.

for $\alpha \in \{A, B\}$. In terms of these operators, the Hamiltonian in Equation (36) becomes:

$$H_f(\pi; 1, 1) = \frac{1}{|\Lambda_L^{\text{red}}|} \sum_{k \in B_L(1,1)} (\hat{a}_k^+, h(k) \hat{a}_k^-),$$

where $(f, g) = \sum_{\alpha} f_{\alpha} g_{\alpha}$, and the Bloch Hamiltonian $h(k)$ is:

$$h(k) = -t \begin{pmatrix} -e^{ik_1} - e^{-ik_1} & 1 + e^{2ik_2} \\ 1 + e^{-2ik_2} & e^{ik_1} + e^{-ik_1} \end{pmatrix}. \quad (39)$$

Its eigenvalues are:

$$e_{\pm}(k) = \pm 2t \sqrt{1 + \frac{1}{2} \cos(2k_1) + \frac{1}{2} \cos(2k_2)}. \quad (40)$$

The eigenvalues are vanishing if and only if $k = (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2})$. The partition function is:

$$Z_{\beta,L}^{\pm}(\pi; 1, 1) = \prod_{k \in B_L(1,1)} \prod_{z=\pm} (1 \pm e^{-\beta e_z(k)}) = \prod_{k \in B_L(1,1)} (2 \pm (e^{-\beta e_-(k)} + e^{\beta e_-(k)})).$$

From this expression, we see that $Z_{\beta,L}^-(\pi; 1, 1) = 0$, due to the presence of zero modes at $k = (\frac{\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2})$, which belong to $B_L(1,1)$ since $L/2$ is even. Instead, the ground state energy $E_{0;L}^+(\pi; 1, 1)$, defined as in Equation (27), is:

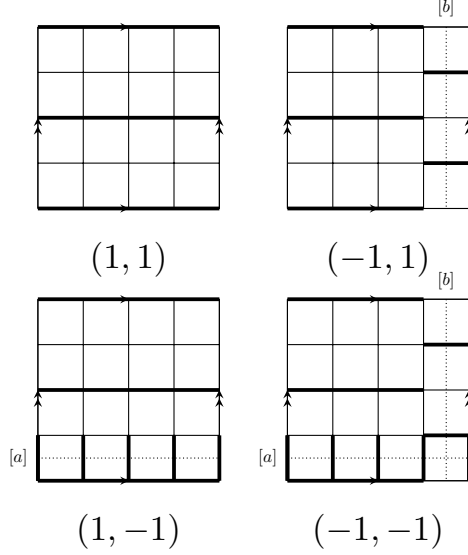
$$E_{0;L}^+(\pi; 1, 1) = \sum_{k \in B_L(1,1)} e_-(k). \quad (41)$$

Let us now consider the ground state energy associated with the π -flux phases with $(a, b) \neq (1, 1)$. The representative spin configurations associated with the remaining cases are illustrated in Figure 6.4.

Consider the case $(a, b) = (-1, 1)$. In order to deal with this case, we just need to do the following replacement:

$$(1, 1) \text{ case} \longrightarrow (-1, 1) \text{ case} \\ a_{L-1,x_2}^+ a_{L,x_2}^- + \text{h.c.} \equiv a_{L-1,x_2}^+ a_{0,x_2}^- + \text{h.c.} \longrightarrow a_{L-1,x_2}^+ a_{L,x_2}^- + \text{h.c.} \equiv -a_{L-1,x_2}^+ a_{0,x_2}^- + \text{h.c.}$$

Thus, the Hamiltonian of the case $(-1, 1)$ can be rewritten as the Hamiltonian of the case $(1, 1)$ if one reabsorbs the minus sign imposing the new rule $a_{L,x_2}^{\pm} \equiv -a_{0,x_2}^{\pm}$, which is equivalent to enforcing anti-periodic boundary conditions in the x_1 direction.

Figure 6.4: All possible π -flux phases, for $L = 4$.

The Brillouin zone corresponding to anti-periodic boundary conditions in x_1 and periodic boundary conditions in x_2 is:

$$B_L(-1, 1) \doteq \left\{ k \in \frac{2\pi}{L} \left(n_1 - \frac{1}{2}, n_2 \right) \mid 0 \leq n_1 \leq L-1, 0 \leq n_2 \leq L/2-1 \right\}. \quad (42)$$

Thanks to the $-\frac{1}{2}$ shift, the system does not admit any zero mode here, which means that the twisted partition function is non-zero. The untwisted/twisted state energies are:

$$E_{0;L}^{\pm}(\pi; -1, 1) = \sum_{k \in B_L(-1, 1)} e_{-}(k).$$

More generally, defining, for $a, b = \pm 1$,

$$B_L(a, b) \doteq \left\{ k \in \frac{2\pi}{L} \left(n_1 - \frac{a-1}{4}, n_2 - \frac{b-1}{4} \right) \mid 0 \leq n_1 \leq L-1, 0 \leq n_2 \leq L/2-1 \right\}, \quad (43)$$

the untwisted/twisted ground state energies of all π -flux phases are, for $(\varepsilon, a, b) \neq (-1, 1, 1)$:

$$E_{0;L}^{\varepsilon}(\pi; a, b) = \sum_{k \in B_L(a, b)} e_{-}(k) = \sum_{k \in B_L(1, 1)} e_{-} \left(k_1 - \frac{\pi}{2L}(a-1), k_2 - \frac{\pi}{2L}(b-1) \right). \quad (44)$$

Therefore, for these choices of labels:

$$\left| E_{0;L}^{\varepsilon}(\pi; a, b) - E_{0;L}^{\varepsilon}(\pi; 1, 1) \right| = \left| \sum_{k \in B_L(1, 1)} \left[e_{-} \left(k_1 - \frac{\pi}{2L}(a-1), k_2 - \frac{\pi}{2L}(b-1) \right) - e_{-}(k_1, k_2) \right] \right|;$$

as shown in [Section 6.A](#), one has, for $0 < \alpha < 1/3$:

$$\left| \sum_{k \in B_L(1, 1)} \left[e_{-} \left(k_1 - \frac{\pi}{2L}(a-1), k_2 - \frac{\pi}{2L}(b-1) \right) - e_{-}(k_1, k_2) \right] \right| \leq \frac{C_{\alpha}}{L^{1-3\alpha}}, \quad (45)$$

which concludes the proof of [Equation \(32\)](#). The case $L/2 = 2n + 1$ can be studied in the same way, we omit the details. ■

It is interesting to observe that, as $L \rightarrow \infty$, the energy bands of $h(k)$ have conical intersections at energy zero, and hence describe a semimetal at half-filling. Let $k_F^- = (\frac{\pi}{2}, \frac{\pi}{2})$, $k_F^+ = (\frac{3\pi}{2}, \frac{\pi}{2})$. In proximity of these points, we can linearize the dispersion relation as, for $\omega = \pm$:

$$e_{\pm}(k_F^{\omega} + q) = q \cdot \nabla_q e_{\pm}(k_F) + O(|q|^2) = \mp 2t|q| + O(|q|^2). \quad (46)$$

Equation (46) describes an isotropic cone, with slope $2t$. This means that the low-energy excitations of $h(k)$ are effectively described by massless Dirac fermions, with Fermi velocity $2t$. This fact will play an important role in the evaluation of the susceptibility, Proposition 68.

6.2.2 Chessboard Estimates

We now adapt the abstract Chessboard argument to the present model. The following version incorporates the columns $\underline{-1}$ forced by the π -flux interfaces; we include the proof because this additional structure is used explicitly below.

Lemma 70 (Column Chessboard Estimate). *Let $L = 4\ell$ and write $F(\underline{X}_1, \dots, \underline{X}_L)$, with $\underline{X}_j \in \mathbb{Z}_2^L$. Suppose that*

$$\begin{aligned} F(\underline{X}_1, \dots, \underline{X}_L) &= F(\underline{X}_L, \underline{X}_1, \dots, \underline{X}_{L-1}) && (\text{cyclicity}) \\ F(\underline{X}_1, \dots, \underline{X}_L) &\geq \frac{1}{2} \left(F(\underline{X}_1, \dots, \underline{X}_{L/2-1}, \underline{-1}, \underline{X}_{L/2-1}, \dots, \underline{X}_1, \underline{-1}) \right. \\ &\quad \left. + F(\underline{X}_L, \dots, \underline{X}_{L/2+1}, \underline{-1}, \underline{X}_{L/2+1}, \dots, \underline{X}_L, \underline{-1}) \right) && (\text{reflection bound}) \end{aligned}$$

where $\underline{-1}$ is the column with all entries equal to -1 . The conclusion of the abstract argument is

$$F(\underline{X}_1, \dots, \underline{X}_L) \geq \frac{1}{L/2} \sum_{j=1}^{L/2} F(\underline{X}_{2j-1}, \underline{-1}, \underline{X}_{2j-1}, \underline{-1}, \dots, \underline{X}_{2j-1}, \underline{-1}). \quad (47)$$

Proof. The proof follows Lemma 37, with minor differences. Let us define the function:

$$G(\underline{X}_1, \dots, \underline{X}_L) \doteq F(\underline{X}_1, \dots, \underline{X}_L) - \frac{1}{L/2} \sum_{j=1}^{L/2} F(\underline{X}_{2j-1}, \underline{-1}, \underline{X}_{2j-1}, \underline{-1}, \dots, \underline{X}_{2j-1}, \underline{-1}). \quad (48)$$

We claim that this function satisfies both properties above. The cyclicity of G follows from the cyclicity of F , and from the fact that the second term is invariant under permutations $\underline{X}_{2j-1} \rightarrow \underline{X}_{2\sigma(j)-1}$ for all $\sigma \in S_{L/2}$. Let us now check the reflection bound. Applying the reflection bound for the first term in G , we have:

$$\begin{aligned} G(\underline{X}_1, \dots, \underline{X}_L) &\geq \frac{1}{2} \left(F(\underline{X}_1, \dots, \underline{X}_{L/2-1}, \underline{-1}, \underline{X}_{L/2-1}, \dots, \underline{X}_1, \underline{-1}) \right. \\ &\quad \left. + F(\underline{X}_L, \dots, \underline{X}_{L/2+1}, \underline{-1}, \underline{X}_{L/2+1}, \dots, \underline{X}_L, \underline{-1}) \right) \\ &\quad - \frac{1}{L/2} \sum_{j=1}^{L/2} F(\underline{X}_{2j-1}, \underline{-1}, \underline{X}_{2j-1}, \underline{-1}, \dots, \underline{X}_{2j-1}, \underline{-1}). \end{aligned} \quad (49)$$

Let:

$$\tilde{F}(\underline{X}_1, \dots, \underline{X}_L) \doteq \frac{1}{L/2} \sum_{j=1}^{L/2} F(\underline{X}_{2j-1}, \underline{-1}, \underline{X}_{2j-1}, \underline{-1}, \dots, \underline{X}_{2j-1}, \underline{-1});$$

from:

$$\begin{aligned} \tilde{F}(\underline{X}_1, \dots, \underline{X}_L) &= \frac{1}{2} \left(\tilde{F}(\underline{X}_1, \dots, \underline{X}_{L/2-1}, \underline{-1}, \underline{X}_{L/2-1}, \dots, \underline{X}_1, \underline{-1}) \right. \\ &\quad \left. + \tilde{F}(\underline{X}_L, \dots, \underline{X}_{L/2+1}, \underline{-1}, \underline{X}_{L/2+1}, \dots, \underline{X}_L, \underline{-1}) \right), \end{aligned}$$

and Equation (49), we have:

$$\begin{aligned} G(\underline{X}_1, \dots, \underline{X}_L) &\geq \frac{1}{2} \left(G(\underline{X}_1, \dots, \underline{X}_{L/2-1}, \underline{-1}, \underline{X}_{L/2-1}, \dots, \underline{X}_1, \underline{-1}) \right. \\ &\quad \left. + G(\underline{X}_L, \dots, \underline{X}_{L/2+1}, \underline{-1}, \underline{X}_{L/2+1}, \dots, \underline{X}_L, \underline{-1}) \right), \end{aligned}$$

which proves the reflection bound for G . Next, we shall prove Equation (47). Let us define:

$$G_* = \min_{\underline{X}_1, \dots, \underline{X}_L \in \mathbb{Z}_2^L} G(\underline{X}_1, \dots, \underline{X}_L).$$

Let $\underline{X}_1^*, \dots, \underline{X}_L^*$ be a minimizing configuration. Then, by the reflection bound:

$$\begin{aligned} G(\underline{X}_1^*, \dots, \underline{X}_L^*) &\geq \frac{1}{2} \left(G(\underline{X}_1^*, \dots, \underline{X}_{L/2-1}^*, \underline{-1}, \underline{X}_{L/2-1}^*, \dots, \underline{X}_1^*, \underline{-1}) \right. \\ &\quad \left. + G(\underline{X}_L^*, \dots, \underline{X}_{L/2+1}^*, \underline{-1}, \underline{X}_{L/2+1}^*, \dots, \underline{X}_L^*, \underline{-1}) \right), \end{aligned}$$

and since $\{\underline{X}_j^*\}$ is a minimizing configuration:

$$G(\underline{X}_1^*, \dots, \underline{X}_L^*) = G(\underline{X}_1^*, \dots, \underline{X}_{L/2-1}^*, \underline{-1}, \underline{X}_{L/2-1}^*, \dots, \underline{X}_1^*, \underline{-1}).$$

Now, by cyclicity,

$$G(\underline{X}_1^*, \dots, \underline{X}_L^*) = G(\underline{X}_1^*, \underline{-1}, \underline{X}_1^*, \dots, \underline{X}_{L/2-1}^*, \underline{-1}, \underline{X}_{L/2-1}^*, \dots, \underline{X}_2^*).$$

Iterating the argument, we ultimately get:

$$G(\underline{X}_1^*, \dots, \underline{X}_L^*) = G(\underline{X}_1^*, \underline{-1}, \underline{X}_1^*, \underline{-1}, \dots, \underline{X}_1^*, \underline{-1}),$$

and recalling the definition of G , Eq. Equation (48), we find:

$$G(\underline{X}_1^*, \dots, \underline{X}_L^*) = 0.$$

Therefore, since $G(\underline{X}_1, \dots, \underline{X}_L) \geq G(\underline{X}_1^*, \dots, \underline{X}_L^*)$, we obtain:

$$F(\underline{X}_1, \dots, \underline{X}_L) \geq \frac{1}{L/2} \sum_{j=1}^{L/2} F(\underline{X}_{2j-1}, \underline{-1}, \underline{X}_{2j-1}, \underline{-1}, \dots, \underline{X}_{2j-1}, \underline{-1}),$$

which concludes the proof of the lemma. ■

We will apply this argument to the twisted/untwisted partition functions of the matter sector. To do this, let us denote by $\phi_{i,j}$ the flux associated with the plaquette (i, j) with $1 \leq i \leq L$; the labelling of the plaquettes coincides with the coordinates of the vertex of the plaquette sitting in the up-right corner. Recall that, see Remark 47, we are identifying the flux through a plaquette with the product of the spins around the same plaquette (same for the non-contractible loops). Let us denote by $\underline{\Phi}_j \in \mathbb{Z}_2^L$ the collection of all magnetic fluxes with column label j . We define:

$$F^\pm(\underline{\Phi}_1, \dots, \underline{\Phi}_L) \doteq - \max_{a,b} \log Z_{\beta,L}^\pm(\underline{\Phi}_1, \dots, \underline{\Phi}_L; a, b) \quad (50)$$

where $Z_{\beta,L}^{\pm}(\underline{\Phi}_1, \dots, \underline{\Phi}_L; a, b)$ is the twisted/untwisted partition function associated with the given configuration of magnetic fluxes, and with fluxes (a, b) on the non-contractible loops. In case zero modes are present for a given (a, b) , $\log Z^-$ is understood as $-\infty$.

Let us now check that the functions $F^{\pm}$ satisfy the hypotheses of [Lemma 70](#), starting from cyclicity. Let $\mathcal{T}_a$ be the unitary operator implementing the lattice translation by $a \in \Lambda_L$:

$$\mathcal{T}_a^* a_x^{\pm} \mathcal{T}_a = a_{x+a}^{\pm}.$$

Calling σ a gauge field configuration associated with the configuration of fluxes $\underline{\Phi}_L, \underline{\Phi}_1, \dots, \underline{\Phi}_{L-1}$, we have:

$$\begin{aligned} F^{\pm}(\underline{\Phi}_L, \underline{\Phi}_1, \dots, \underline{\Phi}_{L-2}, \underline{\Phi}_{L-1}) &= -\max_{a,b} \log Z_{\beta,L}^{\pm}(\underline{\Phi}_L, \dots, \underline{\Phi}_{L-2}, \underline{\Phi}_{L-1}; a, b) \\ &= -\max_{a,b} \log \text{Tr} (\pm 1)^N e^{-\beta H_f(\sigma)} \\ &= -\max_{a,b} \log \text{Tr} \mathcal{T}_{-e_1}^* (\pm 1)^N e^{-\beta H_f(\sigma)} \mathcal{T}_{-e_1} \\ &= -\max_{a,b} \log \text{Tr} (\pm 1)^N e^{-\beta H_f(\tilde{\sigma})}. \end{aligned}$$

where $\tilde{\sigma}$ is a Gauge-field configuration associated with the configuration of fluxes $\underline{\Phi}_1, \underline{\Phi}_2, \dots, \underline{\Phi}_L$ (the periodicity of the torus has been used). This proves cyclicity in [Lemma 70](#). Concerning the reflection estimate, this follows from [Equation \(34\)](#), after taking logarithms. We have, for suitable (a_i, b_i) , $i = 1, 2$:

$$\begin{aligned} -\log Z_{\beta,L}^{\pm}(\underline{\Phi}_1, \dots, \underline{\Phi}_L; a, b) &\geq \frac{1}{2} \left(-\log Z_{\beta,L}^{\pm}(\underline{\Phi}_1, \dots, \underline{\Phi}_{L/2-1}, \underline{\mathbf{1}}, \underline{\Phi}_{L/2-1}, \dots, \underline{\Phi}_1, \underline{\mathbf{1}}; a_1, b_1) \right. \\ &\quad \left. -\log Z_{\beta,L}^{\pm}(\underline{\Phi}_L, \dots, \underline{\Phi}_{L/2+1}, \underline{\mathbf{1}}, \underline{\Phi}_{L/2+1}, \dots, \underline{\Phi}_L, \underline{\mathbf{1}}; a_2, b_2) \right) \\ &\geq \frac{1}{2} \left(-\max_{a,b} \log Z_{\beta,L}^{\pm}(\underline{\Phi}_1, \dots, \underline{\Phi}_{L/2-1}, \underline{\mathbf{1}}, \underline{\Phi}_{L/2-1}, \dots, \underline{\Phi}_1, \underline{\mathbf{1}}; a, b) \right. \\ &\quad \left. -\max_{a,b} \log Z_{\beta,L}^{\pm}(\underline{\Phi}_L, \dots, \underline{\Phi}_{L/2+1}, \underline{\mathbf{1}}, \underline{\Phi}_{L/2+1}, \dots, \underline{\Phi}_L, \underline{\mathbf{1}}; a, b) \right). \end{aligned}$$

Minimizing the left-hand side over a, b , we see that the function [Equation \(50\)](#) satisfies the reflection bound in [Lemma 70](#).

Thus, we may apply [Equation \(47\)](#) to the twisted/untwisted free energies of the Matter sector. This bounds the energy cost of the 0-flux plaquettes by the cost of periodic flux configurations. To this end, let $(\phi_{i,j}^*)$ denote the Chessboard flux configuration,

$$\phi_{i,j}^* = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are odd} \\ -1 & \text{otherwise.} \end{cases}$$

See [Figure 6.5](#) for a graphical representation. The next proposition is a key consequence of the chessboard estimate.

Proposition 71 (Removing monopoles). *Let $L = 4\ell$. Given any configuration Φ of magnetic fluxes, let $k \in \mathbb{N}$ be the number of plaquettes with magnetic flux equal to zero (i.e. $B_{\Lambda} = 1$). Then, the following bound holds true:*

$$-\frac{1}{\beta} \log Z_{\beta,L}^{\pm}(\Phi; a, b) \geq k \Delta_{\beta,L}^{\pm} - \frac{1}{\beta} \log Z_{*}^{\pm} \quad (51)$$

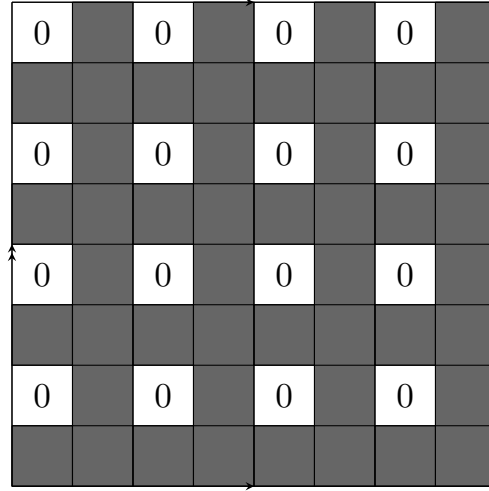


Figure 6.5: Graphical representation of the flux configuration corresponding to $\phi_{i,j}^*$. The white plaquettes correspond to $\phi_{i,j}^* = 1$ while the dark plaquettes correspond to $\phi_{i,j}^* = -1$.

with $Z_*^\pm$ the twisted/untwisted partition functions of the π -flux phase with boundary conditions $(a, b) = (-1, -1)$ and:

$$\Delta_{\beta,L}^\pm = -\frac{1}{\beta L^2} \left(\max_{a,b} \log Z_{\beta,L}^\pm(\Phi^*; a, b) - \log Z_*^\pm \right) \quad (52)$$

where $\Phi^* = \{\phi_{i,j}^*\}$ is the chessboard flux configuration (see Figure 6.5).

Remark 53. This bound is particularly useful, because the quantity $\Delta_{\beta,L}^\pm$ can be explicitly computed as $\beta \rightarrow \infty$, see Proposition 72. We will refer to $\Delta_\infty = \lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \Delta_{\beta,L}^\pm$ as the monopole's mass (the limit is the same for twisted/untwisted partition functions).

Proof. The formula Equation (47) has been obtained via iterative reflections across vertical planes. We could have of course obtained a similar estimate, reflecting across horizontal planes. We do so for all entries of the sum in Equation (47). The final result is:

$$F(\underline{X}_1, \dots, \underline{X}_L) \geq \frac{1}{(L/2)^2} \sum_{i,j=1}^{L/2} F(\underline{Y}_{2j-1,2i-1}, \underline{-1}, \underline{Y}_{2j-1,2i-1}, \underline{-1}, \dots, \underline{Y}_{2j-1,2i-1}, \underline{-1}), \quad (53)$$

where we introduced the vectors:

$$\underline{Y}_{j,i}^T = ((\underline{X}_j)_i, -1, (\underline{X}_j)_i, -1, \dots, (\underline{X}_j)_i, -1).$$

Graphically, the argument of F in the right-hand side of Equation (53) has a Chessboard structure, as in Figure 6.5, with 0 replaced by $0, \pi$ according to whether $(\underline{X}_{2j-1})_{2i-1} = 1, -1$, respectively. Observe that the right-hand side of Equation (53) only depends on the values $(\underline{X}_\ell)_m$ for odd ℓ, m . This reflects our choice of planes intersecting plaquettes with even column label in the first application of Lemma 70. To include all four choices of the parities of rows and columns, we write:

$$F(\underline{X}_1, \dots, \underline{X}_L) = 4 \times \frac{1}{4} F(\underline{X}_1, \dots, \underline{X}_L)$$

and apply Equation (47) to the four functions $(1/4)F(\cdot)$, using cuts associated with even columns–even rows, even columns–odd rows, odd columns–even rows, and odd columns–odd rows. Using cyclicity, the result is:

$$F(\underline{X}_1, \dots, \underline{X}_L) \geq \frac{1}{L^2} \sum_{i,j=1}^L F(\underline{Y}_{i,j}, \underline{-1}, \underline{Y}_{i,j}, \underline{-1}, \dots, \underline{Y}_{i,j}, \underline{-1}). \quad (54)$$

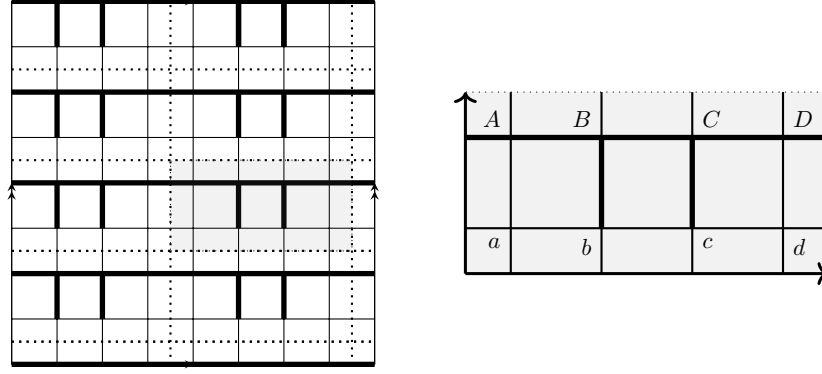


Figure 6.6: Left: gauge field configuration associated with the chessboard flux arrangement Φ^* in Figure 6.5. Solid bonds correspond to spin -1 , while light bonds correspond to spin $+1$. Right: fundamental cell associated with the periodic spin configuration.

Let us now use this estimate to prove Equation (51). Let us choose F as in Equation (50). Denoting by k the number of elements of Φ such that $\phi_{i,j} = 1$, Equation (54) implies:

$$\begin{aligned} -\frac{1}{\beta} \max_{a,b} \log Z_{\beta,L}^{\pm}(\Phi; a, b) &\geq -\frac{k}{\beta L^2} \max_{a,b} \log Z_{\beta,L}^{\pm}(\Phi^*; a, b) - \frac{L^2 - k}{\beta L^2} \log Z_*^{\pm} \\ &\geq -\frac{k}{\beta L^2} \left(\max_{a,b} \log Z_{\beta,L}^{\pm}(\Phi^*; a, b) - \log Z_*^{\pm} \right) - \frac{1}{\beta} \log Z_*^{\pm}, \end{aligned}$$

where Φ^* is the staggered magnetic flux configuration as in Figure 6.5. ■

We conclude the section by computing the asymptotics of $\Delta_{\beta,L}^{\pm}$ as $L \rightarrow \infty, \beta \rightarrow \infty$.

Proposition 72 (Asymptotics of the monopole's mass). *Let $L = 4\ell$. For β, ℓ large enough,*

$$\begin{aligned} \Delta_{\beta,L}^{\pm} &= \frac{t}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 + \frac{1}{2} \cos(k_1) + \frac{1}{2} \cos(k_2)} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 + \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 - \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}} + o(1) \end{aligned} \quad (55)$$

Remark 54. Calling Δ_{∞} the sum of the three integrals in Equation (55), numerical evaluation shows that:

$$\Delta_{\infty} \simeq 0.036t. \quad (56)$$

Proof. Let us consider the Hamiltonian $H_f(\Phi^*; 1, 1)$ associated with a gauge field configuration corresponding to the periodic flux configuration Φ^* depicted in Figure 6.5. Consider the case $(a, b) = (1, 1)$; the final result will not depend on the choice of a, b . A configuration associated with this flux arrangement is depicted in Figure 6.6, together with the corresponding fundamental cell of the fermionic Hamiltonian. The fundamental cell contains eight lattice points, labelled as in Figure 6.6. We denote by $\tilde{\Lambda}_L^{\text{red}}$ the lattice of the centers of the fundamental cells, corresponding with the position of the a sites on the original lattice. Using Fourier transforms, we can diagonalize $H_f(\Phi^*; 1, 1)$ by diagonalizing the corresponding Bloch Hamiltonian $h(k)$:

$$\begin{pmatrix} 0 & 1 & 0 & e^{-4ik_1} & 1 + e^{-2ik_2} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 + e^{-2ik_2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 + e^{-2ik_2} & 0 \\ e^{4ik_1} & 0 & 1 & 0 & 0 & 0 & 0 & 1 + e^{-2ik_2} \\ 1 + e^{2ik_2} & 0 & 0 & 0 & 0 & -1 & 0 & -e^{-4ik_1} \\ 0 & -1 + e^{2ik_2} & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 + e^{2ik_2} & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 + e^{2ik_2} & -e^{4ik_1} & 0 & -1 & 0 \end{pmatrix}.$$

Its eigenvalues are doubly degenerate, and are given by:

$$\begin{aligned} e_{1;\pm}(k) &= \pm 2t \sqrt{1 + \frac{1}{2} \sqrt{1 + \cos(2k_1)^2 + \cos(2k_2)^2}} \\ e_{2;\pm}(k) &= \pm 2t \sqrt{1 - \frac{1}{2} \sqrt{1 + \cos(2k_1)^2 + \cos(2k_2)^2}}. \end{aligned} \quad (57)$$

The quasi-momentum $k = (k_1, k_2)$ is a point on the Brillouin zone,

$$\tilde{B}_L(1, 1) \doteq \left\{ k \in \frac{2\pi}{L}(n_1, n_2) \mid 0 \leq n_1 \leq L/4 - 1, 0 \leq n_2 \leq L/2 - 1 \right\}.$$

More generally, for any $(a, b) \in \mathbb{Z}_2^2$, the energy bands are given by Equation (57), evaluated over the Brillouin zone:

$$\tilde{B}_L(a, b) = \tilde{B}_L(1, 1) - \frac{\pi}{2L}((a-1), (b-1)). \quad (58)$$

We are now ready to compute $\Delta_{\beta, L}^{\pm}$, as given by Equation (52). Let us denote by $e_{\pm}^{\pi}(k)$ the eigenvalues of the Bloch Hamiltonian associated with the π -flux phase, as given by Equation (40). In order to perform the computation, it is convenient to adapt the energy bands Equation (57) to the Brillouin zone of the π -flux phase, Equation (43), using that the functions in Equation (57) are periodic with period $\pi/2$ in the variable k_1 . We get:

$$-\frac{1}{\beta L^2} \log \frac{Z_{\beta, L}^{\pm}(\Phi^*; a, b)}{Z_*^{\pm}} = \frac{1}{\beta L^2} \sum_{k \in B_L(-1, -1)} \log \left(\frac{\prod_{z=\pm} (1 \pm e^{-\beta e_z^{\pi}(k)})}{\prod_{x=1,2} \prod_{y=\pm} (1 \pm e^{-\beta e_{x;y}(k)})^{\frac{1}{2}}} \right).$$

where the factor $1/4$ in the right-hand side is introduced in order to avoid overcounting, due to the fact that every quasi-momentum k in Equation (58) is now counted 4 times. As $\beta, L \rightarrow \infty$ the expression in the right-hand side can be approximated by an integral, independent of the choice of the parity of the partition function, as follows:

$$\begin{aligned} -\frac{1}{\beta L^2} \log \frac{Z_{\beta, L}^{\pm}(\Phi^*; a, b)}{Z_*^{\pm}} &= \frac{t}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 + \frac{1}{2} \cos(k_1) + \frac{1}{2} \cos(k_2)} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 + \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{1 - \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}} + o(1). \end{aligned}$$

This concludes the proof of the proposition. ■

6.3 Suppression of Flux Defects and Stability of the π -Flux Phase

Let us start by proving the statement about the free energy, Equation (4). The lower bound simply follows from:

$$Z_{\beta,L} = \sum_{\Phi} \sum_{a,b} Z_{\beta,L}(\Phi, a, b) \geq \sum_{a,b} Z_{\beta,L}(\pi, a, b) .$$

To prove the upper bound, we proceed as follows. We have:

$$\beta L^2(f_{\beta,L}(\pi) - f_{\beta,L}) = \log \left(\frac{\sum_{k=0}^{L^2/2} \sum_{a,b \in \mathbb{Z}_2^2} \sum_{n(\Phi)=2k} e^{-\beta H_g(\Phi)} (Z_{\beta,L}^+(\Phi, a, b) + Z_{\beta,L}^-(\Phi, a, b))}{\sum_{a,b \in \mathbb{Z}_2^2} e^{-\beta H_g(\pi)} (Z_{\beta,L}^+(\pi, a, b) + Z_{\beta,L}^-(\pi, a, b))} \right) \quad (59)$$

where $n(\Phi)$ is the number of 0-fluxes in the flux configuration Φ ; observe that this number is even, due to the constraint Equation (5), and to the fact that the number of elementary plaquettes is even for L even. By Proposition 71, we have:

$$Z_{\beta,L}^+(\Phi, a, b) + Z_{\beta,L}^-(\Phi, a, b) \leq e^{-\beta 2k \Delta_{\beta,L}^+} Z_*^+ + e^{-\beta 2k \Delta_{\beta,L}^-} Z_*^- \leq e^{-\beta 2k \Delta_{\beta,L}} (Z_*^+ + Z_*^-) . \quad (60)$$

with $\Delta_{\beta,L}$ the minimum between $\Delta_{\beta,L}^+$ and $\Delta_{\beta,L}^-$, and we used that $n(\Phi) = 2k$. Furthermore (recall Equation (12)), using that:

$$H_g(\Phi) = H_g(\pi) - 4k = |P(\Lambda_L)| - 4k , \quad (61)$$

and the positivity of the twisted partition function, we have:

$$(\beta L^2)(f_{\beta,L} - f_{\beta,L}(\pi)) \leq \log \left(1 + 4 \sum_{k=1}^{L^2/2} \binom{L^2}{2k} e^{-\beta 2k(\Delta_{\beta,L}-2)} \right) \leq \log \left(1 + 4 \left((1 + e^{-\beta(\Delta_{\beta,L}-2)})^{L^2} - 1 \right) \right) ,$$

where the first inequality follows by dropping at the denominator all but the largest partition functions, and the factor 4 arises from the sum over boundary conditions at the numerator. This proves the Equation (4). Let us now prove the statement about expectation values, Equation (6). We rewrite the expectation value of the gauge-invariant observable $\mathcal{O}(\sigma)$ as:

$$\langle \mathcal{O} \rangle_{\beta,L} = \frac{1}{\tilde{Z}_{\beta,L}} \sum_{a,b} \sum_{\Phi} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) \mathcal{O}(\Phi; a, b) e^{-\beta H(\Phi; a, b)} \quad (62)$$

where: $\Phi = (\phi_{i,j})$ is the collection of fluxes in the lattice plaquettes; $\mathcal{O}(\Phi; a, b)$ is the observable evaluated over a given gauge field representative of the flux configuration Φ, a, b ; and:

$$\tilde{Z}_{\beta,L} = \sum_{(a,b)} \sum_{\Phi} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) e^{-\beta H(\Phi; a, b)} . \quad (63)$$

Let us consider the partition function in Equation (63). From Equation (60), we obtain the estimate:

$$\frac{\sum_{(a,b)} \sum_{\Phi}^* \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) e^{-\beta H(\Phi; a, b)}}{\sum_{(a,b)} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) e^{-\beta H(\pi; a, b)}} \leq 4((1 + e^{-\beta(\Delta_{\beta,L}-2)})^{L^2} - 1) , \quad (64)$$

where the asterisk denotes the constraint $n(\Phi) \geq 2$. This estimate allows to prove that:

$$\tilde{Z}_{\beta,L} = \left(\sum_{(a,b)} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) e^{-\beta H(\pi; a, b)} \right) (1 + \mathcal{E}_{\beta,L}) \quad \text{with } 0 \leq \mathcal{E}_{\beta,L} \leq 4((1 + e^{-\beta(\Delta_{\beta,L}-2)})^{L^2} - 1). \quad (65)$$

Similarly, we can divide the contribution to the numerator in Equation (62) in a π and non π part:

$$\begin{aligned} \text{Numerator of Equation (62)} &= \sum_{a,b} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) \mathcal{O}(\pi; a, b) e^{-\beta H(\pi; a, b)} \\ &\quad + \sum_{a,b} \sum_{\Phi}^* \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) \mathcal{O}(\Phi; a, b) e^{-\beta H(\Phi; a, b)} \\ &= \sum_{a,b} \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) \mathcal{O}(\pi; a, b) e^{-\beta H(\pi; a, b)} + \mathcal{E}_{\beta, L}^{\mathcal{O}}. \end{aligned} \quad (66)$$

Proceeding as for the partition function, we estimate the last term in Equation (66) as:

$$\begin{aligned} |\mathcal{E}_{\beta, L}^{\mathcal{O}}| &\leq \sum_{a,b} \sum_{\Phi}^* \|\mathcal{O}(\Phi; a, b)\| \text{Tr}_{\mathcal{F}_L} \left(\frac{1 + (-1)^N}{2} \right) e^{-\beta H(\Phi; a, b)} \\ &\leq C_{\mathcal{O}}((1 + e^{-\beta(\Delta_{\beta, L} - 2)})^{L^2} - 1) e^{-\beta H_g(\pi)} \frac{Z_*^+ + Z_*^-}{2}. \end{aligned} \quad (67)$$

Therefore, coming back to Equation (62):

$$\langle \mathcal{O} \rangle_{\beta, L} = \frac{1}{1 + \mathcal{E}_{\beta, L}} \cdot \frac{\left(\sum_{a,b} \text{Tr}_{\mathcal{F}_L} (1 + (-1)^N) \mathcal{O}(\pi; a, b) e^{-\beta H(\pi; a, b)} + \mathcal{E}_{\beta, L}^{\mathcal{O}} \right)}{\sum_{a,b} \text{Tr}_{\mathcal{F}_L} (1 + (-1)^N) e^{-\beta H(\pi; a, b)}}. \quad (68)$$

Using the estimate in Equation (67), we get:

$$\begin{aligned} \frac{\mathcal{E}_{\beta, L}^{\mathcal{O}}}{\sum_{a,b} \text{Tr}_{\mathcal{F}_L} (1 + (-1)^N) e^{-\beta H(\pi; a, b)}} &\leq C_{\mathcal{O}}((1 + e^{-\beta(\Delta_{\beta, L} - 2)})^{L^2} - 1) \frac{Z_*^+ + Z_*^-}{\sum_{a,b} (Z_{\beta, L}^+(\pi; a, b) + Z_{\beta, L}^-(\pi; a, b))} \\ &\leq C_{\mathcal{O}}((1 + e^{-\beta(\Delta_{\beta, L} - 2)})^{L^2} - 1). \end{aligned} \quad (69)$$

Combining the estimates in Equation (69) and Equation (65) with Equation (68), we have:

$$\langle \mathcal{O} \rangle_{\beta, L} = \langle \mathcal{O} \rangle_{\beta, L}^{\pi} + \mathcal{E}_{\mathcal{O}; \beta, L}^{\text{tot}} \quad (70)$$

with:

$$\begin{aligned} |\mathcal{E}_{\mathcal{O}; \beta, L}^{\text{tot}}| &\leq \frac{\mathcal{E}_{\beta, L}}{1 + \mathcal{E}_{\beta, L}} \max_{a,b} \|\mathcal{O}(\pi, a, b)\| + \frac{C_{\mathcal{O}}((1 + e^{-\beta(\Delta_{\beta, L} - 1)})^{L^2} - 2)}{1 + \mathcal{E}_{\beta, L}} \\ &\leq K_{\mathcal{O}}((1 + e^{-\beta(\Delta_{\beta, L} - 2)})^{L^2} - 1). \end{aligned} \quad (71)$$

The energetic closeness of the π -flux phases, Equation (10), has been proved in Proposition 69, therefore the proof of Theorem 2 is concluded. $\blacksquare$

6.4 Low-Momentum Magnetic Susceptibility

6.4.1 Rewriting of the susceptibility

Let us introduce the density operator in Fourier space:

$$\hat{n}(p) = \sum_{x \in \Lambda_L^{\text{red}}} \sum_{i \in I} e^{-ip \cdot (x + r_i)} n_{x, i}, \quad n_{x, i} = a_{x, i}^+ a_{x, i}^-. \quad (72)$$

Let us derive the lattice continuity equation for $\hat{n}(p)$, in imaginary time. We have¹:

$$\begin{aligned}
-iq\partial_s\hat{n}(p, s) &= -iq[H, \hat{n}_p](s) \\
&= -iq \sum_{\substack{x, x', y \in \Lambda_L^{\text{red}} \\ i, j, k \in I}} e^{-ip \cdot (y+r_k)} h_{ij}(x, x') [a_{x,i}^+ a_{x',j}^-, n_{y,k}](s) \\
&= -iq \sum_{\substack{x, x' \in \Lambda_L^{\text{red}} \\ i, j \in I}} (e^{-ip \cdot (x'+r_i)} - e^{-ip \cdot (x+r_j)}) h_{ij}(x, x') (a_{x,i}^+ a_{x',j}^-)(s) \\
&= \frac{iq}{2} \sum_{\substack{x, x' \in \Lambda_L^{\text{red}} \\ i, j \in I}} (h_{ij}(x, x') a_{x,i}^+ a_{x',j}^- - h_{ji}(x', x) a_{x',j}^+ a_{x,i}^-)(s) (e^{-ip \cdot (x+r_i)} - e^{-ip \cdot (x'+r_j)}) \\
&= -\frac{iq}{2} \sum_{\substack{x, x' \in \Lambda_L^{\text{red}} \\ i, j \in I \\ \mu=1,2}} e^{-ip \cdot (x'+r_j)} (h_{ij}(x, x') a_{x,i}^+ a_{x',j}^- - h_{ji}(x', x) a_{x',j}^+ a_{x,i}^-)(s) \eta_\xi(p) (-ip_\mu \xi_\mu)
\end{aligned} \tag{73}$$

recall the definition of the function $\eta_\xi(p)$ in Equation (23). Recalling Equation (22), Equation (73) can be recast as:

$$iq\partial_s\hat{n}(p, s) + \sum_{\mu=1,2} ip_\mu \hat{J}_\mu(p, s) = 0. \tag{74}$$

Therefore,

$$iq\partial_s \left\langle \hat{n}(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} = i \sum_{\mu=1,2} p_\mu \left\langle \hat{J}_\mu(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L}.$$

Let us use the identity in Equation (74) to rewrite in a more convenient way the right-hand side of Equation (24)). Choose $p = (p_1, 0)$. We have:

$$\begin{aligned}
\int_0^\beta ds \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} &= \frac{q}{p_1} \int_0^\beta ds \partial_s \left\langle \hat{n}(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} \\
&= -\frac{q}{p_1} \left\langle [\hat{n}(-p), \hat{J}_1(p)] \right\rangle_{\beta, L},
\end{aligned}$$

where the last step follows from the KMS identity². The commutator is (for general values of p):

$$\begin{aligned}
iq[\hat{n}(-p), \hat{J}_1(p)] &= \frac{q^2}{2} \sum_{\substack{x, x', y \in \Lambda_L^{\text{red}} \\ i, j, k \in I}} e^{ip \cdot (y+r_k - (x'+r_j))} \left(h_{ij}(x, x') [n_{y,k}, a_{x,i}^+ a_{x',j}^-] \eta_\xi(p) \xi_1 \right. \\
&\quad \left. - h_{ji}(x', x) [n_{y,k}, a_{x',j}^+ a_{x,i}^-] \eta_\xi(p) \xi_1 \right) \\
&= \frac{q^2}{2} \sum_{\substack{x, x' \in \Lambda_L^{\text{red}} \\ i, j \in I}} (e^{ip \cdot (x+r_k - (x'+r_j))} - 1) \left(h_{ij}(x, x') a_{x,i}^+ a_{x',j}^- + h_{ji}(x', x) a_{x',j}^+ a_{x,i}^- \right) \eta_\xi(p) \xi_1 \\
&= \sum_{\mu=1,2} ip_\mu \hat{K}_{1,\mu}(p, -p),
\end{aligned}$$

¹Recall the identity:

$$[a_{x,i}^+ a_{x',j}, a_{y,k}^+ a_{y,k}^-] = a_{x,i}^+ a_{y,k}^- \delta_{x',y} \delta_{j,k} - a_{y,k}^+ a_{x',k}^- \delta_{x,y} \delta_{i,k}.$$

²It is easy to check that the KMS identity holds true for the Gibbs state in Equation (2), for physical observables.

recall Equation (22). Thus, for $p = (p_1, 0)$:

$$\left\langle \hat{K}_{1,1}(p, -p) \right\rangle_{\beta, L} = \frac{q}{p_1} \left\langle [\hat{n}_{-p}, \hat{J}_1(p)] \right\rangle_{\beta, L} = - \int_0^\beta ds \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L}. \quad (75)$$

We are interested in the choice of external momenta $p = (0, p_2)$. To begin, observe that, for $p = (0, p_2)$, from Equation (22):

$$\left\langle \hat{K}_{1,1}(p, -p) \right\rangle_{\beta, L} = \left\langle \hat{K}_{1,1}(0, 0) \right\rangle_{\beta, L}.$$

As we will see in the next section, the $\beta, L \rightarrow \infty$ limit of the left-hand side of Equation (75) is continuous in p , thus, we have:

$$\lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \left\langle \hat{K}_{1,1}((0, 0), (0, 0)) \right\rangle_{\beta, L} = - \lim_{p_1 \rightarrow 0} \lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \int_0^\beta ds \left\langle \hat{J}_1((-p_1, 0), s); \hat{J}_1((p_1, 0)) \right\rangle_{\beta, L}. \quad (76)$$

Therefore, we can express the magnetic susceptibility $\chi(p_2)$ (see Equation (25)) as:

$$\frac{1}{p_2^2} \lim_{L \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{1}{L^2} \int_0^\beta ds \left(\left\langle \hat{J}_1((0, -p_2), s); \hat{J}_1((0, p_2)) \right\rangle_{\beta, L} - \left\langle \hat{J}_1((0, 0), s); \hat{J}_1((0, 0)) \right\rangle_{\beta, L} \right). \quad (77)$$

To obtain Equation (77), we implicitly assumed that in the last step of Equation (76) we can first set $p_1 = 0$ in the current-current correlation, and then take $\beta, L \rightarrow \infty$, which will be shown later. The rewriting of Equation (77) will be a convenient starting point for the explicit evaluation of the susceptibility.

6.4.2 Evaluation of $\chi(p_2)$

In order to evaluate the susceptibility, as given by Equation (77), we shall use Theorem 2. We have:

$$\left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L} = \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L}^\pi + \mathfrak{e}_{\beta, L} \quad (78)$$

where: $\mathfrak{e}_{\beta, L}$ is an error term that vanishes as $L, \beta \rightarrow \infty$; the average over the π -flux phases can be written as:

$$\left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\beta, L}^\pi = \sum_{\varepsilon = \pm} \sum_{(a, b) \in \mathbb{Z}_2^2} \left\langle \hat{J}_1(-p, s); \hat{J}_1(p) \right\rangle_{\varepsilon, a, b} z_{\varepsilon, a, b} \quad (79)$$

with, for $(\varepsilon, a, b) \in \mathbb{Z}_2^3$:

$$\langle \mathcal{O} \rangle_{\varepsilon, a, b} \doteq \frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi; a, b)} \mathcal{O}(\pi; a, b)(\varepsilon)^N}{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi; a, b)}(\varepsilon)^N}, \quad z_{\varepsilon, a, b} \doteq \frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi; a, b)}(\varepsilon)^N}{\sum_{(a, b) \in \mathbb{Z}_2^2} \text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi; a, b)}(1 + (-1)^N)}. \quad (80)$$

As we will see, taking the limit $\beta \rightarrow \infty$ and then the limit $L \rightarrow \infty$, all $\langle \mathcal{O} \rangle_{+, a, b}$ converge to a value independent of a, b . In the same order of limits, also $\langle \mathcal{O} \rangle_{-, a, b}$ converge to this value, provided no zero mode is present in the spectrum of the single-particle Hamiltonian. Instead, for choices of a, b for which a zero mode is present, $\langle \mathcal{O} \rangle_{-, a, b} z_{-, a, b} = 0$, provided the order in the fermionic creation/annihilation operators appearing in $\mathcal{O}$ is not too high. This is a consequence of the next lemma.

Lemma 73. *Let $\mathcal{O}$ be an operator in the fermionic algebra, and let H be the second quantization of a Hamiltonian h . Suppose that $\mathcal{O}$ has the following form:*

$$\mathcal{O} = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_r}} O(\underline{i}; \underline{j}) a_{i_1}^+ \cdots a_{i_r}^+ a_{j_r}^- \cdots a_{j_1}^- \quad (81)$$

where a_i and a_i^+ , $i = 1, \dots, L^2$, are the fermionic annihilation and creation operators expressed in the basis of h . Suppose that h has $k > 0$ zero modes, and suppose that $k > r$. Then,

$$\text{Tr}(-1)^N e^{-\beta H} \mathcal{O} = 0.$$

Proof. Let us rewrite the trace as:

$$\sum_{\{q_i\}} \sum_{\{p_j\}} (-1)^{\sum_i q_i + \sum_j p_j} e^{-\beta \sum_i \varepsilon_i q_i} \langle \psi(\{q_i\}, \{p_j\}), \mathcal{O} \psi(\{q_i\}, \{p_j\}) \rangle,$$

where: $\{\varepsilon_i\}$ are the eigenvalues of h ; $\{q_i\}$ are the occupation numbers of the non-zero modes of h , and $\{p_j\}$ are the occupation numbers of the zero modes of h ; and $\langle \psi(\{q_i\}, \{p_j\})$ is the corresponding eigenstate of H (a Slater determinant). We claim that, if the number of zero modes k is larger than r :

$$\sum_{\{p_j\}} (-1)^{\sum_j p_j} \langle \psi(\{q_i\}, \{p_j\}), \mathcal{O} \psi(\{q_i\}, \{p_j\}) \rangle = 0.$$

To see this, it is enough to prove the statement for $\mathcal{O} = n_{b_1} \cdots n_{b_r}$, with $n_b = a_b^+ a_b^-$; the general claim follows by linearity. Let $B = \{b_1, \dots, b_r\}$. We have:

$$\begin{aligned} \sum_{\{p_j\}} (-1)^{\sum_j p_j} \langle \psi(\{q_i\}, \{p_j\}), \mathcal{O} \psi(\{q_i\}, \{p_j\}) \rangle &= \sum_{\{p_j\}} (-1)^{\sum_j p_j} \left[\prod_{\ell \in B} q_\ell \right] \left[\prod_{r \in B} p_r \right] \\ &= \left(\sum_{\{p_j\}: j \notin B} (-1)^{\sum_{j \notin B} p_j} \right) \left(\sum_{\{p_j\}: j \in B} \left[\prod_{\ell \in B} q_\ell \right] \left[\prod_{r \in B} (-1)^{p_r} \right] \right) \end{aligned}$$

which is zero if the set of zero modes is strictly larger than B , true if $k > r$. This concludes the proof of the lemma. $\blacksquare$

Remark 55. Using the path integral representation of non-normalized correlation functions, one can infer the same result by a counting argument. Since the Grassmann measure associated to 0-th Matsubara frequency and the hamiltonian zero modes is flat, in order for the integral to be non-vanishing we need to insert a couple of $\bar{\psi}\psi$ for any flat differential $d\bar{\psi}d\psi$. If h has k zero modes, then there are k flat differentials and therefore any r point correlation function is vanishing for $r < k$.

Let us now consider the special case in which the Hamiltonian h is the lattice Laplacian associated with the π -flux phase, with fluxes a, b on the non-contractible cycles. As discussed in Subsection 6.2.1, zero modes are present if and only if $(a, b) = (1, 1)$. The zero modes take place at $k = (\pi/2, \pi/2), (3\pi/2, \pi/2)$, and each has degeneracy 2 (the 2×2 Bloch Hamiltonian in Equation (39) vanishes for this choice of momentum). Thus, Lemma 73 applies, with $r < 4$. In particular, the operators $\hat{J}_1(-p, s)$, $\hat{J}_1(-p, s)\hat{J}_1(p)$ are of the form written in Equation (81) with $r = 1$ and $r = 2$, respectively.

The plan will be to show that, for these operators,

$$\langle \mathcal{O} \rangle_{\varepsilon, a, b} = \langle \mathcal{O} \rangle_\infty + \mathfrak{e}_{\varepsilon, a, b}^{\beta, L} \quad (82)$$

with $\langle \mathcal{O} \rangle_\infty$ independent of ε, a, b and $\mathfrak{e}_{\varepsilon, a, b}^{\beta, L} = o_L(1)$ as $\beta \rightarrow \infty$. Therefore, from Equation (79):

$$\langle \mathcal{O} \rangle_{\beta, L}^\pi = \langle \mathcal{O} \rangle_\infty + \mathfrak{e}^{\beta, L}, \quad (83)$$

with

$$\begin{aligned} |\mathbf{e}^{\beta,L}| &= \left| \sum_{(\varepsilon,a,b) \neq (-1,1,1)} \mathbf{e}_{\varepsilon,a,b}^{\beta,L} \frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi;a,b)}(\varepsilon)^N}{\sum_{(a,b) \in \mathbb{Z}_2^2} \text{Tr}_{\mathcal{F}_L} e^{-\beta H(\pi;a,b)}(1 + (-1)^N)} \right| \\ &\leq \sum_{(\varepsilon,a,b) \neq (-1,1,1)} |\mathbf{e}_{\varepsilon,a,b}^{\beta,L}| = o_L(1), \end{aligned} \quad (84)$$

where we used the positivity of the twisted partition functions and the vanishing of the partition function associated with $(\varepsilon, a, b) = (-1, 1, 1)$.

Let us now show [Equation \(83\)](#). To this end, we start by considering:

$$\int_0^\beta ds \left\langle \hat{J}_1((0, -p_2), s); \hat{J}_1((0, p_2)) \right\rangle_{-,a,b} \equiv \int_0^\beta ds \left\langle \hat{J}_1((0, -p_2), s) \hat{J}_1((0, p_2)) \right\rangle_{-,a,b} \quad (85)$$

where the identity follows from the fact that the average of $J_1((0, -p_2), s)$ is zero for $p_2 \neq 0$. In order to evaluate this quantity, it is convenient to extend periodically the argument of the integral to all $s \in \mathbb{R}$. Setting $a_x^\sharp(s) = e^{sH} a_x^\sharp e^{-sH}$, let us introduce the fermionic time ordering as, for $s_i \in [0, \beta)$:

$$\mathbf{T} a_{x_1}^{\sharp_1}(s_1) \cdots a_{x_r}^{\sharp_r}(s_r) = \text{sgn}(\pi) a_{x_{\pi(1)}}^{\sharp_{\pi(1)}}(s_{\pi(1)}) \cdots a_{x_{\pi(r)}}^{\sharp_{\pi(r)}}(s_{\pi(r)}), \quad (86)$$

where π is the permutation such that $s_{\pi(1)} \geq s_{\pi(2)} \geq \cdots \geq s_{\pi(n)}$. Equal times ambiguities are solved via normal ordering (they are not important, since they involve a zero measure set of times, and the fermionic operators are bounded). The action of $\mathbf{T}$ is extended to all the fermionic algebra, by linearity. Using this definition, we have:

$$\text{Equation (85)} = \int_{-\beta/2}^{\beta/2} ds \left\langle \mathbf{T} \hat{J}_1((0, -p_2), s); \hat{J}_1((0, p_2)) \right\rangle_{-,a,b} \quad (87)$$

where the argument of the integral is a periodic function over $\mathbb{R}$, with period β ; in $[0, \beta]$, it agrees with the argument of the original integral in [Equation \(85\)](#).

The factor $(-1)^N$ in the definition of $\langle \cdot \rangle_{-,a,b}$ can be taken into account introducing an imaginary chemical potential $\mu = i\pi/\beta$ in the definition of the Gibbs state. The state is quasi-free, and all correlations can be computed using the fermionic Wick's rule. To do this, let us represent the current operator in momentum space, using [Equation \(38\)](#), with $k \in B_L(a, b)$. Observe that, in general, the current operator depends on the gauge field configuration; here, we are considering it evaluated on the periodic configuration that represents the π -flux phase with fluxes (a, b) on the non-contractible cycles. From [Equation \(22\)](#), we have:

$$\hat{J}_\mu(p) = \frac{1}{|\Lambda_L^{\text{red}}|} \sum_{k \in B_L(a,b)} \hat{a}_{k-p,i}^+ j_{\mu;ij}(k-p, k) \hat{a}_{k,j}^- \quad (88)$$

Let $\mu = 1$. Recalling that $(r_i)_1 = 0$ for $i = A, B$, we have, for $p = (0, p_2)$:

$$j_1(k-p, k) = itq\eta_1(p) \begin{pmatrix} e^{-ip_2}(-e^{-i(k_1-p_1)} + e^{ik_1}) & 0 \\ 0 & e^{-i(k_1-p_1)} - e^{ik_1} \end{pmatrix}. \quad (89)$$

The expression in [Equation \(87\)](#) can be computed using the expression for the Euclidean two-point function (see [Equation \(42\)](#)), for $t \neq s$:

$$\langle \mathbf{T} \gamma_t(\hat{a}_{k,i}^-) \gamma_s(\hat{a}_{q,j}^+) \rangle_{-,a,b} = \delta_{k,q} |\Lambda_L^{\text{red}}| \frac{1}{\beta} \lim_{M \rightarrow \infty} \sum_{\substack{\omega \in \mathbb{M}_\beta^{\text{F}} \\ |\omega| \leq M}} e^{-i\omega(t-s)} \left(\frac{1}{-i\omega + h(k) - i\pi/\beta} \right)_{ij} \quad (90)$$

where $\mathbb{M}_\beta^F$ is the set of fermionic Matsubara frequencies, $\mathbb{M}_\beta^F = \frac{2\pi}{\beta}(\mathbb{Z} + 1/2)$. Thus, applying the fermionic Wick's rule, and using that $\mathbb{M}_\beta^F + \pi/\beta = \mathbb{M}_\beta^B$, with $\mathbb{M}_\beta^B$ the set of bosonic Matsubara frequencies, we get:

$$\begin{aligned} \text{Equation (87)} &= -\frac{1}{\beta} \sum_{k \in B_L(a,b)} \sum_{\omega \in \mathbb{M}_\beta^B} \text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k-p) \frac{1}{-i\omega + h(k-p)} j_1(k-p, k) \right) \\ &\stackrel{o(1)}{=} -2|\Lambda_L^{\text{red}}| \int_{\mathbb{B}} \frac{dk}{(2\pi)^2} \int_{\mathbb{R}} \frac{d\omega}{(2\pi)} \text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k-p) \frac{1}{-i\omega + h(k-p)} j_1(k-p, k) \right), \end{aligned} \quad (91)$$

where $\mathbb{B} = [0, 2\pi] \times [0, \pi]$, and the $o(1)$ error terms are due to the approximation of the sums with integrals, which are vanishing as $\beta \rightarrow \infty$ and $L \rightarrow \infty$, irrespectively of the order.

The main term in Equation (91) is independent of a, b ; the choice of a, b only affects the sub-leading error terms. The same computation can be reproduced for all $\langle \cdot \rangle_{\varepsilon, a, b}$, such that no zero modes are present; for $\varepsilon = +$, the two-point function is given by Equation (90), omitting the term $-i\pi/\beta$ at the denominator. All these choices give rise to the same $L, \beta \rightarrow \infty$ limit. This proves Equation (82) and hence Equation (83).

Observe that, for $(\varepsilon, a, b) \neq (-1, 1, 1)$, the momentum-space two-point function is never singular at finite β, L . Also, the integral in the argument of the right-hand side of Equation (91) is continuous in p , thanks to the almost-everywhere continuity and the absolute integrability of the integrand. Similarly, for these choices of (ε, a, b) , the explicit expression of the two-point function shows that the limits $p \rightarrow (0, 0)$ and $L, \beta \rightarrow \infty$ of Equation (87) commute. This allows to check all assumptions we made to obtain the rewriting Equation (77). Thus, for $p = (0, p_2)$, we get, using that $|\Lambda_L^{\text{red}}| = L^2/2$, recall Equation (77):

$$\begin{aligned} \chi(p_2) &= -\frac{1}{p^2} \int_{\mathbb{B}} \frac{dk}{(2\pi)^2} \int_{\mathbb{R}} \frac{d\omega}{(2\pi)} \left[\text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k-p) \frac{1}{-i\omega + h(k-p)} j_1(k-p, k) \right) \right. \\ &\quad \left. - \text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k) \frac{1}{-i\omega + h(k)} j_1(k, k) \right) \right] \\ &\equiv -\frac{1}{p_2^2} (A(p_2) - A(0)). \end{aligned} \quad (92)$$

In order to evaluate this quantity, we proceed similarly to what has been done for the conductivity of graphene, see e.g. [GMP12]. We preliminarily observe that the function:

$$p \mapsto \int_{\mathbb{B}} \frac{dk}{(2\pi)^2} \text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k-p) \frac{1}{-i\omega + h(k-p)} j_1(k-p, k) \right) \quad (93)$$

is even (use the change of variables $k + p \rightarrow k$, and the cyclicity of the trace). Therefore, if the function $A(p_2)$ was differentiable, the limit $\lim_{p_2 \rightarrow 0} p_2 \chi(p_2)$ would be zero. As for graphene, differentiability at zero momentum does not hold; nevertheless, the parity of $A(p_2)$ can be used to simplify the computation of the susceptibility. For $\delta > |p_2|$, we write:

$$\begin{aligned} A(p_2) &= \int_{\mathbb{B}} \int_{\mathbb{R}} \frac{dk}{(2\pi)^2} \frac{d\omega}{(2\pi)} \left[\text{Tr} \left(\frac{1}{-i\omega + h(k)} j_1(k, k-p) \frac{1}{-i\omega + h(k-p)} j_1(k-p, k) \right) \chi_\delta(k) + R_1(p) \right] \\ &\doteq A_\delta(p_2) + R_1(p_2), \end{aligned} \quad (94)$$

where $\chi_\delta(k) = \mathbf{1}(|k_1 - \pi/2| \leq \delta) \mathbf{1}(|k_2 - \pi/2| \leq \delta) + \mathbf{1}(|k_1 - 3\pi/2| \leq \delta) \mathbf{1}(|k_2 - \pi/2| \leq \delta)$ and $R_1(p_2)$ is an error term, which takes into account the integration region associated with k away from the

singularity points $k_F^- = (\pi/2, \pi/2)$, $k_F^+ = (3\pi/2, \pi/2)$. It is easy to see that the function $R_1(p_2)$ is differentiable in p_2 at $p_2 = 0$. Next, in $A_\delta(p_2)$, we compare the integrand with its relativistic approximation. To do this, we use that, for $k = k_F^\alpha + k'$, $\alpha = \pm$ and $|k'| \leq \delta$:

$$\begin{aligned} j_1(k-p, k) &= 2tq \begin{pmatrix} \alpha e^{-ip_2} & 0 \\ 0 & -\alpha \end{pmatrix} + \epsilon_j(k-p, k) \\ j_1(k, k-p) &= 2tq \begin{pmatrix} \alpha e^{ip_2} & 0 \\ 0 & -\alpha \end{pmatrix} + \epsilon_j(k, k-p) \\ h(k_F^\alpha + k') &= -2t \begin{pmatrix} -\alpha k'_1 & -ik'_2 \\ ik'_2 & \alpha k'_1 \end{pmatrix} + \epsilon_h(k), \end{aligned} \quad (95)$$

where, for k such that $|k - k_F^\alpha| \leq 2\delta$:

$$\|\epsilon_j(k-p, k)\| \leq C|k - k_F^\alpha|, \quad \|\epsilon_j(k, k-p)\| \leq C|k - k_F^\alpha|, \quad \|\epsilon_h(k)\| \leq C|k - k_F^\alpha|^2. \quad (96)$$

In $A_\delta(p_2)$, we replace j_1 , h with their leading contributions for k close to either k_F^+ or k_F^- , as given by Equation (95). Correspondingly, we write:

$$A_\delta(p_2) = A_\delta^{\text{rel}}(p_2) + R_2(p_2); \quad (97)$$

the function $A_\delta^{\text{rel}}(p_2)$ takes into account the relativistic approximation of the integrand, while $R_2(p_2)$ is an error term. The relativistic propagator entering in $A_\delta^{\text{rel}}(p_2)$ is, for $k = k' + k_F^\alpha$, $|k'| \leq \delta$ and recalling that $v = 2t$:

$$\left(\frac{1}{-i\omega + h(k)} \right)^{\text{rel}} = \frac{1}{\omega^2 + v^2|k'|^2} \begin{pmatrix} i\omega + v\alpha k'_1 & ivk'_2 \\ -ivk'_2 & i\omega - v\alpha k'_1 \end{pmatrix}. \quad (98)$$

Concerning the error terms, due to the extra factors $|k - k_F^\alpha|$ in the estimates in Equation (96), one can check that $R_2(p_2)$ is differentiable in p at $p = 0$. All together:

$$A(p_2) = A_\delta^{\text{rel}}(p_2) + \tilde{R}(p_2), \quad \tilde{R}(p_2) = R_1(p_2) + R_2(p_2); \quad (99)$$

the function $A_\delta^{\text{rel}}(p_2)$ is even in p_2 , and hence $\tilde{R}(p_2)$ is also even in p_2 . In particular, by differentiability at zero,

$$\frac{1}{p_2}(\tilde{R}(p) - \tilde{R}(0)) \rightarrow 0 \quad \text{as } p_2 \rightarrow 0, \quad (100)$$

which implies:

$$\chi(p_2) = -\frac{1}{p_2^2}(A_\delta^{\text{rel}}(p_2) - A_\delta^{\text{rel}}(0)) + o\left(\frac{1}{p_2}\right). \quad (101)$$

Thus, we are left with evaluating the first term in the right-hand side of Equation (101), which is fully determined by the relativistic approximation at low energy. We will reduce this term to an integral that can be explicitly computed, plus error terms of the form $(1/p_2^2)(R_j(p_2) - R_j(0))$, for suitable $R_j(p_2)$ that are even and differentiable; hence, arguing as before, they give a subleading contribution to $\chi(p_2)$. We have, dropping all primes from now on:

$$\begin{aligned} A_\delta^{\text{rel}}(p_2) &= 16t^2 q^2 \int_{|k_j| \leq \delta} \frac{dk}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{1}{(\omega^2 + v^2|k|^2)(\omega^2 + v^2|k-p|^2)} \\ &\quad \cdot \left(-\omega^2 + v^2 k_1^2 - v^2 k_2(k_2 - p_2) \right) + R_3(p_2), \end{aligned} \quad (102)$$

where $R_3(p_2)$ takes into account the replacement of $e^{\pm ip_2}$ with 1 in Equation (95), and:

$$\frac{1}{p_2^2} \left| R_3(p_2) - R_3(0) \right| \leq C |\log p_2| . \quad (103)$$

Consider the main term in Equation (102). We rewrite it as:

$$\begin{aligned} & 16t^2 q^2 \int_{|k_j| \leq \delta} \frac{dk}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{1}{(\omega^2 + v^2|k|^2)(\omega^2 + v^2|k - p|^2)} \\ & \quad \cdot \left(-\omega^2 + v^2 k_1^2 - v^2 k_2(k_2 - p_2) \right) \\ & = 4q^2 \int_{|k_j| \leq v\delta} \frac{dk}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \frac{1}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} \\ & \quad \cdot \left(-\omega^2 + k_1^2 - k_2(k_2 - vp_2) \right) \\ & = 4q^2 \int_{|k_j| \leq v\delta} \frac{dk}{(2\pi)^2} \int_{|\omega| \leq v\delta} \frac{d\omega}{(2\pi)} \frac{1}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} \\ & \quad \cdot \left(-\omega^2 + k_1^2 - k_2(k_2 - vp_2) \right) + R_4(p_2) , \end{aligned} \quad (104)$$

where $R_4(p_2)$ takes into account the contribution of $|\omega| > v\delta$, and it is smooth and even in p_2 . Concerning the main term, observing that the integration domain is symmetric under exchange of ω and k_1 , we have:

$$\begin{aligned} & 4q^2 \int_{|k_j| \leq v\delta} \frac{dk}{(2\pi)^2} \int_{|\omega| \leq v\delta} \frac{d\omega}{(2\pi)} \frac{-k_2(k_2 - vp_2)}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} \\ & = 4q^2 \int_{|\omega| \leq v\delta} \frac{d\omega}{2\pi} \int_{|k_1| \leq v\delta} \frac{dk_1}{2\pi} \int_{\mathbb{R}} \frac{dk_2}{2\pi} \frac{-k_2(k_2 - vp_2)}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} + R_5(p_2) , \end{aligned} \quad (105)$$

where $R_5(p_2)$ takes into account the contribution of $|k_2| > v\delta$, which we added and subtracted; this error term is smooth and even in p_2 . Let us come back to Equation (101). We have:

$$\begin{aligned} \chi(p_2) & = \frac{4q^2}{p_2^2} \int_{|\omega| \leq v\delta} \frac{d\omega}{2\pi} \int_{|k_1| \leq v\delta} \frac{dk_1}{2\pi} \int_{\mathbb{R}} \frac{dk_2}{2\pi} \left(\frac{k_2(k_2 - vp_2)}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} - \frac{k_2^2}{(\omega^2 + |k|^2)^2} \right) \\ & \quad - \frac{1}{p_2^2} \sum_{j=1}^5 (R_j(p_2) - R_j(0)) . \end{aligned} \quad (106)$$

The integral over k_2 can be evaluated explicitly using residues; it has been done, for instance, in [GMP12, A.9]. We have, replacing p_0 in [GMP12, A.9] with $-vp_2$:

$$\int_{\mathbb{R}} \frac{dk_2}{2\pi} \left(\frac{k_2(k_2 - vp_2)}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} - \frac{k_2^2}{(\omega^2 + |k|^2)^2} \right) = -\frac{v^2 p_2^2}{4(\omega^2 + k_1^2)^{\frac{1}{2}}(v^2 p_2^2 + 4(\omega^2 + k_1^2))} . \quad (107)$$

Thus, we can rewrite the main term in Equation (106) as:

$$\begin{aligned} & \frac{4q^2}{p_2^2} \int_{|\omega| \leq v\delta} \frac{d\omega}{2\pi} \int_{|k_1| \leq v\delta} \frac{dk_1}{2\pi} \int_{\mathbb{R}} \frac{dk_2}{2\pi} \left(\frac{k_2(k_2 - vp_2)}{(\omega^2 + |k|^2)(\omega^2 + |k - vp|^2)} - \frac{k_2^2}{(\omega^2 + |k|^2)^2} \right) \\ & = -\frac{4q^2}{p_2^2} \int_{|\omega| \leq v\delta} \frac{d\omega}{2\pi} \int_{|k_1| \leq v\delta} \frac{dk_1}{2\pi} \frac{v^2 p_2^2}{4(\omega^2 + k_1^2)^{\frac{1}{2}}(v^2 p_2^2 + 4(\omega^2 + k_1^2))} \\ & = -\frac{4vq^2}{|p_2|} \int_{|\omega| \leq \delta/|p_2|} \frac{d\omega}{2\pi} \int_{|k_1| \leq \delta/|p_2|} \frac{dk_1}{2\pi} \frac{1}{4(\omega^2 + k_1^2)^{\frac{1}{2}}(1 + 4(\omega^2 + k_1^2))} \\ & = -\frac{4vq^2}{|p_2|} \int_{\omega^2 + k_1^2 \leq \frac{\delta^2}{p_2^2}} \frac{d\omega}{(2\pi)} \frac{dk_1}{(2\pi)} \frac{1}{4(\omega^2 + k_1^2)^{\frac{1}{2}}(1 + 4(\omega^2 + k_1^2))} + E(p_2) , \end{aligned} \quad (108)$$

where $E(p_2)$ takes into account the change of integration domain, and it is $o(1/|p_2|)$. The main term in Equation (108) can be evaluated explicitly, using that:

$$\begin{aligned} \int_{\omega^2+k_1^2 \leq \frac{\delta^2}{p_2^2}} \frac{d\omega}{(2\pi)} \frac{dk_1}{(2\pi)} \frac{1}{4(\omega^2+k_1^2)^{\frac{1}{2}}(1+4(\omega^2+k_1^2))} &= \frac{1}{2\pi} \int_0^{\delta/|p_2|} x dx \frac{1}{4x(1+4x^2)} \\ &= \frac{1}{16\pi} \int_0^{2\delta/|p_2|} dy \frac{1}{1+y^2} = \frac{1}{16\pi} \arctan(2\delta/|p_2|) . \end{aligned} \quad (109)$$

Thus, putting together Equation (106), Equation (108) and Equation (109), we have:

$$\chi(p_2) = -\frac{vq^2}{8|p_2|} - \frac{1}{p_2^2} \sum_{j=1}^5 (R_j(p_2) - R_j(0)) + \tilde{E}(p_2) , \quad (110)$$

where $\tilde{E}(p_2)$ takes into account the replacement of the arctan with its limit, and it is $o(1/|p_2|)$. Thus, since all $R_j(p_2)$ are even and differentiable, we have:

$$\chi(p_2) = -\frac{vq^2}{8|p_2|} + o\left(\frac{1}{|p_2|}\right) . \quad (111)$$

This concludes the proof of Proposition 68. ■

6.A Asymptotic Expansion of the Ground State Energies

We will treat the cases $L/2 = 2n$ and $L/2 = 2n + 1$ separately, since Lieb's theorem gives different minimizing holonomies in the two cases. Let:

$$e_-(k|a, b) \doteq e_-\left(k_1 - \frac{\pi}{2L}(a-1), k_2 - \frac{\pi}{2L}(b-1)\right) \frac{1}{2t} \quad (112)$$

The quasi-momentum k belongs to the Brillouin zone $B_L(1, 1)$, Equation (37); that is, $k = (2\pi/L)(n_1, n_2)$, with $0 \leq n_1 \leq L-1$, $0 \leq n_2 \leq L/2-1$. Using that $e_-(k_1, k_2) = e_-(k_1 + \pi, k_2)$, we can assume that $0 \leq n_1 \leq L/2-1$, and we count twice the energetic contribution of every quasi-momentum in the evaluation of the total energy. Suppose that $L/2 = 2n$. The parametrization of the lower energy band as function of n_1, n_2 is, for $0 \leq n_i \leq 2n-1$:

$$e_-(k|a, b) = -\sqrt{1 + \frac{1}{2} \cos\left(\frac{\pi n_1}{n} - \frac{\pi}{4n}(a-1)\right) + \frac{1}{2} \cos\left(\frac{\pi n_2}{n} - \frac{\pi}{4n}(b-1)\right)} . \quad (113)$$

For these values of (n_1, n_2) , the energy band is vanishing if and only if $(n_1, n_2) = (n, n)$. To begin, we separate the values of (n_1, n_2) close to (n, n) from the values far from this point. That is, for $0 < \alpha < 1$ to be chosen:

$$\sum_{n_1, n_2: 0 \leq n_i \leq 2n-1} e_-(k|a, b) = \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| \leq n^\alpha} e_-(k|a, b) + \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} e_-(k|a, b) , \quad (114)$$

where here and in the following $|(n_1, n_2) - (n, n)|$ is the Euclidean distance defined mod $2n$. Also, in what follows we will always assume that (n_1, n_2) are summed over the range $0 \leq n_i \leq 2n-1$. We will show that the second term in the right-hand side of Equation (114) is essentially independent of (a, b) , up to subleading terms as $n \rightarrow \infty$, while the first is small.

Consider the first term in the right-hand side of Equation (114). Here we use that, for the considered values of n_1, n_2 , it holds $|e_-(k|a, b)| \leq Cn^{\alpha-1}$, for an α -dependent constant. In what

follows, we will not keep track of this dependence, and we will use the symbol C to denote general multiplicative constants appearing in the estimates, that might change from line to line. Therefore:

$$\sum_{n_1, n_2: |(n_1, n_2) - (n, n)| \leq n^\alpha} |e_-(k|a, b)| \leq C \frac{n^{3\alpha}}{n}, \quad (115)$$

which is $o(1)$ for $\alpha < 1/3$. Consider now the second term in Equation (114). Let us expand $e_-(k|a, b)$ around $e_-(k|1, 1)$, as follows:

$$\begin{aligned} e_-(k|a, b)^2 &= e_-(k|1, 1)^2 + \left[\frac{1}{2} \frac{\pi}{4n} ((a-1) \sin(\pi n_1/n) + (b-1) \sin(\pi n_2/n)) \right. \\ &\quad \left. - \frac{1}{4} \left(\frac{\pi}{4n} \right)^2 ((a-1)^2 \cos(\pi n_1/n) + (b-1)^2 \cos(\pi n_2/n)) \right] + O\left(\frac{1}{n^3}\right). \end{aligned} \quad (116)$$

Observe that, for n large enough, $|e_-(k|a, b)|^2 \geq C(|n - n_1|^2 + |n - n_2|^2)/n^2$, for all a, b , while the argument of the square brackets in Equation (116) is bounded above as $C(|n_1 - n| + |n_2 - n|)/n^2$; thus, for the considered values of n_1, n_2 and for $\alpha > 0$, the first term in the right-hand side of Equation (116) is much larger than the other ones. Taking the square root of Equation (116), we have:

$$\begin{aligned} e_-(k|a, b) &= e_-(k|1, 1) \left(1 + \frac{\pi}{4n} \frac{\frac{(a-1)}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n)}{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)} \right. \\ &\quad \left. - \left(\frac{\pi}{4n} \right)^2 \frac{\frac{(a-1)^2}{2} \cos(\pi n_1/n) + \frac{(b-1)^2}{2} \cos(\pi n_2/n)}{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)} + O\left(\frac{1}{e_-^2(k|1, 1)n^3}\right) \right)^{\frac{1}{2}} \\ &= e_-(k, |1, 1) \left(1 + \frac{1}{2} \frac{\pi}{4n} \frac{\frac{(a-1)}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n)}{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)} \right. \\ &\quad - \frac{1}{2} \left(\frac{\pi}{4n} \right)^2 \frac{\frac{(a-1)^2}{2} \cos(\pi n_1/n) + \frac{(b-1)^2}{2} \cos(\pi n_2/n)}{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)} \\ &\quad \left. - \frac{1}{8} \left(\frac{\pi}{4n} \right)^2 \left(\frac{\frac{(a-1)}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n)}{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)} \right)^2 \right) + r(k|a, b), \end{aligned} \quad (117)$$

where the error term $r(k|a, b)$ collects the Taylor remainder, and it is bounded as, for $|(n_1, n_2) - (n, n)| > n^\alpha$ and $0 < \alpha < 1/3$:

$$\begin{aligned} |r(k|a, b)| &\leq \frac{C}{|e_-(k|1, 1)|n^3} + \frac{C(|n_1 - n|^3 + |n_2 - n|^3)}{n^6 |e_-(k|1, 1)|^5} \\ &\equiv r_1(k|a, b) + r_2(k|a, b). \end{aligned} \quad (118)$$

Consider $r_1(k|a, b)$. Using the lower bound for $e_-(k|1, 1)$, we have:

$$\begin{aligned} \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} r_1(k|a, b) &\leq C \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} \frac{1}{n^2(|n - n_1| + |n - n_2|)} \\ &\leq \frac{C \log n}{n}. \end{aligned} \quad (119)$$

Consider now $r_2(k|a, b)$. We have:

$$\begin{aligned} \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} r_2(k|a, b) &\leq \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} \frac{C(|n_1 - n|^3 + |n_2 - n|^3)}{n(|n_1 - n|^5 + |n_2 - n|^5)} \\ &\leq \frac{C}{n} \sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} \frac{1}{|n_1 - n|^2 + |n_2 - n|^2} \\ &\leq \frac{C \log n}{n}. \end{aligned} \quad (120)$$

All together,

$$\sum_{n_1, n_2: |(n_1, n_2) - (n, n)| > n^\alpha} |r(k|a, b)| \leq \frac{C \log n}{n}. \quad (121)$$

Observe that all the error terms we accumulated so far, namely the ones in Equation (115) and Equation (121), are $o(1)$ as $n \rightarrow \infty$. Our next task will be to show that the sum over n_1, n_2 of the main three terms in Equation (117) is also $o(1)$. We rewrite:

$$\begin{aligned} & e_-(k|a, b) - e_-(k|1, 1) \\ &= \frac{1}{2} \frac{\pi}{4n} \frac{\frac{(a-1)}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} \\ & \quad - \frac{1}{2} \left(\frac{\pi}{4n} \right)^2 \frac{(\frac{a-1}{2})^2 \cos(\pi n_1/n) + (\frac{b-1}{2})^2 \cos(\pi n_2/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} \\ & \quad - \frac{1}{8} \left(\frac{\pi}{4n} \right)^2 \frac{(\frac{a-1}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n))^2}{(1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n))^{3/2}} + r(k|a, b). \end{aligned} \quad (122)$$

Let us now consider the sum over all (n_1, n_2) such that $|n - n_i| > n^\alpha$, for $i = 1, 2$, of the first three terms in the right-hand side of Equation (122). We start from the first term. We have:

$$\begin{aligned} & \frac{\pi}{8n} \sum_{\substack{n_1, n_2=0, \dots, 2n-1 \\ |n-n_i| > n^\alpha}} \frac{\frac{(a-1)}{2} \sin(\pi n_1/n) + \frac{(b-1)}{2} \sin(\pi n_2/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} \\ &= \frac{\pi(a+b-2)}{16n} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n-n_i| > n^\alpha}} \frac{\sin(\pi n_1/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} = 0, \end{aligned} \quad (123)$$

where we used that the integer n_1 belongs to the summation range if and only if $n_1^* = 2n - n_1$ does so, and that the summand evaluated on n_1 is minus the summand evaluated in n_1^* (observe that we could freely add $n_1 = 2n$ to the sum, since the argument of the sum is vanishing). Consider now the second term in Equation (122). We have:

$$\begin{aligned} & -\frac{\pi^2}{32n^2} \sum_{\substack{n_1, n_2=0, \dots, 2n-1 \\ |n-n_i| > n^\alpha}} \frac{(\frac{a-1}{2})^2 \cos(\pi n_1/n) + (\frac{b-1}{2})^2 \cos(\pi n_2/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} \\ &= -\frac{\pi^2((\frac{a-1}{2})^2 + (\frac{b-1}{2})^2)}{32n^2} \sum_{\substack{n_1, n_2=0, \dots, 2n-1 \\ |n-n_i| > n^\alpha}} \frac{\cos(\pi n_1/n)}{\sqrt{1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n)}} \\ &= -\frac{((\frac{a-1}{2})^2 + (\frac{b-1}{2})^2)}{32} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\cos(x)}{\sqrt{1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y)}} + O\left(\frac{n^\alpha}{n}\right). \end{aligned} \quad (124)$$

Finally, consider the third term. We get:

$$\begin{aligned}
& -\frac{\pi^2}{128n^2} \sum_{\substack{n_1, n_2=0, \dots, 2n-1 \\ |n-n_i| > n^\alpha}} \frac{(\frac{a-1}{2} \sin(\pi n_1/n) + \frac{b-1}{2} \sin(\pi n_2/n))^2}{(1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n))^{3/2}} \\
& = -\frac{\pi^2((\frac{a-1}{2})^2 + (\frac{b-1}{2})^2)}{128n^2} \sum_{\substack{n_1, n_2=-n, \dots, n-1 \\ |n-n_i| > n^\alpha}} \frac{\sin(\pi n_1/n)^2}{(1 + \frac{1}{2} \cos(\pi n_1/n) + \frac{1}{2} \cos(\pi n_2/n))^{3/2}} \\
& = -\frac{((\frac{a-1}{2})^2 + (\frac{b-1}{2})^2)}{128} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\sin(x)^2}{(1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y))^{3/2}} + O\left(\frac{n^\alpha}{n}\right) \\
& = \frac{((\frac{a-1}{2})^2 + (\frac{b-1}{2})^2)}{32} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\cos(x)}{\sqrt{1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y)}} + O\left(\frac{n^\alpha}{n}\right).
\end{aligned} \tag{125}$$

Observe that the main term in Equation (125) exactly cancels the main term in Equation (124). Thus, all together we proved Equation (45), for $L/2 = 2n$.

Next, consider the case $L/2 = 2n + 1$. Here, we compare $e_-(k|a, b)$ with $e_-(k|-1, -1)$, using that:

$$\begin{aligned}
\cos\left(2k_1 - \frac{\pi}{L}(a-1)\right) &= \cos\left(2k_1 + \frac{2\pi}{L} - \frac{\pi}{L}(a+1)\right) \\
&= \cos\left(\frac{\pi}{2n+1}(2n_1+1) - \frac{\pi}{2(2n+1)}(a+1)\right).
\end{aligned} \tag{126}$$

The dispersion relation $e_-(k|-1, -1)$ vanishes for $(n_1, n_2) = (n, n)$. As before, we study differently the contributions $|(n_1, n_2) - (n, n)| \leq n^\alpha$, $|(n_1, n_2) - (n, n)| > n^\alpha$. The former are estimated as in Equation (115). Let us consider the latter. We Taylor expand:

$$\begin{aligned}
e_-(k|a, b)^2 &= e_-(k|-1, -1)^2 \\
&+ 4t^2 \left[\frac{1}{2} \frac{\pi}{2(2n+1)} \left((a+1) \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + (b+1) \sin\left(\pi \frac{2n_2+1}{2n+1}\right) \right) \right. \\
&- \frac{1}{4} \left(\frac{\pi}{2(2n+1)} \right)^2 \left((a+1)^2 \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + (b+1)^2 \cos\left(\pi \frac{2n_2+1}{2n+1}\right) \right) \Big] \\
&+ O\left(\frac{1}{n^2}\right).
\end{aligned} \tag{127}$$

The analysis of the error terms is performed as before, and we omit the details. We have, for

$|(n_1, n_2) - (n, n)| > n^\alpha$:

$$\begin{aligned}
e_-(k|a, b) &= e_-(k|-1, -1) \left(1 + \frac{\pi}{2(2n+1)} \frac{\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)}{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)} \right. \\
&\quad - \left(\frac{\pi}{2(2n+1)} \right)^2 \frac{\left(\frac{a+1}{2}\right)^2 \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \left(\frac{b+1}{2}\right)^2 \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)} \\
&\quad \left. + o\left(\frac{1}{e_-^2(k|-1, -1)n^2}\right) \right)^{\frac{1}{2}} \\
&= e_-(k, l|-1, -1) \left(1 + \frac{1}{2} \frac{\pi}{2(2n+1)} \frac{\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)}{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)} \right. \\
&\quad - \frac{1}{2} \left(\frac{\pi}{2(2n+1)} \right)^2 \frac{\left(\frac{a+1}{2}\right)^2 \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \left(\frac{b+1}{2}\right)^2 \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)} \\
&\quad - \frac{1}{8} \left(\frac{\pi}{2(2n+1)} \right)^2 \left(\frac{\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)}{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)} \right)^2 \Bigg) \\
&\quad + \tilde{r}(k|a, b) ,
\end{aligned} \tag{128}$$

where the error term $\tilde{r}(k|a, b)$ is bounded as in Equation (121). Therefore:

$$\begin{aligned}
e_-(k|a, b) - e_-(k|-1, -1) &= \frac{1}{2} \frac{\pi}{2(2n+1)} \frac{\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} \\
&\quad - \frac{1}{2} \left(\frac{\pi}{2(2n+1)} \right)^2 \frac{\left(\frac{a+1}{2}\right)^2 \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \left(\frac{b+1}{2}\right)^2 \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} \\
&\quad - \frac{1}{8} \left(\frac{\pi}{2(2n+1)} \right)^2 \frac{\left(\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)\right)^2}{\left(1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)\right)^{3/2}} \\
&\quad + \tilde{r}(k|a, b) .
\end{aligned} \tag{129}$$

Let us evaluate the sum over all (n_1, n_2) such that $|(n_1, n_2) - (n, n)| > n^\alpha$ of the first three terms in the right-hand side of Equation (129). Consider the first term. We have:

$$\begin{aligned}
&\frac{\pi}{4(2n+1)} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n_i - n| > n^\alpha}} \frac{\frac{(a+1)}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{(b+1)}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} \\
&= \frac{\pi(a+b+2)}{8(2n+1)} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n_i - n| > n^\alpha}} \frac{\sin\left(\pi \frac{2n_1+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} = 0 ,
\end{aligned} \tag{130}$$

where we used that n_1 belongs to the summation range if and only if $n_1^* = 2n - n_1$ does so, and the

argument of the sum takes opposite values at n_1 and n_1^* . Consider the second term. We have:

$$\begin{aligned}
& -\frac{\pi^2}{8(2n+1)^2} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n_i - n| > n^\alpha}} \frac{\left(\frac{a+1}{2}\right)^2 \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \left(\frac{b+1}{2}\right)^2 \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} \\
& = -\frac{\pi^2 \left(\left(\frac{a+1}{2}\right)^2 + \left(\frac{b+1}{2}\right)^2\right)}{8(2n+1)^2} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n_i - n| > n^\alpha}} \frac{\cos\left(\pi \frac{2n_1+1}{2n+1}\right)}{\sqrt{1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)}} \\
& = -\frac{\left(\left(\frac{a+1}{2}\right)^2 + \left(\frac{b+1}{2}\right)^2\right)}{8} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\cos(x)}{\sqrt{1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y)}} + O\left(\frac{n^\alpha}{n}\right). \tag{131}
\end{aligned}$$

Finally, consider the third term. We have:

$$\begin{aligned}
& -\frac{\pi^2}{32(2n+1)^2} \sum_{\substack{n_1, n_2=0, \dots, 2n \\ |n_i - n| > n^\alpha}} \frac{\left(\frac{a+1}{2} \sin\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{b+1}{2} \sin\left(\pi \frac{2n_2+1}{2n+1}\right)\right)^2}{\left(1 + \frac{1}{2} \cos\left(\pi \frac{2n_1+1}{2n+1}\right) + \frac{1}{2} \cos\left(\pi \frac{2n_2+1}{2n+1}\right)\right)^{3/2}} \\
& = -\frac{\left(\left(\frac{a+1}{2}\right)^2 + \left(\frac{b+1}{2}\right)^2\right)}{32} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\sin(x)^2}{\left(1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y)\right)^{3/2}} + O\left(\frac{n^\alpha}{n}\right) \\
& = \frac{\left(\left(\frac{a-1}{2}\right)^2 + \left(\frac{b-1}{2}\right)^2\right)}{8} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dx dy \frac{\cos(x)}{\sqrt{1 + \frac{1}{2} \cos(x) + \frac{1}{2} \cos(y)}} + O\left(\frac{n^\alpha}{n}\right). \tag{132}
\end{aligned}$$

Again, the main terms in Equation (131) and Equation (132) cancel out. This concludes the proof of Equation (45) for $L/2 = 2n + 1$.

Topological order and Anyons

This chapter brings together three ingredients developed in the preceding chapters: the gauge-matter Hilbert-space formalism, Gauss-law constraint, and fixed-background decomposition of [Chapter 5](#) and [Section 5.2](#), the reflection-positive selection and quantitative stability of the π -flux phase established in [Chapter 6](#) and finally the locality and quasi-adiabatic tools reviewed in [Subsection 2.2.4](#) and [Subsection 2.2.6](#). These ingredients are applied here to the massive spinful $\mathbb{Z}_2$ -gauge model introduced, together with its main results, in [Section 1.3](#). The chapter is based on joint work with Sven Bachmann and Marcello Porta [[BGP26](#)]. The precise model and the statements proved below are presented in [Section 7.1](#) and [Section 7.2](#).

Relative to the spinless model of [Chapter 6](#), the staggered mass opens a gap at the Dirac points of the uniform π -flux phase, while the spin degree of freedom allows the inclusion of a weak Hubbard interaction. The original results of the joint work establish that the uniform π -flux sector remains stable and that flux defects have a strictly positive, extensive energy cost. Within this sector, the four low-energy states distinguished by their torus holonomies have an exponentially small energy splitting, are separated from the rest of the spectrum by a uniform gap, and satisfy local topological quantum order. These conclusions persist at sufficiently weak non-zero Hubbard coupling, where an explicit free-fermion diagonalization is no longer available. Finally, the construction of quasi-local loop operators identifies their action on the almost-degenerate ground-state sector and establishes the characteristic $\mathbb{Z}_2$ loop algebra and braiding phases of the toric code. These finite-volume statements do not, by themselves, constitute a construction or classification of thermodynamic-limit superselection sectors, nor of a complete braided or modular tensor category.

The characteristic new idea behind the loop operators is to combine continuous $U(1)$ flux threading with the dynamical $\mathbb{Z}_2$ -gauge structure. For a dual cycle $\mathcal{C}^*$, the microscopic particle-number symmetry defines a uniformly gapped path of flux-threaded Hamiltonians $H_{\mathcal{C}^*}(\phi)$ with $0 \leq \phi \leq \pi$, whose low-energy projections are transported by the quasi-adiabatic propagator $V_{\mathcal{C}^*}(\phi)$. At $\phi = \pi$, the inserted holonomy is $-1 \in \mathbb{Z}_2$ and can therefore be reabsorbed into the dynamical gauge field by applying a large gauge transformation. Consequently, $W_{\mathcal{C}^*} \doteq \hat{X}_{\mathcal{C}^*} V_{\mathcal{C}^*}(\pi)$ closes the flux-insertion path and defines a quasi-local transformation of the almost-degenerate ground-state sector. For a non-contractible dual cycle, this operator changes a $\mathbb{Z}_2$ holonomy and maps one ground state into another. The construction is conceptually related to Oshikawa's flux-insertion argument [[Osh00](#)] and to its extension by Paramakanti and Vishwanath to $\mathbb{Z}_2$ -fractionalized phases [[PV04](#)], where threading a π flux can insert a vison and move in the ground state manifold. Opening these dressed loops creates monopole pairs, and their algebra with the fermionic string operators yields the anyonic braiding relations.

The proof follows the same hierarchy of ingredients. The massive fixed-background Hamiltonians and their holonomy dependence are analyzed in [Section 7.3](#), and the estimates are extended to weak Hubbard interaction in [Subsection 7.3.3](#). The reflection-positive argument of [Chapter 6](#) is adapted in [Subsection 7.3.4](#) to select the uniform π -flux sector and prove the monopole gap. The Gauss-law projection and the spectral estimates then yield the physical almost-degenerate sector and its local indistinguishability in [Section 7.4](#). Finally, the quasi-local loop operators and their braiding algebra are constructed in [Section 7.5](#) and [Subsection 7.5.1](#). The reflection-positive estimates build on [\[GP26\]](#), the interaction analysis uses standard fermionic cluster-expansion methods such as [\[Sal10\]](#), and the spectral-flow construction follows [\[HW05; Bac+11\]](#). These general techniques are inputs, however their model-specific combination and the resulting spectral, topological, and braiding statements constitute the original contributions of the joint work.

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7.1 The Massive Spinful $\mathbb{Z}_2$ -Gauge Model

We now specialize the Gauge–Matter construction of [Chapter 5](#) to the massive spinful model studied in this chapter. The lattice is the square torus Λ_L , with $L \in 4\mathbb{N}$, and the set of internal Fermionic labels is

$$\mathcal{S} \doteq \{\uparrow, \downarrow\}.$$

We retain the notation

$$n_{i,\eta} \doteq a_{i,\eta}^+ a_{i,\eta}^-, \quad n_i \doteq n_{i,\uparrow} + n_{i,\downarrow}, \quad N \doteq \sum_{i \in V(\Lambda_L)} n_i.$$

The Fock space, the Gauge-field Hilbert space, the charge operators $Q_i = A_i(-1)^{n_i}$, and the physical Hilbert space $\mathcal{H}_L^{\text{phys}}$ are precisely those of [Section 5.2](#).

For fixed $t, m > 0$ and $U \in \mathbb{R}$, we focus on the Gauge-invariant Hamiltonian

$$\begin{aligned} H \doteq & -t \sum_{(i,j) \in E(\Lambda_L)} \sum_{\eta \in \mathcal{S}} \left(a_{i,\eta}^+ \hat{\sigma}_{ij}^z a_{j,\eta}^- + \text{h.c.} \right) \\ & + m \sum_{i \in V(\Lambda_L)} \sum_{\eta \in \mathcal{S}} (-1)^{i_1+i_2} n_{i,\eta} + U \sum_{i \in V(\Lambda_L)} \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right). \end{aligned} \quad (1)$$

This is the massive spinful deformation of the hopping model considered in [Chapter 6](#). At $m = U = 0$, the Matter Hamiltonian consists of two identical copies of the massless model. The staggered mass opens a gap at the Dirac points, while the Hubbard term tests the stability of the resulting phase under a genuine many-body interaction. The analysis below is carried out for $|U|$ sufficiently small, uniformly in L .

No electric-field term is present in the Hamiltonian described by [Equation \(1\)](#). Consequently, H decomposes into the fixed-background fibers $H(\boldsymbol{\sigma})$ as in [Equation \(19\)](#). By [Proposition 66](#), their spectral data depend only on the plaquette fluxes and on the two global holonomies. We write -1 for a representative of the uniform π -flux sector; the notation refers to the plaquette fluxes and does not mean that every link variable is equal to -1 .

For later use, if $\Omega \subseteq V(\Lambda_L)$, set

$$Q_\Omega \doteq \prod_{i \in \Omega} Q_i, \quad A_\Omega \doteq \prod_{i \in \Omega} A_i, \quad N_\Omega \doteq \sum_{i \in \Omega} n_i.$$

Gauge invariance gives

$$Q_\Omega H Q_\Omega = H, \quad (2)$$

and, using $Q_\Omega = A_\Omega (-1)^{N_\Omega}$, equivalently

$$A_\Omega H A_\Omega = (-1)^{N_\Omega} H (-1)^{N_\Omega}. \quad (3)$$

These are the only model-specific forms of the Gauge transformation needed below; the general discussion of physical observables, Wilson and 't Hooft strings, Gauss' law, and the resolution into Gauge sectors will not be repeated here.

7.2 Spectral Structure, Local Topological Order, and Braiding

7.2.1 Spectral structure of the model and local topological order

The three main results were announced in [Section 1.3](#). As in the preceding application, we restate them here without renumbering before giving their proofs.

From now on, we shall refer to a *flux* for plaquettes and more generally (contractible) boundaries and a *holonomy* for (non-contractible) cycles that are not boundaries. For any configuration $\boldsymbol{\sigma}$, we can add additional holonomies $(e^{i\theta}, e^{i\phi}) \in U(1) \times U(1)$ with the replacement $\sigma_{ij}^z \rightarrow \sigma_{ij}^z e^{i\theta}$, respectively $\sigma_{ij}^z \rightarrow \sigma_{ij}^z e^{i\phi}$, along any two non-homologous cocycles that are not coboundaries. We denote the corresponding Hamiltonian $H(\boldsymbol{\sigma}, e^{i\theta}, e^{i\phi})$, and we shall denote by $E_{0,L}(\boldsymbol{\sigma}, e^{i\theta}, e^{i\phi})$ its ground state energy which can be expressed as

$$E_{0,L}(\boldsymbol{\sigma}, e^{i\theta}, e^{i\phi}) = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H(\boldsymbol{\sigma}, e^{i\theta}, e^{i\phi})}. \quad (4)$$

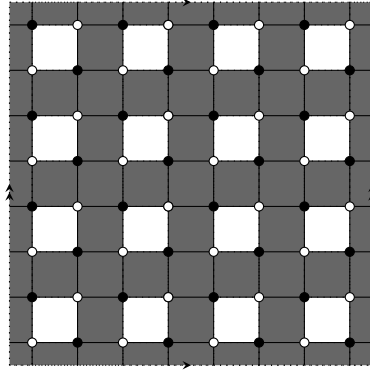


Figure 7.1: The flux configuration corresponding to σ^* is that of a lattice of monopoles, represented here in white, carved on the uniform background of π -fluxes, the dark plaquettes.

Theorem (Spectral structure of the model). *Let $L \in 4\mathbb{N}$. For $|U|$ small enough uniformly in L the following is true.*

1. *For any holonomies $(e^{i\theta}, e^{i\phi}) \in U(1) \times U(1)$, the ground state of $H(-\mathbf{1}, e^{i\theta}, e^{i\phi})$ on $\mathcal{F}_L$ is unique and it is at half-filling. Furthermore, the ground state energy is separated by the rest of the spectrum by a gap $\delta_L \geq m - C|U|^{1/3}$.*
2. *There exist constants $C, c > 0$ depending on t, m , but not on the size of the system, for which:*

$$\left| E_{0,L}(-\mathbf{1}, e^{i\theta}, e^{i\phi}) - E_{0,L}(-\mathbf{1}, e^{i\theta'}, e^{i\phi'}) \right| \leq CL^2 e^{-cL}. \quad (5)$$

3. *Let σ be a background with $2k$ 0-fluxes and holonomies $(e^{i\theta}, e^{i\phi}) \in U(1) \times U(1)$. Then, there is a constant $\Delta_{\beta,L} > 0$, depending on t, m, U , such that:*

$$-\frac{1}{\beta} \log \left(\text{Tr}_{\mathcal{F}_L} (e^{-\beta H(\sigma, e^{i\theta}, e^{i\phi})}) \right) \geq 2k\Delta_{\beta,L} - \frac{1}{\beta} \log \left(\text{Tr}_{\mathcal{F}_L} (e^{-\beta H(-\mathbf{1}, -1, -1)}) \right). \quad (6)$$

The constant $\Delta_{\beta,L}$ is given by:

$$\Delta_{\beta,L} = -\frac{1}{\beta L^2} \max_{(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2} \log \left(\frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(\sigma^*, a, b)}}{\text{Tr}_{\mathcal{F}_L} e^{-\beta H(-\mathbf{1}, -1, -1)}} \right) = \Delta_\infty + o(1) \quad (7)$$

as $\beta, L \rightarrow \infty$, where $\Delta_\infty > 0$. Here, σ^* is a particular the chessboard flux configuration (see Figure 7.1) with holonomies $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, as $\beta \rightarrow \infty$:

$$E_{0,L}(\sigma, e^{i\theta}, e^{i\phi}) \geq 2k\Delta_L + E_{0,L}(-\mathbf{1}, -1, -1) \quad (8)$$

where $\Delta_L = \Delta_\infty + o_L(1)$.

In other words, the global ground state lies in π -flux sector, and monopoles are again massive, as in the case $m = 0$, provided $|U|$ is small enough uniformly in the system's size (but nonuniformly in m). These results are closely related to the massless stability result stated in Theorem 2 and proved in [GP26]. While the monopoles were already shown to be massive there, the presence of the fermionic mass term in the Hamiltonian:

1. yields a global spectral gap above the ground state energy,
2. causes the splitting (see Equation (5)) of the ground state energies to be exponentially small in the system size (rather than power law, recall Equation (45) for the massless of Chapter 6), and indeed the proof will show that $c \rightarrow 0$ as $m \rightarrow 0$, see Equation (51).

We now turn to the ground states and the spectral gap within the π -flux sector.

Theorem (Ground state topological order). *Let $L \in 4\mathbb{N}$, and let $|U|$ be small enough so that $\Delta_\infty > 0$. Let P be the projection onto the span of the four states*

$$|\Omega_{ab}\rangle = \frac{1}{\mathcal{N}} \prod_{i \in V(\Lambda_L)} \left(\frac{1 + Q_i}{2} \right) |\psi_{-\mathbf{1}, a, b}\rangle \otimes |-\mathbf{1}, a, b\rangle, \quad (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \quad (9)$$

where $|\psi_{-\mathbf{1}, a, b}\rangle$ is the ground state of the fermionic Hamiltonian $H(-\mathbf{1}, a, b)$ on $\mathcal{F}_L$, and $\mathcal{N}$ is a normalisation factor. Then:

1. Let $g_L = \inf \text{Spec}(P^\perp H P^\perp) - \sup \text{Spec}(P H P)$. Then

$$g_L \geq \min\{2\delta_L, 2\Delta_L\} - CL^2 e^{-cL} \quad (10)$$

where $\delta_L \geq m - C|U|^{1/3}$.

2. For any fixed gauge-invariant observable $\mathcal{O}$, there are constants $C, c > 0$ such that

$$\left| \langle \Omega_{ab} | \mathcal{O} \Omega_{a'b'} \rangle - \delta_{aa'} \delta_{bb'} \frac{\text{Tr}(P \mathcal{O})}{\text{Tr}(P)} \right| \leq C \mathcal{O} e^{-cL}. \quad (11)$$

In 2. above, a ‘fixed’ observable is meant in the following sense. The tori Λ_L are identified with squares $[-L/2, L/2] \times [-L/2, L/2]$ (recall that $L \in 2\mathbb{N}$) with periodic boundary conditions. There is L_0 such that $\mathcal{O} \in \mathcal{A}_{L_0}$ and therefore $\mathcal{O} \in \mathcal{A}_L$ for all $L \geq L_0$ using the natural embedding $\mathcal{A}_{L_0} \hookrightarrow \mathcal{A}_L$ given by tensoring with the identity. One may also call such an observable a ‘local’ observable, since its distance to the boundary of the cut torus is of order L .

We note that by [Theorem 4 - Item 2](#), the vectors $|\Omega_{ab}\rangle$ correspond to an almost degenerate eigenvalue. From now on, we shall refer to P as the *ground state projection* and the theorem shows that P is *gapped*, this is [Equation \(10\)](#), and *topologically ordered*, which refers to the property in [Equation \(11\)](#). The ground state energy is (almost) fourfold degenerate and isolated from the rest the spectrum by a gap, which remains open in the infinite-volume limit. This degeneracy is not associated with a local order parameter since all local observables are multiples of the identity when restricted to the ground state space. In order to have a concrete picture in mind, we draw in [Figure 7.2](#) below one representative of $([-\mathbf{1}], a, b)$ for all choices (a, b) . Here, $[-\mathbf{1}]$ is the equivalence class of π -flux backgrounds.

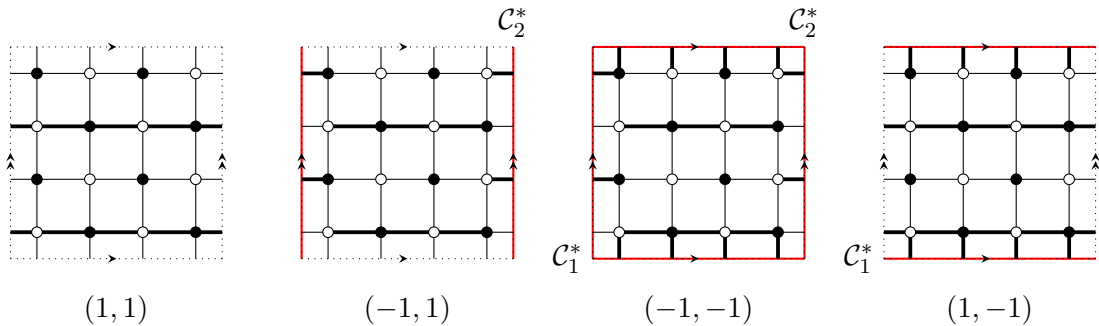


Figure 7.2: A representative $[-\mathbf{1}]$ for all possible π -flux backgrounds, for $L = 4$. A thick line represents an edge with $\sigma_{ij}^z = -1$, while thin lines correspond to $\sigma_{ij}^z = 1$.

Remark 56.

1. The spectral gap g_L above the ground state energy is twice the minimum of the (renormalized) monopole mass Δ_L and of a quantity δ_L , which plays the role of renormalized mass for the fermions. For $|U|$ small enough, it is strictly positive for all L large enough. While the former quasi-particles are neutral, the latter are charged. Following [Lem+24] (and the vast literature therein on gaps in the quantum Hall and anomalous Hall effect) one may ask for the relationship between the two gaps. The exact expression in Equation (104) in the non-interacting limit $U = 0$ indicates a crossover in our model (see Figure 7.3): while the monopole mass remains finite as $m \rightarrow 0$, see [GP26], $\Delta_\infty \rightarrow 0$ as $m \rightarrow \infty$ since the three terms of Equation (104) cancel out in the limit. This is not in contradiction with [Lem+24] since our model does not have an analog of the dipole symmetry.
2. Unlike in the toric code, the map $\mathcal{C} \mapsto \hat{Z}_{\mathcal{C}}$ does not descend to $H_1(\Lambda)$. Indeed, the condition $B_p = -1$ for any plaquette p means that the $\mathbb{Z}_2$ background is not flat: ground states have different eigenvalues with respect to $\hat{Z}_{\mathcal{C}}$ and $\hat{Z}_{\mathcal{C}+\partial\Lambda}$ since

$$\hat{Z}_{\mathcal{C}+\partial\Lambda}|\Omega_{ab}\rangle = (-1)^{|\Lambda|}\hat{Z}_{\mathcal{C}}|\Omega_{ab}\rangle. \quad (12)$$

As pointed out in [CR07], any π -flux background defines in a canonical way a spin structure on the torus. The spin structure space on a given Riemann surface Σ_g is an affine space whose translation space is given by $H_1(\Sigma_g, \mathbb{Z}_2)$: in other words, any two spin structures differ by a $\mathbb{Z}_2$ flat connection. Similarly, here two ground state backgrounds differ by the insertion of a $\mathbb{Z}_2$ holonomy around a non-contractible cycle, but it cannot be defined as the background with a given parallel transport independently on the choice of the representative of the homology cycle.

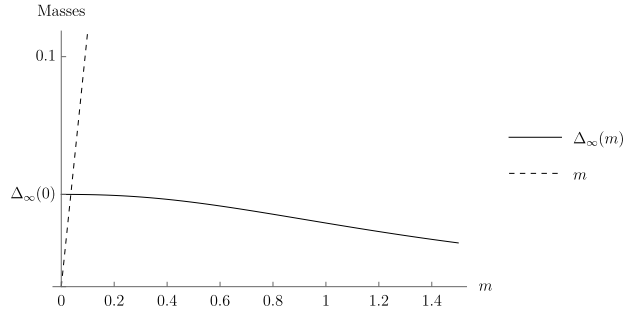


Figure 7.3: Crossover between the monopole and the fermion mass gap (in the thermodynamic limit) as a function of m for $0 \leq m \leq 1.5$ and $t = 1$, $U = 0$. See Figure 7.9 for the behavior of $\Delta_\infty(m)$ for a larger range of m .

7.2.2 Loop operators and braiding properties

We have established that the Hamiltonian has a gapped, topologically ordered ground state space. The elementary excitations come in two types: magnetic monopoles obtained by carving out a 0-flux in the uniform π -flux background on the one hand, and fermionic excitations upon the half-filled fermionic ground state.

We now turn to adiabatic insertion of a flux or a holonomy through the system. We rely on Hastings' quasi-adiabatic evolution reviewed in Subsection 2.2.6 (see also [Bac+11; HW05]) and used here in the context of flux threading as in [Bac+19; Bac+21]. This will allow us to describe the mapping of different ground states onto each other. As is typical in the presence of anyons, this is closely related to the process of creating a pair of monopoles, moving one of them along a

non-contractible loop and fusing them back again cycles through the ground states.

The intersection number $\mathcal{I}$ between two edges $e \in E(\Lambda_L)$ and $e^* \in E(\Lambda_L^*)$ is the map:

$$\mathcal{I} : E(\Lambda_L) \times E(\Lambda_L^*) \rightarrow \{-1, 0, 1\} \quad (13)$$

defined by the convention in Figure 7.4. The intersection number can then be extended to a map

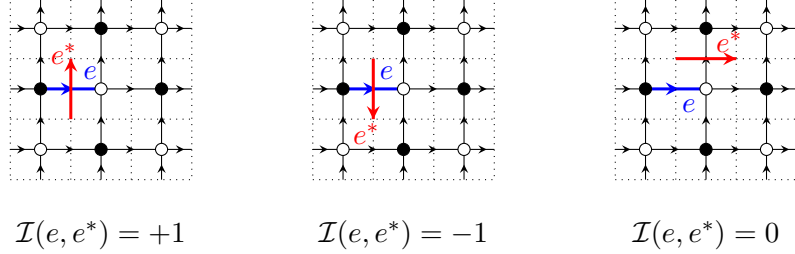


Figure 7.4: Convention for the definition of the intersection number.

on $C_1(\Lambda_L) \times C_1(\Lambda_L^*)$ by adding up the contributions coming from every single edge.

Let now $\mathcal{C}^* \in Z_1(\Lambda_L^*)$. The twisted Hamiltonian is defined as (see Figure 7.5):

$$H_{\mathcal{C}^*}(\phi) = -t \sum_{(i,j) \in E(\Lambda_L)} \sum_{\eta=\uparrow, \downarrow} \left(e^{-i\phi \mathcal{I}[(i,j), \mathcal{C}^*]} a_{i,\eta}^+ \hat{\sigma}_{ij}^z a_{j,\eta}^- + \text{h.c.} \right) + m \sum_{i \in \Lambda_L} (-1)^{|i_1+i_2|} n_i \quad (14)$$

$$+ U \sum_{i \in \Lambda_L} \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right). \quad (15)$$

As we shall see later, see Lemma 78, the flow $H \mapsto H_{\mathcal{C}^*}(\phi)$ is just a unitary transformation whenever $\mathcal{C}^* = \partial\Lambda$ is a trivial cocycle, namely it is a coboundary. As such, it does not affect the spectrum. The case of a non-trivial cocycle $\mathcal{C}^*$ is different and it is described by Theorem 6: the ground state manifold undergoes a non-trivial spectral flow, along which the spectral gap does not close. This ‘parallel transport’ of the ground state space is given by a unitary propagator (see Equation (18) and Equation (19)) generated by the so-called quasi-adiabatic generator.

Definition 74 (Quasi-adiabatic generator). Let $\mathcal{C}^* \in Z_1(\Lambda_L^*)$. The quasi-adiabatic generator $\mathcal{K}_{\mathcal{C}^*}$ is the family

$$\mathcal{K}_{\mathcal{C}^*}(\phi) = \tau_f^\phi \left(\dot{H}_{\mathcal{C}^*}(\phi) \right) = \int_{\mathbb{R}} f(s) e^{isH_{\mathcal{C}^*}(\phi)} \dot{H}_{\mathcal{C}^*}(\phi) e^{-isH_{\mathcal{C}^*}(\phi)} ds, \quad (16)$$

where $f \in L^\infty(\mathbb{R})$ such that $\|f\|_{L^1} = 1$, $\hat{f}(\omega) = \frac{1}{i\omega}$ for $|\omega| > g$ and $|f(t)| \leq \frac{C_k}{1+|t|^k}$ for any $k \in \mathbb{N}$. Here g is gap of H , see Equation (10).

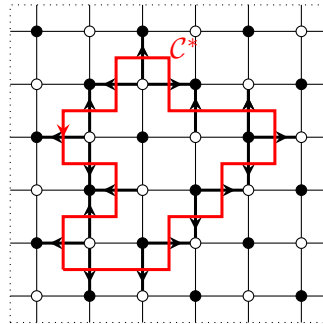


Figure 7.5: Graphical representation of twisting the hamiltonian inserting a flux ϕ inside $\mathcal{C}^*$, oriented counterclockwise. Each bold edge is twisted with a phase of $e^{-i\phi}$.

Remark 57. By definition, the operator $\dot{H}_{\mathcal{C}^*}(\phi)$ is supported on the edges intersecting $\mathcal{C}^*$. With this, it is a standard argument that $\mathcal{K}_{\mathcal{C}^*}(\phi)$ is almost supported on $\mathcal{C}^*$ in the sense that $\mathcal{K}_{\mathcal{C}^*}(\phi)$ can be approximated by an observable strictly supported on a ribbon of width R around $\mathcal{C}^*$, with an error $O(R^{-\infty})$. Indeed, this is true for $e^{isH_{\mathcal{C}^*}(\phi)}\dot{H}_{\mathcal{C}^*}(\phi)e^{-isH_{\mathcal{C}^*}(\phi)}$ for times s of order 1 by the Lieb-Robinson bound (recall Equation (19)), while the contribution to the integral for longer times is small by the decay of f . We refer to [Bac+11, Lemma 4.7] and the review [NSY19] for additional details on the Lieb-Robinson bound (see also to Subsection 2.2.4) and the spectral flow (Subsection 2.2.6) and to [Bac+19] specifically for the locality of $\mathcal{K}_{\mathcal{C}^*}(\phi)$.

Besides its locality, the operator $\mathcal{K}_{\mathcal{C}^*}(\phi)$ generates a parallel transport on the bundle of ground state projections $P_{\mathcal{C}^*}(\phi)$ of $H_{\mathcal{C}^*}(\phi)$:

$$\dot{P}_{\mathcal{C}^*}(\phi) = i[\mathcal{K}_{\mathcal{C}^*}(\phi), P_{\mathcal{C}^*}(\phi)] \quad (17)$$

provided the spectral gap remains open, see [Bac+11, Proposition 2.4]. This is a consequence of the more general fact that the map $O \mapsto \tau_f^\phi(O)$ is an inverse of $-i[H_{\mathcal{C}^*}(\phi), O]$ on the set of off-diagonal operators $O = POP^\perp + P^\perp OP$, applied here to $O = \dot{P}_{\mathcal{C}^*}(\phi)$.

Let now $V_{\mathcal{C}^*}(\phi)$ be the solution of

$$-i\dot{V}_{\mathcal{C}^*}(\phi) = \mathcal{K}_{\mathcal{C}^*}(\phi)V_{\mathcal{C}^*}(\phi) \quad (18)$$

such that $V_{\mathcal{C}^*}(0) = \mathbf{1}$. It is the propagator of parallel transport, namely

$$P_{\mathcal{C}^*}(\phi) = V_{\mathcal{C}^*}(\phi)P_{\mathcal{C}^*}(0)V_{\mathcal{C}^*}(\phi)^*. \quad (19)$$

Definition 75 (Loop Operators). Let $\mathcal{C}^* \in Z_1(\Lambda_L^*)$. We define

$$W_{\mathcal{C}^*} = \hat{X}_{\mathcal{C}^*}V_{\mathcal{C}^*}(\pi). \quad (20)$$

Theorem (Braiding). *Under the same assumptions of Theorem 5 the following is true. Let $\{|\Omega_{ab}\rangle : (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2\}$ be the four ground states of Theorem 5, let P_{ab} be the corresponding orthogonal projectors, and let $P = \sum_{(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2} P_{ab}$.*

1. For any $\mathcal{C}^* \in Z_1(\Lambda_L^*)$

$$[W_{\mathcal{C}^*}, P] \stackrel{L}{=} 0. \quad (21)$$

where $\stackrel{L}{=}$ means that the equality holds up to terms that are smaller, in operator norm, than any power of the system size.

2. For any $\mathcal{C} \in Z_1(\Lambda_L)$, $\mathcal{C}^* \in Z_1(\Lambda_L^*)$,

$$\hat{Z}_{\mathcal{C}}W_{\mathcal{C}^*} = e^{i\pi\mathcal{I}(\mathcal{C}, \mathcal{C}^*)}W_{\mathcal{C}^*}\hat{Z}_{\mathcal{C}}. \quad (22)$$

3. If $\{0, \mathcal{C}_1^*, \mathcal{C}_2^*, \mathcal{C}_1^* + \mathcal{C}_2^*\}$ are representatives of each of the cohomology classes, then

$$P_{a'b'}(W_{\mathcal{C}_1^*})^{p_1}(W_{\mathcal{C}_2^*})^{p_2} \stackrel{L}{=} (W_{\mathcal{C}_1^*})^{p_1}(W_{\mathcal{C}_2^*})^{p_2}P_{ab}, \quad (23)$$

where $a' = (-1)^{p_1}a$, $b' = (-1)^{p_2}b$, for any $p_1, p_2 \in \mathbb{Z}$.

4. For any (a, b) ,

$$(W_{\mathcal{C}_1^*})^*(W_{\mathcal{C}_2^*})^*W_{\mathcal{C}_1^*}W_{\mathcal{C}_2^*}|\Omega_{ab}\rangle \stackrel{L}{=} |\Omega_{ab}\rangle. \quad (24)$$

5. Let $i, j \in \Lambda$ be such that $\text{dist}(i, j)$ is of order L and let $\mathcal{C}_{i,j} \in C_1(\Lambda_L)$ be a 1-chain such that $(i, j) = \partial \mathcal{C}_{i,j}$. Denote $|\xi_{ab}^{ij}\rangle = \mathcal{N}^{-1} a_i^+ \hat{Z}_{\mathcal{C}_{i,j}} a_j^+ |\Omega_{ab}\rangle$, where $\mathcal{N}^{-1}$ ensures that $\|\xi_{ab}^{ij}\| = \|\Omega_{ab}\|$. Then

$$\frac{\langle \xi_{ab}^{ij} | W_{\mathcal{C}^*} \xi_{ab}^{ij} \rangle}{\langle \Omega_{ab} | W_{\mathcal{C}^*} \Omega_{ab} \rangle} \stackrel{L}{=} -1 \quad (25)$$

where $\mathcal{C}^*$ is the circle of radius $\frac{1}{2}\text{dist}(i, j)$ centred at i .

With topological order, all local observables act trivially on the ground state space. Theorem [Item 3](#) exhibits non-trivial global observables, namely the large loop operators $W_{\mathcal{C}_j^*}, j = 1, 2$. When the loops are opened up, a string operator $W_{\mathcal{C}^*}$ creates a pair of monopoles while a string operator $a_i^+ \hat{Z}_{\mathcal{C}_{i,j}} a_j^+$ creates a pair of fermions at their endpoints, and [Item 5](#) establishes the mutual braiding phase -1 characteristic of toric-code anyons. The commutation in [Equation \(24\)](#) is independent of any choice of phases in the definition of the loop operators or ground state vector and gives trivial monopole-monopole braiding.

For any even local observable $\mathcal{O}$ supported away from sites i and j ,

$$\langle \xi_{ab}^{ij} | \mathcal{O} \xi_{ab}^{ij} \rangle \simeq \langle \Omega_{ab} | \mathcal{O} \Omega_{ab} \rangle \quad (26)$$

up to errors that decay superpolynomially in the distance between the support of $\mathcal{O}$ and the sites i, j , and in the system size L . This should be interpreted as the fact that $|\xi_{ab}^{ij}\rangle$ describes (fermionic) excitations that are localized at the sites i, j . In order to derive [Equation \(26\)](#), we first note that for any path $\tilde{\mathcal{C}}_{ij}$ from i to j and such that $\mathcal{C}_{ij} - \tilde{\mathcal{C}}_{ij}$ is a boundary,

$$Z_{\mathcal{C}_{ij}} \Omega_{ab} = z Z_{\tilde{\mathcal{C}}_{ij}} \Omega_{ab} \quad (27)$$

where $|z| = 1$ and it depends on the paths. If $\mathcal{O}$ is a local observable, we can pick $\tilde{\mathcal{C}}_{ij}$ that lies away from its support. The corresponding string operator then commutes with $\mathcal{O}$ and so

$$\langle \xi_{ab}^{ij} | \mathcal{O} \xi_{ab}^{ij} \rangle = \mathcal{N}^{-2} \langle \Omega_{ab} | a_j a_i \mathcal{O} a_i^+ a_j^+ \Omega_{ab} \rangle. \quad (28)$$

If $\mathcal{O}$ is even, then $\langle \Omega_{ab} | a_j a_i \mathcal{O} a_i^+ a_j^+ \Omega_{ab} \rangle = \langle \Omega_{ab} | \mathcal{O} a_j a_i a_i^+ a_j^+ \Omega_{ab} \rangle$. Since the spectral gap implies fast decay of correlations (see [Remark 16 - Item 2](#) and [\[Bac+21, Appendix A\]](#)) and $\mathcal{O}$ is supported away from the sites i, j ,

$$\langle \xi_{ab}^{ij} | \mathcal{O} \xi_{ab}^{ij} \rangle \simeq \mathcal{N}^{-2} \langle \Omega_{ab} | \mathcal{O} P a_j a_i a_i^+ a_j^+ \Omega_{ab} \rangle \quad (29)$$

up to an error that decays with the distance between the support of $\mathcal{O}$ and $\{i, j\}$. Finally, local topological order implies that P can be replaced by the one-dimensional projection onto the span of $|\Omega_{ab}\rangle$ up to an error that decays fast in L , namely

$$\langle \xi_{ab}^{ij} | \mathcal{O} \xi_{ab}^{ij} \rangle \simeq \mathcal{N}^{-2} \langle \Omega_{ab} | \mathcal{O} \Omega_{ab} \rangle \|a_i^+ a_j^+ |\Omega_{ab}\rangle\|^2. \quad (30)$$

This is the claim since the last factor is equal to $\mathcal{N}^2$.

Overall, the finite-volume loop and string algebra reproduces the characteristic toric-code relations. If m denotes a monopole excitation and ϵ a fermionic excitation, the $\mathbb{Z}_2$ composition laws may be summarized formally as $m \times m = 1 = \epsilon \times \epsilon$, with $e = m \times \epsilon$, while the proven loop algebra gives trivial monopole self-braiding and mutual phase -1 between the magnetic and fermionic excitations. These statements establish the toric-code fusion and braiding relations visible in the low-energy sector. They do not, however, amount to a construction or completeness proof of thermodynamic-limit superselection sectors, or to the identification of a full modular tensor category. The model is thus a fermionic reflection-positive realization of local topological order with toric-code loop and braiding relations; see also the discussion in [\[Sop25\]](#).

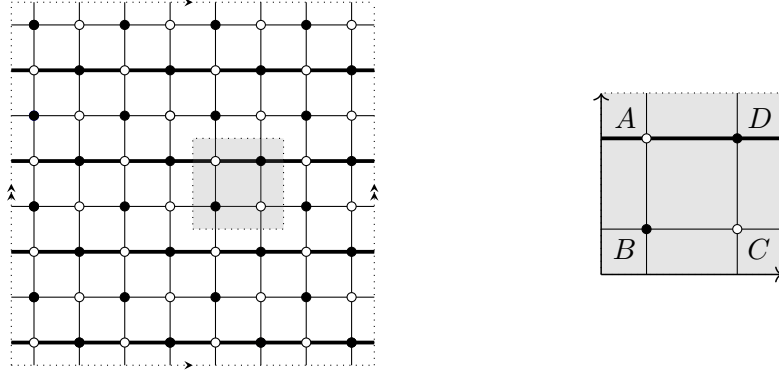


Figure 7.6: On the left, graphical representation of the gauge field configurations associated with the $(1,1)$ π flux phase. The thin bonds correspond to $\sigma^z = +1$, while the solid bonds correspond to spin $\sigma^z = -1$. On the right, definition of the fundamental cell.

7.3 Spectral Structure of the Massive Fermionic Hamiltonian

7.3.1 The Non-Interacting Spectrum

We start by diagonalizing the π -flux Hamiltonian at $U = 0$ with $(1,1)$ holonomies. A particularly simple gauge field configuration associated with this phase is represented in Figure 7.6. Clearly, the corresponding Hamiltonian is not translationally invariant with respect to all lattice translations. In order to recover translation invariance, we introduce the appropriate fundamental cell formed by four lattice sites, labelled by A, B, C, D , as in Figure 7.6. The position of the cell is defined by the coordinate of the B -lattice site. We shall denote by Λ_L^{red} the two-dimensional lattice formed by the positions of the fundamental cells. The lattice spacing between nearest-neighbour fundamental cells in Λ_L^{red} is 2 in both directions.

For vanishing Hubbard interaction, the spin does not play any role and we discuss the computation in the spinless case. If spin was included, all energy levels would simply be doubly degenerate.

Using these coordinates described in Figure 7.6, the Hamiltonian can be rewritten as:

$$H(-1; 1, 1) = -t \sum_{x \in \Lambda_L^{\text{red}}} \left(a_{B,x}^+ a_{A,x}^- + a_{B,x}^+ a_{C,x}^- + a_{A,x}^+ a_{B,x+2e_2}^- - a_{A,x}^+ a_{D,x}^- \right. \quad (31)$$

$$\left. - a_{D,x}^+ a_{A,x+2e_1}^- + a_{D,x}^+ a_{C,x+2e_2}^- + a_{C,x}^+ a_{D,x}^- + a_{C,x}^+ a_{B,x+2e_1}^- + \text{h.c.} \right) \quad (32)$$

$$+ m \sum_{x \in \Lambda_L^{\text{red}}} (n_{A,x} + n_{C,x} - n_{B,x} - n_{D,x}). \quad (33)$$

The Brillouin zone $B_L(1,1)$ is defined as

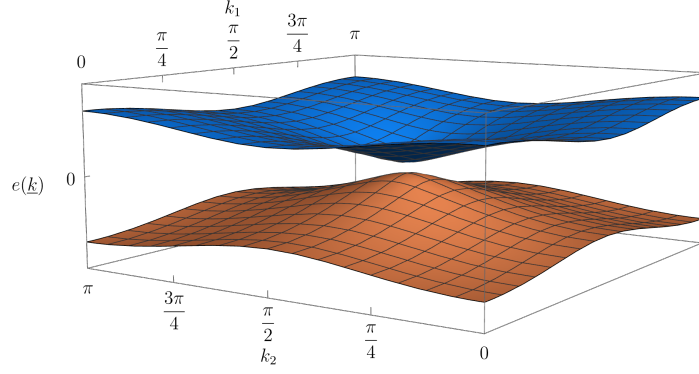
$$B_L(1,1) := \left\{ k \in \frac{2\pi}{L} (n_1, n_2) \mid 0 \leq n_1 \leq L/2 - 1, 0 \leq n_2 \leq L/2 - 1 \right\}, \quad (34)$$

and the momentum-space creation/annihilation operators as:

$$\hat{a}_{\alpha,k}^\pm = \sum_{x \in \Lambda_L^{\text{red}}} e^{\pm i k \cdot x} a_{\alpha,x}^\pm \iff a_{\alpha,x}^\pm = \frac{1}{|\Lambda_L^{\text{red}}|} \sum_{k \in B_L(1,1)} e^{\mp i k \cdot x} \hat{a}_{\alpha,k}^\pm, \quad (35)$$

for $\alpha \in \{A, B, C, D\}$. The Hamiltonian in Equation (33) becomes:

$$H(-1; 1, 1) = \frac{1}{|\Lambda_L^{\text{red}}|} \sum_{k \in B_L(1,1)} (\hat{a}_k^+, h(k) \hat{a}_k^-), \quad (36)$$

Figure 7.7: Energy bands associated with the π -flux phase with a staggered mass Equation (38).

where $(f, g) = \sum_{\alpha} f_{\alpha} g_{\alpha}$, and the Bloch Hamiltonian $h(k)$ is given by

$$h(k) = \begin{pmatrix} m & -t(1 + e^{2ik_2}) & 0 & t(1 + e^{-2ik_1}) \\ -t(1 + e^{-2ik_2}) & -m & -t(1 + e^{-2ik_1}) & 0 \\ 0 & -t(1 + e^{2ik_1}) & m & -t(1 + e^{-2ik_2}) \\ t(1 + e^{2ik_1}) & 0 & -t(1 + e^{2ik_2}) & -m \end{pmatrix}. \quad (37)$$

The above Bloch Hamiltonian coincides with the one found in [Lud+94]. Its (doubly degenerate) eigenvalues are

$$e_{\pm}(k) = \pm 2t \sqrt{\left(\frac{m}{2t}\right)^2 + 1 + \frac{1}{2} \cos(2k_1) + \frac{1}{2} \cos(2k_2)}, \quad (38)$$

and they vanish nowhere in the Brillouin zone whenever $m \neq 0$ (see Figure 7.7), leaving a gap equal to $2m$. The ground state of the system is the Slater determinant obtained by occupying all the Bloch states below energy zero. Since the cardinality of Brillouin zone is equal to $L^2/4$, see Equation (34), the rank of the Fermi projection, namely the number of particles, is equal to $L^2/2$. If the two spin states are reintroduced, the number of particles equals L^2 . In both cases, this corresponds to half-filling.

The case of more general holonomies around non-contractible cycles can be analysed via a change of the boundary conditions, and thus of the allowed momenta in the Brillouin zone. Namely, the Brillouin zone is given by

$$B_L(e^{i\theta}, e^{i\phi}) = \left\{ k \in \frac{2\pi}{L} \left(n_1 + \frac{\theta}{2\pi}, n_2 + \frac{\phi}{2\pi} \right) \mid 0 \leq n_1 \leq L/2 - 1, 0 \leq n_2 \leq L/2 - 1 \right\}. \quad (39)$$

From there, the discussion proceeds exactly as in the (1,1) case. This concludes the proof of Theorem 4 - Item 1, for non-interacting fermions.

7.3.2 Dependence on the Torus Holonomies

The ground state energy of the system for general holonomies is, using the short-hand notation $B(\theta, \phi) \equiv B_L(e^{i\theta}, e^{i\phi})$, $E_0(-\mathbf{1}, \theta, \phi) \equiv E_{0,L}(-\mathbf{1}, e^{i\theta}, e^{i\phi})$:

$$E_0(-\mathbf{1}, \theta, \phi) = 2 \sum_{k \in B(\theta, \phi)} e_{-}(k). \quad (40)$$

In order to prove the bound in Equation (5), we will rewrite Equation (40) using Poisson's summation formula:

$$\frac{1}{L^2} \sum_{k \in B(\theta, \phi)} e_{-}(k) = \sum_{\ell_1, \ell_2 \in \mathbb{Z}} e^{i\theta \ell_1} e^{i\phi \ell_2} e_{\infty}(\ell_1 L, \ell_2 L) \quad (41)$$

where

$$e_{\infty}(\ell_1 L, \ell_2 L) = \int_{\mathbb{T}^2} \frac{d^2 k}{(2\pi)^2} e^{ik_1 \ell_1 L + ik_2 \ell_2 L} e_{-}(k) \quad (42)$$

is the infinite volume ground state energy density (which is independent of the boundary conditions). In this way, we can conveniently rewrite the energy difference as:

$$E_0(-\mathbf{1}, \theta_1, \phi_1) - E_0(-\mathbf{1}, \theta_2, \phi_2) = 2L^2 \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2 \setminus (0,0)} (e^{i\ell_1 \theta_1 + i\ell_2 \phi_1} - e^{i\ell_1 \theta_2 + i\ell_2 \phi_2}) e_{\infty}(\ell_1 L, \ell_2 L). \quad (43)$$

Note that the $\ell_1 = \ell_2 = 0$ term vanishes because the exponentials cancel out.

Proposition 76. *There exist two constants $C, c > 0$ depending only on m and t , such that*

$$\left| \int_{\mathbb{T}^2} \frac{d^2 k}{(2\pi)^2} e^{ikx} e_{-}(k) \right| \leq C e^{-c(|x_1| + |x_2|)}, \quad (44)$$

Hence, by Poisson summation formula in Equation (41),

$$|E_0(-\mathbf{1}, \theta_1, \phi_1) - E_0(-\mathbf{1}, \theta_2, \phi_2)| \leq 4L^2 \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2 \setminus (0,0)} C e^{-c(|\ell_1| + |\ell_2|)L} \leq KL^2 e^{-2cL} \quad (45)$$

uniformly in $\phi_1, \theta_1, \phi_2, \theta_2$. This concludes the proof of Item 2 of Theorem 4 for non-interacting fermions. It remains to prove Proposition 76.

Proof. We first note that

$$e_{-}(k) = \frac{1}{2} \text{Tr}(h(k)p_{-}(k)) \quad (46)$$

where the trace is over the fundamental cell and the factor 2 stands for the rank of $p_{-}(k)$. The Bloch Hamiltonian $h(k)$, which is explicitly given in Equation (37), and the ground state projections $p(k)$ are by definition

$$h(k) = \sum_{x \in \Lambda_L^{\text{red}}} h(x, 0) e^{-ik \cdot x}, \quad p_{-}(k) = \sum_{x \in \Lambda_L^{\text{red}}} p_{-}(x, 0) e^{-ik \cdot x}, \quad (47)$$

where $h(x, y), p_{-}(x, y)$ are the 4×4 -matrix-valued functions given by

$$H(-\mathbf{1}; 1, 1) = \sum_{x, y \in \Lambda_L^{\text{red}}} (a_x^{+}, h(x, y) a_y^{-}), \quad (48)$$

see Equation (33). With this, the integral (see Equation (44)) can be computed as

$$\int_{\mathbb{T}^2} \frac{d^2 k}{(2\pi)^2} e^{ikx} e_{-}(k) = \frac{1}{2} \sum_{y \in \Lambda_L^{\text{red}}} \text{Tr}(h(x - y, 0) p_{-}(y, 0)). \quad (49)$$

By translation invariance, $h(x - y, 0) = h(x, y)$ and it vanishes whenever $|x - y| > 4$. Since the Fermi energy lies in a spectral gap, the Riesz formula for the spectral projector $p_{-} = \frac{1}{2\pi i} \int_{\Lambda} (h - z)^{-1} dz$ together with the Combes-Thomas estimate for the resolvent [AW15, Theorem 10.5] imply:

$$|p_{-}(y, 0)| \leq C e^{-c|y|}. \quad (50)$$

This bound combined with the identity in Equation (49) yields the claim in Equation (44). Finally, the constant $C, c > 0$ can be estimated explicitly in terms of the spectral gap, which is proportional to m and the ‘hopping amplitude’ t of $h(x, y)$:

$$C \propto \frac{1}{m}, \quad c \propto \frac{m}{t}, \quad (51)$$

see again [AW15, Theorem 10.5]. ■

7.3.3 Stability under Weak Hubbard Interactions

We now extend the results of [Subsection 7.3.1](#) and [Subsection 7.3.2](#) to the case of weakly interacting fermions. This will be done using fermionic cluster expansion. The method provides a convergent expansion for the ground state energy,

$$E_0(-\mathbf{1}, \theta, \phi) = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H(-\mathbf{1}, e^{i\theta}, e^{i\phi})} \quad (52)$$

for $|U|$ small enough, uniformly in L . Inspection of the convergent expansion will allow us to prove items 1 and 2 of [Theorem 4](#) for weakly interacting fermions. In the context of topological phases of matter, these methods have been used in [\[GMP17\]](#) to prove the universality of the Hall conductivity. See also [\[Giu+16; GMP19; FGR25\]](#) for the analysis of Hall transitions, via cluster expansion and renormalization group methods. In our setting, inspection of the expansion will allow us to prove the exponential closeness of the approximate ground state energies described in [Equation \(5\)](#). Furthermore, the same technique allows to prove the stability of the fermionic spectral gap, following [\[DS18, Theorem 1\]](#).

Analyticity of the free energy. Here we will discuss the convergent expansion for the free energy, which will be the starting point for the proofs of item 1 and item 2 of [Theorem 4](#) for interacting systems. From now on, we shall write $H(-\mathbf{1}, e^{i\theta}, e^{i\phi}) = H_0 + UV + (U/4)L^2$, where:

$$H_0 = H(-\mathbf{1}, e^{i\theta}, e^{i\phi}) \Big|_{U=0} - \frac{U}{2} \sum_{i,\sigma} n_{i,\sigma}, \quad V = \sum_i n_{i,\uparrow} n_{i,\downarrow}; \quad (53)$$

observe that we included the interaction-dependent quadratic term due to the presence of the $-1/2$ factors in the Hubbard interaction in the definition of the quadratic Hamiltonian H_0 . The additive constant term in the Hamiltonian does not have any relevance for the Gibbs state of the model, and it will be omitted in what follows. The starting point is the Duhamel expansion for the free energy:

$$\frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta(H_0+UV)}}{\text{Tr}_{\mathcal{F}_L} e^{-\beta H_0}} = 1 + \sum_{n \geq 1} (-U)^n \int_0^\beta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \langle \gamma_{t_1}(V) \gamma_{t_2}(V) \cdots \gamma_{t_n}(V) \rangle_{\beta,L}^0 \quad (54)$$

where $\langle \cdot \rangle_{\beta,L}^0$ is the Gibbs state of H_0 at half-filling, and $\gamma_t(\cdot)$ is the imaginary-time evolution,

$$\gamma_t(V) = e^{tH_0} V e^{-tH_0}. \quad (55)$$

[Equation \(54\)](#) can be rewritten as:

$$\frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta(H_0+UV)}}{\text{Tr}_{\mathcal{F}_L} e^{-\beta H_0}} = 1 + \sum_{n \geq 1} \frac{(-U)^n}{n!} \int_{[0,\beta]^n} dt_1 \cdots dt_n \langle \mathbf{T} \gamma_{t_1}(V) \gamma_{t_2}(V) \cdots \gamma_{t_n}(V) \rangle_{\beta,L}^0 \quad (56)$$

where $\mathbf{T}$ is the time-ordering operator. It acts on fermionic monomials as, at non-equal times:

$$\mathbf{T} \gamma_{t_1}(a_{i_1, \sigma_1}^{\varepsilon_1}) \cdots \gamma_{t_n}(a_{i_n, \sigma_n}^{\varepsilon_n}) = \text{sgn}(\pi) \gamma_{t_1}(a_{i_{\pi(1)}, \sigma_{\pi(1)}}^{\varepsilon_{\pi(1)}}) \cdots \gamma_{t_n}(a_{i_{\pi(n)}, \sigma_{\pi(n)}}^{\varepsilon_{\pi(n)}}) \quad (57)$$

where π is the permutation such that $t_{\pi(i)} > t_{\pi(i+1)}$. Whenever two times coincide, the time-ordering operator acts as normal-ordering. Taking the logarithm of the left-hand side and of the right-hand side of [Equation \(56\)](#) one gets:

$$\log \frac{\text{Tr}_{\mathcal{F}_L} e^{-\beta(H_0+UV)}}{\text{Tr}_{\mathcal{F}_L} e^{-\beta H_0}} = \sum_{n \geq 1} \frac{(-U)^n}{n!} \int_{[0,\beta]^n} dt_1 \cdots dt_n \langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \cdots; \gamma_{t_n}(V) \rangle_{\beta,L}^0 \quad (58)$$

where the argument of the integral is the n -th order cumulant. It is well-known that the cumulant expansion can be represented as an expansion over connected Feynman diagrams, by application of the fermionic Wick's rule (see *e.g.* [Sal10] for a review of fermionic perturbation theory). The main ingredient of the expansion is the fermionic two-point function, defined as, for $0 \leq t, t' < \beta$:

$$\begin{aligned} g((t, i, \sigma), (t', i', \sigma')) &= \langle \mathbf{T} \gamma_t(a_{i, \sigma}^-) \gamma_{t'}(a_{i', \sigma'}^+) \rangle_{\beta, L}^0 \\ &= \theta(t > t') \frac{e^{-(t-t')h_0}}{1 + e^{-\beta h_0}}(i, \sigma; i', \sigma') - \theta(t \leq t') \frac{e^{-(t-t')h_0}}{1 + e^{\beta h_0}}(i, \sigma; i', \sigma') \end{aligned} \quad (59)$$

with h_0 the single-particle Hamiltonian associated with H_0 . It is not difficult to see that $g((0, i), (t', j)) = -g((\beta, i), (t', j))$, which allows to extend antiperiodically the two-point function to all times t and t' in $\mathbb{R}$.

The problem with the naive diagrammatic expansion in Equation (58) is that it involves too many addends. In fact, the number of diagrams contributing to the n -th order grows as $(n!)^2$, which beats the $1/n!$ in Equation (58), and does not allow to prove absolute summability of the series. Observe that this apparent factorial divergence of the series is insensitive to the fermionic nature of the particles.

This issue can be avoided using a different expansion, that allows to exploit the anticommutativity of the fermionic operators. Let us rewrite:

$$\gamma_s(V) = \sum_i \gamma_t(V_i) \quad \text{with } V_i = n_{i, \uparrow} n_{i, \downarrow}. \quad (60)$$

The Battle-Brydges-Federbush (BBF) formula states that:

$$\langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \cdots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, L}^0 = \sum_T \alpha_T \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(s) \det [s_{b(f), b(f')} g_{(f, f')}] ; \quad (61)$$

see [DS18, Theorem 6], [GMR21, Appendix D], [GM01, Appendix A3.2] for reviews, proofs, and further references; see also [Che+25, Theorem 4.4] for a recent application in the context of computational quantum physics. Let us explain the meaning of the various objects involved in the identity Equation (61). Every fermionic operator appearing in $\gamma_t(V_i)$ is graphically represented by an oriented line, which exits or enters a vertex labelled by the space-time coordinates (t, i) , depending on whether the fermionic operator creates or destroys a particle, respectively. The lines are further decorated by the spin labels of the corresponding operators. In this way, every operator $\gamma_t(V_i)$ is graphically represented by a vertex with four incident lines, two incoming and two outgoing. The sum in the right-hand side of Equation (61) is over the spanning trees T connecting the n vertices associated with the interaction terms, obtained contracting lines with opposite orientations. The pairing of an incoming line associated with the vertex (t, i) and of an outgoing line associated with the vertex (t', i') , labelled respectively by spin labels σ and σ' , forms an edge $\ell = ((t, i, \sigma), (t', i', \sigma'))$. Associated with every edge, we have a propagator g_ℓ ,

$$g_\ell \equiv g((t, i, \sigma), (t', i', \sigma')) \quad (62)$$

as given by Equation (59). The product in Equation (61) involves the propagators associated with the edges of the tree. In the determinant, $g_{(f, f')}$ is the entry of a $(n+1) \times (n+1)$ dimensional matrix representing the contraction of lines that do not belong to T , where f is the label for a generic incoming line, and f' is the label for a generic outgoing line. From the point of view of Feynman diagrams, these are the loop lines. This matrix element is then multiplied by a number

$s_{b(f),b(f')}$ between 0 and 1, an interpolation parameter, where $b(f), b(f')$ are integers in $[1, n]$, which return the labels of the vertices associated with the lines f, f' . The measure $d\mu_T(\underline{s})$ is a probability measure, supported on sequences $\underline{s}$ whose entries $s_{b(f),b(f')}$ can be written as the scalar product of two vectors in $\mathbb{R}^n$, $u_{b(f)}$ and $u_{b(f')}$, with unit norm. Finally, α_T is either $+1$ or -1 , and it will not play any role in what follows.

Thus, thanks to the BBF formula in Equation (61), we have:

$$\left| \langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \cdots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, L}^0 \right| \leq \sum_T \left[\prod_{\ell \in T} |g_\ell| \right] \int d\mu_T(\underline{s}) \left| \det [s_{b(f),b(f')} g_{(f,f')}] \right|; \quad (63)$$

we will use this bound to show that

$$\frac{1}{\beta} \sum_{i_1, \dots, i_n} \int_{[0, \beta]^n} dt \left| \langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \cdots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, L}^0 \right| \leq L^2 C^n n! \quad (64)$$

an estimate which allows to prove analyticity of the specific free energy in Equation (58) for small $|U|$ (observe that the two-point function depends analytically on U), uniformly in β and L . The existence of the limits as $\beta \rightarrow \infty, L \rightarrow \infty$ follows from the uniform convergence of the series, and from the existence of the pointwise limit of the two-point function, which can be proved easily.

The factorial growth of the bound comes from counting the number of trees connecting the n vertices (observe that this number is much smaller than the number of n -th order Feynman diagrams). The proof of Equation (64) is a consequence of two main ingredients: a good bound for the ℓ^1 norm of the two-point function, and a good bound for the ℓ^∞ norm of the determinant. Let us first discuss the bound for the two-point function. For L fixed, and for any θ, ϕ , the two-point function associated with H_0 can be represented as, using Poisson summation formula:

$$g((t, i), (t', i')) = \sum_{m_1, m_2 \in \mathbb{Z}} e^{i\theta m_1} e^{i\phi m_2} g_\infty((t, i + e_1 m_1 L + e_2 m_2 L), (t', i')) \quad (65)$$

where e_1, e_2 is the standard basis of $\mathbb{R}^2$ and g_∞ is the two-point function on $\mathbb{Z}^2$. By the spectral gap of h_0 , a standard Combes-Thomas estimate (see e.g. [DS18, Appendix B]) allows to prove that, for all t, t' in $\mathbb{R}$ and for all i, i' in $\mathbb{Z}^2$:

$$\|g_\infty((t, i); (t', i'))\| \leq C e^{-c\|(t, i) - (t', i')\|_\beta} \quad (66)$$

with the time-periodic distance:

$$\|(t, i) - (t', i')\|_\beta^2 = \min_{n \in \mathbb{Z}} |t - t' + n\beta|^2 + \|i - i'\|^2. \quad (67)$$

Thus, this bound, combined with Equation (65) allows to prove that the finite-volume two-point function satisfies the estimate (with different constants):

$$\|g((t, i); (t', i'))\| \leq C e^{-c\|(t, i) - (t', i')\|_{\beta, L}} \quad (68)$$

with the space-time periodic distance:

$$\|(t, i) - (t', i')\|_{\beta, L}^2 = \min_{n \in \mathbb{Z}} |t - t' + n\beta|^2 + \min_{n_1, n_2 \in \mathbb{Z}} \|i - i' + n_1 e_1 L + n_2 e_2 L\|^2. \quad (69)$$

This decay estimate immediately implies the finiteness of the ℓ^1 norm of the two-point function,

$$\max_{t', i'} \int_0^\beta dt \sum_i \|g((t, i); (t', i'))\| \leq K \quad (70)$$

uniformly in β, L . Let us now discuss the bound for the determinant in Equation (63). The key tool we shall use is the Gram-Hadamard inequality: if $M = (m_{ij})_{1 \leq i, j \leq K}$ is a Gram matrix, that is if all matrix entries are of the form $m_{ij} = (u_i, w_j)$ for u_i, w_i in a Hilbert space with scalar product $(\cdot, \cdot)$, then:

$$|\det[M]| \leq \prod_{i=1}^K \|u_i\| \|w_i\| \quad (71)$$

with $\|\cdot\|$ the norm induced by the scalar product. This bound is useful if the vectors u_i and w_j have norm independent of K . It is well-known however that the two-point function does not have a good Gram representation, due to the discontinuity of the indicator functions in Equation (59), introduced by the time-ordering. The easiest way to solve this issue is to observe that the two factors multiplying the theta functions in Equation (59) do have a good Gram representation, separately; see e.g. [SS08, Lemma 4.1] and [DS18, Lemma 10]. As shown in [SS08, Theorem 2.4], this fact allows to prove a bound on the determinant in Equation (63), of the form:

$$\left| \det [s_{b(f), b(f')} g_{(f, f')}] \right| \leq C^n \quad (72)$$

where the constant C depends on the Hamiltonian h_0 (and it is finite uniformly in β, L , thanks to the spectral gap). Putting together Equation (63), Equation (70), Equation (72), the bound Equation (64) easily follows.

Proof of item 1 of Theorem 4. Following [DS18, Theorem 1], it is known that the convergence of the above expansion, combined with the spectral gap of the single-particle Hamiltonian h_0 , implies that the many-body Hamiltonian H has a spectral gap, bounded below by $2m - K|U|^{1/3}$ for some $K > 0$, uniformly in L . The same methods allow to prove that, for small U , the many-body ground state is unique and it is at half-filling. These claims are true at $U = 0$, as proved in Subsection 7.3.1. The uniqueness of the ground state follows from the continuity of the eigenvalues of the Hamiltonian as a function of U , and from the stability of the spectral gap for U small. Let us now prove that the ground state is at half-filling. Since the Hamiltonian commutes with the number operator, the unique ground state of H must be an eigenstate of the number operator N . Furthermore, by the convergence of the cluster expansion, $\langle \psi_0, N \psi_0 \rangle$ is continuous in U for $|U|$ small (in fact, analytic). Thus, since $\langle \psi_0, N \psi_0 \rangle$ is integer valued, it must be constant in U , and hence equal to its value at $U = 0$, which is L^2 . This concludes the proof of Item 1 of Theorem 4 for weakly interacting fermions.

Proof of item 2 of Theorem 4 Let us now apply the previous strategy to prove the exponential closeness of the approximate ground state energies, in presence of many-body interactions, Equation (5). The proof is based on the BBF formula in Equation (61) combined with the Poisson formula for the two-point function in Equation (65). Let us write:

$$\begin{aligned} & \frac{1}{\beta} \sum_{i_1, \dots, i_n} \int_{[0, \beta]^n} d\bar{t} \langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \dots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, L}^0 \\ &= \frac{1}{\beta} \sum_{i_1, \dots, i_n} \int_{[0, \beta]^n} d\bar{t} \sum_T \alpha_T \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] \\ &= \sum_T \alpha_T \frac{1}{\beta} \sum_{i_1, \dots, i_n} \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] . \end{aligned} \quad (73)$$

Using the translation-invariance of the π -flux phase, we can rewrite Equation (73) as:

$$(L^2/|\mathcal{C}|) \sum_T \alpha_T \frac{1}{\beta} \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}} \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] \quad (74)$$

where $|\mathcal{C}|$ is the number of sites in the fundamental cell, and $\mathcal{C}(L/2, L/2)$ is the fundamental cell containing the site $(L/2, L/2)$. We then break the innermost sum as:

$$\begin{aligned} & \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}} \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] \\ &= \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}}^* \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] \\ &+ \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}}^{**} \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] , \end{aligned} \quad (75)$$

where the first sum involves vertices with coordinates such that $\|i_j - i_1\| \leq L/3$ for all $j = 2, \dots, n$, while in the second sum at least one vertex is such that $\|i_j - i_1\| > L/3$. Observe that¹ this also implies $\|i_j - i_1\|_L > L/3$, where $\|\cdot\|_L$ denotes the Euclidean distance on the torus. Consider the second term in Equation (75). Let $T' \subseteq T$ be the subtree of T which connects i_1 to i_j . Consider the product of propagators associated with T' . Denoting by $(i_1, i_{f_2}, \dots, i_{f_r}, i_j)$ the path connecting i_1 to i_j in the tree, we have the following:

$$\|i_1 - i_{f_2}\|_L + \dots + \|i_{f_r} - i_j\|_L \geq \|i_j - i_1\|_L \geq L/3. \quad (76)$$

Combined with the exponential decay of the two-point function (see Equation (68)), this allows to extract an exponentially small factor after summing over all vertex coordinates compatible with the constraint in the sum. Repeating the argument used to prove the analyticity of the free energy discussed in the previous paragraph, we obtain:

$$\left| \sum_T \alpha_T \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}}^{**} \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_\ell \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{(f, f')}] \right| \leq C^n n! \beta e^{-cL}. \quad (77)$$

Consider now the first term in Equation (75). Observe that the constraint in the sum implies that $\|i_f - i_{f'}\| \leq 2L/3$ for all branches of the tree. The idea is to replace every propagator in the sum with its infinite volume limit, g_∞ , and controlling the error using the Poisson summation formula in Equation (65). By Equation (65), we have:

$$g((t, i), (t', i')) = g_\infty((t, i), (t', i')) + r((t, i), (t', i')) \quad (78)$$

where, if $\|i - i'\| \leq 2L/3$,

$$\|r((t, i), (t', i'))\| \leq C e^{-(c/6)L - (c/6)\|(t, i) - (t', i')\|_{\beta, L}}. \quad (79)$$

Next, let us define the interpolating propagator as, for $\lambda \in [0, 1]$:

$$g_\lambda((t, i), (t', i')) = \lambda g((t, i), (t', i')) + (1 - \lambda) g_\infty((t, i), (t', i')). \quad (80)$$

¹This is due to the fact that, as vector in $\mathbb{R}^2$, $i_j - i_1$ has norm bounded by $L/\sqrt{2}$. The norm of this vector does not decrease after adding $n_1 e_1 L + n_2 e_2 L$ for integer n_1, n_2 .

Let

$$f_T(\lambda) := \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}}^* \int_{[0, \beta]^n} d\bar{t} \left[\prod_{\ell \in T} g_{\lambda; \ell} \right] \int d\mu_T(\underline{s}) \det [s_{b(f), b(f')} g_{\lambda; (f, f')}] ; \quad (81)$$

for $\lambda = 1$, this is our starting point, computed with a certain choice of holonomies. For $\lambda = 0$, the function $f_T(\lambda)$ does not depend on the boundary conditions. We are thus interested in quantifying the difference:

$$f_T(1) - f_T(0) = \int_0^1 d\lambda \frac{d}{d\lambda} f_T(\lambda) . \quad (82)$$

Whenever the derivative hits the product of propagators on the spanning tree, we get:

$$\sum_{\ell \in T} \left[\prod_{\ell' < \ell} g_{\lambda; \ell'} \right] r_\ell \left[\prod_{\ell' > \ell} g_{\lambda; \ell'} \right] . \quad (83)$$

The sum over all space-time coordinates of this expression can be performed as before, using [Equation \(68\)](#) and [Equation \(79\)](#)). The only difference is the presence of the extra factor $e^{-(c/6)L}$ in the final estimate, coming from [Equation \(79\)](#). Suppose now that the derivative hits the determinant. To begin, observe preliminarily that, as for g and g_∞ , the interpolating propagator g_λ can also be represented as the sum of two terms admitting a Gram representation. To see this, we write:

$$g_\lambda((t, i), (t', i')) = \theta(t > t') A_\lambda^+((t, i), (t', i')) - \theta(t \leq t') A_\lambda^-((t, i), (t', i')) \quad (84)$$

where:

$$A_\lambda^\pm((t, i), (t', i')) = \lambda A^\pm((t, i), (t', i')) + (1 - \lambda) A_\infty^\pm((t, i), (t', i')) . \quad (85)$$

Following [[SS08](#), Lemma 4.1] or [[DS18](#), Lemma 10] (for a more general setting), we have:

$$A^\pm((t, i), (t', i')) = (u_{(t, i)}^\pm, w_{(t', i')}^\pm)_{\mathfrak{h}} , \quad A_\infty^\pm((t, i), (t', i')) = (u_{\infty, (t, i)}^\pm, w_{\infty, (t', i')}^\pm)_{\mathfrak{h}_\infty} \quad (86)$$

for vectors of unit norm in suitable Hilbert spaces $\mathfrak{h}$ and $\mathfrak{h}_\infty$, whose definitions will not be important in what follows. Then, the interpolating functions in [Equation \(85\)](#) inherit the Gram representation:

$$A_\lambda^\pm((t, i), (t', i')) = \left(\sqrt{\lambda} u_{(t, i)}^\pm \oplus \sqrt{1 - \lambda} u_{\infty, (t, i)}^\pm, \sqrt{\lambda} w_{(t', i')}^\pm \oplus \sqrt{1 - \lambda} w_{\infty, (t', i')}^\pm \right)_{\mathfrak{h} \oplus \mathfrak{h}_\infty} . \quad (87)$$

From now on, one can proceed as in [[SS08](#), Theorem 2.4] to show that:

$$\left| \det [s_{b(f), b(f')} g_{\lambda; (f, f')}] \right| \leq C^n . \quad (88)$$

Next, we have to estimate the derivative of the determinant. Let us denote by $G_\lambda(\underline{s})$ the argument of the determinant. By Jacobi's formula:

$$\frac{d}{d\lambda} \det G_\lambda(\underline{s}) = \text{Tr} \left(\text{adj}(G_\lambda(\underline{s})) \frac{d}{d\lambda} G_\lambda(\underline{s}) \right) , \quad (89)$$

where $\text{adj}(\cdot)$ denotes the adjugate matrix, that is the transpose of the matrix of the cofactors. We have:

$$\left(\frac{d}{d\lambda} G_\lambda(\underline{s}) \right)_{f, f'} = s_{b(f), b(f')} r_{f, f'} , \quad (90)$$

with $r_{f, f'}$ satisfying the bound in [Equation \(79\)](#). Therefore:

$$\left| \left(\frac{d}{d\lambda} G_\lambda(\underline{s}) \right)_{f, f'} \right| \leq C e^{-(c/6)L} . \quad (91)$$

Concerning $\text{adj}(G_\lambda(\underline{s}))$, its matrix entries are, up to a sign, the determinants of the minors of the original matrix $G_\lambda(\underline{s})$ after deleting a column and a row. They can all be estimated as in Equation (88). All in all, we have:

$$\left| \frac{d}{d\lambda} \det G_\lambda(\underline{s}) \right| \leq n^2 C^m e^{-(c/6)L}. \quad (92)$$

We are now in the position to estimate Equation (82). From the above considerations we easily get, for suitable constants $C, c > 0$:

$$\left| f_T(1) - f_T(0) \right| = \int_0^1 d\lambda \left| \frac{d}{d\lambda} f_T(\lambda) \right| \leq n! \beta C^m e^{-cL}. \quad (93)$$

Coming back to Equation (75), we proved that:

$$\begin{aligned} & \frac{1}{\beta} \sum_{i_1, \dots, i_n} \int_{[0, \beta]^n} dt \langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \dots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, L}^0 \\ &= (L^2/|\mathcal{C}|) \frac{1}{\beta} \sum_{\substack{i_1, \dots, i_n \\ i_1 \in \mathcal{C}(L/2, L/2)}}^* \int_{[0, \beta]^n} dt \langle \mathbf{T} \gamma_{t_1}(V_{i_1}); \gamma_{t_2}(V_{i_2}); \dots; \gamma_{t_n}(V_{i_n}) \rangle_{\beta, \infty}^0 + \mathfrak{e}_{\beta, L}(n) \end{aligned} \quad (94)$$

where the first term does not depend on the holonomies, while the error term is bounded as:

$$|\mathfrak{e}_{\beta, L}(n)| \leq C^m n! L^2 e^{-cL}. \quad (95)$$

Thus, from Equation (56):

$$\begin{aligned} & \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H(-\mathbf{1}, e^{i\theta}, e^{i\phi})} - \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H(-\mathbf{1}, e^{i\theta'}, e^{i\phi'})} \\ &= \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H_0(-\mathbf{1}, e^{i\theta}, e^{i\phi})} - \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H_0(-\mathbf{1}, e^{i\theta'}, e^{i\phi'})} \\ &+ \sum_{n \geq 1} \frac{(-U)^n}{n!} \int_{[0, \beta]^n} dt_1 \dots dt_n \left(\langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \dots; \gamma_{t_n}(V) \rangle_{\beta, L}^{0; \theta, \phi} \right. \\ &\quad \left. - \langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \dots; \gamma_{t_n}(V) \rangle_{\beta, L}^{0; \theta', \phi'} \right) \end{aligned} \quad (96)$$

where $\langle \cdot \rangle_{\beta, L}^{0; \theta, \phi}$ is the Gibbs state of the quadratic Hamiltonian $H_0(-\mathbf{1}, \theta, \phi)$. By Equation (94) and Equation (95), the sum in Equation (96) can be estimated as:

$$\begin{aligned} & \left| \sum_{n \geq 1} \frac{(-U)^n}{n!} \int_{[0, \beta]^n} dt_1 \dots dt_n \left(\langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \dots; \gamma_{t_n}(V) \rangle_{\beta, L}^{0; \theta, \phi} \right. \right. \\ &\quad \left. \left. - \langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \dots; \gamma_{t_n}(V) \rangle_{\beta, L}^{0; \theta', \phi'} \right) \right| \leq CU \beta L^2 e^{-cL}, \end{aligned} \quad (97)$$

where we used that the main term in Equation (94) cancels in the difference. Furthermore, the difference of the non-interacting free energies has been studied, in the limit $\beta \rightarrow \infty$, in Subsection 7.3.2:

$$\left| \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \left(\log \text{Tr}_{\mathcal{F}_L} e^{-\beta H_0(-\mathbf{1}, \theta, \phi)} - \log \text{Tr}_{\mathcal{F}_L} e^{-\beta H_0(-\mathbf{1}, \theta', \phi')} \right) \right| \leq CL^2 e^{-cL}. \quad (98)$$

In conclusion, the exponential closeness of the many-body ground state energies (see Equation (5)) follows from Equation (97) (in the $\beta \rightarrow \infty$ limit) and from Equation (98). This concludes the proof of part of Theorem 4 for weakly interacting fermions.

7.3.4 Stability of the π -Flux Sector and the Monopole Gap

The proof of Equation (6)–Equation (8) uses the Hamiltonian Reflection-Positivity framework of Chapter 3 and the model-specific Chessboard construction developed in Chapter 6. We do not repeat those arguments. The new point is to verify that both the staggered mass and the Hubbard interaction are compatible with the Lieb reflection; after that verification, the remaining task is the spectral evaluation of the periodic comparison Hamiltonian.

Compatibility of the mass and Hubbard terms with reflection Let P be a bond hyperplane dividing the torus into Λ_L^+ and Λ_L^- , and let θ be the corresponding geometric reflection. We use the antilinear Lieb reflection Θ defined in Equation (29) (recall the discussion Subsection 3.4.1), so that:

$$\Theta(n_{i,\eta}) = 1 - n_{\theta(i),\eta}. \quad (99)$$

Introduce the half-space mass and interaction terms

$$M_{\pm} \doteq m \sum_{i \in \Lambda_L^{\pm}} (-1)^{i_1+i_2} n_i, \quad (100)$$

$$W_{\pm} \doteq U \sum_{i \in \Lambda_L^{\pm}} \left(n_{i,\uparrow} - \frac{1}{2} \right) \left(n_{i,\downarrow} - \frac{1}{2} \right). \quad (101)$$

Lemma (Reflection of the local perturbations). *The mass and Hubbard terms satisfy*

$$\Theta(M_-) = M_+, \quad \Theta(W_-) = W_+.$$

Proof. The reflection exchanges the two sublattices of the square lattice. Hence

$$\Theta(M_-) = m \sum_{i \in \Lambda_L^-} \sum_{\eta \in \mathcal{S}} (-1)^{i_1+i_2} (1 - n_{\theta(i),\eta}) = 2m \sum_{i \in \Lambda_L^-} (-1)^{i_1+i_2} + m \sum_{i \in \Lambda_L^-} \sum_{\eta \in \mathcal{S}} (-1)^{i_1+i_2+1} n_{\theta(i),\eta}.$$

The first term vanishes because each half contains the same number of sites of the two sublattices. Since $(-1)^{\theta(i)_1+\theta(i)_2} = -(-1)^{i_1+i_2}$, the second term is M_+ . Moreover,

$$\Theta \left(n_{i,\eta} - \frac{1}{2} \right) = - \left(n_{\theta(i),\eta} - \frac{1}{2} \right).$$

The two minus signs cancel in the product of the two spin components, which gives $\Theta(W_-) = W_+$. $\blacksquare$

In a fixed vector-potential configuration, decompose

$$H(\boldsymbol{\sigma}) = H_-(\boldsymbol{\sigma}) + H_+(\boldsymbol{\sigma}) + H_{\text{int}}(\boldsymbol{\sigma}),$$

where the first two terms include the corresponding restrictions of the mass and Hubbard interactions. As in Chapter 6, a Gauge transformation makes all vector potentials on the bonds crossing the cut equal to $+1$, so that

$$H_{\text{int}}(\boldsymbol{\sigma}) = -t \sum_{\substack{i \in \Lambda_L^-: \\ i+e_1 \in \Lambda_L^+}} \sum_{\eta \in \mathcal{S}} \left(a_{i,\eta}^+ a_{i+e_1,\eta}^- + a_{i+e_1,\eta}^+ a_{i,\eta}^- \right). \quad (102)$$

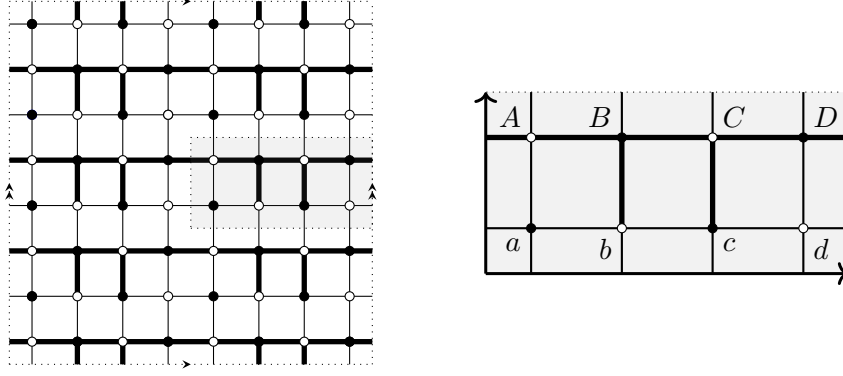


Figure 7.8: Left: gauge field configuration σ^* associated with the chessboard flux arrangement Φ^* . Solid bonds correspond to $\sigma^z = -1$, while light bonds correspond to $\sigma^z = +1$. Right: fundamental cell associated with a translation-invariant configuration.

The crossing term is therefore exactly of the form required by Lieb's Reflection Inequality, while the lemma shows that the two additional local terms are exchanged correctly. Thus

$$[Z_{\beta,L}(H_-, H_+)]^2 \leq Z_{\beta,L}(H_-, \Theta(H_-)) Z_{\beta,L}(\Theta(H_+), \Theta(H_+)).$$

In particular, the partition function remains maximized by the uniform π -flux background in the presence of the staggered mass and the Hubbard interaction. Applying the Chessboard Estimate as in Subsection 6.2.2 reduces the monopole-energy bound to the free energy of the periodic flux configuration Φ^* . It remains to evaluate the corresponding quantity $\Delta_{\beta,L}$ for the massive interacting model, which is the content of the next paragraph.

Lower bound on the monopole mass By Proposition 71, the monopole mass is bounded from below by the free-energy difference between the uniform π -flux background and a periodic Chessboard flux configuration. Let Φ^* denote the latter configuration, depicted in Figure 7.1. Choosing the vector-potential representative σ^* with holonomies $(1, 1)$ as in Figure 7.8, we select a unit cell that makes the system translation invariant; this is possible because $L \in 4\mathbb{N}$. If the corresponding reduced lattice is denoted $\tilde{\Lambda}_L^{\text{red}}$, the Hamiltonian reads

$$\begin{aligned} H(\sigma^*; 1, 1) = & -t \sum_{\mathbf{i} \in \tilde{\Lambda}_L^{\text{red}} \times \{\uparrow, \downarrow\}} \left(a_{a,\mathbf{i}}^+ a_{b,\mathbf{i}}^- + a_{a,\mathbf{i}}^+ a_{A,\mathbf{i}}^- + a_{b,\mathbf{i}}^+ a_{c,\mathbf{i}}^- - a_{b,\mathbf{i}}^+ a_{B,\mathbf{i}}^- + a_{c,\mathbf{i}}^+ a_{d,\mathbf{i}}^- - a_{c,\mathbf{i}}^+ a_{C,\mathbf{i}}^- \right. \\ & + a_{d,\mathbf{i}}^+ a_{D,\mathbf{i}}^- + a_{d,\mathbf{i}}^+ a_{a,\mathbf{i}+4e_1}^- - a_{A,\mathbf{i}}^+ a_{B,\mathbf{i}}^- + a_{A,\mathbf{i}}^+ a_{a,\mathbf{i}+2e_2}^- - a_{B,\mathbf{i}}^+ a_{C,\mathbf{i}}^- \\ & + a_{B,\mathbf{i}}^+ a_{b,\mathbf{i}+2e_2}^- - a_{C,\mathbf{i}}^+ a_{D,\mathbf{i}}^- + a_{C,\mathbf{i}}^+ a_{c,\mathbf{i}+2e_2}^- - a_{D,\mathbf{i}}^+ a_{A,\mathbf{i}+4e_1}^- + a_{D,\mathbf{i}}^+ a_{d,\mathbf{i}+2e_2}^- + \text{h.c.} \Big) \\ & + m \sum_{\mathbf{i} \in \tilde{\Lambda}_L^{\text{red}}} (n_{A,\mathbf{i}} - n_{B,\mathbf{i}} + n_{C,\mathbf{i}} - n_{D,\mathbf{i}} - n_{a,\mathbf{i}} + n_{b,\mathbf{i}} - n_{c,\mathbf{i}} + n_{d,\mathbf{i}}) \\ & + U \sum_{i \in \tilde{\Lambda}_L^{\text{red}}} \sum_{\rho \in I} \left(n_{\rho,i,\uparrow} - \frac{1}{2} \right) \left(n_{\rho,i,\uparrow} - \frac{1}{2} \right) \end{aligned}$$

where I collects the labels of the fundamental cell and we introduced the notation $\mathbf{i} = (i, \eta)$. The Brillouin zone is:

$$\tilde{B}_L(1, 1) := \left\{ k \in \frac{2\pi}{L} (n_1, n_2) \mid 0 \leq n_1 \leq L/4 - 1, 0 \leq n_2 \leq L/2 - 1 \right\}. \quad (103)$$

For $U = 0$, the quantity $\Delta_{\beta,L}$ can be computed by exact diagonalization. Let us consider the spinless case; the presence of the spin will eventually amount to a factor 2 in the final expression.

The Bloch Hamiltonian is:

$$h(k) = \begin{pmatrix} -m & 1 & 0 & e^{-4ik_1} & 1 + e^{-2ik_2} & 0 & 0 & 0 \\ 1 & m & 1 & 0 & 0 & -1 + e^{-2ik_2} & 0 & 0 \\ 0 & 1 & -m & 1 & 0 & 0 & -1 + e^{-2ik_2} & 0 \\ e^{4ik_1} & 0 & 1 & m & 0 & 0 & 0 & 1 + e^{-2ik_2} \\ 1 + e^{2ik_2} & 0 & 0 & 0 & m & -1 & 0 & -e^{-4ik_1} \\ 0 & -1 + e^{2ik_2} & 0 & 0 & -1 & -m & -1 & 0 \\ 0 & 0 & -1 + e^{2ik_2} & 0 & 0 & -1 & m & -1 \\ 0 & 0 & 0 & 1 + e^{2ik_2} & -e^{4ik_1} & 0 & -1 & -m \end{pmatrix}.$$

Its eigenvalues are doubly degenerate, and are given by:

$$e_{1;\pm}(k) = \pm 2t \sqrt{\left(\frac{m}{2t}\right)^2 + 1 + \frac{1}{2} \sqrt{1 + \cos(2k_1)^2 + \cos(2k_2)^2}}$$

$$e_{2;\pm}(k) = \pm 2t \sqrt{\left(\frac{m}{2t}\right)^2 + 1 - \frac{1}{2} \sqrt{1 + \cos(2k_1)^2 + \cos(2k_2)^2}}.$$

We are now ready to compute $\Delta_{\beta,L}$, as given by Equation (7). Let us denote by $e_{\pm}^{\pi}(k)$ the eigenvalues of the Bloch Hamiltonian associated with the π -flux phase, as given by Equation (38). Proceeding as in the proof of [GP26, Proposition 3.13], we have:

$$\begin{aligned} -\frac{1}{\beta L^2} \log \frac{Z_{\beta,L}(\Phi^*; a, b)}{Z_{\beta,L}(-\mathbf{1}; a, b)} &\stackrel{o(1)}{=} \frac{t}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{\left(\frac{m}{2t}\right)^2 + 1 + \frac{1}{2} \cos(k_1) + \frac{1}{2} \cos(k_2)} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{\left(\frac{m}{2t}\right)^2 + 1 + \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}} \\ &\quad - \frac{t}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} dk_1 dk_2 \sqrt{\left(\frac{m}{2t}\right)^2 + 1 - \frac{1}{2} \sqrt{1 + \cos^2(k_1) + \cos^2(k_2)}}. \end{aligned} \quad (104)$$

where $\stackrel{o(1)}{=}$ means that the equality is true up to terms that vanish as $L \rightarrow \infty$. Numerical evaluation shows that the sum of these three dominant terms is positive (see Figure 7.9).

Finally, let us discuss the effect of many-body interactions. By cluster expansion, one could actually prove that $\Delta_{\beta,L}$ is analytic in U for $|U|$ small. This is actually not needed to prove the stability of the monopoles' gap. It is enough to use the general estimate:

$$e^{-\beta \|V\| \|U\|} Z_{\beta,L}^0(\Phi; a, b) \leq Z_{\beta,L}(\Phi; a, b) \leq e^{\beta \|V\| \|U\|} Z_{\beta,L}^0(\Phi; a, b) \quad (105)$$

where $Z_{\beta,L}^0$ is the partition function for $U = 0$ and UV is the Hubbard interaction. This bound follows from the operator inequalities $H_0(\sigma) - |U| \|V\| \leq H(\sigma) \leq H_0(\sigma) + |U| \|V\|$ and from the fact that, for any two self-adjoint operators A, B

$$\text{Tr } e^A \leq \text{Tr } e^B \quad \text{if } A \leq B, \quad (106)$$

a bound that can be proved using the min-max characterization of the eigenvalues; see e.g. [Tas20, Corollary A.8]. The bound in Equation (105), combined with the expression Equation (7), immediately implies that:

$$\Delta_{\beta,L} \geq \Delta_{\beta,L}^0 - C|U|. \quad (107)$$

This concludes the proof of Item 3 of Theorem 4, for holonomies a, b equal to ± 1 .

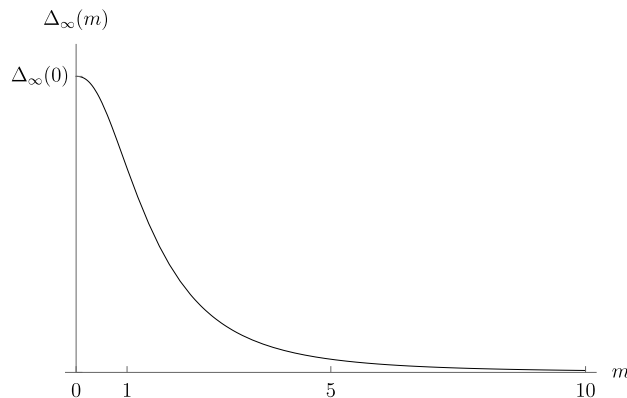


Figure 7.9: Numerical plot of the monopole's mass $\Delta_\infty(m)$ as a function of m ($t = 1$, $U = 0$).

Remark 58. The bound displayed in Equation (107) is useful for small U only. For $U \rightarrow \infty$, we expect that $\Delta_{\beta,L}(U) \rightarrow 0$, as the dominant part of the Hamiltonian is background independent (similarly to what happens in $m \rightarrow \infty$ limit, where the computation can be done exactly).

Extension to twisted Hamiltonians To conclude the proof of Theorem 4, it remains to treat general holonomies. Let $H(\boldsymbol{\sigma}, e^{i\phi}, e^{i\theta})$ denote the Hamiltonian with $\mathbb{Z}_2$ -valued plaquette fluxes specified by $\boldsymbol{\sigma}$ and additional $U(1)$ holonomies $e^{i\phi}$ and $e^{i\theta}$ along two generating non-contractible cycles.² Up to unitary equivalence, this Hamiltonian is obtained by inserting the two phases on the bonds crossed by the corresponding dual cycles, as in Figure 7.10. This changes the global holonomies without changing the plaquette fluxes.

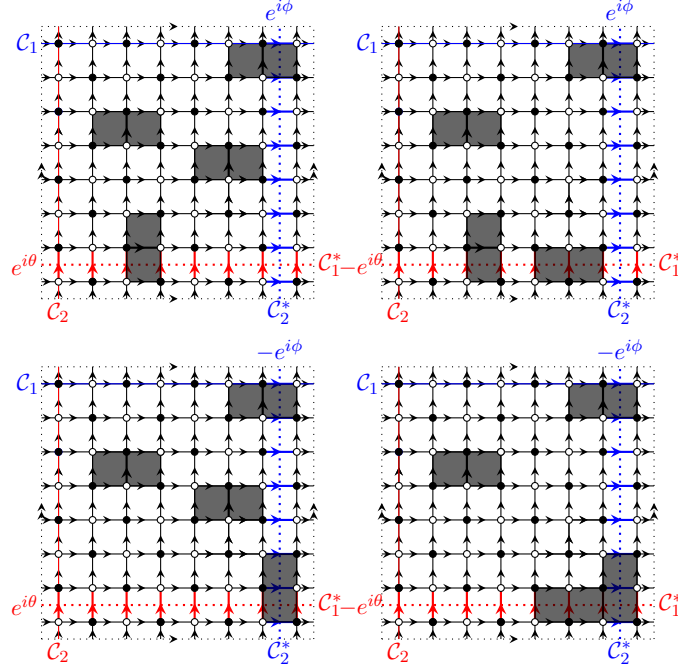
We apply the Chessboard procedure of Subsection 6.2.2, keeping track of the two holonomies. After a horizontal reflection whose plane does not intersect Λ_L along $\mathcal{C}_2^*$, the holonomy along $\mathcal{C}_1$ becomes -1 , while the one along $\mathcal{C}_2$ is unchanged. Indeed, the Lieb reflection reviewed in Chapter 3 contains both complex conjugation and a particle-hole transformation. The latter contributes one minus sign for every reflected pair of hopping terms, and the two bonds crossing the cut contribute the signs fixed by the Gauge choice in Equation (102). The resulting holonomy is therefore $(-1)^{L/2-1} = -1$, because $L/2$ is even. A subsequent vertical reflection similarly changes the holonomy along $\mathcal{C}_2$ to -1 and leaves the first equal to -1 .

After this pair of reflections, the comparison partition functions have $\mathbb{Z}_2$ -valued holonomies and can be represented by $\mathbb{Z}_2$ vector-potential configurations. The remainder of the argument is exactly the model-specific Chessboard construction of Subsection 6.2.2. This proves Item 3 and concludes the proof of Theorem 4. ■

7.4 The Almost-Degenerate Low-Energy Sector and Local Topological Order

Ground states in the physical Hilbert space. The Hamiltonian in Equation (1) has four-fold almost degenerate ground states characterized by the four possible π -flux backgrounds of the torus. Each ground state is obtained by acting with the projection that imposes the Gauss' law on the

²Plaquette fluxes and the two global holonomies are independent Gauge-invariant generators.

Figure 7.10: Possible σ background with $e^{i\theta}$ and $e^{i\phi}$ holonomies around a and b cycles

ground state of the many-body Hamiltonian in the π -flux phase, for a given choice of holonomies:

$$|\Omega_{ab}\rangle = \prod_{i \in V(\Lambda_L)} \left(\frac{1+Q_i}{2} \right) |\psi_{-\mathbf{1},a,b}\rangle \otimes |-\mathbf{1},a,b\rangle = \frac{1}{2^{|\Lambda_L|}} \sum_{\Omega \subset \Lambda_L} A_\Omega (-1)^{N_\Omega} |\psi_{-\mathbf{1},a,b}\rangle \otimes |-\mathbf{1},a,b\rangle \quad (108)$$

where $|-\mathbf{1},a,b\rangle$ are a set of common eigenstates of all $\hat{\sigma}_{ij}^z$ satisfying: $B_p = -1$ for any plaquette $p \in P(\Lambda_L)$, $\hat{Z}_{C_1} = a$ and $\hat{Z}_{C_2} = b$ and $|\psi_{-\mathbf{1},a,b}\rangle$ is the ground state of $H(-\mathbf{1},a,b)$ acting on the fermionic Fock space.

The following proposition states some basic properties about these states. Recall that $\mathcal{H}_L^{\text{phys}}$ is the physical Hilbert space, where Gauss' law is satisfied on each vertex, see Definition 65.

Proposition 77. *The states $|\Omega_{ab}\rangle$ satisfy the following properties:*

1. $|\Omega_{ab}\rangle \in \mathcal{H}_L^{\text{phys}}$
2. $|\Omega_{ab}\rangle$ depends only on the choice of the background holonomies
3. $|\Omega_{ab}\rangle \neq 0$

Proof.

This is immediate since $\prod_{i \in V(\Lambda_L)} \frac{1+Q_i}{2}$ projects onto the eigenspace $+1$ of each Q_i .

2. Let σ, σ' be two different gauge equivalent backgrounds. There exist a subset $\Omega \subset \Lambda_L$ such that $A_\Omega |\sigma\rangle = |\sigma'\rangle$, and the Hamiltonian in the new background is given by $H(\sigma') = (-1)^{N_\Omega} H(\sigma) (-1)^{N_\Omega}$. Thus the ground state of $H(\sigma')$ is

$$|\psi_{\sigma'}\rangle = (-1)^{N_\Omega} |\psi_\sigma\rangle \quad (109)$$

Hence,

$$|\Omega'_{ab}\rangle = \frac{1}{2^{|\Lambda_L|}} \sum_{\Upsilon \subset \Lambda_L} (-1)^{N_\Upsilon} A_\Upsilon |\psi_{\sigma'}\rangle \otimes |\sigma', a, b\rangle \quad (110)$$

$$= \frac{1}{2^{|\Lambda_L|}} \sum_{\Upsilon \subset \Lambda} (-1)^{N_\Upsilon + N_\Omega} A_\Upsilon |\psi_\sigma\rangle \otimes A_\Omega |\sigma, a, b\rangle = |\Omega_{ab}\rangle \quad (111)$$

since $A_\Upsilon A_\Omega = A_{\Upsilon+\Omega}$, $N_\Upsilon + N_\Omega = N_{\Upsilon+\Omega}$ (where the sum of subsets is understood in the $\mathbb{Z}_2$ -valued chain group) and the sum is translation invariant.

3. Using again the right expression of the ground state in [Equation \(108\)](#), we have

$$\|\Omega_{ab}\|^2 = \frac{1}{2^{|\Lambda_L|}} \sum_{\Omega \subset \Lambda_L} \langle -\mathbf{1}, a, b | A_\Omega | -\mathbf{1}, a, b \rangle \left\langle \psi_{-\mathbf{1}, a, b} \left| (-1)^{N_\Omega} \right| \psi_{-\mathbf{1}, a, b} \right\rangle \quad (112)$$

$$= \frac{1}{2^{|\Lambda_L|}} \left\langle \psi_{-\mathbf{1}, a, b} \left| \left(1 + (-1)^{N_{\Lambda_L}} \right) \right| \psi_{-\mathbf{1}, a, b} \right\rangle \quad (113)$$

where we have used that the diagonal matrix elements of A_Ω in the $\hat{\sigma}_{ij}^z$ -basis vanish unless $\Omega = \emptyset$ or $\Omega = \Lambda_L$. Since the ground states are at half-filling by [Theorem 4](#) and $L \in 4\mathbb{N}$, $|\psi_{-\mathbf{1}, a, b}\rangle$ has an even number of particles and so the matrix element is 2. This shows that $|\Omega_{ab}\rangle \neq 0$. ■

Remark 59. The argument used in the computation of the norm of Ω_{ab} can be repeated to rewrite in a convenient way the energy of Ω_{ab} . We have:

$$\begin{aligned} \frac{\langle \Omega_{ab} | H | \Omega_{ab} \rangle}{\|\Omega_{ab}\|^2} &= \langle \psi_{-\mathbf{1}, a, b} | H(-\mathbf{1}, a, b) | \psi_{-\mathbf{1}, a, b} \rangle \\ &= E_{0,L}(-\mathbf{1}, a, b) . \end{aligned} \quad (114)$$

Spectral gap estimate. Let us prove the spectral gap estimate used in [Equation \(10\)](#). In view of [Equation \(114\)](#),

$$\sup \text{Spec}(PHP) = \max_{a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2} E_{0,L}(-\mathbf{1}, a, b) . \quad (115)$$

Then, we write, for any normalized vector $|\xi\rangle$ in the total Hilbert space:

$$\left\langle \xi \left| P^\perp H P^\perp \right| \xi \right\rangle = \left\langle \xi_{-1} \left| P^\perp H P^\perp \right| \xi_{-1} \right\rangle + \left\langle \tilde{\xi} \left| P^\perp H P^\perp \right| \tilde{\xi} \right\rangle \quad (116)$$

where $|\xi_{-1}\rangle$ is the orthogonal projection of $|\xi\rangle$ on the subspace associated with flux π in every plaquette, and $|\tilde{\xi}\rangle = |\xi\rangle - |\xi_{-1}\rangle$. Consider the second term. By [Equation \(8\)](#), we have:

$$\left\langle \tilde{\xi} \left| P^\perp H P^\perp \right| \tilde{\xi} \right\rangle \geq \|\tilde{\xi}\|^2 \left(2\Delta_L + \min_{a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2} E_{0,L}(-\mathbf{1}, a, b) \right) . \quad (117)$$

Consider now the first term in [Equation \(116\)](#). We have:

$$\left\langle \xi_{-1} \left| P^\perp H P^\perp \right| \xi_{-1} \right\rangle \geq \|\xi_{-1}\|^2 \min_{a, b \in \mathbb{Z}_2 \times \mathbb{Z}_2} E_{1,L}(-\mathbf{1}, a, b) , \quad (118)$$

where $E_{1,L}(-\mathbf{1}, a, b)$ is the first eigenvalue above the ground state for the Hamiltonian $H(-\mathbf{1}, a, b)$ acting on $\mathcal{F}_L$. For $U = 0$, the Hamiltonian is gapped, and the spectral gap is m , recall [Equation \(38\)](#).

For $U \neq 0$, the stability of the spectral gap follows from the convergence of the cluster expansion method reviewed in [Subsection 7.3.3](#), see discussion after [Equation \(72\)](#). One has:

$$E_{1,L}(-\mathbf{1}, a, b) \geq E_{0,L}(-\mathbf{1}, a, b) + 2\delta_L, \quad \delta_L \geq m - C|U|^{1/3}. \quad (119)$$

Therefore, putting together [Equation \(115\)](#), [Equation \(116\)](#), [Equation \(118\)](#) and [Equation \(119\)](#), we have, using that $\|\xi\| = 1$:

$$\begin{aligned} \left\langle \xi \left| P^\perp H P^\perp \right| \xi \right\rangle - \sup \text{Spec}(PHP) &\geq 2 \min(\delta_L, \Delta_L) + \min_{a,b} E_{0,L}(-\mathbf{1}, a, b) - \max_{a,b} E_{0,L}(-\mathbf{1}, a, b) \\ &\geq 2 \min(\delta_L, \Delta_L) - CL^2 e^{-cL}, \end{aligned}$$

where in the last step we used the exponential closeness of the approximate ground state energies in [Equation \(5\)](#). This concludes the proof of [Equation \(10\)](#).

Local topological order. Let us now prove topological order defined by the condition in [Equation \(11\)](#). Recalling [Definition 64](#), the algebra of gauge-invariant observables is the centralizer $\mathcal{C}_{\mathcal{A}_L}(\mathcal{Q})$ of $\mathcal{Q}$. It is generated by

$$\{\hat{\sigma}_{ij}^x : (i, j) \in E(\Lambda_L)\} \cup \{\hat{Z}_{\mathcal{C}} : \mathcal{C} \in Z_1(\Lambda_L)\} \cup \{a_{i,\eta}^\pm \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^\pm : i, j \in \Lambda_L, \eta, \eta' \in \{\uparrow, \downarrow\}\}, \quad (120)$$

where $\mathcal{C}_{i,j}$ is any cycle such that $\partial \mathcal{C}_{i,j} = \{i, j\}$, with the convention that $\mathcal{C}_{i,i} = 0$ for all $i \in \Lambda$. We will now show that all these local gauge-invariant observables are first of all diagonal on the ground state space, and secondly that they have the same expectation value in all of the ground states, up to exponentially small errors. But then,

$$\langle \Omega_{ab} | \mathcal{O} \Omega_{ab} \rangle \stackrel{L}{=} \frac{1}{4} \sum_{(a,b) \in \mathbb{Z}_2 \times \mathbb{Z}_2} \langle \Omega_{ab} | \mathcal{O} \Omega_{ab} \rangle = \frac{1}{\text{Tr } P} \text{Tr}(P\mathcal{O}) \quad (121)$$

since $\{|\Omega_{ab}\rangle : (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2\}$ form an orthonormal basis of the range of P .

A monomial in $\hat{\sigma}_{ij}^x$ is naturally identified with $\hat{X}_{\mathcal{C}^*}$ where $\mathcal{C}^*$ is the sum of all edges in the monomial (seen in the $\mathbb{Z}_2$ -valued cohomology group). If $\partial \mathcal{C}^* = 0$, then $\mathcal{C}^*$ is a coboundary $\mathcal{C}^* = \partial \Lambda$ since otherwise it would not be a local observable, and so

$$\hat{X}_{\mathcal{C}^*} |\Omega_{ab}\rangle = (-1)^{N_\Lambda} |\Omega_{ab}\rangle \quad (122)$$

namely, it acts as a purely fermionic observable, and it is therefore diagonal, on the ground state space. As proved below, local observables that are diagonal in the $\hat{\sigma}_{ij}^z$ basis have diagonal elements in the ground-state space that are approximately independent by (a, b) . If $\partial \mathcal{C}^*$ is not empty, then the observable changes the flux from π to 0 at each boundary plaquette and so $\langle \Omega_{a'b'} | \hat{X}_{\mathcal{C}^*} \Omega_{ab} \rangle = 0$ for all $(a', b'), (a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

We are left with discussing gauge invariant observables that are diagonal in the $\hat{\sigma}_{ij}^z$ basis. These observables do not change the background, and hence they are diagonal in the ground state space. We must only prove that the expectation values in all four ground states are equal as $L \rightarrow \infty$. For the $\hat{Z}_{\mathcal{C}}$ observables with $\mathcal{C}$ being a boundary, we write $\mathcal{C} = \partial \sum_i p_i$ for a set of plaquettes $\{p_i : i \in \{1, \dots, M\}\}$ and note that $Z_{\mathcal{C}} |\Omega_{ab}\rangle = \prod_i B_{p_i} |\Omega_{ab}\rangle = (-1)^M |\Omega_{ab}\rangle$ for all $(a, b) \in \mathbb{Z}_2$.

It remains to consider the algebra generated by the ‘open lines’ $a_{i,\eta}^\pm \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^\pm$. Since the states Ω_{ab} have a definite number of fermions, the operators $a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^+$ vanish on the ground state space,

and so do their adjoints. So we turn to $a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^-$. Commuting the operator through the Gauss' law projection, the operator $\hat{Z}_{\mathcal{C}_{i,j}}$ then becomes a phase ± 1 that depends on the background, see [Figure 7.2](#). Hamiltonians in the π -flux phase, differing by the value of the holonomies, can be viewed as being the same Hamiltonian but endowed with different boundary conditions (periodic or antiperiodic). The ± 1 phase produced by the observable is compensated by the change of boundary conditions.

We are left with the evaluation of fermionic monomials in the π -flux state, with (a, b) holonomies. For $U = 0$, this can be done via the application of the fermionic Wick's rule, whose outcome is completely specified by the two-point function in [Equation \(59\)](#) as $\beta \rightarrow \infty$, which is the Fermi projector. The exponential closeness of the expectation values for different (a, b) follows the representation of the finite-volume Fermi projector via Poisson summation formula:

$$P_L^{(a,b)}(x; y) = \sum_{m_1, m_2 \in \mathbb{Z}^2} a^{m_1} b^{m_2} P_\infty(x + m_1 e_1 L + m_2 e_2 L; y), \quad (123)$$

where $P_L^{(a,b)}$ is the Fermi projector for the model in a finite volume and with (a, b) holonomies, while P_∞ is the Fermi projector for the model on $\mathbb{Z}^2$. As in the proof for the exponential closeness of the ground state energies, expectation values for different holonomies are then compared by inspection of the corresponding Wick's rules, using that the term with $m_1 = m_2 = 0$ in [Equation \(123\)](#) is independent of (a, b) , and using that the terms with $m_i \neq 0$ are exponentially suppressed, thanks to the exponential decay of the Fermi projector.

Let us now discuss the case of weakly interacting fermions, $U \neq 0$. The starting point is the Duhamel expansion for the expectation value of the observable,

$$\langle \mathcal{O} \rangle_{\beta, L} = \langle \mathcal{O} \rangle_{\beta, L}^0 + \sum_{n \geq 1} \frac{(-U)^n}{n!} \int_{[0, \beta]^n} dt_1 \cdots dt_n \langle \mathbf{T} \gamma_{t_1}(V); \gamma_{t_2}(V); \cdots; \gamma_{t_n}(V); \mathcal{O} \rangle_{\beta, L}^0 \quad (124)$$

The ground state expectation value is the $\beta \rightarrow \infty$ limit of the above expression. The convergence of the series for small $|U|$, and uniformly in L , follows from fermionic cluster expansion, analogously to what has been reviewed in [Subsection 7.3.3](#) for the free energy. Furthermore, similarly to what has been done in [Subsection 7.3.3](#) to compare energies associated with different boundary conditions, the term-by-term comparison of the cumulants in [Equation \(124\)](#), computed using the BBF formula in [Equation \(61\)](#), combined with the Poisson formula for the two-point function in [Equation \(65\)](#), allows to compare Gibbs states of Hamiltonians with different boundary conditions:

$$|\langle \mathcal{O} \rangle_{\beta, L}^{(a,b)} - \langle \mathcal{O} \rangle_{\beta, L}^{(a',b')}| \leq C_{\mathcal{O}} e^{-cL} \quad (125)$$

for small $|U|$ uniformly in β, L . Taking the $\beta \rightarrow \infty$ limit, the claim in [Equation \(11\)](#) follows. This concludes the proof of [Theorem 5](#). $\blacksquare$

Remark 60. The bound in [Equation \(125\)](#) is effective for observables that are L -independent, or more generally for which the constant $C_{\mathcal{O}}$ depends on L sub-exponentially; e.g. extensive sums of local operators. It does not allow us to consider observables $\mathcal{O}$ that are exponentials of extensive observables (not surprisingly).

Remark 61. Excited eigenstates of H can be constructed by replacing $|\psi_{-1,a,b}\rangle$ in [Equation \(108\)](#) with an excited state of $H(-1, a, b)$. If $U = 0$, these excited states are explicit. For instance,

$$|\zeta_{ab}^{k,k'}\rangle = \prod_{i \in \Lambda_L} \left(\frac{1 + Q_i}{2} \right) a_{k,\eta}^+ a_{k',\eta'}^+ |\psi_{-1,a,b}\rangle \otimes |-\mathbf{1}, a, b\rangle \quad (126)$$

are exact many-body excitations, where $a_{k,\eta}^+$ creates a particle in an eigenstate of the Bloch Hamiltonian in Equation (36) with quasi-momentum k , with appropriate boundary conditions. Due to the fact that the vacuum state is obtained filling all the negative energy eigenstates of the Bloch Hamiltonian, the Pauli principle implies that the operator $a_{k,\eta}^+$ creates a particle with positive energy. More generally, one can generate a family of eigenstates of H by acting on $|\psi_{-1,a,b}\rangle$ with an even number of momentum-space fermionic operators. These states are completely delocalized in configuration space. Instead, if one acts with an odd number of fermionic operators on $|\psi_{-1,a,b}\rangle$, the resulting vector is annihilated by the action of the projector imposing the Gauss' law.

7.5 Quasi-Local Loop Operators and Anyonic Braiding

7.5.1 Adiabatic flux insertion and braiding

We turn to Theorem 6, starting with the properties of the loop operators $W_{\mathcal{C}^*}$, see Definition 75.

Threading a π -Flux through a Contractible Cycle We consider the twisted Hamiltonians as in Equation (15), first in the case of $\mathcal{C}^* = \partial\Lambda$ being a 1-coboundary.

Lemma 78. *Let $\mathcal{C}^* = \partial\Lambda$ be a 1-coboundary. Then*

$$H_{\mathcal{C}^*}(\phi) = e^{i\phi N_\Lambda} H e^{-i\phi N_\Lambda}. \quad (127)$$

Proof. Since both left and right hand sides equal H at $\phi = 0$, it suffices to prove that they satisfy the same differential equation. On the right hand side, we observe that for any $(j, \ell) \in E(\Lambda_L)$,

$$-i \frac{d}{d\phi} (e^{i\phi n_{i,\eta}} a_{j,\eta}^+ a_{\ell,\eta} e^{-i\phi n_{i,\eta}}) = \epsilon(i; j, k) e^{i\phi n_{i,\eta}} a_{j,\eta}^+ a_{\ell,\eta} e^{-i\phi n_{i,\eta}}, \quad \epsilon(i; j, k) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = \ell, \\ 0 & \text{otherwise.} \end{cases} \quad (128)$$

Summing these up, we conclude that the only non-vanishing terms in the derivative of the right hand side are those such that $\mathcal{I}[(i, j), \mathcal{C}^*] = \pm 1$, which is exactly what arises from $\partial_\phi H_{\mathcal{C}^*}(\phi)$ as defined by Equation (15). $\blacksquare$

The observation in Equation (127) implies that $\dot{P}_{\mathcal{C}^*}(\phi) = i[N_\Lambda, P_{\mathcal{C}^*}(\phi)]$ and therefore

$$[\mathcal{K}_{\mathcal{C}^*}(\phi) - N_\Lambda, P_{\mathcal{C}^*}(\phi)] = 0 \quad (129)$$

by Equation (17).

Lemma 79. *Let $\mathcal{C}^* = \partial\Lambda$ be a 1-coboundary. Then $W_{\mathcal{C}^*}$ preserves the ground state manifold of H exactly:*

$$W_{\mathcal{C}^*} P W_{\mathcal{C}^*}^* = P. \quad (130)$$

Proof. Definition 74 and Lemma 78 imply that

$$\mathcal{K}_{\mathcal{C}^*}(\phi) = e^{i\phi N_\Lambda} \mathcal{K}_{\mathcal{C}^*}(0) e^{-i\phi N_\Lambda}. \quad (131)$$

From now on, we will use the notation $\mathcal{K}_{\mathcal{C}^*} \equiv \mathcal{K}_{\mathcal{C}^*}(0)$. It follows that the unitary family $\tilde{V}_{\mathcal{C}^*}(\phi) = e^{i\phi N_\Lambda} e^{-i\phi(N_\Lambda - \mathcal{K}_{\mathcal{C}^*})}$ solves the initial value problem

$$-i\partial_\phi \tilde{V}_{\mathcal{C}^*}(\phi) = \mathcal{K}_{\mathcal{C}^*}(\phi) \tilde{V}_{\mathcal{C}^*}(\phi), \quad \tilde{V}_{\mathcal{C}^*}(0) = \mathbf{1}, \quad (132)$$

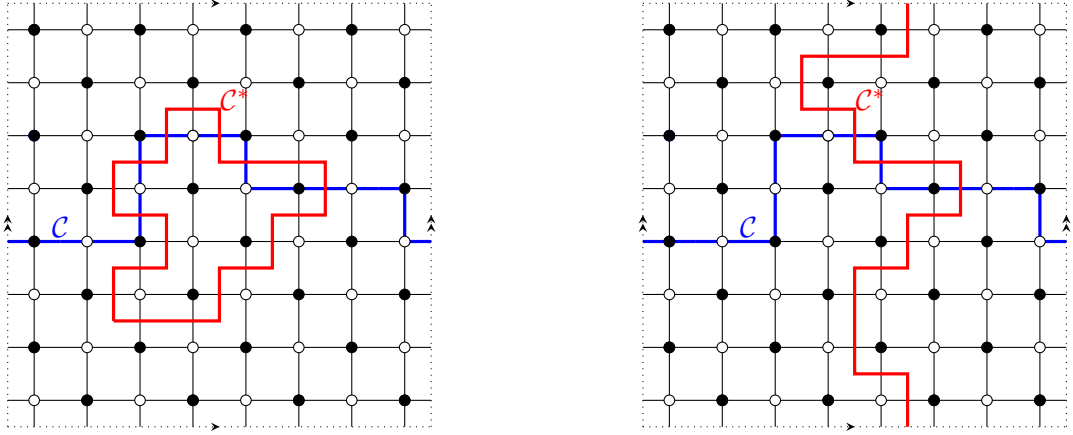


Figure 7.11: If $\mathcal{C}$ is a non-trivial cycle, the number of intersections between $\mathcal{C}^*$ and $\mathcal{C}$ is even if $\mathcal{C}^*$ is a boundary (left) but odd if it is a non-trivial cocycle (right). Notice that the curves' orientation is irrelevant here since $e^{i\pi} = e^{-i\pi}$ so positive and negative intersections give the same contribution mod 2π

and therefore $\tilde{V}_{\mathcal{C}^*}(\phi) = V_{\mathcal{C}^*}(\phi)$ by uniqueness, see Equation (18). Hence $W_{\mathcal{C}^*} = A_{\Lambda} e^{i\pi N_{\Lambda}} e^{-i\pi(N_{\Lambda} - \mathcal{K}_{\mathcal{C}^*})}$. Moreover, $[N_{\Lambda} - \mathcal{K}_{\mathcal{C}^*}, P] = 0$, see Equation (129), implies that

$$[e^{-i\pi(N_{\Lambda} - \mathcal{K}_{\mathcal{C}^*})}, P] = 0. \quad (133)$$

With this, the identity $e^{i\pi N_{\Lambda}} = (-1)^{N_{\Lambda}}$ and the definition of charge in Equation (15) yield

$$W_{\mathcal{C}^*} P W_{\mathcal{C}^*}^* = Q_{\Lambda} P Q_{\Lambda} = P \quad (134)$$

since P is a spectral projector of a gauge invariant Hamiltonian, see Equation (2). $\blacksquare$

While the usefulness of the identity $V_{\mathcal{C}^*}(\phi) = e^{i\phi N_{\Lambda}} e^{-i\phi(N_{\Lambda} - \mathcal{K}_{\mathcal{C}^*})}$ was pointed out in [Bac+20] because the unitary simplifies at $\phi = 2\pi$ by integrality of the spectrum of N_{Λ} , we are interested in applying it here up to only $\phi = \pi$. Physically, this corresponds to a process in which we insert a π -flux with an external $U(1)$ gauge field which is absorbed by the system as a $\mathbb{Z}_2$ gauge transformation, leaving thus the ground state space invariant. This justifies the inclusion of a $\mathbb{Z}_2$ pure gauge transformation $A_{\Lambda} = \hat{X}_{\mathcal{C}^*}$ to $V_{\mathcal{C}^*}(\phi)$ in Definition 75. Of course, this cancellation of the two types of ‘gauge transformation’ is possible only at $\phi = \pi$, since the gauge field is $\mathbb{Z}_2$ -valued, see also the proof just below. This concludes the proof of Item 1 of Theorem 6 in this case.

Since the ground state manifold is degenerate, an adiabatic process preserving such subspace may in principle shuffle its basis in a non universal way. However this does not happen as a consequence of the protection given by topology. As Theorem 5 shows, see Remark 56(2), ground states of H are labelled by pairs $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ of holonomies around two fixed cycles $\mathcal{C}_1, \mathcal{C}_2 \in Z_1(\Lambda_L)$ that are not boundaries:

$$\hat{Z}_{\mathcal{C}_1} |\Omega_{ab}\rangle = a |\Omega_{ab}\rangle, \quad \hat{Z}_{\mathcal{C}_2} |\Omega_{ab}\rangle = b |\Omega_{ab}\rangle. \quad (135)$$

With this, we first recall that $\hat{Z}_{\mathcal{C}_j}$ is gauge invariant because $\mathcal{C}_j$ are cycles, see Equation (7), and note further that they commute with $e^{-i\pi(N_{\Lambda} - \mathcal{K}_{\mathcal{C}^*})}$ since this term is diagonal in $\hat{\sigma}_{ij}^z$ basis. Hence they commute with $W_{\mathcal{C}^*}$. This proves Item 2 of Theorem 6 in this case of ‘contractible cycles’ since the intersection number between $\mathcal{C}$ and $\mathcal{C}^*$ must be even, see Figure 7.11. Moreover,

$$\hat{Z}_{\mathcal{C}_1} W_{\mathcal{C}^*} |\Omega_{ab}\rangle = a W_{\mathcal{C}^*} |\Omega_{ab}\rangle, \quad \hat{Z}_{\mathcal{C}_2} W_{\mathcal{C}^*} |\Omega_{ab}\rangle = b W_{\mathcal{C}^*} |\Omega_{ab}\rangle. \quad (136)$$

Since $W_{\mathcal{C}^*}$ preserves the ground state manifold and the eigenvalues (a, b) uniquely determine the state $|\Omega_{ab}\rangle$, we conclude that

$$W_{\mathcal{C}^*} |\Omega_{ab}\rangle = e^{i\omega_{\mathcal{C}^*}^{ab}} |\Omega_{ab}\rangle \quad (137)$$

for some phase $\omega_{\mathcal{C}^*}^{ab}$ which is in principle dependent on the curve and specific ground state we are acting on, and on the system's size. This concludes the proof of [Item 3](#) of the [Theorem 6](#) for the trivial cohomology class.

With the triviality of $W_{\mathcal{C}^*}$ associated with 1-coboundaries in hand, we now turn to the case of where $\mathcal{C}^*$ are cocycles that are not coboundaries. This will allow us to derive braiding relations.

Threading a π -Holonomy through a Non-Contractible Cycle In differential geometric terms, [Equation \(17\)](#) along non-trivial cocycles describes parallel transport of the Fermi projector around the Jacobian manifold of the (twisted) boundary conditions on the torus since inserting an holonomy around a non-contractible cycle corresponds to changing boundary conditions to the fermions (see [Figure 7.2](#)).

In this case, [Lemma 78](#) cannot be used and the proof of [Lemma 79](#) does not hold. We claim nonetheless that $W_{\mathcal{C}^*}$ still preserve the ground state space: This is the content of the lemma just below. Although its proof is a little technical, the idea is simple: We consider another $\mathcal{C}'^* \in C_1(\Lambda_L^*)$ such that $\mathcal{C}^* - \mathcal{C}'^*$ is a coboundary and such that $\text{dist}(\mathcal{C}^*, \mathcal{C}'^*)$ is of order L . Then, by locality, we prove that (see [Equation \(144\)](#) below) $W_{\mathcal{C}^* - \mathcal{C}'^*} \stackrel{L}{=} W_{\mathcal{C}^*} W_{\mathcal{C}'^*}$. Since the left hand side preserves the ground state space, and all operators are unitary, decay of correlations yields the claim, which is that of [Item 1](#) of [Theorem 6](#):

Lemma 80. *For any $\mathcal{C}^* \in C_1(\Lambda_L^*)$, the operator $W_{\mathcal{C}^*}$ satisfies $\| [W_{\mathcal{C}^*}, P] \| \stackrel{L}{=} 0$.*

Proof. Let $\mathcal{C}'^*$ be another cycle such that

1. $\text{dist}(\mathcal{C}^*, \mathcal{C}'^*) = \rho L$ for some $0 < \rho < 1$,
2. $\mathcal{C}^* - \mathcal{C}'^*$ is a coboundary, namely $\mathcal{C}^* - \mathcal{C}'^* = \partial\Lambda$ for some $\Lambda \subset \Lambda_L$.

see [Figure 7.12](#). [Lemma 79](#) implies that

$$[W_{\mathcal{C}^* - \mathcal{C}'^*}, P] = 0. \quad (138)$$

Since $\dot{H}_{\mathcal{C}^* - \mathcal{C}'^*}(\phi) = \dot{H}_{\mathcal{C}^*}(\phi) + \dot{H}_{-\mathcal{C}'^*}(\phi)$, [Definition 74](#) of the quasi-adiabatic generator yields

$$\mathcal{K}_{\mathcal{C}^* - \mathcal{C}'^*}(\phi) = \tau_f^\phi(\dot{H}_{\mathcal{C}^*}(\phi)) + \tau_f^\phi(\dot{H}_{-\mathcal{C}'^*}(\phi)) \stackrel{L}{=} \mathcal{K}_{\mathcal{C}^*}(\phi) + \mathcal{K}_{-\mathcal{C}'^*}(\phi). \quad (139)$$

The second identity amounts to replacing $\tau_s^{H_{\mathcal{C}^* - \mathcal{C}'^*}(\phi)}(\dot{H}_{\mathcal{C}^*}(\phi))$ with $\tau_s^{H_{\mathcal{C}^*}(\phi)}(\dot{H}_{\mathcal{C}^*}(\phi))$ and similarly with the second term. Here and in the following, we use the notation $\tau_s^A(\mathcal{O}) = e^{isA} \mathcal{O} e^{-isA}$ for any self-adjoint operator A and observable $\mathcal{O}$. This is a consequence of the fact that $\dot{H}_{\mathcal{C}^*}$ is supported on the sites adjacent to $\mathcal{C}^*$ and of the following corollary of the Lieb-Robinson bound (see for example [[Bac+18](#), Lemma 2.2] and the discussion in [Remark 15 - Item 1](#)):

$$\| \tau_t^{H_{X^r}}(\mathcal{O}) - \tau_t^{H_{\Lambda_L}}(\mathcal{O}) \| \leq \int_0^t \left\| [(H_{\Lambda_L} - H_{X^r}), \tau_s^{H_{\Lambda_L}}(\mathcal{O})] \right\| ds \leq C(\mathcal{O}) e^{-c(r-vt)}, \quad (140)$$

where $C(\mathcal{O})$ depends on the norm of $\mathcal{O}$ and the size of its support $X \subset \Lambda_L$, for any $\mathcal{O}$ and any $r \geq 0$, where $X^r = \{x \in \Lambda_L : \text{dist}(x, X) \leq r\}$. In the current application, $\mathcal{O} = \dot{H}_{\mathcal{C}^*}$ whose support has volume $2L$. This linear growth is however controlled by the exponential decay obtained by choosing r proportional to L , which is the reason for Property 1 of $\mathcal{C}'^*$. Note that the error in [Equation \(139\)](#) is only superpolynomial because it is dominated by the tail of the integral defining τ_f^ϕ , see [[Bac+18](#), Section 3.2] for details. By the Lieb-Robinson bound again, the operator $\mathcal{K}_{\mathcal{C}^*}$ is almost localized on $\mathcal{C}^*$ which implies that

$$[\mathcal{K}_{\mathcal{C}^*}, \mathcal{K}_{-\mathcal{C}'^*}] \stackrel{L}{=} 0. \quad (141)$$

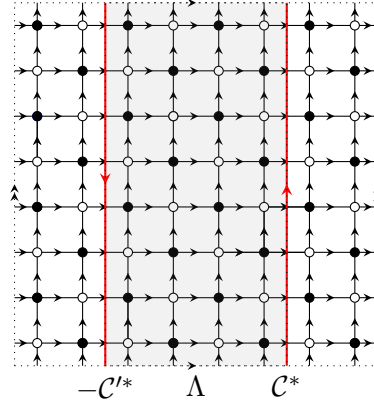


Figure 7.12: The coboundary $\partial\Lambda = \mathcal{C}^* - \mathcal{C}'^*$. The sign is given by the orientation of the dual lattice Λ_L^* .

Recalling in Equation (18) that $V_{\mathcal{C}^*}(\phi)$ is the propagator generated by $\mathcal{K}_{\mathcal{C}^*}(\phi)$, Equation (141) yields that $[V_{\mathcal{C}^*}, \mathcal{K}_{-\mathcal{C}'^*}] \stackrel{L}{=} 0$ and in turn that

$$-i \frac{d}{d\phi} (V_{\mathcal{C}^* - \mathcal{C}'^*} V_{\mathcal{C}^*} V_{-\mathcal{C}'^*}) \stackrel{L}{=} V_{\mathcal{C}^* - \mathcal{C}'^*} (\mathcal{K}_{\mathcal{C}^* - \mathcal{C}'^*} - \mathcal{K}_{\mathcal{C}^*} - \mathcal{K}_{-\mathcal{C}'^*}) V_{\mathcal{C}^*} V_{-\mathcal{C}'^*}. \quad (142)$$

This relation and identity in Equation (139) then imply

$$V_{\mathcal{C}^* - \mathcal{C}'^*}(\phi) \stackrel{L}{=} V_{\mathcal{C}^*}(\phi) V_{-\mathcal{C}'^*}(\phi). \quad (143)$$

Since, moreover, $\hat{X}_{\mathcal{C}^* - \mathcal{C}'^*} = \hat{X}_{\mathcal{C}^*} \hat{X}_{-\mathcal{C}'^*}$, we immediately conclude that

$$W_{\mathcal{C}^* - \mathcal{C}'^*} \stackrel{L}{=} \hat{X}_{\mathcal{C}^*} \hat{X}_{-\mathcal{C}'^*} V_{\mathcal{C}^*}(\pi) V_{-\mathcal{C}'^*}(\pi) \stackrel{L}{=} W_{\mathcal{C}^*} W_{-\mathcal{C}'^*}. \quad (144)$$

In the second equality, we used that $\hat{X}_{-\mathcal{C}'^*}^*$ and $V_{\mathcal{C}^*}(\pi)$ almost commutes because of their almost disjoint supports.

It remains to prove that this and $[W_{\mathcal{C}^* - \mathcal{C}'^*}, P] = 0$ implies that each of the unitary factors $W_{\mathcal{C}^*}, W_{-\mathcal{C}'^*}$ almost commutes with P . As we shall explain just below, this follows from the fact that $\mathcal{C}^*, \mathcal{C}'^*$ are separated by distance of order L and by clustering:

$$\|PAB|\Omega\rangle - PAPB|\Omega\rangle\| \leq C(A, B)f(d(X, Y)) \quad (145)$$

which holds for any normalized ground state Ω in the range of P and any observables A and B supported on X and Y , respectively. Here, $d(X, Y)$ is the distance between the two sets and f is a superpolynomially decaying function (see Remark 16 - Item 2 and [Bac+21, Proposition 3.1]). Since $[P, W_{\mathcal{C}^*}] = ((1 - P)W_{\mathcal{C}^*}P)^* - (1 - P)W_{\mathcal{C}^*}P$, it suffices to show that $\|(1 - P)W_{\mathcal{C}^*}P\| \stackrel{L}{=} 0$ and similarly for the adjoint. For this, we note that

$$1 \geq \|PW_{\mathcal{C}^*}|\Omega\rangle\| \geq \|PW_{-\mathcal{C}'^*}\| \|PW_{\mathcal{C}^*}|\Omega\rangle\| \geq \|PW_{-\mathcal{C}'^*}PW_{\mathcal{C}^*}|\Omega\rangle\| \quad (146)$$

$$\stackrel{L}{=} \|PW_{-\mathcal{C}'^*}W_{\mathcal{C}^*}|\Omega\rangle\| \stackrel{L}{=} 1, \quad (147)$$

where we used clustering, Equation (145), in the second line. More precisely, we applied Equation (145) for $A = W_{-\mathcal{C}'^*}$, $B = W_{\mathcal{C}^*}$, approximately localized in $\mathcal{C}'^*$, $\mathcal{C}^*$. This implies that Equation (145) holds with a constant $C(A, B)$ that grows quadratically with L , which is however controlled by the superpolynomial decay of $f(\cdot)$ and by the order L distance of the two sets. Since this holds for all $|\Omega\rangle = P|\Omega\rangle$, we conclude that $\|(1 - P)W_{\mathcal{C}^*}P\| \stackrel{L}{=} 0$. Repeating the argument with the adjoint yields the claim. $\blacksquare$

We turn to the commutation relations. Let $\mathcal{C} \in Z_1(\Lambda_L)$. Since

$$\hat{Z}_{\mathcal{C}} \hat{X}_{\mathcal{C}^*} = e^{i\pi \mathcal{I}(\mathcal{C}, \mathcal{C}^*)} \hat{X}_{\mathcal{C}^*} \hat{Z}_{\mathcal{C}}, \quad (148)$$

we have that the operator $B_{\mathcal{C}, \mathcal{C}^*}(\phi) = \hat{Z}_{\mathcal{C}} W_{\mathcal{C}^*}(\phi)^* \hat{Z}_{\mathcal{C}} W_{\mathcal{C}^*}(\phi)$ satisfies the equation

$$-i \frac{d}{d\phi} B_{\mathcal{C}, \mathcal{C}^*}(\phi) = 0 \quad (149)$$

since $\hat{Z}_{\mathcal{C}}$ commutes with $\mathcal{K}_{\mathcal{C}^*}$ because the latter contains only $\hat{\sigma}^z$ operators. Moreover, $B_{\mathcal{C}, \mathcal{C}^*}(0) = \hat{Z}_{\mathcal{C}} \hat{X}_{\mathcal{C}^*} \hat{Z}_{\mathcal{C}} \hat{X}_{\mathcal{C}^*} = e^{i\pi \mathcal{I}(\mathcal{C}, \mathcal{C}^*)}$, so that

$$\hat{Z}_{\mathcal{C}} W_{\mathcal{C}^*}(\phi) = e^{i\pi \mathcal{I}(\mathcal{C}, \mathcal{C}^*)} W_{\mathcal{C}^*}(\phi) \hat{Z}_{\mathcal{C}} \quad (150)$$

for all ϕ , a fortiori at $\phi = \pi$ (which is the only case of interest here since $W_{\mathcal{C}^*}(\phi)$ preserves the ground state space only at that particular value). This concludes the proof of [Item 2](#) of [Theorem 6](#). The remaining cases of [Item 3](#) of [Theorem 6](#) follow immediately from this and from the fact that the intersection number of a non-trivial cycle and a non-trivial cocycle is necessarily odd, see [Figure 7.11](#). Therefore

$$\hat{Z}_{\mathcal{C}_1} W_{\mathcal{C}_2^*} |\Omega_{ab}\rangle = -a W_{\mathcal{C}_2^*} |\Omega_{ab}\rangle \quad (151)$$

from which we conclude that

$$W_{\mathcal{C}_2^*} |\Omega_{ab}\rangle \stackrel{L}{=} e^{i\omega_{\mathcal{C}_2^*}^{ab}} |\Omega_{(-a)b}\rangle. \quad (152)$$

The argument is similar with the other possible combinations of cycles and cocycles.

Monopoles are bosons For this section, we denote $H(\phi_1, \phi_2)$ the Hamiltonian twisted by ϕ_1 along $\mathcal{C}_1^*$ and by ϕ_2 along $\mathcal{C}_2^*$. By [Item 1](#) of [Theorem 4](#), this family is uniformly gapped for $(\phi_1, \phi_2) \in \mathbb{T}^2$, with ground state projection $P(\phi_1, \phi_2)$. We let

$$\mathcal{K}_j(\phi_1, \phi_2) = \int f(t) e^{itH(\phi_1, \phi_2)} \partial_{\phi_j} H(\phi_1, \phi_2) e^{-itH(\phi_1, \phi_2)} dt. \quad (153)$$

where f is the function introduced in [Definition 74](#). We denote $\hat{X}_j \equiv \hat{X}_{\mathcal{C}_j^*}$. Then $\hat{X}_1 H(\phi_1, \phi_2) \hat{X}_1 = H(\phi_1 + \pi, \phi_2)$ so that

$$\hat{X}_1 \mathcal{K}_j(\phi_1, \phi_2) \hat{X}_1 = \mathcal{K}_j(\phi_1 + \pi, \phi_2) \quad (154)$$

and similarly for $\hat{X}_2$ with respect to ϕ_2 . Let $\gamma : [0, 1] \rightarrow \mathbb{T}^2$, $s \mapsto (\gamma_1(s), \gamma_2(s))$ be a path and let $V_\gamma(s)$ be the unitary solution of

$$V_\gamma(s) = \mathbf{1} + i \int_0^s \mathcal{K}(\gamma(r)) \cdot \gamma'(r) V_\gamma(r) dr \quad (155)$$

where

$$\mathcal{K}(\gamma(s)) \cdot \gamma'(s) = \mathcal{K}_1(\gamma(s)) \gamma_1'(s) + \mathcal{K}_2(\gamma(s)) \gamma_2'(s). \quad (156)$$

Moreover, [Equation \(154\)](#) implies that

$$\hat{X}_j V_\gamma(s) = V_{\pi e_j + \gamma}(s) \hat{X}_j \quad (157)$$

where $(\pi e_1 + \gamma)(s) = (\pi, 0) + \gamma(s)$ and similarly for $j = 2$.

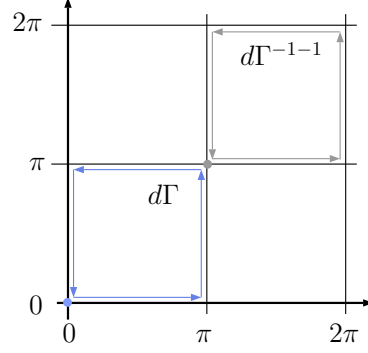


Figure 7.13: In blue, the path $\partial\gamma$ corresponding to the braiding of monopoles on the ground state space (see Equation (161)). Changing the holonomies amounts to changing the initial condition with the grey path corresponding to the vector $\Omega_{-1,-1}$, starting at $(\phi_1, \phi_2) = (\pi, \pi)$.

For any path γ , we shall denote $V_\gamma = V_\gamma(1)$. With this, we recall the Definition 75 of loop operators, observe that $W_{C_j^*} = \hat{X}_j V_{\alpha_j}$, where $\alpha_1(s) = (s\pi, 0)$ and $\alpha_2(s) = (0, s\pi)$ and conclude that

$$(W_{C_2^*})^*(W_{C_1^*})^*W_{C_2^*}W_{C_1^*} = V_{\alpha_2}^*\hat{X}_2V_{\alpha_1}^*\hat{X}_1\hat{X}_2V_{\alpha_2}\hat{X}_1V_{\alpha_1} \quad (158)$$

$$= V_{\alpha_2}^*V_{\pi e_2+\alpha_1}^*V_{\pi e_1+\alpha_2}V_{\alpha_1} = V_{\partial\gamma} \quad (159)$$

where we used Equation (157) and observed that $V_\gamma^* = V_{-\gamma}$ as well as $V_\gamma V_{\gamma'} = V_{\gamma+\gamma'}$ to conclude that the product reduces to the propagator along the boundary of the square Λ drawn in Figure 7.13.

By the gauge invariance of the Hamiltonian,

$$V_\gamma(s)|\Omega_{ab}\rangle = \prod_{i \in \Lambda_L} \left(\frac{1+Q_i}{2} \right) V_{\gamma^{ab}}^\sigma(s) |\psi_{\sigma,a,b}\rangle \otimes |\sigma, a, b\rangle, \quad (160)$$

where $\gamma^{ab}(s) = (\frac{1}{2}(1-a)\pi, \frac{1}{2}(1-b)\pi) + \gamma(s)$ is the shifted path, σ is any configuration in the class $[-1]$ and $V_{\gamma^{ab}}^\sigma(s)$ is defined as $V_\gamma(s)$ but with the Hamiltonian in a fixed flux configuration σ . Since $V_\gamma(s)$ is the spectral flow and $|\psi_{\sigma,a,b}\rangle$ is the unique, gapped, fermionic ground state, the vector $V_{\gamma^{ab}}^\sigma(s)|\psi_{\sigma,a,b}\rangle$ is a fermionic ground state at fluxes $\gamma^{ab}(s)$. By Equation (152), $\langle \Omega_{ab} | (W_{C_2^*})^*(W_{C_1^*})^*W_{C_2^*}W_{C_1^*} \Omega_{ab} \rangle$ is a phase, which we denote $e^{i\omega_{ab}}$, and

$$e^{i\omega_{ab}} \stackrel{L}{=} \langle \Omega_{ab} | (W_{C_2^*})^*(W_{C_1^*})^*W_{C_2^*}W_{C_1^*} \Omega_{ab} \rangle = \langle \psi_{\sigma,a,b} | V_{\partial\gamma^{ab}}^\sigma \psi_{\sigma,a,b} \rangle \quad (161)$$

Let $P^\sigma(\phi_1, \phi_2)$ be the ground state projection of $H(\sigma; \phi_1, \phi_2)$. On the cut torus, there is a smooth family $|\psi^\sigma(\phi_1, \phi_2)\rangle$ of vectors such that $P^\sigma(\phi_1, \phi_2) = |\psi^\sigma(\phi_1, \phi_2)\rangle\langle\psi^\sigma(\phi_1, \phi_2)|$. Upon possibly redefining their phases, we can assume that $|\psi_{\sigma,a,b}\rangle = |\psi^\sigma(\frac{1}{2}(1-a)\pi, \frac{1}{2}(1-b)\pi)\rangle$. Let now

$$|\varphi^\sigma(s)\rangle = V_{\partial\gamma}^\sigma(s)|\psi^\sigma(0)\rangle, \quad (162)$$

where we have now dropped the index ab since it amounts to a choice of one of $(0,0)$, $(\pi,0)$, $(0,\pi)$ or (π,π) as initial condition $\partial\gamma(0)$. The phase of Equation (161) is now given by

$$e^{i\omega} = \langle \psi^\sigma(0) | \varphi^\sigma(1) \rangle. \quad (163)$$

For any $s \in [0, 1]$, there is a $c(s) \in \mathbb{C}$, $|c(s)| = 1$ such that

$$|\varphi^\sigma(s)\rangle = c(s)|\psi^\sigma(s)\rangle. \quad (164)$$

Taking the derivative along the path yields

$$i\mathcal{K}^\sigma(s) \cdot \partial\gamma(s)|\varphi^\sigma(s)\rangle = c'(s)|\psi^\sigma(s)\rangle + c(s)\frac{d}{ds}|\psi^\sigma(s)\rangle. \quad (165)$$

We now take $\langle\psi^\sigma(s)|\cdot\rangle$ to get

$$0 = c'(s) + c(s) \left\langle \psi^\sigma(s) \left| \frac{d}{ds} \psi^\sigma(s) \right. \right\rangle, \quad (166)$$

where we used $\langle\psi^\sigma(s)|\mathcal{K}^\sigma(s)\psi^\sigma(s)\rangle = 0$. Equivalently,

$$\frac{d}{ds} \log c(s) = - \left\langle \psi^\sigma(s) \left| \frac{d}{ds} \psi^\sigma(s) \right. \right\rangle. \quad (167)$$

Hence, since ω is the total change of argument of the function $c(s)$ along the path $\partial\Lambda$,

$$i\omega = - \int_{\partial\gamma} \langle\psi^\sigma|d\psi^\sigma\rangle. \quad (168)$$

We are now in the setting of [ASS83] and so an application of Stokes' theorem yields

$$\int_{\partial\gamma} \langle\psi^\sigma|d\psi^\sigma\rangle = \int_{\gamma} d\langle\psi^\sigma|d\psi^\sigma\rangle = \int_{\gamma} \langle d\psi^\sigma|d\psi^\sigma\rangle = \int_{\gamma} \text{Tr}(dP^\sigma P^\sigma dP^\sigma) \quad (169)$$

$$= \int_{\gamma} \text{Tr}(P^\sigma dP^\sigma \wedge dP^\sigma) \quad (170)$$

namely the braiding phase is the integral of the adiabatic curvature, or equivalently the Hall conductance, over the quarter torus. Since $dP^\sigma = i[\mathcal{K}^\sigma, P^\sigma]$, we conclude that

$$\omega = -i \int_{\Lambda} \text{Tr}(P^\sigma [\mathcal{K}_1^\sigma, \mathcal{K}_2^\sigma]) \quad (171)$$

By [HM15; Bac+18], the curvature is constant. Since the initial fermionic ground states are all time-reversal invariant, the Hall conductance vanishes and so $\omega = 0$, concluding the proof of [Theorem 6, Item 4](#).

Remark 62. The calculation above is in fact completely general under the assumption of a spectral gap. Interestingly, it relates the braiding phase of the dressed monopoles to the Hall conductance of the fermion sector, see also [Bac+20; KS20].

Braiding statistics between monopoles and fermions Let

$$B = \frac{\langle \xi_{ab}^{ij} | W_{\mathcal{C}^*} \xi_{ab}^{ij} \rangle}{\langle \Omega_{ab} | W_{\mathcal{C}^*} \Omega_{ab} \rangle}. \quad (172)$$

As discussed above $|\xi_{ab}^{ij}\rangle$ is obtained creating two fermions on Ω_{ab} , and B is the braiding between one of the fermions and a monopole, see also [Figure 7.14](#).

Since $\mathcal{C}^*$ is a coboundary, the denominator has already been shown (see [Equation \(137\)](#)) to be a phase:

$$\langle \Omega_{ab} | W_{\mathcal{C}^*} \Omega_{ab} \rangle = e^{i\omega_{\mathcal{C}^*}^{ab}}. \quad (173)$$

We claim that

$$\left\| \left\{ W_{\mathcal{C}^*}, a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^+ \right\} \right\| \stackrel{L}{=} 0. \quad (174)$$

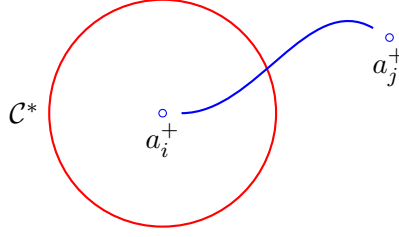


Figure 7.14: A representation of the braiding B . Any choice of the curve connecting i and j crosses $\mathcal{C}^*$ an odd number of times giving a $-$ sign.

With this and $W_{\mathcal{C}^*}|\Omega_{ab}\rangle = e^{i\omega_{\mathcal{C}^*}^{ab}}|\Omega_{ab}\rangle$, we have that

$$B \stackrel{L}{=} -e^{i\omega_{\mathcal{C}^*}^{ab}} \frac{\langle \xi_{ab}^{ij} | \xi_{ab}^{ij} \rangle}{\langle \Omega_{ab} | W_{\mathcal{C}^*} \Omega_{ab} \rangle} = -1 \quad (175)$$

indeed. To prove Equation (174), we recall that $W_{\mathcal{C}^*}$ is a fermion-even operator and almost localized along $\mathcal{C}^*$, so that

$$\|[W_{\mathcal{C}^*}, a_{i,j}^+]\| \stackrel{L}{=} 0. \quad (176)$$

Hence,

$$W_{\mathcal{C}^*} a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^+ \stackrel{L}{=} a_{i,\eta}^+ W_{\mathcal{C}^*} \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^+ = -a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} W_{\mathcal{C}^*} a_{j,\eta'}^+ \stackrel{L}{=} -a_{i,\eta}^+ \hat{Z}_{\mathcal{C}_{i,j}} a_{j,\eta'}^+ W_{\mathcal{C}^*} \quad (177)$$

where we used Equation (148) and the fact that $Z_{\mathcal{C}_{i,j}}$ commutes with $\mathcal{K}_{\mathcal{C}^*}(\phi)$ in the second equality. This concludes the proof Theorem 6-Item 5. $\blacksquare$

Remark 63.

1. Up to a phase, the vector $|\xi_{ab}^{ij}\rangle$ can be represented as:

$$\left(\prod_{i \in V(\Lambda_L)} \frac{1 + Q_i}{2} \right) a_{i,\eta}^+ a_{j,\eta'}^+ |\psi_{-\mathbf{1},a,b}\rangle \otimes |-\mathbf{1}, a, b\rangle. \quad (178)$$

Indeed, $\hat{Z}_{\mathcal{C}_{i,j}} |\psi_{-\mathbf{1},a,b}\rangle \otimes |-\mathbf{1}, a, b\rangle = Z_{\mathcal{C}_{i,j}} |\psi_{-\mathbf{1},a,b}\rangle \otimes |-\mathbf{1}, a, b\rangle$, where $Z_{\mathcal{C}_{i,j}}$ is just a phase. While neither $a_i^+ a_j^+$ nor $\hat{Z}_{\mathcal{C}_{i,j}}$ commute with the $\mathbb{Z}_2$ -charges Q_i , their combination does. Pulling $\hat{Z}_{\mathcal{C}_{i,j}}$ through the projection onto the physical Hilbert space yields the claim.

2. The braiding is the same if one or both of the creation operators are replaced by annihilation operators, because the braiding only takes into account the fermion parity of the excitation, not its $U(1)$ charge.
3. The proof shows explicitly that the braiding is independent of the choice of (a, b) as one would have expected from the topological order condition and the locality of the braiding operation.

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