Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:53
https://doi.org/10.1007/s13398-024-01551-w

ORIGINAL PAPER

Seiberg–Witten differentials on the Hitchin base

Ugo Bruzzo[1,2,3,4] · **Peter Dalakov**[5,6]

Received: 22 February 2023 / Accepted: 2 January 2024

Abstract
In this note we describe explicitly, in terms of Lie theory and cameral data, the covariant (Gauss–Manin) derivative of the Seiberg–Witten differential defined on the weight-one variation of Hodge structures that exists on a Zariski open subset of the base of the Hitchin fibration.

Keywords Hitchin system · Hitchin base · Cameral covers · Donagi–Markman cubic · Balduzzi–Pantev formula · Seiberg–Witten differential

Mathematics Subject Classification 14D20 · 14D07 · 14H70

Contents

Dedicated to Tony Pantev on the occasion of his 60th birthday.

✉ Ugo Bruzzo
bruzzo@sissa.it

1 SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy

2 Departamento de Matemática, Universidade Federal da Paraíba, João Pessoa, Brazil

3 INFN (Istituto Nazionale di Fisica Nucleare), Sezione di Trieste, Trieste, Italy

4 IGAP (Institute for Geometry and Physics), Trieste, Italy

5 American University in Bulgaria, G.Izmirliev Sq.1, 2700 Blagoevgrad, Bulgaria

6 Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Published online: 29 January 2024

1 Introduction

The base of the Hitchin integrable system [14] supports a family of cameral curves, and, as a consequence, carries various Hodge-theoretic and differential-geometric structures [6, 7, 20]. In particular, the Zariski open subset of the base, corresponding to smooth cameral curves with generic ramification carries a weight-one variation of Hodge structures (VHS) with a Seiberg–Witten differential. Our goal in this note is to describe the covariant (Gauss–Manin) derivative of the Seiberg–Witten differential explicitly in terms of Lie theory and cameral data.

We recall now the main ingredients and constructions, starting with the Hodge-theoretic ones.

Let $\mathcal{B}$ be a complex manifold. Recall that a polarised $\mathbb{R}$-VHS of weight $w \in \mathbb{Z}$ on $\mathcal{B}$ consists of data $(\mathcal{V}, \nabla, \mathcal{V}_{\mathbb{R}}, \mathcal{F}^{\bullet}, S)$, where:

- $\mathcal{V}$ is a holomorphic vector bundle on $\mathcal{B}$
- $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathcal{B}}$ is a flat (holomorphic) connection, called *the Gauss–Manin connection*
- $\mathcal{V}_{\mathbb{R}} \subseteq \mathcal{V}$ is a real, ∇-flat subbundle, satisfying $\mathcal{V} = \mathcal{V}_{\mathbb{R}} \otimes \mathcal{O}_{\mathcal{B}}$, called *real structure*
- $\mathcal{F}^{\bullet}$ is a decreasing *filtration* of $\mathcal{V} = \mathcal{F}^0$, *the Hodge filtration*
- $S : \mathcal{V} \otimes \mathcal{V} \to \mathscr{C}^{\infty}_{\mathcal{B}}$ is a non-degenerate, $(-1)^w$-symmetric, ∇-flat pairing, $\mathbb{R}$-valued on $\mathcal{V}_{\mathbb{R}}$, called *polarisation*

such that

(1) $\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \Omega^1_{\mathcal{B}}$ *Griffiths transversality*
(2) $\mathcal{V} = \mathcal{F}^p \oplus \overline{\mathcal{F}^{w+1-p}}$ *Hodge structure*,

or, in terms of the Hodge bundles $\mathcal{H}^{p,w-p} := \mathcal{F}^p \cap \overline{\mathcal{F}^{w-p}}$,

(2') $\mathcal{V} = \bigoplus_p \mathcal{H}^{p,w-p}$
(3) $S(\mathcal{F}^p, \mathcal{F}^{w+1-p}) = 0$
(4) $i^{2p-w} S(v, \overline{v}) > 0$ for $v \in \Gamma(\mathcal{H}^{p,w-p})$, $v \neq 0$.

The notions of polarised $\mathbb{Z}$-VHS or $\mathbb{Q}$-VHS are introduced analogously, by replacing $\mathcal{V}_{\mathbb{R}}$ with appropriate locally constant sheaves $\mathcal{V}_{\mathbb{Z}}$ or $\mathcal{V}_{\mathbb{Q}}$ of $\mathbb{Z}$- or $\mathbb{Q}$-modules, respectively.

The prototypical example is that of a geometric VHS, i.e., one arising from a family of compact Kähler (e.g., projective) manifolds.

By Griffiths Transversality, ∇ induces an $\mathcal{O}_B$-module homomorphism

$$\mathcal{F}^p/\mathcal{F}^{p+1} \longrightarrow \mathcal{F}^{p-1}/\mathcal{F}^p \otimes \Omega^1_{\mathcal{B}}$$

and hence, taking a direct sum over the different p, an $\mathcal{O}_B$-module homomorphism

$$\theta = [\nabla] : \bigoplus_p \mathcal{F}^p/\mathcal{F}^{p+1} \longrightarrow \left(\bigoplus_p \mathcal{F}^p/\mathcal{F}^{p+1} \right) \otimes \Omega^1_{\mathcal{B}},$$

which satisfies $\theta \wedge \theta = 0$.

The pair $\left(E = \bigoplus_p \mathcal{F}^p/\mathcal{F}^{p+1}, \theta\right)$ is an example of a *Higgs bundle* on $\mathcal{B}$. This example played an important rôle in Carlos Simpson's study of Higgs bundles on higher-dimensional varieties [22, 23].

Consider a polarised $\mathbb{Z}$-VHS $(\mathcal{V}, \nabla, \mathcal{V}_{\mathbb{Z}}, \mathcal{V}^{\bullet}, S, \ldots)$ of weight $w = 1$. An *abstract Seiberg–Witten differential* on it is a section $\boldsymbol{\lambda}_{SW} \in H^0(\mathcal{B}, \mathcal{V}^1)$, for which the $\mathcal{O}_{\mathcal{B}}$-module homomorphism

$$T_{\mathcal{B}} \longrightarrow \mathcal{V}^0, \quad v \longmapsto \nabla_v \boldsymbol{\lambda}_{SW}$$

factors through an *isomorphism*

$$T_{\mathcal{B}} \simeq \mathcal{V}^1. \tag{1}$$

Given such data, we obtain a refinement of the weight-1 filtration

$$\mathcal{V}^1 \subseteq \mathcal{V}^0$$

to a weight-3 filtration

$$\underbrace{\mathcal{F}^3}_{=\boldsymbol{\lambda}_{SW}\mathcal{O}_{\mathcal{B}}} \subseteq \underbrace{\mathcal{F}^2}_{\mathcal{V}^1} \subseteq \underbrace{\mathcal{F}^1}_{=(\mathcal{F}^3)^{\perp}} \subseteq \underbrace{\mathcal{F}^0}_{\mathcal{V}^0}.$$

For links to projective special Kähler geometry ("$N = 2$ supergravity") and weight-3 VHS, satisfying the Calabi–Yau condition, one can check [13, §4, §8.3].

Furthermore, given such data, there is an associated fibration of complex tori $\mathcal{J} := \mathcal{V}/(\mathcal{V}^1 + \mathcal{V}_{\mathbb{Z}}) \to \mathcal{B}$, whose vertical bundle is $\mathrm{Vert} = \mathcal{V}/\mathcal{V}^1$. The polarisation S gives rise to an isomorphism $\mathrm{Vert} \simeq (\mathcal{V}^1)^{\vee}$, and hence $\boldsymbol{\lambda}_{SW}$ induces, by composition with the dual of its defining isomorphism $T_{\mathcal{B}} \simeq \mathcal{V}^1$, an isomorphism $i_{\boldsymbol{\lambda}} : \mathrm{Vert} \to T_{\mathcal{B}}^{\vee}$. Such an isomorphism is also induced by a choice of symplectic form on $\mathcal{J}$. There is unique symplectic form $\omega_{\boldsymbol{\lambda}}$ on $\mathcal{J}$, which induces $i_{\boldsymbol{\lambda}}$ and such that the 0-section is Lagrangian.

We next recall the construction of the family of cameral covers over the Hitchin base, and introduce a weight-1 VHS with a Seiberg–Witten differential on it.

First, we fix the following data:

- A simple complex Lie group G of rank l, together with a choice of Borel and Cartan subgroups $T \subset B \subset G$. We denote by $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ the respective Lie algebras and by W the corresponding Weyl group.
- A compact (connected) Riemann surface X of genus $g \geq 2$ (or equivalently, a non-singular proper algebraic curve over $\mathbb{C}$). We do not need to fix a particular projective embedding of X.

Additionally, we choose:

- Homogeneous generators $I_1, \ldots, I_l$ of the ring $\mathbb{C}[\mathfrak{t}]^W \subset \mathbb{C}[\mathfrak{t}]$. We write $d_k = \deg I_k$.
- Simple (positive) roots $\{\alpha_1, \ldots, \alpha_l\}$.

These additional choices are not necessary for the entire discussion, but are needed for the explicit calculation in Theorem A.

Two explicit examples of invariant polynomials—for $SL_3(\mathbb{C})$ and G_2—are given in Eqs. (30) and (32), respectively.

Notice that while $\mathfrak{t}/W$ is a priori just a cone, the choice of generators $\{I_k\}$ allows us to identify it with $\mathbb{C}^l$. Notice also that we may interpret $\{I_k\}$ as elements of $\mathbb{C}[\mathfrak{g}]^G$, via Chevalley's theorem.

The chosen simple roots determine an isomorphism $\mathfrak{t} \simeq \mathbb{C}^l$, $v \mapsto (\alpha_1(v), \ldots, \alpha_l(v))$, using which we further identify $\chi : \mathfrak{t} \to \mathfrak{t}/W$ with a finite map $\mathbf{I} : \mathbb{C}^l \to \mathbb{C}^l$. We may abuse the notation for these maps, e.g., write $\chi = (I_1, \ldots, I_l)$ instead of $\mathbf{I}$, etc.

We proceed by constructing from these data two rank-l vector bundles on X. The first one is $\mathfrak{t} \otimes_{\mathbb{C}} K_X \simeq K_X^{\oplus l}$, whose total space will be denoted by M:

$$M = \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X.$$

The group W acts (fibrewise, via its action on $\mathfrak{t}$) on M. The resulting quotient $\boldsymbol{U}$ is a priori just a cone bundle, but the choice of $\{I_k\}$ allows us to give it the structure of a vector bundle of rank l:

$$\boldsymbol{U} = \mathfrak{t} \otimes_{\mathbb{C}} K_X/W \simeq \bigoplus_{k=1}^{l} K_X^{d_k}. \tag{2}$$

We can also think of $\boldsymbol{U}\backslash\{0\}$ as the $\mathbb{C}^{\times}$-bundle with fibre $\mathfrak{t}/W$, associated to the $\mathbb{C}^{\times}$-bundle $K_X\backslash\{0\}$.

The morphism $\chi : \mathfrak{t} \to \mathfrak{t}/W$ induces a morphism $\chi : M \to \operatorname{tot} \boldsymbol{U}$ of X-varieties (not of vector bundles!):

$$\begin{tikzcd} M = \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X \arrow[rr, "\chi=(I_1,\ldots,I_l)"] \arrow[dr, "\pi"'] & & \operatorname{tot} \boldsymbol{U} \arrow[dl] \\ & X & \end{tikzcd} \tag{3}$$

We write $\mathcal{B}$ for the Hitchin base—the space of global sections of $\boldsymbol{U}$:

$$\mathcal{B} := H^0(X, \boldsymbol{U}) \simeq H^0\left(X, \bigoplus_{k=1}^{l} K_X^{d_k}\right) \simeq \mathbb{C}^{\dim G(g-1)}.$$

Any $b \in \mathcal{B}$ determines a W-cover $p_b : \widetilde{X}_b \to X$ as the pullback of $\chi : M \to \operatorname{tot} \boldsymbol{U}$ via (the evaluation map of) the section b:

$$\begin{tikzcd} \widetilde{X}_b \arrow[r] \arrow[d, "p_b"] & \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X = M \arrow[d, "\chi"] \\ X \arrow[r, "ev_b"] & \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X/W = \operatorname{tot} \boldsymbol{U} \end{tikzcd}$$

This W-cover is called *the cameral cover of X (corresponding to b)*. We may occasionally write $p : \widetilde{X} \to X$ if the point $b \in \mathcal{B}$ is fixed or understood.

By construction $\widetilde{X}_b$ is a closed subscheme of M that can be singular or non-reduced. The cameral cover $\widetilde{X}_b \subset M$ inherits from M a W-action (and thus has lots of automorphisms). For a generic choice of b it is a non-singular ramified Galois W-cover with simple ramification. We write $\mathscr{B} \subseteq \mathcal{B}$ for the open set of generic cameral covers.

The vector bundle $p_b^*\boldsymbol{U}$ is in fact isomorphic to $N_{\widetilde{X}_b/M}$, the normal bundle of $\widetilde{X}_b \subseteq M$, see Sect. 2.1.

Example 1.1 Let $G = SL_2(\mathbb{C})$. Then $W = \mathbb{Z}/2\mathbb{Z}$, $\boldsymbol{U} \simeq K_X^2$, $\mathcal{B} = H^0(X, K_X^2) \simeq \mathbb{C}^{3g-3}$ and $p_b : \widetilde{X}_b \to X$ is a $2:1$ cover. The open set $\mathscr{B} \subseteq \mathcal{B}$ consists of quadratic differentials with simple roots. For $b \in \mathscr{B}$, the cover has genus $g(\widetilde{X}_b) = 4g - 3$.

Example 1.2 Let $G = G_2$. Then $W = D_6$ (dihedral group of order 12) and $\boldsymbol{U} \simeq K_X^2 \oplus K_X^6$. Consequently $\mathcal{B} = H^0(X, K_X^2) \oplus H^0(X, K_X^6) \simeq \mathbb{C}^{14(g-1)}$. The cameral covers $p_b : \widetilde{X}_b \to X$ are $12:1$ covers, with $g(\widetilde{X}_b) = 84(g-1) + 1$.

There is a weight-1 $\mathbb{Z}$-variation of Hodge structures $\mathcal{V}^1 \subseteq \mathcal{V}^0$ over $\mathscr{B} \subseteq \mathcal{B}$, whose fibres are respectively $\mathcal{V}^1_b = H^0(\widetilde{X}_b, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b})^W$ and $\mathcal{V}^0_b = H^1(\widetilde{X}_b, \mathfrak{t})^W$. Intrinsically, it is defined as follows. Let $\Lambda \subseteq \mathfrak{t}$ be the cocharacter lattice and $p : \mathcal{X} \to \mathscr{B}$ the universal cameral curve. Let also p^W_* be the W-invariant pushforward functor. Then we set $\mathcal{V}_{\mathbb{Z}} = R^1 p^W_*(\Lambda)$ and $\mathcal{V} := \mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathscr{B}} \simeq R^1 p^W_*(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^\bullet_{\mathcal{X}/\mathscr{B}})$. The bundle $\mathcal{V}^1 = R^0 p^W_* \left(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^1_{\mathcal{X}/\mathscr{B}}\right)$, and the Hodge filtration is induced by the naive filtration $\Omega^{\bullet \geq 1}_{\mathcal{X}/\mathscr{B}}[-1] \subseteq \Omega^\bullet_{\mathcal{X}/\mathscr{B}}$. The Gauss–Manin connection can be identified with the d_1 differential of the spectral sequence, induced by the Koszul–Leray filtration on $\Omega^\bullet_{\mathcal{X}}$. The polarisation pairing S is given by $S_b(\alpha, \beta) = \left\langle \alpha \cup \beta, \left[\widetilde{X}_b\right]\right\rangle$. For more details, see Sect. 3.2 and the references therein, as well as [13, 8.1] and [9].

On M there is a canonical $\mathfrak{t}$-valued Liouville form λ, see Sect. 2.4. The Liouville form λ determines a Seiberg–Witten differential, $\boldsymbol{\lambda}_{SW} \in \Gamma(\mathscr{B}, \mathcal{V}^1)$, via $\boldsymbol{\lambda}_{SW}(b) = \lambda|_{\widetilde{X}_b}$, and, as in (1), we have that the map

$$\mathcal{B} = T_b\mathscr{B} \ni \mathbf{g} \longmapsto \left(\nabla^{GM}_{\mathbf{g}} \boldsymbol{\lambda}_{SW}\right)_b \in \mathcal{V}^0_b \tag{4}$$

factors through an isomorphism $T_b\mathscr{B} \simeq \mathcal{V}^1_b$, i.e.,

$$\left(\nabla^{GM} \boldsymbol{\lambda}_{SW}\right)_b : \qquad H^0\left(X, \bigoplus_{k=1}^l K_X^{d_k}\right) \xrightarrow{\simeq} H^0(\widetilde{X}_b, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b})^W. \tag{5}$$

In [16, Proposition 2.11], an isomorphism with the same domain and codomain as in (5) is described as the composition of pullback on global sections (by π), contraction with ω and restriction to $\widetilde{X}_b$, see also Proposition 3.1. In [13, Proposition 8.2] it is shown, using a hypercohomology calculation, that the isomorphism described by Hurtubise and Markman coincides with the isomorphism (4). Some of the above relations for $G = SL_2$ are discussed in [5, Proposition 1], see also [21, Eq.(3)].

The above isomorphism can also be considered from an integrable systems viewpoint. Indeed, consider the universal family of generic cameral curves $p : \mathcal{X} \to \mathscr{B} \subseteq \mathcal{B}$. The relative Prym fibration $\mathbf{Prym}_{\mathcal{X}/\mathscr{B}} \to \mathscr{B}$ is in fact an algebraic completely integrable system. The fibre $\mathbf{Prym}_{\widetilde{X}_b}$ over $b \in \mathscr{B}$ is an abelian variety, whose tangent space is Serre dual to $H^0(\widetilde{X}_b, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b})^W$, the right hand side of (5). The isomorphism (5) actually amounts to lifting a tangent vector in $T_{\mathcal{B},b}$ to a vector field along the fibre $\mathbf{Prym}_{\widetilde{X}_b}$ and then pairing it with the symplectic form on the Prym fibration. This is the viewpoint, taken, e.g., by Hurtubise and Markman.

Our goal in this note is to provide an explicit and global (on X and $\widetilde{X}_b$) description of (5) in terms of Lie theory and the covering $p_b : \widetilde{X}_b \to X$.

The simplest case, that of $G = SL_2$, is given in Example 5.1, where we show that Eq. (5) specialises to

$$H^0(X, K_X^2) \ni \mathbf{g} \longmapsto \left(\nabla^{GM}_{\mathbf{g}} \boldsymbol{\lambda}_{SW}\right)_b = \frac{p^*\mathbf{g}}{2\boldsymbol{\alpha}^2} \boldsymbol{\lambda}_{SW}\Big|_{\widetilde{X}_b} \in H^0(\widetilde{X}_b, K_{\widetilde{X}_b})_-^{\mathbb{Z}/2},$$

where $\boldsymbol{\alpha}^2 \in H^0(M, \pi^* K_X^2)$ is the tautological section and $\boldsymbol{\lambda}_{SW}$ is the Liouville (Seiberg–Witten) form. The expression on the right hand side can in fact also be rewritten as $-\frac{p^* g}{2\lambda}$, and in this form it coincides (up to scaling factors) with [21, (4)], who reference Douady–Hubbard [5, §2].

Our main result is a general formula for $\nabla^{GM}_{\mathbf{g}} \boldsymbol{\lambda}_{SW}$ for the case of an arbitrary (complex, simple) group G.

Let $D\mathbf{I}$ be the Jacobi matrix of the adjoint quotient $\mathbf{I} = (I_1, \ldots, I_l) : \mathbb{C}^l \to \mathbb{C}^l$ and ι the natural algebra homomorphism from $\mathrm{Sym}(\mathfrak{t}^\vee)$ into $H^0\left(M, \bigoplus_{n\geq 0} \pi^* K_X^n\right)$, introduced in Eq. (22). Finally, $\boldsymbol{\alpha}_i = \iota(\alpha_i)$ and $\lambda_i = e_i \otimes \boldsymbol{\alpha}_i$, where $\{e_i\}$ is the basis of $\mathfrak{t}$, dual to $\{\alpha_i\}$. In this notation, the Liouville form is $\boldsymbol{\lambda}_{SW} = \sum_i \lambda_i$.

Theorem A *Once the main and additional data are chosen, the isomorphism (5) $\nabla^{GM}\boldsymbol{\lambda}_{SW}$ maps $\mathbf{g} \in T_b\mathscr{B} = H^0(X, \bigoplus_i K_X^{d_i})$ to the section*

$$\left(\nabla_{\mathbf{g}}^{GM}\boldsymbol{\lambda}_{SW}\right)_b = -\sum_{i=1}^{l} \frac{\left(\iota(D\mathbf{I})^{-1}\cdot \pi^*\mathbf{g}\right)_i}{\boldsymbol{\alpha}_i}\lambda_i\Bigg|_{\widetilde{X}_b} = -\,\iota(D\mathbf{I})^{-1}\cdot \pi^*\mathbf{g}\big|_{\widetilde{X}_b}. \tag{6}$$

In particular, for $l = 2$ we have that

$$\left(\nabla^{GM}\boldsymbol{\lambda}_{SW}\right)_b : \; H^0(X, K_X^{d_1} \oplus K_X^{d_2}) \longrightarrow H^0(\widetilde{X}_b, K_{\widetilde{X}_b}^{\oplus 2})^W$$

sends $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ to

$$\left(\nabla_{\mathbf{g}}^{GM}\boldsymbol{\lambda}_{SW}\right)_b = -\left(\frac{\begin{vmatrix}\pi^* g_1 & \iota\partial_2 I_1\\ \pi^* g_2 & \iota\partial_2 I_2\end{vmatrix}}{\boldsymbol{\alpha}_1 \det \iota D\mathbf{I}}\lambda_1 + \frac{\begin{vmatrix}\iota\partial_1 I_1 & \pi^* g_1\\ \iota\partial_1 I_2 & \pi^* g_2\end{vmatrix}}{\boldsymbol{\alpha}_2 \det \iota D\mathbf{I}}\lambda_2\right)\Bigg|_{\widetilde{X}_b} = -\frac{1}{\det \iota D\mathbf{I}}\begin{bmatrix}\begin{vmatrix}\pi^* g_1 & \iota\partial_2 I_1\\ \pi^* g_2 & \iota\partial_2 I_2\end{vmatrix}\\ \\ \begin{vmatrix}\iota\partial_1 I_1 & \pi^* g_1\\ \iota\partial_1 I_2 & \pi^* g_2\end{vmatrix}\end{bmatrix}\Bigg|_{\widetilde{X}_b}. \tag{7}$$

Knowledge of $\boldsymbol{\lambda}_{SW}$ and $\nabla^{GM}\boldsymbol{\lambda}_{SW}$ is essential for describing various geometric structures on $\mathscr{B}$. We mention only two examples as an illustration.

First, for the Hitchin integrable system, the Donagi–Markman cubic [6], which is essentially the infinitesimal period map for the family of Hitchin Pryms, is given by the Balduzzi–Pantev formula [2, Theorem 1]. If we consider the cubic as a global section c of $\mathrm{Sym}^3 T_{\mathscr{B}}^\vee = \mathrm{Sym}^3 \mathcal{B}^\vee \otimes \mathcal{O}_{\mathscr{B}}$, then the Balduzzi–Pantev formula states that the value of c at $b \in \mathscr{B}$ is

$$c_b(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = \frac{1}{2}\sum_{m\in \mathrm{Ram}\, p_b} \mathrm{Res}_m^2\left(p_b^* \,\frac{\mathcal{L}_{\mathbf{g}_1}\mathfrak{D}}{\mathfrak{D}}\bigg|_{\{b\}\times X}\left(\nabla_{\mathbf{g}_2}^{GM}\boldsymbol{\lambda}_{SW}\right)_b \cup \left(\nabla_{\mathbf{g}_3}^{GM}\boldsymbol{\lambda}_{SW}\right)_b\right). \tag{8}$$

Here $\mathfrak{D}$ is the discriminant (see also Sect. 5) and $\mathcal{L}$ denotes Lie derivative. In our previous work [3, Theorem A] we have shown that the Balduzzi–Pantev formula holds along the (good) symplectic leaves of the generalised Hitchin system.

The second example which is worth mentioning is the special Kähler metric g_{SK} on $\mathscr{B}$. It is known that for the case of $G = SL_2(\mathbb{C})$, the special Kähler metric is given by

$$g_{SK}(\mathbf{g}, \mathbf{g})_b = 2\int_{\widetilde{X}_b} |\nabla_{\mathbf{g}}^{GM}\boldsymbol{\lambda}_{SW}|^2, \tag{9}$$

see [11, 2.40], [21, §2.3], [8].

We shall discuss additional applications of Theorem A to various aspects of the geometry of $\mathscr{B}$ in a forthcoming work.

2 Preliminaries

2.1 The embedding of the cameral curve

We are now going to work at a fixed point $b \in \mathcal{B}$ (generic), and hence will write mostly $p : \widetilde{X} \to X$ for the cameral cover. To understand (5) we need to understand $K_{\widetilde{X}}$ and for that we need to know more about the normal bundle N of the closed embedding $\widetilde{X} \subseteq M$. This is not difficult, since $\widetilde{X}$ is in fact the zero locus of a section of a vector bundle on M.

First, notice that the morphism $\chi : M \to \boldsymbol{U}$ (see (3)) induces a tautological section $\sigma \in H^0(M, \pi^*\boldsymbol{U})$ in a standard way, via

$$\begin{tikzcd}
M \arrow[dr, dotted, "\sigma"] \arrow[drr, bend left, "\chi"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \pi^*\boldsymbol{U} \arrow[r] \arrow[d] & \boldsymbol{U} \arrow[d] \\
& M \arrow[r, "\pi"] & X
\end{tikzcd}, \tag{10}$$

which on closed points is simply $\sigma(m) = (m, \chi(m)) \in M \times_X \operatorname{tot} \boldsymbol{U} = \operatorname{tot} \pi^*\boldsymbol{U}$.

Next, the adjunction morphism $\boldsymbol{U} \to \pi_*\pi^*\boldsymbol{U}$ induces on global sections the pullback map $\mathcal{B} = H^0(X, \boldsymbol{U}) \to H^0(M, \pi^*\boldsymbol{U})$, which we write as $b \mapsto \pi^*b$.

Thus the cameral curve $\widetilde{X_b}$ is the zero locus

$$\widetilde{X}_b = \operatorname{zeros}(s_b), \;\; s_b = \sigma - \pi^*(b) \in H^0(M, \pi^*\boldsymbol{U}), \tag{11}$$

i.e., is cut out by the equation(s)

$$\chi(m) = b(\pi(m)) \tag{12}$$

in $M = \operatorname{tot} \mathfrak{t} \otimes_{\mathbb{C}} K_X$. Having fixed basic invariant polynomials $\{I_k\}$, and hence an isomorphism $\boldsymbol{U} \simeq \bigoplus_{k=1}^{l} K_X^{d_k}$, we can express this as the system of equations

$$\left| \begin{array}{l} I_1(m) = b_1(\pi(m)) \\ \vdots \\ I_l(m) = b_l(\pi(m)) \end{array} \right. , \tag{13}$$

for $m \in M$, with $b = (b_1, \ldots, b_l) \in \mathcal{B}$ fixed. These are "global" equations and no choice of local trivialisation is used here: the k-th equation takes values in (the total space of) $K_X^{d_k}$. Another global description is given in Eq. (23).

From Eq. (11) follows

Proposition 2.1 *The normal bundle of* $\widetilde{X}_b \subseteq M$ *is*

$$N_{\widetilde{X}_b/M} \simeq p_b^*\boldsymbol{U} = \mathfrak{t} \otimes_{\mathbb{C}} p_b^* K_X/W \simeq \bigoplus_{k=1}^{l} p_b^* K_X^{d_k}. \tag{14}$$

Proof While in general one uses the Koszul complex to compute the normal bundle, here we have that both $\widetilde{X}_b$ and M are smooth, and moreover, $\widetilde{X}_b$ is a complete intersection. This case is handled by a standard geometric argument, given in, e.g. [10, Proposition 6.15].

The isomorphism $N_{\widetilde{X}_b/M} \simeq p_b^*\boldsymbol{U}$ is induced by the (vertical component of the) differential $ds_b : T_M \to s_b^* T_{\pi^*\boldsymbol{U}}$ of the section $s_b : M \to \operatorname{tot} \pi^*\boldsymbol{U}$. □

Similarly to the above argument, since M is the total space of a vector bundle (namely $\mathfrak{t} \otimes_{\mathbb{C}} K_X$) on X, its tangent bundle T_M is an extension of $\pi^* T_X$ by $\mathfrak{t} \otimes_{\mathbb{C}} \pi^* K_X$. Restricting to $\widetilde{X}$ and combining with the previous result, one gets the diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & T_{\widetilde{X}} \arrow[d] & & \\ 0 \arrow[r] & \mathfrak{t} \otimes_{\mathbb{C}} p^* K_X \arrow[r] & T_M|_{\widetilde{X}} \arrow[r] \arrow[d] & p^* T_X \arrow[r] & 0 \\ & & \mathfrak{t} \otimes_{\mathbb{C}} p^* K_X / W \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{15}$$

Now, consider $\mathbf{g} = (g_1, \ldots, g_l) \in T_{\mathcal{B},b} = \mathcal{B}$, with $g_i \in H^0(X, K_X^{d_i})$. It determines a 1-parameter family of deformations of $\widetilde{X}_b$, given by the equation

$$\chi(m) = b(\pi(m)) + \epsilon \mathbf{g}(\pi(m)), \tag{16}$$

that is, $\left\{\widetilde{X}_{b+\epsilon g}\right\}_\epsilon$. For ϵ in a sufficiently small disk $\Delta_\rho \subseteq \mathbb{C}$ the section $b + \epsilon \mathbf{g} \in \mathcal{B}$ remains generic—which we assume to be the case from now on. The total space of the 1-parameter family is cut out in $M \times \Delta_\rho$ by the Eq. (16).

The section $\mathbf{g}$ determines a section of $N_{\widetilde{X}_b/M} = \mathfrak{t} \otimes_{\mathbb{C}} p_b^* K_X / W$, namely, $p_b^* \mathbf{g}$.

2.2 Local description

It is not hard to describe the objects from the previous section in local coordinates. A choice of a local (analytic) chart ψ on X, identifying an open $U \subseteq X$ with a disk $\Delta \subseteq \mathbb{C}$, determines a local trivialisation of K_X and a compatible bundle chart ϕ on M, identifying $M_U = \pi^{-1}(U) \to U$ with $\mathrm{pr}_1 : \Delta \times \mathfrak{t} \to \Delta$, as usual:

$$\begin{tikzcd} M_U = \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & \Delta \times \mathfrak{t} \arrow[d, "\mathrm{pr}_1"] \\ X \supseteq U \arrow[r, "\psi"] & \Delta \subseteq \mathbb{C} \end{tikzcd} . \tag{17}$$

Such a local chart determines a trivialisation of $K_X^{d_i}$ over U and hence a section $b_i \in H^0(X, K_X^{d_i})$ is represented locally as $(\psi^{-1})^* b_i = \beta_i(z) dz^{\otimes d_i}$ on U, where $\beta_i : \Delta \to \mathbb{C}$ is a holomorphic function.

Using the simple roots as a basis for $\mathfrak{t} \simeq \mathbb{C}^l$, we identify $\widetilde{X}_U = p_b^{-1}(U)$ (via ϕ) with the set of solutions of $\mathbf{I}(\alpha_1, \ldots, \alpha_l) = \boldsymbol{\beta}(z)$ for $(z, \underline{\alpha}) \in \Delta \times \mathbb{C}^l$, giving a local version of Eq. (13).

Next, the trivialisations of $K_X^{d_i}$ $(i = 1 \dots l)$ and the choice of roots provide an induced trivialisation $T_{M_U}\big|_{\widetilde{X}_U}$ and

$$(\psi^{-1})^* T_{M_U}\big|_{\widetilde{X}_U} = (\mathbb{C} \oplus \mathfrak{t}) \otimes_{\mathbb{C}} \mathcal{O}_{\phi(\widetilde{X}_U)} \simeq \mathcal{O}_{\phi(\widetilde{X}_U)} \left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial \alpha_1}, \dots, \frac{\partial}{\partial \alpha_l} \right\rangle \tag{18}$$

and, consequently, a local description of the diagram (15):

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & (\psi^{-1})^* T_{\widetilde{X}_U} \arrow[d] & & \\
0 \arrow[r] & \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_{\phi(\widetilde{X}_U)} \arrow[r, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & (\mathbb{C} \oplus \mathfrak{t}) \otimes_{\mathbb{C}} \mathcal{O}_{\phi(\widetilde{X}_U)} \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -\boldsymbol{\beta}' & D\mathbf{I} \end{pmatrix}"] & \mathcal{O}_{\phi(\widetilde{X}_U)} \arrow[r] & 0 \\
 & & \mathbb{C}^l \otimes_{\mathbb{C}} \mathcal{O}_{\phi(\widetilde{X}_U)} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \quad . \tag{19}$$

Here the bottom vertical map is, in more detail,

$$\begin{pmatrix} -\boldsymbol{\beta}' & D\mathbf{I} \end{pmatrix} = \begin{pmatrix} -\beta_1' & \partial_1 I_1 & \dots & \partial_l I_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\beta_l' & \partial_1 I_l & \dots & \partial_l I_l \end{pmatrix} \in \mathrm{Mat}_{l \times (l+1)} \left(\Gamma \left(\mathcal{O}_{\phi(\widetilde{X}_U)} \right) \right), \tag{20}$$

having rank l everywhere on $\widetilde{X}_U$, under the assumption that $b = (b_1, \dots, b_l) \in \mathcal{B}$ is generic. This is the matrix of the map $\mathrm{pr}_2 \circ ds$ from Proposition 2.1. We write $D\mathbf{I}$ or $D\chi$ for the Jacobi matrix of $\mathbf{I} = (I_1, \dots, I_l) : \mathbb{C}^l \to \mathbb{C}^l$.

Finally, given a tangent vector $\mathbf{g} = (g_1, \dots, g_l) \in T_{\mathcal{B},b} = \mathcal{B}$, with $(\psi^{-1})^* g_i = \gamma_i(z) dz^{\otimes d_i}$ on U, the corresponding 1-parameter (analytic) family of deformations of $\widetilde{X}_b$ is cut out locally (in $\Delta \times \mathbb{C}^l \times \Delta_\rho$) by $\mathbf{I}(\underline{\alpha}) = \boldsymbol{\beta}(z) + \epsilon \boldsymbol{\gamma}(z)$, where $\Delta_\rho \subseteq \mathbb{C}$ is as before.

We may occasionally suppress the pullbacks by ϕ and ψ, except for the cases when there is a risk of confusion, as when discussing (co)roots and some associated objects.

2.3 Objects, associated with roots

Any linear map $\alpha \in \mathfrak{t}^\vee = \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$ determines, by extension of scalars, a vector bundle homomorphism $\mathfrak{t} \otimes_{\mathbb{C}} K_X \to K_X$, denoted by the same letter. Hence, just as χ in Eq. (10), such an α determines a tautological section $\boldsymbol{\alpha} \in H^0(M, \pi^* K_X)$, which on (closed) points maps $m \in M$ to $\boldsymbol{\alpha}(m) = (m, \alpha(m)) \in M \times_X \mathrm{tot}\, K_X$. Furthermore, restricting $\boldsymbol{\alpha}$ to $\widetilde{X} \subset M$ gives a section $\boldsymbol{\alpha}_{\widetilde{X}} \in H^0(\widetilde{X}, p^* K_X)$. Occasionally, we suppress the subscript $\widetilde{X}$, i.e., the restriction.

The section $\boldsymbol{\alpha}$ vanishes along a "hyperplane divisor" $\mathrm{tot}\, (\ker \alpha \otimes_{\mathbb{C}} K_X) \subseteq M$, a rank-$(l-1)$ subbundle of $\mathfrak{t} \otimes_{\mathbb{C}} K_X$. The respective restrictions $\boldsymbol{\alpha}_{i\,\widetilde{X}}$ (of sections arising from roots) vanish along divisors D_{α_i} in $\widetilde{X}$, which are the ramification divisors of $p : \widetilde{X} \to X$.

If we choose a local chart (U, ψ) and $\phi : M_U \simeq \Delta \times \mathfrak{t}$, as in (17), $\boldsymbol{\alpha}$ is represented by $(z, u) \mapsto \alpha(u)dz$, where $\alpha(u) = \langle \alpha, u \rangle$ is the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^\vee$. If we further identify the preimage of $\pi^{-1}(U)$ in tot $\pi^* K_X \to M$ with $\Delta \times \mathfrak{t} \times \mathbb{C}$, via ϕ and a trivialisation of K_X, then the evaluation map of $\boldsymbol{\alpha}$ is represented by

$$\Delta \times \mathfrak{t} \ni (z, u) \longmapsto (z, u, \alpha(u)) \in \Delta \times \mathfrak{t} \times \mathbb{C}.$$

The linear functional $\alpha \in \mathfrak{t}^\vee$ determines a function on $\Delta \times \mathfrak{t}$, that we may denote $\mathrm{pr}_2^*\alpha$ if the distinction from α is important. Furthermore, given the choice of ϕ, we may consider α (or rather, $\mathrm{pr}_2^*\alpha$) a function $\phi^*\alpha \in \mathcal{O}_{M_U}(M_U)$ on M_U. Consequently, upon restriction to $\widetilde{X}_U$, we get a local function $\phi^*\alpha \in \mathcal{O}_{\widetilde{X}_U}(\widetilde{X}_U)$ on the cameral curve. Of course, one should really write $\phi^*\mathrm{pr}_2^*\alpha\big|_{\widetilde{X}_U}$ here.

The distinction between the various objects associated to a root α_i becomes important when one considers their differentials. Since $\pi^* K_X \subseteq \Omega^1_M$, $d\boldsymbol{\alpha}_i \in \Omega^2_M(M)$. At the same time, $d\phi^*\alpha_i \in \Omega^1_{M_U}(M_U)$ and $d(\mathrm{pr}_2^*\alpha_i) \in \Omega^1(\Delta \times \mathfrak{t})$. Naturally, we are going to write $d\alpha_i$ for the penultimate expression, so the distinction between $d\alpha_i$ and $d\boldsymbol{\alpha}_i$ is essential. Finally, we keep in mind that $d\alpha_i = \alpha_i \in \mathrm{Hom}(\mathfrak{t}, \mathbb{C})$, as with any linear map.

The assignment $\alpha_i \mapsto \boldsymbol{\alpha}_i$ determines an (injective) $\mathbb{C}$-algebra homomorphism

$$\iota : \mathrm{Sym}(\mathbf{t}^\vee) \hookrightarrow H^0\left(M, \bigoplus_{n\geq 0} \pi^* K_X^n\right) \tag{21}$$

and, consequently, a homomorphism

$$\mathrm{End}(\mathbb{C}^l) \otimes \mathrm{Sym}(\mathfrak{t}^\vee) \hookrightarrow \mathrm{End}(\mathbb{C}^l) \otimes H^0\left(M, \bigoplus_{n\geq 0} \pi^* K_X^n\right), \tag{22}$$

denoted by ι as well. Given a $\mathrm{Sym}(\mathfrak{t}^\vee)$-valued endomorphism A with non-zero determinant $\det A \in \mathrm{Sym}(\mathfrak{t}^\vee)$, we write $\iota(A)^{-1}$ for the inverse of $\iota(A)$ in the ring of $l \times l$ matrices with coefficients in the field of fractions $\mathrm{Frac}\, H^0\left(M, \bigoplus_{n\geq 0} \pi^* K_X^n\right)$, and in fact, in $\mathrm{End}(\mathbb{C}^l) \otimes H^0\left(M, \bigoplus_{n\geq 0} \pi^* K_X^n\right)\left[\frac{1}{\det \iota(A)}\right]$.

We can, more generally, rewrite the global equations for $\widetilde{X}_b$ as

$$\left|\begin{array}{l} I_1(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l) = \pi^* b_1 \\ \vdots \\ I_l(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_l) = \pi^* b_l \end{array}\right. \tag{23}$$

that is, the linear system $\iota(I_k) = \pi^* b_k$, $k = 1 \ldots l$.

2.4 Liouville form

On $M = \mathrm{tot}\, \mathfrak{t} \otimes_{\mathbb{C}} K_X$ there is a $\mathfrak{t}$-valued 2-form $\omega \in H^0(M, \mathfrak{t} \otimes_{\mathbb{C}} \Omega^2_M)$. Probably the simplest way to introduce it is by setting

$$\omega = -d\lambda,$$

where $\lambda \in H^0(M, \mathfrak{t} \otimes_{\mathbb{C}} \pi^* K_X) \subseteq H^0(M, \mathfrak{t} \otimes_{\mathbb{C}} \Omega^1_M)$ is a tautological section, the "$\mathfrak{t}$-valued Liouville form".

We recall some explicit expressions for λ—although, as usual in symplectic geometry, there are various sign ambiguities in the possible definitions.

The chosen simple roots $\{\alpha_1, \ldots, \alpha_l\}$ form a basis of $\mathfrak{t}^\vee$, and we let $\{e_1, \ldots, e_l\}$ stand for the corresponding dual basis of $\mathfrak{t}$ (consisting of fundamental coweights).

One can then set $\lambda_i = e_i \otimes_{\mathbb{C}} \boldsymbol{\alpha}_i$, a global section of $\mathfrak{t} \otimes_{\mathbb{C}} \pi^* K_X \subseteq \mathfrak{t} \otimes_{\mathbb{C}} \pi^* \Omega^1_M$, and write the Liouville form and the 2-form as

$$\lambda = \sum_{i=1}^{l} \lambda_i = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_l \end{bmatrix}, \ \omega = -\sum_{i=1}^{l} e_i \otimes_{\mathbb{C}} d\boldsymbol{\alpha}_i = \begin{bmatrix} -d\boldsymbol{\alpha}_1 \\ -d\boldsymbol{\alpha}_2 \\ \vdots \\ -d\boldsymbol{\alpha}_l \end{bmatrix}. \tag{24}$$

Finally, if we choose local coordinates as in Eq. (17), we obtain for the pullback of λ and ω to $\Delta \times \mathfrak{t}$

$$(\phi^{-1})^* \lambda = \sum_{i=1}^{n} e_i \otimes_{\mathbb{C}} \alpha_i dz = \begin{bmatrix} \alpha_1 dz \\ \vdots \\ \alpha_l dz \end{bmatrix}, \ (\phi^{-1})^* \omega = \sum_{i=1}^{l} e_i \otimes_{\mathbb{C}} dz \wedge d\alpha_i = \begin{bmatrix} dz \wedge d\alpha_1 \\ \vdots \\ dz \wedge d\alpha_l \end{bmatrix}.$$

3 Background: two results

3.1 A result of Hurtubise and Markman

We begin with the special case of a result of Hurtubise and Markman [16, Proposition 2.11] mentioned in the introduction. We spell out some of the details of their argument for this special case.

Proposition 3.1 *For each generic $b \in \mathcal{B}$, the pullback of global sections via p_b, followed by the isomorphism (14) and contraction with ω induces an isomorphism*

$$\beta : \mathcal{B} = H^0(X, \boldsymbol{U}) \xrightarrow{\simeq} H^0(\widetilde{X}_b, p_b^* \boldsymbol{U})^W \xrightarrow{\simeq} H^0(\widetilde{X}_b, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b})^W, \tag{25}$$

or, using the choice of invariant polynomials $\{I_k\}$, an isomorphism

$$H^0\left(X, \bigoplus_{k=1}^{l} K_X^{d_k}\right) \simeq H^0\left(\widetilde{X}_b, \bigoplus_{k=1}^{l} p_b^* K_X^{d_k}\right)^W \simeq H^0\left(\widetilde{X}_b, K_{\widetilde{X}_b}^{\oplus l}\right)^W.$$

Thus, the isomorphism β (25) is a composition of two maps. The first one is pullback (adjunction) $\mathbf{g} \longmapsto p_b^* \mathbf{g}$, for $\mathbf{g} \in \mathcal{B} = H^0(X, \boldsymbol{U})$. The second one is the map on global sections, induced by the map of bundles

$$N_{\widetilde{X}} \longrightarrow \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}} \tag{26}$$

$$s \longmapsto \omega(\widetilde{s}, \ )|_{\widetilde{X}},$$

where $\widetilde{s}$ is a lift of s to a section of T_M. One may denote this map simply by $\lrcorner\, \omega$ (contraction with ω), but should keep in mind the restriction to $\widetilde{X}$.

The proof of Proposition 3.1 relies on a dimension count, combined with good understanding of the bundle map (26) and the induced map on fibres at $m \in \widetilde{X}$. For that, the cases when m is not a ramification point and when it is one should be considered separately.

Notice that if m is not a ramification point, then $T_{\widetilde{X},m} \nsubseteq \pi^{-1}(\pi(m)) = \mathfrak{t} \otimes_{\mathbb{C}} K_{X,p(m)}$, while $T_{\widetilde{X},m} \subseteq \pi^{-1}(\pi(m))$ if m is a ramification point.

So let us choose a point $m \in M$ and consider the fibre of $\pi : M \to X$, passing through m. We set $L := \pi^{-1}(\pi(m)) = \mathfrak{t} \otimes_{\mathbb{C}} K_{X,\pi(m)} \subseteq M$, and write N_L for the normal bundle of the vector space $L \subseteq M$.

Using the local description of ω, we obtain that $\lrcorner\, \omega$ fits in the following diagram:

$$\begin{tikzcd}
T_{L,m} \otimes N_{L,m} \arrow[r, hook] & T_{M,m}\big|_L \otimes N_{L,m} \arrow[r, "\lrcorner\omega"] \arrow[dr, dotted] & \mathfrak{t} \otimes T^{\vee}_{M,m}\Big|_L \otimes N_{L,m} \\
& & \mathfrak{t} \otimes N^{\vee}_{L,m} \otimes N_{L,m} = \mathfrak{t} \arrow[u, hook]
\end{tikzcd}$$

Since $N_L = T_{X,\pi(m)} \otimes_{\mathbb{C}} \mathcal{O}_L$ and $T_L = L \otimes_{\mathbb{C}} \mathcal{O}_L = \mathfrak{t} \otimes_{\mathbb{C}} K_{X,\pi(m)} \otimes_{\mathbb{C}} \mathcal{O}_L$, there is a canonical trivialisation $T_L \otimes N_L = \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_L$.

Using the normal sequence for $L \subseteq M$, one obtains:

Lemma 3.1 *The map $\lrcorner\, \omega$ induces a trivialisation $T_L \otimes N_L \simeq_\omega \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_L$, which coincides up to sign with the canonical trivialisation. That is,*

$$\begin{tikzcd}
T_L \otimes N_L \arrow[r, "\simeq_\omega"] \arrow[d, equal, "can"'] & \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_L \ . \\
\mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_L \arrow[ur, "-id"] &
\end{tikzcd}$$

Thus, in particular, $\lrcorner\omega$ induces, for any $m \in L$, a W-equivariant isomorphism $T_{L,m} \otimes N_{L,m} \simeq \mathfrak{t}$. A similar result is stated, in a much more general setup, in [16][Theorem 2.8 (5)].

Lemma 3.2 *Consider a point $m \in \widetilde{X}$ that is* not *a ramification point of $p : \widetilde{X} \to X$. The map on fibres, induced by the bundle map* (26) *is an isomorphism*

$$\lrcorner\, \omega : \qquad N_{\widetilde{X},m} \xrightarrow{\simeq} \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X},m}.$$

This is again a local calculation, using the explicit form of ω. Notice that since m is not a ramification point, the composition

$$\begin{tikzcd}
L = T_{L,m} \arrow[r, hook] & T_{M,m} \arrow[r, two heads] & N_{\widetilde{X},m}
\end{tikzcd}$$

is an isomorphism. However, at ramification points the behaviour of $\lrcorner\omega$ is different. In fact, at such points the map (26) is *not* an isomorphism of bundles if $l > 1$, as is clear from the next Lemma.

Lemma 3.3 *Let $m \in \widetilde{X}$ be a ramification point of $p : \widetilde{X} \to X$. Then*

$$\begin{tikzcd}
N_{\widetilde{X},m} \arrow[r, "\lrcorner\omega"] \arrow[dr] & \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X},m} \simeq_\omega T_{L,m} \otimes N_{L,m} \otimes K_{\widetilde{X},m} \\
& T_{\widetilde{X},m} \otimes N_{L,m} \otimes K_{\widetilde{X},m} \arrow[u, hook]
\end{tikzcd}$$

commutes.

This result is shown by a local calculation, which in turn boils down to a linear-algebraic result, using the explicit form of ω. It is also stated in [16, Lemma 2.10].

Proof of Proposition 3.1 The map β, i.e., (25) is a composition of two maps, both of which are injective. Indeed, $H^0(X, \boldsymbol{U}) \hookrightarrow H^0(\widetilde{X}, p^*\boldsymbol{U})$, and the image is contained in $H^0(\widetilde{X}, p^*\boldsymbol{U})^W$. Furthermore, Lemma 3.2 implies that the map (26) induces an injection on global sections, $H^0(\widetilde{X}, \mathfrak{t} \otimes_{\mathbb{C}} p^* K_X/W) \hookrightarrow H^0(\widetilde{X}, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}})$, and it preserves W-invariant sections. Finally, by Serre duality (and the fact that taking duals commutes with taking invariants), we get

$$\mathcal{B} = H^0(X, \boldsymbol{U}) \hookrightarrow H^0(\widetilde{X}, p^*\boldsymbol{U})^W \hookrightarrow H^0(\widetilde{X}, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}})^W \simeq H^1\left(\widetilde{X}, \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{X}}\right)^{W\vee}.$$

But $H^1\left(\widetilde{X}, \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{X}}\right)^W$ is the tangent space of the generalised Prym variety, and by the complete integrability of the Hitchin system, its dimension equals the dimension of the base $\mathcal{B}$. Hence both injections are isomorphisms, and so is their composition. □

3.2 A result of Hertling, Hoevenaars and Posthuma

We introduced earlier a certain weight-1 $\mathbb{Z}$-VHS $(\mathcal{V}, \nabla^{GM}, \mathcal{V}_{\mathbb{Z}}, \mathcal{V}^\bullet, S)$ on $\mathscr{B} \subseteq \mathcal{B}$.

The bundle of lattices $\mathcal{V}_{\mathbb{Z}}$ was defined as $R^1 p_*^W(\Lambda)$, where $p : \mathcal{X} \to \mathscr{B}$ is the universal cameral cover, and the vector bundle $\mathcal{V} = \mathcal{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathscr{B}} \simeq R^1 p_*^W(\mathfrak{t} \otimes_{\mathbb{C}} p^*\mathcal{O}_{\mathscr{B}})$. The relative holomorphic Poincaré Lemma gives a quasi-isomorphism $p^{-1}\mathcal{O}_{\mathscr{B}} \simeq_{quis} \Omega^\bullet_{\mathcal{X}/\mathscr{B}}$, leading to $\mathcal{V} \simeq R^1 p_*^W(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^\bullet_{\mathcal{X}/\mathscr{B}})$.

The Hodge bundles, as for geometric VHS, are determined by the naive filtration of $\Omega^\bullet_{\mathcal{X}/\mathscr{B}}$, see [26, §10.2] and [27, §5.1]. In our case of weight one, $\Omega^{\bullet \geq 1}_{\mathcal{X}/\mathscr{B}}[-1] \subseteq \Omega^\bullet_{\mathcal{X}/\mathscr{B}}$ determines a subbundle $\mathcal{V}^1 \subseteq \mathcal{V}$, as $R^1 p_*^W(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^{\bullet \geq 1}_{\mathcal{X}/\mathscr{B}}[-1]) \simeq R^0 p_*^W(\mathfrak{t} \otimes_{\mathbb{C}} \Omega^1_{\mathcal{X}/\mathscr{B}})$.

The Gauss–Manin connection $\nabla^{GM} : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{\mathscr{B}}$ can be defined in either topological or holomorphic terms. The topological description relies on Ehresmann's theorem, i.e., on the C^∞-local triviality of $p : \mathcal{X} \to \mathscr{B}$. In this case, the homotopy-invariance of de Rham cohomology implies that $\mathcal{V}_{\mathbb{Z}}$ is a locally constant sheaf and ∇^{GM} can be described by a Cartan–Lie formula. For geometric VHS this is described, e.g., in [26, §8.2].

The holomorphic description of ∇^{GM} is discussed in [13, §8], following [18], see also [4] and [27, §5.1]. The Koszul–Leray filtration on $\Omega^\bullet_{\mathcal{X}}$ gives rise to a spectral sequence, for which $(E_1^{\bullet,0}, d_1)$ is identified with $(\Omega^\bullet_{\mathscr{B}}(\mathcal{V}), \nabla^{GM})$.

One has the following result.

Theorem 3.1 [13, Proposition 8.2] *The isomorphisms $\nabla^{GM}\boldsymbol{\lambda}_{SW}$ and β (25) coincide. That is,*

$$\nabla^{GM}_\xi \boldsymbol{\lambda}_{SW} = \beta(\xi),$$

for all tangent vectors $\xi \in T_b\mathscr{B}$ and all $b \in \mathscr{B}$.

The result is proved by an explicit hypercohomology calculation, using the Čech resolution of the relative de Rham complex $(\Omega^\bullet_{\mathcal{X}/\mathscr{B}}, d)$.

4 Proof of Theorem A

We now turn to the proof of our main result, Theorem A. Recall that in the statement of the theorem we use the algebra homomorphism ι from Eq. (22), so $\iota(D\mathbf{I})^{-1}$ is a global meromorphic section of $\underline{End}\left(\mathbb{C}^l \otimes_{\mathbb{C}} \bigoplus_{k\geq 0} \pi^* K_X^k\right)$ with poles along the zeros of $\det \iota D\mathbf{I}$. That is, the homogeneous polynomials $\partial_i I_j \in \mathrm{Sym}^{d_j-1}(\mathfrak{t}^\vee)$ are considered as global sections of $\pi_b^* K_X^{d_j-1}$, or, after restriction to $\widetilde{X}_b$, as sections of $p_b^* K_X^{d_j-1}$.

Using Cramer's formula and the fact that ι is an algebra homomorphism, we can rewrite the right side of (6) as a linear combination of (restrictions of) λ_i with coefficients of the kind

$$\left.\frac{\det\left[\iota\partial_1\mathbf{I}, \ldots, p_b^*\mathbf{g}, \ldots, \iota\partial_l\mathbf{I}\right]}{\boldsymbol{\alpha}_i \det \iota D\mathbf{I}}\right|_{\widetilde{X}_b} \in \mathcal{K}(\widetilde{X}_b),$$

i.e., global meromorphic functions on $\widetilde{X}_b$, since both the numerator and the denominator belong to $H^0\left(\widetilde{X}_b, p_b^* K_X^{\sum_i d_i - l + 1}\right)$.

Now we prove Theorem A. Let us fix $\mathbf{g} \in T_b\mathcal{B} = H^0(X, \bigoplus_i K_X^{d_i})$ and denote by $\mathbf{s} \in H^0(\widetilde{X}_b, \mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b})$ the image of $\mathbf{g}$ under the isomorphism (5). Let us also denote by $\widetilde{\mathbf{s}}$ the section from the right hand side of Eq. (6), i.e.,

$$\widetilde{\mathbf{s}} = -\sum_{i=1}^{l} \left.\frac{\left(\iota(D\mathbf{I})^{-1} \cdot \pi^*\mathbf{g}\right)_i}{\boldsymbol{\alpha}_i}\lambda_i\right|_{\widetilde{X}_b}.$$

This is a meromorphic section of $\mathfrak{t} \otimes_{\mathbb{C}} K_{\widetilde{X}_b}$ with poles at most along the ramification of $p_b : \widetilde{X}_b \to X$. We are going to prove that $\mathbf{s} = \widetilde{\mathbf{s}}$. We use Theorem 3.1 and the representation of the isomorphism β from Eq. (25) is a composition of two maps.

As a first step, we show that

$$\mathbf{s}|_{\widetilde{X}_b \backslash \mathrm{Ram}(p_b)} = \widetilde{\mathbf{s}}|_{\widetilde{X}_b \backslash \mathrm{Ram}(p_b)}. \tag{27}$$

For that we restrict the cameral cover to the complements of the ramification and branch divisors

$$p_b : \widetilde{X}_b \backslash \mathrm{Ram}(p_b) \longrightarrow X \backslash \mathrm{Bra}(p_b)$$

and choose $U \subseteq X\backslash\mathrm{Bra}(p_b)$, biholomorphic to an open disk (via $\psi : U \to \Delta$). In this case, $\widetilde{X}_U \subseteq \widetilde{X} \cap (\det \iota D\mathbf{I} \neq 0)$ has $|W|$ (analytic) connected components, each isomorphic to U, labelled by the different Weyl chambers

$$\widetilde{X}_U = \widetilde{X}_U^1 \coprod \ldots \coprod \widetilde{X}_U^{|W|}.$$

We choose (an analytic) local coordinate z on U and use z (i.e., its pullback $p_b^* z$) as a coordinate on $\widetilde{X}_U \subseteq \widetilde{X}_b\backslash\mathrm{Ram}(p_b)$.

Then, setting $\boldsymbol{\gamma}$ for the coordinate vector of $p_b^*\mathbf{g}$,

$$(\phi^{-1})^*\boldsymbol{\gamma} = \left(-\boldsymbol{\beta}' \;\; D\mathbf{I}|_{\phi(\widetilde{X}_U)}\right)\begin{pmatrix} 0 \\ (D\mathbf{I})|^{-1}_{\phi(\widetilde{X}_U)}((\phi^{-1})^*\boldsymbol{\gamma}) \end{pmatrix} \in \mathcal{O}_\Delta^{\oplus l}(\Delta),$$

i.e., we obtain a lift $\widetilde{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$

$$(\phi^{-1})^*\widetilde{\boldsymbol{\gamma}} = \sum_{i=1}^{l} ((D\mathbf{I})^{-1}|_{\phi(\widetilde{X}_U)}(\phi^{-1})^*\boldsymbol{\gamma})_i \frac{\partial}{\partial\alpha_i} \in \Gamma\left(\phi(\widetilde{X}_U), (\phi^{-1})^* T_M|_{\widetilde{X}_U}\right). \tag{28}$$

Note that the expression for $\widetilde{\boldsymbol{\gamma}}$ is well-defined on $\widetilde{X}_U$: away from ramification, we can solve locally-analytically for α_i in terms of z, so $(D\mathbf{I})^{-1}$, when restricted to a connected component of $\phi(\widetilde{X}_U)$, is actually a section of $\mathrm{End}(\mathbb{C}^l) \otimes_{\mathbb{C}} \mathcal{O}_\Delta^{an}$.

Then, using the lift $\widetilde{\mathbf{g}}$ from Eq. (28), we obtain

$$\widetilde{\mathbf{g}} \lrcorner \omega|_{\widetilde{X}_U} = -\phi^* \left(\sum_{i=1}^{l} \left((D\mathbf{I})|^{-1}_{\phi(\widetilde{X}_U)} \cdot \boldsymbol{\gamma} \right)_i \otimes e_i \otimes [dz] \right) \\ - \sum_{i=1}^{l} \frac{\phi^* \left((D\mathbf{I})|^{-1}_{\phi(\widetilde{X}_U)} \cdot \boldsymbol{\gamma} \right)_i}{\alpha_i} \lambda_i \Bigg|_{\widetilde{X}_U},$$

as $(\phi^{-1})^* \lambda_i = \alpha_i [dz] \otimes e_i$. We write $[dz]$ rather than dz since the cotangent sheaf of $\widetilde{X}_U$ is a quotient of $\Omega^1_{M_U}\Big|_{\widetilde{X}_U}$. This is precisely the expression for $\widetilde{\mathbf{s}}$ from Eq. (6), written locally.

Having shown (27), we now note that the sheaf of meromorphic sections of a holomorphic vector bundle on a smooth curve is trivial (see e.g. [12, p.76]; see also [25, Lemma 31.25.3]). As two meromorphic functions that coincide away from a finite set of points are equal, Eq. (27) shows that $\mathbf{s} = \widetilde{\mathbf{s}}$. Since the two sections $\mathbf{s}$ and $\widetilde{\mathbf{s}}$ are equal, and $\mathbf{s}$ is known to be W-invariant, so is $\widetilde{\mathbf{s}}$. □

5 Examples

5.1 $SL_2(\mathbb{C})$

For completeness, we start with the simplest case of $G = SL_2(\mathbb{C})$. The Cartan subalgebra $\mathfrak{t}$ of diagonal traceless 2×2 matrices is identified with $\mathbb{C}$ via $\alpha(A) = A_{11}$ and we take the $\mathbb{Z}/2\mathbb{Z}$-invariant polynomial $I = \det$, i.e., $I(\alpha) = -\alpha^2$. The cameral (and spectral) curve $\widetilde{X}_b \subseteq \operatorname{tot} K_X^2$ has equation $\boldsymbol{\alpha}^2 = \pi^* b$, for $b \in \mathcal{B} = H^0(X, K_X^2)$. Then, for generic b, the isomorphism (5)

$$\left(\nabla^{GM} \boldsymbol{\lambda}_{SW}\right)_b : \qquad T_b\mathcal{B} = H^0(X, K_X^2) \xrightarrow{\simeq} H^0(\widetilde{X}_b, K_{\widetilde{X}_b})_-^{\mathbb{Z}/2}$$

is given by

$$\mathbf{g} \longmapsto \left(\nabla^{GM}_{\mathbf{g}} \boldsymbol{\lambda}_{SW}\right)_b = \frac{\pi^* \mathbf{g}}{2\boldsymbol{\alpha}^2} \boldsymbol{\lambda}_{SW}\Bigg|_{\widetilde{X}_b} = -\frac{\pi^* \mathbf{g}}{2\sigma} \lambda\Bigg|_{\widetilde{X}_b} = \frac{\pi^* \mathbf{g}}{2\boldsymbol{\lambda}_{SW}}\Bigg|_{\widetilde{X}_b} \tag{29}$$

where $\sigma = -\boldsymbol{\alpha}^2 \in H^0(M, \pi^* K_X^2)$ is the tautological section (10) of $\mathbf{U} = K_X^2$ and $\boldsymbol{\lambda}_{SW} = \boldsymbol{\alpha}$ is the Liouville (Seiberg–Witten) form.

5.2 $SL_3(\mathbb{C})$

Consider $G = SL_3(\mathbb{C})$, with the standard choices of Borel (upper-triangular) and Cartan (diagonal) subgroups. Here $\mathfrak{t} \subseteq \mathfrak{sl}_3(\mathbb{C})$ is the subspace of diagonal traceless 3×3 matrices and $W = S_3$. If we set $\alpha_1(A) = A_{11} - A_{22}$, $\alpha_2(A) = A_{22} - A_{33}$ (two simple positive roots), then we can choose the invariant polynomials to be

$$\left| \begin{array}{l} I_1(\alpha_1, \alpha_2) = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 \\ I_2(\alpha_1, \alpha_2) = -2\alpha_1^3 - 3\alpha_1^2\alpha_2 + 3\alpha_1\alpha_2^2 + 2\alpha_2^3. \end{array} \right. \tag{30}$$

In fact, these are $I_1(A) = -3(A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33})$ and $I_2(A) = -27 \det A$.

Consequently, the cameral curve $\widetilde{X}_b$, corresponding to a generic section $b = (b_1, b_2) \in \mathcal{B} = H^0(X, K_X^2) \oplus H^0(X, K_X^3)$ is cut out in $M = \mathrm{tot}\,(K_X^2 \oplus K_X^3)$ by the equations

$$\left| \begin{array}{l} \boldsymbol{\alpha}_1^2 + \boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_2^2 = \pi^* b_1 \\ -2\boldsymbol{\alpha}_1^3 - 3\boldsymbol{\alpha}_1^2\boldsymbol{\alpha}_2 + 3\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^2 + 2\boldsymbol{\alpha}_2^3 = \pi^* b_2 \end{array} \right. \tag{31}$$

and

$$\left(\nabla_{\mathbf{g}}^{GM}\boldsymbol{\lambda}_{SW}\right)_b = \frac{1}{\det \iota D\mathbf{I}} \begin{bmatrix} 3\boldsymbol{\alpha}_1^2 - 6\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 - 6\boldsymbol{\alpha}_2^2 & 2\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_1 \\ -6\boldsymbol{\alpha}_1^2 - 6\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + 3\boldsymbol{\alpha}_2^2 & -2\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \Bigg|_{\widetilde{X}_b},$$

where

$$\det \iota D\mathbf{I} = 27\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2).$$

5.3 G_2

It is well-known [17, p.103] that the G_2 root system can be embedded in the B_3 root system—and that in fact, this can be done in a way that simple roots of the former are expressed as linear combinations of simple roots of the latter. An explicit description of such an embedding can be obtained by extending the calculations in [1, §4], but we do not need this now. Using this embedding, we can take the $\mathfrak{g}_2$ Cartan subalgebra $\mathfrak{t}$ to consist of diagonal matrices of the form $h = \mathrm{diag}(-a-b, -a, -b, 0, b, a, a+b)$, for $a, b \in \mathbb{C}$. Two simple roots $\alpha_1, \alpha_2 \in \mathfrak{t}^\vee$ are, e.g., b and $a - b$, i.e.,

$$\alpha_1\,(h) = h_{55}, \;\; \alpha_2\,(h) = h_{66} - h_{55}.$$

The six positive roots are then $\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2$. The characteristic polynomial of $h \in \mathfrak{t}$ is

$$\det(h - \lambda E_7) = -\lambda^7 + \lambda^5 2I_1(h) - \lambda^3 I_1^2(h) + \lambda I_2(h),$$

where, if we use α_1 and α_2 as coordinates on $\mathfrak{t}$, we have for the invariants

$$\left| \begin{array}{l} I_1(\alpha_1, \alpha_2) = 3\alpha_1^2 + 3\alpha_1\alpha_2 + \alpha_2^2 \\ I_2(\alpha_1, \alpha_2) = 4\alpha_1^6 + 12\alpha_1^5\alpha_2 + 13\alpha_1^4\alpha_2^2 + 6\alpha_1^3\alpha_2^3 + \alpha_1^2\alpha_2^4. \end{array} \right. \tag{32}$$

The eigenvalues of a matrix from $\mathfrak{g}_2 \subseteq \mathfrak{so}_7$ are $0, \pm\lambda_1, \pm\lambda_2, \pm\lambda_3, \sum_{i=1}^3 \lambda_i = 0$. The two invariants are, respectively, $\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ and $(\lambda_1\lambda_2\lambda_3)^2$.

Consequently, the cameral curve $\widetilde{X}_b$, corresponding to a generic section $(b_1, b_2) \in \mathcal{B} = H^0(X, K_X^2) \oplus H^0(X, K_X^6)$ is cut out in $M = \mathrm{tot}\,(K_X^2 \oplus K_X^6)$ by the equations

$$\left| \begin{array}{l} 3\boldsymbol{\alpha}_1^2 + 3\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_2^2 = \pi^* b_1 \\ 4\boldsymbol{\alpha}_1^6 + 12\boldsymbol{\alpha}_1^5\boldsymbol{\alpha}_2 + 13\boldsymbol{\alpha}_1^4\boldsymbol{\alpha}_2^2 + 6\boldsymbol{\alpha}_1^3\boldsymbol{\alpha}_2^3 + \boldsymbol{\alpha}_1^2\boldsymbol{\alpha}_2^4 = \pi^* b_2. \end{array} \right. \tag{33}$$

Now, identifying the adjoint quotient $\chi : \mathfrak{t} \to \mathfrak{t}/W$ with $\mathbf{I} = (I_1, I_2) : \mathbb{C}^2 \to \mathbb{C}^2$, we obtain that under the isomorphism from Theorem A a section $\mathbf{g} = (g_1, g_2)^T \in H^0(X, K_X^2) \oplus H^0(X, K_X^6) = T_b\mathscr{B}$ is mapped to $\left(\nabla_{\mathbf{g}}^{GM}\boldsymbol{\lambda}_{SW}\right)_b$, i.e.,

$$\frac{1}{\det \iota D\mathbf{I}} \begin{bmatrix} -2\boldsymbol{\alpha}_1^2(6\boldsymbol{\alpha}_1^3 + 13\boldsymbol{\alpha}_1^2\boldsymbol{\alpha}_2 + 9\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^2 + 2\boldsymbol{\alpha}_2^3) & 3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 \\ 2\boldsymbol{\alpha}_1(12\boldsymbol{\alpha}_1^4 + 30\boldsymbol{\alpha}_1^3\boldsymbol{\alpha}_2 + 26\boldsymbol{\alpha}_1^2\boldsymbol{\alpha}_2^2 + 9\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2^3 + \boldsymbol{\alpha}_2^4) & -6\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2 \end{bmatrix} \begin{bmatrix} \pi^* g_1 \\ \pi^* g_2 \end{bmatrix} \Bigg|_{\widetilde{X}_b},$$

where

$$\det \iota D\mathbf{I} = -2\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)(2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)(3\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)(3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2). \tag{34}$$

We see that in all examples det $D\mathbf{I}$ is a constant multiple of the product of all positive roots. In fact, this follows from a classical result of Steinberg [24]. Hence (det $D\mathbf{I})^2$ is proportional to the discriminant $\mathfrak{D}$ of $\mathfrak{g}$—the product of all roots. Being W-invariant, the discriminant can be expressed as a polynomial in the generators of $\mathbb{C}[\mathfrak{t}]^W$—here, I_1 and I_2. We sketch a possible way of obtaining this expression without too much brute force. Using the embedding $\mathfrak{g}_2 \subseteq \mathfrak{so}_7$, we can identify α_1 and α_2 as λ_1 and $\lambda_2 - \lambda_1$, up to reordering λ_i's. Consequently,

$$\begin{aligned}\mathfrak{D} &= \alpha_1^2(\alpha_1+\alpha_2)^2(2\alpha_1+\alpha_2)^2\alpha_2^2(3\alpha_1+\alpha_2)^2(3\alpha_1+2\alpha_2)^2\\ &= \lambda_1^2\lambda_2^2\lambda_3^2(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2.\end{aligned} \tag{35}$$

The product of the first three terms is I_2. The product of the last three terms is a polynomial of degree 6, and hence must be a linear combination of I_1^3 and I_2, and so one checks immediately that

$$\mathfrak{D} = I_2(4I_1^3 - 27I_2).$$

Consequently, the restriction of $\iota\mathfrak{D}$ to the universal cameral cover $\mathcal{X}$ is the pull-back of a section of $\mathcal{O}_{\mathscr{B}} \otimes H^0(X, K_X^{12})$, namely,

$$\mathscr{B} \ni (b_1, b_2) \longmapsto b_2(4b_1^3 - 27b_2).$$

It is the Lie derivative of this section that enters the Balduzzi–Pantev formula and its generalization [2, 3].

We refer the reader to the beautiful papers [15, 19] for additional details on the G_2-Hitchin system, including Langlands duality and the description of Hitchin fibres.

Funding Open access funding provided by Scuola Internazionale Superiore di Studi Avanzati - SISSA within the CRUI-CARE Agreement. U.B.'s research was partly supported by Bolsa de Produtividade 313333/2020-3 of Brazilian CNPq, by PRIN 2017 "Moduli theory and birational classification" and INdAMGNSAGA. P.D. thanks Tony Pantev for helpful discussions related to the project, the Simons Collaboration on Homological Mirror Symmetry for support and the University of Pennsylvania for its hospitality.

References

1. Adler, M., van Moerbeke, P.: Completely integrable systems, Euclidean Lie algebras, and curves. Adv. Math. **38**, 267–317 (1980)
2. Balduzzi, D.: Donagi–Markman cubic for Hitchin systems. Math. Res. Lett. **13**(5–6), 923–933 (2006)
3. Bruzzo, U., Dalakov, P.: Donagi–Markman cubic for the generalized Hitchin system. Int. J. Math. **25**(2), 1450016,20 (2014). arXiv:1308.6788
4. Deligne, P.: Travaux de Griffiths, in Séminaire Bourbaki : vol. 1969/70, exposés 364-381, Séminaire Bourbaki, no. 12 (1971), Talk no. 376, 25 p
5. Douady, A., Hubbard, J.: On the density of Strebel differentials. Invent. Math. **30**(2), 175–179 (1975)
6. Donagi, R., Markman, E.: Cubics, integrable systems, and Calabi–Yau threefolds. In: Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan), Bar-Ilan Univ., Ramat Gan, pp. 199–221 (1996)

7. Donagi, R., Markman, E.: Spectral curves, algebraically completely integrable, Hamiltonian systems, and moduli of bundles. In: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), vol. 1620. LNM, pp. 1–119. Springer, Berlin (1996)
8. Dumas, D., Neitzke, A.: Asymptotics of Hitchin's metric on the Hitchin section. Commun. Math. Phys. **367**(1), 127–150 (2019)
9. Donagi, R., Pantev, T.: Langlands duality for Hitchin systems. Invent. Math. **189**(3), 653–735 (2012)
10. Eisenbud, D., Harris, J.: 3264 and All That—A Second Course in Algebraic Geometry. Cambridge University Press, Cambridge (2016)
11. Fredrickson, L.: Exponential decay for the asymptotic geometry of the Hitchin metric. Commun. Math. Phys. **375**(2), 1393–1426 (2020)
12. Gunning, R.C.: Lectures on Vector Bundles over Riemann Surfaces. University of Tokyo Press; Princeton University Press, Tokyo; Princeton (1967)
13. Hertling, C., Hoevenaars, L., Posthuma, H.: Frobenius manifolds, projective special geometry and Hitchin systems. J. Reine Angew. Math. **649**, 117–165 (2010)
14. Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**(1), 91–114 (1987)
15. Hitchin, N.: Langlands duality and G_2 spectral curves. Q. J. Math. **58**(3), 319–344 (2007)
16. Hurtubise, J.C., Markman, E.: Rank 2-integrable systems of Prym varieties. Adv. Theor. Math. Phys. **2**(3), 633–695 (1998)
17. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9. Springer, New York (1978). Second printing, revised
18. Katz, N.M., Oda, T.: On the differentiation of de Rham cohomology classes with respect to parameters. J. Math. Kyoto Univ. **8**, 199–213 (1968)
19. Katzarkov, L., Pantev, T.: Stable G_2 Higgs bundles and algebraically completely integrable systems. Compos. Math. **92**(1), 43–60 (1994)
20. Markman, E.: Algebraic geometry, integrable systems, and Seiberg–Witten theory. In: Integrability: The Seiberg–Witten and Whitham Equations (Edinburgh, 1998) Gordon and Breach, Amsterdam, pp. 23–41 (2000)
21. Mazzeo, R., Swoboda, J., Weiss, H., Witt, F.: Asymptotic geometry of the Hitchin metric. Commun. Math. Phys. **367**(1), 151–191 (2019)
22. Simpson, C.T.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. Am. Math. Soc. **1**(4), 867–918 (1988)
23. Simpson, C.: Higgs bundles and local systems. Inst. Hautes Études Sci. Publ. Math. **75**, 5–95 (1992)
24. Steinberg, R.: Invariants of finite reflection groups. Can. J. Math. **12**, 616–618 (1960)
25. Various authors. The Stacks Project. https://stacks.math.columbia.edu. Accessed 7 Nov 2023
26. Voisin, C.: Hodge Theory and Complex Algebraic Geometry I. Cambridge Studies in Advanced Mathematics, vol. 76, English edn. Cambridge University Press, Cambridge (2007)
27. Voisin, C.: Hodge Theory and Complex Algebraic Geometry. II. Cambridge Studies in Advanced Mathematics, vol. 76, English edn. Cambridge University Press, Cambridge (2007). Translated from the French by Leila Schneps