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On some viscoelastic problems with memory in time-dependent domains

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Abstract

This thesis is devoted to the study of some dynamic viscoelastic models with memory in time-dependent domains. In the first chapter we consider a domain with prescribed time-dependent cracks, while in the second chapter also the cracks have to be determined. In the third chapter we regard the same viscoelastic material with memory in the contest of a one-dimensional debonding problem, with assigned debonding front.

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Contents

Introduction

This thesis is devoted to the study of some dynamic viscoelastic models with memory in time-dependent domains. More precisely, in the first two chapters we consider the case of a domain with time-dependent cracks, while in the third chapter we study a one-dimensional debonding model (with assigned debonding front).

In the literature there are several results for hyperbolic equations on noncylindrical domains, namely time-dependent domains. For the wave equation, we refer to [3], [51], [14], [15], [25], [40], [45], and [50], where the problem is studied with different techinques (e.g. Galerkin methods, change of variable, semigroup theory). We also refer to [2], [4], [7], [34], and [36], for different evolution equations in time dependent domains.

Moreover, for the particular case of domains with time-dependent cracks one of the first work is [17], while for the debonding problem we refer to [20]. In [17] and [20] the authors considered the wave equation (see also, e.g., [21, 8, 9, 24, 42]), which is used to study elastic materials without damping. In recent years more attention has been paid to materials exhibiting viscoelastic behaviour, in particular to viscoelastic materials with memory.

We now describe in more details the viscoelastic problem with memory we study in this thesis. Let $T > 0$ and let $\{\Omega_t\}_{t\in[-\infty,T]}$ be a time-dependent family of open subset of $\mathbb{R}^d$, with $d \geq 1$. The viscoelastic problem with memory we are considering in this thesis has the form

$$\ddot{u}(t,x) - \operatorname{div}((\mathbb{C}(x)+\mathbb{V}(x))Eu(t,x)) + \operatorname{div}\Big(\int_{-\infty}^{t} \mathrm{e}^{\tau-t}\,\mathbb{V}(x)Eu(\tau,x)\,\mathrm{d}\tau\Big) = f(t,x), \tag{1}$$

for $(t,x) \in \cup_{s\in(-\infty,T)}\{s\}\times\Omega_s$, where u, Eu, and $\ddot{u}$, are the displacement, the symmetric part of its gradient, and its second derivative with respect to time, $\mathbb{C}$ and $\mathbb{V}$ are the elasticity and viscosity tensors, while f is the external load. For the system in (1) the stress is given by

$$\sigma(t,x) = \mathbb{C}(x)Eu(t,x) + \mathbb{V}(x)Eu(t,x) - \int_{-\infty}^{t} \mathrm{e}^{\tau-t}\,\mathbb{V}(x)Eu(\tau,x)\,\mathrm{d}\tau. \tag{2}$$

Moreover, as in [16, 31] we assume that we know the displacement u on $(-\infty,0]$ and we want to solve (1) on $[0,T]$. It is convenient to write (1) in the form

$$\ddot{u}(t,x) - \operatorname{div}(\sigma_0(t,x)) = l_0(t,x) \quad (t,x) \in \cup_{s\in(0,T)}\{s\}\times\Omega_s, \tag{3}$$

where

$$\sigma_0(t,x) = \mathbb{C}(x)Eu(t,x) + \mathbb{V}(x)Eu(t,x) - \int_0^t \mathrm{e}^{\tau-t}\,\mathbb{V}(x)Eu(\tau,x)\,\mathrm{d}\tau, \tag{4}$$

$$l_0(t,x) := f(t,x) - \mathrm{div}F_0(t,x), \tag{5}$$

$$F_0(t,x) := \int_{-\infty}^0 \mathrm{e}^{\tau-t}\,\mathbb{V}(x)Eu_0(\tau,x)\,\mathrm{d}\tau \tag{6}$$

and u_0 is a function that represents the displacement on $(-\infty,0]$, namely $u(t,x) = u_0(t,x)$ for every $(s,x) \in \cup_{s\in(-\infty,0]}\{s\}\times\Omega_s$. Moreover, system (3) is is complemented by suitable initial and boundary conditions.

When the domains are constant in time, viscoelastic problems with memory similar to (1) and (3) have a long history and were studied by Boltzmann ([5], [6]) and Volterra ([48], [49]), while recent results can be found in [26], [31], [32], and [46].

In this thesis we consider the case of time-dependent domains, more precisely we study the particular cases related to crack growth and to a one dimensional debonding problem. In the first case we have $\Omega_t = \Omega\setminus\Gamma_t$, where Ω is an open bounded subset of $\mathbb{R}^d$ and Γ_t is a $(d-1)$-dimensional closed subset of $\overline{\Omega}$ increasing with respect to time, which represents the crack at time $t\in[0,T]$ (see Chapter 1 and 2). In the debonding problem we have $\Omega_t = (0,\ell(t))$, where ℓ is an increasing function, which represents the debonding front at time $t\in[0,T]$ (see Chapter 3).

The thesis is organized in three chapters.

Chapter 1: A viscoelastic problem with prescribed time dependent cracks

This chapter is devoted to the study of the viscoelastic model (3), when $d\geq 2$, $\Omega_t = \Omega\setminus\Gamma_t$, and $\{\Gamma_t\}_{[0.T]}$ is prescribed. This is a preliminary step to solve the dynamic crack problem, where $\{\Gamma_t\}_{[0.T]}$ is not prescribed (see Chapter 2).

The system without damping terms, which reduces to the elastodynamics system, is

$$\ddot{u}(t,x) - \mathrm{div}(\mathbb{C}(x)Eu(t,x)) = f(t,x) \quad t\in[0,T],\, x\in\Omega\setminus\Gamma_t, \tag{7}$$

and is studied in [17, 21, 8, 9, 24]. Regarding viscoelastic materials, the well known Kelvin-Voigt's model leads to the system

$$\ddot{u}(t,x) - \mathrm{div}(\mathbb{C}(x)Eu(t,x)) - \mathrm{div}(\mathbb{V}(x)E\dot{u}(t,x)) = f(t,x) \quad t\in[0,T],\, x\in\Omega\setminus\Gamma_t, \tag{8}$$

which is studied in [17, 47, 10]. As explained in details in the introduction of Chapter 2, this model can not be used in dynamic crack growth because it leads to the Viscoelastic Paradox (i.e., the crack can not increase). This is the reason why we study a different problem, namely the model with memory given by (3).

The existence of a solution u for problem (3) (with prescribed cracks $t\mapsto\Gamma_t$) is given by [43], together to an energy inequality.

In this chapter we present the results obtained in [11], in collaboration with G. Dal Maso, on the uniqueness of the solution u (Theorem 1.1.9) and on its continuous dependence on the data, in particular on the cracks $\{\Gamma_t\}_{t\in[0,T]}$ (Theorem 1.3.2). For the case of the pure elasticity or the Kelvin-Voigt's model the argument adopted to prove uniqueness is based on an change of variable, which allows to recast the original problem in a new one with a more complex equation, but with domain constant in time (see [21, 8, 9, 10]). Unfortunately, this technique can not be applied to (3), due to the difficulties given by the integral term.

To overcome this problem, we consider a new argument, relying on a fixed point argument. More precisely, we write problem (3) in the equivalent form

$$\ddot{u}(t,x) - \operatorname{div}((\mathbb{C}(x) + \mathbb{V}(x))Eu(t,x)) = l_0(t,x) - \operatorname{div}F_u(t,x) \;\; t \in [0,T], \;\; x \in \Omega \setminus \Gamma_t \tag{9}$$

where

$$F_u(t,x) := \int_0^t \mathrm{e}^{\tau-t}\, \mathbb{V}Eu(\tau,x)\,\mathrm{d}\tau. \tag{10}$$

In (9) we regard the viscoelastic problem with memory as an elastic problem with forcing term depending on u. This allows us to estimate u in terms of F_u using the energy inequality for the solution of (7). Then we estimate F_u in terms of u using just the its definition, and uniqueness is obtained from the combined estimate (Subsection 1.2.2). In order to apply this argument, in Subsection 1.2.1 we have to extend the results known in the literature for (7) to more general forcing terms, which will be needed to study (9).

Our second result (Theorem 1.3.2) is the continuous dependence of the solutions of (3) on the cracks. More precisely, we consider a sequence Γ_t^n of time dependent cracks and the solutions u^n of problem (3) with Γ_t replaced by Γ_t^n. Under suitable assumption on the convergence of Γ_t^n to Γ_t we prove that the sequence u^n converges to the solution u of (3). Our assumptions of Γ_t^n are similar to those considered in [21] and [8] to prove the corresponding result for (7).

To prove the continuous dependence we write our problem in the form (9) and we regard u^n as a fixed point for a suitable operator depending on n, which is a contraction if T is small enough. Under this assumption the convergence of u^n is a consequence of a general results on fixed points of contractions (Lemma 1.3.3). To show that its hypotheses are satisfied, we use the continuous dependence on the cracks of the solutions of problem (7) (see [21] and [8]) and we obtain the result for (3) if T is small enough. If T is large we divide the interval $[0,T]$ into smaller intervals, where we can apply the previous result.

The chapter is organized as follows:

- in Section 1.1 we give the precise formulation of problem (3):
  - in Subsection 1.1.1 we make precise the hypotheses on the data and we recall some basic results in crack theory;

  - in Subsection 1.1.2 we define the function spaces involved in the weak formulation of the problem;

- in Section 1.2 we prove the uniqueness for problem (3):

  - in Subsection 1.2.1 we generalized some results of [21] and [8] regarding energy estimates for the wave equations;
  - in Subsection 1.2.2 we prove the uniqueness for (3) (Theorem 1.1.9);

- in Section 1.3 we study the continuous dependence on the data:

  - in Subsection 1.3.1 we prove some preliminary results, extending the results of [21] and [8];
  - in Subsection 1.3.2 we prove the continuous dependence (Theorem 1.3.2).

Chapter 2: Dynamic crack growth in viscoelastic materials with memory

This chapter is devoted to the study of dynamic crack growth in viscoelastic materials governed by system (3). The original results of this chapter are contained in [12].

In the same spirit of [18] and [19], we can describe the problem of dynamic crack growth as following. Given a $(d-1)$-dimensional closed set $\Gamma_{in} \subset \overline{\Omega}$ (the crack at initial time), we want to find a family of $(d-1)$-dimensional closed sets $\{\Gamma_t\}_{t\in[0,T]}$ (the cracks at every time) and a displacement function $u(t)\colon \Omega \setminus \Gamma_t \to \mathbb{R}^d$ such that

i) u satisfy (3) on $\bigcup_{s\in(0,T)}\{s\} \times (\Omega \setminus \Gamma_s)$ with prescribed initial conditions at $t = 0$ and boundary conditions on $\partial\Omega$ and Γ_t;

ii) $\Gamma_0 = \Gamma_{in}$ and $\Gamma_s \subset \Gamma_t$ if $s < t$;

iii) u and $\{\Gamma_t\}_{t\in[0,T]}$ satisfy a dynamic energy-dissipation balance, a dynamic version of Griffith's criterion (see [35], [39], and [37]), namely

$$\mathcal{E}(t) + \mathcal{D}(t) + \mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_{in}) = \mathcal{E}(0) + \mathcal{W}(t), \quad \text{for every } t \in [0,T], \tag{11}$$

where $\mathcal{E}(t)$ is the sum of kinetic and elastic energy at time t, and $\mathcal{D}(t)$ is the energy dissipated by viscosity in the time interval $[0,t]$, $\mathcal{H}^{d-1}(\Gamma_t \setminus \Gamma_{in})$ is the $(d-1)$-dimensional Hausdorff measure of $\Gamma_t \setminus \Gamma_{in}$ (interpreted as the energy dissipated to produce the crack), and $\mathcal{W}(t)$ is the work done by the external forces in the interval $[0,t]$;

iv) $\{\Gamma_t\}_{t\in[0,T]}$ satisfies a maximal dissipation condition, which forces the crack to run as fast as possible (see Definition 2.3.1).

Conditions i) and iii) provide a coupling between the displacement u and the cracks $\{\Gamma_t\}_{t\in[0,T]}$. For this reason, a pair $(u, \{\Gamma_t\}_{t\in[0,T]})$ satisfying conditions i)-iv) is called a solution of the coupled problem.

Note that, if in i) the model with memory is replaced by Kelvin-Voigt's model described by (8), then it is known (see [17] and [47]) that the corresponding solution $(u, \{\Gamma_t\}_{t\in[0,T]})$ of the coupled problem satisfies $\Gamma_t = \Gamma_0$ for every $t \in [0,T]$, namely the crack never grows. This phenomenon is known in the mechanical literature as the Viscoelastic Paradox (see, e.g., [46]). It motivates our choice of the viscoelastic model (3) in i).

In this chapter we study problem i)-iv) when $\Omega \subset \mathbb{R}^2$ (namely $d = 2$) and assuming very strong regularity conditions on the shape of the cracks $\{\Gamma_t\}_{t\in[0,T]}$ and on their dependence on t. Under this a priori assumptions, we prove the existence of a solution of the coupled problem (Theorem 2.3.3).

The proof of this result follows the lines of [18] and [19], where the case of pure elasticity is considered. To deal with the memory term given by (10), we use the results of Chapter 1. In particular the continuous dependence on the data is a fundamental tool for a compactness argument that plays a key role in the proof of Theorem 2.3.3.

The chapter is organized as follows:

- in Section 2.1 we give the precise formulation of the problem and all the preliminary results:
  - in Subsection 2.1.1 we describe the a priori bounds on the geometry of the cracks and on their time evolution;
  - in Subsection 2.1.2 we define the function spaces for the weak formulation of the problem;
  - in Subsection 2.1.3 we extend some preliminary results of Chapter 1;
- in Section 2.2 we deal with the energies involved in the problem:
  - in Subsection 2.2.1 we make precise the formulation of the dynamic energy-dissipation balance;
  - in Subsection 2.2.2 we define the class of cracks $\{\Gamma_t\}_{t\in[0,T]}$ such that the energy-dissipation balance is satisfied, we show that the this class is non-empty, and we prove a compactness result;
- in Section 2.3 we define the maximal dissipation condition and we prove the main result of the chapter (Theorem 2.3.3).

Chapter 3: A viscoelastic problem with prescribed debonding front

This chapter is devoted to the study of a debonding problem for a viscoelstic bar with memory. The reference configuration of the bar is the interval $[0, L]$, for a given $L > 0$. We assume that at every time t the portion $[\ell(t), L]$ is attached to a rigid substrate, while the behaviour of the detached part of the bar is governed by the equation

$$u_{tt}(t,x)-u_{xx}(t,x)+\int_0^t \frac{\mathrm{e}^{\tau-t}}{2} u_{xx}(\tau,x)\,\mathrm{d}\tau = f(t,x)-F_x(t,x), \quad t\in[0,T], x\in(0,\ell(t)). \quad (12)$$

In the previous formula u represents the longitudinal displacement of the reference configuration (extended as $u = 0$ outside $\{(t,x)\,|\,t \in [0,T], x \in (0,\ell(t))\}$), while the right hand side denotes the forcing terms. Equation (12) is a particular case of (3) when $d = 1$, $\Omega_t = (0,\ell(t))$, and $\mathbb{C} = \mathbb{V} = \frac{1}{2}$.

Since the bar is attached on $[\ell(t), L]$, we have that the displacement satisfies $u(t,x) = 0$ for every $t \in [0,T]$ and $x \in [\ell(t), L]$. The number $\ell(t)$ represents the debonding front at time t and, since we assume that the debonding process is irreversible, the function ℓ is increasing. In the debonding problem besides (12) one considers another equation that couples ℓ and u and one has to solve the coupled problem.

In this chapter we consider only (12) assuming that the function ℓ is a prescribed Lipschitz function satisfying $0 \le \dot{\ell}(t) \le 1$. Our main result is the existence and uniqueness of the solution of (12) with prescribed initial and boundary conditions (see Theorem 3.2.18). Moreover, we obtain some regularity results for the solution u and for its energy considered as a function of time (see Theorem 3.3.5).

As done in the previous chapters, in order to solve (12) we study the auxiliary problem without viscosity given by

$$u_{tt}(t,x)-u_{xx}(t,x) = f(t,x)-F_x(t,x), \quad t\in[0,T], x\in(0,\ell(t)). \quad (13)$$

Existence and uniqueness for problem (12) are then obtained by a fixed point argument.

Problem (13) is studied in [20] when $f = F = 0$ and in [42] when $F = 0$. Since the spatial derivative F_x is defined only in the sense of distributions, we cannot apply the previous results directly. However, we are able to prove that the solution of (13) is given by an explicit formula, which extends the classical d'Alembert's formula. We note that this formula is much more complex than to the corresponding one for $F = 0$ (obtained in [42]). Thanks to this formula we also get some extra regularity for the solution, which will be crucial for some final results regarding energy balance.

Using these results we are able to prove in Section 3.3 that the total energy corresponding to (12) is absolutely continuous with respect to time. This is a crucial step in order to define the the equation coupling ℓ and u, and to study in a future work (see [13]) the coupled problem, where also ℓ is unknown.

The chapter is organized as follows:

- in Section 3.1 we give some basic definitions;
- in Section 3.2 we prove existence and uniqueness for the auxiliary problem (13) and the original viscoelstic problem (12), more precisely

- in Subsection 3.2.1 we describe deal with some geometric considerations;
- in Subsection 3.2.2 we prove the representation formula and some regularity results;
- in Subsection 3.2.3 we prove existence and uniqueness for (12);

- in Section 3.3 we study the total energy of the system:
  - in Subsection 3.3.1 we prove that it is absolutely continuous with respect to time;
  - in Subsection 3.3.2 we explain the main ideas that will be used in future works to study the debonding problem with non.prescribed debonding front.

Notations

Basic notation

Let d and m be positive integers.

$\mathbb{R}^{m\times d}$	Space of real $m \times d$ matrices.
$\mathbb{R}^{d\times d}_{sym}$	Space of real symmetric $m \times d$ matrices.
A^T	Transpose of $A \in \mathbb{R}^{d\times d}$.
A^{-1}	Inverse of $A \in \mathbb{R}^{d\times d}$.
A^{sym}	Symmetric part of $A \in \mathbb{R}^{d\times d}$.
I	Identity matrix in $\mathbb{R}^{d\times d}$.
$A : B$	Euclidian scalar product between $A, B \in \mathbb{R}^{m\times d}$.
$a \otimes b$	Tensor product between two vectors $a, b \in \mathbb{R}^d$.
$\mathcal{L}^d$	The d-dimensional Lebesgue measure in $\mathbb{R}^d$.
$\mathcal{H}^{d-1}$	The $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.
$B_r(x)$	The ball of radius r and center x in $\mathbb{R}^d$.
$\max\{\alpha, \beta\}$	Maximum between $\alpha, \beta \in \mathbb{R}$.
$\min\{c_1, c_2\}$	Minimum between $\alpha, \beta \in \mathbb{R}$.
∂_{x_i}	Partial derivatives with respect to the variable x_i.
Du	Jacobian matrix of the function u.
Eu	Jacobian matrix of the function u, namely $Eu \coloneqq \frac{1}{2}(\nabla u + \nabla u^T)$.
$\mathrm{div} T$	Divergence with respect to rows of a tensor T.

Function spaces

Let X, Y be two metric spaces, let $\Omega \subset \mathbb{R}^d$ be an open set, let $(a,b) \subset \mathbb{R}$ be an open interval, let X be a Banach space and let $p \in [1,+\infty]$.

$C^0(X;Y)$	Space of continuous functions from X to Y.
$Lip(X;Y)$	Space of Lipschitz functions from X to Y.
$C^k(\Omega;\mathbb{R}^m)$	Space of $\mathbb{R}^m$-valued functions with k continuous derivatives.
$C^k_c(\Omega;\mathbb{R}^m)$	Space of $C^k(\Omega;\mathbb{R}^m)$ functions with compact support in Ω.
$C^\infty_c(\Omega;\mathbb{R}^m)$	Space of functions that belong to $C^k_c(\Omega;\mathbb{R}^m)$ for every k.
$C^{k,1}(\Omega;\mathbb{R}^m)$	Space of $C^k(\Omega;\mathbb{R}^m)$ functions whose k-derivatives are Lipschitz.
$\mathcal{D}'(\Omega)$	Space of distributions on Ω.
$L^p(\Omega;\mathbb{R}^m)$	Lebesgue space of p-integrable functions.
$W^{k,p}(\Omega;\mathbb{R}^m)$	Sobolev space with k derivatives and p-integrable.
$H^k(\Omega;\mathbb{R}^m)$	Sobolev space with k derivatives and 2-integrable.
$L^p((a,b);X)$	Bochner-Lebesgue space of p-th power integrable functions.
$W^{k,p}((a,b);X)$	Bochner-Sobolev space with k derivatives and p-integrable.
$H^k((a,b);X)$	Bochner-Sobolev space with k derivatives and 2-integrable.
$C^k([a,b];X)$	Bochner space of functions with k continuous derivatives.
$AC([a,b];X)$	Bochner space of absolutely continuous functions.
$C^0_w([a,b];X)$	Bochner space of weakly continuous functions.

Remark 0.0.1. When $m = 1$, we omit $\mathbb{R}^m$ in the previous spaces, e.g. we write $H^k(\Omega)$ instead of $H^k(\Omega;\mathbb{R}^m)$.

Remark 0.0.2. With a slight abuse of notation, we use $\dot{\varphi}$ to denote both the Bochner derivative with respect to time (if $\varphi = \varphi(t,x)$) and the derivative with respect to a single variable (if $\varphi = \varphi(s)$), depending on the context.

Remark 0.0.3. Every function in $W^{k,p}(a,b;X)$ is always identified with its continuous representative on $[a,b]$.

Remark 0.0.4. We make the usual identifications for Bochner spaces, e.g. $L^2(a,b;L^2(\Omega)) \simeq L^2((a,b)\times\Omega)$ and $H^1(a,b;L^2(\Omega)) \cap L^2(a,b;H^1(\Omega)) \simeq H^1((a,b)\times\Omega)$.

Chapter 1

A viscoelastic problem with prescribed time dependent cracks

Contents

In this chapter we study the viscoelastic problem with memory on domains with prescribed time dependents cracks. More precisely, we consider

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) + \operatorname{div}\Big(\int_0^t \mathrm{e}^{\tau - t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big) = \ell(t) \quad \text{in } Q_{cr}, \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^d$ is the reference configuration, $[0,T]$ is the time interval, $\{\Gamma_t\}_{t\in[0,T]}$ is the prescribed crack, $Q_{cr} := \{(x,t) : t \in [0,T],\, x \in \Omega \setminus \Gamma_t\}$, $u(t)$, $Eu(t)$, and $\ddot{u}(t)$ are the displacement at time t, the symmetric part of its gradient, and its second derivative with respect to time, $\mathbb{C}$ and $\mathbb{V}$ are the elasticity and viscosity tensors, and $\ell(t)$ is the external load at time t. Problem (1.0.1) is complemented by initial conditions at $t = 0$ for u and $\dot{u}$ and by boundary conditions on $\partial\Omega$ and Γ_t. See the Introduction for more details on the physical interpretation of (1.0.1).

The chapter is organized ad follows. In Section 1.1 we give the precise formulation of problem (1.0.1). More precisely, we give the hypotheses on the data (in particular on the cracks) and we define the weak formulation of the problem.

Section 1.2 is devoted to the proof of the uniqueness for problem (1.0.1). In particular, in Subsection 1.2.1 we generalized some results of [21] and [8] regarding energy inequalities for the elastodynamic equation (that is problem (1.0.1) without the memory term), namely

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C}+\mathbb{V})Eu(t)) = \ell(t) \quad \text{in } Q_{cr}. \tag{1.0.2}$$

These inequalities will be crucial in order to have some estimates of the norm of the solution of the wave equations in terms of the norm of the data. Using this estimates and a fixed point argument, in Subsection 1.2.2 we prove the uniqueness for (1.0.1) (Theorem 1.1.9).

In Section 1.3 we study the continuous dependence on the data, in particular on cracks. More precisely, in Subsection 1.3.1 we extend the results of [21] and [8] by proving some results concerning the wave when the forcing term is more general and the formulation of the problem is weaker. Finally, in Subsection 1.3.2 we prove the continuous dependence (Theorem 1.3.2), which will be foundamental for the proof of the results of Chapter 2.

All the original results of this chapter are based on [11].

1.1 Formulation of the problem

The reference configuration of our problem is a bounded open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega$ and $\partial_N\Omega$ are disjoint (possibly empty) Borel sets, on which we prescribe Dirichlet and Neumann boundary conditions respectively.

1.1.1 Basic notions

For every $x \in \overline{\Omega}$ the elasticity tensor $\mathbb{C}(x)$ and the viscosity tensor $\mathbb{V}(x)$ are prescribed elements of the space $\mathcal{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ of linear maps from $\mathbb{R}^{d\times d}_{sym}$ into $\mathbb{R}^{d\times d}_{sym}$, where $\mathbb{R}^{d\times d}_{sym}$ is the space of reald $d\times d$ symmetric matrices. The euclidean scalar product between the matrices A and B is denoted by $A : B$. We assume that the functions $\mathbb{C}$, $\mathbb{V}\colon \overline{\Omega} \to \mathcal{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ satisfy the following properties, for suitable constants $\alpha_0 > 0$ and $M_0 > 0$:

(H1) (regularity) $\mathbb{C}$ is of class C^1 and $\max_{x\in\overline{\Omega}}|\mathbb{C}(x)| \leq M_0$;

(H2) (symmetry) $\mathbb{C}(x)A : B = A : \mathbb{C}(x)B$ for every $x \in \overline{\Omega}$ and A, $B \in \mathbb{R}^{d\times d}_{sym}$;

(H3) (coerciveness) $\mathbb{C}(x)A : A \geq \alpha_0|A|^2$ for every $x \in \overline{\Omega}$ and $A \in \mathbb{R}^{d\times d}_{sym}$;

(H4) (regularity) $\mathbb{V}$ is of class C^1 and $\max_{x\in\overline{\Omega}}|\mathbb{V}(x)| \leq M_0$;

(H5) (symmetry) $\mathbb{V}(x)A : B = A : \mathbb{V}(x)B$ for every $x \in \Omega$ and $A, B \in \mathbb{R}^{d\times d}_{sym}$;

(H6) (coerciveness) $\mathbb{V}(x)A : A \geq \alpha_0|A|^2$ for every $x \in \overline{\Omega}$ and $A \in \mathbb{R}^{d\times d}_{sym}$.

Throughout the chapter we study the problem in the time interval $[0, T]$, with $T > 0$. For $t \in [0, T]$ the crack at time t is given by a subset Γ_t of the intersection between $\overline{\Omega}$ and a suitable $d - 1$ dimensional manifold Γ (regarded as the crack path). We assume that

(H7) Γ is a complete $(d-1)$-dimensional C^2 manifold with boundary;

(H8) $\Omega \cap \partial\Gamma = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$, where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensianal Hausdorff measure;

(H9) for every $x \in \Gamma \cap \partial\Omega$ there exists an open neighborhood U_x of x in $\mathbb{R}^d$ such that $U_x \cap (\Omega \setminus \Gamma)$ is the union of two non empty disjoint open sets U_x^+ and U_x^- with Lipschitz boundary;

(H10) Γ_t is closed, $\Gamma_t \subset \Gamma \cap \overline{\Omega}$ for every $t \in [0, T]$, and $\Gamma_s \subset \Gamma_t$ for every $s < t$ (irreversibility of the fracture process).

Moreover we assume that there exist $\Phi, \Psi : [0, T] \times \overline{\Omega} \to \overline{\Omega}$ with the following properties:

(H11) Φ, Ψ are of class $C^{2,1}$;

(H12) $\Psi(t, \Phi(t, y)) = y$ and $\Phi(t, \Psi(t, x)) = x$ for every $x, y \in \overline{\Omega}$;

(H13) $\Phi(t, \Gamma) = \Gamma$, $\Phi(t, \Gamma_0) = \Gamma_t$, and $\Phi(t, y) = y$ for every $t \in [0, T]$ and every y in a neighborhood of $\partial\Omega$;

(H14) $\Phi(0, y) = y$ for every $y \in \overline{\Omega}$;

(H15) for every $y \in \overline{\Omega}$

$$|\dot{\Phi}(t, y)|^2 < \frac{m_{det}(\Psi)\alpha_0}{M_{det}(\Psi)K},$$

where the dot denotes the derivative with respect to t, $m_{det}(\Psi) := \min \det D\Psi$, $M_{det}(\Psi) := \max \det D\Psi$ and K is the constant in Korn's inequality in Lemma 1.1.2 below.

We shall prove that our hypotheses imply that Korn's inequality holds on $\Omega \setminus \Gamma$. We begin with the following technical lemma.

Lemma 1.1.1. *Under hypotheses* (H7)-(H9)*, the set* $\Omega \setminus \Gamma$ *is the union of a finite number of connected open sets with Lipschitz boundary.*

Proof. Since Γ is a C^2 manifold of dimension $d-1$, for every $x \in \Gamma \cap \Omega$ there exists an open neighborhood U_x of x in $\mathbb{R}^d$ such that $U_x \cap (\Omega \setminus \Gamma)$ is the union of two non empty disjoint open sets U_x^+ and U_x^- with Lipschitz boundary.

By our hypothesis on $\Gamma \cap \partial\Omega$ the same property holds, more in general, for every $x \in \Gamma \cap \overline{\Omega}$. Since $\Gamma \cap \overline{\Omega}$ is compact, there exists a finite number of points $x_1, ..., x_m \in \Gamma \cap \overline{\Omega}$ such that

$$\Gamma \cap \overline{\Omega} \subset \cup_{i=1}^m U_{x_i}.$$

Since Ω has Lipschitz boundary, for every $y \in \partial\Omega \setminus \cup_{i=1}^m U_{x_i} \subset \partial\Omega \setminus \Gamma$ there exists an open neighborhood V_y of y in $\mathbb{R}^d$ such that $V_y \cap (\Omega \setminus \Gamma)$ has Lipschitz boundary. By compactness there exists a finite number of points $y_1, ..., y_n \in \partial\Omega \setminus \cup_{i=1}^m U_{x_i}$ such that $\partial\Omega \setminus \cup_{i=1}^m U_{x_i} \subset \cup_{j=1}^n V_{y_j}$.

Since $\overline{\Omega} \setminus (\cup_{i=1}^m U_{x_i} \cup \cup_{j=1}^n V_{y_j})$ is compact and is contained in the open set $\Omega \setminus \Gamma$, there exists an open set W with Lipschitz boundary such that

$$\overline{\Omega} \setminus (\cup_{i=1}^m U_{x_i} \cup \cup_{j=1}^n V_{y_j}) \subset W \subset \Omega \setminus \Gamma$$

Therefore

$$\Omega \setminus \Gamma = W \cup \bigcup_{i=1}^m U_{x_i}^+ \cup \bigcup_{i=1}^m U_{x_i}^- \cup \bigcup_{j=1}^n (V_{y_j} \cap (\Omega \setminus \Gamma)).$$

Since every open sets with Lipschitz boundary is the union of a finite number of connected open sets with Lipschitz boundary, the conclusion follows. □

For every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ Du denotes jacobian matrix in the sense of distributions on $\Omega \setminus \Gamma$ and Eu is its symmetric part, i.e.,

$$Eu := \frac{1}{2}(Du + Du^T).$$

Lemma 1.1.2. *Under hypotheses* (H7)-(H9)*, there exists a constant K, depending only on Ω and Γ, such that*

$$\|Du\|^2 \leq K(\|u\|^2 + \|Eu\|^2) \tag{1.1.1}$$

for every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$, where $\|\cdot\|$ denotes the L^2 norm.

Proof. The result is a consequence of the second Korn's inequality (see, e.g., [41, Theorem 2.4]), applied to the sets with Lipschitz boundary provided by Lemma 1.1.1. □

Remark 1.1.3. Under hypotheses (H7)-(H9), using a localization argument (see the proof of Lemma 1.1.1) we can prove that the trace operator is well defined and continuous from $H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ into $L^2(\partial\Omega; \mathbb{R}^d)$.

1.1.2 Function spaces

We now introduce the function spaces that will be used in the precise formulation of problem (1.0.1). We set

$$V := H^1(\Omega \setminus \Gamma; \mathbb{R}^d) \quad \text{and} \quad H := L^2(\Omega; \mathbb{R}^d). \tag{1.1.2}$$

For every finite dimensional Hilbert space Y the symbols $(\cdot\,,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in the $L^2(\Omega; Y)$, according to the context. The space V is endowed with the norm

$$\|u\|_V := (\|u\|^2 + \|Du\|^2)^{1/2}. \tag{1.1.3}$$

For every $t \in [0, T]$ we define

$$V_t := H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d) \quad \text{and} \quad V_t^D := \{u \in V_t \mid u|_{\partial_D\Omega} = 0\}, \tag{1.1.4}$$

where $u|_{\partial_D\Omega}$ denotes the trace of u on $\partial_D\Omega$. We note that V_t and V_t^D are closed linear subspaces of V.

We define

$$\mathcal{V} := \{v \in L^2(0, T; V) \cap H^1(0, T; H) \,|\, v(t) \in V_t \text{ for a.e. } t \in (0, T)\}, \tag{1.1.5}$$

which is a Hilbert space with the norm

$$\|v\|_{\mathcal{V}} := (\|v\|^2_{L^2(0,T;V)} + \|\dot{v}\|^2_{L^2(0,T;H)})^{\frac{1}{2}}, \tag{1.1.6}$$

where the dot denotes the distibutional derivative with respect to t. Moreover we set

$$\mathcal{V}^D := \{v \in \mathcal{V} \,|\, v(t) \in V_t^D \text{ for a.e. } t \in (0, T)\} \tag{1.1.7}$$

and note that it is a closed linear subspace of $\mathcal{V}$.

Remark 1.1.4. Since $H^1(0, T; H) \hookrightarrow C^0([0, T]; H)$ we have

$$\mathcal{V} \hookrightarrow C^0([0, T], H).$$

In particular $v(0)$ and $v(T)$ are well defined as elements of H, for every $v \in \mathcal{V}$.

We set

$$\tilde{H} := L^2(\Omega; \mathbb{R}^{d\times d}_{sym}). \tag{1.1.8}$$

On the forcing term $\ell(t)$ of (1.0.1) we assume that

$$\ell(t) := f(t) - \mathrm{div} F(t), \tag{1.1.9}$$

where

$$f \in L^2(0, T; H) \quad \text{and} \quad F \in H^1(0, T; \tilde{H}) \tag{1.1.10}$$

are prescribed function.

Remark 1.1.5. As usual the divergence of a matrix valued function is the vector valued function whose components are obtained taking the divergence of the rows.

As for the Dirichlet boundary condition on $\partial_D\Omega$, it is obtained by prescribing a function

$$u_D \in H^2(0,T;\, H) \cap H^1(0,T;\, V_0). \tag{1.1.11}$$

We impose that for a.e. $t \in [0,T]$ the trace of the solution $u(t)$ is equal to the trace $u_D(t)$ on $\partial_D\Omega$, i.e.,

$$u(t) - u_D(t) \in V_t^D.$$

About the initial data we fix

$$u^0 \in V_0 \quad \text{and} \quad u^1 \in H. \tag{1.1.12}$$

Moreover, we assume the compatibility condition

$$u^0 - u_D(0) \in V_0^D. \tag{1.1.13}$$

We are now in a position to give the precise definition of solution of problem (1.0.1).

Definition 1.1.6 (Solution for visco-elastodynamics with cracks)**.** We say that u is a weak solution of problem (1.0.1) of visco-elastodynamics on the cracked domains $\Omega \setminus \Gamma_t$, with external load $\ell = f - \mathrm{div}F$, Dirichlet boundary condition u_D on $\partial_D\Omega$, natural Neumann boundary condition on $\partial_N\Omega \cup \Gamma_t$, and initial conditions u^0 and u^1, if

$$u \in \mathcal{V} \quad \text{and} \quad u - u_D \in \mathcal{V}^D, \tag{1.1.14}$$

$$\begin{aligned}
&-\int_0^T (\dot u(t), \dot\varphi(t))\,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{V})Eu(t), E\varphi(t))\,\mathrm{d}t - \int_0^T\int_0^t \mathrm{e}^{\tau-t}(\mathbb{V}Eu(\tau), E\varphi(t))\,\mathrm{d}\tau\mathrm{d}t\\
&= \int_0^T (f(t), \varphi(t))\,\mathrm{d}t + \int_0^T (F(t), E\varphi(t))\,\mathrm{d}t,\\
&\text{for all } \varphi \in \mathcal{V}^D \text{ with } \varphi(0) = \varphi(T) = 0,
\end{aligned} \tag{1.1.15}$$

$$u(0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot u(0) = u^1 \quad \text{in } (V_0^D)^*, \tag{1.1.16}$$

where $(V_0^D)^*$ denotes the topological dual of V_t^D for $t = 0$.

Remark 1.1.7. If u satisfy (1.1.14) and (1.1.15), it is possible to prove that $\dot u \in H^1(0,T;(V_0^D)^*)$ (see [43, Remark 4.6]), which implies $\dot u \in C^0([0,T];(V_0^D)^*)$. In particular $\dot u(0)$ is well defined as an element of $(V_0^D)^*$.

Remark 1.1.8. Under suitable regularity assumptions, u is a solution in the sense of Definition 1.1.6 if and only if $u(0) = u^0$, $\dot u(0) = u^1$, and for every $t \in [0,T]$

$$\ddot u(t) - \mathrm{div}((\mathbb{C}+\mathbb{V})Eu(t)) + \mathrm{div}\Big(\int_0^t \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big) = f(t) - \mathrm{div}F(t) \qquad \text{in } \Omega \setminus \Gamma_t,$$

$$u(t) = u_D(t) \qquad \text{on } \partial_D\Omega,$$

$$\Big((\mathbb{C}+\mathbb{V})Eu(t) - \int_0^t \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big)\nu = F(t)\nu \qquad \text{on } \partial_N\Omega,$$

$$\Big((\mathbb{C}+\mathbb{V})Eu(t) - \int_0^t \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big)^{\pm}\nu = F(t)^{\pm}\nu \qquad \text{on } \Gamma_t,$$

where ν is the unit normal and the symbol $\pm$ denotes suitable limits on each side of Γ_t.

The last two conditions represent the natural Neumann boundary conditions on $\partial_N\Omega$ and on the faces of Γ_t.

To describe the boundedness properties of the solutions of problem (1.1.14)-(1.1.16), we introduce the space

$$\mathcal{V}^\infty := \{v \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H) \,|\, v(t) \in V_t \text{ for a.e. } t \in (0,T)\}, \tag{1.1.17}$$

which is a Banach space with the norm

$$\|v\|_{\mathcal{V}^\infty} := \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;H)}. \tag{1.1.18}$$

As for the continuity properties, it is convenient to introduce the space of weakly continuous functions with values in a Banach space X with topological dual X^*, defined by

$$C^0_w([0,T];X) := \{v : [0,T] \to X \,|\, t \mapsto \langle h,\, v(t)\rangle \text{ is continuous for every } h \in X^*\}.$$

We are now in position to state one of the main results of the chapter.

Theorem 1.1.9. *Assume* (H1)-(H15) *and* (1.1.10)-(1.1.13). *Then there exists a unique solution of problem* (1.1.14)-(1.1.16). *Moreover* $u \in \mathcal{V}^\infty$, $u \in C^0_w([0,T];V)$, *and* $\dot{u} \in C^0_w([0,T];H)$.

The existence of a solution is proved in [43] under much weaker assumptions on the cracks Γ_t. The uniqueness will be proved in the next section.

1.2 Uniqueness

In our proof of Theorem 1.1.9 we shall use some known results about existence and uniqueness for the system of elastodynamics on cracked domains, where the memory terms is not present. We set

$$\mathbb{A} := \mathbb{C} + \mathbb{V} \tag{1.2.1}$$

and we consider $\mathbb{A}$ as the elasticity tensor of the auxiliary problem defined below.

1.2.1 The auxiliary problem

Definition 1.2.1 (Solution for elastodynamics with cracks)**.** We say that v is a weak solution of problem (1.0.2) of elastodynamics on the cracked domains $\Omega \setminus \Gamma_t$, with external load $\ell = f - \mathrm{div} F$, Dirichlet boundary condition u_D on $\partial_D \Omega$, natural Neumann boundary condition on $\partial_N \Omega \cup \Gamma_t$, and initial conditions u^0 and u^1, if

$$v \in \mathcal{V} \quad \text{and} \quad v - u_D \in \mathcal{V}^D, \tag{1.2.2}$$

$$- \int_0^T (\dot{v}(t), \dot{\varphi}(t))\,\mathrm{d}t + \int_0^T (\mathbb{A} E v(t), E\varphi(t))\,\mathrm{d}t = \int_0^T (f(t), \varphi(t))\,\mathrm{d}t$$
$$+ \int_0^T (F(t), E\varphi(t))\,\mathrm{d}t \quad \text{for all } \varphi \in \mathcal{V}^D \text{ with } \varphi(0) = \varphi(T) = 0, \tag{1.2.3}$$

$$v(0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot{v}(0) = u^1 \quad \text{in } (V_0^D)^*. \tag{1.2.4}$$

The following technical lemma will be used in the proof of Theorem 1.2.3.

Lemma 1.2.2. *Let v be a weak solution according to Definition 1.2.1 satisfying $\dot{v}(0) = 0$ in the sense of $(V_0^D)^*$. Then (1.2.3) holds for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) \in V_0^D$ and $\varphi(t) = 0$ in a neighborhood of T, even if the condition $\varphi(0) = 0$ is not satisfied.*

Proof. Let φ as in the statement. For every $\varepsilon > 0$, we define

$$\varphi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon}\varphi(0) & \text{for } t \in [0, \varepsilon], \\ \varphi(t - \varepsilon) & \text{for } t \in (\varepsilon, T]. \end{cases}$$

Then $\varphi_\varepsilon \in \mathcal{V}^D$ and $\varphi_\varepsilon(0) = \varphi_\varepsilon(T) = 0$, for ε small enough (1.2.3) holds for φ_ε. We observe that

$$\begin{aligned} \int_0^T (\dot{v}(t), \dot{\varphi}_\varepsilon(t))\,\mathrm{d}t =& \frac{1}{\varepsilon}\int_0^\varepsilon (\dot{v}(t), \varphi(0))\,\mathrm{d}t + \int_\varepsilon^T (\dot{v}(t), \dot{\varphi}(t-\varepsilon))\,\mathrm{d}t \\ &\to \int_0^T (\dot{v}(t), \dot{\varphi}(t))\,\mathrm{d}t \end{aligned}$$

as $\varepsilon \to 0$, where we have used the initial condition in the first term and the continuity of translations in the second one. In a similar way we can pass to the limit in the other terms of equation (1.2.3). □

We are now in a position to state the existence and uniqueness result for the solutions of elastodynamics with cracks.

Theorem 1.2.3. *Assume* (H1)-(H15) *and* (1.1.10)-(1.1.13)*. Then there exists a unique solution v of problem* (1.2.2)-(1.2.4)*. Moreover $v \in \mathcal{V}^\infty$, $v \in C^0_w([0,T]; V)$, and $\dot{v} \in C^0_w([0,T]; H)$.*

Proof. In the case $F = 0$ the existence result, together with an energy bound, is proved in [8] and [47] (a previous result in the scalar case is proved in [17]). When F is present, the same proof can be repeated with obvious modifications (for instance it is enough to repeat the arguments of [43] with $\mathbb{V} = 0$).

As for uniqueness, it can be proved as in [24, Example 4.2 and Theorem 4.3]. Since in that paper the initial conditions are given in a different sense, we have to replace [24, Proposition 2.10] by our Lemma 1.2.2.

The uniqueness result and the existence of a solution with bounded energy imply that the solution satisfies $v \in \mathcal{V}^\infty$. This fact, together with the continuity of v in H and $\dot{v} \in (V_0^D)^*$ (Remark 1.1.7), implies that $v \in C_w^0([0,T];V)$ and $\dot{v} \in C_w^0([0,T];H)$ (see, e.g., [30, Chapitre XVIII, §5, Lemme 6]). □

We now deal with some energetic considerations that will be crucial in the next section. For every $v \in C_w^0([0,T];V)$, with $\dot{v} \in C_w^0([0,T];H)$, the energy of v is defined for every $t \in [0,T]$ as

$$\mathcal{E}_v(t) := \frac{1}{2}\|\dot{v}(t)\|^2 + \frac{1}{2}(\mathbb{A}Ev(t), Ev(t)). \tag{1.2.5}$$

Under the same assumption on v, when $u_D = 0$ the work done by the external forces on the displacement v in the time interval $[0,t] \subset [0,T]$ can be written as

$$\begin{aligned}\mathcal{W}_v(t) := \int_0^t (f(s), \dot{v}(s))\,\mathrm{d}s - \int_0^t (\dot{F}(s), Ev(s))\,\mathrm{d}s \\ + (F(t), Ev(t)) - (F(0), Ev(0)),\end{aligned} \tag{1.2.6}$$

see for instance [43, Remarks 5.9 and 5.11].

We now give an inequality regarding the energy of the system.

Theorem 1.2.4. *Under the assumptions of Theorem 1.2.3, if $u_D = 0$, then the unique solution v of problem (1.2.2)-(1.2.4) satisfies the energy inequality*

$$\mathcal{E}_v(t) \le \mathcal{E}_v(0) + \mathcal{W}_v(t) \quad \textit{for all } t \in [0,T]. \tag{1.2.7}$$

For a proof we refer to [24, Corollary 3.2] and [43, Remark 5.11].

Proposition 1.2.5. *Under the assumptions of Theorem 1.2.3, suppose in addition that $u_D = 0$ and $u^0 = 0$. Then there exists a positive constants A, depending on the constant K in Korn's inequality (1.1.1) and on the constant α_0 in (H3), but not on T, f, F, and u^1, such that the solution v of problem (1.2.2)-(1.2.4) satisfies*

$$\|v\|_{\mathcal{V}^\infty} \le A(1+T)\Big(\|u^1\| + \|F\|_{L^\infty(0,T;\tilde{H})} + T^{1/2}(\|\dot{F}\|_{L^2(0,T;\tilde{H})} + \|f\|_{L^2(0,T;H)})\Big). \tag{1.2.8}$$

Proof. Under our assumption we have

$$\mathcal{W}_v(t) := \int_0^t (f(s), \dot{v}(s))\mathrm{d}s - \int_0^t (\dot{F}(s), Ev(s))\,\mathrm{d}s + (F(t), Ev(t)),$$

$$\mathcal{E}_v(0) = \frac{1}{2}\|u^1\|^2.$$

Recalling (H3), (H6), and (1.2.7) we have

$$\begin{aligned}\frac{1}{2}\|\dot{v}(t)\|^2 + \frac{\alpha_0}{2}\|Ev(t)\|^2 \leq\ & T^{1/2}\|\dot{F}\|_{L^2(0,T;\tilde{H})}\|Ev\|_{L^\infty(0,T;\tilde{H})} \\ &+ \|F\|_{L^\infty(0,T;\tilde{H})}\|Ev\|_{L^\infty(0,T;\tilde{H})} \\ &+ T^{1/2}\|f\|_{L^2(0,T;H)}\|\dot{v}\|_{L^\infty(0,T;H)} + \frac{1}{2}\|u^1\|^2.\end{aligned}$$

for all $t \in [0,T]$. We set

$$S := \sup_{t\in[0,T]} (\|\dot{v}(t)\|^2 + \|Ev(t)\|^2)^{1/2}.$$

From the previous inequality we obtain

$$\min\{1/2,\, \alpha_0/2\}S \leq T^{1/2}\|\dot{F}\|_{L^2(0,T;\tilde{H})} + \|F\|_{L^\infty(0,T;\tilde{H})} \tag{1.2.9}$$

$$+ T^{1/2}\|f\|_{L^2(0,T;H)} + \|u^1\|. \tag{1.2.10}$$

Since $v(t) = \int_0^t \dot{v}(s)\,\mathrm{d}s$ we have $\sup_{t\in[0,T]}\|v(t)\| \leq TS$. Using Korn's inequality (1.1.1) we obtain $\sup_{t\in[0,T]}\|Dv(t)\| \leq K^{1/2}S$. Therefore

$$\|u\|_{\mathcal{V}^\infty} \leq S + K^{1/2}S + TS,$$

which, together with (1.2.9), gives (1.2.8). □

1.2.2 Proof of the uniquenes

Let $\mathcal{L} : \mathcal{V}^\infty \longrightarrow H^1(0,T;\tilde{H})$ be the linear operator defined by

$$(\mathcal{L}u)(t) := \int_0^t \mathrm{e}^{\tau - t}\mathbb{V}Eu(\tau)\,\mathrm{d}\tau \tag{1.2.11}$$

for every $u \in \mathcal{V}^\infty$ and $t \in [0,T]$. Since

$$(\dot{\widehat{\mathcal{L}u}})(t) = \mathbb{V}Eu(t) - \int_0^t \mathrm{e}^{\tau - t}\mathbb{V}Eu(\tau)\,\mathrm{d}\tau,$$

it is easy to check that $\mathcal{L}$ is bounded. Indeed we have

$$\|\mathcal{L}u\|_{L^\infty(0,T;\tilde{H})} \leq T\|\mathbb{V}\|_\infty\|u\|_{\mathcal{V}^\infty}, \tag{1.2.12}$$

$$\|\dot{\widehat{\mathcal{L}u}}\|_{L^2(0,T;\tilde{H})} \leq (T^{1/2} + T^{3/2})\|\mathbb{V}\|_\infty\|u\|_{\mathcal{V}^\infty}. \tag{1.2.13}$$

Corollary 1.2.6. *Under the assumptions of Theorem 1.2.3 there exists a positive constant B, depending on the constant K in Korn's inequality* (1.1.1) *and on the constant α_0 in (H3), but not on T and $\mathbb{V}$, such that, if u satisfies* (1.2.2)-(1.2.4) *with u^0, u^1, u_D, and f replaced by zero and F replaced by $\mathcal{L}u$, then*

$$\|u\|_{\mathcal{V}^\infty} \le B(T + T^3)\|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty}. \tag{1.2.14}$$

Proof. By Proposition 1.2.5, (1.2.12), and (1.2.13) we have

$$\|u\|_{\mathcal{V}^\infty} \le A\Big((1+T)T + (T^{1/2} + T^{3/2})^2\Big)\|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty},$$

which implies (1.2.14). □

We are now in a position to prove the existence and uniqueness result.

Proof of Theorem 1.1.9. The existence result is obtained in [11] under more general hypotheses. To prove uniqueness, we assume by contradiction that there exist two distinct solution u_1 and u_2 of problem (1.1.14)-(1.1.16). Then $u := u_1 - u_2$ is a solution of the same problem with u^0, u^1, u_D, f, and F replaced by zero. Therefore u satisfies (1.2.2)-(1.2.4) with u^0, u^1, u_D, and f replaced by zero and F replaced by $\mathcal{L}u$. By Theorem 1.2.3 this implies that $u \in C_w([0,T];V)$ and $\dot u \in C_w([0,T];H)$.

We set

$$t_0 := \inf\{t \in [0,T] \,|\, u(t) \neq 0\}.$$

Since u is not identically zero, we have $t_0 < T$. We fix $\delta \in (0, T - t_0)$ such that

$$B(\delta + \delta^3)\|\mathbb{V}\|_\infty < 1, \tag{1.2.15}$$

where B is the constant in (1.2.14), and we define $t_1 := t_0 + \delta$. In order to study the problem on $[t_0, t_1]$ we define the spaces $\mathcal{V}^D_{t_0,t_1}$ and $\mathcal{V}^\infty_{t_0,t_1}$ as $\mathcal{V}^D$ and $\mathcal{V}^\infty$ (see (1.1.7) and (1.1.17)), with 0 and T replaced by t_0 and t_1.

It is clear that $u \in \mathcal{V}^D_{t_0,t_1}$ and since $Eu(\tau) = 0$ for every $\tau \in [0, t_0]$ we have

$$\begin{aligned} &-\int_{t_0}^{t_1} (\dot u(t), \dot\varphi(t))\,\mathrm{d}t + \int_{t_0}^{t_1} (\mathbb{A}Eu(t), E\varphi(t))\,\mathrm{d}t \\ &-\int_{t_0}^{t_1}\int_{t_0}^{t} \mathrm{e}^{\tau - t}(\mathbb{V}Eu(\tau), E\varphi(t))\,\mathrm{d}\tau\mathrm{d}t = 0 \end{aligned}$$

for every $\varphi \in \mathcal{V}^D_{t_0,t_1}$ such that $\varphi(t_0) = \varphi(t_1) = 0$. Moreover, since $u \in C_w([0,T];V)$, $\dot u \in C_w([0,T];H)$, and u is identically zero on $[0, t_0]$, we have that $u(t_0) = 0$ and $\dot u(t_0) = 0$. By (1.2.14), applied with 0 and T replaced by t_0 and t_1, we have

$$\|u\|_{\mathcal{V}^\infty_{t_0,t_1}} \le B(\delta + \delta^3)\|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty_{t_0,t_1}}.$$

Using (1.2.15) we obtain $u = 0$ on $[t_0, t_1]$. This contradicts the definition of t_0 and concludes the proof. □

This section concludes the proof of existence and uniqueness. Moreover, we have proved some inequalities concerning the enrgy of the system that will be useful for the next section.

1.3 Continuous dependence on the data

In this section we consider a sequence $\{\Gamma_t^n\}_{t\in[0,T]}$ of time dependent cracks and we want to study the convergence, as $n \to +\infty$, of the solutions of the corresponding viscoelastic problems. For completeness we assume that also the other data of the problem depend on n.

1.3.1 Preliminary results for the continuous dependence

For every $n \in \mathbb{N}$, let $\mathbb{C}^n$, $\mathbb{V}^n : \overline{\Omega} \to \mathcal{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$, let Γ^n be a $(d-1)$-dimensional C^2 manifold, let $\{\Gamma_t^n\}_{t\in[0,T]}$ be a family of closed subsets of Γ^n, and let Φ^n, $\Psi^n : [0,T]\times\overline{\Omega} \to \overline{\Omega}$. We assume that

(H16) $\mathbb{C}^n$, $\mathbb{V}^n$ satisfy (H1)-(H6) with constants α_0 and M_0 independent of n;

(H17) Γ^n and $\{\Gamma_t^n\}_{t\in[0,T]}$ satisfy (H7)-(H10);

(H18) Φ^n, Ψ^n satisfy (H11)-(H15) (with Γ and Γ_t replaced by Γ^n and Γ_t^n), the latter with the constant K that appears in (1.3.7).

Let $\mathbb{R}^{d\times d}$ be the space of $d \times d$ real matrices. For every pair of normed spaces X and Y let $\mathcal{L}(X;Y)$ be the space of linear and continuous maps between X and Y. For every $x \in \overline{\Omega}$ it is convenient to consider the extensions $\mathbb{C}_e(x)$, $\mathbb{V}_e(x)$, $\mathbb{C}_e^n(x)$, $\mathbb{V}_e^n(x) \in \mathcal{L}(\mathbb{R}^{d\times d}; \mathbb{R}^{d\times d}_{sym})$ of the linear maps $\mathbb{C}(x)$, $\mathbb{V}(x)$, $\mathbb{C}^n(x)$, $\mathbb{V}^n(x)$ defined as

$$\mathbb{C}_e^n(x)[A] := \mathbb{C}^n(x)[A_{sym}] \quad \text{and} \quad \mathbb{V}_e^n(x)[A] := \mathbb{V}^n(x)[A_{sym}] \quad \text{for all } A \in \mathbb{R}^{d\times d}, \tag{1.3.1}$$

$$\mathbb{C}_e(x)[A] := \mathbb{C}(x)[A_{sym}] \quad \text{and} \quad \mathbb{V}_e(x)[A] := \mathbb{V}(x)[A_{sym}] \quad \text{for all } A \in \mathbb{R}^{d\times d}, \tag{1.3.2}$$

where A_{sym} is the symmetric part of the matrix A. Moreover we set

$$\mathbb{A}_e^n := \mathbb{C}_e^n + \mathbb{V}_e^n \quad \text{and} \quad \mathbb{A}_e := \mathbb{C}_e + \mathbb{V}_e. \tag{1.3.3}$$

For technical reasons we use a change of variable which maps Γ_0^n into Γ_0. This is done by means of diffeomorphisms Θ^n, $\Xi^n : \overline{\Omega} \to \overline{\Omega}$ such that

(H19) Θ^n and Ξ^n are of class $C^{2,1}$;

(H20) $\Theta^n(\Xi^n(x)) = x$ and $\Xi^n(\Theta^n(x)) = x$ for every $x \in \overline{\Omega}$;

(H21) $\det D\Theta^n(x) > 0$ for every $x \in \overline{\Omega}$;

(H22) $\Theta^n(\Gamma\cap\overline{\Omega})=\Gamma^n\cap\overline{\Omega}$, and $\Theta^n(\Gamma_0)=\Gamma^n_0$;

(H23) $\Theta^n(\partial_D\Omega)=\partial_D\Omega$ and $\Theta^n(\partial_N\Omega)=\partial_N\Omega$.

We now introduce the function spaces that will be used in the formulation of the n-th viscoelastic problem. For every $n\in\mathbb{N}$ and $t\in[0,T]$ let V^n, V^n_t, and $V^{n,D}_t$ be defined as V, V_t, and V^D_t (see (1.1.2) and (1.1.4)) with Γ and Γ_t replaced by Γ^n and Γ^n_t. Let $\mathcal{V}^n$, $\mathcal{V}^{n,D}$, and $\mathcal{V}^{n,\infty}$ be defined as $\mathcal{V}$, $\mathcal{V}^D$, and $\mathcal{V}^\infty$ (see (1.1.5), (1.1.7), and (1.1.17)) with V_t and V^D_t replaced by V^n_t and $V^{n,D}_t$.

For every $n\in\mathbb{N}$ we fix

$$u^{0,n}\in V^n_0,\quad u^{1,n}\in H,\quad u^n_D\in H^2(0,T;\,H)\cap H^1(0,T;\,V^n_0),\tag{1.3.4}$$

$$f^n\in L^2(0,T;H),\quad F^n\in H^1(0,T;\tilde{H}),\tag{1.3.5}$$

and we suppose that $u^{0,n}$ and u^n_D satisfy the compatibility condition

$$u^{0,n}-u^n_D(0)\in V^{n,D}_0.\tag{1.3.6}$$

Now we give the detailed regularity and convergence hypotheses on the data. First of all we assume that there exists a constant $K>0$ such that for every $n\in\mathbb{N}$ the following Korn inequality is satisfied:

$$\|Dv\|^2\leq K(\|v\|^2+\|Ev\|^2)\quad\text{for every} v\in H^1(\Omega\setminus\Gamma^n;\mathbb{R}^d).\tag{1.3.7}$$

We set $\underline{H}=L^2(\Omega,\mathbb{R}^{d\times d})$. Concerning the convergence of our data we assume that

$$\|\Phi^n-\Phi\|_{C^2}\to 0,\qquad \|\Psi^n-\Psi\|_{C^2}\to 0,\tag{1.3.8}$$

$$\|\mathbb{C}^n-\mathbb{C}\|_{C^1}\to 0,\qquad \|\mathbb{V}^n-\mathbb{V}\|_{C^1}\to 0\tag{1.3.9}$$

$$\|u^n_D-u_D\|_{H^2(0,T;\,H)}\to 0,\qquad \|Du^n_D-Du_D\|_{H^1(0,T;\underline{H})}\to 0,\tag{1.3.10}$$

$$\|f^n-f\|_{L^2(0,T;H)}\to 0,\qquad \|F^n-F\|_{H^1(0,T;\tilde{H})}\to 0,\tag{1.3.11}$$

$$\|u^{0,n}-u^0\|\to 0,\qquad \|Du^{0,n}-Du^0\|\to 0,\qquad \|u^{1,n}-u^1\|\to 0,\tag{1.3.12}$$

$$\|\Theta^n-Id\|_{C^2}\to 0,\qquad \|\Xi^n-Id\|_{C^2}\to 0.\tag{1.3.13}$$

It follows from (H19)-(H21) and (1.3.13) that

$$m_{det}(\Psi^n)\to m_{det}(\Psi)\quad\text{and}\quad M_{det}(\Psi^n)\to M_{det}(\Psi^n)\quad\text{as } n\to\infty.\tag{1.3.14}$$

For every $n\in\mathbb{N}$ we consider the solution u^n of the problem

$$u^n\in\mathcal{V}^n\quad\text{and}\quad u^n-u^n_D\in\mathcal{V}^{n,D},\tag{1.3.15}$$

$$
\begin{aligned}
&-\int_0^T (\dot u^n(t),\dot\varphi(t))\,\mathrm{d}t+\int_0^T ((\mathbb{C}^n+\mathbb{V}^n)Eu^n(t),E\varphi(t))\,\mathrm{d}t\\
&-\int_0^T\int_0^t \mathrm{e}^{\tau-t}(\mathbb{V}^nEu^n(\tau),E\varphi(t))\,\mathrm{d}\tau\mathrm{d}t=\int_0^T (f^n(t),\varphi(t))\,\mathrm{d}t\\
&+\int_0^T (F^n(t),E\varphi(t))\,\mathrm{d}t \quad \text{for all } \varphi\in\mathcal{V}^{n,D} \text{ with } \varphi(0)=\varphi(T)=0, \qquad (1.3.16)\\
&u^n(0)=u^{0,n} \quad \text{in } H \quad \text{and} \quad \dot u^n(0)=u^{1,n} \quad \text{in } (V_0^{D,n})^*. \qquad (1.3.17)
\end{aligned}
$$

We also consider the solution v^n of the problem

$$
\begin{aligned}
&v^n\in\mathcal{V}^n \quad \text{and} \quad v^n-u_D^n\in\mathcal{V}^{n,D}, \qquad (1.3.18)\\
&-\int_0^T (\dot v^n(t),\dot\varphi(t))\,\mathrm{d}t+\int_0^T (\mathbb{A}^nEv^n(t),E\varphi(t))\,\mathrm{d}t=\int_0^T (f^n(t),\varphi(t))\,\mathrm{d}t\\
&+\int_0^T (F^n(t),E\varphi(t))\,\mathrm{d}t \quad \text{for all } \varphi\in\mathcal{V}^{n,D} \text{ with } \varphi(0)=\varphi(T)=0, \qquad (1.3.19)\\
&v^n(0)=u^{0,n} \quad \text{in } H \quad \text{and} \quad \dot v^n(0)=u^{1,n} \quad \text{in } (V_0^{D,n})^*. \qquad (1.3.20)
\end{aligned}
$$

Remark 1.3.1. The notion of convergence for u^n as $n\to\infty$ can't be given directly because they don't belong to the same space. To overcome this problem we need to embed V^n into a common space. This will be done using the standard embedding

$$V^n\hookrightarrow H\times \underline{H}$$

given by $v\mapsto(v,\,Dv)$, where the distrubutional gradient Dv on $\Omega\setminus\Gamma^n$ is regarded as a function defined a.e. on Ω, which belongs to $\underline{H}$.

We are now in a position to state one the main result of this section.

Theorem 1.3.2. *Assume (H1)-(H23), (1.1.10)-(1.1.13), and (1.3.4)-(1.3.13). Let u be the solution of (1.1.14)-(1.1.16) and let (for every $n\in\mathbb{N}$) u^n be the solution of (1.3.15)-(1.3.17). Then*

$$(u^n(t),\,Du^n(t),\,\dot u^n(t))\to(u(t),\,Du(t),\,\dot u(t)) \qquad \text{in } H\times\underline{H}\times H$$

for every $t\in[0,T]$. Moreover there exists a constant $C>0$ such that

$$\|u^n(t)\|+\|Du^n(t)\|+\|\dot u^n(t)\|\le C$$

for every $n\in\mathbb{N}$ and $t\in[0,T]$.

The proof is based on the following lemma.

Lemma 1.3.3. *Let X a complete metric space, let G_n, $G\colon X \to X$ with $n \in \mathbb{N}$ be maps with same contraction constant $\lambda \in (0,1)$, and let x_n, x be the corresponding fixed points. Suppose that $G_n(y) \to G(y)$ for every $y \in X$. Then $x_n \to x$.*

Proof. We have $d(x_n, x) = d(G_n(x_n), G(x)) \leq d(G_n(x_n), G_n(x)) + d(G_n(x), G(x)) \leq \lambda d(x_n, x) + d(G_n(x), G(x))$, hence

$$(1-\lambda) d(x_n, x) \leq d(G_n(x), G(x)) \to 0$$

as $n \to +\infty$. □

In order to apply the previous lemma we will identify u_n and u with the fixed points of suitable operators defined in the Banach space

$$\mathcal{W} := L^2((0,T); H \times \underline{H} \times H), \tag{1.3.21}$$

where on $H \times \underline{H} \times H$ we consider the Hilbert product norm defined by

$$\|(h_1, h_2, h_3)\|_{H \times \underline{H} \times H} := \left(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2\right)^{1/2} \tag{1.3.22}$$

for every $(h_1, h_2, h_3) \in H \times \underline{H} \times H$.

In order to define the sequence of maps whose fixed points are $(u^n, Du^n, \dot{u}^n)$ and $(u, Du, \dot{u})$, we consider the linear operators

$$\mathcal{T}^n : \mathcal{W} \longrightarrow H^1(0,T;\tilde{H}) \quad \text{and} \quad \mathcal{T} : \mathcal{W} \longrightarrow H^1(0,T;\tilde{H}) \tag{1.3.23}$$

defined as

$$(\mathcal{T}^n w)(t) := \int_0^t \mathrm{e}^{\tau - t} \mathbb{V}_e^n w_2(\tau)\, \mathrm{d}\tau \quad \text{and} \quad (\mathcal{T} w)(t) := \int_0^t \mathrm{e}^{\tau - t} \mathbb{V}_e w_2(\tau)\, \mathrm{d}\tau, \tag{1.3.24}$$

where $w(t) = (w_1(t), w_2(t), w_3(t))$ and $\mathbb{V}_e^n$, $\mathbb{V}_e$ are as in (1.3.1) and (1.3.2). Arguing as in (1.2.12) and (1.2.13) we get that

$$\|\mathcal{T} w\|_{L^\infty(0,T;\tilde{H})} \leq T^{1/2} \|\mathbb{V}\|_\infty \|w\|_{\mathcal{W}}, \tag{1.3.25}$$

$$\|\dot{\widehat{\mathcal{T} w}}\|_{L^2(0,T;\tilde{H})} \leq (1+T) \|\mathbb{V}\|_\infty \|w\|_{\mathcal{W}}, \tag{1.3.26}$$

and the same estimate holds for $\mathcal{T}^n w$ with $\mathbb{V}$ replaced by $\mathbb{V}^n$, namely

$$\|\mathcal{T}^n w\|_{L^\infty(0,T;\tilde{H})} \leq T^{1/2} \|\mathbb{V}^n\|_\infty \|w\|_{\mathcal{W}}, \tag{1.3.27}$$

$$\|\dot{\widehat{\mathcal{T}^n w}}\|_{L^2(0,T;\tilde{H})} \leq (1+T) \|\mathbb{V}^n\|_\infty \|w\|_{\mathcal{W}}. \tag{1.3.28}$$

Let
$$\mathcal{G}:\mathcal{W}\to\mathcal{W}$$
be the operator defined for every $w\in\mathcal{W}$ by
$$\mathcal{G}(w)=(z,Dz,\dot{z}),\tag{1.3.29}$$
where z is the solution of problem (1.2.2)-(1.2.4) with F replaced by $F+\mathcal{T}w$. From the definition of $\mathcal{G}$ it follows that $(u,Du,\dot{u})$ is a fixed point of map $\mathcal{G}$ if and only if u is the solution of the problem considered in Theorem 1.3.2.

Similarly, let
$$\mathcal{G}^n:\mathcal{W}\to\mathcal{W}$$
be the operator defined for every $w\in\mathcal{W}$ by
$$\mathcal{G}^n(w)=(z^n,Dz^n,\dot{z}^n),\tag{1.3.30}$$
where z^n is the solution of problem (1.3.18)-(1.3.20) with F replaced by $F^n+\mathcal{T}^nw$. From the definition of $\mathcal{G}^n$ it follows that u^n is the solution of problem (1.3.15)-(1.3.17) if and only if $(u^n,Du^n,\dot{u}^n)$ is a fixed point of map $\mathcal{G}^n$.

The following lemma provides a uniform Lipschitz estimate for the operators $\mathcal{G}^n$.

Proposition 1.3.4. *There exist a positive constants B, independent of n and T, such that*
$$\|\mathcal{G}^n(w_1)-\mathcal{G}^n(w_2)\|_{\mathcal{W}}\leq B(T+T^3)\|w_1-w_2\|_{\mathcal{W}},\tag{1.3.31}$$
for every w_1, $w_2\in\mathcal{W}$.

Proof. Let us fix w_1, $w_2\in\mathcal{W}$ and set $w:=w_1-w_2$. We observe that
$$\mathcal{G}^n(w_1)-\mathcal{G}^n(w_2)=(z^n,Dz^n,\dot{z}^n)$$
where z^n is the solution of problem (1.3.15)-(1.3.17) with F^n replaced by $\mathcal{T}^nw$ and u^n_D, f^n, $u^{0,n}$, $u^{1.n}$ replaced by zero. From Theorem 1.2.5 and from the uniform bound of the data there exists a positive constants A, independent of n and T, such that
$$\|z^n\|_{\mathcal{V}^\infty}\leq A(1+T)\|\mathcal{T}^nw\|_{L^\infty(0,T;\tilde{H})}+A(T^{1/2}+T^{3/2})\|\dot{\widehat{\mathcal{T}^nw}}\|_{L^2(0,T;\tilde{H})}.\tag{1.3.32}$$
Using (1.3.25) and (1.3.26) we get
$$\|(z^n,Dz^n,\dot{z}^n)\|_{\mathcal{W}}\leq A\Big((1+T)T+(T^{1/2}+T^{3/2})^2\Big)\|\mathbb{V}^n\|_\infty\|w\|_{\mathcal{W}}\tag{1.3.33}$$
which gives (1.3.31) taking into account (1.3.9). $\square$

To apply Lemma 1.3.3 we have to prove that

$$\mathcal{G}^n(w) \to \mathcal{G}(w) \quad \text{in } \mathcal{W},$$

for every $w \in \mathcal{W}$. In order to prove this we will use the results for the wave equation developed in [9]. Unfortunately these results can not be applied directly because they are obtained under the assumptions:

(a) $\Gamma_0^n = \Gamma_0$ for all $n \in \mathbb{N}$,

(b) the forcing terms belong to $L^2(0,T;H)$.

To overcome the difficulties due to (a) we need some preliminary results. The first one is an uniform bound of the solution of problems (1.3.18)-(1.3.20).

Proposition 1.3.5. *Assume (H1)-(H23), (1.3.4)-(1.3.10), and (1.3.12)-(1.3.13). Let let v^n be the solution of (1.3.18)-(1.3.20). Then the there exists a positive constant C such that*

$$\|v^n\|_{\mathcal{V}^{n,\infty}} \leq C \quad \textit{for every } n \in \mathbb{N}. \tag{1.3.34}$$

Proof. We note that

$$v_0^n(t) := v^n(t) - u^{0,n} + u_D^n(0) - u_D^n(t)$$

is the solution of (1.3.18)-(1.3.20) with u^0 replaced by 0, $u^{1,n}$ replaced by $u^{1,n} - \dot{u}_D^n(0)$, u_D^n replaced by 0, f^n replaced by $f^n - \ddot{u}_D^n$, and F^n replaced by $F^n - \mathbb{A}^n E u_D^n - \mathbb{A}^n E(u^{n,0} - u_D^n(0))$. Then we can apply Proposition 1.2.5 and (1.3.8)-(1.3.13) to obtain that

$$\|v_0^n\|_{\mathcal{V}^n} \leq C.$$

By (1.3.10) and (1.3.12) we get (1.3.34). □

The next proposition deals with the case of solution of (1.3.18)-(1.3.20) when F^n is replaced by 0.

Proposition 1.3.6. *Assume (H1)-(H23), (1.3.4)-(1.3.10), and (1.3.12)-(1.3.13). Given $g \in L^2(0,T;\, H)$, let v^n be the solution of (1.3.18)-(1.3.20) with f^n replaced by g and F^n replaced by 0. Let v be the solution in (1.2.2)-(1.2.4) with f replaced by g and F replaced by 0. Then for every $t \in [0,T]$ we have*

$$(v^n(t),\, Dv^n(t),\, \dot{v}^n(t)) \to (v(t),\, Dv(t),\, \dot{v}(t)) \quad \textit{in } H \times \underline{H} \times H. \tag{1.3.35}$$

In order to prove this proposition it is convenient to use the following elementary result, whose proof, based on a change of variables, is omitted (for a similar result see [21, Lemma A.7]).

Lemma 1.3.7. *For every $n \in \mathbb{N}$ let h^n, $h \in H$ and let Λ^n, $\Lambda : \overline{\Omega} \to \overline{\Omega}$ be C^1 diffeomorphisms. Assume that $h^n \to h$ in H and $\Lambda^n \to \Lambda$ in C^1. Assume also that $\det D\Lambda^n(x) > 0$ and $\det D\Lambda(x) > 0$ for every $x \in \overline{\Omega}$ and $n \in \mathbb{N}$. Then*

$$h^n \circ \Lambda^n \to h \circ \Lambda$$

as $n \to \infty$ in H.

Proof of Proposition 1.3.6. To overcome the difficulty due to the fact that we may have $\Gamma_0^n \neq \Gamma_0$, by a change of variables we transform our problem into a problem with new cracks $\hat{\Gamma}_t^n$ satisfying $\hat{\Gamma}_0^n = \Gamma_0$ for every n, to which we can apply the results of [8] and [9].

For every n and t we define $\hat{\Gamma}_t^n := \Xi^n(\Gamma_t^n) \subset \Gamma$ and observe that $\hat{\Gamma}_t^n$ satisfies (H10). The vector spaces $\hat{V}_t^n$ and $\hat{V}_t^{n,D}$ are defined as V_t^n and $V_t^{n,D}$ (see (1.1.4)) with Γ_t replaced by $\hat{\Gamma}_t^n$, while $\hat{\mathcal{V}}^n$ and $\hat{\mathcal{V}}^{n,D}$ are defined as $\mathcal{V}^n$ and $\mathcal{V}^{n,D}$ (see (1.1.5) and (1.1.7)) with V_t and V_t^D replaced by $\hat{V}_t^n$ and $\hat{V}_t^{n,D}$.

For every $t \in [0,T]$ let $\hat{v}^n(t) := v^n(t) \circ \Theta^n$, $\hat{u}_D^n(t) := u_D^n(t) \circ \Theta^n$, $\hat{u}^{0,n} := u^{0,n} \circ \Theta^n$, $\hat{u}^{1,n} := u^{1,n} \circ \Theta^n$, and $\hat{g}^n(t) := g(t) \circ \Theta^n$. It is easy to see that $\hat{v}^n \in \hat{\mathcal{V}}^n$, $\hat{v}^n - \hat{u}_D^n \in \hat{\mathcal{V}}^{n,D}$, $\hat{v}^n(0) = \hat{u}^{0,n}$, $\dot{\hat{v}}^n(0) = \hat{u}^{1,n}$.

To write the equation satisfied by $\hat{v}^n$ we introduce

$$\hat{\mathbb{A}}^n : \overline{\Omega} \to \mathcal{L}(\mathbb{R}^{d\times d}; \mathbb{R}^{d\times d})$$

defined as

$$\hat{\mathbb{A}}^n(y)[A] := \mathbb{A}_e^n(\Theta^n(y))[AD\Xi^n(\Theta^n(y))](D\Xi^n(\Theta^n(y)))^T \quad \text{for all } A \in \mathbb{R}^{d\times d}, \tag{1.3.36}$$

where $\mathbb{A}^n$ is defined in (1.3.3). We note that $\hat{\mathbb{A}}^n$ is of class C^1, with equibounded C^1 norm. Moreover it is symmetric on $\mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$.

Setting $h^n(x) := \nabla[\det D\Xi^n(x)]$, we introduce

$$\mathbb{L}^n : \overline{\Omega} \to \mathcal{L}(\mathbb{R}^{d\times d}; \mathbb{R}^d)$$

defined as

$$\mathbb{L}^n(y)[A] = \mathbb{A}_e^n(\Theta^n(y))[AD\Xi^n(\Theta^n(y))]h^n(\Theta^n(y)) \det D\Theta^n(y) \quad \text{for all } A \in \mathbb{R}^{d\times d}.$$

Let $\varphi \in \hat{\mathcal{V}}^{n,D}$ with $\varphi(0) = \varphi(T) = 0$. Using $(\varphi(t) \circ \Xi^n) \det D\Xi^n$ as test function in the equation for $v^n(t)$ we get

$$-\int_0^T (\dot{\hat{v}}^n(t), \dot{\varphi}(t))\, \mathrm{d}t + \int_0^T (\hat{\mathbb{A}}^n D\hat{v}^n(t), D\varphi(t))\, \mathrm{d}t + \int_0^T (\mathbb{L}^n D\hat{v}^n(t), \varphi(t))\, \mathrm{d}t =$$

$$\int_0^T (\hat{g}^n(t), \varphi(t))\, \mathrm{d}t.$$

By Proposition (1.3.5) the sequence $||v^n||_{\mathcal{V}^n}$ is bounded and in particular $\|Dv^n(t)\|$ is uniformly bounded with respect to n and t. By the definition of $\hat{v}^n$ and (1.3.13) also $\|D\hat{v}^n(t)\|$ is uniformly bounded with respect to n and t. Since $\det D\Xi^n \to 1$ in $C^1(\overline{\Omega})$, we have $\nabla[\det D\Xi^n] \to 0$ in $C^0(\overline{\Omega}, \mathbb{R}^d)$, which implies that $\mathbb{L}^n \to 0$ uniformly as $n \to +\infty$. From this fact and the uniform bound on $\|D\hat{v}^n(t)\|$ we get

$$\|\mathbb{L}^n D\hat{v}^n(t)\| \to 0 \text{ as } n \to +\infty, \tag{1.3.37}$$

uniformly in t. Therefore, setting

$$\hat{f}^n := \hat{g}^n - \mathbb{L}^n D\hat{v}^n, \tag{1.3.38}$$

we conclude that

$$\hat{v}^n \in \hat{\mathcal{V}}^n \quad \text{and} \quad \hat{v}^n - \hat{u}^n_D \in \hat{\mathcal{V}}^{n,D}, \tag{1.3.39}$$

$$-\int_0^T (\dot{\hat{v}}^n(t), \dot{\varphi}(t))\,\mathrm{d}t + \int_0^T (\hat{\mathbb{A}}^n D\hat{v}^n(t), D\varphi(t))\,\mathrm{d}t = \int_0^T (\hat{f}^n(t), \varphi(t))\,\mathrm{d}t,$$

$$\text{for all } \varphi \in \hat{\mathcal{V}}^{n,D} \text{ such that } \varphi(0) = \varphi(T) = 0, \tag{1.3.40}$$

$$\hat{v}^n(0) = \hat{u}^{0,n} \quad \text{in } H \quad \text{and} \quad \dot{\hat{v}}^n(0) = \hat{u}^{1,n} \quad \text{in } (V_0^D)^*. \tag{1.3.41}$$

In order to apply the results of [9] we define

$$\hat{\Phi}^n(t,y) := \Xi^n(\Phi^n(t,\Theta^n(y))),\;\; \hat{\Psi}^n(t,x) := \Xi^n(\Psi^n(t,\Theta^n(x))).$$

We observe that $\hat{\Phi}^n$ and $\hat{\Psi}^n$ satisfy (H11)-(H14) with Γ_t replaced by $\hat{\Gamma}^n_t$. Since in general $\hat{\mathbb{A}}^n[A] \neq \hat{\mathbb{A}}^n[A_{sym}]$ for some $A \in \mathbb{R}^{d\times d}$, we cannot apply the results of [8].

However it is possible to use the results of [9] which hold under more general assumptions involving the tensor

$$\hat{\mathbb{B}}^n(t,y)[A] := \hat{\mathbb{A}}^n(\hat{\Phi}^n(t,y))[AD\hat{\Psi}^n(t,\hat{\Phi}^n(t,y))]D\hat{\Psi}^n(t,\hat{\Phi}^n(t,y))^T$$
$$-A\dot{\hat{\Psi}}^n(t,\hat{\Phi}^n(t,y)) \otimes \dot{\hat{\Psi}}^n(t,\hat{\Phi}^n(t,y)),$$

for all $A \in \mathbb{R}^{d\times d}$, $t \in [0,T]$, $y \in \overline{\Omega}$. We claim that there exists two constants c_0, $c_1 > 0$ (independent of n) such that, for n large enough, we have

$$(\hat{\mathbb{B}}^n(t)D\varphi,\, D\varphi) \geq c_0\|\varphi\|^2_{V_0} - c_1\|\varphi\|^2 \tag{1.3.42}$$

for all $\varphi \in V_0$ and $t \in [0,T]$. This is the hypothesis on $\hat{\mathbb{B}}^n$ required in [9].

To prove the claim we use (H3), (H15), and (1.3.13) (which are satisfied uniformly in n) and by standard computations (see, for instance, [9, Section 1.2]) we obtain

$$(\hat{\mathbb{B}}^n(t)D\varphi,\, D\varphi) \geq \int_\Omega |D\varphi(y)D\Xi^n(\Theta^n(y))D\Psi^n(t,\Phi^n(t,\Theta^n(y)))|^2\omega^n(t,y)\,dy$$

$$- \alpha_0 \min_{[0,T]\times\overline{\Omega}} \{\det D\Xi^n \det D\Psi^n\} \int_\Omega |\varphi(\Xi^n(\Psi^n(t,y)))|^2 \, dy \qquad (1.3.43)$$

where

$$\omega^n(t,y) := \frac{\alpha_0 m_{det}(\Psi^n)}{K M_{det}(\Psi^n)} \min_{\overline{\Omega}}\{\det D\Xi^n\} \min_{\overline{\Omega}}\{\det D\Theta^n\} - |\dot{\Phi}^n(t,\Theta^n(y))|^2,$$

while $m_{det}(\Psi^n)$, $M_{det}(\Psi^n)$, α_0, and K are the constants that appear in (H15), (H16), and (1.3.7).

Since the inverse of the matrices $D\Xi^n(x)D\Psi^n(t,\Phi^n(t,x))$ are bounded uniformly with respect to n, t, and x, there exists a constant $\beta > 0$ such that

$$\int_\Omega |D\varphi(y) D\Xi^n(\Theta^n(y)) D\Psi^n(t,\Phi^n(t,\Theta^n(y)))|^2 \omega^n(t,y)\, dy \geq \beta \int_\Omega |D\varphi(y)|^2 \omega^n(t,y)\, dy$$

for all n and t. Moreover by (1.3.8) and (1.3.13) there exists a constant $\gamma > 0$ such that

$$\alpha_0 \min_{[0,T]\times\overline{\Omega}} \{\det D\Xi^n \det D\Psi^n\} \int_\Omega |\varphi(\Xi^n(\Psi^n(t,y)))|^2 \, dy \leq \gamma \int_\Omega |\varphi(y)|^2 \, dy$$

for all n and t. Therefore (1.3.43) gives

$$(\hat{\mathbb{B}}^n(t) D\varphi, \, D\varphi) \geq \beta \int_\Omega |D\varphi(y)|^2 \omega^n(t,y)\, dy - \gamma \int_\Omega |\varphi(y)|^2 \, dy. \qquad (1.3.44)$$

To conclude the proof of the claim, we define

$$\omega(t,y) := \frac{\alpha_0 m_{det}(\Psi)}{K M_{det}(\Psi)} - |\dot{\Phi}(t,y)|^2.$$

By (1.3.8), (1.3.13) and (1.3.14), we have $\omega^n \to \omega$ uniformly on $[0,T]\times\overline{\Omega}$. By (H15) and by continuity there exists $\varepsilon > 0$ such that $\omega(t,y) \geq 2\varepsilon$ for all $(t,y) \in [0,T]\times\overline{\Omega}$. By uniform convergence there exists n_ε such that $\omega^n(t,y) \geq \varepsilon$ for all $(t,y) \in [0,T]\times\overline{\Omega}$ and for all $n > n_\varepsilon$. This inequality together with (1.3.44) implies (1.3.42) and concludes the proof of the claim.

By (1.3.8) and (1.3.13) we get $\hat{\Phi}^n \to \Phi$ and $\hat{\Psi}^n \to \Psi$ in C^2, while (1.3.36) and (1.3.13) give $\hat{\mathbb{A}}^n \to \mathbb{A}$ in C^1. Moreover applying Lemma 1.3.7 to the functions and their derivatives we can prove that $\hat{u}^{0,n} \to u^0$ in V_0, $\hat{u}^{1,n} \to u^1$ in H, $\hat{u}^n_D \to u_D$ in $H^2(0,T;H)\cap H^1(0,T;V_0)$, and $\hat{g}^n \to g$ in $L^2(0,T;H)$. Using (1.3.37) and (1.3.38) we have that $\hat{f}^n \to g$ in $L^2(0,T;H)$. We are now in a position to apply [9, Theorem 1.4.1] to problem (1.3.39)-(1.3.41) and we obtain

$$(\hat{v}^n(t), \, D\hat{v}^n(t), \, \dot{\hat{v}}^n(t)) \to (v(t), \, Dv(t), \, \dot{v}(t)) \quad \text{ in } H \times \underline{H} \times H$$

for every $t \in [0,T]$. Since

$$v^n(t,\cdot) = \hat{v}^n(t,\Xi^n(\cdot)), \quad Dv^n(t,\cdot) = D\hat{v}^n(t,\Xi^n(\cdot)) D\Xi^n(\cdot),$$

$$\dot{v}^n(t,\cdot) = \dot{\hat{v}}^n(t, \Xi^n(\cdot)),$$

using Lemma 1.3.7 we get (1.3.35) for every $t \in [0,T]$. □

To use Proposition 1.3.6 in the proof of the convergence $\mathcal{G}^n(w) \to \mathcal{G}(w)$ we need the following approximation result.

Lemma 1.3.8. *Let $G \in H^1((0,T);\, \tilde{H})$. For every $\varepsilon > 0$ there exists a compact neighborhood K_ε of $\Gamma \cap \overline{\Omega}$ and $G_\varepsilon \in H^1((0,T);\, \tilde{H})$ such that $G_\varepsilon(t) \in C_c^\infty(\Omega \setminus K_\varepsilon;\, \mathbb{R}^{d\times d}_{sym})$ for every $t \in [0,T]$ and*

$$\|G_\varepsilon - G\|_{L^\infty(0,T;\tilde{H})} + \|\dot{G}_\varepsilon - \dot{G}\|_{L^2(0,T;\tilde{H})} < \varepsilon.$$

Remark 1.3.9. By (H22) and (1.3.13) for every $\varepsilon > 0$ there exists n_ε such that $\Gamma^n \subset K_\varepsilon$, for $n > n_\varepsilon$. From the properties of G_ε follows that

$$(G_\varepsilon(t),\, Ev) = -(\mathrm{div} G_\varepsilon(t),\, v) \tag{1.3.45}$$

for all $t \in [0,T]$ and for all $v \in V_n$, for $n > n_\varepsilon$.

Proof of Lemma 1.3.8. Given a partition of $[0,T]$, we can consider the piecewise affine interpolation of the values of F at the nodes. It is well known that this interpolation converges in $H^1(0,T;\tilde{H})$ to F as the fineness of the partition tends to zero. To conclude, it is enough to approximate in $\tilde{H}$ the values of F at the nodes by elements of $C_c^\infty(\Omega\setminus\Gamma;\, \mathbb{R}^{d\times d}_{sym})$ and to consider the corresponding piecewise affine interpolation. □

Proposition 1.3.10. *Assume (H1)-(H23) and (1.3.7)-(1.3.13). Let v^n be the solution of (1.3.18)-(1.3.20) and let v be the solution of (1.2.2)-(1.2.4). Then for every $t \in [0,T]$ we have*

$$(v^n(t),\, Dv^n(t),\, \dot{v}^n(t)) \to (v(t),\, Dv(t),\, \dot{v}(t)) \quad \text{in } H \times \underline{H} \times H. \tag{1.3.46}$$

Moreover

$$(v^n,\, Dv^n,\, \dot{v}^n) \to (v,\, Dv,\, \dot{v}) \quad \text{in } \mathcal{W} = L^2((0,T); H \times \underline{H} \times H). \tag{1.3.47}$$

Proof. Let $\varepsilon > 0$, let G_ε the function in Lemma 1.3.8 with $G = F$. Let v^n_ε solution of (1.3.18)-(1.3.20) with f^n and F^n replaced by f and G_ε, let v^ε solution of (1.2.2)-(1.2.4) with F replaced by G_ε. By (1.3.11) there exists n_ε such that

$$\|f^n - f\|_{L^2(0,T;H)} + \|F^n - G_\varepsilon\|_{L^\infty(0,T;\tilde{H})} + \|\dot{F}^n - \dot{G}_\varepsilon\|_{L^2(0,T;\tilde{H})} < \varepsilon \tag{1.3.48}$$

for every $n > n_\varepsilon$. The function $v^n - v^n_\varepsilon$ is the solution of problem (1.3.18)-(1.3.20) with f^n and F^n replaced by $f^n - f$ and $F^n - G_\varepsilon$ and u^n_D, f^n, $u^{n,0}$, $u^{1.n}$ replaced by zero. Then by Proposition 1.2.5 there exists a constant $C(T)$ depending on T (independent of n and ε) such that

$$\|v^n - v^n_\varepsilon\|_{\mathcal{V}^{n,\infty}} \le C(T)\varepsilon. \tag{1.3.49}$$

for every $n > n_\varepsilon$. Similarly we can prove

$$\|v - v_\varepsilon\|_{\mathcal{V}^\infty} \leq C(T)\varepsilon. \tag{1.3.50}$$

Changing the value of n_ε, by (1.3.45) we have that v_ε^n is the solution of (1.3.18)-(1.3.20) with f^n replaced by $g_\varepsilon := f - \mathrm{div} G_\varepsilon$ and F^n replaced by 0, while v_ε is the solution of (1.2.2)-(1.2.4) with f replaced by $g_\varepsilon := f - \mathrm{div} G_\varepsilon$ and F replaced by 0. By Proposition 1.3.6 for every $t \in [0, T]$ we have

$$(v_\varepsilon^n(t),\, Dv_\varepsilon^n(t),\, \dot{v}_\varepsilon^n(t)) \to (v_\varepsilon(t),\, Dv_\varepsilon(t),\, \dot{v}_\varepsilon(t)) \quad \text{in } H \times \underline{H} \times H. \tag{1.3.51}$$

Since

$$\begin{aligned} &\|(v^n(t),\, Dv^n(t),\, \dot{v}^n(t)) - (v(t),\, Dv(t),\, \dot{v}(t))\| \leq \|v^n - v_\varepsilon^n\|_{\mathcal{V}^{n,\infty}} \\ &+ \|(v_\varepsilon^n(t),\, Dv_\varepsilon^n(t),\, \dot{v}_\varepsilon^n(t)) - (v_\varepsilon(t),\, Dv_\varepsilon(t),\, \dot{v}_\varepsilon(t))\| + \|v - v_\varepsilon\|_{\mathcal{V}^\infty}, \end{aligned}$$

by (1.3.49)-(1.3.51) we get

$$\limsup_{n \to +\infty} \|(v^n(t),\, Dv^n(t),\, \dot{v}^n(t)) - (v(t),\, Dv(t),\, \dot{v}(t))\| \leq 2C(T)\varepsilon$$

for every $t \in [0, T]$. By the arbitrareness of ε we obtain (1.3.46). Finally, using the estimate in Proposition 1.3.5 and the Dominated Convergence Theorem we obtain (1.3.47). □

Corollary 1.3.11. *Assume (H1)-(H23) and* (1.3.7)-(1.3.13)*. Then for every $w \in \mathcal{W}$ we have*

$$\mathcal{G}^n(w) \to \mathcal{G}(w) \quad \mathcal{W}.$$

Proof. By (1.3.9) we get

$$\mathcal{T}^n w \to \mathcal{T} w$$

in $H^1(0, T; \tilde{H})$ for every $w \in \mathcal{W}$. The result follows from Proposition 1.3.10 with F^n and F replaced by $F^n + \mathcal{T}^n w$ and $F + \mathcal{T} w$. □

1.3.2 Proof of the continuous dependence

As a consequence of Lemma 1.3.3, Proposition 1.3.4, and Corollary 1.3.11 we obtain the continuous dependence result when T is small enough.

Theorem 1.3.12. *Assume that $B(T + T^3) < 1$, where B is the constant in Proposition 1.3.4. Then the conclusion of Theorem 1.3.2 holds.*

Proof. By Corollary 1.3.11

$$\mathcal{G}^n(w) \to \mathcal{G}(w)$$

in $\mathcal{W}$ for every $w \in \mathcal{W}$. By Proposition 1.3.4 the maps $\mathcal{G}^n$ have the same contraction constant $B(T+T^3)<1$. Then we are in a position to apply Lemma 1.3.3 and we get

$$w^n := (u^n,\, Du^n,\, \dot{u}^n) \to (u,\, Du,\, \dot{u}) =: w \quad \text{in } \mathcal{W} = L^2((0,T); H \times \underline{H} \times H). \tag{1.3.52}$$

From this convergence and (1.3.9), we obtain $\mathcal{T}^n w^n \to \mathcal{T} w$ in $H^1(0,T;\tilde{H})$ and we can apply Proposition 1.3.10, with forcing term F^n and F replaced by $F^n + \mathcal{T}^n w^n$ and $F + \mathcal{T} w$. Since

$$F^n + \mathcal{T}^n w^n \to F + \mathcal{T} w$$

in $H^1(0,T;\tilde{H})$ we get

$$(u^n(t),\, Du^n(t),\, \dot{u}^n(t)) \to (u(t),\, Du(t),\, \dot{u}(t)) \quad \text{ in } H \times \underline{H} \times H$$

for every $t \in [0,T]$. We can apply Proposition 1.3.5 with F^n replaced by $F^n + \mathcal{T}^n w^n$ and we obtain that there exists a constant $C > 0$ such that

$$\|u^n(t)\| + \|Du^n(t)\| + \|\dot{u}^n(t)\| \leq C$$

for every $n \in \mathbb{N}$ and $t \in [0,T]$. □

We are now in a position prove Theorem 1.3.2 without additional assumptions on T.

Proof of Theorem 1.3.2. There exists $k \in \mathbb{N}$ such that $T_0 := T/k$ satisfies $B(T_0 + T_0^3) < 1$. By Theorem 1.3.12 we have

$$(u^n(t),\, Du^n(t),\, \dot{u}^n(t)) \to (u(t),\, Du(t),\, \dot{u}(t)) \quad \text{ in } H \times \underline{H} \times H \;\text{ for all } t \in [0,T_0], \tag{1.3.53}$$

and moreover

$$(u^n,\, Du^n,\, \dot{u}^n) \to (u,\, Du,\, \dot{u}) \quad \text{in } L^2((0,T_0); H \times \underline{H} \times H). \tag{1.3.54}$$

If $k = 1$ the proof is finished, otherwise we consider the problem on the interval $[T_0, 2T_0]$.

Note that

$$u^n(T_0) \in V^n \text{ and } \dot{u}^n(T_0) \in H$$

are well defined, because $u \in C^0_w([0,T_0]; V^n)$ and $\dot{u} \in C^0_w([0,T_0]; H)$. Since $u^n(t) \in V^n_t$ for a.e. $t \in (0,T_0)$, it easy to see that $u^n(T_0) \in V^n_{T_0}$.

In order to study the problem on $[T_0,\, 2T_0]$ we define the spaces $\mathcal{V}_{T_0,2T_0}$, $\mathcal{V}^D_{T_0,2T_0}$, $\mathcal{V}^\infty_{T_0,2T_0}$, $\mathcal{V}^n_{T_0,2T_0}$, $\mathcal{V}^{n,D}_{T_0,2T_0}$, $\mathcal{V}^{n,\infty}_{T_0,2T_0}$, and $\mathcal{W}_{T_0.2T_0}$ as $\mathcal{V}$, $\mathcal{V}^D$, $\mathcal{V}^\infty$, $\mathcal{V}^n$, $\mathcal{V}^{n,D}$, $\mathcal{V}^{n,\infty}$, and $\mathcal{W}$ with 0 and T replaced by T_0 and $2T_0$.

For every $t \in [T_0, 2T_0]$ we set

$$G(t) := F(t) + \int_0^{T_0} \mathrm{e}^{\tau - t} \mathbb{V} E u(\tau) \mathrm{d}\tau,$$

$$G^n(t) := F^n(t) + \int_0^{T_0} \mathrm{e}^{\tau - t} \mathbb{V}^n E u^n(\tau) \mathrm{d}\tau.$$

Let v be the solution of the problem

$$\begin{aligned}
& v \in \mathcal{V}_{T_0,2T_0} \quad \text{and} \quad v - u_D \in \mathcal{V}^D_{T_0,2T_0}, \\
& -\int_{T_0}^{2T_0} (\dot{v}(t), \dot{\varphi}(t))\, \mathrm{d}t + \int_{T_0}^{2T_0} (\mathbb{A} E v(t), E\varphi(t))\, \mathrm{d}t - \int_{T_0}^{2T_0} \int_{T_0}^{t} \mathrm{e}^{\tau - t} (\mathbb{V} E v(\tau), E\varphi(t))\, \mathrm{d}\tau \mathrm{d}t \\
& = \int_{T_0}^{2T_0} (f(t), \varphi(t))\, \mathrm{d}t + \int_{T_0}^{2T_0} (G(t), E\varphi(t))\, \mathrm{d}t \\
& \text{for every } \varphi \in \mathcal{V}^D_{T_0,2T_0} \text{with } \varphi(T_0) = \varphi(2T_0) = 0, \\
& v(T_0) = u(T_0) \quad \text{in } H \quad \text{and} \quad \dot{v}(T_0) = \dot{u}(T_0) \quad \text{in } (V^D_{T_0})^*.
\end{aligned}$$

For every $n \in \mathbb{N}$ let v^n be the solution of the problem

$$\begin{aligned}
& v^n \in \mathcal{V}^n_{T_0,2T_0} \quad \text{and} \quad v^n - u^n_D \in \mathcal{V}^{n,D}_{T_0,2T_0}, \\
& -\int_{T_0}^{2T_0} (\dot{v}^n(t), \dot{\varphi}(t))\, \mathrm{d}t + \int_{T_0}^{2T_0} (\mathbb{A}^n E v^n(t), E\varphi(t))\, \mathrm{d}t - \int_{T_0}^{2T_0} \int_{T_0}^{t} \mathrm{e}^{\tau - t} (\mathbb{V}^n E v^n(\tau), E\varphi(t))\, \mathrm{d}\tau \mathrm{d}t \\
& = \int_{T_0}^{2T_0} (f^n(t), \varphi(t))\, \mathrm{d}t + \int_{T_0}^{2T_0} (G^n(t), E\varphi(t))\, \mathrm{d}t \\
& \text{for every } \varphi \in \mathcal{V}^{n,D}_{T_0,2T_0} \text{with } \varphi(T_0) = \varphi(2T_0) = 0, \\
& v^n(T_0) = u^n(T_0) \quad \text{in } H \quad \text{and} \quad \dot{v}^n(T_0) = \dot{u}^n(T_0) \quad \text{in } (V^{n,D}_{T_0})^*.
\end{aligned}$$

We note that, by the definition of G and G^n, the restrictions of u and u^n to $[T_0, 2T_0]$ satisfy the problems for v and v^n. By uniqueness we have that $v = u$ and $v^n = u^n$ on $[T_0, 2T_0]$.

For every $x \in \overline{\Omega}$ and $[T_0, 2T_0]$ we define

$$\Phi_{T_0}(t,x) := \Phi(t, \Psi(T_0,x)),\ \ \Psi_{T_0}(t,x) := \Psi(t, \Phi(T_0,x))$$

$$\Phi^n_{T_0}(t,x) := \Phi^n(t, \Psi^n(T_0,x))\ \ \Psi^n_{T_0}(t,x) := \Psi^n(t, \Phi^n(T_0,x))$$

which satisfy (H11)-(H15), (1.3.8) with 0 and T replaced by T_0 and $2T_0$. For every $x \in \overline{\Omega}$ we define

$$\Theta^n_{T_0}(x) := \Phi^n(T_0, \Theta^n(\Psi(T_0,x))),\ \ \Xi^n_{T_0}(x) := \Phi(T_0, \Xi^n(\Psi^n(T_0,x)))$$

and we observe that they satisfy (H19)-(H23) and (1.3.13) with 0 and T replaced by T_0 and $2T_0$.

By (1.3.53) we have that $(u^n(T_0), Du^n(T_0), \dot{u}^n(T_0)) \to (u(T_0), Du(T_0), \dot{u}(T_0))$ in $H \times \underline{H} \times H$ while (1.3.9), (1.3.11), and (1.3.54) give $G^n \to G$ in $H^1(0, T; \tilde{H})$. We are now in a position to apply Theorem 1.3.12 on $[T_0, 2T_0]$ to obtain

$$(u^n(t),\, Du^n(t),\, \dot{u}^n(t)) \to (u(t),\, Du(t),\, \dot{u}(t)) \quad \text{in } H \times \underline{H} \times H,$$

for all $t \in [T_0, 2T_0]$. Moreover there exists a constant $C > 0$ such that

$$\|u^n(t)\| + \|Du^n(t)\| + \|\dot{u}^n(t)\| \leq C$$

for every $n \in \mathbb{N}$ and $t \in [T_0, 2T_0]$. The conclusion can be obtained by repeating this process a finite number of times. $\square$

Chapter 2

Dynamic crack growth in viscoelastic materials with memory

Contents

We now study the problem of planar crack growth for the viscoelastic system when the crack evolution is not prescribed. More precisely, as explained in the Introduction, we consider

$$\ddot{u}(t,x) - \operatorname{div}(\sigma_0(t,x)) = \ell_0(t) \qquad t \in [0,T],\ x \in \Omega \setminus \Gamma_t, \tag{2.0.1}$$

$$u(t,x) = u_D(t,x) \qquad t \in [0,T],\ x \in \partial_D \Omega, \tag{2.0.2}$$

$$\sigma_0(t,x)\nu = F_0(t,x)\nu \qquad t \in [0,T],\ x \in \partial_N \Omega, \tag{2.0.3}$$

$$\sigma_0^{\pm}(t,x)\nu = F_0^{\pm}(t,x)\nu \qquad t \in [0,T],\ x \in \Gamma_t, \tag{2.0.4}$$

$$u(0,x) = u^0(x) \quad \text{and} \quad \dot{u}(0,x) = u^1(x) \qquad x \in \Omega \setminus \Gamma_t \tag{2.0.5}$$

for every $t \in [0,T]$, where σ is the stress tensor defined as

$$\sigma(t,x) := \mathbb{C}(x)Eu(t,x) + \mathbb{V}(x)Eu(t,x) - \int_0^t \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau,x)\,\mathrm{d}\tau, \tag{2.0.6}$$

u_D is the Dirichlet condition, u^0 is the initial condition for the displacement, u^1 is the initial condition for the velocity, ν is the unit normal, and the symbol $\pm$ in (2.0.4) denotes suitable limits on each side of Γ_t. The reference configuration is an open subset of the plane, namely $\Omega \subset \mathbb{R}^2$, and the cracks $\{\Gamma_t\}_{t\in[0,T]}$ is an unknown family of 1-dimensional increasing closed subsets of $\overline{\Omega}$ such that $\Gamma_0 = \Gamma_{in}$, where Γ_{in} is a prescribed initial crack. When the cracks are prescribed for every time, we refer to Chapter 1. We couple the previous system with conditions on the cracks:

a) an energy dissipation balance (see Definition 2.2.3): the sum of the kinetic and elastic energies and of the energies dissipated by viscosity and crack growth balances the work done by the forces acting on the system;

b) a maximal dissipation condition, depending on a parameter $\eta > 0$ (see Definition 2.3.1), which forces the crack to run as fast as possible.

The main result of this chapter is that, given initial and boundary conditions satisfying suitable hypotheses, there exists a couple $(u, \{\Gamma_t\}_{t\in[0,T]})$ satisfying the viscoelastic equation with memory and conditions a) and b) (see Theorem 2.3.3). The proof follows the lines of [19] and use the results of Chapter 1.

The chapter is organized as follows. In Section 2.1 we give the precise formulation of the problem. More precisely, in Subsection 2.1.1 we describe the geometry of the cracks while in Subsection 2.1.2 we define the function spaces for the weak formulation and in Subsection 2.1.3 we extend some preliminary results of Chapter 1. In Section 2.2 is divided in Subsection 2.2.1, where we define the dynamic energy-dissipation balance, and Subsection 2.2.2, where we define the class of cracks satisfying the energy-dissipation balance. Finally, in Section 2.3 we define the maximal dissipation condition and we prove the main result.

The original results of this chapter are based on [12].

2.1 Formulation of the problem

The reference configuration of our problem is a bounded open set $\Omega \subset \mathbb{R}^2$, with Lipschitz boundary $\partial\Omega$ and we assume that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega$ and $\partial_N\Omega$ are disjoint (possibly empty) Borel sets, on which we prescribe Dirichlet and Neumann boundary conditions respectively. Moreover, we fix a time interval $[0,T]$, with $T > 0$.

2.1.1 The geometry of the cracks

We give a precise definition of the admissible cracks of our model using a suitable class of curves. The following definitions and results are based on [18] and [19]. The curves are always parameterized using the arc-length parameter s and for a given curve $\gamma : [a_\gamma, b_\gamma] \to \mathbb{R}^2$ we define $\Gamma^\gamma := \gamma([a_\gamma,\, b_\gamma])$ and $\Gamma^\gamma_s := \gamma([a_\gamma,\, s])$, for every $s \in [a_\gamma,\, b_\gamma]$. When it is clear from the context we omit the dependence on γ and we write Γ and Γ_s instead of Γ^γ and Γ^γ_s. In order to describe the initial crack, we fix a curve $\gamma_0 : [a_0,\, 0] \to \overline{\Omega}$ such that $\gamma_0(a_0) \in \partial\Omega$, $\gamma_0(s) \in \Omega$ for every $s \in (a_0, 0]$ and we define the initial crack as

$$\Gamma_0 := \gamma_0([a_0, 0]).$$

We suppose that γ_0 is of class $C^{3,1}$ and that it is transversal to $\partial\Omega$ at $\gamma_0(a_0)$ (there exists an isosceles triangle contained in $\overline{\Omega}$ with vertex in $\gamma_0(a_0)$ and axis parallel to $\gamma_0'(a_0)$). We fix two constants $r > 0$ and $L > 0$ and we now define the space of admissible crack paths.

Definition 2.1.1. Let $\mathcal{G}_{r,L}$ be the space of simple curves $\gamma : [a_0, b_\gamma] \to \overline{\Omega}$ of class $C^{3,1}$, with $a_0 < 0 \le b_\gamma$, such that

(a) $\gamma(s) = \gamma_0(s)$ for every $s \in [a_0,\, 0]$,

(b) $|\gamma'(s)| = 1$ for every $s \in [a_0, b_\gamma]$,

(c) the two open disks of radius r tangent to Γ at $\gamma(s)$ do not intersect Γ,

(d) $\mathrm{dist}(\gamma([0, b_\gamma]), \partial\Omega) \ge 2r$,

(e) $|\gamma^{(3)}(s)| \le L$, $|\gamma^{(3)}(s_2) - \gamma^{(3)}(s_1)| \le L|s_2 - s_1|$ for any s, s_1, $s_2 \in [a_0, b_\gamma]$,

where $\gamma^{(i)}$ denotes the $i-$th derivative of γ.

We fix γ_0, r, and L such that $\mathcal{G}_{r,L} \neq \emptyset$.

Remark 2.1.2. By (a) and (d) we have $|a_0| \ge 2r$. Condition (c) implies $|\gamma^{(2)}(s)| \le 1/r$ for every $s \in [a_0,\, b_\gamma]$.

Definition 2.1.3. Let γ_k be a sequence of curves in $\mathcal{G}_{r,L}$ and let $\gamma \in \mathcal{G}_{r,L}$. We say that γ_k converges uniformly to γ if $b_{\gamma_k} \to b_\gamma$ and for every $b \in (0, b_\gamma)$ we have $\gamma_k|_{[a_0,b]} \to \gamma|_{[a_0,b]}$ uniformly in $[a_0, b]$.

Lemma 2.1.4. *There exist two constants $\hat{r}$ and $\hat{L}$, with $0 < \hat{r} < r$ and $\hat{L} > L$, depending only on r and L, such that for every $\gamma\colon [a_0, b_\gamma] \to \overline{\Omega}$ with $\gamma \in \mathcal{G}_{r,L}$ there exists an extension*

$$\hat{\gamma}\colon [a_0, b_\gamma + \hat{r}] \to \overline{\Omega}$$

of γ with $\hat{\gamma} \in \mathcal{G}_{\hat{r},\hat{L}}$, whose image will be indicated by $\hat{\Gamma}$. Moreover, the extension can be chosen in such a way that the uniform convergence of γ_k implies the uniform convergence of the corresponding extensions $\hat{\gamma}_k$.

Lemma 2.1.5. *Let γ_k be a sequence of curves in $\mathcal{G}_{r,L}$. Then there exist a subsequence, not relabelled, and a curve $\gamma \in \mathcal{G}_{r,L}$ such that γ_k converges to γ uniformly.*

For a proof of the previous two lemmas see [19].

We have to describe the dependence of the crack length on the time. We fix two constants $\mu > 0$ and $M > 0$ which bound the speed of the crack tip and some higher order derivatives of the crack length, respectively.

Definition 2.1.6. Let $T_0 < T_1$. The class $\mathcal{S}^{reg}_{\mu,M}(T_0, T_1)$ is composed of all nonnegative functions satisfying the following conditions:

$$s \in C^{3,1}([T_0, T_1]), \tag{2.1.1}$$

$$0 \leq \dot{s}(t) \leq \mu \tag{2.1.2}$$

$$|\ddot{s}(t)| \leq M, \ \ |\dddot{s}(t)| \leq M, \ \ |\dddot{s}(t_1) - \dddot{s}(t_2)| \leq M|t_1 - t_2|, \tag{2.1.3}$$

for $t, t_1, t_2 \in [T_0, T_1]$, where dots denote derivatives with respect to time. We denote by $\mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$ the set of all functions $s \in C^0([T_0, T_1])$ such that there exists a finite subdivision

$$T_0 = \tau_0 < \tau_1 < ... < \tau_k = T_1$$

for which $s|_{[\tau_{j-1},\tau_j]} \in \mathcal{S}^{reg}_{\mu,M}(\tau_{j-1}, \tau_j)$. The minimal set $\{\tau_0, \tau_1, ..., \tau_k\}$ for which this property holds is denoted by $sing(s)$.

Given $0 \leq T_0 < T_1 \leq T$, $\gamma \in \mathcal{G}_{r,L}$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$, with $s(T_1) \leq b_\gamma$, the time dependent cracks corresponding to these functions are given by

$$\Gamma^\gamma_{s(t)} := \gamma([a_0, s(t)]) \quad \text{for all } t \in [T_0, T_1],$$

and the corresponding cracked domains are

$$\Omega^\gamma_{s(t)} := \Omega \setminus \Gamma^\gamma_{s(t)} \quad \text{for all } t \in [T_0, T_1].$$

For simplicity of notation we sometimes denote $\Gamma^\gamma_{s(t)}$ by $\Gamma_{s(t)}$, when γ is clear from the context.

In [8], [9], and [11] the cracks are described using a family of time-dependent diffeomorphism $\Phi, \Psi : [0, T] \times \overline{\Omega} \to \overline{\Omega}$. Thanks to the following result it is possible to obtain the same maps also in our case. For a proof see [18, Lemma 2.8].

Lemma 2.1.7. *Let $\varepsilon > 0$ and let $\rho \in (0, \hat{r}/2)$, where $\hat{r}$ is the constant that appears in Lemma 2.1.4. Then there exists two constants $\delta \in (0, \rho/\mu)$ and $C > 0$ depending only on r, L, μ, M, ε, and ρ, with the following property: for every $\gamma \in \mathcal{G}_{r,L}$, for every $t_0 < t_1$, and for every $s \in \mathcal{S}^{reg}_{\mu,M}(t_0, t_1)$, with $t_1 - t_0 \leq \delta$, $s(t_1) \leq b_\gamma$, we can define two functions $\Phi, \Psi \colon [t_0, t_1] \times \overline{\Omega} \to \overline{\Omega}$ of class $C^{2,1}$ with the following properties:*

(a) *for every* $t \in [t_0, t_1]$ *we have* $\Phi(t, \overline{\Omega}) = \overline{\Omega}$, $\Phi(t, \hat{\Gamma}) = \hat{\Gamma}$ *(where* $\hat{\Gamma}$ *is the set that appears in Lemma 2.1.4),* $\Phi(t, \Gamma_{s(t_0)}) = \Gamma_{s(t)}$, *and* $\Phi(t, y) = y$ *on* $\overline{\Omega} \setminus B(\gamma(s(t_0)), 2\rho)$;

(b) $\Phi(t_0, y) = y$ *for every* $y \in \overline{\Omega}$;

(c) *for every* $t \in [t_0, t_1]$, $\Psi(t, \cdot)$ *is the inverse of* $\Phi(t, \cdot)$ *on* $\overline{\Omega}$;

(d) *for every* $t \in [t_0, t_1]$ *we have*

$$1 - \varepsilon \leq \det D\Phi(t, y) \leq 1 + \varepsilon,$$

$$1 - \varepsilon \leq \det D\Psi(t, y) \leq 1 + \varepsilon$$

for every $x, y \in \overline{\Omega}$, *where* D *denotes the spatial jacobian matrix.*

(e) *for every* $t \in [t_0, t_1]$ *we have* $|\partial_t \Phi(t, y)| \leq \mu(1 + \varepsilon)$ *for every* $y \in \overline{\Omega}$;

(f) *the absolute values of all partial derivatives of* Φ *and of* Ψ *of order less than or equal to two, as well as the Lipschitz constants of all second derivatives, are bounded by* C;

(g) *if* $\gamma_k \in \mathcal{G}_{r,L}$ *converges to* γ *uniformly, if* $s_k \in \mathcal{S}^{reg}_{\mu,M}(t_0, t_1)$ *converges to* s *uniformly, with* $s_k(t_1) \leq b_{\gamma_k}$ *for every* k, *then the corresponding diffemorphisms satisfy* $\Phi_k(t, x) \to \Phi(t, x)$ *for every* $t \in [t_0, t_1]$ *and for every* $x \in \overline{\Omega}$.

2.1.2 The functional spaces for the viscoelastic problem

We now define the functional spaces that will be used in order to give the definition of weak solution of the viscoelastic problem.

We define $\mathbb{R}^{2\times 2}$ as the space of real 2×2 matrix and $\mathbb{R}^{2\times 2}_{sym}$ as the space of real 2×2 symmetric matrices. The euclidean scalar product between the matrices A and B is denoted by $A : B$. For every $A \in \mathbb{R}^{2\times 2}$ the symmetric part $A_{sym} \in \mathbb{R}^{2\times 2}$ is defined as $A_{sym} = \frac{1}{2}(A + A^T)$, where A^T denotes the transpose matrix of A. For any pair of vector spaces we define $\mathcal{L}(X; Y)$ as the space of linear and continuous maps form X into Y. Let $0 < \lambda < \Lambda$ be two fixed constants. We now define the space of tensors that will be used in the chapter.

Definition 2.1.8. We define $\mathcal{E}(\lambda, \Lambda)$ as the set of all maps $\mathbb{L} : \overline{\Omega} \to \mathcal{L}(\mathbb{R}^{2\times 2}; \mathbb{R}^{2\times 2})$ of class C^2 such that for every $x \in \overline{\Omega}$ we have

$$\mathbb{L}(x)A = \mathbb{L}(x)A_{sym} \in \mathbb{R}^{2\times 2}_{sym} \quad \text{for every } A \in \mathbb{R}^{2\times 2}, \tag{2.1.4}$$

$$\mathbb{L}(x)A : B = \mathbb{L}(x)B : A \quad \text{for every } A,\, B \in \mathbb{R}^{2\times 2}, \tag{2.1.5}$$

$$\lambda |A_{sym}|^2 \leq \mathbb{L}(x)A : A \leq \Lambda |A_{sym}|^2 \quad \text{for every } A \in \mathbb{R}^{2\times 2}. \tag{2.1.6}$$

We now fix the following maps

$$\mathbb{C}, \mathbb{V} \in \mathcal{E}(\lambda, \Lambda), \quad \mathbb{A} := \mathbb{C} + \mathbb{V} \tag{2.1.7}$$

where $\mathbb{C}(x)$ and $\mathbb{V}(x)$ respectively represent the elasticity and viscosity tensor at the point $x \in \overline{\Omega}$.

Given $\gamma \in \mathcal{G}_{r,L}$, $0 \leq T_0 < T_1 \leq T$, and $s \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$, with $s(T_1) \leq b_\gamma$, we now introduce the function spaces that will be used in the precise formulation of problem (2.0.1)-(2.0.5).

We recall that

$$\Gamma := \gamma([a_0, b_\gamma]; \mathbb{R}^2). \tag{2.1.8}$$

For every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^2)$ Du denotes jacobian matrix in the sense of distributions on $\Omega \setminus \Gamma$ and Eu is its symmetric part, i.e.,

$$Eu := \frac{1}{2}(Du + Du^T).$$

The following lemma is an extension of the second Korn's inequality (see, e.g., [41]) to the case of cracked domain. For a proof see, e.g., [11].

Lemma 2.1.9. *Let $\gamma \in \mathcal{G}_{r,L}$ and let $\Gamma := \gamma([a_0, b_\gamma]; \mathbb{R}^2)$. Then there exists a constant K, depending only on Ω and Γ, such that*

$$\|Du\|^2 \leq K(\|u\|^2 + \|Eu\|^2) \tag{2.1.9}$$

for every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^2)$, where $\|\cdot\|$ denotes the L^2 norm.

Remark 2.1.10. Let $\gamma \in \mathcal{G}_{r,L}$ and let $\Gamma := \gamma([a_0, b_\gamma]; \mathbb{R}^2)$. Then, using a localization argument (see, e.g., [11]), we can prove that the trace operator is well defined and continuous from $H^1(\Omega \setminus \Gamma; \mathbb{R}^2)$ into $L^2(\partial\Omega; \mathbb{R}^2)$.

We set

$$V^\gamma := H^1(\Omega \setminus \Gamma; \mathbb{R}^d), \quad H := L^2(\Omega; \mathbb{R}^d), \quad \text{and} \quad \underline{H} := L^2(\Omega; \mathbb{R}^{d\times d}) \tag{2.1.10}$$

Since $\mathcal{L}^2(\Gamma) = 0$, we have the embedding

$$V^\gamma \hookrightarrow H \times \underline{H} \tag{2.1.11}$$

given by $v \mapsto (v, Dv)$ and we can see the distrubutional gradient Dv on $\Omega \setminus \Gamma$ as a function defined a.e. on Ω, which belongs to $\underline{H}$.

For every finite dimensional Hilbert space Y the symbols $(\cdot\,,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in the $L^2(\Omega; Y)$, according to the context. The space V^γ is endowed with the norm

$$\|u\|_{V^\gamma} := (\|u\|^2 + \|Du\|^2)^{1/2}. \tag{2.1.12}$$

For every $\overline{s} \in [a_0, b_\gamma]$ we define

$$V_{\overline{s}}^{\gamma} := H^1(\Omega \setminus \Gamma_{\overline{s}}; \mathbb{R}^2) \quad \text{and} \quad V_{\overline{s}}^{\gamma,D} := \{u \in V_{\overline{s}}^{\gamma} \mid u|_{\partial_D\Omega} = 0\}, \tag{2.1.13}$$

where $\Gamma_{\overline{s}} = \gamma([a_0, \overline{s}])$ and $u|_{\partial_D\Omega}$ denotes the trace of u on $\partial_D\Omega$. We note that $V_{\overline{s}}^{\gamma}$ and $V_{\overline{s}}^{\gamma,D}$ are closed linear subspaces of V^γ. For every $t \in [T_0, T_1]$ the spaces $V_{s(t)}^{\gamma}$ and $V_{s(t)}^{\gamma,D}$ are defined as in (2.1.13) with $\overline{s} = s(t)$.

We define

$$\mathcal{V}_{\gamma,s}(T_0, T_1) := \{v \in L^2(T_0, T_1; V^\gamma) \cap H^1(T_0, T_1; H) \,|\, v(t) \in V_{s(t)}^{\gamma} \text{ for a.e. } t \in (T_0, T_1)\}, \tag{2.1.14}$$

which is a Hilbert space with the norm

$$\|v\|_{\mathcal{V}_{\gamma,s}} := (\|v\|^2_{L^2(T_0,T_1;V^\gamma)} + \|\dot{v}\|^2_{L^2(T_0,T_1;H)})^{\frac{1}{2}}, \tag{2.1.15}$$

where the dot denotes the distibutional derivative with respect to t. Moreover we set

$$\mathcal{V}_{\gamma,s}^{D}(T_0, T_1) := \{v \in \mathcal{V}_{\gamma,s}(T_0, T_1) \,|\, v(t) \in V_{s(t)}^{D} \text{ for a.e. } t \in (T_0, T_1)\}, \tag{2.1.16}$$

which is a closed linear subspace of $\mathcal{V}_{\gamma,s}(T_0, T_1)$ and we define

$$\mathcal{V}_{\gamma,s}^{\infty}(T_0, T_1) := \{v \in L^\infty(T_0, T_1; V^\gamma) \cap W^{1,\infty}(T_0, T_1; H) \,|\, v(t) \in V_{s(t)}^{\gamma} \text{ for a.e. } t \in (T_0, T_1)\}, \tag{2.1.17}$$

which is a Banach space with the norm

$$\|v\|_{\mathcal{V}_{\gamma,s}^{\infty}} := \|v\|_{L^\infty(T_0,T_1;V^\gamma)} + \|\dot{v}\|_{L^\infty(T_0,T_1;H)}. \tag{2.1.18}$$

Moreover, it is convenient to introduce the space of weakly continuous functions with values in a Banach space X with topological dual X^*, defined by

$$C_w^0([T_0, T_1]; X) := \{v : [T_0, T_1] \to X \,|\, t \mapsto \langle h, v(t) \rangle \text{ is continuous for every } h \in X^*\}.$$

When it is clear from the context we will omit the dependence on γ or s in the functional spaces, writing V, $V_{s(t)}$, $V_{s(t)}^{D}$, $\mathcal{V}(T_0, T_1)$, $\mathcal{V}^D(T_0, T_1)$, and $\mathcal{V}^\infty(T_0, T_1)$ instead of V^γ, $V_{s(t)}^{\gamma}$, $V_{s(t)}^{\gamma,D}$, $\mathcal{V}_{\gamma,s}(T_0, T_1)$, $\mathcal{V}_{\gamma,s}^{D}(T_0, T_1)$, and $\mathcal{V}_{\gamma,s}^{\infty}(T_0, T_1)$.

Since

$$H^1(T_0, T_1; H) \hookrightarrow C^0([T_0, T_1]; H), \tag{2.1.19}$$

we have

$$\mathcal{V}(T_0, T_1) \hookrightarrow C^0([T_0, T_1], H). \tag{2.1.20}$$

In particular $v(T_0)$ and $v(T_1)$ are well defined elements of H, for every $v \in \mathcal{V}(T_0, T_1)$.

We set

$$\tilde{H} := L^2(\Omega; \mathbb{R}^{2\times 2}_{sym}). \tag{2.1.21}$$

On the forcing term $\ell(t)$ of (2.0.1) we assume that

$$\ell(t) := f(t) - \mathrm{div}F(t), \tag{2.1.22}$$

where

$$f \in L^2(0,T;H) \quad \text{and} \quad F \in H^1(0,T;\tilde{H}) \tag{2.1.23}$$

are prescribed function and the divergence of a matrix valued function is the vector valued function whose components are obtained taking the divergence of the rows.

The Dirichlet boundary condition on $\partial_D\Omega$ is obtained by prescribing a function

$$u_D \in H^2(0,T;\, H) \cap H^1(0,T;\, V_0). \tag{2.1.24}$$

where V_0 is $V_{\overline{s}}$ for $\overline{s} = 0$. It is not restrictive to assume that for every $t \in [0,T]$

$$u_D(t) = 0 \quad \text{a.e. on } \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) \geq r\}. \tag{2.1.25}$$

Moreover we will prescribe the natural Neumann boundary condition on $\partial_N\Omega \cup \Gamma_t$.

We are now in a position to give the definition of weak solution for the viscoelastic problem.

Definition 2.1.11 (Solution for visco-elastodynamics with cracks)**.** Let $\gamma \in \mathcal{G}_{r,L}$, $0 \leq T_0 < T_1 \leq T$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1) \leq b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25). Let $u^0 \in V_{s(T_0)}$, such that $u^0 - u_D(T_0) \in V^D_{s(T_0)}$ and let $u^1 \in H$. We say that u is a weak solution of the problem of visco-elastodynamics on the cracked domains $\Omega \setminus \Gamma_{s(t)}$, $t \in [T_0, T_1]$, with initial conditions u^0 and u^1, if

$$u \in \mathcal{V}(T_0,T_1) \quad \text{and} \quad u - u_D \in \mathcal{V}^D(T_0,T_1), \tag{2.1.26}$$

$$\begin{aligned} &-\int_{T_0}^{T_1} (\dot{u}(t), \dot{\varphi}(t))\,\mathrm{d}t + \int_{T_0}^{T_1} (\mathbb{A}Eu(t), E\varphi(t))\,\mathrm{d}t \\ &-\int_{T_0}^{T_1}\int_{T_0}^{t} \mathrm{e}^{\tau-t}(\mathbb{V}Eu(\tau), E\varphi(t))\,\mathrm{d}\tau\mathrm{d}t = \int_{T_0}^{T_1} (f(t), \varphi(t))\,\mathrm{d}t \\ &+\int_{T_0}^{T_1} (F(t), E\varphi(t))\,\mathrm{d}t \quad \text{for all } \varphi \in \mathcal{V}^D(T_0,T_1) \text{ with } \varphi(T_0) = \varphi(T_1) = 0, \end{aligned} \tag{2.1.27}$$

$$u(T_0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot{u}(T_1) = u^1 \quad \text{in } (V^D_{s(T_0)})^*, \tag{2.1.28}$$

where $(V^D_{s(T_0)})^*$ denotes the topological dual of $V^D_{s(T_0)}$.

Remark 2.1.12. If u satisfy (2.1.26) and (2.1.27), it is possible to prove that $\dot{u} \in H^1(0,T;(V^D_{s(T_0)})^*)$ (see [43, Remark 4.6]), which implies $\dot{u} \in C^0([T_0,T_1];(V^D_{s(T_0)})^*)$. In particular $\dot{u}(T_0)$ is well defined as an element of $(V^D_{s(T_0)})^*$.

Remark 2.1.13. In the case of smooth functions problem (2.1.26)-(2.1.28) is satisfied in a stronger sense. Namely, u and $\{\Gamma_{s(t)}\}_{t\in[T_0,T_1]}$ satisfy

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C}+\mathbb{V})Eu(t)) + \operatorname{div}\Big(\int_{T_0}^{t} \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big) = \ell(t) \qquad \text{in } \Omega\setminus\Gamma_{s(t)}, \tag{2.1.29}$$

$$u(t) = u_D(t) \qquad \text{on } \partial_D\Omega, \tag{2.1.30}$$

$$\Big((\mathbb{C}+\mathbb{V})Eu(t) - \int_{T_0}^{t} \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big)\nu = F(t)\nu \qquad \text{on } \partial_N\Omega, \tag{2.1.31}$$

$$\Big((\mathbb{C}+\mathbb{V})Eu(t) - \int_{T_0}^{t} \mathrm{e}^{\tau-t}\,\mathbb{V}Eu(\tau)\,\mathrm{d}\tau\Big)^{\pm}\nu = F(t)^{\pm}\nu \qquad \text{on } \Gamma_{s(t)}, \tag{2.1.32}$$

$$u(T_0) = u^0 \quad \text{and} \quad \dot{u}(T_0) = u^1 \tag{2.1.33}$$

for every $t\in[T_0,T_1]$, where

$$\ell(t) := f(t) - \operatorname{div}F(t), \tag{2.1.34}$$

ν is the unit normal, and the symbol $\pm$ in (2.1.32) denotes suitable limits on each side of $\Gamma_{s(t)}$.

2.1.3 A more general result on existence, uniqueness, and continuous dependence

Existence of the solution for the viscoelastic problem (2.1.26)-(2.1.28) is given by [43] for $\Omega\subset\mathbb{R}^d$ with $d\geq 1$ and under more general assumptions on the regularity of the cracks. Uniqueness and continuous dependence on the data are proved in [11] under the assumption that the constant μ, which controls the speed of the crack tip in Definition 2.1.6, satisfies

$$0 < \mu < \mu_0, \tag{2.1.35}$$

where the constant μ_0 is not explicitly defined in terms of the data of the problem. Using the fact that $d = 2$ in our work, we will prove that uniqueness and continuous dependence can be obtained under the explicit assumption

$$0 < \mu < \sqrt{\lambda}/2, \tag{2.1.36}$$

where λ are the constants that appears in Defintion 2.1.8 respectively.

In order to prove this results, we have to define an auxiliary problem, which can be interpreted as the elastodynamics problem with elasticity tensor replaced by $\mathbb{A}$.

Definition 2.1.14 (Solution for elastodynamics with cracks)**.** Let $\gamma\in\mathcal{G}_{r,L}$, $0\leq T_0 < T_1\leq T$, $s\in\mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1)\leq b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25). Let $u^0\in V_{s(T_0)}$, such that $u^0 - u_D(T_0)\in V^D_{s(T_0)}$ and let $u^1\in H$. We say that v is a weak solution of the problem of elastodynamics on the cracked domains $\Omega \setminus \Gamma_{s(t)}$, $t \in [T_0, T_1]$, with initial conditions u^0 and u^1, if

$$v \in \mathcal{V}(T_0,T_1) \quad \text{and} \quad v - u_D \in \mathcal{V}^D(T_0,T_1), \tag{2.1.37}$$

$$-\int_{T_0}^{T_1} (\dot v(t), \dot\varphi(t))\,\mathrm{d}t + \int_{T_0}^{T_1} (\mathbb{A}Ev(t), E\varphi(t))\,\mathrm{d}t = \int_{T_0}^{T_1} (f(t), \varphi(t))\,\mathrm{d}t$$

$$+\int_{T_0}^{T_1} (F(t), E\varphi(t))\,\mathrm{d}t \quad \text{for all } \varphi \in \mathcal{V}^D(T_0,T_1) \text{ with } \varphi(T_0) = \varphi(T_1) = 0, \tag{2.1.38}$$

$$v(T_0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot v(T_1) = u^1 \quad \text{in } (V^D_{s(T_0)})^*, \tag{2.1.39}$$

Remark 2.1.15. In the case of smooth functions problem (2.1.37)-(2.1.39) is satisfied in a stronger sense. Namely, v and $\{\Gamma_{s(t)}\}_{t\in[T_0,T_1]}$ satisfy

$$\ddot v(t) - \mathrm{div}(\mathbb{A}Ev(t)) = \ell(t) \qquad \text{in } \Omega \setminus \Gamma_{s(t)}, \tag{2.1.40}$$

$$v(t) = u_D(t) \qquad \text{on } \partial_D\Omega, \tag{2.1.41}$$

$$(\mathbb{A}Ev(t))\nu = F(t)\nu \qquad \text{on } \partial_N\Omega, \tag{2.1.42}$$

$$(\mathbb{A}Ev(t))^{\pm}\nu = F(t)^{\pm}\nu \qquad \text{on } \Gamma_{s(t)}, \tag{2.1.43}$$

$$v(T_0) = u^0 \quad \text{and} \quad \dot v(T_0) = u^1 \tag{2.1.44}$$

for every $t \in [T_0, T_1]$, where $\ell(t) := f(t) - \mathrm{div}F(t)$, ν is the unit normal, and the symbol $\pm$ in (2.1.43) denotes suitable limits on each side of $\Gamma_{s(t)}$.

Existence and uniqueness for the system of elastodynamics with cracks (2.1.37)-(2.1.39) under the assumption (2.1.36) is given by [19], where the authors consider a slight different formulation of the problem which is stronger in time. The proof, which is based on a localization argument, works also for the formulation given in Definition 2.1.14. Then we can state the following result.

Theorem 2.1.16. *Let $\gamma \in \mathcal{G}_{r,L}$, $0 \le T_0 < T_1 \le T$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1) \le b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0 \in V_{s(T_0)}$, such that $u^0 - u_D(T_0) \in V^D_{s(T_0)}$ and let $u^1 \in H$. Then there exists a unique solution v of problem (2.1.37)-(2.1.39). Moreover $v \in \mathcal{V}^\infty(T_0,T_1)$, $v \in C^0_w([T_0,T_1];V)$, and $\dot v \in C^0_w([T_0,T_1];H)$.*

With the following result we obtain a better regularity with respect to time.

Proposition 2.1.17. *Under the same assumption of Theorem 2.1.16, let v be the unique solution of problem (2.1.37)-(2.1.39). Then*

$$v \in C^0([T_0,T_1],V) \cap C^1([T_0,T_1],H). \tag{2.1.45}$$

Proof. In the case $F = 0$, a solution for the elastodynamics with cracks in the sense of [19] is also a solution in the sense of Definition 2.1.14. By uniqueness, the two solutions coincide. In particular, we get that, if $F = 0$, the solution is in $C^0([T_0, T_1], V) \cap C^1([T_0, T_1], H)$.

If the forcing term F is not zero, we can use same approximation argument used in [11, Lemma 5.7]. Then for every $\varepsilon > 0$ there exists $F_\varepsilon \in H^1(0, T, \tilde{H})$ such that

$$F_\varepsilon(t) \in C_c^\infty(\Omega \setminus \Gamma; \mathbb{R}^{d\times d}_{sym}) \tag{2.1.46}$$

for every $t \in [0, T]$ and

$$\|F_\varepsilon - F\|_{L^\infty(0,T;\tilde{H})} + \|\dot{F}_\varepsilon - \dot{F}\|_{L^2(0,T;\tilde{H})} < \varepsilon. \tag{2.1.47}$$

We define v_ε as the solution of the elastodynamic problem in Definition 2.1.14 with F replaced by F_ε. Since F_ε is regular in space we have that

$$(F_\varepsilon(t), E\psi) = -(\mathrm{div} F_\varepsilon(t), \psi) \tag{2.1.48}$$

for all $t \in [0, T]$ and for all $\psi \in V$. It follows that v_ε is a solution in the sense of Definition 2.1.14 with f and F respectively replaced by $f - \mathrm{div} F_\varepsilon$ and 0. By the results of [19] we have that

$$v_\varepsilon \in C^0([T_0, T_1], V) \cap C^1([T_0, T_1], H). \tag{2.1.49}$$

Using the continuous dependence on the forcing terms given by [11, Proposition 4.5] and (2.1.47), we obtain that

$$\sup_{t\in[0,T]} \|v_\varepsilon(t) - v(t)\|_V + \sup_{t\in[0,T]} \|\dot{v}_\varepsilon(t) - \dot{v}(t)\| \to 0 \quad \text{as } \varepsilon \to 0.$$

In particular, we get that $v \in C^0([T_0, T_1], V) \cap C^1([T_0, T_1], H)$. □

We now fix the notation that will be useful in order to give the main results concerning continuous dependence on the data.

Let $0 \le T_0 < T_1 \le T$, let $\gamma_k \in \mathcal{G}_{r,L}$ be a sequence of cracks paths, and let $s_k \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$, with $s_k(T_1) \le b_{\gamma_k}$, be a sequence of crack lengths, we define V^{γ_k}, $\|\cdot\|_{V^{\gamma_k}}$, $V^{\gamma_k}_{s_k(t)}$, $V^{\gamma_k,D}_{s_k(t)}$, $\mathcal{V}_{\gamma_k,s_k}(T_0, T_1)$, $\|\cdot\|_{\mathcal{V}_{\gamma_k\cdot s_k}}$, $\mathcal{V}^D_{\gamma_k,s_k}(T_0, T_1)$ as in (2.1.10)-(2.1.16) with Γ and $\Gamma_{s(t)}$ replaced by

$$\Gamma^{\gamma_k} := \gamma_k([a_0, b_{\gamma_k}]) \quad \text{and} \quad \Gamma^{\gamma_k}_{s_k(t)} := \gamma_k([a_0, s_k(t)]). \tag{2.1.50}$$

Let $u^0_k \in V^{\gamma_k}_{s_k(T_0)}$, with $u^0_k - u_D(T_0) \in V^{\gamma_k,D}_{s_k(T_0)}$, $u^1_k, \in H$,

$$f_k \in L^2(0, T; H) \quad \text{and} \quad F_k \in H^1(0, T; \tilde{H}). \tag{2.1.51}$$

We define u_k as the weak solution of k-th viscoelastic problem on the cracked domains $\Omega \setminus \Gamma^{\gamma_k}_{s_k(t)}$, $t \in [T_0, T_1]$, that is

$$u_k \in V_{\gamma_k,s_k}(T_0, T_1) \quad \text{and} \quad u_k - u_D \in \mathcal{V}^D_{\gamma_k,s_k}(T_0, T_1), \tag{2.1.52}$$

$$
\begin{aligned}
&-\int_{T_0}^{T_1}(\dot u_k(t),\dot\varphi(t))\,\mathrm{d}t+\int_{T_0}^{T_1}(\mathbb{A}Eu_k(t),E\varphi(t))\,\mathrm{d}t\\
&-\int_{T_0}^{T_1}\int_{T_0}^{t}\mathrm{e}^{\tau-t}(\mathbb{V}Eu_k(\tau),E\varphi(t))\,\mathrm{d}\tau\mathrm{d}t=\int_{T_0}^{T_1}(f_k(t),\varphi(t))\,\mathrm{d}t\\
&+\int_{T_0}^{T_1}(F_k(t),E\varphi(t))\,\mathrm{d}t\quad\text{for all }\varphi\in\mathcal{V}^D_{\gamma_k,s_k}(T_0,T_1)\text{ with }\varphi(T_0)=\varphi(T_1)=0, && (2.1.53)\\
&u_k(T_0)=u^0_k\quad\text{in }H\quad\text{and}\quad\dot u_k(T_1)=u^1_k\quad\text{in }(V^{\gamma_k,D}_{s_k(T_0)})^*. && (2.1.54)
\end{aligned}
$$

Moreover, we define v_k as the weak solution of k-th problem of elastodynamics on the cracked domains $\Omega\setminus\Gamma^{\gamma_k}_{s_k(t)}$, $t\in[T_0,T_1]$, that is

$$
\begin{aligned}
&v_k\in\mathcal{V}_{\gamma_k,s_k}(T_0,T_1)\quad\text{and}\quad v_k-u_D\in\mathcal{V}^D_{\gamma_k,s_k}(T_0,T_1), && (2.1.55)\\
&-\int_{T_0}^{T_1}(\dot v_k(t),\dot\varphi(t))\,\mathrm{d}t+\int_{T_0}^{T_1}(\mathbb{A}Ev_k(t),E\varphi(t))\,\mathrm{d}t=\int_{T_0}^{T_1}(f_k(t),\varphi(t))\,\mathrm{d}t\\
&+\int_{T_0}^{T_1}(F_k(t),E\varphi(t))\,\mathrm{d}t\quad\text{for all }\varphi\in\mathcal{V}_{\gamma_k,s_k}(T_0,T_1)\text{ with }\varphi(T_0)=\varphi(T_1)=0, && (2.1.56)\\
&v_k(T_0)=u^0_k\quad\text{in }H\quad\text{and}\quad\dot v_k(T_1)=u^1_k\quad\text{in }(V^{\gamma_k,D}_{s_k(T_0)})^*. && (2.1.57)
\end{aligned}
$$

We now state the result concernig continuous dependence on the data for the problem of elastodynamics. It will be used to prove the same result for the viscoelastic problem.

Theorem 2.1.18. *Let $\gamma\in\mathcal{G}_{r,L}$, $0\le T_0<T_1\le T$, $s\in\mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1)\le b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0\in V^\gamma_{s(T_0)}$, with $u^0-u_D(T_0)\in V^{\gamma,D}_{s(T_0)}$ and let $u^1\in H$. Let $\gamma_k\in\mathcal{G}_{r,L}$, let $s_k\in\mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s_k(T_1)\le b_{\gamma_k}$. Let $u^0_k\in V^{\gamma_k}_{s_k(T_0)}$, with $u^0_k-u_D(T_0)\in V^{\gamma_k,D}_{s_k(T_0)}$, $u^1_k,\in H$, and assume (2.1.51). Let v be the weak solution of problem (2.1.37)-(2.1.39) on the cracked domains $\Omega\setminus\Gamma^\gamma_{s(t)}$, $t\in[T_0,T_1]$. Let v_k the weak solution problem (2.1.55)-(2.1.57) on the cracked domains $\Omega\setminus\Gamma^{\gamma_k}_{s_k(t)}$, $t\in[T_0,T_1]$. Assume that*

$$\|f_k-f\|_{L^2(0,T;H)}\to 0,\qquad\|F_k-F\|_{H^1(0,T;\tilde H)}\to 0, \tag{2.1.58}$$

$$s_k\to s\quad\text{uniformly},\qquad\gamma_k\to\gamma\quad\text{uniformly}, \tag{2.1.59}$$

$$u^0_k\to u^0\quad\text{in }H,\qquad Du^0_k\to Du^0\quad\text{in }\underline{H},\qquad u^1_k\to u^1\quad\text{in }H. \tag{2.1.60}$$

Then

$$v_k(t)\to v(t)\quad\text{in }H, \tag{2.1.61}$$

$$Dv_k(t)\to Dv(t)\quad\text{in }\underline{H}, \tag{2.1.62}$$

$$\dot v_k(t)\to\dot v(t)\quad\text{in }H, \tag{2.1.63}$$

for every $t\in[T_0,T_1]$.

Proof. In the case $f_k = f$, $F_k = F = 0$ for any $k \in \mathbb{N}$, it is a consequence of [19, Theorem 3.5]. In the general case, the result follows from the same approximation argument used in [11, Lemma 5.7, Proposition 5.9]. $\square$

Now we are in a position to obtain the same results for the viscoelastic system.

Theorem 2.1.19. *Let $\gamma \in \mathcal{G}_{r,L}$, $0 \le T_0 < T_1 \le T$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1) \le b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0 \in V_{s(T_0)}$, such that $u^0 - u_D(T_0) \in V^D_{s(T_0)}$ and let $u^1 \in H$. Then there exists a unique solution u of problem (2.1.26)-(2.1.28). Moreover $u \in \mathcal{V}^\infty(T_0,T_1)$, $u \in C^0_w([T_0,T_1];V)$, and $\dot u \in C^0_w([T_0,T_1];H)$.*

Proof. We can not apply directly [11, Theorem 4.10] because in general (2.1.35) is not satisfied. However, assuming (2.1.36) instead of (2.1.35) we can repeat all arguments of the proof of that theorem, which is based on existence and uniqueness for elastodynamics with cracks (in our case given by Theorem 2.1.16 and Theorem 2.1.18) and on a fixed point argument. $\square$

Proposition 2.1.20. *Under the same assumption of Theorem 2.1.19, let u be the unique solution of problem (2.1.26)-(2.1.28). Then $u \in C^0([T_0,T_1],V) \cap C^1([T_0,T_1],H)$.*

Proof. It is enough to apply Proposition 2.1.17 with $F(t)$ replaced by

$$F(t) + \int_{T_0}^t \mathrm{e}^{\tau - t}\mathbb{V}Eu(\tau)\mathrm{d}\tau,$$

for all $t \in [T_0,T_1]$. $\square$

The following theorem provides the continuous dependence on the data for the solution of the viscoelastic problem.

Theorem 2.1.21. *Let $\gamma \in \mathcal{G}_{r,L}$, $0 \le T_0 < T_1 \le T$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s(T_1) \le b_\gamma$, and assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0 \in V^\gamma_{s(T_0)}$, such that $u^0 - u_D(T_0) \in V^{\gamma,D}_{s(T_0)}$ and let $u^1 \in H$. Let $\gamma_k \in \mathcal{G}_{r,L}$, let $s_k \in \mathcal{S}^{piec}_{\mu,M}(T_0,T_1)$, with $s_k(T_1) \le b_{\gamma_k}$. Let $u^0_k \in V^{\gamma_k}_{s_k(T_0)}$, such that $u^0_k - u_D(T_0) \in V^{\gamma_k,D}_{s_k(T_0)}$, $u^1_k, \in H$, and assume (2.1.51). Let u be the weak solution of problem (2.1.26)-(2.1.28) on the cracked domains $\Omega \setminus \Gamma^\gamma_{s(t)}$, $t \in [T_0,T_1]$. Let u_k the weak solution problem (2.1.52)-(2.1.54) on the cracked domains $\Omega \setminus \Gamma^{\gamma_k}_{s_k(t)}$, $t \in [T_0,T_1]$. Assume that*

$$\|f_k - f\|_{L^2(0,T;H)} \to 0, \qquad \|F_k - F\|_{H^1(0,T;\tilde H)} \to 0, \tag{2.1.64}$$

$$s_k \to s \quad \text{uniformly}, \qquad \gamma_k \to \gamma \quad \text{uniformly}, \tag{2.1.65}$$

$$u^0_k \to u^0 \quad \text{in } H, \qquad Du^0_k \to Du^0 \quad \text{in } \underline{H}, \qquad u^1_k \to u^1 \quad \text{in } H. \tag{2.1.66}$$

Then

$$u_k(t) \to u(t) \quad \text{in } H, \tag{2.1.67}$$

$$Du_k(t) \to Du(t) \quad \text{in } \underline{H}, \tag{2.1.68}$$

$$\dot{u}_k(t) \to \dot{u}(t) \quad \text{in } H, \tag{2.1.69}$$

for every $t \in [T_0, T_1]$*. Moreover there exists a constant* $C > 0$ *such that*

$$\|u_k(t)\| + \|Du_k(t)\| + \|\dot{u}_k(t)\| \leq C$$

for every $k \in \mathbb{N}$ *and* $t \in [T_0, T_1]$*.*

Proof. As in the proof of Theorem 2.1.19, we cannot apply directly [11, Theorem 6.1], because in general (2.1.35) is not satisfied. However, assuming (2.1.36) instead of (2.1.35) we can repeat all arguments of the proof of that theorem, which is based on the continuous dependence on the data for elastodynamics with cracks (in our case given by Theorem 2.1.18) and on a results concerning the convergence of fixed points of a sequence of functions (see [11, Lemma 4.2]). □

2.2 Energy balance

In this section we study the problem of the dynamic energy-dissipation balance on a given cracked domain $\Omega \setminus \Gamma^\gamma_{s(t)}$ for a solution of a viscoelastic problem.

2.2.1 Dynamic dissipation energy balance

Let $\gamma \in \mathcal{G}_{r,L}$, $0 \leq T_0 < T_1 \leq T$, $s \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$, with $s(T_1) \leq b_\gamma$. It is convenient to define the operator

$$\mathcal{L}_{T_0} \colon \mathcal{V}(T_0, T_1) \to H^1(T_0, T_1; H), \tag{2.2.1}$$

$$(\mathcal{L}_{T_0} u)(t) := \int_{T_0}^t \mathrm{e}^{\tau - t} \mathbb{V} E u(\tau) \mathrm{d}\tau, \tag{2.2.2}$$

for all $u \in \mathcal{V}(T_0, T_1)$, for all $t \in [T_0, T_1]$. Since

$$(\dot{\widehat{\mathcal{L}_{T_0} u}})(t) = \mathbb{V} E u(t) - \int_{T_0}^t \mathrm{e}^{\tau - t} \mathbb{V} E u(\tau) \, \mathrm{d}\tau,$$

it is easy to check that $\mathcal{L}_{T_0}$ is bounded. Indeed, using the Hölder inequality it is possible to prove that

$$\|\mathcal{L}_{T_0} u\|_{L^\infty(T_0, T_1; \tilde{H})} \leq (T_1 - T_0)^{1/2} \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}(T_0, T_1)}, \tag{2.2.3}$$

$$\|\dot{\widehat{\mathcal{L}_{T_0} u}}\|_{L^2(T_0, T_1; \tilde{H})} \leq (1 + T_1 - T_0) \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}(T_0, T_1)}. \tag{2.2.4}$$

Assume (2.1.7), (2.1.23)-(2.1.25) and let $v \in C^0([T_0, T_1], V) \cap C^1([T_0, T_1], H)$. For every $t \in [T_0, T_1]$ the sum of kinetic and elastic energy is given by

$$\mathcal{E}_v(t) = \frac{1}{2} \|\dot{v}(t)\|^2 + \frac{1}{2} (\mathbb{C} E v(t), E v(t)). \tag{2.2.5}$$

For an interval $[t_1, t_2] \subset [T_0, T_1]$ the dissipation due to viscosity between time t_1 and t_2 is given by

$$\begin{aligned}\mathcal{D}_v(t_1, t_1) =& \frac{1}{2}(\mathbb{V}Ev(t_2), Ev(t_2)) - \frac{1}{2}(\mathbb{V}Ev(t_1), Ev(t_1)) \\ & - ((\mathcal{L}_{T_0}v)(t_2), Ev(t_2)) + ((\mathcal{L}_{T_0}v)(t_1), Ev(t_1)) \\ & + \int_{t_1}^{t_2} (\mathbb{V}Ev(t), Ev(t))\mathrm{d}t - \int_{t_1}^{t_2} ((\mathcal{L}_{T_0}v)(t), Ev(t))\mathrm{d}t. \end{aligned} \tag{2.2.6}$$

Moreover, we assume that the energy dissipated in the process of crack production on the interval $[t_1, t_2]$ is proportional to $s(t_2) - s(t_1)$, which represent the length of the crack increment. For simplicity we take the proportionality constant equal to one. Finally, the work done between time t_1 and t_2 by the boundary and volume forces is

$$\begin{aligned}\mathcal{W}_v(t_1, t_2) =& \int_{t_1}^{t_2} \Big((f(t), \dot{v}(t) - \dot{u}_D(t)) + ((\mathbb{C} + \mathbb{V})Ev(t), E\dot{u}_D(t)) - ((\mathcal{L}_{T_0}v)(t), E\dot{u}_D(t))\Big)\mathrm{d}t \\ & - \int_{t_1}^{t_2} (\dot{F}(t), Ev(t) - Eu_D(t))\mathrm{d}t - \int_{t_1}^{t_2} (\dot{v}(t), \ddot{v}_D(t))\mathrm{d}t + (\dot{v}(t_2), \dot{u}_D(t_2)) \\ & - (\dot{v}(t_1), \dot{u}_D(t_1)) + (F(t_2), Ev(t_2) - Eu_D(t_2)) - (F(t_1), Ev(t_1) - Eu_D(t_1)). \end{aligned} \tag{2.2.7}$$

Remark 2.2.1. When $F = F_0$ as in (6) and all terms are regular enough, formulas (2.2.6) and (2.2.7) can be obtained from (1) in $(-\infty, T]$, using the explicit expression of the stress tensor (2) and integrating by parts. For more details when viscosity is not present see also to [18, Section 3] and [19, Section 4].

Remark 2.2.2. We stress that (2.2.6) and (2.2.7) make sense for every weak solution of problem (2.1.26)-(2.1.27), thanks to Proposition 2.1.20.

We now define the class of cracks whose solutions of the viscoelastic problem satisfy the dynamic energy-dissipation balance.

2.2.2 The class of admissible cracks

Definition 2.2.3. Let $0 \le T_0 < T_1 \le T$, $s_0 \ge 0$, and $\overline{\gamma} \in \mathcal{G}_{r,L}$, with $b_{\overline{\gamma}} = s_0$, and assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0 \in V_{s_0}^{\overline{\gamma}}$, such that $u^0 - u_D(T_0) \in V_{s_0}^{\overline{\gamma},D}$ and let $u^1 \in H$. The class

$$\mathcal{B}^{reg}(T_0, T_1) = \mathcal{B}^{reg}(T_0, T_1, s_0, \overline{\gamma}, \mathbb{C}, \mathbb{V}, f, F, u_D, u^0, u^1)$$

is composed of all pairs (γ, s), with

$$\gamma \in \mathcal{G}_{r,L}, \tag{2.2.8}$$

$$\gamma|_{[a_0,s_0]} = \overline{\gamma}|_{[a_0,s_0]}, \tag{2.2.9}$$

$$s \in \mathcal{S}^{reg}_{\mu,M}([T_0,T_1]). \tag{2.2.10}$$

$$s(T_0) = s^0, \quad s(T_1) \le b_\gamma, \tag{2.2.11}$$

such that the unique weak solution u of the viscoelastic problem (2.1.26)-(2.1.28) satisfies the energy-dissipation balance

$$\mathcal{E}_u(t_2) - \mathcal{E}_u(t_1) + \mathcal{D}_u(t_1,t_2) + s(t_2) - s(t_1) = \mathcal{W}_u(t_1,t_2) \tag{2.2.12}$$

for every interval $[t_1,t_2] \subset [T_0,T_1]$. Similarly, the class

$$\mathcal{B}^{piec}(T_0,T_1) = \mathcal{B}^{piec}(T_0,T_1,s_0,\overline{\gamma},\mathbb{C},\mathbb{V},f,F,u_D,u^0,u^1)$$

is defined in the same way replacing $s \in \mathcal{S}^{reg}_{\mu,M}([T_0,T_1])$ by $s \in \mathcal{S}^{piec}_{\mu,M}([T_0,T_1])$.

The class $\mathcal{B}^{reg}(T_0,T_1)$ is nonempty, as clarified by the following result, whose proof follows the lines of [23, Lemma 1] and [22, Proposition 2.7].

Proposition 2.2.4. *Under the assumption of Definition 2.2.3, the pair $(\overline{\gamma},s)$, with $s(t) = s_0$ for every $t \in [T_0,T_1]$, belongs to $\mathcal{B}^{reg}(T_0,T_1)$.*

Proof. We prove the result in the case of homogeneous boundary condition, i.e. $u_D = 0$. Indeed, the case of non-homogeneous data can be obtained considering the equation for $u - u_D$. It is convenient to extend our data on $[0,2T]$ by setting $f(t) = 0$ and $F(t) = F(T)$ for $t \in (T,2T]$. It is clear that $f \in L^2(0,2T,H)$, $F \in H^1(0,2T,\tilde{H})$, and that, by uniqueness, the solution u of the viscoelastic problem on $[T_0,2T]$ is an extension of the solution on $[T_0,T_1]$. Since the domain is constant with respect to time we deduce from (2.1.26)-(2.1.27) that $u \in H^2([T_0,2T];(V^D_{s_0})^*)$ and

$$\langle \ddot{u}(t),\varphi\rangle + ((\mathbb{C}+\mathbb{V})Eu(t),E\varphi) - (\mathcal{L}_{T_0}u(t),E\varphi) = (f(t),\varphi) + (F(t),E\varphi). \tag{2.2.13}$$

for all $\varphi \in V^D_{s_0}$ and for a.e. $t \in [T_0,2T]$.

Given a Banach space X and a function $r : [T_0,2T] \to X$, for every $h > 0$ we define $\sigma^h r, \delta^h r : [T_0,2T-h] \to X$ by

$$\sigma^h r(t) := r(t+h) + r(t), \tag{2.2.14}$$

$$\delta^h r(t) := r(t+h) - r(t). \tag{2.2.15}$$

For a.e. $t \in [T_0,2T-h]$ we have $\sigma^h u(t), \delta^h u(t) \in V^D_{s_0}$. We consider (2.2.13) at time t and a time $t+h$, in both cases with $\varphi = \delta^h u(t)$. We sum the two expressions and we integrate on $[t_1,t_2] \subseteq [T_0,T_1]$. We get

$$\int_{t_1}^{t_2} (K_h(t) + E_h(t) + D_h(t))\,\mathrm{d}t = \int_{t_1}^{t_2} L_h(t)\,\mathrm{d}t, \tag{2.2.16}$$

where the terms that appear in (2.2.16) are defined as

$$
\begin{aligned}
K_h(t) &:= \langle \sigma^h \ddot{u}(t), \delta^h u(t) \rangle, \\
E_h(t) &:= ((\mathbb{C}+\mathbb{V})\sigma^h Eu(t), \delta^h Eu(t)), \\
D_h(t) &:= -(\sigma^h [\mathcal{L}_{T_0} u(t)], \delta^h Eu(t)), \\
L_h(t) &:= (\sigma^h f(t), \delta^h u(t)) + (\sigma^h F(t), \delta^h Eu(t)).
\end{aligned}
$$

We have that

$$
\begin{aligned}
\int_{t_1}^{t_2} K_h(t)\,\mathrm{d}t &= -\int_{t_1}^{t_2} (\sigma^h \dot{u}(t), \delta^h \dot{u}(t))\,\mathrm{d}t + (\sigma^h \dot{u}(t_2), \delta^h u(t_2)) - (\sigma^h \dot{u}(t_1), \delta^h u(t_1)) \\
&= -\int_{t_1}^{t_2} (\|\dot{u}(t+h)\|^2 \mathrm{d}t - \|\dot{u}(t)\|^2)\mathrm{d}t + (\sigma^h \dot{u}(t_2), \delta^h u(t_2)) - (\sigma^h \dot{u}(t_1), \delta^h u(t_1)) \\
&= -\int_{t_1+h}^{t_2+h} \|\dot{u}(t)\|^2 \mathrm{d}t + \int_{t_1}^{t_2} \|\dot{u}(t)\|^2 \mathrm{d}t + (\sigma^h \dot{u}(t_2), \delta^h u(t_2)) - (\sigma^h \dot{u}(t_1), \delta^h u(t_1)) \\
&= -\int_{t_2}^{t_2+h} \|\dot{u}(t)\|^2 \mathrm{d}t + \int_{t_1}^{t_1+h} \|\dot{u}(t)\|^2 \mathrm{d}t + (\sigma^h \dot{u}(t_2), \delta^h u(t_2)) - (\sigma^h \dot{u}(t_1), \delta^h u(t_1))
\end{aligned}
$$

and dividing by h we get

$$
\int_{t_1}^{t_2} \frac{K_h(t)}{h}\,\mathrm{d}t = -⨍_{t_2}^{t_2+h} \|\dot{u}(t)\|^2 \mathrm{d}t + ⨍_{t_1}^{t_1+h} \|\dot{u}(t)\|^2 \mathrm{d}t + (\sigma^h \dot{u}(t_2), \frac{\delta^h u(t_2)}{h}) - (\sigma^h \dot{u}(t_1), \frac{\delta^h u(t_1)}{h}).
$$

Then

$$
\int_{t_1}^{t_2} \frac{K_h(t)}{h}\,\mathrm{d}t \to -\|\dot{u}(t_2)\|^2 + \|\dot{u}(t_1)\|^2 + 2\|\dot{u}(t_2)\|^2 - 2\|\dot{u}(t_1)\|^2 = \|\dot{u}(t_2)\|^2 - \|\dot{u}(t_1)\|^2, \tag{2.2.17}
$$

as $h \to 0^+$, where we have used the fact that $u \in C^1([T_0, 2T], H)$. Moreover

$$
\begin{aligned}
\int_{t_1}^{t_2} E_h(t)\,\mathrm{d}t &= \int_{t_1}^{t_2} ((\mathbb{C}+\mathbb{V})Eu(t+h), Eu(t+h))\mathrm{d}t - \int_{t_1}^{t_2} ((\mathbb{C}+\mathbb{V})Eu(t), Eu(t))\mathrm{d}t \\
&= \int_{t_1+h}^{t_2+h} ((\mathbb{C}+\mathbb{V})Eu(t), Eu(t))\mathrm{d}t - \int_{t_1}^{t_2} ((\mathbb{C}+\mathbb{V})Eu(t), Eu(t))\mathrm{d}t \\
&= \int_{t_2}^{t_2+h} ((\mathbb{C}+\mathbb{V})Eu(t), Eu(t))\mathrm{d}t - \int_{t_1}^{t_1+h} ((\mathbb{C}+\mathbb{V})Eu(t), Eu(t))\mathrm{d}t
\end{aligned}
\tag{2.2.18}
$$

which give us

$$
\int_{t_1}^{t_2} \frac{E_h(t)}{h}\,\mathrm{d}t \to ((\mathbb{C}+\mathbb{V})Eu(t_2), Eu(t_2)) - ((\mathbb{C}+\mathbb{V})Eu(t_1), Eu(t_1)) \tag{2.2.19}
$$

as $h \to 0^+$, where we have used the fact that $u \in C^0([T_0, 2T], V)$. Regarding the term D_h we have

$$\begin{aligned}
-\int_{t_1}^{t_2} D_h(t)\,\mathrm{d}t &= \int_{t_1}^{t_2} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t+h))\mathrm{d}t - \int_{t_1}^{t_2} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t \\
&= \int_{t_1+h}^{t_2+h} (\sigma^{-h}[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t - \int_{t_1}^{t_2} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t \\
&= \int_{t_1+h}^{t_2+h} (\mathcal{L}_{T_0}u(t-h) - \mathcal{L}_{T_0}u(t+h), Eu(t))\mathrm{d}t \\
&\quad - \int_{t_1}^{t_1+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t + \int_{t_2}^{t_2+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t \\
&= \int_{t_1}^{t_2} (\mathcal{L}_{T_0}u(t) - \mathcal{L}_{T_0}u(t+2h), Eu(t+h))\mathrm{d}t \\
&\quad - \int_{t_1}^{t_1+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t + \int_{t_2}^{t_2+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t, \quad (2.2.20)
\end{aligned}$$

which give us

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{D_h(t)}{h}\,\mathrm{d}t &= -\int_{t_1}^{t_2} \Big(\frac{\mathcal{L}_{T_0}u(t) - \mathcal{L}_{T_0}u(t+2h)}{h}, Eu(t+h)\Big)\mathrm{d}t \\
&\quad + \fint_{t_1}^{t_1+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t - \fint_{t_2}^{t_2+h} (\sigma^h[\mathcal{L}_{T_0}u(t)], Eu(t))\mathrm{d}t \\
&\to 2\int_{t_1}^{t_2} ((\dot{\widehat{\mathcal{L}_{T_0}u}})(t), Eu(t))\mathrm{d}t \\
&\quad + 2(\mathcal{L}_{T_0}u(t_1), Eu(t_1)) - 2(\mathcal{L}_{T_0}u(t_2), Eu(t_2)) \\
&= 2\int_{t_1}^{t_2} (\mathbb{V}Eu(t) - \mathcal{L}_{T_0}u(t), Eu(t))\mathrm{d}t \\
&\quad + 2(\mathcal{L}_{T_0}u(t_1), Eu(t_1)) - 2(\mathcal{L}_{T_0}u(t_2), Eu(t_2)), \quad \text{as } h \to 0^+, \qquad (2.2.21)
\end{aligned}$$

where we have used again that $u \in C^0([T_0, 2T], V)$.

With similar arguments, we have that

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{L_h(t)}{h}\mathrm{d}t &\to 2\int_{t_1}^{t_2} (f(t), \dot{u}(t))\mathrm{d}t - 2\int_{t_1}^{t_2} (\dot{F}(t), Eu(t))\mathrm{d}t \\
&\quad + 2(F(t_2), Eu(t_1)) - 2(F(t_1), Eu(t_1)), \quad \text{as } h \to 0^+. \qquad (2.2.22)
\end{aligned}$$

Dividing by h Equation (2.2.16) and using Equations (2.2.17), (2.2.19), (2.2.21), and (2.2.22), we get the following identity

$$\|\dot{u}(t_2)\|^2 + ((\mathbb{C}+\mathbb{V})Eu(t_2), Eu(t_2)) + 2\int_{t_1}^{t_2} (\mathbb{V}Eu(t) - \mathcal{L}_{T_0}u(t), Eu(t))\mathrm{d}t$$

$$
\begin{aligned}
&- 2(\mathcal{L}_{T_0}u(t_2), Eu(t_2)) = \|\dot u(t_1)\|^2 + ((\mathbb{C}+\mathbb{V})Eu(t_1), Eu(t_1)) - 2(\mathcal{L}_{T_0}u(t_1), Eu(t_1))\\
&+ 2\int_{t_1}^{t_2}(f(t), \dot u(t))\mathrm{d}t - 2\int_{t_1}^{t_2}(\dot F(t), Eu(t))\mathrm{d}t + 2(F(t_2), Eu(t_1)) - 2(F(t_1), Eu(t_1)),
\end{aligned}
\tag{2.2.23}
$$

that is the energy-dissipation balance (2.2.12) when $u_D = 0$ and $s(t) = s_0$ for all $t \in [T_0, T_1]$. $\square$

The following remark deals with the concatenation of solutions on adjacent time intervals.

Remark 2.2.5. Under the assumption of Definition 2.2.3, let $0 \le T_0 < T_1 < T_2 \le T$,

$$(\gamma_1, s_1) \in \mathcal{B}^{piec}(T_0, T_1, s_0, \overline{\gamma}, \mathbb{C}, \mathbb{V}, f, F, u_D, u^0, u^1),$$

$$(\gamma_2, s_2) \in \mathcal{B}^{piec}(T_1, T_2, s_1(T_1), \gamma_1, \mathbb{C}, \mathbb{V}, f, F, u_D, u(T_1), \dot u(T_1)).$$

Let $s\colon [T_0, T_2] \to \mathbb{R}$ be defined as

$$s(t) := \begin{cases} s_1(t) & \text{if } t \in [T_0, T_1], \\ s_2(t) & \text{if } t \in [T_1, T_2]. \end{cases} \tag{2.2.24}$$

Then $(\gamma_2, s) \in \mathcal{B}^{piec}(T_0, T_2, s_0, \overline{\gamma}, \mathbb{C}, \mathbb{V}, f, F, u_D, u^0, u^1)$.

Using the continuous dependence Theorem 2.1.21 we are in a position to prove a compactness result for $\mathcal{B}^{reg}$, which will be useful for the proof of the main result of the chapter (see Theorem 2.3.3).

Theorem 2.2.6. *Under the assumption of Definition 2.2.3, let $(\gamma_k, s_k) \in \mathcal{B}^{reg}(T_0, T_1)$. Then there exists a not relabelled subsequence and there exists $(\gamma, s) \in \mathcal{B}^{reg}(T_0, T_1)$ such that $\gamma_k \to \gamma$ uniformly (in the sense of Definition 2.1.3) and $s_k \to s$ in $C^3([T_0, T_1])$.*

Proof. By Lemma 2.1.5 there exists a subsequence (not relabelled) γ_k and $\gamma \in \mathcal{G}_{r,L}$ such that $\gamma_k \to \gamma$ uniformly (in the sense of Definition 2.1.3). By Ascoli-Arzelà Theorem there exists $s \in C^3([T_0, T_1])$ and a further subsequence s_k converging to s in $C^3([T_0, T_1])$. Moreover, if we pass to the limit ad $k \to +\infty$ in the conditions in Definition 2.1.6 for s_k, we get that $s \in \mathcal{S}^{reg}_{\mu,M}([T_0, T_1])$. We defined u as the solution of the viscoelastic problem (2.1.26)-(2.1.28) on the time-dependent cracked domain $t \mapsto \Omega \setminus \Gamma^{\gamma}_{s(t)}$ with $t \in [T_0, T_1]$ and we define u_k as the solution of the viscoelastic problem on the time-dependent cracked domain $t \mapsto \Omega \setminus \Gamma^{\gamma_k}_{s_k(t)}$ with $t \in [T_0, T_1]$. Since $(\gamma_k, s_k) \in \mathcal{B}^{reg}(T_0, T_1)$ we have

$$
\begin{aligned}
&\frac{1}{2}\|\dot u_k(t_2)\|^2 + \frac{1}{2}((\mathbb{C}+\mathbb{V})Eu_k(t_2), Eu_k(t_2)) - (\mathcal{L}_{T_0}u_k(t_2), Eu_k(t_2))\\
&-\frac{1}{2}\|\dot u_k(t_1)\|^2 - \frac{1}{2}((\mathbb{C}+\mathbb{V})Eu_k(t_1), Eu_k(t_1)) + (\mathcal{L}_{T_0}u_k(t_1), Eu_k(t_1))
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t_1}^{t_2}(\mathbb{V}Eu_k(t),Eu_k(t))\mathrm{d}t-\int_{t_1}^{t_2}(\mathcal{L}_{T_0}u_k(t),Eu_k(t))\mathrm{d}t+s_k(t_2)-s_k(t_1)\\
&=\int_{t_1}^{t_2}\Big((f(t),\dot u_k(t)-\dot u_D(t))+((\mathbb{C}+\mathbb{V})Eu_k(t),E\dot u_D(t))-(\mathcal{L}_{T_0}u_k(t),E\dot u_D(t))\Big)\mathrm{d}t\\
&-\int_{t_1}^{t_2}(\dot F(t),Eu_k(t)-Eu_D(t))\mathrm{d}t+(F(t_2),Eu_k(t_2)-Eu_D(t_2))-(F(t_1),Eu_k(t_1)-Eu_D(t_1))\\
&-\int_{t_1}^{t_2}(\dot u_k(t),\ddot u_D(t))\mathrm{d}t+(\dot u_k(t_2),\dot u_D(t_2))-(\dot u_k(t_1),\dot u_D(t_1)),
\end{aligned}\tag{2.2.25}
$$

for every interval $[t_1,t_2]\subset[T_0,T_1]$. Using Theorem 2.1.21 and the bounds (2.2.3)-(2.2.4), we can pass to the limit as $k\to+\infty$ in (2.2.25) and we get the energy-dissipation balance (2.2.12) for u. This proves that $(\gamma,s)\in\mathcal{B}^{reg}(T_0,T_1)$ and concludes the proof. □

2.3 Existence for the coupled problem

In this section we prove an existence result for the crack evolution (described by the functions γ and s). In order to do this we define a maximal dissipation condition (see also [18] and [19]), which forces the crack tip to choose a path which allows for a maximal speed.

Definition 2.3.1. Assume (2.1.7), (2.1.23)-(2.1.25) and (2.1.36). Let $u^0\in V_0$, such that $u^0-u_D(0)\in V_0^D$, and let $u^1\in H$. Given $\eta>0$ we say that $(\gamma,s)\in\mathcal{B}^{piec}(0,T)$ satisfies the $\eta-$maximal dissipation condition on $[0,T]$ if there exists no $(\hat\gamma,\hat s)\in\mathcal{B}^{piec}(0,\tau_1)$, for some $\tau_1\in(0,T]$, such that

(M1) $sing(\hat s)\subset sing(s)$,

(M2) $\hat s(t)=s(t)$ and $\hat\gamma(\hat s(t))=\gamma(s(t))$ for every $t\in[0,\tau_0]$, for some $\tau_0\in[0,\tau_1)$,

(M3) $\hat s(t)>s(t)$ for every $t\in(\tau_0,\tau_1]$ and $\hat s(\tau_1)>s(\tau_1)+\eta$.

Remark 2.3.2. We refer to the discussion in [19, Section 1] for some comments on the presence of the parameter $\eta>0$.

We are now in position to prove the main result of the chapter. The proof follows the lines of [18] and [19], devoted to the case of elastodynamics without viscosity terms.

Theorem 2.3.3. *Under the assumption of Definition 2.3.1, for every $\eta>0$ there exists a pair $(\gamma,s)\in\mathcal{B}^{piec}(0,T)$ satisfying the η-maximal dissipation condition on $[0,T]$.*

Proof. Let us fix $\eta>0$ and a finite subdivision $0=T_0<T_1<...<T_k=T$ of the time interval $[0,T]$ such that

$$T_i-T_{i-1}<\frac{\eta}{\mu},\tag{2.3.1}$$

for every $i \in \{0,1,2,...,k\}$. We will define the solution using a recursive procedure on each subinterval $[T_{i-1}, T_i]$, for every $i \in \{0,1,2,...,k\}$. In order to define this procedure, we set

$$\mathcal{X}_1 := \Big\{(\gamma, s) \in \mathcal{B}^{piec}(0, T_1, 0, \gamma_0, \mathbb{C}, \mathbb{V}, f, F, u_D, u^0, u^1) \,|\, s \in \mathcal{S}^{reg}_{\mu,M}(0, T_1),\ s(0) = 0\Big\}, \tag{2.3.2}$$

where γ_0 is the function that appears in Definition 2.1.1. By Proposition 2.2.4 we have that $(\gamma_0, 0) \in \mathcal{X}_1$ and in particular we have $\mathcal{X}_1 \neq \emptyset$. Moreover, we choose $(\gamma_1, s_1) \in \mathcal{X}_1$ such that

$$\int_{T_0}^{T_1} s_1(t)\,\mathrm{d}t = \max_{(\gamma,s)\in\mathcal{X}_1} \int_{T_0}^{T_1} s(t)\,\mathrm{d}t,$$

where the existence of (γ_1, s_1) is guaranteed by Lemma 2.3.4 below. If $k = 1$, we define $(\gamma, s) := (\gamma_1, s_1)$ and we have to prove that this couple satisfies the η-maximal dissipation condition. Otherwise, we fix $i \in \{2, ..., k\}$ and we set

$$\begin{aligned}\mathcal{X}_i := \Big\{(\gamma, s) \in \mathcal{B}^{piec}(0, T_i, 0, \gamma_0, \mathbb{C}, \mathbb{V}, f, F, u_D, u^0, u^1) \,|\, s|_{[T_{i-1},T_i]} \in \mathcal{S}^{reg}_{\mu,M}(T_{i-1}, T_i),\\ s(t) = s_{i-1}(t), \gamma(s(t)) = \gamma_{i-1}(s_{i-1}(t))\,\forall\, t \in [0, T_{i-1}]\Big\}.\end{aligned} \tag{2.3.3}$$

We note that $\mathcal{X}_i \neq \emptyset$. Indeed, if we define $\tilde{s}_{i-1}$ as

$$\tilde{s}_{i-1}(t) := \begin{cases} s_{i-1}(t) & \text{for } t \in [0, T_{i-1}],\\ s_{i-1}(T_{i-1}) & \text{for } t \in [T_{i-1}, T_i],\end{cases}$$

we can apply Proposition 2.2.4 and Remark 2.2.5 to obtain $(\gamma_{i-1}, \tilde{s}_{i-1}) \in \mathcal{X}_i$. Assume that the pair $(\gamma_{i-1}, s_{i-1}) \in \mathcal{X}_{i-1}$ has already been defined, then we choose $(\gamma_i, s_i) \in \mathcal{X}_i$ such that

$$\int_{T_{i-1}}^{T_i} s_i(t)\,\mathrm{d}t = \max_{(\gamma,s)\in\mathcal{X}_i} \int_{T_{i-1}}^{T_i} s(t)\,\mathrm{d}t, \tag{2.3.4}$$

where the existence of (γ_i, s_i) is guaranteed by Lemma 2.3.4 below.

We now define $(\gamma, s) := (\gamma_k, s_k)$, where (γ_k, s_k) is the the pair defined in the final step of the procedure defined above. It remains to prove that (γ, s) satisfies the η-maximal dissipation condition on the interval $[0, T]$. Assume, by contradiction that there exist $0 \le \tau_0 < \tau_1 \le T$ and $(\hat{\gamma}, \hat{s}) \in \mathcal{B}^{piec}(0, \tau_1)$ such that:

(i) $sing(\hat{s}) \subset sing(s) \subset \{T_1, ..., T_{k-1}\}$

(ii) $s(t) = \hat{s}(t)$ and $\gamma(s(t)) = \hat{\gamma}(\hat{s}(t))$ for every $t \in [0, \tau_0]$,

(iii) $s(t) < \hat{s}(t)$ for every $t \in (\tau_0, \tau_1]$ and $\hat{s}(\tau_1) > s(\tau_1) + \eta$.

Since $\tau_0 < T$, there exists an index $j \in \{1, ..., k\}$ such that $\tau_0 \in [T_{j-1}, T_j)$. We claim that $\tau_1 > T_j$. Indeed, the using the monotonicity of s and the points (ii) and (iii), we have that

$$\hat{s}(\tau_1) > s(\tau_1) + \eta \geq s(\tau_0) + \eta = \hat{s}(\tau_0) + \eta, \tag{2.3.5}$$

and in particular $\hat{s}(\tau_1) - \hat{s}(\tau_0) > \eta$. On the other hand, since $\hat{s} \in \mathcal{S}^{piec}_{\mu,M}(0, \tau_1)$ we have

$$\hat{s}(\tau_1) - \hat{s}(\tau_0) \leq \mu(\tau_1 - \tau_0) \tag{2.3.6}$$

which together with the previous inequality give us $\tau_1 - \tau_0 > \eta/\mu$. Since the subdivision of the interval was choosen such that $T_{i-1} - T_i < \eta/\mu$ for every $i \in \{1, ..., k\}$, we get that $\tau_1 > T_j$.

Using (i) we have that $\hat{s}|_{[T_{j-1},T_j]} \in \mathcal{S}^{reg}_{\mu,M}(T_{j-1}, T_j)$ and taking (ii) into account we get that $(\hat{\gamma}, \hat{s}) \in \mathcal{X}_j$. By construction $s = s_j$ on $[T_{j-1}, T_j]$, where s_j is the function defined in (2.3.4) for $i = j$. As a consequence of (iii) we get $\hat{s}(t) > s(t) = s_j(t)$ for every $t \in (\tau_0, T_j]$, which contradicts (2.3.4).

□

We close this section with the following Lemma used to prove Theorem 2.3.3. The proof follows the lines of [19, Lemma 5.3] with obvious modifications.

Lemma 2.3.4. *For every $i = 1, ..., k$ there exists $(\gamma_i, s_i) \in \mathcal{X}_i$ such that*

$$\int_{T_{i-1}}^{T_i} s_i(t)\,\mathrm{d}t = \max_{(\gamma,s)\in\mathcal{X}_i} \int_{T_{i-1}}^{T_i} s(t)\,\mathrm{d}t, \tag{2.3.7}$$

where $\mathcal{X}_i$ is the space defined in (2.3.2) and (2.3.3).

Proof. Let $i \in \{1, ..., k\}$ be fixed and let us define

$$S := \sup_{(\gamma,s)\in\mathcal{X}_i} \int_{T_{i-1}}^{T_i} s(t)\,\mathrm{d}t.$$

For every $n \in \mathbb{N}$ let $(\gamma^n, s^n) \in \mathcal{X}_i$ be such that

$$\int_{T_{i-1}}^{T_i} s^n(t)\,\mathrm{d}t \geq S - \frac{1}{n}. \tag{2.3.8}$$

Let u_{i-1} be the unique solution of the viscoelastic system (2.1.26)-(2.1.28) on the time-dependent domain $t \mapsto \Omega \setminus \Gamma^{\gamma_{i-1}}_{s_{i-1}(t)}$ for $t \in [0, T_{i-1}]$. By Theorem 2.2.6 there exists a (not relabelled) subsequence of $(\gamma^n, s^n|_{[T_{i-1},T_i]})$ and an element

$$(\gamma_i, \tilde{s}) \in \mathcal{B}^{reg}(T_{i-1}, T_i, s_{i-1}(T_{i-1}), \gamma_{i-1}, \mathbb{C}, \mathbb{V}, f, F, u_D, u_{i-1}(T_{i-1}), \dot{u}_{i-1}(T_{i-1})),$$

such that $\gamma^n \to \gamma_i$ uniformly (in the sense of Definition 2.1.3) and $s^n|_{[T_{i-1},T_i]} \to s$ in $C^3([T_{i-1},T_i])$. We now define

$$s_i(t) := \begin{cases} s_{i-1}(t) & \text{for } t \in [0,T_{i-1}], \\ \tilde{s}(t) & \text{for } t \in [T_{i-1},T_i]. \end{cases}$$

By definition of $\mathcal{X}_i$, we have that $\gamma^n(s_{i-1}(t)) = \gamma^n(s^n(t)) = \gamma_{i-1}(s_{i-1}(t))$ for all $t \in [0,T_{i-1}]$. Passing to the limit as $n \to +\infty$ we obtain that $\gamma_i(s_i(t)) = \gamma_{i-1}(s_{i-1}(t))$ for all $t \in [0,T_{i-1}]$, which together to Remark 2.2.5, give us $(\gamma_i, s_i) \in \mathcal{X}_i$. Finally, passing to the limit as $n \to +\infty$ in Equation (2.3.8), we get Equation (2.3.7) and this concludes the proof. $\square$

Chapter 3

A viscoelastic problem with prescribed debonding front

Contents

In this chapter we consider the one-dimensional debonding model for a viscoelastic material described in the introduction, with assigned debonding front.

Given $T > 0$ and a positive and prescribed increasing function $\ell\colon [0,T] \to \mathbb{R}$, the problem considered in this chapter is a particular case of (3), namely

$$
\begin{aligned}
&u_{tt}(t,x) - u_{xx}(t,x) + \int_0^t \frac{\mathrm{e}^{\tau-t}}{2} u_{xx}(t,x)\,\mathrm{d}t = f(t,x) - F_x(t,x), && (t,x) \in \Omega^\ell, && (3.0.1)\\
&u(t,0) = u_D(t), && t \in (0,T), && (3.0.2)\\
&u(t,\ell(t)) = 0, && t \in (0,T), && (3.0.3)\\
&u(0,x) = u^0(x), && x \in (0,\ell_0), && (3.0.4)\\
&u_t(0,x) = u^1(x), && x \in (0,\ell_0). && (3.0.5)
\end{aligned}
$$

where $\Omega^\ell = \{(t,x) \in \mathbb{R}^2 \,|\, t \in (0,T), x \in (0,\ell(t))\}$, u is the longitudinal displacement (extended as $u = 0$ on $((0,T) \times (0,\ell(T))) \setminus \Omega^\ell)$, f and F are the forcing terms, u_D is the Dirichlet condition, and u^0 and u^1 are the initial conditions. For more details on this model and is physical interpretation we refer to the Introduction of this thesis. The aim of this chapter is to prove existence, uniqueness and regularity results for problem (3.0.1)-(3.0.5). In order to study this problem, we first work on the auxiliary problem without damping term given by

$$u_{tt}(t,x) - u_{xx}(t,x) = f(t,x) - F_x(t,x), \qquad (t,x) \in \Omega^\ell, \tag{3.0.6}$$

$$u(t,0) = u_D(t), \qquad t \in (0,T), \tag{3.0.7}$$

$$u(t,\ell(t)) = 0, \qquad t \in (0,T), \tag{3.0.8}$$

$$u(0,x) = u^0(x), \qquad x \in (0,\ell_0), \tag{3.0.9}$$

$$u_t(0,x) = u^1(x), \qquad x \in (0,\ell_0). \tag{3.0.10}$$

and, similarly to the previous chapters, we use fixed point arguments to get the results for (3.0.1)-(3.0.5).

The chapter is organized as follows. In Section 3.1 we give the basic definition in order to define the weak formulation of (3.0.1)-(3.0.5) and (3.0.6)-(3.0.10). Section 3.2 is divided in Subsection 3.2.1, where we describe some geometric considerations on the debonding, and Subsection 3.2.2, where we prove the representation formula, and in Subsection 3.2.3 we prove existence and uniqueness for (3.0.1)-(3.0.5). Finally, in Section 3.3 we deal with some energetic results. More precisely, in Subsection 3.3.1 we prove that the total energy (as a function of time) is absolutely continuous, while in Subsection 3.3.2 we give the main ideas in order to study the debonding problem with a non-prescribed debonding front (this case will be studied in a future research project).

The original results of this chapter will be contained [13] (work in preparation).

3.1 Preliminary results

In this chapter we set the notations and we give the main definition following the same presentation of [20] and [42]. We fix $T > 0$, $\ell_0 > 0$ and a function

$$\ell\colon [0,T] \to \mathbb{R} \tag{3.1.1}$$

such that

$$\ell \in C^{0,1}([0,T]), \tag{3.1.2}$$

$$\ell(0) = \ell_0, \text{ and } 0 \leq \dot{\ell}(t) \leq 1 \text{ for a.e. } t \in [0,T]. \tag{3.1.3}$$

We introduce the following two auxiliary functions

$$\varphi,\, \psi\colon [0,T] \to \mathbb{R} \tag{3.1.4}$$

defined for every $t \in [0,T]$ as

$$\varphi(t) := t - \ell(t) \quad \text{and} \quad \psi(t) := t + \ell(t). \tag{3.1.5}$$

We set $L := \ell(T)$. The function ψ is invertible and we can define

$$\omega\colon [\ell_0, \ell_0 + L] \mapsto \mathbb{R}, \quad \omega(y) := \varphi \circ \psi^{-1}(y), \tag{3.1.6}$$

which is a Lipschitz function whose derivative satisfies for a.e. $y \in [\ell_0, \ell_0 + L]$

$$0 \le \dot\omega(y) = \frac{1 - \dot\ell(\psi^{-1}(y))}{1 + \dot\ell(\psi^{-1}(y))} \le 1. \tag{3.1.7}$$

For every $t^* \in (0,T]$ we define

$$\Omega_{t^*} := \{(t,x) \,|\, t \in (0,t^*),\, x \in (0,\ell(t))\} \tag{3.1.8}$$

and

$$Q_{t^*} := \{(t,x) \,|\, t \in (0,t^*),\, x \in (0,L)\}. \tag{3.1.9}$$

In the case $t^* = T$ we omit the dependence on t^* and we simply write Ω and Q instead of Ω_T and Q_T. We stress that, unlike the previous chapters, here Ω is a space-time domain and not a spatial domain.

Regarding the boundary and initial conditions we assume

$$u^0 \in H^1(0,\ell_0), \tag{3.1.10}$$

$$u^1 \in L^2(0,\ell_0), \tag{3.1.11}$$

$$u_D \in H^1(0,T), \tag{3.1.12}$$

with the compatibility conditions

$$u^0(0) = u_D(0) \quad \text{and} \quad u^0(\ell_0) = 0. \tag{3.1.13}$$

For the forcing term we set

$$f \in L^2(0,T;L^2(0,L)),\ F \in H^1(0,T;L^2(0,L)) \text{ and } f = F = 0 \text{ a.e. on } Q \setminus \Omega. \tag{3.1.14}$$

Remark 3.1.1. We recall the following identifications: $L^2(0,T;L^2(0,L)) \simeq L^2(Q)$ and $H^1(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L)) \simeq H^1(Q)$.

Remark 3.1.2. With a slight abuse of notation, we use $\dot w$ to denote both the Bochner derivative with respect to time (if $w = w(t,x)$) and the derivative with respect to a single variable (if $w = w(s)$), depending on the context.

Remark 3.1.3. Let X be a Banach space. As done before, we identify a function in $H^1(0,T;X)$ with its continuous representative on $[0,T]$.

Definition 3.1.4 (Solution of the viscoelastic debonding problem)**.** We say that a function $u \in H^1(\Omega)$ is a solution of the viscoelastic problem (3.0.1)-(3.0.5) if equation (3.0.1) holds in the sense of $\mathcal{D}'(\Omega)$, the boundary conditions (3.0.2)-(3.0.3) are intended in the sense of traces and the initial conditions (3.0.4)-(3.0.5) are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.

We recall that $C_c^\infty(\Omega)$ is the set of smooth functions with compact support on Ω.

Remark 3.1.5. Let u be as in Definition 3.1.4. Then (3.0.1) in the sense of $\mathcal{D}'(\Omega)$ means

$$\begin{aligned}
&-\int_\Omega u_t(t,x)\phi_t(t,x)\,\mathrm{d}t\mathrm{d}x + \int_\Omega u_x(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x - \int_\Omega\int_0^t \frac{\mathrm{e}^{\tau-t}}{2}u_x(\tau,x)\,\mathrm{d}\tau\,\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x\\
&= \int_\Omega f(t,x)\phi(t,x)\,\mathrm{d}t\mathrm{d}x + \int_\Omega F(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x, \text{ for all } \phi \in C_c^\infty(\Omega). \qquad (3.1.15)
\end{aligned}$$

Remark 3.1.6. Taking into account condition (3.0.3), a solution u in the sense of Definition 3.1.4 is extended on Q (still denoting it by u), by setting $u = 0$ on $Q \setminus \Omega$ and the extensions belong to $H^1(Q)$. In particular, the term $(t,x) \mapsto \int_0^t \frac{\mathrm{e}^{\tau-t}}{2}u_x(\tau,x)\,\mathrm{d}\tau$ is well defined for a.e. $(t,x) \in \Omega$.

Remark 3.1.7. The initial conditions in Definition 3.1.4 are well posed. Indeed, a solution $u \in H^1(\Omega)$ must satisfies $u \in H^1((0,T)\times(0,\ell_0)) \simeq H^1(0,T;L^2(0,\ell_0)) \cap L^2(0,T;H^1(0,\ell_0))$ which implies $u \in C^0(0,T;L^2(0,\ell_0))$. Moreover, using (3.0.1), one can prove that $u_t \in H^1(0,T;H^{-1}(0,\ell_0)) \hookrightarrow C^0(0,T;H^{-1}(0,\ell_0))$ (see also [20])

3.2 Existence, uniqueness, and representation formula

In order to prove existence and uniqueness for the viscoelastic problem (3.0.1)-(3.0.5) we have to consider the auxiliary problem (3.0.6)-(3.0.10).

3.2.1 The auxiliary problem

Definition 3.2.1 (Solution of the elastic debonding problem)**.** We say that a function $v \in H^1(\Omega)$ is a solution of elastic problem (3.0.6)-(3.0.10) if equation (3.0.6) holds in the sense of $\mathcal{D}'(\Omega)$, the boundary conditions (3.0.7)-(3.0.8) are intended in the sense of traces and the initial conditions (3.0.9)-(3.0.10) are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.

Remark 3.2.2. Let v be as in Definition 3.2.1. Then (3.0.6) in the sense of $\mathcal{D}'(\Omega)$ means

$$\begin{aligned}
&-\int_\Omega u_t(t,x)\phi_t(t,x)\,\mathrm{d}t\mathrm{d}x + \int_\Omega u_x(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x =\\
&\int_\Omega f(t,x)\phi(t,x)\,\mathrm{d}t\mathrm{d}x + \int_\Omega F(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x, \text{ for all } \phi \in C_c^\infty(\Omega). \qquad (3.2.1)
\end{aligned}$$

Remark 3.2.3. The initial conditions in Defintion 3.2.1 are well posed for the same argument used in Remark 3.1.7.

Remark 3.2.4. Taking into account condition (3.0.8), a solution v in the sense of Definition 3.2.1 can be extended on Q (still denoting it by v), by setting $v = 0$ on $Q \setminus \Omega$ and the extensions belong to $H^1(Q)$.

In order to prove existence and uniqueness for problem (3.0.6)-(3.0.10) we find an explicit formula for the solution v on a particular subset of Ω. Since in (3.0.6) the term F_x does not belongs to $L^2(0,T;L^2(0,L))$, we can not apply the formula of [20, 42] to our case, but we have to compute a new one. Moreover, we stress that this result is a natural extension of the classical D'Alembert's formula.

We define some subsets that will be used to define the explicit formula. We fix the following sets

$$\Omega_1' := \{(t,x) \in \Omega \,|\, t \leq x,\, t+x \leq \ell_0\}, \tag{3.2.2}$$

$$\Omega_2' := \{(t,x) \in \Omega \,|\, t > x,\, t+x < \ell_0\}, \tag{3.2.3}$$

$$\Omega_3' := \{(t,x) \in \Omega \,|\, t < x,\, t+x > \ell_0\}, \tag{3.2.4}$$

$$\Omega' := \Omega_1' \cup \Omega_2' \cup \Omega_3', \tag{3.2.5}$$

and the following functions

$$\gamma_1(\tau;t,x) = \begin{cases} x-t+\tau & (t,x) \in \Omega_1', \\ |x-t+\tau| & (t,x) \in \Omega_2', \\ x-t+\tau & (t,x) \in \Omega_3', \end{cases} \tag{3.2.6}$$

$$\hat\gamma_1(\tau;t,x) = \begin{cases} x-t+\tau & (t,x) \in \Omega_1', \\ \max\{x-t+\tau,\, 0\} & (t,x) \in \Omega_2', \\ x-t+\tau & (t,x) \in \Omega_3', \end{cases} \tag{3.2.7}$$

$$\gamma_2(\tau;t,x) = \begin{cases} x+t-\tau & (t,x) \in \Omega_1', \\ x+t-\tau & (t,x) \in \Omega_2', \\ \tau-\omega(x+t) & (t,x) \in \Omega_3',\ \tau \leq \psi^{-1}(x+t), \\ x+t-\tau & (t,x) \in \Omega_3'\ \tau > \psi^{-1}(x+t), \end{cases} \tag{3.2.8}$$

$$\hat\gamma_2(\tau;t,x) = \begin{cases} x+t-\tau & (t,x) \in \Omega_1', \\ x+t-\tau & (t,x) \in \Omega_2', \\ \ell(\psi^{-1}(x+t)) & (t,x) \in \Omega_3',\ \tau \leq \psi^{-1}(x+t), \\ x+t-\tau & (t,x) \in \Omega_3'\ \tau > \psi^{-1}(x+t), \end{cases} \tag{3.2.9}$$

Moreover, for every $(t,x) \in \Omega'$ we define

$$R(t,x) := \{(\tau,y) \in \Omega' \,|\, 0 < \tau < t,\ \gamma_1(\tau;t,x) < y < \gamma_2(\tau;t,x)\}, \tag{3.2.10}$$

$$T^+(t,x) := \{(\tau,y) \in \Omega' \,|\, 0 < \tau < t,\ x < y < \hat\gamma_2(\tau;t,x)\}, \tag{3.2.11}$$

$$T^-(t,x) := \{(\tau,y) \in \Omega' \,|\, 0 < \tau < t,\ \hat\gamma_1(\tau;t,x) < y < x\}, \tag{3.2.12}$$

$$T_0^+(t,x) := \{(\tau,y) \in \Omega' \,|\, 0 < \tau < \psi^{-1}(x+t),\ \gamma_2(\tau;t,x) < y < \hat\gamma_2(\tau;t,x)\} \tag{3.2.13}$$

$$T_0^-(t,x) := \{(\tau,y) \in \Omega' \,|\, 0 < \tau < t-x,\ \hat\gamma_1(\tau;t,x) < y < \gamma_1(\tau;t,x)\} \tag{3.2.14}$$

Remark 3.2.5. We note that $T_0^+(t,x) = \emptyset$ if $(t,x) \in \Omega_1 \cup \Omega_2$ and $T_0^-(t,x) = \emptyset$ if $(t,x) \in \Omega_1 \cup \Omega_3$.

We denote by A the solution of (3.0.6)-(3.0.10) in the case $f = F = 0$ and we recall that in [20] is proved that A can be written as

$$A(t,x) = a_1(x+t) + a_2(t-x), \quad \text{for every } (t,x) \in \Omega, \tag{3.2.15}$$

for two suitable functions a_1, $a_2 \in H^1(\mathbb{R})$. In general it is not easy to give an explicitic representation of a_1 and a_2 due to the superpositions of waves generated by "bouncing" against the sets $\{(t,x) \,|\, t \in [0,T],\ x = \ell(t)\}$ and $\{(t,x) \,|\, t \in [0,T],\ x = 0\}$ but in [20] is proved that if we consider the restriction of A to Ω' we have

$$A(t,x) = \begin{cases} \dfrac{u^0(x+t) + u^0(x-t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} u^1(s)\,\mathrm{d}s, & (t,x) \in \Omega_1', \\ \dfrac{u^0(x+t) - u^0(t-x)}{2} + \dfrac{1}{2}\displaystyle\int_{t-x}^{x+t} u^1(s)\,\mathrm{d}s + u_D(t-x), & (t,x) \in \Omega_2', \\ \dfrac{u^0(x-t) - u^0(-\omega(x+t))}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{-\omega(x+t)} u^1(s)\,\mathrm{d}s, & (t,x) \in \Omega_3'. \end{cases} \tag{3.2.16}$$

So we have that $a_1\colon (0, 2t_0^\ell) \to \mathbb{R}$ is defined as

$$a_1(z) = \begin{cases} \dfrac{1}{2}u^0(z) + \dfrac{1}{2}\displaystyle\int_0^z u^1(s)\,\mathrm{d}s, & z \in (0,\ell_0], \\ -\dfrac{1}{2}u^0(-\omega(z)) + \dfrac{1}{2}\displaystyle\int_0^{-\omega(z)} u^1(s)\,\mathrm{d}s, & z \in (\ell_0, 2t_0^\ell), \end{cases} \tag{3.2.17}$$

while $a_2\colon (-\ell_0, \ell_0) \to \mathbb{R}$ is is given by

$$a_2(z) = \begin{cases} \dfrac{1}{2}u^0(-z) - \dfrac{1}{2}\displaystyle\int_0^{-z} u^1(s)\,\mathrm{d}s, & z \in (-\ell_0, 0], \\ u_D(z) - \dfrac{1}{2}u^0(z) - \dfrac{1}{2}\displaystyle\int_0^z u^1(s)\,\mathrm{d}s, & z \in (0,\ell_0), \end{cases} \tag{3.2.18}$$

where we set

$$t_0^\ell := \min\{T, \inf\{t \in [\ell_0, T] \,|\, t = \ell(t)\}\}, \tag{3.2.19}$$

with the convention $\inf\{\emptyset\} = +\infty$. Moreover, in [42] is proved that the solution of (3.0.6)-(3.0.10) with $F = 0$ is given by the sum between A and the integral of f on a suitable moving domain, according to the Duhamel's principle. We will extend this results to the case $F \neq 0$ and we will prove (in Theorem 3.2.10) that the solution v of (3.0.6)-(3.0.10) is

$$\begin{aligned} v(t,x) = &\frac{1}{2}\int_{R(t,x)} f(\tau,y)\,\mathrm{d}\tau\mathrm{d}y - \frac{1}{2}\int_{T^+(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + \frac{1}{2}\int_{T^-(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y \\ &- \frac{1}{2}\int_{T_0^+(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + \frac{1}{2}\int_{T_0^-(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + A(t,x) + B(t,x),\ (t,x) \in \Omega' \end{aligned} \tag{3.2.20}$$

where the term B is defined as

$$B(t,x) = \begin{cases} -\dfrac{1}{2}\displaystyle\int_x^{x+t} F(0,y)\,\mathrm{d}y + \dfrac{1}{2}\int_{x-t}^{x} F(0,y)\,\mathrm{d}y, & (t,x) \in \Omega_1', \\ \dfrac{1}{2}\displaystyle\int_0^{x} F(0,y)\,\mathrm{d}y + \dfrac{1}{2}\int_0^{t-x} F(0,y)\,\mathrm{d}y - \dfrac{1}{2}\int_x^{x+t} F(0,y)\,\mathrm{d}y, & (t,x) \in \Omega_2', \\ \dfrac{1}{2}\displaystyle\int_{x-t}^{x} F(0,y)\,\mathrm{d}y - \dfrac{1}{2}\int_x^{\ell(\psi^{-1}(x+t))} F(0,y)\,\mathrm{d}y & \\ -\dfrac{1}{2}\displaystyle\int_{-\omega(x+t)}^{\ell(\psi^{-1}(x+t))} F(0,y)\,\mathrm{d}y, & (t,x) \in \Omega_2', \end{cases} \tag{3.2.21}$$

We need some technical lemma regarding the regularity of some terms that appear in the representation formula for v on Ω'.

Lemma 3.2.6. *Let us assume (3.1.1)-(3.1.7) and (3.1.10)-(3.1.14). Let $A\colon \Omega' \to \mathbb{R}$ be the function defined in (3.2.16) and let $B\colon \Omega' \to \mathbb{R}$ be the function defined in (3.2.21). Then A and B are continuous on $\overline{\Omega}'$ and they belongs to $H^1(\Omega')$. Moreover, setting $A \equiv B \equiv 0$ outside $\overline{\Omega}$ we have*

$$A,\, B \in C^0([0, \frac{\ell_0}{2}]; H^1(0,+\infty)) \cap C^1([0, \frac{\ell_0}{2}]; L^2(0,+\infty)) \tag{3.2.22}$$

Proof. The regularity of A is proved in [42, Lemma 1.10]. The proof for B easily follows from similar considerations. □

Since we have to deal with integral functions, we recall the following result.

Theorem 3.2.7 (Leibniz differentiation rule)**.** *Let $\phi \in C^{0,1}([0,T])$ be nondecreasing and let $a \leq \phi(0)$. Consider the set $\Omega_T^\phi := \{(t,x) \in \mathbb{R}^2 \,|\, 0 \leq t \leq T,\, a \leq y \leq \phi(t)\}$ and let $h\colon \Omega_T^\phi \to \mathbb{R}$ be a measurable function such that:*

i) for every $t \in [0,T]$ it holds $h(t,\cdot) \in L^1(a,\phi(t))$,
ii) for a.e. $y \in [a,\phi(T)]$ it holds $h(\cdot,y) \in AC(I_y)$, where $I_y = \{t \in [0,T] \,|\, y \leq \phi(t)\}$,
c) the partial derivative $\dfrac{\partial h}{\partial t}(t,y) := \lim_{\varepsilon\to 0} \dfrac{h(t+\varepsilon,y)-h(t,y)}{\varepsilon}$ (which for a.e. $y \in [a,\phi(T)]$ is well defined for a.e. $t \in I_y$) is summable in Ω_T^ϕ.

Then the function $H(t) := \displaystyle\int_a^{\phi(t)} h(t,y)dy$ belongs to $AC([0,T])$ and for a.e. $t \in [0,T]$

$$\dot{H}(t) := H(t,\phi(t))\dot{\phi}(t) + \int_a^{\phi(t)} \frac{\partial h}{\partial t}(t,y)dy \tag{3.2.23}$$

For a proof of previous theorem see, e.g., [42, Theorem A.8]. From this results we obtain the following regularity results for integral functions in the plane.

Lemma 3.2.8. *Let us assume (3.1.1)-(3.1.7). Let $g \in L^2(\Omega')$ and for every $(t,x) \in \Omega'$ let*

$$H_1(t,x) := \int_{R(t,x)} g(\tau,y)\,\mathrm{d}\tau\mathrm{d}y, \quad H_2(t,x) := \int_{T^+(t,x)} g(\tau,y)\,\mathrm{d}\tau\mathrm{d}y, \tag{3.2.24}$$

$$H_3(t,x) := \int_{T^-(t,x)} g(\tau,y)\,\mathrm{d}\tau\mathrm{d}y, \quad H_4(t,x) := \int_{T_0^+(t,x)} g(\tau,y)\,\mathrm{d}\tau\mathrm{d}y, \tag{3.2.25}$$

$$H_5(t,x) := \int_{T_0^-(t,x)} g(\tau,y)\,\mathrm{d}\tau\mathrm{d}y. \tag{3.2.26}$$

Then H_1, H_2, H_3, H_4, and H_5 are continuous on $\overline{\Omega}'$ and they belongs to $H^1(\Omega')$. Moreover, setting $H_i \equiv 0$ for $i = 1,...,5$ outside $\overline{\Omega}$ we have

$$H_i \in C^0([0,\frac{\ell_0}{2}];H^1(0,+\infty)) \cap C^1([0,\frac{\ell_0}{2}];L^2(0,+\infty)) \quad \text{for } i = 1,...,5. \tag{3.2.27}$$

Proof. The proof of claim for H_1 is given by [42, Lemma 1.1]. With small modifications, the same proof works also for $i = 2,...,5$. □

Moreover, we need the following result regarding slicing for non cylindrical domains.

Lemma 3.2.9. *Let $A \subseteq \mathbb{R}^2$ be open and let $w \in L^2(A)$ such that*

$$\int_A w(x,y)\phi_x(x,y)\,\mathrm{d}x\mathrm{d}y = 0 \quad \text{for any } \phi \in C_c^\infty(A). \tag{3.2.28}$$

Then, for a.e. $y \in \pi_2(A) := \{y \in \mathbb{R} \,|\, \exists\, x \in \mathbb{R} \text{ s.t. } (x,y) \in A\}$ *we have*

$$\int_{A_y} w(x,y)\dot{\alpha}(x)\,\mathrm{d}x = 0 \quad \textit{for any } \alpha \in C_c^\infty(A_y), \tag{3.2.29}$$

where $A_y := \{x \in \mathbb{R} \,|\, (x,y) \in A\}$.

Proof. Let L_n^1, L_n^2 be a sequence of open intervals such that the $\bigcup_{n\in\mathbb{N}}(L_n^1 \times L_n^2) = A$. For a fixed $n \in \mathbb{N}$ we denote by $\{\alpha_n^k\}_{k\in\mathbb{N}}$ a sequence dense in $C_c^\infty(L_n^1)$ with respect to the $C^1(L_n^1)$ convergence. For every $\beta_n \in C_c^\infty(L_n^2)$ and for every $k \in \mathbb{N}$ we define $\phi_n^k(x,y) := \alpha_n^k(x)\beta_n(y)$, which belongs to $C_c^\infty(A)$ and we use it as test function in (3.2.28) obtaining

$$\int_{L_n^2}\int_{L_n^1} w(x,y)\dot{\alpha}_n^k(x)\,\mathrm{d}x\beta_n(y)\,\mathrm{d}y = 0. \tag{3.2.30}$$

By the arbitrariness of β_n we get that there exist $I_{n,k}^2 \subset L_n^2$ such that $\mathcal{L}^1(I_{n,k}^2) = 0$ and for all $y \in L_n^2 \setminus I_{n,k}^2$

$$\int_{L_n^1} w(x,y)\dot{\alpha}_n^k(x)\,\mathrm{d}x = 0. \tag{3.2.31}$$

We now define $I_n^2 := \bigcup_k I_{n,k}^2 \subset L_n^2$ which satisfies $\mathcal{L}^1(I_n^2) = 0$ and for every $y \in L_n^2 \setminus I_n^2$ we have

$$\int_{L_n^1} w(x,y)\dot{\alpha}_n^k(x)\,\mathrm{d}x = 0, \quad \text{for every } k \in \mathbb{N}. \tag{3.2.32}$$

Since $\{\alpha_n^k\}_{k\in\mathbb{N}}$ is a dense in $C_c^\infty(L_n^1)$ with respect to the $C^1(L_n^1)$ convergence, we can pass to limit as $k \to +\infty$ an we get that for every $y \in L_n^2 \setminus I_n^2$ it holds

$$\int_{L_n^1} w(x,y)\dot{\alpha}_n(x)\,\mathrm{d}x = 0 \quad \text{for every } \alpha_n \in C_c^\infty(L_n^1). \tag{3.2.33}$$

In order to extend the results to a generic test function, it is enough to use the localization property of distribution. More precisely, let us define $I^2 := \bigcup_n I_n^2$ which satisfies $\mathcal{L}^1(I^2) = 0$, let $y \in \pi_2(A) \setminus I^2$ and consider $\alpha \in C_c^\infty(A_y)$. We know that $supp(\alpha) \subset K$ for some compact $K \subset A_y$ and we note that the set $\{A_y \cap L_n^1\}_{n\in\mathbb{N}}$ is an open cover of K. By compactness of K we can find a finite subcover and considering a smooth partition of unity subordinate to this subcover, we can conclude. □

3.2.2 The representation formula and more regularity

It is convenient to consider a localized problem.

Theorem 3.2.10 (Local existence and uniqueness)**.** *Let us assume* (3.1.1)-(3.1.7) *and* (3.1.10)-(3.1.14)*. A function $v \in H^1(\Omega')$ is a solution of the elastic problem* (3.0.6)-(3.0.10) *on the domain Ω' (in the sense of Definition 3.2.1) if and only if*

$$\begin{aligned} v(t,x) = \frac{1}{2}\int_{R(t,x)} f(\tau,y)\,\mathrm{d}\tau\mathrm{d}y - \frac{1}{2}\int_{T^+(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + \frac{1}{2}\int_{T^-(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y \\ - \frac{1}{2}\int_{T_0^+(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + \frac{1}{2}\int_{T_0^-(t,x)} F_\tau(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + A(t,x) + B(t,x), \end{aligned} \tag{3.2.34}$$

for a.e. $(t,x) \in \Omega'$.

Proof. It is easy to check that (3.2.34) satisfies (3.0.6)-(3.0.10) on the domain Ω' using Lemma 3.2.6 and Lemma 3.2.8 to compute the spatial and time derivatives of the right hand side of (3.2.34).

It remains to prove that if $v \in H^1(\Omega')$ is a solution in Ω' of (3.0.6)-(3.0.10) then (3.2.34) holds. First of all, it is convenient to make some changes in the right hand side of (3.0.6). Indeed, taking into account (3.1.14) we have that $F_t \in L^2(0,T;L^2(0,L))$ and we obtain that (3.0.6) is equivalent to

$$v_{tt} - v_{xx} = g + F_t - F_x \quad \text{in } \mathcal{D}'(\Omega') \tag{3.2.35}$$

where $g := f - F_t \in L^2(0,T;L^2(0,L))$. We define the change of variable

$$\begin{cases} \xi = t - x, \\ \eta = t + x. \end{cases} \tag{3.2.36}$$

Then the function $\tilde{v}(\xi,\eta) := v(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$ belongs to $H^1(\tilde{\Omega}')$, where $\tilde{\Omega}'$ is the image of Ω' through (3.2.36), and satisfies

$$4\tilde{v}_{\xi\eta} = \tilde{g} + 2\tilde{F}_\xi \quad \text{in } \mathcal{D}'(\tilde{\Omega}'), \tag{3.2.37}$$

where $\tilde{g}(\xi,\eta) := g(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$ and $\tilde{F}(\xi,\eta) := F(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2})$. Equation (3.2.37) can be written as

$$-4\int_{\tilde{\Omega}'} \tilde{v}_\eta(\xi,\eta)\phi_\xi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta = \int_{\tilde{\Omega}'} \tilde{g}(\xi,\eta)\phi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta - 2\int_{\tilde{\Omega}'} \tilde{F}(\xi,\eta)\phi_\xi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta \tag{3.2.38}$$

for every $\phi \in C_c^\infty(\tilde{\Omega}')$. In particular, identity (3.2.38) is valid for every $\phi \in C_c^\infty(\tilde{\Omega}'_1)$, where $\tilde{\Omega}'_1$ is the image of Ω'_1 through (3.2.36), that is

$$-4\int_{\tilde{\Omega}'_1} \tilde{v}_\eta(\xi,\eta)\phi_\xi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta = \int_{\tilde{\Omega}'_1} \tilde{g}(\xi,\eta)\phi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta - 2\int_{\tilde{\Omega}'_1} \tilde{F}(\xi,\eta)\phi_\xi(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta \tag{3.2.39}$$

for every $\phi \in C_c^\infty(\tilde\Omega_1')$. Defined $\tilde G(\xi,\eta) := \int_{-\eta}^{\xi} \tilde g(z,\eta)\,\mathrm{d}z$ we can apply Lemma 3.2.9 to the function $-4\tilde v_\eta + \tilde G + 2\tilde F$ and we get that for a.e. $\eta \in (0,\ell_0)$

$$\int_{\tilde\Omega_{1,\eta}'} [-4\tilde v_\eta(\xi,\eta) + \tilde G(\xi,\eta) + 2\tilde F(\xi,\eta)]\dot\alpha(\xi)\,\mathrm{d}\xi = 0 \quad \text{for any } \alpha \in C_c^\infty(\tilde\Omega_{1,\eta}'), \tag{3.2.40}$$

where $\tilde\Omega_{1,\eta}' := \{\xi \in \mathbb{R} \,|\, (\xi,\eta) \in \tilde\Omega_1'\} = (-\eta, 0)$. Using the fundamental lemma of calculus of variations and Fubini's Theorem we have that

$$-4\tilde v_\eta(\xi,\eta) + \tilde G(\xi,\eta) + 2\tilde F(\xi,\eta) = C(\eta) \quad \text{for a.e. } (\xi,\eta) \in \tilde\Omega_1', \tag{3.2.41}$$

where $C \in L^2(\mathbb{R})$ is an arbitrary function. This give us that

$$\tilde v(\xi,\eta) = \int_{-\xi}^{\eta} \frac{1}{4}[\tilde G(\xi,s) + 2\tilde F(\xi,s)]\,\mathrm{d}s + \Phi(\eta) + \Psi(\xi) \quad \text{for a.e. } (\xi,\eta) \in \tilde\Omega_1', \tag{3.2.42}$$

where $\Phi \in H^1(\mathbb{R})$ and $\Psi \in L^2(\mathbb{R})$ are arbitrary functions. Using the inverse of (3.2.36) we have

$$\begin{aligned} 2v(t,x) = &\int_{R(t,x)} f(\tau,y)\,\mathrm{d}\tau\mathrm{d}y - \int_{T^+(t,x)} \partial_t F(\tau,y)\,\mathrm{d}\tau\mathrm{d}y + \int_{T^-(t,x)} \partial_t F(\tau,y)\,\mathrm{d}\tau\mathrm{d}y \\ &- \int_x^{x+t} F(0,y)\,\mathrm{d}y + \int_{x-t}^{x} F(0,y)\,\mathrm{d}y + 2\Psi(x-t) + 2\Phi(x+t) \quad \text{for a.e. } (t,x) \in \Omega_1'. \end{aligned} \tag{3.2.43}$$

Taking into account that $v \in H^1(\Omega')$, $\Phi \in H^1(\mathbb{R})$ and using Lemma 3.2.8, we get from (3.2.43) that $\Psi \in H^1(\mathbb{R})$. Moreover, using again Lemma 3.2.8, we have that the right hand side of equation (3.2.43) is continuous on $\overline{\Omega_1'}$ and we can use it as continuous representative for v. In particular, equation (3.2.43) is true for every $(t,x) \in \overline{\Omega_1'}$. In order to determine Φ and Ψ we have to use the initial conditions (3.0.9) and (3.0.10) (in the sense of Definition 3.2.1). Using again regularity property given by Lemma 3.2.8 and imposing (3.0.9)-(3.0.10) we get

$$\begin{cases} \Psi(x) + \Phi(x) = u^0(x), & \text{for every } x \in (0,\ell_0), \\ -\dot\Psi(x) + \dot\Phi(x) = u^1(x), & \text{for a.e. } x \in (0,\ell_0). \end{cases} \tag{3.2.44}$$

which give us that

$$\Psi(x-t) + \Phi(x+t) = \frac{u^0(x+t) + u^0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u^1(s)\,\mathrm{d}s, \quad \text{for any } (t,x) \in \Omega_1'. \tag{3.2.45}$$

Thanks to (3.2.45) we have that (3.2.34) is proved on Ω_1'.

In order to obtain (3.2.34) also on Ω_2' and Ω_3', it is enough to follow the same computations done on Ω_1' with small modifications and using the boundary conditions (3.0.7)-(3.0.8) in order to determine the arbitrary functions. □

Remark 3.2.11 (Regularity)**.** As a consequence of Theorem 3.2.10 and taking into account Lemmas 3.2.6 and 3.2.8, we get that the solution v of problem (3.0.6)-(3.0.10) admits a representative, given by the right-hand side of (3.2.34) and still denoted by v, which is continuous on $\overline{\Omega'}$ and belongs to $C^0([0, \frac{\ell_0}{2}]; H^1(0, +\infty)) \cap C^1([0, \frac{\ell_0}{2}]; L^2(0, +\infty))$ (where we extend $v \equiv 0$ outside Ω'). In particular $v(0)$ and $v_t(0)$ are well defined as elements of $H^1(0, +\infty)$ and $L^2(0, +\infty)$. Moreover, the continuous representative satisfies the boundary and initial conditions in a sense stronger that Definition 3.2.1, namely

$$\begin{cases} v(t,0) = u_D(t) & \text{for every } t \in (0,T), \\ v(t,\ell(t)) = 0 & \text{for every } t \in (0,T), \\ v(0,x) = u^0(x) & \text{for every } x \in (0,\ell_0), \\ v_t(0,x) = u^1(x) & \text{for a.e. } x \in (0,\ell_0), \end{cases}$$

and for $t \to 0^+$ we have

$$\begin{cases} v(t) \to u^0 & \text{in } H^1(0,\ell_0), \\ v_t(t) \to u^1 & \text{in } L^2(0,\ell_0). \end{cases}$$

We are in a position to prove existence and uniqueness of the solution of (3.0.6)-(3.0.10) on the whole domain Ω.

Theorem 3.2.12 (Existence and uniqueness for the elastic problem)**.** *Let us assume* (3.1.1)-(3.1.7) *and* (3.1.10)-(3.1.14)*. Then, there exists a unique solution v of the elastic problem* (3.0.6)-(3.0.10) *on the domain Ω (in the sense of Definition 3.2.1). Moreover v has a continuous representative on Ω, still denoted by v, and (setting $v \equiv 0$ outside Ω), it holds*

$$v \in C^0([0,T]; H^1(0,+\infty)) \cap C^1([0,T]; L^2(0,+\infty)). \tag{3.2.46}$$

Proof. Thanks to Theorem 3.2.10 we know that there exists a unique function satisfying (3.0.6)-(3.0.10) on the domain $\Omega' \subset \Omega$ and in particular on $\Omega_{\ell_0/2} \subset \Omega'$, where $\Omega_{\ell_0/2}$ is defined as in (3.1.8) with $t^* = \ell_0/2$, namely

$$\Omega_{\ell_0/2} = \{(t,x) \,|\, t \in (0,\ell_0/2),\, x \in (0,\ell(t))\}. \tag{3.2.47}$$

We denote the solution on $\Omega_{\ell_0/2}$ as $v_{\ell_0/2}$. Thanks to Remark 3.2.11 we have that $v_{\ell_0/2}$ admits a representative (still denoted by $v_{\ell_0/2}$) continuous on $\overline{\Omega}_{\ell_0/2}$ and such that $v_{\ell_0/2} \in C^0([0, \frac{\ell_0}{2}]; H^1(0,+\infty)) \cap C^1([0, \frac{\ell_0}{2}]; L^2(0,+\infty))$. In particular $v_{\ell_0/2}(\ell_0/2)$ and $(v_{\ell_0/2})_t(\ell_0/2)$ are well defined elements of $H^1(0,+\infty)$ and $L^2(0,+\infty)$, respectively. We consider the elastic problem (3.0.6)-(3.0.10) on

$$\Omega_{\ell_0} \setminus \overline{\Omega}_{\ell_0/2} = \{(t,x) \,|\, t \in (\ell_0/2, \ell_0),\, x \in (0,\ell(t))\} \tag{3.2.48}$$

with initial condition u^0 and u^1 replaced by $v_{\ell_0/2}(\ell_0/2)$ and $(v_{\ell_0/2})_t(\ell_0/2)$. We can apply again Theorem 3.2.10 and we get that there exists a unique solution denoted by v_{ℓ_0}. We can repeat this procedure on

$$\Omega_{k\ell_0/2} \setminus \overline{\Omega}_{(k-1)\ell_0/2} = \{(t,x) \,|\, t \in ((k-1)\ell_0/2, k\ell_0/2),\, x \in (0,\ell(t))\} \quad \text{for } k = 3, ..., \lceil \frac{2T}{\ell_0} \rceil. \tag{3.2.49}$$

and we denote by $v_{k\ell_0/2}$ the solution the elastic problem (3.0.6)-(3.0.10) on $\Omega_{k\ell_0/2} \setminus \overline{\Omega}_{(k-1)\ell_0/2}$ for $k = 3, ..., \left\lceil \frac{2T}{\ell_0} \right\rceil$ with initial condition u^0 and u^1 replaced by $v_{(k-1)\ell_0/2}((k-1)\ell_0/2)$ and $(v_{(k-1)\ell_0/2})_t((k-1)\ell_0/2)$. For $(t,x) \in \Omega$ we define

$$v(t,x) := v_{k\ell_0/2}(t,x) \tag{3.2.50}$$

for a suitable $k = 1, ..., \left\lceil \frac{2T}{\ell_0} \right\rceil$ such that $(t,x) \in \Omega_{k\ell_0/2} \setminus \overline{\Omega}_{(k-1)\ell_0/2}$ (with the convention that $\Omega_{(k-1)\ell_0/2} = \emptyset$ if $k=1$). It is easy to see that $v \in H^1(\Omega)$ and that it is a solution of (3.0.6)-(3.0.10) on Ω. Moreover, tanking into account Remark 3.2.11, we get that

$$v \in C^0([0,T]; H^1(0,+\infty)) \cap C^1([0,T]; L^2(0,+\infty)). \tag{3.2.51}$$

In order to prove uniqueness it not restrictive to assume $u^0 = u^1 = u_D = f = F = 0$, thanks to the linearity of the problem. From (3.2.34) we get that $v_{\ell_0/2} = 0$ on $\Omega_{\ell_0/2}$. Iterating this argument we get that $v_{k\ell_0/2} = 0$ for every k and in particular $v = 0$ on Ω. This proves the uniqueness of the solution and concludes the proof. □

Remark 3.2.13. From the proof of Theorem 3.2.12 we have that the solution v of the elastic problem (3.0.6)-(3.0.10) on the domain Ω (in the sense of Definition 3.2.1) can be represented on each

$$\Omega_{k\ell_0/2} \setminus \overline{\Omega}_{(k-1)\ell_0/2} = \{(t,x) \,|\, t \in ((k-1)\ell_0/2, k\ell_0),\, x \in (0,\ell(t))\} \quad \text{for } k = 3, ..., \lceil \frac{2T}{\ell_0} \rceil.$$

with a formula similar to (3.2.34).

3.2.3 Proof of the existence and uniqueness for the viscoelastic problem

We are now in a position to prove existence and uniqueness for the viscoelastic problem (3.0.1)-(3.0.5).

It is convenient to define the operator

$$\mathcal{L}_{deb}\colon H^1(Q) \to H^1(0,T; L^2(0,L)), \tag{3.2.52}$$

$$(\mathcal{L}_{deb}u)(t) := \frac{1}{2}\int_0^t \mathrm{e}^{\tau - t} u_x(\tau)\mathrm{d}\tau, \tag{3.2.53}$$

for all $u \in H^1(Q)$, for all $t \in [0,T]$.

Remark 3.2.14. From Remark 3.1.6 we have that if $u \in H^1(\Omega)$ is a solution of viscoelastic problem (3.0.1)-(3.0.5), then it is extended as $u \in H^1(Q)$. In particular, $\mathcal{L}_{deb}u$ is well defined.

Taking into account that for a.e. $t \in [0,T]$

$$(\dot{\widehat{\mathcal{L}_{deb}u}})(t) = \frac{1}{2}u_x(t) - (\mathcal{L}_{deb}u)(t), \tag{3.2.54}$$

we can prove that $\mathcal{L}_{deb}$ is bounded. More precisely, we have

$$\|\mathcal{L}_{deb}u\|_{L^\infty(0,T;L^2(0,L))} \leq \frac{T^{1/2}}{2}\|u\|_{H^1(Q)}, \tag{3.2.55}$$

$$\|\dot{\widehat{\mathcal{L}_{deb}u}}\|_{L^2(0,T;L^2(0,L))} \leq \frac{(1+T)}{2}\|u\|_{H^1(Q)}. \tag{3.2.56}$$

We now prove a particular energy estimate for elastic problem (3.0.6)-(3.0.10) that will be used in a fixed point argument in the proof of existence and uniqueness of viscoelastic problem (3.0.1)-(3.0.5).

Proposition 3.2.15. *Let us assume (3.1.1)-(3.1.7) and (3.1.10)-(3.1.14). Moreover suppose that $u_D = u^0 = u^1 = f = 0$. Then the corresponding solution $v \in H^1(\Omega)$ of the elastic problem (3.0.6)-(3.0.10) satisfies the following inequality*

$$\frac{1}{2}\|\dot{v}(t)\|^2_{L^2(0,\ell(t))} + \frac{1}{2}\|v_x(t)\|^2_{L^2(0,\ell(t))} \leq (F(t), v_x(t))_{L^2(0,\ell(t))} - \int_0^t (\dot{F}(s), v_x(s))_{L^2(0,\ell(s))}\,\mathrm{d}s,$$

for every $t \in [0,T]$.

The proof is based on a time discretization argument, in the spirit of [38, Theorem 1.8].

Proof. Let $n \in \mathbb{N}$ and define the sequence $t_k^n := kT/n$ for $k = 0, 1, ..., n$. For every $k = 1, ..., n$ we denote by v_k^n the unique solution of the elastic problem on the cylinder $Q_k^n := (t_{k-1}^n, t_k^n) \times (0, \ell(t_{k-1}^n))$ defined as

$$
\begin{aligned}
&(v_k^n)_{tt}(t,x) - (v_k^n)_{xx}(t,x) = -F_x(t,x), && (t,x) \in Q_k^n, && (3.2.57)\\
&v_k^n(t,0) = 0, && t \in (t_{k-1}^n, t_k^n), && (3.2.58)\\
&v_k^n(t,\ell(t_{k-1}^n)) = 0, && t \in (t_{k-1}^n, t_k^n), && (3.2.59)\\
&v_k^n(t_{k-1}^n, x) = u_{k-1}^n(t_{k-1}^n, x), && x \in (0, \ell(t_{k-1}^n)), && (3.2.60)\\
&(v_k^n)_t(t_{k-1}^n, x) = (u_{k-1}^n)_t(t_{k-1}^n, x), && x \in (0, \ell(t_{k-1}^n)). && (3.2.61)
\end{aligned}
$$

with the conventions that $u_0^n(0,x) = 0$, $(u_0^n)_t(0,x) = 0$ for $x \in (0,\ell_0)$. Moreover, we extend v_k^n to 0 in $((t_{k-1}^n, t_k^n) \times (0,\ell(T))) \setminus Q_k$ and the extension (still denoted by v_k^n) belongs to $H^1((t_{k-1}^n, t_k^n) \times (0,\ell(T)))$. By known results we have that

$$u \in C^0([t_{k-1}^n, t_k^n]; H^1(0,\ell(T))) \cap C^1([t_{k-1}^n, t_k^n]; L^2(0,\ell(T))) \tag{3.2.62}$$

and the initial conditions (3.2.60)-(3.2.61) are satisfied in $C^0([t^n_{k-1},t^n_k];H^1(0,\ell(T)))$ and $C^0([t^n_{k-1},t^n_k];L^2(0,\ell(T)))$, respectively. Moreover, for every $t\in[t^n_{k-1},t^n_k]$ it holds:

$$\begin{aligned}&\frac{1}{2}\|\dot v^n_k(t)\|^2_{L^2(0,\ell(t^n_{k-1}))}+\frac{1}{2}\|(v^n_k)_x(t)\|^2_{L^2(0,\ell(t^n_{k-1}))}=\frac{1}{2}\|\dot v^n_k(t^n_{k-1})\|^2_{L^2(0,\ell(t^n_{k-1}))}\\&+\frac{1}{2}\|(v^n_k)_x(t^n_{k-1})\|^2_{L^2(0,\ell(t^n_{k-1}))}+(F(t),(v^n_k)_x(t))_{L^2(0,\ell(t^n_{k-1}))}-(F(t^n_{k-1}),(v^n_k)_x(t^n_{k-1}))_{L^2(0,\ell(t^n_{k-1}))}\\&-\int_{t^n_{k-1}}^t(\dot F(s),(v^n_k)_x(s))_{L^2(0,\ell(t^n_{k-1}))}\,\mathrm{d}s.\end{aligned}\tag{3.2.63}$$

By summing (3.2.63) for $j\in\{2,...,k\}$ we get that for every $t\in[t^n_{k-1},t^n_k]$

$$\begin{aligned}\frac{1}{2}\|\dot v^n_k(t)\|^2_{L^2(0,\ell(T))}+\frac{1}{2}\|(v^n_k)_x(t)\|^2_{L^2(0,\ell(T))}&=-\sum_{j=1}^{k-1}\int_{t^n_{j-1}}^{t_j}(\dot F(s),(v^n_j)_x(s))_{L^2(0,\ell(t^n_{j-1}))}\,\mathrm{d}s\\-\int_{t^n_{k-1}}^t(\dot F(s),(v^n_k)_x(s))_{L^2(0,\ell(t^n_{k-1}))}\,\mathrm{d}s&+(F(t),(v^n_k)_x(t))_{L^2(0,\ell(T))},\end{aligned}\tag{3.2.64}$$

where we have used the initial conditions and the fact that $v^n_k=0$ outside Q_k. We define

$$v^n(t):=\begin{cases}v^n_k(t), & t\in[t^n_{k-1},t^n_k)\text{ for a suitable }k,\\ v^n_k(T), & t=T,\end{cases}$$

and (3.2.64) can be written as

$$\frac{1}{2}\|\dot v^n(t)\|^2_{L^2(0,\ell(t))}+\frac{1}{2}\|v^n_x(t)\|^2_{L^2(0,\ell(t))}=(F(t),v^n_x(t))_{L^2(0,\ell(t))}-\int_0^t(\dot F(s),v^n_x(s))_{L^2(0,\ell(s))}\,\mathrm{d}s,\tag{3.2.65}$$

for every $t\in[0,T]$. Moreover, it is easy to check that v^n belongs to $C^0([0,T];H^1_0(0,\ell(T)))\cap C^1([0,T];L^2(0,\ell(T)))$, satisfies the initial conditions (3.2.60)-(3.2.61) and for every $\phi\in C^\infty_c((0,T)\times(0,\ell(T)))$ with $supp(\phi)\subset\cup_{k=1}^n[t^n_{k-1},t^n_k)\times(0,\ell(t^n_{k-1}))$ it holds

$$-\int_Q v^n_t(t,x)\phi_t(t,x)\,\mathrm{d}t\mathrm{d}x+\int_Q v^n_x(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x=\int_Q F(t,x)\phi_x(t,x)\,\mathrm{d}t\mathrm{d}x.\tag{3.2.66}$$

Applying Grönwall Lemma to (3.2.65) we get that there exists a constant C independent on t and n such that

$$\frac{1}{2}\|\dot v^n(t)\|^2_{L^2(0,\ell(t))}+\frac{1}{2}\|v^n_x(t)\|^2_{L^2(0,\ell(t))}\le C\tag{3.2.67}$$

for every $t\in[0,T]$. Then, there exists a function v such that, up to subsequences, $v^n\rightharpoonup v$ weakly in $L^2(0,T;H^1_0(0,\ell(T)))\cap H^1(0,T;L^2(0,\ell(T)))$. Passing to the limit as $n\to+\infty$ in (3.2.66) and in the initial and boundary conditions, we get that v is the (unique) solution of the elastic problem (3.0.6)-(3.0.10) (given by Theorem 3.2.12). We can integrate (3.2.65) between two arbitrary times $0 \leq \alpha \leq \beta \leq T$ and using standard semicontinuity results as $n \to +\infty$ we get that

$$\int_\alpha^\beta \Big(\frac{1}{2}\|\dot v(t)\|^2_{L^2(0,\ell(T))}+\frac{1}{2}\|v_x(t)\|^2_{L^2(0,\ell(T))}\Big)\,\mathrm{d}t \leq \int_\alpha^\beta (F(t),v_x(t))_{L^2(0,\ell(T))}\,\mathrm{d}t - \int_\alpha^\beta\int_0^t (\dot F(s),v_x(s))_{L^2(0,\ell(s))}\,\mathrm{d}s\,\mathrm{d}t. \tag{3.2.68}$$

Since α and β are arbitrary, we get

$$\frac{1}{2}\|\dot v(t)\|^2_{L^2(0,\ell(T))}+\frac{1}{2}\|v_x(t)\|^2_{L^2(0,\ell(T))}\leq (F(t),v_x(t))_{L^2(0,\ell(T))} - \int_0^t (\dot F(s),v_x(s))_{L^2(0,\ell(s))}\,\mathrm{d}s, \tag{3.2.69}$$

for a.e. $t \in (0,T)$. To prove the previous inequality for every time $t^\star \in [0,T]$ it is enough to consider a sequence $t_k \to t^\star$ such that (3.2.69) is satisfied for every t_k. Passing to the limit as $k \to +\infty$ and taking into account that $v \in C^0([0,T];H^1(0,\ell(T)))\cap C^1([0,T];L^2(0,\ell(T)))$ we can conclude. □

Remark 3.2.16. The argument in the proof of Proposition 3.2.15 can be used also to prove existence of a solution of time-dependent problems in dimension bigger than one. See [38, Theorem 1.8] for more details.

Lemma 3.2.17. *Under the assumptions of Proposition 3.2.15 we have that*

$$\|v\|_{H^1(\Omega)}\leq 2T^{1/2}(2+T)(T^{1/2}\|\dot F\|_{L^2(0,T;L^2(0,\ell(T)))}+\|F\|_{L^\infty(0,T;L^2(0,\ell(T)))}). \tag{3.2.70}$$

Proof. From Proposition 3.2.15 we have that for every $t \in [0,T]$

$$\frac{1}{2}\|\dot v(t)\|^2_{L^2(0,\ell(T))}+\frac{1}{2}\|v_x(t)\|^2_{L^2(0,\ell(T))}\leq \|F(t)\|_{L^2(0,\ell(T))}\|v_x(t)\|_{L^2(0,\ell(T))} + \int_0^T \|\dot F(t)\|_{L^2(0,\ell(T))}\|v_x(t)\|_{L^2(0,\ell(T))}\,\mathrm{d}s, \tag{3.2.71}$$

which implies that for every $t \in [0,T]$

$$\frac{1}{2}\|\dot v(t)\|^2_{L^2(0,\ell(T))}+\frac{1}{2}\|v_x(t)\|^2_{L^2(0,\ell(T))}\leq M_u(T^{1/2}\|\dot F\|_{L^2(0,T;L^2(0,\ell(T)))}+\|F\|_{L^\infty(0,T;L^2(0,\ell(T)))}),$$

where $M_v := \sup_{t\in[0,T]}\Big(\|\dot v(t)\|^2_{L^2(0,\ell(T))}+\|v_x(t)\|^2_{L^2(0,\ell(T))}\Big)^{\frac{1}{2}}$. We get

$$M_v^2 \leq 2M_v(T^{1/2}\|\dot F\|_{L^2(0,T;L^2(0,\ell(T)))}+\|F\|_{L^\infty(0,T;L^2(0,\ell(T)))}).$$

Finally, since $\|v\|_{H^1(\Omega)}\leq T^{1/2}(2+T)M_v$, we obtain (3.2.70). □

Theorem 3.2.18 (Existence and uniqueness for the viscoelastic problem)**.** *Let us assume* (3.1.1)-(3.1.7) *and* (3.1.10)-(3.1.14)*. Then, there exists a unique solution u of the viscoelastic problem* (3.0.1)-(3.0.5) *on the domain Ω (in the sense of Definition 3.1.4). Moreover u has a continuous representative on Ω, still denoted by u, and (setting $u \equiv 0$ outside Ω), it holds*

$$u \in C^0([0,T]; H^1(0,+\infty)) \cap C^1([0,T]; L^2(0,+\infty)). \tag{3.2.72}$$

Proof. By definition, we have that a function u is a solution of the viscoelastic problem (3.0.1)-(3.0.5) (in the sense of Definition 3.1.4) if and only if

$$\begin{aligned}
&u_{tt}(t,x) - u_{xx}(t,x) = f(t,x) - \partial_x(F(t,x) + (\mathcal{L}_{deb}u)(t,x)), && (t,x) \in \Omega,\\
&u(t,0) = u_D(t), && t \in (0,T),\\
&u(t,\ell(t)) = 0, && t \in (0,T),\\
&u(0,x) = u^0(x), && x \in (0,\ell_0),\\
&u_t(0,x) = u^1(x), && x \in (0,\ell_0),
\end{aligned}$$

where $\mathcal{L}_{deb}$ is defined in (3.2.52). This means that u is a solution of the elastic problem (3.0.6)-(3.0.10) (in the sense of Definition 3.2.1) with forcing term F replaced by $F + \mathcal{L}_{deb}u$. Let

$$\mathcal{R} : H^1(\Omega) \to H^1(\Omega) \tag{3.2.73}$$

be the operator defined for every $w \in H^1(\Omega)$ by $\mathcal{R}(w) = z$, where z is a solution of elastic problem (3.0.6)-(3.0.10) with F replaced by $F + \mathcal{L}_{deb}w$. The operator $\mathcal{R}$ is well posed as consequence of Theorem 3.2.12. From the definition it follows that u is a fixed point of map $\mathcal{R}$ if and only if u is the solution of the viscoelastic problem considered in (3.0.1)-(3.0.5). In order to get existence and uniqueness of the solution, we have to prove that the operator $\mathcal{R}$ is a contraction.

By definition of $\mathcal{R}$ and linearity of $\mathcal{L}_{deb}$, we have that for every w_1, $w_2 \in H^1(\Omega)$ the function $\mathcal{R}(w_1) - \mathcal{R}(w_2)$ is the solution of elastic problem (3.0.6)-(3.0.10) with F replaced by $\mathcal{L}_{deb}(w_1 - w_2)$ and $u_D = u^0 = u^1 = f = 0$. We apply Lemma 3.2.17 obtaining that

$$\begin{aligned}
\|\mathcal{R}(w_1) - \mathcal{R}(w_2)\|_{H^1(\Omega)} \le{}& 2T(2+T)\|\dot{\widehat{\mathcal{L}_{deb}(w_1 - w_2)}}\|_{L^2(0,T;L^2(0,\ell(T)))}\\
&+ 2T^{1/2}(2+T)\|\mathcal{L}_{deb}(w_1 - w_2)\|_{L^\infty(0,T;L^2(0,\ell(T)))}.
\end{aligned} \tag{3.2.74}$$

We can combine the previous inequality with (3.2.55) and (3.2.56), to get

$$\|\mathcal{R}(w_1) - \mathcal{R}(w_2)\|_{H^1(\Omega)} \le (T(1+T)(2+T) + T(2+T))\|w_1 - w_2\|_{H^1(\Omega)}. \tag{3.2.75}$$

If T is satisfies $(T(1+T)(2+T) + T(2+T)) < 1$ then the map $\mathcal{R}$ is a contraction and the Banach-Caccioppoli fixed point theorem give us that there exists a unique fixed point and the proof is finished. Otherwise, it is enough to consider a partition $0 = T_0 < T_1 < ... < T_N = T$ such that $T_k - T_{k-1}$ satisfies

$$((T_k - T_{k-1})(1 + T_k - T_{k-1})(2 + T_k - T_{k-1}) + (T_k - T_{k-1})(2 + T_k - T_{k-1})) < 1,$$

for every $k = 1, ..., N$. Then there exists a unique solution $u_1 \in H^1(\Omega_{T_1})$, where $\Omega_{T_1} = \{(t,x) \in \Omega \,|\, t \in [0,T_1]\}$ for the viscoelastic problem (3.0.1)-(3.0.5) on the time interval $[0,T_1]$. We consider the viscoelastic problem (3.0.1)-(3.0.5) on the time interval $[T_1,T_2]$ with initial conditions $u(T_1) = u_1(T_1)$ and $u_t(T_1) = (u_1)_t(T_1)$ and we apply again the fixed point argument to get a unique solution $u_2 \in H^1(\Omega_{T_1,T_2})$, where $\Omega_{T_1.T_2} = \{(t,x) \in \Omega \,|\, t \in [T_1,T_2]\}$ for the viscoelastic problem on the time interval $[T_1,T_2]$. It easy to see that the function defined as

$$\tilde{u} := \begin{cases} u_1 & \text{in } \Omega_{T_1}, \\ u_2 & \text{in } \Omega_{T_1,T_2}. \end{cases} \tag{3.2.76}$$

belongs to $H^1(\Omega_{T_2})$, where $\Omega_{T_2} = \{(t,x) \in \Omega \,|\, t \in [0,T_2]\}$, and that is the unique solution of viscoelastic problem on the time interval $[0,T_2]$. Repeting this procedure a finite number of times, we have that there exists a unique solution of (3.0.1)-(3.0.5) on Ω. Finally, (3.2.72) it is a conseguence of (3.2.46) and that fact that $\mathcal{L}_{deb} u \in H^1(0,T;L^2(0,L))$. □

3.3 Energetic analysis

In this section we study the total energy of problem (3.0.1)-(3.0.5) and (3.0.6)-(3.0.10).

3.3.1 Regularity of the total energy

Definition 3.3.1. Let us assume (3.1.1)-(3.1.7) and (3.1.10)-(3.1.14). Let v be the solution of (3.0.6)-(3.0.10). Then we define the total energy of the elastic problem at time $t \in [0,T]$ as

$$\mathcal{E}_v^{tot}(t) := \mathcal{E}_v(t) - \mathcal{W}_v(t) \tag{3.3.1}$$

where $\mathcal{E}_v(t)$ is the sum of kinetic and elastic energy at time t, namely

$$\mathcal{E}_v(t) := \frac{1}{2}\int_0^{\ell(t)} v_t^2(t,x) + v_x^2(t,x)\,\mathrm{d}x, \tag{3.3.2}$$

while $\mathcal{W}_v(t)$ is the work done by the external loads and the boundary conditions in the time interval $[0,t]$, that is

$$\begin{aligned}\mathcal{W}_v(t) = \int_0^t \int_0^{\ell(s)} f(s,x)v_t(s,x)\,\mathrm{d}x\mathrm{d}s - \int_0^t \int_0^{\ell(s)} F_t(s,x)v_x(s,x)\,\mathrm{d}x\mathrm{d}s \\ + \int_0^{\ell(t)} F(t,x)v_x(t,x)\,\mathrm{d}x - \int_0^{\ell_0} F(0,x)u_x^0(0,x)\,\mathrm{d}x\end{aligned}$$

$$-\int_0^t \dot{u}_D(s)\beta(s)\mathrm{d}s \tag{3.3.3}$$

where $\beta(s) := \dot{u}^0(s) + u^1(s) - F(0,s) + \int_0^s (f(\tau, -\tau+s) - \partial_1 F(\tau, -\tau+s))\mathrm{d}\tau - \dot{u}_D(s)$ and $\partial_1 F$ denotes the derivative of F with respect to the first variable.

Remark 3.3.2. When all data are regular it is possible to perform suitable integrations by parts that prove that the work $\mathcal{W}_v$ in Definition 3.3.1 coincide with the classical one from mechanics (see e.g. [46]).

Theorem 3.3.3. *Let us assume* (3.1.1)-(3.1.7) *and* (3.1.10)-(3.1.14). *Let v be the solution of* (3.0.6)-(3.0.10). *Then the total energy for the elastic problem belongs to $AC([0,T])$ and for a.e. $t \in (0, \ell_0/2)$ we have*

$$\begin{aligned}\dot{\mathcal{E}}_v^{tot}(t) = \frac{\dot{\ell}(t)}{2}\frac{\dot{\ell}(t)-1}{1+\dot{\ell}(t)}\Bigg[\int_0^t (f(\tau,\tau+\ell(t)-t) + \partial_1 F(\tau,\tau+\ell(t)-t))\mathrm{d}\tau \\ - \dot{u}^0(\ell(t)-t) + u^1(\ell(t)-t) + F(0,\ell(t)-t)\Bigg]^2.\end{aligned} \tag{3.3.4}$$

Proof. In order to prove that $\mathcal{E}_v^{tot} \in AC([0,T])$ it is enough to prove (3.3.4) on $[0, \ell_0/2]$ and then repeat the same argument on $[(k-1)\ell_0/2, k\ell_0/2]$ for $k = 1, ..., \left\lceil \frac{2T}{\ell_0} \right\rceil$. Taking into account the representation formula (3.2.34) and Lemmas 3.2.6 and 3.2.8, we have that

$$v_t(t,x) = \alpha_1(x+t) + \alpha_2(x-t) + h_1(t,x) + h_2(t,x), \tag{3.3.5}$$

$$v_x(t,x) = \alpha_1(x+t) - \alpha_2(x-t) + h_1(t,x) - h_2(t,x) + F(t,x), \tag{3.3.6}$$

for a.e. $(t,x) \in \Omega_{\ell_0/2} = \{(t,x) \in \Omega \,|\, t \in (0, \ell_0/2)\}$, where

$$\alpha_1(z) = \begin{cases} \frac{1}{2}\dot{u}^0(z) + \frac{1}{2}u^1(z) - \frac{1}{2}F(0,z), & z \in (0, \ell_0], \\ -\frac{1}{2}F(0,-\omega(z))\dot{\omega}(z) + \frac{1}{2}\dot{u}^0(-\omega(z))\dot{\omega}(z) - \frac{1}{2}u^1(-\omega(z))\dot{\omega}(z), & z \in (\ell_0, 2t_0^\ell), \end{cases}$$

$$\alpha_2(z) = \begin{cases} -\frac{1}{2}\dot{u}^0(-z) - \frac{1}{2}u^1(-z) + \frac{1}{2}F(0,-z) + \dot{u}_D(-z), & z \in (-\ell_0, 0], \\ -\frac{1}{2}\dot{u}^0(z) + \frac{1}{2}u^1(z) + \frac{1}{2}F(0,z), & z \in (0, \ell_0), \end{cases}$$

with $t_0^\ell := \min\{T, \inf\{t \in [\ell_0, T] \,|\, t = \ell(t)\}\}$ and

$$h_1(t,x)=\begin{cases}\frac{1}{2}\int_0^t f(\tau,-\tau+x+t)\,\mathrm{d}\tau-\frac{1}{2}\int_0^t \partial_1 F(\tau,-\tau+x+t)\,\mathrm{d}\tau, & (t,x)\in\Omega_1',\\ \frac{1}{2}\int_0^t f(\tau,-\tau+x+t)\,\mathrm{d}\tau-\frac{1}{2}\int_0^t \partial_1 F(\tau,-\tau+x+t)\,\mathrm{d}\tau, & (t,x)\in\Omega_2',\\ \frac{1}{2}\int_{\psi^{-1}(x+t)}^t f(\tau,-\tau+x+t)\,\mathrm{d}\tau-\frac{1}{2}\int_0^{\psi^{-1}(x+t)} f(\tau,\tau-\omega(x+t))\,\mathrm{d}\tau\dot\omega(x+t) & \\ -\frac{1}{2}\int_{\psi^{-1}(x+t)}^t \partial_1 F(\tau,-\tau+x+t)\,\mathrm{d}\tau-\frac{1}{2}\int_0^{\psi^{-1}(x+t)} \partial_1 F(\tau,\tau-\omega(x+t))\,\mathrm{d}\tau\dot\omega(x+t), & (t,x)\in\Omega_3',\end{cases}$$

while

$$h_2(t,x)=\begin{cases}\frac{1}{2}\int_0^t f(\tau,\tau+x-t)\,\mathrm{d}\tau+\frac{1}{2}\int_0^t \partial_1 F(\tau,\tau+x-t)\,\mathrm{d}\tau, & (t,x)\in\Omega_1',\\ \frac{1}{2}\int_{t-x}^t f(\tau,\tau+x-t)\,\mathrm{d}\tau-\frac{1}{2}\int_0^{t-x} f(\tau,-\tau+t-x)\,\mathrm{d}\tau & \\ +\frac{1}{2}\int_{t-x}^t \partial_1 F(\tau,\tau+x-t)\,\mathrm{d}\tau+\frac{1}{2}\int_0^{t-x} \partial_1 F(\tau,-\tau+t-x)\,\mathrm{d}\tau, & (t,x)\in\Omega_2',\\ \frac{1}{2}\int_0^t f(\tau,\tau+x-t)\,\mathrm{d}\tau+\frac{1}{2}\int_0^t \partial_1 F(\tau,\tau+x-t)\,\mathrm{d}\tau, & (t,x)\in\Omega_3'.\end{cases}$$

We can substitute the expression in (3.3.5) and (3.3.6) in $\mathcal{E}_v$ and we get

$$\begin{aligned}\mathcal{E}_v(t)&=\frac{1}{2}\int_0^{\ell(t)} v_t^2(t,x)+((v_x(t,x)-F(t,x))+F(t,x))^2\,\mathrm{d}x\\ &=\frac{1}{2}\int_0^{\ell(t)}(\alpha_1(x+t)+\alpha_2(x-t)+h_1(t,x)+h_2(t,x))^2\,\mathrm{d}x\\ &+\frac{1}{2}\int_0^{\ell(t)}(\alpha_1(x+t)-\alpha_2(x-t)+h_1(t,x)-h_2(t,x))^2\,\mathrm{d}x\\ &+\frac{1}{2}\int_0^{\ell(t)} 2F(t,x)v_x(t,x)-F^2(t,x)\,\mathrm{d}x\end{aligned}$$

which implies

$$\mathcal{E}_v^{tot}(t)=A(t)+B(t),$$

where

$$\begin{aligned}A(t)=&\int_0^{\ell(t)}(\alpha_1(x+t)+h_1(t,x))^2\,\mathrm{d}x\\ &+\int_0^{\ell(t)}(\alpha_2(x-t)+h_2(t,x))^2\,\mathrm{d}x,\end{aligned}\tag{3.3.7}$$

and

$$B(t) = -\int_0^t \int_0^{\ell(s)} f(s,x)v_t(s,x)\,\mathrm{d}x\mathrm{d}s + \int_0^t \int_0^{\ell(s)} F_t(s,x)v_x(s,x)\,\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^{\ell_0} F(0,x)u^0_x(0,x) + \int_0^t \dot{u}_D(s)\beta(s)\mathrm{d}s\,\mathrm{d}x - \frac{1}{2}\int_0^{\ell(t)} F^2(t,x)\mathrm{d}x. \tag{3.3.8}$$

Using Theorem 3.2.7, it is easy to check that B in (3.3.8) is absolutely continuous. In order to study the regularity of (3.3.7), we perform a suitable change of variable, that is

$$A(t) = \int_t^{\ell(t)+t} (\alpha_1(y) + h_1(t, y-t))^2\,\mathrm{d}y + \int_{-t}^{\ell(t)-t} (\alpha_2(y) + h_2(t, y+t))^2\,\mathrm{d}y. \tag{3.3.9}$$

Taking into account the definitions of α_1, α_2, h_1, h_2, we have that right hand side of (3.3.9) satisfies the conditions of Theorem 3.2.7. Finally, by deriving A and B it is easy to check that we get (3.3.4). □

We deal now with the case of the viscoelastic problem (3.0.1)-(3.0.5).

Definition 3.3.4. Let us assume (3.1.1)-(3.1.7) and (3.1.10)-(3.1.14). Let u be the solution of (3.0.1)-(3.0.5). Then we define the total energy of the elastic problem at time $t \in [0,T]$ as

$$\mathcal{E}^{tot}_u(t) := \mathcal{E}_u(t) + \mathcal{D}_u(t) - \mathcal{W}_u(t) \tag{3.3.10}$$

where $\mathcal{E}_u(t)$ is the sum of kinetic and elastic energy at time t, namely

$$\mathcal{E}_u(t) := \frac{1}{2}\int_0^{\ell(t)} u_t^2(t,x) + u_x^2(t,x)\,\mathrm{d}x, \tag{3.3.11}$$

the term $\mathcal{D}_u(t)$ is the dissipation due to viscosity in $[0,t]$ which is defined as

$$\mathcal{D}_u(t) := \int_0^t \int_0^{\ell(s)} (\frac{1}{2}u_x(s,x) - (\mathcal{L}_{deb}u)(s,x))u_x(s,x)\,\mathrm{d}x\mathrm{d}s - \int_0^{\ell(t)} (\mathcal{L}_{deb}u)(t,x)u_x(t,x)\,\mathrm{d}x, \tag{3.3.12}$$

while $\mathcal{W}_u(t)$ is the work done by the external loads and the boundary conditions in the time interval $[0,t]$, that is

$$\mathcal{W}_u(t) = \int_0^t \int_0^{\ell(s)} f(s,x)u_t(s,x)\,\mathrm{d}x\mathrm{d}s - \int_0^t \int_0^{\ell(s)} F_t(s,x)u_x(s,x)\,\mathrm{d}x\mathrm{d}s \tag{3.3.13}$$
$$+ \int_0^{\ell(t)} F(t,x)u_x(t,x)\,\mathrm{d}x - \int_0^{\ell_0} F(0,x)u^0_x(0,x)\,\mathrm{d}x \tag{3.3.14}$$
$$- \int_0^t \dot{u}_D(s)\xi(s)\mathrm{d}s \tag{3.3.15}$$

where

$$\begin{aligned}\xi(s) &:= \dot u^0(s) + u^1(s) - F(0,s) - \dot u_D(s)\\ &\quad + \int_0^s (f(\tau,-\tau+s) - \partial_1 F(\tau,-\tau+s) - \frac{1}{2}u_x(\tau,-\tau+s) + (\mathcal{L}_{deb}u)(\tau,-\tau+s))\mathrm{d}\tau\end{aligned}$$

and $\partial_1 F$ denotes the derivative of F with respect to the first variable.

Theorem 3.3.5. *Let us assume* (3.1.1)-(3.1.7) *and* (3.1.10)-(3.1.14)*. Let u be the solution of* (3.0.1)-(3.0.5)*. Then the total energy for the elastic problem belongs to $AC([0.T])$ and for a.e. $t \in [0, \ell_0/2]$ we have*

$$\begin{aligned}\dot{\mathcal{E}}^{tot}_u(t) = \frac{\dot\ell(t)}{2}\frac{\dot\ell(t)-1}{1+\dot\ell(t)}\Bigg[& F(0,\ell(t)-t) - \dot u^0(\ell(t)-t) + u^1(\ell(t)-t)\\ & + \int_0^t (f(\tau,\tau+\ell(t)-t) + \partial_1 F(\tau,\tau+\ell(t)-t))\mathrm{d}\tau\\ & + \int_0^t (\frac{1}{2}u_x(\tau,\tau+\ell(t)-t) - (\mathcal{L}_{deb}u)(\tau,\tau+\ell(t)-t))\mathrm{d}\tau\Bigg]^2. \end{aligned} \tag{3.3.16}$$

Proof. The result follows replacing F with $F + \mathcal{L}_{deb}u$ in Theorem 3.3.3 and taking into account (3.2.54) and that $(\mathcal{L}_{deb}u)(0,x) = 0$. □

3.3.2 Applications and future research

In this subsection we give the main ideas to define the coupled problem, namely the problem where both the displacement u and the debonding evolution $\ell\colon [0,T] \to [\ell_0,+\infty)$ are unknown, following [20] and [42]. We have no claims of completeness and the aim of this subsection is only to show, without technical details, what will be studied in the future work [13]. In this subsection we assume (3.1.1)-(3.1.7) and (3.1.10)-(3.1.14) and moreover

$$\dot\ell(t) < 1 \quad \text{a.e. } t \in [0,T]. \tag{3.3.17}$$

The definition of coupled problem is based on energetic considerations regarding the debondig ℓ. Let $0 < c_1 < c_2$ and let $\kappa\colon [0,T] \to [c_1,c_2]$ a bounded measurable function. Given $0 \le x_1 < x_2$ we define the energy dissipated to debond the segment $[x_1,x_2]$ as

$$\mathcal{D}_{deb}(x_1,x_2) := \int_{x_1}^{x_2} \kappa(x)\,\mathrm{d}x, \tag{3.3.18}$$

where κ represents the local toughness. This means that the energy dissipeted in the debonding process on the interval $[0,t]$, for every time $t \in (0,T]$, is given by

$$\mathcal{D}_{deb}(\ell_0,\ell(t)) = \int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x. \tag{3.3.19}$$

Inspired by the case of crack theory (see the Introduction and Chapter 2), we require that our model satisfies the Griffith's criterion, namely the following energy balance holds:

$$\mathcal{E}_u^{tot}(t) + \mathcal{D}_{deb}(\ell_0, \ell(t)) = \mathcal{E}_u^{tot}(0) \quad \text{for every } t \in (0, T], \tag{3.3.20}$$

where $\mathcal{E}_u^{tot}$ is the total energy for the viscoelastic system (3.0.1)-(3.0.5) given in Definition 3.3.4. Using Theorem 3.3.5 we can derive (3.3.20) with respect to time obtaining

$$\kappa(\ell(t))\dot{\ell}(t) = G(t)\dot{\ell}(t) \quad \text{for a.e. } t \in (0, T), \tag{3.3.21}$$

where

$$\begin{aligned} G(t) := -\frac{1}{2}\frac{\dot{\ell}(t)-1}{1+\dot{\ell}(t)}\bigg[& F(0, \ell(t)-t) - \dot{u}^0(\ell(t)-t) + u^1(\ell(t)-t) \\ & + \int_0^t (f(\tau, \tau+\ell(t)-t) + \partial_1 F(\tau, \tau+\ell(t)-t))\mathrm{d}\tau \\ & + \int_0^t (\frac{1}{2}u_x(\tau, \tau+\ell(t)-t) - (\mathcal{L}_{deb}u)(\tau, \tau+\ell(t)-t))\mathrm{d}\tau \bigg]^2, \end{aligned} \tag{3.3.22}$$

is the dynamic energy release rate for the viscoelastic system, which is the energy dissipated by the system (per unit lenght). Equation (3.3.21) gives us an ordinary differential equation for ℓ. The function $\ell(t) \equiv \ell_0$ is clearly a solution so, in order to avoid trivial cases, we postulate in our model a maximum dissipation principle that forces the debonding front to move with the maximum speed allowed by the energy balance. A more precise definition of dynamic energy release and maximum dissipation principle will be given in [13] (see also [20] and [42]). It is possible to prove that these conditions are equivalent to the following system: for a.e. t it holds

$$\begin{cases} 0 \le \dot{\ell}(t), \\ G(t) \le \kappa(\ell(t)), \\ \big[G(t) - \kappa(\ell(t))\big]\dot{\ell}(t) = 0. \end{cases} \tag{3.3.23}$$

The first condition means that the debonding can only increase, while the second one asserts that the dynamic energy release rate is bounded by the local toughness. The last conditions states that the debonding front increase with non null speed only when the energy release rate is critical, that is $G(t) = \kappa(\ell(t))$. The conditions (3.3.23) can be used to write an explicit ordinary differential equation for ℓ (depending also on the displacement u) which, coupled with system (3.0.1)-(3.0.5), can be used to study the dynamic evolution of debonding when both u and ℓ are not given.

Thanks to the results of this chapter (in particular Theorem 3.3.5) in [13] we will be in a position to prove existence and uniqueness for the coupled problem, following the ideas of [20, 42].

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