Math. Z. (2018) 290:699–710
https://doi.org/10.1007/s00209-017-2037-2

Mathematische Zeitschrift

The DT/PT correspondence for smooth curves

Andrea T. Ricolfi[1]

Received: 5 September 2017 / Accepted: 19 November 2017 / Published online: 11 January 2018

Abstract We show a version of the DT/PT correspondence relating local curve counting invariants, encoding the contribution of a fixed smooth curve in a Calabi–Yau threefold. We exploit a local study of the Hilbert–Chow morphism about the cycle of a smooth curve. We compute, via Quot schemes, the global Donaldson–Thomas theory of a general Abel–Jacobi curve of genus 3.

Keywords Donaldson–Thomas invariants · Hilbert–Chow morphism

Mathematics Subject Classification 14N35 · 14C05

Contents

✉ Andrea T. Ricolfi
atricolfi@gmail.com

1 Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

1 Introduction

Let Y be a smooth, projective Calabi–Yau threefold. The Donaldson–Thomas (DT, for short) invariants of Y are enumerative invariants attached to the Hilbert schemes

$$I_m(Y,\beta)=\{Z\subset Y\,|\,\chi(\mathscr{O}_Z)=m,\ [Z]=\beta\},$$

viewed as moduli spaces of ideal sheaves [18]. These numbers are insensitive to small deformations of the complex structure of Y, but they do change when we perturb the stability condition that defines them: the rules that govern these changes are the so called *wall-crossing formulas*. It is in this spirit that one can interpret the "DT/PT correspondence", an equality of generating functions

$$\mathsf{DT}_\beta(q)=\mathsf{DT}_0(q)\cdot\mathsf{PT}_\beta(q) \tag{1.1}$$

first conjectured in [14] and later proved by Bridgeland [4] and Toda [19]. The left hand side of (1.1) is the Laurent series encoding the Donaldson–Thomas invariants of Y in the class $\beta\in H_2(Y,\mathbb{Z})$, whereas PT_β encodes the Pandharipande–Thomas (PT, for short) invariants of Y, defined through the moduli space of stable pairs [14,15],

$$P_m(Y,\beta)=\{(F,s)\,|\,\chi(F)=m,\ [\operatorname{Supp}F]=\beta\}.$$

Recall that a pair (F,s), consisting of a one-dimensional coherent sheaf $F\in\operatorname{Coh}(Y)$ and a section $s\in H^0(Y,F)$, is said to be *stable* when F is pure and the cokernel of $s:\mathscr{O}_Y\to F$ is zero-dimensional. Finally, the formula

$$\mathsf{DT}_0(q)=M(-q)^{\chi(Y)},$$

proven in [2,9,10], determines the zero-dimensional DT theory of Y. Here $M(q)=\prod_{k>0}(1-q^k)^{-k}$ is the MacMahon function, the generating function of plane partitions.

1.1 Main result

We prove a variant of (1.1) in this note. Let $C\subset Y$ be a smooth curve of genus g embedded in class β. For integers $n\geq 0$, we define "local" DT invariants

$$\mathsf{DT}_{n,C}=\int_{I_n(Y,C)}\nu_I\ \mathrm{d}\chi$$

where $\nu_I: I_{1-g+n}(Y,\beta)\to\mathbb{Z}$ is the Behrend function [1] on the Hilbert scheme and $I_n(Y,C)\subset I_{1-g+n}(Y,\beta)$ is the closed subset parametrizing subschemes $Z\subset Y$ containing C as their maximal purely 1-dimensional subscheme. For instance, a generic point of $I_n(Y,C)$ represents a subscheme consisting of C along with n distinct points in $Y\setminus C$. Similarly, we consider $P_n(Y,C)\subset P_{1-g+n}(Y,C)$, the closed subset parametrizing stable pairs (F,s) such that $\operatorname{Supp}F=C$. The local invariants on the stable pair side

$$\mathsf{PT}_{n,C}=\int_{P_n(Y,C)}\nu_P\ \mathrm{d}\chi$$

have been studied in [15]. By smoothness of C, the local moduli space $P_n(Y,C)$ can be identified with the symmetric product $\operatorname{Sym}^n C$. Let us form the generating functions

$$\mathsf{DT}_C(q)=\sum_{n\geq 0}\mathsf{DT}_{n,C}\,q^{1-g+n},\qquad \mathsf{PT}_C(q)=\sum_{n\geq 0}\mathsf{PT}_{n,C}\,q^{1-g+n}.$$

We say that the DT/PT correspondence holds for C if one has

$$\mathsf{DT}_C(q) = \mathsf{DT}_0(q) \cdot \mathsf{PT}_C(q). \tag{1.2}$$

We can view (1.2) as a wall-crossing formula relating the local curve counting invariants attached to C.

Let $n_{g,C}$ be the BPS number of $C \subset Y$ [15]. Using the known value of the PT side [15, Section 3],

$$\mathsf{PT}_C(q) = n_{g,C} \cdot q^{1-g}(1+q)^{2g-2},$$

we proved that the DT/PT correspondence (1.2) holds for C smooth and *rigid* [16, Section 5]. The goal of this note is to extend the result to all smooth curves.

Theorem 1.1 *Let Y be a smooth, projective Calabi–Yau threefold, $C \subset Y$ a smooth curve. Then the DT/PT correspondence* (1.2) *holds for C.*

In fact, the conclusion of the theorem holds for all Cohen–Macaulay curves, by recent work of Oberdieck [12]. While he works with motivic Hall algebras, our method involves a local study of the Hilbert–Chow morphism, and builds upon previous calculations [16], especially the weighted Euler characteristic of the Quot scheme $\mathrm{Quot}_n(\mathscr{I}_C)$, as we explain in Sect. 2.

The local invariants do not depend on any scheme structure one may put on $I_n(Y, C)$ and $P_n(Y, C)$. However, on the DT side, we will exploit the Hilbert–Chow morphism to endow $I_n(Y, C)$ with a natural scheme structure, and we will prove that it agrees with the Quot scheme studied in [16]. So, in the local theory, we can think of the moduli spaces

$$\mathrm{Quot}_n(\mathscr{I}_C) \qquad \text{and} \qquad \mathrm{Sym}^n\, C$$

as sitting on opposite sides of the wall separating DT and PT theory from one another. *Conventions.* All schemes are defined over $\mathbb{C}$. The Calabi–Yau condition for us is simply the existence of a trivialization of the canonical line bundle. The Hilbert–Chow morphism $\mathrm{Hilb}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is the one constructed by Rydh [17].

2 The DT/PT correspondence

In this section we outline our strategy to deduce Theorem 1.1.

Let Y be a smooth projective variety, not necessarily Calabi–Yau. We consider the Hilbert–Chow morphism

$$\mathrm{Hilb}_1(Y) \to \mathrm{Chow}_1(Y) \tag{2.1}$$

constructed in [17], sending a 1-dimensional subscheme of Y to its fundamental cycle. We recall its definition in Sect. 3.1. Let $I_m(Y, \beta) \subset \mathrm{Hilb}_1(Y)$ be the component parametrizing subschemes $Z \subset Y$ such that

$$\chi(\mathscr{O}_Z) = m \in \mathbb{Z}, \qquad [Z] = \beta \in H_2(Y, \mathbb{Z}).$$

Similarly, we let $\mathrm{Chow}_1(Y, \beta) \subset \mathrm{Chow}_1(Y)$ be the component parametrizing 1-cycles of degree β. Then (2.1) restricts to a morphism

$$\mathsf{h}_m : I_m(Y, \beta) \to \mathrm{Chow}_1(Y, \beta).$$

Definition 2.1 Fix an integer $n \geq 0$. For a Cohen–Macaulay curve $C \subset Y$ of arithmetic genus g embedded in class β, we let

$$I_n(Y, C) \subset I_{1-g+n}(Y, \beta)$$

denote the scheme-theoretic fibre of h_{1-g+n}, over the cycle of C.

Remark 2.1 We will use that (2.1) is an isomorphism around normal schemes, at least in characteristic zero [17, Cor. 12.9]. Thus, for a smooth curve $C \subset Y$, we will identify Chow with Hilb locally around the cycle $[C] \in \mathrm{Chow}_1(Y)$ and the ideal sheaf $\mathscr{I}_C \in \mathrm{Hilb}_1(Y)$. For this reason, we do not need the representability of the global Chow functor, as around the point $[C] \in \mathrm{Chow}_1(Y, \beta)$ we can work with the Hilbert scheme $I_{1-g}(Y, \beta)$ instead.

Consider the Quot scheme

$$\mathrm{Quot}_n(\mathscr{I}_C)$$

parametrizing quotients of length n of the ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$. We proved in [16, Lemma 5.1] that the association $[\theta : \mathscr{I}_C \twoheadrightarrow \mathscr{E}] \mapsto \ker \theta$ defines a closed immersion

$$\mathrm{Quot}_n(\mathscr{I}_C) \hookrightarrow I_{1-g+n}(Y, \beta). \tag{2.2}$$

More precisely, for a scheme S, an S-valued point of the Quot scheme is a flat quotient $\mathscr{E} = \mathscr{I}_{C\times S}/\mathscr{I}_Z$, and in the short exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_Z \to \mathscr{O}_{C\times S} \to 0$$

living over $Y \times S$, the middle term is S-flat, so Z defines an S-point of $I_{1-g+n}(Y, \beta)$. The S-valued points of the image of (2.2) consist precisely of those flat families $Z \subset Y \times S \to S$ such that Z contains $C \times S$ as a closed subscheme. This will be used implicitly in the proof of Theorem 2.1.

The schemes $I_n(Y, C)$ and $\mathrm{Quot}_n(\mathscr{I}_C)$ have the same $\mathbb{C}$-valued points: they both parametrize subschemes $Z \subset Y$ consisting of C together with "n points", possibly embedded. The first step towards Theorem 1.1 is the following result, whose proof is postponed to the next section.

Theorem 2.1 *Let Y be a smooth projective variety, $C \subset Y$ a smooth curve of genus g. Then $I_n(Y, C) = \mathrm{Quot}_n(\mathscr{I}_C)$ as subschemes of $I_{1-g+n}(Y, \beta)$.*

As an application of Theorem 2.1, in Sect. 4 we compute the reduced Donaldson–Thomas theory of a general Abel–Jacobi curve of genus 3.

To proceed towards Theorem 1.1, we need to examine the local structure of the Hilbert scheme around subschemes $Z \subset Y$ whose maximal purely 1-dimensional subscheme $C \subset Z$ is smooth. The result, given below, will be proven in the next section.

Theorem 2.2 *Let Y be a smooth projective variety, $C \subset Y$ a smooth curve of genus g. Then, locally analytically around $I_n(Y, C)$, the Hilbert scheme $I_{1-g+n}(Y, \beta)$ is isomorphic to $I_n(Y, C) \times \mathrm{Chow}_1(Y, \beta)$.*

Roughly speaking, this means that the Hilbert–Chow morphism, locally about the cycle

$$[C] \in \mathrm{Chow}_1(Y, \beta),$$

behaves like a fibration with typical fibre $I_n(Y, C)$. To obtain this, we first identify Chow with Hilb locally around C, cf. Remark 2.1. We then need to trivialize the universal curve

$\mathscr{C} \to$ Hilb, which can be done since smooth maps are analytically locally trivial (on the source). However, even if we had $\mathscr{C} = C \times$ Hilb, we would not be done: the fibre of Hilbert–Chow (which is the Quot scheme by Theorem 2.1) depends on the embedding of the curve into Y, not just on the abstract curve. So to prove Theorem 2.2 we need to trivialize (locally) the embedding of the universal curve into $Y \times$ Hilb. This is taken care of by a local-analytic version of the tubular neighborhood theorem. After this step, Theorem 2.2 follows easily.

Granting Theorems 2.1 and 2.2, we can prove the DT/PT correspondence for smooth curves. So now we assume C is a smooth curve embedded in class β in a smooth, projective Calabi–Yau threefold Y.

Proof of Theorem 1.1 By [17, Cor. 12.9], the Hilbert–Chow morphism

$$\mathsf{h}_{1-g} : I_{1-g}(Y, \beta) \to \mathrm{Chow}_1(Y, \beta)$$

is (in characteristic zero) an isomorphism over the locus of normal schemes. Under this local identification, the cycle $[C]$ corresponds to the ideal sheaf $\mathscr{I}_C$. We let $\nu(\mathscr{I}_C)$ be the value of the Behrend function on $I_{1-g}(Y, \beta)$ at the point corresponding to $\mathscr{I}_C$. Since the Behrend function can be computed locally analytically [1, Prop. 4.22], Theorem 2.2 implies the identity

$$\nu_I\big|_{I_n(Y,C)} = \nu(\mathscr{I}_C) \cdot \nu_{I_n(Y,C)},$$

where ν_I is the Behrend function of $I = I_{1-g+n}(Y, \beta)$. After integration, we find

$$\mathsf{DT}_{n,C} = \nu(\mathscr{I}_C) \cdot \widetilde{\chi}(I_n(Y, C)),$$

where $\widetilde{\chi}(I_n(Y, C))$, by Theorem 2.1, agrees with the weighted Euler characteristic of the Quot scheme $\mathrm{Quot}_n(\mathscr{I}_C)$. But we proved in [16, Thm. 5.2] that the relation

$$\mathsf{DT}_{n,C} = \nu(\mathscr{I}_C) \cdot \widetilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C))$$

is equivalent to the C-local DT/PT correspondence (1.2), so the theorem follows. □

As observed in [16], the C-local DT/PT correspondence says that the local invariants are determined by the topological Euler characteristic of the corresponding moduli space, along with the BPS number of the fixed smooth curve $C \subset Y$. The latter can be computed as

$$n_{g,C} = \nu(\mathscr{I}_C).$$

For any integer $n \geq 0$, the explicit formulas are

$$\mathsf{DT}_{n,C} = n_{g,C} \cdot (-1)^n \chi(I_n(Y, C)),$$
$$\mathsf{PT}_{n,C} = n_{g,C} \cdot (-1)^n \chi(P_n(Y, C)).$$

In particular, the local invariants differ by the Euler characteristic of the corresponding moduli space by the *same* constant.

3 Proofs

It remains to prove Theorems 2.1 and 2.2. For Theorem 2.1, we need to review some definitions and results from [17].

3.1 The fibre of Hilbert–Chow

Rydh has developed a powerful theory of *relative cycles* and has defined a Hilbert–Chow morphism

$$\mathrm{Hilb}_r(X/S) \to \mathrm{Chow}_r(X/S) \tag{3.1}$$

for every algebraic space X locally of finite type over an arbitrary scheme S. For us X is always a scheme, projective over S.

We quickly recall the definition of (3.1). First of all, the Hilbert scheme $\mathrm{Hilb}_r(X/S)$ parametrizes S-subschemes of X that are proper and of dimension r over S, but not necessarily equidimensional, while $\mathrm{Chow}_r(X/S)$ parametrizes *equidimensional*, proper relative cycles of dimension r. We refer to [17, Def. 4.2] for the definition of *relative cycles* on X/S. Cycles have a (not necessarily equidimensional) support, which is a locally closed subset $Z \subset X$. Rydh shows [17, Prop. 4.5] that if α is a relative cycle on $f : X \to S$ with support Z, then, for every $r \geq 0$, on the same family there is a unique *equidimensional* relative cycle α_r with support

$$Z_r = \{x \in Z \,|\, \dim_x Z_{f(x)} = r\} \subset Z.$$

Cycles are called equidimensional when their support is equidimensional over the base. The essential tool for the definition of (3.1) is the *norm family*, defined by the following result.

Theorem 3.1 [17, Thm. 7.14] *Let $X \to S$ be a locally finitely presented morphism, $\mathcal{F}$ a finitely presented $\mathscr{O}_X$-module which is flat over S. Then there is a canonical relative cycle $\mathcal{N}_{\mathcal{F}}$ on X/S, with support equal to* $\mathrm{Supp}\,\mathcal{F}$. *This construction commutes with arbitraty base change. When $Z \subset X$ is a subscheme which is flat and of finite presentation over S, we write $\mathcal{N}_Z = \mathcal{N}_{\mathscr{O}_Z}$.*

The Hilbert–Chow functor (3.1) is defined by $Z \mapsto (\mathcal{N}_Z)_r$.

Even though we do not recall here the full definition of relative cycle, the main idea is the following. For a locally closed subset $Z \subset X$, Rydh defines a *projection of X/S adapted to Z* to be a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{p} & X \\ \downarrow & & \\ B & & \downarrow \\ \downarrow & & \\ T & \xrightarrow{g} & S \end{array} \tag{3.2}$$

where $U \to X \times_S T$ is étale, $B \to T$ is smooth and $p^{-1}(Z) \to B$ is finite. A relative cycle α on X/S with support $Z \subset X$ is the datum, for every projection adapted to Z, of a proper family of *zero*-cycles on U/B, which Rydh defines as a morphism

$$\alpha_{U/B/T} : B \to \Gamma^\star(U/B)$$

to the scheme of divided powers. We refer to [17, Def. 4.2] for the additional compatibility conditions that these data should satisfy.

Let now $\mathcal{F}$ be a flat family of coherent sheaves on X/S. If $\mathsf{p} = (U, B, T, p, g)$ denotes a projection of X/S adapted to $\mathrm{Supp}\,\mathcal{F} \subset X$ as in (3.2), then the *zero*-cycle defining the norm family $\mathcal{N}_{\mathcal{F}}$ at p is

$$(\mathcal{N}_{\mathcal{F}})_{U/B/T} = \mathcal{N}_{p^*\mathcal{F}/B},$$

constructed in [17, Cor. 7.9]. For us $\mathcal{F}$ will always be a structure sheaf, so it will be easy to compare these zero-cycles.

If $Z \subset X$ is a subscheme that is smooth over S, then the norm family $\mathcal{N}_Z$ is an example of a *smooth* relative cycle, cf. [17, Def. 8.11]. The next result states an equivalence, in characteristic zero, between smooth relative cycles and subschemes smooth over the base.

Theorem 3.2 [17, Thm. 9.8] *If S is of characteristic zero, then for every smooth relative cycle α on X/S there is a unique subscheme $Z \subset X$, smooth over S, such that $\alpha = \mathcal{N}_Z$.*

We can now prove Theorem 2.1. We fix Y to be a smooth projective variety, $C \subset Y$ a smooth curve of genus g in class β, and we denote by $I_n(Y, C)$ the fibre over $[C]$ of the Hilbert–Chow morphism

$$I_{1-g+n}(Y, \beta) \to \mathrm{Chow}_1(Y, \beta),$$

as in Definition 2.1.

Proof of Theorem 2.1 We need to show that

$$I_n(Y, C) = \mathrm{Quot}_n(\mathscr{I}_C)$$

as subschemes of $I_{1-g+n}(Y, \beta)$. Let S be a scheme over $\mathbb{C}$, and set $X = Y \times_{\mathbb{C}} S$. Then a family

$$Z \subset X \to S$$

in the Hilbert scheme is an S-valued point of $I_n(Y, C)$ when $(\mathcal{N}_Z)_1 = \mathcal{N}_{C\times S}$. The closed immersion (2.2) from the Quot scheme to the Hilbert scheme factors through $I_n(Y, C)$. Indeed, any S-point $\mathscr{I}_{C\times S} \twoheadrightarrow \mathscr{I}_{C\times S}/\mathscr{I}_Z$ of the Quot scheme gives a closed immersion $C \times S \hookrightarrow Z$ whose relative ideal is zero-dimensional over S, thus we have $(\mathcal{N}_Z)_1 = (\mathcal{N}_{C\times S})_1 = \mathcal{N}_{C\times S}$, where in the second equality we used that $\mathcal{N}_{C\times S}$ is equidimensional of dimension one over S. So we obtain a closed immersion

$$\iota : \mathrm{Quot}_n(\mathscr{I}_C) \hookrightarrow I_n(Y, C).$$

For every scheme S, we therefore have an injective map of sets

$$\iota(S) : \mathrm{Quot}_n(\mathscr{I}_C)(S) \hookrightarrow I_n(Y, C)(S),$$

and since $\iota(\mathrm{Spec}\ \mathbb{C})$ is a bijection, so far ι is just a bijective closed immersion. We need to show $\iota(S)$ is onto, and for the moment we deal with the case where S is a fat point. In other words, assume S is the spectrum of a local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Let $Z \subset X \to S$ be an S-valued point of $I_n(Y, C)$. Consider the finite subscheme $F \subset Y \subset X$ given by the support of $\mathscr{I}_C/\mathscr{I}_{Z_0}$, where Z_0 is the closed fibre of $Z \to S$. Form the open set $V = X \setminus F \subset X$. Then we have, as relative cycles on V/S,

$$(\mathcal{N}_Z)_1\big|_V = \mathcal{N}_{C\times S}\big|_V = \mathcal{N}_{(C\times S)\cap V}.$$

We claim the left hand side equals the relative cycle $\mathcal{N}_{Z\cap V}$. For sure, these two cycles have the same support, as $Z \cap V = Z_1 \cap V$, and they are determined by the same set of projections; indeed, being equidimensional of dimension one, they are determined by (compatible data of) relative zero-cycles for every projection $\mathsf{p}_{V/S} = (U, B, T, p, g)$ such that B/T is smooth of relative dimension one. Let us focus on $(\mathcal{N}_Z)_1$ first. Here $r = 1$ is the maximal relative dimension of a point in Z, so the zero-cycle corresponding to a projection $\mathsf{p}_{X/S}$ as in (3.2), and adapted to Z_1, is the same as the one defined by the norm family of Z (cf. the proof of [17, Prop. 4.5]), namely $\mathcal{N}_{p^*\mathscr{O}_{Z/B}}$. Now we restrict to the open subset $i : V \to X$. By definition of pullback, the zero-cycle attached to a projection $\mathsf{p}_{V/S}$ (adapted to $Z_1 \cap V$) is the cycle corresponding to the projection $(U, B, T, i \circ p, g)$ for the full family Z/S, namely

$$\mathcal{N}_{(i\circ p)^*\mathscr{O}_{Z/B}} = \mathcal{N}_{p^*\mathscr{O}_{Z\cap V/B}}.$$

The latter is precisely the zero-cycle defining the norm family of $Z \cap V/S$ at the same projection $\mathsf{p}_{V/S}$, so the claim is proved,

$$\mathcal{N}_{Z\cap V} = (\mathcal{N}_Z)_1\big|_V.$$

By the equivalence between smooth cycles and smooth subschemes stated in Theorem 3.2, we conclude that $Z \cap V$ and $(C \times S) \cap V$ are the same (smooth) family over S. Moreover, the closure

$$\overline{(C \times S) \cap V} \subset Z$$

equals $C \times S$, because the open subscheme $(C \times S) \cap V \subset C \times S$ is fibrewise dense (intersecting with V is only deleting a finite number of points in the special fibre). We have thus reconstructed a closed immersion $C \times S \hookrightarrow Z$, giving a well-defined S-valued point of $\mathrm{Quot}_n(\mathscr{I}_C)$. So $\iota(S)$ is onto, and thus a bijection, whenever S is a fat point. This implies ι is étale, by a simple application of the formal criterion for étale maps. The theorem follows because we already know ι is a bijective closed immersion. □

3.2 Local triviality of Hilbert–Chow

In this section we prove Theorem 2.2. The main tool used in the proof is the following local analytic version of the tubular neighborhood theorem.

Lemma 3.1 *Let S be a scheme, $j : X \to Y$ a closed immersion over S. Assume X and Y are both smooth over S, of relative dimension d and n respectively. Then j is locally analytically isomorphic to the standard linear embedding $\mathbb{C}^d \times S \to \mathbb{C}^n \times S$.*

Proof Let $x \in X$ and $y = j(x) \in Y$. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the ideal sheaf of X in Y. The relative smoothness of X, given that of Y, is characterized by the Jacobian criterion [3, Section 8.5], asserting that the short exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to j^*\Omega_{Y/S} \to \Omega_{X/S} \to 0$$

is split locally around $x \in X$. According to *loc. cit.* this is also equivalent to the following: whenever we choose local sections $t_1, \ldots, t_n$ and $g_1, \ldots, g_N$ of $\mathscr{O}_{Y,y}$ such that $\mathrm{d}\,t_1, \ldots, \mathrm{d}\,t_n$ constitute a free generating system for $\Omega_{Y/S,y}$ and $g_1, \ldots, g_N$ generate $\mathscr{I}_y$, after a suitable relabeling we may assume $g_{d+1}, \ldots, g_n$ generate $\mathscr{I}$ about y and

$$\mathrm{d}\,t_1, \ldots, \mathrm{d}\,t_d, \mathrm{d}\,g_{d+1}, \ldots, \mathrm{d}\,g_n$$

generate $\Omega_{Y/S}$ locally around y. In particular, $f_i = t_i \circ j$, for $i = 1, \ldots, d$, define a local system of parameters at x. By this choice of local basis for $\Omega_{Y/S}$ around y, we can find open neighborhoods $x \in U \subset X$ and $y \in V \subset Y$ fitting in a commutative diagram

$$\begin{array}{ccc} U & \xhookrightarrow{\ j\ } & V \\ {\scriptstyle \text{ét}}\downarrow & & \downarrow{\scriptstyle \text{ét}} \\ \mathbb{A}^d_S & \hookrightarrow & \mathbb{A}^n_S \end{array}$$

where the vertical maps are defined by the local systems of parameters $(f_1, \dots, f_d)$ and $(t_1, \dots, t_d, g_{d+1}, \dots, g_n)$ respectively, and the lower immersion is defined by sending $t_i \mapsto f_i$ for $i = 1, \dots, d$ and $g_k \mapsto 0$. Using the analytic topology, the inverse function theorem allows us to translate the étale maps into local analytic isomorphisms, and the statement follows. □

Note that Lemma 3.1 does not hold globally. For a closed immersion $X \subset Y$ of smooth complex *projective* varieties, it is not true in general that one can find a global tubular neighborhood. The obstruction lies in $\mathrm{Ext}^1(N_{X/Y}, T_X)$.

Before the proof of Theorem 2.2, we introduce the following notation. If $Z \subset Y$ is a 1-dimensional subscheme corresponding to a point in the fibre $I_n(Y, C)$ of Hilbert–Chow, we can attach to Z its "finite part", the finite subset $F_Z \subset Z$ which is the support of the maximal zero-dimensional subsheaf of $\mathscr{O}_Z$, namely the quotient $\mathscr{I}_C / \mathscr{I}_Z$.

Proof of Theorem 2.2 By [17, Cor. 12.9] the Hilbert–Chow map is a local isomorphism around normal schemes, so we may identify an open neighborhood of the cycle of C in the Chow scheme with an open neighborhood U of $[C]$ in the Hilbert scheme $I_{1-g}(Y, \beta)$. We then consider the Hilbert–Chow map

$$\mathsf{h} = \mathsf{h}_{1-g+n} : I_{1-g+n}(Y, \beta) \to \mathrm{Chow}_1(Y, \beta)$$

and we fix a point in the fibre $[Z_0] \in I_n(Y, C)$. It is easy to reduce to the case where the finite part $F_0 = F_{Z_0} \subset Z_0$ is confined on C, that is, Z_0 has only embedded points. We need to show that the Hilbert scheme is locally analytically isomorphic to $U \times I_n(Y, C)$ about $[Z_0]$. By Lemma 3.1, the universal embedding $\mathscr{C} \subset Y \times U$, locally around the finite set of points $F_0 \subset C \subset \mathscr{C}$, is locally analytically isomorphic to the embedding of the zero section $C \times U \subset C \times U \times \mathbb{C}^2$ of the trivial rank 2 bundle. In particular we can find, in $C \times U \times \mathbb{C}^2$ and in $Y \times U$, analytic open neighborhoods V and V' of F_0, fitting in a commutative diagram

$$\begin{array}{ccccc} (C \times U) \cap V & \hookrightarrow & V & \xrightarrow{\text{open}} & C \times U \times \mathbb{C}^2 \\ \downarrow \wr & & \downarrow \wr & & \\ \mathscr{C} \cap V' & \hookrightarrow & V' & \xrightarrow{\text{open}} & Y \times U \end{array}$$

where the vertical maps are analytic isomorphisms. Now consider the open subset

$$A = \left\{ (Z, u) \in I_n(Y, C) \times U \,\middle|\, F_Z \subset V_u \right\} \subset I_n(Y, C) \times U.$$

Letting φ denote the isomorphism $V \xrightarrow{\sim} V'$, given a pair $(Z, u) \in A$ we can look at $Z' = \mathscr{C}_u \cup \varphi(F_Z)$, which is a new subscheme of Y, mapping to u under Hilbert–Chow. The association $(Z, u) \mapsto Z'$ defines an isomorphism between A and the open subset $B \subset \mathsf{h}^{-1}(U)$ parametrizing subschemes $Z' \subset Y$ such that $F_{Z'}$ is contained in V'_u, where u is the image of $[Z']$ under Hilbert–Chow. Note that $[Z_0] \in B$ corresponds to $(Z_0, C) \in A$ under this isomorphism. The theorem is proved. □

4 The DT theory of an Abel–Jacobi curve

In this section we fix a non-hyperelliptic curve C of genus 3, embedded in its Jacobian

$$Y = (\mathrm{Jac}\, C, \Theta)$$

via an Abel–Jacobi map. We let $\beta = [C] \in H_2(Y, \mathbb{Z})$ be the corresponding curve class. For $n \geq 0$, we let

$$\mathcal{H}_C^n \subset I_{n-2}(Y, \beta)$$

be the component of the Hilbert scheme parametrizing subschemes $Z \subset Y$ whose fundamental cycle is algebraically equivalent to $[C]$.

Let $-1 : Y \to Y$ be the automorphism $y \mapsto -y$, and let $-C$ denote the image of C. As C is non-hyperelliptic, the cycle of C is not algebraically equivalent to the cycle of $-C$ [6]. The Hilbert scheme $I_{n-2}(Y, \beta)$ consists of two connected components, which are interchanged by -1. Moreover, the Abel–Jacobi embedding $C \subset Y$ has unobstructed deformations, and there is an isomorphism $Y \xrightarrow{\sim} \mathcal{H}_C^0$ given by translations [8].

Example 4.1 As remarked in [7, Example 2.3], the morphism

$$\mathcal{H}_C^1 \to \mathcal{H}_C^0 \times Y$$

sending $T_x(C) \cup y \mapsto (T_x(C), y)$, where T_x denotes translation by x, is the Albanese map. It can be easily checked that $\mathcal{H}_C^1$ is isomorphic to the blow-up

$$\mathrm{Bl}_{\mathcal{U}}(\mathcal{H}_C^0 \times Y),$$

where $\mathcal{U}$ is the universal family. In particular, $\mathcal{H}_C^1$ is smooth of dimension 6.

The quotient of the Hilbert scheme by the translation action of Y gives a Deligne–Mumford stack $I_m(Y, \beta)/Y$. In fact, since the Y-action is free, this is an algebraic space. The *reduced* Donaldson–Thomas invariants

$$\mathsf{DT}_{m,\beta}^Y = \int_{I_m(Y,\beta)/Y} \nu \,\mathrm{d}\chi \in \mathbb{Q}$$

were introduced in [5] for arbitrary abelian threefold. We consider their generating function

$$\mathsf{DT}_\beta(p) = \sum_{m \in \mathbb{Z}} \mathsf{DT}_{m,\beta}^Y p^m.$$

We state the following result as a corollary of Theorem 2.1.

Corollary 4.1 *Let $C \subset Y$ be non-hyperelliptic, embedded in class β. Then*

$$\mathsf{DT}_\beta(p) = 2p^{-2}(1+p)^4.$$

Proof As the Hilbert–Chow morphism is an isomorphism around normal schemes, we have an isomorphism

$$I_{-2}(Y, \beta) \xrightarrow{\sim} \mathrm{Chow}_1(Y, \beta).$$

On the other hand, the Hilbert scheme is the disjoint union of two copies of $\mathcal{H}_C^0$, where $\mathcal{H}_C^0 \cong Y$ because C is not hyperelliptic. Focusing on the component parametrizing translates of C, the Hilbert–Chow morphism $\mathcal{H}_C^n \to \mathcal{H}_C^0$ induces an isomorphism

$$Y \times \mathrm{Quot}_n(\mathscr{I}_C) \xrightarrow{\sim} \mathcal{H}_C^n$$

by Theorem 2.1. This shows that the quotient space $\mathcal{H}_C^n/Y$ is isomorphic to the Quot *scheme* $\mathrm{Quot}_n(\mathscr{I}_C)$. Keeping into account the second component of $I_{n-2}(Y, \beta)$, still isomorphic to $\mathcal{H}_C^n$, we find

$$\mathsf{DT}_{n-2,\beta}^Y = 2 \cdot \widetilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C)),$$

where $\widetilde{\chi}$ denotes the Behrend weighted Euler characteristic. Then

$$\begin{aligned}\mathsf{DT}_\beta(p) &= \sum_{n\geq 0} \mathsf{DT}^Y_{n-2,\beta} p^{n-2} = 2p^{-2}\sum_{n\geq 0}\widetilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C))p^n \\ &= 2p^{-2}(1+p)^4,\end{aligned}$$

where the last equality follows from [16, Prop. 5.1]. □

If one considers homology classes of type $(1, 1, d)$ for all $d \geq 0$, on an arbitrary abelian threefold Y, one has the formula

$$\sum_{d\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{DT}^Y_{m,(1,1,d)}(-p)^m q^d = -K(p,q)^2, \tag{4.1}$$

where K is the Jacobi theta function

$$K(p,q) = (p^{1/2} - p^{-1/2})\prod_{m\geq 1}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

Relation (4.1) was conjectured in [5] and proved in [11,13]. Corollary 4.1 confirms the coefficient of q via Quot schemes, when Y is the Jacobian of a general curve. Indeed, in this case the Abel–Jacobi class is of type $(1, 1, 1)$.

The *local* DT theory of a general Abel–Jacobi curve C of genus 3 is determined as follows. Using again the isomorphism $Y \cong \mathcal{H}^0_C$, we can compute the BPS number

$$n_{3,C} = \nu(\mathscr{I}_C) = -1,$$

thus the DT/PT correspondence at C (Theorem 1.1) yields

$$\mathsf{DT}_C(q) = \mathsf{PT}_C(q) = -q^{-2}(1+q)^4.$$

In other words, the global theory is related to the local one by

$$\mathsf{DT}_\beta(q) = -2\cdot\mathsf{DT}_C(q).$$

Acknowledgements Open access funding provided by Max Planck Society. I would like to thank my advisors Martin G. Gulbrandsen and Lars H. Halle, and Letterio Gatto, for many valuable discussions. Thanks to David Rydh for sharing many useful comments and a key example that was useful to understand the norm family. Finally, I wish to thank Richard Thomas for patiently sharing his insights and commenting on a previous draft of this work.

References

1. Behrend, K.: Donaldson–Thomas type invariants via microlocal geometry. Ann. Math. **2**(170), 1307–1338 (2009)
2. Behrend, K., Fantechi, B.: Symmetric obstruction theories and Hilbert schemes of points on threefolds. Algebra Number Theory **2**, 313–345 (2008)
3. Bosch, S.: Algebraic Geometry and Commutative Algebra. Springer, London (2013)
4. Bridgeland, T.: Hall algebras and curve counting invariants. J. Am. Math. Soc. **24**(4), 969–998 (2011)
5. Bryan, J., Oberdieck, G., Pandharipande, R., Yin, Q.: Curve counting on Abelian surfaces and threefolds. Algebraic. Geom. (2016). arXiv:1506.00841v3 (to appear)

6. Ceresa, G.: C is not algebraically equivalent to C^- in its Jacobian. Ann. Math. (2) **117**(2), 285–291 (1983)
7. Gulbrandsen, M.G.: Counting sheaves on Calabi–Yau and abelian threefolds. In: Laza, R., Schütt, M., Yui, N. (eds.) Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds. Fields Inst. Commun., vol. 67, pp. 535–548. Springer, New York (2013)
8. Lange, H., Sernesi, E.: On the Hilbert scheme of a Prym variety. Ann. Mat. Pura Appl. (4) **183**(3), 375–386 (2004)
9. Levine, M., Pandharipande, R.: Algebraic cobordism revisited. Invent. Math. **176**(1), 63–130 (2009)
10. Li, J.: Zero dimensional Donaldson–Thomas invariants of threefolds. Geom. Topol. **10**, 2117–2171 (2006)
11. Oberdieck, G.: Gromov–Witten theory of K3 $\times$ $\mathbb{P}^1$ and quasi-Jacobi forms. Int. Math. Res. Not. arXiv:1605.05238v1 (2016) (to appear)
12. Oberdieck, G.: On reduced stable pair invariants. Math. Z. arXiv:1605.04631 (2016) (to appear)
13. Oberdieck, G., Shen, J.: Curve counting on elliptic Calabi–Yau threefolds via derived categories. J. Eur. Math. Soc. (2016). arXiv:1608.07073v2 (to appear)
14. Pandharipande, R., Thomas, R.P.: Curve counting via stable pairs in the derived category. Invent. Math. **178**(2), 407–447 (2009)
15. Pandharipande, R., Thomas, R.P.: Stable pairs and BPS invariants. J. Am. Math. Soc. **23**(1), 267–297 (2010)
16. Ricolfi, A.: Local contributions to Donaldson–Thomas invariants. Int. Math. Res. Not. https://doi.org/10.1093/imrn/rnx046 (to appear)
17. Rydh, D.: Families of cycles and the Chow scheme. Ph.D. thesis, KTH, Stockholm (2008)
18. Thomas, R.P.: A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on K3 fibrations. J. Differ. Geom. **54**(2), 367–438 (2000)
19. Toda, Y.: Curve counting theories via stable objects. I: DT/PT correspondence. J. Am. Math. Soc. **23**(4), 1119–1157 (2010)