Differentiability in Measure of the Flow Associated with a Nearly Incompressible BV Vector Field

DIFFERENTIABILITY IN MEASURE OF THE FLOW ASSOCIATED TO A NEARLY INCOMPRESSIBLE BV VECTOR FIELD

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ABSTRACT. We study the regularity of the flow $\boldsymbol{X}(t,y)$ which represents (in the sense of Smirnov or as regular Lagrangian flow of Ambrosio) a solution $\rho \in L^\infty(\mathbb{R}^{d+1})$ of the transport PDE

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{b}) = 0,$$

with $\boldsymbol{b} \in L^1_t \mathrm{BV}_x$. We prove that $\boldsymbol{X}$ is differentiable in measure in the sense of Ambrosio-Malý, i.e.

$$\frac{\boldsymbol{X}(t,y+rz) - \boldsymbol{X}(t,y)}{r} \underset{r\to 0}{\to} W(t,y)z \quad \text{in measure},$$

where derivative $W(t,y)$ is a BV function satisfying the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t,y) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)} W(t-,y),$$

where $(D\boldsymbol{b})_y(\mathrm{d}t)$ is the disintegration of the measure $\int D\boldsymbol{b}(t,\cdot)\,\mathrm{d}t$ with respect to the partition given by the trajectories $\boldsymbol{X}(t,y)$ and the Jacobian $J(t,y)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t,y) = (\operatorname{div} \boldsymbol{b})_y(\mathrm{d}t) = \operatorname{Tr}(D\boldsymbol{b})_y(\mathrm{d}t).$$

The proof of this regularity result is based on the theory of Lagrangian representations and proper sets introduced by Bianchini and Bonicatto (2019), on the construction of explicit approximate tubular neighborhoods of trajectories, and on estimates that take into account the local structure of the derivative of a BV vector field.

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1. Introduction

We consider a vector field $\boldsymbol{b} : \mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}^d$ of class $L^1_t \,\mathrm{BV}_x$, and a solution $\rho \in C([0,T], L^\infty_{\mathrm{w}}(\mathbb{R}^d))$ to the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{b}) = 0, \qquad (t,x) \in (0,T)\times\mathbb{R}^d. \tag{1.1}$$

We assume that $\boldsymbol{b}$ and ρ are compactly supported. From the results of [16], it follows that ρ has a unique representation is terms of characteristics, i.e. absolutely continuous solutions to the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \boldsymbol{b}(t,\gamma(t)), \quad t \in (0,T).$$

More precisely, there exists a unique flow $\boldsymbol{X} : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$, defined for $\rho(0,\cdot)\mathcal{L}^d$-a.e. $y \in \mathbb{R}^d$, such that

$$\rho(t,\cdot) = \boldsymbol{X}(t,\cdot)_\sharp(\rho(0,\cdot)\mathcal{L}^d),$$

which means that, for every test function $\varphi \in C^\infty_c((0,T)\times\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(t,x)\rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(t,\boldsymbol{X}(t,y))\rho(0,y)\,\mathrm{d}y.$$

For the precise statement, see Theorem 3.5 of Section 3.2. The appropriate notion of flow for ODEs driven by rough (non-Lipschitz continuous) vector fields, introduced in the seminal papers [40, 8], is the one of *regular Lagrangian flow*, which consists of a measurable selection of characteristics such that $\boldsymbol{X}(t,\cdot)_\sharp\mathcal{L}^d \le C\mathcal{L}^d$ holds (see [9] for further information).

The main result of this paper is the *differentiability in measure* of the flow $\boldsymbol{X}$ (in the sense of Ambrosio-Malý, see [13]). Let $(D\boldsymbol{b})_y$ be the rescaled conditional probabilities associated with the disintegration of $D\boldsymbol{b}$ along the trajectories of $\boldsymbol{X}$: i.e., if

$$\mathcal{F} = \bigcup_{y\in F} X((0,T),y),$$

where F is a σ-compact set where $\rho(0,\cdot)$ is concentrated, then (up to a negligible $\rho(0,\cdot)\mathcal{L}^d$-set)

$$D\boldsymbol{b}\llcorner_{\mathcal{F}} = \int_F (D\boldsymbol{b})_y(\mathrm{d}t)\mathcal{L}^d(\mathrm{d}y).$$

Equivalently for every test function $\varphi \in C_c(\mathbb{R}^+\times\mathbb{R}^d)$

$$\int_{\mathcal{F}} \varphi D\boldsymbol{b} = \int_F \Big[\int_{\mathbb{R}^+} \varphi(t,X(t,y))(D\boldsymbol{b})_y(\mathrm{d}t)\Big]\mathcal{L}^d(\mathrm{d}y).$$

Similarly, for the divergence $\operatorname{div}\boldsymbol{b}$, we can write

$$\operatorname{div}\boldsymbol{b}\llcorner_{\mathcal{F}} = \int_F (\operatorname{div}\boldsymbol{b})_y(\mathrm{d}t)\mathcal{L}^d(\mathrm{d}y), \quad (\operatorname{div}\boldsymbol{b})_y = \operatorname{Tr}(D\boldsymbol{b})_y.$$

Our main theorem is as follows.

Theorem 1.1 (Differentiability in measure of the flow associated to a BV vector field)**.** *The flow $\boldsymbol{X} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is differentiable in measure at any time $T > 0$: i.e., for every $\varepsilon > 0$, we have*

$$\lim_{r\to 0} \mathcal{L}^{2d}\Big(\Big\{(y,z) \in \mathbb{R}^d \times B^d_1(0) : \Big|\frac{\boldsymbol{X}(T,y+rz) - \boldsymbol{X}(T,y)}{r} - W(T,y)\cdot z\Big| > \varepsilon\Big\}\Big) = 0, \tag{1.2}$$

for some matrix valued function $W(T,y)$. Moreover, the matrix $W(t,y)$ satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t,y) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)}W(t-,y), \quad W(0,y) = y, \tag{1.3}$$

and the Jacobian $J(t,y)$ satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}J(t,y) = (\operatorname{div}\boldsymbol{b})_y = \operatorname{Tr}(D\boldsymbol{b})_y(\mathrm{d}t), \quad J(0,y) = 1. \tag{1.4}$$

In the statement $W(t-,y)$, $J(t-,y)$ are the left limits of $W(\cdot,y)$, $J(\cdot,y)$, whose existence follows from the fact that they solve their respective ODE with measure r.h.s. and then are BV functions of time; we also notice that in general $W(\cdot,y)$, $J(\cdot,y)$ are discontinuous due to the singular measure $(D\boldsymbol{b})_y$.

We also remark that the convergence in measure expressed by formula (1.2) can be written equivalently as

$$\lim_{r\to 0}\int_{\mathbb{R}^d}\int_{B_1^d(0)} 1 \wedge \left|\frac{\boldsymbol{X}(T,y+rz)-\boldsymbol{X}(t,y)}{r} - W(T,y)\cdot z\right| \mathrm{d}y\,\mathrm{d}z = 0.$$

1.1. **Uniqueness and regularity of the flow associated to a rough velocity field.** The study of the well-posedness of transport equations driven by rough velocity fields started with the pioneering paper [40], where DiPerna and Lions introduced the notion of renormalized solution and proved existence and uniqueness for (1.1) in the case of Sobolev $W^{1,p}$ vector fields (with $p \in [1,\infty]$) with bounded divergence (or divergence in a suitable L^p space). Ambrosio extended the theory to BV vector fields with bounded divergence in [8] (see also [31, 48]). More recently, Bianchini and Bonicatto proved a uniqueness result in the more general case of nearly incompressible BV vector fields (see [16]), obtaining, as a consequence, a positive answer to Bressan's compactness conjecture (see[26]). A locally integrable vector field is called *nearly incompressible* if there exists a solution $C^{-1} \le \rho(t,x) \le C$ for $\mathcal{L}^{d-1}$-a.e. $(t,x) \in (0,T)\times\mathbb{R}^d$ to the continuity equation (1.1); such assumption is implied by the stronger condition $\operatorname{div}\boldsymbol{b} \in L^\infty$. We refer the reader to [9, 10, 37] and the references therein for a more comprehensive overview of this area of research.

In case $\boldsymbol{b} \in L^1_t W^{1,1}_x$ and divergence-free (plus some growth assumptions), in [46], Le Bris and Lions proved that, if $\boldsymbol{X}(t,y)$ is the unique regular Lagrangian flow generated by $\boldsymbol{b}$, then there exists a limit for the incremental ratio

$$\frac{\boldsymbol{X}(t,y+\varepsilon r)-\boldsymbol{X}(t,y)}{\varepsilon} \underset{\varepsilon\to 0}{\to} W(t,y,r) \quad \text{in measure},$$

and $W(t,y,r)$ is a renormalized solution to

$$\partial_t W(t,y,r) = \nabla_y \boldsymbol{b}(\boldsymbol{X}(t,y))W(t,y,r), \quad W(0,y,r) = r, \quad \dot{\boldsymbol{X}} = \boldsymbol{b}(t,\boldsymbol{X}),$$

or, equivalently, any renormalized solution to

$$\partial_t\varphi(t,x,w) + \boldsymbol{b}(t,x)\cdot\nabla_x\varphi(t,x,w) + (\nabla\boldsymbol{b}(t,x)\cdot w)\cdot\nabla_w\varphi(t,x,w) = 0$$

is given by

$$\varphi(t,\boldsymbol{X}(t,y),W(t,y,r)) = \varphi(0,y,r).$$

In [13], Ambrosio and Malý proved that $W(t,y,r) = W(t,y)r$, and compared this *differentiability in measure* to other notions of differentiability. As it turns out (see [13, Section 5]), this property is much weaker than *approximate differentiability* (see [11, Section 3.6]).

Approximate differentiability of regular Lagrangian flows generated by $W^{1,p}$ vector fields, with $p > 1$, was first obtained by Ambrosio, Lecumberry and Maniglia in [12]. In [33], Crippa and De Lellis improved this result by proving a quantitative estimate of Lusin-Lipschitz type for the flow generated by a $L^1_t W^{1,p}_x$ vector field with bounded divergence, with $p > 1$: for every ε, one can remove a set of measure ε and $\boldsymbol{X}(t=T)$ on the remaining set coincides with a Lipschitz continuous function having Lipschitz constant $e^{C/\varepsilon}$. Their approach is based on a priori estimates for a functional measuring a "logarithmic distance" between two flows associated to the same vector field (see also [45, 23, 44, 49, 20, 34, 53, 57, 58, 55, 54] for related results that rely on this strategy). However, as noted in [28], this approach cannot be used to prove a regularity result for the flow associated to a BV vector field.

A quantitative Lusin-Lipschitz regularity results for the flow $\boldsymbol{X}$ associated to a vector field $\boldsymbol{b}$ implies lower bounds on the mixing scale of passive scalars driven by $\boldsymbol{b}$ through the transport equation (1.1) (see [56]). In particular, extending the result by Crippa and De Lellis to the case $p = 1$ would give a positive answer to the well-known

Bressan's mixing conjecture proposed in [27] (see also [50, 51, 6, 5, 7, 35, 36, 41, 62, 47, 43, 15, 29, 30, 32, 59] for related results).

For the special case of bounded autonomous divergence-free vector fields $\boldsymbol{b} \in \mathrm{BV}(\mathbb{R}^2;\mathbb{R}^2)$ with compact support, in [22], Bonicatto and Marconi proved a Lusin-Lipschitz regularity result and showed that the Lipschitz constant grows at most linearly in time. In this setting, the analysis is facilitated by the Hamiltonian structure of the vector fields (see [3, 4, 2, 19, 17, 21]).

In the present paper, we establish the differentiability in measure for a nearly incompressible vector field $\boldsymbol{b} \in L^1_t \mathrm{BV}_x$. Our approach is based on the localization of the problem (which relies on the theory of proper sets introduced in [16]): we exploit the local structure of the vector field $\boldsymbol{b}$ to prove differentiability in measure locally; then, Theorem 1.1 is obtained by suitably combining the local estimates.

1.2. **Notations.** For an integer $d \geq 1$, the d-dimensional Euclidean real vector space is denoted by $\mathbb{R}^d$. We write the component of a d-dimensional point or vector as $x = (x_1, \dots, x_d)$; we also write $x_{\not{i},\not{j},\dots}$ to denote the point obtained by removing the coordinate component $i, j, \dots$ from x. The unit vector along the i-coordinate is $\mathbf{e}_i$.

The d-dimensional ball in $\mathbb{R}^d$ of radius r centered at x is written as $B^d_r(x)$. Given a curve $t \mapsto \gamma(t) \in \mathbb{R}^d$, we write

$$\bigcup_t \gamma(t) + B^d_R(0) = \big\{(t,x) : t \in [0,T], |x - \gamma(t)| < r\big\}.$$

The relative closure of the set A in the topological space B is denoted by $\mathrm{clos}(A, B)$; we also write $\mathrm{clos}\, A$ when the ambient topological space is clear. Similarly, the interior of a set A is written as $\mathrm{int}\, A$ or $\mathrm{int}(A, B)$. The boundary is denoted by $\mathrm{Fr}\, A$ or $\mathrm{Fr}(A, B)$ or, sometimes, by the standard notation $\partial\Omega$. We write $A \Subset B$ if $\mathrm{clos}\, A$ is a compact set contained in B.

$\mathbb{I}$ is the identity matrix, the minimum between two quantities a, b is denoted by $a \wedge b$, and the maximum by $a \vee b$.

The d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$.

If X is a set and $\mathscr{A}$ is a σ-algebra on X, we will call $(X, \mathscr{A})$ a *measure space*. A measure μ is *concentrated* on a set $C \subset X$ if $|\mu|(X \setminus C) = 0$. Let μ be a measure on $(X, \mathscr{A})$ and $A \in \mathscr{A}$. We define the *restriction* $\mu \llcorner_A$ *of* μ *to* A as the measure on $\mathscr{A}$ given by $\mu\llcorner_A(E) := \mu(A \cap E)$ for any $E \subset \mathscr{A}$.

The σ-algebra generated by open sets is called *Borel σ-algebra* and will be denoted by $\mathscr{B}(X)$. Let X, Y be two metric spaces, μ a measure on $(X, \mathscr{B}(X))$ and $f : X \to Y$ a Borel function. We define the *push-forward* of μ with respect to f as the measure on $(Y, \mathscr{B}(Y))$ given by $f_\sharp\mu(B) := \mu(f^{-1}(B))$ for all $B \in \mathscr{B}(Y)$. In particular, for a Borel map $g : Y \to \mathbb{R}$ it holds that

$$\int_Y g(y)(f_\sharp\mu)(\mathrm{d}y) = \int_X (g \circ f)(x)\mu(\mathrm{d}x).$$

The *disintegration of a measure* μ with respect to a partition $\{A_\alpha\}_\alpha$ is written as

$$\mu = \int \mu_\alpha f_\sharp\mu(\mathrm{d}\alpha),$$

where f is the partition function, i.e. $f^{-1}(\alpha) = A_\alpha$ (see [42, Section 452]).

The Lebesgue spaces $L^p(X, \mu; Y)$ are defined in the usual way; if $X = \mathbb{R}^d$ and $\mu = \mathcal{L}^d$, we just write $L^p(\mathbb{R}^d; Y)$; if, moreover, $Y = \mathbb{R}$, we write $L^p(\mathbb{R}^d)$. We use the standard notation for Sobolev spaces. We denote by $[\mathcal{M}_{loc}(X)]^m$ and by $[\mathcal{M}(X)]^m$, respectively, the space of $\mathbb{R}^m$-valued *Radon measures* and the space of $\mathbb{R}^m$-valued *finite Radon measures.* The space $[\mathcal{M}(X)]^m$ is a Banach space with the norm $\|\mu\|_{\mathcal{M}} := |\mu|(X)$, where $|\mu|$ is the *total variation* of the measure μ. In the case $m = 1$, we denote the set of *signed Radon measures*, *positive Radon measures*, and *finite Radon measures* by $\mathcal{M}(X)$, $\mathcal{M}^+(X)$, and $\mathcal{M}_b(X)$ respectively (see [11, Chapter 1]).

We say that $\boldsymbol{b} \in L^1(\Omega; \mathbb{R}^m)$ has *bounded variation* in Ω, and we write $\boldsymbol{b} \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ if $D\boldsymbol{b}$ is representable by a $\mathbb{R}^{m\times d}$-valued measure with finite total variation in Ω. Endowed with the norm $\|\boldsymbol{b}\|_{\mathrm{BV}(\Omega)} = \int_\Omega |u| \,\mathrm{d}x + |D\boldsymbol{b}|(\Omega) = \|\boldsymbol{b}\|_{L^1(\Omega)} + \|D\boldsymbol{b}\|_{\mathcal{M}(\Omega)}$, the space $\mathrm{BV}(\Omega; \mathbb{R}^m)$ is a Banach space (see [11, Chapter 3]).

Given a Banach space X, by $L^p([0,T]; X)$ we denote the Lebesgue-Bochner space of strongly measurable maps $f : [0,T] \to X$ with $\|f\|^p_{L^p([0,T];X)} := \int_0^T \|f\|^p_X \,\mathrm{d}t < \infty$. For the sake of brevity, we often write $L^p_t X_x$ to indicate $L^p([0,T]; X)$. We add the subscript loc to denote properties which holds locally.

For a vector field $\boldsymbol{b} : \mathbb{R}^{d+1} \to \mathbb{R}^d$, sometimes we also use the notation $\boldsymbol{b}(t) : \mathbb{R}^d \to \mathbb{R}^d$; moreover, for the vector field $\boldsymbol{b} \in L^1_t \mathrm{BV}_x$, we write $D\boldsymbol{b}$ to denote the measure

$$\int \varphi(t,x) D\boldsymbol{b}(\mathrm{d}t\,\mathrm{d}x) = \int \Big[\int \varphi(t,x) D\boldsymbol{b}(t,\mathrm{d}x) \Big] \mathrm{d}t,$$

while $D\boldsymbol{b}(t)$ denotes the space derivative of $\boldsymbol{b}$ at time t. Similar notations are used for $|D\boldsymbol{b}|$.

We write $f(x\pm)$ to denote the right/left limit of f in x (when such limit exists, e.g. in case $f \in \mathrm{BV}(\mathbb{R})$, see [11]).

If A is a Borel set of positive measure, we write the average integral of $f \in L^1(\mu)$ as

$$\fint_A f(x)\mu(\mathrm{d}x) = \frac{1}{\mu(A)} \int_A f(x)\mu(\mathrm{d}x).$$

We say that $\gamma : (t^-, t^+) \mapsto \mathbb{R}^d$ is a *characteristic* of the vector field $\boldsymbol{b} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ if it is an absolutely continuous function such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \boldsymbol{b}(t,\gamma(t)) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (t^-, t^+). \tag{1.5}$$

If the ODE above generates a flow, we use the notation $\boldsymbol{X}(t,s,y)$ for the solution to (1.5) with initial data y at time s. The graph of $\boldsymbol{X}$ in a time interval (s,t) is denoted by $\boldsymbol{X}((t,s),y)$, and when we restrict the curve to some open set Ω we will use the notation $\boldsymbol{X}(t,t^-(y),y)$, with $y \in \partial\Omega$ and $X(t^-(y),t^-(y),y) = y$; the exit time is $t^+(y)$. For the sets (perturbed proper sets) we are using all quantities are well defined.

If K is a compact set of initial data, we use the notation $\mathcal{K}$ to denote the union of its trajectories,

$$\mathcal{K} = \bigcup_{y \in K} \boldsymbol{X}((t^+(y), t^-(y)), y).$$

2. Structure of the paper

The proof of our main result is quite technical. In this section, we outline its structure and the reason of the technicalities. Moreover, we provide a sketch of the proof under the stronger assumption $\boldsymbol{b} \in L^1_t W^{1,1}_x$ (which makes the argument much easier) and show where the difficulties for the BV case lie.

In Section 3, we present some preliminary results that are needed in the proof of our main theorem. In Section 3.1, we collect some technical results on the existence of open sets $\Omega \subset [0,T] \times \mathbb{R}^d$ with particularly nice properties for the vector field $(1,\boldsymbol{b})$, the so-called *proper sets*, introduced in [16]. Roughly speaking, these are open sets where the problem can be meaningfully localized. Since the argument of the proof is based on the analysis of local properties of the vector field $\boldsymbol{b}$, the tool of proper sets plays a fundamental role. The main results are Lemma 3.2, which states that there are sufficiently many of them, and Theorem 3.4, which allows us to perturb them so that there are finitely many “time-flat” boundary regions where the majority of the flow of $(1,\boldsymbol{b})$ is entering or leaving. The motivation for this construction is that it is much easier to state the differentiability of the flow $\boldsymbol{X}$ when it is parameterized by its crossing point y on a flat surface; we acknowledge that it is also possible avoid it, but we decided to use perturbed proper sets since this tool has already been established in the literature (see [16]).

Section 3.2 deals with Smirnov's decomposition of $(1,\boldsymbol{b})$, which is stated in Theorem 3.5: i.e., thanks to the *superposition principle*, which has been established by Ambrosio in [8] (see also [60] in the context of a general normal 1-current and [61]), every non-negative (possibly measure-valued) solution to the PDE (1.1) can be written as a superposition of solutions obtained via propagation along the characteristics of $\boldsymbol{b}$ (such representation is also called a *Lagrangian representation*, see [16, Section 5]). Theorem 3.5 is used to construct L^∞ solutions ρ satisfying (1.1) by considering the curves $\gamma_{\mathfrak{a}}$ of the decomposition which start from 0 and arrive to T, and such that the Jacobian of the transformation $\gamma_{\mathfrak{a}}(0) \mapsto \gamma_{\mathfrak{a}}(t)$ is uniformly bounded.

In Section 3.3, we observe that our main theorem also gives the differentiability in measure of the Smirnov decomposition of $(1,\boldsymbol{b})$: by a countable partition of the set of curves $\{\gamma_a\}_{\mathfrak{a}}$ used in the Smirnov decomposition, one can find countably many L^∞-solutions $\rho_i \mathcal{L}^{d+1}$, $i \in \mathbb{N}$, of (1.1) defined for $t \in (t_i^-, t_i^+)$ such that

$$\sum_{i \in \mathbb{N}} \rho_i \geq 1 \quad \mathcal{L}^{d+1}\text{-a.e.},$$

and apply Theorem 1.1 to this set of trajectories. Finally, in this section, we also select the curves for which we address the differentiability in order to have a uniform control of the rescaled conditional probabilities $(D\boldsymbol{b})_y$

and $(\operatorname{div} \boldsymbol{b})_y$ and to have $y \mapsto \boldsymbol{X}(\cdot, y)$ continuous in C^0. The precise statement is in Proposition 3.6, which is an application of Lusin's theorem.

2.1. $\boldsymbol{b} \in L^1_t W^{1,1}_x$. We sketch the proof of differentiability in measure for the case $\boldsymbol{b} \in L^1_t W^{1,1}_x$. Under this assumption, we can directly estimate

$$\lim_{r \searrow 0} \int_{\mathbb{R}^d} \int_{B^d_1(0)} 1 \wedge \Big| \frac{\boldsymbol{X}(T, y + rz) - \boldsymbol{X}(t, y)}{r} - W(T, y) z \Big| \rho(0, y) \, \mathrm{d}z \, \mathrm{d}y,$$

where $W(t, y)$ solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t, y) = \nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y)) W(t, y). \tag{2.1}$$

Here we make use of the fact that the rescaled conditional probabilities $(D\boldsymbol{b})_y$ are given by $\nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y)) J(t, y)$ due to the change of variable $(t, x) \mapsto (t, \boldsymbol{X}(t, y))$ and Fubini's theorem. We remark that, by Fubini's theorem, we also have $\nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y)) \in L^1(0, T)$, so that the ODE (6.1) is well-defined.

Being $W(t, y)z$ a Lipschitz continuous function in z and an absolutely continuous (a.c.) function in t, we can use the following estimate for Lipschitz semigroups (see [24, Lemma 4] or [25, Theorem 2.9], applied here as in Corollary A.2): for $\rho(0, \cdot)\mathcal{L}^d$-a.e. $y \in \mathbb{R}^d$, if $t^+(y, rz) \in [0, T]$ is the exit time of the trajectory $\boldsymbol{X}(t, y + rz)$ from the set

$$\bigcup_t \boldsymbol{X}(t, y) + B^d_R(0) = \big\{(t, x) : t \in [0, T], |x - \boldsymbol{X}(t, y)| < R\big\},$$

then

$$\begin{aligned} &\big|\boldsymbol{X}(t^+(y, rz), y + rz) - \boldsymbol{X}(t^+(y, rz), y) - W(t^+(y, rz), y) rz\big| \\ &\leq e^{\int_0^T |D\boldsymbol{b}|(t, \boldsymbol{X}(t, y)) \, \mathrm{d}t} \int_0^{t^+(y, rz)} \big|\boldsymbol{b}(t, \boldsymbol{X}(t, y + rz)) - \boldsymbol{b}(t, \boldsymbol{X}(t, y)) \\ &\qquad - \nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y))(\boldsymbol{X}(t, y + rz) - \boldsymbol{X}(t, y))\big| \, \mathrm{d}t. \end{aligned} \tag{2.2}$$

This estimate follows from integrating of the infinitesimal error at time t

$$\boldsymbol{b}(t, \boldsymbol{X}(t, y + rz)) - \boldsymbol{b}(t, \boldsymbol{X}(t, y)) - \nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y))(\boldsymbol{X}(t, y + rz) - \boldsymbol{X}(t, y))$$

along the trajectory, and multiplying it by the Lipschitz constant $e^{\int_0^T |D\boldsymbol{b}|(t, \boldsymbol{X}(t, y)) \, \mathrm{d}t}$ of the semigroup generated by (2.1). Since we are considering trajectories $\{\boldsymbol{X}(\cdot, y)\}_{y \in K}$ such that

$$\int_0^T |\nabla \boldsymbol{b}|(t, \boldsymbol{X}(t, y)) \, \mathrm{d}t \leq M, \qquad \text{for all } y \in K,$$

for some fixed M (this is part of the statement of Proposition 3.6, see discussion above), we have the exponential factor in (2.2) is bounded by e^M and the Jacobian is controlled by

$$J(t, y) \in [1/\bar{C}, \bar{C}], \tag{2.3}$$

and then, integrating for all $(y, z) \in K \times B^d_1(0)$, we obtain the bound

$$\begin{aligned} &\int_K \int_{B^d_1(0)} \big|\boldsymbol{X}(t^+(y, rz), y + rz) - \boldsymbol{X}(t^+(y, rz), y) - W(t^+(y, rz), y) rz\big| \, \mathrm{d}z \, \mathrm{d}y \\ &\underset{(2.2)}{\leq} e^M \int_K \int_{B^d_1(0)} \int_0^{t^+(y, rz)} \big|\boldsymbol{b}(t, \boldsymbol{X}(t, y + rz)) - \boldsymbol{b}(t, \boldsymbol{X}(t, y)) \\ &\qquad - \nabla \boldsymbol{b}(t, \boldsymbol{X}(t, y))(\boldsymbol{X}(t, y + rz) - \boldsymbol{X}(t, y))\big| \, \mathrm{d}t \, \mathrm{d}z \, \mathrm{d}y \\ &\underset{(2.3)}{\leq} C_d \Big(\frac{R}{r}\Big)^d \bar{C}^2 e^M \int_{\mathcal{K}} \fint_{B^d_R(0)} \int_0^T \big|\boldsymbol{b}(t, x + w) - \boldsymbol{b}(t, x) - \nabla \boldsymbol{b}(t, x) w\big| \, \mathrm{d}t \, \mathrm{d}w \, \mathrm{d}x \\ &\leq C_d \Big(\frac{R}{r}\Big)^{d+1} \bar{C}^2 e^M r \int_{\mathbb{R}^d} \fint_{B^d_R(0)} \int_0^T |\nabla \boldsymbol{b}(t, x + w) - \nabla \boldsymbol{b}(t, x)| \, \mathrm{d}t \, \mathrm{d}w \, \mathrm{d}x, \end{aligned} \tag{2.4}$$

where C_d is a dimensional constant and

$$\mathcal{K} = \bigcup_{y \in K} \boldsymbol{X}([0,T], y).$$

The last integral in (2.4) converges to 0 due to the continuity of translations in L^1, and this shows that the set of trajectories starting in $B^d_r(y)$ and exiting the cylinder $\boldsymbol{X}(t,y) + B^d_{rR}(0)$ with

$$R = 2e^M r \tag{2.5}$$

can be made arbitrarily small and, for the remaining ones, the double integral converges to 0. This yields the convergence in measure.

2.2. $\boldsymbol{b} \in L^1_t \mathrm{BV}_x$**.** The argument above also highlights what is the key difficulty of the BV case: the dependence $\mathbb{R}^d \ni y \mapsto (D\boldsymbol{b})_y \in \mathcal{M}(\mathbb{R})$ is only *weakly* continuous, and then (2.4) gives only a bound in terms of the constant $|D\boldsymbol{b}|_y(0,T) \leq M$, and the last integral of (2.4) does not converge to 0. The present paper deals precisely with how to remove this difficulty.

The following diagram represents a general scheme of the proof and outlines its various components as well as the relations among them:

Section 4: the analysis of the linearized ODE (1.3) with useful estimates.

Section 6: the local differentiability for the a.c. $D^{\text{a.c.}}\boldsymbol{b}$.

Section 5: the general argument on how to prove differentiability in measure from a local approximate version of differentiability in measure.

Section 7: the local differentiability for the singular part $D^{\text{sing}}\boldsymbol{b}$.

Section 8: the construction of a suitable partition into proper sets by piecing together the estimates for the singular and a.c. parts of $D\boldsymbol{b}$.

The sections are almost independent from each other, and their arrangement in the paper could be altered. We first study the ODE (Section 4) to obtain some useful bounds on $W(t,y)$, and then present the local-to-global argument (Section 5), in order to have a clear picture of the local estimates one has to prove. As one can imagine, the most complex part of the paper is the one concerning local estimates for the singular part $D^{\text{sing}}\boldsymbol{b}$.

In the remaining part of this introduction, we present a detailed description of these core sections. According to the notations of Section 1.2, we write (t^-, t^+) for the interval of time a trajectory spends inside an open set Ω (and $(t_i^-(y), t_i^+(y))$ if the trajectory is $\boldsymbol{X}(t,y)$ and the open set is Ω_i). When we are considering a single proper set Ω, trajectories are parameterized by their entrance point y, and are considered distinct after reentering. This is in accord with the property of proper sets that the restriction of a Lagrangian representation to a proper set is still a Lagrangian representation (see [16, Section 5]).

In Section 4, we study the ODE (1.3) for the Jacobian matrix $W(t,y)$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t,y) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)}W(t-,y), \quad W(0,y)=y,$$

Since this is not the classical setting, we provide a constructive proof of the well-posedness theorem (Theorem 4.1) based on the convergence of an Euler scheme. An interesting observation (Remark 4.3) is that if we require the ODE for W to be time invertible, i.e. that $W(T-t,y)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}W(T-t,y) = -\frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t+,y)}W(T-t+,y), \quad W(0,y)=y,$$

the rank-one property of the vector field is needed (see [1]). This remark could be used in the case of $2d$-autonomous vector fields to have another proof of Alberti's rank-one theorem, because in this case the well-posedness does not require rank-one (see [4]), although clearly there are much simpler proof of rank-one property in the literature (see, e.g., [52, 39, 38]).

The core of the proof is in the next four sections: in order, first, we present the argument to prove the differentiability in measure if there exists a partition into perturbed proper sets where suitable properties are satisfied (Section 5), then these properties are proved for the a.c. part of the derivative (Section 6) and for the singular part (Section 7), and finally the partition is constructed (Section 8).

The local-to-global argument is in Section 5: we prove that the existence of a partition into (perturbed) proper sets where approximate local differentiability assumptions are satisfied implies a global result on differentiability in measure. In the beginning (page 17), the key assumptions on the partition into perturbed proper sets are stated, which can be explained as follows: apart from the smallness of a measure μ_P controlling the total error (Assumption (1)) and the fact that the trajectories considered for the differentiability are sufficiently close (Assumptions (2) and (3)), the key assumption is that there exists an approximate flow $\tilde{\boldsymbol{X}}(r,y;t,z)$ which approximates both the perturbation $\boldsymbol{X}(t,y+z)-\boldsymbol{X}(t,y)$, when the latter quantity has $\mathbb{R}^d$-norm smaller than r, and also the derivative $W(t,y)z$ (Assumptions (5) and (7)). Moreover, the approximate flow $\tilde{\boldsymbol{X}}$ has a controlled growth, as in Assumption (6). The reason why we need to introduce this approximate flow $\tilde{\boldsymbol{X}}$ is because $y \mapsto (D\boldsymbol{b})_y$ is only weakly continuous, as we explained before in Section 2.1: so we choose a flow $\tilde{\boldsymbol{X}}$ that solves an ODE for which the convergence of $\dot{\tilde{\boldsymbol{X}}}$ to $(D\boldsymbol{b})_y$ is in mass and not in the weak sense (or, equivalently, their difference in norm is small). This comparison works only at the initial and final time (as shown also in Assumption (7), where the comparison is directly between $\boldsymbol{X}(t,y+z)-\boldsymbol{X}(t,y)$ and $W(t,y)z$). There are some additional technical assumptions, in particular that the estimates are valid only after removing some trajectories (Assumption (4)), which is also the reason why we obtain only differentiability in measure (instead of approximate differentiability).

The argument to pass from these local assumptions to a global differentiability result is presented in Proposition 5.1. First, we remove all trajectories which do not satisfy the previous estimates in some of the sets Ω_i of the partition: these are controlled by the measure μ_P, which is assumed to be small (Step 1-3 of the proof). Second, we control the perturbations $\boldsymbol{X}(t,y+z)-\boldsymbol{X}(t,y)$ which do not remain close to 0 (i.e. $\boldsymbol{X}(t,y+z)$ not close to $\boldsymbol{X}(t,y)$) for all $t\in[0,T]$ (Step 5-11 of the proof): the idea here is that, in order to exit the ball $B^d_R(\boldsymbol{X}(t,y))$, a trajectory has first to growth much more of the approximate flow $\tilde{\boldsymbol{X}}(R,y;t,z)$, and a suitable choice of the initial distance r and of R yields a control on these runaway trajectories (similar to (2.5)). For the remaining ones, a suitable comparison with the linearized flow $W(t,y)z$ holds. This yields the differentiability in measure (Step 12-13).

Sections 6 and 7 show that it is possible to construct proper sets where the local estimates required at the beginning of Section 5 are satisfied. The analysis of the absolutely continuous part is roughly the same as the one sketched in Section 2.1 for the $\boldsymbol{b}\in L^1_tW^{1,1}_x$ case; as an additional error term, the mass of the singular part $D^{\mathrm{sing}}\boldsymbol{b}$ inside the proper set also appears. The analysis of the singular part is instead the core of the paper, and requires many technical estimates. The first step is to consider a small neighborhood of a Lebesgue point of the singular part of the derivative (Section 7.1). This allows us to write $D\boldsymbol{b}\simeq\bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|$ (by Alberti's rank-one theorem), and to use the latter measure to build an approximate vector field whose flow is $\tilde{\boldsymbol{X}}$. The definition of the approximate vector field $\tilde{\boldsymbol{b}}^H(r,y;t,w)$ is in Section 7.2, and its explicit expression is in formula (7.5), namely assuming $\bar{\eta}=\mathbf{e}_1$

and $\bar\xi = \bar\xi_1 \mathbf{e}_1 + \bar\xi_2 \mathbf{e}_2$,

$$\tilde{\boldsymbol{b}}^H(r,y;t,w) = \frac{\bar\xi}{\mathcal{L}^{d-1}(Q^H(r))} \begin{cases} -|D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [w_1,0] \times Q^H(r)) & \text{if } w_1 \leq 0, \\ |D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [0,w_1] \times Q^H(r)) & \text{if } w_1 > 0, \end{cases}$$

where

$$Q^H(r) = [-Hr, Hr] \times B_r^{d-2}(0).$$

The choice of H follows the ideas of [8, 16]. The parameter H needs to be sufficiently large, while $r \ll 1$ in order to be inside the neighborhood of the Lebesgue points of $D^{\mathrm{sing}}\boldsymbol{b}$. How close $\tilde{\boldsymbol{b}}^H$ is to $D\boldsymbol{b}$ is estimated in Proposition 7.1. The choice of $\tilde{\boldsymbol{b}}^H$ is based on the following considerations. First, the derivative depends essentially on first coordinate w_1, so that $\tilde{\boldsymbol{b}}^H$ depends only on the first coordinate. Secondly, the solution $\tilde{\boldsymbol{X}}^H$ to the ODE (7.11), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{X}}_1^H(r,y;t,z) = \tilde{\boldsymbol{b}}_1^H(r,y;t,\tilde{\boldsymbol{X}}_1^H(r,y;t,z)),$$

has the property that the flux across the cylinder

$$\bigcup_t \boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,z_1)] \times Q^H(r),$$

is small (see Lemma 7.5). An important consequence of the control of the flux across the boundary is that the quantity

$$\int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}_1^H(r,y;t,z_1))\,\mathrm{d}t \quad \text{is close in mass to } (D\boldsymbol{b})_y(t^-(y),t^+(y)),$$

where $(t^-(y),t^+(y))$ is the interval of time where the trajectory is inside the perturbed proper set Ω. This is the main difference in using the approximate flow $\tilde{\boldsymbol{X}}^H(r,y;t,z)$ instead of the linearized flow $W(t,y)z$. The precise estimate is in Proposition 7.7, Section 7.4.3.

The next step is to prove that the approximate vector field $\tilde{\boldsymbol{X}}^H(r,y;t,w)$ is close to the perturbation $\boldsymbol{X}(t,y+w) - \boldsymbol{X}(t,y)$. The components not along $\bar\xi$ are the easiest ones to estimate (see Lemma 7.9). The component along $\bar\eta = \mathbf{e}_1$ is analyzed in two steps. First we assume that $\bar\xi_1 = \bar\eta \cdot \bar\xi \leq 0$, i.e. the flow $\tilde{\boldsymbol{X}}^H$ is a contraction (Lemma 7.2). In this case, the analysis relies again on the estimate (2.2) and it is done in Proposition 7.3. The case $\bar\eta \cdot \bar\xi > 0$ is studied in Section 7.8. The key observation here is that the control on the Jacobian $J \in [1/\bar C, \bar C]$ implies that we can change coordinates from the initial point to the end point, so that reversing time we come back to the contractive case: the key point is Point (4), page 44. The main difficulty concerns the components along the direction of $\bar\xi$ perpendicular to $\bar\eta$ (which we choose to be $\bar\xi \cdot \mathbf{e}_2 = \xi_2$): in this case the approximate flow $\tilde{\boldsymbol{X}}_2^H$ is not Lipschitz continuous, so that we cannot use estimate (2.2). The idea is to exploit the fact that $\tilde{\boldsymbol{X}}^H(r,y;t,x)$ depends only on w_1, and we have a control on $\boldsymbol{X}_1(t,y+w) - \boldsymbol{X}_1(t,y) - \tilde{\boldsymbol{X}}_1^H(r,y;t,z)$: this allows to prove that removing a small set of trajectories we still have that $\boldsymbol{X}_2(t,y+z) - \boldsymbol{X}_2(t,y) - \tilde{\boldsymbol{X}}_2^H(r,y;t,z)$ is small (see Proposition 7.11). The final step is to show how $\tilde{\boldsymbol{X}}^H(r,y;t,z)$ is close to the $W(t,y)z$; this is analyzed in Section 7.6: first, we can replace $(D\boldsymbol{b})_y$ with $\bar\xi \otimes \bar\eta |D\boldsymbol{b}|_y$ with a controlled error; then, the explicit solution to the ODE

$$\dot{\tilde{W}}(t,y) = \bar\xi \otimes \bar\eta |D\boldsymbol{b}|_y(\mathrm{d}t) \frac{\tilde W(t-,y)}{\tilde J(t-,y)}, \quad \text{where} \quad \tilde J(t,y) = \det(\tilde W(t,y)),$$

is

$$\tilde W(t,y) = \mathbb{I} + \bar\xi \otimes \bar\eta |D\boldsymbol{b}|_y(t^-(y),t),$$

which turns out to be close to the perturbed flow $\tilde{\boldsymbol{X}}^H$. This concludes the estimates, which are collected in Sections 7.7 and 7.8.

Finally, Section 8 concerns the construction of a disjoint partition of $[0,T] \times \mathbb{R}^d$ into perturbed proper sets as required in Section 5 and is based on the analysis of the absolutely continuous part (Section 6) and the singular part (Section 7) of the derivative $D\boldsymbol{b}$. First, we cover a large portion of the singular part $D^{\mathrm{sing}}\boldsymbol{b}$ with disjoint perturbed proper sets so that the required estimates holds, and then the remaining part. This is done in Theorem 8.1 and Proposition 8.2. The proof of our main theorem is thus concluded.

In Appendix A, we give a proof of the estimate (2.2) in our setting.

3. Preliminaries and setting of the problem

In this section we collect some preliminary information on proper sets and the decomposition of a BV vector field; then we present the setting of our problem.

3.1. **Proper sets.** The analysis of open sets Ω such that $\boldsymbol{b}\llcorner_\Omega$ maintains suitable regularity properties has been carried out in [16]. In this section, we present the main definitions are results.

Definition 3.1 (Proper sets)**.** An open bounded set $\Omega \subset \mathbb{R}^{d+1}$ is called $\rho(1,\boldsymbol{b})$-*proper* if the following properties hold.

(1) $\partial\Omega$ has finite $\mathcal{H}^d$-measure and it can be written as
$$\partial\Omega = \bigcup_{i\in\mathbb{N}} U_i \cup N,$$
where N is a closed set with $\mathcal{H}^d(N) = 0$ and $\{U_i\}_{i\in\mathbb{N}}$ are countably many C^1-hypersurfaces such that the following holds: for every $(t,x) \in U_i$, there exists a ball $B_r^{d+1}(t,x)$ such that $\partial\Omega \cap B_r^{d+1}(t,x) = U_i$.

(2) If the functions $\varphi^{\delta,\pm}$ are given by
$$\varphi^{\delta,+}(t,x) := \max\Big\{1 - \frac{\operatorname{dist}((t,x),\Omega)}{\delta}, 0\Big\}, \qquad \varphi^{\delta,-}(t,x) := \min\Big\{\frac{\operatorname{dist}((t,x),\mathbb{R}^{d+1}\setminus\Omega)}{\delta}, 1\Big\}, \tag{3.1}$$
then
$$\lim_{\delta\searrow 0} |\rho(1,\boldsymbol{b})\cdot\nabla\varphi^{\delta,\pm}|\mathcal{L}^{d+1} = |\rho(1,\boldsymbol{b})\cdot\mathbf{n}|\mathcal{H}^d\llcorner_{\partial\Omega}, \qquad \text{weakly-star in } \mathcal{M}_b(\mathbb{R}^{d+1}).$$

It is possible to prove that almost all balls and cylinders
$$\mathrm{Cyl}_{t,x}^{r,L} = \big\{(\tau,y) : |\tau - t| < Lr, |y - x - \boldsymbol{b}(t,x)(\tau - t)| < r\big\}$$
are proper sets (see [16, Lemma 4.10]).

Lemma 3.2. *For every (t,x) consider the family of balls $\{B_r^{d+1}(t,x)\}_{r>0}$ and the family of cylinders $\{\mathrm{Cyl}_{t,x}^{r,L}\}_{r>0}$ with $L > 0$ fixed. Then for $\mathcal{L}^1$-a.e. $r > 0$ the ball $B_r^{d+1}(t,x)$ and the cylinder $\mathrm{Cyl}_{t,x}^{r,L}$ are proper sets.*

The finite union of proper balls and proper cylinders is proper. More generally, it can be showed that, if Ω_1, Ω_2 are proper sets with $\mathcal{H}^d\big(\operatorname{Fr}(\partial\Omega_1\cap\partial\Omega_2, \partial\Omega_1\cup\partial\Omega_2)\big) = 0$, then their union $\Omega_1\cup\Omega_2$ is a proper set (see [16, Proposition 4.11]) and their difference $\Omega_1\setminus\Omega_2$ is also a proper set. We prove the last claim in the following lemma.

Lemma 3.3. *Let Ω_1, Ω_2 be proper sets such that*
$$\mathcal{H}^d\big(\operatorname{Fr}(\partial\Omega_1\cup\partial\Omega_2, \partial\Omega_1\cap\partial\Omega_2)\big) = 0.$$
Then $\Omega_1\setminus\Omega_2$ is a proper set.

Proof of Lemma 3.3. If Ω is proper, so is $\operatorname{int}(\mathbb{R}^{d+1}\setminus\Omega)$ since the conditions to be proper are given on $\partial\Omega = \partial(\operatorname{int}(\mathbb{R}^{d+1}\setminus\Omega))$. Thus, by writing
$$\Omega_1\setminus\Omega_2 = \operatorname{int}\big(\mathbb{R}^{d+1}\setminus\big(\operatorname{int}(\mathbb{R}^{d+1}\setminus\Omega_1)\cup\Omega_2\big)\big), \tag{3.2}$$
the result follows from [16, Proposition 4.11]. □

Furthermore, it is possible consider a perturbation Ω^ε of a proper set Ω in order to have a large part of the inflow and outflow of $\rho(1,\boldsymbol{b})$ across $\partial\Omega^\varepsilon$ occurring on finitely many time-constant hyperplanes, i.e. regions of the boundary $\partial\Omega^\varepsilon$ such that their outer normal is $\mathbf{n} = (\pm1, 0)$. We shall call S_1 the union of the hyperplanes of inflow and S_2 the union of the hyperplanes of outflow. More precisely, the following theorem holds true (see [16, Theorem 4.18])

Theorem 3.4 (Perturbed proper sets)**.** *Let $\Omega \subset \mathbb{R}^{d+1}$ be a $\rho(1,\boldsymbol{b})$-proper set. For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

(1) *$\Omega \subset \Omega^\varepsilon \subset \Omega + B_\varepsilon^{d+1}(0)$;*

(2) *if*

$$\partial\Omega_1^\varepsilon = \big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (1,0) \text{ in a neigborhood of } (t,x)\big\},$$

then $\partial\Omega_1^\varepsilon$ *is made of Lebesgue points of* $\rho(1,\boldsymbol{b})$ *and*

$$\Big|\int_{\partial\Omega_1^\varepsilon} \rho\mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\boldsymbol{b})\cdot\mathbf{n}]^+\mathcal{H}^d\Big| < \varepsilon;$$

(3) *if*

$$\partial\Omega_2^\varepsilon = \big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (-1,0) \text{ in a neigborhood of } (t,x)\big\},$$

then $\partial\Omega_2^\varepsilon$ *is made of Lebesgue points of* $\rho(1,\boldsymbol{b})$ *and*

$$\Big|\int_{\partial\Omega_2^\varepsilon} \rho\mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\boldsymbol{b})\cdot\mathbf{n}]^-\mathcal{H}^d\Big| < \varepsilon.$$

3.2. **Decomposition of** BV **vector fields.** The following result summarizes [16, Main Theorem 1 pag. 18] applied to the PDE

$$\mathrm{div}_{t,x}(1,\boldsymbol{b}) = \mathrm{div}\,\boldsymbol{b}(t) = \mu,$$

$\mu \in \mathcal{M}(\mathbb{R}^{d+1})$. The validity of the assumptions for proving [16, Main Theorem 1, pag. 18] are shown in [16, Theorem 11.6 p. 128, Theorem 8.9 p. 105, Main Theorem 2 p. 18].

Theorem 3.5 (Partition via characteristics)**.** *Let* $\boldsymbol{b} \in L^1_t \mathrm{BV}_x$ *be a compactly supported vector field. Then there exists a Borel map* $\mathtt{f} : \mathbb{R}^{d+1} \to \mathfrak{A} \subset \mathbb{R}$*, named a* partition via characteristics *of* $(1,\boldsymbol{b})$*, such that*

(1) $\mathtt{f}^{-1}(\mathfrak{a})$ *is the graph of some characteristic* $\gamma_a : I_\mathfrak{a} \to \mathbb{R}^d$ *of* $\boldsymbol{b}$*, where* $I_\mathfrak{a}$ *is an open interval of* $\mathbb{R}$*;*
(2) $\mathtt{f}$ *disintegrates* $\mathcal{L}^{d+1}$*:*

$$\mathcal{L}^{d+1}\llcorner_{B^{d+1}_R(0)} = \int (\mathbb{I},\gamma_\mathfrak{a})_\sharp(w_\mathfrak{a}(\mathrm{d}t)\mathcal{L}^1(\mathrm{d}t) m(\mathrm{d}\mathfrak{a}), \quad m = \mathtt{f}_\sharp\mathcal{L}^{d+1}\llcorner_{B^{d+1}_R(0)}, \tag{3.3}$$

and $w_\mathfrak{a}\llcorner_{I_\mathfrak{a}} > 0$*;*

(3) *when* $w_\mathfrak{a}$ *is extended to* 0 *outside* $I_\mathfrak{a}$*, then it is a* BV *function,*

$$D_t w_\mathfrak{a} = \mu_\mathfrak{a}, \qquad \mu_\mathfrak{a} \in \mathcal{M}(\mathbb{R}), \tag{3.4}$$

and

$$\mathrm{div}\,\boldsymbol{b} = \int (\mathbb{I},\gamma_\mathfrak{a})_\sharp\mu_\mathfrak{a} m(\mathrm{d}\mathfrak{a}), \quad |\mathrm{div}\,\boldsymbol{b}| = \int (\mathbb{I},\gamma_a)_\sharp|\mu_\mathfrak{a}| m(\mathrm{d}\mathfrak{a}); \tag{3.5}$$

(4) *if* $\rho \in L^\infty((0,T)\times\mathbb{R}^d)$ *satisfies the PDE*

$$\mathrm{div}_{\,t,x}(\rho(1,\boldsymbol{b})) = \nu,$$

where $\nu \in \mathcal{M}(\mathbb{R}^{d+1})$*, then* $\mathtt{f}$ *is a partition via characteristics as above also for* $\rho(1,\boldsymbol{b})$ *(with the requirement* $\rho(t,\gamma_\mathfrak{a}(t))w_\mathfrak{a}\llcorner_{I_\mathfrak{a}} > 0$*), i.e. the same results as above are true replacing*

$$D_t w_\mathfrak{a} = \mu_\mathfrak{a} \quad \text{with} \quad D_t(\rho(t,\gamma_\mathfrak{a}(t), w_\mathfrak{a}(t)) = \nu_\mathfrak{a}$$

and μ*,* $\mu_\mathfrak{a}$ *with* ν*,* $\nu_\mathfrak{a}$ *in* (3.5)*.*

A possible choice of $\mathtt{f}$ is to take countably many sets $\{t = t_i\}_{i\in\mathbb{N}}$ and define $\mathtt{f}(\gamma) = \gamma(t_i)$. This choice is more suitable when one wants to construct a flux from the partition via characteristics. Indeed, with this choice, the function $w_\mathfrak{a}$ becomes naturally the Jacobian $J(t,y)$, where $\gamma(t_i) = y$ and (3.4) is the equation for the evolution of J.

A corollary of formula (3.3) is that, given a proper set, we can estimate the flux across its boundary as follows (see [16, Proposition 5.11]):

$$\rho(1,\boldsymbol{b})\cdot\mathbf{n}\mathcal{H}^d + \mathrm{div}_{t,x}\,\rho(1,\boldsymbol{b}) = \int D_t(\rho(t,\gamma_\mathfrak{a}(t))\chi_{\gamma_a^{-1}(\Omega)}(t)) m(\mathrm{d}\mathfrak{a}),$$

where $\mathbf{n}$ is the inner normal to Ω. In particular, from [16, Theorem 6.8 and Proposition 6.10], we obtain that, for $N \subset \partial\Omega$,

$$m(\{\mathfrak{a} : \mathrm{Graph}\, \gamma_a \cap N \neq \emptyset\}) \leq \int_N |\rho(1, \boldsymbol{b}) \cdot \mathbf{n}| \mathcal{H}^d, \tag{3.6}$$

i.e. the flux through N controls the measure of trajectories crossing N.

3.3. **Setting of the problem.** We consider the set of trajectories starting from $t = 0$ and arriving to $t = T$ living inside the ball of radius R_0 and such that $J(t, y) \in [1/\bar{C}, \bar{C}]$. By an elementary partition argument, the partition via characteristics of $(1, \boldsymbol{b})$ can be decomposed as a countable union of such a sets by varying $\bar{C}$ and the initial and final time (here for definiteness we have assume them to be $0, T$, respectively). We can define $\rho = 1/J$ and obtain a solution to $\mathrm{div}_{t,x}(\rho(1, \boldsymbol{b})) = 0$ which is nearly incompressible in $[0, T] \times \mathbb{R}^d$.

We denote with $\mathcal{K}_0$ a compact set made of these trajectories, i.e.

$$\mathcal{K}_0 = \bigcup_{y \in K_0} \boldsymbol{X}([0, T], y). \tag{3.7}$$

Being $y \mapsto \boldsymbol{X}(\cdot, y)$ a Borel function, the above sets are compact, and K_0 can be parameterized by the initial data, i.e. $K_0 = \mathcal{K}_0 \cap \{t = 0\}$.

Since the values of $\boldsymbol{b}$ outside $(0, T)$ are not important, we assume that $\boldsymbol{b}(t) = 0$ for $t \notin (0, T)$, and also outside the ball of radius $2R_0$. We will often write $\mathbb{R}^{d+1}$ in the estimates, even if we are working in the ball of radius $2R_0$. We disintegrate

$$D\boldsymbol{b}\llcorner_{\mathcal{K}_0}(\mathrm{d}t\, \mathrm{d}x) = \int (D\boldsymbol{b})_y(\mathrm{d}t)\mathcal{L}^d(\mathrm{d}y) + (D\boldsymbol{b}\llcorner_{\mathcal{K}_0})^r,$$

where $(D\boldsymbol{b}\llcorner_{\mathcal{K}_0})^r$ is the part of $D\boldsymbol{b}$ whose image measure is not absolutely continuous. Being the flow defined for $\mathcal{L}^d$-a.e. $y \in K_0$, we can assume that $(D\boldsymbol{b}\llcorner_{\mathcal{K}_0})^r = 0$ by removing a negligible set of trajectories.

Since

$$\|D\boldsymbol{b}\| = \int_0^T |D\boldsymbol{b}|(B^d_{R_0}(0))\mathrm{d}t < \infty,$$

then, for every M,

$$M\mathcal{L}^d(\{y \in B^d_{R_0}(0) : |(D\boldsymbol{b})_y|([0, T]) > M\}) < \|D\boldsymbol{b}\|,$$

by Chebyshev's inequality; so, if

$$\varepsilon_M = \frac{\|D\boldsymbol{b}\|}{M}, \tag{3.8}$$

then there exists a compact set $K_1 \subset K_0$ of trajectories such that

$$\mathcal{L}^d(K_0 \setminus K_1) < \varepsilon_M \quad \text{and} \quad |D\boldsymbol{b}|_y([0, T]) \leq M \quad \forall y \in K_1.$$

We also define $\mathcal{K}_1 \subset \mathcal{K}_0$ as the union of the graphs of the trajectories starting in K_1, as in (3.7).

We observe that, by the monotonicity properties of measures, if $\mathcal{K}'$ is another compact set of trajectories such that $\mathcal{K}' \cap \mathcal{K}_1 = \emptyset$, then

$$\lim_{r \to 0} |D\boldsymbol{b}\llcorner_{\mathcal{K}'}|(\mathcal{K}_1 + B^d_r(0)) = 0.$$

Summing up, we are in the following situation.

Proposition 3.6. *We can restrict to a compact set of trajectories $K_1 \subset K_0$ such that*

(1) $\mathcal{L}^d(K_0 \setminus K_1) < \varepsilon_M$*;*
(2) $\boldsymbol{X}\llcorner_{K_1}$ *is continuous;*
(3) *we have*

$$D\boldsymbol{b}\llcorner_{\mathcal{K}_1}(\mathrm{d}t\, \mathrm{d}x) = \int_{K_1} (D\boldsymbol{b})_y(\mathrm{d}t)\mathcal{L}^d(\mathrm{d}y), \quad |(D\boldsymbol{b})_y|([0, T]) = |(D\boldsymbol{b})_y((0, T))| \leq M, \tag{3.9}$$

where $\mathcal{K}_1 = \boldsymbol{X}([0, T], K_1)$*;*

(4) *the Jacobian $J(t,y)$ satisfies*

$$J(t,y) \in \Big(\frac{1}{\bar{C}}, \bar{C}\Big). \tag{3.10}$$

for some constant $\bar{C}$.

In Point (3) above we have observed that $D\boldsymbol{b}\llcorner_{\mathcal{K}_1} = D\boldsymbol{b}\llcorner_{\mathcal{K}_1\cap(0,T)}$ because $D\boldsymbol{b}(\{t \in N\}) = 0$ for every $\mathcal{L}^1$-negligible set $N \subset \mathbb{R}^1$, which implies that $(D\boldsymbol{b})_y\llcorner_{[0,T]} = (D\boldsymbol{b})_y\llcorner_{(0,T)}$ for $\mathcal{L}^d$-a.e. y.

4. The ODE satisfied by the derivative of the flow

We consider the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t,y) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)} W(t-,y), \qquad W(0,y) = y, \tag{4.1}$$

where the Jacobian $J(t,y)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t,y) = (\operatorname{div} \boldsymbol{b})_y(\mathrm{d}t) = \operatorname{Tr}(D\boldsymbol{b})_y(\mathrm{d}t), \qquad J(0,y) = 1, \tag{4.2}$$

and, by assumption,

$$J(t,y) \in \Big(\frac{1}{\bar{C}}, \bar{C}\Big). \tag{4.3}$$

In this section the variable y is a fixed parameter.

Theorem 4.1. *Then there exists a unique left continuous solution $t \mapsto W(t,y)$ to the Cauchy problem* (4.1) *such that*

$$|W(t,y)| \leq e^{\bar{C}|D\boldsymbol{b}|([0,t))}, \quad \text{Tot.Var.}(W(\cdot,y),[0,T]) \leq \bar{C}|D\boldsymbol{b}|([0,t))e^{\bar{C}|D\boldsymbol{b}|([0,t))}.$$

Moreover, it is the limit of every sequence of Euler scheme solutions $W^{\delta t}(t,y)$ corresponding to a partition $\{[t_i,t_{i+1})\}$ of $[0,T)$ as $\delta t = \max_i |t_{i+1} - t_i| \to 0$.

Proof. For the sake of brevity, we use the notation

$$\widetilde{(D\boldsymbol{b})_y}(\mathrm{d}t) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)},$$

so that the ODE is

$$\begin{cases} \dot{W}(t,y) = \widetilde{(D\boldsymbol{b})_y}(\mathrm{d}t) W(t-,y), \\ W(0,y) = y. \end{cases}$$

By the assumptions on the disintegration and near incompressibility, we have

$$|\widetilde{(D\boldsymbol{b})_y}|((0,T)) \leq \bar{C}M. \tag{4.4}$$

As a first step, we prove existence of a solution to the ODE by means of an Euler scheme (see [14]). Secondly, we prove uniqueness by a Gronwall-type argument.

Step 1. Construction of a solution. The construction of a solution is done by the Euler method: for every partition of $[0,T)$ made of intervals $\{[t_i,t_{i+1})\}_{0\leq i\leq I}$, such that $t_0 = 0$, $t_I = T$, and $\delta t = \max_i\{t_i - t_{i-1}\}$, we define the approximate solution $W^{\delta t}$ as follows:

$$\begin{aligned} W^{\delta t}(t,y) &= \overset{\curvearrowright}{\prod_{t_i \leq t}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_i-,y)}\Big) \\ &= \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_i-,y)}\Big) \cdot \ldots \cdot \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_1,t_2))}{J(t_2-,y)}\Big) \cdot \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([0,t_1))}{1}\Big), \end{aligned} \tag{4.5}$$

where we have used the fact that $J(0,y) = 1$ and, with an abuse of notation, we denote by δt the partition with point $\{t_i\}_i$; later we will also denote a sequence of functions depending on the partitions $\{t_i^n\}_i$ with the apex δt_n.

The function $W^{\delta t}$ is piece-wise constant, right continuous, and its jump at each t_i is given by

$$W^{\delta t}(t_i+,y) = W^{\delta t}(t_i,y) = W^{\delta t}(t_i-,y) + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_i-,y)} W^{\delta t}(t_i-,y).$$

We have that $W^{\delta t}$ is uniformly bounded: indeed

$$\begin{aligned} |W^{\delta t}(t,y)| &\underset{(4.5)}{\leq} \overset{\curvearrowright}{\prod_{t_i \leq t}} \Big| \mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_i-,y)} \Big| \underset{(4.3)}{\leq} \prod_{i=0}^{I} \big(1 + \bar{C}|(D\boldsymbol{b})_y([t_{i-1},t_i))|\big) \\ &\leq e^{\sum_{i=0}^{I} \bar{C}|(D\boldsymbol{b})_y([t_{i-1},t_i))|} \underset{(3.9)}{\leq} e^{\bar{C}M}. \end{aligned} \tag{4.6}$$

Moreover its total variation is controlled by

$$\begin{aligned} \sum_i |W^{\delta t}(t_i,y) - W^{\delta t}(t_{i-1},y)| &\underset{(4.5)}{=} \sum_i \Big| \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_i-,y)} W^{\delta t}(t_{i-1},y) \Big| \\ &\underset{\substack{(4.6),\\(4.3)}}{\leq} \bar{C}e^{\bar{C}M} \sum_i |(D\boldsymbol{b})_y([t_{i-1},t_i))| \underset{(3.9)}{\leq} \bar{C}Me^{\bar{C}M}. \end{aligned}$$

Therefore, by Helly's Compactness Theorem (see [25, Theorem 2.3]), for every sequence of intervals such that $\delta t \to 0$ there is a subsequence δt_n such that $W^{\delta t_n}(t,y) \to W(t,y)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, and the function $W(\cdot,y) \in \mathrm{BV}((0,T),\mathbb{R}^{d\times d})$.

By the estimate on the total variation, for every $t < \tau$ we have

$$\begin{aligned} &|W^{\delta t}(\tau,y) - W^{\delta t}(t,y)| \\ &= \Big| \overset{\curvearrowright}{\prod_{t_i \leq \tau}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) - \overset{\curvearrowright}{\prod_{t_i \leq t}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) \Big| \\ &\leq \Big| \overset{\curvearrowright}{\prod_{t_i \leq t}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) \Big| \Big| \overset{\curvearrowright}{\prod_{t < t_i \leq \tau}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) - \mathbb{I} \Big| \\ &= \Big| \overset{\curvearrowright}{\prod_{t_i \leq t}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) \Big| \Big| \sum_{t < t_i \leq \tau} \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big(\overset{\curvearrowright}{\prod_{t_j > t}^{t_{i-1}}} \Big(\mathbb{I} + \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big) \Big) \Big| \\ &\leq \prod_{t_i \leq \tau} \Big(\mathbb{I} + \Big| \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big| \Big) \Big(\sum_{t < t_i \leq \tau} \Big| \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1}-,y)} \Big| \Big) \\ &\leq \bar{C}e^{\bar{C}M} |(D\boldsymbol{b})_y|((t-\delta t,\tau)), \end{aligned} \tag{4.7}$$

where we have used the estimate

$$\overset{\curvearrowright}{\prod_i} (\mathbb{I} + A_i) - \mathbb{I} = \sum_i A_i \overset{\curvearrowright}{\prod_{j<i}} (\mathbb{I} + A_j) \tag{4.8}$$

in the third line, and the Jacobian bound (4.3) with the fact that $\max_i\{t_i - t_{i-1}\} \leq \delta$ in the last line. In particular, if $W^{\delta t_n}(t,y) \to W(t,y)$ for a fixed t, then

$$\limsup_{\delta t_n \to 0} |W^{\delta t_n}(\tau,y) - W^{\delta t_n}(t,y)| = \limsup_{\delta t_n \to 0} |W^{\delta t_n}(\tau,y) - W(t,y)| \underset{(4.7)}{\leq} \bar{C}e^{\bar{C}M} |(D\boldsymbol{b})_y|([t,\tau)). \tag{4.9}$$

Being the set of times for which $W^{\delta t_n}(t,y)$ is convergent dense in $[0,T]$, it follows by letting $t \nearrow \tau$ that the limit $W^{\delta t_n}$ exists for every t and moreover $t \mapsto W(t,y)$ is left continuous by (4.9): clearly $W(0,y) = \mathbb{I}$. A similar result can be stated for $J(t,y)$: defining

$$J^{\delta t_n}(t,y) = J(t_{i-1}-,y) \quad \text{if } t \in [t_{i-1},t_i),$$

then we have

$$J^{\delta t_n}(t,y) \underset{\delta t_n \to 0}{\to} J(t-,y). \tag{4.10}$$

In this case, the proof is elementary.

Hence we can pass to the limit to the approximate ODE for $W^{\delta t_n}$: its equation is

$$\dot{W}^{\delta t_n} = \Big(\sum_{t_i} \widetilde{(D\boldsymbol{b})_y}([t_{i-1},t_i))\delta_{t_i}(\mathrm{d}t)\Big) W^{\delta t_n}(t-,y) = \widetilde{(D\boldsymbol{b})_y}^{\delta t_n}(\mathrm{d}t) W^{\delta t_n}(t-,y), \tag{4.11}$$

where, as in the previous equation, the matrix valued measure $\widetilde{(D\boldsymbol{b})_y}^{\delta t_n}(\mathrm{d}t)$ is defined as

$$\widetilde{(D\boldsymbol{b})_y}^{\delta t_n}(\mathrm{d}t) = \sum_{t_i} \frac{(D\boldsymbol{b})_y([t_{i-1},t_i))}{J(t_{i-1},y)} \delta_{t_i}(\mathrm{d}t).$$

We write the ODE (4.11) in integral form:

$$\begin{aligned} W^{\delta t_n}(t,y) &= \mathbb{I} + \int_{[0,t)} \widetilde{(D\boldsymbol{b})_y}^{\delta t_n}(\mathrm{d}t) W^{\delta t_n}(t-,y) \\ &= \mathbb{I} + \int_{[0,t_i<t)} (D\boldsymbol{b})_y(\mathrm{d}t) \frac{W^{\delta t_n}(t,y)}{J^{\delta t_n}(t,y)} \\ &= \mathbb{I} + \int_{[0,t)} (D\boldsymbol{b})_y(\mathrm{d}t) \frac{W^{\delta t_n}(t,y)}{J^{\delta t_n}(t,y)} - (D\boldsymbol{b})_y([t_i,t)) \frac{W^{\delta t_n}(t_i,y)}{J^{\delta t_n}(t,y)}, \end{aligned} \tag{4.12}$$

where we observed that $W^{\delta t_n}(t,y)$ is equal to $W^{\delta t_n}(t_{i-1},y)$ in every interval $[t_{i-1},t_i)$ by (4.5) so that

$$\widetilde{(D\boldsymbol{b})_y}([t_{i-1},t_i)) W^{\delta t_n}(t_i-,y) = (D\boldsymbol{b})_y([t_{i-1},t_i)) \frac{W^{\delta t_n}(t_{i-1},y)}{J^{\delta t_n}(t_{i-1},y)} = \int_{[t_{i-1},t_i)} (D\boldsymbol{b})_y(\mathrm{d}t) \frac{W^{\delta t_n}(t_{i-1},y)}{J^{\delta t_n}(t_{i-1},y)}, \tag{4.13}$$

and we have to leave out the final interval for which $t \in [t_{i-1},t_i)$.

From the pointwise convergence, we obtain that

$$\int_{[0,t)} (D\boldsymbol{b})_y(\mathrm{d}t) \frac{W^{\delta t_n}(t,y)}{J^{\delta t_n}(t,y)} \to \int_{[0,t)} (D\boldsymbol{b})_y(\mathrm{d}t) \frac{W(t,y)}{J(t,y)},$$

while, from $\delta t_n \to 0$ and the boundedness of $W(t,y)/J(t,y)$,

$$(D\boldsymbol{b})_y([t_i,t)) \frac{W^{\delta t_n}(t_i,y)}{J^{\delta t_n}(t_i,y)} \to 0.$$

Hence for every $\delta t \to 0$ there exists a subsequence converging to a solution.

Step 2. Uniqueness of the solution. The uniqueness of the solution can be proved by observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}|W(t,y)| \leq |\widetilde{(D\boldsymbol{b})_y}|(\mathrm{d}t)|W(t-,y)|,$$

which gives

$$\begin{aligned} D_t \log|W(t,y)| &= \frac{1}{|W(t,y)|} D_t^{\mathrm{cont}}|W(t,y)| + \sum_i \log\Big(\frac{|W(\tau_i+,y)|}{|W(\tau_i-,y)|}\Big)\delta_{\tau_i}(\mathrm{d}t) \\ &\leq |\widetilde{(D\boldsymbol{b})_y}^{\mathrm{cont}}|(\mathrm{d}t) + \sum_i \Big(\frac{|W(\tau_i+,y)|}{|W(\tau_i-,y)|} - 1\Big)\delta_{\tau_i} \\ &\leq |\widetilde{(D\boldsymbol{b})_y}|(\mathrm{d}t), \end{aligned}$$

where we have allowed the initial data to be general, and the τ_i's denote the jump set of $W(\cdot,y)$, a subset of the set where the jump part of $\widetilde{(D\boldsymbol{b})_y}(\mathrm{d}t)$ is concentrated.

Thus, we conclude that

$$|W(t,y)| \leq |W(t,0)| e^{|\widetilde{(D\boldsymbol{b})_y}|([0,t))}, \tag{4.14}$$

which gives the uniqueness. □

Remark 4.2 (Time reversibility of the ODE). We note that the ODE is time reversible. Being $\boldsymbol{b}(t)$ a BV function, by Alberti's rank-one theorem we can write for the singular part of $(D\boldsymbol{b})_y$ as follows:

$$(D\boldsymbol{b})_y^{\mathrm{sing}} = \xi(t,y)\cdot\eta^T(t,y)|(D\boldsymbol{b})_y|(\mathrm{d}t), \qquad \eta^T(t,y)\cdot\xi(t,y)|(D\boldsymbol{b})_y|(\{t\}) = J(t,y) - J(t-,y). \tag{4.15}$$

The ODE for $W(T-t,y)$ is then

$$\begin{aligned}\dot W(T-t,y) &= -D_t^{\mathrm{cont}} W(T-t,y) - \sum_i (W(\tau_i+,y) - W(\tau_i-,y))\delta_{T-\tau_i}(\mathrm{d}t)\\ &= -\frac{(D\boldsymbol{b})_y^{\mathrm{cont}}(T-t)}{J(T-t,y)} W(T-t,y) - \sum_i \xi(\tau_i,y)\cdot\eta^T(\tau_i,y)\frac{|(D\boldsymbol{b})_y|(\tau_i)}{J(\tau_i-,y)} W(\tau_i-,y).\end{aligned}$$

By (4.15), we have the relations

$$\begin{gathered}\Big(\mathbb{I} + \frac{\xi(t,y)\cdot\eta^T(t,y)|D\boldsymbol{b}|_y(\{t\})}{J(t-,y)}\Big)^{-1} = \Big(\mathbb{I} - \frac{\xi(t,y)\cdot\eta^T(t,y)|D\boldsymbol{b}|_y(\{t\})}{J(t+,y)}\Big),\\ J(t+,y) = J(t-,y) + \eta^T(t,y)\cdot\xi(t,y)|D\boldsymbol{b}|_y(\{t\}),\end{gathered}$$

so that

$$\begin{aligned}W(t_i-,y) &= \Big(\mathbb{I} + \frac{\xi\cdot\eta^T|(D\boldsymbol{b})_y|(t_i)}{J(t_i-,y)}\Big)^{-1} W(t_i+,y)\\ &= \Big(\mathbb{I} - \frac{\xi\cdot\eta^T|(D\boldsymbol{b})_y|(t_i)}{J(t_i+,y)}\Big) W(t_i+,y),\end{aligned}$$

which is

$$W(\tau_i-,y) - W(\tau_i+,y) = -\frac{(D\boldsymbol{b})_y(\tau_i)}{J(\tau_i+,y)} W(\tau_i+,y) = -\frac{(D\boldsymbol{b})_y(\tau_i)}{J(\tau_i-,y)} W(\tau_i-,y).$$

In particular we have that

$$(D\boldsymbol{b})_y(\tau_i)\Big(\frac{W(\tau_i+,y)}{J(\tau_i+,y)} - \frac{W(\tau_i-,y)}{J(\tau_i-,y)}\Big) = 0.$$

Substituting we conclude that

$$\dot W(T-t,y) = -(D\boldsymbol{b})_y^{\mathrm{cont}}(T-y)W(T-t,y) - \sum_i \xi(\tau_i,y)\cdot\eta^T(t_i,y)\frac{|(D\boldsymbol{b})_y|(\tau_i)}{J(\tau_i+,y)} W(\tau_i+,y),$$

which is the ODE

$$\dot W(T-t,y) = (D\boldsymbol{b})_y(T-t)W((T-t)+,y).$$

Remark 4.3 (Time reversibility and rank-one property). We remark that, in turn, the invertibility of the ODE does not imply that the vector field satisfies the rank-one property. The invertibility condition is that for the singular part

$$\Big(\mathbb{I} + \frac{A}{J^-}\Big)\Big(\mathbb{I} - \frac{A}{J^+}\Big) = \mathbb{I}, \quad J^+ - J^- = \mathrm{Tr}A, \quad A = (D\boldsymbol{b})_y(\tau_i),$$

which is equivalent to

$$A^2 = (\mathrm{Tr}A)A, \tag{4.16}$$

However, it turns out that the above condition is valid also for the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\end{pmatrix},$$

which is not of rank one.

In the 2×2 case, on the other hand, where the proof of the existence of the flow is independent from the rank one property (see [17]), condition (4.16) is a characterization of rank-one matrices (since it is equivalent to $\det A = 0$).

5. Local-to-global argument

The key idea of our proof is to build the derivative in measure by patching together local estimates. In this section, we show how the existence of a partition into (perturbed) proper sets where an approximate differentiability in measure property is satisfied leads to a global estimate on the differentiability in measure.

We assume that there is a finite partition $\{\Omega_i\}$ of $[0,T]\times B^d_{R_0}(0)$ into disjoint (perturbed) proper sets (up to the negligible set made of their boundaries) such that the following local estimates hold true.

(1) **Measure controlling the total error:** there exists a measure μ_P with

$$\mu_P(\mathbb{R}^{d+1}) < \varepsilon_P$$

for some $\varepsilon_P > 0$.

(2) **Removal of a small set of initial points:** in each set Ω_i of the partition, there exists a set of initial point $S'_{i,1} \subset S_{i,1} \cap \mathcal{K}_0$ whose co-measure is

$$\mathcal{H}^d((S_{i,1} \cap \mathcal{K}_0) \setminus S'_{i,1}) < \mu_P(\Omega_i). \tag{5.1}$$

The set $S_{i,1}$ is the boundary part of the (perturbed) proper set Ω_i defined in Theorem 3.4, with $\varepsilon = \mu_P(\Omega_i)$. Moreover, up to a $\mathcal{H}^d$-negligible set, $S_{i,2}$ is contained in $\cup_j S_{j,1} \cup \{t = 0,T\}$ up to a $\mathcal{H}^d$-negligible set: this means that the trajectories exiting one (perturbed) proper set from $S_{i,2}$ are entering another (perturbed) proper set trough $S_{j,1}$ (or are initial-final points). An equivalent way of expressing (5.1) is to say that the measure of trajectories we remove is less than $\mu_P(\Omega)$.

(3) **Cylinders where the linear approximation is constructed:** there exists R_i such that for every $y_i \in S'_{i,1}$ the set $\boldsymbol{X}(t,y_i) + B^d_{R_i}(0)$ is contained in Ω_i. In particular, $y_i + B^d_{R_i}(0) \subset S_{i,1}$, and similarly for the exit point.

(4) **Bad set of trajectories for the linear approximation:** for every $y_i \in S'_{i,1}$ and $r_i \leq R_i$ there exists a set of initial points $E_{1,i}(r_i,y_i) \subset B^d_{r_i}(0) \cap (\mathcal{K}_0 - y_i)$ such that

$$\int_{S'_{i,1}} \mathcal{L}^d(E_{1,i}(r_i,y_i))\,\mathrm{d}y_i < \mathcal{L}^d(B^d_{r_i}(0))\mu_P(\Omega_i).$$

(5) **Error estimate for the flow generated by an approximate vector field:** for every $y_i \in S'_{i,1}$ and $r_i \leq R_i$, there exists an approximated vector field $\tilde{\boldsymbol{b}}_i(r_i,y_i;t,w_i)$ such that the flow $\tilde{\boldsymbol{X}}$ generated by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{X}}_i(r_i,y_i;t,z) = \tilde{\boldsymbol{b}}_i(r_i,y_i;t,\tilde{\boldsymbol{X}}(r_i,y_i;t,z)), \quad t \in (t^-_i(y_i), t^+_i(y_i)), \\ \boldsymbol{X}(r_i,y_i;t^-_i(y_i,z),z) = z, \end{cases}$$

satisfies

$$\int_{S'_{i,1}} \int_{(B^d_{r_i}(0)\cap(\mathcal{K}_0-y_i))\setminus E_{1,i}(r_i,y_i)} \big\|\boldsymbol{X}(\cdot,t^-_i(y_i),y_i+z_i) - \boldsymbol{X}(\cdot,t^-_i(y_i),y_i) - \tilde{\boldsymbol{X}}_i(r_i,y_i;\cdot,z_i)\big\|_{C^0(t^-_i(y_i),t^+_i(y_i,z_i))}\,\mathrm{d}z_i\,\mathrm{d}y_i$$
$$< r_i\mathcal{L}^d(B^d_{r_i}(0))\mu_P(\Omega_i), \tag{5.2}$$

where $t^+_i(y_i,z_i)$ is the exit time of the trajectory $\boldsymbol{X}_i(\cdot,t^-_i(y_i),y_i+z_i)$ from the cylinder $\boldsymbol{X}((t^+_i(y_i),t^-_i(y_i)),y_i) + B^d_{r_i}(0)$.

(6) **Control on the approximate flow:** the approximated solution $\tilde{\boldsymbol{X}}(r_i,y_i;t,z_i)$ satisfies for $r'_i \leq r_i$

$$\int_{S'_{i,1}} \int_{B^d_{r'_i}(0)} \|\tilde{\boldsymbol{X}}(r_i,y_i;\cdot,z_i) - z_i\|_{C^0(t^-_i(y_i),t^+_i(y_i))}\,\mathrm{d}z_i\,\mathrm{d}y \leq Cr'_i\mathcal{L}^d(B^d_{r'_i}(0))|D\boldsymbol{b}|(\Omega_i). \tag{5.3}$$

(7) **Comparison with the linearized flow:** let $E_{2,i}(r_i, y_i)$ be the initial set of the trajectories starting in $(\boldsymbol{X}(t_i^-(y_i), y_i) + B^d_{r_i}(0)) \cap \mathcal{K}_0$ which exit before $t_i^+(y_i)$ from $\boldsymbol{X}((t_i^-(y_i), t_i^+(y_i)), y_i) + B^d_{r_i}(0)$: then the remaining trajectories satisfy

$$\int_{S'_{i,1}} \int_{(B^d_{r_i}(0)\cap(\mathcal{K}_0 - y_i))\setminus(E_{1,i}(r_i,y_i)\cup E_{2,i}(r_i,y_i))} \Big| \boldsymbol{X}(t_i^+(y_i), t_i^-(y_i), y_i + z_i) - \boldsymbol{X}(t_i^+(y_i), t_i^-(y_i), y_i) - W(t_i^-(y_i), t_i^+(y_i), y_i) z_i \Big| \, \mathrm{d}z_i \, \mathrm{d}y_i < r_i \mathcal{L}^d(B^d_{r_i}(0)) \mu_P(\Omega_i), \tag{5.4}$$

where $W(t_i^-(y_i), t_i^+(y_i), y_i)$ is the solution $W(\cdot, t_i^-(y), y)$ to the linearized ODE (4.1) with initial data $W(t_i^-(y), t_i^-(y), y) = y$.

With the above assumptions, we proceed to prove the differentiability in measure of the flow.

Proposition 5.1. *Under Assumptions* (1)–(7)*, the following properties hold:*

(1) *there is a set $K_2 \subset K_1$ of initial points of co-measure*

$$\mathcal{L}^d(K_1 \setminus K_2) \le \bar{C}\varepsilon_P;$$

(2) *for every $y \in K_2$ there is a set $E_y \cup F_y \cup G_y \subset B^d_r(0) \cap (\mathcal{K}_0 - y_i)$ whose total measure is*

$$\int_{K_2} \mathcal{L}^d(E_y \cup F_y \cup G_y) \, \mathrm{d}y < \mathcal{O}(1) \varepsilon_P^{1/(d+2)} \mathcal{L}^d(B^d_r(0)),$$

where the factor $\mathcal{O}(1)$ depends only on M, $\bar{C}$ and d;

(3) *in the remaining set, we have*

$$\int_{K_2} \int_{(B^d_r(0)\cap(\mathcal{K}_0 - y))\setminus(E_y\cup F_y\cup G_y)} \big| \boldsymbol{X}(T, y+z) - \boldsymbol{X}(T, y) - W(T, y) z \big| \, \mathrm{d}y \, \mathrm{d}z \le \mathcal{O}(1) \varepsilon_P^{1/(d+2)} r \mathcal{L}^d(B^d_r(0)).$$

Together with Point (1) of Proposition 3.6, this gives the differentiability in measure of Theorem 1.1, under the assumptions (1)–(7) above. In the following sections, we will show how to construct the partition and obtain the estimates.

Proof. The proof is organized into several steps. The idea is that one uses the comparison with the linear flow when the perturbed trajectory $\boldsymbol{X}(t, y+z)$ is not exiting the cylinder $\boldsymbol{X}(t, y) + B^d_r(0)$, while the estimate using the approximated flow controls how many trajectories are exiting from $\boldsymbol{X}(t, y) + B^d_R(0)$, $0 < r < R$. Then, a suitable choice of r, R allows to prove the claim.

(1) **Removal of trajectories which are not inside $S'_{i,1}$.** We remove trajectories of K_0 for which $\boldsymbol{X}(t_i^-(y), y) \notin S'_{i,1}$ (and we control also the trajectories not entering in $S_{1,i}$ or leaving from $S_{2,i}$, i.e. the ones which cross on the lateral boundaries, because of the last part of Point (2) of the assumptions: by nearly incompressibility and formula (3.6), the measure of trajectories we remove is less than

$$\bar{C} \sum_i \mathcal{L}^d((S_{i,1} \cap \mathcal{K}_0) \setminus S'_{i,1}) \underset{\text{Point (2)}}{\le} \bar{C} \sum_i \mu_P(\Omega_i) \le \bar{C}\mu_P(\mathbb{R}^{d+1}) \underset{\text{Point (1)}}{<} \bar{C}\varepsilon_P. \tag{5.5}$$

Thus we restrict to a compact set $K_2 \subset K_1$ whose co-measure in K_2 is bounded by $\bar{C}\varepsilon_P$. This set is the set of Point (1) of the statement.

(2) **Choice of the radius of the cylinders and definition of the partition of sets crossed by a trajectory.** Let $\bar{R} = \min_i R_i$ and, for each $y \in K_2$, let i_y be the sequence of proper sets Ω_i which the trajectory $\boldsymbol{X}(t, y)$ is crossing. We will abuse the notation, writing $(i-1)_y$ for the predecessor of i_y, 1_y for the initial i, $0_y = 0$, and so on; we also note that one trajectory may cross a given Ω_i several times, however from [16, Corollary 6.9] the number of crossings is finite for $\mathcal{L}^d$-a.e. $y \in K_2$, so there are only finitely many indexes i_y. The exit time of a trajectory $\boldsymbol{X}(t, y)$ from Ω_{i_y} will be denoted by t_{i_y}.

(3) **Removal of the set of perturbations which do not behave mildly.** For every $y \in K_2$, remove all $z \in B^d_{\bar{R}}(0) \cap (K_0 - y)$ such that

$$\boldsymbol{X}(t_{(i-1)_y}, y + z) - \boldsymbol{X}(t_{(i-1)_y}, y) \in E_{1,i_y}(\bar{R}, \boldsymbol{X}(t_{(i-1)_y}, y)). \tag{5.6}$$

This means that at time $t_{(i-1)_y}$ we remove the trajectories which do not satisfy (5.2) while in Ω_{i_y}. Here we have used the notation

$$(t^-_{i_y}(y), t^+_{i_y}(y)) = (t_{(i-1)_y}, t_{i_y}). \tag{5.7}$$

The i ranges from 0_y to $\bar{i}_y(z)$ corresponding to the index of the set Ω_i such that the trajectory $\boldsymbol{X}(t, y+z)$ is exiting for the first time from $\boldsymbol{X}(t,y) + B^d_{\bar{R}}(0)$ within Ω_i.
This new set

$$E_y = \big\{ z \in B^d_{\bar{R}}(0) \cap (K_0 - y) : \exists i_y \text{ s.t. (5.6) holds} \big\}$$

has measure bounded by (we use the nearly incompressibility property of the map $z \mapsto \boldsymbol{X}(t, y+z) - \boldsymbol{X}(t,y)$, which is the Lagrangian flow of the vector field $(t,z) \mapsto \boldsymbol{b}(t, \boldsymbol{X}(t,y)+z) - \boldsymbol{b}(t, \boldsymbol{X}(t,y)))$

$$\begin{aligned}
\int_{K_2} \mathcal{L}^d(E_y)\,\mathrm{d}y & \underset{(3.10)}{\leq} \bar{C} \int_{K_2} \Big[\sum_{i_y} \mathcal{L}^d(E_{1,i_y}(\bar{R}, \boldsymbol{X}(t_{i_y}, y))) \Big] \mathrm{d}y \\
\text{(3.10) and Fubini } & \leq \bar{C}^2 \sum_i \int_{S'_{i,1}} \mathcal{L}^d(E_{1,i}(\bar{R}, y_i))\,\mathrm{d}y_i \\
\text{Point (4) } & < \bar{C}^2 \sum_i \mathcal{L}^d(B^d_{\bar{R}}(0)) \mu_P(\Omega_i) \\
\text{Point (1) } & < \bar{C}^2 \mathcal{L}^d(B^d_{\bar{R}}(0)) \varepsilon_P.
\end{aligned} \tag{5.8}$$

(4) **Change of coordinate for the disintegration.** The disintegration formula of (5.4), Point (7) of page 18, is computed in the coordinates y_i on the surface $S_{1,i}$. When using instead the coordinates y at $t = 0$, we have to replace

$$W(t, t^-_y(y_i), y_i) \mapsto W(t,y), \qquad \text{where} \quad y_i = \boldsymbol{X}(t^-_i(y_i), y). \tag{5.9}$$

Indeed this is just the composition properties for the solution to (4.1).

(5) **Estimate on the growth of the perturbation.** We now use the estimate of Equation (5.4) up to the last time $t_{(\bar{i}-1)_y}(z)$ such that the trajectory remains at distance $\bar{R}$ from $\boldsymbol{X}(t,y)$, i.e. when crossing $\Omega_{(\bar{i}-1)_y}$. We define, for $1_y \leq i_y \leq (\bar{i}-1)_y$,

$$\Delta_{i_y}(y,z) = \boldsymbol{X}(t_{i_y}, y+z) - \boldsymbol{X}(t_{i_y}, y), \quad W_{i_y}(y) = W(t_{i_y}, t_{(i-1)_y}, y).$$

Let us set the initial data as

$$\Delta_{0_y}(y,z) = z,$$

and consider the difference equation

$$\Delta_{i_y}(y,z) = W_{i_y}(y) \Delta_{(i-1)_y}(y,z) + [\Delta_{i_y}(y,z) - W_{i_y}(y)\Delta_{(i-1)_y}(y,z)].$$

By Duhamel's formula for difference equation, i.e.

$$a_n = b_n a_{n-1} + c_n, \qquad a_n = \Big(\prod_{j=1}^n b_j \Big) a_0 + \sum_{k=1}^n \Big(\prod_{j=k+1}^n b_j \Big) c_k, \tag{5.10}$$

we obtain

$$\begin{aligned}
|\Delta_{i_y}(y,z)| & \underset{(5.10)}{\leq} \Big| \prod_{j_y=1_y}^{i_y} W_{j_y}(y) \Big| |z| + \sum_{k_y=1_y}^{i_y} \Big| \prod_{j_y=(k+1)_y}^{i_y} W_{j_y}(y) \Big| \big| \Delta_{k_y}(y,z) - W_{k_y}(y) \Delta_{(k-1)_y}(y,z) \big| \\
(y \in K_1) & \underset{\text{Thm. 4.1}}{\leq} e^{\bar{C}M} \Big(|z| + \sum_{k_y=1_y}^{i_y} \big| \Delta_{k_y}(y,z) - W_{k_y}(y) \Delta_{(k-1)_y}(y,z) \big| \Big).
\end{aligned} \tag{5.11}$$

(6) **Choice of the initial radius r.** Let M' be a constant to be chosen later, and set

$$r = \frac{e^{-\bar{C}M - \bar{C}M'}}{4} \bar{R}. \tag{5.12}$$

(7) **Estimate on the trajectories with large growth.** Consider first the trajectories such that

$$\max_{0_y,\dots,(\bar{i}-1)_y(z)} |\Delta_{i_y}(y,z)| \geq 2e^{\bar{C}M} r = \frac{e^{-\bar{C}M'}}{2}\bar{R} = r', \qquad |z| \leq r. \tag{5.13}$$

From (5.11), we obtain the estimate

$$\sum_{k_y=1_y}^{(\bar{i}-1)_y(z)} \big|\Delta_{k_y}(y,z) - W_{k_y}(y)\Delta_{(k-1)_y}(y,z)\big| \geq r = \frac{e^{-\bar{C}M-\bar{C}M'}}{4}\bar{R}. \tag{5.14}$$

(8) **Measure of trajectories with large growth.** Thus, using (5.4), we get

$$\begin{aligned}
&\int_{K_2} \mathcal{L}^d\big(\big\{z \in (B^d_r(0) \cap (K_0 - y)) \setminus E_y : |\Delta_{(\bar{i}-1)_y}| \geq 2e^{\bar{C}M} r\big\}\big)\,\mathrm{d}y \\
&\underset{(5.14)}{\leq} \frac{1}{r}\int_{K_2}\int_{(B^d_r(0)\cap(\mathcal{K}_0-y))\setminus E_y} \sum_{k_y=1_y}^{(\bar{i}-1)_y(z)} \big|\Delta_{k_y}(y,z) - W_{k_y}(y)\Delta_{(k-1)_y}(y,z)\big|\,\mathrm{d}z\,\mathrm{d}y \\
\text{Fubini, (3.10)}\ &\leq \frac{\bar{C}^2}{r}\sum_i \int_{S'_{i,1}} \Big[\int_{(B^d_{\bar{R}}(0)\cap(\mathcal{K}_0-y_i))\setminus(E_{1,i}(\bar{R},y_i)\cup E_{2,i}(\bar{R},y_i))} \\
&\qquad \Big|\boldsymbol{X}(t_i^+(y_i),t_i^-(y_i),y_i+z_i) - \boldsymbol{X}(t_i^+(y_i),t_i^-(y_i),y_i) - W(t_i^+(y_i),t_i^-(y_i))z_i\Big|\,\mathrm{d}z_i\Big]\,\mathrm{d}y_i \\
&\underset{(5.4)}{\leq} \frac{\bar{C}^2}{r}\sum_i \bar{R}\mathcal{L}^d(B^d_{\bar{R}}(0))\mu_P(\Omega_i) \leq \bar{C}^2\Big(\frac{\bar{R}}{r}\Big)^{d+1}\mathcal{L}^d(B^d_r(0))\mu_P(\mathbb{R}^{d+1}) \\
&\underset{(5.12)}{\leq} \bar{C}^2(4e^{\bar{C}(M+M')})^{d+1}\varepsilon_P\mathcal{L}^d(B^d_r(0)).
\end{aligned}$$

In the third line, we have used that the trajectories under consideration are not exiting $\boldsymbol{X}(t,y)+B^d_{\bar{R}}(0)$ in Ω_i, see the definition of $E_{2,i}(r_i,y_i)$ in Assumption (7).

Hence, we can remove a set $F_y \subset (B^d_r(0)\cap(K_0-y))\setminus E_y$ such that

$$\int_{K_2} \mathcal{L}^d(F_y)\mathrm{d}y \leq \bar{C}^2(4e^{\bar{C}(M+M')})^{d+1}\varepsilon_P\mathcal{L}^d(B^d_r(0)) \tag{5.15}$$

and all trajectories in $(B^d_r(0)\cap(K_0-y))\setminus(E_y\cup F_y)$ remain inside $\boldsymbol{X}(t,y)+B^d_{r'}(0)$ up to $(\bar{i}-1)_y(z)$, with r' defined in (5.13).

(9) **Estimate of the trajectories exiting at $\bar{i}_y(z)$.** For the trajectories for which

$$|\Delta_{(\bar{i}-1)_y(z)}(y,z)| < r' \underset{(5.13)}{=} \frac{e^{-\bar{C}M'}}{2}\bar{R} \tag{5.16}$$

and exit at $\bar{i}_y(z)$, we can write that

$$\begin{aligned}
\frac{1}{2}\bar{R} &\leq \Big(1 - \frac{e^{-\bar{C}M'}}{2}\Big)\bar{R} \\
&\underset{(5.16)}{\leq} \bar{R} - |\Delta_{\bar{i}_y-1}(y,z)| \\
&\leq \big\|\boldsymbol{X}(\cdot,y+z) - \boldsymbol{X}(\cdot,y) - \Delta_{(\bar{i}_y-1)(y)}(y,z)\big\|_{C^0(t^-_{\bar{i}_y}(y),t^+_{\bar{i}_y}(y,\Delta_{(\bar{i}_y-1)(y)}(y,z)))},
\end{aligned} \tag{5.17}$$

where $t_{\bar{i}_y}(y,\Delta_{(\bar{i}_y-1)(y)}(y,z))$ is the exit time of the trajectory $\boldsymbol{X}(t,y+z)$ from $\boldsymbol{X}(t,y)+B^d_{\bar{R}}(0)$.

(10) **Measure of exiting trajectories.** We have the estimate

$$\begin{aligned}
&\int_{K_2} \mathcal{L}^d\big(\big\{z \in (B^d_r(0) \cap (K_0 - y)) \setminus (E_y \cup F_y) : \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y)\|_{C([0,T])} \geq \bar{R}\big\}\big)\, \mathrm{d}y \\
&\underset{(5.17)}{\leq} \frac{2}{\bar{R}} \int_{K_2} \int_{(B^d_r(0)\cap(K_0-y))\setminus(E_y\cup F_y)} \big\|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y) - \Delta_{(\bar{i}-1)_y(z)}(y,z)\big\|_{C^0(t^-_{\bar{i}_y(z)}(y), t^+_{\bar{i}_y(z)}(y, \Delta_{\bar{i}_y - 1}(y,z)))} \,\mathrm{d}z\, \mathrm{d}y \\
&\underset{(5.16),\text{Fubini}}{\leq} \frac{2\bar{C}^2}{\bar{R}} \sum_i \int_{S'_{i,1}} \int_{(B^d_{r'}(0)\cap(\mathcal{K}_0-y))\setminus E_{1,i}} \big\|\boldsymbol{X}(\cdot, y_i+z_i) - \boldsymbol{X}(\cdot, y_i) - z_i\big\|_{C^0(t^-_i(y_i), t^+_i(y_i,z_i))} \,\mathrm{d}y_i\, \mathrm{d}z_i \\
&\leq \frac{2\bar{C}^2}{\bar{R}} \sum_i \int_{S'_{i,1}} \int_{(B^d_{r'}(0)\cap(\mathcal{K}_0-y))\setminus E_{1,i}} \big\|\boldsymbol{X}(\cdot, y_i+z_i) - \boldsymbol{X}(\cdot, y_i) - \tilde{\boldsymbol{X}}_i(\bar{R}, y_i; \cdot, z_i)\big\|_{C^0(t^-_i(y_i), t^+_i(y_i,z_i))} \,\mathrm{d}y_i\, \mathrm{d}z_i \\
&\quad + \frac{2\bar{C}^2}{\bar{R}} \sum_i \int_{S'_{i,1}} \int_{(B^d_{r'}(0)\cap(\mathcal{K}_0-y))\setminus E_{1,i}} \|\tilde{\boldsymbol{X}}_i(\bar{R}, y_i; \cdot, z_i) - z_i\|_{C^0(t^-(y_i), t^+_{\bar{i}}(y, \Delta_{\bar{i}_y}(y,z)))} \,\mathrm{d}z_i\, \mathrm{d}y_i \\
&\underset{(5.2),(5.3)}{\leq} \frac{2\bar{C}^2}{\bar{R}} \sum_i \bar{R}\mathcal{L}^d(B^d_{\bar{R}}(0))\mu_P(\Omega_i) + \frac{2\bar{C}^2}{\bar{R}} \sum_i r' \mathcal{L}^d(B^d_{r'}(0))|D\boldsymbol{b}|(\Omega_i) \\
&\underset{(5.12),(5.13)}{\leq} 2\bar{C}^2(4e^{\bar{C}(M+M')})^d \mathcal{L}^d(B^d_r(0))\varepsilon_P + 2^d\bar{C}^2 e^{\bar{C}(dM-M')}\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega).
\end{aligned}$$

Choosing (see also (5.23))

$$e^{-\bar{C}M'} = (\varepsilon_P)^{1/(d+2)}, \tag{5.18}$$

we obtain

$$\begin{aligned}
&\int_{K_2} \mathcal{L}^d\big(\big\{z \in (B^d_r(0) \cap (K_0 - y)) \setminus (E_y \cup F_y) : \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y)\|_{C([0,T])} \geq \bar{R}\big\}\big)\, \mathrm{d}y \\
&\leq \mathcal{O}(1)\varepsilon_P^{1/(d+2)}\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{5.19}$$

The factor $\mathcal{O}(1)$ depends only on M, $\bar{C}$ and d.

(11) **Final estimate of the exiting trajectories.** Thus, we can remove the set of trajectories

$$Gy = \Big\{z \in (B^d_r(0) \cap (K_0 - y)) \setminus (E_y \cup F_y) : \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y)\|_{C([0,T])} \geq \bar{R}\Big\}$$

of measure

$$\int_{K_2} \mathcal{L}^d(G_y)\mathrm{d}y \underset{(5.19)}{\leq} \mathcal{O}(1)\varepsilon_P^{1/(d+2)}\mathcal{L}^d(B^d_r(0)), \tag{5.20}$$

and the remaining trajectories lie inside $\boldsymbol{X}(t,y) + B^d_R(0)$. The total set of trajectories $E_y \cup F_y \cup G_y$ we remove from $(B^d_r(0) \cap (K_0 - y))$ has measure

$$\begin{aligned}
\int_{K_2} \mathcal{L}^d(E_y \cup F_y \cup G_y)\,\mathrm{d}y &\underset{(5.8),(5.15),(5.20)}{\leq} \bar{C}^2\mathcal{L}^d(B^d_R(0))\varepsilon_P \\
&\qquad + \bar{C}^2(4e^{M+M'})^{d+1}\mathcal{L}^d(B^d_r(0))\varepsilon_P + \mathcal{O}(1)\varepsilon_P^{1/(d+2)}\mathcal{L}^d(B^d_r(0)) \\
&< \mathcal{O}(1)\varepsilon_p^{1/(d+2)}\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{5.21}$$

The factor $\mathcal{O}(1)$ depends only on M, $\bar{C}$ and d. This proves Point (2) of the statement.

(12) **Comparison with linear flow.** We estimate now the difference of the trajectory with the composition of the linear maps $(\mathbb{I} + B_i)$: we have

$$\Big|\Delta_{i_y} - \Big(\prod_{j_y=1_y}^{i_y} W_{j_y}\Big)z\Big| \leq \Big|\Delta_{i_y} - \Delta_{(i-1)_y} - W_{i_y}\Delta_{(i-1)_y}\Big| + |W_{i_y}|\Big|\Delta_{(i-1)_y} - \Big(\prod_{j_y=1_y}^{(i-1)_y} W_{j_y}\Big)z\Big|.$$

Hence, using $\prod_{j_y=1_y}^{i_y} W_{i_y} = W(t_i^+(y_i), y)$, again the solution formula (5.10) gives (note that in this case the initial data is 0)

$$\begin{aligned}\left|\Delta_{i_y} - W(t_{i_y}, y)z\right| &\leq \sum_{k_y=1_y}^{i_y} \bigg(\prod_{j_y=(k+1)_y}^{i_y} |W_{j_y}| \left|\Delta_{k_y} - W_{k_y}\Delta_{(k-1)_y}\right| \bigg) \\ &\leq e^{\bar{C}M} \bigg(\sum_{k_y=1_y}^{i_y} \left|\Delta_{k_y} - W_{k_y}\Delta_{(k-1)_y}\right| \bigg).\end{aligned} \tag{5.22}$$

(13) **Estimate of the error with respect to the linear flow.** Integrating as in the previous points

$$\begin{aligned}&\int_{K_2} \int_{(B_r^d(0)\cap(K_0-y))\setminus(E_y\cup F_y\cup G_y)} \left|\Delta_{i_y} - W(t_{j_y}, y)z\right| \mathrm{d}y\,\mathrm{d}z \\ &\underset{(5.22)}{\leq} e^{\bar{C}M} \int_{K_2} \int_{(B_r^d(0)\cap(K_0-y))\setminus(E_y\cup F_y\cup G_y)} \sum_{k_y=1_y}^{i_y} \left|\Delta_{k_y} - W_{k_y}\Delta_{(k-1)_y}\right| \mathrm{d}z\,\mathrm{d}y \\ \text{Fubini, (4.3)} &\leq \bar{C}^2 e^{\bar{C}M} \sum_i \int_{S'_{1,i}} \int_{(B_{\bar{R}}^d\cap(\mathcal{K}_0-y))\setminus(E_{1,i}(\bar{R},y_i)\cup E_{2,i}(\bar{R},y_i))} \Big| \boldsymbol{X}(t_i^+(y_i), t_i^-(y_i), y_i + z_i) \\ &\qquad - \boldsymbol{X}(t_i^+(y_i), t_i^-(y_i), y_i) - W(t_i^+(y_i), t_i^-(y_i), y_i) z_i \Big| \mathrm{d}z_i\,\mathrm{d}y_i \\ &\underset{(5.4),(5.9)}{\leq} \bar{C}^2 e^{\bar{C}M} \bar{R}\mathcal{L}^d(B_R^d(0))\varepsilon_P \\ &\underset{(5.12)}{<} \bar{C}^2 4^{d+1} e^{\bar{C}((d+2)M+(d+1)M')} r\mathcal{L}^d(B_r^d(0))\varepsilon_P \\ &\underset{(5.18)}{=} \mathcal{O}(1)\varepsilon_P^{1/(d+2)} r\mathcal{L}^d(B_r^d(0)).\end{aligned} \tag{5.23}$$

The factor $\mathcal{O}(1)$ depends only on M, $\bar{C}$ and d.

In particular we can choose i_y as the last index i_y^{last}, for which $t_{i_y^{\mathrm{last}}} = T$ and

$$\Delta_{i_y^{\mathrm{last}}}(y, z) = \boldsymbol{X}(T, y+z) - \boldsymbol{X}(T, y),$$

obtaining the last point of the statement. □

Remark 5.2. We observe that the estimate gives some sort of differentiability in measure even with i_y depending on y. This is not surprising since the sets $S_{i,1}$ are subsets of finitely many sets $\{t = \mathrm{const}\}$. However, the set K_2 *depends* on the partition: indeed, the derivative $W(t, y)$ has discontinuities; thus, at any time τ_i of discontinuity, we have, in general,

$$\int_{(B_r^d(0)\cap(K_0-y))\setminus(E_y\cup F_y)} \left|\boldsymbol{X}(\tau_i, y+z) - \boldsymbol{X}(\tau_i, y) - Az\right| \mathrm{d}y\,\mathrm{d}z = \mathcal{O}(1)$$

for every linear map A. As an example one may consider the vector field in $(t, x_1, x_2) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$

$$\boldsymbol{b}(t, x) = \begin{cases} (1, 0) & \text{if } x_1 < 0, \\ (1, 1) & \text{if } x_1 \geq 0, \end{cases} \qquad \boldsymbol{X}(t, y) = y + (t, [t + y_1]^+),$$

so that at any time T the set of trajectories for which the differential cannot be computed is $y_1 = -T$. Thus for every T the set of trajectories which have to be removed is different.

In next two sections we will show how to prove Assumptions (2)-(7) in two cases:

(1) when one takes into account only the a.c. part of $D\boldsymbol{b}$ (Section 6);
(2) in the Lebesgue points of the singular part of $D\boldsymbol{b}$ (Section 7).

The choice of the measure μ_P will be obtained by piecing together these two cases.

6. Local estimate with the absolutely continuous part

We fix a perturbed proper set Ω_i and in it we consider the following vector field:

$$\tilde{\boldsymbol{b}}(t, y_i, w_i) = (D\boldsymbol{b})^{\mathrm{a.c.}}_{y_i}(\boldsymbol{X}(t, y_i))w_i.$$

In order to make the notation lighter, going forward we will neglect the index i. The measure μ_P will be defined at the end.

(1) **Control of the derivative.** First, for $M > 0$ chosen in Proposition 3.6, we have

$$|(D\boldsymbol{b})^{\mathrm{a.c.}}_y|(t^-(y), t^+(y)) \leq M$$

because $|(D\boldsymbol{b})^{\mathrm{a.c.}}_y|\llcorner_{\Omega_i} \leq |(D\boldsymbol{b})_y|$.

(2) **Cylinders where the linear flow is constructed.** By choosing $R \ll 1$, we can also assume that the cylinder $\boldsymbol{X}(t, y) + B^d_R(0)$ has bases inside the entering and exiting flat parts of Ω_i: again we can assume that we remove a set of trajectories of measure smaller than $\varepsilon \mathcal{L}^{d+1}(\Omega)$, where $\varepsilon \to 0$ when $R \to 0$. Let $S'_1 \subset S_1$ be the set of initial data of the remaining trajectories: the choice of R corresponds to Point (3) of page 17. In order to satisfy Point (2) of page 17, we will choose $\varepsilon < \varepsilon_P/(2T\mathcal{L}^d(B^d_{R_0}(0))$.

(3) **Choice of the approximated flow.** For the a.c. part we can use directly the linearized flow as approximated flow $\tilde{\boldsymbol{X}}_i$, because we have a good control on the error. Inside the ball of radius R_i we compare the flow with the linear flow

$$\begin{cases} \dot{W}^{\mathrm{a.c.}}(t, y) = \frac{(D\boldsymbol{b})^{\mathrm{a.c.}}_y(\mathrm{d}t)}{J(t,y)} W^{\mathrm{a.c.}}(t, y), \\ W^{\mathrm{a.c.}}(t^-(y), y) = y. \end{cases} \tag{6.1}$$

This flow has Lipschitz constant bounded by e^M by Point (1) above and Theorem 4.1, and moreover if $W(t, y)$ is the solution to

$$\begin{cases} \dot{W}(t, y) = \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)} W(t-, y), \\ W(t^-(y), y) = y, \end{cases}$$

then, by Duhamel's formula and the same estimate as in the proof of Theorem 4.1, we have

$$|W(t, y) - W^{\mathrm{a.c.}}(t, y)| \leq \bar{C} e^{\bar{C}M} |(D\boldsymbol{b})^{\mathrm{sing}}_y|(t^-(y), t^+(y)). \tag{6.2}$$

Hence, (5.3) of Point (6) of page 17 holds with $C = e^{\bar{C}M}$ for both $W^{\mathrm{a.c.}}, W$.

(4) **Comparison with Lipschitz flow.** We can compare the evolution of a trajectory with the evolution of the Lipschitz linear flow as follows:

$$\begin{aligned} &\big|\boldsymbol{X}(t, y + rz) - \boldsymbol{X}(t, y) - W^{\mathrm{a.c.}}(t, y)z\big| \\ &\leq e^{\bar{C}M} \int_{t^-(y)}^{t} \Big|\boldsymbol{b}(s, \boldsymbol{X}(s, y + z)) - \boldsymbol{b}(s, \boldsymbol{X}(s, y)) - (D\boldsymbol{b})^{\mathrm{a.c.}}_y(s)(\boldsymbol{X}(s, y + z) - \boldsymbol{X}(s, y))\Big| \mathrm{d}s, \end{aligned} \tag{6.3}$$

where we have used the estimate for the Lipschitz flow $z \mapsto W^{\mathrm{a.c.}}(t, y)z$ given by Corollary A.2 and Point (1) of page 50.

(5) **Estimate up to exit time.** Integrating the above estimate w.r.t. the initial data in the ball $y + B^d_R(0)$ we obtain up to the exit time $t^+(y, z)$

$$\begin{aligned} &\int_{B^d_R(0)\cap(\mathcal{K}_0 - y)} \big\|\boldsymbol{X}(t^+(y, z), t^-(y), y + z) - \boldsymbol{X}(t^+(y, z), y) - W^{\mathrm{a.c.}}(t^+(y, z), y)z\big\|_{C^0(t^-(y), t^+(y,z))} \mathrm{d}z \\ &\underset{(6.3)}{\leq} e^{\bar{C}M} \int_{B^d_R(0)\cap(\mathcal{K}_0 - y)} \int_{t^-(y)}^{t^+(y,z)} \Big|\boldsymbol{b}(s, \boldsymbol{X}(s, y + z)) - \boldsymbol{b}(s, \boldsymbol{X}(s, y)) \\ &\qquad\qquad - (D\boldsymbol{b})^{\mathrm{a.c.}}_y(s)(\boldsymbol{X}(s, y + z) - \boldsymbol{X}(s, y))\Big| \mathrm{d}s \, \mathrm{d}z \\ &\underset{(3.10}{\leq} \bar{C} e^{\bar{C}M} \int_{t^-(y)}^{t^+(y)} \int_{B^d_R(0)} \Big|\boldsymbol{b}(s, \boldsymbol{X}(s, y) + w) - \boldsymbol{b}(s, \boldsymbol{X}(s, y)) - (D\boldsymbol{b})^{\mathrm{a.c.}}_y(s, \boldsymbol{X}(s, y))w\Big| \mathrm{d}s \, \mathrm{d}w. \end{aligned} \tag{6.4}$$

(6) **Integral over all trajectories.** The last integral can be evaluated after integrating with respect to y as follows:

$$
\begin{aligned}
&\bar{C}e^{\bar{C}M}\int_{S_1'}\int_{t^-(y)}^{t^+(y)}\int_{B^d_R}\Big|\boldsymbol{b}(s,\boldsymbol{X}(s,y)+w)-\boldsymbol{b}(s,\boldsymbol{X}(s,y))-(D\boldsymbol{b})^{\mathrm{a.c.}}_y(s,\boldsymbol{X}(s,y))w\Big|\,\mathrm{d}s\,\mathrm{d}w\,\mathrm{d}y \\
&\underset{(3.10)}{\leq} \bar{C}^2e^{\bar{C}M}\int_{B^d_R}|w|\int_0^1\|(D\boldsymbol{b})(t,\cdot+\lambda w)-(D\boldsymbol{b})^{\mathrm{a.c.}}(t,\cdot)\|_{\mathcal{M}(\Omega)}\,\mathrm{d}\lambda\,\mathrm{d}w \\
&\leq \bar{C}^2e^{\bar{C}M}R\omega(R)\mathcal{L}^d(B^d_R(0))|D\boldsymbol{b}|^{\mathrm{a.c.}}(\Omega)+\bar{C}^2e^{\bar{C}M}R\mathcal{L}^d(B^d_R(0))|(D\boldsymbol{b})^{\mathrm{sing}}|(\Omega),
\end{aligned}
\tag{6.5}
$$

where ω is the modulus of continuity in L^1 of the a.c. part of $D\boldsymbol{b}$.

Conclusion. We now show that Assumptions (2-7) hold with the choice of a suitable measure.

More precisely:

(1) concerning **Point (4) of page 17**, we set

$$E_1(r,y)=\emptyset;$$

(2) concerning **Point (5) of page 17**, by Point (3) and Point (6) above we have

$$
\begin{aligned}
&\int_{S_1'}\int_{B^d_R(0)}\big\|\boldsymbol{X}(\cdot,y+z)-\boldsymbol{X}(\cdot,y)-W(t,y)z\big\|_{C^0(t^-(y),t^+(y,z))}\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(6.4),(6.5)}{\leq} \bar{C}^2e^{\bar{C}M}R\omega(R)\mathcal{L}^d(B^d_R(0))|D\boldsymbol{b}|(\Omega)+\bar{C}^2e^{\bar{C}M}\ R\mathcal{L}^d(B^d_R(0))|(D\boldsymbol{b})^{\mathrm{sing}}|(\Omega) \\
&\quad+\int_{S_1'}\int_{B^d_R(0)}|W(t,y)-W^{\mathrm{a.c.}}(t,y)||z|\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(6.2)}{\leq} \bar{C}^2e^{\bar{C}M}R\omega(R)\mathcal{L}^d(B^d_R(0))|D\boldsymbol{b}|(\Omega)+2\bar{C}^2e^{\bar{C}M}R\mathcal{L}^d(B^d_R(0))|(D\boldsymbol{b})^{\mathrm{sing}}|(\Omega),
\end{aligned}
\tag{6.6}
$$

which shows (5.2) if $R\ll 1$ and a suitable choice of μ_P;

(3) the above estimate implies also **estimate (5.4) of Point (7) of page 18**, in the Lebesgue points of the a.c. part of $D\boldsymbol{b}$, if the diameter of Ω is sufficiently small: indeed if

$$\varepsilon_P=\bar{C}^2e^{\bar{C}M}R\omega(R)\mathcal{L}^d(B^d_R(0))+2\bar{C}^2e^{\bar{C}M}R\mathcal{L}^d(B^d_R(0))\frac{|(D\boldsymbol{b})^{\mathrm{sing}}|(\Omega)}{|D\boldsymbol{b}|(\Omega)}.$$

then the measure μ_P will be

$$\mu_P=\varepsilon_P(\mathcal{L}^{d+1}+|D\boldsymbol{b}|).$$

Note that $\varepsilon_P\ll 1$ as $R\to 0$ and $|D\boldsymbol{b}|(\Omega)\to 0$ if it is a Lebesgue point for the a.c. part of $|D\boldsymbol{b}|$.

This concludes the analysis of the a.c. part of $D\boldsymbol{b}$.

7. Local estimates with the singular part

The analysis of the singular part is more complicated and depends on the choice of several parameters: in particular, we will need the set E_1 of Point (4) of page 17, which collects the perturbed trajectories which do not behave mildly. As before, we will neglect the index i; moreover, here we assume that the perturbed proper set is in a small neighborhood of a Lebesgue point of the singular part $D^{\mathrm{sing}}\boldsymbol{b}$. We will first compute our estimates in the case of "contracting" flow, i.e. $\operatorname{div}\boldsymbol{b}<0$. Then, we will show how to deduce the general case from this.

7.1. Localization and coordinates.

Let $\bar{\varepsilon}\ll 1$ be given. For every Lebesgue point of the singular part $D^{\mathrm{sing}}\boldsymbol{b}$ of $D\boldsymbol{b}$, we can choose Ω as follows.

(1) **Entering and exiting sets:** the (perturbed) proper set Ω is a proper small perturbation of a ball centered in the Lebesgue point, such that the set of trajectories N_Ω not entering from S_1 and not leaving from S_2 has η-measure

$$\eta(N_\Omega)<\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega),\tag{7.1}$$

where η denotes the Lagrangian representation of $\rho(1,\boldsymbol{b})\mathcal{L}^{d+1}$ as in [16, Definition 3.1].

(2) **Lebesgue point of the derivative:** by the rank-one property of the singular part of the derivative of BV functions [1], there exist vectors $\bar{\xi}, \bar{\eta}$ such that

$$|D\boldsymbol{b} - \bar{\xi} \otimes \bar{\eta}^T |D\boldsymbol{b}||(\Omega) < \bar{\varepsilon}|D\boldsymbol{b}|(\Omega).$$

We assume that $\bar{\xi} = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$, $\bar{\eta} = \mathbf{e}_1$, by a linear change of coordinates. With the above choice of $\bar{\xi}, \bar{\eta}$ we have

$$\big|D\boldsymbol{b} - \bar{\xi} \otimes \mathbf{e}_1 |D\boldsymbol{b}|\big|(\Omega), \big|D\boldsymbol{b} - \bar{\xi} \otimes \mathbf{e}_1 |D_1 \boldsymbol{b}_{1,2}|\big|(\Omega) < \bar{\varepsilon}|D\boldsymbol{b}|(\Omega). \tag{7.2}$$

In particular almost all of the derivative occurs when moving along the 1-direction, and the variation lies in the 1,2-directions. Hence the other components have small derivative.

(3) **Contraction in time:** by reversing time if necessary, we assume that

$$\bar{\eta} \cdot \bar{\xi} = \bar{\xi}_1 \leq 0. \tag{7.3}$$

This implies that the flow is essentially contracting forward in time. However the nearly incompressibility (3.10) yields that the contraction is controlled, as we will see later on.

The proof of the needed estimates is divided into several subsections.

7.2. **Construction of the approximate vector field.** Let

$$H = \frac{1}{\sqrt{\bar{\varepsilon}}} \gg 1, \tag{7.4}$$

and define

$$Q^H(r) = [-Hr, Hr] \times B_r^{d-2}(0) = r^{d-1} Q^H(1).$$

Sometimes will will consider it as embedded into $\mathbb{R}^d$: in this case its definition refers to the coordinates $(x_2, x_{\cancel{1},\cancel{2}}) \in \mathbb{R} \times \mathbb{R}^{d-2}$.

Let $\tilde{\boldsymbol{b}}^H$ be the approximate vector fields defined by

$$\tilde{\boldsymbol{b}}^H(r, y; t, w) = \frac{\bar{\xi}}{\mathcal{L}^{d-1}(Q^H(r))} \begin{cases} -|D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [w_1, 0] \times Q^H(r)) & \text{if } w_1 \leq 0, \\ |D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [0, w_1] \times Q^H(r)) & \text{if } w_1 > 0. \end{cases} \tag{7.5}$$

Notice that it depends only on the first component w_1.

In order to simplify the notation, we will often assume $w_1 \geq 0$, mainly when we need to integrate in intervals $[0, w_1]$; the other case gives exactly the same estimates, as one can check.

We begin with a series of estimates for the vector field $\tilde{\boldsymbol{b}}^H$.

Proposition 7.1. *The following estimates hold:*

(1) *there exists $R_\Omega > 0$ and $K_\Omega \subset S_1$ compact such that*

$$\mathcal{H}^d((S_1 \cap \mathcal{K}_0) \setminus K_\Omega) \leq \mathcal{O}(\bar{\varepsilon})(\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)), \tag{7.6}$$

and each trajectory starting in $y \in K_\Omega$ satisfies $\boldsymbol{X}(t,y) + B_R^d(0) \subset \Omega$ for all $t \in (t^-(y), t^+(y))$, and $\boldsymbol{X}(t^+(y), y) + B_R^d(0) \subset S_2$;

(2) *it holds*

$$\int_{K_\Omega} \int_{B_r^d(0)} \big|\boldsymbol{b}(t, \boldsymbol{X}(t,y) + w) - \boldsymbol{b}(t, \boldsymbol{X}(t,y)) - \tilde{\boldsymbol{b}}^H(r, y; t, w)\big| \,\mathrm{d}w\,\mathrm{d}y \leq 5\bar{C}\sqrt{\bar{\varepsilon}} r \mathcal{L}^d(B_r^d(0))|D\boldsymbol{b}|(\Omega); \tag{7.7}$$

(3) *finally*

$$\int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{B_{r'}^d(0)} |\tilde{\boldsymbol{b}}^H(r, y; t, w)| \,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}y \leq \bar{C} r' \mathcal{L}^d(B_{r'}^d(0))|D\boldsymbol{b}|(\Omega). \tag{7.8}$$

In particular from the first point, if

$$\sqrt{2 + H^2} r < 2Hr < R \quad \text{i.e.} \quad r < \frac{\sqrt{\bar{\varepsilon}}}{2} R,$$

then the set

$$\boldsymbol{X}((t^-(y),t^+(y)),y)+[-r,r]\times Q^H(r)\subset\Omega. \tag{7.9}$$

The choice of R_i of Point (3) of page 17 is done at this step by setting $R_i=\sqrt{\varepsilon}R/2$.

Proof. The first point follows with the same reasoning as in Point (2) of page 23: let K_Ω be a compact set of initial points $y\in S_1\cap\mathcal{K}_1$ with $\boldsymbol{X}(t,y)+B^d_R(0)\subset\Omega$, $t\in(t^-(y),t^+(y))$ and $\boldsymbol{X}(t^+(y),y)+B^d_R(0)\subset S_2$. We can assume that the amount of trajectories we are neglecting is of order

$$\mathcal{H}^d((S_1\cap\mathcal{K}_0)\setminus K_\Omega)\frac{|D\boldsymbol{b}|(\Omega)}{M}\le\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega)$$

for $R\ll 1$. This is (7.6).

For the second point of the statement, by (7.5) we can estimate

$$\begin{aligned}
&\int_{K_\Omega}\int_{B^d_r(0)}\big|\boldsymbol{b}(t,\boldsymbol{X}(t,y)+w)-\boldsymbol{b}(t,\boldsymbol{X}(t,y))-\tilde{\boldsymbol{b}}^H(r,y;t,w)\big|\,\mathrm{d}w\,\mathrm{d}y\\
\text{triangle ineq. }&\le\int_{K_\Omega}\int_{B^d_r(0)}|\boldsymbol{b}_{\cancel{1,2}}(t,\boldsymbol{X}(t,y)+w)-\boldsymbol{b}_{\cancel{1,2}}(t,\boldsymbol{X}(t,y))|\,\mathrm{d}w\,\mathrm{d}y\\
&\quad+\mathcal{L}^d(B^d_r(0))\int_{K_\Omega}\Big|\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y))-\fint_{Q^H(r)}\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+z)\,\mathrm{d}z\Big|\,\mathrm{d}y\\
&\quad+\int_{K_\Omega}\int_{B^d_r(0)}\Big|\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+w)-\fint_{Q^H(r)}\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+w_1\mathbf{e}_1+z)\mathrm{d}z\Big|\,\mathrm{d}w\,\mathrm{d}y\\
&\quad+\int_{K_\Omega}\int_{B^d_r(0)}\Big|\fint_{Q^H(r)}\big(\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+w_1\mathbf{e}_1+z)-\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+z)\big)\,\mathrm{d}z\\
&\qquad\qquad-\frac{\operatorname{sign}(w_1)\bar{\xi}}{\mathcal{L}^{d-1}(Q^H(r))}|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0\wedge w_1,0\vee w_1]\times Q^H(r))\Big|\,\mathrm{d}w\,\mathrm{d}y\\
\text{bring aver. out }&\le\int_{K_\Omega}\int_{B^d_r(0)}|\boldsymbol{b}_{\cancel{1,2}}(t,\boldsymbol{X}(t,y)+w)-\boldsymbol{b}_{\cancel{1,2}}(t,\boldsymbol{X}(t,y))|\,\mathrm{d}w\,\mathrm{d}y\\
&\quad+\mathcal{L}^d(B^d_R(0))\fint_{Q^H(r)}\Big[\int_{K_\Omega}|\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y))-\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+z)|\,\mathrm{d}y\Big]\,\mathrm{d}z\\
&\quad+\fint_{Q^H(r)}\int_{K_\Omega}\int_{B^d_r(0)}\big|\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+w)-\boldsymbol{b}_{1,2}(t,\boldsymbol{X}(t,y)+w_1\mathbf{e}_1+z)\big|\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}z\\
&\quad+\frac{1}{\mathcal{L}^{d-1}(Q^H(r))}\int_{K_\Omega}\int_{B^d_r(0)}\big|D_1\boldsymbol{b}_{1,2}(\boldsymbol{X}(t,y)+[0\wedge w_1,0\vee w_1]\times Q^H(r))\\
&\qquad\qquad-\bar{\xi}|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0\wedge w_1,0\vee w_1]\times Q^H(r))\big|\,\mathrm{d}w\,\mathrm{d}y\\
\text{estim. transl. BV }&\le\bar{C}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}_{\cancel{1,2}}|(\Omega)+\bar{C}Hr\mathcal{L}^d(B^d_r(0))|D_2\boldsymbol{b}_{1,2}|(\Omega)+\bar{C}(1+H)r\mathcal{L}^d(B^d_r(0))|D_2\boldsymbol{b}_{1,2}|(\Omega)\\
&\quad+\bar{C}\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}r|D\boldsymbol{b}|(\Omega)\\
&\underset{(7.2),(7.4)}{\le}2\bar{C}r\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)+\frac{\bar{C}}{\sqrt{\bar{\varepsilon}}}r\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)+\frac{2\bar{C}}{\sqrt{\bar{\varepsilon}}}r\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)\\
&\le 5\bar{C}\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega).
\end{aligned}$$

This yields (7.7).

Finally, for a fixed $w\in B^1_r(0)$, $w_1\ge 0$, recalling that $t^\pm(y)$ are the entering/exiting times for the trajectory $\boldsymbol{X}(t,y)$ and using (7.5)

$$\begin{aligned}
\int_{K_\Omega}\int_{t^-(y)}^{t^+(y)}|\tilde{\boldsymbol{b}}^H(r,y;t,w)|\mathrm{d}t\mathrm{d}y&\underset{(7.5)}{=}\int_{K_\Omega}\int_{t^-(y)}^{t^+(y)}\frac{|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0,w_1]\times Q^H(r))}{\mathcal{L}^{d-1}(Q^H(r))}\,\mathrm{d}t\,\mathrm{d}y\\
\text{near. incompr. }&\le\bar{C}|w_1||D\boldsymbol{b}|(\Omega).
\end{aligned}\tag{7.10}$$

Thus integrating we get for every $r' \leq r$ ($|w_1| \leq r$)

$$\int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{B^d_{r'}(0)} |\tilde{\boldsymbol{b}}^H(r, y; t, w)| \, \mathrm{d}w \, \mathrm{d}t \, \mathrm{d}y \leq \bar{C} r' \mathcal{L}^d(B^d_{r'0}(0)) |D\boldsymbol{b}|(\Omega).$$

This concludes the proof of (7.8). □

7.3. **Estimate on the first component $\mathbf{e}_1$.** The ODE for the first component is

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\boldsymbol{X}}^H_1(r, y; t, z) = \tilde{\boldsymbol{b}}^H_1(r, y; t, \tilde{\boldsymbol{X}}^H_1(r, y; t, z)), \qquad \tilde{\boldsymbol{X}}^H_1(r, y; t^+(y), z) = z. \tag{7.11}$$

The first observation is that by the choice (7.3), it holds for $z_1 \leq z_2$

$$(\tilde{\boldsymbol{b}}^H_1(r, y; t, z_1) - \tilde{\boldsymbol{b}}^H_1(r, y; t, z_2))(z_1 - z_2) \underset{(7.5)}{=} \frac{\bar{\xi}_1}{\mathcal{L}^{d-1}(Q^H(r))} |D\boldsymbol{b}|(\boldsymbol{X}(t, y) + [z_1, z_2] \times Q^H(r))(z_1 - z_2) \underset{(7.3)}{\leq} 0.$$

We have thus proved the following result.

Lemma 7.2. *The first component $\tilde{\boldsymbol{X}}^H_1(r, y; t, z)$ is a contraction w.r.t. the initial data z, and $\tilde{\boldsymbol{X}}^H_1(r, y; t, 0) = 0$.*

Hence in particular solutions with initial data $w_1 \geq 0$ remain positive.

We can use thus Bressan's estimate on Lipschitz flow to compare $\tilde{\boldsymbol{X}}^H_1$ with the real flow, Corollary A.2 and Point 2 of Page 2. For a.e. trajectory $\boldsymbol{X}(t, y')$, let $\cup_i (t^-_i(y, y'), t^+_i(y, y'))$ be the set of time where it belongs to the cylinder $\boldsymbol{X}(t, y) + B^d_r(0)$, so that for every $t \in (t^-_i(y, y'), t^+_i(y, y'))$ it holds

$$\begin{aligned}
&\big|\boldsymbol{X}_1(t, y') - \boldsymbol{X}_1(t, y) - \tilde{\boldsymbol{X}}^H_1\big(r, y; t, \boldsymbol{X}_1(t^-_i(y, y'), y') - \boldsymbol{X}_1(t^-_i(y, y'), y)\big)\big| \\
\text{Corollary A.2 } &\leq \int_{t^-_i(y,y')}^{t} \big|\boldsymbol{b}_1(s, \boldsymbol{X}(s, y')) - \boldsymbol{b}_1(s, \boldsymbol{X}(s, y)) - \tilde{\boldsymbol{b}}^H_1\big(r, y; s, \boldsymbol{X}_1(s, y') - \boldsymbol{X}_1(s, y)\big)\big| \, \mathrm{d}s \\
&\leq \int_{t^-_i(y,y')}^{t^+_i(y,y')} \big|\boldsymbol{b}_1(s, \boldsymbol{X}(s, y')) - \boldsymbol{b}_1(s, \boldsymbol{X}(s, y)) - \tilde{\boldsymbol{b}}^H_1\big(r, y; s, \boldsymbol{X}_1(s, y') - \boldsymbol{X}_1(s, y)\big)\big| \, \mathrm{d}s.
\end{aligned}$$

Integrating over all $y' \in K_\Omega$ we obtain

$$\begin{aligned}
&\int_{K_\Omega} \big\|\boldsymbol{X}_1(t, y') - \boldsymbol{X}_1(t, y) - \tilde{\boldsymbol{X}}^H_1\big(r, y; t, \boldsymbol{X}_1(t^-_i(y, y'), y') - \boldsymbol{X}_1(t^-_i(y, y'), y)\big)\big\|_{L^\infty(t^-(y,y'),t^+(y,y'))} \mathrm{d}y' \\
&\leq \int_{K_\Omega} \int_{t^-_i(y,y')}^{t^+_i(y,y')} \big|\boldsymbol{b}_1(s, \boldsymbol{X}(s, y')) - \boldsymbol{b}_1(s, \boldsymbol{X}(s, y)) - \tilde{\boldsymbol{b}}^H_1\big(r, y; s, \boldsymbol{X}_1(s, y') - \boldsymbol{X}_1(s, y)\big)\big| \, \mathrm{d}s \, \mathrm{d}y' \\
\text{near. inc. as in (5.8) } &\leq \bar{C} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y)} \big|\boldsymbol{b}_1(s, \boldsymbol{X}(s, y) + z) - \boldsymbol{b}_1(s, \boldsymbol{X}(s, y)) - \tilde{\boldsymbol{b}}^H_1(r, y; s, z)\big| \, \mathrm{d}s \, \mathrm{d}z.
\end{aligned}$$

Integrating over all $y \in K_\Omega$ we obtain

$$
\begin{aligned}
&\int_{K_\Omega}\int_{\mathbb{R}^d} \big\|\boldsymbol{X}_1(t,y') - \boldsymbol{X}_1(t,y) - \tilde{\boldsymbol{X}}_1^H(r,y;t,\boldsymbol{X}_1(t^-(y,y'),y') - \boldsymbol{X}_1(t^-(y,y'),y))\big\|_{L^\infty(t_i^-(y,y'),t_i^+(y,y'))}\,\mathrm{d}y'\,\mathrm{d}y \\
&\leq \bar{C}\int_{K_\Omega}\int_{B^d_r(0)}\int_{t^-(y)}^{t^+(y)} \big|\boldsymbol{b}_1(s,\boldsymbol{X}(s,y)+z) - \boldsymbol{b}_1(s,\boldsymbol{X}(s,y)) - \tilde{\boldsymbol{b}}_1^H(r,y;s,z)\big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.5)}{=} \bar{C}\int_{K_\Omega}\int_{B^d_r(0)}\int_{t^-(y)}^{t^+(y)} \Big|\boldsymbol{b}_1(s,\boldsymbol{X}(s,y)+z) - \boldsymbol{b}_1(s,\boldsymbol{X}(s,y)) \\
&\qquad\qquad - \frac{\mathrm{sign}(z_1)\bar{\xi}_1}{\mathcal{L}^{d-1}(Q^H(r))}|D\boldsymbol{b}|(\boldsymbol{X}(s,y)+[0\wedge z_1, 0\vee z_1]\times Q^H(r))\Big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\leq \bar{C}^2\int_{\Omega}\int_{B^d_r(0)} \Big|\boldsymbol{b}_1(s,x+z) - \boldsymbol{b}_1(s,x) - \frac{\mathrm{sign}(z_1)\bar{\xi}_1}{\mathcal{L}^{d-1}(Q^H(r))}|D\boldsymbol{b}(s)|(x+[0\wedge z_1, 0\vee z_1]\times Q^H(r))\Big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\leq \bar{C}^2\int_{\Omega}\int_{B^d_r(0)} \Big|\boldsymbol{b}_1(s,x+z) - \fint_{Q^H(r)} \boldsymbol{b}_1(s,x+z_1\mathbf{e}_1+z^\perp)\,\mathrm{d}z^\perp\Big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \bar{C}^2\int_{\Omega}\int_{B^d_r(0)} \Big|\boldsymbol{b}_1(s,x) - \fint_{Q^H(r)} \boldsymbol{b}_1(s,x+z^\perp)\,\mathrm{d}z^\perp\Big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\quad + \bar{C}^2\int_{\Omega}\int_{B^d_r(0)} \Big|\fint_{Q^H(r)} \boldsymbol{b}_1(s,x+z_1\mathbf{e}_1+z^\perp) - \boldsymbol{b}_1(s,x+z^\perp)\,\mathrm{d}z^\perp \\
&\qquad\qquad - \frac{\mathrm{sign}(w_1)\bar{\xi}_1}{\mathcal{L}^{d-1}(Q^H(r))}|D\boldsymbol{b}(s)|(x+[0\wedge z_1, 0\vee z_1]\times Q^H(r))\Big|\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.2)}{\leq} 2\bar{C}^2\bar{\varepsilon}(Hr+r)\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega) \\
&\leq 3\bar{C}^2\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega),
\end{aligned}
\tag{7.12}
$$

where we observed that the trajectories in the comparison we never exit the set Ω. We collect the above estimate in the following Proposition.

Proposition 7.3. *We have the following estimate: if $\boldsymbol{X}(t,y') \in B^d_R(\boldsymbol{X}(t,y))$ for $t \in (t_i^-(y,y'), t_i^+(y,y'))$, then*

$$
\begin{aligned}
&\int_{K_\Omega}\int_{\mathbb{R}^d} \big\|\boldsymbol{X}_1(t,y') - \boldsymbol{X}_1(t,y) - \tilde{\boldsymbol{X}}_1^H(r,y;t,\boldsymbol{X}_1(t_i^-(y,y'),y') - \boldsymbol{X}_1(t_i^-(y,y'),y))\big\|_{L^\infty(t_i^-(y,y'),t_i^+(y,y'))}\,\mathrm{d}y'\,\mathrm{d}y \\
&\qquad \leq 3\bar{C}^2\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega).
\end{aligned}
\tag{7.13}
$$

From Proposition 7.3, by means of the Chebyshev's inequality, we obtain the following result.

Corollary 7.4. *The set*

$$
\begin{aligned}
E_1^1(y,r) = \big\{&z \in B^d_r(0)\cap(K_\Omega - y) : \\
&\big\|\boldsymbol{X}_1(\cdot,y+z) - \boldsymbol{X}_1(\cdot,y) - \tilde{\boldsymbol{X}}_1^H(r,y;t_i^-(y,y'),z)\big\|_{C^0(t_i^-(y,y'),t_i^+(y,y'))} \geq (\bar{\varepsilon})^{1/4}r\big\}
\end{aligned}
\tag{7.14}
$$

has measure bounded by

$$
\begin{aligned}
&\int_{K_\Omega}\mathcal{L}^d(E_1(y,r))\,\mathrm{d}y \\
&\leq \frac{1}{(\bar{\varepsilon})^{1/4}r}\int_K\int_{B^d_r(0)} \|\boldsymbol{X}_1(\cdot,y+z) - \boldsymbol{X}_1(\cdot,y) - \tilde{\boldsymbol{X}}_1^H(r,y;t_i^-(y,y'),x)\|_{C^0(t_i^-(y,y'),t_i^+(y,y'))}\,\mathrm{d}z\,\mathrm{d}y \\
&< 4\bar{C}^2(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega).
\end{aligned}
\tag{7.15}
$$

7.4. **Comparison with the disintegration.** Aim of this section is to compare $\tilde{\boldsymbol{b}}^H(r,y;t,w)$ with the disintegrated measure $(D\boldsymbol{b}_{1,2})_y w_1$. Being these measures singular, the estimate is done by considering their time integral: this reflects the fact that we want to compare the flow generated by $\tilde{\boldsymbol{b}}^H$ and $(D\boldsymbol{b})_y$, not the vector fields themselves.

Here we need to consider the flow $\tilde{\boldsymbol{X}}_1^H$ generated by the vector field (7.11). The proof of the final theorem requires several steps, which are listed below.

7.4.1. *Estimate of the flow across the lateral boundary.* We have the following

Lemma 7.5. *The flow $\Phi_L(y)$ across the lateral boundary of the set*

$$\bigcup_{t\in(t^-(y),t^+(y))} \boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times Q^H(r),$$

for every $0 \le w_1 \le r$, can be estimated as

$$\int_{K_\Omega} \Phi_L(y)\mathrm{d}y \le \bar{C}\Big(C_d\sqrt{\bar{\varepsilon}} + \frac{4r\sqrt{\bar{\varepsilon}}}{|w_1|}\Big)|w_1|\mathcal{L}^{d-1}(Q^H(r))|D\boldsymbol{b}|(\Omega), \tag{7.16}$$

where C_d is a constant depending only on the dimension d of the space.

Proof. Write as before

$$\mathcal{K}_\Omega = \bigcup_{y\in K_\Omega} X((t^-(y),t^+(y)),y),$$

and estimate:

(1) the flow across the surface $\{0\}\times Q^H(r)$, whose normal is $(0,-\mathbf{e}_1)$: by using nearly incompressibility

$$\begin{aligned}
&\int_{K_\Omega}\int_{t^-(y)}^{t^+(y)}\int_{Q^H(r)} |\boldsymbol{b}_1(t,\boldsymbol{X}(t,y)+z) - \boldsymbol{b}_1(t,\boldsymbol{X}(t,y))|\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad \le \bar{C}\int_{\mathcal{K}_\Omega}\int_{Q^H(r)} |\boldsymbol{b}_1(t,x+z) - \boldsymbol{b}_1(t,x)|\,\mathrm{d}z\,\mathrm{d}x \\
&\qquad \le \bar{C}\int_{Q^H(r)}\Big(\int_\Omega |D_\gamma \boldsymbol{b}_1|\,\mathrm{d}x\,\mathrm{d}t\Big)|z|\,\mathrm{d}z \\
&\qquad \underset{(7.2)}{\le} \bar{C}Hr\mathcal{L}^{d-1}(Q^H(r))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega) \\
&\qquad \underset{(7.4)}{\le} \bar{C}\frac{r\sqrt{\bar{\varepsilon}}}{|w_1|}|w_1|\mathcal{L}^{d-1}(Q^H(r))|D\boldsymbol{b}|(\Omega);
\end{aligned} \tag{7.17}$$

(2) the flow across the surface $\{(t, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)), t \in (t^-(y), t^+(y))\} \times Q^H(r)$, whose normal is $(-\tilde{\boldsymbol{b}}_1^H(r,y;t,\tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)), \mathbf{e}_1)$:

$$\begin{aligned}
&\int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{Q^H(r)} \big| \boldsymbol{b}_1(t, \boldsymbol{X}(t,y) + \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)\mathbf{e}_1 + z) \\
&\qquad\qquad - \boldsymbol{b}_1(t,\boldsymbol{X}(t,y)) - \tilde{\boldsymbol{b}}_1^H(r,y;t,\tilde{\boldsymbol{X}}_1^H(r,y;t,w_1))\big| \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
&\underset{(7.12)}{\leq} \int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{Q^H(r)} \Big| \boldsymbol{b}_1(t, \boldsymbol{X}(t,y) + \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)\mathbf{e}_1 + z) \\
&\qquad\qquad - \fint_{Q^H(r)} \boldsymbol{b}_1(t, \boldsymbol{X}(t,y) + \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)\mathbf{e}_1 + z')\mathrm{d}z' \Big| \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_{\mathcal{K}_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{Q^H(r)} \Big| \boldsymbol{b}_1(t,\boldsymbol{X}(t,y)) - \fint_{Q^H(r)} \boldsymbol{b}_1(s, \boldsymbol{X}(t,y) + z')\mathrm{d}z' \Big| \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int_{\mathcal{K}_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{Q^H(r)} \Big| \fint_{Q^H(r)} \boldsymbol{b}_1(s, \boldsymbol{X}(t,y) + \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)\mathbf{e}_1 + z') - \boldsymbol{b}_1(s, \boldsymbol{X}(t,y)+z')\,\mathrm{d}z' \\
&\qquad\qquad - \frac{\bar{\xi}_1}{\mathcal{L}^{d-1}(Q^H(r))} |D\boldsymbol{b}(s)|(\boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times Q^H(r)) \Big| \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
&\leq \bar{C}(|w_1| + 2Hr)\mathcal{L}^{d-1}(Q^H(r))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega) \\
&\leq \bar{C}\Big(\bar{\varepsilon} + \frac{2r\sqrt{\bar{\varepsilon}}}{|w_1|}\Big)|w_1|\mathcal{L}^{d-1}(Q^H(r))|D\boldsymbol{b}|(\Omega) \\
&\leq \bar{C}\Big(\frac{3r\sqrt{\bar{\varepsilon}}}{|w_1|}\Big)|w_1|\mathcal{L}^{d-1}(Q^H(r))|D\boldsymbol{b}|(\Omega);
\end{aligned} \tag{7.18}$$

(3) the flow across the surface $(0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)) \times [-Hr, Hr] \times \partial B_r^{d-2}(0)$: if $\boldsymbol{b}^\perp$ is the component not 1 or 2, then as above

$$\begin{aligned}
&\int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{(0,\tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)) \times [-Hr,Hr] \times \partial B_r^{d-2}(0)} |\boldsymbol{b}^\perp(t, \boldsymbol{X}(t,y)+z) - \boldsymbol{b}^\perp(t,\boldsymbol{X}(t,y))| \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t \\
\text{near. inc.} \quad &\leq \bar{C} \int_{\mathcal{K}_\Omega} \int_{(0,\tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)) \times [-Hr,Hr] \times \partial B_r^{d-2}(0)} |\boldsymbol{b}^\perp(t,x+z) - \boldsymbol{b}^\perp(t,x)| \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t \\
\tilde{\boldsymbol{X}}_1 \text{ contr.} \quad &\leq \bar{C} \int_{(0,w_1) \times [-Hr,Hr] \times \partial B_r^{d-2}(0)} \Big(\int_\Omega |D\boldsymbol{b}^\perp| \,\mathrm{d}x\,\mathrm{d}t \Big) |z| \,\mathrm{d}z \\
(|z| \leq 2Hr) \quad &\leq \bar{C}(2Hr)^2 |w_1| \mathcal{H}^{d-3}(\partial B_r^{d-2}(0)) \bar{\varepsilon} |D\boldsymbol{b}|(\Omega) \\
&= 4\bar{C}H \frac{r\mathcal{H}^{d-3}(\partial B_r^{d-2}(0))}{\mathcal{L}^{d-2}(B_r^{d-2}(0))} |w_1| (Hr\mathcal{L}^{d-2}(B_r^{d-2}(0))) \bar{\varepsilon} |D\boldsymbol{b}|(\Omega) \\
&\underset{(7.4)}{\leq} \bar{C}C_d \sqrt{\bar{\varepsilon}} |w_1| \mathcal{L}^{d-1}(Q^H(r)) |D\boldsymbol{b}|(\Omega);
\end{aligned} \tag{7.19}$$

(4) the flow across the surface $[0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times \{Hr\} \times B_r^{d-2}(0)$, whose normal is $\mathbf{e}_2$:

$$
\begin{aligned}
& \int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{[0,\tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times B_r^{d-2}(0)} |\boldsymbol{b}_2(t, \boldsymbol{X}(t,y) + Hr\mathbf{e}_2 + z) - \boldsymbol{b}_2(t, \boldsymbol{X}(t,y))| \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}t \\
\tilde{\boldsymbol{X}}_1 \text{ contr. } & \leq \int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{[0,w_1] \times B_r^{d-2}(0)} |\boldsymbol{b}_2(t, \boldsymbol{X}(t,y) + Hr\mathbf{e}_2 + z) - \boldsymbol{b}_2(t, \boldsymbol{X}(t,y) + Hr\mathbf{e}_2)| \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}t \\
& \quad + \int_{K_\Omega} \int_{t^-(y)}^{t^+(y)} \int_{[0,w_1] \times B_r^{d-2}(0)} |\boldsymbol{b}_2(t, \boldsymbol{X}(t,y) + Hr\mathbf{e}_2) - \boldsymbol{b}_2(t, \boldsymbol{X}(t,y))| \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}t \\
\text{near. inc. } & \leq \bar{C} \int_{[0,w_1] \times B_r^{d-2}(0)} \Big(\int_\Omega |D\boldsymbol{b}_2| \, \mathrm{d}x \, \mathrm{d}t \Big) |z| \, \mathrm{d}z + \bar{C} Hr |w_1| \mathcal{L}^{d-2}(B_r^{d-2}(0)) |D_2 \boldsymbol{b}_2|(\Omega) \\
& \leq \bar{C}(|w_1| + r)|w_1| \mathcal{L}^{d-2}(B_r^{d-2}(0)) |D\boldsymbol{b}|(\Omega) + \bar{C} Hr |w_1| \mathcal{L}^{d-2}(B_r^{d-2}(0)) \bar{\varepsilon} |D\boldsymbol{b}|(\Omega) \\
& \leq \bar{C} \Big(\frac{1}{H} + \frac{\bar{\varepsilon}}{2} \Big) |w_1| \mathcal{L}^{d-1}(Q^H(r)) |D\boldsymbol{b}|(\Omega) \\
& \leq \bar{C} C_d \sqrt{\bar{\varepsilon}} |w_1| \mathcal{L}^{d-1}(Q^H(r)) |D\boldsymbol{b}|(\Omega).
\end{aligned}
\tag{7.20}
$$

The same for the surface $[0, w_1] \times \{-Hr\} \times B_r^{d-2}(0)$.

Summing up the estimates (7.17), (7.18), (7.19) and (7.20) we obtain the statement (7.16). □

7.4.2. *First selection of initial point in order to have continuity of the flow and disintegration.* Consider a compact set $K_{\Omega,1} \subset K_\Omega$ of trajectories $\boldsymbol{X}(t,y)$, where $y \mapsto \boldsymbol{X}(t,y)$ is continuous in C^0, and such that the disintegration $y \mapsto (D\boldsymbol{b})_y$ is weakly continuous in the sense of measures and $m \llcorner_{K_{\Omega,1}} = \mathcal{L}^d \llcorner_{K_{\Omega,1}}$, which means that the singular part of m has measure 0 on K_{Ω_1}.

Since we have

$$
\int_N |D\boldsymbol{b}(t)|(\mathbb{R}^d) \mathrm{d}t = \int_{\mathbb{R}^d} |(D\boldsymbol{b})_y|(N) m(\mathrm{d}y),
$$

then it follows that if $\mathcal{L}^1(N) = 0$ then

$$
|(D\boldsymbol{b})_y|(N) = 0 \qquad m\text{-a.e. y}.
$$

In particular we can assume that the initial and end sets $\{t^-(y)\}_{y \in K_{\Omega,1}}, \{t^+(y)\}_{y \in K_{\Omega,1}}$ have measure 0 w.r.t. $(D\boldsymbol{b})_y$, and thus in $K_{\Omega,1}$ it holds

$$
y \mapsto (D\boldsymbol{b})_y((t^-(y), t^+(y)) \quad \text{is continuous with continuity modulus } \omega_{\mathrm{dis}}.
$$

We can also take a second compact set $K_{\Omega,2}$ made of Lebesgue points of $K_{\Omega,1}$ and such that the limits

$$
\frac{\mathcal{L}^d(B_r^d(y) \cap K_{\Omega,1})}{\mathcal{L}^d(B_r^d(0))} \to 1,
$$

$$
\fint_{B_r^d(0) \cap K_{\Omega,1}} \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(t,y)\|_{C^0((t^-(y),t^+(y))} \mathrm{d}z \to 0,
$$

$$
\fint_{B_r^d(0)} \big| (D\boldsymbol{b})_{y+z}((t^-(y), t^+(y))) - (D\boldsymbol{b})_y((t^-(y), t^+(y))) \big| m(\mathrm{d}z) \to 0
$$

are uniform with continuity modulus $\omega_{\mathrm{dis}}(r)$ (eventually changing ω_{dis} of (7.22)). The total error can be taken

$$
\mathcal{H}^d(K_\Omega \setminus K_{\Omega,2}) < \bar{\varepsilon} \mathcal{L}^{d+1}(\Omega)
$$

by Egorov and Lusin Theorems.

We thus have proved the following lemma.

Lemma 7.6. *There exist two compact sets $K_{\Omega,2} \subset K_{\Omega,1} \subset K_\Omega$ such that the following holds:*

(1) *their difference in measure is small, i.e.*

$$
\mathcal{H}^d(K_\Omega \setminus K_{\Omega,2}) < \bar{\varepsilon} \mathcal{L}^{d+1}(\Omega); \tag{7.21}
$$

(2) *the maps $K_{\Omega,1} \ni y \mapsto \boldsymbol{X}(t,y)$ is continuous in C^0 with modulus of continuity ω_{dis};*

(3) *for every $y \in K_{\Omega,1}$ it holds*

$$(D\boldsymbol{b})_y(\{t^\pm(y)\}) = 0$$

and

$$y \mapsto (D\boldsymbol{b})_y((t^-(y), t^+(y))) \quad \text{is continuous with modulus } \omega_{\mathrm{dis}}; \tag{7.22}$$

(4) *the compact set $K_{\Omega,2}$ is made of Lebesgue points of $K_{\Omega,1}$ such that*

$$\Big|\frac{\mathcal{L}^d(B^d_r(y) \cap K_{\Omega,1})}{\mathcal{L}^d(B^d_r(0))} - 1\Big| \leq \omega_{\mathrm{dis}}(r), \tag{7.23a}$$

$$\fint_{B^d_r(0)\cap K_{\Omega,1}} \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(t,y)\|_{C^0((t^-(y),t^+(y))}\mathrm{d}z \leq \omega_{\mathrm{dis}}(r), \tag{7.23b}$$

$$\fint_{B^d_r(0)} \big|(D\boldsymbol{b})_{y+z}((t^-(y), t^+(y))) - (D\boldsymbol{b})_y((t^-(y), t^+(y)))\big| m(\mathrm{d}z) \leq \omega_{\mathrm{dis}}(r). \tag{7.23c}$$

7.4.3. *Comparison of approximate flow with the disintegration.* Aim of this part is to prove the following results.

Proposition 7.7. *If $r \leq \hat{r}(\bar{\varepsilon})$, it holds*

$$\begin{aligned}\int_{K_{\Omega,2}} \int_{B^d_r(0)} \Big| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))\,\mathrm{d}t - (D\boldsymbol{b}_{1,2})_y((0,t))w\Big| \,\mathrm{d}w\,\mathrm{d}y \\ \leq C_d\bar{C}\sqrt{\bar{\varepsilon}} r\mathcal{L}^d(B^b_r(0))[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] + \bar{C}|D\boldsymbol{b}|(\Omega \setminus \mathcal{K}_0) r\mathcal{L}^d(B^d_r(0)).\end{aligned} \tag{7.24}$$

Proof. We estimate the difference of the approximate vector fields $\tilde{\boldsymbol{b}}^H_i(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))$ and the disintegration $(D_1\boldsymbol{b}_i)_y w_1$ for a fixed w_1 (> 0 for definiteness), with $i = 1,2$. The proof is given in several steps.

Step 1. By using (7.5)

$$\begin{aligned}
&\int_{K_{\Omega,2}} \Big| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H_{1,2}(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))\,\mathrm{d}t - (D_1\boldsymbol{b}_{1,2})_y((t^-(y),t^+(y))w_1\Big|\,\mathrm{d}y \\
&\underset{(7.5)}{=} \int_{K_{\Omega,2}} \Big| \int_{t^-(y)}^{t^+(y)} \bar{\xi}\frac{|D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r))}{\mathcal{L}^{d-1}(Q^H(r))}\,\mathrm{d}t - (D_1\boldsymbol{b}_{1,2})_y((t^-(y),t^+(y))w_1\Big|\,\mathrm{d}y \\
&\leq \int_{K_{\Omega,2}} \Big| \int_{t^-(y)}^{t^+(y)} \frac{1}{\mathcal{L}^{d-1}(Q^H(r))}\Big[\bar{\xi}|D\boldsymbol{b}|(\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r)) \\
&\qquad - (D\boldsymbol{b})_{1,2}(\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r))\Big]\,\mathrm{d}t\Big|\,\mathrm{d}y \\
&\quad + |w_1|\int_{K_{\Omega,2}} \Big|\frac{1}{|w_1|\mathcal{L}^{d-1}(Q^H(r))}\Big[\int_{t^-(y)}^{t^+(y)} (D\boldsymbol{b})_{1,2}(\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r))\,\mathrm{d}t \\
&\qquad - \int_{K_0} (D_1\boldsymbol{b}_{1,2})_z\big(\cup_t \{\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r)\}\big)\mathrm{d}z\Big]\Big|\,\mathrm{d}y \\
&\quad + |w_1|\int_{K_{\Omega,2}} \Big|\frac{1}{|w_1|\mathcal{L}^{d-1}(Q^H(r))}\int_{K_0} (D_1\boldsymbol{b}_{1,2})_z\big(\cup_t \{\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r)\}\big)\,\mathrm{d}z \\
&\qquad - (D_1\boldsymbol{b}_{1,2})_y((t^-(y),t^+(y))\Big|\,\mathrm{d}y, \\
&\underset{(7.2),(3.10)}{\leq} \bar{\varepsilon}|w_1||D\boldsymbol{b}|(\Omega) + \bar{C}|w_1||D\boldsymbol{b}|(\Omega \setminus \mathcal{K}_0) \\
&\quad + \bar{C}|w_1|\int_{K_{\Omega,2}} \Big|\frac{1}{|w_1|\mathcal{L}^{d-1}(Q^H(r))}\int_{K_0} (D_1\boldsymbol{b}_{1,2})_z\big(\cup_t \{\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1)] \times Q^H(r)\}\big)\,\mathrm{d}z \\
&\qquad - (D_1\boldsymbol{b}_{1,2})_y((t^-(y),t^+(y))\Big|\,\mathrm{d}y,
\end{aligned} \tag{7.25}$$

where in the last step we use the definition of disintegration.

We estimate the integral

$$\int_{K_0} (D_1 \boldsymbol{b}_{1,2})_z \big(\{ \cup_t \boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times Q^H(r) \} \big) \, \mathrm{d}z$$

by

$$\int_{K_0} [\dots] \, \mathrm{d}z = \Big\{ \int_{K^{\mathrm{in}}_{\Omega,1}(y)} + \int_{K_\Omega \setminus K^{\mathrm{in}}_{\Omega,1}(y)} \Big\} [\dots] \, \mathrm{d}z \tag{7.26}$$

where $K^{\mathrm{in}}_{\Omega,1}(y)$ are the trajectories in $K_{\Omega,1}$ which remain inside $\cup_t \boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times Q^H(r)$. Recall that we denote with $\mathcal{K}_\Omega$ the union of the graph of the trajectories starting in K_Ω, and the same with $\mathcal{K}_{\Omega,1}$.

Step 2. We write the last term of (7.25) as

$$\begin{aligned}
&\int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \int_{K_\Omega} [\dots] \, \mathrm{d}z - (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big| \, \mathrm{d}y \\
&\underset{(7.26)}{=} \int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \Big\{ \int_{K^{\mathrm{in}}_{\Omega,1}(y)} + \int_{K_\Omega \setminus K^{\mathrm{in}}_{\Omega,1}} \Big\} [\dots] \mathrm{d}z - (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big| \, \mathrm{d}y \\
&\le \int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \int_{K^{\mathrm{in}}_{\Omega,1}(y)} [\dots] \, \mathrm{d}z - (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big| \, \mathrm{d}y \\
&\quad + \int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \int_{K_\Omega \setminus K^{\mathrm{in}}_{\Omega_1}} [\dots] \, \mathrm{d}z \Big| \, \mathrm{d}y.
\end{aligned}$$

We have:

(1) *term $K^{\mathrm{in}}_{\Omega,1}(y)$:* in this set the measure $(D_1 \boldsymbol{b}_i)_z$ are continuous by (7.22) and the trajectories remain inside the set by the definition of $K^{\mathrm{in}}_{\Omega,1}(y)$, so that

$$\begin{aligned}
&\int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \int_{K^{\mathrm{in}}_{\Omega,1}(y)} [\dots] m(\mathrm{d}z) - (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big| \, \mathrm{d}y \\
\text{traj. are inside} \quad &\underset{m \llcorner K_{\Omega,1} \ll \mathcal{L}^d}{=} \int_{K_{\Omega,2}} \Big| \frac{1}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \int_{K^{\mathrm{in}}_{\Omega,1}(y)} (D_1 \boldsymbol{b}_{1,2})_z((t^-(y), t^+(y))) \, \mathrm{d}z \\
&\qquad\qquad - (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big| \, \mathrm{d}y \\
(D_1 \boldsymbol{b}_2)_z \text{ continuous} \quad &\le \int_{K_{\Omega,2}} \Big[\frac{\mathcal{L}^d(K^{\mathrm{in}}_{\Omega,1}(y))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \omega_{\mathrm{dis}}(Hr) + \Big| (D_1 \boldsymbol{b}_{1,2})_y((t^-(y), t^+(y))) \Big(\frac{\mathcal{L}^d(K^{\mathrm{in}}_{\Omega,1}(y))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} - 1 \Big) \Big| \Big] \, \mathrm{d}y \\
(D_1 \boldsymbol{b}_2)_y \text{ bounded} \quad &\le \int_{K_{\Omega,2}} \Big[\frac{\mathcal{L}^d(K^{\mathrm{in}}_{\Omega,1}(y))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \omega_{\mathrm{dis}}(Hr) + M \Big(1 - \frac{\mathcal{L}^d(K^{\mathrm{in}}_{\Omega,1}(y))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \Big) \Big] \, \mathrm{d}y \\
\text{see below} \quad &\le \int_{K_{\Omega,2}} \Big[\frac{\mathcal{L}^d(K^{\mathrm{in}}_{\Omega,1}(y))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \omega_{\mathrm{dis}}(Hr) + M \min \Big\{ 1, \frac{\omega_{\mathrm{dis}}(2Hr) \mathcal{L}^d(B^d_{2Hr}(0))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \Big\} \Big] \, \mathrm{d}y \\
&\quad + M \int_{K_{\Omega,2}} \frac{(\text{exiting flow})}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \, \mathrm{d}y \\
K^{\mathrm{in}}_{\Omega,1}(y) \subset S_1 \cap B^d_{2Hr}(y) \quad &\underset{(7.16)}{\le} 2M \min \Big\{ 1, \frac{\omega_{\mathrm{dis}}(2Hr) \mathcal{L}^d(B^d_{2Hr}(0))}{|w_1| \mathcal{L}^{d-1}(Q^H(r))} \Big\} \mathcal{L}^d(K_{\Omega,2}) \\
&\quad + M \bar{C} \Big(C_d \sqrt{\bar{\varepsilon}} + \frac{4r\sqrt{\bar{\varepsilon}}}{|w_1|} \Big) |D\boldsymbol{b}|(\Omega),
\end{aligned}$$

where we have observed that

$$\begin{aligned}
&z \in [0, w_1] \times Q^H(r) \setminus K^{\mathrm{in}}_{\Omega,1} \\
&\quad \subset (B^d_{2Hr}(0) \setminus K_{\Omega,1}) \cup (K_{\Omega,1} \cap (\text{trajectories exiting from} \bigcup_t \boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}_1^H(r,y;t,w_1)] \times Q^H(r))),
\end{aligned}$$

and by (7.23a)

$$\mathcal{L}^d(B^d_{2Hr}(0) \setminus K_{\Omega,1}) \leq \omega_{\mathrm{dis}}(2Hr)\mathcal{L}^d(B^d_{2Hr}(0));$$

(2) *term* $K_\Omega \setminus K^{\mathrm{in}}_{\Omega_1}$*:* these trajectories satisfy $|(D_1\boldsymbol{b}_i)_z| \leq M$ and exit, so that

$$\begin{aligned}
&\int_{K_{\Omega,2}} \Big| \frac{1}{|w_1|\mathcal{L}^{d-1}(Q^H(r))} \int_{K_\Omega \setminus K^{\mathrm{in}}_{\Omega_1}} [\dots]\mathrm{d}z \Big| \,\mathrm{d}y \\
&\leq \frac{M}{|w_1|\mathcal{L}^{d-1}(Q^H(r))} \int_{K_{\Omega,2}} [m\text{-measure of exiting/entering trajectories}]\,\mathrm{d}y \\
&\leq \frac{M}{|w_1|\mathcal{L}^{d-1}(Q^H(r))} \int_{K_{\Omega,2}} [\text{flow on the boundary}]\,\mathrm{d}y \\
&\underset{(7.16)}{\leq} \frac{M}{|w_1|\mathcal{L}^{d-1}(Q^H(r))} \Big[\bar{C}\Big(C_d\sqrt{\bar\varepsilon} + \frac{4r\sqrt{\bar\varepsilon}}{|w_1|}\Big)|w_1|\mathcal{L}^{d-1}(Q^H(r))|D\boldsymbol{b}|(\Omega)\Big] \\
&\leq M\bar{C}\Big(C_d\sqrt{\bar\varepsilon} + \frac{4r\sqrt{\bar\varepsilon}}{|w_1|}\Big)|D\boldsymbol{b}|(\Omega);
\end{aligned}$$

Finally, collecting all estimates,

$$\begin{aligned}
&\int_{K_{\Omega,2}} \Big| \frac{1}{|w_1|\mathcal{L}^{d-1}(Q^H(r))} \int_{K_\Omega} [\dots] m(\mathrm{d}z) - (D_1\boldsymbol{b}_2)_y((t^-(y), t^+(y))) \Big| \,\mathrm{d}y \\
&\leq 2M \min\Big\{1, \frac{\omega_{\mathrm{dis}}(2Hr)\mathcal{L}^d(B^d_{2Hr}(0))}{|w_1|\mathcal{L}^{d-1}(Q^H(r))}\Big\}\mathcal{L}^d(K_{\Omega,2}) + 2M\bar{C}\Big(C_d + \frac{4r}{|w_1|}\Big)\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega).
\end{aligned} \tag{7.27}$$

Step 3. We thus have

$$\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \Big| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r, y; t, \tilde{\boldsymbol{X}}^H_1(r, y; t, w_1))\,\mathrm{d}t - (D_1\boldsymbol{b}_{1,2})_y((t^-(y), t^+(y)))w_1 \Big| \,\mathrm{d}w\,\mathrm{d}y \\
&\underset{(7.25),(7.27)}{\leq} \int_{B^d_r(0)} |w_1| \Big[\bar\varepsilon|D\boldsymbol{b}|(\Omega) + \bar{C}|D\boldsymbol{b}|(\Omega \setminus \mathcal{K}_0) \\
&\qquad + 2M \min\Big\{1, \frac{\omega_{\mathrm{dis}}(2Hr)\mathcal{L}^d(B^d_{2Hr}(0))}{|w_1|\mathcal{L}^{d-1}(Q^H(r))}\Big\}\mathcal{L}^d(K_{\Omega,2}) \\
&\qquad + 2M\bar{C}\Big(C_d + \frac{4r}{|w_1|}\Big)\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega)\Big]\,\mathrm{d}w \\
(|w_1| \leq r) \quad &\leq 2M \int_{B^d_r(0)} \min\{|w_1|, C_d\omega_{\mathrm{dis}}(2Hr)H^{d-1}r\}\mathcal{L}^d(K_{\Omega,2})\,\mathrm{d}w \\
&\quad + 2M\bar{C}(C_d + 4)r\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) \\
&\quad + r\bar\varepsilon|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) + \bar{C}|D\boldsymbol{b}|(\Omega \setminus \mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{7.28}$$

Step 4. Observing that

$$\int_{B^d_r(0)} \min\{|w_1|, C_d\omega_{\mathrm{dis}}(2Hr)H^{d-1}r\}\,\mathrm{d}w \leq C_d\mathcal{L}^d(B^d_r(0))\omega_{\mathrm{dis}}(2Hr)H^{d-1}r, \tag{7.29}$$

we obtain

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \bigg| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))\mathrm{d}t - (D_1\boldsymbol{b}_2)_y((t^-(y),t^+(y)))w_1\bigg| \,\mathrm{d}w\,\mathrm{d}y \\
&\underset{(7.28)}{\leq} 2M \int_{B^d_r(0)} \min\{|w_1|, C_d\omega_{\mathrm{dis}}(2Hr)H^{d-1}r\}\mathcal{L}^d(K_{\Omega,2})\,\mathrm{d}w \\
&\quad + 2M\bar{C}(C_d+4)r\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) \\
&\quad + r\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)) \\
&\underset{(7.29)}{\leq} 2MC_d\mathcal{L}^d(B^d_r(0))\omega_{\mathrm{dis}}(2Hr)H^{d-1}r\mathcal{L}^d(K_{\Omega,2}) \\
&\quad + 2M\bar{C}(C_d+4)r\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) \\
&\quad + r\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{7.30}
$$

Conclusion. For $r < \hat{r} = \hat{r}(\bar{\varepsilon}) < \bar{r}$ such that

$$
\omega_{\mathrm{dis}}(2H\hat{r})H^{d-1}\mathcal{L}^d(K_{\Omega,2}) = \omega\Big(\frac{2\hat{r}}{\sqrt{\bar{\varepsilon}}}\Big)\frac{1}{(\bar{\varepsilon})^{(d-1)/2}}\mathcal{L}^d(K_{\Omega,2}) < \sqrt{\bar{\varepsilon}}\mathcal{L}^{d+1}(\Omega), \tag{7.31}
$$

we obtain

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \bigg| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))\,\mathrm{d}t - (D_1\boldsymbol{b}_2)_y((t^-(y),t^+(y)))w_1\bigg| \,\mathrm{d}w\,\mathrm{d}y \\
&\underset{(7.30),(7.31)}{\leq} 2MC_dr\sqrt{\bar{\varepsilon}}\mathcal{L}^d(B^d_r(0))\mathcal{L}^{d+1}(\Omega) \\
&\quad + 2M\bar{C}(C_d+4)r\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) \\
&\quad + r\bar{\varepsilon}|D\boldsymbol{b}|(\Omega)\mathcal{L}^d(B^d_r(0)) + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)) \\
&\leq r\mathcal{L}^d(B^d_r(0))\sqrt{\bar{\varepsilon}}\Big[2MC_d\mathcal{L}^{d+1}(\Omega) + 2M\bar{C}(C_d+6)|D\boldsymbol{b}|(\Omega)\Big] + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)) \\
&\leq C_d\bar{C}\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^b_r(0))[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{7.32}
$$

We have removed a set of trajectories of measure

$$
\underset{(7.6),(7.21)}{<} 2\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega)
$$

and the estimate holds for

$$
r \leq \hat{r} = \hat{r}(\bar{\varepsilon}). \qquad \square
$$

This concludes the comparison with the disintegration, which will be used when analyzing the approximate flow with the linear one in Point (7) of page 18. Using (7.2) we have also

Corollary 7.8. *It holds*

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \bigg| \int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}^H_1(r,y;t,w_1))\,\mathrm{d}t - (D\boldsymbol{b})_y((0,t))w\bigg| \,\mathrm{d}w\,\mathrm{d}y \\
&\qquad \leq C_d(1+\bar{C})r\mathcal{L}^d(B^b_r(0))\sqrt{\bar{\varepsilon}}[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] + \bar{C}|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0)r\mathcal{L}^d(B^d_r(0)).
\end{aligned} \tag{7.33}
$$

7.5. **Estimates on the approximated flow.** The approximated flow is defined by solving the ODE

$$
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{X}}^H(r,y;t,z) = \tilde{\boldsymbol{b}}^H(r,y;t,\tilde{\boldsymbol{X}}^H(r,y;t,z)),
$$

in the time interval of interest for $\boldsymbol{X}(t,y)$, i.e. $t \in (t^-(y),t^+(y))$, with initial data z at $t^-(y)$. We recall that

$$
\tilde{\boldsymbol{b}}^H(r,y;t,z) \underset{(7.5)}{=} \frac{\bar{\xi}}{\mathcal{L}^{d-1}(Q^H(r))}|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0,w_1]\times Q^H(r)).
$$

The first component $\tilde{\boldsymbol{X}}^H_1(r,y;t,z)$ has already been studied in Section 7.3.

7.5.1. *The part not along* $\mathbf{e}_1, \mathbf{e}_2$. The component of $\tilde{\boldsymbol{b}}$ along the direction $\mathbf{e}_{\not y,\not z}$ is clearly 0 because $\bar{\xi}$ lies in the 1, 2-plane by assumptions, so that

$$\tilde{\boldsymbol{X}}^H_{\not y,\not z}(r, y; t, z) = z_{\not y,\not z}.$$

In particular this flow is perfectly 1-Lipschitz.

We can use Corollary A.2 to compare the real flow $\boldsymbol{X}_{\not y,\not z}(t, y + z) - \boldsymbol{X}_{\not y,\not z}(t, y)$ with the approximate flow until the exit time $t^+(y; z)$ from the ball $B^d_r(0)$.

Lemma 7.9. *It holds*

$$\int_{K_{\Omega,2}} \int_{B^d_r(0)} \big\| \boldsymbol{X}_{\not y,\not z}(\cdot, y + z) - \boldsymbol{X}_{\not y,\not z}(\cdot, y) - z_{\not y,\not z}\big\|_{C^0(t^-(y),t^+(y,z))} \,\mathrm{d}z \,\mathrm{d}y < \bar{C}^2 r \mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega), \tag{7.34}$$

where $t^+(y, z)$ *is the exit time from the ball* $X(t^+(y, z), y) + B^d_r(0)$ *or it coincides with the final time* $t^+(y)$.

Proof. Corollary A.2 with $\tilde{\boldsymbol{b}} = 0$, $L = 1$ gives for all $t \in [t^-(y), t^+(y; z)]$

$$\big|\boldsymbol{X}_{\not y,\not z}(t, y + z) - \boldsymbol{X}_{\not y,\not z}(t, y) - z_{\not y,\not z}\big| \le \int_{t^-(y)}^{t} |\boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y + z)) - \boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y))| \,\mathrm{d}s. \tag{7.35}$$

Let $t(y, z) \in [t^-(y), t^+(y, z)]$ be such that

$$\big|\boldsymbol{X}_{\not y,\not z}(t(y, z), y + z) - \boldsymbol{X}_{\not y,\not z}(t(y, z), y) - z_{\not y,\not z}\big| = \big\|\boldsymbol{X}_{\not y,\not z}(\cdot, y + z) - \boldsymbol{X}_{\not y,\not z}(\cdot, y) - z_{\not y,\not z}\big\|_{C^0([t^-(y),t^+(y,z)])}. \tag{7.36}$$

Integrating w.r.t. $z \in B^d_r(0)$ and $y \in K_{\Omega,2}$ we obtain

$$\begin{aligned}
& \int_{K_{\Omega,2}} \int_{B^d_r(0)} \big\|\boldsymbol{X}_{\not y,\not z}(\cdot, y + z) - \boldsymbol{X}_{\not y,\not z}(\cdot, y) - z_{\not y,\not z}\big\|_{C^0} \,\mathrm{d}z \,\mathrm{d}y \\
& \underset{(7.36)}{=} \int_{K_{\Omega,2}} \int_{B^d_r(0)} \big|\boldsymbol{X}_{\not y,\not z}(t(y, z), y + z) - \boldsymbol{X}_{\not y,\not z}(t(y, z), y) - z_{\not y,\not z}\big| \,\mathrm{d}z \,\mathrm{d}y \\
& \underset{(7.35)}{\le} \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t(y,z)} |\boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y + z)) - \boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y))| \,\mathrm{d}s \,\mathrm{d}z \,\mathrm{d}y \\
t(y, z) \le t^+(y, z) \quad & \le \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y,z)} |\boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y + z)) - \boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y))| \,\mathrm{d}s \,\mathrm{d}z \,\mathrm{d}y \\
\text{near. incompr. w.r.t. } z \quad & \le \bar{C} \int_{K_{\Omega,2}} \int_{t^-(y)}^{t^+(y)} \int_{B^d_r(0)} |\boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y) + w) - \boldsymbol{b}_{\not y,\not z}(s, \boldsymbol{X}(s, y))| \,\mathrm{d}w \,\mathrm{d}s \,\mathrm{d}y \\
\text{near. incompr. w.r.t. } y \quad & \le \bar{C}^2 \int_{B^d_r(0)} \int_{\Omega} |\boldsymbol{b}_{\not y,\not z}(s, x + w) - \boldsymbol{b}_{\not y,\not z}(s, x)| \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}w \\
& \underset{(7.2)}{<} \bar{C}^2 r \mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega).
\end{aligned}$$

which is (7.34). □

Corollary 7.10. *If*

$$E^2_1(y, r) = \big\{z \in B^d_r(0) : \|\boldsymbol{X}_{\not y,\not z}(\cdot, y + z) - \boldsymbol{X}_{\not y,\not z}(\cdot, y) - z_{\not y,\not z}\|_{C^0} \ge \sqrt{\bar{\varepsilon}} r\big\} \tag{7.37}$$

then

$$\int_{K_{\Omega,2}} \mathcal{L}^d(E^2_1(y, r))\mathrm{d}y < \bar{C}^2 \mathcal{L}^d(B^d_r(0))\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega). \tag{7.38}$$

Proof. Indeed by Chebyshev

$$\begin{aligned}
\int_{K_{\Omega,2}} \mathcal{L}^d(E^2_1(y, r))\mathrm{d}y &\le \frac{1}{\sqrt{\bar{\varepsilon}} r} \int_{K_{\Omega,2}} \int_{B^d_r(0)} \big\|\boldsymbol{X}_{\not y,\not z}(\cdot, y + z) - \boldsymbol{X}_{\not y,\not z}(\cdot, y) - z_{\not y,\not z}\big\|_{C^0} \,\mathrm{d}z \,\mathrm{d}y \\
&\underset{(7.34)}{<} \bar{C}^2 \mathcal{L}^d(B^d_r(0))\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega).
\end{aligned}$$

which is the statement. □

Defining

$$E_1(r,y) = E_1^1(r,y) \cup E_1^2(r,y) \tag{7.39}$$

where $E_1^1(r,y)$ is defined in (7.14) and $E_1^2(r,y)$ in (7.37), we conclude that

$$\int_{K_{\Omega,2}} \mathcal{L}^d(E_1(y,r))\mathrm{d}y \underset{(7.15,(7.38)}{\leq} 5\bar{C}^2(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B_r^d(0))|D\boldsymbol{b}|(\Omega) \tag{7.40}$$

for $\bar{\varepsilon} \leq 1$.

The estimate (7.40) gives Point (4) of page 17.

7.5.2. *The part along* $\mathbf{e}_2$. The part along $\mathbf{e}_2$ satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{X}}_2^H(r,y;t,z) = \tilde{\boldsymbol{b}}_2^H(r,y;t,\tilde{\boldsymbol{X}}_2^H(r,y;t,z)).$$

Since $\tilde{\boldsymbol{b}}^H$ depends only on z_1, the solution is for $t \in [t^-(y), t^+(y,z)]$ ($t^+(y,z)$ being the exit time from $B_r^d(0)$)

$$\tilde{\boldsymbol{X}}_2^H(r,y;t,z) = z_2 + \int_{t^-(y)}^t \tilde{\boldsymbol{b}}_2^H(r,y;s,\tilde{\boldsymbol{X}}_1^H(r,y;t,z_1))\,\mathrm{d}s. \tag{7.41}$$

By (7.8), we have that, for every Borel function $z \mapsto t(y,z) \leq t^+(y)$,

$$\begin{aligned}
\int_{K_\Omega}\int_{B_r^d(z)} |\tilde{\boldsymbol{X}}_2^H(r,y;t(y,z),z) - z_2|\,\mathrm{d}z\,\mathrm{d}y &\underset{(7.41)}{\leq} \int_{K_\Omega}\int_{B_r^d(z)} \bigg|\int_{t^-(y)}^{t(y,z)} \tilde{\boldsymbol{b}}_2^H(r,y;s,\tilde{\boldsymbol{X}}_1^H(r,y;t,z_1))\,\mathrm{d}s\bigg|\,\mathrm{d}z\,\mathrm{d}y \\
\tilde{\boldsymbol{X}}_1^H \text{ contraction} &\leq \int_{K_\Omega}\int_{B_r^d(z)} \bigg|\int_{t^-(y)}^{t(y,z)} \tilde{\boldsymbol{b}}_2^H(r,y;s,z_1)\,\mathrm{d}s\bigg|\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.8)}{\leq} \bar{C}r\mathcal{L}^d(B_r^d(0))|D\boldsymbol{b}|(\Omega),
\end{aligned}$$

which in particular is equivalent to

$$\int_{K_\Omega}\int_{B_r^d(z)} \|\tilde{\boldsymbol{X}}_2^H(r,y;\cdot,z) - z_2\|_{C^0}\,\mathrm{d}z\,\mathrm{d}y \leq \bar{C}r\mathcal{L}^d(B_r^d(0))|D\boldsymbol{b}|(\Omega). \tag{7.42}$$

Here we can allow the time $t(y,z)$ to be larger than the exit time from the cylinder $\boldsymbol{X}(t,y) + B_r^d(0)$ because of the particular form of the flow $\tilde{\boldsymbol{X}}^H$: the first component is a contraction, and the second depends only on the first. Clearly it will be meaningless when exiting $\boldsymbol{X}(t,y) + [-r,r] \times Q^H(r)$ because of the form of $\tilde{\boldsymbol{b}}^H$.

In the following proposition, we estimate the quantity

$$\big|\boldsymbol{X}_2(t,y+z) - \boldsymbol{X}_2(t,y) - \tilde{\boldsymbol{X}}_2^H(r,y;t,z)\big|.$$

Proposition 7.11. *If $E_1(r,y)$ is the set defined in* (7.39)*, then it holds*

$$\int_{K_{\Omega,2}}\int_{B_r^d(0)\setminus E_1(r,y)} \big\|\boldsymbol{X}_2(\cdot,y+z) - \boldsymbol{X}_2(\cdot,y) - \tilde{\boldsymbol{X}}_2^H(r,y;\cdot,z)\big\|_{C^0}\,\mathrm{d}z\,\mathrm{d}y < 7\bar{C}r\mathcal{L}^d(B_r^d(0))(\bar{\varepsilon})^{1/4}|D\boldsymbol{b}|(\Omega). \tag{7.43}$$

This proposition corresponds to Point (5) of page 17, Equation (5.2).

Proof. In this case the flow $\tilde{\boldsymbol{X}}_2^H$ is not Lipschitz (take for example a single jump discontinuity), so we cannot use Corollary A.2 and instead proceed as follows.

Let $t_2(y,z) \in [t^-(y), t^+(y,z)]$ be the time where

$$\big|\boldsymbol{X}_2(t_2(y,z),y+z) - \boldsymbol{X}_2(t_2(y,z),y) - \tilde{\boldsymbol{X}}_2^H(r,y;t_2(y,z),z)\big| = \big\|\boldsymbol{X}_2(\cdot,y+z) - \boldsymbol{X}_2(\cdot,y) - \tilde{\boldsymbol{X}}_2^H(r,y;\cdot,z)\big\|_{C^0}.$$

The above quantity can be written as

$$
\begin{aligned}
&\big|\boldsymbol{X}_2(t_2(y,z),y+z)-\boldsymbol{X}_2(t_2(y,z),y)-\tilde{\boldsymbol{X}}_2^H(r,y;t_2(y,z),z)\big| \\
&= \Big|\int_{t^-(y)}^{t_2(y,z)} \big[\boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z))-\boldsymbol{b}_2(s,\boldsymbol{X}(s,y))-\tilde{\boldsymbol{b}}_2^H(r,y;s,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1))\big]\,\mathrm{d}s\Big| \\
\text{triangle ineq.}\quad &\le \Big|\int_{t^-(y)}^{t_2(y,z)} \Big[\boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z))-\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}_1(s,y+z)\mathbf{e}_1+w)\,\mathrm{d}w\Big]\,\mathrm{d}s\Big| \\
&\quad+\Big|\int_{t^-(y)}^{t_2(y,z)} \Big[\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}_1(s,y+z)\mathbf{e}_1+w)\,\mathrm{d}w \\
&\qquad\qquad-\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)\mathbf{e}_1+w)\,\mathrm{d}w\Big]\,\mathrm{d}s\Big| \\
&\quad+\Big|\int_{t^-(y)}^{t_2(y,z)} \frac{1}{\mathcal{L}^{d-1}(Q^H(r))}\big[(D_1\boldsymbol{b}_2)(\boldsymbol{X}(t,y)+[0,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)]\times Q^H(r)) \\
&\qquad\qquad-\bar{\xi}_2|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)]\times Q^H(r))\big]\,\mathrm{d}t\Big| \\
&\quad+\Big|\int_{t^-(y)}^{t_2(y,z)} \Big[\boldsymbol{b}_2(s,\boldsymbol{X}(s,y))-\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+w)\,\mathrm{d}w\Big]\,\mathrm{d}s\Big|.
\end{aligned}
\tag{7.44}
$$

We have used

$$
\begin{aligned}
&\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)\mathbf{e}_1+w)\,\mathrm{d}w-\fint_{Q^H(r)}\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+w)\,\mathrm{d}w \\
&\qquad=\frac{1}{\mathcal{L}^{d-1}(Q^H(r))}(D_1\boldsymbol{b}_2)(\boldsymbol{X}(t,y)+[0,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)]\times Q^H(r)).
\end{aligned}
$$

Integrating the third term w.r.t. y and using (7.2), we get

$$
\begin{aligned}
&\int_{K_{\Omega,2}}\int_{B_r^d(0)}\Big|\int_{t^-(y)}^{t_2(y,z)}\frac{1}{\mathcal{L}^{d-1}(Q^H(r))}\big[D_1\boldsymbol{b}_2(\boldsymbol{X}(t,y)+[0,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)]\times Q^H(r)) \\
&\qquad\qquad-\bar{\xi}_2|D\boldsymbol{b}|(\boldsymbol{X}(t,y)+[0,\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)]\times Q^H(r))\big]\,\mathrm{d}t\Big|\,\mathrm{d}y \\
&\le\int_{K_{\Omega,2}}\int_{B_r^d(0)}\int_{t^-(y)}^{t_2(y,z)}\frac{1}{\mathcal{L}^{d-1}(Q^H(r))}|(D_1\boldsymbol{b}_2)-\bar{\xi}_2|D\boldsymbol{b}||(\boldsymbol{X}(t,y)+[0,w_1]\times Q^H(r))\big]\,\mathrm{d}t\,\mathrm{d}y \\
&\le\bar{C}\int_{B_r^d(0)}\int_{\Omega}\frac{1}{\mathcal{L}^{d-1}(Q^H(r))}|(D_1\boldsymbol{b}_2)-\bar{\xi}_2|D\boldsymbol{b}||((t,x)+[0,w_1]\times Q^H(r))\,\mathrm{d}t\,\mathrm{d}x \\
&\underset{(7.2)}{\le}\bar{C}r\mathcal{L}^d(B_r^d(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega),
\end{aligned}
\tag{7.45}
$$

where we used the fact that $\tilde{\boldsymbol{X}}_1^H(r,y;s,z_1)\le w_1$ in the first inequality and $|w_1|\le r$ in the last one.

Integrating we obtain for the fourth term

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \Big| \int_{t^-(y)}^{t_2(y,z)} \Big[\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)) - \fint_{Q^H(r)} \boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+w)\,\mathrm{d}w\Big]\,\mathrm{d}s\Big|\,\mathrm{d}z\,\mathrm{d}y \\
&\leq \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y)} \fint_{Q^H(r)} \big|\boldsymbol{b}_2(s,\boldsymbol{X}(s,y)) - \boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+w)\big|\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
\text{near. incompr.}\quad &\leq \bar{C}\mathcal{L}^d(B^d_r(0)) \fint_{Q^H(r)} \int_\Omega |\boldsymbol{b}_2(t,x+w) - \boldsymbol{b}_2(t,x)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \\
(|w| \leq (1+H)r)\quad &\leq \bar{C}(1+H)r\mathcal{L}^d(B^d_r(0))|D_2\boldsymbol{b}_2|(\Omega) \\
&\underset{(7.2)}{\leq} 2\bar{C}Hr\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega) \\
&\underset{(7.4)}{<} 2\bar{C}r\mathcal{L}^d(B^d_r(0))\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega).
\end{aligned} \tag{7.46}
$$

The above estimate is the same for the first term:

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)} \Big| \int_{t^-(y)}^{t_2(y,z)} \Big[\boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z)) - \fint_{Q^H(r)} \boldsymbol{b}_2(s,\boldsymbol{X}_1(s,y+z)\mathbf{e}_1+w)\,\mathrm{d}w\Big]\mathrm{d}s\Big|\,\mathrm{d}z\,\mathrm{d}y \\
&\leq \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y,z)} \fint_{Q^H(r)} \big|\boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z)) - \boldsymbol{b}_2(s,\boldsymbol{X}_1(s,y+z)\mathbf{e}_1+w)\big|\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&\leq \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y,z)} \fint_{Q^{H+1}(r)} \big|\boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z)) - \boldsymbol{b}_2(s,\boldsymbol{X}(s,y+z)+w)\big|\,\mathrm{d}w\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
\text{near. incompr.}\quad &\leq \bar{C}\mathcal{L}^d(B^d_r(0)) \fint_{Q^{H+1}(r)} \int_\Omega |\boldsymbol{b}_2(t,x+w) - \boldsymbol{b}_2(t,x)|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}w \\
&\leq \bar{C}(2+H)r\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega) \\
&< 2\bar{C}r\mathcal{L}^d(B^d_r(0))\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega),
\end{aligned} \tag{7.47}
$$

where in the third step we have used

$$
\boldsymbol{X}_1(t,y+z)\mathbf{e}_1 + Q^H(r) \subset \boldsymbol{X}(t,y+z) + Q^{H+1}(r),
$$

valid until the exit time $t_2(y,z)$.

Recalling that for $z \in B^d_r(0) \setminus E_1(r,y)$ it holds

$$
\big\|\boldsymbol{X}_1(\cdot,y+z) - \boldsymbol{X}_1(\cdot,y) - \tilde{\boldsymbol{X}}^H_1(r,y;\cdot,z_1)\big\|_{C^0} < (\bar{\varepsilon})^{1/4}r, \tag{7.48}
$$

we obtain (the interval in the second line may have the extremals exchanged depending on s, here for definiteness we assume $\tilde{\boldsymbol{X}}^H_1(r,y;s,z) \leq \boldsymbol{X}_1(s,y+z) - \boldsymbol{X}_1(s,y)$)

$$
\begin{aligned}
&\fint_{Q^H(r)} \big|\boldsymbol{b}_2(s,\boldsymbol{X}_1(s,y+z)\mathbf{e}_1+w) - \boldsymbol{b}_2(s,\boldsymbol{X}(s,y)+\tilde{\boldsymbol{X}}^H_1(r,y;s,z_1)\mathbf{e}_1+w)\big|\,\mathrm{d}w \\
&\leq \frac{1}{\mathcal{L}^{d-1}(Q^H(r))}|D_1\boldsymbol{b}_2|\big(\boldsymbol{X}(s,y) + [\tilde{\boldsymbol{X}}^H_1(r,y;s,z_1), \boldsymbol{X}_1(s,y+z) - \boldsymbol{X}_1(s,y)] \times Q^H(r)\big) \\
&\underset{(7.48)}{\leq} \frac{1}{\mathcal{L}^{d-1}(Q^H(r))}|D_1\boldsymbol{b}_2|(\boldsymbol{X}(s,y) + [-(\bar{\varepsilon})^{1/4}r, (\bar{\varepsilon})^{1/4}r] \times Q^H(r)).
\end{aligned} \tag{7.49}
$$

Then integrating the second term

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)\setminus E_1(r,y)} \Big| \int_{t^-(y)}^{t_2(y,z)} \Big[\fint_{Q^H(r)} \boldsymbol{b}_2(s, \boldsymbol{X}_1(s, y+z)\mathbf{e}_1 + w)\,\mathrm{d}w \\
&\qquad\qquad - \fint_{Q^H(r)} \boldsymbol{b}_2(s, \boldsymbol{X}(s,y) + z_1\mathbf{e}_1 + w)\,\mathrm{d}w \Big]\,\mathrm{d}s \Big|\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.49)}{\leq} \int_{K_{\Omega,2}} \int_{B^d_r(0)} \int_{t^-(y)}^{t^+(y,z)} \frac{|D_1\boldsymbol{b}_2|(\boldsymbol{X}(s,y) + [-(\bar\varepsilon)^{1/4}r, (\bar\varepsilon)^{1/4}r] \times Q^H(r))}{\mathcal{L}^{d-1}(Q^H(r))}\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}y \\
&< 2\bar C(\bar\varepsilon)^{1/4} r \mathcal{L}^d(B^d_r(0))|D_1\boldsymbol{b}_2|(\Omega),
\end{aligned}
\tag{7.50}
$$

where we used the fact that $B^d_r(0)\setminus E_1(r,y) \subset B^d_r(0)$.

Finally, collecting all estimates,

$$
\begin{aligned}
&\int_{K_{\Omega,2}} \int_{B^d_r(0)\setminus E_1(r,y)} \big\|\boldsymbol{X}_2(\cdot, y+z) - \boldsymbol{X}_2(\cdot,y) - \tilde{\boldsymbol{X}}^H_2(r,y;\cdot,z)\big\|_{C^0}\,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.44),(7.47),(7.50),(7.45),(7.46)}{<} 2\bar C(\bar\varepsilon)^{1/4} r\mathcal{L}^d(B^d_r(0))|D_1\boldsymbol{b}_2|(\Omega) + \bar C r\mathcal{L}^d(B^d_r(0))\bar\varepsilon|D\boldsymbol{b}|(\Omega) \\
&\qquad + 2\bar C r\mathcal{L}^d(B^d_r(0))\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega) + 2\bar C r\mathcal{L}^d(B^d_r(0))\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega) \\
&< 7\bar C r\mathcal{L}^d(B^d_r(0))(\bar\varepsilon)^{1/4}|D\boldsymbol{b}|(\Omega),
\end{aligned}
\tag{7.51}
$$

which is the statement. □

7.6. **The linearized ODE.** We compare now the approximate flux $\tilde{\boldsymbol{X}}^H(r,y;t,w)$ with the linearized flow of Section 4, namely the solution to

$$
\dot W(t,y) = (D\boldsymbol{b})_y(\mathrm{d}t)\frac{W(t-,y)}{J(t^-,y)}, \quad W(t^-(y),y) = \mathbb{I},\ t \in (t^-(y), t^+(y)).
$$

Let $\tilde W(t,y)$ be the solution to the following approximated ODE (whenever it exists, i.e. whenever $J(t,y) \geq c > 0$)

$$
\dot{\tilde W}(t,y) = \bar\xi \otimes \bar\eta|D\boldsymbol{b}|_y(\mathrm{d}t)\frac{\tilde W(t-,y)}{\tilde J(t-,y)}, \quad \tilde W(t^-(y),y) = \mathbb{I},
\tag{7.52}
$$

where $\tilde J(t,y) = \det(\tilde W(t,y))$. Clearly due to the simple form of the r.h.s. one gets

$$
\dot{\tilde J}(t,y) = \bar\xi\cdot\bar\eta|D\boldsymbol{b}|_y(\mathrm{d}t) = \bar\xi_1|D\boldsymbol{b}|_y(\mathrm{d}t), \quad \tilde J(t^-(y),y) = 1.
\tag{7.53}
$$

Lemma 7.12. *There exists a set $K_{\Omega_3} \subset K_\Omega$ with co-measure*

$$
\mathcal{H}^d(K_\Omega \setminus K_{\Omega,3}) \leq \sqrt{\bar\varepsilon}
\tag{7.54}
$$

such that

$$
\|J(\cdot,y) - \tilde J(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))} \leq \sqrt{\bar\varepsilon}.
\tag{7.55}
$$

In this set the solution to (7.52) *is defined for all $t \in [t^-(y), t^+(y)]$ and it holds*

$$
\int_{K_{\Omega,3}} \|W(\cdot,y) - \tilde W(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))}\mathrm{d}y \leq 3\bar C^2 e^{3\bar C M}\sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega).
\tag{7.56}
$$

Proof. We can write

$$
\frac{\mathrm{d}}{\mathrm{d}t}(J(t,y) - \tilde J(t,y)) = (\operatorname{div}\boldsymbol{b})_y(\mathrm{d}t) - \bar\xi\cdot\bar\eta|D\boldsymbol{b}|_y(\mathrm{d}t)
\tag{7.57}
$$

Hence integrating in K_Ω one obtains

$$
\int_{K_\Omega} \|J(\cdot,y) - \tilde J(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))}\,\mathrm{d}y \underset{(7.57)}{\leq} \int_{K_\Omega}\int_{t^-(y)}^{t^+(y)} |(D\boldsymbol{b})_y - \bar\xi\otimes\bar\eta|D\boldsymbol{b}|_y|(\mathrm{d}t)\,\mathrm{d}y \underset{(7.2)}{<} \bar\varepsilon|D\boldsymbol{b}|(\Omega).
$$

Hence by Chebyshev inequality we can remove a set of trajectories of measure $< \sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega)$ and in the remaining set the estimate (7.55) holds:

$$\|J(\cdot,y) - \tilde{J}(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))} \leq \sqrt{\varepsilon}.$$

In particular we deduce that

$$\frac{1}{2\bar{C}} \leq \frac{1}{\bar{C}} - \sqrt{\varepsilon} \leq \tilde{J} \leq \bar{C} + \sqrt{\varepsilon} \leq 2\bar{C},$$

so that the solution $\tilde{W}(t,y)$ does exist on this set, and in the same way as in (4.14) one gets

$$|\tilde{W}(t,y)| \leq e^{2\bar{C}M}. \tag{7.58}$$

Write for these trajectories

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(W(t,y) - \tilde{W}(t,w)) &= (D\boldsymbol{b})_y(\mathrm{d}t)\frac{W(t-,y)}{J(t-,y)} - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y(\mathrm{d}t)\frac{\tilde{W}(t-,y)}{\tilde{J}(t-,y)} \\
&= \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)}(W(t-,y) - \tilde{W}(t-,y)) + (D\boldsymbol{b})_y(\mathrm{d}t)\frac{\tilde{W}(t-,y)}{J(t-,y)} - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y(\mathrm{d}t)\frac{\tilde{W}(t-,y)}{\tilde{J}(t-,y)} \\
&= \frac{(D\boldsymbol{b})_y(\mathrm{d}t)}{J(t-,y)}(W(t-,y) - \tilde{W}(t-,y)) + [(D\boldsymbol{b})_y(\mathrm{d}t) - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y(\mathrm{d}t)]\frac{\tilde{W}(t-,y)}{J(t-,y)} \\
&\quad + \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y(\mathrm{d}t)\Big(\frac{1}{J(t-,y)} - \frac{1}{\tilde{J}(t-,y)}\Big)\tilde{W}(t-,y).
\end{aligned}$$

Integrating in time and using Duhamel Formula together with (4.4), (7.58) one gets

$$\begin{aligned}
\|W(\cdot,y) - \tilde{W}(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))} \leq{}& \bar{C}e^{3\bar{C}M}\int_{t^-(y)}^{t^+(y)} |(D\boldsymbol{b})_y - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y|(\mathrm{d}t) \\
&+ 2\bar{C}^2e^{3\bar{C}M}\|J(\cdot,y) - \tilde{J}(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))}|D\boldsymbol{b}|_y(t^-(y),t^+(y)).
\end{aligned} \tag{7.59}$$

Integrating in $K_{\Omega,3}$ one deduce

$$\int_{K_{\Omega,3}} \|W(\cdot,y) - \tilde{W}(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))}\,\mathrm{d}y \underset{(7.59),(7.2),(7.55)}{\leq} 3\bar{C}^2e^{3\bar{C}M}\sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega).$$

□

The solution to (7.52) can be computed explicitly when $y \in K_{\Omega,3}$: the equation for the first component is

$$\dot{\tilde{W}}_{11}(t,y) = \bar{\xi}_1|D\boldsymbol{b}|_y(\mathrm{d}t)\frac{\tilde{W}_{11}(t-,y)}{\tilde{J}(t-,y)}, \quad \tilde{W}_{11}(t^-(y),y) = 1,$$

whose unique solution (Theorem 4.1) is clearly

$$\tilde{W}_{11}(t,y) = \tilde{J}(t,y).$$

The only component non constant beside $\tilde{W}_{11}$ is the $\tilde{W}_{12}(t,y)$, which satisfies

$$\dot{\tilde{W}}_{12}(t,y) = \bar{\xi}_2|D\boldsymbol{b}|_y(\mathrm{d}t)\frac{\tilde{W}_{11}(t-,y)}{\tilde{J}(t-,y)} = \bar{\xi}_2|D\boldsymbol{b}|_y(\mathrm{d}y), \quad \tilde{W}_{12}(t^-(y),y) = 0,$$

whose solution is

$$\tilde{W}_{12}(t,y) = \xi_2|D\boldsymbol{b}|_y(t^-(y),t).$$

Hence, we obtain the following result.

Lemma 7.13. *If $y \in K_{\Omega,3}$, the explicit solution to* (7.52) *is given by*

$$\tilde{W}(t,y) = \mathbb{I} + \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y(t^-(y),t). \tag{7.60}$$

7.7. **Collecting the estimates for the singular part.** Here we prove that the assumptions of Section 5 are verified with the measure with a suitable measure μ_P, which will be given in Section 8, when collecting all estimates.

More precisely:

(1) **Point (2) of page 17:** there exists a set of trajectories $S_1' = K_{\Omega,3}$ such that

$$\mathcal{H}^d(S_1 \setminus S_1') \underset{(7.21),(7.54)}{<} \bar{\varepsilon}\mathcal{L}^{d+1}(\Omega) + \sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega) < (\bar{\varepsilon})^{1/4}[|D\boldsymbol{b}|(\Omega) + \mathcal{L}^{d+1}(\Omega)]; \tag{7.61}$$

(2) **Point (6) of page 17:** there exists an approximated solution $\tilde{\boldsymbol{X}}^H(r,y;t,w)$ such that for all $t \in (t^-(y), t^+(y))$, $r' \leq r$

$$\begin{aligned}\int_{K_{\Omega,3}} \int_{B^d_{r'}(z)} \|\tilde{\boldsymbol{X}}^H(r,y;\cdot,w) - w\|_{C^0} \,\mathrm{d}w\,\mathrm{d}y &\underset{(7.11)}{\leq} \int_{K_\Omega} \int_{B^d_{r'}(0)} \int_{t^-(y)}^{t^+(y)} |\tilde{\boldsymbol{b}}^H(r,y;s,w)| \,\mathrm{d}s\,\mathrm{d}w\,\mathrm{d}y \\ &\underset{(7.8)}{\leq} \bar{C}r'\mathcal{L}^d(B^d_{r'}(0))|D\boldsymbol{b}|(\Omega).\end{aligned} \tag{7.62}$$

(3) **Point (4) of page 17:** for every $y \in K_{\Omega,3}$ there exists a set of initial points $E_1(r,y) \subset B^d_r(0)$ such that

$$\int_{K_{\Omega,3}} \mathcal{L}^d(E_1(r,y))\,\mathrm{d}y \underset{(7.40)}{<} 5\bar{C}^2(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega); \tag{7.63}$$

(4) **Formula** (5.2) **of Point (5), page 17:** for the remaining trajectories it hold

$$\begin{aligned} &\int_{K_{\Omega,3}} \int_{B^d_r(0)\setminus E_1(r,y)} \|\boldsymbol{X}(\cdot,y+z) - \boldsymbol{X}(\cdot,y) - \tilde{\boldsymbol{X}}^H(r,y;\cdot,z)\|_{C^0(t^-(y),t^+(y,z))}\,\mathrm{d}z\,\mathrm{d}y \\ (7.13),(7.34),(7.43)\quad &< 3\bar{C}^2\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega) + \bar{C}^2r\mathcal{L}^d(B^d_r(0))\bar{\varepsilon}|D\boldsymbol{b}|(\Omega) + 7\bar{C}r\mathcal{L}^d(B^d_r(0))(\bar{\varepsilon})^{1/4}|D\boldsymbol{b}|(\Omega) \\ &< 11\bar{C}^2(\bar{\varepsilon})^{1/4}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega); \end{aligned} \tag{7.64}$$

(5) **Comparison of approximate solution with the linearised flow:** recalling that

$$\tilde{\boldsymbol{X}}^H(r,y;t,z) - z = \int_{t^-(y)}^t \tilde{\boldsymbol{b}}^H(r,y;s,\tilde{\boldsymbol{X}}^H_1(r,y;s,z))\,\mathrm{d}s, \tag{7.65}$$

for the approximate vector field

$$\begin{aligned} &\int_{K_{\Omega,3}} \int_{B^d_r(0)} \big|(\tilde{\boldsymbol{X}}^H(r,y;t^+(y),w) - w) - W(t^+(y),t^-(y),y)w\big|\,\mathrm{d}w\,\mathrm{d}y \\ &\underset{(7.60}{\leq} \int_{K_{\Omega,3}} \int_{B^d_r(0)} \big|(\tilde{\boldsymbol{X}}^H(r,y;t^+(y),w) - w) - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y((t^-(y),t^+(y)))w_1\big|\,\mathrm{d}w\,\mathrm{d}y \\ &\quad + \int_{B^d_r(0)} |w|\mathrm{d}w \int_{K_{\Omega,3}} \|W(\cdot,y) - \tilde{W}(\cdot,y)\|_{L^\infty(t^-(y),t^+(y))}\,\mathrm{d}y \\ &\underset{(7.65),(7.56)}{=} \int_{K_{\Omega,3}} \int_{B^d_r(0)} \left|\int_{t^-(y)}^{t^+(y)} \tilde{\boldsymbol{b}}^H(r,y;s,\tilde{\boldsymbol{X}}^H(r,y;s,w))\,\mathrm{d}s - \bar{\xi}\otimes\bar{\eta}|D\boldsymbol{b}|_y((t^-(y),t^+(y)))w_1\right|\mathrm{d}w\,\mathrm{d}y \\ &\quad + 3\bar{C}^2e^{3\bar{C}M}\sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega) \\ &\underset{(7.24),(7.2)}{\leq} C_d\bar{C}\sqrt{\bar{\varepsilon}}r\mathcal{L}^d(B^b_r(0))[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] + \bar{C}r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0) \\ &\qquad + 3\bar{C}^2e^{3\bar{C}M}\sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega). \end{aligned} \tag{7.66}$$

(6) **Point (7) of page 18:** if $E_2(r,y)$ is the set of trajectories which exit from $\boldsymbol{X}(t,y) + B^d_r(0)$ before $t^+(y)$, then

$$
\begin{aligned}
\int_{K_{\Omega,2}} \int_{B^d_r(0)\setminus(E_1(r,y)\cup E_2(r,y))} & \big|\boldsymbol{X}(t^+(y), y+z) - \boldsymbol{X}(t^+(y),y) - W(t,y)z\big| \mathrm{d}z\mathrm{d}y \\
\underset{(7.64),(7.66)}{<} & \; 11C^2(\bar\varepsilon)^{1/4} r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega) \\
& + C_d(1+\bar C) r\mathcal{L}^d(B^d_r(0))(\bar\varepsilon)^{1/4}[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] \\
& + \bar C r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0) + 3\bar C^2 e^{3\bar C M}\sqrt{\varepsilon}|D\boldsymbol{b}|(\Omega) \\
< 16\bar C^2 C_d r\mathcal{L}^d(B^d_r(0))(\bar\varepsilon)^{1/4} & [\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] \\
& + \bar C r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0).
\end{aligned}
\tag{7.67}
$$

This concludes the local estimates in the case the singular part is contracting, i.e. $\bar\xi\cdot\bar\eta \le 0$.

7.8. **The time reverse case.** To study the case $\bar\xi\cdot\bar\eta > 0$, we use the estimates we have already proved by reversing time or, equivalently, by changing variables and using as initial set the set S_2 instead of the set S_1. In order to have more flexibility in the proof, we will choose the parameter H determining $Q^H(r)$ later.

We proceed as follows.

(1) First of all, we will consider as initial points S_2' the image of the set $K_{\Omega,3}$, i.e. $S_2 \cap \mathcal{K}_{\Omega,3}$: by the near incompressibility and the fact that up to $\bar C\bar\varepsilon\mathcal{L}^{d+1}(\Omega)$ all trajectories start from S_1 and leave from S_2 for a perturbed proper set, we obtain that

$$
\begin{aligned}
\mathcal{H}^d(S_2\setminus S_2') & \underset{(7.61),(7.1)}{<} \bar\varepsilon\mathcal{L}^{d+1}(\Omega) + \bar C(\bar\varepsilon)^{1/4}(|D\boldsymbol{b}|(\Omega) + \mathcal{L}^{d+1}(\Omega)) \\
& \underset{\bar\varepsilon\ll 1}{<} 2\bar C(\bar\varepsilon)^{1/4}(|D\boldsymbol{b}|(\Omega) + \mathcal{L}^{d+1}(\Omega)).
\end{aligned}
\tag{7.68}
$$

(2) We estimate of the flow exiting or entering the sets

$$
\bigcup_t \big\{\boldsymbol{X}(t,y) + [0, \tilde{\boldsymbol{X}}^H_1(r,y;t,r)] \times Q^H(r)\big\} \quad \text{or} \quad \bigcup_t \big\{\boldsymbol{X}(t,y) + [\tilde{\boldsymbol{X}}^H_1(r,y;t,-r), 0] \times Q^H(r)\big\}. \tag{7.69}
$$

One can repeat the analysis of Lemma 7.5, letting the dependence w.r.t. H be explicit, and obtain that the flow $\Phi_L(y)$ is controlled by

$$
\int_{K_\Omega} \Phi_L(y)\,\mathrm{d}y \underset{(7.17),(7.18),(7.19),(7.20)}{\le} C_d\bar C(1+H)r\mathcal{L}^{d-1}(Q^H_r)\bar\varepsilon|D\boldsymbol{b}|(\Omega) \le C_d\bar C H^2\bar\varepsilon r^d|D\boldsymbol{b}|(\Omega). \tag{7.70}
$$

As usual, the constant C_d may increase line by line. Hence, if

$$
E^+(r,y) = \Big\{y' \in \boldsymbol{X}(t^+(y),y) + B^d_r(0), y' \text{ end point of a trajectory crossing the boundary of (7.69)}\Big\},
$$

we have

$$
\int_{K_\Omega} \mathcal{L}^d(E^+(r,y))\,\mathrm{d}y \underset{(7.70),\text{ near. incompr.}}{\le} C_d\bar C^2 H^2\bar\varepsilon r^d|D\boldsymbol{b}|(\Omega). \tag{7.71}
$$

(3) By Chebyshev inequality applyed to (7.71), we remove a set of initial points $Z_1 \subset K_{\Omega,3}$ of measure

$$
\mathcal{H}^d(Z_1) < \bar C\sqrt{\bar\varepsilon}|D\boldsymbol{b}|(\Omega), \tag{7.72}
$$

for the rest of the trajectories $\boldsymbol{X}(t,y)$, $y \in K_{\Omega,3}\setminus Z_1$, the flow crossing the boundary of $\bigcup_t\{\boldsymbol{X}(t,y) + [0,\tilde{\boldsymbol{X}}^H_1(r,y;t,r)]\times Q^H(r)\}$ or $\bigcup_t\{\boldsymbol{X}(t,y) + [\tilde{\boldsymbol{X}}^H_1(r,y;t,-r),0]\times Q^H(r)\}$ is controlled by (one H in (7.70) has been incorporated in $Q^H(r)$)

$$
\Phi_L(y) \le C_d\bar C\sqrt{\bar\varepsilon}Hr\mathcal{L}^{d-1}(Q^H_r). \tag{7.73}
$$

The set S_2'' of final points is thus the image of $K_{\Omega,3} \setminus Z_1$, which satisfies by near incompressibility

$$\begin{aligned}\mathcal{H}^d(S_2 \setminus S_2'') &\underset{(7.68),(7.72)}{<} 2\bar{C}(\bar{\varepsilon})^{1/4}[|D\boldsymbol{b}|(\Omega) + \mathcal{L}^{d+1}(\Omega)] + \bar{C}^2\sqrt{\bar{\varepsilon}}|D\boldsymbol{b}|(\Omega) \\ &< 3\bar{C}(\bar{\varepsilon})^{1/4}[|D\boldsymbol{b}|(\Omega) + \mathcal{L}^{d+1}(\Omega)].\end{aligned} \tag{7.74}$$

This is Point (2) of page 17 for the case $\bar{\xi} \cdot \bar{\eta} > 0$.

(4) Using the bound (7.73), we can estimate the size of $\tilde{\boldsymbol{X}}_1(r, y; t^+(y), r)$: by the balance

$$\text{final area} + \text{lateral flow} \geq \frac{1}{\bar{C}}\text{initial trajectories},$$

one gets

$$\tilde{\boldsymbol{X}}_1^H(r, y; t^+(y), r)\mathcal{L}^{d-1}(Q^H(r)) + C_d\bar{C}H\sqrt{\bar{\varepsilon}}r\mathcal{L}^{d-1}(Q^H(r)) \geq \frac{1}{\bar{C}}r\mathcal{L}^{d-1}(Q^H(r)),$$

where we have used (7.73). The above equation gives for $y \in K_{\Omega,3} \setminus Z_1$ that

$$\tilde{\boldsymbol{X}}_1^H(r, y; t^+(y), r) \geq \frac{1}{\bar{C}}(1 - C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H)r = r^+(r). \tag{7.75}$$

Note that $r^+(r) \geq \mathcal{O}(1)\bar{C}^{-1}r$ by the choice of H in the next points, as one has to expect from the near incompressibility (4.3).

(5) We can thus estimate the subset $E_1^+(r, y)$ of $B^d_{r^+(r)}(\boldsymbol{X}(t^+(y), y))$ coming from trajectories crossing the boundary of (7.69):

$$\begin{aligned}\int_{K_{\Omega,3}\setminus Z_1} \mathcal{L}^d(E_1^+(r, y))\mathrm{d}y &\underset{(7.71)}{\leq} C_d\bar{C}^2H^2\bar{\varepsilon}r^d|D\boldsymbol{b}|(\Omega) \\ &= C_d\bar{C}^2H^2\bar{\varepsilon}\Big(\frac{r}{r^+(r)}\Big)^d\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega) \\ &\underset{(7.75)}{=} \frac{C_d\bar{C}^{d+2}\bar{\varepsilon}H^2}{(1 - C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H)^d}\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega).\end{aligned} \tag{7.76}$$

We also estimates the set $\boldsymbol{X}(t^+(y), y) + E_2^+(r, y) \subset \boldsymbol{X}(t^+(y), y) + B^d_{r^+(r)}(0)$ of trajectories arriving from points which do not belongs to $\mathcal{K}_{\Omega,3}$:

$$\int_{K_{\Omega,3}} \mathcal{L}^d(E_2^+(r, y))\,\mathrm{d}y \underset{\text{for } r \ll 1}{\leq} \bar{\varepsilon}\mathcal{L}^d(B^d_{r^+(r)}(0))\mathcal{L}^{d+1}(\Omega), \tag{7.77}$$

where we have observed that $\mathcal{H}^d \llcorner_{S_2}$-a.e. point in $S_2 \cap \mathcal{K}_{\Omega,3}$ is a Lebesgue point.

Finally we have that if

$$E^+(r, y) = E_1^+(r, y) \cup E_2^+(r, y)$$

then

$$\begin{aligned}\int_{K_{\Omega,3}\setminus Z_1} \mathcal{L}^d(E^+(r, y))\,\mathrm{d}y &\underset{(7.76),(7.77)}{\leq} \frac{C_d\bar{C}^{d+2}\bar{\varepsilon}H^2}{(1 - C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H)^d}\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega) \\ &\quad + \bar{\varepsilon}\mathcal{L}^d(B^d_{r^+(r)}(0))\mathcal{L}^{d+1}(\Omega).\end{aligned}$$

(6) The remaining trajectories in $\boldsymbol{X}(t^+(y), y) + B^d_{r^+(r)}(0)$ are arriving from some set which we denote as

$$\boldsymbol{X}(t^-(y), y) + A(y) \subset \boldsymbol{X}(t^-(y), y) + [-r, r] \times Q^H(r) \subset \mathcal{K}_{\Omega,3} \cap (y + B^d_{(1+H)r}(0)),$$

and are not crossing the boundary of (7.69); hence these trajectories cannot arrive from $E_2((1 + H)r, y))$, being the latter defined as the set of trajectories in $\boldsymbol{X}(t^-(y), y) + B^d_{(1+H)r}(0)$ which exit $\boldsymbol{X}(t, y) + B^d_r(0)$

before $t^+(y)$. Thus by changing the coordinates from the initial points y, z at time $t^-(y)$ to the end points at time $t^+(y)$ and using the near incompressibility we can write

$$\begin{aligned}
&\int_{K_{\Omega,3}} \int_{A(y)\setminus E_1((1+H)r,y)} \big|\boldsymbol{X}(t,y+z) - \boldsymbol{X}(t,y) - W(t^+(y),t^-(y),y)z\big| \,\mathrm{d}y\,\mathrm{d}z \\
&\underset{(4.3),(7.79)}{\geq} \frac{1}{\bar{C}^2} \int_{\boldsymbol{X}(K_{\Omega,3})} \int_{B^d_{r^+(r)}(0)\setminus E(r^+(r),y')} \big|w - W(t^+(y),t^-(y),y)(\boldsymbol{X}^{-1}(y'+w) - \boldsymbol{X}^{-1}(y'))\big| \,\mathrm{d}w\,\mathrm{d}y' \\
&\underset{\substack{|(D\boldsymbol{b})_y|\leq M,\\ \text{Thm. 4.1}}}{\geq} \frac{e^{-\bar{C}M}}{\bar{C}^2} \int_{\boldsymbol{X}(K_{\Omega,3})} \int_{B^d_{r^+(r)}(0)\setminus E(r^+(r),y')} \big|\boldsymbol{X}^{-1}(y'+w) - \boldsymbol{X}^{-1}(y') - W^{-1}(t^-(y),t^+(y),y)w\big| \,\mathrm{d}w\,\mathrm{d}y'.
\end{aligned} \tag{7.78}$$

For shortness we have used the notation $y(y')$ inverting the function

$$\boldsymbol{X}(t^+(y(y')),y(y')) = y'.$$

The set $E(r^+(r),y)$ is the set not covered by the trajectories starting in $\boldsymbol{X}(t^-(y),y) + B^d_{(1+H)r}(0)$, which satisfy the estimate for which we can use (7.67): using again that the exiting trajectories have already been counted in $E_1^+(r^+(r),y)$

$$\begin{aligned}
E(r^+(r),y) &= E_1^+(r,y) \cup E_2^+(r,y) \cup \Big[\big(\boldsymbol{X}(t^+(y),y+E_1((1+H)r,y) \\
&\qquad \cup E_2((1+H)r,y)) - \boldsymbol{X}(t^+(y),y)\big) \cap B^d_{r^+(r)}(0)\Big] \\
&= E_1^+(r,y) \cup E_2^+(r,y) \\
&\qquad \cup \Big[\big(\boldsymbol{X}(t^+(y),y+E_1((1+H)r,y)) - \boldsymbol{X}(t^+(y),y)\big) \cap B^d_{r^+(r)}(0)\Big].
\end{aligned} \tag{7.79}$$

(7) Noting that

$$W^{-1}(t^+(y),t^-(y),y) = W(t^-(y),t^+(y),\boldsymbol{X}(t^+(y),y)) = W(t^-(y'),t^+(y'),y'),$$

the bound (7.67) with r replaced with $(1+H)r$ gives

$$\begin{aligned}
&\frac{e^{-\bar{C}M}}{\bar{C}^2} \int_{\boldsymbol{X}(K_{\Omega,3}\setminus Z_1)} \int_{B^d_{r^+(r)}(0)\setminus E(r^+(r),y')} \big|\boldsymbol{X}^{-1}(y'+w) - \boldsymbol{X}^{-1}(y') - W(t^-(y'),t^+(y'),y')w\big| \,\mathrm{d}w\,\mathrm{d}y' \\
&\underset{(7.78)}{\leq} \int_{K_{\Omega,3}\setminus Z_1} \int_{A(y)\setminus E_1((1+H)r,y)} \big|\boldsymbol{X}(t,y+z) - \boldsymbol{X}(t,y) - W(t^+(y),t^-(y),y)z\big| \,\mathrm{d}y\,\mathrm{d}z \\
&\underset{A(y)\subset[-r,r]\times Q^H(r)}{\leq} \int_{K_{\Omega,3}\setminus Z_1} \int_{B^d_{(1+H)r}(0)\setminus(E_1((1+H)r,y)\cup E_2((1+H)r,y))} \big|\boldsymbol{X}(t,y+z) - \boldsymbol{X}(t,y) - W(t^+(y),t^-(y),y)z\big| \,\mathrm{d}y\,\mathrm{d}z \\
&\underset{(7.67)}{\leq} 16\bar{C}^2 C_d r \mathcal{L}^d(B^d_r(0))(\bar{\varepsilon})^{1/4}[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] + \bar{C} r \mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0) \\
&= 16\bar{C}^2 C_d (1+H)^{d+1}\Big(\frac{r}{r^+(r)}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))(\bar{\varepsilon})^{1/4}[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] \\
&\quad + \bar{C}\Big(\frac{r}{r^+(r)}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0) \\
&\underset{(7.75)}{=} 16 C_d \frac{(\bar{C})^{d+3}(1+H)^{d+1}}{(1 - C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H)^d}(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_{r^+(r)}(0))[\mathcal{L}^{d+1}(\Omega) + |D\boldsymbol{b}|(\Omega)] \\
&\quad + \bar{C}\Big(\frac{\bar{C}}{1 - C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0).
\end{aligned} \tag{7.80}$$

(8) Choosing

$$H = (\bar{\varepsilon})^{-1/16(d+1)} \gg 1, \tag{7.81}$$

we obtain that

$$r^+(r) \underset{(7.75)}{=} \frac{1}{\bar{C}}(1 - C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)})r$$

and the estimate of (7.76) becomes

$$\int_{K_\Omega} \mathcal{L}^d(E^+(r,y))\,\mathrm{d}y = \frac{C_d(\bar{C})^{d+2}(\bar{\varepsilon})^{(8d+7)/(8d+8)}}{(1-C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega). \tag{7.82}$$

The image of the set $E_1((1+H)r,y)$ is controlled by (7.63): hence using the nearly incompressibility its image has are controlled by

$$\int_{K_{\Omega,2}} \mathcal{L}^d(\boldsymbol{X}(t^+(y),t^-(y),E_1(r,y)))\,\mathrm{d}y \underset{(7.63)}{\leq} 5\bar{C}^3(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega), \tag{7.83}$$

so that we conclude with

$$\begin{aligned}\int_{K_{\Omega,3}} \mathcal{L}^d(E(r^+,y))\,\mathrm{d}y &\underset{(7.82),(7.83)}{\leq} \frac{C_d(\bar{C})^{d+2}(\bar{\varepsilon})^{(8d+7)/(8d+8)}}{(1-C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_{r^+(r)}(0)|D\boldsymbol{b}|(\Omega) \\ &\qquad + 5\bar{C}^3(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_{(1+H)r}(0)|D\boldsymbol{b}|(\Omega) \\ ((1+H)\leq 2H) \ &= \frac{C_d(\bar{C})^{d+2}(\bar{\varepsilon})^{(4d+3)/(8d+8)}}{(1-C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_{r^+(r)}(\boldsymbol{X}(t^+(y),y)))|D\boldsymbol{b}|(\Omega) \\ &\qquad + 5\bar{C}^3 2^d(\bar{\varepsilon})^{(3d+4)/(16d+16)}\mathcal{L}^d(B^d_{r^+(r)}(\boldsymbol{X}(t^+(y),y)))|D\boldsymbol{b}|(\Omega).\end{aligned} \tag{7.84}$$

Up to pushing the measure $\mathcal{L}^d(\mathrm{d}y)$ to the end points $\boldsymbol{X}(t^+(y),t^-(y),y))$ (thus multiplying the r.h.s. by $\bar{C}$ when integrating in $\mathcal{L}^d(\mathrm{d}y'))$, the estimate (7.84) corresponds to **Point (3) of page 17**, as well as the evaluation of the measure of $E_2(r,y)$ of **Point (7) for the expanding case.**

We note in particular that the fraction of $E_2(r,y)$ can be made small around a large set of initial points: this is what is proved here for the final points, but the argument can be repeated also in the contractive case.

(9) The remaining trajectories start in $B^d_{(1+H)r}(y)\setminus(E_1((1+H)r,y)\cup E_2((1+H)r,y))$, because of the choice of $E(r^+,y)$ and the assumptions that they do not leave $B^d_{(1+H)r}(0)$. Hence we can use (7.80) together with (7.81) to obtain

$$\begin{aligned}&\int_{\boldsymbol{X}(K_\Omega)}\int_{B^d_{r^+(r)}\setminus E(r^+,y')} \big|\boldsymbol{X}^{-1}(y'+z)-\boldsymbol{X}^{-1}(y)-W(t^-(y'),t^+(y'),y)z\big|\,\mathrm{d}y'\,\mathrm{d}z \\ &\underset{(7.80)}{\leq} 16C_d\frac{(\bar{C})^{d+3}(1+H)^{d+1}}{(1-C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H)^d}(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_{r^+(r)}(0))[\mathcal{L}^{d+1}(\Omega)+|D\boldsymbol{b}|(\Omega)] \\ &\quad + \bar{C}\Big(\frac{\bar{C}}{1-C_d\bar{C}^2\sqrt{\bar{\varepsilon}}H}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0) \\ &\underset{(7.81)}{=} 16C_d\frac{2^{d+1}\bar{C}^{d+3}(\bar{\varepsilon})^{3/16}}{(1-C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_{r^+(r)}(0))[\mathcal{L}^{d+1}(\Omega)+|D\boldsymbol{b}|(\Omega)] \\ &\quad + \bar{C}\Big(\frac{\bar{C}}{1-C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)}}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0).\end{aligned} \tag{7.85}$$

Since the trajectories not in $E(r^+,y)$ are not exiting, we can just use Point (7) of page 18 for the previous two points: (5.3) follows from the properties of the disintegration applied to the linear flow $W(\cdot,y)$.

This concludes the proof that the assumptions of Section 5 hold around Lebesgue points of the singular part of the derivative.

8. Construction of a suitable partition into proper sets

The differentiability in measure follows from the estimates in the previous sections if we can find a suitable partition into perturbed proper sets such that the assumptions of Section 5 hold.

Theorem 8.1. *For every open set $\Omega \supset \mathcal{K}_0$ there exists a finite partition $\{\Omega_i^{\mathrm{sing}}\}_{i=1}^N \cup \Omega^{\mathrm{rem}}$ of the compact set $\mathcal{K}_0 \Subset [0,T] \times \mathbb{R}^d$ made of disjoint perturbed proper sets, such that*

$$\mathcal{K}_0 \subset \bigcup_i \bar{\Omega}_i^{\mathrm{sing}} \cup \Omega^{\mathrm{rem}} \subset \Omega,$$

and such that Points (2), (3) of Theorem 3.4 hold with ε replaced by $\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega_i)$, Ω_i^{sing} and satisfies the assumptions of Section 7.1 and $|D^{\mathrm{sing}}\boldsymbol{b}|(\Omega^{\mathrm{rem}}) < \bar{\varepsilon}$.

Proof. Fix Ω open neighborhood of $\mathcal{K}_0$, and let Ω' be another open set such that

$$\mathcal{K} \subset \Omega' \subset \bar{\Omega}' \subset \Omega.$$

The construction of a disjoint covering is done as follows.

(1) Consider a Lebesgue negligible set S where $|D^{\mathrm{sing}}\boldsymbol{b}|$ is concentrated. By Besicovitch's theorem (see [11, Theorem 2.17]), we can cover S with countably many disjoint closed proper balls such that the estimates of Section 7.7 and Section 7.8 hold: these are collected in Point (2) of page 48 in the proof that the partition satisfies the assumptions of Section 5.
(2) Hence we can consider finitely many closed proper balls $\{B_{r_i}^{d+1}(t_i,x_i)\}_{i=1}^N$, contained in Ω', such that

$$|D^{\mathrm{sing}}\boldsymbol{b}|\Big(\Omega' \setminus \bigcup_{i=1}^N B_{r_i}^{d+1}(t_i,x_i)\Big) < \bar{\varepsilon}.$$

(3) Being these balls at positive distance from one another and from $\mathbb{R}^{d+1} \setminus \Omega$, we can perturb them into disjoint proper balls $\{\Omega_i^{\mathrm{sing}}\}_{i=1}^N$, $\Omega_i^{\mathrm{sing}} \subset \Omega$, such that the estimates of Point (2) of page 48 for the singular part $D^{\mathrm{sing}}\boldsymbol{b}$ hold with ε replaced by

$$\min\left\{\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega_i^{\mathrm{sing}}),\ \bar{\varepsilon}\frac{\mathcal{L}^{d+1}(\Omega' \setminus \cup_{i=1}^N \Omega_i^{\mathrm{sing}})}{N}\right\}. \tag{8.1}$$

(4) The complement of the union of the closure of these perturbed proper balls is the set

$$\Omega^{\mathrm{rem}} = \Omega' \setminus \bigcup_{i=1}^N \bar{\Omega}_i^{\mathrm{sing}}.$$

In order to show that the sets $\{\Omega_i^{\mathrm{sing}}, \Omega^{\mathrm{rem}}\}$ satisfy the statement, we just need to prove that Ω^{rem} is a perturbed proper set. To this end, we need to estimate the trajectories in $\mathcal{K}$ which cross $\partial\Omega^{\mathrm{rem}}$ outside $\{t = 0, T\} \cup \cup_i S_i^{\mathrm{sing}} \cup \cup_j S_j^{\mathrm{a.c.}}$: indeed these are the non flat parts of the boundary of Ω^{rem} from which a trajectory in $\mathcal{K}$ may enter. We observe that these trajectories are leaving one of the Ω_i^{sing}, $\Omega_j^{\mathrm{a.c.}}$ not from some flat parts, so that their total estimate is bounded by (8.1) by $\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega^{\mathrm{rem}})$. □

We conclude this section by proving that the partition constructed in Theorem 8.1 satisfies the assumptions of Section 5 with a suitable measure μ_P. This will conclude the proof of Theorem 1.1.

Proposition 8.2. *The partition into perturbed proper sets $\{\Omega_i^{\mathrm{sing}}\}_{i=1}^N \cup \{\Omega^{\mathrm{rem}}\}$ satisfies the assumptions of Section 5 with*

$$\mu_P = C_d(\bar{\varepsilon})^{3/16}(\mathcal{L}^d + |D\boldsymbol{b}|)C_d|D^{\mathrm{sing}}\boldsymbol{b}|\llcorner_{\Omega^{\mathrm{rem}}} + C_d|D\boldsymbol{b}|(\Omega \setminus \mathcal{K}_0), \tag{8.2}$$

whose total mass is of order $\bar{\varepsilon}^{3/16}$.

Proof. We consider separately Ω_i^{sing} and Ω^{rem}.

(1) **Estimates for Ω^{rem}.** We can use the comparison with the a.c. part $D^{\mathrm{a.c.}}\boldsymbol{b}$ of $D\boldsymbol{b}$ in order to obtain the estimates for every perturbed proper set $\Omega_j^{\mathrm{a.c.}}$:
   (a) **Point (2) of page 17:** by Point (2) of page 23, restrict to a set of initial data $S'_{1,j}$ whose co-measure is small than $\bar{\varepsilon}\mathcal{L}^{d+1}(\Omega^{\mathrm{rem}})$;

(b) **Points (4), (5), (6), (7) of page 18:** By (6.6), we have the following error estimate with respect to the linear flow $W(\cdot, y)$ solving (6.1):

$$\begin{aligned}&\int_{S'_{1,j}} \int_{B^d_R(0)} \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y) - W(\cdot, y)z\|_{C^0(t^-(y), t^+(y,z))} \,\mathrm{d}z \,\mathrm{d}y \\ &\underset{(6.6)}{\leq} \bar{C}^2 e^M R\omega(R)\mathcal{L}^d(B^d_R(0))|D\boldsymbol{b}|(\Omega^{\mathrm{rem}}) + \bar{C}^2 e^M R\mathcal{L}^d(B^d_R(0))|(D\boldsymbol{b})^{\mathrm{sing}}|(\Omega^{\mathrm{rem}}) \\ &\leq C_d \bar{C}^2 R\mathcal{L}^d(B^d_R(0))\big(\bar{\varepsilon}|D\boldsymbol{b}|(\Omega^{\mathrm{rem}}) + |D^{\mathrm{sing}}\boldsymbol{b}|(\Omega^{\mathrm{rem}})\big)\end{aligned} \tag{8.3}$$

for $R \ll 1$.

(2) **Estimates for Ω^{sing}_i.** By construction, the sets Ω^{sing}_i satisfy the assumptions of Section 7.1. Moreover, we prove that the following properties hold true.

(a) **Point (2) of page 17:** There exists a set of trajectories $S'_{1,i}$ such that, for $\bar{\varepsilon} \ll 1$,

$$\mathcal{H}^d(S_{1,i} \setminus S'_{1,i}) \underset{(7.61),(7.74)}{\leq} 3(\bar{\varepsilon})^{1/4}(|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i) + \mathcal{L}^{d+1}(\Omega^{\mathrm{sing}}_i));$$

(b) **Point (6) of page 17:** There exists an approximated vector solution $\tilde{X}^H(r, y; t, w)$ such that, for all t, $r' \leq r$,

$$\int_{S'_{1,i}} \int_{B^d_{r'}(z)} \|\tilde{X}^H(r, y; \cdot, z) - z\|_{C^0} \,\mathrm{d}w \,\mathrm{d}y \underset{(7.62)}{\leq} 2\bar{C}r'\mathcal{L}^d(B^d_{r'}(0))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i). \tag{8.4}$$

We have used the estimates on the norm of conditional probabilities for the disintegration, since in the time reverse case the approximate vector field is exactly the disintegration $(D\boldsymbol{b})_y$ (see Point (9) of page 46).

(c) **Point (4) of page 17:** For every $y \in S'_{1,i}$, there exists a set of initial points $E_1(r, y) \subset B^d_r(0)$ such that

$$\int_{S'_{1,i}} \mathcal{L}^d(E_1(r, y)) \,\mathrm{d}y \underset{(7.63)}{\leq} (\bar{\varepsilon})^{1/2}\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i)$$

or

$$\begin{aligned}\int_{S'_{1,i}} \mathcal{L}^d(E_1(r, y)) \,\mathrm{d}y &\underset{(7.84)}{\leq} \frac{C_d(\bar{C})^{d+2}(\bar{\varepsilon})^{(4d+3)/(8d+8)}}{(1 - C_d\bar{C}(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_r(y)))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i) \\ &\quad + 5\bar{C}^3 2^d(\bar{\varepsilon})^{(4d+3)/(16d+16)}\mathcal{L}^d(B^d_r(y)))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i) \\ &\leq C_d(\bar{\varepsilon})^{3/16}\mathcal{L}^d(B^d_r(y)))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i).\end{aligned}$$

(d) **Point (5) of page 17:** For the remaining trajectories, we have

$$\begin{aligned}&\int_{S'_{1,i}} \int_{B^d_r(0)\setminus E_1(r,y)} \|\boldsymbol{X}(\cdot, y+z) - \boldsymbol{X}(\cdot, y) - \tilde{X}^H(r, y; \cdot, z)\|_{C^0(t^-(y), t^+_y(z))} \,\mathrm{d}z \,\mathrm{d}y \\ &\underset{(7.85)}{\leq} 11\bar{C}^2(\bar{\varepsilon})^{1/4} r\mathcal{L}^d(B^d_r(0))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i) \\ &\quad + 14C_d M \frac{2^{d+1}\bar{C}^{d+3}(\bar{\varepsilon})^{(3d+3)/(16d+16)}}{(1 - C_d\bar{C}(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d}\mathcal{L}^d(B^d_{r^+(r)}(0))[\mathcal{L}^{d+1}(\Omega^{\mathrm{sing}}_i) + |D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i)] \\ &\quad + \bar{C}\Big(\frac{\bar{C}}{1 - C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)}}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i \setminus \mathcal{K}_0) \\ &\leq C_d(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_r(y))[\mathcal{L}^{d+1}(\Omega^{\mathrm{sing}}_i) + |D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i)] + C_d\mathcal{L}^d(B^d_r(y))|D\boldsymbol{b}|(\Omega^{\mathrm{sing}}_i \setminus \mathcal{K}_0).\end{aligned} \tag{8.5}$$

(e) **Point (7) of page 18:** By the choice of the singular point,

$$
\begin{aligned}
&\int_{S'_{1,i}} \int_{B^d_r(0)\setminus(E_1(r,y)\cup E_2(r,y))} \big|\boldsymbol{X}(t^+(y), y+z) - \boldsymbol{X}(t^+(y), y) - W((t^-(y), t^+(y)), y)z\big| \,\mathrm{d}z\,\mathrm{d}y \\
&\underset{(7.67),(7.85)}{\leq} 14 C_d \bar{C}^2 r \mathcal{L}^d(B^d_r(0))(\bar{\varepsilon})^{1/4}[\mathcal{L}^{d+1}(\Omega_i^{\mathrm{sing}}) + |D\boldsymbol{b}|(\Omega_i^{\mathrm{sing}})] \\
&\quad + 14 C_d M \frac{2^{d+1}\bar{C}^{d+3}(\bar{\varepsilon})^{(3d+3)/(16d+16)}}{(1 - C_d\bar{C}(\bar{\varepsilon})^{(8d+7)/(16d+16)})^d} \mathcal{L}^d(B^d_{r^+(r)}(0))[\mathcal{L}^{d+1}(\Omega_i^{\mathrm{sing}}) + |D\boldsymbol{b}|(\Omega_i^{\mathrm{sing}})] \\
&\quad + \bar{C}\Big(\frac{\bar{C}}{1 - C_d\bar{C}^2(\bar{\varepsilon})^{(8d+7)/(16d+16)}}\Big)^{d+1} r^+(r)\mathcal{L}^d(B^d_{r^+(r)}(0))|D\boldsymbol{b}|(\Omega_i^{\mathrm{sing}} \setminus \mathcal{K}_0) \\
&\leq C_d(\bar{\varepsilon})^{1/4}\mathcal{L}^d(B^d_r(y))[\mathcal{L}^{d+1}(\Omega_i^{\mathrm{sing}}) + |D\boldsymbol{b}|(\Omega_i^{\mathrm{sing}})] + C_d\mathcal{L}^d(B^d_r(y))|D\boldsymbol{b}|(\Omega_i^{\mathrm{sing}} \setminus \mathcal{K}_0).
\end{aligned}
$$

We then define the measure μ_P as

$$
\mu_P = C_d(\bar{\varepsilon})^{3/16}[\mathcal{L}^d + |D\boldsymbol{b}|]C_d|D^{\mathrm{sing}}\boldsymbol{b}|_{\llcorner\Omega^{\mathrm{rem}}} + C_d|D\boldsymbol{b}|(\Omega\setminus\mathcal{K}_0),
$$

whose total mass is less than $\mathcal{O}(\bar{\varepsilon}^{3/16})$. This gives the **Point (1) of page 17**. □

Appendix A. Bressan's lemma on the approximation of Lipschitz continuous flows

A key tool in the proof of our main result is the following lemma (see [24, Lemma 4] or [25, Theorem 2.9]): given an absolutely continuous curve γ and a Lipschitz continuous semigroup S_t, we can estimate the distance between γ and the trajectory of the semigroup starting at $\gamma(0)$.

Lemma A.1. *If $t \mapsto \gamma(t)$ is an a.c. curve, and S_t is a semigroup such that there exists an L^1-function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$
\|S_t - S_s\| \leq \int_s^t f(\tau)\,\mathrm{d}\tau,
$$

then

$$
|\gamma(T) - S_T\gamma(0)| \leq L\int_0^T \liminf_{h\searrow 0} \frac{|\gamma(t+h) - S_h(\gamma(t))|}{h}\,\mathrm{d}t.
$$

Proof. Consider the curve

$$
t \mapsto X(t) = S_{T-t}\gamma(t).
$$

We have

$$
\begin{aligned}
|X(t+h) - X(t)| &= |S_{T-t-h}\gamma(t+h) - S_{T-t}\gamma(t)| \\
&= |S_{T-t-h}(\gamma(t+h) - S_h\gamma(t)| \leq L|\gamma(t+h) - S_h\gamma(t)|.
\end{aligned}
$$

Using the assumption on S_t, we have

$$
|X(t+h) - X(t)| \leq L\int_t^{t+h} (|\dot{\gamma}|(s) + f(s))\,\mathrm{d}s,
$$

so that the curve is absolutely continuous. Moreover, in each point of differentiability,

$$
|\dot{X}|(t) = \lim_{h\searrow 0}\frac{|X(t+h) - X(t)|}{h} \leq L\liminf_{h\searrow 0}\frac{|\gamma(t+h) - S_h\gamma(t)|}{h},
$$

which concludes the proof. □

We use Lemma A.1 to compare a nearly incompressible flow $\boldsymbol{X}(t,y)$ generated by a vector field $\boldsymbol{b}$ with a L-Lipschitz flow $\boldsymbol{Y}(t,y)$ generated by the vector field $\tilde{\boldsymbol{b}}$ that satisfies

$$
\lim_{h\to 0}\frac{\boldsymbol{Y}(t+h,x) - x}{h} = \tilde{\boldsymbol{b}}(t,x) \tag{A.1}
$$

on a set of full $\mathcal{L}^{d+1}$-measure.

Corollary A.2. *If $\boldsymbol{X}(t,y)$ is a nearly incompressible flow generated by $\boldsymbol{b}(t,x)$ and $\boldsymbol{Y}(t,y)$ is a L-Lipschitz continuous semigroup such that* (A.1) *holds for $\mathcal{L}^{d+1}$-a.e. $(t,x) \in [0,T] \times \mathbb{R}^d$, then, for $\mathcal{L}^d$-a.e. $y \in \mathbb{R}^d$,*

$$|\boldsymbol{X}(t,y) - \boldsymbol{Y}(t,y)| \leq L \int_0^T |\boldsymbol{b}(t, \boldsymbol{X}(t,y)) - \tilde{\boldsymbol{b}}(t, \boldsymbol{X}(t,y))| \,\mathrm{d}t.$$

Proof. Lemma A.1 yields

$$\begin{aligned} |\boldsymbol{X}(T,y) - \boldsymbol{Y}(T,y)| &\leq L \int_0^T \liminf_{h \searrow 0} \frac{|\boldsymbol{X}(t+h,y) - \boldsymbol{Y}(t+h, \boldsymbol{X}(t,y))|}{h} \,\mathrm{d}t \\ &= L \int_0^T \liminf_{h \searrow 0} \left| \boldsymbol{b}(t, \boldsymbol{X}(t,y)) - \frac{\boldsymbol{Y}(t+h, \boldsymbol{X}(t,y)) - \boldsymbol{X}(t,y)}{h} \right| \mathrm{d}t. \end{aligned}$$

By the nearly incompressibility and Fubini theorem, for $\mathcal{L}^d$-a.e. trajectory the above limit is equal for $\mathcal{L}^1$-a.e. t to $|\boldsymbol{b}(t, \boldsymbol{X}(t,y)) - \tilde{\boldsymbol{b}}(t, \boldsymbol{X}(t,y))|$. We thus proved the claim. □

We can show that the assumption (A.1) holds in the following two cases (which are relevant to Sections 6 and 7 respectively):

(1) the linear flow generated by a matrix $A(t) \in L^1((0,t))$;
(2) the solution to the differential inclusion
$$\dot{x} \in -A(t,x),$$
with $A(t)$ a quasi monotone operator defined in $\mathbb{R}^d$ and such that $|A(t,0)| \in L^1$.

The first case is elementary; the second one is analyzed in [18].

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