Nicola Gigli · Luca Tamanini

Second order differentiation formula on $\mathsf{RCD}^*(K, N)$ spaces

Abstract. The aim of this paper is to prove a second order differentiation formula for $H^{2,2}$ functions along geodesics in $\mathsf{RCD}^*(K, N)$ spaces with $K \in \mathbb{R}$ and $N < \infty$. This formula is new even in the context of Alexandrov spaces, where second order differentiation is typically related to semiconvexity.

　　　We establish this result by showing that W_2-geodesics can be approximated up to second order, in a sense which we shall make precise, by entropic interpolations. In turn this is achieved by proving new, even in the smooth setting, estimates concerning entropic interpolations which we believe are interesting on their own. In particular we obtain:

- equiboundedness of densities along entropic interpolations,
- local equi-Lipschitz continuity of Schrödinger potentials,
- uniform weighted L^2 control of the Hessian of such potentials.

Finally, the techniques adopted in this paper can be used to show that in the RCD setting the viscous solution of the Hamilton–Jacobi equation can be obtained via a vanishing viscosity method, as in the smooth case.

　　　With respect to a previous version, where the space was assumed to be compact, in this paper the second order differentiation formula is proved in full generality.

Keywords. Optimal transport, metric geometry, RCD spaces, entropic interpolation, Schrödinger problem

Contents

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1. Introduction

In the last ten years there has been a great interest in the study of metric measure spaces with Ricci curvature bounded from below: see for instance [49], [60], [61], [33], [5], [6], [31], [7], [53], [54], [34], [28], [30], [43], [9], [52], [15], [14]. The starting points of this research line have been the seminal papers [49] and [60], [61] which linked lower Ricci bounds on metric measure spaces to properties of entropy-like functionals in connection with W_2-geometry. Later [5] it emerged that also Sobolev calculus is linked to W_2-geometry and building on this the original definition of CD spaces by Lott–Sturm–Villani has evolved into that of RCD spaces ([6], [31]).

An example of a link between Sobolev calculus and W_2-geometry is the following result (a minor variant of a statement in [28]). It says that we can safely take one derivative of a $W^{1,2}(X)$ function along an optimal geodesic test plan $\boldsymbol{\pi}$, i.e. a test plan satisfying

$$\iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) = W_2^2\big((\mathrm{e}_0)_*\boldsymbol{\pi}, (\mathrm{e}_1)_*\boldsymbol{\pi}\big).$$

Theorem 1.1 (First order differentiation formula). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space,* $K \in \mathbb{R}$, $\boldsymbol{\pi}$ *an optimal geodesic test plan with bounded support* (*equivalently, such that* $\{\gamma_t : t \in [0, 1], \ \gamma \in \mathrm{supp}(\boldsymbol{\pi})\} \subset X$ *is bounded*) *and* $f \in W^{1,2}(X)$. *Then the map* $[0, 1] \ni t \mapsto f \circ \mathrm{e}_t \in L^2(\boldsymbol{\pi})$ *is in* $C^1([0, 1], L^2(\boldsymbol{\pi}))$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \mathrm{e}_t) = \langle \nabla f, \nabla \phi_t \rangle \circ \mathrm{e}_t$$

for every $t \in [0, 1]$, *where* $\mathrm{e}_t : C([0, 1], X) \to X, \gamma \mapsto \gamma_t$, *is the evaluation map and* ϕ_t *is any function such that for some distinct* $t, s \in [0, 1]$, *the function* $-(s - t)\phi_t$ *is a Kantorovich potential from* $(\mathrm{e}_t)_*\boldsymbol{\pi}$ *to* $(\mathrm{e}_s)_*\boldsymbol{\pi}$.

Recall that on $\mathrm{RCD}(K, \infty)$ spaces every W_2-geodesic (μ_t) between measures with bounded density and support is such that $\mu_t \leq C\mathfrak{m}$ for every $t \in [0, 1]$ and some $C > 0$ [54], so that between two such measures there always exists a (unique) optimal geodesic test plan with bounded support. Thus the theorem also says that we can find 'many' C^1 functions on RCD spaces. We remark that such C^1 regularity—which was crucial in [28]—is non-trivial even if the function f is assumed to be Lipschitz and that statements about C^1 smoothness are quite rare in metric geometry.

Furthermore, projecting from $\boldsymbol{\pi}$ to $\mu_t := (e_t)_* \boldsymbol{\pi}$ one can see that Theorem 1.1 immediately implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = \int \langle \nabla f, \nabla \phi_t \rangle \, \mathrm{d}\mu_t \tag{1.1}$$

and one might think of this identity as an 'integrated' version of the basic formula

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t) = \mathrm{d}f(\gamma_t')$$

valid in the smooth framework; at the technical level the proof of the claim has to do with the fact that the geodesic (μ_t) solves the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \mu_t + \mathrm{div}(\nabla \varphi_t \mu_t) = 0, \tag{1.2}$$

where the φ_t's are appropriate choices of Kantorovich potentials (see also [32] in this connection), and with the fact that $\nabla \varphi_t = \nabla \phi_t$ (see Lemma A.7 below).

In [29], the first author developed a second order calculus on RCD spaces, in particular defining the space $H^{2,2}(\mathrm{X})$ and for $f \in H^{2,2}(\mathrm{X})$ the Hessian $\mathrm{Hess}(f)$ (see [29] and the Appendix). It is then natural to ask whether an 'integrated' version of the second order differentiation formula

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\gamma_t) = \mathrm{Hess}(f)(\gamma_t', \gamma_t') \quad \text{for } \gamma \text{ geodesic}$$

holds in this framework. In this paper we provide an affirmative answer to this question:

Theorem 1.2 (Second order differentiation formula). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* RCD*(K, N) *space,* $K \in \mathbb{R}$ *and* $N < \infty$, $\boldsymbol{\pi}$ *an optimal geodesic test plan with bounded support and* $f \in H^{2,2}(\mathrm{X})$. *Then the map* $[0, 1] \ni t \mapsto f \circ e_t \in L^2(\boldsymbol{\pi})$ *is in* $C^2([0, 1], L^2(\boldsymbol{\pi}))$ *and*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(f \circ e_t) = \mathrm{Hess}(f)(\nabla \phi_t, \nabla \phi_t) \circ e_t \tag{1.3}$$

for every $t \in [0, 1]$, *where* ϕ_t *is as in Theorem* 1.1.

An equivalent formulation, which is the one we shall actually prove (see Theorem 5.13) and is more in the spirit of (1.1), is the following:

Theorem 1.3 (Second order differentiation formula (2nd form)). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* RCD*(K, N) *space,* $K \in \mathbb{R}$ *and* $N < \infty$, *let* $\mu_0, \mu_1 \in \mathscr{P}_2(\mathrm{X})$ *with compact supports be such that* $\mu_0, \mu_1 \leq C\mathfrak{m}$ *for some* $C > 0$, *and let* (μ_t) *be the unique* W_2-*geodesic connecting* μ_0 *to* μ_1. *Also, let* $f \in H^{2,2}(\mathrm{X})$. *Then the map*

$$[0, 1] \ni t \mapsto \int f \, \mathrm{d}\mu_t \in \mathbb{R}$$

belongs to $C^2([0, 1])$ *and*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \, \mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla \phi_t, \nabla \phi_t) \, \mathrm{d}\mu_t \tag{1.4}$$

for every $t \in [0, 1]$, *where* ϕ_t *is any function such that for some distinct* $t, s \in [0, 1]$, *the function* $-(s - t)\phi_t$ *is a Kantorovich potential from* μ_t *to* μ_s.

Let us comment on the assumptions in Theorems 1.2 and 1.3:

- The first order differentiation formula is valid on general $\mathrm{RCD}(K, \infty)$ spaces, while for the second order one we assume finite-dimensionality. This is due to the strategy of our proof, which among other things uses the Li–Yau inequality; it is therefore unknown whether such assumption is really needed.
- There exist optimal geodesic test plans without bounded support (if $K = 0$ or the densities of the initial and final marginals decay sufficiently fast) but in this case the functions ϕ_t appearing in the statement(s) are not Lipschitz. It then seems hard to get $\mathrm{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ \mathrm{e}_t \in L^1(\boldsymbol{\pi})$ and thus we cannot really hope for anything like (1.3), (1.4) to hold: this explains the need of the assumption on bounded supports.

Having at our disposal such second order differentiation formula is interesting not only at the theoretical level, but also for applications to the geometry of RCD spaces. For instance, the proofs of both the splitting theorem [28] and of the 'volume cone implies metric cone' statement [22] in this setting can be greatly simplified by using that formula (in this connection, see [63] for comments about the splitting). Also, one aspect of the theory of RCD spaces which is not yet clear is whether they have constant dimension: for Ricci-limit spaces this is known to be true by a result of Colding–Naber [21] which uses second order derivatives along geodesics in a crucial way. Thus our result is necessary to replicate the Colding–Naber argument in the non-smooth setting (but not sufficient: they also use a calculation with Jacobi fields which as of today does not have a non-smooth counterpart).[1]

Let us discuss the strategy of the proof. Our starting point is a related second order differentiation formula obtained in [29], available under proper regularity assumptions:

Theorem 1.4. *Let (μ_t) be a W_2-absolutely continuous curve solving the continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(X_t \mu_t) = 0$$

for some vector fields $(X_t) \subset L^2(T\mathrm{X})$ in the following sense: for every $f \in W^{1,2}(\mathrm{X})$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \, \mathrm{d}\mu_t = \int \langle \nabla f, X_t \rangle \, \mathrm{d}\mu_t.$$

Assume that

(i) $t \mapsto X_t \in L^2(T\mathrm{X})$ *is absolutely continuous,*

(ii) $\sup_t \{\|X_t\|_{L^2} + \|X_t\|_{L^\infty} + \|\nabla X_t\|_{L^2}\} < \infty$.

Then for $f \in H^{2,2}(\mathrm{X})$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is $C^{1,1}$ and the formula

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int f \, \mathrm{d}\mu_t = \int \Big(\mathrm{Hess}(f)(X_t, X_t) + \big\langle \nabla f, \tfrac{\mathrm{d}}{\mathrm{d}t}X_t + \nabla_{X_t} X_t \big\rangle \Big) \, \mathrm{d}\mu_t \qquad (1.5)$$

holds for a.e. $t \in [0, 1]$.

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[1] Added in proof: Brué–Semola [13] recently obtained the constant dimension property by other means.

If the vector fields X_t are of gradient type, so that $X_t = \nabla\phi_t$ for every t and the 'acceleration' a_t is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t + \frac{|\nabla\phi_t|^2}{2} =: a_t,$$

then (1.5) reads

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int f\,\mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla\phi_t, \nabla\phi_t)\,\mathrm{d}\mu_t + \int \langle\nabla f, \nabla a_t\rangle\,\mathrm{d}\mu_t. \tag{1.6}$$

In the case of geodesics, the functions φ_t appearing in (1.2) solve (in a sense which we will not make precise here) the Hamilton–Jacobi equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t + \frac{|\nabla\varphi_t|^2}{2} = 0, \tag{1.7}$$

thus in this case the acceleration a_t is identically 0. Hence if the vector fields $(\nabla\varphi_t)$ satisfy the regularity requirements (i), (ii) in the last theorem we would easily be able to establish Theorem 1.2. However in general this is not the case; informally speaking, this has to do with the fact that for solutions of the Hamilton–Jacobi equations we do not have sufficiently strong second order estimates.

In order to establish Theorem 1.2 it is therefore natural to look for suitable 'smooth' approximations of geodesics for which we can apply Theorem 1.4 above and then pass to the limit in formula (1.5). Given that the lack of smoothness of W_2-geodesics is related to the lack of smoothness of solutions of (1.7), also in line with the classical theory of viscous approximation for the Hamilton–Jacobi equation there is a quite natural thing to try: solve, for $\varepsilon > 0$, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\varepsilon = \frac{|\nabla\varphi_t^\varepsilon|^2}{2} + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon, \quad \varphi_0^\varepsilon := \varphi,$$

where φ is a given, fixed, Kantorovich potential for the geodesic (μ_t), and then solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t^\varepsilon - \mathrm{div}(\nabla\varphi_t^\varepsilon\mu_t^\varepsilon) = 0, \quad \mu_0^\varepsilon := \mu_0.$$

This plan can actually be pursued and following the ideas in this paper one can show that if the space $(X, d, \mathfrak{m})$ is RCD*(K, N) and the geodesic (μ_t) is made up of measures with equibounded densities, then as $\varepsilon \downarrow 0$:

(i) the curves (μ_t^ε) W_2-uniformly converge to the geodesic (μ_t) and the measures μ_t^ε have equibounded densities,
(ii) the functions φ_t^ε are equi-Lipschitz and converge both uniformly and in the $W^{1,2}$-topology to the only viscous solution (φ_t) of (1.7) with φ as initial datum; in particular the continuity equation (1.2) for the limit curve holds.

These convergence results are based on Hamilton's gradient estimates and the Li–Yau inequality and are sufficient to pass to the limit in the term with the Hessian in (1.6). For these curves the acceleration is given by $a_t^\varepsilon = -\frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon$ and thus it remains to prove that the quantity

$$\varepsilon \int \langle \nabla f, \nabla \Delta \varphi_t^\varepsilon \rangle \, \mathrm{d}\mu_t^\varepsilon$$

goes to 0 in some sense. However, there appears to be *no hope of obtaining this by PDE estimates*. The problem is that this kind of viscous approximation can produce in the limit a curve which is not a geodesic if φ is not c-concave: briefly, this happens if a shock appears in Hamilton–Jacobi. Since there is no hope for formula (1.4) to be true for non-geodesics, we see that there is little chance of obtaining it via such viscous approximation.

We therefore use another way of approximating geodesics: the slowing down of entropic interpolation. Let us briefly describe what this is in the familiar Euclidean setting.

Fix two probability measures $\mu_0 = \rho_0 \mathcal{L}^d$, $\mu_1 = \rho_1 \mathcal{L}^d$ on $\mathbb{R}^d$. The Schrödinger functional equations are

$$\rho_0 = f \, \mathsf{h}_1 g, \qquad \rho_1 = g \, \mathsf{h}_1 f, \tag{1.8}$$

the unknown being the Borel functions $f, g : \mathbb{R}^d \to [0, \infty)$, where $\mathsf{h}_t f$ is the heat flow starting at f evaluated at time t. It turns out that in great generality these equations admit a solution which is unique up to the trivial transformation $(f, g) \mapsto (cf, g/c)$ for some constant $c > 0$. Such a solution can be found in the following way: Let R be the measure on $(\mathbb{R}^d)^2$ whose density with respect to $\mathcal{L}^{2d}$ is given by the heat kernel $\mathsf{r}_t(x, y)$ at time $t = 1$ and minimize the Boltzmann–Shannon entropy $H(\gamma \,|\, \mathsf{R})$ among all transport plans γ from μ_0 to μ_1. The Euler equation for the minimizer forces it to be of the form $f \otimes g \, \mathsf{R}$ for some Borel functions $f, g : \mathbb{R}^d \to [0, \infty)$, where $f \otimes g(x, y) := f(x)g(y)$ (we shall re-prove this known result in Proposition 2.1). Then the fact that $f \otimes g \, \mathsf{R}$ is a transport plan from μ_0 to μ_1 is equivalent to (f, g) solving (1.8).

Once we have found a solution of (1.8) we can use it in conjunction with the heat flow to interpolate from ρ_0 to ρ_1 by defining

$$\rho_t := \mathsf{h}_t f \, \mathsf{h}_{1-t} g.$$

This is called *entropic interpolation*. Now we slow down the heat flow: fix $\varepsilon > 0$ and by mimicking the above find $f^\varepsilon, g^\varepsilon$ such that

$$\rho_0 = f^\varepsilon \, \mathsf{h}_{\varepsilon/2} g^\varepsilon, \qquad \rho_1 = g^\varepsilon \, \mathsf{h}_{\varepsilon/2} f^\varepsilon$$

(the factor $1/2$ plays no special role, but is convenient in computations). Then define

$$\rho_t^\varepsilon := \mathsf{h}_{t\varepsilon/2} f^\varepsilon \, \mathsf{h}_{(1-t)\varepsilon/2} g^\varepsilon.$$

A remarkable and non-trivial fact here is that as $\varepsilon \downarrow 0$ the curves $(\rho_t^\varepsilon \mathcal{L}^d)$ of measures converge to the W_2-geodesic from μ_0 to μ_1.

The first connections between Schrödinger equations and optimal transport have been obtained by Mikami [50] for the quadratic cost on $\mathbb{R}^d$; later Mikami–Thieullen [51] showed that a link persists even for more general cost functions. The statement we have

just made about convergence of entropic interpolations to displacement ones has been proved by Léonard [46]. Actually, Léonard worked in much higher generality: as it is perhaps clear from the presentation, the construction of entropic interpolation can be done in great generality, as only a heat kernel is needed. He also provided a basic intuition about why such convergence is in place: the basic idea is that if the heat kernel admits the asymptotic expansion $\varepsilon \log r_\varepsilon(x, y) \sim -d^2(x, y)/2$ (in the sense of large deviations), then the rescaled entropy functionals $\varepsilon H(\cdot \mid R_\varepsilon)$ converge to $\frac{1}{2} \int d^2(x, y) d\cdot$ (in the sense of Γ-convergence). We refer to [48] for a deeper discussion of this topic, historical remarks and much more, and to [20] and [26] for more recent developments about the link between optimal transport and the Schrödinger problem.

Starting from these intuitions and results, working in the setting of RCD*(K, N) spaces we gain new information about the convergence of entropic interpolations to displacement ones. In order to state our results, it is convenient to introduce the Schrödinger potentials

$$\varphi_t^\varepsilon := \varepsilon \log(\mathsf{h}_{t\varepsilon/2} f^\varepsilon), \qquad \psi_t^\varepsilon := \varepsilon \log(\mathsf{h}_{(1-t)\varepsilon/2} g^\varepsilon).$$

In the limit $\varepsilon \downarrow 0$ these will converge to forward and backward Kantorovich potentials along the limit geodesic (μ_t) (see below). In this direction, it is worth noticing that while for $\varepsilon > 0$ there is a tight link between potentials and densities, as we trivially have

$$\varphi_t^\varepsilon + \psi_t^\varepsilon = \varepsilon \log \rho_t^\varepsilon,$$

in the limit this becomes the well known (weaker) relation between forward/backward Kantorovich potentials and measures (μ_t):

$$\varphi_t + \psi_t \begin{cases} = 0 & \text{on } \mathrm{supp}(\mu_t), \\ \leq 0 & \text{on X} \end{cases}$$

(see e.g. [64, Remark 7.37], paying attention to the different sign convention). By direct computation one can verify that (φ_t^ε), (ψ_t^ε) solve the Hamilton–Jacobi–Bellman equations

$$\frac{d}{dt}\varphi_t^\varepsilon = \frac{1}{2}|\nabla\varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon, \qquad -\frac{d}{dt}\psi_t^\varepsilon = \frac{1}{2}|\nabla\psi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta\psi_t^\varepsilon, \tag{1.9}$$

thus introducing the functions

$$\vartheta_t^\varepsilon := \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$$

it is not hard to check that

$$\frac{d}{dt}\rho_t^\varepsilon + \mathrm{div}(\nabla\vartheta_t^\varepsilon \, \rho_t^\varepsilon) = 0 \tag{1.10}$$

and

$$\frac{d}{dt}\vartheta_t^\varepsilon + \frac{|\nabla\vartheta_t^\varepsilon|^2}{2} = a_t^\varepsilon, \quad \text{where} \quad a_t^\varepsilon := -\frac{\varepsilon^2}{8}(2\Delta \log \rho_t^\varepsilon + |\nabla \log \rho_t^\varepsilon|^2).$$

With this, our main results about entropic interpolations can be summarized as follows. Under the assumptions that the metric measure space (X, d, $\mathfrak{m}$) is RCD*(K, N), $N < \infty$, and that ρ_0, ρ_1 with bounded supports belong to $L^\infty(X)$ we have:

- Zeroth order:

  - *Bound:* For some $C > 0$ depending on K, N, ρ_0, ρ_1 we have $\rho_t^\varepsilon \leq C$ for every $\varepsilon \in (0, 1)$ and $t \in [0, 1]$.
  - *Convergence:* The curves $(\rho_t^\varepsilon \mathfrak{m})$ W_2-uniformly converge to the unique W_2-geodesic (μ_t) from μ_0 to μ_1 and setting $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$ we have $\rho_t^\varepsilon \overset{*}{\rightharpoonup} \rho_t$ in $L^\infty(X)$ for all $t \in [0, 1]$.

- First order:

  - *Bound:* For any $t \in (0, 1]$ the functions $\{\varphi_t^\varepsilon\}_{\varepsilon \in (0,1)}$ are locally equi-Lipschitz, and similarly for the ψ's.
  - *Convergence:* For every sequence $\varepsilon_n \downarrow 0$ there is a subsequence—not relabelled—such that for any $t \in (0, 1]$ the functions $\varphi_t^{\varepsilon_n}$ converge both locally uniformly and in $W_{\mathrm{loc}}^{1,2}(X)$ to a function φ_t such that $-t\varphi_t$ is a Kantorovich potential from μ_t to μ_0, and similarly for the ψ's.

- Second order: For every $\delta \in (0, 1/2)$ we have

  - *Bound:*

$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} (|\mathrm{Hess}(\vartheta_t^\varepsilon)|_{\mathsf{HS}}^2 + \varepsilon^2 |\mathrm{Hess}(\log \rho_t^\varepsilon)|_{\mathsf{HS}}^2) \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty,$$
$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} (|\Delta\vartheta_t^\varepsilon|^2 + \varepsilon^2 |\Delta\log \rho_t^\varepsilon|^2) \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty. \tag{1.11}$$

  Notice that since in general the Laplacian is not the trace of the Hessian, there is no direct link between these two bounds.

  - *Convergence:* For every function $h \in W^{1,2}(X)$ with $\Delta h \in L^\infty(X)$ we have

$$\lim_{\varepsilon \downarrow 0} \iint_\delta^{1-\delta} \langle \nabla h, \nabla a_t^\varepsilon \rangle \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 0. \tag{1.12}$$

With the exception of the convergence $\rho_t^\varepsilon \mathfrak{m} \to \mu_t$, all these results are new even on smooth manifolds (in fact, even on $\mathbb{R}^d$) and have been partially used in our recent paper [38], where further analogies between entropic interpolations/Schrödinger potentials on the one hand and W_2-geodesics/Kantorovich potentials on the other one are investigated within the RCD framework, in particular in connection with a Benamou–Brenier-like formulation of these problems. Such analogies have been first pointed out in [20] and [26] in the Euclidean setting, and these papers have been our source of inspiration for [38].

The zeroth and first order bounds are both consequences of the Hamilton–Jacobi–Bellman equations (1.9) satisfied by the φ's and ψ's and can be obtained from Hamilton's gradient estimate and the Li–Yau inequality. The facts that the limit curve is the W_2-geodesic and that the limit potentials are Kantorovich potentials are consequences of the fact that we can pass to the limit in the continuity equation (1.10) and that the limit

potentials satisfy the Hamilton–Jacobi equation. In this regard it is key that we approximate at the same time both the 'forward' potentials ψ and the 'backward' one φ: see the proof of Proposition 5.4 and recall that the simple viscous approximation may converge to curves which are not W_2-geodesics.

Notice that these zeroth and first order convergences are sufficient to pass to the limit in the term with the Hessian in (1.6). As already mentioned, also viscous approximation could produce the same kind of convergence.

The crucial advantage of dealing with entropic interpolation (which has no counterpart in viscous approximation) is thus in the second order bounds and convergence results which show that the term with the acceleration in (1.6) vanishes in the limit and thus eventually allows us to prove our main result, Theorem 1.2. In this direction, we informally point out that since the geodesic equation is of second order, in searching for an approximation procedure it is natural to look for one producing some sort of second order convergence.

The limiting property (1.12) is mostly a consequence—although perhaps non-trivial—of the bound (1.11) (see in particular Lemma 4.10 and the proof of Theorem 5.13), thus let us focus on how to get (1.11). The starting point here is a formula due to Léonard [44], who realized that there is a connection between entropic interpolation and lower Ricci bounds: he computed the second order derivative of the entropy along entropic interpolations and in this direction our contribution has been the rigorous proof in the RCD framework of his formal computations, thus getting

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(\mu_t^\varepsilon \,|\, \mathfrak{m}) = \int \rho_t^\varepsilon \,\mathrm{d}\big(\Gamma_2(\vartheta_t^\varepsilon) + \tfrac{\varepsilon^2}{4}\Gamma_2(\log \rho_t^\varepsilon)\big) = \frac{1}{2} \int \rho_t^\varepsilon \,\mathrm{d}\big(\Gamma_2(\varphi_t^\varepsilon) + \Gamma_2(\psi_t^\varepsilon)\big),$$
(1.13)

where Γ_2 is the 'iterated carré du champ' operator defined as

$$\Gamma_2(f) := \mathbf{\Delta}\frac{|\nabla f|^2}{2} - \langle \nabla f, \nabla \Delta f \rangle$$

(in the setting of RCD spaces some care is needed when handling this object; see also the Appendix for an explanation of the distinction between $\mathbf{\Delta}$ and Δ, but let us neglect this issue here).

Observe that if $h : [0, 1] \to \mathbb{R}^+$ is a convex function, then $-\frac{h(0)}{t} \leq h'(t) \leq \frac{h(1)}{1-t}$ for any $t \in (0, 1)$ and thus

$$\int_\delta^{1-\delta} h''(t) \,\mathrm{d}t = h'(1 - \delta) - h'(\delta) \leq \frac{h(1)}{1 - \delta} + \frac{h(0)}{\delta}. \tag{1.14}$$

If we assume for simplicity that $K = 0$ we have $\Gamma_2 \geq 0$, so that (1.13) tells us in particular that $t \mapsto H(\mu_t^\varepsilon \,|\, \mathfrak{m})$ is convex for any $\varepsilon > 0$, and if we also assume that $\mathfrak{m}(X) = 1$, that function is non-negative. Therefore (1.14) shows that for any $\delta \in (0, 1/2)$,

$$\sup_{\varepsilon \in (0,1)} \int_\delta^{1-\delta} \int \rho_t^\varepsilon \,\mathrm{d}\big(\Gamma_2(\vartheta_t^\varepsilon) + \tfrac{\varepsilon^2}{4}\Gamma_2(\log \rho_t^\varepsilon)\big) \,\mathrm{d}t \leq \frac{H(\mu_1 \,|\, \mathfrak{m})}{1 - \delta} + \frac{H(\mu_0 \,|\, \mathfrak{m})}{\delta} < \infty.$$
(1.15)

Recalling the Bochner inequalities ([23], [9], [29])

$$\Gamma_2(\eta) \geq |\mathrm{Hess}(\eta)|_{\mathsf{HS}}^2 \mathfrak{m}, \quad \Gamma_2(\eta) \geq \frac{(\Delta\eta)^2}{N}\mathfrak{m},$$

we see that (1.11) follows from (1.15). Then with some work (see Lemma 4.10 and Theorem 5.13 for the details) starting from (1.11) we can deduce (1.12), which in turn ensures that the term with the acceleration in (1.6) vanishes in the limit $\varepsilon \downarrow 0$, thus leading to our main result, Theorem 1.2.

Structure of the paper. In Section 2 we prove the solvability of the Schrödinger system (1.8) in great generality and deduce some properties of the solutions. In Section 3 Hamilton's gradient estimate and the Li–Yau Laplacian estimate are recalled and adapted to future purposes. Section 4 is devoted to a deeper investigation of entropic interpolation and the associated Schrödinger potentials; in particular, we establish the zeroth, first and second order bounds presented before and show that the entropy is C^2 along entropic interpolations with explicit formulas for the first and second derivatives. The zeroth, first and second order convergences are then proved in Section 5 and, relying on them and on the previous results, the main theorem as well as some equivalent formulations are deduced. Finally, in Appendix A the reader can find all the relevant notions, results and bibliographic references related to calculus and optimal transport on RCD spaces.

2. The Schrödinger problem

Let (X, τ) be a Polish space, let $\mu_0, \mu_1 \in \mathscr{P}(X)$ and let R be a non-negative Radon measure on X^2. Recall that $\boldsymbol{\gamma} \in \mathscr{P}(X^2)$ is called a *transport plan* for μ_0, μ_1 provided $\pi_*^0\boldsymbol{\gamma} = \mu_0$ and $\pi_*^1\boldsymbol{\gamma} = \mu_1$, where $\pi^0, \pi^1 : X^2 \to X$ are the canonical projections. We are interested in finding a transport plan of the form

$$\boldsymbol{\gamma} = f \otimes g \, \mathsf{R}$$

for certain Borel functions $f, g : X \to [0, \infty)$, where $f \otimes g(x, y) := f(x)g(y)$. As we shall see in this short section, in great generality this problem can be solved in a unique way and the plan $\boldsymbol{\gamma}$ can be found as the minimum of

$$\boldsymbol{\gamma}' \mapsto H(\boldsymbol{\gamma}' \,|\, \mathsf{R})$$

among all transport plans from μ_0 to μ_1, where $H(\,\cdot\,|\,\cdot\,)$ is the Boltzmann–Shannon entropy. For an appropriate choice of the reference measure R (which will also be our choice in the following), this minimization problem is called the *Schrödinger problem*; we refer to [48] for a survey on this topic.

Let us first recall the definition of the relative entropy functional in the case of a reference measure with possibly infinite mass (see [47] for more details). Given a σ-finite measure ν on a Polish space (Y, τ'), there exists a measurable function $W : Y \to [0, \infty)$ such that

$$z_W := \int e^{-W} \, \mathrm{d}\nu < \infty.$$

Introducing the probability measure $\nu_W := z_W^{-1} e^{-W} \nu$, for any $\sigma \in \mathscr{P}(Y)$ such that $\int W \, d\sigma < \infty$ the *Boltzmann–Shannon entropy* is defined as

$$H(\sigma \mid \nu) := H(\sigma \mid \nu_W) - \int W \, d\sigma - \log z_W \tag{2.1}$$

where $H(\sigma \mid \nu_W)$ is in turn defined as

$$H(\sigma \mid \tilde{\nu}) := \begin{cases} \int \rho \log \rho \, d\tilde{\nu} & \text{if } \sigma = \rho\tilde{\nu}, \\ +\infty & \text{if } \sigma \not\ll \tilde{\nu}, \end{cases}$$

for all $\tilde{\nu} \in \mathscr{P}(Y)$; notice that Jensen's inequality and the fact that $\tilde{\nu} \in \mathscr{P}(Y)$ imply that $\int \rho \log \rho \, d\tilde{\nu}$ is well defined and non-negative, in particular the definition makes sense. The definition is meaningful, because if $\int W' \, d\sigma < \infty$ for another function W' such that $z_{W'} < \infty$, then

$$H(\sigma \mid \nu_W) - \int W \, d\sigma - \log z_W = H(\sigma \mid \nu_{W'}) - \int W' \, d\sigma - \log z_{W'}.$$

Hence $H(\cdot \mid \nu)$ is well defined for all $\sigma \in \mathscr{P}(Y)$ such that $\int W \, d\sigma < \infty$ for some non-negative measurable function W with $z_W < \infty$.

The following proposition collects the basic properties of the minimizer of the Schrödinger problem; we emphasize that point (i) of the statement is already known (see in particular [45], [12] and [55]) and there are similarities between point (ii) and some results in [12]. A complete proof has already been presented in [37] for the compact case; here we adapt the arguments to our more general case. Notice that Radon measures on Polish spaces are always σ-finite, hence the above discussion about the Boltzmann–Shannon entropy applies.

For sake of notation, by $L^p(X)$ we shall always mean $L^p(X, \mathfrak{m})$; when integrability with respect to a different measure is considered, this will always be specified.

Proposition 2.1. *Let* $(X, \tau, \mathfrak{m})$ *be a Polish space equipped with a non-negative Radon measure* $\mathfrak{m}$ *and let* R *be a non-negative Radon measure on* X^2 *such that* $\pi_*^0 R = \pi_*^1 R = \mathfrak{m}$ *and*

$$\mathfrak{m} \otimes \mathfrak{m} \ll R \ll \mathfrak{m} \otimes \mathfrak{m}.$$

Let $\mu_0 = \rho_0 \mathfrak{m}$ *and* $\mu_1 = \rho_1 \mathfrak{m}$ *be Borel probability measures and assume that there exists a Borel function* $B : X \to [0, \infty)$ *such that*

$$\int_{X^2} e^{-B(x)-B(y)} \, dR(x, y) < \infty, \quad \int B \, d\mu_0 < \infty, \quad \int B \, d\mu_1 < \infty. \tag{2.2}$$

Then the following hold:

(i) *Assume that*

$$H(\mu_0 \otimes \mu_1 \mid R) < \infty. \tag{2.3}$$

Then:

(i-a) *There exists a unique minimizer* $\boldsymbol{\gamma}$ *of* $H(\cdot \mid R)$ *among all transport plans from* μ_0 *to* μ_1.

(i-b) $\gamma = f \otimes g\mathsf{R}$ *for appropriate Borel functions* $f, g : \mathrm{X} \to [0, \infty)$ *which are* $\mathfrak{m}$-*a.e. unique up to the trivial transformation* $(f, g) \mapsto (cf, g/c)$ *for some* $c > 0$.

(ii) *Assume that* $\rho_0, \rho_1 \in L^\infty(\mathrm{X})$ *and that for some* $c > 0$,

$$\mathsf{R} \geq c\mathfrak{m} \otimes \mathfrak{m} \quad \text{in } P_0 \times P_1, \tag{2.4}$$

where $P_0 := \{\rho_0 > 0\}$ *and* $P_1 := \{\rho_1 > 0\}$. *Then:*

(ii-a) *The bound* (2.3) *holds.*

(ii-b) *The functions* f, g *given by* (i-b) *above are in* $L^1 \cap L^\infty(\mathrm{X})$ *with*

$$\|f\|_{L^\infty(\mathrm{X})}\|g\|_{L^1(\mathrm{X})} \leq \frac{\|\rho_0\|_{L^\infty(\mathrm{X})}}{c} \quad \text{and} \quad \|f\|_{L^1(\mathrm{X})}\|g\|_{L^\infty(\mathrm{X})} \leq \frac{\|\rho_1\|_{L^\infty(\mathrm{X})}}{c} \tag{2.5}$$

and γ *is the only transport plan which can be written as* $f' \otimes g'\mathsf{R}$ *for* $f', g' :$ $\mathrm{X} \to [0, \infty)$ *Borel.*

Proof. (i-a) Existence follows by the direct method of calculus of variations: the class of transport plans is not empty, narrowly compact (see e.g. [4]) and $H(\,\cdot\,|\mathsf{R})$ is well defined there: indeed, by assumption $\int W \, d\sigma < \infty$ with $W(x, y) := B(x) + B(y)$ for all transport plans σ. Moreover by (2.1) we have

$$H(\sigma \mid \mathsf{R}) = H(\sigma \mid \mathsf{R}_W) - \int B \, d\mu_0 - \int B \, d\mu_1 - \log z_W,$$

so that $H(\,\cdot\,|\mathsf{R})$ is narrowly lower semicontinuous on the class of transport plans.

Since $H(\,\cdot\,|\mathsf{R})$ is strictly convex, uniqueness is equivalent to the existence of a transport plan γ from μ_0 to μ_1 with finite entropy with respect to R and by (2.3) we get (i-a).

(i-b) The uniqueness part of the claim is trivial, so we concentrate on existence. Finiteness of entropy in particular implies that $\gamma \ll \mathsf{R}$. Put $p := \frac{d\gamma}{d\mathsf{R}}$ and let $P_0 := \{\rho_0 > 0\}$ and $P_1 := \{\rho_1 > 0\}$. We start by claiming that

$$p > 0 \quad \mathfrak{m} \otimes \mathfrak{m}\text{-a.e. on } P_0 \times P_1. \tag{2.6}$$

Since $\mathfrak{m} \otimes \mathfrak{m}$ and R are mutually absolutely continuous, the claim makes sense. Arguing by contradiction assume that $\mathsf{R}(Z) > 0$, where $Z := (P_0 \times P_1) \cap \{p = 0\}$. Let $s := \frac{d(\mu_0 \otimes \mu_1)}{d\mathsf{R}}$ and for $\lambda \in (0, 1)$ define $\Phi(\lambda) : \mathrm{X}^2 \to \mathbb{R}$ by

$$\Phi(\lambda) := \frac{u(p + \lambda(s - p)) - u(p)}{\lambda}, \quad \text{where} \quad u(z) := z \log z.$$

The convexity of u implies that $\Phi(\lambda) \leq u(s) - u(p) \in L^1(\mathrm{X}^2, \mathsf{R})$ (recall (2.3)) and that $\Phi(\lambda)$ is decreasing as $\lambda \downarrow 0$. Moreover, on Z we have $\Phi(\lambda) \downarrow -\infty$ R-a.e. as $\lambda \downarrow 0$, thus the monotone convergence theorem ensures that

$$\lim_{\lambda \downarrow 0} \frac{H(\gamma + \lambda(\mu_0 \otimes \mu_1 - \gamma) \mid \mathsf{R}) - H(\gamma \mid \mathsf{R})}{\lambda} = -\infty.$$

Since $\boldsymbol{\gamma} + \lambda(\mu_0 \otimes \mu_1 - \boldsymbol{\gamma})$ is a transport plan from μ_0 to μ_1 for $\lambda \in (0, 1)$, this is in contradiction with the minimality of $\boldsymbol{\gamma}$, which implies that the left-hand side is non-negative, hence Z is R-negligible, as desired.

Now pick $h \in L^\infty(X^2, \boldsymbol{\gamma})$ such that $\pi_*^0(h\boldsymbol{\gamma}) = \pi_*^1(h\boldsymbol{\gamma}) = 0$ and $\varepsilon \in (0, \|h\|_{L^\infty(X^2, \boldsymbol{\gamma})}^{-1})$. Then $(1 + \varepsilon h)\boldsymbol{\gamma}$ is a transport plan from μ_0 to μ_1 and since hp is well defined R-a.e. we have

$$\|u((1 + \varepsilon h)p)\|_{L^1(X^2, R)} = \int |(1 + \varepsilon h)p \log((1 + \varepsilon h)p)| \, dR$$

$$\leq \int (1 + \varepsilon h)p \, |\log p| \, dR + \int (1 + \varepsilon h) \, |\log(1 + \varepsilon h)| \, d\boldsymbol{\gamma}$$

$$\leq \|1 + \varepsilon h\|_{L^\infty(X^2, \boldsymbol{\gamma})} \|p \log p\|_{L^1(X^2, R)} + \|(1 + \varepsilon h) \log(1 + \varepsilon h)\|_{L^\infty(X^2, \boldsymbol{\gamma})},$$

so that $u((1 + \varepsilon h)p) \in L^1(X^2, R)$. Then again by the monotone convergence theorem

$$\lim_{\varepsilon \downarrow 0} \frac{H((1 + \varepsilon h)\boldsymbol{\gamma} \mid R) - H(\boldsymbol{\gamma} \mid R)}{\varepsilon} = \int \lim_{\varepsilon \downarrow 0} \frac{u((1 + \varepsilon h)p) - u(p)}{\varepsilon} \, dR$$

$$= \int hp(\log p + 1) \, dR.$$

By the minimality of $\boldsymbol{\gamma}$, the left-hand side is non-negative, so after running the same computation with $-h$ in place of h and noticing that the choice of h yields $\int hp \, dR = \int h \, d\boldsymbol{\gamma} = 0$ we obtain

$$\int hp \log p \, dR = 0 \quad \forall h \in L^\infty(X^2, \boldsymbol{\gamma}) \text{ with } \pi_*^0(h\boldsymbol{\gamma}) = \pi_*^1(h\boldsymbol{\gamma}) = 0. \qquad (2.7)$$

The rest of the argument is better understood by introducing the spaces $V, {}^\perp W \subset L^1(X^2, \boldsymbol{\gamma})$ and $V^\perp, W \subset L^\infty(X^2, \boldsymbol{\gamma})$ as follows:

$$V := \{f \in L^1(X^2, \boldsymbol{\gamma}) : f = \varphi \oplus \psi \text{ for some } \varphi \in L^0(X, \mathfrak{m}_{|P_0}), \ \psi \in L^0(X, \mathfrak{m}_{|P_1})\},$$

$$W := \{h \in L^\infty(X^2, \boldsymbol{\gamma}) : \pi_*^0(h\boldsymbol{\gamma}) = \pi_*^1(h\boldsymbol{\gamma}) = 0\},$$

$$V^\perp := \left\{h \in L^\infty(X^2, \boldsymbol{\gamma}) : \int fh \, d\boldsymbol{\gamma} = 0 \ \forall f \in V\right\},$$

$${}^\perp W := \left\{f \in L^1(X^2, \boldsymbol{\gamma}) : \int fh \, d\boldsymbol{\gamma} = 0 \ \forall h \in W\right\},$$

where $\varphi \oplus \psi$ is defined as $\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y)$. Notice that the Euler equation (2.7) reads $\log p \in {}^\perp W$ and our conclusion is $\log p \in V$; hence it is sufficient to show that ${}^\perp W \subset V$.

Claim 1. *V is a closed subspace of $L^1(X^2, \boldsymbol{\gamma})$.*

We start by claiming that $f \in V$ if and only if $f \in L^1(X^2, \boldsymbol{\gamma})$ and

$$f(x, y) + f(x', y') = f(x, y') + f(x', y) \quad \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}\text{-a.e. } (x, x', y, y') \in P_0^2 \times P_1^2. \tag{2.8}$$

Indeed, the 'only if' part follows trivially from $\gamma \ll \mathfrak{m} \otimes \mathfrak{m}$ and the definition of V. For the 'if' we apply Fubini's theorem to get the existence of $x' \in P_0$ and $y' \in P_1$ such that

$$f(x, y) + f(x', y') = f(x, y') + f(x', y) \quad \mathfrak{m} \otimes \mathfrak{m}\text{-a.e. } x, y \in P_0 \times P_1.$$

Thus $f = f(\cdot, y') \oplus (f(x', \cdot) - f(x', y'))$, as desired.

Now notice that since (2.6) yields $(\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1} \ll \gamma$, we see that the condition (2.8) is closed with respect to $L^1(X^2, \gamma)$ convergence.

Claim 2. $V^\perp \subset W$.

Let $h \in L^\infty(X^2, \gamma) \setminus W$, so that either the first or second marginal of $h\gamma$ is non-zero, say the first. Since $\pi^0_* \gamma = \mu_0$ we have $\pi^0_*(h\gamma) = f_0 \mu_0$ for some $f_0 \in L^\infty(X, \mu_0) \setminus \{0\}$. Then the function $f := f_0 \oplus 0 = f_0 \circ \pi^0$ belongs to V and we have

$$\int hf \, d\gamma = \int f_0 \circ \pi^0 \, d(h\gamma) = \int f_0 \, d\pi^0_*(h\gamma) = \int f_0^2 \, d\mu_0 > 0,$$

so that $h \notin V^\perp$.

Claim 3. $^\perp W \subset V$.

Let $f \in L^1(X^2, \gamma) \setminus V$, use the fact that V is closed and the Hahn–Banach theorem to find $h \in L^\infty(X^2, \gamma) \sim L^1(X^2, \gamma)^*$ such that $\int fh \, d\gamma \neq 0$ and $\int \tilde{f}h \, d\gamma = 0$ for every $\tilde{f} \in V$. Thus $h \in V^\perp$ and hence $h \in W$ by the previous step. The fact that $\int fh \, d\gamma \neq 0$ shows that $f \notin \,^\perp W$, as desired.

(ii-a) The bounds (2.2) and (2.4) imply that $\int e^{-B(x)-B(y)} \, d(\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1} < \infty$, which together with (2.2) again shows $H(\mu_0 \otimes \mu_1 \,|\, (\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1})$ is well defined. The assumption that $\rho_0, \rho_1 \in L^\infty(X)$ then ensures that $H(\mu_0 \otimes \mu_1 \,|\, (\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1})$ is finite, hence the claim follows by direct computations:

$$H(\mu_0 \otimes \mu_1 \,|\, \mathsf{R})$$
$$= H(\mu_0 \otimes \mu_1 \,|\, (\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1}) + \int \log\left(\frac{d\big((\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1}\big)}{d\mathsf{R}}\right) \rho_0 \otimes \rho_1 \, d(\mathfrak{m} \otimes \mathfrak{m})$$
$$\leq H(\mu_0 \otimes \mu_1 \,|\, (\mathfrak{m} \otimes \mathfrak{m})_{|P_0 \times P_1}) - \log c < \infty.$$

(ii-b) Let σ be a transport plan from μ_0 to μ_1 such that $\sigma = f' \otimes g'\mathsf{R}$ for suitable non-negative Borel functions f', g'. We claim that in this case $f', g' \in L^\infty(X)$, leading in particular to the claim in the statement about γ.

By disintegrating R with respect to π^0, from $\pi^0_*(f' \otimes g'\mathsf{R}) = \rho_0 \mathfrak{m}$ and $\mathsf{R}_0 = \mathfrak{m}$ we get

$$f'(x) \int g'(y) \, d\mathsf{R}_x(y) = \rho_0(x) < \infty \quad \text{for } \mathfrak{m}\text{-a.e. } x, \tag{2.9}$$

whence $g' \in L^1(X, \mathsf{R}_x)$ for $\mathfrak{m}$-a.e. x. Notice that the sets where f' and g' are positive must coincide with P_0 and P_1 respectively, up to $\mathfrak{m}$-negligible sets, so that nothing changes in

(2.9) if we restrict the integral to P_1. Moreover, since from (2.4) we have $R_x \geq c\mathfrak{m}$ in P_1 for $\mathfrak{m}$-a.e. $x \in P_0$, we see that $g' \in L^1(X)$ with

$$c\|g'\|_{L^1(X)} \leq \int g'(y)\,dR_x(y) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in P_0$$

and thus (2.9) yields

$$f' \leq \frac{\|\rho_0\|_{L^\infty(X)}}{c\|g'\|_{L^1(X)}} \quad \mathfrak{m}\text{-a.e. in } P_0,$$

which is the first inequality in (2.5), because in $X \setminus P_0$ we already know that f' vanishes $\mathfrak{m}$-a.e. By interchanging the roles of f' and g', the same conclusion follows for g'.

For the uniqueness of $\boldsymbol{\gamma}$, put $\varphi := \log f'$, $\psi := \log g'$ and notice that, by what we have just proved, they are bounded from above. On the other hand,

$$\int \varphi \oplus \psi \, d\sigma = H(\sigma \mid R) > -\infty$$

because, as already remarked in the proof of (i), (2.2) implies that $H(\cdot \mid R)$ is well defined on $\mathrm{Adm}(\mu_0, \mu_1)$. From these two facts we infer that

$$\varphi \circ \pi^0, \psi \circ \pi^1 \in L^1(X^2, \sigma). \tag{2.10}$$

Putting for brevity $p' := f' \otimes g'$ and arguing as before to justify the passage to the limit in the integrand we get

$$\frac{d}{d\lambda} H\big((1-\lambda)\sigma + \lambda\boldsymbol{\gamma} \mid R\big)\big|_{\lambda=0^+} = \int (p - p') \log p' \, dR$$

$$= \int \varphi \oplus \psi \, d(\boldsymbol{\gamma} - \sigma)$$

$$\text{(by (2.10))} \quad = \int \varphi \, d\pi_*^0(\boldsymbol{\gamma} - \sigma) + \int \psi \, d\pi_*^1(\boldsymbol{\gamma} - \sigma)$$

$$\text{(because } \sigma \text{ and } \boldsymbol{\gamma} \text{ have the same marginals)} \quad = 0.$$

This equality and the convexity of $H(\cdot \mid R)$ yield $H(\sigma \mid R) \leq H(\boldsymbol{\gamma} \mid R)$, and since $\boldsymbol{\gamma}$ is the unique minimum of $H(\cdot \mid R)$ among transport plans from μ_0 to μ_1, we conclude that $\sigma = \boldsymbol{\gamma}$. $\qquad\square$

The above result is valid in the very general framework of Polish spaces. We shall now restate it in the form we shall need in the context of RCD spaces.

Recall that on a finite-dimensional RCD*(K, N) space (X, d, $\mathfrak{m}$), $\mathfrak{m}$ satisfies the volume growth condition (A.20), so that we can choose $W = d^2(\cdot, \bar{x})$ for any $\bar{x} \in X$ in (2.1). Setting $z := \int e^{-d^2(\cdot, \bar{x})} d\mathfrak{m}$ and

$$\tilde{\mathfrak{m}} := z^{-1} e^{-d^2(\cdot, \bar{x})} \mathfrak{m},$$

the definition (2.1) becomes

$$H(\mu \mid \mathfrak{m}) = H(\mu \mid \tilde{\mathfrak{m}}) - \int d^2(\cdot, \bar{x}) \, d\mu - \log z \tag{2.11}$$

and this shows that $H(\,\cdot\,|\,\mathfrak{m})$ is well defined on $\mathscr{P}_2(X)$ and W_2-lower semicontinuous. Let us also recall that on RCD spaces there is a well defined heat kernel $\mathsf{r}_\varepsilon[x](y)$ (see (A.3) and (A.4)). Choosing $\mathsf{r}_{\varepsilon/2}$ in the following statement is convenient for the computations later on.

Theorem 2.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$ *endowed with a non-negative Radon measure* $\mathfrak{m}$. *For* $\varepsilon > 0$ *define* $\mathsf{R}^{\varepsilon/2} \in \mathscr{P}(X^2)$ *as*

$$\mathrm{d}\mathsf{R}^{\varepsilon/2}(x, y) := \mathsf{r}_{\varepsilon/2}[x](y)\, \mathrm{d}\mathfrak{m}(x)\, \mathrm{d}\mathfrak{m}(y).$$

Also, let $\mu_0, \mu_1 \in \mathscr{P}(X)$ *be Borel probability measures with bounded densities and supports. Then there exist and are uniquely* $\mathfrak{m}$*-a.e. determined* (*up to multiplicative constants*) *two Borel non-negative functions* $f^\varepsilon, g^\varepsilon : X \to [0, \infty)$ *such that* $f^\varepsilon \otimes g^\varepsilon \mathsf{R}^{\varepsilon/2}$ *is a transport plan from* μ_0 *to* μ_1. *In addition,* $f^\varepsilon, g^\varepsilon$ *belong to* $L^\infty(X)$ *and their supports are included in* $\mathrm{supp}(\mu_0)$ *and* $\mathrm{supp}(\mu_1)$ *respectively.*

Proof. Start by observing that $\mathsf{R}_0^{\varepsilon/2} = \mathsf{R}_1^{\varepsilon/2} = \mathfrak{m}$ and if we set $B := \mathsf{d}^2(\cdot, \bar{x})$ with any $\bar{x} \in X$, then the second and third conditions in (2.2) are automatically satisfied; for the first one notice that

$$\int_{X^2} e^{-B \oplus B}\, \mathrm{d}\mathsf{R}^{\varepsilon/2} = \int (e^{-\mathsf{d}^2(y, \bar{x})}\, \mathrm{d}\mathsf{R}_x^{\varepsilon/2}(y)) e^{-\mathsf{d}^2(x, \bar{x})}\, \mathrm{d}\mathfrak{m}(x),$$

$e^{-\mathsf{d}^2(y, \bar{x})} \leq 1$, $\mathsf{R}_x^{\varepsilon/2}$ is a probability measure and recall (A.20). Hence Proposition 2.1, the fact that the Gaussian estimates (A.5) on the heat kernel imply that there are constants $0 < c_\varepsilon \leq C_\varepsilon < \infty$ such that

$$c_\varepsilon \mathfrak{m} \otimes \mathfrak{m} \leq \mathsf{R}^{\varepsilon/2} \leq C_\varepsilon \mathfrak{m} \otimes \mathfrak{m}$$

in $P_0 \times P_1$, and the fact that $f^\varepsilon \otimes g^\varepsilon \mathsf{R}^{\varepsilon/2}$ is a transport plan from μ_0 to μ_1, yield the conclusion. $\square$

3. Hamilton's and Li–Yau's estimates

Here we recall Hamilton's gradient estimate and Li–Yau Laplacian estimates for $\log \mathsf{h}_t u$, where u is a non-negative function.

Let us start with the following result, which we shall frequently use later on without explicit mention:

Proposition 3.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$, $t > 0$ *and* $u_0 \in L^2 \cap L^\infty(X)$ *be non-negative and not identically zero. Put* $u_t := \mathsf{h}_t u_0$. *Then* $\log u_t \in \mathrm{Test}_{\mathrm{loc}}^\infty(X)$.

Proof. By (A.7), $u_t \in \mathrm{Test}^\infty(X)$, and by (A.5), u_t is locally bounded away from 0. Since $\log$ is smooth on $(0, \infty)$, the conclusion easily follows from (A.6). $\square$

We now recall Hamilton's gradient estimate on $\mathrm{RCD}(K, \infty)$ spaces, which is known to be true from [42]:

Theorem 3.2 (Hamilton's gradient estimate). *Let* $(X, d, \mathfrak{m})$ *be an* RCD*(K, ∞) *space with* $K \in \mathbb{R}$ *and let* $u_0 \in L^p \cap L^\infty(X)$ *be positive with* $p \in [1, \infty)$. *Put* $u_t := \mathsf{h}_t u_0$ *for all* $t > 0$. *Then*

$$t|\nabla \log u_t|^2 \leq (1 + 2K^- t) \log\left(\frac{\|u_0\|_{L^\infty(X)}}{u_t}\right) \quad \mathfrak{m}\text{-}a.e.$$

for all $t > 0$, *where* $K^- := \max\{0, -K\}$.

Proof. In [42] this result has been stated for proper RCD(K, ∞) spaces; still, the assumption that bounded sets are relatively compact is never used so that the proof works in general RCD spaces. We remark that in [42] the authors refer to [31], [3], [6] and [56] for the various calculus rules and that in those references no properness assumption is made. □

In the finite-dimensional case, thanks to the Gaussian estimates for the heat kernel we can easily obtain a bound independent of the L^∞ norm of the initial datum:

Theorem 3.3. *Let* $(X, d, \mathfrak{m})$ *be an* RCD*(K, N) *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Then there is a constant* C *depending on* K, N *only such that for any* $u_0 \in L^1(X)$ *non-negative, not identically* 0 *and with bounded support the inequality*

$$|\nabla \log(u_t)|^2 \leq C\left(1 + \frac{1}{t}\right)\left(1 + t + \frac{D_0^2(x)}{t}\right) \quad \mathfrak{m}\text{-}a.e. \tag{3.1}$$

holds for all $t > 0$, *where* $u_t := \mathsf{h}_t u$ *and*

$$D_0(x) := \sup_{y \in \operatorname{supp}(u_0)} d(x, y).$$

In particular, for every $0 < \delta \leq T < \infty$ *and* $\bar{x} \in X$ *there is a constant* $C_{\delta,T} > 0$ *depending on* $K, N, \delta, T, \bar{x}$ *and* $\operatorname{diam}(\operatorname{supp}(u_0))$ *such that for every* $\varepsilon \in (0, 1)$,

$$\varepsilon |\nabla \log(u_{\varepsilon t})| \leq C_{\delta,T}(1 + d(\cdot, \bar{x})) \quad \forall t \in [\delta, T]. \tag{3.2}$$

Proof. Recall the representation formula (A.4),

$$u_t(x) = \int u_0(y) r_t[y](x) \, d\mathfrak{m}(y) = \int_{\operatorname{supp}(u_0)} u_0(y) r_t[y](x) \, d\mathfrak{m}(y) \quad \forall x \in X,$$

and that for the transition probability densities $r_t[y](x)$ we have the Gaussian estimates (A.5), which can be simplified as

$$\frac{C_0}{\mathfrak{m}(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x, y)}{3t} - C_2 t\right) \leq r_t[x](y) \leq \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(y))} e^{C_2 t} \quad \forall x, y \in X,$$

for appropriate constants C_0, C_1, C_2 depending only on K, N. Therefore,

$$\|u_t\|_{L^\infty} = \sup_x u_t(x) \leq C_1 e^{C_2 t} \int_{\operatorname{supp}(u_0)} \frac{u(y)}{\mathfrak{m}(B_{\sqrt{t}}(y))} \, d\mathfrak{m}(y),$$

$$\inf_x u_{2t}(x) \geq C_0 e^{-2C_2 t} e^{-D_0^2(x)/t} \int_{\operatorname{supp}(u_0)} \frac{u(y)}{\mathfrak{m}(B_{\sqrt{2t}}(y))} \, d\mathfrak{m}(y) > 0.$$

Since $\mathfrak{m}$ is uniformly locally doubling we know that

$$\mathfrak{m}(B_{\sqrt{2t}}(y)) \leq \mathfrak{m}(B_{\sqrt{t}}(y))C_3 e^{C_4\sqrt{t}} \quad \forall y \in X, \ t > 0,$$

where C_3, C_4 only depend on K, N. As a consequence,

$$\frac{\|u_t\|_{L^\infty}}{u_{2t}(x)} \leq C_5 e^{3C_2 t + C_4\sqrt{t} + D_0^2(x)/t} \quad \forall x \in X, \ t > 0.$$

We now apply Proposition 3.2 with u_t in place of u_0 (notice that the assumptions are fulfilled) to get

$$t|\nabla \log(u_{2t})|^2 \leq (1 + 2K^- t) \log\left(\frac{\|u_t\|_{L^\infty}}{u_{2t}}\right)$$

$$\leq (1 + 2K^- t)(\log C_5 + 3C_2 t + C_4\sqrt{t} + D_0^2(x)/t)$$

$\mathfrak{m}$-a.e., which is (equivalent to) the bound (3.1). The last statement is now obvious, noticing that $D_0(x) \leq D_0(\bar{x}) + d(x, \bar{x})$ for any $\bar{x} \in \mathrm{supp}(u_0)$. $\qquad\square$

A further result that we shall need is the Li–Yau inequality in the form proved by Baudoin and Garofalo (see [25] for the case of finite mass and [40] for the general one).

Theorem 3.4 (Li–Yau inequality). *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$ *and let* $u_0 \in L^p(X)$ *for some* $p \in [1, \infty)$ *be non-negative. Put* $u_t := h_t u_0$ *for all* $t > 0$. *Then*

$$|\nabla \log u_t|^2 \leq e^{-2Kt/3} \frac{\Delta u_t}{u_t} + \frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}} \quad \mathfrak{m}\text{-a.e.} \tag{3.3}$$

for all $t > 0$, *where* $\frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}$ *is understood as* $\frac{N}{2t}$ *when* $K = 0$.

We restate the above inequality in the form that we shall use:

Theorem 3.5. *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Then for every* $0 < \delta \leq T < \infty$ *and* $\bar{x} \in X$ *there exists a constant* $C_{\delta,T} > 0$ *depending on* $K, N, \delta, T, \bar{x}$ *and* $\mathrm{diam}(\mathrm{supp}(u_0))$ *such that for any* $u_0 \in L^1(X)$ *non-negative, not identically zero and with bounded support and for any* $\varepsilon \in (0, 1)$,

$$\varepsilon \Delta \log(h_{\varepsilon t}(u_0)) \geq -C_{\delta,T}(1 + d^2(\cdot, \bar{x})) \quad \forall t \in [\delta, T]. \tag{3.4}$$

Proof. Rewrite the Li–Yau inequality (3.3) as

$$e^{-2Kt/3}\left(\underbrace{\frac{\Delta u_t}{u_t} - |\nabla \log u_t|^2}_{=\Delta \log u_t}\right) \geq (1 - e^{-2Kt/3})|\nabla \log u_t|^2 - \frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}$$

and use Hamilton's gradient estimate (3.2) to control $|\nabla \log u_t|^2$ on the right-hand side.

$\qquad\square$

4. The Schrödinger problem: properties of the solutions

4.1. The setting

Let us fix once for all the assumptions and notations which we shall use from now on.

Setting 4.1. $(X, d, \mathfrak{m})$ is an RCD*(K, N) space with $K \in \mathbb{R}$ and $N \in [1, \infty)$, and $\mu_0 = \rho_0 \mathfrak{m}$, $\mu_1 = \rho_1 \mathfrak{m}$ are two absolutely continuous Borel probability measures with bounded densities and supports.

For any $\varepsilon > 0$ we consider the couple $(f^\varepsilon, g^\varepsilon)$ given by Theorem 2.2 normalized in such a way that

$$\int \log(\mathsf{h}_{\varepsilon/2} f^\varepsilon) \rho_1 \, d\mathfrak{m} = 0. \tag{4.1}$$

Then we set $\rho_0^\varepsilon := \rho_0$, $\rho_1^\varepsilon := \rho_1$, $\mu_0^\varepsilon := \mu_0$, $\mu_1^\varepsilon := \mu_1$ and

$$\begin{cases} f_t^\varepsilon := \mathsf{h}_{\varepsilon t/2} f^\varepsilon, \\ \varphi_t^\varepsilon := \varepsilon \log f_t^\varepsilon, \\ \text{for } t \in (0, 1], \end{cases} \quad \begin{cases} g_t^\varepsilon := \mathsf{h}_{\varepsilon(1-t)/2} g^\varepsilon, \\ \psi_t^\varepsilon := \varepsilon \log g_t^\varepsilon, \\ \text{for } t \in [0, 1), \end{cases} \quad \begin{cases} \rho_t^\varepsilon := f_t^\varepsilon g_t^\varepsilon, \\ \mu_t^\varepsilon := \rho_t^\varepsilon \mathfrak{m}, \\ \vartheta_t^\varepsilon := \frac{1}{2}(\psi_t^\varepsilon - \varphi_t^\varepsilon), \\ \text{for } t \in (0, 1). \end{cases}$$

In order to investigate the time behaviour of the functions just defined, let us introduce the weighted L^2 and $W^{1,2}$ spaces. The weight we will always consider is e^{-V} with $V = M d^2(\cdot, \bar{x})$; because of (A.20), $e^{-V}\mathfrak{m}$ has finite mass for every $M > 0$. For $L^2(X, e^{-V}\mathfrak{m})$ no comments are required. The weighted Sobolev space is defined as

$$W^{1,2}(X, e^{-V}\mathfrak{m}) := \{ f \in W_{\text{loc}}^{1,2}(X) : f, |Df| \in L^2(X, e^{-V}\mathfrak{m}) \}$$

where $|Df|$ is the local minimal weak upper gradient already introduced. Since V is locally bounded, $W^{1,2}(X, e^{-V}\mathfrak{m})$ turns out to coincide with the Sobolev space built over the metric measure space $(X, d, e^{-V}\mathfrak{m})$, thus motivating the choice of the notation. The advantage of dealing with $L^2(X, e^{-V}\mathfrak{m})$ and $W^{1,2}(X, e^{-V}\mathfrak{m})$ is that they are Hilbert spaces, unlike $L_{\text{loc}}^2(X)$ and $W_{\text{loc}}^{1,2}(X)$.

As two different reference measures on X might be considered from now on, namely $\mathfrak{m}$ and $e^{-V}\mathfrak{m}$, to avoid possible misunderstandings it is worth stressing that the notations $L^2(X)$ and $W^{1,2}(X)$ will always mean $L^2(X, \mathfrak{m})$, $W^{1,2}(X, \mathfrak{m})$ respectively.

Let us now begin with a couple of quantitative estimates for f_t^ε, g_t^ε and ρ_t^ε.

Lemma 4.2. *Under Setting 4.1, defining*

$$v_s := \inf_{y \in \text{supp}(\rho_0) \cup \text{supp}(\rho_1)} \mathfrak{m}(B_{\sqrt{s}}(y)), \quad V_s := \sup_{y \in \text{supp}(\rho_0) \cup \text{supp}(\rho_1)} \mathfrak{m}(B_{\sqrt{s}}(y)), \tag{4.2}$$

for any $\bar{x} \in X$ there exist positive constants $C_1, \ldots, C_9$ depending on $K, N, \rho_0, \rho_1, \bar{x}$ only such that:

(i) *For any $\varepsilon > 0$ and $t \in (0, 1]$ we have*

$$\frac{C_1}{V_{\varepsilon t/2}} \|f^\varepsilon\|_{L^1(X)} \exp\left(-\frac{C_2 d^2(\cdot, \bar{x})}{\varepsilon t} - \frac{C_3}{\varepsilon t}\right)$$

$$\leq f_t^\varepsilon \leq \frac{C_4}{v_{\varepsilon t/2}} \|f^\varepsilon\|_{L^1(X)} \exp\left(-\frac{C_5 d^2(\cdot, \bar{x})}{\varepsilon t} + \frac{C_6}{\varepsilon t}\right) \quad (4.3)$$

and analogously for g_t^ε and $t \in [0, 1)$.

(ii) *For any $\varepsilon \in (0, 1)$ and $t \in [0, 1]$ we have*

$$\rho_t^\varepsilon \leq \frac{C_7}{\varepsilon^{N/2}} \exp\left(\frac{C_8 - C_9 d^2(\cdot, \bar{x})}{\varepsilon}\right). \quad (4.4)$$

Proof. (i) This is a direct consequence of the representation formula (A.4), the Gaussian estimates (A.5) and the fact that ρ_0 and f^ε have the same support.

(ii) We shall indicate by C a constant depending only on $K, N, \rho_0, \rho_1, \bar{x}$ whose value might change in various occurrences. Start from $\rho_t^\varepsilon = f_t^\varepsilon g_t^\varepsilon \leq f_t^\varepsilon \|g_t^\varepsilon\|_{L^\infty} \leq f_t^\varepsilon \|g^\varepsilon\|_{L^\infty}$, then use the bounds (4.3), (2.5) and notice that the constant c appearing in (2.5) is $\geq (C/V_{\varepsilon/2})e^{-C/\varepsilon}$ to obtain

$$\rho_t^\varepsilon \leq \frac{C V_{\varepsilon/2}}{v_{\varepsilon t/2}} \exp\left(\frac{C - C d^2(\cdot, \bar{x})}{\varepsilon t}\right).$$

Noticing that the Bishop–Gromov inequality (A.18) ensures that for every $s \in [0, 1]$ we have $V_s \leq C\mathfrak{m}(B_1(\bar{x}))$ and $v_s \geq C\mathfrak{m}(B_1(\bar{x}))s^{N/2}$, we obtain the claim for $t \in [1/2, 1]$. The case $t \in [0, 1/2]$ follows by a symmetric argument. □

The following proposition collects the basic properties of the functions defined in Setting 4.1 and the respective 'PDEs' solved:

Proposition 4.3. *Under Setting 4.1, the following holds. All the functions are well defined and for any $\varepsilon > 0$:*

(a) $f_t^\varepsilon, g_t^\varepsilon, \rho_t^\varepsilon \in \text{Test}^\infty(X)$ *for all $t \in \mathfrak{I}$, where $\mathfrak{I}$ is the respective domain of definition (for (ρ_t^ε) we pick $\mathfrak{I} = (0, 1)$);*

(b) $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon \in \text{Test}_{\text{loc}}^\infty(X)$ *for all $t \in \mathfrak{I}$, where $\mathfrak{I}$ is the respective domain of definition.*

For any $\varepsilon > 0$, $\mathcal{C} \subset \mathfrak{I}$ compact and $\bar{x} \in X$ there exists $M = M(K, N, \rho_0, \rho_1, \mathcal{C}, \bar{x}) > 0$ such that all the curves $(f_t^\varepsilon), (g_t^\varepsilon), (\rho_t^\varepsilon)$ belong to $AC(\mathcal{C}, W^{1,2}(X))$ and $(\varphi_t^\varepsilon), (\psi_t^\varepsilon), (\vartheta_t^\varepsilon)$ to $AC(\mathcal{C}, W^{1,2}(X, e^{-V}\mathfrak{m}))$, where $\mathfrak{I}$ is the respective domain of definition (for (ρ_t^ε) we pick $\mathfrak{I} = (0, 1)$) and $V = M d^2(\cdot, \bar{x})$; their time derivatives are given by the following expressions for a.e. $t \in [0, 1]$:

$$\frac{d}{dt} f_t^\varepsilon = \frac{\varepsilon}{2} \Delta f_t^\varepsilon, \qquad \frac{d}{dt} g_t^\varepsilon = -\frac{\varepsilon}{2} \Delta g_t^\varepsilon,$$

$$\frac{d}{dt} \varphi_t^\varepsilon = \frac{1}{2} |\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon, \quad -\frac{d}{dt} \psi_t^\varepsilon = \frac{1}{2} |\nabla \psi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \psi_t^\varepsilon,$$

$$\frac{d}{dt} \rho_t^\varepsilon + \text{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) = 0, \qquad \frac{d}{dt} \vartheta_t^\varepsilon + \frac{|\nabla \vartheta_t^\varepsilon|^2}{2} = -\frac{\varepsilon^2}{8} (2\Delta \log \rho_t^\varepsilon + |\nabla \log \rho_t^\varepsilon|^2).$$

Moreover, for every $\varepsilon > 0$ we have:

(i)

$$\sup_{t\in\mathcal{C}}\{\|h_t^\varepsilon\|_{L^\infty(X)} + \mathrm{Lip}(h_t^\varepsilon) + \|\Delta h_t^\varepsilon\|_{W^{1,2}(X)}\} < \infty \tag{4.5}$$

if (h_t^ε) is equal to any of (f_t^ε), (g_t^ε), (ρ_t^ε), and

$$\sup_{t\in\mathcal{C}}\{\|e^{-V}h_t^\varepsilon\|_{L^\infty(X)} + \|e^{-V}\mathrm{lip}(h_t^\varepsilon)\|_{L^\infty(X)} + \|\Delta h_t^\varepsilon\|_{W^{1,2}(X,e^{-V}\mathfrak{m})}\} < \infty \tag{4.6}$$

if (h_t^ε) is equal to any of (φ_t^ε), (ψ_t^ε), $(\vartheta_t^\varepsilon)$; in both cases, $\mathcal{C}$ is a compact subset of the respective domain of definition $\mathfrak{I}$ (for (ρ_t^ε) we pick $\mathfrak{I} = (0, 1)$),

(ii) $\mu_t^\varepsilon \in \mathscr{P}_2(X)$ for every $t \in [0, 1]$ and $(\rho_t^\varepsilon) \in C([0, 1], L^2(X))$,

(iii) we have $f_t^\varepsilon \to f^\varepsilon$ and $g_t^\varepsilon \to g^\varepsilon$ in $L^2(X)$ as $t \downarrow 0$ and $t \uparrow 1$ respectively.

Proof. **Properties of** (f_t^ε), (g_t^ε)**.** Recalling (A.7) we see that $f_{t_0}^\varepsilon \in \mathrm{Test}^\infty(X)$ for any $t_0 > 0$. Then the maximum principle for the heat flow, the fact that it is a contraction in $W^{1,2}(X)$ and the Bakry–Émery gradient estimates (A.9) together with the Sobolev-to-Lipschitz property imply that (4.5) holds for (f_t^ε). The fact that $(f_t^\varepsilon) \in AC(\mathcal{C}, W^{1,2}(X))$ and that it solves the stated scaled heat equation is trivial. The fact that $f_t^\varepsilon \to f^\varepsilon$ in $L^2(X)$ as $t \downarrow 0$ follows from the L^2 continuity of the heat flow.

Properties of (φ_t^ε), (ψ_t^ε)**.** By Proposition 3.1 we know that $\varphi_t^\varepsilon \in \mathrm{Test}_{\mathrm{loc}}^\infty(X)$ and from the chain and Leibniz rules we see that

$$\nabla\varphi_t^\varepsilon = \varepsilon\frac{\nabla f_t^\varepsilon}{f_t^\varepsilon},$$

$$\Delta\varphi_t^\varepsilon = \varepsilon\left(\frac{\Delta f_t^\varepsilon}{f_t^\varepsilon} - \frac{|\nabla f_t^\varepsilon|^2}{(f_t^\varepsilon)^2}\right),$$

$$\nabla\Delta\varphi_t^\varepsilon = \varepsilon\left(\frac{\nabla\Delta f_t^\varepsilon}{f_t^\varepsilon} - \frac{\Delta f_t^\varepsilon\nabla f_t^\varepsilon}{(f_t^\varepsilon)^2} - \frac{\nabla|\nabla f_t^\varepsilon|^2}{(f_t^\varepsilon)^2} + \frac{2|\nabla f_t^\varepsilon|^2\nabla f_t^\varepsilon}{(f_t^\varepsilon)^3}\right).$$

These identities, (4.5) for (f_t^ε), estimate (A.8) and (4.3) imply that for any $\bar{x}$ there is $M > 0$ such that for $V := M\mathrm{d}^2(\cdot, \bar{x})$ the bound (4.6) for (φ_t^ε) holds, as claimed. Similarly, we see that $|\nabla\varphi_t^\varepsilon|^2 \in L_{\mathrm{loc}}^2((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$.

The expressions for $\nabla\varphi_t^\varepsilon$, $\Delta\varphi_t^\varepsilon$ and the equation for (f_t^ε) also show that $\mathfrak{m}$-a.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\varepsilon = \frac{1}{2}|\nabla\varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon \tag{4.7}$$

for a.e. t and since the right-hand side belongs to $L_{\mathrm{loc}}^2((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$, this shows at once that $(\varphi_t^\varepsilon) \in AC_{\mathrm{loc}}((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$ and that (4.7) holds when the left-hand side is understood as the limit of difference quotients in $W^{1,2}(X, e^{-V}\mathfrak{m})$, as claimed.

The same arguments apply to ψ_t^ε.

Properties of (ρ_t^ε), $(\vartheta_t^\varepsilon)$**.** The bound (4.5) for (f_t^ε), (g_t^ε) and the Leibniz rules for the gradient and Laplacian give the bound (4.5) for (ρ_t^ε) and also show that $(\rho_t^\varepsilon) \in AC_{\mathrm{loc}}((0, 1), L^2(X))$. To see that this curve is absolutely continuous with values

in $W^{1,2}(X)$ notice that

$$\frac{d}{dt}\rho_t^\varepsilon = \frac{\varepsilon}{2}(g_t^\varepsilon \Delta f_t^\varepsilon - f_t^\varepsilon \Delta g_t^\varepsilon)$$

and recall (4.5) for $f_t^\varepsilon, g_t^\varepsilon$. The stated equation for (ρ_t^ε) is now a matter of direct computation:

$$\frac{\varepsilon}{2}(g_t^\varepsilon \Delta f_t^\varepsilon - f_t^\varepsilon \Delta g_t^\varepsilon) = \frac{\varepsilon}{2}\rho_t^\varepsilon(\Delta \log f_t^\varepsilon + |\nabla \log f_t^\varepsilon|^2 - \Delta \log g_t^\varepsilon - |\nabla \log g_t^\varepsilon|^2)$$

$$= \rho_t^\varepsilon \frac{1}{\varepsilon}\left(\frac{|\nabla \varphi_t^\varepsilon|^2}{2} - \frac{|\nabla \psi_t^\varepsilon|^2}{2} + \frac{\varepsilon}{2}\Delta \varphi_t^\varepsilon - \frac{\varepsilon}{2}\Delta \psi_t^\varepsilon\right)$$

$$= \rho_t^\varepsilon(-\langle \nabla \vartheta_t^\varepsilon, \nabla \log \rho_t^\varepsilon \rangle - \Delta \vartheta_t^\varepsilon)$$

$$= -\langle \nabla \vartheta_t^\varepsilon, \nabla \rho_t^\varepsilon \rangle - \rho_t^\varepsilon \Delta \vartheta_t^\varepsilon = -\operatorname{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon).$$

It is clear that $\rho_t^\varepsilon \geq 0$ for every ε, t, hence the identity

$$\int \rho_t^\varepsilon \, d\mathfrak{m} = \int \mathsf{h}_{\varepsilon t/2} f^\varepsilon \mathsf{h}_{\varepsilon(1-t)/2} g^\varepsilon \, d\mathfrak{m} = \int f^\varepsilon \mathsf{h}_{\varepsilon/2} g^\varepsilon \, d\mathfrak{m} = \int \rho_0^\varepsilon \, d\mathfrak{m} = 1$$

shows that $\mu_t^\varepsilon \in \mathscr{P}(X)$. The fact that μ_t^ε has finite second moment is a direct consequence of the Gaussian bound (4.4) and the volume growth estimate (A.20).

For the L^2 continuity of ρ_t^ε in $t = 0, 1$, by the L^2 continuity of the heat flow and the fact that $f^\varepsilon, g^\varepsilon \in L^\infty$ (Theorem 2.2) we see that $\rho_t^\varepsilon \to f^\varepsilon \mathsf{h}_{\varepsilon/2} g^\varepsilon$ and $\rho_t^\varepsilon \to \mathsf{h}_{\varepsilon/2} f^\varepsilon g^\varepsilon$ as $t \to 0, 1$ respectively. Hence all we have to check is that

$$\rho_0 = f^\varepsilon \mathsf{h}_{\varepsilon/2} g^\varepsilon, \qquad \rho_1 = g^\varepsilon \mathsf{h}_{\varepsilon/2} f^\varepsilon, \tag{4.8}$$

but as already noticed in the proof of Theorem 2.2, these are equivalent to the fact that $f^\varepsilon \otimes g^\varepsilon R^{\varepsilon/2}$ is a transport plan from μ_0 to μ_1; hence, (4.8) holds by the very choice of $(f^\varepsilon, g^\varepsilon)$ made.

Finally, the fact that $(\vartheta_t^\varepsilon)$ belongs to $AC_{\text{loc}}((0, 1), W^{1,2}(X, e^{-V}\mathfrak{m}))$ and satisfies the bound (4.6) is a direct consequence of the analogous property for $(\varphi_t^\varepsilon), (\psi_t^\varepsilon)$. The equation for its time derivative follows by direct computation:

$$\frac{d}{dt}\vartheta_t^\varepsilon + \frac{|\nabla \vartheta_t^\varepsilon|^2}{2} = -\frac{|\nabla \psi_t^\varepsilon|^2}{4} - \frac{\varepsilon}{4}\Delta \psi_t^\varepsilon - \frac{|\nabla \varphi_t^\varepsilon|^2}{4} - \frac{\varepsilon}{4}\Delta \varphi_t^\varepsilon$$

$$+ \frac{|\nabla \psi_t^\varepsilon|^2}{8} + \frac{|\nabla \varphi_t^\varepsilon|^2}{8} - \frac{\langle \nabla \psi_t^\varepsilon, \nabla \varphi_t^\varepsilon \rangle}{4}$$

$$= -\frac{\varepsilon^2}{4}\Delta \log \rho_t^\varepsilon - \frac{1}{8}(|\nabla \psi_t^\varepsilon|^2 + |\nabla \varphi_t^\varepsilon|^2 + 2\langle \nabla \varphi_t^\varepsilon, \nabla \psi_t^\varepsilon \rangle)$$

$$= -\frac{\varepsilon^2}{8}(2\Delta \log \rho_t^\varepsilon + |\nabla \log \rho_t^\varepsilon|^2),$$

hence the proof is complete.   □

Using the terminology adopted in the literature (see [48]) we shall refer to:

- φ_t^ε and ψ_t^ε as *Schrödinger potentials*, in connection with Kantorovich ones;
- $(\mu_t^\varepsilon)_{t \in [0,1]}$ as *entropic interpolation*, in analogy with displacement interpolation.

4.2. Uniform estimates for the densities and the potentials

We start by collecting information about quantities which remain bounded as $\varepsilon \downarrow 0$.

Proposition 4.4 (Locally uniform Lipschitz and Laplacian controls for the potentials). *Under Setting 4.1, the following holds. For all $\delta \in (0, 1)$ and $\bar{x} \in X$ there exists $C > 0$ which only depends on $K, N, \delta, \bar{x}$ such that*

$$\text{lip}(\varphi_t^\varepsilon) \leq C(1 + \mathsf{d}(\cdot, \bar{x})) \quad \mathfrak{m}\text{-a.e.,} \tag{4.9a}$$

$$\Delta\varphi_t^\varepsilon \geq -C(1 + \mathsf{d}^2(\cdot, \bar{x})) \quad \mathfrak{m}\text{-a.e.,} \tag{4.9b}$$

for every $t \in [\delta, 1]$ and $\varepsilon \in (0, 1)$. Furthermore, for all $M > 0$ there exists $C' > 0$ which only depends on $K, N, \delta, \bar{x}, M$ such that

$$\int |\Delta\varphi_t^\varepsilon| e^{-M\mathsf{d}^2(\cdot, \bar{x})} \, \mathrm{d}\mathfrak{m} \leq C' \tag{4.10}$$

for every $t \in [\delta, 1]$ and $\varepsilon \in (0, 1)$. Analogous bounds hold for the ψ_t^ε's in the time interval $[0, 1 - \delta]$.

Proof. Fix $\delta \in (0, 1)$ and $\bar{x} \in X$ as in the statement and notice that the bound (3.2) yields

$$|\nabla\varphi_t^\varepsilon| = \varepsilon|\nabla \log(\mathsf{h}_{\varepsilon t/2} f^\varepsilon)| \leq C(1 + \mathsf{d}(\cdot, \bar{x})) \quad \forall t \in [\delta, 1], \; \varepsilon \in (0, 1).$$

Thus recalling the Sobolev-to-Lipschitz property (A.10) we obtain the bound (4.9a). The bound (4.9b) is a restatement of (3.4). Finally, let $M > 0$ and χ a 1-Lipschitz cut-off function with bounded support; notice that $|h| = h + 2h^-$, whence

$$\int \chi e^{-M\mathsf{d}^2(\cdot, \bar{x})} |\Delta\varphi_t^\varepsilon| \, \mathrm{d}\mathfrak{m} = \int \chi e^{-M\mathsf{d}^2(\cdot, \bar{x})} \Delta\varphi_t^\varepsilon \, \mathrm{d}\mathfrak{m} + 2 \int \chi e^{-M\mathsf{d}^2(\cdot, \bar{x})} (\Delta\varphi_t^\varepsilon)^- \, \mathrm{d}\mathfrak{m}.$$

Integration by parts and the facts that $|\nabla\mathsf{d}^2(\cdot, \bar{x})| = 2\mathsf{d}(\cdot, \bar{x})$, $|\nabla\chi| \leq 1$ and $0 \leq \chi \leq 1$ then imply that

$$\int \chi e^{-M\mathsf{d}^2(\cdot, \bar{x})} |\Delta\varphi_t^\varepsilon| \, \mathrm{d}\mathfrak{m} \leq \int e^{-M\mathsf{d}^2(\cdot, \bar{x})} |\nabla\varphi_t^\varepsilon| \, \mathrm{d}\mathfrak{m} + 2M \int \mathsf{d}(\cdot, \bar{x}) e^{-M\mathsf{d}^2(\cdot, \bar{x})} |\nabla\varphi_t^\varepsilon| \, \mathrm{d}\mathfrak{m}$$

$$+ 2 \int e^{-M\mathsf{d}^2(\cdot, \bar{x})} (\Delta\varphi_t^\varepsilon)^- \, \mathrm{d}\mathfrak{m},$$

and taking into account (4.9a) and (4.9b), the bound (4.10) follows.

For ψ_t^ε the argument is the same. $\qquad\square$

The gradient estimates that we have just obtained together with the Gaussian bounds on $f_t^\varepsilon, g_t^\varepsilon, \rho_t^\varepsilon$ that we previously proved have the following direct implication, which we shall frequently use later on to justify our computations:

Lemma 4.5. *Under Setting* 4.1, *the following holds. For any* $\varepsilon \in (0, 1)$ *and* $t \in (0, 1)$ *let* h_t^ε *denote any of* $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon, \log \rho_t^\varepsilon$ *and, for any* $n \in \mathbb{N}$, *let* H_t^ε *denote any of the functions*

$$\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n, \quad \rho_t^\varepsilon \log(\rho_t^\varepsilon)|\nabla h_t^\varepsilon|^n, \quad |\nabla \rho_t^\varepsilon| \, |\nabla h_t^\varepsilon|^n, \quad \Delta \rho_t^\varepsilon |\nabla h_t^\varepsilon|^n, \quad \rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle. \tag{4.11}$$

Then $H_t^\varepsilon \in L^1(\mathrm{X})$ *for every* $\varepsilon, t \in (0, 1)$. *Moreover for every* $\delta \in (0, 1/2)$ *we have*

$$\lim_{R \to \infty} \sup_{t \in [\delta, 1-\delta]} \int_{\mathrm{X} \setminus B_R(\bar{x})} |H_t^\varepsilon| \, d\mathfrak{m} = 0 \quad \forall \varepsilon \in (0, 1), \, \bar{x} \in \mathrm{X}. \tag{4.12}$$

Finally, $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathfrak{m}$ *is continuous.*

Proof. **General considerations.** We shall repeatedly use the fact that if h_1 has Gaussian decay and h_2 has polynomial growth, i.e.

$$h_1 \le c_1 \exp(-c_2 \mathsf{d}^2(\cdot, \bar{x})), \quad h_2 \le c_3(1 + \mathsf{d}^{c_4}(\cdot, \bar{x}))$$

for some $c_1, \ldots, c_4 > 0$, $\bar{x} \in \mathrm{X}$, then their product $h_1 h_2$ belongs to $L^1 \cap L^\infty(\mathrm{X})$: the L^∞ bound is obvious, the one for the L^1 norm is a direct consequence of the volume growth (A.20) and explicit computations.

For the continuity of $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathfrak{m}$, notice that Proposition 4.3 implies that all the maps $(0, 1) \ni t \mapsto |\nabla h_t^\varepsilon| \in L^2(\mathrm{X}, e^{-V}\mathfrak{m})$ and $(0, 1) \ni t \mapsto \rho_t^\varepsilon, |\nabla \rho_t^\varepsilon|, \Delta \rho_t^\varepsilon \in L^2(\mathrm{X})$ are continuous (for $\Delta \rho_t^\varepsilon$ use the fact that $\Delta \rho_t^\varepsilon = g_t^\varepsilon \Delta f_t^\varepsilon + f_t^\varepsilon \Delta g_t^\varepsilon + 2\langle \nabla f_t^\varepsilon, \nabla g_t^\varepsilon \rangle$ and the continuity of $(0, 1) \ni t \mapsto \Delta f_t^\varepsilon, \Delta g_t^\varepsilon \in L^2(\mathrm{X})$). Hence all the functions in (4.11), with the possible exception of the last one, are continuous from $(0, 1)$ to $L^0(\mathrm{X})$ equipped with the topology of convergence in measure on bounded sets. Therefore the continuity of $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathfrak{m}$ for these maps will follow as soon as we show that they are, locally in $t \in (0, 1)$, uniformly dominated by an $L^1(\mathrm{X})$ function. Given that such domination also gives (4.12), we shall focus on proving it.

Finally, we shall consider only the case $h_t^\varepsilon = \varphi_t^\varepsilon$, as the estimates for ψ_t^ε can be obtained by symmetric arguments and the ones for $\vartheta_t^\varepsilon, \log \rho_t^\varepsilon$ follow from the identities $\vartheta_t^\varepsilon = (\psi_t^\varepsilon - \varphi_t^\varepsilon)/2$ and $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$.

Study of $\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n$**.** By (4.9a) we know that $|\nabla \varphi_t^\varepsilon|$ has linear growth locally uniform in $t \in (0, 1)$; hence $|\nabla \varphi_t^\varepsilon|^n$ has polynomial growth locally uniform in $t \in (0, 1)$. Since ρ_t^ε has Gaussian bounds by (4.4), we deduce that $\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n$ is, locally in $t \in (0, 1)$, uniformly dominated.

Study of $\rho_t^\varepsilon \log(\rho_t^\varepsilon)|\nabla h_t^\varepsilon|^n$**.** Writing $\log \rho_t^\varepsilon = \log f_t^\varepsilon + \log g_t^\varepsilon$ and using (4.3) we see that $|\log \rho_t^\varepsilon|$ has quadratic growth locally uniform in $t \in (0, 1)$. Thus the claim follows as before.

Study of $|\nabla \rho_t^\varepsilon| \, |\nabla h_t^\varepsilon|^n$**.** Notice that $|\nabla \rho_t^\varepsilon| = \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|$ and observe that from $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$ and (4.9a) we see that $|\nabla \log \rho_t^\varepsilon|$ has linear growth locally uniform in $t \in (0, 1)$.

Study of $|\Delta \rho_t^\varepsilon| \, |\nabla h_t^\varepsilon|^n$**.** Write

$$|\Delta \rho_t^\varepsilon| \le f_t^\varepsilon |\Delta g_t^\varepsilon| + g_t^\varepsilon |\Delta f_t^\varepsilon| + 2\varepsilon^{-2} \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| \, |\nabla \psi_t^\varepsilon|$$

and notice that the term $\rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \psi_t^\varepsilon|$ can be handled as before and that by (A.7) and the maximum principle for the heat flow we know that

$$\Delta f_t^\varepsilon, \Delta g_t^\varepsilon \text{ are bounded in } L^\infty(X) \text{ locally uniformly in } t \in (0, 1). \quad (4.13)$$

Hence the conclusion follows from the Gaussian bounds (4.3).

Study of $\rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle$. Notice that

$$|\nabla \Delta \varphi_t^\varepsilon| \leq \varepsilon \left(\frac{|\nabla \Delta f_t^\varepsilon|}{f_t^\varepsilon} + \frac{|\Delta f_t^\varepsilon| |\nabla f_t^\varepsilon|}{(f_t^\varepsilon)^2} + \frac{|\nabla |\nabla f_t^\varepsilon|^2|}{(f_t^\varepsilon)^2} + \frac{2|\nabla f_t^\varepsilon|^3}{(f_t^\varepsilon)^3} \right)$$

$$\leq \frac{1}{f_t^\varepsilon} \left(\varepsilon |\nabla \Delta f_t^\varepsilon| + |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon| + 2|\nabla \varphi_t^\varepsilon| |\text{Hess}(f_t^\varepsilon)|_{\text{HS}} + 2\varepsilon^{-2} f_t^\varepsilon |\nabla \varphi_t^\varepsilon|^3 \right)$$

and therefore, using also $2 g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 |\text{Hess}(f_t^\varepsilon)|_{\text{HS}} \leq g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4 + g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2$, we get

$$\rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta \varphi_t^\varepsilon|$$
$$\leq \varepsilon g_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta f_t^\varepsilon| + g_t^\varepsilon |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon|^2 + g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4 + g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 + 2\varepsilon^{-2} \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4.$$

By what we have already proved, the last term on the right-hand side is dominated locally uniformly in $t \in (0, 1)$. Similarly, the term $g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4$ is, locally in t, dominated thanks to the Gaussian bounds on g_t^ε; domination for $g_t^\varepsilon |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon|^2$ then follows using (4.13). Writing $\nabla \Delta f_t^\varepsilon = \nabla h_{t-\delta} \Delta f_\delta^\varepsilon$ for any $t \geq \delta > 0$ and using (A.7) and the Bakry–Émery estimates (A.9) we see that

$$|\nabla \Delta f_t^\varepsilon| \text{ is, locally in } t, \text{ uniformly bounded in } L^\infty(X), \quad (4.14)$$

thus local uniform domination for $\varepsilon g_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta f_t^\varepsilon|$ follows.

It remains to consider the term $g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2$. We know from (A.13) that $|\text{Hess}(f_t^\varepsilon)|_{\text{HS}} \in L^2(X)$ and from (4.3) that $g_t^\varepsilon \in L^\infty(X)$. This is sufficient to conclude that $\rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle \in L^1(X)$. To prove (4.12), thanks to the dominations previously obtained, it is enough to show that

$$\lim_{R \to \infty} \int (1 - \chi_R) g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 \, d\mathfrak{m} = 0 \quad \text{locally uniformly in } t \in (0, 1), \quad (4.15)$$

where for any $R > 0$ the function χ_R is a cut-off given by Lemma A.2. From (A.16) we have

$$\int (1 - \chi_R) g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 \, d\mathfrak{m}$$
$$\leq \int \left(\Delta((1 - \chi_R) g_t^\varepsilon) \frac{|\nabla f_t^\varepsilon|^2}{2} + (1 - \chi_R) g_t^\varepsilon (\langle \nabla f_t^\varepsilon, \nabla \Delta f_t^\varepsilon \rangle - K|\nabla f_t^\varepsilon|^2) \right) d\mathfrak{m}.$$

By (4.14), the already noticed fact that $|\nabla f_t^\varepsilon|$ is also uniformly bounded in $L^\infty(X)$ locally in $t \in (0, 1)$ and the Gaussian bounds (4.3) on g_t^ε we see that the second summand in the last integral is, locally in $t \in (0, 1)$, dominated by an $L^1(X)$ function.

For the first summand we write

$$\Delta((1 - \chi_R)g_t^\varepsilon) = -g_t^\varepsilon \Delta \chi_R - 2\langle \nabla \chi_R, \nabla g_t^\varepsilon \rangle + (1 - \chi_R)\Delta g_t^\varepsilon, \quad |\nabla f_t^\varepsilon| = \varepsilon^{-1} f_t^\varepsilon |\nabla \varphi_t^\varepsilon|,$$

and use the properties of χ_R given by Lemma A.2 and those of g_t^ε, f_t^ε, $|\nabla \varphi_t^\varepsilon|$ that we already mentioned to deduce that $\Delta((1 - \chi_R)g_t^\varepsilon)$ is bounded in $L^\infty(X)$ and $|\nabla f_t^\varepsilon|^2$ is dominated in $L^1(X)$, both locally uniformly in $t \in (0, 1)$. Hence (4.15) follows from the fact that $1 - \chi_R, |\nabla \chi_R|, \Delta \chi_R$ are identically 0 on $B_R(\bar{x})$.

It remains to prove that $t \mapsto \int \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathfrak{m}$ is continuous and thanks to (4.12) to this end it is sufficient to show that for any $R > 0$ the map $t \mapsto \int \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathfrak{m}$ is continuous. To see this, notice that

$$\int \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathfrak{m}$$

$$= -\int \left[(\chi_R \langle \nabla \rho_t^\varepsilon, \nabla \varphi_t^\varepsilon \rangle + \rho_t^\varepsilon \langle \nabla \chi_R, \nabla \varphi_t^\varepsilon \rangle) \Delta \varphi_t^\varepsilon + \chi_R \rho_t^\varepsilon |\Delta \varphi_t^\varepsilon|^2 \right] d\mathfrak{m}, \quad (4.16)$$

and that the maps $t \mapsto \rho_t^\varepsilon, \varphi_t^\varepsilon$ are continuous with values in $W^{1,2}(X)$, $W^{1,2}(X, e^{-V}\mathfrak{m})$ respectively. Also, writing $\Delta \varphi_t^\varepsilon = \varepsilon \frac{\Delta f_t^\varepsilon}{f_t^\varepsilon} - \varepsilon |\nabla \varphi_t^\varepsilon|^2$, using the continuity of $t \mapsto f_t^\varepsilon, \Delta f_t^\varepsilon \in L^2(X)$, the bound (4.13) and the fact that f_t^ε is bounded from below on $\mathrm{supp}(\chi_R)$ by a positive constant depending continuously on t and taking into account what we previously proved, we see that the integrand on the right-hand side of (4.16) is continuous as a map with values in $L^0(X, \mathfrak{m}_{|\mathrm{supp}(\chi_R)})$ and, locally in t, uniformly dominated by an $L^1(X)$ function. This is sufficient to conclude the proof. □

Proposition 4.6 (Uniform L^∞ bound on the densities). *Under Setting 4.1, the following holds. For every $\bar{x} \in X$ there exist constants $C, C' > 0$ which depend on $K, N, \bar{x}, \rho_0, \rho_1$ such that*

$$\rho_t^\varepsilon \leq C e^{-C' d^2(\cdot, \bar{x})} \quad \mathfrak{m}\text{-a.e.} \tag{4.17}$$

for every $t \in [0, 1]$ and every $\varepsilon \in (0, 1)$.

Proof. From (4.4) and direct manipulation we see that there are constants $c, c', r > 0$ depending on $K, N, \bar{x}, \rho_0, \rho_1$ only such that

$$\rho_t^\varepsilon(x) \leq c e^{-c' d^2(x, \bar{x})} \quad \forall x \notin B_r(\bar{x}), \ \varepsilon \in (0, 1), \ t \in [0, 1], \tag{4.18}$$

hence it is sufficient to show that there exists a constant $M > 0$ depending on $K, N, \bar{x}, \rho_0, \rho_1$ only such that

$$\|\rho_t^\varepsilon\|_{L^\infty(X)} \leq M \quad \forall \varepsilon \in (0, 1), \ t \in [0, 1]. \tag{4.19}$$

For later purposes it will be useful to observe that from (4.18) and the volume growth estimate (A.20) it follows that there is $R > r$ such that

$$\int_{X \setminus B_R(\bar{x})} (\rho_t^\varepsilon)^p d^2(\cdot, \bar{x}) \, d\mathfrak{m} \leq 1 \quad \forall \varepsilon \in (0, 1), \ t \in [0, 1], \ p \geq 2. \tag{4.20}$$

Now fix $\varepsilon > 0$. We know from Proposition 4.3 that $(\rho_t^\varepsilon) \in C([0, 1], L^2(X)) \cap AC_{\mathrm{loc}}((0, 1), L^2(X))$ and by the maximum principle for the heat equation $\rho_t^\varepsilon \leq C_\varepsilon$ for all $t \in [0, 1]$, thus for any $p > 2$ the function $E_p : [0, 1] \to [0, \infty)$ defined by

$$E_p(t) := \int (\rho_t^\varepsilon)^p \, \mathrm{d}\mathfrak{m}$$

belongs to $C([0, 1]) \cap AC_{\mathrm{loc}}((0, 1))$. An application of the dominated convergence theorem show that its derivative can be computed by passing to the limit in the integrand, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(t) = p \int (\rho_t^\varepsilon)^{p-1} \frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} = -p \int (\rho_t^\varepsilon)^{p-1} \operatorname{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \, \mathrm{d}\mathfrak{m}.$$

Then the definition of ϑ_t^ε, (4.9a), (4.9b) and (4.4) justify integration by parts, whence

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(t) = p(p-1) \int (\rho_t^\varepsilon)^{p-1} \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$$

$$= (p-1) \int \langle \nabla (\rho_t^\varepsilon)^p, \nabla \vartheta_t^\varepsilon \rangle \mathrm{d}\mathfrak{m} = -(p-1) \int (\rho_t^\varepsilon)^p \Delta \vartheta_t^\varepsilon \, \mathrm{d}\mathfrak{m},$$

and recalling that $\vartheta_t^\varepsilon = \psi_t^\varepsilon - \frac{\varepsilon}{2} \log \rho_t^\varepsilon$ we obtain (the integrals are well defined for the same reasons as above)

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(t) = -(p-1) \int (\rho_t^\varepsilon)^p \Delta \psi_t^\varepsilon \, \mathrm{d}\mathfrak{m} + \frac{\varepsilon}{2}(p-1) \int (\rho_t^\varepsilon)^p \Delta \log \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m}. \qquad (4.21)$$

Now notice (the same arguments as above justify integration by parts) that

$$\int (\rho_t^\varepsilon)^p \Delta \log \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} = -p \int (\rho_t^\varepsilon)^{p-1} \langle \nabla \rho_t^\varepsilon, \nabla \log \rho_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$$

$$= -p \int (\rho_t^\varepsilon)^{p-2} |\nabla \rho_t^\varepsilon|^2 \, \mathrm{d}\mathfrak{m} \leq 0$$

and choose $\delta := \frac{1}{2}$ and $T := 1$ in (3.4) to get the existence of a constant $c'' > 0$ depending on K, N, $\bar{x}$ and the diameters of the supports of ρ_0, ρ_1 such that $\Delta \psi_t^\varepsilon \geq -c''(1 + \mathrm{d}^2(\cdot, \bar{x}))$ for any $t \in [0, 1/2]$. Thus from (4.21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(t) \leq c''(p-1) E_p(t) + c''(p-1) \int (\rho_t^\varepsilon)^p \mathrm{d}^2(\cdot, \bar{x}) \, \mathrm{d}\mathfrak{m} \quad \text{a.e. } t \in [0, 1/2],$$

and recalling (4.20) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} E_p(t) \leq c''(p-1) E_p(t) + c''(p-1) \int_{B_R(\bar{x})} (\rho_t^\varepsilon)^p \mathrm{d}^2(\cdot, \bar{x}) \, \mathrm{d}\mathfrak{m} + 1 \leq c'''(p-1) E_p(t) + 1$$

for a.e. $t \in [0, 1/2]$. Then Grönwall's lemma gives

$$E_p(t) \leq \left(E_p(0) + \frac{1}{c'''(p-1)} \right) e^{c'''(p-1)} \quad \forall t \in [0, 1/2].$$

Passing to the p-th roots, writing $E_p(t) = \|\rho_t^\varepsilon\|_{L^{p-1}(\mu_t^\varepsilon)}^{p-1}$ and observing that since μ_t^ε is a probability measure, we have $\|h\|_{L^p(\mu_t^\varepsilon)} \uparrow \|h\|_{L^\infty(\mu_t^\varepsilon)}$ as $p \to \infty$, we obtain

$$\|\rho_t^\varepsilon\|_{L^\infty} \leq e^{c'''}\|\rho_0\|_{L^\infty} \quad \forall t \in [0, 1/2].$$

Switching the roles of ρ_0 and ρ_1 we get analogous control for $t \in [1/2, 1]$, whence the claim (4.19) with $M := e^{c'''} \max\{\|\rho_0\|_{L^\infty}, \|\rho_1\|_{L^\infty}\}$.   □

4.3. Entropy along entropic interpolations

In [44] Léonard computed the first and second derivatives of the relative entropy along entropic interpolations: here we are going to show that his computations are fully justifiable in our setting. As we shall see later on, these formulas will be the crucial tool for showing that the acceleration of entropic interpolation goes to 0 in a suitable weak sense.

We start by noticing that a form of Bochner inequality for the Schrödinger potentials can be deduced. Observe that in general the object $\Gamma_2(\varphi_t^\varepsilon)$ is not a well defined measure, because in some sense it can have both infinite positive mass and infinite negative mass; this is not due to the generality of the framework we are working in, but to the fact that even in the Euclidean space φ_t^ε, $|\nabla \varphi_t^\varepsilon|$, $\Delta \varphi_t^\varepsilon$ need not be integrable. Nevertheless, thanks to Lemma 4.5, the action of $\Gamma_2(\varphi_t^\varepsilon)$ on ρ_t^ε can still be defined: we will put

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle := \int \left(\tfrac{1}{2} \Delta \rho_t^\varepsilon |\nabla h_t^\varepsilon|^2 - \rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle \right) d\mathfrak{m},$$

where h_t^ε is any of φ_t^ε, ψ_t^ε, ϑ_t^ε, $\log \rho_t^\varepsilon$, and notice that Lemma 4.5 ensures that the integral on the right-hand side is well defined and finite. We then have:

Lemma 4.7. *Under Setting 4.1, for any $\varepsilon > 0$ and $t \in (0, 1)$ we have*

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle \geq \int \left(|\mathrm{Hess}(h_t^\varepsilon)|_{\mathsf{HS}}^2 + K|\nabla h_t^\varepsilon|^2 \right) \rho_t^\varepsilon \, d\mathfrak{m}, \tag{4.22a}$$

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle \geq \int \left((\Delta h_t^\varepsilon)^2/N + K|\nabla h_t^\varepsilon|^2 \right) \rho_t^\varepsilon \, d\mathfrak{m}, \tag{4.22b}$$

where h_t^ε is any of φ_t^ε, ψ_t^ε, ϑ_t^ε, $\log \rho_t^\varepsilon$.

Proof. Fix $\varepsilon > 0$, $t \in (0, 1)$ and, for given $\bar{x} \in X$ and $R > 0$, let $\chi_R \in \mathrm{Test}^\infty(X)$ be a cut-off function with support in $B_{R+1}(\bar{x})$ and such that $\chi_R \equiv 1$ in $B_R(\bar{x})$. Then we know that $\chi_{R+1}\varphi_t^\varepsilon \in \mathrm{Test}^\infty(X)$ and thus (A.16) holds for it:

$$\Gamma_2(\chi_{R+1}\varphi_t^\varepsilon) \geq \left(|\mathrm{Hess}(\chi_{R+1}\varphi_t^\varepsilon)|_{\mathsf{HS}}^2 + K|\nabla(\chi_{R+1}\varphi_t^\varepsilon)|^2 \right)\mathfrak{m}.$$

Multiplying both sides by $\chi_R \rho_t^\varepsilon$, integrating over X and using the locality of the various differential operators involved we obtain

$$\int \left(\tfrac{1}{2} \Delta(\chi_R \rho_t^\varepsilon)|\nabla \varphi_t^\varepsilon|^2 - \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \right) d\mathfrak{m}$$
$$\geq \int \chi_R \rho_t^\varepsilon \left(|\mathrm{Hess}(\varphi_t^\varepsilon)|_{\mathsf{HS}}^2 + K|\nabla \varphi_t^\varepsilon|^2 \right) d\mathfrak{m}. \tag{4.23}$$

By monotone convergence,

$$\lim_{R \to \infty} \int \chi_R \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 \, d\mathfrak{m} = \int \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 \, d\mathfrak{m},$$

$$\lim_{R \to \infty} \int \chi_R \rho_t^\varepsilon |\mathrm{Hess}(\varphi_t^\varepsilon)|_{\mathsf{HS}}^2 \, d\mathfrak{m} = \int \rho_t^\varepsilon |\mathrm{Hess}(\varphi_t^\varepsilon)|_{\mathsf{HS}}^2 \, d\mathfrak{m},$$

and thus the right-hand side of (4.23) tends to the right-hand side of (4.22a). Now notice that

$$\Delta(\chi_R \rho_t^\varepsilon) = \chi_R \Delta \rho_t^\varepsilon + 2\langle \nabla \rho_t^\varepsilon, \nabla \chi_R \rangle + \rho_t^\varepsilon \Delta \chi_R$$

and the choice of χ_R implies that $|\chi_R|, |\nabla \chi_R|, |\Delta \chi_R|$ are uniformly bounded and $\mathfrak{m}$-a.e. converge to $1, 0, 0$ respectively as $R \to \infty$. Hence Lemma 4.5 and the dominated convergence theorem show that the left-hand side of (4.23) converges to $\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle$, proving (4.22a) for $h_t^\varepsilon = \varphi_t^\varepsilon$. The other claims follow by similar means taking (A.17) into account. $\square$

Now we can motivate Léonard's computations, getting formulas for the first and second derivatives of entropy along entropic interpolations.

Proposition 4.8. *Under Setting* 4.1, *the following holds. For any* $\varepsilon > 0$ *the map* $t \mapsto H(\mu_t^\varepsilon \mid \mathfrak{m})$ *belongs to* $C([0, 1]) \cap C^2((0, 1))$, *and for every* $t \in (0, 1)$,

$$\frac{d}{dt} H(\mu_t^\varepsilon \mid \mathfrak{m}) = \int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, d\mathfrak{m} = \frac{1}{2\varepsilon} \int (|\nabla \psi_t^\varepsilon|^2 - |\nabla \varphi_t^\varepsilon|^2) \rho_t^\varepsilon \, d\mathfrak{m}, \tag{4.24a}$$

$$\frac{d^2}{dt^2} H(\mu_t^\varepsilon \mid \mathfrak{m}) = \langle \Gamma_2(\vartheta_t^\varepsilon), \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{4} \langle \Gamma_2(\log \rho_t^\varepsilon), \rho_t^\varepsilon \rangle = \frac{1}{2} \langle \Gamma_2(\varphi_t^\varepsilon), \rho_t^\varepsilon \rangle + \frac{1}{2} \langle \Gamma_2(\psi_t^\varepsilon), \rho_t^\varepsilon \rangle. \tag{4.24b}$$

Proof. By Lemma 4.5, the middle terms and the right-hand sides in (4.24a) and (4.24b) exist, are finite and continuously depend on $t \in (0, 1)$. Also, the equality between the middle terms and the right-hand sides follows trivially from the relations $\vartheta_t^\varepsilon = (\psi_t^\varepsilon - \varphi_t^\varepsilon)/2$ and $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$.

Thus it is sufficient to show that $t \mapsto H(\mu_t^\varepsilon \mid \mathfrak{m})$ is in $C([0, 1]) \cap C^2((0, 1))$ and that (4.24a) and (4.24b) hold for a.e. $t \in (0, 1)$.

Lemma 4.5 ensures that $t \mapsto H(\mu_t^\varepsilon \mid \mathfrak{m})$ is continuous in $(0, 1)$. To check continuity in $t = 0, 1$, thanks to the fact that $(\rho_t^\varepsilon) \in C([0, 1], L^2(X))$ by Proposition 4.3 and arguing as in the proof of Lemma 4.5, it is sufficient to show that $\rho_t^\varepsilon \log \rho_t^\varepsilon$ is dominated by an $L^1(X)$ function. To see this, write

$$\rho_t^\varepsilon \log \rho_t^\varepsilon = g_t^\varepsilon f_t^\varepsilon \log f_t^\varepsilon + f_t^\varepsilon g_t^\varepsilon \log g_t^\varepsilon$$

and notice that for $t \in [0, 1/2]$ the bound (4.3) ensures that the function g_t^ε is uniformly bounded above by a Gaussian and that $\log g_t^\varepsilon$ has quadratic growth. On the other hand, we know by Theorem 2.2 that $f_0^\varepsilon = f^\varepsilon$ is in L^∞, thus the maximum principle for the heat flow and the fact that $z \mapsto z \log z$ is bounded from below show the L^∞ norms of $f_t^\varepsilon, f_t^\varepsilon \log f_t^\varepsilon$ are uniformly bounded in $t \in [0, 1/2]$. As discussed in the proof of Lemma 4.5, this is sufficient to conclude the proof and a similar argument yields the desired bound for $t \in [1/2, 1]$.

Now fix $\varepsilon > 0$ and for $R > 0$ let $\chi_R \in \mathrm{Test}^\infty(\mathrm{X})$ be a cut-off function as given by Lemma A.2. Notice that Lemma 4.5 implies that

$$\int \chi_R \rho_t^\varepsilon \log \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} \to \int \rho_t^\varepsilon \log \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} \quad \text{as } R \to \infty \text{ for every } t \in (0,1). \qquad (4.25)$$

Also, Proposition 4.3 tells us that $(\rho_t^\varepsilon) \in AC_{\mathrm{loc}}((0,1), L^2(\mathrm{X}))$ and that it is, locally in $t \in (0,1)$ and in space, uniformly bounded away from 0 and ∞. Therefore, for $u(z) := z \log z$ the map $(0,1) \ni t \mapsto \chi_R u(\rho_t^\varepsilon) \in L^2(\mathrm{X})$ is absolutely continuous. In particular, so is $\int \chi_R u(\rho_t^\varepsilon) \, \mathrm{d}\mathfrak{m}$ and it is then clear that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R u(\rho_t^\varepsilon) \, \mathrm{d}\mathfrak{m} = \int \chi_R (\log \rho_t^\varepsilon + 1) \frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} \quad \text{a.e. } t.$$

Using the formula for $\frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon$ provided by Proposition 4.3 we then get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R u(\rho_t^\varepsilon) \, \mathrm{d}\mathfrak{m} = -\int \chi_R (\log \rho_t^\varepsilon + 1) \, \mathrm{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \, \mathrm{d}\mathfrak{m}$$

$$= \int \langle \nabla(\chi_R (\log \rho_t^\varepsilon + 1)), \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m}$$

$$= \int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m} + \int \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle (\log \rho_t^\varepsilon + 1) \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m}.$$

Since $|\nabla \chi_R|$ is uniformly bounded and identically 0 on $B_R(\bar{x})$, Lemma 4.5 shows that the last expression in the above identity converges to $\int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$ as $R \to \infty$ locally uniformly in $t \in (0,1)$. This fact, (4.25) and the initial discussion give $C^1((0,1))$ regularity for $t \mapsto H(\mu_t^\varepsilon \mid \mathfrak{m})$ and (4.24a).

For (4.24b), from Proposition 4.3 we know that $(\rho_t^\varepsilon) \in AC_{\mathrm{loc}}((0,1), W^{1,2}(\mathrm{X}))$ and $(\vartheta_t^\varepsilon) \in AC_{\mathrm{loc}}((0,1), W^{1,2}(\mathrm{X}, e^{-V}\mathfrak{m}))$ with $V = M\mathrm{d}^2(\cdot, \bar{x})$ for some $\bar{x} \in \mathrm{X}$ and $M > 0$ sufficiently large. Hence $(0,1) \ni t \mapsto \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \in L^2(\mathrm{X})$ is absolutely continuous. In particular, so is $\int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m} = \int \chi_R \left(\left\langle \nabla \frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \right\rangle + \left\langle \nabla \rho_t^\varepsilon, \nabla \frac{\mathrm{d}}{\mathrm{d}t} \vartheta_t^\varepsilon \right\rangle \right) \mathrm{d}\mathfrak{m} \quad \text{a.e. } t.$$

Thus from the formulas for $\frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon$, $\frac{\mathrm{d}}{\mathrm{d}t} \vartheta_t^\varepsilon$ provided in Proposition 4.3 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m} = \underbrace{\int -\chi_R \langle \nabla(\mathrm{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon)), \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}}_{A_t(R)}$$

$$+ \underbrace{\int \chi_R \left\langle \nabla \rho_t^\varepsilon, \nabla \left(-\tfrac{1}{2} |\nabla \vartheta_t^\varepsilon|^2 - \tfrac{\varepsilon^2}{4} \Delta \log \rho_t^\varepsilon - \tfrac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \right) \right\rangle \mathrm{d}\mathfrak{m}}_{B_t(R)}.$$

Now notice that a few integrations by parts and the Leibniz rule give

$$A_t(R) = \int \operatorname{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m} + \int \chi_R \operatorname{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \Delta \vartheta_t^\varepsilon \, \mathrm{d}\mathfrak{m}$$

$$= \int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m} - \int \chi_R \rho_t^\varepsilon \langle \nabla \vartheta_t^\varepsilon, \nabla \Delta \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$$

and

$$B_t(R)$$
$$= \int \left(\frac{1}{2} |\nabla \vartheta_t^\varepsilon|^2 \operatorname{div}(\chi_R \nabla \rho_t^\varepsilon) - \frac{\varepsilon^2}{4} \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \Delta \log \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \operatorname{div}(\chi_R \nabla \rho_t^\varepsilon) \right) \mathrm{d}\mathfrak{m}$$

$$= \int \left(\frac{1}{2} \chi_R \Delta \rho_t^\varepsilon |\nabla \vartheta_t^\varepsilon|^2 - \frac{\varepsilon^2}{4} \chi_R \rho_t^\varepsilon \langle \nabla \log \rho_t^\varepsilon, \nabla \Delta \log \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} \chi_R \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 \right) \mathrm{d}\mathfrak{m}$$

$$+ \int \left(\frac{1}{2} |\nabla \vartheta_t^\varepsilon|^2 \langle \nabla \chi_R, \nabla \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \langle \nabla \chi_R, \nabla \rho_t^\varepsilon \rangle \right) \mathrm{d}\mathfrak{m}.$$

Since $|\nabla \chi_R|$ is uniformly bounded and identically 0 on $B_R(\bar{x})$, Lemma 4.5 gives

$$A_t(R) + B_t(R) \to \langle \Gamma_2(\vartheta_t^\varepsilon), \rho_t^\varepsilon \rangle + \tfrac{\varepsilon^2}{4} \langle \Gamma_2(\log \rho_t^\varepsilon), \rho_t^\varepsilon \rangle \qquad \text{as } R \to \infty$$

locally uniformly in $t \in (0, 1)$.

This fact, the convergence of $\int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$ to $\int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathfrak{m}$ as $R \to \infty$ (which is also a consequence of Lemma 4.5) and the initial discussion give the conclusion. $\square$

As a first consequence of the formulas just obtained, we show that some quantities remain bounded as $\varepsilon \downarrow 0$:

Lemma 4.9 (Bounded quantities). *Under Setting 4.1, for any $\bar{x} \in \mathrm{X}$ we have*

$$\sup_{\varepsilon \in (0,1),\, t \in [0,1]} \int \mathrm{d}^2(\cdot, \bar{x}) \rho_t^\varepsilon \, \mathrm{d}\mathfrak{m} < \infty, \tag{4.26a}$$

$$\sup_{\varepsilon \in (0,1),\, t \in [0,1]} |H(\mu_t^\varepsilon \,|\, \mathfrak{m})| < \infty, \tag{4.26b}$$

$$\sup_{\varepsilon \in (0,1)} \iint_0^1 (|\nabla \vartheta_t^\varepsilon|^2 + \varepsilon^2 |\nabla \log \rho_t^\varepsilon|^2) \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty, \tag{4.26c}$$

and for any $\delta \in (0, 1/2)$,

$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} \left(|\mathrm{Hess}(\vartheta_t^\varepsilon)|_{\mathsf{HS}}^2 + \varepsilon^2 |\mathrm{Hess}(\log \rho_t^\varepsilon)|_{\mathsf{HS}}^2 \right) \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty, \tag{4.27a}$$

$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} (|\Delta \vartheta_t^\varepsilon|^2 + \varepsilon^2 |\Delta \log \rho_t^\varepsilon|^2) \rho_t^\varepsilon \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty. \tag{4.27b}$$

Proof. (4.26a) follows from (4.17) and the volume growth (A.20). As regards (4.26b), notice that (4.26a) and (2.11) give a uniform lower bound on $H(\mu_t^\varepsilon \,|\, \mathfrak{m})$; for the upper bound notice that (4.17) implies uniform quadratic growth of $\log \rho_t^\varepsilon$.

Let us now pass to (4.26c) and observe that Proposition 4.4 together with (4.26a) yields

$$\sup_{\varepsilon \in (0,1)} \iint_{1/2}^{1} |\nabla \varphi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} + \iint_{0}^{1/2} |\nabla \psi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty. \qquad (4.28)$$

As a second step, notice that (4.24a) gives

$$\iint_{0}^{1/2} |\nabla \varphi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = \iint_{0}^{1/2} |\nabla \psi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} - 2\varepsilon \int_{0}^{1/2} \frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m}) \, \mathrm{d}t$$

$$= \iint_{0}^{1/2} |\nabla \psi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} + 2\varepsilon(H(\mu_0 \,|\, \mathfrak{m}) - H(\mu_t^{\varepsilon} \,|\, \mathfrak{m}))$$

so that taking into account (4.26b) and (4.28) we see that the right-hand side is uniformly bounded for $\varepsilon \in (0, 1)$. Using again (4.28) we deduce that

$$\sup_{\varepsilon \in (0,1)} \iint_{0}^{1} |\nabla \varphi_t^{\varepsilon}|^2 \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} < \infty.$$

A symmetric argument provides the analogous bound for (ψ_t^{ε}) and thus recalling that $\vartheta_t^{\varepsilon} = \frac{1}{2}(\psi_t^{\varepsilon} - \varphi_t^{\varepsilon})$ and $\varepsilon \log \rho_t^{\varepsilon} = \psi_t^{\varepsilon} + \varphi_t^{\varepsilon}$ we obtain (4.26c).

Now use the fact that $\vartheta_t^{\varepsilon} = -\varphi_t^{\varepsilon} + \frac{\varepsilon}{2} \log \rho_t^{\varepsilon}$ in conjunction with (4.24a) to get

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m})_{|t=\delta} = - \int \langle \nabla \rho_\delta^{\varepsilon}, \nabla \varphi_\delta^{\varepsilon} \rangle \, \mathrm{d}\mathfrak{m} + \frac{\varepsilon}{2} \int \langle \nabla \rho_\delta^{\varepsilon}, \nabla \log \rho_\delta^{\varepsilon} \rangle \, \mathrm{d}\mathfrak{m}$$

$$= \int \rho_\delta^{\varepsilon} \Delta \varphi_\delta^{\varepsilon} \, \mathrm{d}\mathfrak{m} + \frac{\varepsilon}{2} \int \frac{|\nabla \rho_\delta^{\varepsilon}|^2}{\rho_\delta^{\varepsilon}} \, \mathrm{d}\mathfrak{m} \geq \int \rho_\delta^{\varepsilon} \Delta \varphi_\delta^{\varepsilon} \, \mathrm{d}\mathfrak{m}.$$

Recalling the lower bound (4.9b) and (4.26a), we find that for some constant C_δ independent of ε,

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m})_{|t=\delta} \geq -C_\delta \quad \forall \varepsilon \in (0, 1),$$

and an analogous argument starting from $\vartheta_t^{\varepsilon} = \psi_t^{\varepsilon} - \frac{\varepsilon}{2} \log \rho_t^{\varepsilon}$ yields $\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m})_{|t=1-\delta} \leq C_\delta$ for every $\varepsilon \in (0, 1)$. Therefore

$$\sup_{\varepsilon \in (0,1)} \int_{\delta}^{1-\delta} \frac{\mathrm{d}^2}{\mathrm{d}t^2} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m}) = \sup_{\varepsilon \in (0,1)} \left(\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m})_{|t=1-\delta} - \frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t^{\varepsilon} \,|\, \mathfrak{m})_{|t=\delta} \right) < \infty.$$

The bounds (4.27a) and (4.27b) then follow from this last inequality in conjunction with (4.24b), (4.26c) and the weighted Bochner inequalities (4.22a) and (4.22b) respectively.
                                                                                              □

With the help of the previous lemma we can now prove that some crucial quantities vanish in the limit $\varepsilon \downarrow 0$; as we shall see in the proof of our main Theorem 5.13, this is what we will need to prove that the acceleration of entropic interpolation goes to 0 as $\varepsilon \downarrow 0$.

Lemma 4.10 (Vanishing quantities). *Under Setting* 4.1, *for any* $\delta \in (0, 1/2)$ *we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 0, \tag{4.29a}$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 0, \tag{4.29b}$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}| \, |\nabla \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 0, \tag{4.29c}$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^3 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = 0. \tag{4.29d}$$

Proof. For (4.29a) we notice that

$$\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} \leq \varepsilon \sqrt{1 - 2\delta} \sqrt{\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}|^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}}$$

and that, by (4.27b), the last square root is uniformly bounded in $\varepsilon \in (0, 1)$.

For (4.29b) we start by observing that Lemma 4.11 below applies to ρ_t^{ε}, because by Proposition 4.3, $\rho_t^{\varepsilon} \in \mathrm{Test}^{\infty}(X) \cap L^1(X)$ and

$$\Delta \rho_t^{\varepsilon} = f_t^{\varepsilon} \Delta g_t^{\varepsilon} + g_t^{\varepsilon} \Delta f_t^{\varepsilon} + 2\langle \nabla f_t^{\varepsilon}, \nabla g_t^{\varepsilon} \rangle \in L^1(X).$$

Hence, from the identity $\rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^2 = -\rho_t^{\varepsilon} \Delta \log \rho_t^{\varepsilon} + \Delta \rho_t^{\varepsilon}$ and the fact that $\int \Delta \rho_t^{\varepsilon} \, \mathrm{d}\mathfrak{m} = 0$ we get

$$\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} = -\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} \Delta \log \rho_t^{\varepsilon} \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}$$

$$\leq \varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}$$

and then use (4.29a).

For (4.29c) we observe that

$$\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}| \, |\nabla \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}$$

$$\leq \sqrt{\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\Delta \log \rho_t^{\varepsilon}|^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}} \sqrt{\varepsilon^2 \iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^2 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}},$$

and use the fact that the first square root on the right-hand side is bounded (by (4.27b)) and the second one goes to 0 (by (4.29b)).

To prove (4.29d) we start again from the identity $\rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^2 = -\rho_t^{\varepsilon} \Delta \log \rho_t^{\varepsilon} + \Delta \rho_t^{\varepsilon}$ to get

$$\iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}|^3 \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}$$

$$= -\iint_{\delta}^{1-\delta} \rho_t^{\varepsilon} \Delta(\log \rho_t^{\varepsilon}) |\nabla \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m} + \iint_{\delta}^{1-\delta} \Delta \rho_t^{\varepsilon} |\nabla \log \rho_t^{\varepsilon}| \, \mathrm{d}t \, \mathrm{d}\mathfrak{m}.$$

After multiplication by ε^2 we see that the first integral on the right-hand side vanishes as $\varepsilon \downarrow 0$ thanks to (4.29c). For the second integral we start by noticing that an application of the dominated convergence theorem ensures that

$$\iint_\delta^{1-\delta} \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| \, dt \, d\mathfrak{m} = \lim_{\eta \downarrow 0} \iint_\delta^{1-\delta} \Delta \rho_t^\varepsilon \sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} \, dt \, d\mathfrak{m}, \qquad (4.30)$$

then we observe that for every $\eta > 0$ the map $z \mapsto \sqrt{\eta + z}$ is in $C^1([0, \infty))$ and Lipschitz continuous there, and since $|\nabla \log \rho_t^\varepsilon|^2 \in W^{1,2}(X, e^{-V}\mathfrak{m})$ for $V = M\mathsf{d}^2(\cdot, \bar{x})$ and suitable $\bar{x}$, M (recall Proposition 4.3) we deduce that $\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} \in W^{1,2}(X, e^{-V}\mathfrak{m})$ as well. Thus by the chain rule for gradients, the Leibniz rule (A.15) and also using a cut-off argument in conjunction with Lemma 4.5 to justify integration by parts, we see that

$$\left| \iint_\delta^{1-\delta} \Delta \rho_t^\varepsilon \sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} \, dt \, d\mathfrak{m} \right|$$

$$= \left| \iint_\delta^{1-\delta} \frac{\rho_t^\varepsilon}{2\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2}} \langle \nabla \log \rho_t^\varepsilon, \nabla |\nabla \log \rho_t^\varepsilon|^2 \rangle \, dt \, d\mathfrak{m} \right|$$

$$= \left| \iint_\delta^{1-\delta} \frac{\rho_t^\varepsilon}{\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2}} \operatorname{Hess}(\log \rho_t^\varepsilon)(\nabla \log \rho_t^\varepsilon, \nabla \log \rho_t^\varepsilon) \, dt \, d\mathfrak{m} \right|$$

$$\leq \iint_\delta^{1-\delta} \rho_t^\varepsilon |\operatorname{Hess}(\log \rho_t^\varepsilon)|_{\mathsf{HS}} |\nabla \log \rho_t^\varepsilon| \, dt \, d\mathfrak{m},$$

and since this true for any $\eta > 0$, from (4.30) we obtain

$$\varepsilon^2 \left| \iint_\delta^{1-\delta} \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| \, dt \, d\mathfrak{m} \right| \leq \varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\operatorname{Hess}(\log \rho_t^\varepsilon)|_{\mathsf{HS}} |\nabla \log \rho_t^\varepsilon| \, dt \, d\mathfrak{m}$$

$$\leq \sqrt{\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\operatorname{Hess}(\log \rho_t^\varepsilon)|_{\mathsf{HS}}^2 \, dt \, d\mathfrak{m}} \sqrt{\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 \, dt \, d\mathfrak{m}}.$$

In this last expression the first square root is uniformly bounded in $\varepsilon \in (0, 1)$ by (4.27a), while the second one vanishes as $\varepsilon \downarrow 0$ thanks to (4.29b). □

Lemma 4.11. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$ *and* $h \in D(\Delta) \cap L^1(X)$ *with* $\Delta h \in L^1(X)$. *Then*

$$\int \Delta h \, d\mathfrak{m} = 0.$$

Proof. Let $\bar{x} \in X$, $R > 0$ and $\chi_R \in \operatorname{Test}^\infty(X)$ be a cut-off function as given by Lemma A.2. Then

$$\left| \int \chi_R \Delta h \, d\mathfrak{m} \right| = \left| \int \Delta \chi_R h \, d\mathfrak{m} \right| = \left| \int_{X \setminus B_R(\bar{x})} \Delta \chi_R h \, d\mathfrak{m} \right| \leq \|\Delta \chi_R\|_{L^\infty(X)} \int_{X \setminus B_R(\bar{x})} h \, d\mathfrak{m}.$$

Since Lemma A.2 ensures that $\|\Delta \chi_R\|_{L^\infty(X)}$ is uniformly bounded in R, the conclusion follows by letting $R \to \infty$ in the above. □

5. From entropic to displacement interpolations

5.1. Compactness

Starting from the uniform estimates discussed in Section 4, let us first prove that when we pass to the limit as $\varepsilon \downarrow 0$, up to subsequences Schrödinger potentials and entropic interpolations converge in a suitable sense to limit potentials and interpolations.

To formulate the result we need to introduce the Banach space $(C(X, e^{-V}), \|\cdot\|_{C(X,e^{-V})})$, where $V = M d^2(\cdot, \bar{x})$ for some $\bar{x} \in X$ and $M > 0$: the norm $\|\cdot\|_{C(X,e^{-V})}$ is defined as

$$\|f\|_{C(X,e^{-V})} := \sup_{x \in X} |f(x)| e^{-V(x)}$$

and $C(X, e^{-V}) := \{f \in C(X) : \|f\|_{C(X,e^{-V})} < \infty\}$.

Proposition 5.1 (Compactness for measures). *Under Setting* 4.1, *the following holds. For any sequence* $\varepsilon_n \downarrow 0$ *there exists a subsequence, not relabelled, such that the curves* $(\mu_t^{\varepsilon_n})$ *uniformly converge in* $(\mathscr{P}_2(X), W_2)$ *to a limit curve* (μ_t) *belonging to* $AC([0, 1], (\mathscr{P}_2(X), W_2))$. *Moreover, there is* $C > 0$ *such that*

$$\mu_t \leq C\mathfrak{m} \quad \forall t \in [0, 1], \tag{5.1}$$

and setting $\rho_t := \frac{d\mu_t}{d\mathfrak{m}}$ *we have*

$$\rho_t^{\varepsilon_n} \stackrel{*}{\rightharpoonup} \rho_t \quad \text{in } L^\infty(X) \quad \forall t \in [0, 1]. \tag{5.2}$$

Proof. Fix $\varepsilon \in (0, 1)$; we want to apply Theorem A.5 to (μ_t^ε) and $(\nabla \vartheta_t^\varepsilon)$. The continuity of $t \mapsto \rho_t^\varepsilon \in L^2(X)$ granted by Proposition 4.3 yields weak continuity of (μ_t), and (A.23a) is a consequence of (4.17). The bound (4.26c) yields (A.23b) and from the formula for $\frac{d}{dt}\rho_t^\varepsilon$ given in Proposition 4.3 and again the L^2 continuity of (ρ_t^ε) on $[0, 1]$ it easily follows that (μ_t) and $(\vartheta_t^\varepsilon)$ solve the continuity equation in the sense of Theorem A.5. The conclusion of that theorem ensures that (μ_t^ε) is W_2-absolutely continuous with

$$\int_0^1 |\dot{\mu}_t^\varepsilon|^2 \, dt = \iint_0^1 |\nabla \vartheta_t^\varepsilon|^2 \rho_t^\varepsilon \, dt \, d\mathfrak{m}.$$

The bound (4.26c) implies that the right-hand side is uniformly bounded in $\varepsilon \in (0, 1)$, and since $\{(\mu_t^\varepsilon)\}_\varepsilon$ is tight and 2-uniformly integrable by (4.17) (hence W_2-compact), this is sufficient to ensure the compactness of the family $\{(\mu_t^\varepsilon)\}_\varepsilon$ in $C([0, 1], (\mathscr{P}_2(X), W_2))$ and, by the lower semicontinuity of the kinetic energy, the fact that any limit curve (μ_t) is absolutely continuous. The bound (5.1) is then a direct consequence of the uniform bound (4.19), and the convergence property (5.2) comes from the weak convergence of the measures and the uniform bound on the densities. $\qquad \square$

Proposition 5.2 (Compactness for potentials). *Under Setting* 4.1, *the following holds. For any sequence* $\varepsilon_n \downarrow 0$ *there exists a subsequence, not relabelled, such that for all* $\bar{x} \in X$ *and* $M > 0$, *putting* $V := M d^2(\cdot, \bar{x})$ *we have:*

(i) *For every* $\delta \in (0, 1)$ *there exists* $C > 0$ *which only depends on* $K, N, \delta, \bar{x}$ *such that*

$$|\varphi_t^\varepsilon|, |\psi_{1-t}^\varepsilon| \leq C(1 + d^2(\cdot, \bar{x})) \quad \forall t \in [\delta, 1], \ \varepsilon \in (0, 1). \tag{5.3}$$

(ii) *The curves $(\varphi_t^{\varepsilon_n})$, $(\psi_t^{\varepsilon_n})$ converge locally uniformly on $\mathcal{I}$ with values in $L^1(X, e^{-V}\mathfrak{m})$
    to limit curves (φ_t), $(\psi_t) \in AC_{\mathrm{loc}}(\mathcal{I}, L^1(X, e^{-V}\mathfrak{m}))$ respectively, where $\mathcal{I} := (0, 1]$
    for the φ's, and $\mathcal{I} := [0, 1)$ for the ψ's.*

(iii) *For all $t \in \mathcal{I}$, the functions $\varphi_t^{\varepsilon_n}$, $\psi_t^{\varepsilon_n}$ also converge in $C(X, e^{-V})$ to φ_t, ψ_t.*

(iv) *For any $\delta \in (0, 1)$ there exists $C > 0$ which only depends on K, N, δ, $\bar{x}$ such that*

$$\sup_{t \in [\delta, 1]} \mathrm{lip}(\varphi_t) + \sup_{t \in [0, 1-\delta]} \mathrm{lip}(\psi_t) \le C(1 + \mathrm{d}(\cdot, \bar{x})) \quad \mathfrak{m}\text{-}a.e. \tag{5.4}$$

(v) *Finally, up to passing to a suitable subsequence to obtain the existence of limit measures μ_t as in Proposition 5.1 above, for every $t \in (0, 1)$ we have*

$$\begin{aligned}
\varphi_t + \psi_t &\le 0 \quad \text{on } X, \\
\varphi_t + \psi_t &= 0 \quad \text{on } \mathrm{supp}(\mu_t).
\end{aligned} \tag{5.5}$$

Similarly, the curves $(\vartheta_t^{\varepsilon_n})$ and functions $\vartheta_t^{\varepsilon_n}$ converge in $(0, 1)$ to the limit curve $t \mapsto \vartheta_t := \frac{1}{2}(\psi_t - \varphi_t)$ and to the functions ϑ_t in the same sense as above.

Proof. (i) We start by claiming that for all $\varepsilon > 0$ and $t, s \in (0, 1]$ with $t < s$,

$$\|\varphi_t^\varepsilon - \varphi_s^\varepsilon\|_{L^1(X, e^{-V}\mathfrak{m})} \le \iint_t^s e^{-V}\left(\frac{|\nabla \varphi_r^\varepsilon|^2}{2} + \frac{\varepsilon}{2}|\Delta \varphi_r^\varepsilon|\right) \mathrm{d}r \, \mathrm{d}\mathfrak{m}. \tag{5.6}$$

Indeed, by Proposition 4.3 we know that $(\varphi_t^\varepsilon) \in AC([\delta, 1], W^{1,2}(X, e^{-V'}\mathfrak{m}))$ with $V' := M'\mathrm{d}^2(\cdot, \bar{x})$ and $M' = M'(\delta)$ sufficiently large, for any $\delta \in (0, 1)$. Thus for any cut-off function $\chi_R \in \mathrm{Test}^\infty(X)$ with $\chi_R \equiv 1$ on $B_R(\bar{x})$ and support in $B_{R+1}(\bar{x})$ we have $(\chi_R \varphi_t^\varepsilon) \in AC_{\mathrm{loc}}((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$, and since $W^{1,2}(X, e^{-V}\mathfrak{m}) \subset L^1(X, e^{-V}\mathfrak{m})$ (because $e^{-V}\mathfrak{m}$ is a finite measure) a fortiori this is true for $(\chi_R e^{-V}\varphi_t^\varepsilon)$. From the formula for $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\varepsilon$ (Proposition 4.3) this implies

$$\|\chi_R(\varphi_t^\varepsilon - \varphi_s^\varepsilon)\|_{L^1(X, e^{-V}\mathfrak{m})} \le \iint_t^s \chi_R e^{-V}\left(\frac{|\nabla \varphi_r^\varepsilon|^2}{2} + \frac{\varepsilon}{2}|\Delta \varphi_r^\varepsilon|\right) \mathrm{d}r \, \mathrm{d}\mathfrak{m},$$

so that (5.6) follows by letting $R \to \infty$ and using the monotone convergence theorem. Denoting by C_δ a constant depending on K, N, ρ_0, ρ_1, $\bar{x}$, δ, but independent of ε, t, whose value might change in various occurrences, estimates (4.9a) and (4.10) give

$$\|\varphi_t^\varepsilon - \varphi_s^\varepsilon\|_{L^1(X, e^{-V}\mathfrak{m})} \le C_\delta |s - t| \quad \forall \varepsilon \in (0, 1), \ t, s \in [\delta, 1]. \tag{5.7}$$

Now we observe that from (4.9a) and the fact that X is a geodesic space it follows that

$$|\varphi_t^\varepsilon(x) - \varphi_t^\varepsilon(\bar{x})| \le C_\delta \mathrm{d}(x, \bar{x})(1 + \mathrm{d}(x, \bar{x})) \le C_\delta(1 + \mathrm{d}^2(x, \bar{x})) \quad \forall \varepsilon \in (0, 1), \ t \in [\delta, 1]. \tag{5.8}$$

This already tells us that φ_t^ε has quadratic growth (with constants possibly depending on t, ε). For μ with finite second moment, integrate (5.8) with respect to μ in x to get

$$\left|\varphi_t^\varepsilon(\bar{x}) - \int \varphi_t^\varepsilon \, \mathrm{d}\mu\right| \le \int |\varphi_t^\varepsilon(x) - \varphi_t^\varepsilon(\bar{x})| \, \mathrm{d}\mu(x) \le C_\delta \int (1 + \mathrm{d}^2(x, \bar{x})) \, \mathrm{d}\mu(x), \tag{5.9}$$

then pick $t := 1$, $\mu := \mu_1$ and recall that the normalization chosen for $(f^\varepsilon, g^\varepsilon)$ in Setting 4.1 reads $\int \varphi_1^\varepsilon \, d\mu_1 = 0$ to deduce that $\sup_{\varepsilon \in (0,1)} |\varphi_1^\varepsilon(\bar{x})| < \infty$; thus (5.8) gives

$$|\varphi_1^\varepsilon| \leq C_\delta(1 + d^2(x, \bar{x})) \quad \forall \varepsilon \in (0, 1),$$

which in turn implies $\sup_{\varepsilon \in (0,1)} \|\varphi_1^\varepsilon\|_{L^1(X, e^{-V}\mathfrak{m})} < \infty$. This bound in conjunction with (5.7) gives

$$\|\varphi_t^\varepsilon\|_{L^1(X, e^{-V}\mathfrak{m})} \leq C_\delta \quad \forall \varepsilon \in (0, 1), \ t \in [\delta, 1],$$

so that picking $\mu := e^{-V}\mathfrak{m}$ in (5.9) we see that $|\varphi_t^\varepsilon(\bar{x})| \leq C_\delta$ for all $\varepsilon \in (0, 1)$, $t \in [\delta, 1]$ and in conclusion (5.8) gives (5.3) for the φ_t^ε's.

Following the same lines, the bound (5.3) for ψ_t^ε will follow provided we are able to show that for some measures μ^ε with uniformly bounded second moment w have

$$\sup_{t \in [0, 1-\delta]} \sup_{\varepsilon \in (0,1)} \left| \int \psi_t^\varepsilon \, d\mu^\varepsilon \right| < \infty$$

for any $\delta \in (0, 1)$. We pick $\mu^\varepsilon := \mu_{1/2}^\varepsilon$: (4.26a) gives a uniform bound on the second moment, while multiplying by $\rho_{1/2}^\varepsilon$ the identity $\varphi_{1/2}^\varepsilon + \psi_{1/2}^\varepsilon = \varepsilon \log \rho_{1/2}^\varepsilon$ and integrating we get

$$\int \varphi_{1/2}^\varepsilon \, d\mu_{1/2}^\varepsilon + \int \psi_{1/2}^\varepsilon \, d\mu_{1/2}^\varepsilon = H(\mu_{1/2}^\varepsilon \,|\, \mathfrak{m})$$

and the conclusion follows from (4.26b) and (5.3) in conjunction with (4.26a).

(ii) By Ascoli–Arzelà's theorem, for given $\bar{x} \in X$ and $C > 0$ the set of functions φ on X such that

$$|\varphi| \leq C(1 + d^2(\cdot, \bar{x})), \quad \text{lip}(\varphi) \leq C(1 + d(\cdot, \bar{x}))$$

is a compact subset of $C(X, e^{-V}\mathfrak{m})$. Thus for any $\delta \in (0, 1)$ the estimates (5.3) and (4.9a) show that $\{\varphi_t^\varepsilon : \varepsilon \in (0, 1), \ t \in [\delta, 1]\}$ is compact in $C(X, e^{-V}\mathfrak{m})$ and thus a fortiori also compact in $L^1(X, e^{-V}\mathfrak{m})$. This fact, (5.7), the arbitrariness of $\delta \in (0, 1)$ and again Ascoli–Arzelà's theorem give the claim. Similarly for the ψ's.

(iii) We know that for any $t \in (0, 1)$ we have $\varphi_t^{\varepsilon_n} \to \varphi_t$ in $L^1(X, e^{-V}\mathfrak{m})$. We also noticed that for any $t \in (0, 1)$ the family $\{\varphi_t^{\varepsilon_n}\}_n$ is compact in $C(X, e^{-V}\mathfrak{m})$, thus the claim follows. Similarly for the ψ's.

(iv) We know that for any $x \in X$ we have $\text{lip}(\varphi_t)(x) \leq \underline{\lim}_{r \downarrow 0} \text{Lip}(\varphi_t|_{B_r(x)})$, and since X is geodesic, $\text{Lip}(\varphi_t^\varepsilon|_{B_r(x)}) = \sup_{B_r(x)} \text{lip}(\varphi_t^\varepsilon)$. Thus the claim follows from the bound (4.9a) and the fact that $\text{Lip}(\varphi_t|_{B_r(x)}) \leq \underline{\lim}_{n \to \infty} \text{Lip}(\varphi_t^{\varepsilon_n}|_{B_r(x)})$, which in turn is a trivial consequence of the local uniform convergence we already proved. Similarly for the ψ's.

(v) For the inequality in (5.5) we pass to the limit in the identity

$$\varphi_t^\varepsilon + \psi_t^\varepsilon = \varepsilon \log \rho_t^\varepsilon \tag{5.10}$$

recalling the uniform bound (4.19). To get the identity in (5.5) we multiply both sides of (5.10) by ρ_t^ε and integrate to obtain

$$\int (\varphi_t^\varepsilon + \psi_t^\varepsilon)\rho_t^\varepsilon \, d\mathfrak{m} = \varepsilon H(\mu_t^\varepsilon \,|\, \mathfrak{m}).$$

Letting $\varepsilon = \varepsilon_n \downarrow 0$ we see that the right-hand side goes to 0 by (4.26b); then we use the fact that $W_2(\mu_t^{\varepsilon_n}, \mu_t) \to 0$ and the functions $\varphi_t^{\varepsilon}, \psi_t^{\varepsilon}$ have uniform quadratic growth and converge locally uniformly to φ_t, ψ_t respectively to deduce that the left-hand side goes to $\int (\varphi_t + \psi_t) \, \mathrm{d}\mu_t$. This is sufficient to conclude the proof.   □

5.2. Identification of the limit curve and potentials

We now show that the limit interpolation is the geodesic from μ_0 to μ_1 and the limit potentials are Kantorovich potentials. We shall make use of the following simple lemma valid on general metric measure spaces:

Lemma 5.3. *Let* $(Y, d_Y, \mathfrak{m}_Y)$ *be a complete separable metric measure space endowed with a non-negative measure* $\mathfrak{m}_Y$ *which is finite on bounded sets and assume that* $W^{1,2}(Y)$ *is separable. Let* $\boldsymbol{\pi}$ *be a test plan and* $f \in W^{1,2}(Y)$. *Then* $t \mapsto \int f \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}$ *is absolutely continuous and*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int f \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi} \right| \le \int |\mathrm{d}f|(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \quad a.e.\ t \in [0, 1], \tag{5.11}$$

where the exceptional set can be chosen to be independent of f.

   Moreover, if $(f_t) \in AC([0, 1], L^2(Y)) \cap L^{\infty}([0, 1], W^{1,2}(Y))$, *then the map* $t \mapsto \int f_t \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}$ *is also absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(\int f_s \circ \mathrm{e}_s \, \mathrm{d}\boldsymbol{\pi} \right)_{|s=t} = \int \left(\frac{\mathrm{d}}{\mathrm{d}s} f_s |_{s=t} \right) \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi} + \frac{\mathrm{d}}{\mathrm{d}s} \left(\int f_t \circ \mathrm{e}_s \, \mathrm{d}\boldsymbol{\pi} \right)_{|s=t} \quad a.e.\ t \in [0, 1].$$

Proof. The absolute continuity of $t \mapsto \int f \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}$ and the bound (5.11) are trivial consequences of the definitions of test plans and Sobolev functions. The fact that the exceptional set can be chosen independently of f follows from the separability of $W^{1,2}(Y)$ and standard approximation procedures, carried out, for instance, in [29].

   For the second part, we start by noticing that the second derivative on the right-hand side exists for a.e. t thanks to what we have just proved, so that the claim makes sense. The absolute continuity follows from the fact that for any $t_0, t_1 \in [0, 1]$ with $t_0 < t_1$,

$$\left| \int (f_{t_1} \circ \mathrm{e}_{t_1} - f_{t_0} \circ \mathrm{e}_{t_0}) \, \mathrm{d}\boldsymbol{\pi} \right| \le \left| \int (f_{t_1} \circ \mathrm{e}_{t_1} - f_{t_1} \circ \mathrm{e}_{t_0}) \, \mathrm{d}\boldsymbol{\pi} \right| + \left| \int (f_{t_1} - f_{t_0}) \, \mathrm{d}(\mathrm{e}_{t_0})_* \boldsymbol{\pi} \right|$$

$$\le \int\!\!\int_{t_0}^{t_1} |\mathrm{d}f_{t_1}|(\gamma_t) |\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma) + \int\!\!\int_{t_0}^{t_1} \left| \frac{\mathrm{d}}{\mathrm{d}t} f_t \right| \, \mathrm{d}t \, \mathrm{d}(\mathrm{e}_{t_0})_* \boldsymbol{\pi}$$

and our assumptions on (f_t) and $\boldsymbol{\pi}$. Now fix a point t of differentiability for (f_t) and observe that the fact that $\frac{f_{t+h} - f_t}{h}$ strongly converges in $L^2(Y)$ to $\frac{\mathrm{d}}{\mathrm{d}t} f_t$ and $(\mathrm{e}_{t+h})_* \boldsymbol{\pi}$ weakly converges to $(\mathrm{e}_t)_* \boldsymbol{\pi}$ as $h \to 0$ and the densities are equibounded is sufficient to get

$$\lim_{h \to 0} \int \frac{f_{t+h} - f_t}{h} \circ \mathrm{e}_{t+h} \, \mathrm{d}\boldsymbol{\pi} = \int \frac{\mathrm{d}}{\mathrm{d}t} f_t \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi} = \lim_{h \to 0} \int \frac{f_{t+h} - f_t}{h} \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}.$$

Hence the conclusion comes from dividing by h the trivial identity

$$\int (f_{t+h} \circ e_{t+h} - f_t \circ e_t) \, d\boldsymbol{\pi} = \int (f_t \circ e_{t+h} - f_t \circ e_t) \, d\boldsymbol{\pi} + \int (f_{t+h} \circ e_t - f_t \circ e_t) \, d\boldsymbol{\pi}$$

$$+ \int \Big((f_{t+h} - f_t) \circ e_{t+h} - (f_{t+h} - f_t) \circ e_t \Big) \, d\boldsymbol{\pi}$$

and letting $h \to 0$. $\qquad \square$

We now prove that in the limit the potentials evolve according to the Hopf–Lax semigroup (recall formula (A.25)).

Proposition 5.4 (Limit curve and potentials). *Under Setting* 4.1, *the following holds. The limit curve (μ_t) given by Proposition* 5.1 *is unique* (*i.e. independent of the sequence $\varepsilon_n \downarrow 0$*) *and is the only W_2-geodesic connecting μ_0 to μ_1.*

For any $\bar{x} \in X$, $M > 0$ and any limit curve (φ_t) given by Proposition 5.2, (φ_t) *is in $AC_{\mathrm{loc}}((0, 1], C(X, e^{-V})) \cap L^\infty_{\mathrm{loc}}((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$, where $V := M\mathrm{d}^2(\cdot, \bar{x})$, and for any $t_0, t_1 \in (0, 1]$ with $t_0 < t_1$ we have*

$$-\varphi_{t_1} = Q_{t_1 - t_0}(-\varphi_{t_0}), \tag{5.12a}$$

$$\int \varphi_{t_0} \, d\mu_{t_0} - \int \varphi_{t_1} \, d\mu_{t_1} = \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}), \tag{5.12b}$$

and $-(t_1 - t_0)\varphi_{t_1}$ is a Kantorovich potential from μ_{t_1} to μ_{t_0}. Similarly, for V as above and any limit curve (ψ_t) given by Proposition 5.2, (ψ_t) *belongs to $AC_{\mathrm{loc}}([0, 1), C(X, e^{-V})) \cap L^\infty_{\mathrm{loc}}([0, 1), W^{1,2}(X, e^{-V}\mathfrak{m}))$ and for every $t_0, t_1 \in [0, 1)$ with $t_0 < t_1$ we have*

$$-\psi_{t_0} = Q_{t_1 - t_0}(-\psi_{t_1}), \tag{5.13a}$$

$$\int \psi_{t_1} \, d\mu_{t_1} - \int \psi_{t_0} \, d\mu_{t_0} = \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}), \tag{5.13b}$$

and $-(t_1 - t_0)\psi_{t_0}$ is a Kantorovich potential from μ_{t_0} to μ_{t_1}.

Proof. **Inequality $\leq$ in** (5.12a). Pick $x, y \in X$ and $r > 0$, define

$$\nu_x^r := \frac{1}{\mathfrak{m}(B_r(x))} \mathfrak{m}|_{B_r(x)}, \qquad \nu_y^r := \frac{1}{\mathfrak{m}(B_r(y))} \mathfrak{m}|_{B_r(y)}$$

and $\boldsymbol{\pi}^r$ as the lifting of the only W_2-geodesic from ν_x^r to ν_y^r (recall Theorem A.6(i)). Since ν_x^r, ν_y^r have compact support and $\boldsymbol{\pi}^r$ is an optimal geodesic plan from ν_x^r to ν_y^r, there exist $\bar{x} \in X$ and $R > 0$ sufficiently large such that

$$\mathrm{supp}((e_t)_* \boldsymbol{\pi}^r) \subset B_R(\bar{x}) \quad \forall t \in [0, 1]. \tag{5.14}$$

Let χ be a Lipschitz cut-off function with bounded support such that $\chi \equiv 1$ in $B_R(\bar{x})$. Then, let $\varepsilon \in (0, 1)$ and $0 < t_0 < t_1 \leq 1$, put $\tilde{\varphi}_t^\varepsilon := \chi \varphi_t^\varepsilon$ and observe that $(\tilde{\varphi}_t^\varepsilon) \in$

$AC_{\mathrm{loc}}((0, 1], L^2(X)) \cap L^\infty_{\mathrm{loc}}((0, 1], W^{1,2}(X))$ by Proposition 4.3 and the compactness of the support of χ; thus, by Lemma 5.3 applied to $\boldsymbol{\pi}^r$ and $t \mapsto \tilde{\varphi}^\varepsilon_{(1-t)t_0+tt_1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \tilde{\varphi}^\varepsilon_{(1-t)t_0+tt_1} \circ e_t \, \mathrm{d}\boldsymbol{\pi}^r$$
$$\geq \int \left((t_1 - t_0) \frac{\mathrm{d}}{\mathrm{d}s} \tilde{\varphi}^\varepsilon_s|_{s=(1-t)t_0+tt_1}(\gamma_t) - |\mathrm{d}\tilde{\varphi}^\varepsilon_{(1-t)t_0+tt_1}|(\gamma_t)|\dot{\gamma}_t| \right) \mathrm{d}\boldsymbol{\pi}^r(\gamma).$$

As (5.14) implies that $\chi(\gamma_t) = 1$ for all $t \in [0, 1]$ for $\boldsymbol{\pi}^r$-a.e. γ, $\tilde{\varphi}^\varepsilon$ can be replaced by φ^ε in the inequality above and, recalling the expression for $\frac{\mathrm{d}}{\mathrm{d}t}\varphi^\varepsilon_t$ and using Young's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi^\varepsilon_{(1-t)t_0+tt_1} \circ e_t \, \mathrm{d}\boldsymbol{\pi}^r \geq \int \left(\varepsilon \frac{t_1 - t_0}{2} \Delta\varphi^\varepsilon_{(1-t)t_0+tt_1}(\gamma_t) - \frac{1}{2(t_1 - t_0)}|\dot{\gamma}_t|^2 \right) \mathrm{d}\boldsymbol{\pi}^r(\gamma).$$

Integrating in time and recalling that $\boldsymbol{\pi}^r$ is optimal we get

$$\int \varphi^\varepsilon_{t_1} \, \mathrm{d}\nu^r_y - \int \varphi^\varepsilon_{t_0} \, \mathrm{d}\nu^r_x \geq -\frac{1}{2(t_1 - t_0)} W^2_2(\nu^r_y, \nu^r_x) + \iint_0^1 \varepsilon \frac{t_1 - t_0}{2} \Delta\varphi^\varepsilon_{(1-t)t_0+tt_1} \circ e_t \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}^r.$$

Let $\varepsilon \downarrow 0$ along the sequence (ε_n) for which $(\varphi^{\varepsilon_n}_t)$ converges to our given (φ_t) in the sense of Proposition 5.2 and use the uniform bound (4.10) and the fact that $\boldsymbol{\pi}^r$ has bounded compression to deduce that

$$\int \varphi_{t_1} \, \mathrm{d}\nu^r_y - \int \varphi_{t_0} \, \mathrm{d}\nu^r_x \geq -\frac{1}{2(t_1 - t_0)} W^2_2(\nu^r_y, \nu^r_x);$$

finally letting $r \downarrow 0$ we conclude from the arbitrariness of $x \in X$ that

$$-\varphi_{t_1}(y) \leq Q_{t_1-t_0}(-\varphi_{t_0})(y) \quad \forall y \in X. \tag{5.15}$$

Inequality $\geq$ in (5.12a). To prove the opposite inequality we fix $\bar{x} \in X$, $r > 1$, again $0 < t_0 < t_1 \leq 1$ and let $\tilde{R} > r$, to be fixed later. Let $\chi_{\tilde{R}} \in \mathrm{Test}^\infty(X)$ be given by Lemma A.2, define the vector field $X^\varepsilon_t := \chi_{\tilde{R}} \nabla \varphi^\varepsilon_t$ and apply Theorem A.4 to $((t_1 - t_0)X^\varepsilon_{(1-t)t_1+tt_0})$: the inequality

$$\mathrm{div}\, X^\varepsilon_t \geq \chi_{\tilde{R}} \Delta\varphi^\varepsilon_t - |\nabla\chi_{\tilde{R}}| \, |\nabla\varphi^\varepsilon_t|$$

and the bounds (4.9a), (4.9b) on $\nabla\varphi^\varepsilon_t$, $\Delta\varphi^\varepsilon_t$ ensure that the theorem is applicable and we obtain existence of the regular Lagrangian flow F^ε. Notice that from (4.9a) we know that $|X^\varepsilon_t| \leq C'(1 + \mathrm{d}(\cdot, \bar{x}))$ for all $t \in [t_0, 1]$ for some $C' < \infty$ independent of $\tilde{R}, \varepsilon$, therefore for $\mathfrak{m}$-a.e. x we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}(F^\varepsilon_t(x), \bar{x}) \leq \mathrm{ms}_t(F^\varepsilon_\cdot(x)) \overset{(A.22)}{=} (t_1 - t_0)|X^\varepsilon_{(1-t)t_1+tt_0}|(F^\varepsilon_t(x)) \leq C'(1 + \mathrm{d}(F^\varepsilon_t(x), \bar{x}))$$

for a.e. t and thus Grönwall's Lemma implies the existence of R independent of $\tilde{R}, \varepsilon$ such that for $\mathfrak{m}$-a.e. x,

$$x \in B_r(\bar{x}) \implies F_t(x) \in B_R(\bar{x}) \, \forall t \in [0, 1]. \tag{5.16}$$

We now fix $\tilde{R} := R$ and put $\boldsymbol{\pi}^\varepsilon := \mathfrak{m}(B_r(\bar{x}))^{-1}(F_\cdot^\varepsilon)_* \mathfrak{m}_{|B_r(\bar{x})}$, where $F_\cdot^\varepsilon : \mathrm{X} \to C([0, 1], \mathrm{X})$ is the $\mathfrak{m}$-a.e. defined map which sends x to $t \mapsto F_t^\varepsilon(x)$, and observe that the bound (A.21) and the identity (A.22) provided by Theorem A.4 coupled with the estimates (4.9a), (4.9b) on $\nabla\varphi_t^\varepsilon$, $\Delta\varphi_t^\varepsilon$ and the fact that $\chi_R \in \mathrm{Test}^\infty(\mathrm{X})$ ensure that $\boldsymbol{\pi}^\varepsilon$ is a test plan with

$$\sup_{\varepsilon \in (0,1)} \iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma) < \infty, \quad (\mathrm{e}_t)_* \boldsymbol{\pi}^\varepsilon \le C\mathfrak{m} \quad \forall t \in [0, 1], \, \varepsilon \in (0, 1), \quad (5.17)$$

for some $C < \infty$. Now put $\tilde{\varphi}_t^\varepsilon := \chi_R \varphi_t^\varepsilon$ and notice that the definition of $\boldsymbol{\pi}^\varepsilon$ and (5.16) ensures that for every $t \in [0, 1]$ we have $\tilde{\varphi}_t^\varepsilon = \varphi_t^\varepsilon \, (\mathrm{e}_t)_* \boldsymbol{\pi}$-a.e. Moreover we have $(\tilde{\varphi}_t^\varepsilon) \in AC_{\mathrm{loc}}((0, 1], L^2(\mathrm{X})) \cap L_{\mathrm{loc}}^\infty((0, 1], W^{1,2}(\mathrm{X}))$, thus by Lemma 5.3 applied to $\boldsymbol{\pi}^\varepsilon$ and $t \mapsto \tilde{\varphi}_{(1-t)t_1 + tt_0}^\varepsilon$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi_{(1-t)t_1+tt_0}^\varepsilon \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon = \frac{\mathrm{d}}{\mathrm{d}t} \int \tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon$$

$$= \int (t_0 - t_1) \frac{\mathrm{d}}{\mathrm{d}s} \tilde{\varphi}_s^\varepsilon |_{s=(1-t)t_1+tt_0} \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon + \frac{\mathrm{d}}{\mathrm{d}s} \int \tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon \circ \mathrm{e}_s \, \mathrm{d}\boldsymbol{\pi}^\varepsilon |_{s=t}$$

$$= \int (t_0 - t_1) \frac{\mathrm{d}}{\mathrm{d}s} \varphi_s^\varepsilon |_{s=(1-t)t_1+tt_0} \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon + (t_1 - t_0) \int \mathrm{d}\tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon (X_t^\varepsilon) \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon$$

$$= \int \left(\frac{t_0 - t_1}{2} |\mathrm{d}\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon + (t_1 - t_0) |\mathrm{d}\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 \right) \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon$$

$$= \int \left(\frac{t_1 - t_0}{2} |\mathrm{d}\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon \right) \circ \mathrm{e}_t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon.$$

Integrating in time and recalling (A.22) we deduce

$$\int (\varphi_{t_0}^\varepsilon \circ \mathrm{e}_1 - \varphi_{t_1}^\varepsilon \circ \mathrm{e}_0) \, \mathrm{d}\boldsymbol{\pi}^\varepsilon = \iint_0^1 \left(\frac{1}{2(t_1-t_0)} |\dot{\gamma}_t|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon(\gamma_t) \right) \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}^\varepsilon(\gamma). \quad (5.18)$$

Now, as before, we let $\varepsilon \downarrow 0$ along the sequence (ε_n) for which $(\varphi_t^{\varepsilon_n})$ converges to our given (φ_t) in the sense of Proposition 5.2: the first property in (5.17) ensures that $(\boldsymbol{\pi}^\varepsilon)$ is tight in $\mathscr{P}(C([0, 1], \mathrm{X}))$ (because $\gamma \mapsto \int_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t$ has locally compact sublevels and $(\mathrm{e}_0)_* \boldsymbol{\pi}^\varepsilon = \mathfrak{m}(B_r(\bar{x}))^{-1} \mathfrak{m}_{|B_r(\bar{x})})$ and thus up to passing to a subsequence, not relabelled, we can assume that $(\boldsymbol{\pi}^{\varepsilon_n})$ weakly converges to some $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], \mathrm{X}))$. The second property in (5.17) and the bound (4.10) imply that the term with the Laplacian in (5.18) vanishes in the limit and thus taking into account the lower semicontinuity of the 2-energy we deduce that

$$\int (\varphi_{t_0} \circ \mathrm{e}_1 - \varphi_{t_1} \circ \mathrm{e}_0) \, \mathrm{d}\boldsymbol{\pi} \ge \frac{1}{2(t_1-t_0)} \iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi} \ge \frac{1}{2(t_1-t_0)} \int \mathrm{d}^2(\gamma_0, \gamma_1) \, \mathrm{d}\boldsymbol{\pi}(\gamma).$$

Now notice that (5.15) implies that

$$\frac{\mathrm{d}^2(\gamma_0, \gamma_1)}{2(t_1 - t_0)} \ge \varphi_{t_0}(\gamma_1) - \varphi_{t_1}(\gamma_0) \quad (5.19)$$

for any curve γ, hence the above gives

$$\int (\varphi_{t_0} \circ e_1 - \varphi_{t_1} \circ e_0)\, d\boldsymbol{\pi} \geq \frac{1}{2(t_1 - t_0)} \int d^2(\gamma_0, \gamma_1)\, d\boldsymbol{\pi}(\gamma) \geq \int (\varphi_{t_0} \circ e_1 - \varphi_{t_1} \circ e_0)\, d\boldsymbol{\pi},$$

thus forcing the inequalities to be equalities. In particular, equality in (5.19) holds for $\boldsymbol{\pi}$-a.e. γ and since $(e_0)_* \boldsymbol{\pi} = \mathfrak{m}|_{B_r(\bar{x})}$, this is the same as saying that for $\mathfrak{m}$-a.e. $y \in B_r(\bar{x})$ equality holds in (5.15). Since both sides of (5.15) are continuous in y, we deduce that equality holds for any $y \in B_r(\bar{x})$ and the arbitrariness of r allows us to conclude that equality actually holds for any $y \in X$.

Other properties of φ_t. From Proposition 5.2 we already know that, for any $\bar{x} \in X$ and $M > 0$, $(\varphi_t) \in AC_{\mathrm{loc}}((0, 1], L^1(X, e^{-V}\mathfrak{m})) \cap L^\infty_{\mathrm{loc}}((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$, where $V := M d^2(\cdot, \bar{x})$. Since φ_t is a real-valued function for all $t \in (0, 1]$, (5.12a) tells us that for all $x \in X$, $t \mapsto \varphi_t(x)$ satisfies (A.26) for a.e. $t \in (0, 1]$; taking (5.4) into account, this implies that for all $\delta \in (0, 1)$ and $t_0, t_1 \in [\delta, 1]$ with $t_0 < t_1$,

$$\|\varphi_{t_1} - \varphi_{t_0}\|_{C(X, e^{-V})} \leq \sup_{x \in X} \int_{t_0}^{t_1} \frac{(\mathrm{lip}(\varphi_t)(x))^2}{2}\, dt \leq \left(\sup_{x \in X} C_\delta (1 + d(x, \bar{x})) e^{-V(x)} \right) |t_1 - t_0|,$$

whence $(\varphi_t) \in AC_{\mathrm{loc}}((0, 1], C(X, e^{-V})) \cap L^\infty_{\mathrm{loc}}((0, 1], W^{1,2}(X, e^{-V}\mathfrak{m}))$.

Extracting a further subsequence—not relabelled—we can assume that the curves $(\mu_t^{\varepsilon_n})$ converge to a limit curve (μ_t) as in Proposition 5.1. We claim that for any $t_0, t_1 \in (0, 1]$ with $t_0 < t_1$,

$$-\int \varphi_{t_1}\, d\mu_{t_1} + \int \varphi_{t_0}\, d\mu_{t_0} \geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}) \tag{5.20}$$

and start by observing that since $(\varphi_t) \in C((0, 1], C(X, e^{-V}))$ and $(\mu_t) \in AC([0, 1], (\mathscr{P}_2(X), W_2))$, by the uniform estimates (5.3) both sides in (5.20) are continuous in t_0, t_1, hence it is sufficient to prove (5.20) for $t_0, t_1 \in (0, 1)$.

Now fix $\bar{x} \in X$ and $R > 0$, let $\chi_R \in \mathrm{Test}^\infty(X)$ be a cut-off function as in Lemma A.2 and observe that by Proposition 4.3, $t \mapsto \int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon\, d\mathfrak{m}$ belongs to $C((0, 1]) \cap AC_{\mathrm{loc}}((0, 1))$ with

$$-\frac{d}{dt} \int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon\, d\mathfrak{m}$$

$$= \int \left(\chi_R \left(-\frac{|\nabla \varphi_t^\varepsilon|^2}{2} - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon - \langle \nabla \varphi_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \right) \rho_t^\varepsilon + \varphi_t^\varepsilon \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon \right) d\mathfrak{m}$$

$$= \int \left(\chi_R \left(\frac{|\nabla \vartheta_t^\varepsilon|^2}{2} - \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \right) \rho_t^\varepsilon + \varphi_t^\varepsilon \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon \right) d\mathfrak{m} \tag{5.21}$$

for a.e. $t \in (0, 1)$, where we have also used the identity $\varphi_t^\varepsilon = \frac{\varepsilon}{2} \log \rho_t^\varepsilon - \vartheta_t^\varepsilon$. By (5.3), (4.26a) and Lemma 4.5 it is readily verified that $\int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon\, d\mathfrak{m} \to \int \varphi_t^\varepsilon \rho_t^\varepsilon\, d\mathfrak{m}$ as $R \to \infty$

for any $t \in (0, 1)$ and the right side of (5.21) converges as $R \to \infty$ locally uniformly in $t \in (0, 1)$. Hence after an integration in t and letting $R \to \infty$ in (5.21) we obtain

$$-\int \varphi_{t_1}^{\varepsilon} \, d\mu_{t_1}^{\varepsilon} + \int \varphi_{t_0}^{\varepsilon} \, d\mu_{t_0}^{\varepsilon} = \iint_{t_0}^{t_1} \left(\frac{|\nabla \vartheta_t^{\varepsilon}|^2}{2} - \frac{\varepsilon^2}{8} |\nabla \log \rho_t^{\varepsilon}|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^{\varepsilon} \right) \rho_t^{\varepsilon} \, dt \, d\mathfrak{m}.$$

As already noticed in the proof of Proposition 5.1, (μ_t^{ε}) and $(\nabla \vartheta_t^{\varepsilon})$ satisfy the assumptions of Theorem A.5, thus from the theorem we deduce that

$$\iint_{t_0}^{t_1} \frac{|\nabla \vartheta_t^{\varepsilon}|^2}{2} \rho_t^{\varepsilon} \, dt \, d\mathfrak{m} = \frac{1}{2} \int_{t_0}^{t_1} |\dot{\mu}_t^{\varepsilon}|^2 \, dt \geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}^{\varepsilon}, \mu_{t_1}^{\varepsilon}).$$

Therefore

$$-\int \varphi_{t_1}^{\varepsilon} \, d\mu_{t_1}^{\varepsilon} + \int \varphi_{t_0}^{\varepsilon} \, d\mu_{t_0}^{\varepsilon}$$

$$\geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}^{\varepsilon}, \mu_{t_1}^{\varepsilon}) + \iint_{t_0}^{t_1} \left(-\frac{\varepsilon^2}{8} |\nabla \log \rho_t^{\varepsilon}|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^{\varepsilon} \right) \rho_t^{\varepsilon} \, dt \, d\mathfrak{m}.$$

We now pass to the limit as $\varepsilon = \varepsilon_n \downarrow 0$: we know from Proposition 5.1 that $W_2(\mu_t^{\varepsilon_n}, \mu_t) \to 0$ and together with (5.3) this also shows that the left-hand side trivially converges to the left-hand side of (5.20). The contribution of the term with $|\nabla \log \rho_t^{\varepsilon}|$ vanishes by (4.29b) and so does the one with $\Delta \varphi_t^{\varepsilon}$ by (4.10) and (4.17). Hence (5.20) is proved.

Now notice that (5.12a) can be rewritten as

$$-(t_1 - t_0)\varphi_{t_1} = ((t_1 - t_0)\varphi_{t_0})^c,$$

so that in particular $-(t_1 - t_0)\varphi_{t_1}$ is c-concave and $(-(t_1 - t_0)\varphi_{t_1})^c \geq (t_1 - t_0)\varphi_{t_0}$. Hence both (5.12b) and the fact that $-(t_1 - t_0)\varphi_{t_1}$ is a Kantorovich potential follow from

$$\frac{1}{2} W_2^2(\mu_{t_0}, \mu_{t_1}) \geq \int -(t_1 - t_0)\varphi_{t_1} \, d\mu_{t_1} + \int (-(t_1 - t_0)\varphi_{t_1})^c \, d\mu_{t_0}$$

$$\geq \int -(t_1 - t_0)\varphi_{t_1} \, d\mu_{t_1} + \int (t_1 - t_0)\varphi_{t_0} \, d\mu_{t_0} \overset{(5.20)}{\geq} \frac{1}{2} W_2^2(\mu_{t_0}, \mu_{t_1}).$$

Then (5.13b) and the other claims about (ψ_t) are proved in the same way.

(μ_t) **is a geodesic.** Let $[t_0, t_1] \subset (0, 1)$, pick $t \in [0, 1]$ and put $t_0' := (1 - t)t_1 + t t_0$. We know that $-(t_1 - t_0)\varphi_{t_1}$ and $-t(t_1 - t_0)\varphi_{t_1}$ are Kantorovich potentials from μ_{t_1} to μ_{t_0} and from μ_{t_1} to $\mu_{t_0'}$ respectively and thus by Theorem A.6(ii) we deduce

$$W_2^2(\mu_{t_0}, \mu_{t_1}) = \int |d((t_1 - t_0)\varphi_{t_1})|^2 \, d\mu_{t_1}$$

$$= \frac{1}{t^2} \int |d((t_1 - t_0')\varphi_{t_1})|^2 \, d\mu_{t_1} = \frac{(t_1 - t_0)^2}{(t_1 - t_0')^2} W_2^2(\mu_{t_1}, \mu_{t_0'}).$$

Swapping the roles of t_0, t_1 and using the ψ's in place of the φ's we then get

$$W_2(\mu_{t_1'}, \mu_{t_0'}) = \frac{t_1' - t_0'}{t_1 - t_0} W_2(\mu_{t_1}, \mu_{t_0}) \quad \forall [t_0', t_1'] \subset [t_0, t_1] \subset (0, 1).$$

This shows that the restriction of (μ_t) to any interval $[t_0, t_1] \subset (0, 1)$ is a constant speed geodesic. Since (μ_t) is continuous on the whole $[0, 1]$, this gives the conclusion. Since in this situation the W_2-geodesic connecting μ_0 to μ_1 is unique (recall Theorem A.6(i)), by the arbitrariness of the subsequences chosen we also deduce the uniqueness of the limit curve (μ_t). $\qquad\square$

Remark 5.5 (The vanishing viscosity limit). The part of the last proposition concerning the properties of the φ_t^ε's is valid in a context wider than the one provided by the Schrödinger problem: we could restate the result by saying that if (φ_t^ε) solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^\varepsilon = \frac{1}{2}|\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon \tag{5.22}$$

and φ_0^ε uniformly converges to some φ_0, then φ_t^ε uniformly converges to $\varphi_t := -Q_t(-\varphi_0)$.

In this connection, it is worth recalling that in [2] and [24] a theory has been developed of viscosity solutions for some first order Hamilton–Jacobi equations on metric spaces. This theory applies in particular to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = \frac{1}{2}\mathrm{lip}(\varphi_t)^2 \tag{5.23}$$

whose only viscosity solution is given by the formula $\varphi_t := -Q_t(-\varphi_0)$.

Therefore, we have just proved that if one works not only on a metric space, but on a metric measure space which is an $\mathrm{RCD}^*(K, N)$ space, then the solutions of the viscous approximation (5.22) converge to the unique viscosity solution of (5.23), in accordance with the classical case.

Remark 5.6. It is not clear whether the 'full' families φ_t^ε, ψ_t^ε converge as $\varepsilon \downarrow 0$ to a unique limit. This is related to the non-uniqueness of the Kantorovich potentials in the classical optimal transport problem.

We shall now make use of the following lemma. It could be directly deduced from the results obtained by Cheeger [16]; however, the additional regularity assumptions on both the space and the function allow for a 'softer' argument based on the metric Brenier's theorem, which we propose.

Lemma 5.7. *Let* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$ *and let* $\phi : Y \to \mathbb{R} \cup \{-\infty\}$ *be a c-concave function not identically* $-\infty$. *Let* Ω *be the interior of the set* $\{\phi > -\infty\}$. *Then* ϕ *is locally Lipschitz on* Ω *and*

$$\mathrm{lip}(\phi) = |\mathrm{d}\phi| \quad \mathfrak{m}\text{-a.e. on } \Omega.$$

Proof. Lemma 3.3 in [36] shows that ϕ is locally Lipschitz on Ω and that $\partial^c\phi(x) \neq \emptyset$ for every $x \in \Omega$. The same lemma also implies that for $K \subset \Omega$ compact, the set $\bigcup_{x \in K} \partial^c\phi(x)$ is bounded. Recalling that $\partial^c\phi$ is the set of $(x, y) \in Y^2$ such that

$$\phi(x) + \phi^c(y) = \tfrac{1}{2}d^2(x, y)$$

and that ϕ, ϕ^c are upper semicontinuous, we see that $\partial^c\phi$ is closed. Hence for $K \subset \Omega$ compact the set $\bigcup_{x \in K} \partial^c\phi(x)$ is compact and non-empty and thus by the Kuratowski–Ryll-Nardzewski Borel selection theorem we deduce the existence of a Borel map $T : \Omega \to Y$ such that $T(x) \in \partial^c\phi(x)$ for every $x \in \Omega$.

Pick $\mu \in \mathscr{P}_2(Y)$ with $\mathrm{supp}(\mu) \subset\subset \Omega$ and $\mu \leq C\mathfrak{m}$ for some $C > 0$ and set $\nu := T_*\mu$. By construction, μ, ν both have bounded support, T is an optimal map and ϕ is a Kantorovich potential from μ to ν.

Hence Theorem A.6(iii) applies and since $\mathrm{lip}(\phi) = \max\{|D^+\phi|, |D^-\phi|\}$, by the arbitrariness of μ it is sufficient to show that $|D^+\phi| = |D^-\phi|$ $\mathfrak{m}$-a.e. This easily follows from the fact that $\mathfrak{m}$ is doubling and ϕ is Lipschitz [5, Proposition 2.7]. □

We can now show that the weighted energies of the Schrödinger potentials converge to the weighted energy of the limit ones:

Proposition 5.8. *Under Setting* 4.1*, the following holds. Let $\varepsilon_n \downarrow 0$ be a sequence such that $(\varphi_t^{\varepsilon_n})$, $(\psi_t^{\varepsilon_n})$ converge to limit curves (φ_t), (ψ_t) as in Proposition* 5.2 *and let $V := Md^2(\cdot, \bar{x})$ with $\bar{x} \in X$ and $M > 0$ arbitrary. Then for every $\delta \in (0, 1)$ we have*

$$\begin{aligned}
\lim_{n \to \infty} \iint_\delta^1 e^{-V} |d\varphi_t^{\varepsilon_n}|^2 \, dt \, d\mathfrak{m} &= \iint_\delta^1 e^{-V} |d\varphi_t|^2 \, dt \, d\mathfrak{m}, \\
\lim_{n \to \infty} \iint_0^{1-\delta} e^{-V} |d\psi_t^{\varepsilon_n}|^2 \, dt \, d\mathfrak{m} &= \iint_0^{1-\delta} e^{-V} |d\psi_t|^2 \, dt \, d\mathfrak{m}.
\end{aligned} \tag{5.24}$$

Proof. Fix $\delta \in (0, 1)$ and argue as in the proof of Proposition 5.2 to deduce that $t \mapsto \int e^{-V} \varphi_t^\varepsilon \, d\mathfrak{m}$ is absolutely continuous in $[\delta, 1]$ (see in particular (5.7)) and that

$$\int e^{-V}(\varphi_1^\varepsilon - \varphi_\delta^\varepsilon) \, d\mathfrak{m} = \frac{1}{2} \iint_\delta^1 e^{-V}(|d\varphi_t^\varepsilon|^2 + \varepsilon\Delta\varphi_t^\varepsilon) \, dt \, d\mathfrak{m}.$$

Choosing $\varepsilon := \varepsilon_n$, letting $n \to \infty$ and using the uniform bounds (4.10), (5.3) and the volume growth estimate (A.20) we obtain

$$\lim_{n \to \infty} \frac{1}{2} \iint_\delta^1 e^{-V} |d\varphi_t^{\varepsilon_n}|^2 \, dt \, d\mathfrak{m} = \lim_{n \to \infty} \int e^{-V}(\varphi_1^{\varepsilon_n} - \varphi_\delta^{\varepsilon_n}) \, d\mathfrak{m} = \int e^{-V}(\varphi_1 - \varphi_\delta) \, d\mathfrak{m}. \tag{5.25}$$

Combining (A.26) and (5.12a) we see that for any $x \in X$,

$$\frac{d}{dt}\varphi_t(x) = \frac{1}{2}(\mathrm{lip}(\varphi_t)(x))^2 \quad \text{a.e. } t \in [0, 1].$$

By Fubini's theorem, the same identity holds for $\mathscr{L}^1 \otimes \mathfrak{m}$-a.e. $(t, x) \in [\delta, 1] \times X$. The identity (5.12a) also shows that φ_t is a multiple of a c-concave function, thus the conclusion of Lemma 5.7 is valid for φ_t and recalling that $(\varphi_t) \in AC_{\text{loc}}((0, 1], L^1(X, e^{-V}\mathfrak{m}))$ by Proposition 5.2 we deduce that

$$\int e^{-V}(\varphi_1 - \varphi_\delta) \, d\mathfrak{m} = \int_\delta^1 \frac{d}{dt} \int e^{-V}\varphi_t \, d\mathfrak{m} \, dt = \iint_\delta^1 e^{-V} \frac{|d\varphi_t|^2}{2} \, dt \, d\mathfrak{m},$$

which together with (5.25) gives the first limit in (5.24). The proof of the second one is analogous. □

As a direct consequence of (5.24) and the local equi-Lipschitz bounds (4.9a) we obtain the following result. To state it, let us introduce $L^2(T^*X, e^{-V}\mathfrak{m}) = \{\omega \in L^0(T^*X) : |\omega| \in L^2(X, e^{-V}\mathfrak{m})\}$; an analogous definition can be given for $L^2((T^*)^{\otimes 2}X)$.

Corollary 5.9. *Under Setting 4.1, the following holds. Let $\varepsilon_n \downarrow 0$ be a sequence such that $(\varphi_t^{\varepsilon_n})$, $(\psi_t^{\varepsilon_n})$ converge to limit curves (φ_t), (ψ_t) as in Proposition 5.2. Then for every $\delta \in (0, 1)$, $\bar{x} \in X$ and $M > 0$ we have*

$$\begin{aligned}
(d\varphi_t^{\varepsilon_n}) &\to (d\varphi_t) &&\text{in } L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m})), \\
(d\psi_t^{\varepsilon_n}) &\to (d\psi_t) &&\text{in } L^2([0, 1-\delta], L^2(T^*X, e^{-V}\mathfrak{m})), \\
(d\varphi_t^{\varepsilon_n} \otimes d\varphi_t^{\varepsilon_n}) &\to (d\varphi_t \otimes d\varphi_t) &&\text{in } L^2([\delta, 1], L^2((T^*)^{\otimes 2}X, e^{-V}\mathfrak{m})), \\
(d\psi_t^{\varepsilon_n} \otimes d\psi_t^{\varepsilon_n}) &\to (d\psi_t \otimes d\psi_t) &&\text{in } L^2([0, 1-\delta], L^2((T^*)^{\otimes 2}X, e^{-V}\mathfrak{m})), \\
(d\varphi_t^{\varepsilon_n} \otimes d\psi_t^{\varepsilon_n}) &\to (d\varphi_t \otimes d\psi_t) &&\text{in } L^2([\delta, 1-\delta], L^2((T^*)^{\otimes 2}X, e^{-V}\mathfrak{m})),
\end{aligned} \tag{5.26}$$

where $V := M d^2(\cdot, \bar{x})$.

Proof. Closeness of the differential implies that $d\varphi_t^{\varepsilon_n} \rightharpoonup d\varphi_t$ in $L^2(T^*X, e^{-V}\mathfrak{m})$ for all $t \in (0, 1]$. This and the fact that $(d\varphi_t^{\varepsilon_n})$ is equibounded in $L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m}))$, as a direct consequence of (4.9a), ensure that $(d\varphi_t^{\varepsilon_n}) \rightharpoonup (d\varphi_t)$ in $L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m}))$. Given that the first limit in (5.24) gives convergence of the $L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m}))$ norms, we deduce strong convergence. This establishes the first limit.

Observe that for every $\omega \in L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m}))$ the fact that $e^{-V}|d\varphi_t^{\varepsilon_n}|$ is uniformly bounded in $L^\infty([\delta, 1] \times X)$ for every $M > 0$ in the definition of V and the strong L^2 convergence just proved ensure that $\langle d\varphi_t^{\varepsilon_n}, \omega_t \rangle \to \langle d\varphi_t, \omega_t \rangle$ in $L^2([\delta, 1] \times X, dt \otimes e^{-V}\mathfrak{m})$. It follows that for any $\omega_1, \omega_2 \in L^2([\delta, 1], L^2(T^*X, e^{-V}\mathfrak{m}))$ we have

$$\iint_\delta^1 e^{-V} \langle d\varphi_t^{\varepsilon_n}, \omega_{1,t} \rangle \langle d\varphi_t^{\varepsilon_n}, \omega_{2,t} \rangle \, dt \, d\mathfrak{m} \to \iint_\delta^1 e^{-V} \langle d\varphi_t, \omega_{1,t} \rangle \langle d\varphi_t, \omega_{2,t} \rangle \, dt \, d\mathfrak{m}$$

and thus it remains to prove that

$$\iint_\delta^1 e^{-V} |d\varphi_t^{\varepsilon_n} \otimes d\varphi_t^{\varepsilon_n}|_{\text{HS}}^2 \, dt \, d\mathfrak{m} \to \iint_\delta^1 e^{-V} |d\varphi_t \otimes d\varphi_t|_{\text{HS}}^2 \, dt \, d\mathfrak{m}.$$

Since $|v \otimes v|_{\text{HS}}^2 = |v|^4$ this is a direct consequence of the fact that $|d\varphi_t^{\varepsilon_n}|$ is uniformly bounded and converges to $|d\varphi_t|$ in $L^2([\delta, 1] \times X, dt \otimes e^{-V}\mathfrak{m})$. Hence also the third limit is established.

The other claims follow by analogous arguments. □

The estimates for the φ's tell us nothing about their regularity as $t \downarrow 0$ and similarly we know little so far about the ψ's for $t \uparrow 1$. We will now see in which sense the limit functions φ_0, ψ_1 exist. This is not needed for the proof of our main result, but we believe it is relevant on its own.

Thus fix $\varepsilon_n \downarrow 0$ such that $\varphi_t^{\varepsilon_n} \to \varphi_t$ for $t \in (0, 1]$ and $\psi_t^{\varepsilon_n} \to \psi_t$ for $t \in [0, 1)$ as in Proposition 5.2. Then define $\varphi_0, \psi_1 : X \to \mathbb{R} \cup \{-\infty\}$ as

$$\varphi_0(x) := \inf_{t \in (0, 1]} \varphi_t(x) = \lim_{t \downarrow 0} \varphi_t(x), \quad \psi_1(x) := \inf_{t \in [0, 1)} \psi_t(x) = \lim_{t \uparrow 1} \psi_t(x). \quad (5.27)$$

That the inf's are equal to the stated limits is a consequence of (5.12a), (5.13a), which directly imply that for every $x \in X$ the maps $t \mapsto \varphi_t(x)$ and $t \mapsto \psi_{1-t}(x)$ are non-decreasing.

The main properties of φ_0, ψ_1 are collected in the following proposition:

Proposition 5.10. *Under Setting 4.1 and for φ_0, ψ_1 defined by (5.27) the following hold.*

(i) *The functions $-\varphi_t$ (resp. $-\psi_t$) Γ-converge to $-\varphi_0$ (resp. $-\psi_1$) as $t \downarrow 0$ (resp. $t \uparrow 1$).*
(ii) *For every $t \in (0, 1]$ we have*

$$Q_t(-\varphi_0) = -\varphi_t, \quad Q_t(-\psi_1) = -\psi_{1-t}.$$

(iii) *We have*

$$\varphi_0(x) = \begin{cases} -\psi_0(x) & \text{if } x \in \text{supp}(\rho_0), \\ -\infty & \text{otherwise,} \end{cases} \quad \psi_1(x) = \begin{cases} -\varphi_1(x) & \text{if } x \in \text{supp}(\rho_1), \\ -\infty & \text{otherwise,} \end{cases}$$

(iv) *We have*

$$\int \varphi_0 \rho_0 \, d\mathfrak{m} + \int \psi_1 \rho_1 \, d\mathfrak{m} = \tfrac{1}{2} W_2^2(\mu_0, \mu_1).$$

(v) *Define φ_0^ε on $\{\rho_0 > 0\}$ as $\varphi_0^\varepsilon := \varepsilon \log(f^\varepsilon)$ and let $\varepsilon_n \downarrow 0$ be such that $\varphi_t^{\varepsilon_n}, \psi_t^{\varepsilon_n}$ converge to φ_t, ψ_t as $n \to \infty$ as in Proposition 5.2. Then the functions $\rho_0 \varphi_0^{\varepsilon_n}$, set to be 0 on $X \setminus \{\rho_0 > 0\}$, converge to $\rho_0 \varphi_0$ in $L^\infty(X)$ as $n \to \infty$. With the analogous definition of $\rho_1 \psi_1^{\varepsilon_n}$, these converge to $\rho_1 \psi_1$ in $L^\infty(X)$ as $n \to \infty$.*

Proof. We shall prove the claims for φ_0 only, as those for ψ_1 follow along similar lines.

(i) For the $\Gamma\text{-}\overline{\lim}$ inequality we simply observe that by definition $-\varphi_0(x) = \lim_{t \downarrow 0} -\varphi_t(x)$. To prove the $\Gamma\text{-}\underline{\lim}$ inequality, use the fact that $-\varphi_t \geq -\varphi_s$ for $0 < t \leq s$ and the continuity of φ_s: for given (x_t) converging to x we have

$$\underline{\lim_{t \downarrow 0}} -\varphi_t(x_t) \geq \underline{\lim_{t \downarrow 0}} -\varphi_s(x_t) = -\varphi_s(x) \quad \forall s > 0.$$

The conclusion follows by letting $s \downarrow 0$.

(ii) From $-\varphi_0 \geq -\varphi_s$ we deduce that

$$Q_t(-\varphi_0) \geq Q_t(-\varphi_s) \overset{(5.12a)}{=} -\varphi_{t+s} \quad \forall s \in (0, 1]$$

and thus letting $s \downarrow 0$ and using the continuity of $(0, 1] \ni t \mapsto \varphi_t(x)$ for all $x \in X$ we obtain $Q_t(-\varphi_0)(x) \geq -\varphi_t(x)$ for all $x \in X$. For the opposite inequality, notice that the right inequality of (4.3) gives

$$\varphi_t^\varepsilon \leq \varepsilon \log C_4 - \varepsilon \log v_{\varepsilon t/2} + \varepsilon \log \|f^\varepsilon\|_{L^1(X)} - \frac{C_5 d^2(\cdot, \bar{x})}{t} + \frac{C_6}{t} \tag{5.28}$$

for all $t \in (0, 1]$ with C_4, C_5, C_6 depending on $K, N, \rho_0, \rho_1, \bar{x}$ only and $v_{\varepsilon t/2}$ as in (4.2). We now claim that for every $\varepsilon \in (0, 1)$ we have

$$\varepsilon \log v_\varepsilon \geq -C, \qquad \varepsilon \log \|f^\varepsilon\|_{L^1(X)} \leq C \tag{5.29}$$

for some constant C depending on $K, N, \rho_0, \rho_1, \bar{x}$ only. Indeed, from (A.19) we see that letting D be the diameter of $\mathrm{supp}(\rho_0)$ and $c = c(D)$ a constant depending only on D we have

$$\mathfrak{m}(B_{\sqrt{\varepsilon}}(x)) \geq c^{\log_2(D/\sqrt{\varepsilon})+1} \mathfrak{m}(\mathrm{supp}(\rho_0)) \quad \forall x \in \mathrm{supp}(\rho_0).$$

Thus $v_\varepsilon \geq C^{\log_2(D/\sqrt{\varepsilon})+1} \mathfrak{m}(\mathrm{supp}(\rho_0))$ and thus the first inequality in (5.29) follows. For the second one we start by noticing that the first inequality in (A.5), the identity $\int f^\varepsilon \otimes g^\varepsilon \, d\mathsf{R}^{\varepsilon/2} = 1$ and the fact that the supports of $f^\varepsilon, g^\varepsilon$ coincide with those of ρ_0, ρ_1 respectively give

$$\varepsilon \log(\|f^\varepsilon\|_{L^1(X)} \|g^\varepsilon\|_{L^1(X)}) = \varepsilon \log \int_{\mathrm{supp}(\rho_0) \times \mathrm{supp}(\rho_1)} f^\varepsilon \otimes g^\varepsilon \, d(\mathfrak{m} \otimes \mathfrak{m})$$
$$\leq \varepsilon \log(C_1 \mathfrak{m}(B)) + \bar{D}^2 + C_2 \varepsilon^2 \tag{5.30}$$

for every $\varepsilon \in (0, 1)$, where $\bar{D} = \mathrm{diam}(\mathrm{supp}(\rho_0) \cup \mathrm{supp}(\rho_1))$ and B is the 1-neighbourhood of $\mathrm{supp}(\rho_0) \cup \mathrm{supp}(\rho_1)$. Then recall the normalization (4.1), the identity $\log \rho_1 = \log g^\varepsilon + \log(\mathsf{h}_{\varepsilon/2} f^\varepsilon)$ and use Jensens' inequality for $-\log$ to obtain

$$H(\mu_1 \mid \mathfrak{m}) = \int \rho_1 \log \rho_1 \, d\mathfrak{m} = \int \log(g^\varepsilon) \rho_1 \, d\mathfrak{m} \leq \log \int g^\varepsilon \rho_1 \, d\mathfrak{m}$$
$$\leq \log(\|g^\varepsilon\|_{L^1(X)} \|\rho_1\|_{L^\infty(X)}),$$

whence $\log \|g^\varepsilon\|_{L^1(X)} \geq H(\mu_1 \mid \mathfrak{m}) - \log \|\rho_1\|_{L^\infty(X)}$ for all $\varepsilon \in (0, 1)$, which together with (5.30) gives the second inequality in (5.29).

Therefore passing to the limit in (5.28) as $\varepsilon = \varepsilon_n \downarrow 0$ and recalling the local uniform convergence of $\varphi_t^{\varepsilon_n}$ to φ_t gives $-\varphi_t \geq -\tilde{C}/t + C_5 d^2(\cdot, \bar{x})/t$ for every $t \in (0, 1]$, where $\tilde{C} \geq 0$ depends on $K, N, \rho_0, \rho_1, \bar{x}$ only. It follows that

$$-\varphi_t \geq \frac{C_5 d^2(\cdot, \bar{x})}{2t} \geq \frac{C_5}{2} d^2(\cdot, \bar{x}) \quad \forall t \in (0, 1], \ x \notin B_{\sqrt{2C_5/\tilde{C}}}(\bar{x}). \tag{5.31}$$

Now fix $x \in X$ and a sequence $t_n \downarrow 0$; the bound (5.31) implies that there are $y_n \in X$ such that

$$Q_t(-\varphi_{t_n})(x) = \frac{d^2(x, y_n)}{2t} - \varphi_{t_n}(y_n)$$

and that these y_n range in a bounded set. Thus up to a subsequence we can assume that $y_n \to y$ for some $y \in X$, so that taking into account the $\Gamma\text{-}\underline{\lim}$ inequality previously proved we get

$$
\frac{\mathrm{d}^2(x, y)}{2t} - \varphi_0(y) \leq \varliminf_{n \to \infty} \frac{\mathrm{d}^2(x, y_n)}{2t} - \varphi_{t_n}(y_n) = \varliminf_{n \to \infty} Q_t(-\varphi_{t_n})(x)
$$

$$
\overset{(5.12a)}{=} \varliminf_{n \to \infty} -\varphi_{t_n+t}(x) = -\varphi_t(x),
$$

which shows that $Q_t(-\varphi_0)(x) \leq -\varphi_t(x)$, as desired.

(iii) For any $t \in (0, 1]$ we have $\varphi_0 \leq \varphi_t \overset{(5.5)}{\leq} -\psi_t$, so that letting $t \downarrow 0$ and using the continuity of $[0, 1) \ni t \mapsto \psi_t(x)$ for all $x \in X$ we deduce that

$$
\varphi_0 \leq -\psi_0 \quad \text{on X.}
$$

Now notice that the fact that $-\varphi_0 \leq \Gamma\text{-}\underline{\lim}(-\varphi_t)$ implies that

$$
\varphi_0(\gamma_0) \geq \varlimsup_{t \downarrow 0} \varphi_t(\gamma_t) \quad \forall \gamma \in C([0, 1], X). \tag{5.32}
$$

Let $\boldsymbol{\pi}$ be the lifting of the W_2-geodesic (μ_t) (recall Theorem A.6(i)); taking into account that the evaluation maps $\mathrm{e}_t : C([0, 1], X) \to X$ are continuous and that $\mathrm{supp}(\boldsymbol{\pi})$ is a compact subset of $C([0, 1], X)$, being given by constant speed geodesics running from the compact set $\mathrm{supp}(\rho_0)$ to the compact $\mathrm{supp}(\rho_1)$, it is easy to see that for every $\gamma \in \mathrm{supp}(\boldsymbol{\pi})$ and $t \in [0, 1]$ we have $\gamma_t \in \mathrm{supp}(\mu_t)$, and vice versa, for every $x \in \mathrm{supp}(\mu_t)$ there is $\gamma \in \mathrm{supp}(\boldsymbol{\pi})$ with $\gamma_t = x$.

Thus let $x \in \mathrm{supp}(\rho_0) = \mathrm{supp}(\mu_0)$ and find $\gamma \in \mathrm{supp}(\boldsymbol{\pi})$ with $\gamma_0 = x$; from the fact that $\gamma_t \in \mathrm{supp}(\mu_t)$ and (5.5) we get

$$
\varphi_0(x) \overset{(5.32)}{\geq} \varlimsup_{t \downarrow 0} \varphi_t(\gamma_t) = \varlimsup_{t \downarrow 0} -\psi_t(\gamma_t),
$$

and since the continuity of $[0, 1) \ni t \mapsto \psi_t \in L^1(X, e^{-V}\mathfrak{m})$ and the uniform local Lipschitz continuity of the ψ_t's (both coming from Proposition 5.2) imply local uniform convergence of ψ_t to ψ_0, we conclude $\varphi_0(x) \geq \psi_0(x)$.

Thus it remains to prove that $\varphi_0 = -\infty$ outside $\mathrm{supp}(\rho_0)$. To this end, we notice again that the supports of f^ε, g^ε coincide with those of ρ_0, ρ_1 and use the second inequality in (A.5) to get

$$
f_t^\varepsilon(x) = \mathsf{h}_{\varepsilon t/2} f^\varepsilon(x) = \int f^\varepsilon(y) \mathsf{r}_{\varepsilon t/2}(x, y) \, \mathrm{d}\mathfrak{m}(y) \leq \frac{c_1}{v_{\varepsilon t/2}} e^{-c_2 \frac{\mathrm{d}^2(x, \mathrm{supp}(\rho_0))}{3\varepsilon t} + c_3 \varepsilon t} \int f^\varepsilon \, \mathrm{d}\mathfrak{m},
$$

$$
g_t^\varepsilon(x) = \mathsf{h}_{\varepsilon(1-t)/2} g^\varepsilon(x) = \int g^\varepsilon(y) \mathsf{r}_{\varepsilon(1-t)/2}(x, y) \, \mathrm{d}\mathfrak{m}(y) \leq \frac{c_4}{v_{\varepsilon(1-t)/2}} \int g^\varepsilon \, \mathrm{d}\mathfrak{m},
$$

for every $t \in (0, 1)$ and constants $c_i > 0$ depending on K, N, ρ_0, ρ_1, $\bar{x}$ only. From these bounds, the identity $\rho_t^\varepsilon = f_t^\varepsilon g_t^\varepsilon$ and the estimates (5.30) and (5.29) we deduce that

$$
\varlimsup_{\varepsilon \downarrow 0} \varepsilon \log(\rho_t^\varepsilon(x)) \leq c_5 - c_6 \frac{\mathrm{d}^2(x, \mathrm{supp}(\rho_0))}{t} \quad \forall x \in X, \ t \in (0, 1). \tag{5.33}
$$

Now let $\varepsilon_n \downarrow 0$ be such that $\varphi_t^{\varepsilon_n}, \psi_t^{\varepsilon_n}$ converge to φ_t, ψ_t as in Proposition 5.2 and put $S(x) := \sup_{\varepsilon \in (0,1), t \in [0,1/2]} |\psi_t^\varepsilon(x)| < \infty$ (recall (5.3)). The inequality

$$\varphi_t(x) = \lim_{n \to \infty} \varphi_t^{\varepsilon_n}(x) \leq \overline{\lim_{n \to \infty}} \varepsilon_n \log(\rho_t^{\varepsilon_n}(x)) - \lim_{n \to \infty} \psi_t^{\varepsilon_n}(x)$$

$$\overset{(5.33)}{\leq} S(x) + c_5 - c_6 \frac{\mathrm{d}^2(x, \operatorname{supp}(\rho_0))}{t}$$

shows that if $x \notin \operatorname{supp}(\rho_0)$ we have $\varphi_0(x) = \lim_{t \downarrow 0} \varphi_t(x) = -\infty$, as desired.

(iv) By (iii) we have

$$\int \varphi_0 \rho_0 \, \mathrm{d}\mathfrak{m} + \int \psi_1 \rho_1 \, \mathrm{d}\mathfrak{m} = - \int \psi_0 \rho_0 \, \mathrm{d}\mathfrak{m} - \int \varphi_1 \rho_1 \, \mathrm{d}\mathfrak{m},$$

so that taking into account the weak continuity of $t \mapsto \mu_t$, the fact that the measures μ_t have equibounded supports and the continuity of $t \mapsto \varphi_t$ (resp. $t \mapsto \psi_t$) for t close to 1 (resp. close to 0) in the topology of local uniform convergence (direct consequence of the continuity in $L^1(X, e^{-V}\mathfrak{m})$ and the uniform local Lipschitz estimates provided by Proposition 5.2), we get

$$\int \varphi_0 \rho_0 \, \mathrm{d}\mathfrak{m} + \int \psi_1 \rho_1 \, \mathrm{d}\mathfrak{m} = \lim_{t \downarrow 0} - \int \psi_t \rho_t \, \mathrm{d}\mathfrak{m} - \int \varphi_{1-t} \rho_{1-t} \, \mathrm{d}\mathfrak{m}$$

$$\overset{(5.5)}{=} \lim_{t \downarrow 0} \int \varphi_t \rho_t \, \mathrm{d}\mathfrak{m} - \int \varphi_{1-t} \rho_{1-t} \, \mathrm{d}\mathfrak{m} \overset{(5.12b)}{=} \tfrac{1}{2} W_2^2(\mu_0, \mu_1).$$

(v) Since $\rho_0 \in L^\infty(X)$, we also have $\rho_0 \log(\rho_0) \in L^\infty(X)$. The claim then follows from the identity $\rho_0 \varphi_0^\varepsilon = \varepsilon \rho_0 \log \rho_0 - \rho_0 \psi_0^\varepsilon$, the compactness of $\operatorname{supp}(\rho_0)$, the local uniform convergence of $\psi_0^{\varepsilon_n}$ to ψ_0 as $n \to \infty$ and the fact that $\psi_0 = -\varphi_0$ on $\operatorname{supp}(\rho_0)$. $\qquad \square$

Remark 5.11 (Entropic and transportation cost). For $\varepsilon > 0$ the *entropic cost* from ρ_0 to ρ_1 is defined as

$$\mathcal{T}_\varepsilon(\rho_0, \rho_1) := \inf H(\boldsymbol{\gamma} \,|\, \mathsf{R}^{\varepsilon/2}),$$

the infimum being taken over all transport plans $\boldsymbol{\gamma}$ from $\mu_0 := \rho_0 \mathfrak{m}$ to $\mu_1 := \rho_1 \mathfrak{m}$. Hence with our notation

$$\mathcal{T}_\varepsilon(\rho_0, \rho_1) = H(f^\varepsilon \otimes g^\varepsilon \mathsf{R}^{\varepsilon/2} \,|\, \mathsf{R}^{\varepsilon/2})$$

$$= \frac{1}{\varepsilon} \int \varphi_0^\varepsilon \oplus \psi_1^\varepsilon f^\varepsilon \otimes g^\varepsilon \, \mathrm{d}\mathsf{R}^{\varepsilon/2} = \frac{1}{\varepsilon} \left(\int \varphi_0^\varepsilon \rho_0 \, \mathrm{d}\mathfrak{m} + \int \psi_1^\varepsilon \rho_1 \, \mathrm{d}\mathfrak{m} \right)$$

and by (iv), (v) of the previous proposition we get

$$\lim_{\varepsilon \downarrow 0} \varepsilon \, \mathcal{T}_\varepsilon(\rho_0, \rho_1) = \tfrac{1}{2} W_2^2(\mu_0, \mu_1).$$

In other words, after the natural rescaling the entropic cost converges to the quadratic transportation cost, thus establishing another link between the Schrödinger problem and the transport problem.

We emphasize that although this argument is new, the result is not, not even on RCD*(K, N) spaces: Léonard [46] proved that the same limit holds in a very abstract setting provided the heat kernel satisfies the appropriate large deviation principle $\varepsilon \log(\mathsf{r}_{\varepsilon/2}[x](y)) \sim -\mathsf{d}^2(x, y)/2$. Since recently such asymptotic behaviour for the heat kernel on RCD*(K, N) spaces has been proved by Jiang–Li–Zhang [41], Léonard's result applies. Thus in this remark we simply wanted to show an alternative proof of that limiting property.

5.3. Proof of the main theorem

We start with the following simple continuity statement:

Lemma 5.12. *Under Setting* 4.1, *let* $t \mapsto \mu_t = \rho_t \mathfrak{m}$ *be the* W_2-*geodesic from* μ_0 *to* μ_1 *and* $(\varphi_t)_{t \in (0,1]}$ *and* $(\psi_t)_{t \in [0,1)}$ *any couple of limit functions given by Proposition* 5.2. *Then the maps*

$$(0, 1] \ni t \mapsto \rho_t \, \mathrm{d}\varphi_t \in L^2(T^*X),$$

$$[0, 1) \ni t \mapsto \rho_t \, \mathrm{d}\psi_t \in L^2(T^*X),$$

$$(0, 1] \ni t \mapsto \rho_t \, \mathrm{d}\varphi_t \otimes \mathrm{d}\varphi_\in L^2((T^*)^{\otimes 2}X),$$

$$[0, 1) \ni t \mapsto \rho_t \, \mathrm{d}\psi_t \otimes \mathrm{d}\psi_\in L^2((T^*)^{\otimes 2}X)$$

are all continuous in the strong topologies.

Proof. By Lemma A.8, for any $p < \infty$ we have $\rho_s \to \rho_t$ in $L^p(X)$ as $s \to t$ and in particular $\sqrt{\rho_s} \to \sqrt{\rho_t}$ as $s \to t$. Moreover, the compactness of the supports of ρ_0 and ρ_1 implies that there exist $\bar{x} \in X$ and $R > 0$ such that $\mathrm{supp}(\rho_t) \subset B_R(\bar{x})$ for all $t \in [0, 1]$. Consider a Lipschitz cut-off function χ with support in $B_{R+1}(\bar{x})$ such that $\chi \equiv 1$ in $B_R(\bar{x})$. The closedness of the differential and the fact that $\varphi_s \to \varphi_t$ weakly in $W^{1,2}(X, e^{-V}\mathfrak{m})$ as $s \to t > 0$ (as a consequence of $(\varphi_t) \in C((0, 1], C(X, e^{-V})) \cap L^\infty_{\mathrm{loc}}((0, 1), W^{1,2}(X, e^{-V}\mathfrak{m}))$, see Proposition 5.4 and the notation therein) imply that $\mathrm{d}\varphi_s \to \mathrm{d}\varphi_t$ weakly in $L^2(T^*X, e^{-V}\mathfrak{m})$ and thus $\chi \mathrm{d}\varphi_s \to \chi \mathrm{d}\varphi_t$ in $L^2(T^*X)$. Together with the previous claim about the densities, the fact that the latter are uniformly bounded in $L^\infty(X)$ and how χ is constructed, this is sufficient to conclude that $t \mapsto \sqrt{\rho_t} \, \mathrm{d}\varphi_t \in L^2(T^*X)$ is weakly continuous.

We now claim that $t \mapsto \sqrt{\rho_t} \, \mathrm{d}\varphi_t \in L^2(T^*X)$ is strongly continuous; to prove this, we show that their $L^2(T^*X)$ norms are constant. Recall that by Proposition 5.4, for $t \in (0, 1]$ the function $-(1 - t)\psi_t$ is a Kantorovich potential from μ_t to μ_1, while from (5.5) and the locality of the differential we get $|\mathrm{d}\varphi_t| = |\mathrm{d}\psi_t|$ μ_t-a.e., thus by Theorem A.6(iii) we have

$$\int |\mathrm{d}\varphi_t|^2 \rho_t \, \mathrm{d}\mathfrak{m} = \frac{1}{(1-t)^2} \int |\mathrm{d}(1-t)\psi_t|^2 \rho_t \, \mathrm{d}\mathfrak{m} = \frac{1}{(1-t)^2} W_2^2(\mu_t, \mu_1) = W_2^2(\mu_0, \mu_1).$$

Multiplying $\sqrt{\rho_t}\mathrm{d}\varphi_t$ by $\sqrt{\rho_t}$ and using again the $L^2(X)$ strong continuity of $\sqrt{\rho_t}$ and the uniform $L^\infty(X)$ bound we conclude that $t \mapsto \rho_t \mathrm{d}\varphi_t \in L^2(T^*X)$ is strongly continuous, as desired.

To prove the strong continuity of $t \mapsto \rho_t \, d\varphi_t \otimes d\varphi_t \in L^2((T^*)^{\otimes 2}X)$ we argue as in Corollary 5.9: the strong continuity of $t \mapsto \sqrt{\rho_t} d\varphi_t \in L^2(T^*X)$ and the fact that these are, locally in $t \in (0, 1]$, uniformly bounded (thanks again to $\mathrm{supp}(\rho_t) \subset B_R(\bar{x})$ for all $t \in [0, 1]$) imply both that $t \mapsto \|\rho_t d\varphi_t \otimes d\varphi_t\|_{L^2((T^*)^{\otimes 2}X)}$ is continuous and that $t \mapsto \rho_t d\varphi_t \otimes d\varphi_t \in L^2((T^*)^{\otimes 2}X)$ is weakly continuous.

The claims about the ψ_t's follow in the same way. □

We now have all the tools needed to prove our main result. Notice that we shall not make explicit use of Theorem 1.4 but rather re-prove it for (the restriction to $[\delta, 1 - \delta]$ of) entropic interpolations.

Theorem 5.13. *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Let* $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ *be such that* $\mu_0, \mu_1 \leq C\mathfrak{m}$ *for some* $C > 0$, *with compact supports and let* (μ_t) *be the unique* W_2-*geodesic connecting* μ_0 *to* μ_1. *Also, let* $h \in H^{2,2}(X)$. *Then the map*

$$[0, 1] \ni t \mapsto \int h \, d\mu_t \in \mathbb{R}$$

belongs to $C^2([0, 1])$, *and the following formulas hold for every* $t \in [0, 1]$:

$$
\begin{aligned}
\frac{d}{dt} \int h \, d\mu_t &= \int \langle \nabla h, \nabla \phi_t \rangle \, d\mu_t, \\
\frac{d^2}{dt^2} \int h \, d\mu_t &= \int \mathrm{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t,
\end{aligned}
\tag{5.34}
$$

where ϕ_t *is any function such that for some* $s \neq t$ *in* $[0, 1]$, *the function* $-(s - t)\phi_t$ *is a Kantorovich potential from* μ_t *to* μ_s.

Proof. For the given μ_0, μ_1 we introduce the notation of Setting 4.1 and then find $\varepsilon_n \downarrow 0$ such that $(\varphi_t^{\varepsilon_n})$, $(\psi_t^{\varepsilon_n})$ converge to limit curves (φ_t), (ψ_t) as in Proposition 5.2.

By Lemma A.7 the choice of the ϕ_t's in the statement does not affect the right-hand sides in (5.34), so we shall prove the formulas for $\phi_t := \psi_t$, which is admissible thanks to Proposition 5.4 whenever $t < 1$. The case $t = 1$ can be achieved by swapping the roles of μ_0, μ_1, or equivalently with the choice $\phi_t = -\varphi_t$ which is admissible for $t > 0$.

Fix $h \in H^{2,2}(X)$ with compact support and for $t \in [0, 1]$ set

$$\mathsf{H}_n(t) := \int h \, d\mu_t^{\varepsilon_n}, \qquad \mathsf{H}(t) := \int h \, d\mu_t.$$

The bound (4.19) implies that the H_n's are uniformly bounded and the convergence in (5.2) yields $\mathsf{H}_n(t) \to \mathsf{H}(t)$ for any $t \in [0, 1]$.

Since $(\rho_t^{\varepsilon_n}) \in AC_{\mathrm{loc}}((0, 1), W^{1,2}(X))$ we have $\mathsf{H}_n \in AC_{\mathrm{loc}}((0, 1))$, and recalling the formula for $\frac{d}{dt} \rho_t^\varepsilon$ given by Proposition 4.3,

$$\frac{d}{dt} \mathsf{H}_n(t) = \int h \frac{d}{dt} \rho_t^{\varepsilon_n} \, d\mathfrak{m} = -\int h \, \mathrm{div}(\rho_t^{\varepsilon_n} \nabla \vartheta_t^{\varepsilon_n}) \, d\mathfrak{m} = \int \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} \, d\mathfrak{m}. \tag{5.35}$$

The fact that $\vartheta_t = \frac{\psi_t - \varphi_t}{2}$, the compactness of supp(h) and the bounds (4.19) and (4.9a) ensure that $\left|\frac{d}{dt}H_n(t)\right|$ is uniformly bounded in n and $t \in [t_0, t_1] \subset (0, 1)$ and the compactness of supp(h) also allows us to use the convergence properties (5.26) and (5.2), which yields

$$\iint\int_{t_0}^{t_1} \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} \, dt \, d\mathfrak{m} \to \iint\int_{t_0}^{t_1} \langle \nabla h, \nabla \vartheta_t \rangle \rho_t \, dt \, d\mathfrak{m}.$$

This is sufficient to pass to the limit in the distributional formulation of $\frac{d}{dt}H_n(t)$ and taking into account that $H \in C([0, 1))$ we have just proved that $H \in AC_{loc}((0, 1))$ with

$$\frac{d}{dt}H(t) = \int \langle \nabla h, \nabla \vartheta_t \rangle \rho_t \, d\mathfrak{m} \tag{5.36}$$

for a.e. $t \in [0, 1]$. Recalling $\vartheta_t = (\psi_t - \varphi_t)/2$, (5.5) and the locality of the differential we see that

$$\nabla \vartheta_t = \nabla \psi_t \quad \rho_t \mathfrak{m}\text{-a.e.} \quad \forall t \in [0, 1), \tag{5.37}$$

and thus by Lemma 5.12 the right-hand side of (5.36) is continuous in $t \in [0, 1)$, which implies $H \in C^1([0, 1))$ and the first identity in (5.34) for any $t \in [0, 1)$.

For the second derivative we assume in addition that $h \in \text{Test}^\infty(X)$. Then we recall that $(\rho_t^{\varepsilon_n}) \in AC_{loc}((0, 1), W^{1,2}(X))$ and $(\vartheta_t^{\varepsilon_n}) \in AC_{loc}((0, 1), W^{1,2}(X, e^{-V}\mathfrak{m}))$ with $V = Md^2(\cdot, \bar{x})$ for some $\bar{x} \in X$ and $M > 0$ sufficiently large. Consider the rightmost expression of (5.35) to get $\frac{d}{dt}H_n(t) \in AC_{loc}((0, 1))$ and

$$\frac{d^2}{dt^2}H_n(t) = \int \left(\langle \nabla h, \nabla \frac{d}{dt}\vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} + \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \frac{d}{dt}\rho_t^{\varepsilon_n} \right) d\mathfrak{m}$$

for a.e. t, so that defining the 'acceleration' a_t^ε as

$$a_t^\varepsilon := -\left(\frac{\varepsilon^2}{4} \Delta \log \rho_t^\varepsilon + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \right)$$

and recalling the formula for $\frac{d}{dt}\vartheta_t^\varepsilon$ given by Proposition 4.3 we have

$$\frac{d^2}{dt^2}H_n(t) = \int \left[\langle \nabla h, \nabla\left(-\tfrac{1}{2}|\nabla \vartheta_t^{\varepsilon_n}|^2 + a_t^{\varepsilon_n}\right) \rangle \rho_t^{\varepsilon_n} - \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \text{div}(\rho_t^{\varepsilon_n} \nabla \vartheta_t^{\varepsilon_n}) \right] d\mathfrak{m}$$

$$= \int \left(-\tfrac{1}{2}\langle \nabla h, \nabla|\nabla \vartheta_t^{\varepsilon_n}|^2 \rangle + \langle \nabla(\langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle), \nabla \vartheta_t^{\varepsilon_n} \rangle + \langle \nabla h, \nabla a_t^{\varepsilon_n} \rangle \right) \rho_t^{\varepsilon_n} \, d\mathfrak{m}$$

(by (A.15)) $\quad = \int \text{Hess}(h)(\nabla \vartheta_t^{\varepsilon_n}, \nabla \vartheta_t^{\varepsilon_n}) \rho_t^{\varepsilon_n} \, d\mathfrak{m} - \int (\Delta h + \langle \nabla h, \nabla \log \rho_t^{\varepsilon_n} \rangle) a_t^{\varepsilon_n} \rho_t^{\varepsilon_n} \, d\mathfrak{m}.$

Since $\vartheta_t^\varepsilon = (\psi_t^\varepsilon - \varphi_t^\varepsilon)/2$ and $\text{Hess}(h) \in L^2(T^{*\otimes 2}X)$ with compact support, by (5.26) and (5.2) we see that

$$\int \text{Hess}(h)(\nabla \vartheta_t^{\varepsilon_n}, \nabla \vartheta_t^{\varepsilon_n}) \rho_t^{\varepsilon_n} \, d\mathfrak{m} \xrightarrow{n \to \infty} \int \text{Hess}(h)(\nabla \vartheta_t, \nabla \vartheta_t) \rho_t \, d\mathfrak{m} \quad \text{in } L^1_{loc}(0, 1)$$

and since $|\nabla h|$, $\Delta h \in L^{\infty}(X)$, by Lemma 4.10 we deduce that

$$\int (\Delta h + \langle \nabla h, \nabla \log \rho_t^{\varepsilon_n} \rangle) a_t^{\varepsilon_n} \rho_t^{\varepsilon_n} \, d\mathfrak{m} \to 0 \quad \text{in } L^1_{\text{loc}}(0,1).$$

Hence we can pass to the limit in the distributional formulation of $\frac{d^2}{dt^2} \mathsf{H}_n$ to find that $\frac{d}{dt} \mathsf{H} \in AC_{\text{loc}}((0,1))$ and

$$\frac{d^2}{dt^2} \mathsf{H}(t) = \int \text{Hess}(h)(\nabla \vartheta_t, \nabla \vartheta_t) \rho_t \, d\mathfrak{m} \tag{5.38}$$

for a.e. t. Using again (5.37) and Lemma 5.12 we conclude that the right-hand side of (5.38) is continuous on $[0,1)$, so that $\mathsf{H} \in C^2([0,1))$ and the second identity in (5.34) holds for every $t \in [0,1)$.

It remains to remove the assumption that $h \in \text{Test}^{\infty}(X)$ and has compact support. To this end we claim that functions in $\text{Test}^{\infty}(X)$ with compact support are dense in $H^{2,2}(X)$. To see this, let χ_R be as in Lemma A.2 and notice that the Leibniz rules for the gradient and the Laplacian easily give $\chi_R h \to h$ in $W^{1,2}(X)$ and $\Delta(\chi_R h) \to \Delta h$ in $L^2(X)$ as $R \to \infty$ for every $h \in \text{Test}^{\infty}(X)$. Hence by (A.13) we also have $\chi_R h \to h$ in $H^{2,2}(X)$. Taking into account that $\text{Test}^{\infty}(X)$ is dense in $H^{2,2}(X)$ (recall (A.14)), our claim follows.

Now let $h \in H^{2,2}(X)$ be arbitrary and let $(h_k) \subset \text{Test}^{\infty}(X)$ with bounded support $H^{2,2}$ converge to h. We can choose the ϕ_t's to be uniformly Lipschitz (e.g. by taking $\phi_t := \psi_t$ for $t \geq 1/2$, $\phi_t := -\varphi_t$ for $t < 1/2$ and recalling Proposition 4.4 and using a cut-off argument). The uniform L^{∞} estimates (A.24), the fact that all the densities ρ_t are supported in a compact set B independent of $t \in [0,1]$ and the L^2 convergence of $h_k, \nabla h_k, \text{Hess}(h_k)$ to $h, \nabla h, \text{Hess}(h)$ respectively imply that as $k \to \infty$ we have

$$\int h_k \, d\mu_t \to \int h \, d\mu_t,$$

$$\int \langle \nabla h_k, \nabla \phi_t \rangle \, d\mu_t \to \int \langle \nabla h, \nabla \phi_t \rangle \, d\mu_t,$$

$$\int \text{Hess}(h_k)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t \to \int \text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t,$$

uniformly in $t \in [0,1]$. This is sufficient to pass to the limit in the formulas (5.34) for the h_k's to get the one for h, and also ensures the C^2 regularity of $t \mapsto \int h \, d\mu_t$. $\qquad \square$

5.4. Related differentiation formulas

In this last part we collect some direct consequences of Theorem 5.13. For the notion of covariant derivative, related calculus rules as well as for the definition of the space $H_C^{1,2}(X)$ we refer to [29].

Theorem 5.14. *Let* $(X, d, \mathfrak{m})$ *be an* $\text{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Then:*

(i) *Let $\boldsymbol{\pi}$ be an optimal geodesic test plan with bounded support and $h \in H^{2,2}(X)$. Then the map $[0, 1] \ni t \mapsto h \circ e_t \in L^2(\boldsymbol{\pi})$ is in $C^2([0, 1], L^2(\boldsymbol{\pi}))$ and*

$$\frac{d}{dt}(h \circ e_t) = \langle \nabla h, \nabla \phi_t \rangle \circ e_t, \qquad \frac{d^2}{dt^2}(h \circ e_t) = \mathrm{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ e_t,$$

for every $t \in [0, 1]$, where ϕ_t is any function such that for some $s \neq t$ in $[0, 1]$, the function $-(s - t)\phi_t$ is a Kantorovich potential from $(e_t)_* \boldsymbol{\pi}$ to $(e_s)_* \boldsymbol{\pi}$.

(ii) *Let $\boldsymbol{\pi}$ be an optimal geodesic test plan with bounded support and $W \in H_C^{1,2}(X)$. Then the map $[0, 1] \ni t \mapsto \langle W, \nabla \phi_t \rangle \circ e_t \in L^2(\boldsymbol{\pi})$ is in $C^1([0, 1], L^2(\boldsymbol{\pi}))$ and*

$$\frac{d}{dt}(\langle W, \nabla \phi_t \rangle \circ e_t) = (\nabla W : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t$$

for every $t \in [0, 1]$, where ϕ_t is as in (i).

(iii) *Let μ_0, μ_1, ϕ_t be as in Theorem 5.13 and $W \in H_C^{1,2}(TX)$. Then the map*

$$[0, 1] \ni t \mapsto \int \langle W, \nabla \phi_t \rangle \, d\mu_t \in \mathbb{R}$$

belongs to $C^1([0, 1])$ and for every $t \in [0, 1]$,

$$\frac{d}{dt} \int \langle W, \nabla \phi_t \rangle \, d\mu_t = \int \nabla W : (\nabla \phi_t \otimes \nabla \phi_t) \, d\mu_t.$$

Proof. (i) Start by observing that for $\boldsymbol{\pi}$ as in the assumptions and $\Gamma \subset C([0, 1], X)$ Borel with $\boldsymbol{\pi}(\Gamma) > 0$, the curve $t \mapsto \boldsymbol{\pi}(\Gamma)^{-1}(e_t)_* \boldsymbol{\pi}_{|\Gamma}$ fulfills the assumptions of Theorem 5.13 with the same ϕ_t's as in the current setting (and in particular, the ϕ_t's can be chosen independent of Γ). Then, taking into account Lemma 5.12 it is easy to check that the maps $[0, 1] \ni t \mapsto h \circ e_t, \langle \nabla h, \nabla \phi_t \rangle \circ e_t, (\mathrm{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ e_t) \in L^2(\boldsymbol{\pi})$ are all continuous, and in particular with uniformly, in $t \in [0, 1]$, bounded $L^2(\boldsymbol{\pi})$ norms.

Also, Theorem 5.13 applied to $t \mapsto \boldsymbol{\pi}(\Gamma)^{-1}(e_t)_* \boldsymbol{\pi}_{|\Gamma}$ gives, after integration and an application of Fubini's theorem, that for any $t, s \in [0, 1]$ with $t < s$ we have

$$\int \eta(h \circ e_s - h \circ e_t) \, d\boldsymbol{\pi} = \int \eta \int_t^s \langle \nabla h, \nabla \phi_r \rangle \circ e_r \, dr \, d\boldsymbol{\pi},$$

$$\int \eta(\langle \nabla h, \nabla \phi_s \rangle \circ e_s - \langle \nabla h, \nabla \phi_t \rangle \circ e_t) \, d\boldsymbol{\pi} = \int \eta \int_t^s \mathrm{Hess}(h)(\nabla \phi_r, \nabla \phi_r) \circ e_r \, dr \, d\boldsymbol{\pi},$$

for every η of the form $\eta = \chi_\Gamma$ with $\Gamma \subset C([0, 1], X)$ Borel, where the integrals (here and below) are understood in the Bochner sense. Then the fact that the linear span of such η's is dense in $L^2(\boldsymbol{\pi})$ forces the equalities

$$h \circ e_s - h \circ e_t = \int_t^s \langle \nabla h, \nabla \phi_r \rangle \circ e_r \, dr,$$

$$\langle \nabla h, \nabla \phi_s \rangle \circ e_s - \langle \nabla h, \nabla \phi_t \rangle \circ e_t = \int_t^s \mathrm{Hess}(h)(\nabla \phi_r, \nabla \phi_r) \circ e_r \, dr,$$

which is the claim.

(ii) By (i) and the Leibniz rule for the covariant derivative (see [29]) we see that the claim holds for $W = \sum_{i=1}^{n} f_i \nabla g_i$ with $n \in \mathbb{N}$ and $(f_i), (g_i) \in \text{Test}^{\infty}(X)$. These vector fields are dense in the space $H_C^{1,2}(TX)$, hence the claim follows by noticing that if $W_n \to W$ in $H_C^{1,2}(TX)$ and the ϕ_t's are chosen uniformly Lipschitz (which as discussed in the proof of Theorem 5.13 is always admissible) then $\langle W_n, \nabla \phi_t \rangle \to \langle W, \nabla \phi_t \rangle$ and $\nabla W_n : (\nabla \phi_t \otimes \nabla \phi_t) \to \nabla W : (\nabla \phi_t \otimes \nabla \phi_t)$ in $L^2(X)$ as $n \to \infty$. Therefore, since $(e_t)_* \boldsymbol{\pi} \le C\mathfrak{m}$ for every $t \in [0, 1]$ and some $C > 0$, we have

$$\langle W_n, \nabla \phi_t \rangle \circ e_t \to \langle W, \nabla \phi_t \rangle \circ e_t,$$
$$(\nabla W_n : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t \to (\nabla W : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t$$

in $L^2(\boldsymbol{\pi})$ uniformly in $t \in [0, 1]$. The conclusion follows.

(iii) This is a direct consequence of (ii) and integration with respect to $\boldsymbol{\pi}$. □

Appendix. Reminders about analysis on RCD spaces

In this appendix we recall the basic definitions and properties of the various objects that we used in the body of the paper. We also provide detailed bibliographical references.

A.1. Sobolev calculus on RCD spaces

We denote by $C([0, 1], (X, d))$, or simply $C([0, 1], X)$, the space of continuous curves with values in the metric space (X, d), and for $t \in [0, 1]$ the *evaluation map* $e_t :$ $C([0, 1], (X, d)) \to X$ is defined as $e_t(\gamma) := \gamma_t$. For the notion of *absolutely continuous curve* in a metric space and of *metric speed* see for instance [4, Section 1.1]. The collection of absolutely continuous curves on $[0, 1]$ is denoted $AC([0, 1], (X, d))$, or simply by $AC([0, 1], X)$.

We denote by $\mathscr{P}(X)$ the space of Borel probability measures on (X, d) and by $\mathscr{P}_2(X) \subset \mathscr{P}(X)$ the subclass of those with finite second moment.

Let $(X, d, \mathfrak{m})$ be a complete and separable metric measure space endowed with a Borel non-negative measure which is finite on bounded sets.

For the definition of *test plans*, of the *Sobolev class* $S^2(X)$ and of *minimal weak upper gradient* $|Df|$ see [5] (and the previous works [16], [57] for alternative—but equivalent—definitions of Sobolev functions). The local counterpart of $S^2(X)$ is introduced as follows: $L_{\text{loc}}^2(X)$ is defined as the space of functions $f \in L^0(X)$ such that for every compact set $\Omega \subset X$ there exists a function $g \in L^2(X)$ such that $f = g$ $\mathfrak{m}$-a.e. in Ω, and the *local Sobolev class* $S_{\text{loc}}^2(X)$ is then defined as

$$S_{\text{loc}}^2(X) := \{f \in L^0(X) : \forall \Omega \subset\subset X \; \exists g \in S^2(X), \; f = g \; \mathfrak{m}\text{-a.e. in } \Omega\}. \qquad (A.1)$$

The local minimal weak upper gradient of a function $f \in S_{\text{loc}}^2(X)$ is denoted by $|Df|$, omitting the locality feature, and defined for all $\Omega \subset\subset X$ as $|Df| := |Dg|$ $\mathfrak{m}$-a.e. in Ω,

where g is as in (A.1). The definition depends neither on Ω nor on the choice of g by locality of the minimal weak upper gradient.

The Sobolev space $W^{1,2}(\mathrm{X})$ (resp. $W^{1,2}_{\mathrm{loc}}(\mathrm{X})$) is defined as $L^2(\mathrm{X}) \cap S^2(\mathrm{X})$ (resp. $L^2_{\mathrm{loc}} \cap S^2_{\mathrm{loc}}(\mathrm{X})$). When endowed with the norm $\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||Df|\|^2_{L^2}$, $W^{1,2}(\mathrm{X})$ is a Banach space. The *Cheeger energy* is the convex and lower semicontinuous functional $E : L^2(\mathrm{X}) \to [0, \infty]$ given by

$$E(f) := \begin{cases} \dfrac{1}{2} \displaystyle\int |Df|^2 \, d\mathfrak{m} & \text{for } f \in W^{1,2}(\mathrm{X}), \\ +\infty & \text{otherwise.} \end{cases}$$

$(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* (see [31]) if $W^{1,2}(\mathrm{X})$ is Hilbert. In this case E is a Dirichlet form and its infinitesimal generator Δ, which is a closed self-adjoint operator on $L^2(\mathrm{X})$, is called the *Laplacian* on $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$, with domain denoted by $D(\Delta) \subset W^{1,2}(\mathrm{X})$. The flow (h_t) associated to E is called the *heat flow* (see [5]), and for any $f \in L^2(\mathrm{X})$ the curve $t \mapsto \mathsf{h}_t f \in L^2(\mathrm{X})$ is continuous on $[0, \infty)$, locally absolutely continuous on $(0, \infty)$ and the only solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{h}_t f = \Delta \mathsf{h}_t f, \quad \mathsf{h}_t f \to f \quad \text{as } t \downarrow 0.$$

If moreover $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ is an RCD(K, ∞) space (see [6]), the following a priori estimates hold true for every $f \in L^2(\mathrm{X})$ and $t > 0$:

$$E(\mathsf{h}_t f) \leq \frac{1}{4t} \|f\|^2_{L^2(\mathrm{X})}, \tag{A.2a}$$

$$\|\Delta \mathsf{h}_t f\|^2_{L^2(\mathrm{X})} \leq \frac{1}{2t^2} \|f\|^2_{L^2(\mathrm{X})}. \tag{A.2b}$$

Still within the RCD framework, there exists the *heat kernel*, a function

$$(0, \infty) \times \mathrm{X}^2 \ni (t, x, y) \mapsto r_t[x](y) = r_t[y](x) \in (0, \infty) \tag{A.3}$$

such that

$$\mathsf{h}_t f(x) = \int f(y) r_t[x](y) \, d\mathfrak{m}(y) \quad \forall t > 0 \tag{A.4}$$

for every $f \in L^2(\mathrm{X})$. For every $x \in \mathrm{X}$ and $t > 0$, $r_t[x]$ is a probability density and thus (A.4) can be used to extend the heat flow to $L^1(\mathrm{X})$ and shows that the flow is *mass preserving* and satisfies the *maximum principle*, i.e.

$$f \leq c \ \mathfrak{m}\text{-a.e.} \implies \mathsf{h}_t f \leq c \ \mathfrak{m}\text{-a.e.}, \ \forall t > 0.$$

For finite-dimensional RCD*(K, N) spaces (introduced in [31]; for the distinction between RCD and RCD* conditions see [11] and [14]) the heat kernel satisfies *Gaussian*

estimates, i.e. for every $\delta > 0$ there are positive constants $C_1 = C_1(K, N, \delta)$ and $C_2 = C_2(K, N, \delta)$ such that for all $x, y \in \mathrm{X}$ and $t > 0$,

$$\frac{1}{C_1 \mathfrak{m}(B_{\sqrt{t}}(y))} \exp(-\frac{\mathrm{d}^2(x, y)}{(4 - \delta)t} - C_2 t) \leq \mathrm{r}_t[x](y) \leq \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(y))} \exp\left(-\frac{\mathrm{d}^2(x, y)}{(4 + \delta)t} + C_2 t\right) \tag{A.5}$$

(see [41]), which adapts the approach of [58], [59] to the RCD setting.

For general metric measure spaces, the *differential* is a well defined linear map d from $S^2(\mathrm{X})$ with values in the *cotangent module* $L^2(T^*\mathrm{X})$ (see [29]); it is a closed operator when seen as an unbounded operator on $L^2(\mathrm{X})$. If $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian, which from now on we shall always assume, the cotangent module is canonically isomorphic to its dual, the *tangent module* $L^2(T\mathrm{X})$, and the isomorphism sends the differential $\mathrm{d}f$ to the gradient ∇f. Replacing the language of L^2 normed modules with the L^0's (see [29]), the differential can be extended to $\mathrm{d} : S^2_{\mathrm{loc}}(\mathrm{X}) \to L^0(T^*\mathrm{X})$, where $L^0(T^*\mathrm{X})$ denotes the family of (measurable) 1-forms. The dual of $L^0(T^*\mathrm{X})$ as an L^0 normed module is denoted by $L^0(T\mathrm{X})$, it is canonically isomorphic to $L^0(T^*\mathrm{X})$ and its elements are called *vector fields*. Further, $L^2_{\mathrm{loc}}(T^*\mathrm{X}) \subset L^0(T^*\mathrm{X})$ (resp. $L^2_{\mathrm{loc}}(T\mathrm{X}) \subset L^0(T\mathrm{X})$) is defined as the collection of 1-forms ω such that $|\omega| \in L^2_{\mathrm{loc}}(\mathrm{X})$ (resp. vector fields v such that $|v| \in L^2_{\mathrm{loc}}(\mathrm{X})$).

The *divergence* of a vector field is defined as (minus) the adjoint of the differential, i.e. $v \in L^2(T\mathrm{X})$ (resp. $v \in L^2_{\mathrm{loc}}(T\mathrm{X})$) *has divergence* in $L^2(\mathrm{X})$ (resp. in $L^2_{\mathrm{loc}}(\mathrm{X})$), and write $v \in D(\mathrm{div})$ (resp. $v \in D(\mathrm{div}_{\mathrm{loc}})$) if there is $g \in L^2(\mathrm{X})$ (resp. $g \in L^2_{\mathrm{loc}}(\mathrm{X})$) such that

$$\int f g \, \mathrm{d}\mathfrak{m} = - \int \mathrm{d}f(v) \, \mathrm{d}\mathfrak{m} \quad \forall f \in W^{1,2}(\mathrm{X}).$$

(resp. for all Lipschitz functions f with bounded support). In this case g is unique and is denoted $\mathrm{div}(v)$. A function $f \in W^{1,2}_{\mathrm{loc}}(\mathrm{X})$ *has Laplacian* in $L^2_{\mathrm{loc}}(\mathrm{X})$, and we shall write $f \in D(\Delta_{\mathrm{loc}})$, if there exists $g \in L^2_{\mathrm{loc}}(\mathrm{X})$ such that

$$\int \phi g \, \mathrm{d}\mathfrak{m} = - \int \langle \nabla \phi, \nabla f \rangle \, \mathrm{d}\mathfrak{m} \quad \forall \phi \text{ Lipschitz with bounded support,}$$

and in this case, since g is unique, we set $\Delta f := g$. It can be verified that

$$f \in D(\Delta_{\mathrm{loc}}) \quad \text{if and only if} \quad \nabla f \in D(\mathrm{div}_{\mathrm{loc}}) \quad \text{and in this case} \quad \Delta f = \mathrm{div}(\nabla f),$$

in accordance with the smooth case.

As regards the properties of d, div, Δ, the differential satisfies the following calculus rules which we shall use extensively without further notice:

$$\begin{aligned}
|\mathrm{d}f| &= |Df| \quad \mathfrak{m}\text{-a.e.} & &\forall f \in S^2(\mathrm{X}), \\
\mathrm{d}f &= \mathrm{d}g \quad \mathfrak{m}\text{-a.e. on } \{f = g\} & &\forall f, g \in S^2(\mathrm{X}), \\
\mathrm{d}(\varphi \circ f) &= \varphi' \circ f \, \mathrm{d}f & &\forall f \in S^2(\mathrm{X}), \ \varphi : \mathbb{R} \to \mathbb{R} \text{ Lipschitz,} \\
\mathrm{d}(fg) &= g \, \mathrm{d}f + f \, \mathrm{d}g & &\forall f, g \in L^\infty \cap S^2(\mathrm{X}),
\end{aligned}$$

where it is part of the properties that $\varphi \circ f, fg \in S^2(X)$ for φ, f, g as above. For the divergence, we have

$$\mathrm{div}(fv) = \mathrm{d}f(v) + f\,\mathrm{div}(v) \quad \forall f \in W^{1,2}(X),\ v \in D(\mathrm{div}) \text{ such that } |f|, |v| \in L^\infty(X),$$

where it is understood in particular that $fv \in D(\mathrm{div})$ for f, v as above, and for the Laplacian,

$$\Delta(\varphi \circ f) = \varphi'' \circ f |\mathrm{d}f|^2 + \varphi' \circ f \Delta f,$$
$$\Delta(fg) = g\Delta f + f\Delta g + 2\langle \nabla f, \nabla g \rangle,$$

where in the first equality we assume that $f \in D(\Delta)$, $\varphi \in C^2(\mathbb{R})$ are such that $f, |\mathrm{d}f| \in L^\infty(X)$ and $\varphi', \varphi'' \in L^\infty(\mathbb{R})$, and in the second that $f, g \in D(\Delta) \cap L^\infty(X)$ and $|\mathrm{d}f|, |\mathrm{d}g| \in L^\infty(X)$, and it is part of the claims that $\varphi \circ f, fg$ are in $D(\Delta)$. On $S^2_{\mathrm{loc}}(X)$ as well as on $D(\mathrm{div}_{\mathrm{loc}})$ and $D(\Delta_{\mathrm{loc}})$ the same calculus rules hold with slight adaptations. For sake of information, we present the chain rule for the differential and the Laplacian, as they will be widely exploited without further mention.

Lemma A.1 (Calculus rules). *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *with* $K \in \mathbb{R}$. *Then:*

(i) *For all* $f \in S^2_{\mathrm{loc}}(X)$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that for all* $C \subset\subset X$ *there exists* $I_C \subset\subset \mathbb{R}$ *with* $\mathscr{L}^1(f(C) \setminus I_C) = 0$ *and* $\varphi_{|I_C}$ *Lipschitz, we have*

$$\mathrm{d}(\varphi \circ f) = \varphi' \circ f,$$

where it is part of the statement that $\varphi \circ f \in S^2_{\mathrm{loc}}(X)$ *for* φ, f *as above; analogous statements hold for the gradient.*

(ii) *For all* $f \in D(\Delta_{\mathrm{loc}})$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that* $f, |\mathrm{d}f| \in L^\infty_{\mathrm{loc}}(X)$ *and* $\varphi', \varphi'' \in L^\infty(\mathbb{R})$ *we have*

$$\Delta(\varphi \circ f) = \varphi'' \circ f |\mathrm{d}f|^2 + \varphi' \circ f \Delta f,$$

where it is part of the claims that $\varphi \circ f \in D(\Delta_{\mathrm{loc}})$.

The proof is based on the locality of such differentiation operators and the analogous properties of their global counterparts defined on $S^2(X)$, $D(\Delta)$.

Besides this notion of L^2-valued Laplacian, we shall also need that of measure-valued Laplacian [31]. A function $f \in W^{1,2}(X)$ *has measure-valued Laplacian*, and we write $f \in D(\mathbf{\Delta})$, if there exists a Borel (signed) measure μ whose total variation is finite on bounded sets and

$$\int g\,\mathrm{d}\mu = -\int \langle \nabla g, \nabla f \rangle \,\mathrm{d}\mathfrak{m} \quad \forall g \text{ Lipschitz with bounded support.}$$

In this case μ is unique and denoted $\mathbf{\Delta} f$. This notion is compatible with the previous one in the sense that

$$f \in D(\mathbf{\Delta}),\ \mathbf{\Delta} f \ll \mathfrak{m} \text{ and } \frac{\mathrm{d}\mathbf{\Delta} f}{\mathrm{d}\mathfrak{m}} \in L^2(X) \iff f \in D(\Delta) \text{ and in this case } \Delta f = \frac{\mathrm{d}\mathbf{\Delta} f}{\mathrm{d}\mathfrak{m}}.$$

On RCD(K, ∞) spaces, the vector space of 'test functions' (see [56])

$$\mathrm{Test}^\infty(\mathrm{X}) := \{f \in D(\Delta) \cap L^\infty(\mathrm{X}) : |\nabla f| \in L^\infty(\mathrm{X}), \ \Delta f \in L^\infty \cap W^{1,2}(\mathrm{X})\}$$

is an algebra dense in $W^{1,2}(\mathrm{X})$ for which

$$f \in \mathrm{Test}^\infty(\mathrm{X}) \text{ and } \varphi \in C^\infty(\mathbb{R}) \implies \varphi \circ f \in \mathrm{Test}^\infty(\mathrm{X}). \tag{A.6}$$

Combining the Gaussian estimates on RCD$^*(K, N)$ spaces, $N < \infty$, with the results in [56] we see that

$$f \in L^2 \cap L^\infty(\mathrm{X}), \ t > 0 \implies \mathsf{h}_t(f) \in \mathrm{Test}^\infty(\mathrm{X}). \tag{A.7}$$

The fact that $\mathrm{Test}^\infty(\mathrm{X})$ is an algebra is based on the property

$f \in \mathrm{Test}^\infty(\mathrm{X}) \implies |\mathrm{d}f|^2 \in W^{1,2}(\mathrm{X})$ with

$$\int \left|\mathrm{d}(|\mathrm{d}f|^2)\right|^2 \mathrm{d}\mathfrak{m} \leq \left\||\mathrm{d}f|\right\|_{L^\infty}^2 \left(\left\||\mathrm{d}f|\right\|_{L^2} \left\||\mathrm{d}\Delta f|\right\|_{L^2} + |K| \left\||\mathrm{d}f|\right\|_{L^2}^2\right) \tag{A.8}$$

and actually a further regularity property of test functions is that

$$f \in \mathrm{Test}^\infty(\mathrm{X}) \implies |\mathrm{d}f|^2 \in D(\boldsymbol{\Delta}),$$

so that it is possible to introduce the *measure-valued* Γ_2 *operator* [56] as

$$\Gamma_2(f) := \boldsymbol{\Delta}\frac{|\mathrm{d}f|^2}{2} - \langle \nabla f, \nabla \Delta f\rangle \mathfrak{m} \quad \forall f \in \mathrm{Test}^\infty(\mathrm{X}).$$

By construction, the assignment $f \mapsto \Gamma_2(f)$ is a quadratic form.

An important property of the heat flow on RCD(K, ∞) spaces is the *Bakry–Émery contraction estimate* (see [6]):

$$|\mathrm{d}\mathsf{h}_t f|^2 \leq e^{-2Kt}\mathsf{h}_t(|\mathrm{d}f|^2) \quad \forall f \in W^{1,2}(\mathrm{X}), \ t \geq 0. \tag{A.9}$$

We also recall that RCD(K, ∞) spaces have the *Sobolev-to-Lipschitz* property ([6], [28]):

$$f \in W^{1,2}(\mathrm{X}), |\mathrm{d}f| \in L^\infty(\mathrm{X}) \implies \exists \tilde{f} = f \ \mathfrak{m}\text{-a.e. with } \mathrm{Lip}(\tilde{f}) \leq \left\||\mathrm{d}f|\right\|_{L^\infty}, \tag{A.10}$$

and thus we shall typically identify Sobolev functions with bounded differentials with their Lipschitz representatives; in particular this will be the case for functions in $\mathrm{Test}^\infty(\mathrm{X})$.

A well-known consequence of lower Ricci curvature bounds (see e.g. [17], [18], [19]) is the existence of 'good cut-off functions', typically intended as cut-offs with bounded Laplacian; for our purposes the following result will be sufficient:

Lemma A.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* RCD$^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Then for all* $R > 0$ *and* $x \in \mathrm{X}$ *there exists a function* $\chi_R : \mathrm{X} \to \mathbb{R}$ *satisfying:*

(i) $0 \leq \chi_R \leq 1$, $\chi_R \equiv 1$ *on* $B_R(x)$ *and* $\mathrm{supp}(\chi_R) \subset B_{R+1}(x)$;
(ii) $\chi_R \in \mathrm{Test}^\infty(\mathrm{X})$.

Moreover, there exist constants $C, C' > 0$ depending on K, N only such that

$$\left\||\nabla \chi_R|\right\|_{L^\infty(X)} \leq C, \qquad \|\Delta \chi_R\|_{L^\infty(X)} \leq C'. \qquad (A.11)$$

The proof can be obtained by following verbatim the arguments in [52, Lemma 3.1] (inspired by [8], see also [35] for an alternative approach): there the authors are interested in cut-off functions such that $\chi \equiv 1$ on $B_R(x)$ and $\text{supp}(\chi) \subset B_{2R}(x)$; for this reason they fix $R > 0$ and then claim that for all $x \in X$ and $0 < r < R$ there exists a cut-off function χ satisfying (i), (ii) and (A.11) with C, C' also depending on R. However, as far as one is concerned with cut-off functions χ where the distance between $\{\chi = 0\}$ and $\{\chi = 1\}$ is always equal to 1, the proof of [52] in the case $R = 1$ applies and does not affect (A.11). As regards the assumption $N < \infty$, this cannot be avoided either in [52] or in [35]; in [8] the construction of cut-off functions is carried out in RCD(K, ∞) spaces, but it only allows one to separate relatively compact sets, and balls in an RCD(K, ∞) space need not be relatively compact.

A direct consequence of the existence of such cut-off functions is that

$$\{f \in L^2_{\text{loc}}(X) : \forall \Omega \subset\subset X \ \exists g \in \text{Test}^\infty(X), \ f = g \ \mathfrak{m}\text{-a.e. in } \Omega\}$$
$$= \{f \in D(\Delta_{\text{loc}}) \cap L^\infty_{\text{loc}}(X) : |\nabla f| \in L^\infty_{\text{loc}}(X), \ \Delta f \in W^{1,2}_{\text{loc}}(X)\}. \qquad (A.12)$$

Indeed, the '$\subset$' inclusion is obvious, while for the opposite one if f belongs to the second set and $\Omega \subset X$ is a bounded open set, consider a cut-off function $\chi \in \text{Test}^\infty(X)$ with compact support and $\chi \equiv 1$ on Ω; it is clear that $\chi f \in \text{Test}^\infty(X)$ and $\chi f \equiv f$ on Ω. We shall call the set in (A.12) the space of *local* test functions and denote it $\text{Test}^\infty_{\text{loc}}(X)$.

The existence of the space of test functions and the language of L^2 normed L^∞-modules allow us to introduce the spaces $W^{2,2}(X)$ as follows (see [29]). We first consider the tensor product $L^2((T^*)^{\otimes 2}X)$ of $L^2(T^*X)$ with itself. The pointwise norm on that module is denoted $|\cdot|_{\text{HS}}$ (as in the smooth case it coincides with the Hilbert–Schmidt norm) and '$:$' is the scalar product inducing it. Then we say that a function $f \in W^{1,2}(X)$ belongs to $W^{2,2}(X)$ if there exists $A \in L^2((T^*)^{\otimes 2}X)$ symmetric, i.e. such that $A(v_1, v_2) = A(v_2, v_1)$ $\mathfrak{m}$-a.e. for every $v_1, v_2 \in L^2(TX)$, for which

$$\int h A(\nabla g, \nabla g) \, \mathrm{d}\mathfrak{m} = \int \left(-\langle \nabla f, \nabla g \rangle \, \text{div}(h \nabla g) - h \left\langle \nabla f, \nabla \frac{|\nabla g|^2}{2} \right\rangle \right) \mathrm{d}\mathfrak{m}$$

for all $g, h \in \text{Test}^\infty(X)$. In this case A is unique, called the *Hessian* of f and denoted by $\text{Hess}(f)$. The space $W^{2,2}(X)$ endowed with the norm

$$\|f\|^2_{W^{2,2}(X)} := \|f\|^2_{L^2(X)} + \|\mathrm{d}f\|^2_{L^2(T^*X)} + \|\text{Hess}(f)\|^2_{L^2((T^*)^{\otimes 2}X)}$$

is a separable Hilbert space which contains $\text{Test}^\infty(X)$ and in particular is dense in $W^{1,2}(X)$. It is proved in [29] that $D(\Delta) \subset W^{2,2}(X)$ with

$$\int |\text{Hess}(f)|^2_{\text{HS}} \, \mathrm{d}\mathfrak{m} \leq \int [(\Delta f)^2 - K|\nabla f|^2] \, \mathrm{d}\mathfrak{m} \quad \forall f \in D(\Delta). \qquad (A.13)$$

The space $H^{2,2}(X)$ is defined as the closure of $D(\Delta)$ in $W^{2,2}(X)$ and following the arguments in [29, Proposition 4.3.18] it is not difficult to see that

$$\text{Test}^\infty(X) \text{ is dense in } H^{2,2}(X). \tag{A.14}$$

It is unknown whether $H^{2,2}(X) = W^{2,2}(X)$ or not. We recall that

$$d\langle \nabla f, \nabla g \rangle = \text{Hess}(f)(\nabla g, \cdot) + \text{Hess}(g)(\nabla f, \cdot) \quad \forall f, g \in \text{Test}^\infty(X) \tag{A.15}$$

and that the Hessian is a local operator, i.e. $\text{Hess}(f) = \text{Hess}(g)$ $\mathfrak{m}$-a.e. on $\{f = g\}$. Using this latter fact, for $f \in \text{Test}^\infty_{\text{loc}}(X)$ we can define $\text{Hess}(f)$ as the element in the L^0 completion of $L^2((T^*)^{\otimes 2}X)$ defined by

$$\text{Hess}(f) := \text{Hess}(g) \quad \mathfrak{m}\text{-a.e. on } \{f = g\} \quad \forall g \in \text{Test}^\infty(X).$$

The *Bochner inequality* on $\text{RCD}(K, \infty)$ spaces takes the form of an inequality between measures ([29]; see also the previous contributions [56], [62]):

$$\Gamma_2(f) \geq \left(|\text{Hess}(f)|^2_{\text{HS}} + K|df|^2\right)\mathfrak{m} \quad \forall f \in \text{Test}^\infty(X), \tag{A.16}$$

and if the space is $\text{RCD}^*(K, N)$ for some finite N then also ([23], [9])

$$\Gamma_2(f) \geq \left((\Delta f)^2/N + K|df|^2\right)\mathfrak{m} \quad \forall f \in \text{Test}^\infty(X). \tag{A.17}$$

Notice that since the Laplacian is in general not the trace of the Hessian, the former does not trivially imply the latter (in this connection, see [39]).

As regards the geometric features of finite-dimensional $\text{RCD}^*(K, N)$ spaces, we recall the *Bishop–Gromov inequality* in the form we shall need (see [60], [61]): for any $x \in \text{supp}(\mathfrak{m})$ and any $0 < r \leq R < \infty$,

$$\begin{aligned}
\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} &\geq \frac{\int_0^r \sinh(t\sqrt{-K/(N-1)})^{N-1}\,dt}{\int_0^R \sinh(t\sqrt{-K/(N-1)})^{N-1}\,dt}, \\
\frac{\mathfrak{s}_r(x)}{\mathfrak{s}_R(x)} &\geq \left(\frac{\sinh(r\sqrt{-K/(N-1)})}{\sinh(R\sqrt{-K/(N-1)})}\right)^{N-1}
\end{aligned} \tag{A.18}$$

(with standard adaptations and caveat if $K \geq 0$) where

$$\mathfrak{s}_r(x) := \varlimsup_{\delta \downarrow 0} \frac{1}{\delta}\mathfrak{m}(\overline{B_{r+\delta}(x)} \setminus B_r(x)).$$

A couple of interesting consequences are the following: $\mathfrak{m}$ is uniformly locally doubling with an explicit expression for the local doubling constant, i.e. for all $x \in X$ and $r > 0$,

$$\mathfrak{m}(B_{2r}(x)) \leq 2^N \cosh\left(2\sqrt{\frac{-K}{N-1}}r\right)^{N-1} \mathfrak{m}(B_r(x)); \tag{A.19}$$

and for all $x \in X$ there exists a constant $C > 0$ depending on x (and on K, N) such that the following volume growth condition is satisfied:

$$\mathfrak{m}(B_r(x)) \leq Ce^{Cr} \quad \forall r > 0. \tag{A.20}$$

We conclude the section recalling the notion of regular Lagrangian flow, introduced by Ambrosio–Trevisan [10] as the generalization to RCD spaces of the analogous concept existing on $\mathbb{R}^d$ and proposed by Ambrosio [1]:

Definition A.3 (Regular Lagrangian flow). Given $(v_t) \in L^1([0, 1], L^2(TX))$, the function $F : [0, 1] \times X \to X$ is a *regular Lagrangian flow* for (v_t) provided:

(i) $[0, 1] \ni t \mapsto F_t(x)$ is continuous for every $x \in X$.
(ii) For every $f \in \mathrm{Test}^\infty(X)$ and $\mathfrak{m}$-a.e. x the map $t \mapsto f(F_t(x))$ belongs to $W^{1,1}([0, 1])$ and
$$\frac{\mathrm{d}}{\mathrm{d}t} f(F_t(x)) = \mathrm{d}f(v_t)(F_t(x)) \quad \text{a.e. } t \in [0, 1].$$

(iii) We have
$$(F_t)_* \mathfrak{m} \le C\mathfrak{m} \quad \forall t \in [0, 1]$$
for some constant $C > 0$.

In [10] the authors prove that under suitable assumptions on the v_t's, the regular Lagrangian flow exists and is unique. We shall use the following formulation of their result (weaker than the one provided in [10]):

Theorem A.4. *Let* $(X, d, \mathfrak{m})$ *be an* RCD(K, ∞) *space and* $(v_t) \in L^1([0, 1], L^2(TX))$ *be such that* $v_t \in D(\mathrm{div})$ *for a.e. t and*

$$\mathrm{div}(v_t) \in L^1([0, 1], L^2(X)), \quad (\mathrm{div}(v_t))^- \in L^1([0, 1], L^\infty(X)).$$

Then there exists a unique, up to $\mathfrak{m}$-a.e. equality, regular Lagrangian flow F for (v_t). For that flow, the quantitative bound

$$(F_t)_* \mathfrak{m} \le \exp\left(\int_0^1 \|(\mathrm{div}(v_t))^-\|_{L^\infty(X)} \, \mathrm{d}t \right) \mathfrak{m} \tag{A.21}$$

holds for every $t \in [0, 1]$, and for $\mathfrak{m}$-a.e. x the curve $t \mapsto F_t(x)$ is absolutely continuous and its metric speed $\mathrm{ms}_t(F_.(x))$ at time t satisfies

$$\mathrm{ms}_t(F_.(x)) = |v_t|(F_t(x)) \quad a.e. \ t \in [0, 1]. \tag{A.22}$$

To be precise, (A.22) is not explicitly stated in [10]; its proof is anyway not hard and can be obtained, for instance, following the arguments in [29].

A.2. *Optimal transport on* RCD *spaces*

It is well known that on $\mathbb{R}^d$, curves of measures which are W_2-absolutely continuous are in correspondence with appropriate solutions of the *continuity equation* [4]. It has been proved in [32] that the same connection holds on arbitrary metric measure spaces $(X, d, \mathfrak{m})$ provided the measures are controlled by $C\mathfrak{m}$ for some $C > 0$; the formulation of the result which we shall need is:

Theorem A.5 (Continuity equation and W_2-AC curves). *Let* $(X, d, \mathfrak{m})$ *be infinitesimally Hilbertian,* $(\mu_t) \subset \mathscr{P}(X)$ *be weakly continuous and* $t \mapsto v_t \in L^0(TX)$ *be a family of vector fields, possibly defined only for a.e.* $t \in [0, 1]$. *Assume that the map* $t \mapsto \int |v_t|^2 \, d\mu_t$ *is Borel and*

$$\mu_t \le C\mathfrak{m} \quad \forall t \in [0, 1] \text{ for some } C > 0, \tag{A.23a}$$

$$\int_0^1 \int |v_t|^2 \, d\mu_t \, dt < \infty \tag{A.23b}$$

and that the continuity equation

$$\frac{d}{dt}\mu_t + \operatorname{div}(v_t \mu_t) = 0$$

is satisfied in the following sense: for any $f \in W^{1,2}(X)$ *the map* $[0, 1] \ni t \mapsto \int f \, d\mu_t$ *is absolutely continuous and*

$$\frac{d}{dt} \int f \, d\mu_t = \int df(v_t) \, d\mu_t \quad a.e. \, t.$$

Then $(\mu_t) \in AC([0, 1], (\mathscr{P}(X), W_2))$ *and*

$$|\dot{\mu}_t|^2 = \int |v_t|^2 \, d\mu_t \quad a.e. \, t \in [0, 1].$$

Recall that given $f : X \to \mathbb{R}$ the *upper* and *lower slopes* $|D^+ f|, |D^- f| : X \to [0, \infty]$ are defined as 0 at isolated points and elsewhere by

$$|D^+(f)|(x) := \varlimsup_{y \to x} \frac{(f(y) - f(x))^+}{d(x, y)}, \quad |D^- f|(x) := \varlimsup_{y \to x} \frac{(f(y) - f(x))^-}{d(x, y)}.$$

Similarly, the *local Lipschitz constant* $\operatorname{lip}(f) : X \to [0, \infty]$ is defined as 0 at isolated points and elsewhere as

$$\operatorname{lip}(f)(x) := \max\{|D^+ f|(x), |D^- f|(x)\} = \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If f is Lipschitz, then its *Lipschitz constant* is denoted by $\operatorname{Lip} f$. We also recall that the *c-transform* $\varphi^c : X \to \mathbb{R} \cup \{-\infty\}$ of a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$\varphi^c(x) := \inf_{y \in X} \left(d^2(x, y)/2 - \varphi(y) \right)$$

and that φ is said to be *c-concave* if $\varphi = \psi^c$ for some ψ. Also, given $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is a *Kantorovich potential* from μ_0 to μ_1 if it is *c*-concave and

$$\int \varphi \, d\mu_0 + \int \varphi^c \, d\mu_1 = \tfrac{1}{2} W_2^2(\mu_0, \mu_1).$$

It is worth recalling that on general complete and separable metric spaces (X, d), for any $\mu_0, \mu_1 \in \mathscr{P}(X)$ with bounded support there exists a Kantorovich potential from μ_0 to μ_1 which is Lipschitz and bounded. This can be obtained starting from an arbitrary Kantorovich potential ψ and then defining

$$\varphi(x) := \min\left\{C, \inf_{y \in X} \big(d^2(x, y)/2 - \psi^c(y)\big)\right\}$$

for C sufficiently large.

Moreover, we recall the following version of Brenier–McCann theorem on RCD spaces ((i) comes from [27] and [54], (ii) from [6] and [31], (iii) from [5], and (iv) from [36]).

Theorem A.6. *Let* $(X, d, \mathfrak{m})$ *be an* RCD(K, ∞) *space and let* $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ *have bounded support and satisfy* $\mu_0, \mu_1 \leq C\mathfrak{m}$ *for some* $C > 0$. *Also, let* φ *be a Kantorovich potential for the couple* (μ_0, μ_1) *which is locally Lipschitz on a neighbourhood of* supp(μ_0). *Then:*

(i) *There exists a unique geodesic* (μ_t) *from* μ_0 *to* μ_1, *it satisfies*

$$\mu_t \leq C'\mathfrak{m} \quad \forall t \in [0, 1] \quad \text{for some } C' > 0, \tag{A.24}$$

   and has a unique lifting $\boldsymbol{\pi}$, *i.e. there is a unique measure* $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], X))$ *such that* $(e_t)_* \boldsymbol{\pi} = \mu_t$ *for every* $t \in [0, 1]$ *and* $\iint_0^1 |\dot{\gamma}_t|^2 \, dt \, d\boldsymbol{\pi}(\gamma) = W_2^2(\mu_0, \mu_1)$.

(ii) *For every* $f \in W^{1,2}(X)$ *the map* $t \mapsto \int f \, d\mu_t$ *is differentiable at* $t = 0$ *and*

$$\frac{d}{dt} \int f \, d\mu_t{}_{|t=0} = -\int df(\nabla\varphi) \, d\mu_0.$$

(iii) *We have*

$$|d\varphi|(\gamma_0) = |D^+\varphi|(\gamma_0) = d(\gamma_0, \gamma_1) \quad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma.$$

(iv) *If the space is* RCD*(K, N) *for some* $N < \infty$, *then* (i)–(iii) *hold with* μ_1 *only assumed to be with bounded support, with the caveat that* (A.24) *holds in the form: for every* $\delta \in (0, 1/2)$ *there is* $C_\delta > 0$ *such that* $\mu_t \leq C_\delta'\mathfrak{m}$ *for every* $t \in [0, 1 - \delta]$.

A related property is that although Kantorovich potentials are not uniquely determined by the initial and final measures, their gradients are. This is expressed by the following result, which also says that if we are at an intermediate point of a geodesic and move to one extreme or the other, then the two corresponding velocities are opposite of each other (see [28, Lemmas 5.8 and 5.9] for the proof):

Lemma A.7. *Let* $(X, d, \mathfrak{m})$ *be an* RCD(K, ∞) *space with* $K \in \mathbb{R}$ *and* $(\mu_t) \subset \mathscr{P}_2(X)$ *a* W_2-*geodesic such that* $\mu_t \leq C\mathfrak{m}$ *for every* $t \in [0, 1]$ *for some* $C > 0$. *For* $t \in [0, 1]$ *let* $\phi_t, \phi_t' : X \to \mathbb{R}$ *be locally Lipschitz functions such that for some* $s, s' \neq t$ *the functions* $-(s - t)\phi_t$ *and* $-(s' - t)\phi_t'$ *are Kantorovich potentials from* μ_t *to* μ_s *and from* μ_t *to* $\mu_{s'}$ *respectively. Then*

$$\nabla\phi_t = \nabla\phi_{t'} \quad \mu_t\text{-a.e.}$$

On RCD spaces, W_2-geodesics made up of measures with bounded density also have the weak continuity property of the densities expressed by the following lemma. The proof follows by a simple argument involving Young's measures and the continuity of the entropy along a geodesic (see [28, Corollary 5.7]):

Lemma A.8. *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space with* $K \in \mathbb{R}$ *and* $(\mu_t) \subset \mathscr{P}_2(X)$ *a* W_2-*geodesic such that* $\mu_t \leq C\mathfrak{m}$ *for every* $t \in [0, 1]$ *for some* $C > 0$. *Let* ρ_t *be the density of* μ_t. *Then for any* $t \in [0, 1]$ *and any sequence* $(t_n)_{n \in \mathbb{N}} \subset [0, 1]$ *converging to* t *there exists a subsequence* $(t_{n_k})_{k \in \mathbb{N}}$ *such that*

$$\rho_{t_{n_k}} \to \rho_t \quad \mathfrak{m}\text{-}a.e. \quad as \ k \to \infty.$$

Finally, we recall some properties of the *Hopf–Lax semigroup* in metric spaces, also in connection with optimal transport. For $f : X \to \mathbb{R} \cup \{+\infty\}$ and $t > 0$ we define the function $Q_t f : X \to \mathbb{R} \cup \{-\infty\}$ as

$$Q_t f(x) := \inf_{y \in X} \left(\frac{\mathrm{d}^2(x, y)}{2t} + f(y) \right) \tag{A.25}$$

and set $t_* := \sup\{t > 0 : Q_t f(x) > -\infty \text{ for some } x \in X\}$; note that actually t_* does not depend on x, since if $Q_t f(x) > -\infty$, then $Q_s f(y) > -\infty$ for all $s \in (0, t)$ and all $y \in X$. With this notation we have the following result (see [5]):

Proposition A.9. *Let* (X, d) *be a length space and* $f : X \to \mathbb{R} \cup \{+\infty\}$. *Then for all* $x \in X$ *the map* $(0, t_*) \ni t \mapsto Q_t f(x)$ *is locally Lipschitz and*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t f(x) + \frac{1}{2} (\mathrm{lip}(Q_t f)(x))^2 = 0 \quad a.e. \ t \in (0, t_*). \tag{A.26}$$

References

[1] Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. Invent. Math. **158**, 227–260 (2004)   Zbl 1075.35087   MR 2096794

[2] Ambrosio, L., Feng, J.: On a class of first order Hamilton–Jacobi equations in metric spaces. J. Differential Equations **256**, 2194–2245 (2014)   Zbl 1283.49027   MR 3160441

[3] Ambrosio, L., Gigli, N., Mondino, A., Rajala, T.: Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure. Trans. Amer. Math. Soc. **367**, 4661–4701 (2015)   Zbl 1317.53060   MR 3335397

[4] Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures. 2nd ed., Lectures in Math. ETH Zürich, Birkhäuser, Basel (2008) Zbl 1145.35001   MR 2401600

[5] Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. **195**, 289–391 (2014) Zbl 1312.53056 MR 3152751

[6] Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163**, 1405–1490 (2014) Zbl 1304.35310 MR 3205729

[7] Ambrosio, L., Gigli, N., Savaré, G.: Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds. Ann. Probab. **43**, 339–404 (2015) Zbl 1307.49044 MR 3298475

[8] Ambrosio, L., Mondino, A., Savaré, G.: On the Bakry–Émery condition, the gradient estimates and the local-to-global property of RCD*(K, N) metric measure spaces. J. Geom. Anal. **26**, 24–56 (2016) Zbl 1335.35088 MR 3441502

[9] Ambrosio, L., Mondino, A., Savaré, G.: Nonlinear diffusion equations and curvature conditions in metric measure spaces. Mem. Amer. Math. Soc. **262**, no. 1270, v+121 pp. (2019) Zbl 07161621 MR 4044464

[10] Ambrosio, L., Trevisan, D.: Well posedness of Lagrangian flows and continuity equations in metric measure spaces. Anal. PDE **7**, 1179–1234 (2014) Zbl 1357.49058 MR 3265963

[11] Bacher, K., Sturm, K.-T.: Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. J. Funct. Anal. **259**, 28–56 (2010) Zbl 1196.53027 MR 2610378

[12] Borwein, J. M., Lewis, A. S., Nussbaum, R. D.: Entropy minimization, DAD problems, and doubly stochastic kernels. J. Funct. Anal. **123**, 264–307 (1994) Zbl 0815.15021 MR 1283029

[13] Bruè, E., Semola, D.: Constancy of the dimension for RCD(K, N) spaces via regularity of Lagrangian flows. Comm. Pure Appl. Math. **73**, 1141–1204 (2020) Zbl 1442.35054 MR 4156601

[14] Cavalletti, F., Milman, E.: The globalization theorem for the curvature dimension condition. arXiv:1612.07623 (2016)

[15] Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. **208**, 803–849 (2017) Zbl 1375.53053 MR 3648975

[16] Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**, 428–517 (1999) Zbl 0942.58018 MR 1708448

[17] Cheeger, J., Colding, T. H.: On the structure of spaces with Ricci curvature bounded below. I. J. Differential Geom. **46**, 406–480 (1997) Zbl 0902.53034 MR 1484888

[18] Cheeger, J., Colding, T. H.: On the structure of spaces with Ricci curvature bounded below. II. J. Differential Geom. **54**, 13–35 (2000) Zbl 1027.53042 MR 1815410

[19] Cheeger, J., Colding, T. H.: On the structure of spaces with Ricci curvature bounded below. III. J. Differential Geom. **54**, 37–74 (2000) Zbl 1027.53043 MR 1815411

[20] Chen, Y., Georgiou, T. T., Pavon, M.: On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint. J. Optim. Theory Appl. **169**, 671–691 (2016) Zbl 1344.49072 MR 3489825

[21] Colding, T. H., Naber, A.: Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. Ann. of Math. (2) **176**, 1173–1229 (2012) Zbl 1260.53067 MR 2950772

[22] De Philippis, G., Gigli, N.: From volume cone to metric cone in the nonsmooth setting. Geom. Funct. Anal. **26**, 1526–1587 (2016) Zbl 1356.53049 MR 3579705

[23] Erbar, M., Kuwada, K., Sturm, K.-T.: On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. Invent. Math. **201**, 993–1071 (2015) Zbl 1329.53059 MR 3385639

[24] Gangbo, W., Święch, A.: Metric viscosity solutions of Hamilton–Jacobi equations depending on local slopes. Calc. Var. Partial Differential Equations **54**, 1183–1218 (2015) Zbl 1355.49024 MR 3385197

[25] Garofalo, N., Mondino, A.: Li–Yau and Harnack type inequalities in RCD*(K, N) metric measure spaces. Nonlinear Anal. **95**, 721–734 (2014) Zbl 1286.58016 MR 3130557

[26] Gentil, I., Léonard, C., Ripani, L.: About the analogy between optimal transport and minimal entropy. Ann. Fac. Sci. Toulouse Math. (6) **26**, 569–600 (2017) Zbl 1380.49067 MR 3669966

[27] Gigli, N.: Optimal maps in non branching spaces with Ricci curvature bounded from below. Geom. Funct. Anal. **22**, 990–999 (2012) Zbl 1257.53055 MR 2984123

[28] Gigli, N.: The splitting theorem in non-smooth context. arXiv:1302.5555 (2013)

[29] Gigli, N.: Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below. Mem. Amer. Math. Soc. **251**, no. 1196, vi+166 pp. (2018) Zbl 1404.53056 MR 3756920

[30] Gigli, N.: An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature. Anal. Geom. Metric Spaces **2**, 169–213 (2014) Zbl 1310.53031 MR 3210895

[31] Gigli, N.: On the differential structure of metric measure spaces and applications. Mem. Amer. Math. Soc. **236**, no. 1113, vi+91 pp. (2015) Zbl 1325.53054 MR 3381131

[32] Gigli, N., Han, B.: The continuity equation on metric measure spaces. Calc. Var. Partial Differential Equations **53**, 149–177 (2013) Zbl 1316.35076 MR 3336316

[33] Gigli, N., Kuwada, K., Ohta, S.-i.: Heat flow on Alexandrov spaces. Comm. Pure Appl. Math. **66**, 307–331 (2013) Zbl 1267.58014 MR 3008226

[34] Gigli, N., Mosconi, S.: The Abresch–Gromoll inequality in a non-smooth setting. Discrete Contin. Dynam. Systems **34**, 1481–1509 (2014) Zbl 1280.53038 MR 3121629

[35] Gigli, N., Mosconi, S.: The abstract Lewy–Stampacchia inequality and applications. J. Math. Pures Appl. (9) **104**, 258–275 (2015) Zbl 1318.49016 MR 3365829

[36] Gigli, N., Rajala, T., Sturm, K.-T.: Optimal maps and exponentiation on finite-dimensional spaces with Ricci curvature bounded from below. J. Geom. Anal. **26**, 2914–2929 (2016) Zbl 1361.53036 MR 3544946

[37] Gigli, N., Tamanini, L.: Second order differentiation formula on compact RCD*(K, N) spaces. Rend. Lincei Mat. Appl. **29**, 377–386 (2018) Zbl 1394.53050 MR 3797990

[38] Gigli, N., Tamanini, L.: Benamou–Brenier and duality formulas for the entropic cost on RCD*(K, N) spaces. Probab. Theory Related Fields **176**, 1–34 (2020) Zbl 1432.49063 MR 4055185

[39] Han, B.: Ricci tensor on RCD*(K, N) spaces. J. Geom. Anal. **28**, 1295–1314 (2018) Zbl 1395.53046 MR 3790501

[40] Jiang, R.: The Li–Yau inequality and heat kernels on metric measure spaces. J. Math. Pures Appl. (9) **104**, 29–57 (2015) Zbl 1316.53049 MR 3350719

[41] Jiang, R., Li, H., Zhang, H.-C.: Heat kernel bounds on metric measure spaces and some applications. Potential Anal. **44**, 601–627 (2016) Zbl 1339.53043 MR 3489857

[42] Jiang, R., Zhang, H.: Hamilton's gradient estimates and a monotonicity formula for heat flows on metric measure spaces. Nonlinear Anal. **131**, 32–47 (2016) Zbl 1329.53060 MR 3427968

[43] Ketterer, C.: Cones over metric measure spaces and the maximal diameter theorem. J. Math. Pures Appl. (9) **103**, 1228–1275 (2015) Zbl 1317.53064 MR 3333056

[44] Léonard, C.: On the convexity of the entropy along entropic interpolations. In: Measure Theory in Non-Smooth Spaces, De Gruyter, 194–242 (2017) MR 3701740

[45] Léonard, C.: Minimizers of energy functionals. Acta Math. Hungar. **93**, 281–325 (2001) Zbl 1002.49017 MR 1925356

[46] Léonard, C.: From the Schrödinger problem to the Monge–Kantorovich problem. J. Funct. Anal. **262**, 1879–1920 (2012) Zbl 1236.49088 MR 2873864

[47] Léonard, C.: Some properties of path measures. In: Séminaire de Probabilités XLVI, Lecture Notes in Math. 2123, Springer, 207–230 (2014) Zbl 1390.60279 MR 3330819

[48] Léonard, C.: A survey of the Schrödinger problem and some of its connections with optimal transport. Discrete Contin. Dynam. Systems **34**, 1533–1574 (2014) Zbl 1277.49052 MR 3121631

[49] Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) **169**, 903–991 (2009) Zbl 1178.53038 MR 2480619

[50] Mikami, T.: Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. Probab. Theory Related Fields **129**, 245–260 (2004) Zbl 1061.58034 MR 2063377

[51] Mikami, T., Thieullen, M.: Optimal transportation problem by stochastic optimal control. SIAM J. Control Optim. **47**, 1127–1139 (2008) Zbl 1175.93245 MR 2407010

[52] Mondino, A., Naber, A.: Structure theory of metric-measure spaces with lower Ricci curvature bounds I. J. Eur. Math. Soc. **21**, 1809–1854 (2019) Zbl 07060417 MR 3945743

[53] Rajala, T.: Local Poincaré inequalities from stable curvature conditions on metric spaces. Calc. Var. Partial Differential Equations **44**, 477–494 (2012) Zbl 1250.53040 MR 2915330

[54] Rajala, T., Sturm, K.-T.: Non-branching geodesics and optimal maps in strong CD(K, ∞)-spaces. Calc. Var. Partial Differential Equations **50**, 831–846 (2012) Zbl 1296.53088 MR 3216835

[55] Rüschendorf, L., Thomsen, W.: Note on the Schrödinger equation and I-projections. Statist. Probab. Lett. **17**, 369–375 (1993) Zbl 0780.60036 MR 1237783

[56] Savaré, G.: Self-improvement of the Bakry–Émery condition and Wasserstein contraction of the heat flow in RCD(K, ∞) metric measure spaces. Discrete Contin. Dynam. Systems **34**, 1641–1661 (2014) Zbl 1275.49087 MR 3121635

[57] Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoamer. **16**, 243–279 (2000) Zbl 0974.46038 MR 1809341

[58] Sturm, K.-T.: Heat kernel bounds on manifolds. Math. Ann. **292**, 149–162 (1992) Zbl 0747.58048 MR 1141790

[59] Sturm, K.-T.: Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. Osaka J. Math. **32**, 275–312 (1995) Zbl 0854.35015 MR 1355744

[60] Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**, 65–131 (2006) Zbl 1105.53035 MR 2237206

[61] Sturm, K.-T.: On the geometry of metric measure spaces. II. Acta Math. **196**, 133–177 (2006) Zbl 1106.53032 MR 2237207

[62] Sturm, K.-T.: Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes. J. Funct. Anal. **275**, 793–829 (2018) Zbl 1419.58020 MR 3807777

[63] Tamanini, L.: Analysis and geometry of RCD spaces via the Schrödinger problem. PhD thesis, Univ. Paris Nanterre and SISSA (2017)

[64] Villani, C.: Optimal Transport: Old and New. Grundlehren Math. Wiss. 338, Springer, Berlin (2009) Zbl 1156.53003 MR 2459454