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On the distribution of the van der Corput sequences

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Abstract. For an integer $p \geq 2$, let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ be the p-adic van der Corput sequence. For intervals $[0,\alpha) \subset \mathbb{T}$ and for positive integers N, consider the geometrically-shifted discrepancy function $D_{p,N,\alpha}(t) = \sum_{n=0}^{N-1} \mathcal{X}_{[0,\alpha)}(x_n + t) - N\alpha$. In this paper, we give a characterization of the asymptotic behavior of $\|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}$ for $N \to \infty$ that depends on the p-adic expansion of α.

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1. Introduction and statement of the result. Let $\mathbb{N}$ denote the set of all non-negative integers, and let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the one dimensional torus. For every $x \in \mathbb{R}$, let $\{x\}$ represent the fractional part of x and let $\lfloor x \rfloor$ represent the integer part of x. Also, set $\|x\| = \min(\{x\}, 1-\{x\})$ as the distance from x to $\mathbb{Z}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be uniformly distributed in $\mathbb{T}$ if for any interval $I \subseteq \mathbb{T}$ with measure $|I|$, it holds

$$\lim_{N\to\infty} \frac{\#\{x_n \in I \mid 0 \leq n \leq N-1\}}{N} = |I|.$$

We introduce the concept of discrepancy as a quantitative counterpart of the uniform distribution. Namely, for $N \in \mathbb{N}$, we define the discrepancy of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ with respect to intervals as

$$D_N(\{x_n\}_{n\in\mathbb{N}}) := \sup_{I\subset\mathbb{T},\, I \text{ interval}} \left| \sum_{n=0}^{N-1} \mathcal{X}_I(x_n) - N|I| \right|,$$

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where $\mathcal{X}_I \colon \mathbb{T} \to \{0,1\}$ is the indicator function of the interval $I \subset \mathbb{T}$.

In 1935, J.G. van der Corput conjectured that for any sequence of real numbers $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$, the quantity $D_N(\{x_n\}_{n\in\mathbb{N}})$ is unbounded with respect to N. This conjecture was first proved ten years later by van Aardenne-Ehrenfest [14] with a first lower bound, and in 1954, Roth [10] significantly improved this estimate with one of the most relevant results in discrepancy theory. Later in 1972, Schmidt [12] gave the best possible estimate (up to a multiplicative constant). Namely, he showed that for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$, it holds the lower bound

$$\limsup_{N\to\infty} \frac{D_N(\{x_n\}_{n\in\mathbb{N}})}{\log(N)} > 10^{-2}.$$

We formally introduce the object of our studies. Let $\mathbb{N}_+$ denote the set of all positive natural numbers, and consider $p \geq 2$. For $n \in \mathbb{N}$ with p-adic expansion

$$n = \sum_{j=1}^{\eta} n_j p^{j-1} \text{ with } \eta \in \mathbb{N}_+ \text{ and } 0 \leq n_j < p \text{ for all } j,$$

we define the p-adic van der Corput sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$ as the sequence described by

$$x_n = \sum_{j=1}^{\eta} n_j p^{-j}.$$

In 1935, van der Corput [15] showed that the discrepancy of the dyadic van der Corput sequence is optimal with respect to the order of magnitude in N, and it is well-known that the same holds for all p-adic van der Corput sequences. In particular, Faure [4] generalized p-adic van der Corput sequences and further investigated their qualities. Since its low-discrepancy property, the distribution of the van der Corput sequence has been studied under various points of view. We refer the reader to [6] for a detailed survey on all the generalizations of such sequence and the concerning properties.

We introduce a major tool in discrepancy theory. Consider a fixed anchored interval $[0,\alpha) \subset \mathbb{T}$ and, for $N \in \mathbb{N}_+$, define the discrepancy function of the p-adic van der Corput sequence $\{x_n\}_{n\in\mathbb{N}}$ with respect to $[0,\alpha)$ as

$$D_{p,N,\alpha} := \sum_{n=0}^{N-1} \mathcal{X}_{[0,\alpha)}(x_n) - N\alpha.$$

In 1972, Schmidt [11] first showed that if α has finite dyadic expansion, then $D_{2,N,\alpha}$ is bounded, and in 1978, Shapiro [13] proved the converse implication through ergodic theory methods. Later in 1980, Hellekalek [7] generalized the results for p-adic van der Corput sequences.

In 1983, Faure [5] gave an explicit formula for $D_{p,N,\alpha}$. For completeness, we report his result as follows. Let $N \in \mathbb{N}_+$, and write the p-adic expansions $\alpha = \sum_{j=0}^{\infty} \alpha_j p^{-j-1}$ and $N = \sum_{i=0}^{\infty} N_i p_i$. Now, set $V(N,\alpha) = \sum_{i=0}^{\infty} \inf(N_i, \alpha_i)$

and denote $W(N,\alpha)$ as the number of couples $(-,+)$ in the sequences of the signs of $\{\alpha_i - N_i\}_{i\geq 0}$ without considering the zeros. Then it holds

$$D_{p,N,\alpha} = -\sum_{j=0}^{\infty}\sum_{i=0}^{j} N_i\alpha_j p^{i-j-1} + V(N,\alpha) + W(N,\alpha).$$

Further useful estimates for $D_{2,N,\alpha}$ can be found in [3].

It is now time to introduce the main tool of this paper.

Definition 1.1. Consider a fixed interval $[0,\alpha) \subset \mathbb{T}$. For $N \in \mathbb{N}_+$, we define the geometrically-shifted discrepancy function of the p-adic van der Corput sequence $\{x_n\}_{n\in\mathbb{N}}$ with respect to $[0,\alpha)$ as

$$D_{p,N,\alpha}(t) := \sum_{n=0}^{N-1} \mathcal{X}_{[0,\alpha)}(x_n + t) - N\alpha. \tag{1.1}$$

One can interpret (1.1) as the usual discrepancy function of the p-adic van der Corput sequence with respect to a shifted interval $[-t,-t+\alpha) \subset \mathbb{T}$. In particular, $D_{p,N,\alpha}(0) = D_{p,N,\alpha}$.

The aim of this paper is to investigate the asymptotic behavior of $\|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}$, i.e., the root mean square average of the discrepancy function over all possible intervals of length α in $\mathbb{T}$.

We remark that the quantity

$$\left(\int_0^1 \|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}^2 \, d\alpha\right)^{\frac{1}{2}}$$

is exactly the periodic L^2-discrepancy introduced by Lev [9] in 1995. Moreover, we directly deduce from Lemma 2.2 that the latter quantity coincides with the diaphony introduced by Zinterhof [16] in 1976 (see [1, Proposition 2] for a generalization of such equality in higher dimensions). As a matter of fact, the periodic L^2-discrepancy is a geometric interpretation of diaphony, and it is also closely related to the worst-case integration error of quasi-Monte Carlo integration rules (see, for example, [8]).

We need a last definition in order to state our main result.

Definition 1.2. For α with p-adic expansion $\alpha = \sum_{\ell=1}^{\infty} \alpha_\ell p^{-\ell}$ and for N with p-adic expansion $N = \sum_{j=1}^{\eta} N_j p^{j-1}$ (with $N_\eta \neq 0$), we define

$$\mathcal{L}_{p,N,\alpha} := \{1 \leq \ell \leq \lfloor \log_p(N) \rfloor \text{ such that } \alpha_\ell \notin \{0, p-1\} \text{ or } \alpha_{\ell-1} \neq \alpha_\ell\}.$$

In other words, $\mathcal{L}_{p,N,\alpha}$ represents the set of the indices $\ell \in [1, \eta-1]$ such that the coefficient α_ℓ does not lie on the extremities $\{0, p-1\}$ or it belongs to $\{0, p-1\}$ but it is different from the previous coefficient. We further refer to Faure's [5, Theorem 1] on generalized van der Corput sequences for similarities between the quantities involved.

Then, in relation to the p-adic expansion of α, we can give an asymptotic characterization of $\|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}$ with respect to N. We state the main result of this paper.

Theorem 1.3. *For $\alpha \in (0,1)$ and $p \in \mathbb{N}_+$, we have*

$$0 < \limsup_{N\to\infty} \frac{\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})}}{\#\mathcal{L}_{p,N,\alpha}} < \infty.$$

It directly follows a characterization of the intervals for which the L^2-norm of the geometrically-shifted discrepancy function is bounded.

Corollary 1.4. *Consider $\alpha \in (0,1)$ and $p \in \mathbb{N}_+$, then $\|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}$ is bounded with respect to N if and only if α has finite p-adic expansion.*

2. Auxiliary results for the proof. From now on, let $\{x_n\}_{n\in\mathbb{N}}$ denote the p-adic van der Corput sequence. Moreover, improper sums (as $\sum_{i=1}^{0} 1$) will be conventionally considered as zeros.

Notice that for any $\eta \in \mathbb{N}$, we have

$$\{x_n\}_{n=0}^{p^\eta-1} = \left\{\sum_{j=1}^{\eta} n_j p^{-j} \text{ such that } 0 \le n_j < p \text{ for all } j\right\} = \left\{\frac{k}{p^\eta}\right\}_{k=0}^{p^\eta-1}.$$

More precisely, for any $\eta \in \mathbb{N}$, the first p^η points in the p-adic van der Corput sequence are p^η equidistant points in $\mathbb{T}$ and, trivially, the point 0 is the first of them. We extend this fact in the following lemma.

Lemma 2.1. *Given $h \in \mathbb{N}_+$ and $N \in \mathbb{N}_+$ such that N has p-adic expansion*

$$N = \sum_{j=h}^{\eta} N_j p^{j-1} \textit{ with } 0 \le N_j < p \textit{ for all } j,$$

then $\{x_n\}_{n=N}^{N+p^{h-1}-1}$ are p^{h-1} equidistant points in $\mathbb{T}$, moreover the point $\sum_{j=h}^{\eta} N_j p^{-j}$ is one of them.

Proof. We can uniquely write all the numbers k such that $N \le k \le N+p^{h-1}-1$ as p-adic expansions

$$k = \sum_{j=h}^{\eta} N_j p^{j-1} + \sum_{j=1}^{h-1} \beta_j p^{j-1} \text{ where } 0 \le \beta_j < p \text{ for all } j.$$

This consequently means that all the elements x_k in $\{x_n\}_{n=N}^{N+p^{h-1}-1}$ can uniquely be written as

$$x_k = \sum_{j=h}^{\eta} N_j p^{-j} + \sum_{j=1}^{h-1} \beta_j p^{-j} \text{ where } 0 \le \beta_j < p \text{ for all } j.$$

The first sum is then a constant term for all these x_k, while the second sum is the same one that appears for the first p^{h-1} elements of the van der Corput sequence. Therefore $\{x_n\}_{n=N}^{N+p^{h-1}-1}$ are p^{h-1} equidistant points of $\mathbb{T}$, and $\sum_{j=0}^{\eta} N_j p^{-j}$ is one of them. □

Now, we need to recall some fundamental properties of Fourier analysis. For $f \in L^2(\mathbb{T})$, we define the Fourier transform of f as the function $\mathcal{F}f\colon \mathbb{Z} \to \mathbb{C}$ described by

$$(\mathcal{F}f)(k) = \int_{\mathbb{T}} f(x) \exp(-2\pi i x k)\, dx \qquad \forall k \in \mathbb{Z}.$$

For $y \in \mathbb{R}$, we use the notations $\tau_y f = f(\cdot - y)$ and $e_y(k) = \exp(2\pi i y k)$. It is then a well-known property of Fourier transforms that $\mathcal{F}(\tau_y f) = e_{-y}\mathcal{F}f$. Therefore, considering indicator functions of intervals, we get

$$\mathcal{F}(\mathcal{X}_{[0,\alpha)}(x_n + \cdot)) = \mathcal{F}(\tau_{-x_n}\mathcal{X}_{[0,\alpha)}) = e_{x_n}\,\mathcal{F}\mathcal{X}_{[0,\alpha)}. \tag{2.1}$$

We show a fundamental lemma that works as starting point for our results.

Lemma 2.2. *Consider an interval* $[0,\alpha) \subset \mathbb{T}$, *then*

$$\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\pi k|^{-2} |\sin(\pi\alpha k)|^2 \left|\sum_{n=0}^{N-1} \exp(2\pi i k x_n)\right|^2.$$

Proof. It is obvious that $D_{p,N,\alpha}(\cdot)$ is in $L^2(\mathbb{T})$, so that we can apply Parseval's identity and, by (2.1), we get

$$\begin{aligned}\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} |\mathcal{F}D_{p,N,\alpha}(k)|^2 \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\sum_{n=0}^{N-1} \exp(2\pi i k x_n)\right|^2 \left|\mathcal{F}\mathcal{X}_{[0,\alpha)}(k)\right|^2\end{aligned}$$

since it easily follows that $\mathcal{F}D_{p,N,\alpha}(0) = 0$. Moreover, we have

$$\begin{aligned}\mathcal{F}\mathcal{X}_{[0,\alpha)}(k) &= \int_{\mathbb{T}} \mathcal{X}_{[0,\alpha)}(x) \exp(-2\pi i k x) dx = (-2\pi i k)^{-1}(\exp(-2\pi i k\alpha) - 1) \\ &= (2\pi i k)^{-1} \exp(-\alpha\pi i k)\,[\exp(+\alpha\pi i k) - \exp(-\alpha\pi i k)] \\ &= (\pi k)^{-1} \exp(-\alpha\pi i k)\,[\sin(\pi\alpha k)]\,,\end{aligned}$$

so that we get our initial claim. □

3. The proof of the main result. The results in this section are inspired by [2], and we follow its notation. For any positive integer q, we define the function $\delta_q\colon \mathbb{Z} \to \{0,1\}$ as

$$\delta_q(h) = \begin{cases} 1 & \text{if } h \equiv 0 \pmod{q}, \\ 0 & \text{if } h \not\equiv 0 \pmod{q}. \end{cases}$$

Moreover, the sum of a geometric series gives

$$\delta_q(h) = \frac{1}{q}\sum_{s=0}^{q-1} \exp\left(2\pi i h \frac{s}{q}\right). \tag{3.1}$$

Now, consider $N \in \mathbb{N}$ with p-adic expansion $N = \sum_{j=1}^{\eta} N_j p^{j-1}$ and recall that, by Lemma 2.2, we have

$$\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\pi k|^{-2} |\sin(\pi\alpha k)|^2 \left|\sum_{n=0}^{N-1} \exp(2\pi i k x_n)\right|^2 . \quad (3.2)$$

Because of Lemma 2.1, we are able to split $\{x_n\}_{n=0}^{N-1}$ in disjoint sets of equidistant points. More precisely, we get

$$\{x_n\}_{n=0}^{N-1} = \\ = \bigsqcup_{\nu=0}^{\eta-1} \bigsqcup_{d=0}^{N_{\eta-\nu}-1} \left\{ sp^{-\eta+\nu+1} + dp^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} N_{\eta-\mu} p^{-\eta+\mu} \mid 0 \le s \le p^{\eta-\nu-1} - 1 \right\},$$

where as the parameter ν increases, we consider (exponentially) smaller sets of equidistant points in $\mathbb{T}$, while the parameter d is less significant but nonetheless necessary.

Hence, we can also split the sum in the previous modulus in several parts, namely

$$\sum_{n=0}^{N-1} \exp(2\pi i k x_n) = \\ = \sum_{\nu=0}^{\eta-1} \sum_{d=0}^{N_{\eta-\nu}-1} \sum_{s=0}^{p^{\eta-\nu-1}-1} \exp\left(2\pi i k\left(sp^{-\eta+\nu+1} + dp^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} N_{\eta-\mu} p^{-\eta+\mu}\right)\right).$$

Although the long expression, we are just reordering and writing explicitly all these sets because each one of them represents a regular polygon lying on the unit circle in the complex plane. Hence, from (3.1), we further get

$$\sum_{n=0}^{N-1} \exp(2\pi i k x_n) = \\ = \sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1} \delta_{p^{\eta-\nu-1}}(k) \sum_{d=0}^{N_{\eta-\nu}-1} \exp\left(2\pi i k\left(dp^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} N_{\eta-\mu} p^{-\eta+\mu}\right)\right). \quad (3.3)$$

Before giving the proof of our main result, let us first show a simpler one.

Proposition 3.1. *Let $\lambda \in \mathbb{N}_+$ and let $[0, \alpha)$ be such that $\alpha = \sum_{\ell=1}^{\lambda} \alpha_\ell p^{-\ell}$ with $0 \le \alpha_\ell < p$ for all ℓ (i.e., α has finite p-adic expansion), then $\|D_{p,N,\alpha}(\cdot)\|_{L^2(\mathbb{T})}$ is bounded with respect to N by a constant depending only on p and λ.*

Proof. It is easy to see that

$$|\sin(\pi\alpha k)| \le 1 - \delta_{p^\lambda}(k). \quad (3.4)$$

Moreover, notice that for $a, b \in \mathbb{N}_+$ such that $a \le b$, we have

$$(1 - \delta_{p^a}(k))\, \delta_{p^b}(k) = 0 \quad \forall k \in \mathbb{Z}. \quad (3.5)$$

Consider $\eta \in \mathbb{N}_+$ such that $\eta \geq \lambda$, and let $\sum_{j=1}^{\eta} N_j p^{j-1}$ be the p-adic expansion of N. Starting with (3.2) and by (3.3)-(3.4)-(3.5), it follows that

$$\begin{aligned}\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} &\leq \pi^{-2} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2}(1-\delta_{p^\lambda}(k))^2 \Bigg| \sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1}\delta_{p^{\eta-\nu-1}}(k) \cdot \\ &\qquad \cdot \sum_{d=0}^{N_{\eta-\nu}-1} \exp\Big(2\pi i k\Big(dp^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} N_{\eta-\mu}p^{-\eta+\mu}\Big)\Big)\Bigg|^2 \\ &\leq \pi^{-2} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2} \Bigg| \sum_{\nu=\eta-\lambda}^{\eta-1} p^{\eta-\nu-1}\delta_{p^{\eta-\nu-1}}(k) \cdot \\ &\qquad \cdot \sum_{d=0}^{N_{\eta-\nu}-1} \exp\Big(2\pi i k\Big(dp^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} N_{\eta-\mu}p^{-\eta+\mu}\Big)\Big)\Bigg|^2 .\end{aligned}$$

With the change of variable $\tau = \eta - \nu - 1$, and since $0 \leq N_j < p$ for all j, we finally get

$$\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \leq \pi^{-2} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2} \left| (p-1)\sum_{\tau=0}^{\lambda-1} p^\tau \delta_{p^\tau}(k)\right|^2 ,$$

which is clearly bounded by a constant depending only on p and λ. □

It is time to prove the main result of this paper.

Proof of Theorem 1.3. We break the proof in two parts, first we show that the right inequality holds.

As before, let $\sum_{j=1}^{\eta} N_j p^{j-1}$ (with $N_\eta \neq 0$) be the p-adic expansion of N. Let k be a positive integer, we respectively define $L(k)$ and $G(k)$ as the lowest and the greatest exponents (plus 1) appearing in the p-adic expansion of k, namely $k = \sum_{m=L(k)}^{G(k)} \beta_m p^{m-1}$ with $\beta_{L(k)} \neq 0$ and $\beta_{G(k)} \neq 0$.

In the case of $L(k) \leq \eta$, starting with (3.3) and recalling that $0 \leq N_j < p$ for all j, we get

$$\begin{aligned}\left|\sum_{n=0}^{N-1} \exp(2\pi i k x_n)\right| &\leq (p-1)\left(\sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1}\delta_{p^{\eta-\nu-1}}(k)\right) \\ &= (p-1)\left(\sum_{\nu=\eta-L(k)}^{\eta-1} p^{\eta-\nu-1}\right) \\ &= (p-1)\left(\frac{p^{L(k)}-1}{p-1}\right) = p^{L(k)} - 1\end{aligned}$$

since k is a multiple of $p^{\eta-\nu-1}$ if and only if $L(k) - 1 \geq \eta - \nu - 1$. Trivially it holds $N \leq p^\eta$, so that for any choice of $k \in \mathbb{Z}$, we get

$$\left|\sum_{n=0}^{N-1} \exp(2\pi i k x_n)\right| \leq p^{\min(L(k),\eta)}.$$

Also, it is a well-known inequality that

$$\pi\|x\| \geq |\sin(\pi x)| \geq 2\|x\| \qquad \forall x \in \mathbb{R}, \tag{3.6}$$

therefore, starting with (3.2), we get

$$\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2}\|\alpha k\|^2 p^{2\min(L(k),\eta)}. \tag{3.7}$$

Now, write $\mathcal{L}_{p,N,\alpha}$ as an increasing sequence of indices $\{\ell_q\}_{q=1}^{\#\mathcal{L}_{p,N,\alpha}}$ and, for the sake of construction, add $\ell_0 = 0$ and $\ell_{1+\#\mathcal{L}_{p,N,\alpha}} = \eta$. Notice that all the indices ℓ (if any) in between two consecutive ℓ_q are such that the respective coefficients α_ℓ are either all 0 or all $p-1$.

In the case of $\ell_q < L(k) \leq G(k) \leq \ell_{q+1}$, we proceed to show that

$$\|\alpha k\| \leq p^{G(k)-\ell_{q+1}+1}.$$

In fact, suppose without loss of generality that all the ℓ such that $\ell_q < \ell < \ell_{q+1}$ are zeros (the other case is similar), so that we get

$$\begin{aligned}\|\alpha k\| &= \|\sum_{\ell=1}^{\infty}\alpha_\ell p^{-\ell}\cdot\sum_{m=L(k)}^{G(k)}\beta_m p^{m-1}\|\\ &= \|(\sum_{\ell=1}^{\ell_q}+\sum_{\ell=\ell_{q+1}}^{\infty})\alpha_\ell p^{-\ell}\cdot\sum_{m=L(k)}^{G(k)}\beta_m p^{m-1}\|\\ &= \|\sum_{\ell=\ell_{q+1}}^{\infty}\alpha_\ell p^{-\ell}\cdot\sum_{m=L(k)}^{G(k)}\beta_m p^{m-1}\| \leq \Big\{\sum_{\ell=\ell_{q+1}}^{\infty}\alpha_\ell p^{-\ell}\cdot\sum_{m=L(k)}^{G(k)}\beta_m p^{m-1}\Big\}\\ &= \sum_{\ell=\ell_{q+1}}^{\infty}\alpha_\ell p^{-\ell}\cdot\sum_{m=L(k)}^{G(k)}\beta_m p^{m-1} \leq p^{-\ell_{q+1}+1}p^{G(k)} = p^{G(k)-\ell_{q+1}+1}.\end{aligned}$$

In the case $L(k) \leq \ell_q < G(k) \leq \ell_{q+1}$, we trivially get $\|\alpha k\| \leq 1$. Then it is useful to define

$$f_q(r,s) = \begin{cases} p^{s-\ell_{q+1}+1} & \text{if} \quad \ell_q < r \leq s \leq \ell_{q+1},\\ 1 & \text{else.}\end{cases}$$

Moreover, notice that $\#\{k \in \mathbb{N} \mid L(k) = r, G(k) = s\} \leq p^{s-r+1}$ and that trivially $p^{G(k)-1} \leq |k| < p^{G(k)}$.

Finally, taking into account what we have gathered so far and reordering the sum in (3.7), we get

$$\begin{aligned}&\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2}\|\alpha k\|^2 p^{2\min(L(k),\eta)} \leq\\ &\leq 2\Big(\sum_{q=0}^{\#\mathcal{L}_{p,N,\alpha}}\sum_{G=\ell_q+1}^{\ell_{q+1}}+\sum_{G=\eta+1}^{\infty}\Big)\sum_{L=1}^{G} p^{G-L+1}p^{-2G+2}f_q(L,G)^2\, p^{2\min(L,\eta)},\end{aligned}$$

so, writing explicitly all the sums, it follows that

$$\begin{aligned}\frac{1}{2}\|D_{p,N,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \leq & \sum_{q=0}^{\#\mathcal{L}_{p,N,\alpha}} \sum_{G=\ell_q+1}^{\ell_{q+1}} \Big(\sum_{L=1}^{\ell_q} p^{G-L+1} p^{-2G+2} p^{2L} \\ & + \sum_{L=\ell_q+1}^{G} p^{G-L+1} p^{-2G+2} p^{2G-2\ell_{q+1}+2} p^{2L} \Big) \\ & + \sum_{G=\eta+1}^{\infty} \sum_{L=1}^{G} p^{G-L+1} p^{-2G+2} p^{2\min(L,\eta)} \\ \leq & \sum_{q=0}^{\#\mathcal{L}_{p,N,\alpha}} \sum_{G=\ell_q+1}^{\ell_{q+1}} \Big(p^3 \sum_{L=1}^{\ell_q} p^{-G+L} + p^5 \sum_{L=\ell_q+1}^{G} p^{G+L-2\ell_{q+1}} \Big) \\ & + p^3 \sum_{G=\eta+1}^{\infty} \Big(\sum_{L=1}^{\eta} p^{-G+L} + \sum_{L=\eta+1}^{G} p^{-G-L+2\eta} \Big) \\ \leq & \, 2p^5 \sum_{q=0}^{\#\mathcal{L}_{p,N,\alpha}} \sum_{G=\ell_q+1}^{\ell_{q+1}} \big(p^{-G+\ell_q} + p^{2G-2\ell_{q+1}} \big) \\ & + 2p^3 \sum_{G=\eta+1}^{\infty} \big(p^{-G+\eta} + p^{-G+\eta} \big) \\ \leq & \, 4p^3 + 2p^5 \sum_{q=0}^{\#\mathcal{L}_{p,N,\alpha}} 3 \leq 7p^5 \#\mathcal{L}_{p,N,\alpha},\end{aligned}$$

so that our initial claim on the right inequality holds.

Now, we proceed to show that the left inequality holds. Notice that for $\ell_q \in \mathcal{L}_{p,N,\alpha}$ it holds

$$\|p^{\ell_q-2}\alpha\| = \|p^{\ell_q-2} \sum_{\ell=1}^{\infty} \alpha_\ell p^{-\ell}\| = \|p^{\ell_q-2} \sum_{\ell=\ell_q-1}^{\infty} \alpha_\ell p^{-\ell}\| = \|\sum_{n=1}^{\infty} \alpha_{\ell_q+n-2} p^{-n}\|,$$

and, from Definition 1.2, it follows that

$$\sum_{n=1}^{\infty} \alpha_{\ell_q+n-2} p^{-n} \in \left(\frac{1}{p^2}, \frac{p^2-1}{p^2} \right),$$

because the couple $(\alpha_{\ell_q-1}, \alpha_{\ell_q}) \notin \{(0,0),(p-1,p-1)\}$. Therefore

$$\|p^{\ell_q-2}\alpha\| \geq p^{-2}. \tag{3.8}$$

Now, for $\theta \in \{0,1,2,3\}$ and for $N \in \mathbb{N}$, consider

$$\mathcal{L}^{\theta}_{p,N,\alpha} := \{\ell \text{ such that } \ell \in \mathcal{L}_{p,N,\alpha} \text{ and } \ell \equiv \theta \pmod 4\},$$

then for each $\eta \in \mathbb{N}$, there is a $\theta_\eta \in \{0,1,2,3\}$ such that

$$\#\mathcal{L}^{\theta_\eta}_{p,p^\eta,\alpha} \geq \frac{1}{4} \#\mathcal{L}_{p,p^\eta,\alpha} \tag{3.9}$$

and, without loss of generality, we assume $\theta_\eta \equiv 0$ for all η.

Also, for any $\eta \in \mathbb{N}$ such that $\eta \equiv 0 \pmod 4$, define

$$M_\eta = \sum_{m=1}^{\frac{\eta}{4}} p^{4m-3} = \sum_{n=1}^{\eta} \Delta_n \, p^{n-1} \quad \text{where } \Delta_n = \delta_4(n-2),$$

then notice that $p^{\eta-3} \le M_\eta < p^{\eta-2}$, and therefore $\#\mathcal{L}^0_{p,M_\eta,\alpha} = \#\mathcal{L}^0_{p,p^{\eta-3},\alpha}$.

Now, write $\mathcal{L}^0_{p,M_\eta,\alpha}$ as an increasing sequence of indices $\{\ell_g\}_{g=1}^{\#\mathcal{L}^0_{p,M_\eta,\alpha}}$ and finally, starting with (3.2) and by (3.6)–(3.8), we get

$$\begin{aligned} &\|D_{p,M_\eta,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \ge \\ &\ge 4(\pi p)^{-2} \sum_{\ell_g \in \mathcal{L}^0_{p,M_\eta,\alpha}} p^{-2(\ell_g-2)} \left| \sum_{n=0}^{M_\eta-1} \exp(2\pi i p^{\ell_g-2} x_n) \right|^2 . \end{aligned} \tag{3.10}$$

Notice, by construction, that for ν and ℓ_g such that $0 \le \eta-\nu-1 \le \ell_g-2$ (or equivalently $\delta_{p^{\eta-\nu-1}}(p^{\ell_g-2}) = 1$), we have that the fractional part

$$\left\{ p^{\ell_g-2} \sum_{\mu=0}^{\nu-1} \Delta_{\eta-\mu} p^{-\eta+\mu} \right\} \in \left(\frac{1}{p^4}, \frac{1}{p^3} \right). \tag{3.11}$$

Finally, starting with (3.3), we get

$$\begin{aligned} &\sum_{n=0}^{M_\eta-1} \exp(2\pi i p^{\ell_g-2} x_n) = \\ &= \sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1} \delta_{p^{\eta-\nu-1}}(p^{\ell_g-2}) \sum_{d=0}^{\Delta_{\eta-\nu}-1} \exp\left(2\pi i p^{\ell_g-2} \left(d p^{-\eta+\nu} + \sum_{\mu=0}^{\nu-1} \Delta_{\eta-\mu} p^{-\eta+\mu}\right)\right) \\ &= \sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1} \delta_{p^{\eta-\nu-1}}(p^{\ell_g-2}) \, \Delta_{\eta-\nu} \exp\left(2\pi i p^{\ell_g-2} \sum_{\mu=0}^{\nu-1} \Delta_{\eta-\mu} \, p^{-\eta+\mu}\right), \end{aligned}$$

so that, taking the imaginary part, we get

$$\begin{aligned} &\left| \sum_{n=0}^{M_\eta-1} \exp(2\pi i p^{\ell_g-2} x_n) \right| \ge \\ &\ge \left| \sum_{\nu=0}^{\eta-1} p^{\eta-\nu-1} \delta_{p^{\eta-\nu-1}}(p^{\ell_g-2}) \, \Delta_{\eta-\nu} \sin\left(2\pi p^{\ell_g-2} \sum_{\mu=0}^{\nu-1} \Delta_{\eta-\mu} p^{-\eta+\mu}\right) \right| . \end{aligned} \tag{3.12}$$

Recall that the sine function $\sin(2\pi\cdot)$ is increasing and positive in the interval $(0, \frac{1}{4})$ so, from (3.11), it follows that

$$\sin\left(2\pi p^{\ell_g-2} \sum_{\mu=0}^{\nu-1} \Delta_{\eta-\mu} p^{-\eta+\mu}\right) \ge \sin(2\pi p^{-4})$$

whenever $\eta-\nu-1 \le \ell_g-2$. On the other hand, whenever $\eta-\nu-1 > \ell_g-2$, we trivially get $\delta_{p^{\eta-\nu-1}}(p^{\ell_g-2}) = 0$.

In conclusion, from (3.12), we get

$$\left|\sum_{n=0}^{M_\eta-1}\exp(2\pi i p^{\ell_g-2}x_n)\right| \geq \sin(2\pi p^{-4})\left|\sum_{\nu=\eta-\ell_g+1}^{\eta-1} p^{\eta-\nu-1}\Delta_{\eta-\nu}\right|$$
$$\geq \sin(2\pi p^{-4})\sum_{\tau=0}^{\ell_g-2} p^{\tau}\Delta_{\tau+1} \geq \sin(2\pi p^{-4})p^{-1}p^{\ell_g-2}.$$

Therefore, (3.10) leads us to

$$\|D_{p,M_\eta,\alpha}(\cdot)\|^2_{L^2(\mathbb{T})} \geq 4\pi^{-2}\sin(2\pi p^{-4})p^{-3}\sum_{\ell_g\in\mathcal{L}^0_{p,M_\eta,\alpha}} p^{-2(\ell_g-2)}p^{2(\ell_g-2)}$$
$$= 4\pi^{-2}\sin(2\pi p^{-4})p^{-3}\#\mathcal{L}^0_{p,M_\eta,\alpha},$$

and, by (3.9), we have

$$\#\mathcal{L}^0_{p,M_\eta,\alpha} = \#\mathcal{L}^0_{p,p^{\eta-3},\alpha} \geq 1 + \#\mathcal{L}^0_{p,p^\eta,\alpha} \geq 1 + \frac{1}{4}\#\mathcal{L}_{p,p^\eta,\alpha}.$$

Hence, letting $\eta \to \infty$, we obtain that our initial claim holds. □

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