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Shielding of breathers for the focusing nonlinear Schrödinger equation

Gregorio Falqui [a,b,*], Tamara Grava [c,d,e], Christian Puntini [f]

[a] Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via R. Cozzi 55, 20125 Milano, Italy
[b] INFN, Sezione di Milano, Italy
[c] INFN, Sezione di Trieste, Italy
[d] SISSA, via Bonomea 265, 34136, Trieste, Italy
[e] School of Mathematics, University of Bristol, Fry Building, Bristol, BS8 1UG, UK
[f] Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

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ABSTRACT

We study a deterministic gas of breathers for the Focusing Nonlinear Schrödinger equation. The gas of breathers is obtained from a N-breather solution in the limit $N \to \infty$. The limit is performed at the level of scattering data by letting the N-breather spectrum to fill uniformly a suitable compact domain of the complex plane in the limit $N \to \infty$. The corresponding norming constants are interpolated by a smooth function and scaled as $1/N$. For particular choices of the domain and the interpolating function, the gas of breathers behaves as finite breathers solution. This extends the *shielding effect* discovered in Bertola et al. (2023) for a soliton gas also to a breather gas.

1. Introduction

The hallmark of integrability of a non-linear dispersive wave equation is the existence of pseudo-particle solutions, that is, coherent nonlinear modes that interact elastically. Recently several investigations involving very large number of solitons have been carried out, both on the mathematical and the physical side [1–7]. Coherent and random superpositions of solitons were considered.

Random nonlinear superpositions of solitons are more closely related to the notion of a soliton gas in an infinite statistical ensemble of interacting solitons that was first introduced by Zakharov by considering the Korteweg de Vries (KdV) equation in [8]. More recently this concept has been extended to the Focusing Nonlinear Schrödinger (FNLS) equation by El and Tovbis in [1], and has been studied in the context of the modified Korteweg–de Vries (mKdV) equation with nonzero boundary conditions by Zhang and Ling in [9]. Moreover, in [10] the problem of both a deterministic and a random distribution of N soliton solutions of FNLS was discussed and in [11] a central limit theory for a soliton gas has been derived. In [10] it was proved that a particular configuration of a deterministic soliton gas surprisingly yields a one-soliton solution. Such a phenomenon was termed *soliton shielding* and further studied in [12].

In this paper we study the corresponding problem considering the case of a *gas of breathers*. Generally speaking, a breather is a nonlinear wave in which energy concentrates in a localized and oscillatory way. The first breather type solution of FNLS was found in [13,14] and called *Kuznetsov–Ma breather*. It is localized in the spatial variable x and periodic in the temporal variable t. In particular, in [14] the initial value problem for FNLS with initial data a perturbed plane wave solution with non-vanishing boundary conditions at $x \to \pm\infty$ was shown to yield asymptotic states consisting of periodically "pulsating" solitary waves (plus a residual of small amplitude dispersive waves).

Different kind of such pulsating localized solution to FNLS were subsequently discovered. In [15] a rational solution of this kind, called *Peregrine breather* was found. It is localized in both spatial and temporal variables (x, t). In [16] an x-periodic and t-localized new solution, called *Akhmediev breather* was discovered. In 1988 Tajiri and Watanabe [17] found a travelling breather, now called *Tajiri–Watanabe breather*, whose speed depends on the spectrum of the associated linear problem. Such solutions have been shown to be a model of modulational instability by Zakharov and Gelash in 2013 [18,19].

The Kuznetsov–Ma, Peregrine and Akhmediev breathers find applications as prototypical models of rogue waves in nonlinear optics and the theory of gravity water waves [20–24]. Furthermore the Peregrine breather emerges in a universal way in the semiclassical limit of the focusing nonlinear Schrödinger equation [25]. Higher order breathers solution of FNLS obtained using Darboux transformations from the rational Peregrine breather in [26,27] were further studied in [4] and observed in a water-wave tank to be approximation of "super rogue waves" with an amplitude of up to 5 times the background in [28].

* Corresponding author at: Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, via R. Cozzi 55, 20125 Milano, Italy.
E-mail address: gregorio.falqui@unimib.it (G. Falqui).

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This analysis was pushed forward by Bilman, Buckingham, Ling, Miller and Wang [29–33] where breathers and solitons of infinite order have been considered, and a universal profile emerges after an appropriate scaling.

The starting point of our study is the fact that N-breather solutions to FNLS can be obtained from the Zakharov–Shabat linear spectral problem, reformulated as a Riemann–Hilbert problem for a 2×2 matrix $M(z;x,t)$, z being the spectral parameter, with suitable scattering data [4,34]. In particular, for the Kuznetsov–Ma and the Tajiri–Watanabe N-breather, one needs $4N$ poles where N poles are free parameters located in the upper half complex plane minus the unit half disk, while the remaining $3N$ poles are obtained by suitable symmetries. The spectral data are completed by assigning to each of the first N poles a norming constant, while the remaining norming constants are obtained by symmetry requirements. Our main aim is to study the limit of the N-breather solution when the number N goes to infinity and the location of the poles accumulates uniformly on a bounded domain $\mathcal{D}_1$ of the complex plane (and its "symmetric" counterparts as required by the very notion of breather solution). The corresponding norming constants are interpolated by a smooth function and rescaled by $1/N$. In this way we arrive at the inverse problem for a gas of breathers and we show the existence of its solution. For certain choices of the domain $\mathcal{D}_1$ and the interpolation function, the behaviour of the gas of breathers corresponds to a n-breather solution, n being determined by the geometry of the above-mentioned domain of pole accumulation $\mathcal{D}_1$.

The layout of the present paper is the following.

- In the second section we briefly review the characteristic properties of the inverse spectral problem for breather solutions, as well as the associated Riemann–Hilbert problem, and frame the Kuznetsov–Ma and the Tajiri–Watanabe 1-breather solution within this scheme.
- In section three we transform the Riemann–Hilbert problem for N poles into a Riemann–Hilbert problem with a jump matrix defined on closed contours encircling the accumulation domains of the poles and describe its solution.
- We finally pass to the $N \to \infty$ limit, showing the existence of the solution of the limiting inverse problem. We illustrate the shielding effect for particular choices of the domain $\mathcal{D}_1$ and we explicitly illustrate our procedure and results for the cases when the limiting solution reduces to a $n = 1$ and $n = 2$ shielding breather, corresponding to the choice of the domain $\mathcal{D}_1$ (respectively) a disk and an eight-shaped figure.

2. A brief review of the IST for breathers

In this section we report and summarize the main steps of the inverse scattering transform for the N-breather solution of the FNLS equation

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0\,, \tag{1}$$

with non-zero boundary conditions at $x = \pm\infty$ originally introduced in [13] and further developed in [34,35].

In order to remove the possible time-dependence of the boundary conditions, we can rewrite Eq. (1) as

$$i\psi_t + \psi_{xx} + 2\left(|\psi|^2 - \psi_0^2\right)\psi = 0\,, \tag{2}$$

the nonzero boundary conditions as $x \to \pm\infty$ being

$$\lim_{x\to\pm\infty} \psi(x,t) = \psi_\pm, \quad |\psi_\pm| =: \psi_0 \neq 0. \tag{3}$$

The additional term $-2\psi_0^2\psi$ in (2) that has been added in order to make the boundary conditions (3) independent of time can be removed by the rescaling $\widetilde{\psi}(x,t) = e^{2i\psi_0^2 t}\psi(x,t)$.

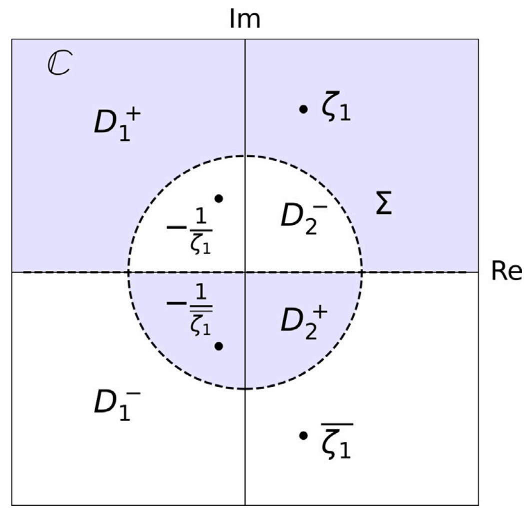

Fig. 1. The complex plane divided as in (6) and 4 poles satisfying (7). Fixing ζ_1 in D_1^+ determines the other three "fellow" points of the spectrum, $\zeta_2 = -1/\bar{\zeta}_1, \zeta_3 = \bar{\zeta}_1, \zeta_4 = -1/\zeta_1$.

As it well known, the FNLS Eq. (1) admits the scaling invariance

$$\psi(x,t) \to \sigma\psi(\sigma x, \sigma^2 t), \quad \forall\, \sigma > 0, \tag{4}$$

hence, we may assume that $\psi_- = \psi_0 = 1$ and ψ_+ is an arbitrary constant with $|\psi_+| = 1$. The nonlinear Schrödinger equation is the compatibility condition of the Zakharov–Shabat linear spectral problem [36]

$$\begin{aligned} &\partial_x f = \mathcal{L} f, \qquad \mathcal{L} = ik\sigma_3 + \Psi, \quad \Psi = \begin{pmatrix} 0 & \psi \\ -\overline{\psi} & 0 \end{pmatrix}, \\ &i\partial_t f = \mathcal{B} f, \qquad \mathcal{B} = -2ik^2\sigma_3 + i\sigma_3(\Psi_x - \Psi^2 - 1) - 2k\Psi\ , \end{aligned} \tag{5}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\overline{\psi}$ stands for complex conjugate. The direct and inverse spectral problems with non zero boundaries conditions have been developed by Biondini and Kovačič in [34], extending the seminal work of Kuznetsov in 1977 [13]. Here, we summarize the inverse spectral problem for reflectionless potentials in the complex z-plane where $z = k + \sqrt{k^2+1}$.

The spectral z-complex plane is partitioned into the following domains:

$$\begin{aligned} D^+ &= \left\{ z \in \mathbb{C}\ :\ \left(|z|^2 - 1\right)\Im(z) > 0 \right\}, \\ D^- &= \left\{ z \in \mathbb{C}\ :\ \left(|z|^2 - 1\right)\Im(z) < 0 \right\}, \end{aligned} \tag{6}$$

with boundary $\Sigma = \mathbb{R} \cup S^1 = \mathbb{R} \cup \{z \in \mathbb{C}\ :\ |z| = 1\}$. We observe that D^+ is the union of the exterior to the unit disk in the upper half plane D_1^+ and the interior of the unit disk in the lower half plane D_2^+. The domain D^-, obtained from D^+ by reflection along the real axis, is equally partitioned into an unbounded domain D_1^- in the lower half plane and a bounded domain D_2^- in the upper half plane (see Fig. 1).

The N-breather solution is obtained from the following scattering data:

the discrete spectrum $\mathcal{Z} = \{\zeta_1, \ldots, \zeta_{2N}, \overline{\zeta_1}, \ldots, \overline{\zeta_{2N}}\}$ of the Zakharov–Shabat linear spectral problem [34] with

$$\begin{aligned} &\zeta_j \in D_1^+, \quad \zeta_{N+j} = -\frac{1}{\overline{\zeta}_j} \in D_2^+, \\ &\overline{\zeta_j} \in D_1^-, \quad \overline{\zeta_{N+j}} = -\frac{1}{\zeta_j} \in D_2^-, \end{aligned} \qquad \text{with } j = 1, \ldots, N. \tag{7}$$

The points of the discrete spectrum satisfy, with respect to the boundary values $\psi_\pm$ in (3), the "theta" condition :

$$\arg\left(\frac{\psi_-}{\psi_+}\right) = 4\sum_{j=1}^{N} \arg(\zeta_j). \tag{8}$$

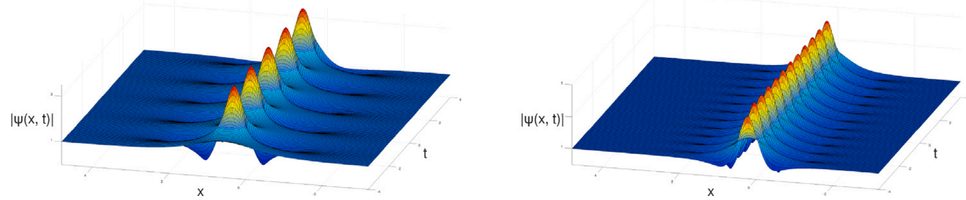

Fig. 2. Two different Kuznetsov–Ma breathers with $Z = 2$, $\kappa = 0$, $\phi = 0$ (left) and $Z = 3$, $\kappa = 1$, $\phi = 0$ (right). In both cases, $\psi(x,t) \to 1$ as $x \to \pm\infty$.

Condition (8) is termed "theta" as it relates the discrete spectrum, the elements of the scattering matrix and the asymptotic phase difference of the corresponding FNLS solution, originally denoted by θ in [37] . As we are assuming $\psi_- = 1$, (8) becomes, in our case,

$$-\arg\left(\psi_+\right) = 4\sum_{j=1}^{N} \arg(\zeta_j), \quad (\mathrm{mod}\ 2\pi). \tag{9}$$

To complete the scattering data, we need to add the norming constants. For each point of the spectrum $\{\zeta_j\}_{j=1}^N$ in D_1^+, we chose a nonzero complex number, obtaining the sequence $\{C_j\}_{j=1}^N$. The remaining $3N$ norming constants are obtained by symmetry as follows: for points in $D_2^+ \subset D^+$ one sets

$$C_{N+j} = -\left(\frac{1}{z_j}\right)^2 \overline{C_j}, \quad \text{for } j = 1,\dots,N, \tag{10}$$

while the $2N$ norming constants associated with the poles in D^- are obtained from the previous ones as

$$C_{2N+\alpha} = -\overline{C_\alpha}, \quad \text{for } \alpha = 1,\dots,2N. \tag{11}$$

The N-breather solution is recovered by solving the following Riemann–Hilbert problem.

Riemann–Hilbert problem A (Rational N-breather problem):
To find a 2×2 matrix $M(z;x,t)$ with the following properties:

- $M(z;x,t)$ is a meromorphic matrix in $\mathbb{C}$, with simple poles in the set $\mathcal{Z} = \{\zeta_\alpha, \overline{\zeta_\alpha}\}_{\alpha=1}^{2N}$ and $z = 0$.
- $M(z;x,t)$ satisfies the following residue conditions

$$\begin{aligned}
\mathrm{Res}_{z=\zeta_\alpha} M(z;x,t) &= \lim_{z\to\zeta_\alpha} M(z;x,t)\begin{bmatrix} 0 & 0 \\ C_\alpha e^{-2i\theta(\zeta_\alpha;x,t)} & 0\end{bmatrix},\\
\mathrm{Res}_{z=\overline{\zeta_\alpha}} M(z;x,t) &= \lim_{z\to\overline{\zeta_\alpha}} M(z;x,t)\begin{bmatrix} 0 & -\overline{C_\alpha} e^{2i\overline{\theta(\zeta_\alpha;x,t)}} \\ 0 & 0\end{bmatrix},
\end{aligned} \tag{12}$$

for $\alpha = 1,\dots,2N$, with

$$\theta(z;x,t) = \frac{1}{2}\left(z+\frac{1}{z}\right)\left(x - \left(z-\frac{1}{z}\right)t\right). \tag{13}$$

- $M(z;x,t)$ satisfies the following asymptotic conditions:

$$M(z;x,t) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z\to\infty, \tag{14}$$

$$M(z;x,t) = \frac{i}{z}\sigma_1 + \mathcal{O}(1), \quad \text{as } z\to 0, \tag{15}$$

where σ_1 is the Pauli matrix $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

For further use we note that

$$\overline{\theta(z;x,t)} = \theta(\overline{z};x,t), \quad \text{and} \quad \theta\left(-\frac{1}{z};x,t\right) = -\theta(z;x,t). \tag{16}$$

From the solution of the above Riemann–Hilbert problem A for the 2×2 matrix M, one recovers the N-breather solution of the focusing nonlinear Schrödinger equation by the relation

$$\psi(x,t) = -i\lim_{z\to\infty}(zM_{12}(z;x,t)), \tag{17}$$

where M_{12} is the 1,2 entry of M.

The conditions (12) show that the matrix M is meromorphic in the complex plane; in particular, for $z \in D_+$, the second column of M is analytic while the first column has a first order pole at $\zeta_\alpha \in D_+$, $\alpha = 1,\dots,2N$. On the other hand the second column of M is analytic in D_- and it has a first order pole at $\overline{\zeta_\alpha} \in D_-$, $\alpha = 1,\dots,2N$.

Taking into account the asymptotic conditions (14), the 2×2 matrix $M(z;x,t)$ takes the form

$$M(z;x,t) = \mathbb{I} + \frac{i}{z}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \sum_{\alpha=1}^{2N}\frac{\begin{bmatrix}A_\alpha(x,t) & 0\\ B_\alpha(x,t) & 0\end{bmatrix}}{z-\zeta_\alpha} + \sum_{\alpha=1}^{2N}\frac{\begin{bmatrix}0 & E_\alpha(x,t)\\ 0 & F_\alpha(x,t)\end{bmatrix}}{z-\overline{\zeta_\alpha}}, \tag{18}$$

where the coefficients $A_\alpha(x,t)$, $B_\alpha(x,t)$, $E_\alpha(x,t)$, $F_\alpha(x,t)$ are to be determined from the residue conditions (12) as follows. Let us introduce the notation

$$c_\alpha = C_\alpha e^{-2i\theta(\zeta_\alpha;x,t)}, \quad \alpha = 1\dots,2N.$$

From the residue condition (12) of the N-breather solution and the ansatz (18) we obtain

$$\begin{aligned}
\begin{bmatrix}A_\beta\\ B_\beta\end{bmatrix} &= \begin{bmatrix}\frac{ic_\beta}{\zeta_\beta}\\ c_\beta\end{bmatrix} + c_\beta\sum_{\alpha=1}^{2N}\frac{\begin{bmatrix}E_\alpha\\ F_\alpha\end{bmatrix}}{\zeta_\beta - \overline{\zeta_\alpha}},\\
\begin{bmatrix}E_\beta\\ F_\beta\end{bmatrix} &= -\begin{bmatrix}\overline{c_\beta}\\ \frac{i\overline{c_\beta}}{\overline{\zeta_\beta}}\end{bmatrix} - \overline{c_\beta}\sum_{\alpha=1}^{2N}\frac{\begin{bmatrix}A_\alpha\\ B_\alpha\end{bmatrix}}{\overline{\zeta_\beta} - \zeta_\alpha},
\end{aligned} \tag{19}$$

for $\beta = 1,\dots,2N$. This is a system of $2N$ equations in $2N$ unknowns $A_\alpha, B_\alpha, E_\alpha$ and F_α (see Appendix A for a comprehensive solution).

From (17) and (19), the N-breather solution of the focusing nonlinear Schrödinger equation is finally recovered as

$$\psi(x,t) = -i\lim_{z\to\infty} zM_{12}(z;x,t) = 1 - i\sum_{\alpha=1}^{2N} E_\alpha(x,t). \tag{20}$$

2.1. Kuznetsov–Ma and Tajiri–Watanabe breathers

We report the formulæ of the Kuznetsov–Ma [13,14] and the Tajiri–Watanabe [17,38] breather solutions. The formulæ presented here have been derived from the expression $\psi(x,t) = 1 - i\,(E_1(x,t) + E_2(x,t))$ where $E_1(x,t)$ and $E_2(x,t)$ solve the linear system (76) in Appendix A. The Kuznetsov–Ma breather is obtained from the data (ζ_1, C_1) where the single pole ζ_1 is purely imaginary

$$\zeta_1 = iZ, \quad Z > 1 \quad \text{and } C_1 = e^{\kappa + i\phi}, \quad \text{with } \kappa,\ \phi \text{ real}. \tag{21}$$

The remaining poles and norming constants are obtained by symmetry according to (7), (10) and (11) respectively. The Kuznetsov–Ma breather solution is given by

$$\psi_{1KM}(x,t) = \frac{\cosh(\chi(x)) - \frac{1}{2}c_+\left(1 + c_-^2/c_+^2\right)\sin(s(t)) + ic_-\cos(s(t))}{\cosh(\chi(x)) - A\sin(s(t))}, \tag{22}$$

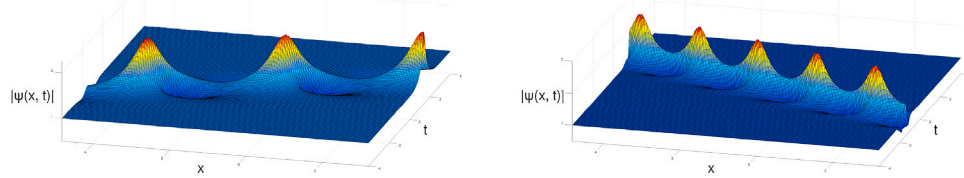

Fig. 3. Two different Tajiri–Watanabe breathers, left $Z = 2$, $\kappa = 0$, $\phi = 0$ and $\alpha = \pi/6$ and right $Z = 3$, $\kappa = 0$, $\phi = 0$ and $\alpha = -\pi/8$.

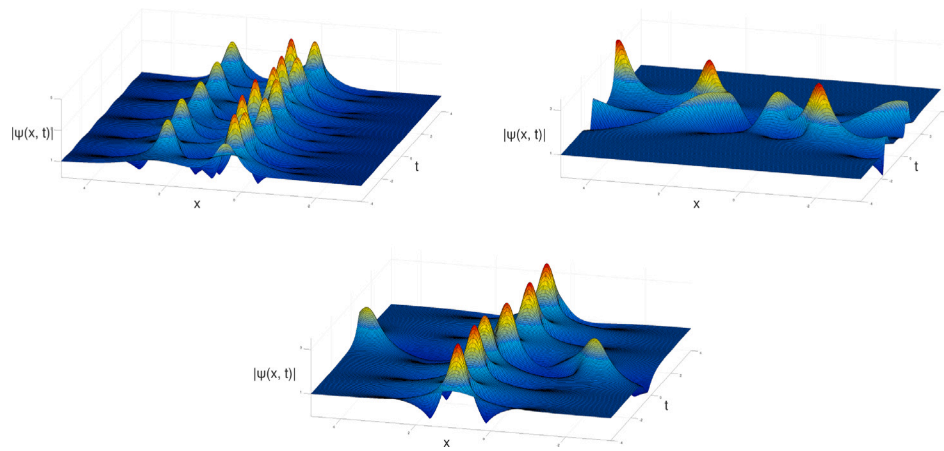

Fig. 4. The three possible kinds of 2-breather solutions. Top left: a KM-type 2-breather with $\zeta_1 = 2i$ and $\zeta_2 = 3i$. Top right: a TW-type 2-breather with $\zeta_1 = 1+2i$ and $\zeta_2 = -2+i$. Bottom: a 2-breather of mixed type with $\zeta_1 = 2i$ and $\zeta_2 = 1+i$. In all three plots we set $C_1 = 1$ and $C_2 = 2$.

with

$$
\begin{aligned}
&c_\pm = Z \pm \frac{1}{Z}, \quad A = \frac{2}{c_+} < 1,\\
&\chi(x) = c_- x + c_0 + \kappa, \quad c_0 = \ln\left(\frac{c_+}{2Zc_-}\right),\\
&s(t) = c_+ c_- t - \phi.
\end{aligned}
\tag{23}
$$

We remark that the Kuznetsov–Ma breather is a stationary breather, since it is localized in the spatial variable x and periodic in the temporal variable t, with period $T = \frac{2\pi}{c_+c_-} = \frac{2\pi}{Z^2-1/Z^2}$ (see Fig. 2).

The Tajiri–Watanabe breather is instead obtained by choosing

$$
\begin{aligned}
&\zeta_1 = iZe^{i\alpha}, \quad \text{with } Z > 1 \text{ and } -\frac{\pi}{2} < \alpha < \frac{\pi}{2},\\
&C_1 = e^{\kappa + i\phi}, \quad \text{with } \kappa \text{ and } \phi \text{ real.}
\end{aligned}
\tag{24}
$$

The corresponding formula for the solution is

$$
\psi_{1TW}(x,t) = \frac{\cosh(\chi - 2i\alpha) + \frac{B}{2}\left[d_+(Z^2\sin(s-2\alpha) - \sin(s)) + i\,d_-(Z^2\cos(s-2\alpha) - \cos(s))\right]}{e^{2i\alpha}(\cosh(\chi) + B\left[Z^2\sin(s-2\alpha) - \sin(s)\right])},
\tag{25}
$$

with

$$
\begin{aligned}
&c_\pm = Z \pm \frac{1}{Z}, \quad d_\pm = Z^2 \pm \frac{1}{Z^2},\\
&B = \frac{2\cos(\alpha)}{|1 - Z^2 e^{-2i\alpha}|\left(Z + \frac{1}{Z}\right)},\\
&\chi(x,t) = c_- x\cos(\alpha) + d_+ t\sin(2\alpha) + c_0' + \kappa,\\
&s(x,t) = c_+ x\sin(\alpha) - d_- t\cos(2\alpha) + \phi,\\
&c_0' = -\ln\left(\frac{2\cos(\alpha)|1 - Z^2e^{-2i\alpha}|}{Z + \frac{1}{Z}}\right).
\end{aligned}
\tag{26}
$$

Let us observe that

$$
\psi_{1TW}(x,t) \to 1 \text{ as } x \to -\infty, \quad \psi_{1TW}(x,t) \to e^{-4i\alpha}, \text{ as } x \to +\infty.
$$

The Tajiri–Watanabe breather is a non-stationary breather, since the peak does not remain localized at a fixed value in x but rather moves with a speed given by equation $\chi(x,t) = 0$ (see Fig. 3).

We have limited ourselves to display formulas for 1-breather solutions. There exist N-breather solutions for every natural N which can be obtained from (20) by solving the linear system (74). We report in Fig. 4 the plot of three different 2-breather solutions obtained via different choices of pole locations.

As it is well known in the theory of integrable systems, the 2-breather solution is not the sum of two 1-breather solutions. Such decomposition is true for large times $t \to \pm\infty$ when the 2-breather solution is asymptotically close to the sum of two 1-breather solutions, plus o(1) [39].

Remark 1. In the limit $Z \to 1$ the Tajiri–Watanbe breather converges to the Akhmediev breather

$$
\psi_{1A}(x,t) = \frac{\cosh(2t\sin(2\alpha) + c_0' + \kappa - 2i\alpha) - \cos(\alpha)\cos(2x\sin(\alpha) + \phi - \alpha)}{e^{2i\alpha}(\cosh(2t\sin(2\alpha) + c_0' + \kappa) - \cos(\alpha)\cos(2x\sin(\alpha) + \phi - \alpha))},
\tag{27}
$$

with

$$
c_0' = -\ln\left(2\cos(\alpha)|\sin(\alpha)|\right). \tag{28}
$$

In the further limit $\alpha \to 0$ one recovers the Peregrine breather.

3. Gas of breathers and shielding

This section contains our main result. We consider a N-breather solution with the first N poles $\zeta_1, \ldots, \zeta_N$ filling uniformly a "parent" bounded domain $\mathcal{D}_1$ in D_1^+, with the corresponding norming constants interpolated by a smooth function and rescaled by N, and we let N go to infinity. Moreover, we assume that $\mathcal{D}_1$ has a smooth boundary. In order to perform the limit $N \to \infty$ on the inverse problem for $M(z;x,t)$, we transform the residue conditions (12) into jump conditions for a suitably defined Riemann–Hilbert problem.

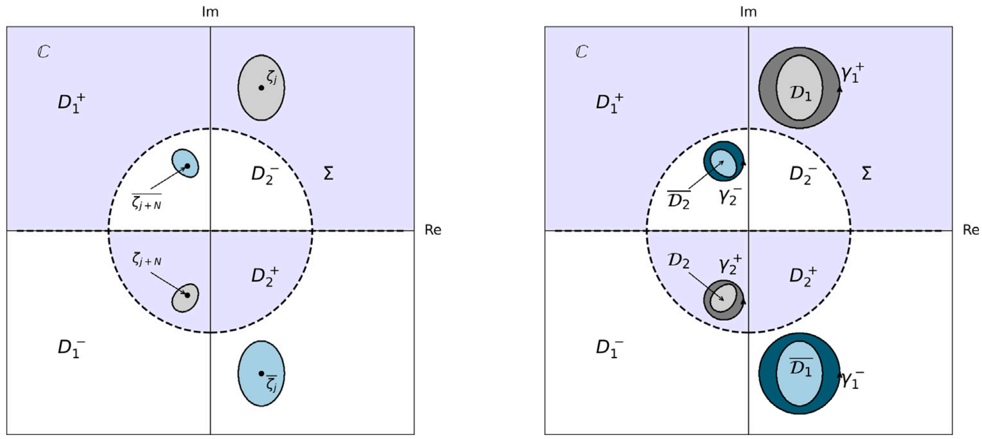

Fig. 5. The complex plane with the four domains $\mathcal{D}_1$, $\mathcal{D}_2$, $\overline{\mathcal{D}_1}$, $\overline{\mathcal{D}_2}$ and the four contours γ_1^+, γ_2^+, γ_1^-, γ_2^-.

For the purpose we introduce a close anticlockwise oriented contour γ_1^+ in D_1^+ that encircles $\mathcal{D}_1$ and we call Γ_1^+ the domain with boundary γ_1^+. The other domains are obtained by symmetry: in particular, $\mathcal{D}_2$ is the domain obtained from $\mathcal{D}_1$ by the relation

$$z \in \mathcal{D}_1 \iff -\frac{1}{\overline{z}} \in \mathcal{D}_2\,, \tag{29}$$

while $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ in D^- are their complex conjugates.

We introduce a closed anticlockwise oriented contour γ_2^+ that encircles $\mathcal{D}_2$ in D_2^+ and we define as Γ_2^+ the finite domain with boundary γ_2^+.

Analogously we define $\gamma_1^- = -\overline{\gamma_1^+}$ and $\gamma_2^- = -\overline{\gamma_2^+}$ as the closed anticlockwise oriented contours that encircle $\overline{\mathcal{D}_1}$ and $\overline{\mathcal{D}_2}$ respectively. Also in this case we call Γ_1^- and Γ_2^- the domains whose boundaries are γ_1^- and γ_2^- respectively (see Fig. 5 for the geometry of these contours).

We reformulate the Riemann–Hilbert problem A as a Riemann–Hilbert problem for a matrix $Y^N(z;x,t)$ that is obtained from $M(z;x,t)$ by removing the poles but paying the price of putting a jump.

More precisely, we define the matrix $Y^N(z;x,t)$ as

$$Y^N(z;x,t) = \begin{cases} M(z;x,t) & \text{for } z \in \mathbb{C} \setminus \left\{\Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_1^- \cup \Gamma_2^-\right\}, \\ M(z;x,t)J^N(z;x,t) & \text{for } z \in \Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_1^- \cup \Gamma_2^-, \end{cases} \tag{30}$$

with the jump matrix $J^N(z;x,t)$ having the following expression

$$J^N(z;x,t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -e^{-2i\theta(z)} \displaystyle\sum_{j=1}^{N} \frac{C_j}{z-\zeta_j} & 1 \end{bmatrix} & \text{for } z \in \Gamma_1^+, \\ \begin{bmatrix} 1 & 0 \\ -e^{2i\theta(z)} \displaystyle\sum_{j=N+1}^{2N} \frac{C_j}{z-\zeta_j} & 1 \end{bmatrix} & \text{for } z \in \Gamma_2^+, \\ \begin{bmatrix} 1 & e^{2i\theta(z)} \displaystyle\sum_{j=1}^{N} \frac{\overline{C_j}}{z-\overline{\zeta_j}} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \Gamma_1^-, \\ \begin{bmatrix} 1 & e^{2i\theta(z)} \displaystyle\sum_{j=N+1}^{2N} \frac{\overline{C_j}}{z-\overline{\zeta_j}} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \Gamma_2^-, \end{cases} \tag{31}$$

where we used the shorthand notation $\theta(z) = \theta(z;x,t)$.

It is immediate to check that the requirement of $Y^N(z;x,t)$ to be analytic in $\mathbb{C}\backslash\{\boldsymbol{\gamma} \cup \{0\}\}$, where $\boldsymbol{\gamma} = \{\gamma_1^+ \cup \gamma_2^+ \cup \gamma_1^- \cup \gamma_2^-\}$, is equivalent to the residue conditions (12).

Therefore, we obtain the following Riemann–Hilbert problem for the matrix function $Y^N(z;x,t)$.

Riemann–Hilbert problem B (jump N-breather problem):
To find a 2×2 matrix $Y^N(z;x,t)$ with the following properties:

- $Y^N(z;x,t)$ is analytic in $\mathbb{C}\backslash\{\boldsymbol{\gamma}\cup\{0\}\}$.
- $Y^N(z;x,t)$ satisfies to the jump condition
$$Y_+^N(z;x,t) = Y_-^N(z;x,t)J^N(z;x,t) \tag{32}$$
for $z \in \boldsymbol{\gamma}$. The subscript $Y_\pm(z)$ denotes the left/right boundary values of $Y(z)$ as $z \to \boldsymbol{\gamma}$ in a non tangential direction.
- $Y^N(z;x,t)$ fulfils the asymptotic conditions
$$\begin{aligned} &Y^N(z;x,t) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z\to\infty, \\ &Y^N(z;x,t) = \frac{i}{z}\sigma_1 + \mathcal{O}(1), \quad \text{as } z \to 0\,. \end{aligned} \tag{33}$$

Finally, the N-breather solution of the focusing nonlinear Schrödinger equation is recovered from the matrix $Y^N(z;x,t)$ by the relation

$$\psi(x,t) = -i \lim_{z\to\infty} zY_{12}^N(z;x,t)\,. \tag{34}$$

3.1. Gas of breathers

Now we have reformulated the inverse problem in such a way that inspecting the jump matrix (31), it is possible to perform the limit $N\to\infty$ by assuming that the point spectrum $\zeta_1,\dots\zeta_N$ of the breathers accumulates uniformly inside the domain $\mathcal{D}_1 \subset \Gamma_1^+$ as $N\to\infty$ (see Fig. 6). Namely, we assume that the spectral breather density converges to the uniform measure of the domain $\mathcal{D}_1$, that is

$$\frac{\mathcal{A}_1}{N} \overset{N\to\infty}{\to} d^2w, \tag{35}$$

where $\mathcal{A}_1$ is the area of $\mathcal{D}_1$ and d^2w is the infinitesimal cartesian area.

We interpolate the norming constants as

$$\begin{aligned} C_j &= \frac{\mathcal{A}_1}{\pi N}\beta_1\left(\zeta_j,\overline{\zeta_j}\right), \\ C_{N+j} &= \frac{\mathcal{A}_1}{\pi N}\beta_2\left(\zeta_{N+j},\overline{\zeta_{N+j}}\right), \end{aligned} \qquad j=1,\dots,N\,, \tag{36}$$

where $\beta_1(\cdot,\cdot)$ and $\beta_2(\cdot,\cdot)$ are smooth functions, and we use the notation $\beta_1\left(\zeta_j,\overline{\zeta_j}\right)$ to stress the fact that the function β_1 is not analytic. Since the norming constants satisfy the conditions (10) and (11), in particular

$$C_{N+j} = -\left(\frac{1}{\zeta_j}\right)^2 \overline{C_j}, \quad j=1,\dots,N, \tag{37}$$

it follows that

$$\beta_2\left(\zeta_{N+j},\overline{\zeta_{N+j}}\right) = -\left(\frac{1}{\overline{\zeta_j}}\right)^2 \overline{\beta_1\left(\zeta_j,\overline{\zeta_j}\right)}$$

or, equivalently using the relation $\zeta_{N+j} = -\frac{1}{\overline{\zeta_j}}$,

$$\beta_2\left(z,\overline{z}\right) = -z^2\,\overline{\beta_1\left(-\frac{1}{\overline{z}},-\frac{1}{z}\right)}, \quad z \in \mathcal{D}_2. \tag{38}$$

In this way the sum in the off-diagonal entries of the jump matrix J^N in (31) becomes a Riemann sum and we have

$$\begin{aligned}
\sum_{j=1}^{N} \frac{C_j}{(z-\zeta_j)} &= \sum_{j=1}^{N} \frac{\mathcal{A}_1}{\pi N}\frac{\beta_1(\zeta_j,\overline{\zeta_j})}{(z-\zeta_j)} \underset{N\to\infty}{\longrightarrow} \iint_{\mathcal{D}_1} \frac{\beta_1(w,\overline{w})}{z-w}\frac{d^2 w}{\pi},\\
\sum_{j=N+1}^{2N} \frac{C_j}{(z-\zeta_j)} &= \sum_{j=N+1}^{2N} \frac{\mathcal{A}_1}{\pi N}\frac{\beta_2(\zeta_j,\overline{\zeta_j})}{z-\zeta_j} \underset{N\to\infty}{\longrightarrow} \iint_{\mathcal{D}_2} \frac{\beta_2(w,\overline{w})}{z-w}\frac{d^2 w}{\overline{w}^2 w^2 \pi},
\end{aligned} \tag{39}$$

where the second relation follows from the fact the uniform area measure in $\mathcal{D}_1$ becomes, under the coordinate transformation $w \to -\frac{1}{\overline{w}}$, the measure $\frac{d^2 w}{\overline{w}^2 w^2}$ in $\mathcal{D}_2$. Since the "theta" condition

$$-\arg\left(\psi_+\right) = 4\sum_{n=1}^{N} \arg\left(\zeta_n\right) \tag{40}$$

must hold, we assume that the domain $\mathcal{D}_1$ (and consequentially $\overline{\mathcal{D}_1}$, $\mathcal{D}_2$ and $\overline{\mathcal{D}_2}$) is chosen such that (40) holds also when passing to the limit $N \to \infty$.

Therefore the Riemann–Hilbert problem B, when $N \to \infty$, transforms into the following .

Riemann–Hilbert problem C (jump $N \to \infty$ problem):

To seek for a 2×2 matrix $Y^\infty(z;x,t)$ with the following properties:

- $Y^\infty(z;x,t)$ is analytic in $\mathbb{C}\backslash\{\boldsymbol{\gamma} \cup \{0\}\}$.
- $Y^\infty(z;x,t)$ fulfils the jump condition $\mathbb{C}\backslash\{\boldsymbol{\gamma} \cup \{0\}\}$.

$$Y_+^\infty(z;x,t) = Y_-^\infty(z;x,t)(\mathbb{I} + J^\infty(z;x,t)), \quad z \in \boldsymbol{\gamma}, \tag{41}$$

with

$$\begin{aligned}
J^\infty(z;x,t) &= \begin{bmatrix} 0 & e^{2i\theta(z)} J_{12}^\infty(z;x,t) \\ -e^{-2i\theta(z)} J_{21}^\infty(z;x,t) & 0 \end{bmatrix},\\
J_{21}^\infty(z;x,t) &= \iint_{\mathcal{D}_1} \frac{\beta_1(w,\overline{w})d^2 w}{\pi(w-z)} \mathbf{1}_{\gamma_1^+}(z) + \iint_{\mathcal{D}_2} \frac{\beta_2(w,\overline{w})d^2 w}{\pi(w-z)\overline{w}^2 w^2} \mathbf{1}_{\gamma_2^+}(z),\\
J_{12}^\infty(z;x,t) &= \iint_{\overline{\mathcal{D}_1}} \frac{\beta_1^*(w,\overline{w})d^2 w}{\pi(w-z)} \mathbf{1}_{\gamma_1^-}(z) + \iint_{\overline{\mathcal{D}_2}} \frac{\beta_2^*(w,\overline{w})d^2 w}{\pi(w-z)\overline{w}^2 w^2} \mathbf{1}_{\gamma_2^-}(z),
\end{aligned} \tag{42}$$

where $\beta_1^*(w,\overline{w}) = \overline{\beta_1(\overline{w},w)}$, $\beta_2^*(w,\overline{w}) = \overline{\beta_2(\overline{w},w)}$, and $\mathbf{1}_\gamma$ is the characteristic function of the contour γ.

- $Y^\infty(z;x,t)$ satisfies the asymptotic conditions

$$\begin{cases} Y^\infty(z;x,t) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), & \text{as } z \to \infty,\\ Y^\infty(z;x,t) = \frac{i}{z}\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} + \mathcal{O}(1), & \text{as } z \to 0. \end{cases} \tag{43}$$

Theorem 1 (*Existence of the Solution ψ_∞*). *There is a unique solution $Y^\infty(z;x,t)$ to the limiting Riemann–Hilbert problem C, which determines a solution $\psi_\infty(x,t)$ to the focusing NLS equation via*

$$\psi_\infty(x,t) = -i \lim_{z\to\infty} z Y_{12}^\infty(z;x,t). \tag{44}$$

Moreover, ψ_∞ is a classical solution to the focusing NLS equation, which belongs to the class $C^\infty(\mathbb{R}\times\mathbb{R}^+)$.

The proof of the above theorem is reported in Appendix B.

While the properties of the class of solutions obtained by solving the above Riemann–Hilbert problem for Y^∞ is not explored, in the section

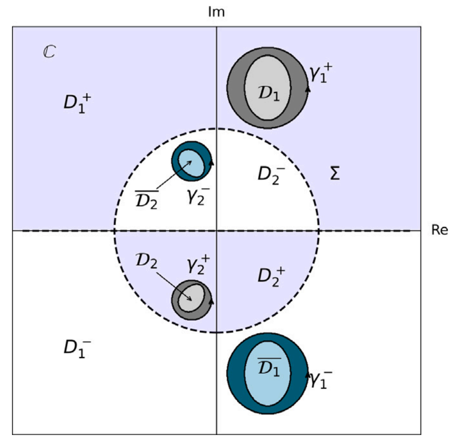

Fig. 6. The complex plane with the four domains $\mathcal{D}_1$, $\mathcal{D}_2$, $\overline{\mathcal{D}_1}$, $\overline{\mathcal{D}_2}$.

below we can get an insight for particular choices of the domain $\mathcal{D}_1$ and the breather density β_1. Using Green's theorem and the Cauchy theorem we can solve the problem exactly and reduce it to a finite number of breathers.

Remark 2. In the case of a gas of Akhmediev breathers, we can let the point $\zeta_1,\dots\zeta_N$ accumulate on an arc $\mathcal{L}_1$ of the unit circle. The corresponding gas of Akhmediev breathers is described by the solution of the Riemann–Hilbert problem C where the jump matrix $J^\infty(z;x,t)$ has entries

$$\begin{aligned}
J_{21}^\infty(z;x,t) &= \int_{\mathcal{L}_1} \frac{\beta_1(w,\overline{w})dw}{\pi(w-z)} \mathbf{1}_{\gamma_1^+}(z) + \int_{\mathcal{L}_2} \frac{\beta_2(w,\overline{w})dw}{\pi(w-z)} \mathbf{1}_{\gamma_2^+}(z),\\
J_{12}^\infty(z;x,t) &= \int_{\overline{\mathcal{L}_1}} \frac{\beta_1^*(w,\overline{w})d^2 w}{\pi(z-w)} \mathbf{1}_{\gamma_1^-}(z) + \int_{\overline{\mathcal{L}_2}} \frac{\beta_2^*(w,\overline{w})dw}{\pi(z-w)} \mathbf{1}_{\gamma_2^-}(z),
\end{aligned} \tag{45}$$

where $z \in \mathcal{L}_2$ if $-z \in \mathcal{L}_1$. The contours γ_1^+ and γ_2^+ encircle anti-clockwise the arcs $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively and similarly the contours γ_1^- and γ_2^- encircle anti-clockwise the arcs $\overline{\mathcal{L}_1}$ and $\overline{\mathcal{L}_2}$, respectively. We assume that the arcs $\mathcal{L}_1$, $\mathcal{L}_2$, $\overline{\mathcal{L}_1}$ and $\overline{\mathcal{L}_2}$ do not intersect. By a transformation as in (30), the above Riemann–Hilbert problem can be mapped to a Riemann–Hilbert problem with jumps on the contours $\mathcal{L}_1, \mathcal{L}_2, \overline{\mathcal{L}_1}$ and $\overline{\mathcal{L}_2}$. The initial data associated to such gas of breathers is expected to be step-like oscillatory, namely for $x \to -\infty$ one expects to have a travelling wave and for $x \to +\infty$ one expects that the initial data is approaching a nonzero constant of modulus one. This is a generalization of the soliton gas considered in [12].

3.2. Shielding of breathers on quadrature domains

In this section we take a particular class of domains $\mathcal{D}_1$ and interpolating function β_1 so that the gas of breathers turns into a finite set of breathers. Let us start by considering the class of quadrature domains [40]

$$\mathcal{D}_1 := \left\{ z \in \mathbb{C} \text{ s.t. } \left|(z-d_0)^m - d_1\right| < \rho \right\}, \quad m \in \mathbb{N}, \tag{46}$$

with $d_0 \in D_1^+$ and $|d_1|$, $\rho > 0$ chosen so that $\mathcal{D}_1 \subset D_1^+$. The boundary of $\mathcal{D}_1$ is described by the Schwartz function [40]:

$$\begin{aligned}
&\overline{z} = S_1(z),\\
&S_1(z) = \overline{d_0} + \left(\overline{d_1} + \frac{\rho^2}{(z-d_0)^m - d_1}\right)^{\frac{1}{m}}.
\end{aligned} \tag{47}$$

The domain $\mathcal{D}_2$ is defined by symmetry:

$$\mathcal{D}_2 = \left\{ z \in \mathbb{C} \text{ s.t. } -\frac{1}{\overline{z}} \in \mathcal{D}_1 \right\}, \tag{48}$$

that gives

$$\mathcal{D}_2 := \left\{ z \in \mathbb{C} \text{ s.t. } \left|(-\frac{1}{z} - \overline{d_0})^m - \overline{d_1}\right| < \rho \right\}, \quad m \in \mathbb{N}, \tag{49}$$

with boundary given by the relation

$$\overline{z} = S_2(z), \quad S_2(z) = -\frac{1}{d_0 + \left(d_1 + \frac{\rho^2}{(-\frac{1}{z}-\overline{d_0})^m - \overline{d_1}}\right)^{\frac{1}{m}}}. \tag{50}$$

Recalling that $\psi_- = \psi_0 = 1$, we choose the asymptotic value ψ_+ to be as simple as possible, namely $\psi_+ = 1$, leading to the "theta" condition (8)

$$4 \sum_{j=1}^{N} \arg(\zeta_j) = 0, \quad (\text{mod } 2\pi). \tag{51}$$

This condition is satisfied if ζ_j lies on the imaginary axis, or pairs of points ζ_j are symmetric with respect to the imaginary axis. This can be achieved by taking the domain $\mathcal{D}_1$ symmetric with respect to the imaginary axis. A possible choice to realize this condition is

$$\begin{aligned} &\mathfrak{I}(d_0) > 1,\ \mathfrak{R}(d_0) = 0, \mathfrak{R}(d_1) = 0, \quad m \text{ odd}, \\ &\mathfrak{I}(d_0) > 1,\ \mathfrak{R}(d_0) = 0,\ \mathfrak{I}(d_1) = 0, \quad m \text{ even}, \end{aligned} \tag{52}$$

where $\mathfrak{R}(z)$ and $\mathfrak{I}(z)$ are, respectively, the real and imaginary parts of the complex number z. We also remark that the constants d_0, d_1 and ρ have to be tuned in such a way that the domain $\mathcal{D}_1$ lies in D_1. We chose the interpolating function β_1 of the form

$$\beta_1(z, \overline{z}) = n(\overline{z} - \overline{d_0})^{n-1} r(z) = \overline{\partial}\left((\overline{z} - \overline{d_0})^n\, r(z)\right), \tag{53}$$

with $r(z)$ analytic in $\mathcal{D}_1$. Then

$$\beta_2(z, \overline{z}) = -n\left(-\frac{1}{\overline{z}} - d_0\right)^{n-1} z^2\, \overline{r\left(-\frac{1}{\overline{z}}\right)}, \quad z \in \mathcal{D}_2 \tag{54}$$

and we note that by the Schwartz reflection principle $\overline{r\left(-\frac{1}{\overline{z}}\right)}$ is analytic in $\mathcal{D}_2$.

Let us insert our choices in the expression (42) for the jump matrix J^∞ in (42). For $z \notin \mathcal{D}_1$ we obtain

$$\begin{aligned} \iint_{\mathcal{D}_1} \frac{\beta_1(w, \overline{w})}{\pi(z - w)} d^2 w &= \frac{1}{2\pi i} \oint_{\partial \mathcal{D}_1} \frac{r(w)(\overline{w} - \overline{d_0})^n}{z - w}\, dw \\ &= \frac{1}{2\pi i} \oint_{\partial \mathcal{D}_1} \frac{(S_1(w) - \overline{d_0})^n r(w)}{z - w}\, dw \end{aligned} \tag{55}$$

where S_1 is the Schwartz function (47). The first equality in (55) is due to Green's theorem in its complex form, while in the second one we use the fact that the boundary $\partial \mathcal{D}_1$ of $\mathcal{D}_1$ is described by $\overline{z} = S_1(z)$.

The next step is to consider the case $m = n$ so that the integral (55) can be evaluated using the residue theorem, thus showing that a n-breather solution emerges via this relation between the quadrature domain defined by formulas (46) and (47) and the interpolating function.

Indeed using the residue theorem to the contour integral (55) at the n poles $\{\lambda_1, \dots, \lambda_n\}$ given by the solution of the equation $(z - d_0)^n = d_1$, it follows that:

$$\begin{aligned} \iint_{\mathcal{D}_1} \frac{\beta_1(w, \overline{w})}{\pi(z - w)} d^2 w &= \frac{1}{2\pi i} \oint_{\partial \mathcal{D}_1} \frac{(S_1(w) - \overline{d_0})^n r(w)}{z - w}\, dw \\ &= \rho^2 \sum_{j=1}^{n} \frac{r(\lambda_j)}{\prod_{k \neq j}(\lambda_j - \lambda_k)} \frac{1}{(z - \lambda_j)}, \quad \text{for } z \notin \mathcal{D}_1. \end{aligned} \tag{56}$$

Comparing the above relation with jump matrix (31) for the N-breather solution yields the norming constants

$$C_j = \frac{\rho^2 r(\lambda_j)}{\prod_{k \neq j}(\lambda_j - \lambda_k)}. \tag{57}$$

In particular, it is evident that the emergence of the n-breather solution is due to the term $n(\overline{z} - \overline{d_0})^{n-1}$ in (53), while the function $r(z)$ influences the value of the norming constants, due to (57), hence contributing only to the "shape" of the n-breather. Moreover, we remark that the first equality in (56) does not depend on the choice $m = n$. On the other hand, the choice $m = n$ ensured the validity of the second equality of (56).

Since the n poles $\{\lambda_1, \dots, \lambda_n\}$ are the solution of the equation $(z - d_0)^n = d_1$, they have the form

$$\lambda_k = d_0 + \sqrt[n]{|d_1|}\, e^{i\left(\frac{\arg(d_1)}{n} + \frac{2(k-1)\pi}{n}\right)} \text{ for } k = 1, \dots, n, \tag{58}$$

and hence from (52) the "theta" condition $4\sum_{j=1}^{n} \arg(\lambda_j) = 0, \bmod 2\pi$ is satisfied.

The integral over $\mathcal{D}_2$ gives a result compatible with the symmetry constraints. Indeed we have, also taking (54) into account,

$$\begin{aligned} \iint_{\mathcal{D}_2} \frac{\beta_2(w, \overline{w}) d^2 w}{\pi(w - z)\overline{w}^2 w^2} &= -n \iint_{\mathcal{D}_2} \frac{\left(-\frac{1}{\overline{w}} - d_0\right)^{n-1} w^2\, \overline{r\left(-\frac{1}{\overline{w}}\right)} d^2 w}{\pi(w - z)\overline{w}^2 w^2} \\ = -\int_{\partial \mathcal{D}_2} \frac{\left(-\frac{1}{\overline{w}} - d_0\right)^n \overline{r\left(-\frac{1}{\overline{w}}\right)} dw}{2i\pi(w - z)} &= \int_{\partial \mathcal{D}_2} \frac{\left(-\frac{1}{S_2(w)} - d_0\right)^n \overline{r\left(-\frac{1}{\overline{w}}\right)} dw}{2i\pi(w - z)} \\ = -\int_{\partial \mathcal{D}_2} \frac{\left(d_1 + \frac{\rho^2}{(-\frac{1}{w} - \overline{d_0})^n - \overline{d_1}}\right) \overline{r\left(-\frac{1}{\overline{w}}\right)} dw}{2i\pi(w - z)} & \\ = -\frac{\rho^2}{(-\overline{d_0})^n - \overline{d_1}} \sum_{j=1}^{n} \frac{\lambda_{j+n}^n \overline{r(-\frac{1}{\overline{\lambda_{j+n}}})}}{\prod_{k \neq j}(\lambda_{j+n} - \lambda_{k+n})} \frac{1}{(z - \lambda_{j+n})}. & \end{aligned} \tag{59}$$

In this equation λ_{n+j}, $j = 1, \dots, n$ are the n zeros of the equation $(-\frac{1}{z} - \overline{d_0})^n - \overline{d_1} = 0$,. We remark that the second equality holds thanks to the fact that $\overline{r\left(-\frac{1}{\overline{z}}\right)}$ is analytic because of Schwartz reflection principle.

From (59) it is immediate to verify that

$$\lambda_{n+j} = -\frac{1}{\overline{\lambda_j}}, \quad j = 1, \dots, n, \tag{60}$$

and the norming constants C_{n+j} given by

$$C_{n+j} = -\frac{\rho^2}{(-\overline{d_0})^n - \overline{d_1}} \lambda_{j+n}^n \frac{\overline{r(-\frac{1}{\overline{\lambda_{j+n}}})}}{\prod_{k \neq j}(\lambda_{j+n} - \lambda_{k+n})} \tag{61}$$

satisfy the symmetry

$$C_{n+j} = -\left(\frac{1}{\overline{\lambda_j}}\right)^2 \overline{C_j}, \quad j = 1, \dots, n. \tag{62}$$

Adding up the two integrals over $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain, up to a sign, the entry 2,1 of the jump matrix (31).

Therefore we can conclude that the breather gas solution $\psi_\infty(x, t)$ in (44) coincides with the n-breather solution $\psi_n(x, t)$ in (34) with spectrum $\{\lambda_1, \dots, \lambda_{2n}\}$, with $\lambda_{n+j} = -\frac{1}{\overline{\lambda_j}}$, for $j = 1, \dots, n$, and corresponding norming constants given by (57) and (62). We summarize the section with the following theorem

Theorem 2. *The gas breather solution (44) obtained from the Riemann–Hilbert problem C is reduced to a n-breather solution when the domain $\mathcal{D}_1$ is a quadrature domain of the form*

$$\mathcal{D}_1 := \left\{ z \in \mathbb{C} \text{ s.t. } \left|(z - d_0)^n - d_1\right| < \rho \right\}, \quad n \in \mathbb{N},$$

where the constants d_0, d_1, ρ and n are subject to the constraints (52) and tuned in such a way that $\mathcal{D}_1 \subset D_1$ and the interpolating function is of the form

$$\beta_1(z, \overline{z}) = n(\overline{z} - \overline{d_0})^{n-1} r(z)$$

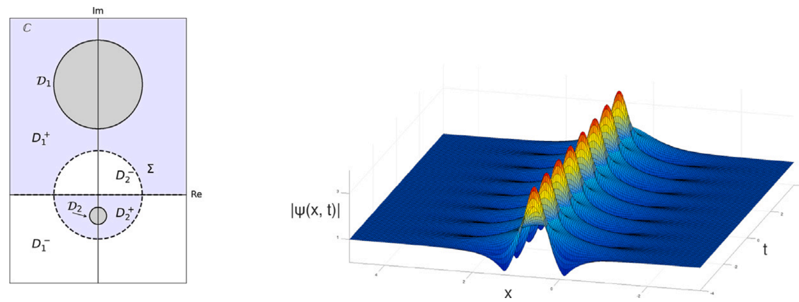

Fig. 7. Left: The 1-fold domain of accumulation: a disk of radius $\rho = 1$ centred at $\frac{5}{2}i$. Right: the "shielding" Kuznetsov–Ma breather. Parameters: $\lambda_1 = \frac{5}{2}i$, $C_1 = 1$, $r(z) = 1$.

where $r(z)$ is analytic in $\mathcal{D}_1$. The spectrum of the n-breather solution is given by the zeros $\{\lambda_1, \dots, \lambda_n\}$ of the polynomial $(z-d_0)^n - d_1 = 0$ and the norming constants are

$$C_j = \frac{\rho^2 r(\lambda_j)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}.$$

3.3. Examples

We end this Section by considering the simplest instances of quadrature domains leading to effective $n = 1$ and $n = 2$ breathers.

3.3.1. The $n = 1$ case: the Kutznesov-Ma breather

In the particular case $n = 1$, $\mathcal{D}_1$ is the disk of radius ρ centred at $d_0 + d_1 = i\varsigma$ with $\varsigma > 1$

$$\mathcal{D}_1 := \left\{ z \in \mathbb{C} \text{ s.t. } \left|z - i\varsigma\right| < \rho \right\}, \tag{63}$$

whose boundary is described by the Schwarz function

$$\overline{z} = S_{disk}(z), \quad S_{disk}(z) = -i\varsigma + \frac{\rho^2}{(z - i\varsigma)}. \tag{64}$$

We obtain exactly the Riemann–Hilbert problem (32), (33) for $N = 1$ with $\mathcal{Z} = \{\lambda_1 = i\varsigma,\ \lambda_2 = -i/\varsigma\}$, $C_1 = \rho^2 r(i\varsigma)$ and $C_2 = -\left(\frac{i}{\varsigma}\right)^2 \overline{\rho^2 r(i\varsigma)}$ (see Fig. 7).

Writing $C_1 = e^{\kappa_0 + i\phi_0}$ we obtain

$$\psi_1(x,t) = \frac{\cosh(\widetilde{\chi}) - \frac{1}{2}\widetilde{c_+}\left(1 + \frac{\widetilde{c_-}^2}{\widetilde{c_+}^2}\right)\sin(\widetilde{s}) + i\widetilde{c_-}\cos(\widetilde{s})}{\cosh(\widetilde{\chi}) - A\sin(\widetilde{s})} \tag{65}$$

with

$$\begin{gathered} A = \frac{2}{\widetilde{c_+}} < 1, \quad \widetilde{c_\pm} = \varsigma \pm \frac{1}{\varsigma}, \\ \widetilde{\chi}(x,t) = \widetilde{c_-}x + \widetilde{c_0} + \kappa_0, \quad \widetilde{c_0} = \ln\left(\frac{\widetilde{c_+}}{2\varsigma\widetilde{c_-}}\right), \\ s(x,t) = \widetilde{c_+}\widetilde{c_-}t - \phi_0. \end{gathered} \tag{66}$$

3.3.2. The $n = 2$ case: the Tajiri–Watanabe and Kutznesov–Ma breathers

We consider poles accumulating on $\mathcal{D}_1$, where now $\mathcal{D}_1$ is the 2-fold domain given by

$$\mathcal{D}_1 = \left\{ z \in \mathbb{C}\ s.t. \left|(z - i\varsigma)^2 - d_1\right| < \rho \right\}, \tag{67}$$

and $\mathcal{D}_2$ obtained by symmetry

$$\mathcal{D}_2 = \left\{ z \in \mathbb{C}\ s.t. \left|\left(-\frac{1}{\overline{z}} - i\varsigma\right)^2 - d_1\right| < \rho \right\}, \tag{68}$$

with $\varsigma > 1$ ($d_0 = i\varsigma$) and d_1 real. From the Eqs. (57) and (58), we obtain

$$\begin{aligned} &\lambda_1 = i\varsigma + \sqrt{d_1}, \quad \lambda_2 = i\varsigma - \sqrt{d_1}, \\ &C_1 = \frac{\rho^2 r(\lambda_1)}{\lambda_1 - \lambda_2}, \quad C_2 = \frac{\rho^2 r(\lambda_2)}{\lambda_2 - \lambda_1}. \end{aligned} \tag{69}$$

Note that when $d_1 < 0$ the eigenvalues λ_1 and λ_2 are purely imaginary so that one obtains 2-breather solution of Kutznesov–Ma type, while when $d_1 > 0$ one obtain a 2-breather solution of Tajiri–Watanabe type because the eigenvalues λ_1 and λ_2 are complex.

Two examples of breathers are obtained in Fig. 8 with $r(z) = 1$.

4. Conclusions and outlook

This paper was devoted to study a gas of N-breather solutions of the FNLS equation in a suitably taylored limit $N \to \infty$. Namely, we started from the well known Riemann–Hilbert problem leading to N-breather solutions [13,34] and considered families of such solutions with poles accumulating in specific domains, namely quadrature domains of a specific kind. Then we performed a specific $N \to \infty$ limit according to the following prescriptions:

(i) The soliton spectrum $\{\zeta_j\}$, $j = 1, \dots, N$ is chosen to be uniformly distributed in the "parent" domain $\mathcal{D}_1$, the spectrum distribution in the other *three* domains being obtained by the well-known symmetry properties of FNLS breather spectrum.

(ii) The norming constants C_j, $j = 1, \dots N$ is interpolated by a smooth function $\beta_1(z, \bar{z})$ and asymptotically scale with $1/N$.

We had to carefully consider the so-called *theta condition*, and we restricted our study to the simplest case, namely the one for which $4\sum_{j=1}^{N} \arg(\zeta_j) = 0$, which implies no phase shift for the $\pm\infty$ asymptotic values of the FNLS solution. This can be accomplished by cherrypicking a parent domain $\mathcal{D}_1$ symmetric with respect to the positive imaginary axis.

We showed that, under such circumstance, the *soliton shielding* phenomenon discovered in [10] can happen also for gas of breathers. Indeed, when the domain $\mathcal{D}_1$ is the n-fold quadrature domain given by the inequality $\left|(z-d_0)^n - d_1\right| < \rho$, and the interpolating function is of the kind $\beta_1(z, \bar{z}) = n(\overline{z} - \overline{d_0})^{n-1} r(z)$ (for some analytic function $r(z)$), we have shown that the limiting solution is the n Kuznetsov–Ma and/or Tajiri–Watanabe breather. We illustrated our results in the simplest case with $n = 1, 2$.

It is fair to say that our results require some stringent conditions to hold. Unravelling the analytical properties of a general gas of breathers remains an open problem. In particular, the question on how to obtain general n combinations of Kuznetsov–Ma and Tajiri–Watanabe shielding breathers by relaxing our condition of equal phases at $x = \pm\infty$

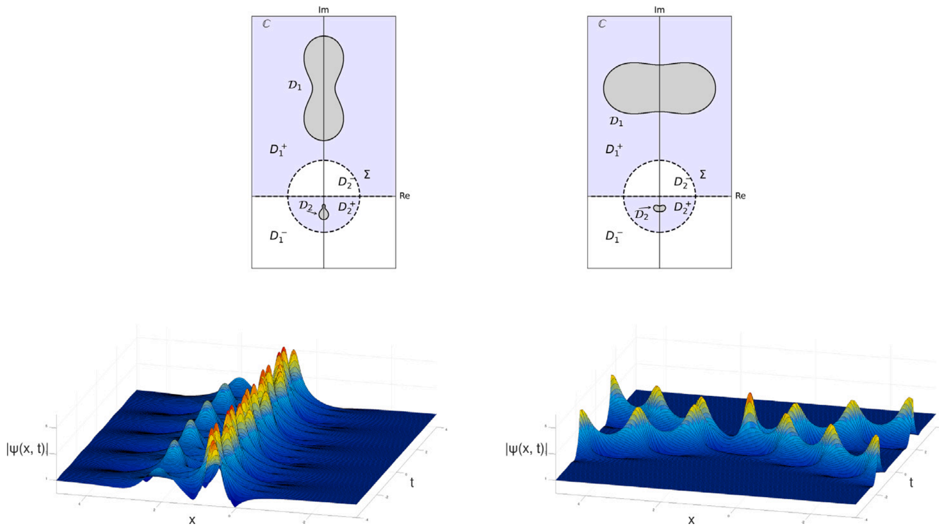

Fig. 8. 2-fold domains on which the poles accumulate and the corresponding "shielding" 2-breather. Top left: $\rho = \sqrt{6/5}$, $d_0 = i\varsigma = 3i$, $d_1 = -1$ and $r(z) = 1$. Top right: $\rho = \sqrt{2}$, $d_0 = i\varsigma = 3i$, $d_1 = 1$ and $m = 2$. Below the corresponding 2-breather solutions with effective parameters $\lambda_{1,2} = (2i, 4i)$, $C_{1,2} = (-\frac{3}{5}i, \frac{3}{5}i)$ (left), $\lambda_{1,2} = 1 \pm 3i$, $C_{1,2} = \pm 1$ (right) as per (69).).

should be further examined. Also, Akhmediev's breathers were not considered here. For a gas of such breathers, it is expected that the corresponding initial data be step-like oscillatory. Some preliminary steps along these lines have been undertaken in [41].

CRediT authorship contribution statement

Gregorio Falqui: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Tamara Grava:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Christian Puntini:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Determination of the breather parameters

The system of Eqs. (19) is further simplified, using the symmetry of the matrix M (see formula (2).24 [34]) between the first column $M_1(z)$ and the second column $M_2(z)$:

$$M_1(z) = \frac{i}{z} M_2\left(-\frac{1}{z}\right). \tag{70}$$

The symmetry condition (70) gives the following relation of the matrix entries of $M(z)$ in (18)

$$M_1(z) = \begin{bmatrix} 1 \\ \frac{i}{z} \end{bmatrix} + \sum_{\alpha=1}^{2N} \frac{\begin{bmatrix} A_\alpha \\ B_\alpha \end{bmatrix}}{z-\zeta_\alpha} = \frac{i}{z} M_2\left(-\frac{1}{z}\right) = \begin{bmatrix} 1 \\ \frac{i}{z} \end{bmatrix} - \sum_{\alpha=1}^{2N} \frac{i}{\zeta_\alpha} \frac{\begin{bmatrix} E_\alpha \\ F_\alpha \end{bmatrix}}{z + \frac{1}{\zeta_\alpha}}, \tag{71}$$

therefore

$$\sum_{j=1}^{N} \frac{\begin{bmatrix} A_{j+N} \\ B_{j+N} \end{bmatrix}}{z-\zeta_{j+N}} = -\sum_{j=1}^{N} \frac{i}{\zeta_j} \frac{\begin{bmatrix} E_j \\ F_j \end{bmatrix}}{z-\zeta_{j+N}}$$

and

$$\sum_{j=1}^{N} \frac{\begin{bmatrix} A_j \\ B_j \end{bmatrix}}{z-\zeta_j} = -\sum_{j=1}^{N} \frac{i}{\zeta_{j+N}} \frac{\begin{bmatrix} E_{j+N} \\ F_{j+N} \end{bmatrix}}{z-\zeta_j},$$

or equivalently

$$\begin{bmatrix} A_{j+N} \\ B_{j+N} \end{bmatrix} = -\frac{i}{\zeta_j} \begin{bmatrix} E_j \\ F_j \end{bmatrix}, \quad \begin{bmatrix} A_j \\ B_j \end{bmatrix} = -\frac{i}{\zeta_{j+N}} \begin{bmatrix} E_{j+N} \\ F_{j+N} \end{bmatrix}. \tag{72}$$

Inserting the above relation into the system (19) we obtain after some algebra the following linear system of equations for the constants E_α, $\alpha = 1, \dots, 2N$:

$$E_\alpha - i\overline{c_\alpha} \sum_{j=1}^{N} \left(\frac{1}{\zeta_{j+N}} \frac{E_{j+N}}{\overline{\zeta_\alpha} - \zeta_j} + \frac{1}{\zeta_j} \frac{E_j}{\overline{\zeta_\alpha} - \zeta_{j+N}} \right) = -\overline{c_\alpha}. \tag{73}$$

Using the symmetries (7), (10), (11) and (16) the above system of equations can be written in the form

$$\begin{bmatrix} 1_N + \overline{X} & -\overline{Y} D \\ D^{-1} Y & 1_N + D^{-1} X D \end{bmatrix} \begin{bmatrix} E_1 \\ \vdots \\ E_{2N} \end{bmatrix} = - \begin{bmatrix} \overline{c_1} \\ \vdots \\ \overline{c_{2N}} \end{bmatrix} \tag{74}$$

where 1_N is the N-dimensional identity and

$$X_{kj} = i \frac{c_k}{1 + \zeta_k \zeta_j}, \quad Y_{kj} = i \frac{c_k}{\zeta_k - \overline{\zeta_j}}, \quad D = \operatorname{diag}(\zeta_1, \dots, \zeta_N).$$

Invertibility of the system of Eqs. (74) follows from the solvability of the Riemann–Hilbert problem B for the N-breather solution proved in Appendix B. For $N = 1$ the 1-breather solution (20),

$$\psi(x,t) = 1 - i(E_1 + E_2), \tag{75}$$

is obtained by solving the linear system

$$\begin{bmatrix} 1-i\dfrac{\overline{c_1}}{1+\overline{\zeta_1}^2} & i\dfrac{\overline{c_1}\zeta_1}{\overline{\zeta_1}-\zeta_1} \\ i\dfrac{c_1}{\zeta_1(\zeta_1-\overline{\zeta_1})} & 1+i\dfrac{c_1}{\zeta_1^2+1} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = -\begin{bmatrix} \overline{c_1} \\ -\frac{c_1}{\zeta_1^2} \end{bmatrix}. \tag{76}$$

The solution of such system for complex $\zeta_1 \in D_1^+$ and $c_1 \in \mathbb{C}$ gives the Tajiri–Watanabe breather. In the case ζ_1 is pure imaginary and $\zeta_1 \in D_1^+$ one gets the Kutznesov–Ma breather.

Appendix B. Proof of Theorem 1

In this appendix we show the solvability of the Riemann–Hilbert problem B for N-breather and the Riemann–Hilbert problem C for the breather gas. We follow the steps in [42]. Since the proof is identical in the two cases, we consider only the latter case. This proves Theorem 1. The Riemann–Hilbert problem C can be written in the form

$$\begin{cases} Y_+^\infty(z;x,t) - Y_-^\infty(z;x,t) = Y_-^\infty(z;x,t)J^\infty(z;x,t), & z\in\boldsymbol{\gamma}, \\ Y^\infty(z;x,t) = \mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right), & \text{as } z\to\infty, \\ Y^\infty(z;x,t) = \frac{i}{z}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \mathcal{O}(1), & \text{as } z\to 0, \end{cases} \tag{77}$$

which is equivalently written using the Sokhotski–Plemelj integral formula as

$$Y^\infty(z;x,t) = \mathbb{I} + \frac{i}{z}\sigma_2 + \frac{1}{2\pi i}\int_{\boldsymbol{\gamma}} \frac{Y_-^\infty(s;x,t)J^\infty(s;x,t)}{s-z}ds,$$

where we recall that $\boldsymbol{\gamma}$ is the union of all the contours $\gamma_1^\pm$ and $\gamma_2^\pm$. We can obtain an integral equation by taking the boundary value $Y_-^\infty(z)$ as z approaches non tangentially the oriented contour $\boldsymbol{\gamma}$ from the right:

$$Y_-^\infty(\xi) = \mathbb{I} + \frac{i}{\xi}\sigma_2 + \lim_{\substack{z\to\xi \\ z\in \text{ right side of } \gamma}} \left(\frac{1}{2\pi i}\int_{\boldsymbol{\gamma}} \frac{Y_-^\infty(s)J^\infty(s)}{s-\xi}ds\right). \tag{78}$$

We define the integral operator C_{J^∞} as

$$C_{J^\infty}(h)(\xi) = C_-(hJ^\infty)(\xi), \tag{79}$$

where C_- is the Cauchy projection operator acting on $L^2(\boldsymbol{\gamma}, ds)\otimes Mat(2\times 2,\mathbb{C})$ to itself, namely

$$C_-(h)(\xi) = \lim_{\substack{z\to\xi \\ z\in \text{ right side of } \gamma}} \left(\frac{1}{2\pi i}\int_{\boldsymbol{\gamma}} \frac{h(s)}{s-z}ds\right), \tag{80}$$

here $L^2(\boldsymbol{\gamma}, ds)$ is the space of square integrable functions on the contour $\boldsymbol{\gamma}$. Then the integral Eq. (78) takes the form

$$\left[1 - C_{J^\infty}\right] Y_-^\infty(\xi) = \mathbb{I} + \frac{i}{\xi}\sigma_2, \tag{81}$$

where 1 is the identity in $L^2(\boldsymbol{\gamma}, \text{ş}) \otimes Mat(2\times 2,\mathbb{C})$. Then Theorem 9.3 from [42](p.984) guarantees that the operator $1 - C_{J^\infty}$ is invertible as an operator acting from $L^2(\boldsymbol{\gamma}, ds)\otimes Mat(2\times 2,\mathbb{C})$ to itself. Invertibility is guaranteed from the fact that $1 - C_{J^\infty}$ is a Fredholm integral operator with zero index and the kernel of $1 - C_{J^\infty}$ is $\{0\}$. In particular this last point is obtained by applying the vanishing lemma [42]. Indeed suppose that exists $\hat{Y}_- \in L^2(\boldsymbol{\gamma}, \text{ş})\otimes Mat(2\times 2,\mathbb{C})$ such that $(1-C_{J^\infty})\hat{Y}_- = 0$. Then the quantity

$$\hat{Y}(x,t;z) = \frac{1}{2\pi i}\int_{\boldsymbol{\gamma}} \frac{\hat{Y}_-(s)J^\infty(s;x,t)}{s-z}ds \tag{82}$$

solves the following Riemann–Hilbert problem:

- $\hat{Y}(z;x,t)$ is analytic in $\mathbb{C}\backslash\boldsymbol{\gamma}$;
- $\hat{Y}_+(\xi) = \hat{Y}_-(\xi)(\mathbb{I} + J^\infty(\xi))$ for $\xi\in\boldsymbol{\gamma}$;
- $\hat{Y}(z;x,t) = \mathcal{O}(z^{-1})$ as $z\to\infty$.

To show that such problem has only the zero solution we use the following properties:

- the jump matrix satisfies the Schwartz symmetry: $J^\infty(z;x,t) = -\overline{J^\infty(\overline{z};x,t)}^t$;
- the contour $\boldsymbol{\gamma}$ is symmetric with respect to the real axis $\mathbb{R}$ (up to the orientation).

We define $H(z) = \hat{Y}(z)\overline{\hat{Y}(\overline{z})}^t$, then clearly $\overline{H(\overline{z})}^t = H(z)$, $H(z) = \mathcal{O}(z^{-2})$ as $z\to\infty$ and $H(z)$ is analytic in $\mathbb{C}\backslash\boldsymbol{\gamma}$. We consider the integral on the real axis and apply contour deformation on the upper half space

$$\begin{aligned}\int_{-\infty}^{\infty} H_+(z)dz &= \int_{\gamma_2^-\cup\gamma_1^+} H_-(z)dz \\ &= \int_{\gamma_2^-\cup\gamma_1^+} \hat{Y}_+(z)(\mathbb{I}+J^\infty(z))^{-1}((\mathbb{I}+\overline{J^\infty(\overline{z})})^{-1})^t\overline{\hat{Y}_+(\overline{z})}^t dz \\ &= \int_{\gamma_2^-\cup\gamma_1^+} \hat{Y}_+(z)\overline{\hat{Y}_+(\overline{z})}^t dz = 0.\end{aligned}$$

The second identity is obtained using the boundary values of $\hat{Y}(z)$. In the third identity we have used the Schwartz reflection of the jump matrix J^∞ and the fourth identity is obtained using the fact that $\hat{Y}(z)$ is analytic inside $\gamma_2^-\cup\gamma_1^+$. We conclude that

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} \hat{Y}_+(z)\overline{\hat{Y}_+(\overline{z})}^t dz \\ &= \int_{-\infty}^{\infty} \begin{bmatrix} |\hat{Y}_{11}(z)|^2 + |\hat{Y}_{12}(z)|^2 & * \\ * & |\hat{Y}_{21}(z)|^2 + |\hat{Y}_{22}(z)|^2 \end{bmatrix} dz,\end{aligned}$$

which implies $\hat{Y}_{ij}(z) = 0$, $i,j = 1,2$ and $z\in\mathbb{R}$. Due to analyticity of $\hat{Y}(z)$ in $\mathbb{C}\backslash\boldsymbol{\gamma}$ and the unique continuation theorem, then $\hat{Y}$ is identically zero. So the null space is empty and the operator $1 - C_{J^\infty}$ is invertible showing the existence of the solution to the Riemann–Hilbert problem for $Y^\infty(z;x,t)$. The proof that the solution $\psi_\infty(x,t)$ is in $C^\infty(\mathbb{R}\times\mathbb{R}^+)$ relies on the proof of existence of the matrix $\partial^n Y^\infty(z;x,t)$ for $n\in\mathbb{N}$ with the following properties:

1. $\partial^n Y^\infty(z;x,t)$ is holomorphic for $z\in\mathbb{C}\setminus\boldsymbol{\gamma}$.
2. $\partial^n Y^\infty(z;x,t) = \mathcal{O}\left(z^{-1}\right)$, as $z\to\infty$
3. For $z\in\boldsymbol{\gamma}$, the boundary values of $\partial^n Y^\infty(z;x,t)$ satisfy the jump relation

$$\begin{aligned}&(\partial^n Y^\infty(z;x,t))_+ = (\partial^n Y^\infty(z;x,t))_-\, J^\infty(z,;x,t) + \mathcal{F}^{(n)}(z;x,t), \\ &\mathcal{F}^{(n)}(z;x,t) := \sum_{\ell=1}^{n}\binom{n}{\ell}\partial^{n-j}Y^\infty(z;x,t)(z;x,t)\partial^j J^\infty(z;x,t).\end{aligned} \tag{83}$$

Here $\partial^n Y^\infty(z;x,t)$ is the partial derivative with respect to a parameter, in our case t or x. The details on the existence of the solution to such problem are given in [11,43].

Data availability

No data was used for the research described in the article.

References

[1] G.A. El, A. Tovbis, Spectral theory of soliton and breather gases for the focusing nonlinear Schrödinger equation, Phys. Rev. E 101 (5) (2020) 052207.
[2] M. Girotti, T. Grava, R. Jenkins, K.T.R. McLaughlin, Rigorous asymptotics of a KdV soliton gas, Comm. Math. Phys. 384 (2021) 733–784.
[3] M. Girotti, T. Grava, R. Jenkins, K.T.R. McLaughlin, A. Minakov, Soliton versus the gas: Fredholm determinants, analysis, and the rapid oscillations behind the kinetic equation, Comm. Pure Appl. Math. 76 (11) (2023) 3233–3299.
[4] D. Bilman, P.D. Miller, A robust inverse scattering transform for the focusing nonlinear Schrödinger equation, Comm. Pure Appl. Math. 72 (8) (2019) 1722–1805.
[5] A.A. Gelash, D. Agafontsev, P. Suret, S. Randoux, Solitonic model of the condensate, Phys. Rev. E 104 (4) (2021) 044213.
[6] A.A. Gelash, D. Agafontsev, V.E. Zakharov, G.A. El, S. Randoux, P. Suret, Bound state soliton gas dynamics underlying the spontaneous modulational instability, Phys. Rev. Lett. 123 (23) (2019) 234102.

[7] G. Roberti, G.A. El, A. Tovbis, F. Copie, P. Suret, S. Randoux, Numerical spectral synthesis of breather gas for the focusing nonlinear Schrödinger equation, Phys. Rev. E 103 (4) (2021) 042205.

[8] V.E. Zakharov, Kinetic equation for solitons, Sov. Phys. JETP 33 (3) (1971) 538–540.

[9] X. Zhang, L. Ling, A modified Korteweg–de Vries equation soliton gas under the nonzero background. Preprint https://arxiv.org/pdf/2407.05384.

[10] M. Bertola, T. Grava, G. Orsatti, Soliton shielding of the focusing nonlinear Schrödinger equation, Phys. Rev. Lett. 130 (12) (2023) 1.

[11] M. Girotti, T. Grava, K.T.R. McLaughlin, J. Najnudel, Law of Large Numbers and Central Limit Theorem for random sets of solitons of the focusing nonlinear Schrödinger equation. Preprint https://arxiv.org/pdf/2411.17036.

[12] M. Bertola, T. Grava, G. Orsatti, $\bar{\partial}$-Problem for focusing nonlinear Schrödinger equation and soliton shielding, 2024, Preprint https://arxiv.org/pdf/2409.14825.

[13] E.A. Kuznetsov, Solitons in a parametrically unstable plasma, Akad. Nauk. SSSR Dokl. 236 (1977) 575–577.

[14] Y-C. Ma, The perturbed plane-wave solutions of the cubic Schrödinger equation, Stud. Appl. Math. 60 (1) (1979) 43–58.

[15] D.H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, ANZIAM J. 25 (1) (1983) 16–43.

[16] N.N. Akhmediev, V.M. Eleonskii, N.E. Kulagin, Exact first-order solutions of the nonlinear Schrödinger equation, Theoret. and Math. Phys. 72 (2) (1987) 809–818.

[17] M. Tajiri, Y. Watanabe, Breather solutions to the focusing nonlinear Schrödinger equation, Phys. Rev. E 57 (3) (1998) 3510.

[18] V.E. Zakharov, A.A. Gelash, Nonlinear stage of modulation instability, Phys. Rev. Lett. 111 (5) (2013) 054101.

[19] V.E. Zakharov, A.A. Gelash, Freak waves as a result of modulation instability, Procedia IUTAM 9 (2013) 165–175.

[20] P.A.E.M. Janssen, Nonlinear four-wave interactions and freak waves, J. Phys. Oceanogr. 33 (4) (2003) 863–884.

[21] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves in the Ocean, Springer Science & Business Media, Berlin, 2008.

[22] K. Dysthe, K. Trulsen, Note on breather type solutions of the NLS as models for freak-waves, Phys. Scr. T 82 (1999) 48.

[23] A. Costa, A.R. Osborne, D.T. Resio, S. Alessio, E. Chrivì, E. Saggese, K. Bellomo, C.E. Long, Soliton turbulence in shallow water ocean surface waves, Phys. Rev. Lett. 113 (10) (2014) 108501.

[24] A.A. Gelash, Formation of rogue waves from a locally perturbed condensate, Phys. Rev. E 97 (2) (2018) 022208.

[25] M. Bertola, A. Tovbis, Universality for the focusing nonlinear Schrdinger equation at the gradient catastrophe point: rational breathers and poles of the tritronquée solution to painlevé i, Comm. Pure Appl. Math. 66 (5) (2013) 678–752.

[26] N.N. Akhmediev, A. Ankiewicz, M. Taki, Waves that appear from nowhere and disappear without a trace, Phys. Lett. A 37 (6) (2009) 675–678.

[27] N.N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation, Phys. Rev. E 80 (2009) 026601.

[28] A. Chabchoub, N. Hoffmann, M. Onorato, N.N. Akhmediev, Super rogue waves: Observation of a higher-order breather in water waves, Phys. Rev. X 2 (1) (2012) 011015.

[29] D. Bilman, P.D. Miller, Broader universality of rogue waves of infinite order, Phys. D 435 (2022) 133289.

[30] D. Bilman, L. Ling, P.D. Miller, Extreme superposition: Rogue waves of infinite order and the Painlevé-III hierarchy, Duke Math. J. 169 (2020) 671–760.

[31] D. Bilman, R. Buckingham, Large-order asymptotics for multiple-pole solitons of the focusing nonlinear Schrödinger equation, J. Nonlinear Sci. 29 (2019) 2185–2229.

[32] D. Bilman, P.D. Miller, Extreme superposition: High-order fundamental rogue waves in the far-field regime, Mem. Amer. Math. Soc. 300 (2024) 1505.

[33] D. Bilman, R. Buckingham, D.-S. Wang, Far-field asymptotics for multiple-pole solitons in the large-order limit, J. Differential Equations 297 (2021) 320–369.

[34] G. Biondini, G. Kovacic, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys. 55 (3) (2014) 031506.

[35] G. Biondini, E. Fagerstrom, B. Prinari, Inverse scattering transform for the defocusing nonlinear Schrödinger equation with fully asymmetric non-zero boundary conditions, Phys. D 333 (2016) 117–136.

[36] A. Shabat, V.E. Zakharov, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP 34 (1) (1972) 62.

[37] L. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, vol. 23, Springer Science & Business Media, Berlin, 2007.

[38] D. Agafontsev, A.A. Gelash, S. Randoux, P. Suret, Multisoliton interactions approximating the dynamics of breather solutions, Stud. Appl. Math. 152 (2) (2024) 810–834.

[39] S. Li, G. Biondini, Soliton interactions and degenerate soliton complexes for the focusing nonlinear Schrödinger equation with nonzero background, Eur. Phys. J. Plus 133 (10) (2018) 400.

[40] B. Gustafsson, Quadrature identities and the schottky double, Acta Appl. Math. 1 (1983) 209–240.

[41] W. Weng, G. Zhang, B.A. Malomed, Z. Yan, Breather gas and shielding for the focusing nonlinear Schrödinger equation with nonzero backgrounds. Preprint https://arxiv.org/pdf/2501.03481.

[42] X. Zhou, The Riemann-Hilbert problem and inverse scattering, SIAM J. Math. Anal. 20 (1989) 966–986.

[43] P. Deift, X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. a case study, Acta Math. 188 (2) (2002) 163–262.