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Quantum Fisher information in non-interacting systems after a sudden quench

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Abstract

In a recent study concerning the multipartite entanglement structure in states satisfying the eigenstate thermalization hypothesis (ETH), a measure known as the quantum Fisher information (QFI) was observed to detect differences in the entanglement content between thermal states of the same temperature; thus establishing a new, distinct way to distinguish between thermal states of different types (pure or mixed). In this work, we try to extend the previous analysis to the class of integrable models–i.e., systems which do not thermalize in the conventional way. We start this investigation with the simplest case, that of non-interacting systems. We derive exact analytic expressions of the QFI for a state described by the generalized Gibbs ensemble (GGE) and for a quenched pure state. We then show through various examples that this hierarchy of entanglement content persists.

Acknowledgments

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Contents

Chapter 1

Introduction

Quantum information theory has become an indispensible tool in studying complex many-body systems, providing novel insights into their properties and illuminating connections to well-known concepts in statistical mechanics [1–5]. For the past few decades, this line of research has been growing steadily, touching upon wide-ranging phenomena such as decoherence [6], quantum teleportation [7, 8] and information scrambling [9–11]. The vast applicability of its tools along with the advent of new technologies (like quantum computing) has put it front and center not just to physicists but also to the general public, and it can be expected to remain as an active field in the upcoming years [12].

In lieu of this, a promising area that continually receives a new flurry of interest is the study of entanglement and its relation to quantum dynamics [13–17]. Entanglement has already been established as a powerful resource for understanding various many-body phenomena. However, most of these studies deal with the von Neumann entropy measure known more commonly as *entanglement entropy*, which is a measure of the bipartite entanglement of the system [18]. Due to the inherent complexity in the classification and measurement of multipartite entanglement, it is only recently that we are beginning to understand and observe its importance in the overall behavior of many-body systems [19–25].

In particular, recent studies have shown that the entanglement structure in the steady state, characterized by the object called the quantum Fisher information (QFI), which measures the degree of multipartite entanglement in the system [26, 27], is a property which can differentiate supposedly identical "thermal" states with different initial conditions or represented by different ensembles [24, 25]. It should be noted that this is a remarkable result since these "thermal" states are indistinguishable by measurements of local observables, and it has been shown to be true even in non-equilibrium settings. Nonetheless, despite the significant progress being made in tackling this general problem, most, if not all of the current analysis are done within the confines of non-integrable models.

In contrast to an isolated non-integrable system with a Hamiltonian $\hat{H}$, whose steady-state can be described by the familiar canonical or *Gibbs* ensemble, $\hat{\rho} \propto \exp\left\{-\beta\hat{H}\right\}$, where the inverse temperature β is fixed by the initial energy density of the system, integrable systems are characterized by states described by the so-called generalized *Gibbs* ensemble (GGE) [28],

$$\hat{\rho}_{GGE} = Z^{-1} \exp\left(-\sum_k \lambda_k \hat{Q}_k\right), \tag{1.1}$$

where Z is a normalization constant (partition function), and which involves an extensive number of conserved quantities, $\hat{Q}_k$, and their associated Lagrange multipliers λ_k which are fixed by initial conditions. The construction of the steady-state $\hat{\rho}_{GGE}$ can sometimes be difficult and depends on the specific model being considered, since the process of identifying the set of conserved quantities $\hat{Q}_k$ is not clear-cut [29, 30]. Nevertheless, all integrable systems are characterized by a constrained dynamics due to the presence of these additional conservation laws. Generally, steady states of integrable systems and their corresponding GGE are indistinguishable by local measurements of observables and their correlations [31, 32]. Thus it begs the question whether as for non-integrable systems the information lost in constructing the effective ensemble describing the long-time steady state (GGE) makes its appearance in the study of multipartite entanglement, implying that the latter is always greater in the pure steady state than in the effective GGE.

Motivated by this question, we extend the work from Reference [25] to integrable systems whose states are described by the generalized Gibbs ensemble (GGE). We start by deriving analytic expressions for the QFI of states described by the GGE (F^{GGE}) for the simplest class of integrable models: systems which can be represented in terms of free bosonic or fermionic quasi-particles. In the derivations, we use certain class of observable $\hat{O}$ which is diagonal in the quantum numbers. We then compare this to that obtained from a quenched pure state F^{pure}, showing that in general $F^{GGE} \leq F^{pure}$.

Chapter 2

Quantum Fisher information in non-interacting systems

Here we aim to provide a self-contained discussion of the theoretical framework used in this work along with its main results. We start the chapter by giving a brief overview on the topic of quantum quenches. Next, we introduce the definition of the quantum Fisher information we will use in this work, and then proceed to the derivation of the QFI in non-interacting systems.

2.1 Non-equilibrium dynamics of many-body systems

We consider an isolated quantum many-body system, described by the free-particle Hamiltonian $\hat{H}$,

$$\hat{H} = \sum_k \varepsilon_k \hat{\gamma}_k^\dagger \hat{\gamma}_k, \tag{2.1}$$

where $\hat{\gamma}_k^\dagger$ and $\hat{\gamma}_k$ are the usual creation and annihilation operators for either bosonic or fermionic modes. To study its non-equilibrium dynamics, the system is initialized at time $t = 0$, in a pure state, $|\psi_0\rangle$, which could be taken to be an eigenstate of another distinct Hamiltonian, $\hat{H}_0$. Since its isolated, its dynamics is governed by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle, \tag{2.2}$$

where $|\psi(t)\rangle$ is the state at time t. By defining an orthonormal basis that diagonalizes the Hamiltonian H, we can express the unitary time evolution of $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |n\rangle, \tag{2.3}$$

where $c_n = \langle n|\psi_0\rangle$ is the overlap between the eigenstates of the Hamiltonian $\hat{H}$ and the initial state, and E_n is its corresponding energy. Assuming that $\hat{H}$ does not commute with the initial Hamiltonian $\hat{H}_0$, the ensuing dynamics is non-trivial. The state of affairs we have just described is more commonly known as a quantum quench, and in this work we will consider a quantum quench protocol wherein a parameter, for example an external field, h is changed globally and instantly from its initial value h_0 to h.

2.2 Quantum Fisher information

2.2.1 Background

The QFI was originally introduced in quantum metrology and quantifies the maximal precision a phase or parameter θ can be estimated from a state $\hat{\rho}$ upon measuring the observable $\hat{\mathcal{O}}$ [26, 33, 34]. Over M independent measurements, the QFI bounds the variance of the phase, $(\Delta\theta)^2$ by the so-called quantum Cramer-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{MF_Q(\hat{\mathcal{O}}, \hat{\rho})}. \tag{2.4}$$

More recently, the QFI is being studied due to its usefulness in probing the multipartite entanglement structure of quantum states [26,27,33]. For example, in spin systems the QFI density optimized over a class of operators consisting of all possible linear combinations of Pauli matrices, one obtains that

$$f_Q > k, \tag{2.5}$$

with k being an integer, implies that the state is $(k+1)$-entangled.

However, the most relevant feature of the QFI for this work is in its recent application for the establishment of a hierarchy in the entanglement structure of states subjected through a quantum quench, which is remarkable since these steady states are indistinguishable from each other through measurements of local observables [25]. Specifically, in Ref. [25], the following inequality has been observed:

$$F_Q(\hat{O}, \rho_\beta) \leq F_Q^{ETH} \leq F_Q^{\infty}, \tag{2.6}$$

where F_Q^{Gibbs} is the QFI obtained from a state in the canonical Gibbs ensemble with an and F_Q^{ETH} is the QFI obtained from a generic state which thermalizes according to the eigenstate thermalization hypothesis (ETH) with the same effective temperature as ρ_β , and F_Q^{∞} is the QFI calculated on the steady state attained by a system starting with a microcanonical superposition as time $t = 0$. Our goal will be to figure out whether this result can be extended to integrable systems, that is, we check if the following inequality holds:

$$F_Q^{GGE} \overset{?}{\leq} F_Q^{\infty}. \tag{2.7}$$

2.2.2 QFI in the quenched pure state and in the GGE

For a pure initial state, $\hat{\rho} = |\psi\rangle \langle\psi|$, the QFI has a simple form given by

$$F_Q^{pure}(\hat{\mathcal{O}}, \hat{\rho}; t) = 4\left\langle (\Delta\hat{\mathcal{O}}(t))^2 \right\rangle = 4\left[\left\langle \hat{\mathcal{O}}^2(t) \right\rangle - \left\langle \hat{\mathcal{O}}(t) \right\rangle^2\right]. \tag{2.8}$$

To calculate the long-time limit of $F_Q^{pure}(t)$, we first recognize that the expectation value of an observable $\hat{\mathcal{O}}$ at any time t, can be generally expressed as

$$\langle\hat{\mathcal{O}}(t)\rangle = \bar{\mathcal{O}} + \delta\mathcal{O}(t), \tag{2.9}$$

where

$$\bar{\mathcal{O}} = \sum_n |c_n|^2 \mathcal{O}_{nn}, \tag{2.10}$$

$$\delta\mathcal{O}(t) = \sum_{m \neq n} \exp^{-i(E_n - E_m)t/hbar} c_n^* c_n \mathcal{O}_{mn}, \tag{2.11}$$

with $\mathcal{O}_{mn} = \langle m|\hat{\mathcal{O}}|n\rangle$. Using the expansion above, we can recast the fluctuation of the operator at any time t as

$$\langle \Delta\hat{\mathcal{O}}^2(t)\rangle = \langle \hat{\mathcal{O}}^2\rangle_d + \delta O^2(t), \tag{2.12}$$

where $\langle \Delta\hat{\mathcal{O}}^2\rangle_d$ corresponds to a stationary value and is equivalent to the fluctuation of the operator within the *diagonal ensemble* [31, 32],

$$\langle \Delta\hat{\mathcal{O}}^2\rangle_d = \sum_n c_n^2 \langle n|\hat{\mathcal{O}}^2|n\rangle - \left[\sum_n c_n|^2 \langle n|\hat{\mathcal{O}}|n\rangle\right]^2, \tag{2.13}$$

and a time dependent part ($\delta O^2(t)$) which we write explicitly:

$$\begin{aligned} \delta O^2(t) = &\sum_l \sum_{m \neq n} e^{-i(E_n - E_m)t/\hbar} (c_m^* c_n \mathcal{O}_{ml}\mathcal{O}_{ln} - 2|c_l|^2 c_m^* c_n \mathcal{O}_{ll}\mathcal{O}_{mn}) \\ &- \sum_{m \neq n} \sum_{m' \neq n'} -e^{-i(E_n + E_{n'} - E_m - E_{m'})} c_m^* c_n c_{m'}^* c_{n'} \mathcal{O}_{mn}\mathcal{O}_{m'n'}. \end{aligned} \tag{2.14}$$

For non-integrable systems, one can assume a non-degenerate energy spectrum so that $\delta O^2(t)$ vanishes in the long time limit due to dephasing. However, we are not afforded this luxury when working with non-interacting systems. Degeneracies in single-particle energy levels as well as in the gap excitations exists and must be taken into account. However, with our choice of operators we can limit the non-vanishing terms. Specifically, we choose observables $\hat{\mathcal{O}}$ which are diagonal in k and which can be written as a single sum. This restriction will invariably lead to vanishing off-diagonal terms $\mathcal{O}_{mn, m \neq n}$ where these degenerices arise, and we are left with a positive definite term we call $\delta O^2_{degen} \propto \langle n|\hat{\mathcal{O}}^2|n\rangle$. Thus, with these considerations, we can express the QFI in the quenched pure state as

$$F_Q^\infty(\hat{\mathcal{O}}, \hat{\rho}) = \lim_{t \to \infty} F_Q^{pure}(\hat{\mathcal{O}}, \hat{\rho}; t) \geq 4\langle \Delta\hat{\mathcal{O}}^2\rangle_d, \tag{2.15}$$

where we emphasize that the equality holds in the absence of degeneracies in the single-particle energy spectrum ($\varepsilon_{k_1} = \varepsilon_{k_2}, k_1 \neq k_2$) as well as an absence of degenerate gap excitations ($\varepsilon_{k_1} - \varepsilon_{k_2} = \varepsilon_{q_1} - \varepsilon_{q_2}$).

Lastly, for a generic mixed state, $\rho = \sum_n \rho_n |n\rangle\langle n|$, the QFI can be defined through the eigenvalues of the density matrix and the square of the matrix element of the operator $\hat{\mathcal{O}}$ [3],

$$F_Q(\hat{\mathcal{O}}^{(p)}, \hat{\rho}) = 2\sum_m \sum_n \frac{(\rho_{mm} - \rho_{nn})^2}{\rho_{mm} + \rho_{nn}} |\langle m|\hat{\mathcal{O}}|n\rangle|^2, \tag{2.16}$$

where $\rho_{mm} = \langle m|\hat{\rho}|m\rangle$. Hence, to calculate the QFI in the state described by the gge, we use the density matrix prescribed by Eqn.(1.1).

2.2.3 QFI in fermionic systems after quench

Let us consider the case wherein the observable $\hat{\mathcal{O}}$ consists of fermionic operators (which we now designate as $\hat{\mathcal{O}}^f$ for clarity). We introduce a simple yet generic, observable $\hat{\mathcal{O}}_1^f$,

$$\hat{\mathcal{O}}_1^f = \sum_k \left(A_k \hat{\gamma}_k^\dagger + B_k \hat{\gamma}_k^\dagger \hat{\gamma}_k^\dagger + C_k \hat{\gamma}_k^\dagger \hat{\gamma}_k + h.c. \right). \tag{2.17}$$

We note here that the observable presented above encompasses a large number of single-body fermionic operators since higher order operators vanish or can be straightforwardly rewritten as bilinear operators via Wick's theorem. As such, if $|n\rangle$ is a many-body eigenstate of the Hamiltonian $\hat{H}$, we get the following matrix elements:

$$\langle n|\hat{\mathcal{O}}_1^f|n\rangle = \sum_k C_k \langle n|\hat{\gamma}_k^\dagger \hat{\gamma}_k|n\rangle, \tag{2.18}$$

$$\langle n|(\hat{\mathcal{O}}_1^f)^2|n\rangle = \sum_k |A_k|^2 + \sum_{k,k'} |C_k|^2 \langle n|\hat{\gamma}_k^\dagger \hat{\gamma}_k \hat{\gamma}_{k'}^\dagger \hat{\gamma}_{k'}|n\rangle, \tag{2.19}$$

wherein we recall that $\langle n|\hat{\gamma}_k^\dagger \hat{\gamma}_k|n\rangle = n_k = [0,1]$ is the occupation at a particular mode k for a specific many-body eigenstate $|n\rangle$. From these expectation values, it follows immediately that the fluctuations of the operator $\hat{\mathcal{O}}_1^f$ with respect to the diagonal-ensemble is simply

$$\langle \Delta(\hat{\mathcal{O}}_1^f)^2 \rangle_d = \sum_k \left[|A_k|^2 + |C_k|^2 \langle n_k \rangle (1 - \langle n_k \rangle) \right]. \tag{2.20}$$

Hence, from Eqns.(2.13) and (2.15) we express the QFI in the quenched pure state as

$$F_Q^\infty(\hat{\mathcal{O}}_1^f, \hat{\rho}) = 4 \sum_k \left[|A_k|^2 + C_k^2 \langle n_k \rangle (1 - \langle n_k \rangle) \right]. \tag{2.21}$$

On the other hand, to facilitate our calculation for the QFI in the GGE, we first note that the sum with respect to the different eigenstates $|n\rangle$ and $|m\rangle$ in Eqn.(2.16), can be recast into a configuration sum with respect to the set of occupation numbers $\{m_k\}$ and $\{n_k\}$; that is, let us use the set of occupation numbers $\{n_k\}$ to label each eigenstate $|n\rangle$ and then rewrite the sum as

$$\sum_n \to \sum_{\{n_k\}} = \sum_{n_{k_1}} \sum_{n_{k_2}} \cdots \sum_{n_{k_N}}. \tag{2.22}$$

The eigenvalues of the GGE density matrix are given by $\rho_{mm} = \langle m|\hat{\rho}_{GGE}|m\rangle = Z^{-1} \exp(\sum_k \lambda_k m_k)$, while the square of the matrix element of the operator $\hat{\mathcal{O}}_1^f$ with respect to the many-body eigenstates can be expressed as

$$\begin{aligned} |\langle m|\hat{\mathcal{O}}_1^f|n\rangle|^2 = & \sum_k |A_k|^2 \, \delta_{\{m_k\},\{n_k|n_k \to 1-n_k\}} \\ & + \sum_k \left((A_K)^2 + (A_k^*)^2 \right) \sqrt{n_k(1-n_k)} \\ & \times \delta_{\{m_k\},\{n_k|n_k \to 1-n_k\}} \\ & + \sum_{k,k'} C_k C_{k'} n_k n_{k'} \, \delta_{\{m_k\},\{n_k\}}, \end{aligned} \tag{2.23}$$

wherein the Kronecker-deltas $\delta_{\{m_k\},\{n_k|n_k\to n_k+c\}}$ acts on the set of occupation numbers $\{n_k\}$, i.e., it designates a relationship between the two sets of occupation numbers $\{m_k\}$ and $\{n_k\}$. Explicitly, we can expand the Kronecker-delta, $\delta_{\{m_k\},\{n_k\}}$ as products of individual Kronecker-deltas for each distinct occupation number m_k, whence,

$$\delta_{\{m_k\},\{n_k|n_l\to n_k+c\}} = \delta_{m_{k_1},n_{k_1}} \cdots \delta_{m_k,n_k+c} \cdots \delta_{m_{k_N},n_{k_N}}, \tag{2.24}$$

for a set of N distinct occupational modes. Inserting the square of the matrix element into Eqn. (2.16), yields

$$\begin{aligned} F_Q^{GGE}(\hat{\mathcal{O}}_1^f, \hat{\rho}) = 2 \sum_{\{n_k\}} \sum_k \Big[\frac{(\rho_{\{n_k|n_k\to 1-n_k\}} - \rho_{\{n_k\}})^2}{\rho_{\{n_k|n_k\to 1-n_k\}} + \rho_{\{n_k\}}} \\ \times |A_k|^2 \Big] + 2 \sum_{\{n_k\}} \sum_k \frac{(\rho_{\{n_k|n_k\to 1-n_k\}} - \rho_{\{n_k\}})^2}{\rho_{\{n_k|n_k\to 1-n_k\}} + \rho_{\{n_k\}}} \\ \times \left(\left(A_k^2 + (A_k^*)^2\right) \sqrt{n_k(1-n_k)} \right). \end{aligned} \tag{2.25}$$

wherein we have already performed the sum with respect to the set of quantum numbers $\{m_k\}$, which simply enforces Kronecker deltas in Eqn. (2.23); that is, we shifted the occupation numbers in the terms containing the probabilities $\rho_{\{m_k\}}$. Then, upon evaluation of the last configuration sum ($\{n_k\}$), we immediately see that the second term in the previous equation vanishes since the occupation numbers n_k only take values of zero or one, thus we get

$$F_Q^{GGE}(\hat{\mathcal{O}}_1^f, \hat{\rho}) = 4 \sum_k \tanh^2\left(\frac{\lambda_k}{2}\right) |A_k|^2, \tag{2.26}$$

where as mentioned earlier, λ_k are fixed from the initial condition, $\langle n_k \rangle_0 = \mathrm{Tr}[\hat{\rho}_{GGE}\hat{n_k}]$, which for a fermionic system is given by $\lambda_k = \ln\left(\frac{1-\langle n_k\rangle}{\langle n_k\rangle}\right)$.

It is straightforward from Eqns.(2.21) and (2.26) that the inequality, $F_Q^{GGE} \leq F_Q^{\infty}$, holds true since the QFI contribution from any mode k, which we can define as $F_Q = \sum_k F_{Q,k}$, is always larger in the quenched pure state.

As an exercise, we used the expressions obtained above to calculate the QFI density in a transverse field Ising chain subjected to a quantum quench. This model is described by the Hamiltonian, $H_{Ising} = -J\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - Jh\sum_i \hat{\sigma}_i^z$, where $\hat{\sigma}_i^{x,z}$ are Pauli matrices and h is the transverse field [35,36]. This model can be diagonalized using a Jordan-Wigner rotation and a subsequent Bogoliubov transformation to a non-interacting system of fermionic quasi-particles with an energy dispersion of $\varepsilon_k = 2\sqrt{1+h^2-2h\cos(k)}$. We initialize the system in the ground state of the Hamiltonian with transverse field h_0 and change the magnetic field at $t=0$ from h_0 to h. The occupation at each mode $\langle n_k\rangle$ can be derived as a function of the quench parameters using the relation [36],

$$\langle n_k \rangle_{ising} = \frac{1}{2}\left(1 - \cos\left(\theta_k\right)\right), \tag{2.27}$$

where $\theta_k = 4(1+h_0h-(h_0+h)\cos(k))/(\varepsilon_k\varepsilon_k^0)$ with ε_k^0 denoting the energy at mode k before the quench. The QFI densities calculated using a the operator

$\hat{\mathcal{O}}_1^f$ with constant coefficient $|A_k|^2 = C_k^2 = 1$, are presented in Fig.2.1. As predicted, the inequality $F_Q^{GGE} \leq F_Q^{\infty}$ holds true for the two types of quench protocols considered.

To provide another example with a different choice of operator, we extend our analyses to operators of the form,

$$\hat{\mathcal{O}}_2^f = \sum_k \left(D_k \hat{\gamma}_k^\dagger \hat{\gamma}_{k+k_1}^\dagger + E_k \hat{\gamma}_k^\dagger \hat{\gamma}_{k+k_1} + h.c. \right), \tag{2.28}$$

with k_1 fixed. We find, using the methodologies outlined above, the QFI for the quenched pure state as

$$\begin{aligned} F_Q^{\infty}(\hat{\mathcal{O}}_2^f, \hat{\rho}) = 4 \sum_k \Big[&|D_k|^2 + \left(|D_k|^2 - E_k^2 \right) \\ &\times \left(2\langle n_k \rangle \langle n_{k+k_1} \rangle - \langle n_k \rangle - \langle n_{k+k_1} \rangle \right) \Big]. \end{aligned} \tag{2.29}$$

While in the GGE, F_Q^{GGE} can be initially expressed as

$$\begin{aligned} F_Q^{GGE}(\hat{\mathcal{O}}_2^f, \hat{\rho}) = 2 \sum_{\{n_k\}} \sum_k & \frac{\left(\rho_{\{1-n_k, 1-n_{k+k_1}\}'} - \rho_{\{n_k\}} \right)^2}{\rho_{\{1-n_k, 1-n_{k+k_1}\}'} + \rho_{\{n_k\}}} \\ \times \left[|D_k|^2 + \left(|D_k|^2 - E_k^2 \right) \left(2 n_k n_{k+k_1} - n_k - n_{k+k_1} \right) \right], & \end{aligned} \tag{2.30}$$

which, upon the evaluation of the configuration sum in $\{n_k\}$ yields

$$\begin{aligned} F_Q^{GGE}(\hat{\mathcal{O}}_2^f, \hat{\rho}) = 2 &\sum_k |D_k|^2 \Theta_1(\lambda_k, \lambda_{k+k_1}) \\ &- 2 \sum_k \left(|D_k|^2 - E_k^2 \right) \Theta_2(\lambda_k, \lambda_{k+k_1}), \end{aligned} \tag{2.31}$$

where the coefficients, Θ_1 and Θ_2 are given by the following expressions:

$$\begin{aligned} \Theta_1(\lambda_k, \lambda_{k+k_1}) &= 2 - \frac{4}{\cosh(\lambda_k) + \cosh(\lambda_{k+k_1})}, \\ \Theta_2(\lambda_k, \lambda_{k_k+1} &= \frac{(e^{-\lambda_k} - e^{\lambda_{k+k_1}})^2}{(1+e^{-\lambda_k})^2 (1+e^{-\lambda_{k+k_1}})} \\ &+ \frac{(e^{-\lambda_k} - e^{\lambda_{k+k_1}})^2}{(1+e^{-\lambda_k})(1+e^{-\lambda_{k+k_1}})^2}. \end{aligned} \tag{2.32}$$

Applying these expressions onto the transverse field Ising model , the QFI densities are obtained in the quenched pure state and in the state described by the GGE. We present these results in Fig. 2.2. We still observe the inequality of $F_Q^{GGE} \leq F_Q^{\infty}$ for all values of the quench parameter h.

2.2.4 QFI in bosonic systems after quench

Unlike in the fermionic case, there is no simple way to write all the relevant bosonic operators into a single sum in k. As such, for this problem we would be considering a generic pth-order, bosonic operator which we write as

$$\hat{\mathcal{O}}_p^b = \sum_k \sum_{q=0}^{p} C_{p,q}(k) (\hat{b}_k^\dagger)^{p-q} (\hat{b}_k)^q. \tag{2.33}$$

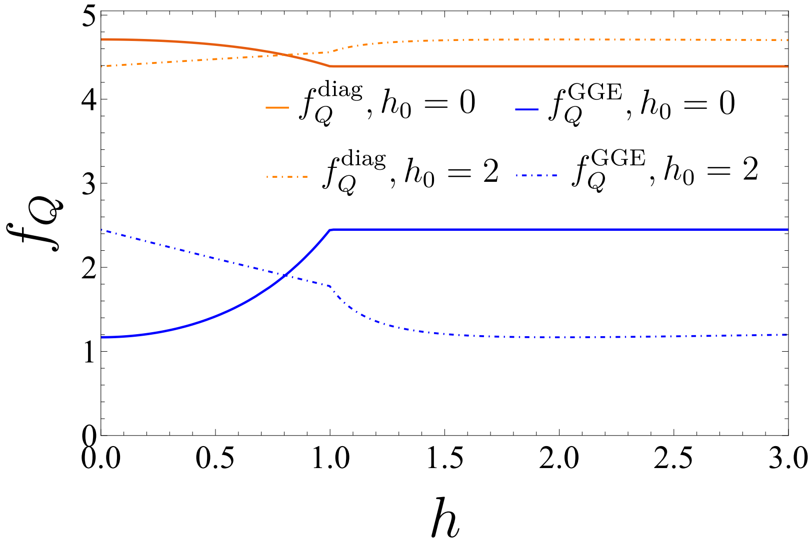

Figure 2.1: QFI density in the transverse field Ising chain various quench protocols. Here we consider the sudden quenches with initial quench parameters $h_0 = 0$ and $h_0 = 2$, while $J = 1$, and $h \geq 0$. The occupation at each mode k is described by Eqn.(2.27), and the observable used to calculate the QFI is $\hat{\mathcal{O}}_1^f$, given by Eqn.(2.17) with constant coefficients, $A_k = C_k = 1$. We considered a system with periodic boundary conditions and $L = 10^3$.

For the particular operator we have chosen, the matrix element $\langle m|\hat{\mathcal{O}}_p^b|n\rangle$ can be straightforwardly obtained:

$$\begin{aligned}\hat{\mathcal{O}}_{p,mn}^b \equiv& \langle m|\hat{\mathcal{O}}_p^b|n\rangle = \sum_k \sum_{q=0}^{p} C_{p,q}(k) \, \langle m| \left(\hat{b}_k^\dagger\right)^{p-q} \left(\hat{b}_k\right)^q |n\rangle, \\ =& \sum_k \sum_{q=0}^{p} c_{p,q}(k) \sqrt{(n_k)(n_k-1)\cdots(n_k-q+1)} \\ & \times \langle m| \left(\hat{b}_k^\dagger\right)^{p-q} |\{n_k|n_k \to n_k - q\}\rangle, \\ =& \sum_k \sum_{q=0}^{p} c_{p,q}(k) \sqrt{(n_k)\cdots(n_k-q+1)} \sqrt{(n_k-q+1)\cdots(n_k+p-2q)} \\ & \times \langle m|\{n_k|n_k \to n_k - q\}\rangle, \\ =& \sum_k \sum_{q=0}^{p} c_{p,q}(k) \sqrt{\frac{(n_k)!}{(n_k-q)!}} \sqrt{\frac{(n_k+p-2q)!}{(n_k-q)!}} \, \langle m|\{n_k|n_k \to n_k - q\}\rangle,\end{aligned} \tag{2.34}$$

where we have made use of the relations

$$(n)(n-1)(n-2)\cdots(n-a) = \frac{n!}{(n-a-1)!}, \tag{2.35}$$

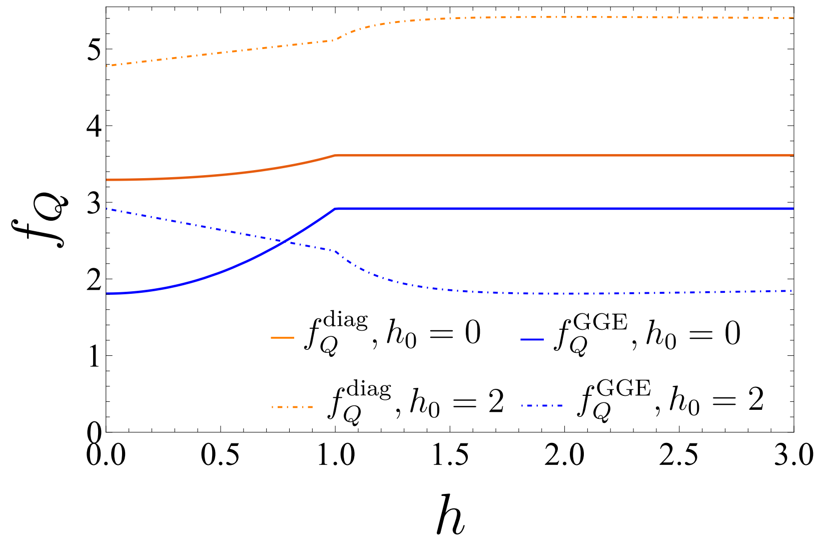

Figure 2.2: QFI density in the transverse field Ising chain using an observable which enables mode-mixing. We use an observable mode-mixing terms with coefficients $E_k^2/|D_k|^2 = 2$. The sudden quench protocol studied are for by $h_0 = 0$ and $h_0 = 2.0$. The occupation at each mode k is described by Eqn.(2.27).

and

$$(n)(n+1)(n+2)\cdots(n+b) = \frac{(n+b)!}{(n-1)!} \tag{2.36}$$

to simplify the products inside the square root in Eqn. (2.34). Thus, we arrive at the general expression for the matrix element,

$$\langle m|\hat{\mathcal{O}}_p^b|n\rangle = \sum_k \sum_{q=0}^{p} \left[c_{p,q}(k) \sqrt{\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2}}\, \delta_{\{m_k\},\{n_k|n_k\to n_k+p-2q\}} \right], \tag{2.37}$$

It immediately follows then that

$$\langle m|\hat{\mathcal{O}}_p^b|m\rangle = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \sum_k \left[c_{p,p/2}(k) \left(\frac{(n_k)!}{(n_k-p/2)!} \right) \right] & \text{if } p \text{ is even,} \end{cases}. \tag{2.38}$$

and

$$\langle m|(\hat{\mathcal{O}}_p^b)^2|m\rangle = \begin{cases} \sum_k \sum_{q=0}^{p} |C_{p,q}(k)|^2 \left(\frac{(m_k)!(m_k+p-2q)!}{((m_k-q)!)^2} \right) & \text{if } p \text{ is odd,} \\ \sum_k \sum_{q=0}^{p} |C_{p,q}(k)|^2 \left(\frac{(m_k)!(m_k+p-2q)!}{((m_k-q)!)^2} \right) + \langle m|\hat{\mathcal{O}}_p^b|m\rangle^2 & \text{if } p \text{ is even.} \end{cases} \tag{2.39}$$

Therefore, the QFI in the quenched pure state has the expression,

$$F_Q^{\infty}(\hat{\mathcal{O}}_p^b, r\hat{h}o) = 4 \sum_k \sum_{q=0}^{p} \sum_{\{n_k\}} |C_{p,q}(k)|^2 \left[\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2} |d_{n,0}|^2 \right]. \tag{2.40}$$

On the other hand, for the QFI in the GGE, we use the following expression for the square of the matrix element,

$$|\langle m|\hat{\mathcal{O}}^{(p)}|n\rangle|^2 = \begin{cases} \sum_k \sum_{q=0}^{p} \left[|c_{p,q}|^2 \left(\frac{(n_k)!(n_k+p-2q)!}{((n_K-q)!)^2}\right) \delta_{\{m_k\},\{n_k|n_k\to n_k+p-2q\}}\right] & \text{if } p \text{ is odd,} \\ \sum_k \sum_{q=0}^{p} \left[|c_{p,q}|^2 \left(\frac{(n_k)!(n_k+p-2q)!}{((n_K-q)!)^2}\right) \delta_{\{m_k\},\{n_k|n_k\to n_k+p-2q\}}\right] & \\ + \sum_{k\neq k'} \left[c_{p,p/2}(k) c_{p,p/2}(k') \left(\frac{(n_k)!(n_{k'})!}{(n_k-p/2)!(n_{k'}-p/2)!}\right) \delta_{\{m_k\},\{n_k\}}\right] & \text{if } p \text{ is even.} \end{cases} \tag{2.41}$$

which, upon into our definition of the QFI for a generic mixed state, yields

$$\mathcal{F}_{GGE}(\hat{\mathcal{O}}^{(p)}, \hat{\rho}) = 2 \sum_k \sum_{q=0}^{p} \sum_{\{n_k\}} |c_{p,q}(k)|^2 \left[\frac{\left(\rho_{\{n_k+p-2q\}'} - \rho_{\{n_k\}}\right)^2}{\rho_{\{n_k+p-2q\}'} + \rho_{\{n_k\}}} \left(\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2} \right) \right]. \tag{2.42}$$

Here we reiterate that in the previous line we have already performed the configuration sum with respect to the set of quantum numbers $\{m_k\}$, which simply enforces the action of the Kronecker delta in Eqn. (2.41). This shifts the occupation numbers in the terms containing the probabilities $\rho_{\{m_k\}}$. In lieu of this, the prime in the subscript $\{n_k+p-2q\}'$ indicates that the occupation for a single mode (at k) is shifted; explicitly, this means that

$$\rho_{\{n_k\}} = \frac{1}{Z} \exp\left[- \sum_{\{n_k\}} \lambda_k n_k \right] = \frac{1}{Z} e^{-\lambda_{k_1} n_{k_1}} e^{-\lambda_{k_2} n_{k_2}} \cdots e^{-\lambda_k n_k} \cdots e^{-\lambda_{k_N} n_{k_N}}, \tag{2.43a}$$

$$\rho_{\{n_k+\zeta\}'} = \frac{1}{Z} \exp\left[- \sum_{\{n_k+\zeta\}'} \lambda_k n_k \right] = \frac{1}{Z} e^{-\lambda_{k_1} n_{k_1}} e^{-\lambda_{k_2} n_{k_2}} \cdots e^{-\lambda_k (n_k+\zeta)} \cdots e^{-\lambda_{k_N} n_{k_N}}. \tag{2.43b}$$

Moreover, it can be easily verified that the following relation holds:

$$\frac{\rho_{\{m_k\}} - \rho_{\{m_k+\zeta\}'}}{\rho_{\{m_k\}} + \rho_{\{m_k+\zeta\}'}} = \frac{e^{-\lambda_k m_k} - e^{-\lambda_k (m_k+\zeta)}}{e^{-\lambda_k m_k} + e^{-\lambda_k (m_k+\zeta)}} = \frac{1 - e^{-\zeta\lambda_k}}{1 + e^{-\zeta\lambda_k}} = \tanh\left(\frac{\zeta\lambda_k}{2}\right). \tag{2.44}$$

Taking all the previous considerations into account, we can write the QFI in the generalized Gibbs ensemble in the following way:

$$F_Q^{GGE}(\hat{\mathcal{O}}_p^b,\hat{\rho}) = 2\sum_k\sum_{q=0}^{p}\sum_{\{n_k\}}|c_{p,q}(k)|^2\Bigg[\left(\frac{\rho_{\{n_k+p-2q\}'}-\rho_{\{n_k\}}}{\rho_{\{n_k+p-2q\}'}+\rho_{\{n_k\}}}\right)\left(\rho_{\{n_k+p-2q\}'}-\rho_{\{n_k\}}\right),$$
$$\times\left(\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2}\right)\Bigg] \tag{2.45}$$

$$=2\sum_k\sum_{q=0}^{p}\sum_{\{n_k\}}|c_{p,q}(k)|^2\left[\tanh\left(\frac{\lambda_k(p-2q)}{2}\right)\left(\rho_{\{n_k\}}-\rho_{\{n_k+p-2q\}'}\right).\right.$$
$$\left(\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2}\right) \tag{2.46}$$

Finally, using the relation,

$$\rho_{\{n_k+\zeta\}'} = e^{-\lambda_k\zeta}\rho_{\{n_k\}}, \tag{2.47}$$

we arrive at the simplified expression for the QFI:

$$\mathcal{F}_{GGE}(\hat{\mathcal{O}}^{(p)},\hat{\rho}) = 2\sum_k\sum_{q=0}^{p}\sum_{\{n_k\}}|c_{p,q}(k)|^2\Bigg[\tanh\left(\frac{\lambda_k(p-2q)}{2}\right)\left(1-e^{-\lambda_k(p-2q)}\right)$$
$$\left(\frac{(n_k)!(n_k+p-2q)!}{((n_k-q)!)^2}\right)\rho_{\{n_k\}}\Bigg]. \tag{2.48}$$

As an application of our results, we calculate the QFI densities in a one-dimensional chain of harmonic oscillators, which is defined by the Hamiltonian, $\hat{H}_{HO} = 1/2\sum_n\left[\pi_n^2+\omega^2\varphi_n^2+(\varphi_{n+1}-\varphi_n)^2\right]$ [37]. Here φ_n and π_n are the position and momentum operators of a harmonic oscillator at site n with frequency ω. This model is easily diagonalizable in terms of free-bosonic quasiparticles which has the following dispersion relation,

$$\varepsilon_k = \sqrt{\omega^2+2(1-\cos(k))}. \tag{2.49}$$

For our quench protocol, we let the system initialze at the ground state of the Hamiltonian with $\omega_0 = 1$, then at time $t = 0$ the system is driven such that $\omega > \omega_0$. The occupation at each mode k can be derived as follows:

$$\langle n_k\rangle_{HO} = \frac{1}{4}\left(\frac{\varepsilon_k}{\varepsilon_{k,0}}+\frac{\varepsilon_{k,0}}{\varepsilon_k}\right)-\frac{1}{2}, \tag{2.50}$$

where $\varepsilon_{k,0}$ is the energy at mode k in the initial Hamiltonian (with frequency ω_0). For our observable, we use a linear bosonic operator, with constant coefficients, $C_{1,0}(k) = C_{1,1}(k) = 1$

$$\hat{\mathcal{O}}_1^b = \sum_k \hat{b}_k^\dagger + \hat{b}_k. \tag{2.51}$$

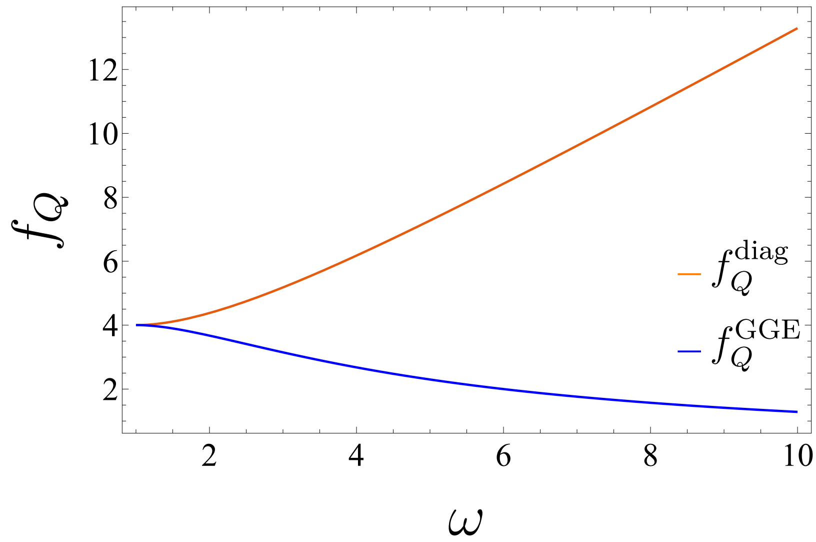

Figure 2.3: QFI density in the one-dimensional harmonic oscillator chain using a linear operator observable.

With this observable, the QFI in the quenched pure state and in the GGE can be simplified into the following expressions:

$$F_Q^{\infty}(\hat{\mathcal{O}}_1^b, \hat{\rho}) = 8\sum_k \left[\langle n_k\rangle + \frac{1}{2}\right], \tag{2.52}$$

$$F_Q^{GGE}(\hat{\mathcal{O}}_1^b, \hat{\rho}) = 8\sum_k \left[\frac{1}{4}\left(\frac{1}{\langle n_k\rangle} + \frac{\langle n_k\rangle}{\langle n_k\rangle + 1} - 1\right)\left(\langle n_k\rangle + \frac{\langle n_k\rangle}{2\langle n_k\rangle + 1}\right)\right]. \tag{2.53}$$

Chapter 3

Conclusions

In this work, we have derived exact analytic expressions for the quantum Fisher information in generic non-interacting systems subject to a sudden quench protocol. Using generic observables which are diagonal in the quasiparticle modes, we have shown through the QFI density that even in non-interacting systems, the QFI detects the difference between a pure state and a mixed state described by the GGE, even if they are constrained by the same integrals of motion, which in this case are associated with the initial occupation at each mode $\langle n_k \rangle_0$.

The significant discrepancies found between the QFI in a quenched pure state and that obtained a mixed state described by the GGE suggests a more fundamental difference between these two descriptions that warrants a more comprehensive study.

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