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Reduced basis stabilization for the unsteady Stokes and Navier-Stokes equations

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Abstract

In the Reduced Basis approximation of Stokes and Navier-Stokes problems, the Galerkin projection on the reduced spaces does not necessarily preserved the inf-sup stability even if the snapshots were generated through a stable full order method. Therefore, in this work we aim at building a stabilized Reduced Basis (RB) method for the approximation of unsteady Stokes and Navier-Stokes problems in parametric reduced order settings. This work extends the results presented for parametrized steady Stokes and Navier-Stokes problems in a work of ours [1]. We apply classical residual-based stabilization techniques for finite element methods in full order, and then the RB method is introduced as Galerkin projection onto RB space. We compare this approach with supremizer enrichment options through several numerical experiments. We are interested to (numerically) guarantee the parametrized reduced inf-sup condition and to reduce the online computational costs.

Keywords: reduced basis method, *offline-online stabilization*, RB inf-sup stability

1 Introduction

In the finite element (FE) simulation of incompressible flows using a standard Galerkin formulation there are two possible sources of instabilities. One reason could be due to the presence of convection term which for high Reynolds number creates instability in numerical solution. Another source of instability could be due to the inappropriate choice of interpolating functions for velocity and pressure. Starting from early 70s, different researchers [40, 12, 23, 24, 31] proposed several stabilized schemes to overcome stability issues. For instance, Hughes and Brooks [26, 9, 10] proposed to add artificial diffusion term acting only in the streamline direction and named this type of formulation as Streamline Upwind/Petrov Galerkin (SUPG) formulation. An extension of SUPG formulation is given by Hughes et al. [29] and is named as Galerkin Least Square (GALS) formulation. Later on Douglas-Wang [15] introduced the change of sign in GALS formulation. A penalty method in which pressure is eliminated by penalizing the continuity equation and then retained in boundary condition was introduced by Hughes et al. [30]. Hughes et al. [28] used equal order interpolation for velocity and pressure by perturbing

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the pressure test function with a gradient term to achieve the stability. A symmetric version of this method was given by Hughes and Franca [27]. The SUPG method, first applied by Brooks and Hughes [10] to solve numerically the incompressible Navier-Stokes equations with high Reynolds number was later on extended by various researchers [18, 15, 32, 22, 17, 46].

Similarly the RB method for the Stokes [35] and Navier-Stokes [34] problems requires the fulfillment of discrete inf-sup condition for reduced velocity and pressure spaces, respectively. In this paper we are not considering the convection dominated case, but we only focus on the inf-sup stability at reduced order level. Previous works based on supremizer enrichment to cure the reduced inf-sup condition are given by Rovas [41], Rozza et al. [42, 44]. Supremizer enrichment approach consists in the introduction of the inner pressure supremizer for the velocity-pressure stability of the RB spaces. Several works on RB method for Stokes and Navier-Stokes problems using the pressure stabilization via the inner pressure supremizer operator are given by [37, 43, 13, 14, 36, 47, 3, 2, 45].

In our recent work on steady Stokes and Navier-Stokes problems [1, 25], we proposed to use the classical residual based stabilization methods (such as SUPG, GALS and Douglas-Wang, mentioned above) to deal with the inf-sup stability. This work is the continuation of proposed method to unsteady problems in parametric reduced order setting. We study the *offline-online stabilization* [38] method, based on performing the Galerkin projections in both *offline* and *online* stage with respect to the consistent stabilized formulations, and the *offline-only stabilization*, consisting in using the stabilized formulations only during the *offline* stage and then projecting with respect to the standard formulation during the *online* stage. We also guarantee the *online* computational savings by reducing the dimension of the *online* RB system, i.e, we show that with this approach it is possible to get the stable RB solution without the supremizer enrichment into velocity space.

This work has two parts: unsteady Stokes problem and unsteady Navier-Stokes problem. Further organization of this paper is as it follows: In section 2 after recalling the unsteady Stokes problem, we present stabilized FE formulation and then its stabilized reduced basis (RB) formulation. Then we present some numerical results for unsteady Stokes problem in section 3 showing the error comparison between different stabilization and supremizer options.

In section 4 we follow a similar pattern for unsteady Navier-Stokes problem. We first define the full order FE formulation, followed by stabilized FE formulation, and then, we project onto RB space. Finally, we show some numerical results and discussions in section 5. The outcome of this work is summarized in section 6.

2 Unsteady parametrized Stokes problem

Let $\Omega \subset \mathbb{R}^2$, be a reference configuration, and we assume that current configuration $\Omega_o(\boldsymbol{\mu})$ can be obtained as the image of map $\boldsymbol{T}(.;\boldsymbol{\mu}) : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $\Omega_o(\boldsymbol{\mu}) = \boldsymbol{T}(\Omega;\boldsymbol{\mu})$. The unsteady parametrized Stokes problem in current configuration reads as follows: find $\boldsymbol{u}_o(t;\boldsymbol{\mu}) \in \boldsymbol{V}$ and $p_o(t;\boldsymbol{\mu}) \in Q$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{u}_o - \nu\Delta\boldsymbol{u}_o + \nabla p_o = \boldsymbol{0} & \text{in } \Omega_o(\boldsymbol{\mu}) \times (0,T)\,, \\ \text{div } \boldsymbol{u}_o = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \times (0,T)\,, \\ \boldsymbol{u}_o = \boldsymbol{g}_D & \text{on } \Gamma_{D,o}(\boldsymbol{\mu}) \times (0,T)\,, \\ \boldsymbol{u}_o|_{t=0} = \boldsymbol{u_0} & \text{on } \Gamma_{W,o}(\boldsymbol{\mu}), \end{cases} \tag{1}$$

where $(0,T)$ with $T > 0$ is the time interval of interest, $\boldsymbol{u_0} \in L^2(\Omega)$ and ν is the viscosity of fluid. The boundary $\partial\Omega_o(\boldsymbol{\mu})$ is divided into two parts in such a way that $\partial\Omega_o(\boldsymbol{\mu}) = \Gamma_{D,o}(\boldsymbol{\mu}) \cup \Gamma_{W,o}(\boldsymbol{\mu})$, where $\Gamma_{D,o}(\boldsymbol{\mu})$ is the Dirichlet boundary with non-homogeneous data and $\Gamma_{W,o}(\boldsymbol{\mu})$ denotes the Dirichlet boundary with zero data.

We multiply (1) by velocity and pressure test functions $\boldsymbol{v}$ and q, respectively then integrating by parts, and tracing everything back onto the reference domain Ω, we obtain the following parametrized formulation of problem (1):

for a given $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}(t;\boldsymbol{\mu}) \in \boldsymbol{V}$ and $p(t;\boldsymbol{\mu}) \in Q$ such that

$$\begin{cases} m(\frac{\partial}{\partial t}\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + b(\boldsymbol{v}, p; \boldsymbol{\mu}) = F(\boldsymbol{v}; \boldsymbol{\mu}) & \forall v \in \boldsymbol{V}, t > 0, \\ b(\boldsymbol{u}, q; \boldsymbol{\mu}) = G(q; \boldsymbol{\mu}) & \forall q \in Q, t > 0, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u_0}. \end{cases} \tag{2}$$

We define the spaces $\boldsymbol{V} = L^2(\mathbb{R}^+; [H^1(\Omega)]^2) \cap C^0(\mathbb{R}^+; [L^2(\Omega)]^2)$ for velocity and $Q = L^2(\mathbb{R}^+; L^2_0(\Omega))$ for pressure on reference domain. Here, $H^1(\Omega)$ and $L^2(\Omega)$ are equipped with H^1-seminorm and L^2-norm respectively. Bilinear forms in (2) are

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) &= \int_\Omega \frac{\partial \boldsymbol{u}}{\partial x_i} \kappa_{ij}(x;\boldsymbol{\mu}) \frac{\partial \boldsymbol{v}}{\partial x_j} d\boldsymbol{x}, \qquad b(\boldsymbol{v}, q; \boldsymbol{\mu}) = -\int_\Omega q \chi_{ij}(x;\boldsymbol{\mu}) \frac{\partial v_j}{\partial x_i} d\boldsymbol{x}, \\ m(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) &= \int_\Omega \pi(\boldsymbol{x};\boldsymbol{\mu}) \boldsymbol{u}_i \boldsymbol{v}_i d\boldsymbol{x}. \end{aligned} \tag{3}$$

We define the terms F and G in (2) as:

$$\begin{aligned} F(\boldsymbol{v}; \boldsymbol{\mu}) &= -a(\boldsymbol{l}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}), \\ G(q; \boldsymbol{\mu}) &= -b(\boldsymbol{l}(\boldsymbol{\mu}), q; \boldsymbol{\mu}), \end{aligned} \tag{4}$$

where we denote by $\boldsymbol{l}(\boldsymbol{\mu})$ a parametrized lifting function such that $\boldsymbol{l}(\boldsymbol{\mu})|_{\Gamma_{D_g}} = \boldsymbol{g}_D(\boldsymbol{\mu})$.

The tensors $\boldsymbol{\kappa}$, $\boldsymbol{\chi}$ and scalar π encoding both physical and geometrical parametrization are defined as follows

$$\begin{aligned} \boldsymbol{\kappa}(x;\boldsymbol{\mu}) &= \nu (J_T(x;\boldsymbol{\mu}))^{-1} (J_T(x;\boldsymbol{\mu}))^{-T} |J_T(X;\boldsymbol{\mu})|, \\ \boldsymbol{\chi}(x;\boldsymbol{\mu}) &= (J_T(x;\boldsymbol{\mu}))^{-1} |J_T(X;\boldsymbol{\mu})|, \\ \pi(\boldsymbol{x};\boldsymbol{\mu}) &= |J_T(X;\boldsymbol{\mu})|, \end{aligned} \tag{5}$$

where $J_T \in \mathbb{R}^{2\times 2}$ is the Jacobian matrix of the map $\boldsymbol{T}(.;\boldsymbol{\mu})$, and $|J_T|$ denotes the determinant.

2.1 Semi-discrete Finite Element formulation

The mixed Galerkin finite element semi-discretization [19, 20] of (2) is defined as follows:

for a given $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_h(t;\boldsymbol{\mu}) \in \boldsymbol{V}_h \subset \boldsymbol{V}$ and $p_h(t;\boldsymbol{\mu}) \in Q_h \subset Q$ such that

$$\begin{cases} m(\frac{\partial}{\partial t}\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + a(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t > 0, \\ b(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall q_h \in Q_h, t > 0, \\ \boldsymbol{u}_h|_{t=0} = \boldsymbol{u}_{0,h}. \end{cases} \tag{6}$$

We consider a partition of the interval $[0, T]$ into K sub-intervals of equal length $\Delta t = T/K$ and $t^k = k\Delta t, 0 \le k \le K$. Applying the implicit Euler time discretization we obtain the following time discrete problem:

for a given $\boldsymbol{\mu} \in \mathbb{P}$, and $(\boldsymbol{u}_h^{k-1}(\boldsymbol{\mu}), p_h^{k-1}(\boldsymbol{\mu}))$, find $\boldsymbol{u}_h^k(t;\boldsymbol{\mu}) \in \boldsymbol{V}_h$ and $p_h^k(t;\boldsymbol{\mu}) \in Q_h$ such that

$$\begin{cases} \frac{1}{\Delta t} m(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + a(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h^k; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \\ + \frac{1}{\Delta t} m(\boldsymbol{u}_h^{k-1}, \boldsymbol{v}_h; t^{k-1}; \boldsymbol{\mu}) & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h^k, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall q_h \in Q_h, \\ \boldsymbol{u}_h^0 = \boldsymbol{u}_{0,h}. \end{cases} \tag{7}$$

We provide the algebraic formulation of the semi-discrete problem (6). The resulting ODE system is as follows:

$$\begin{bmatrix} M(\boldsymbol{\mu}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{U}}(t;\boldsymbol{\mu}) \\ \dot{\boldsymbol{P}}(t;\boldsymbol{\mu}) \end{bmatrix} + \begin{bmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t;\boldsymbol{\mu}) \\ \boldsymbol{P}(t;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix} \tag{8}$$

for the vectors $\boldsymbol{U} = (u_h^{(1)}, ..., u_h^{(\mathcal{N}_u)})^T, \boldsymbol{P} = (p_h^{(1)}, ..., p_h^{(\mathcal{N}_p)})^T$, where for $1 \leq i, j \leq \mathcal{N}_u$ and $1 \leq k \leq \mathcal{N}_p$. Let $\{\boldsymbol{\phi}_i^h\}_{i=1}^{\mathcal{N}_u}$ and $\{\psi_j^h\}_{j=1}^{\mathcal{N}_p}$ be basis functions of $\boldsymbol{V}_h$ and Q_h respectively. We define the matrices

$$\begin{aligned} (M(\boldsymbol{\mu}))_{ij} &= m(\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \qquad & (A(\boldsymbol{\mu}))_{ij} &= a(\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \\ (B(\boldsymbol{\mu}))_{ki} &= b(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}), & (\bar{\boldsymbol{f}}(\boldsymbol{\mu}))_i &= F(\boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \\ & & (\bar{\boldsymbol{g}}(\boldsymbol{\mu}))_k &= G(\psi_k^h; \boldsymbol{\mu}), \end{aligned} \tag{9}$$

A key assumption for an efficient ROM evaluation is the capability to decouple the construction stage of the reduced order space (*offline*) from evaluation stage (*online*). We require that the matrices and vectors appearing in (9) can be written as

$$\begin{aligned} M(\boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \Theta_q^a(\boldsymbol{\mu}) M^q, \quad A(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_a} \Theta_q^a(\boldsymbol{\mu}) A^q, \qquad & B(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_b} \Theta_q^b(\boldsymbol{\mu}) B^q, \\ \bar{\boldsymbol{f}}(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_f} \Theta_q^f(\boldsymbol{\mu}) \bar{\boldsymbol{f}}^q, & \bar{\boldsymbol{g}}(\boldsymbol{\mu}) &= \sum_{q=1}^{Q_g} \Theta_q^g(\boldsymbol{\mu}) \bar{\boldsymbol{g}}^q. \end{aligned} \tag{10}$$

After applying the time discretization with implicit Euler scheme, the resulting algebraic formulation of (7) is

$$\begin{aligned} \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} + A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}(t^k;\boldsymbol{\mu}) \end{bmatrix} &= \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix} \\ &+ \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}. \end{aligned} \tag{11}$$

For a stable solution the FE spaces $\boldsymbol{V}_h$ and Q_h have to fulfill the following parametrized inf-sup stability condition (LBB) [39]:

$$\exists \beta_0(\boldsymbol{\mu}) > 0 : \beta_h(\boldsymbol{\mu}) = \inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu})}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{Q_h}} \geq \beta_0(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu} \in \mathbb{P}. \tag{12}$$

This relation holds if, e.g., the Taylor-Hood ($\mathbb{P}_2/\mathbb{P}_1$) FE spaces are chosen. It is important to mention that condition (12) does not hold in case of equal order FE spaces $(\mathbb{P}_k/\mathbb{P}_k), k \geq 1$ and for lowest order element ($\mathbb{P}_1/\mathbb{P}_0$). Therefore, in such situations we need to introduce some additional stabilization terms, as in the following.

2.2 Stabilized Finite Element formulation

Let us modify equation (6) by adding the stabilization terms. We read the modified formulation as follows: for a given $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_h(t;\boldsymbol{\mu}) \in \boldsymbol{V}_h$ and $p_h(t;\boldsymbol{\mu}) \in Q_h$ such that

$$\begin{cases} m(\dfrac{\partial}{\partial t}\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + a(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) + b(\boldsymbol{v}_h, p_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, t > 0, \\ b(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{ut,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall q_h \in Q_h, t > 0, \\ \boldsymbol{u}_h|_{t=0} = \boldsymbol{u}_{0,h}, \end{cases} \tag{13}$$

where $s_h^{ut,q}(.,.;\boldsymbol{\mu})$, $s_h^{u,q}(.,.;\boldsymbol{\mu})$ and $s_h^{p,q}(.,.;\boldsymbol{\mu})$ are the stabilization terms. For a detail discussion on the choice of stabilization terms, we refer to recent work of ours [1] and references therein. In this case we prefer to chose the stabilization technique given by Hughes et al. [28]:

$$s_h^{ut,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial}{\partial t}\boldsymbol{u}_h, \nabla q_h), \tag{14}$$

$$s_h^{u,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_h, \nabla q_h), \tag{15}$$

and

$$s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, \nabla q_h), \tag{16}$$

Therefore, the stabilized algebraic system can be written as:

$$\begin{bmatrix} M(\boldsymbol{\mu}) & \mathbf{0} \\ \tilde{M}(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{U}}(t;\boldsymbol{\mu}) \\ \dot{\boldsymbol{P}}(t;\boldsymbol{\mu}) \end{bmatrix} + \begin{bmatrix} A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ \tilde{B}(\boldsymbol{\mu}) & -S(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t;\boldsymbol{\mu}) \\ \boldsymbol{P}(t;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix}, \tag{17}$$

where $\tilde{M}(\boldsymbol{\mu}), \tilde{B}(\boldsymbol{\mu})$ and $-S(\boldsymbol{\mu})$ contains the stabilization effects [28], and defined as follows:

$$\begin{aligned} &\left(\tilde{M}(\boldsymbol{\mu})\right)_{ki} = s_h^{ut,q}(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}), \qquad \left(\tilde{B}(\boldsymbol{\mu})\right)_{ki} = b(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}) + s_h^{u,q}(\boldsymbol{\phi}_i^h, \psi_k^h; \boldsymbol{\mu}), \\ &(S(\boldsymbol{\mu}))_{ij} = s_h^{p,q}(\psi_j^h, \psi_i^h; \boldsymbol{\mu}), \qquad \text{for} \quad 1 \le i, j \le \mathcal{N}_u, 1 \le k \le \mathcal{N}_p, \end{aligned} \tag{18}$$

After applying the time discretization with implicit Euler scheme, the system (17) becomes

$$\begin{aligned} &\begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} + A(\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ \tilde{B}(\boldsymbol{\mu}) + \dfrac{\tilde{M}(\boldsymbol{\mu})}{\Delta t} & -S(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}(t^k;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix} \\ &\qquad + \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \\ \dfrac{\tilde{M}(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}. \end{aligned} \tag{19}$$

The stabilized formulation requires the FE spaces to fulfill the following modified inf-sup condition [8, 11, 4] after adding some additional stabilization terms:

$$\exists \beta_0(\boldsymbol{\mu}) > 0 : \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu})}{\|\nabla \boldsymbol{v}_h\|} + s_h^{p,q}(q_h, q_h; \boldsymbol{\mu})^{1/2} \ge \beta_0(\boldsymbol{\mu}) \|q_h\|, \forall q_h \in Q_h. \tag{20}$$

2.3 Reduced Basis formulation

In this section we present the RB formulation of the unsteady Stokes problem formulated in section 2.1. Let us define the parameter sample $s_N = \{\boldsymbol{\mu}^1, ..., \boldsymbol{\mu}^N\}$, where $\boldsymbol{\mu}^n \in \mathbb{P}$. The reduced basis approximation is based on an N-dimensional reduced basis spaces $\boldsymbol{V}_N$ and Q_N generated by a sampling procedure which combines spatial snapshots in time and parameter space in an optimal way. In particular, in our case we have used the POD-Greedy algorithm [21] for snapshots selection to generate the reduced spaces. Reduced basis velocity and pressure spaces are

$$\boldsymbol{V}_N = \text{span}\left\{\text{POD}(\boldsymbol{u}_h(t^k;\boldsymbol{\mu}^n)), 1 \le k \le K, 1 \le n \le N_u\right\}, \tag{21}$$

$$Q_N = \text{span}\left\{\text{POD}(p_h(t^k;\boldsymbol{\mu}^n)), 1 \le k \le K, 1 \le n \le N_p\right\}. \tag{22}$$

We introduce the supremizer operator $T^{\boldsymbol{\mu}} : Q_h \to \boldsymbol{V}_h$ defined as follows:

$$(T^{\boldsymbol{\mu}} q_h, \boldsymbol{v}_h)_{\boldsymbol{V}} = b(\boldsymbol{v}_h, q_h; \boldsymbol{\mu}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{23}$$

which is evaluated for $\boldsymbol{\mu} = \boldsymbol{\mu}^n$ and the corresponding pressure snapshot $q_h^k := p_h(t^k; \boldsymbol{\mu}^n)$, $n = 1, \ldots, N$, to obtain N supremizer snapshots. Afterwards, the RB velocity space $\boldsymbol{V}_N$ is enriched with the supremizer snapshots. We denote the enriched RB velocity space by $\tilde{\boldsymbol{V}}_N$, defined as:

$$\tilde{\boldsymbol{V}}_N = \text{span}\left\{\text{POD}(\boldsymbol{u}_h(t^k; \boldsymbol{\mu}^n)), 1 \le n \le N_u; \text{POD}(T^{\boldsymbol{\mu}} q_h(t^k; \boldsymbol{\mu}^n)), 1 \le n \le N_s\right\}, \tag{24}$$

where $N_s \le N_p$ denotes the number of supremizer snapshots. Now the reduced basis formulation corresponding to semi-discrete FE formulation (6) can be written as: for any $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_N(t; \boldsymbol{\mu}) \in \boldsymbol{V}_N$ and $p_N(t; \boldsymbol{\mu}) \in Q_N$ such that

$$\begin{cases} m(\dfrac{\partial}{\partial t}\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) + a(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall\, q_N \in Q_N, \\ \boldsymbol{u}_N|_{t=0} = \boldsymbol{u}_{0,N}. \end{cases} \tag{25}$$

In the online stage, the algebraic formulation of resulting reduced order approximation for any $\boldsymbol{\mu} \in \mathbb{P}$ is given by

$$\begin{aligned} \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} + A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^k; \boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^k; \boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix} \\ + \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^{k-1}; \boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^{k-1}; \boldsymbol{\mu}) \end{bmatrix}, \end{aligned} \tag{26}$$

where the reduced order matrices are defined as:

$$\begin{aligned} M_N(t; \boldsymbol{\mu}) = Z_{u,s}^T M(t; \boldsymbol{\mu}) Z_{u,s}, \quad A_N(\boldsymbol{\mu}) = Z_{u,s}^T A(\boldsymbol{\mu}) Z_{u,s}, \quad B_N(\boldsymbol{\mu}) = Z_p^T B(\boldsymbol{\mu}) Z_{u,s}, \\ \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) = Z_{u,s}^T \bar{\boldsymbol{f}}(\boldsymbol{\mu}), \quad \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) = Z_p^T \bar{\boldsymbol{g}}(\boldsymbol{\mu}), \end{aligned} \tag{27}$$

with $Z_{u,s}$ being the velocity snapshot matrix including the supremizer solutions, Z_p denotes the pressure snapshot matrix. Moreover, thanks to the affine parametric dependence (10), we need to store only the matrices and vectors

$$A_N^q = Z_{u,s}^T A^q Z_{u,s}, \quad B_N^q = Z_p^T B^q Z_{u,s}, \quad \bar{\boldsymbol{f}}_N^q = Z_{u,s}^T \bar{\boldsymbol{f}}^q, \quad \bar{\boldsymbol{g}}_N^q = Z_p^T \bar{\boldsymbol{g}}^q. \tag{28}$$

The store data structures do not depend explicitly on time because the temporal dependence is stored in the multiplicative factors $\Theta(t; \boldsymbol{\mu})$. Therefore, $A^q, B^q, \bar{\boldsymbol{f}}^q, \bar{\boldsymbol{g}}^q$ are independent of both $\boldsymbol{\mu}$ and t.

2.4 Stabilized Reduced Basis formulation

In this section we present the stabilized RB model for unsteady Stokes problem derived from the stabilized FE problem (13). The stabilized RB approximation of velocity and pressure field obtained by means of Galerkin projection on reduced spaces reads:

for any $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_N(t; \boldsymbol{\mu}) \in \boldsymbol{V}_N$ and $p_N(t; \boldsymbol{\mu}) \in Q_N$ such that

$$\begin{cases} m(\dfrac{\partial}{\partial t}\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) + a(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) + b(\boldsymbol{v}_N, p_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_N \in \boldsymbol{V}_N, t > 0, \\ b(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{ut,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall\, q_N \in Q_N, t > 0, \\ \boldsymbol{u}_N|_{t=0} = \boldsymbol{u}_{0,N}, \end{cases} \tag{29}$$

where $s_N^{ut,q}(.,.;\boldsymbol{\mu})$, $s_N^{u,q}(.,.;\boldsymbol{\mu})$ and $s_N^{p,q}(.,.;\boldsymbol{\mu})$ are the reduced order stabilization terms defined as:

$$s_N^{ut,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial}{\partial t}\boldsymbol{u}_N, \nabla q_N), \tag{30}$$

$$s_N^{u,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (-\nu \Delta \boldsymbol{u}_N, \nabla q_N), \tag{31}$$

and

$$s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_N, \nabla q_N), \tag{32}$$

Finally, we write the reduced order stabilized formulation of unsteady FE stabilized Stokes problem (19) in compact form as:

$$\begin{aligned} \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} + A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ \tilde{B}_N(\boldsymbol{\mu}) + \dfrac{\tilde{M}_N(\boldsymbol{\mu})}{\Delta t} & -S_N(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^k;\boldsymbol{\mu}) \end{bmatrix} &= \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix} \\ &+ \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \\ \dfrac{\tilde{M}_N(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}. \end{aligned} \tag{33}$$

where $\tilde{M}_N(\boldsymbol{\mu})$, $\tilde{B}_N(\boldsymbol{\mu})$ and $S_N(\boldsymbol{\mu})$ are RB stabilization matrices defined as:

$$\tilde{M}_N(\boldsymbol{\mu}) = Z_p^T \tilde{M}(\boldsymbol{\mu}) Z_{u,s}, \quad \tilde{B}_N(\boldsymbol{\mu}) = Z_p^T \tilde{B}(\boldsymbol{\mu}) Z_{u,s}, \quad S_N(\boldsymbol{\mu}) = Z_p^T S(\boldsymbol{\mu}) Z_p, \tag{34}$$

We also define the reduced order generalized inf-sup condition

$$\exists \beta_{0,N}(\boldsymbol{\mu}) > 0 : \sup_{\boldsymbol{v}_N \in \boldsymbol{V}_N} \frac{b(\boldsymbol{v}_N, q_N; \boldsymbol{\mu})}{\|\nabla \boldsymbol{v}_N\|} + s_N^{p,q}(q_N, q_N; \boldsymbol{\mu})^{1/2} \geq \beta_{0,N}(\boldsymbol{\mu}) \|q_N\|, \forall q_N \in Q_N, \tag{35}$$

where $s_N^{p,q}(.,.;\boldsymbol{\mu})$ is due to the addition of stabilization terms in RB formulation.

For a detailed discussion about the combination of supremizer and stabilization approaches to fulfill the reduced inf-sup condition (35), we refer to our recent work [1]. Here, we discuss and compare the following options using unstable FE pair $\mathbb{P}_k/\mathbb{P}_k$:

- for *offline-online stabilization* with supremizer we solve the stabilized system (19) in the *offline* stage and stabilized RB system (33) in the *online* stage; and the velocity space in this case is enriched with supremizer solutions, given by (24);

- for *offline-online stabilization* without supremizer we solve the stabilized system (19) in the *offline* stage and stabilized RB system (33) in the *online* stage; but the velocity space in this case is given by (21);

- for *offline-only stabilization* with supremizer we solve the stabilized system (19) in the *offline* stage and non-stabilized RB system (26) in the *online* stage; and the velocity space in this case is enriched with supremizer solutions, given by (24);

3 Numerical results and discussion

In this section, we present several numerical results for stabilized reduced order model for unsteady Stokes problem developed in section 2.

We set the parametrized domain $\Omega_o(\boldsymbol{\mu}) = (0, \mu_2) \times (0, 1)$, where we define $\boldsymbol{\mu} = (\mu_1, \mu_2)$ such that μ_1 is physical parameter (kinematic viscosity of fluid) and μ_2 is geometrical parameter (length of domain). Main goal is to see the effect of geometrical parameter on the velocity and pressure. Parametrized domain is shown in Fig. 1.

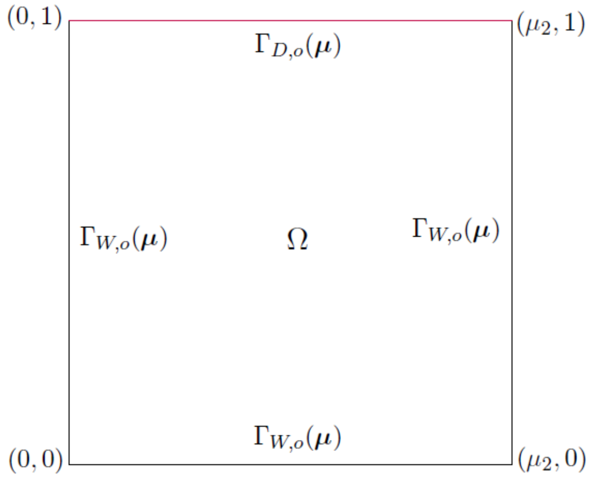

Figure 1: Parametrized domain

We consider a partition of the boundary $\partial\Omega$ into $\Gamma_{D,o}(\boldsymbol{\mu}) \cup \Gamma_{W,o}(\boldsymbol{\mu})$, where we have the homogeneous Dirichlet condition on $\Gamma_{W,o}(\boldsymbol{\mu})$ and non-homogeneous Dirichlet condition on $\Gamma_{D,o}(\boldsymbol{\mu})$.

3.1 Numerical results for $\mathbb{P}_k/\mathbb{P}_k$ (for $k = 1, 2$)

The aim of present subsection is to show and discuss some numerical results for unsteady parametrized Stokes problem (3) using Franca-Hughes stabilization [28].

In Table 1 we show the details of parameter ranges in *offline*, *online* stages; and other information about the *offline* stage.

In Fig. 2 we show the RB solutions for velocity and pressure at different time steps using the *offline-online stabilization* without supremizer. We observe that as the time increases, both velocity and pressure fields are converging to steady state solutions. We have similar results with *offline-online stabilization* with supremizer that we do not show here.

Figure 3 shows the error between FE and RB solutions for velocity (left) and pressure (right), respectively. From these plots we observe that the *offline-online stabilization* with and without supremizer show the same convergence behavior in case of velocity but in case of pressure, supremizer is improving the *offline-online stabilization* up to one order of magnitude. We have similar results for $\mathbb{P}_1/\mathbb{P}_1$ FE pair. This property will be much important in case of coupling conditions in multi-physics involving pressure, for example, since we may guarantee a better accuracy. In contrast, the *offline-only stabilization* with supremizer option has poor performance for both velocity and pressure. From table 1 we see that the computation time of *offline-online stabilization* without supremizer is less than the computation time of *offline-online stabilization* with supremizer in both *offline* and *online stages*.

Number of Parameters	2: μ_1(viscosity), μ_2(domain's length)
μ_1 range *offline*	[0.25,0.75]
μ_2 range *offline*	[1,2]
μ_1 value *online*	0.57
μ_2 value *online*	1.78
Final time	0.2
Time step Δt	0.02
N_{train}	25
N_{max}	25
Stabilization coefficient δ	0.05
FE degrees of freedom	6222 ($\mathbb{P}_1/\mathbb{P}_1$) 18300 ($\mathbb{P}_2/\mathbb{P}_2$)
RB dimension	$N_u = N_s = N_p = 30$
Computation time ($\mathbb{P}_2/\mathbb{P}_1$)	1780s (*offline*), 300s (*online*) with supremizer
Offline time ($\mathbb{P}_1/\mathbb{P}_1$)	1046s (*offline-online stabilization* with supremizer) 738s (*offline-online stabilization* without supremizer) 980s (*offline-only stabilization* with supremizer)
Offline time ($\mathbb{P}_2/\mathbb{P}_2$)	2260s (*offline-online stabilization* with supremizer) 1945s (*offline-online stabilization* without supremizer) 1730s (*offline-only stabilization* with supremizer)
Online time ($\mathbb{P}_1/\mathbb{P}_1$)	103s (*offline-online stabilization* with supremizer) 82s (*offline-online stabilization* without supremizer) 81s (*offline-only stabilization* with supremizer)
Online time ($\mathbb{P}_2/\mathbb{P}_2$)	242s (*offline-online stabilization* with supremizer) 180s (*offline-online stabilization* without supremizer) 90s (*offline-only stabilization* with supremizer)

Table 1: Stokes problem: Computational details of unsteady Stokes problem (3).

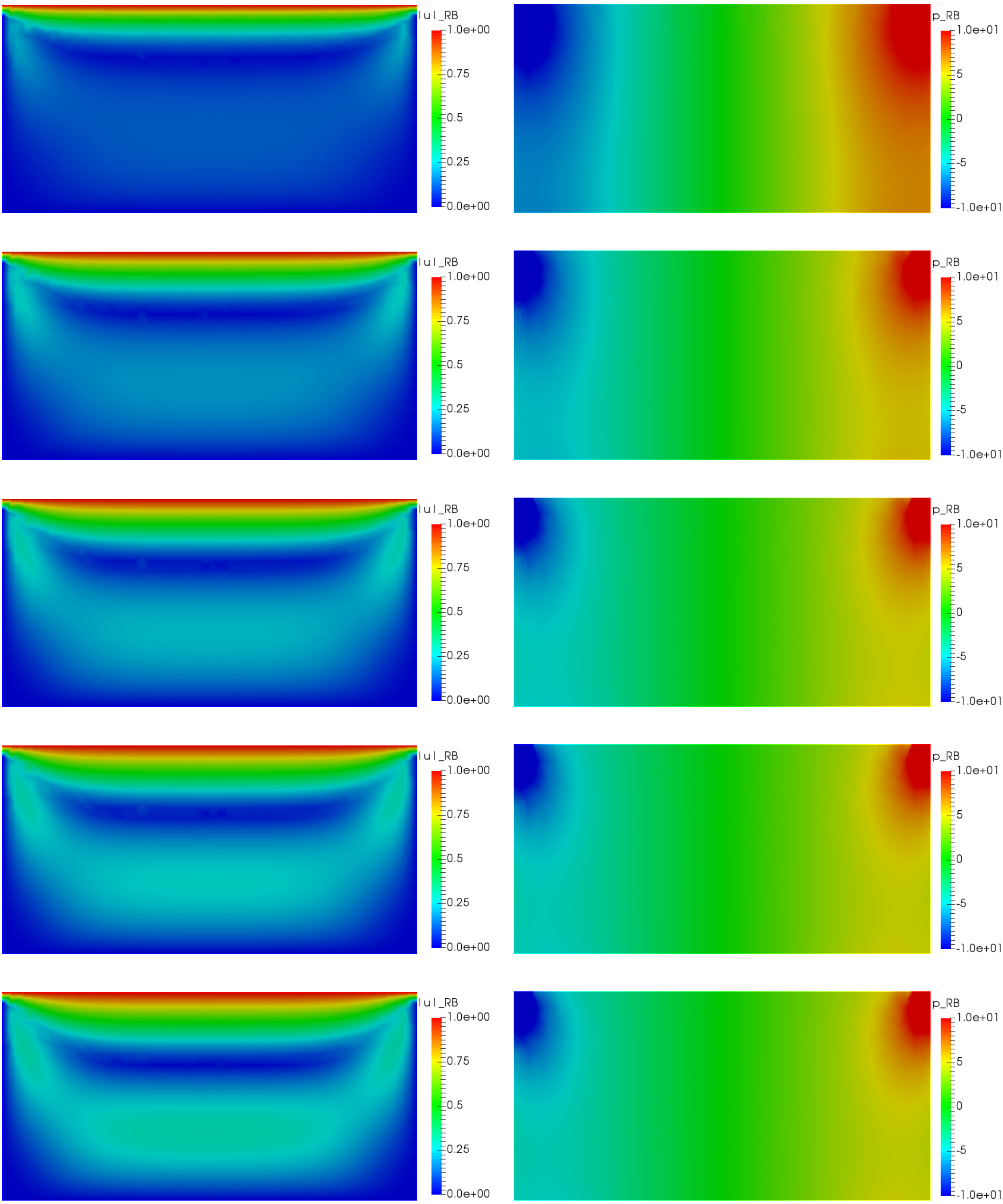

Figure 2: Stokes problem: Franca-Hughes stabilization with $\mathbb{P}_2/\mathbb{P}_2$ FE pair; RB solutions for Velocity field (left) and Pressure field (right) at different time step from top to bottom; $t = 0.02, 0.04, 0.06, 0.1, 0.12,\ N_u = N_p = 30$.

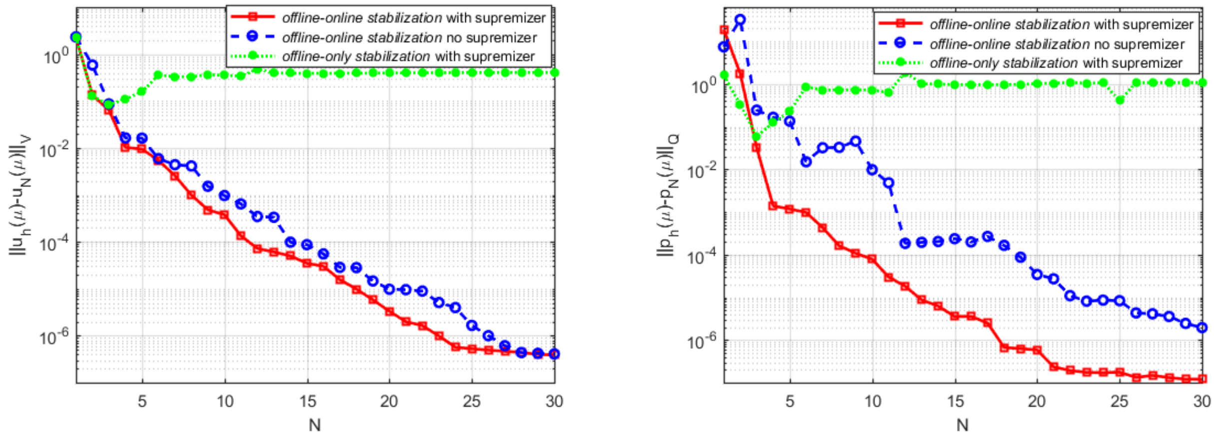

Figure 3: Stokes problem: Franca-Hughes stabilization with $\mathbb{P}_2/\mathbb{P}_2$ on cavity flow; L^2-error in time for velocity (left) and pressure (right) with stabilization coefficient $\delta = 0.05$ and $\Delta t = 0.02$.

3.2 Numerical results for $\mathbb{P}_1/\mathbb{P}_0$

In this subsection we show some results for the error comparison between the different stabilization options using the lowest order FE pair $\mathbb{P}_1/\mathbb{P}_0$. The choice of stabilization term in equation (13) for lowest order element is as follows [39]:

$$s_h^{p,q}(q_h;\boldsymbol{\mu}) := \delta \sum_{\sigma\in\Gamma_h} h_\sigma \int_\sigma [p_h]_\sigma \, [q_h]_\sigma \,, \tag{36}$$

where Γ_h is the set of all edges σ of the triangulation except for those belonging to the boundary $\partial\Omega$, h_σ is the length of σ and $[q_h]_\sigma$ denotes its jump across σ.

The motivation in doing this case is to support the *offline-online stabilization*, i.e, we want to show, by doing different numerical experiments that the *offline-online stabilization* is the best way to stabilize whatever the stabilization we chose. For instance, in subsection 3.1 we chose the Franca-Hughes stabilization, which has different stabilization terms as compared to this subsection.

We plot the L^2- error in time for velocity and pressure, respectively in Fig. 4. These results further strengthen our claim that the *offline-online stabilization* is the best way to stabilize.

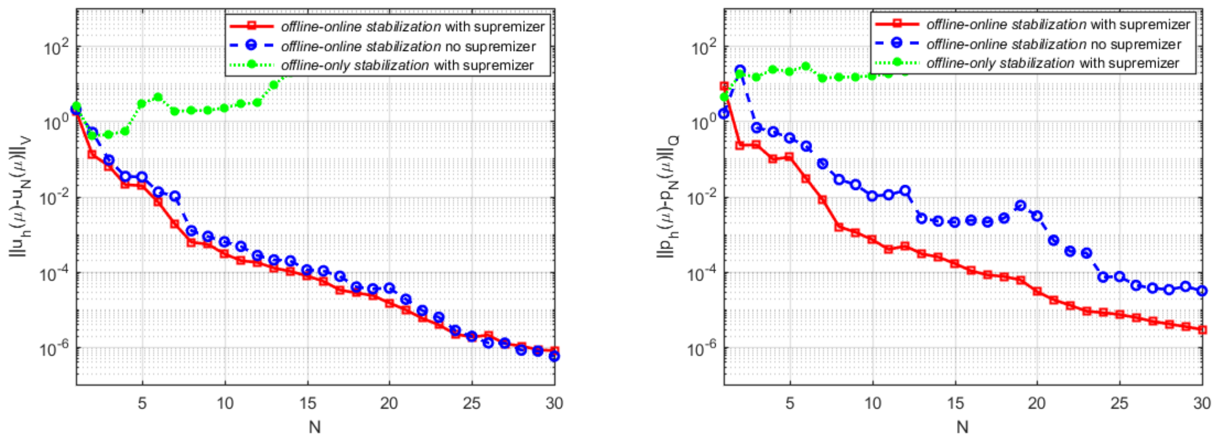

Figure 4: Stokes problem: L^2-error in time for velocity (left) and pressure (right) with stabilization coefficient $\delta = 0.05$ and $\Delta t = 0.02$. using $\mathbb{P}_1/\mathbb{P}_0$.

3.3 Sensitivity on Δt

Consistently stabilized FE methods have complications while working with small time steps. These complications are reported in [5, 7] and references therein. The analysis found in [7] established that

$$\Delta t > \delta h^2,$$

is a sufficient condition to avoid instabilities. Later on a detailed study and series of numerical experiments are performed in [6] and it is established that the fully discrete problem (13) is conditionally stable with the condition

$$\Delta t/\delta h^2 \geq \delta, \tag{37}$$

where Δt is the time step, δ is the stabilization coefficient independent of the spatial grid size h.

In this subsection we present some numerical results to see the variation of Δt on the error between FE and RB solutions. We use the *offline-online stabilization* without supremizer to plot the error between FE and RB solution for velocity (left) and pressure (right) in Fig. 5. We fix the value of stabilization coefficient $\delta = 0.05$

From these error plots, we observe that $\Delta t = 0.02$ (in this case, not generally) is the best value. If we decrease the value of Δt, keeping δ and h fixed, i.e, we are decreasing the left hand side of (37), which increases the error.

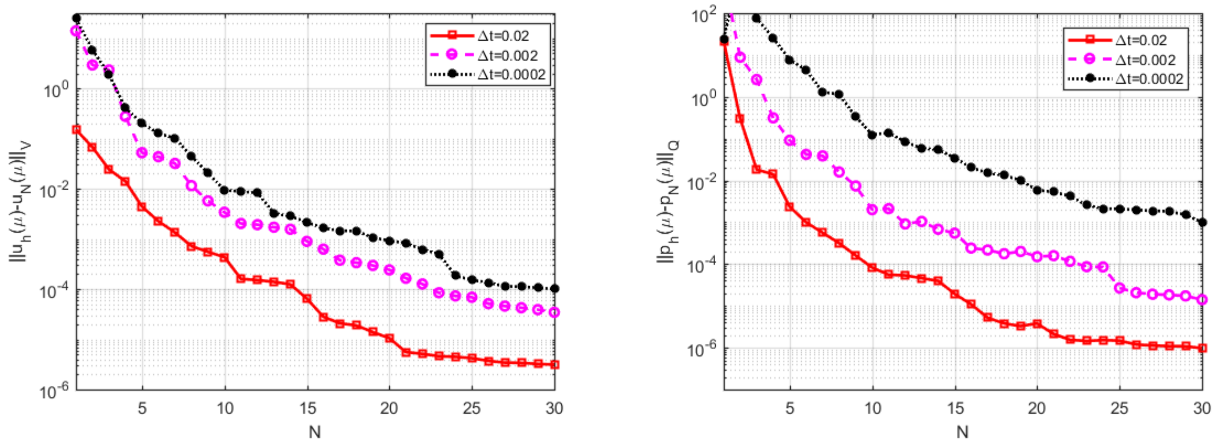

Figure 5: Stokes problem: Franca-Hughes stabilization; L^2-error in time for Velocity (left) and pressure (right) using $\mathbb{P}_2/\mathbb{P}_2$ and $\delta = 0.05$, $\Delta t = 0.02, 0.002, 0.0002$. *offline-online stabilization* without supremizer.

4 Parametrized unsteady Navier-Stokes problem

In this section, we develop a stabilized RB method using SUPG stabilization method for the approximation of unsteady Navier-Stokes problem in reduced order parametric setting. Let $\Omega \subset \mathbb{R}^2$, be a reference configuration, and we assume that current configuration $\Omega_o(\boldsymbol{\mu})$ can be obtained as the image of map $\boldsymbol{T}(.;\boldsymbol{\mu}) : \mathbb{R}^2 \to \mathbb{R}^2$, i.e. $\Omega_o(\boldsymbol{\mu}) = \boldsymbol{T}(\Omega;\boldsymbol{\mu})$. First we define the unsteady Navier-Stokes problem on a domain $\Omega_o(\boldsymbol{\mu})$ in $\mathbb{R}^2$. We consider the fluid flow in a region $\Omega_o(\boldsymbol{\mu})$, bounded by walls and driven by a body force $\boldsymbol{f}(\boldsymbol{\mu})$. The fluid velocity and pressure are the functions $\boldsymbol{u_o}(t;\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathbb{P}, 0 \leq t \leq T$ and $p_o(t;\boldsymbol{\mu})$ for $0 < t \leq T$, respectively which satisfies

$$\begin{cases} \dfrac{\partial \boldsymbol{u_o}}{\partial t} - \nu \Delta \boldsymbol{u}_o + \boldsymbol{u_o} \cdot \nabla \boldsymbol{u_o} + \nabla p_o = \boldsymbol{f}(\boldsymbol{\mu}) & \text{in } \Omega_o(\boldsymbol{\mu}) \times (0,T)\,, \\ \mathrm{div} \boldsymbol{u}_{\mathrm{o}} = 0 & \text{in } \Omega_o(\boldsymbol{\mu}) \times (0,T)\,, \\ \boldsymbol{u}_o = \boldsymbol{g} & \text{on } \partial\Omega \times (0,T)\,, \\ \boldsymbol{u}_o|_{t=0} = \boldsymbol{u_0} & \text{in } \Omega_o(\boldsymbol{\mu}). \end{cases} \tag{38}$$

By multiplying (38) with velocity and pressure test functions $\boldsymbol{v}$ and q, respectively, integrating by parts, and tracing everything back onto the reference domain Ω, we obtain the following parametrized weak formulation of (38):

for a given $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}(t;\boldsymbol{\mu}) \in \boldsymbol{V}$ and $p(t;\boldsymbol{\mu})) \in Q$ such that

$$\begin{cases} m(\dfrac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}; \boldsymbol{\mu}) + a(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}) + b(\boldsymbol{v}, p; \boldsymbol{\mu}) = F(\boldsymbol{v}; \boldsymbol{\mu}) & \forall\, v \in \boldsymbol{V}, t > 0, \\ b(\boldsymbol{u}, q; \boldsymbol{\mu}) = G(q; \boldsymbol{\mu}) & \forall\, q \in Q, t > 0, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u_0}, \end{cases} \tag{39}$$

where the bilinear forms are given in (3) and trilinear form is defined as:

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}; \boldsymbol{\mu}) = \int_\Omega u_i \chi_{ji}(x; \boldsymbol{\mu}) \frac{\partial v_m}{\partial x_j} w_m d\boldsymbol{x}. \tag{40}$$

The tensors $\boldsymbol{\kappa}$, $\boldsymbol{\chi}$ and scalar π are given by (5).

4.1 Discrete Finite Element formulation

As in the previous part for unsteady Stokes problem, let us now discretize problem (39). Consider $\{T_h\}_{h>0}$ be the triangulations and h denotes a discretization parameter [19, 20]. Let $\boldsymbol{V}_h$ and Q_h be two finite dimensional spaces such that $\boldsymbol{V}_h \subset \boldsymbol{H}^1(\Omega)$ and $Q_h \subset L^2_0(\Omega)$. We use implicit Euler scheme for time derivative term. We consider a partition of the interval $[0,T]$ into K sub-intervals of equal length $\Delta t = T/K$ and $t^k = k\Delta t, 0 \leq k \leq K$. We approximate the time derivative in the $(k) - th$ time layer as

$$\frac{\partial \boldsymbol{u}_h(t^k)}{\partial t} \approx \frac{\boldsymbol{u}_h^k - \boldsymbol{u}_h^{k-1}}{\Delta t}, \tag{41}$$

where Δt is a constant time step. We define the semi discrete FE approximation problem of (39) while using (41) in (39) we get as follows:
for a given $\boldsymbol{\mu} \in \mathbb{P}$, and $(\boldsymbol{u}_h^{k-1}(\boldsymbol{\mu}), p_h^{k-1}(\boldsymbol{\mu}))$, find $\boldsymbol{u}_h^k(t;\boldsymbol{\mu}) \in \boldsymbol{V}_h$ and $p_h^k(t;\boldsymbol{\mu}) \in Q_h$ such that

$$\begin{cases} \frac{1}{\Delta t} m(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + a(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + c(\boldsymbol{u}_h^k, \boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) \\ +b(\boldsymbol{v}_h, p_h^k; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) + \frac{1}{\Delta t} m(\boldsymbol{u}_h^{k-1}, \boldsymbol{v}_h; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h^k, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall\, q_h \in Q_h, \\ \boldsymbol{u}_h^0 = \boldsymbol{u}_{0,h}. \end{cases} \tag{42}$$

The algebraic formulation of (42) can be written as:

$$\begin{aligned} \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} + A(\boldsymbol{\mu}) + C(\boldsymbol{u}(t^k;\boldsymbol{\mu});\boldsymbol{\mu}) & B^T(\boldsymbol{\mu}) \\ B(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}(t^k;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}(\boldsymbol{\mu}) \end{bmatrix} \\ + \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}, \end{aligned} \tag{43}$$

where the matrices corresponding to bilinear forms, and the vectors are given in (9). The matrix corresponding to nonlinear form is defined as:

$$(C(\boldsymbol{u}(t;\boldsymbol{\mu});\boldsymbol{\mu}))_{ij} = \sum_{m=1}^{\mathcal{N}_u} \boldsymbol{u}_h^m(t;\boldsymbol{\mu}) c(\boldsymbol{\phi}_m^h, \boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h; \boldsymbol{\mu}), \tag{44}$$

where $\boldsymbol{\phi}_i^h$ and $\boldsymbol{\psi}_j^h$, are the basis functions of $\boldsymbol{V}_h$ and Q_h respectively. As in previous case, we impose the affine parametric dependence on these matrices and vectors and we skip the detail here.

4.2 Stabilized Finite Element formulation

In this section we give the stabilized formulation of time-dependent Navier-Stokes equations defined in previous section. We use the SUPG stabilization method [10] first in full order, and then, we project on reduced spaces to fulfill the reduced inf-sup condition.

The stabilized FE formulation of (39) read as: for a given $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}(t;\boldsymbol{\mu}) \in \boldsymbol{V}$ and $p(t;\boldsymbol{\mu})) \in Q$ such that

$$\begin{cases} \frac{1}{\Delta t} m(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + a(\boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) + c(\boldsymbol{u}_h^k, \boldsymbol{u}_h^k, \boldsymbol{v}_h; \boldsymbol{\mu}) & \\ + b(\boldsymbol{v}_h, p_h^k; \boldsymbol{\mu}) - s_h^{ut,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) - s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) = F(\boldsymbol{v}_h; \boldsymbol{\mu}) + \frac{1}{\Delta t} m(\boldsymbol{u}_h^{k-1}, \boldsymbol{v}_h; \boldsymbol{\mu}) & \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h^k, q_h; \boldsymbol{\mu}) - s_h^{ut,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) - s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) = G(q_h; \boldsymbol{\mu}) & \forall\, q_h \in Q_h, \\ \boldsymbol{u}_h^0 = \boldsymbol{u}_{0,h}. & \end{cases} \tag{45}$$

where $s_h^{ut,v}(.,.;\boldsymbol{\mu})$, $s_h^{p,v}(.,.;\boldsymbol{\mu})$, $s_h^{ut,q}(.,.;\boldsymbol{\mu})$ and $s_h^{p,q}(.,.;\boldsymbol{\mu})$ are the stabilization terms [39] defined as:

$$s_h^{ut,v}(\boldsymbol{u}_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u}_h + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h, \boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h), \tag{46}$$

$$s_h^{p,v}(p_h, \boldsymbol{v}_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, \boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h), \tag{47}$$

$$s_h^{ut,q}(\boldsymbol{u}_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u}_h + \boldsymbol{u}_h \cdot \nabla \boldsymbol{u}_h, \nabla q_h), \tag{48}$$

$$s_h^{p,q}(p_h, q_h; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_h, \nabla q_h), \tag{49}$$

where δ is the stabilization coefficient. The stabilized algebraic formulation of (45) reads as:

$$\begin{aligned} &\begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} + A(\boldsymbol{\mu}) + \tilde{C}(\boldsymbol{u}(t^k;\boldsymbol{\mu});\boldsymbol{\mu}) & \tilde{B}^T(\boldsymbol{\mu}) \\ \tilde{B}(\boldsymbol{\mu}) + \dfrac{\tilde{M}(\boldsymbol{\mu})}{\Delta t} & -S(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}(t^k;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \tilde{\bar{\boldsymbol{f}}}(\boldsymbol{\mu}) \\ \tilde{\bar{\boldsymbol{g}}}(\boldsymbol{\mu}) \end{bmatrix} \\ &\qquad + \begin{bmatrix} \dfrac{M(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \\ \dfrac{\tilde{M}(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}, \end{aligned} \tag{50}$$

where $\tilde{B}, \tilde{B}^T, \tilde{M}$ and $\tilde{C}$, are the sum of original matrices in formulation (43) and the SUPG stabilization matrices. Similarly $\tilde{\bar{\boldsymbol{f}}}$ and $\tilde{\bar{\boldsymbol{g}}}$ are vectors on right hand side which are sum of original vectors in formulation (43) and SUPG stabilization terms [16]. These matrices and vectors can be written similar to Stokes case (18).

4.3 Reduced Basis formulation

A reduced order approximation of velocity and pressure field is obtained by means of Galerkin projection on the RB spaces $\boldsymbol{V}_N, Q_N$ and $\tilde{\boldsymbol{V}}_N$, defined in (21), (22) and (24), respectively.

In the *online* stage, the resulting reduced order approximation of (42) is as follows: for any parameter $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_N^k(t;\boldsymbol{\mu}) \in \boldsymbol{V}_N$ and $p_N^k(t;\boldsymbol{\mu}) \in Q_N$ such that

$$\begin{cases} \frac{1}{\Delta t} m(\boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) + a(\boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) + c(\boldsymbol{u}_N^k, \boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) & \\ +b(\boldsymbol{v}_N, p_N^k; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) + \frac{1}{\Delta t} m(\boldsymbol{u}_N^{k-1}, \boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N^k, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N, \\ \boldsymbol{u}_N^0 = \boldsymbol{u}_{0,N}. & \end{cases} \tag{51}$$

The algebraic formulation of (51) can be written as

$$\begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} + A_N(\boldsymbol{\mu}) + C_N(\boldsymbol{u}_N(t^k;\boldsymbol{\mu});\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^k;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) \end{bmatrix} + \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}, \tag{52}$$

where, the reduced order matrices are defined as:

$$\begin{gathered} M_N(\boldsymbol{\mu}) = Z_{u,s}^T M(\boldsymbol{\mu})_{u,s}, \quad A_N(\boldsymbol{\mu}) = Z_{u,s}^T A(\boldsymbol{\mu}) Z_{u,s}, \quad B_N(\boldsymbol{\mu}) = Z_p^T B(\boldsymbol{\mu}) Z_u, \\ C_N(.;\boldsymbol{\mu}) = Z_u^T C(.;\boldsymbol{\mu}) Z_u, \quad \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) = Z_u^T \bar{\boldsymbol{f}}(\boldsymbol{\mu}), \quad \bar{\boldsymbol{g}}_N(\boldsymbol{\mu}) = Z_p^T \bar{\boldsymbol{g}}(\boldsymbol{\mu}). \end{gathered} \tag{53}$$

4.4 Stabilized Reduced Basis formulation

We write the stabilized formulation of (51) as follows: for any parameter $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_N^k(t;\boldsymbol{\mu}) \in \boldsymbol{V}_N$ and $p_N^k(t;\boldsymbol{\mu}) \in Q_N$ such that

$$\begin{cases} \frac{1}{\Delta t} m(\boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) + a(\boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) + c(\boldsymbol{u}_N^k, \boldsymbol{u}_N^k, \boldsymbol{v}_N; \boldsymbol{\mu}) & \\ +b(\boldsymbol{v}_N, p_N^k; \boldsymbol{\mu}) - s_N^{ut,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) - s_N^{p,v}(p_N, \boldsymbol{v}_N; \boldsymbol{\mu}) = F(\boldsymbol{v}_N; \boldsymbol{\mu}) + \frac{1}{\Delta t} m(\boldsymbol{u}_N^{k-1}, \boldsymbol{v}_N; \boldsymbol{\mu}) & \forall \boldsymbol{v}_N \in \boldsymbol{V}_N, \\ b(\boldsymbol{u}_N^k, q_N; \boldsymbol{\mu}) - s_N^{ut,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) - s_N^{p,q}(p_N, q_N; \boldsymbol{\mu}) = G(q_N; \boldsymbol{\mu}) & \forall q_N \in Q_N, \\ \boldsymbol{u}_N^0 = \boldsymbol{u}_{0,N}. & \end{cases} \tag{54}$$

where $s_N^{ut,v}(.,.;\boldsymbol{\mu})$, $s_N^{p,v}(.,.;\boldsymbol{\mu})$, $s_N^{ut,q}(.,.;\boldsymbol{\mu})$ and $s_N^{p,q}(.,.;\boldsymbol{\mu})$ are reduced order stabilization terms defined as:

$$s_N^{ut,v}(\boldsymbol{u}_N, \boldsymbol{v}_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u}_N + \boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N, \boldsymbol{u}_N \cdot \nabla \boldsymbol{v}_N), \tag{55}$$

$$s_N^{p,v}(p_N, \boldsymbol{v}_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_N, \boldsymbol{u}_N \cdot \nabla \boldsymbol{v}_N), \tag{56}$$

$$s_N^{ut,q}(\boldsymbol{u}_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u}_N + \boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N, \nabla q_N), \tag{57}$$

$$s_h^{p,q}(p_N, q_N; \boldsymbol{\mu}) := \delta \sum_K h_K^2 \int_K (\nabla p_N, \nabla q_N), \tag{58}$$

The algebraic formulation of (54) can be written as:

$$\begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} + A_N(\boldsymbol{\mu}) + \tilde{C}_N(\boldsymbol{u}_N(t^k;\boldsymbol{\mu});\boldsymbol{\mu}) & \tilde{B}_N^T(\boldsymbol{\mu}) \\ \tilde{B}_N(\boldsymbol{\mu}) + \dfrac{\tilde{M}_N(\boldsymbol{\mu})}{\Delta t} & -S_N(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^k;\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^k;\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_N(\boldsymbol{\mu}) \\ \bar{\tilde{\boldsymbol{g}}}_N(\boldsymbol{\mu}) \end{bmatrix} + \begin{bmatrix} \dfrac{M_N(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \\ \dfrac{\tilde{M}_N(\boldsymbol{\mu})}{\Delta t} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_N(t^{k-1};\boldsymbol{\mu}) \\ \boldsymbol{P}_N(t^{k-1};\boldsymbol{\mu}) \end{bmatrix}, \tag{59}$$

where $\tilde{B}_N, \tilde{B}_N^T, \tilde{M}_N$ and $\tilde{C}_N$ are RB stabilized matrices, and can be obtained similarly as (53).

5 Numerical results and discussion

In this section we apply the stabilized RB model for unsteady Navier-Stokes problem presented in section 4 and subsections therein to *lid-driven cavity* flow problem on parametrized domain shown in Fig. 1. We first show some numerical results for only physical parameterization in subsection 5.1, and then, we show numerical results for both physical and geometrical parametrization in subsection 5.2. In both cases we compare and discuss the three options; (i) *offline-online stabilization* with supremizer, (ii) *offline-online stabilization* without supremizer, (iii) *offline-only stabilization* with supremizer.

5.1 Results for physical parameter case only

The parameter in this case is only the physical parameter, i.e, the Reynolds number and is denoted by μ. The details of computation is summarized in Table 2.

Physical parameter	μ (Reynolds number)
Range of μ	[100,200]
Online μ (example)	130
FE degrees of freedom	5934 ($\mathbb{P}_1/\mathbb{P}_1$)
RB dimension	$N_u = N_s = N_p = 30$
Offline time ($\mathbb{P}_1/\mathbb{P}_1$)	40612s (*offline-online stabilization* with supremizer) 38781s (*offline-online stabilization* without supremizer)
Online time ($\mathbb{P}_1/\mathbb{P}_1$)	4640s (*offline-online stabilization* with supremizer) 4040s (*offline-online stabilization* without supremizer)
Time step	0.02
Final time	0.5

Table 2: Navier-Stokes problem with physical parameter only: Computational details of unsteady Navier-Stokes problem without Empirical Interpolation.

Figure 6 plots the L^2-error in time for velocity (left) and pressure (right) using $\mathbb{P}_1/\mathbb{P}_1$ FE pair. Similarly results for velocity and pressure using $\mathbb{P}_2/\mathbb{P}_2$ FE pair are shown in Fig. 7. In all numerical results presented in this section, we observe that the *offline-online stabilization* without supremizer has better performance for velocity in terms of error. However, in case of pressure, our results show that supremizer is still improving the error but on the other hand addition of supremizer is computationally expensive. The *offline-only stabilization* is not accurate also in this case.

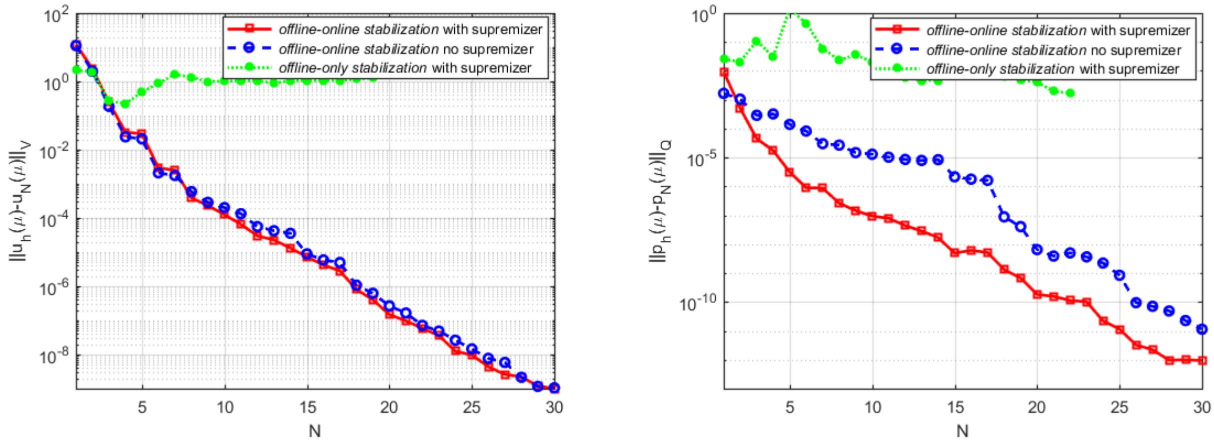

Figure 6: Navier-Stokes problem with SUPG stabilization; physical parametrization on cavity flow; Error between FE and RB solution for velocity (left) and pressure (right) using $\mathbb{P}_1/\mathbb{P}_1$.

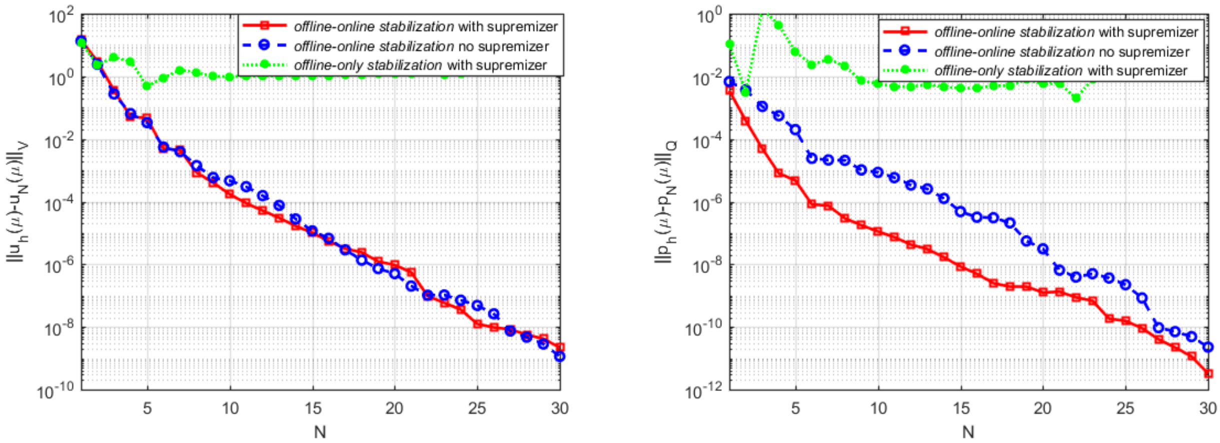

Figure 7: Navier-Stokes problem with SUPG stabilization; physical parametrization on cavity flow; Error between FE and RB solution for velocity (left) and pressure (right) using $\mathbb{P}_2/\mathbb{P}_2$.

5.2 Results for physical and geometrical parameters

In this section we present some numerical results for unsteady Navier Stokes problem with physical and geometrical parameters. The computation details are presented in Table 3. We recall that we are not using any "hyper-reduction" technique to improve online performance at the moment. Our interest at the moment is in a preliminary testing of accuracy and stability.

Figure 8 illustrates the error between FE and RB solution for velocity (left) and pressure (right) using $\mathbb{P}_1/\mathbb{P}_1$ FE pair. We observe that the error between two solutions, obtained by using *offline-online stabilization* with/without supremizer is negligible in case of velocity. However, in case of pressure, supremizer has better performance. We have similar results for $\mathbb{P}_2/\mathbb{P}_2$ FE pair that we do not show here.

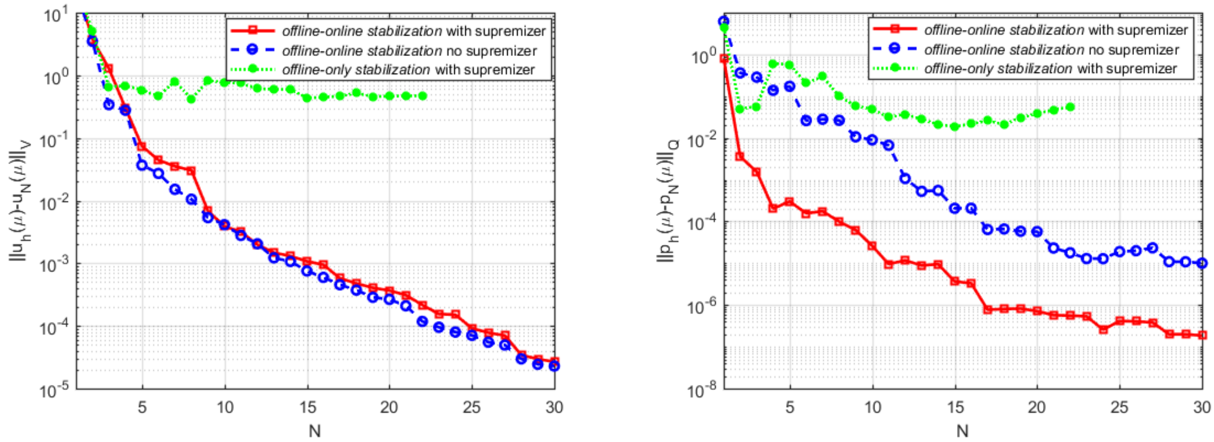

Figure 8: Navier-Stokes problem with SUPG stabilization using $\mathbb{P}_1/\mathbb{P}_1$: Velocity (left) and pressure (right) error for physical and geometrical parameters on cavity flow.

Physical parameter	μ_1 (Reynolds number)
Geometrical parameter	μ_2 (horizontal length of domain)
Range of μ_1	[100,200]
Range of μ_2	[1.5,3]
μ_1 *online* (example)	130
μ_2 *online* (example)	2
FE degrees of freedom	6222 ($\mathbb{P}_1/\mathbb{P}_1$)
RB dimension	$N_u = N_s = N_p = 30$
Offline time ($\mathbb{P}_1/\mathbb{P}_1$)	$44693s$ (*offline-online stabilization* with supremizer) $40153s$ (*offline-online stabilization* without supremizer)
Online time ($\mathbb{P}_1/\mathbb{P}_1$)	$5169s$ (*offline-online stabilization* with supremizer) $4724s$ (*offline-online stabilization* without supremizer)
Time step	0.02
Final time	0.5

Table 3: Computational details for unsteady Navier-Stokes problem with physical and geometrical parameters: stabilization and computational reduction.

6 Concluding remarks

In this work we have developed a stabilized RB method for the approximation of unsteady parametrized Stokes and Navier-Stokes problem. We have extended the analysis carried out in our previous work [1] to the unsteady problems. The RB formulation is built, using the classical residual based stabilization technique in full order during the *offline* stage and, then, projecting on the RB space. We have compared our approach with the existing approaches based on supremizers [44] through numerical experiments. In particular, the comparison between *offline-online stabilization* with/without supremizer and *offline-only stabilization* for unsteady Stokes and Navier-Stokes problems is presented. Our results in this work are consistent with those of the steady Stokes and Navier-Stokes case [1]. On the basis of numerical results the main observations are as it follows:

- *offline-online stabilization* is the most appropriate way to perform RB stabilization (if needed) for unsteady Stokes and Navier-Stokes problems;

- using residual based stabilization, velocity is still better using *offline-online stabilization* (without supremizer) even if pressure is improved in its accuracy by the supremizer enrichment;

- *offline-only stabilization* is not accurate. As in [1], this is due to the lack of consistency between the full and reduced order schemes, which occurs when solving the stabilized system during the *offline* stage and non-stabilized system during the *online* stage;

- in terms of CPU time, the Taylor-Hood FE pair ($\mathbb{P}_2/\mathbb{P}_1$) is more expensive than ($\mathbb{P}_1/\mathbb{P}_1$) stabilized but less expensive than ($\mathbb{P}_2/\mathbb{P}_2$) stabilized (see, for instance Table 1);

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