



Physics Area – PhD course in Theoretical Particle Physics

Bootstrap and non-perturbative methods across dimensions

Candidate:
Alessandro Piazza

Advisor:
Prof. Marco Serone
Prof. Agnese Bissi

Academic Year 2025–2026

*To my family
and my friends*

*I wanted to learn about elementary
particles by studying boiling water
– Alexander Polyakov*

Acknowledgements

First of all, I would like to thank my supervisors Agnese Bissi and Marco Serone. Your way of thinking about physics has shaped me as a researcher, and I am grateful for the freedom you gave me to explore my ideas. Thank you, Marco, for your guidance during these years, for all the discussions, in physics and beyond, and for giving me the opportunity to work on such diverse topics. Thank you, Agnese, for your guidance, your friendliness, and your many useful pieces of advice on a career in physics.

The work I carried out during these years would not have been possible without the bright collaborators I had the chance to work with. Many thanks to Alessandro Vichi, Daniele Pavarini, Denis Karateev, Emilio Trevisani, Gabriele Ferretti, Jiaxin Qiao, Lorenzo Di Pietro, Marco Meineri, Marten Reehorst, Nicola Dondi, Ning Su, Shai Chester, Stefanos Kousvos, and Tomas Reis. I have learned something from each of you. Thank you, Stefanos, for being a really good friend and for all the advice you gave me, both in physics and in life. Thank you, Emilio, for your warmth and cheerfulness. Thank you, Ning, for explaining so many things about the bootstrap to me and for patiently answering all my questions. Thank you, Lorenzo, for teaching me what it is like to live in hyperbolic space.

I acknowledge the use of the computational resources of the Ulysses HPC Cluster at SISSA, the Resnick High Performance Computing Center at Caltech, the Imperial College Research Computing Service and the Scientific IT and Application Support Center of EPFL for the work conducted during my PhD. I am grateful for all the IT and HPC support I received for this purpose.

Thanks to all the SISSA and ICTP professors and postdocs for the lively environment you keep creating. It has been a privilege to be part of this community and to benefit from its many seminars, journal clubs, and informal discussions.

I am grateful to SISSA for allowing me to travel and take part in many interesting conferences and fruitful visits to scientific institutions. I am grateful for the opportunity this gave me to travel across the world and experience different cultures. I am particularly grateful for being able to attend the yearly bootstrap conferences, which have been such a source of inspiration and exchange of ideas. Thanks to Francesco B., Francesco R., Ilija, Petar, and Stefanos for the fun, the gossip, and the drinks we shared at those and other occasions.

Thank you, Amartya, Anant, Gabriel, and Marco: these four years would not have been the same without you. Thank you for your friendship, all the fun we had, and for sharing the ups and the downs of a PhD. We have gone through a lot together, and I count myself lucky to have had you around. As we are sadly all parting ways, I wish you a bright future and I hope we will maintain our ties. Thanks to all the other PhD students at SISSA and ICTP for the fun moments we shared. A special thanks to Alfredo, Beniamino, Davide, Marina, Reza, and Shreansh.

Thank you, Andrea, Bruno, Nicolò, Sara, and Veronica. The years go by and we all move to different places, but I always felt that distance didn't change our enduring friendship. Thank you, Veronica, for your amazing hospitality during my visits to Switzerland. Thank you, Eleonora and Umberto, for keeping our friendship alive throughout these years and for making my trips back home a special moment.

Finally, I am truly grateful to my family. I thank my parents, Fabrizio and Paola, and my sister Beatrice for their constant affection, for always supporting me in my choices and encouraging me throughout these years in Trieste. None of this would have been possible without you. I am grateful to my grandparents Franco, Seve and Vanna for their never-ending kindness and warmth, which I have always carried with me. Grazie nonna Seve per il tuo affetto senza fine: ti ricorderò sempre, con il tuo sorriso solare e le dolci parole.

Abstract

This thesis collects several explorations in formal aspects of quantum and conformal field theories using non-perturbative methods. The common thread is the *bootstrap* philosophy, in which fundamental principles and mathematical consistency, rather than a microscopic model, are used to constrain the space of physical theories. Alongside it, we use *resurgence* as a complementary method that reconstructs non-perturbative information from the perturbative expansion. We apply these ideas to a range of theories across two, three and four dimensions.

In two dimensions, we study the large central charge expansion of conformal field theories. We show that the forbidden singularities of the semi-classical expansion are artefacts of the perturbative series, and that judicious resummation reconstructs the exact correlators. In three dimensions, we apply the numerical conformal bootstrap to the simplest candidate for a deconfined quantum critical point, finding evidence that it is critical and predicting its low-lying spectrum. Bridging three and four dimensions, we consider four-dimensional Yang–Mills theory in a rigid Anti-de Sitter background and bootstrap the conserved currents of its boundary theory to probe proposed mechanisms for the transition to a confining theory in flat space. Finally, in four dimensions, we extend the S-matrix bootstrap to the scattering of two particles of unequal mass, featuring non-analyticities that do not arise for identical particles, and derive bounds on several couplings and on the scalar scattering length.

Declaration

This thesis is based on the following published works

- A. Bissi, N. Dondi, A. Piazza, T. Reis and M. Serone, *On the $1/c$ expansion in 2d CFTs with degenerate operators*, *JHEP* **03** (2025) 092 [[2412.04387](#)] [[1](#)],
- S.M. Chester, A. Piazza, M. Reehorst and N. Su, *Bootstrapping the simplest deconfined quantum critical point*, *Phys. Rev. D* **113** (2026) L081701 [[2507.06283](#)] [[2](#)],
- L. Di Pietro, S.R. Kousvos, M. Meineri, A. Piazza, M. Serone and A. Vichi, *A bootstrap study of confinement in AdS*, *JHEP* **08** (2026) 057 [[2512.00150](#)] [[3](#)],

and the following pre-print

- G. Ferretti, D. Karateev, A. Piazza and M. Serone, *Bounds on scattering amplitudes of non-identical scalar particles in 4d* [[2609.09282](#)] [[4](#)].

During my PhD, I also co-authored the following works which are not included

- S.R. Kousvos, A. Piazza and A. Vichi, *Exploring replica-Potts CFTs in two dimensions*, *JHEP* **11** (2024) 030 [[2405.19416](#)] [[5](#)],
- A. Piazza, M. Serone and E. Trevisani, *The random free field scalar theory*, *JHEP* **08** (2025) 072 [[2409.10608](#)] [[6](#)],
- S.M. Chester, D. Pavarini and A. Piazza, *Bootstrapping holographic theories in the planar limit*, to appear [[7](#)].

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Chapter 1

A journey across dimensions

Quantum Field Theory (QFT) is the universal framework of modern theoretical physics. It underlies the Standard Model, our best understanding of the fundamental particles and their interactions, and its applications range from collider physics to cosmology and down to low-energy condensed-matter systems. Conformal Field Theories (CFTs) play a special role in the space of QFTs: realised at the fixed points of the renormalisation-group flow, they govern the physics at both long and short distances. They arise as descriptions of the critical points of statistical and quantum systems, and as the holographic duals of stringy theories of quantum gravity. In all these guises they are tightly constrained by symmetry and mathematical consistency.

Despite their central role, quantum field theories are notoriously hard to study. When the relevant physics takes place near a weakly coupled fixed point, perturbation theory is the most powerful tool available, and it yields some of the most precise predictions in all of physics. Many phenomena of interest, however, are intrinsically strongly coupled, from confinement in gauge theories to the critical behaviour of continuous phase transitions, and there perturbation theory breaks down. Several non-perturbative approaches exist to bridge this gap, some of which will be considered here.

This thesis collects explorations in formal aspects of quantum field theories using non-perturbative methods. The unifying theme is the *bootstrap*: the use of fundamental principles and mathematical consistency to constrain physical theories, without committing to a microscopic model.

The idea is an old one. The scattering matrix, introduced by Wheeler [8] and Heisenberg, is a general, model-independent object that encodes the outcome of a scattering process. In the 1950s and 1960s it became the object of a programme, associated above all with Chew [9, 10], which sought to determine the S-matrix of the strong interactions of the hadrons from general principles: unitarity, causality, crossing symmetry, and the requirement of maximal analyticity. The ambitious slogan of *nuclear democracy* held that no hadron is more elementary than the others, so that consistency of the general principles alone should fix the spectrum and the interactions of these particles. The programme never reached that goal, but it left us with many important results and formal developments: dispersion relations and the Mandelstam

representation [11], the Froissart–Martin bound on the high-energy growth of amplitudes [12, 13], the proof of crossing from axiomatic field theory [14], Regge theory [15], and the Veneziano amplitude [16], which was soon recognised as the scattering amplitude of a relativistic string and gave birth to string theory [17]. With the advent of quantum chromodynamics (QCD) and asymptotic freedom in the early 1970s, the strong interactions acquired a Lagrangian description and the S-matrix programme was largely set aside.

The bootstrap philosophy did not disappear. In the 1970s, Polyakov [18] and Ferrara, Gatto and Grillo [19] proposed to constrain conformal field theories using conformal symmetry and the associativity of the Operator Product Expansion (OPE). The idea found its first solvable realisation in two dimensions with the seminal work of Belavin, Polyakov and Zamolodchikov [20]. Two-dimensional CFTs were then understood in great detail, see [21] for a comprehensive account. The higher-dimensional problem resisted for decades, until the numerical conformal bootstrap of Rattazzi, Rychkov, Tonni and Vichi in 2008 [22], which opened the way to systematic bootstrap studies of CFTs in any dimension. We refer to [23, 24] for a review of the developments that followed; see also [25] for more recent updates.

The work of 2008 changed how the bootstrap is understood today. The aim is no longer to single out *the* theory that solves the constraints, but to chart the *space of theories*, carving out the regions consistent with conformal symmetry, unitarity and the associativity of the OPE. This perspective is powerful because the approach is agnostic to microscopic realisations of the CFT. Except for instances where specific tools like supersymmetry or integrability are available, it is one of the few flexible and general theoretical tools available for strongly coupled theories. The consistency conditions can be recast as a semi-definite optimisation problem that rules out inconsistent values of the operator dimensions and OPE coefficients [26–28]. When the constraints are strong enough, they confine a theory to a small *island* in the allowed space. The flagship example is the three-dimensional Ising model, whose critical exponents the bootstrap now determines more precisely than any other method [29–31]. In parallel, an *analytic* conformal bootstrap has developed, deriving results in closed form by studying particular limits of the crossing equations. See [32, 33] for an account of the many developments in this direction.

A bridge connects the conformal bootstrap back to its origins in scattering theory: quantum field theory in Anti-de Sitter (AdS) space. A QFT placed in a rigid AdS background has boundary correlation functions with the same structure as those of a conformal theory, a convergent operator product expansion, crossing symmetry and unitarity, so the conformal bootstrap can be applied to them even though the bulk theory is not conformal. This idea has its roots in the AdS/CFT correspondence [34–36], but was later put forward in [37–40] as a generic framework that applies also to ordinary QFTs in a rigid AdS background. The AdS radius is a free parameter: as it is sent to infinity the boundary theory approaches its flat-space limit and the boundary correlators reduce to the flat-space S-matrix. AdS thus interpolates between the two programmes, and turns the rigid machinery of the numerical conformal bootstrap into a tool for bounding ordinary, non-conformal quantum field theories.

At the flat-space end of this bridge, the S-matrix programme itself was revived in the same non-perturbative spirit [41–43]. Rather than seeking the unique amplitude of the strong

interactions, one now carves out the space of scattering amplitudes consistent with analyticity, crossing and unitarity, and derives rigorous bounds on their observables such as couplings and scattering lengths. See [44] for a review. This programme has seen both primal and dual formulations, which are complementary. The primal method, in the original formulation of [42], fills in the allowed region by constructing explicit scattering amplitudes that satisfy these fundamental principles. The dual method, in the more recent formulations of [45–47], rules out amplitudes that are inconsistent with the general principles.

A different kind of non-perturbative method also runs through this thesis. Unlike the bootstrap, which does not refer to a specific theory, the *resurgence* programme attempts to recover non-perturbative information starting from the perturbative expansion. Although its applications in physics are typically tailored to a specific theory, the framework is general and flexible, and its power again rests on principles of mathematical consistency. In this sense it is a non-perturbative method akin to the bootstrap. Perturbative expansions in quantum field theory are generically asymptotic, with zero radius of convergence, as first argued by Dyson [48]. Their coefficients grow factorially and the series must be resummed to define a function. This divergence is not a defect, although it is hard to handle when only a few orders of the perturbative expansion are known. The large-order growth of the perturbative series is governed by the exponentially small effects that no finite order of perturbation theory can see. The theory of resurgence, developed by Écalle [49], turns this relation into a systematic tool: it organises the perturbative series together with its non-perturbative sectors into a single formal series, and reconstructs the exponentially small corrections from the perturbative data through Borel summation and the Stokes phenomenon. The effects it reconstructs vanish to all orders in the coupling, and are therefore invisible order by order in perturbation theory. These techniques have since found wide application across quantum field theory and string theory; see [50–54] for reviews and lecture notes on the subject.

The thesis is built as a journey across dimensions. Each chapter applies one of the methods above to a problem in two, three or four dimensions, and shows how non-perturbative techniques deepen our understanding of theory space and of strongly coupled phenomena. We now summarise their content.

We begin in **Chapter 2** with the background material used throughout. We introduce asymptotic series and Borel summation, which underlie the resurgence techniques of Chapter 3. We then give a light review of conformal field theories in general dimensions and, in particular, of the *numerical conformal bootstrap* applied in Chapters 4 and 5. Finally, we recall the *primal S -matrix bootstrap* for identical particles, one generalisation of which is developed in Chapter 6.

Chapter 3 opens the journey in two dimensions. We demonstrate the use of resurgence in the context of $2d$ CFTs and their large central charge c expansion. Although the resummation methods are typically applied to perturbative expansions in specific theories, here we take a model-independent stance and discuss general properties of families of non-unitary CFTs with parametrically large central charge. On general grounds, the $\text{AdS}_3/\text{CFT}_2$ correspondence maps the $1/c$ expansion of a correlation function to the semi-classical, weak-gravity expansion

around AdS_3 , and it is asymptotic. Choosing simple external operators that keep the calculation under control, we compute this expansion to all orders and study it in great detail. From the divergent perturbative series alone we then reconstruct the exact correlator, non-perturbative part included. In doing so we show that the *forbidden singularities*, which seem to spoil the semiclassical result, are not physical, but artefacts of the asymptotic expansion that disappear once the exponentially small effects are restored. We also show that, although the large- c expansion describes a non-unitary family of theories, its resummation reproduces the exact answer all the way down to small central charge, reaching the simplest unitary theories, the minimal models. Finally, we discuss future directions, in particular extensions to generic unitary CFTs.

In **Chapter 4** we move to three dimensions and to the numerical conformal bootstrap. We study the CP^2 model, a three-dimensional gauge theory that is the simplest candidate for a *deconfined quantum critical point*, a continuous phase transition that falls outside the standard Landau paradigm. Using as input the scaling dimension of the lightest monopole operator, known accurately from the large- N expansion, we bootstrap the theory and find that our bounds are saturated by a candidate spectrum agreeing with both the large- N and the lattice predictions. This gives evidence that the CP^2 model is critical, and lets us predict several of its operator dimensions, including spinning ones that match an independent large-charge description.

Chapter 5 bridges three and four dimensions by considering four-dimensional Yang–Mills theory in a rigid AdS background. This turns a question about confinement into a question about the boundary theory, to which the conformal bootstrap applies. As the radius of AdS increases, the theory approaches the flat-space limit, where Yang–Mills confines. With the relevant boundary conditions, the boundary theory cannot survive to that limit, and must undergo a transition at a finite radius; we ask through which mechanism this happens. Bootstrapping the correlators of the boundary conserved currents, we rule out one of the proposed mechanisms and find evidence favouring another, in which a scalar operator becomes marginal rather than the boundary global symmetry being spontaneously broken.

Chapter 6 closes the journey in four dimensions with the S-matrix bootstrap. We extend the method to the scattering of two particles of different mass, a case motivated by low-energy processes such as pion–kaon and pion–nucleon scattering in QCD. The unequal masses bring a new analytic feature, a *pseudo-physical* region where unitarity gives no constraint, and handling it is the central difficulty. Constructing amplitudes consistent with analyticity, crossing and unitarity, we bound several couplings and the scattering length. These bounds are one-sided in general, but become two-sided when the two masses are equal, and also away from that point once additional assumptions are made.

Despite sharing common techniques and methodologies, the physical problems and settings discussed in this thesis are somewhat disconnected. For this reason, each chapter introduces its setup and the questions we address, and concludes with its own outlook on future directions. Additional technical material is collected in several appendices at the end of this thesis.

Chapter 2

Background material

2.1 Asymptotic series and Borel summation

Perturbative expansions in Quantum Mechanics (QM) and QFT are generically divergent. Following an argument due to Dyson [48], a perturbative series in a coupling g cannot have a finite radius of convergence whenever the point $g = 0$ is non-analytic, as happens for instance when the theory is physically unstable for $g < 0$. The divergence is typically factorial, and reflects the exponentially small non-perturbative effects that perturbation theory does not capture. This section collects the definitions and results on asymptotic series, Borel summation and resurgence used in Chapter 3. We refer to [54] for a pedagogical review and to [50–53], for derivations and further details.

A formal power series in a perturbative parameter ϵ ,

$$\tilde{f}(\epsilon) = \sum_{n=0}^{+\infty} f_n \epsilon^n, \quad (2.1.1)$$

is the *asymptotic series* of a function $f(\epsilon)$, written $f \sim \tilde{f}$, if for every N

$$f(\epsilon) - \sum_{n=0}^{N-1} f_n \epsilon^n = O(\epsilon^N), \quad \epsilon \rightarrow 0. \quad (2.1.2)$$

If the series is convergent, the partial sums converge to $f(\epsilon)$ for any ϵ within its domain of convergence. When the series is asymptotic, the partial sums approximate f for small ϵ , but taking more terms at fixed ϵ does not improve the result indefinitely. Different functions can also share the same asymptotic series, since a term of the form $e^{-S/\epsilon}$ is invisible to (2.1.2) order by order in ϵ and leaves $\tilde{f}$ unchanged. More technically, the series is said to be of *Gevrey-1* type if $|f_n| \leq C S^{-n} n!$ for some constants C, S as $n \rightarrow +\infty$. For a divergent Gevrey-1 series the partial sums first approach f and then diverge, so there is an *optimal truncation* $N_* \simeq |S/\epsilon|$ beyond which the approximation becomes worse. The associated error can be estimated to be of order $e^{-|S/\epsilon|}$, the same size as the non-perturbative effects perturbation theory omits.

Borel summation assigns a function to a divergent series. The standard *Borel transform* of $\tilde{f}$ is $\sum_{n \geq 0} f_n t^n / n!$, which for a Gevrey-1 series is analytic in a disc around the origin of

the *Borel plane*, the division by $n!$ cancelling the factorial growth of the coefficients. More generally, the *Borel–Le Roy transform* divides f_n by $\Gamma(n + 1 + b)$ for a real parameter b , the standard case being $b = 0$. Changing b can alter the nature of the Borel singularities (e.g. a pole into a logarithmic branch cut) but neither their position nor the resummed result, and is often a convenient trick. For the purposes of Chapter 3, we find it convenient to adopt the choice $b = -1$, for which the *Borel function*, denoted by a hat, reads

$$\hat{f}(t) = \sum_{n=1}^{+\infty} \frac{f_n}{(n-1)!} t^{n-1}. \quad (2.1.3)$$

We assume that $\hat{f}$ can be analytically continued over the whole complex plane. If it has no singularities on the positive real axis and grows no faster than an exponential, the Laplace transform

$$s_0(\tilde{f}) = f_0 + \int_0^{+\infty} dt e^{-t/\epsilon} \hat{f}(t), \quad \epsilon > 0, \quad (2.1.4)$$

is well defined and has $\tilde{f}$ as its asymptotic series. More generally, if $\hat{f}$ has no singularities along the ray $e^{i\theta}\mathbb{R}_+$ in the Borel plane it is useful to define the Laplace transform in the direction θ ,

$$s_\theta(\tilde{f}) = f_0 + \int_0^{e^{i\theta}\infty} dt e^{-t/\epsilon} \hat{f}(t), \quad \operatorname{Re}\left(\frac{e^{i\theta}}{\epsilon}\right) > 0. \quad (2.1.5)$$

The *Borel summation* or *resummation* of $\tilde{f}$ is $f^B = s_0(\tilde{f})$ for $\epsilon > 0$ and $f^B = s_\pi(\tilde{f})$ for $\epsilon < 0$. Whether f^B reproduces the original f is a difficult problem: the Nevanlinna–Sokal theorem [55, 56] gives sufficient analyticity conditions for $f^B = f$, singling out a unique representative among the functions sharing the asymptotic series $\tilde{f}$, but in physical contexts these conditions are in practice hard to establish.

When $\hat{f}$ has singularities on the positive real axis the integral in (2.1.5) is ambiguous and the series is not Borel summable. One then deforms the contour above or below the axis, defining the *lateral* resummations $s_{0\pm} = s_{\pm\delta}$ with $0 < \delta \ll 1$. Their difference, called the *Stokes discontinuity*, measures the non-summability and is exponentially small,

$$(s_{0+} - s_{0-})(\tilde{f}) \neq 0, \quad (2.1.6)$$

of order $e^{-S/\epsilon}$, with S the position of the singularity nearest the origin. This ambiguity is a manifestation of the *Stokes phenomenon* and is a convenient way to uncover non-perturbative effects. The same information is contained in the large-order behaviour of the coefficients. If $\hat{f}$ has a singularity at $S = \rho e^{i\phi}$ in the Borel plane, the discontinuity of the Borel sum across the ray at angle ϕ contains terms of the form

$$(s_{\phi+} - s_{\phi-})(\tilde{f}) \approx 2\pi i C_0 \left(\frac{\epsilon}{S}\right)^\beta e^{-S/\epsilon}, \quad |\epsilon| \ll 1, \quad (2.1.7)$$

and correspondingly the coefficients grow as

$$f_n \approx C_0 S^{-n} \Gamma(n - \beta). \quad (2.1.8)$$

The leading large-order behaviour is fixed by the singularity closest to the origin, wherever it sits in the complex plane, so the scale $e^{-S/\epsilon}$ of the effect perturbation theory misses is encoded in the divergence of the series itself.

The discontinuity (2.1.6) can itself often be expressed as the Borel sum of another asymptotic series, which may again be ambiguous, and so on. The mathematical theory of *resurgence*, developed by Écalle [49], provides the systematic framework for such resurgent structures. The divergent series $\tilde{f}_0$ is promoted to a *trans-series*, a sum over perturbative sectors weighted by non-perturbative factors,

$$\tilde{F}(\epsilon) = \sum_{n \geq 0} \sigma_n e^{-S_n/\epsilon} \tilde{f}_n(\epsilon), \quad (2.1.9)$$

where the S_n are the locations of the Borel singularities and the σ_n are *Stokes constants*. In the simplest case a single leading singularity generates the tower $S_n = nS$. Although the individual sectors $\tilde{f}_n$ need not be Borel summable, the ambiguities cancel across the trans-series, so that $\tilde{F}$ reconstructs an unambiguous function. In this way the non-perturbative sectors “resurge” from the perturbative one, and the exponentially small terms are tied to the perturbative coefficients through their large-order behaviour. In the context of QM and QFT, the term with $S_n = 0$ usually plays the role of the standard perturbative series, while the terms with $S_n \neq 0$ play the role of non-perturbative sectors, and S_n can sometimes be identified with the action of non-trivial path-integral saddles.

The Airy function. The Airy function provides a standard illustration of these ideas [54]. The Airy equation $y''(z) = z y(z)$ can be solved by the semi-classical ansatz

$$\tilde{y}_{\pm}(z) = e^{\pm \frac{2}{3} z^{3/2}} \frac{z^{-1/4}}{2\sqrt{\pi}} \tilde{\psi}_{\pm}(z), \quad \tilde{\psi}_{\pm}(z) = \sum_{n \geq 0} c_n^{\pm} z^{-3n/2}, \quad (2.1.10)$$

where the two branches correspond to the two independent solutions. The Borel transforms of the series $\tilde{\psi}_{\pm}$ can be written in terms of hypergeometric functions,

$$\hat{\psi}_{\pm}(t) \propto {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \pm \frac{3t}{4z^{3/2}}\right), \quad (2.1.11)$$

analytic in the cut plane with a branch point at $t_{\pm} = \pm 4z^{3/2}/3$. For real positive z the series $\tilde{\psi}_{+}$ is not Borel summable, and along the rays $\arg z = \pm 2\pi/3$ the same holds for $\tilde{\psi}_{-}$. The complex plane splits into three wedges bounded by these Stokes lines, and the Borel sums $s(\tilde{\psi}_{\pm})$ define the two solutions differently in each wedge.

The transition between wedges is governed by the resurgent relation that the Stokes jump of $\tilde{\psi}_{+}$ is proportional to $\tilde{\psi}_{-}$ and vice versa. This can be obtained, for example, using the connection formula of the hypergeometric functions in (2.1.11). Writing a generic solution in a reference wedge as $c_{+}y_{+} + c_{-}y_{-}$, crossing a Stokes line or the branch cut at the origin acts linearly on the coefficients (c_{+}, c_{-}) through the connection matrices

$$S_{+} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad S_{-} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad (2.1.12)$$

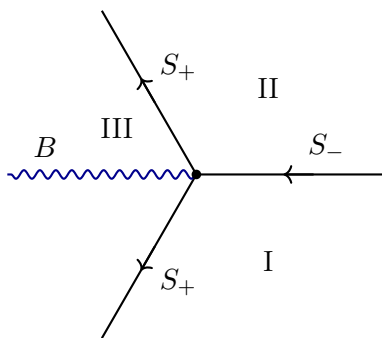


Figure 2.1: The complex z -plane for the Airy function, adapted from [54]. The Stokes lines at $\arg z = 0, \pm 2\pi/3$ divide the plane into three wedges (I, II, III), while the branch cut of the formal series lies along $\arg z = \pi$. Crossing an oriented Stokes line multiplies the coefficients (c_+, c_-) by S_- or S_+ , according to whether its arrow points towards or away from the origin, and crossing the branch cut by B . Encircling the origin returns to the starting wedge, so the ordered product of the jumps is trivial, reproducing (2.1.13).

where $S_{\pm}$ act at the two Stokes lines and B at the branch cut. Since $z = 0$ is a regular point of the Airy differential equation, the total monodromy is trivial (see Figure 2.1),

$$S_- S_+ B S_+ = I, \quad (2.1.13)$$

and this constraint, together with the Stokes jumps, assembles the multivalued sectors into the single-valued solutions.

In terms of the wedge solutions $y_{\pm}$, the standard Airy functions read

$$\text{Ai}(x) = \begin{cases} y_- , & x > 0, \\ y_- + i y_+ , & x < 0, \end{cases} \quad \text{Bi}(x) = \begin{cases} y_+ + i y_- , & x < 0, \\ 2y_+ + i y_- , & x > 0. \end{cases} \quad (2.1.14)$$

The connection matrices (2.1.12) are exactly those that control the ${}_2F_1$ appearing in the $1/c$ expansion of Virasoro blocks studied in Chapter 3, where a similar resurgent analysis is carried out explicitly (Appendix A.2).

2.2 Conformal bootstrap

The conformal bootstrap constrains the space of CFTs by imposing conformal symmetry, unitarity and the associativity of the OPE on correlation functions. Rather than solving a specific model, it asks which sets of observables are consistent with these principles, and in this way carves out the space of consistent theories. This section reviews the ingredients used in Chapters 4 and 5, restricted for simplicity to the bootstrap of four-point functions of identical scalar operators. We follow the reviews [23, 24, 57, 58] and refer to them for the material omitted here.

Conformal field theories describe the fixed points of the renormalisation group, and thus govern the universal behaviour of statistical and quantum systems at criticality and the short-

and long-distance limits of QFTs. A CFT can be characterised through its *CFT data*, the dimensions and OPE coefficients of its operators, defined below. The bootstrap treats these data as the unknowns and the consistency conditions of the theory as non-perturbative *sum rules*, which is what makes it applicable to strongly coupled theories where perturbation theory breaks down.

Conformal symmetry and primaries. A conformal transformation is a coordinate change that preserves angles, that is one that rescales the metric by a local factor $\Omega(x)$. For Euclidean flat space $\mathbb{R}^d$ with $d \geq 2$, such transformations are compositions of translations, rotations, dilatations, and special conformal transformations. The corresponding generators are denoted by P_μ , $M_{\mu\nu}$, D and K_μ , and the resulting conformal group has dimension $(d+1)(d+2)/2$. The dilatation acts as a grading, since

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad (2.2.1)$$

so that P_μ and K_μ raise and lower the eigenvalues of the dilatation operator. In Euclidean signature the conformal algebra is isomorphic to that of $SO(d+1, 1)$ ($SO(d, 2)$ in Lorentzian signature). In $d = 2$, the conformal group admits a local infinite-dimensional extension into the Virasoro algebra.

Local operators fall into irreducible representations of the conformal group, labelled by a *scaling dimension* Δ , the eigenvalue of D , and by a representation of the rotation group $SO(d)$. For simplicity, we will discuss only traceless symmetric tensors of *spin* ℓ . The *twist* is defined as $\tau = \Delta - \ell$. In a unitary theory, the spectrum of D is bounded from below and each representation contains a local operator of lowest dimension, the *primary* $\mathcal{O}_{\Delta, \ell}(x)$, annihilated by the lowering generators,

$$[K_\mu, \mathcal{O}_{\Delta, \ell}(0)] = 0. \quad (2.2.2)$$

The remaining operators of the multiplet, the *descendants*, are obtained by acting with P_μ , and are simply the derivatives of the primary. The transformation of a primary under the full conformal group is fixed by Δ and ℓ . The set of primary dimensions is the *spectrum* of the theory, which in $d \geq 3$ is generically discrete.

A convenient organising principle is radial quantisation, in which one foliates space by spheres surrounding the origin and treats the dilatation as the generator of radial evolution, in place of the usual time translation. A state is prepared by inserting operators inside a sphere, and the *state-operator correspondence* identifies each local operator inserted at the origin with a definite state on the sphere. Up to a Weyl transformation, a CFT radially quantised on $\mathbb{R}^d$ is equivalent to a CFT on a cylinder $\mathbb{R} \times \mathbb{S}^{d-1}$, where the dilatation operator becomes the Hamiltonian of “time” translation along the cylinder and scaling dimensions become the energy levels. This establishes a one-to-one map between states in the Hilbert space of the CFT and local operators (acting on the vacuum). The primary condition (2.2.2) selects the lowest-energy state of each conformal family.

Correlators. Conformal symmetry constrains correlation functions of primaries through their covariance under the conformal group. For two- and three-point functions the constraint is so strong that the functional form is fixed up to constants. The two-point functions of scalar primaries are diagonal and fixed by their dimension,

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{(x_{12}^2)^\Delta}, \quad x_{ij} \equiv x_i - x_j, \quad (2.2.3)$$

where we adopt the standard unit normalisation. The three-point function of scalars is fixed up to a single constant,

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{\lambda_{123}}{(x_{12}^2)^{\Delta_{123}} (x_{13}^2)^{\Delta_{132}} (x_{23}^2)^{\Delta_{231}}}, \quad \Delta_{ijk} \equiv \frac{\Delta_i + \Delta_j - \Delta_k}{2} \quad (2.2.4)$$

and similarly for two scalars and one spinning operator.¹ The coefficients λ_{ijk} , called *OPE coefficients*, together with the spectrum (Δ, ℓ) , form the CFT data.

The dynamical content of the theory is encoded in the *operator product expansion*, which follows from the state-operator correspondence: the product of two nearby operators is equivalent, inside any correlator, to a sum over the operators of the theory,

$$\mathcal{O}_i(x) \times \mathcal{O}_j(0) \sim \sum_k \lambda_{ijk} C_{ijk}(x, \partial) \mathcal{O}_k(0), \quad (2.2.5)$$

where the sum runs over primaries, the differential operators C_{ijk} resum the descendant contributions and are fixed by conformal symmetry, and the λ_{ijk} are the same coefficients that appear in (2.2.4). The expansion converges inside correlation functions [60].

The first correlator not fixed by symmetry alone is the four-point function. For four identical scalars $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$ it takes the form

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{g(u, v)}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}}, \quad u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (2.2.6)$$

where u and v are the two conformally invariant *cross ratios*. Conformal symmetry no longer fixes the correlator but reduces it to a function of two variables $g(u, v)$. It is often convenient to trade u, v for complex variables $z, \bar{z}$ through $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$, which amounts to using conformal transformations to place the four points in a plane. The residual freedom of two real parameters is precisely what is left undetermined by symmetry.

The OPE can be used to reduce higher point functions to infinite sums of two- and three-point functions. It is then clear that the CFT data, together with the existence of a convergent OPE, is enough to fix all correlation functions of a CFT.

Conformal blocks. Applying the OPE (2.2.5) to the pairs (12) and (34) in the four-point function organises $g(u, v)$ as a sum over the primaries exchanged in the $\mathcal{O} \times \mathcal{O}$ channel,

$$g(u, v) = \sum_{\mathcal{O}'} \lambda_{\mathcal{O}\mathcal{O}\mathcal{O}'}^2 g_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}(u, v), \quad (2.2.7)$$

¹For two or more spinning primaries, there are multiple linearly independent tensor structures, see e.g. [59] for a systematic discussion.

where each *conformal block* $g_{\Delta,\ell}$ resums the contribution of a primary of dimension Δ and spin ℓ together with all of its descendants. The blocks depend on the space-time dimension d , the scaling dimension Δ , the spin ℓ and the cross-ratios.² They are entirely fixed by conformal symmetry. Concretely, since the exchanged conformal family is an eigenstate of the quadratic conformal Casimir, the block is the corresponding eigenfunction of the Casimir differential operator acting on the cross-ratios, singled out by the boundary condition that it reproduce the leading OPE behaviour as $x_1 \rightarrow x_2$. Closed-form expressions are available in even dimensions [61–63], while in odd dimensions no such expression is known in general. Nevertheless, the blocks admit efficient and convergent recursion relations [27, 64] that allow for fast numerical evaluation. Convergence is particularly good when the blocks are expressed in terms of a radial coordinate $\rho(z)$ [65] that maps the domain of interest to the unit disk, in which the block expansion converges geometrically. Moreover, the blocks admit a useful expansion as meromorphic functions of the scaling dimension Δ : the block has poles at the locations where a descendant becomes null, and keeping a finite number of them recasts the block, to any desired accuracy, as a rational function of Δ multiplied by a positive prefactor [27, 28]. See [66] for recent algorithmic improvements to this numerical approximation. This rational approximation, and the positivity of the prefactor on the unitary range of dimensions, is what makes the numerical methods described later powerful.

Crossing equation. The associativity of the OPE turns the freedom in $g(u, v)$ into a strong constraint. The four-point function can be computed by taking the OPE in the (12)(34) channel or in the (14)(23) channel, and the two decompositions must agree, see Figure 2.2. Equating them gives the *crossing equation*, which can be written as a sum rule on the CFT data

$$\sum_{\mathcal{O}'} \lambda_{\mathcal{O}\mathcal{O}\mathcal{O}'}^2 F_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}^{\Delta_{\mathcal{O}}}(z, \bar{z}) = 0, \quad F_{\Delta, \ell}^{\Delta_{\mathcal{O}}}(z, \bar{z}) = v^{\Delta_{\mathcal{O}}} g_{\Delta, \ell}(u, v) - u^{\Delta_{\mathcal{O}}} g_{\Delta, \ell}(v, u), \quad (2.2.8)$$

where the sum runs over the operators exchanged in the $\mathcal{O} \times \mathcal{O}$ OPE. This sum rule is not trivially satisfied, i.e. the $F_{\Delta, \ell}^{\Delta_{\mathcal{O}}}$ is not automatically zero. An equivalent way to state this, is that Euclidean correlation functions of identical operators are invariant under permutation of the operators, so that exchanging $x_2 \leftrightarrow x_4$ in (2.2.6)

$$\frac{g(u, v)}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} = \frac{g(v, u)}{(x_{14}^2 x_{23}^2)^{\Delta_{\mathcal{O}}}}. \quad (2.2.9)$$

Plugging the conformal block decomposition (2.2.7) in this equation gives (2.2.8).

The identity operator, present in every theory with $\lambda_{\mathcal{O}\mathcal{O}\mathbb{1}} = 1$, contributes a known inhomogeneous term to (2.2.6). Already the $z \rightarrow 0$ limit of (2.2.8) shows that infinitely many further operators must appear in the OPE in order to reproduce the identity contribution in one OPE channel. A systematic analysis of this property, the *light-cone bootstrap*, shows that the OPE must contain an infinite tower of large-spin operators with twist accumulating at $2\Delta_{\mathcal{O}} + 2n$ [67, 68] with n an integer, and it initiated the analytic conformal bootstrap reviewed

²For a four-point function of identical scalars they are also independent of the dimension of the external primary $\Delta_{\mathcal{O}}$.

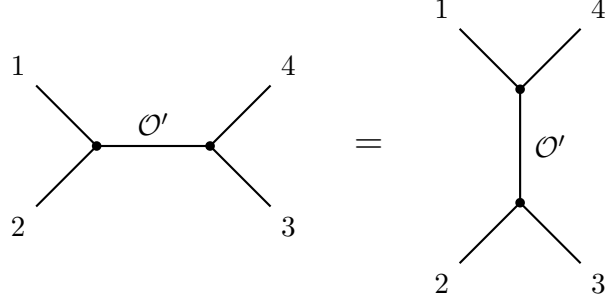


Figure 2.2: The crossing equation for a four-point function of identical scalars: the conformal-block decomposition in the (12)(34) channel equals the one in the (14)(23) channel. Consistency of the two expansions constrains the CFT data.

in [32, 33]. For four identical scalars the third, (13)(24) channel OPE gives no independent constraint. The crossing equation can be imposed at any point where both channel expansions converge. In practice, for the numerical bootstrap, one works near the symmetric configuration $z = \bar{z} = 1/2$, where the convergence of both channels is fastest.

The crossing equation leads to important constraints when combined with *unitarity*. In Euclidean signature unitarity is the property of reflection positivity: the norm of a state, which is a particular correlator of operators and their reflected conjugates, must be non-negative. Positivity of these norms makes the space of states a Hilbert space and has two consequences central to the bootstrap. First, imposing that the norms of the descendants of a primary are non-negative bounds its dimension from below, giving the *unitarity bounds* [69, 70]

$$\Delta \geq \frac{d-2}{2} \quad (\ell = 0), \quad \Delta \geq \ell + d - 2 \quad (\ell \geq 1), \quad (2.2.10)$$

saturated by the free scalar and by conserved currents respectively. Second, the OPE coefficients of real operators are real, so that the coefficients appearing in (2.2.8) are non-negative,

$$\lambda_{\mathcal{O}\mathcal{O}\mathcal{O}'}^2 \geq 0. \quad (2.2.11)$$

The bootstrap problem is thus to determine which spectra of dimensions, subject to (2.2.10), admit non-negative coefficients solving the crossing equation (2.2.8).

Numerical bootstrap. The crossing equation (2.2.8) is an infinite set of constraints, one for each value of the cross-ratios. The constraint involves infinitely many unknowns, which are the spectrum and the OPE coefficients of the theory. The numerical bootstrap makes it tractable by discretising the first one and reducing the unknowns to a finite-dimensional subspace. See [58] for a pedagogical review of the numerical conformal bootstrap. In its simplest form, one Taylor-expands the crossing equation around the symmetric point $z = \bar{z} = 1/2$ and keeps derivatives up to a fixed order Λ , which reduces it to a finite-dimensional vector equation

$$\sum_{\mathcal{O}'} \lambda_{\mathcal{O}\mathcal{O}\mathcal{O}'}^2 \vec{F}_{\Delta_{\mathcal{O}'}, \ell_{\mathcal{O}'}}^{\Delta_{\mathcal{O}}} = 0, \quad (\vec{F}_{\Delta, \ell}^{\Delta_{\mathcal{O}}})^{m, n} = \partial_z^m \partial_{\bar{z}}^n F_{\Delta, \ell}^{\Delta_{\mathcal{O}}}(z, \bar{z}) \Big|_{z=\bar{z}=1/2}, \quad (2.2.12)$$

with $m + n \leq \Lambda$. Because the coefficients are non-negative by (2.2.11), this states that a set of vectors, one per exchanged operator, sums to zero with positive weights.

To decide whether a given spectrum assumption is consistent, one looks for a linear functional $\vec{\alpha}$ acting on (2.2.12). If $\vec{\alpha}$ can be chosen positive on the identity contribution and non-negative on every operator allowed by the assumption,

$$\vec{\alpha} \cdot \vec{F}_{0,0}^{\Delta_{\mathcal{O}}} > 0, \quad \vec{\alpha} \cdot \vec{F}_{\Delta,\ell}^{\Delta_{\mathcal{O}}} \geq 0 \text{ for all allowed } (\Delta, \ell), \quad (2.2.13)$$

then no positive combination of vectors can vanish, and the assumed spectrum is excluded: $\vec{\alpha}$ plays the role of a separating plane. Recall that $(\Delta, \ell) = (0, 0)$ corresponds to the identity operator which always appears in the $\mathcal{O} \times \mathcal{O}$ OPE. Searching for such a functional while scanning the assumptions and the scaling dimension of the external operator $\Delta_{\mathcal{O}}$ maps out the boundary between allowed and excluded spectra. A closely related construction bounds an OPE coefficient: isolating a chosen operator $\mathcal{O}^*$ and normalising $\vec{\alpha} \cdot \vec{F}_{\Delta_{\mathcal{O}^*}, \ell_{\mathcal{O}^*}}^{\Delta_{\mathcal{O}}} = 1$ while keeping positivity on the rest, the crossing equation gives

$$\lambda_{\mathcal{O}\mathcal{O}\mathcal{O}^*}^2 \leq -\vec{\alpha} \cdot \vec{F}_{0,0}^{\Delta_{\mathcal{O}}}, \quad (2.2.14)$$

and minimising the right-hand side yields the optimal upper bound [71, 72]. A lower bound follows analogously when $\mathcal{O}^*$ is isolated in the spectrum.

The remaining difficulty is that the positivity conditions in (2.2.13) must hold for a continuum of dimensions at each spin. Using the rational approximation of the conformal blocks, each condition becomes the statement that a polynomial in Δ is non-negative on a half-line, which can be rewritten as a positive semi-definite matrix condition. In practice one also truncates the exchanged spins at a maximum $\ell_{\max}$ and checks stability of the bound in this parameter. The search for $\vec{\alpha}$ is then a *polynomial matrix program* (PMP), a particular class of *semi-definite program* (SDP) [26–28]. These problems can be solved efficiently using the SDP solver SDPB [73, 74], which was written precisely for the purposes of the numerical conformal bootstrap.

At a point where a bound is saturated the optimal functional becomes unique (at finite Λ and $\ell_{\max}$), and the operators on which $\vec{\alpha} \cdot \vec{F}_{\Delta,\ell}$ vanishes are exactly those that can appear in the crossing equation with non-zero coefficient. This *extremal functional* [72, 75] therefore reads off the spectrum of the theory saturating the bound, and the OPE coefficients follow by solving the now finite linear system. Extremal solutions often approximate known CFTs very well, which is one reason the boundaries of the allowed regions are physically interesting.

The output of these methods is a set of rigorous statements about CFT data. One obtains upper bounds on the dimension of the first operator in a given OPE channel, whose finiteness already proves that such an operator must exist, and two-sided bounds on OPE coefficients, both as functions of the external dimensions. Such bounds frequently display kinks at the location of physical theories, see Figure 2.3, which was how the three-dimensional Ising model was first isolated by these methods [29].

Mapping out the region of CFT data compatible with crossing and unitarity yields *allowed regions* whose boundaries are saturated by special, extremal theories. In favourable cases, under

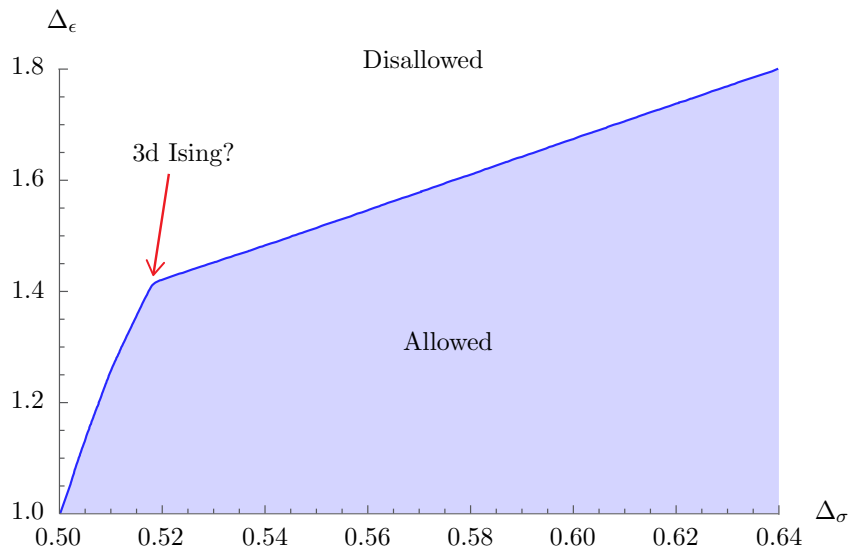


Figure 2.3: Upper bound on the dimension Δ_ϵ of the lowest-dimension scalar in the $\sigma \times \sigma$ OPE, with σ a real scalar primary of a unitary three-dimensional CFT with a $\mathbb{Z}_2$ symmetry, as a function of Δ_σ . The region above the curve is excluded by crossing and unitarity, the shaded region below is allowed. The bound develops a kink close to the values realised by the 3d Ising CFT, $(\Delta_\sigma, \Delta_\epsilon) \approx (0.518, 1.412)$. Figure taken from the lectures [57]. The bound was computed in [76] at derivative order $\Lambda = 24$.

additional assumptions, these regions close into small isolated *islands* that determine a CFT to high precision [30]. The search in high-dimensional space of CFT data can be performed more efficiently using smart algorithms, such as the OPE scan method [77], the *navigator* function [78] (both used in Chapters 4 and 5), and the *skydiving* algorithm [79].

Derivative functionals have historically been the most common tool for carving out the space of theories in the numerical conformal bootstrap, but they are not the only possibility. The strategy as described above can be applied to any linear functional that acts on conformal blocks and satisfies certain positivity properties. In particular, for 1d CFTs, analytic functionals have been constructed in [40, 80–82] to solve analytically certain optimisation problems on CFT data, and later generalised in any dimension [83]. An alternative basis of functionals based on the dispersive sum rules of [84] has been derived in [85]. Analytic functionals have seen recent numerical implementations in [86, 87], while the dispersive functionals have seen applications in supersymmetric theories [7, 88–90].

We have restricted our discussions for simplicity to four-point functions of identical scalar operators. The extension to different scalars was introduced in [28] and requires considering mixed systems of correlators to exploit the positivity of the OPE coefficients. A specific application of mixed correlators bootstrap, with the addition of continuous global symmetries, will be discussed in Section 4.3. The technology for dealing with spinning operators was spelled out in the foundational works [59, 91], see also [92, 93]. Following these, the numerical bootstrap was generalised to fermions in three [94–97] and four dimensions [98], and finally to spin-1

currents [99–103] and the stress tensor [31, 104] in three dimensions. An application of spin-1 non-Abelian currents bootstrap will be the subject of Chapter 5.

2.3 Primal S-matrix bootstrap

The S-matrix bootstrap constrains scattering amplitudes non-perturbatively, relying on the first-principle constraints of analyticity, crossing and unitarity. This section reviews the *primal* formulation for the two-to-two scattering of identical massive scalars, which Chapter 6 generalises to two unequal particles. We follow the lectures [105] and the works [41, 42, 106], and refer to [107] for a modern review of the analyticity properties of scattering amplitudes from a bootstrap perspective.

Scattering amplitudes and analyticity. Consider the two-to-two scattering of identical scalar particles of mass m in $d \geq 3$ dimensions. Writing the S-matrix as $\mathbb{S} = \mathbb{1} + i\mathbb{T}$, the dynamics is encoded in the interacting part $\mathbb{T}$, whose matrix elements on 2-particle free asymptotic states give the scattering amplitude $T(p_1, p_2, p_3, p_4)$, up to an overall momentum-conserving delta-function. Poincaré invariance reduces the connected scattering amplitude to a function $T(p_1, p_2, p_3, p_4) = T(s, t, u)$ of the Mandelstam invariants

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2, \quad s + t + u = 4m^2. \quad (2.3.1)$$

In the centre-of-mass frame s is the squared total energy and the scattering angle is $\cos \theta = 1 + 2t/(s - 4m^2)$. *Crossing symmetry* is the statement that exchanging an incoming and an outgoing particle relates the same function evaluated in different channels. In this setup, this amounts to exchanging the Mandelstam variables and leads to a crossing equation

$$T(s, t, u) = T(t, s, u) = T(u, t, s). \quad (2.3.2)$$

For two-to-two scattering this can be established from the axioms of quantum field theory [14].

The physical amplitude is the boundary value of an analytic function of the two complex invariants $T(s, t) \equiv T(s, t, 4m^2 - s - t)$, approached from above the cut, namely

$$T(s, t) = \lim_{\epsilon \rightarrow 0^+} T(s + i\epsilon, t). \quad (2.3.3)$$

For fixed physical t the amplitude satisfies dispersion relations in s , with at most two subtractions for $|t| < 4m^2$. As a function of the scattering angle at fixed energy it is analytic inside the *Lehmann ellipse* [108] with foci at $\cos \theta = \pm 1$, while its discontinuity is analytic in a larger ellipse. Combining these with unitarity, Martin showed that the amplitude is analytic in the domain $|t| < 4m^2$ together with the cut s -plane [13, 107]. Together with real analyticity,

$$T(s^*, t^*) = T(s, t)^*, \quad (2.3.4)$$

these establish the analytic structure in a neighbourhood of the physical region.

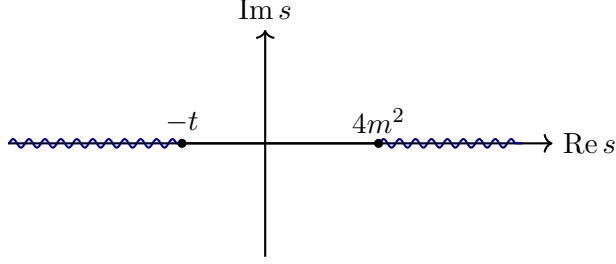


Figure 2.4: Analytic structure of $T(s, t)$ in the complex s -plane at fixed physical $t < 0$ under maximal analyticity: the s -channel cut from $s = 4m^2$ and the u -channel cut ending at $s = -t$. The amplitude is real on the segment between the two cuts.

These properties give the amplitude a dispersive representation. At fixed t , the analyticity in s , together with the polynomial boundedness at large energy encoded in the Froissart bound [12, 109], yields a twice-subtracted fixed- t dispersion relation

$$T(s, t) = c_0(t) + c_1(t)s + \frac{s^2}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \left[\frac{\text{Im } T(s', t)}{s' - s} + \frac{\text{Im } T(s', t)}{s' - u} \right], \quad (2.3.5)$$

where the two terms in the integrand are the s - and u -channel cuts and $u = 4m^2 - s - t$. The amplitude is thus reconstructed from its discontinuity, plus some extra data encoded in the c_0, c_1 functions due to the double subtraction. Assuming in addition maximal analyticity in both variables leads to the Mandelstam representation [11], a double dispersion relation over a double spectral density $\rho(s, t)$, which makes crossing manifest.

The bootstrap program of [42] goes further and assumes *maximal analyticity* (or Mandelstam analyticity): on the physical sheet the only singularities of T are the two-particle unitarity cuts $s, t, u \geq 4m^2$ and, if present, bound-state poles at $0 < s, t, u < 4m^2$. This is stronger than what has been proven, but is consistent with unitarity and with perturbation theory for the scattering of the lightest particle. As shown in Figure 2.4, in the s -plane at fixed physical t it leaves the amplitude holomorphic away from the s - and u -channel cuts and real on the segment between them, so that (2.3.4) follows by the Schwarz reflection principle. The assumption fails for heavier particles, whose amplitudes can develop further singularities.

Partial waves and unitarity. Decomposing the two-particle states into angular-momentum eigenstates diagonalises the S-matrix and gives the partial amplitudes $S_\ell(s)$. For identical scalars Bose symmetry keeps only even spins $\ell = 0, 2, 4, \dots$, and the projection is

$$T_\ell(s) \propto \int_{-1}^1 dz (1 - z^2)^{\frac{d-4}{2}} C_\ell^{(d-3)/2}(z) T(s, t(s, z), u(s, z)), \quad S_\ell(s) = 1 + iT_\ell(s), \quad (2.3.6)$$

with $z = \cos \theta$ and C_ℓ the Gegenbauer polynomials, which reduce to the Legendre polynomials in $d = 4$. The decomposition can be inverted and the full amplitude is recovered by summing the partial amplitudes against $C_\ell(z)$. The forward limit relates the amplitude to the total cross section through the optical theorem, $\text{Im } T(s, 0) \propto \sqrt{s(s - 4m^2)} \sigma_{\text{tot}}(s)$.

Unitarity of the S-matrix, $\mathbb{S}\mathbb{S}^\dagger = \mathbb{1}$, expresses conservation of probability. Restricted to a single partial wave it requires the two-by-two matrix of in and out states to be positive semi-definite,

$$\begin{pmatrix} 1 & S_\ell^*(s) \\ S_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad (2.3.7)$$

which is equivalent to $|S_\ell(s)|^2 \leq 1$, or $2\text{Im} T_\ell(s) \geq |T_\ell(s)|^2$. In the elastic regime the inequality is saturated, $|S_\ell(s)| = 1$, and thus the partial amplitude can be written in terms of a real phase shift $\delta_\ell(s)$ as $S_\ell = e^{2i\delta_\ell}$. Above the first multi-particle threshold, $s \geq 16m^2$, inelastic channels open and dropping their positive contribution leaves the inequality. A weaker but linear constraint, obtained by keeping only the term linear in T_ℓ , is *positivity*,

$$\text{Im} T_\ell(s) \geq 0. \quad (2.3.8)$$

This is the only constraint surviving at weak coupling. When combined with dispersion relations such as (2.3.5), positivity can be used to derive rigorous bounds on various observables, sometimes even analytic [110].

Primal ansatz. The primal strategy exploits the fact that an analytic function bounded on the boundary of its domain is constrained throughout the interior. For the lightest particle the cuts of the amplitude coincide with physical kinematics, where unitarity bounds the partial amplitudes by $|S_\ell| \leq 1$. The maximum modulus principle then turns this into constraints on the amplitude, and hence on the observables, inside the domain of analyticity [42].

The mechanism is cleanest in two dimensions [41], where the amplitude depends on the single variable s and unitarity reads $|S(s)| \leq 1$ directly on the cut. Suppose the theory has a bound state, appearing as a pole of S whose residue measures the square of the coupling. Mapping the cut s -plane to the unit disc through a conformal map $s \rightarrow \rho(s)$, with the pole sent to the origin, S becomes a function bounded by one on the boundary circle with a simple pole at the centre. The maximum modulus principle applied to $\rho S(\rho)$ then bounds the residue, and hence the coupling, by its value for the optimal amplitude $S = 1/\rho$. A non-perturbative bound on a physical coupling is then easily established. The higher-dimensional bootstrap is the numerical implementation of the same idea in the more complicated setting of analytic functions of several variables.

To make this concrete one maps the cut plane to a disc. The ρ -variable

$$\rho(s) = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}} \quad (2.3.9)$$

is a conformal map that sends the s -plane cut along $s \geq 4m^2$ to the unit disc, the two-particle threshold cut to the boundary circle, and the freely chosen reference point $s_0 < 4m^2$ to the origin (Figure 2.5). Any function analytic in the cut plane thus has a convergent Taylor expansion in ρ inside the disc.

For the full amplitude one relaxes the constraint $s + t + u = 4m^2$ and treats s, t, u as independent, mapping each to its own ρ -variable so that all cuts lie outside of a poly-disc. The

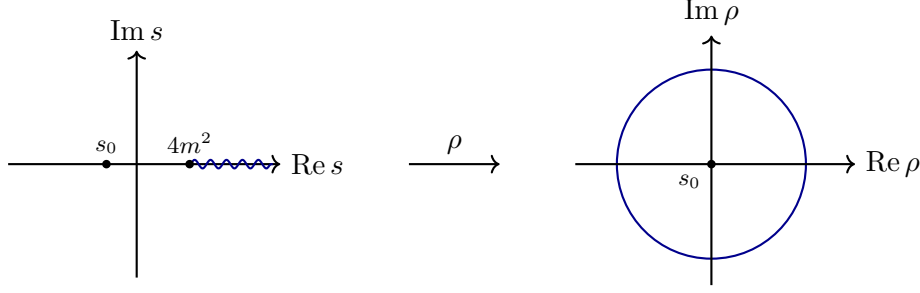


Figure 2.5: The ρ -variable (2.3.9) maps the cut s -plane to the unit disc, sending the two-particle cut to the boundary circle, where unitarity is imposed, and the reference point s_0 to the origin.

amplitude is then written as

$$T(s, t, u) = \sum_{a, b, c \geq 0} \alpha_{a, b, c} \rho(s)^a \rho(t)^b \rho(u)^c, \quad (2.3.10)$$

with real coefficients $\alpha_{a, b, c}$, plus explicit pole terms if bound states are present. Crossing symmetry (2.3.2) is imposed by taking the $\alpha_{a, b, c}$ totally symmetric in (a, b, c) , and the constraint $s + t + u = 4m^2$ can be used to remove residual redundancies. Any amplitude obeying maximal analyticity and crossing can be written in the form (2.3.10), as a consequence of the analytic extension from the constraint surface to the full poly-disc guaranteed by Cartan's theorem B on Stein manifolds [42]. Numerically one truncates the sum at a maximum degree

$$a + b + c \leq N_{\max}, \quad (2.3.11)$$

leaving a finite set of parameters that is systematically enlarged.

Observables and numerical optimisation. Different theories are distinguished by their *observables*. A convenient set is given by the derivatives of the amplitude at the crossing-symmetric point,

$$\lambda_{k, l} \equiv \frac{m^{d-4+2(k+l)}}{k! l!} \partial_s^k \partial_t^l T\left(\frac{4}{3}m^2, \frac{4}{3}m^2, \frac{4}{3}m^2\right), \quad (2.3.12)$$

which are real and dimensionless. These observables are not all independent, because of the crossing symmetry of the amplitude [106]. The lowest, $\lambda_{0,0}$, corresponds at leading order in perturbation theory to the quartic coupling in a Lagrangian description. Other observables, such as the scattering length read off from the threshold expansion of the partial amplitudes, may be used equivalently. In the ansatz (2.3.10) every such observable is a linear function of the coefficients $\alpha_{a, b, c}$.

The primal bootstrap imposes the full non-linear condition (2.3.7) as a numerical optimisation problem. To impose unitarity one evaluates the partial amplitudes (2.3.6) of the ansatz on the physical cut, where they too are linear in the coefficients,

$$S_\ell(s) = 1 + i \vec{\alpha} \cdot \vec{X}_\ell(s), \quad (2.3.13)$$

with $\vec{X}_\ell(s)$ known kinematic functions, computed by performing the partial-wave projection (2.3.6) on each term of the ansatz (2.3.10). The unitarity condition (2.3.7) then becomes a positive semi-definite constraint on a matrix that is linear in $\vec{\alpha}$,

$$\begin{pmatrix} 2 - \vec{\alpha} \cdot \text{Im } \vec{X}_\ell(s) & \vec{\alpha} \cdot \text{Re } \vec{X}_\ell(s) \\ \vec{\alpha} \cdot \text{Re } \vec{X}_\ell(s) & \vec{\alpha} \cdot \text{Im } \vec{X}_\ell(s) \end{pmatrix} \succeq 0, \quad (2.3.14)$$

imposed over a grid of energies $s \geq 4m^2$ and for spins up to a cutoff $\ell \leq L_{\text{max}}$. Maximising or minimising a chosen observable (2.3.12) subject to (2.3.14) is an SDP, linear in the unknowns $\vec{\alpha}$ with semi-definite constraints. The resulting program can be solved efficiently and at high precision using e.g. SDPB [73, 74]. The computation depends on the truncations N_{max} , L_{max} and on the number of sampled energies. These are increased until the result stabilises. As we increase N_{max} , we fill the space of allowed theories from the inside with explicit S-matrices that optimise a given observable. Extrapolating the bounds to $N_{\text{max}} \rightarrow \infty$ carves out the allowed theory space in a subspace of observables.

The construction makes two-particle unitarity, crossing and maximal analyticity manifest. It does not build in multi-particle thresholds or elastic unitarity. The ρ -expansion is not expected to converge point-wise on the boundary once such thresholds are present, and these constraints are recovered, if at all, only in the large- N_{max} limit or by enlarging the ansatz [107].

Chapter 3

$2d$: large central charge and degenerate operators

This chapter applies the theory of resurgence to the large central charge c expansion of two-dimensional conformal field theories, and is organised as follows. We start in Section 3.1 by reviewing the large- c expansion of Virasoro blocks and its holographic interpretation; we state the main questions we would like to address and our main results. In Section 3.2 we recall the four-point correlators of degenerate operators that we study, in the two configurations of four heavy operators (4H) and two heavy plus two light operators (2H2L). We carry out the resurgent analysis of the 4H correlator in Section 3.3 and of the 2H2L correlator in Section 3.4, where we also show how the construction reaches the unitary minimal models. Open directions are collected in Section 3.5. In Appendix A, gathered at the end of the thesis, we collect the asymptotic expansions of the relevant hypergeometric functions, their resurgent structure, and a self-contained summary of the new large-parameter expansions of ${}_2F_1$ that we worked out. This chapter is based on [1].

3.1 Introduction

Two-dimensional conformal field theories are an incredibly rich class of theories. Due to the power of conformal symmetry in $2d$, entire classes of strongly coupled models can be analytically solved [20]. Within the AdS/CFT correspondence [34–36], $2d$ CFTs with large central charge c are expected to be related to a semi-classical gravitational theory on AdS_3 with $c = 3\ell/2G_N$, G_N and ℓ being Newton’s constant and the AdS_3 length of the $3d$ theory, respectively [111]. The $1/c$ expansion in a $2d$ CFT is then mapped to a gravitational semi-classical expansion around $3d$ asymptotic AdS space-times, and as such it is expected to be factorially divergent.

Virasoro primary states with conformal dimension $\Delta = h + \bar{h}$, h and $\bar{h}$ being the holomorphic and anti-holomorphic conformal weights, are associated to specific fields in AdS_3 . The $\Delta = 0$ identity state corresponds to pure AdS_3 . $2d$ states with fixed Δ as $c \rightarrow \infty$ can be seen as local perturbations of AdS_3 , and are typically denoted “light” states. States with $\Delta \propto c$ are denoted “heavy”: they can no longer be considered as probes and back-react on the metric.

For $\Delta < c/12$, they correspond to matter fields with mass $m^2 = \Delta(\Delta - 2)/\ell^2$ and give rise to AdS_3 backgrounds with a deficit/excess angle $\Delta\phi = 2\pi\sqrt{1 - 12\Delta/c}$. States with $\Delta > c/12$ are instead associated with Banados-Teitelboim-Zanelli (BTZ) black holes [112]. CFTs with a finite number of states with $\Delta < c/12$ and large c are particularly interesting because they can admit a dual description in terms of a semi-classical theory of gravity (holographic CFTs) [37].

Correlation functions of local operators in CFTs can be expanded using the operator product expansion in terms of products of holomorphic and anti-holomorphic Virasoro conformal blocks $\mathcal{F}_h$ and $\overline{\mathcal{F}}_{\bar{h}}$.¹ In contrast to the $\text{SL}(2, \mathbb{R})$ global blocks, the analytic form of the Virasoro blocks is unknown.² A convergent power series in the conformal cross-ratio z for Virasoro blocks was provided long ago by Zamolodchikov [115, 116]. Starting from this expression, obtaining a $1/c$ expansion is in principle straightforward, as already pointed out in [117, 118]. For four-point correlators on the plane, the most studied case, we schematically have

$$\mathcal{F}_h(h_i, c; z) \sim \sum_{k=0}^{\infty} \frac{f_k(h, h_i; z)}{c^k}, \quad (3.1.1)$$

where h_i ($i = 1, 2, 3, 4$) and h denote the conformal weights of the external and the exchanged operators, respectively. The coefficients f_k are functions of h, h_i and z , whose explicit form can be recursively computed but whose closed form is generally hard to determine. In writing (3.1.1) we have assumed that h and h_i do not scale with c , in which case f_0 coincides with the $\text{SL}(2, \mathbb{R})$ global block. We do not have an equality sign in (3.1.1) because the $1/c$ expansion is generally asymptotic.

Over the years, great progress has been achieved in finding approximate expressions for the Virasoro blocks. This was mostly motivated by the fact that, in holographic CFTs, such expressions can shed light and give a new perspective on the properties of $3d$ gravitational theories. Special attention has been paid to the Virasoro identity block exchange, $h = 0$ in (3.1.1), and in the cases in which two or all four external operators have $h_i \propto c$ and are heavy. When all operator dimensions scale with c , Virasoro blocks are expected to exponentiate [119] (see [120] for a proof). A form of exponentiation is also expected for arbitrary exchanged operators, in particular for the identity exchange. For four identical external heavy operators with $h_i = \eta c$, a useful parametrization of the $1/c$ expansion of the identity block $\mathcal{F}_{1,1} = \mathcal{F}_{h=0}$ is given by

$$\mathcal{F}_{1,1}(h_i, c; z) \sim e^{-cS(\eta, z)} A_{4\text{H}}(\eta, z) \left(1 + \sum_{k=1}^{\infty} \frac{g_k(\eta, z)}{c^k} \right), \quad (4\text{H}). \quad (3.1.2)$$

The leading order terms S and $A_{4\text{H}}$ define a semi-classical approximation to the block and their form is in general unknown, like all other sub-leading terms. For specific external heavy operators (twist fields), $\mathcal{F}_{1,1}$ has been shown to govern the Rényi entropy of two intervals in holographic CFTs and the expression of S for $z \rightarrow 0$ has been determined [121]. For $\mathbb{Z}_2$ twist fields, with $h_i = c/32$, the semi-classical block S can be determined exactly [122].

¹This is the $2d$ analogue of the conformal block decomposition (2.2.7) that holds in general dimension.

²Although the form of the blocks is unknown, the crossing kernel relating Virasoro blocks in different channels is explicitly known [113, 114].

The setup which has been more extensively studied is the one of a four-point correlator between two light (L) and two heavy (H) generic states. In this case, the identity block can be written as

$$\mathcal{F}_{1,1}(h_i, c; z) \sim z^{-2h_H} A_{2H2L}(\eta, h_L, z) \left(1 + \sum_{k=1}^{\infty} \frac{h_k(\eta, h_L, z)}{c^k} \right), \quad (2H2L), \quad (3.1.3)$$

where for simplicity we have taken identical light and heavy fields, with conformal weights $h_H = \eta c$ and h_L , respectively. The semi-classical block in this case is identified with the leading order term A_{2H2L} , whose form has been worked out in [117, 123]. From the gravity side, A_{2H2L} has been shown to correspond to a Witten diagram on geodesics over a conical geometry, the latter created by the heavy operators [124]. Its form can also be determined by taking appropriate limits of the crossing kernels [125].

The identity block exchange in 2H2L correlators, in the appropriate channel, is roughly expected to correspond on the AdS_3 side to the all-order perturbative series of graviton exchange between the probe light state interacting with the source (point-like for $\Delta_H < c/12$ or a BTZ black hole for $\Delta_H > c/12$).³ When $\Delta_H > c/12$, assuming the eigenstate thermalization hypothesis, the four-point function to leading order corresponds to the thermal two-point function of the light states in the presence of a BTZ black hole, with mass, spin and temperature governed by the conformal weights of the heavy state [123]. In Minkowski signature this correlator, when approximated with the exchange of the semi-classical identity block, decays exponentially to zero at late-times, a signal of information loss [126]. This is expected since in this approximation the exponential decay is given by quasi-normal modes associated with the relaxation to equilibrium of the excited BTZ black hole [127]. The identity exchange in an out-of-time-ordered 2H2L correlator has also been studied, in connection with chaotic properties of $2d$ CFTs [128, 129].

The very same semi-classical A_{2H2L} in (3.1.3), in Euclidean signature, has been shown to be singular away from OPE singularities of the correlators, which are the only ones expected to occur. In analogy to the phenomenon of information loss, such forbidden singularities should be an artefact of the semi-classical expansion and are expected to be somehow smoothed out at the “quantum” level. Understanding how such singularities cancel motivated the study of the first $1/c$ corrections to (3.1.3), computed in [130–132]. A universal behaviour of the identity block close to the forbidden singularities has been proposed in [133] (see also [134] for a related analysis). When one or more of the external operators are degenerate, a more in-depth analysis is possible [135]. By performing an analysis of Stokes phenomena of the identity block for external degenerate operators, it has been argued in [133] that non-perturbative effects of order $\exp(-c)$ are responsible for the resolution of both forbidden singularities and information loss in the Euclidean and Lorentzian regimes, respectively. See also [136] for a numerical study and [137] for a proposal of how forbidden singularities can be resolved. Although it was already suspected that an explanation of what forbidden singularities are and how they are

³In $3d$ there are no propagating gravitons, but we do have gravitational interactions due to off-shell gravitons. In this sense, the situation is analogous to gauge interactions between charged particles in $2d$ gauge theories.

resolved could be obtained by a systematic analysis of the $1/c$ asymptotic series (3.1.3) (see e.g. [133, 134]), such an analysis has not been performed yet.

The theory of resurgence [49] is the appropriate tool to investigate this problem as it provides, among other things, a convenient way to discover non-perturbative effects. In the context of $2d$ QFTs, these techniques have mostly been applied on integrable models with asymptotically free weak coupling limits [138–146]. On the other hand, in the context of CFT specifically, the investigation has mostly been in four and three dimensions, namely for $\mathcal{N} = 4$ SYM and the large charge expansion of sigma models, see e.g. [147–151]. For resurgence of the large c expansion in two dimensional CFT, there were some early investigations in [133] and recent study in [152, 153].

The aim of this chapter is to perform a resurgent analysis of the $1/c$ expansion for four-point correlators of the type 4H and 2H2L. Since a rigorous study requires a detailed knowledge of the terms g_k and h_k appearing in (3.1.2) and (3.1.3), which is hard to achieve in general, in this chapter we focus our attention on correlators involving degenerate operators of the kind $\phi_{2,1}$ (notation as in [21]).⁴ As well known, such correlators satisfy a second-order differential equation and the Virasoro blocks can be determined in terms of hypergeometric functions. We focus on the cases of four heavy $\phi_{2,1}$ degenerate operators (4H) and two heavy $\phi_{2,1}$ degenerate plus two arbitrary light operators (2H2L). For the above specific correlators, starting from the all-order $1/c$ series associated with the identity block (related with the all-order graviton corrections around the $3d$ background formed by the heavy states), we answer positively the following questions:

- Can we see that the identity block cannot be the full answer and other non-perturbative terms should be present?
- Do forbidden singularities appear also for 4H correlators?
- How are these globally resolved for both 4H and 2H2L?
- More ambitiously, can we reconstruct the full correlators?

Since the resulting holographic theories are necessarily non-unitary, we do not discuss information loss and consider correlators only in the Euclidean regime.

Summary of the results. Four-point correlators of $\phi_{2,1}$ degenerate fields involve two Virasoro blocks, $\mathcal{F}_{1,1}$ associated to the exchange of the identity operator $\phi_{1,1} = \mathbb{1}$ and $\mathcal{F}_{3,1}$ associated to the exchange of the degenerate $\phi_{3,1}$ field. Both are exactly determined in terms of hypergeometric functions ${}_2F_1(\alpha, \beta, \gamma; z)$, where α, β, γ scale with c . We show how such hypergeometric functions can be rewritten in terms of other ${}_2F_1$ -s where only one out of the three parameters scales with c , see (A.1.42) and (A.1.50). We then determine the whole asymptotic series of the latter hypergeometric function, reported in (A.1.29). To the best of our knowledge, these results present new asymptotic expansions for large parameters of ${}_2F_1$ which can be useful

⁴See [152] for a first analysis in this direction.

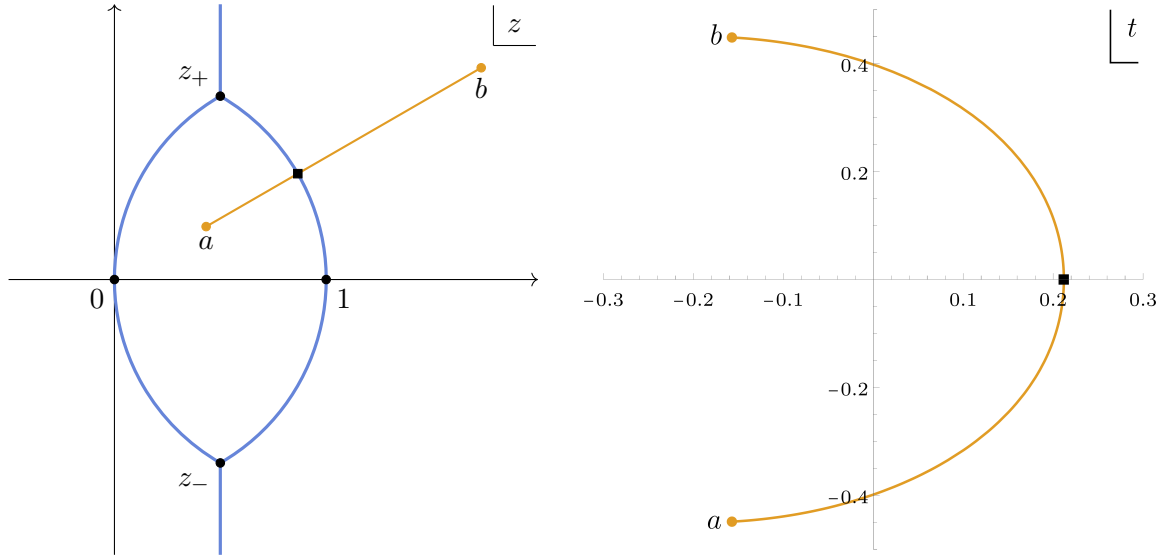


Figure 3.1: (left) Stokes lines in the z -plane for the 4H correlator (blue lines). The points $z_{\pm} = \exp(\pm i\pi/3)$ are forbidden singularities. The orange line is $z = \rho \exp(i\pi/6)$ with $1/2 \leq \rho \leq 2$. (right) The position of the leading Borel singularity as z varies along the straight orange line in the left panel. When the orange line intersects the blue Stokes line (squared bullet), the Borel singularity is on the positive real axis in the t -plane. See Figure 3.3 for a more detailed version and explanation of this figure.

more broadly in mathematics and physics. For the reader's convenience, we have collected these results in Appendix A.3.

Using the above results, we determine to all orders the terms g_k and h_k appearing in (3.1.2) and (3.1.3). The coefficients admit a simple expression using appropriate maps, $z \rightarrow r(z)$ (4H) or $z \rightarrow v(z)$ (2H2L), see (3.3.5) and (3.4.2) for their explicit form. In both cases, the forbidden singularities are non-analytic points of the above maps. At those points, all terms in (3.1.2) and (3.1.3) diverge.

Having an analytic expression for the entire asymptotic series, we determine its associated Borel function, its large order behaviour (see (3.3.22) and (3.4.10)) and its Borel summability, which crucially depends on z . The $1/c$ asymptotic series is Borel summable for generic values of z , when all singularities of the Borel function associated with the two asymptotic series (3.1.2) and (3.1.3) are away from the real positive axis in the Borel t -plane. For special values of z , however, some singularities approach the real axis and give rise to a Stokes phenomenon, see (3.3.13) (4H) and (3.4.13) (2H2L). This occurs in a codimension-one locus in the complex plane forming a web of Stokes lines, see Figure 3.1 (4H) and Figure 3.2 (2H2L) for an illustration. Forbidden singularities are nothing else than points emanating Stokes lines. Remarkably, they are not exclusive to 2H2L correlators but feature also in 4H ones. Interestingly enough, they only appear in the sub-leading term $A_{4H}(z)$ in (3.1.2), the leading term $S(z)$ being smooth at forbidden singularities.⁵ We conjecture that, in analogy to 2H2L correlators, forbidden singularities in $A_{4H}(z)$ persist for more general 4H correlators.

⁵This is consistent with $S(z)$ computed for the identity block with $h_i = c/32$ fields in [122].

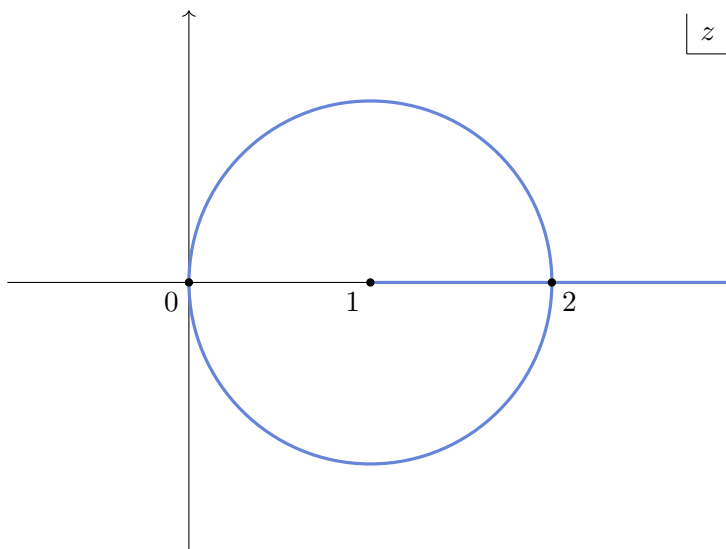


Figure 3.2: Stokes lines in the z -plane for the 2H2L correlator (blue lines). The point $z = 2$ is a forbidden singularity. See Figure 3.6 for a more detailed version and explanation of this figure.

In exact WKB terminology, we get two simple turning points (three Stokes lines converging) for 4H, z_+ and z_- in Figure 3.1, and one double turning point (four Stokes lines converging) for the 2H2L correlators, $z = 2$ in Figure 3.2. Forbidden singularities in both cases correspond to an artefact of the asymptotic expansion. We show how Stokes line discontinuities combine with the semi-classical branch cut to turn a forbidden singularity into a perfectly regular point, with trivial monodromy, see (3.3.31) (4H) and (3.4.22) (2H2L).

The Airy function, as we reviewed in Section 2.1, provides a simple example of this phenomenon. The exact function is analytic at $z = 0$, but its asymptotic expression, being an expansion in powers of $z^{-3/2}$, is ill-defined there. The point $z = 0$ is a simple turning point and is the analogue of a forbidden singularity. The combined monodromies due to Stokes lines jumps and branch cuts appearing in the asymptotic representation of the function nicely cancel, reflecting the actual analytic nature of the point $z = 0$.

The $\mathcal{F}_{3,1}$ block is completely determined in terms of the Stokes discontinuity of the identity block. Putting together the identity block and its Stokes discontinuity, and combining holomorphic and anti-holomorphic components, as in (3.3.36), the full four-point function is reconstructed by demanding trivial monodromy around $z = 0$ and $z = 1$.

For both correlators 4H and 2H2L, two scenarios are possible:

- i) $c > 0$, $h_{2,1} < 0$.
- ii) $c < 0$, $h_{2,1} > 0$.

Starting from the identity block, the two cases are quite distinct. In i) there exist regions in z where the identity block expansion Borel resums to the exact result. It should be noted that in this case, the identity block is sub-leading in the OPE limit compared to the $\phi_{3,1}$ exchange,

	$c > 0$	$c < 0$
$s(\tilde{\mathcal{F}}_{1,1}) = \mathcal{F}_{1,1}$	✓	✗
Stokes jumps	✓	✓
Identity leading as $z \rightarrow 0$	✗	✓
Full correlator from $\tilde{\mathcal{F}}_{1,1}$	✓	✓

Table 3.1: Summary of the key properties of the asymptotic expansion $\tilde{\mathcal{F}}_{1,1}$ of the identity block for both $c > 0$ and $c < 0$. The results apply for both the 4H and the 2H2L correlators. The relation $s(\tilde{\mathcal{F}}_{1,1}) = \mathcal{F}_{1,1}$ is understood to apply only in a sector of the z -complex plane delimited by Stokes lines. See the main text for further explanations.

since $h_{3,1} < 0$. In case ii) the identity block dominates in the OPE limit ($h_{3,1} > 0$), but the asymptotic series of the identity block now never reconstructs the exact block, because the latter includes non-perturbative e^{-c} terms which would be missed. The Borel resummation of the series gives rise in this case to the unique combination of the identity and $\mathcal{F}_{3,1}$ block with suitable analytic properties in the $1/c$ plane, as expected from general arguments. We summarize the key properties of the two cases in Table 3.1. The accuracy of the semi-classical approximation of the identity block in the z -plane matches the expectations coming from an analysis of Stokes and anti-Stokes lines, see Figures 3.5 and 3.7 for both $c > 0$ and $c < 0$ for 4H and 2H2L.

Interestingly enough, the Borel summability to the exact result of the $1/c$ expansion for positive central charge extends down to $c < 1$. We can then interpret the correlators of unitary minimal models with $c < 1$ as the $1/c$ resummations in the “deep quantum gravity” regime of an AdS_3 theory in the presence of exotic negative energy sources!⁶

3.2 Four-point functions with degenerate operators

In this section, we review basic facts about the Virasoro conformal blocks and the form of four-point functions in the presence of degenerate fields and set up our notations. We use conventions as in [21]. For simplicity, we consider non-chiral CFTs with $c_L = c_R = c$.

Degenerate operators $\phi_{r,s}$ are Virasoro primaries having a null descendant at level rs . By the Kac formula, these operators have holomorphic conformal weight

$$h_{r,s} = -\frac{b^2}{4}(r^2 - 1) - \frac{1}{4b^2}(s^2 - 1) - \frac{1}{2}(rs - 1), \quad (3.2.1)$$

where r, s are positive integers and b is a convenient parametrization for the central charge:

$$c = 13 + 6\left(b^2 + \frac{1}{b^2}\right). \quad (3.2.2)$$

The operators $\phi_{r,s}$ can be part of a unitary CFT only if $0 < c < 1$, but they exist for any value of c if the unitarity constraint is relaxed. We focus on the simplest case of four-point functions

⁶This is different from the connections between certain minimal models and $3d$ gravity pointed out in [122, 154].

on the plane/sphere involving the degenerate field $\phi_{2,1}$. The fusion rule of the field with itself contains just two Virasoro primaries

$$\phi_{2,1} \times \phi_{2,1} \sim \phi_{1,1} + \phi_{3,1}, \quad (3.2.3)$$

$\phi_{1,1} = \mathbb{1}$ being the identity operator. The operators involved have holomorphic weights

$$h_{2,1} = -\frac{3}{4}b^2 - \frac{1}{2}, \quad h_{1,1} = 0, \quad h_{3,1} = -2b^2 - 1. \quad (3.2.4)$$

We take the large c limit by sending $|b^2| \rightarrow +\infty$. For simplicity, we consider the case in which b^2 is real, though it is not difficult to extend our analysis for complex values.⁷ In both cases the heavy field H is identified with the degenerate field as

$$H(z, \bar{z}) = \phi_{2,1}(z)\bar{\phi}_{2,1}(\bar{z}), \quad (3.2.5)$$

which is a scalar with scaling dimension $\Delta_H = 2h_{2,1}$ proportional to c . For $b^2 \rightarrow +\infty$, $c > 0$ and $\Delta_H < 0$. For $b^2 \rightarrow -\infty$, $c < 0$ while $\Delta_H > 0$. We will separately consider both cases in the next sections.

The fusion rule (3.2.3) implies that in an OPE expansion, the four-point function is given by the sum of the product of two Virasoro blocks. Moreover, since H has a null descendant at level 2, the four-point function satisfies a second-order equation of hypergeometric type (both in z and $\bar{z}$). This allows for an exact determination of the two blocks, which are nothing but a basis of solutions of such differential equation. Demanding trivial monodromy at $z = 0$ and $z = 1$ finally allows us to reconstruct the entire correlator.

The two correlators we will analyse are

$$\mathcal{A}_{4H} = \langle H(\infty)H(1)H(z, \bar{z})H(0) \rangle, \quad (3.2.6)$$

$$\mathcal{A}_{2H2L} = \langle L(\infty)L(1)H(z, \bar{z})H(0) \rangle. \quad (3.2.7)$$

In the 2H2L correlator, the field L is a generic light scalar operator with scaling dimension $\Delta_L = 2h_L$. It is convenient to parametrize its conformal weight as

$$h_L(b^2) = \gamma + \frac{\gamma(1-\gamma)}{b^2}, \quad (3.2.8)$$

with γ constant. The explicit form of these correlators is given by

$$\mathcal{A}_{4H} = |\mathcal{F}_{1,1}|^2 + \lambda_{4H}|\mathcal{F}_{3,1}|^2, \quad (3.2.9)$$

$$\mathcal{A}_{2H2L} = |\mathcal{F}_{1,1}|^2 + \lambda_{2H2L}|\mathcal{F}_{3,1}|^2. \quad (3.2.10)$$

Here $\mathcal{F}_{1,1}$ and $\mathcal{F}_{3,1}$ are the Virasoro conformal blocks, associated with the exchange of the operators in the fusion rule (3.2.3). λ are the product of OPE coefficients entering the s -channel decomposition of the corresponding four-point function. The two-point functions are unit normalized and in the s -channel OPE limit, $z \rightarrow 0$, we normalize $\mathcal{F}_h \approx z^{-2h_{2,1}+h}$. The

⁷As we will see, Stokes discontinuities will anyhow force us to give an imaginary component to b^2 .

explicit form of Virasoro blocks is given in terms of ${}_2F_1$ hypergeometric functions (see e.g. [155]). Their expression, together with the OPE coefficients squared, is

$$(4H) \quad \begin{cases} \mathcal{F}_{1,1} = z^{\frac{3}{2}b^2+1}(1-z)^{-\frac{1}{2}b^2} {}_2F_1(-b^2, 1+b^2, 2+2b^2; z), \\ \mathcal{F}_{3,1} = z^{-\frac{1}{2}b^2}(1-z)^{\frac{3}{2}b^2+1} {}_2F_1(-b^2, 1+b^2, -2b^2; z), \\ \lambda_{4H} = \frac{\Gamma(-b^2)^2 \Gamma(2b^2+1)^2 \Gamma(2b^2+2)^2}{\pi^2 \left(\csc(\pi b^2)^2 - \csc(2\pi b^2)^2 \right) \Gamma(3b^2+2)^2}, \end{cases} \quad (3.2.11)$$

$$(2H2L) \quad \begin{cases} \mathcal{F}_{1,1} = z^{\frac{3}{2}b^2+1}(1-z)^\gamma {}_2F_1(2\gamma, 1+b^2, 2+2b^2; z), \\ \mathcal{F}_{3,1} = z^{-\frac{1}{2}b^2}(1-z)^\gamma {}_2F_1(-b^2, 2\gamma-1-2b^2, -2b^2; z), \\ \lambda_{2H2L} = -\frac{\Gamma(-b^2)^2 \Gamma(2+2b^2)^2 \Gamma(1-2\gamma) \Gamma(-1-2b^2+2\gamma)}{\Gamma(-2b^2)^2 \Gamma(1+b^2)^2 \Gamma(2+2b^2-2\gamma) \Gamma(2\gamma)}, \end{cases} \quad (3.2.12)$$

where γ is defined in (3.2.8). We show in Appendix A.1 how to get an analytic closed form expression for the asymptotic expansion of the above blocks for $|b^2| \rightarrow \infty$ using some appropriate rewriting of the ${}_2F_1$ functions. More precisely, we will perform an expansion in $|\epsilon| \ll 1$, where

$$\frac{1}{\epsilon} = b^2 + \frac{3}{2}. \quad (3.2.13)$$

The offset by $3/2$ is useful to simplify various expressions that will follow. Note that the operator placement for (3.2.7) is not the conventional one to obtain the thermal correlator in the large c limit, but rather its crossing symmetric.

The large central charge limit can also be taken by sending $b^2 \rightarrow 0$. In this case, the role of the heavy and light fields is exchanged, and $\mathcal{A}_{4H}$ turns into a four-point function of light operators. In $\mathcal{A}_{2H2L}$, unitarity is now broken only by the light probe and can be considered a milder breaking. We do not discuss this case, because the asymptotic expansion of hypergeometric functions with small parameters is more complicated and we currently lack a closed-form expression.

3.3 4H correlator with $\phi_{2,1}$

The correlator of four $\phi_{2,1}$ fields is known, and given by (3.2.9) and (3.2.11). As anticipated in the introduction, we consider this correlator starting from the $1/c$ asymptotic expansion of the identity block and see what we can learn.

The major and non-trivial step of the analysis is the determination of the whole large c expansion, which can be found in Appendix A.1. From the asymptotic series of the identity block, we can “discover” the $\phi_{3,1}$ exchange when the series becomes non-Borel summable. This happens at specific values of z which draw Stokes lines in the complex z -plane. These asymptotic series also have forbidden singularities which are not compatible with the full correlator. We show that the non-perturbative effects at the Stokes lines resolve these singularities and finally we show how to fix the whole four-point function. We discuss first the case $\epsilon > 0$.

We start by expanding the conformal block for identity exchange shown in (3.2.11) in powers of ϵ . We find that

$$\tilde{\mathcal{F}}_{1,1}(z; \epsilon) = e^{-\frac{1}{\epsilon} S_1(z)} A_1(z) \tilde{F}_1(z; \epsilon), \quad (3.3.1)$$

where

$$\begin{aligned} S_1(z) &= S(z) + \frac{1}{2} \log(1-z) - \frac{3}{2} \log z, & A_1(z) &= A(z) z^{-\frac{5}{4}} (1-z)^{\frac{3}{4}}, \\ \tilde{F}_1 &= f_0 + \sum_{n=1}^{\infty} f_n(r) \epsilon^n, & f_0 &= 1, & f_n(r) &= \sum_{k=1}^n (-1)^{n+k} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \frac{(\frac{1}{6})_k (\frac{5}{6})_k}{k!} r^k, \end{aligned} \quad (3.3.2)$$

$$(3.3.3)$$

with S and A given by

$$S(z) = \frac{1}{2} \log \left(\frac{27}{16} \frac{z^2}{(1-z)^2} \frac{1-r(z)}{r(z)} \right), \quad A(z) = \frac{e^{S(z)}}{(1-z+z^2)^{\frac{1}{4}}}, \quad (3.3.4)$$

and the map $r(z)$ given by

$$r(z) = \frac{1}{2} + \frac{(z+1)(2z-1)(2-z)}{4(1-z+z^2)^{\frac{3}{2}}}. \quad (3.3.5)$$

In (3.3.3) $\{n\}_k$ are the Stirling numbers of the second kind and $(a)_k$ denotes the Pochhammer symbol. The derivation of (3.3.1) is quite non-trivial. We refer the reader to Appendix A.1.5 for a detailed derivation of how the map $r(z)$ is found, and to Appendix A.1.4 for the form of the coefficients f_n . It is quite remarkable that the coefficients f_n have a simple form when expressed in terms of the variable $r(z)$ in (3.3.5).

The map $r(z)$ satisfies the following properties:

$$r(0) = 0, \quad r(1) = 1, \quad r(1-z) = 1-r(z). \quad (3.3.6)$$

It is easy to show that only for $z = 0, 1$ we have $r = 0, 1$, respectively. Furthermore, for any real value of z , $r(z)$ is contained in $(0, 1)$. Around each of these points $r(z)$ behaves as

$$r(z) = \frac{27}{16} z^2 + \mathcal{O}(z^3), \quad 1-r(z) = \frac{27}{16} (1-z)^2 + \mathcal{O}((1-z)^3). \quad (3.3.7)$$

Meanwhile, $r(z)$ is singular at

$$z_{\pm} = e^{\pm \frac{i\pi}{3}}. \quad (3.3.8)$$

While $S(z)$ is not singular at $z_{\pm}$, the leading factor $A(z)$ is indeed singular, as every perturbative correction f_n is. In fact, the higher the order in perturbation theory, the higher the degree of singularity. Both the full four-point Euclidean correlator and the Virasoro blocks can be singular at $z = 0, 1, \infty$, when operators collide, but nowhere else. This makes $z_{\pm}$ examples of “forbidden singularities”, in the terminology of [133], with the novelty that they are identified for correlators with four heavy operators. Notice that they are elusive in an analysis around $z = 0$, as the one presented in [152].

3.3.1 Borel summation and discontinuity

As derived in Appendix A.2, the Borel transform of the $\tilde{F}_1$ series is given by

$$\hat{F}_1(r; t) = \frac{5r}{36} {}_2F_1\left(\frac{7}{6}, \frac{11}{6}, 2; r(1 - e^{-t})\right). \quad (3.3.9)$$

For $\epsilon > 0$, when $0 < z < 1$ and thus $0 < r < 1$, we have that $F_1^B = F_1$, as shown in Appendix A.2. It follows that the Borel summation of the asymptotic series of the identity block recovers the full block:

$$\mathcal{F}_{1,1}^B = \mathcal{F}_{1,1}, \quad 0 < z < 1. \quad (3.3.10)$$

The validity of (3.3.10) can be extended into the z complex plane up to the points where the series $\tilde{F}_1$ becomes non-Borel summable. To find these points, observe that the function (3.3.9) has singularities, among others (see (A.2.6)), at

$$t_k(r) = \log\left(\frac{r}{r-1}\right) + 2\pi i k, \quad k \in \mathbb{Z}. \quad (3.3.11)$$

When $r > 1$ we see that t_0 becomes positive and real turning the series non-Borel summable. The discontinuity of the lateral summation will then be a non-perturbative effect proportional to

$$(s_{0+} - s_{0-})\tilde{\mathcal{F}}_{1,1}(z; \epsilon) \propto e^{-\frac{1}{\epsilon}t_0(r)} = \left(\frac{r}{r-1}\right)^{-\frac{1}{\epsilon}}. \quad (3.3.12)$$

By specializing (A.2.12) for $a = 1/6$ and $b = 5/6$, we get the full result

$$(s_{0+} - s_{0-})\tilde{\mathcal{F}}_{1,1}(z; \epsilon) = -ie^{\pm \frac{i\pi}{\epsilon}} s_0\left(\tilde{\mathcal{F}}_{1,1}(1 - z; \epsilon)\right). \quad (3.3.13)$$

Note how the non-perturbative factor (3.3.12) is naturally absorbed into $\tilde{\mathcal{F}}_{1,1}$ evaluated at the crossing point $z \rightarrow 1 - z$, up to a phase. The sign ambiguity of the phase in (3.3.13) depends on whether z is in the upper or lower half-plane, since

$$\left(\frac{r-1}{r}\right)^{\frac{1}{\epsilon}-1} = -\left(\frac{1-r}{r}\right)^{\frac{1}{\epsilon}-1} \begin{cases} e^{\frac{i\pi}{\epsilon}} & \text{Im } z > 0 \\ e^{-\frac{i\pi}{\epsilon}} & \text{Im } z < 0 \end{cases}, \quad (3.3.14)$$

which is necessary to retrieve $\tilde{\mathcal{F}}_{1,1}(1 - z)$. Observe that if r is such that $\tilde{F}_1(z)$ is not Borel summable, then $\tilde{F}_1(1 - z)$ certainly is, because $t_0 \rightarrow -t_0$ when $z \rightarrow 1 - z$. For convenience we define

$$\tilde{\mathcal{G}}(z; \epsilon) = \tilde{\mathcal{F}}_{1,1}(1 - z; \epsilon). \quad (3.3.15)$$

We now determine in which regions of the z plane Stokes jumps can occur. At a generic point in the z complex plane, both $\tilde{\mathcal{F}}_{1,1}(z)$ and $\tilde{\mathcal{G}}(z)$ are Borel summable to the functions $\mathcal{F}_{1,1}^B$ and $\mathcal{G}^B$. It is useful to combine them in a vector

$$V(z; \epsilon) = \begin{pmatrix} \mathcal{F}_{1,1}^B(z; \epsilon) \\ \mathcal{G}^B(z; \epsilon) \end{pmatrix}. \quad (3.3.16)$$

By symmetry, when $r < 0$, $\tilde{\mathcal{G}}$ is not Borel summable while $\tilde{\mathcal{F}}_{1,1}$ is. We find

$$(s_{0+} - s_{0-})\tilde{\mathcal{G}} = -ie^{\mp \frac{i\pi}{\epsilon}} s_0 \left(\tilde{\mathcal{F}}_{1,1} \right). \quad (3.3.17)$$

These results determine the Stokes jumps at a fixed value of $r(z)$. As we move around the z plane, the line $r \in (1, +\infty)$ is mapped to three lines in the z -plane. We have

$$r\left(\frac{1}{2} + ia\right) = \frac{1}{2} + i\frac{a(9 + 4a^2)}{(3 - 4a^2)^{\frac{3}{2}}} > 1 \quad \text{if} \quad a < -\frac{\sqrt{3}}{2}, \quad (3.3.18)$$

leading to a line from $-i\infty$ to z_- . Moreover, $r > 1$ also when $z = e^{i\theta}$ with $\theta \in (-\pi/3, \pi/3)$, which leads to two circle arcs connecting 1 to $z_{\pm}$. When z crosses one of these lines, t_0 becomes positive and real. Thus the Borel summations along the positive real axis before and after z crossing the line differ, and they differ precisely by Stokes jump calculated when z lies precisely at the line, i.e. (3.3.13). The Stokes jump in the Borel plane now appears as Stokes lines in the z plane emanating from the forbidden singularities and ending at the actual singularities. We can establish the region D_0 in which (3.3.10) holds, by continuing from $0 < z < 1$ until we reach either a Stokes line or a branch cut of the $r(z)$ map, and thus

$$D_0 = \left\{ z \in \mathbb{C} : \text{Re}(z) < \frac{1}{2} \vee |z| < 1 \right\}. \quad (3.3.19)$$

See Appendix A.1.5 for further discussion. We can also consider the Stokes lines where $\tilde{\mathcal{G}}$ is not Borel summable which, from $z \rightarrow 1 - z$, correspond to a vertical line from z_+ to $+i\infty$ and two circle arcs connecting 0 to $z_{\pm}$. We then define the region

$$E_0 = \{ z \in \mathbb{C} : |z| < 1 \wedge |1 - z| < 1 \}, \quad (3.3.20)$$

between the two circle arcs, which is a result of continuing the interval $(0, 1)$ up until we reach a Stokes line of $\tilde{\mathcal{F}}_{1,1}$ or $\tilde{\mathcal{G}}$, see Figure 3.3 (shaded region).

We report the whole web of Stokes lines in Figure 3.3, where we also report the branch cuts of the $r(z)$ map itself, emanating from the singularities at $z_{\pm}$. At the branch cut, the map $r(z)$ is discontinuous, obeying

$$r(z + \delta z) = 1 - r(z - \delta z) = r(1 - (z - \delta z)), \quad (3.3.21)$$

for z in the branch cut and δz an infinitesimal displacement orthogonal to it. To avoid overlapping these cuts with the Stokes lines, we place them by convention slightly to the right/left, as in Figure 3.3.⁸

An important feature which can be read from the Borel transform is the large-order behaviour of the asymptotic series. As recalled in (2.1.7) and (2.1.8), the singularities of the Borel transform closest to the origin dictate the dominant asymptotic behaviour of the series. For real series, typically either a single singularity $t = t_0$ dominates, leading to a series such as $t_0^{-n} n!$,

⁸Alternatively, we could keep the branch cuts parallel to the imaginary axis and assign a small imaginary part to ϵ to slightly rotate the Stokes lines.

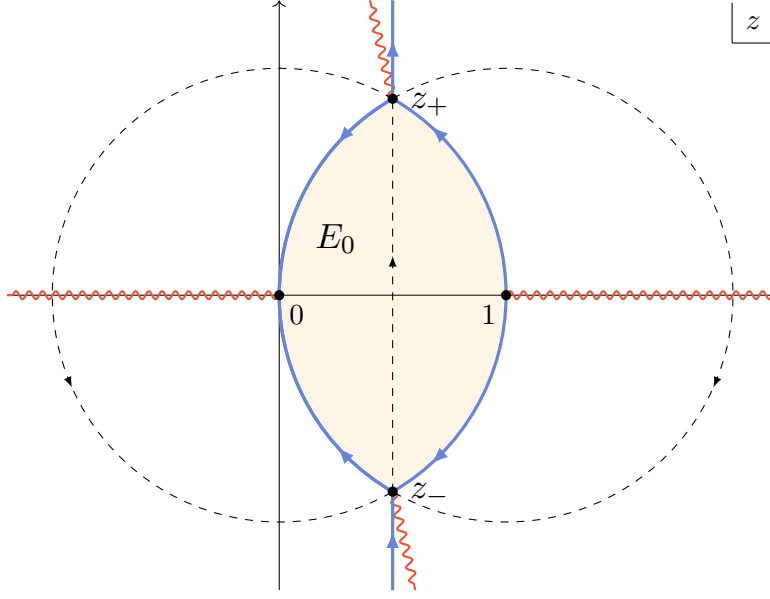


Figure 3.3: Branch cuts (red wiggly lines), Stokes lines (blue) and anti-Stokes lines (dashed) in the z -plane for the 4H asymptotic series $(\tilde{\mathcal{F}}_{1,1}, \tilde{\mathcal{G}})$. Blue lines ingoing and outgoing from $z_{\pm}$ indicate the Stokes lines where $\tilde{\mathcal{F}}_{1,1}$ and $\tilde{\mathcal{G}}$ jump, respectively. A refined version of the left panel of Figure 3.1. See Section 3.3.2 for details.

or two complex conjugate singularities, which lead to an oscillatory modulation of the factorial divergence. For most complex values of z , one of the t_k singularities dominates. However, for real z we have two equidistant t_k singularities at $\log(\frac{r}{1-r}) \pm \pi i$. (A.2.29) reduces in this case to

$$f_n \approx -\frac{(n-1)!}{\pi} \left(\frac{r}{1-r} \right) \left| \log \frac{r}{r-1} \right|^{-n} \cos \left(n \arg \left(\log \frac{r}{r-1} \right) \right). \quad (3.3.22)$$

Notice that if one takes $z \rightarrow 0$, using (3.3.7), the oscillatory behaviour is damped out and we retrieve the scaling $(\log z^2)^{-n} n!$ which is implicit in the saddle points identified in [152]. We refer to Appendix A.2 for further details of the large order behaviour analysis. As we discuss below, we can already see that this series is always Borel summable for real ϵ . Thus, non-Borel summability and resurgence must appear at complex z .

It is useful to look at the $z \rightarrow 0$ limit to identify what $\tilde{\mathcal{G}}$ is. We get

$$\lim_{z \rightarrow 0} \tilde{\mathcal{G}} \propto z^{-\frac{1}{2\epsilon} + \frac{3}{4}} (1 + \dots). \quad (3.3.23)$$

Denoting by h_X the conformal weight of the exchanged operator, we should have as $z \rightarrow 0$ $\tilde{\mathcal{G}} \propto z^{-2h_{2,1} + h_X} (1 + \dots)$. By matching we get

$$h_X = -\frac{2}{\epsilon} + 2 = -2b^2 - 1 \quad \Rightarrow \quad h_X = h_{3,1}, \quad (3.3.24)$$

and hence we see that $\tilde{\mathcal{G}}$ includes the asymptotic expansion of the conformal block of the $\phi_{3,1}$ field. We can obtain an exact expression for the resummation of $\mathcal{G}$ by taking (3.3.10) at $z \rightarrow 1 - z$. This holds for $0 < z < 1$, but now we also know we can extend it to the entire

region contained between the circle arcs, see Figure 3.3, where neither $\tilde{\mathcal{F}}_{1,1}$ nor $\tilde{\mathcal{G}}$ have a Stokes jump. Then, using simple hypergeometric identities we get $\mathcal{G}^B$. Then we have

$$\begin{aligned}\mathcal{F}_{1,1}^B(z) &= \mathcal{F}_{1,1}(z), \\ \mathcal{G}^B(z) &= \frac{1}{2\sin(\frac{\pi}{\epsilon})}\mathcal{F}_{1,1}(z) + k(\epsilon)\mathcal{F}_{3,1}(z), \quad \epsilon > 0, \quad z \in E_0,\end{aligned}\tag{3.3.25}$$

where

$$k(\epsilon) = \frac{\Gamma(\frac{2}{\epsilon} - 2)\Gamma(\frac{2}{\epsilon} - 1)}{\Gamma(\frac{1}{\epsilon} - \frac{1}{2})\Gamma(\frac{3}{\epsilon} - \frac{5}{2})}.\tag{3.3.26}$$

We recognize $\mathcal{F}_{3,1}$ as the block of the $\phi_{3,1}$ field, which we have then “discovered” from the Stokes discontinuity of the identity block.

Conversely, we can invert the relation above and state that, while $\mathcal{F}_{1,1}(z)$ is given by $\mathcal{F}_{1,1}^B(z)$ for $z \in D_0$, $\mathcal{F}_{3,1}(z)$ is given by the resummation of a trans-series containing both $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{F}}_{1,1}$. Notice that since $\mathcal{F}_{1,1}(z)$ is regular across Stokes jumps but the resummation of $\tilde{\mathcal{F}}_{1,1}$ is not, once we cross a Stokes lines $\mathcal{F}_{1,1}(z)$ becomes itself also a trans-series. That is

$$\mathcal{F}_{1,1}(z - \delta z) = \mathcal{F}_{1,1}^B(z - \delta z) = \mathcal{F}_{1,1}^B(z + \delta z) \mp ie^{\pm \frac{i\pi}{\epsilon}} \mathcal{G}^B(z + \delta z) = \mathcal{F}_{1,1}(z + \delta z),\tag{3.3.27}$$

where δz is some small displacement from the Stokes line such that $z - \delta z$ is in E_0 and $z + \delta z$ is to the right of the Stokes lines, while the signs are for upper/lower half plane, respectively. Thus, generically in the complex plane, both blocks are given by trans-series which include both $\tilde{\mathcal{F}}_{1,1}$ and $\tilde{\mathcal{G}}$.

3.3.2 Stokes lines and monodromy matrices

It is useful to organise the Stokes jumps in the graphical notation of Figure 3.4 (adapted from [156]). The Stokes lines become noted as “incoming” or “outgoing” lines, with respect to the turning point at $z_{\pm}$. As discussed in (3.3.21), there are also branch cuts emanating from the forbidden singularities. At these lines,

$$V(z - \delta z) = S_- V(z + \delta z), \quad V(z - \delta z) = S_+ V(z + \delta z), \quad V(z - \delta z) = B V(z + \delta z),\tag{3.3.28}$$

for each line. Here we define δz such that in the orientation of Figure 3.4, $z \pm \delta z$ is above/below the line. In this convention, the matrices are

$$S_+ = \begin{pmatrix} 1 & 0 \\ -ie^{\mp \frac{i\pi}{\epsilon}} & 1 \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & -ie^{\pm \frac{i\pi}{\epsilon}} \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & ie^{\pm \frac{i\pi}{\epsilon}} \\ ie^{\mp \frac{i\pi}{\epsilon}} & 0 \end{pmatrix}, \quad \epsilon > 0,\tag{3.3.29}$$

where in the phases we take the upper/lower sign according to whether we are in upper/lower half plane. There are two other branch cuts, at $(-\infty, 0)$ and $(1, +\infty)$, due to the leading terms $S(z)$ and $A(z)$. These are simply

$$\begin{aligned}V(z + i0) &= B_0 V(z - i0), \quad B_0 = \begin{pmatrix} -ie^{\frac{3i\pi}{\epsilon}} & 0 \\ 0 & -ie^{-\frac{i\pi}{\epsilon}} \end{pmatrix}, \quad z \in (-\infty, 0), \\ V(z - i0) &= B_1 V(z + i0), \quad B_1 = \begin{pmatrix} -ie^{-\frac{i\pi}{\epsilon}} & 0 \\ 0 & -ie^{\frac{3i\pi}{\epsilon}} \end{pmatrix}, \quad z \in (1, +\infty).\end{aligned}\tag{3.3.30}$$

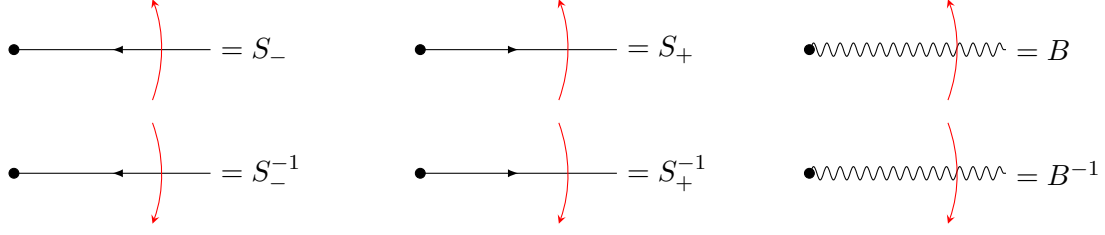


Figure 3.4: Local connection matrices as we pass an oriented Stokes line (straight lines with arrows) or a branch cut (wavy lines) emanating from $z_{\pm}$ (adapted from figure 2 in [156]).

The final graph is Figure 3.3, including the branch cuts (3.3.30). We also include the anti-Stokes lines, which are the lines at which $\mathcal{F}_{1,1}^B$ and $\mathcal{G}^B$ are of comparable size, defined in Appendix A.1. Their arrows are placed such that when you cross an outgoing line clockwise you go from a region where $\tilde{\mathcal{F}}_{1,1}$ is exponentially enhanced to suppressed, and the opposite for clockwise crossing an ingoing line.⁹

We now determine the monodromies around the singular points of V , $z = 0, 1, z_{\pm}$. We do not need to determine the monodromy around $z = \infty$ since this is fixed by the other ones. We start with $z = z_{\pm}$. Under a 2π rotation around $z_{\pm}$, the total monodromy is

$$\begin{aligned} M_{z=z_+} &= S_- S_+ B S_+ = I, \\ M_{z=z_-} &= S_- B S_- S_+ = I. \end{aligned} \quad (3.3.31)$$

Trivial monodromy is a necessary condition for $z_{\pm}$ to be a regular point. We then see how concretely and globally Stokes discontinuities resolve forbidden singularities. Note that, modulo the phase factors $-e^{\pm \frac{i\pi}{\epsilon}}$ the connection matrices (3.3.29) are the same as those appearing in the Airy function, which are also the universal Stokes lines for simple turning points in quantum mechanics in exact WKB [157], see a brief discussion in Section 2.1 and [54] for a pedagogical introduction.¹⁰ Even though forbidden singularities persist to all orders in perturbation theory, once we include all non-perturbative effects they are resolved. This is a crucial result from this analysis. Furthermore, these non-perturbative effects can all be retrieved from the large order behaviour of the perturbative part.

We now compute the monodromy matrices around 1 and 0. The monodromy of the resummed functions can of course be derived using the known discontinuity formula of the hypergeometric function, but crucially we “reconstruct” them here from products of connection matrices. Moreover, this can be seen as a non-trivial check of the formalism. Under a $2\pi i$ rotation around $z = 1$, starting slightly above $(0, 1)$, leads to

$$V \rightarrow M_{z=1} V, \quad M_{z=1} = (S_-)^{-1} B_1 (S_-)^{-1} = \begin{pmatrix} -ie^{-\frac{i\pi}{\epsilon}} & 1 + e^{\frac{2i\pi}{\epsilon}} \\ 0 & -ie^{\frac{3i\pi}{\epsilon}} \end{pmatrix}. \quad (3.3.32)$$

⁹Briefly, the “non-perturbative” effect $\tilde{\mathcal{G}}$ is suppressed (or enhanced) relative to $\tilde{\mathcal{F}}_{1,1}$ as $e^{-\frac{1}{\epsilon}\Delta S(z)}$ when $\Delta S(z)$ has a positive (or negative) real part, with ΔS being the difference between the semi-classical parts of each. When it is purely imaginary then $|\tilde{\mathcal{F}}_{1,1}| \sim |\tilde{\mathcal{G}}|$. The lines of $\text{Re}[\Delta S(z)] = 0$ are the anti-Stokes lines.

¹⁰The connection matrices appearing in [54, 156] act on the coefficients of the functions, rather than the function themselves, as here. For this reason they are given by the transpose of those in (3.3.29) (without the phase factors), but the two actions are completely equivalent.

The leftmost S_- is in the lower half plane, and the rightmost in the upper one. Note that generically the monodromy matrix M depends on the starting region, since matrices M_i and M_j starting in two different adjacent regions i and j are related by $M_j = S_{ij}M_iS_{ij}^{-1}$, with S_{ij} the product of connection matrices $S_{\pm}$ and B_i associated with lines separating regions i and j . Further, by replacing the asymptotic series with its summation in (3.3.32), one can check that

$$\left(1 + e^{\frac{2i\pi}{\epsilon}}\right)\mathcal{G}^B = -2i\pi e^{i\pi b^2} z^{\frac{3}{2}b^2+1} (1-z)^{\frac{3}{2}b^2+1} \sin(\pi b^2) {}_2F_1(b^2+1, 3b^2+2; 2b^2+2; 1-z), \quad (3.3.33)$$

which reproduces the exact discontinuity of the original hypergeometric representation of $\mathcal{F}_{1,1}$. The analysis at $z = 0$ is symmetric under $z \rightarrow 1 - z$,

$$M_{z=0} = (S_+)^{-1} B_0 (S_+)^{-1} = \begin{pmatrix} -ie^{\frac{3i\pi}{\epsilon}} & 0 \\ 1 + e^{\frac{2i\pi}{\epsilon}} & -ie^{\frac{-i\pi}{\epsilon}} \end{pmatrix}, \quad (3.3.34)$$

where the first and second S_+ matrices in (3.3.34) are in the upper and lower half planes respectively.

3.3.3 Fixing the four-point function

The above analysis applies verbatim to the anti-holomorphic sector starting from the asymptotic expansion of

$$\widetilde{\mathcal{F}}_{1,1}(\bar{z}) = \left(\widetilde{\mathcal{F}}_{1,1}(z)\right)^*. \quad (3.3.35)$$

Putting together holomorphic and anti-holomorphic sectors, we finally show how to determine the entire four-point function by demanding its single-valuedness around $z = 0, 1$.

We write the four-point function as an *a priori* unfixed quadratic form, up to a z -independent normalization factor,

$$\mathcal{A} = g(\epsilon) \bar{V}_i C_{ij} V_j, \quad C = \begin{pmatrix} 1 & c_1 \\ c_1^* & c_2 \end{pmatrix}, \quad c_1 \in \mathbb{C}, \quad c_2 \in \mathbb{R}. \quad (3.3.36)$$

For the reader more familiar with resurgence, one can take the asymptotic form of the above expression and consider a trans-series

$$\widetilde{\mathcal{A}} = g(\epsilon) \left(\mathcal{C}_0 \widetilde{\mathcal{F}}_{1,1} \widetilde{\mathcal{F}}_{1,1} + \mathcal{C}_1 \widetilde{\mathcal{F}}_{1,1} \widetilde{\mathcal{G}} + \bar{\mathcal{C}}_1 \widetilde{\mathcal{G}} \mathcal{F}_{1,1} + \mathcal{C}_2 \widetilde{\mathcal{G}} \widetilde{\mathcal{G}} \right), \quad (3.3.37)$$

where the $\mathcal{C}_i$ are piece-wise constant in the z complex plane (and also piece-wise constant in the ϵ -plane). In E_0 , we should have

$$\mathcal{C}_0 = 1, \quad \mathcal{C}_1 = c_1, \quad \mathcal{C}_2 = c_2. \quad (3.3.38)$$

At Stokes lines, these parameters change to “counter-act” the discontinuities of Borel summation in $\widetilde{\mathcal{F}}_{1,1}$ and $\widetilde{\mathcal{G}}$, keeping $\mathcal{A}^B$ continuous and ensuring that

$$\mathcal{A}^B = \mathcal{A}, \quad (3.3.39)$$

in the entire complex z -plane.

Then, knowing the Stokes constants in the preceding section, we fix the matrix C , or trans-series parameters, with physical principles. Namely, we ensure that the sequence of Stokes jumps around the singularities should bring the trans-series back to itself. This is a resurgence rephrasing of the fact that the monodromy around $z = 0, 1$ does not spoil the single-valuedness of the four-point function. After we determine the trans-series parameters in some region, they are automatically fixed in all regions, since they must change according to Stokes jumps, keeping $\mathcal{A}$ continuous.

We determine the matrix C , by convention, for $z \in E_0$ which includes the interval $(0, 1)$. We impose

$$\bar{V}M_{z=0}^\dagger CM_{z=0}V = \bar{V}M_{z=1}^\dagger CM_{z=1}V = \bar{V}CV, \quad (3.3.40)$$

where $M_{z=0}$ and $M_{z=1}$ are the monodromy matrices (3.3.34) and (3.3.32). We get as solution

$$c_1 = -\frac{1}{2 \sin(\frac{\pi}{\epsilon})}, \quad c_2 = 1. \quad (3.3.41)$$

We fix the factor $g(\epsilon)$ by demanding the correct normalization of the identity block. This requires to consider values of $\epsilon > 1$, in which case the identity becomes the leading contribution in the correlator. We then have

$$\mathcal{A} \approx z^{\frac{3}{\epsilon} - \frac{5}{2}}, \quad \epsilon > 1. \quad (3.3.42)$$

Since $z = 0$ is a regular point of $\mathcal{F}_{1,1}^B$ and the perturbative series trivializes to $1 + \mathcal{O}(z)$ at small z , we can quickly derive $\mathcal{F}_{1,1}^B \approx z^{\frac{3}{2\epsilon} - \frac{5}{4}}$ even for ϵ close to 1. For $\mathcal{G}$, however, the origin in z is a singular point so we must first resum the asymptotic series and then use

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{\epsilon}; 1-r\right) = r^{\frac{1}{\epsilon}-1} \left(\frac{1}{2} \csc\left(\frac{\pi}{\epsilon}\right) + \mathcal{O}(r) \right) + \left(-\frac{\pi \csc(\frac{\pi}{\epsilon}) \Gamma(\frac{1}{\epsilon})}{\Gamma(2 - \frac{1}{\epsilon}) \Gamma(\frac{1}{\epsilon} - \frac{5}{6}) \Gamma(\frac{1}{\epsilon} - \frac{1}{6})} + \mathcal{O}(r) \right). \quad (3.3.43)$$

While the identity block dominates for $\epsilon > 1$, it is not sufficient to look at $\mathcal{F}_{1,1}^B$ since, as shown in (3.3.25), $\mathcal{G}$ contains a contribution to the $|\mathcal{F}_{1,1}|^2$ part of the four-point function. Using the $z \rightarrow 0$ limits as given by (3.3.43) and matching with the behaviour (3.3.42) allows us to fix $g(\epsilon)$ and the whole correlator (3.3.36). We finally get

$$g(\epsilon) = \frac{4 \sin^2(\frac{\pi}{\epsilon})}{1 - 2 \cos(\frac{2\pi}{\epsilon})}, \quad C = \begin{pmatrix} 1 & -\frac{1}{2} \csc(\frac{\pi}{\epsilon}) \\ -\frac{1}{2} \csc(\frac{\pi}{\epsilon}) & 1 \end{pmatrix}, \quad \epsilon > 0. \quad (3.3.44)$$

It is now easy to check that the correlator given by (3.3.36) and (3.3.44) reproduce the known correlator (3.2.9) and (3.2.11) in the ordinary Virasoro block basis. Note how the less “physical”, but more “mathematical”, V -basis gives rise to simpler formulas, where crossing symmetry is manifest.

3.3.4 Negative central charge

The analysis in the previous sections assumed $\epsilon > 0$. We started with this case because there are regions in the z plane where the identity block is fully reconstructable from its asymptotic

series, which makes the analysis more conventional and easier to follow. Moreover, as we will see in Section 3.4.5, the reconstruction extends up to $c < 1$ and allows us to reach the unitary minimal models. However, since for $\epsilon > 0$ the conformal weight of the other exchanged highest weight state $h_{3,1} < 0$, the identity block is *not* the leading contribution in any OPE channel. Physically speaking, one might then legitimately question the motivation to start the analysis from the identity block. For this reason, it is important to discuss the $\epsilon < 0$ case, where $h_{3,1} > 0$ and the identity operator provides the dominant contribution. The above analysis valid for $\epsilon > 0$ readily applies for $\epsilon < 0$ (and in fact can be extended for any complex value of ϵ).

For $\epsilon < 0$, the resummation of the asymptotic series in terms of hypergeometric functions is given by (A.2.24). The difference with $\epsilon > 0$ is not unexpected, since we cross multiple Stokes phenomena, due to the multitude of singularities (A.2.6) that we encounter, as we move from the positive to the negative real t axis. As a consequence, the resummation of $\tilde{\mathcal{F}}_{1,1}$ no longer corresponds to $\mathcal{F}_{1,1}$ in E_0 , while the resummation of $\mathcal{G}$ does match $\mathcal{F}_{3,1}$, up to an overall factor. More precisely, we have

$$\begin{aligned}\mathcal{F}_{1,1}^B(z; \epsilon) &= \mathcal{F}_{1,1}(z; \epsilon) + \frac{2 \sin(\frac{\pi}{\epsilon})}{2 \cos(\frac{2\pi}{\epsilon}) - 1} k(\epsilon) \mathcal{F}_{3,1}(z; \epsilon), \\ \mathcal{G}^B(z; \epsilon) &= \frac{4 \sin(\frac{\pi}{\epsilon})^2}{1 - 2 \cos(\frac{2\pi}{\epsilon})} k(\epsilon) \mathcal{F}_{3,1}(z; \epsilon),\end{aligned}\quad \epsilon < 0, \quad z \in E_0, \quad (3.3.45)$$

where $k(\epsilon)$ is defined in (3.3.26). Although perhaps unfamiliar, it should not be surprising that the Borel resummation f^B of the asymptotic series $\tilde{f}$ of a function f does not reproduce the function f . Equivalence classes of functions which differ by non-perturbative terms give rise to the same asymptotic expansion and the Borel resummed function f^B picks up the unique representative in the equivalence class with given analyticity properties, see the discussion at the beginning of Appendix A.2. In our case, this representative is given by the specific linear combination reported in the right-hand side of the first relation in (3.3.45).¹¹ Inverting this relation, we see that the identity block has a trans-series representation even for $0 < z < 1$, while $\mathcal{F}_{3,1}$ can be obtained from the resummation of a series.

Like for $\epsilon > 0$, the relations (3.3.45) are subject to Stokes jumps as we move out of E_0 . The analysis is identical to the one we performed for $\epsilon > 0$, the only change being that the relevant axis in the Borel t plane is now the negative, and not the positive one. We need to avoid that the singularities t_k of (3.3.11) become negative and real, as discussed in Appendix A.2. This is at the origin of the different summability conditions with respect to the $\epsilon > 0$ case, i.e. $\tilde{\mathcal{F}}_{1,1}$ is non-Borel summable when $r < 0$ and $\tilde{\mathcal{G}}$ is non-Borel summable when $r > 1$. The jumps themselves remain the same, as shown in the same appendix. Consequently, we need to switch the rules for the Stokes lines, while the monodromies due to branch cuts B , B_0 and B_1 remain

¹¹The fact that $\mathcal{F}_{1,1}^B \neq \mathcal{F}_{1,1}$ for $\epsilon < 0$ is related to the c parameter of the ${}_2F_1(a, b, c; z)$ appearing in the identity block (3.2.9), which turns negative. A similar comment applies to the resummation of the block $\mathcal{F}_{3,1}$ for $\epsilon > 0$.

the same as for $\epsilon > 0$. We then have

$$S_+ = \begin{pmatrix} 1 & -ie^{\pm \frac{i\pi}{\epsilon}} \\ 0 & 1 \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & 0 \\ -ie^{\mp \frac{i\pi}{\epsilon}} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & ie^{\pm \frac{i\pi}{\epsilon}} \\ ie^{\mp \frac{i\pi}{\epsilon}} & 0 \end{pmatrix}, \quad \epsilon < 0. \quad (3.3.46)$$

Since the Stokes graph is unaltered, the products of connection matrices that construct the monodromy matrices are unchanged. Then (3.3.31) applies, even with the new Stokes matrices, and the monodromy matrices $M_{z=z_{\pm}}$ are trivial. Hence, forbidden singularities are resolved in the same way as in the $\epsilon > 0$ case. The monodromy matrices around $z = 0$ and $z = 1$ read

$$\begin{aligned} M_{z=0} &= (S_+)^{-1} B_0 (S_+)^{-1} = \begin{pmatrix} -ie^{\frac{3i\pi}{\epsilon}} & 1 + e^{\frac{2i\pi}{\epsilon}} \\ 0 & -ie^{-\frac{i\pi}{\epsilon}} \end{pmatrix}, \\ M_{z=1} &= (S_-)^{-1} B_1 (S_-)^{-1} = \begin{pmatrix} -ie^{-\frac{i\pi}{\epsilon}} & 0 \\ 1 + e^{\frac{2i\pi}{\epsilon}} & -ie^{\frac{3i\pi}{\epsilon}} \end{pmatrix}. \end{aligned} \quad (3.3.47)$$

Finally, we can determine the full correlator. As in the $\epsilon > 0$ case, we write the amplitude as

$$\mathcal{A} = g(\epsilon) \bar{V}_i C_{ij} V_j, \quad (3.3.48)$$

and determine the matrix C demanding trivial monodromy around $z = 0$ and $z = 1$. We get

$$C = \begin{pmatrix} 1 & \frac{1}{2} \csc\left(\frac{\pi}{\epsilon}\right) \\ \frac{1}{2} \csc\left(\frac{\pi}{\epsilon}\right) & 1 \end{pmatrix}, \quad \epsilon < 0. \quad (3.3.49)$$

The determination of $g(\epsilon)$ is trivial for $\epsilon < 0$. The identity block dominates at small ϵ and $\mathcal{G}$ corresponds only to $\mathcal{F}_{3,1}$, up to normalization. Since only $\mathcal{F}_{1,1}^B$ contributes to the identity block, we have

$$g(\epsilon) = 1, \quad \epsilon < 0. \quad (3.3.50)$$

One can check that (3.3.48), (3.3.49) and (3.3.50) reproduce the correct result (3.2.9), which holds for both signs of ϵ .

In Figure 3.5, we compare $\mathcal{F}_{1,1}$ with its semi-classical approximation

$$\mathcal{F}_{1,1}^{\text{sc}} = e^{-\frac{1}{\epsilon} S_1} A_1, \quad (3.3.51)$$

for $\epsilon > 0$ (left panel) and $\epsilon < 0$ (right panel).

For $\epsilon > 0$, $\mathcal{F}_{1,1} = \mathcal{F}_{1,1}^B$ for $z \in D_0$ and $\mathcal{F}_{1,1}^{\text{sc}}$ provides a good approximation. Observe that anti-Stokes lines in D_0 are “deactivated”, as there is no other saddle contribution. After we cross the Stokes lines and $z \notin D_0$, $\mathcal{F}_{1,1}$ becomes a trans-series but $\mathcal{F}_{1,1}^{\text{sc}}$ remains the leading contribution in a disc around $z = 1$ until we cross the anti-Stokes line, in which case $\tilde{\mathcal{G}}$ becomes the dominant contribution and the approximation breaks down.

For $\epsilon < 0$, $\mathcal{F}_{1,1}$ is always a combination of $\mathcal{F}_{1,1}^B$ and $\mathcal{G}^B$. We see that $\mathcal{F}_{1,1}^{\text{sc}}$ provides a good approximation when z is in the left half of region E_0 and beyond, in a disc around $z = 0$. However, as we cross anti-Stokes lines (both sides), $\mathcal{G}^B$ dominates and the semi-classical approximation breaks down. Interestingly enough, the rightmost Stokes line triggers a Stokes

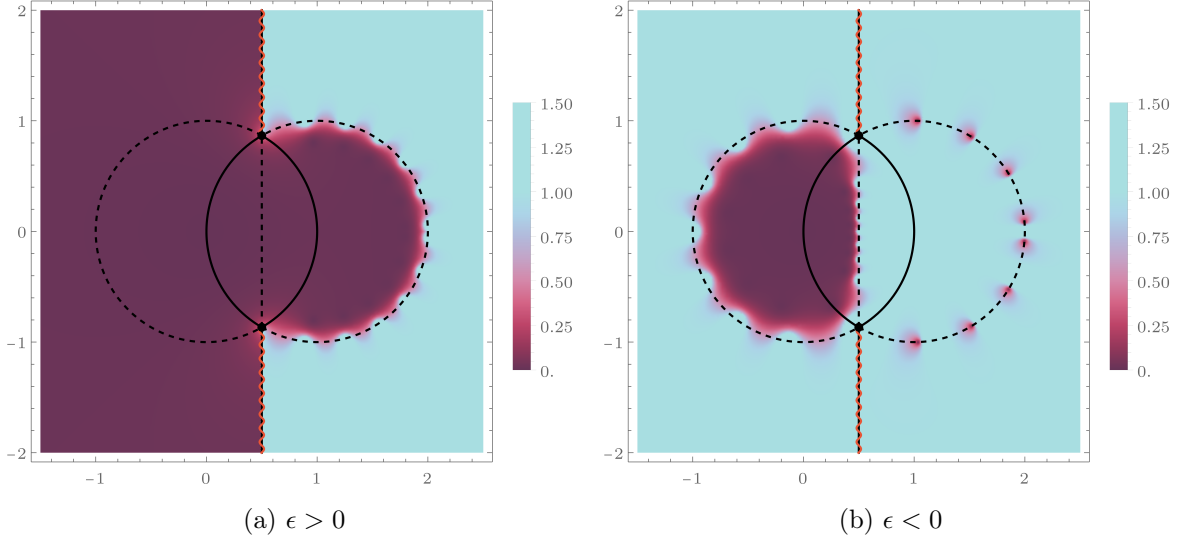


Figure 3.5: Density plots of $|1 - \mathcal{F}_{1,1}^{\text{sc}}/\mathcal{F}_{1,1}|$ as a function of z . The black lines are the Stokes (solid) and the anti-Stokes (dashed) lines. Wiggled lines are branch cuts of the semi-classical approximation emanating from $z = z_{\pm}$ (black dots). The density plot was generated with $\epsilon = \pm 0.10137$ (corresponding to central charge $c \simeq 82$ and $c \simeq -34$, respectively).

jump of $\tilde{\mathcal{G}}$ given by S_- in (3.3.46). As a consequence, when we cross the outer anti-Stokes line on the right, $\tilde{\mathcal{F}}_{1,1}$ becomes dominant again but the semi-classical approximation remains poor since the prefactor has changed.

3.4 2H2L correlator with $\phi_{2,1}$

Having established the analysis in the 4H case, the 2H2L case will follow a very similar path. We use the results of Appendix A.2 to derive the asymptotic expansion of the identity block, study in detail its Borel summability and Stokes phenomena, verify that the non-perturbative Stokes jump resolve forbidden singularities, and, lastly, fix the four-point function. As in the previous section, we find it useful to first start with $\epsilon > 0$ and then discuss the case $\epsilon < 0$.

By using the analysis of Appendix A.1.6, we can write the identity block in (3.2.12) in the form (3.3.1), but now with prefactors¹²

$$S_1(z) = -\frac{3}{2} \log(z), \quad A_1(z) = z^{-\frac{5}{4}}(1 - v(z))^\gamma, \quad (3.4.1)$$

where the map v is given by

$$v(z) = \frac{z^2}{(2 - z)^2}. \quad (3.4.2)$$

See Appendix A.1.6 for a detailed derivation of how the map $v(z)$ is found. The same $v(z)$ governs the form of the coefficients in the asymptotic series:

$$\tilde{F}_1 = f_0 + \sum_{n=1}^{+\infty} f_n(v) \epsilon^n \quad f_0 = 1, \quad f_n(v) = \sum_{k=1}^n (-1)^{n+k} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \frac{(\gamma)_k (\gamma + 1/2)_k}{k!} v^k, \quad (3.4.3)$$

¹²The exponential of S_1 is just a re-writing of the kinematical factor $z^{-2h_{2,1}}$ in (3.1.3).

which is obtained from (A.1.29) with the identification $a \rightarrow \gamma$, $b \rightarrow \gamma + \frac{1}{2}$, $z \rightarrow v(z)$. Like in the 4H case, it is remarkable that the coefficients f_n have a simple form when expressed in terms of the variable $v(z)$ in (3.4.2).

It is useful to match our results for S_1 and A_1 with the form of the semi-classical blocks for a generic 2H2L correlator found in [117, 123]. For two heavy operators with conformal weight $h_H = \eta c(1 + \mathcal{O}(c^{-1}))$ and two light operators with weight h_L , the block associated with identity exchange has the semi-classical limit

$$\lim_{c \rightarrow \infty} \mathcal{F}(h_i, 0, c; z) = z^{-2h_H} (1-w)^{h_L(1-\frac{1}{a})} \left(\frac{w}{az}\right)^{-2h_L}, \quad (3.4.4)$$

where

$$w = 1 - (1-z)^a, \quad a = \sqrt{1-24\eta}. \quad (3.4.5)$$

This block has forbidden singularities at

$$z_n = 1 - e^{\frac{2\pi i}{a}n}, \quad n \in \mathbb{Z}_{\neq 0}. \quad (3.4.6)$$

From (3.4.1) we have

$$\tilde{\mathcal{F}}_{1,1}(z; \epsilon) = e^{\frac{1}{4}c \log z} \left(\frac{4(1-z)}{(2-z)^2}\right)^{h_L} \left(1 + \mathcal{O}(c^{-1})\right), \quad (3.4.7)$$

where we use $\gamma = h_L + \mathcal{O}(c^{-1})$ and have rewritten the expression in terms of the central charge c . The expression (3.4.7) matches (3.4.4) with $\eta = -1/8$ and $a = 2$. For $a = 2$, we get from (3.4.6) a single forbidden singularity at $z = 2$, which is precisely the singular point of the map $v(z)$. Like in the 4H case discussed before, in addition to the semi-classical block, every single order in perturbation theory is more and more singular at $z = 2$. Unlike the map $r(z)$, $v(z)$ does not have branch cuts. The singular behaviour of perturbation theory had already been observed in [133], where it was conjectured a form for the f_n as a polynomial in z divided by a power of $(z(2-z))^2$. We see in (3.4.3) that the f_n are polynomials in v .

3.4.1 Borel summation and discontinuity

The Borel transform of (3.4.3) is given by

$$\hat{F}_1(v; t) = \gamma(\gamma + \frac{1}{2}) v {}_2F_1(\gamma + 1, \gamma + \frac{3}{2}; 2; v(1 - e^{-t})), \quad (3.4.8)$$

with z -dependent singularities given by

$$t_k(v) = \log\left(\frac{v}{v-1}\right) + 2\pi i k, \quad k \in \mathbb{Z}, \quad (3.4.9)$$

which become real and positive for $v > 1$. We can read from the Borel transform the large order behaviour, which, for example, for $0 < z < 1$ is given by

$$f_n \approx \frac{\Gamma(n + 2\gamma - \frac{1}{2})}{\Gamma(\gamma)\Gamma(\gamma + \frac{1}{2})} \frac{v}{(1-v)^{2\gamma + \frac{1}{2}}} \left| \log \frac{v}{v-1} \right|^{\frac{1}{2} - 2\gamma - n} 2 \sin\left((n + 2\gamma - \frac{1}{2}) \arg(\log \frac{v}{v-1}) - 2\pi\gamma\right), \quad (3.4.10)$$

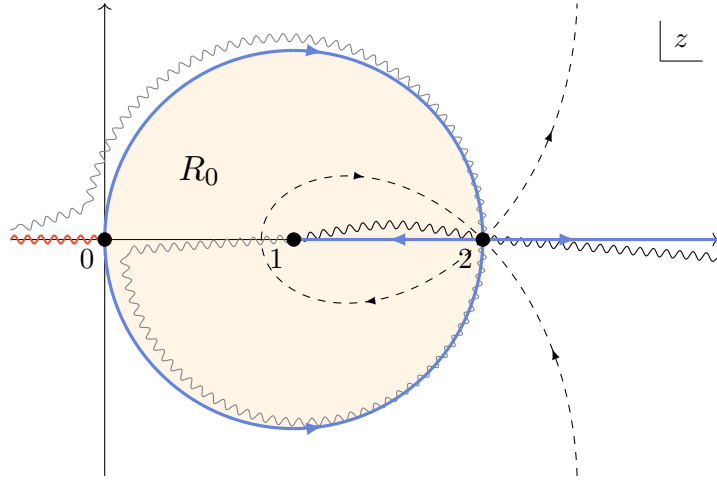


Figure 3.6: Branch cuts (wiggly lines), Stokes lines (blue) and anti-Stokes lines (dashed) in the z -plane for the 2H2L asymptotic series $(\tilde{\mathcal{F}}_{1,1}, \tilde{\mathcal{G}})$. Starting from $z = 2$, outgoing arrows denote lines where $\tilde{\mathcal{F}}_{1,1}$ has a Stokes discontinuity (and S_2 has a branch cut), while ongoing arrows denote lines where $\tilde{\mathcal{G}}$ jumps. The wiggled black and grey lines respectively denote branch cuts $A_{1,2}(z)$, while the red one denotes an additional branch cut from $S_{1,2}(z)$. Refined version of Figure 3.2.

as follows from (A.2.29).

We see that $z(v)$ maps $v \in (1, +\infty)$ to two intervals, $(1, 2)$ and $(2, +\infty)$. Outside of these intervals, $\tilde{\mathcal{F}}_{1,1}$ is Borel summable and (3.3.10) also holds for $\epsilon > 0$. We then have

$$\mathcal{F}_{1,1}^B = \mathcal{F}_{1,1}, \quad z \notin (1, +\infty). \quad (3.4.11)$$

When $z \in (1, +\infty)$, there is a Stokes jump because $t_0(v)$ lies in the positive real axis. Thus the jump is non-perturbative in $1/c$, again with the form

$$(s_{0+} - s_{0-})\tilde{\mathcal{F}}_{1,1}(z; \epsilon) \propto e^{-\frac{1}{\epsilon}t_0(v)} = \left(\frac{v}{v-1}\right)^{-\frac{1}{\epsilon}}. \quad (3.4.12)$$

Its full expression can be derived using the results of Appendix A.2, which give

$$(s_{0+} - s_{0-})\tilde{\mathcal{F}}_{1,1}(z; \epsilon) = i\sqrt{2}e^{i\pi\gamma \pm \frac{i\pi}{\epsilon}} \sqrt{\sin(2\pi\gamma)} s_0(\tilde{\mathcal{G}}(z + i0; \epsilon)), \quad z \in (1, +\infty), \quad (3.4.13)$$

where $\pm$ in the phase is taken depending on whether $z \leq 2$. Unlike the 4H case, the series $\tilde{\mathcal{G}}$ is related to the genuinely new series $\tilde{F}_2$, described in detail in Appendix A.2. The prefactors are given by

$$\begin{aligned} \tilde{\mathcal{G}}(\epsilon; z) &= e^{\pm(i\pi\gamma + \frac{i\pi}{\epsilon})} e^{-\frac{1}{\epsilon}S_1(z)} A_1(z) \tilde{F}_2(\epsilon; v(z)) \approx e^{-\frac{1}{\epsilon}S_2(z)} \epsilon^{\frac{1}{2}-2\gamma} A_2(z) (1 + \mathcal{O}(\epsilon)), \\ S_2(z) &= S_1(z) + \log\left(\frac{v}{1-v}\right), \quad A_2(z) = z^{-5/4} \frac{v}{(v-1)^{\gamma+\frac{1}{2}}} 4^\gamma \sqrt{\frac{\Gamma(1-2\gamma)}{2\Gamma(2\gamma)}}, \end{aligned} \quad (3.4.14)$$

where we choose $\pm$ in the phase according to whether z is in the upper/lower half plane. After combining with the prefactors in $\tilde{F}_2$ this ambiguity is resolved, as can be seen in the second line.

This is analogous to the point discussed in (3.3.14). $\tilde{\mathcal{G}}$ has Stokes jumps when $v \in (-\infty, 0)$, which in the z -plane maps to two half unit circles centred around $z = 1$, as in Figure 3.6. Then the maximal continuation of $(0, 1)$ until a Stokes line of either $\tilde{\mathcal{F}}_{1,1}$ or $\tilde{\mathcal{G}}$ is the region

$$R_0 = \{ z \in \mathbb{C} : |z - 1| < 1 \wedge z \notin (1, 2) \}. \quad (3.4.15)$$

Repeating the exercise of extracting the conformal weight by taking $z \rightarrow 0$, we discover once again that $\tilde{\mathcal{G}}$ is the perturbative part of the $(3, 1)$ block. However, $F_2^B(\epsilon; v)$ is a sum of both solutions of the hypergeometric equation, as discussed in Appendix A.2, meaning that $\mathcal{G}^B$ is a linear combination of both blocks, much like in the 4H case. We find

$$\begin{aligned} \mathcal{F}_{1,1}^B &= \mathcal{F}_{1,1}, \\ \mathcal{G}^B &= -\frac{ie^{-i\pi\gamma}}{\sqrt{\sin(2\pi\gamma)}} \left(h(\epsilon)\mathcal{F}_{3,1} + \frac{1}{\sqrt{2}} \frac{\sin(2\pi\gamma)}{\sin(\frac{\pi}{\epsilon})} \mathcal{F}_{1,1} \right), \quad \epsilon > 0, \quad z \in R_0, \end{aligned} \quad (3.4.16)$$

where

$$h(\epsilon) = \frac{16^{\frac{1}{\epsilon}-1} \Gamma(\frac{1}{\epsilon} - 1) \Gamma(\frac{1}{\epsilon})}{\sqrt{2} \Gamma(2\gamma) \Gamma(-2\gamma + \frac{2}{\epsilon} - 1)}. \quad (3.4.17)$$

$\tilde{\mathcal{G}}$ becomes non-Borel summable when $v < 0$, and its Stokes jump is given by $\tilde{\mathcal{F}}_{1,1}$ up to some overall constants, as we discuss in detail in the next section.

3.4.2 Stokes lines and monodromy matrices

It is convenient to present the Stokes jumps using a graphical notation for Stokes lines similar to the 4H case. This case requires some additional care since $S_2(z)$ has a branch along the Stokes line of $\tilde{\mathcal{F}}_{1,1}$, while $A_1(z)$ and $A_2(z)$ have branch cuts which overlap with both the real axis and the Stokes lines. While the former is easy to incorporate in the Stokes jump matrices, for the latter we place them as in Figure 3.6.¹³ Applying the inverse map $z(v)$ to $(1, \infty)$ and $(-\infty, 0)$ results in Figure 3.6. In the convention of Figure 3.4, the Stokes jumps in Figure 3.6 are

$$S_+ = \begin{pmatrix} 1 & 0 \\ -e^{\frac{i\pi}{\epsilon} - i\pi\gamma} \sqrt{2\sin(2\pi\gamma)} & e^{\frac{2i\pi}{\epsilon}} \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & ie^{i\pi\gamma - \frac{i\pi}{\epsilon}} \sqrt{2\sin(2\pi\gamma)} \\ 0 & e^{-\frac{2i\pi}{\epsilon}} \end{pmatrix}, \quad (3.4.19)$$

which can be derived from the Stokes jump of $\tilde{F}_{1,2}$ discussed in Appendix A.2. There are two types of branch cuts emanating from $z = 2$, those of $A_1(z)$ (drawn in black in Figure 3.6) and those of $A_2(z)$ (drawn in gray). They are given by, respectively,

$$B_1 = \begin{pmatrix} e^{-2i\pi\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2i\pi\gamma} \end{pmatrix}, \quad (3.4.20)$$

¹³This is equivalent to redefining the one-loop factors with a small angle α such that

$$A_1(z) \propto z^{-5/4} e^{i\pi\alpha\gamma} \left(e^{-i\pi\alpha(1-v(z))} \right)^\gamma, \quad A_2(z) \propto \frac{z^{-5/4} v(z)}{e^{i\pi\alpha(\gamma+\frac{1}{2})} (e^{-i\pi\alpha(v(z)-1)})^{\gamma+\frac{1}{2}}}. \quad (3.4.18)$$

when crossing the branch cut anti-clockwise with respect to $z = 2$. Lastly there is a branch cut at $(-\infty, 0)$ due to the $\log(z)$ factor in the origin,

$$B_3 = -ie^{\frac{3i\pi}{\epsilon}} I. \quad (3.4.21)$$

In contrast to the singular points $z_{\pm}$ in the 4H correlator, where three Stokes lines meet, at $z = 2$ four Stokes lines meet. The point $z = 2$ corresponds to a double turning point, where two simple turning points merge. This is evident from the analysis in Appendix A.1.2, where one can see that $\sqrt{Q_0}$ in (A.1.10) has a double zero at $z = 2$ for the values $a_1 = 0$, $b_1 = 1$, $c_1 = 2$. As is known in exact WKB, $q + 2$ Stokes lines end at a turning point of order q .

By taking care of all branch cuts and Stokes lines, we determine the total monodromy in making a 2π rotation around $z = 2$. We get

$$M_{z=2} = B_2 S_+ B_1 S_- B_2 S_+ B_1 S_- = I, \quad (3.4.22)$$

as it should be for a regular point, and importantly establishes how non-perturbative effects resolve the forbidden singularity at $z = 2$. The monodromies around $z = 0, 1$, starting from $z \in (0, 1)$, are

$$\begin{aligned} M_{z=0} &= S_+ B_3^{-1} S_+ = \begin{pmatrix} ie^{-\frac{3i\pi}{\epsilon}} & 0 \\ -2ie^{-\frac{i\pi}{\epsilon} - i\pi\gamma} \sqrt{2\sin(2\pi\gamma)} \cos\left(\frac{\pi}{\epsilon}\right) & ie^{\frac{i\pi}{\epsilon}} \end{pmatrix}, \\ M_{z=1} &= B_1 S_- B_2 = \begin{pmatrix} e^{-2i\pi\gamma} & -ie^{i\pi\gamma - \frac{i\pi}{\epsilon}} \sqrt{2\sin(2\pi\gamma)} \\ 0 & -e^{-\frac{2i\pi}{\epsilon} + 2i\pi\gamma} \end{pmatrix}. \end{aligned} \quad (3.4.23)$$

It can be checked that, if we replace $\tilde{\mathcal{F}}_{1,1}$ and $\tilde{\mathcal{G}}$ by their resummed hypergeometric forms using (A.2.7) and (A.2.15), the matrix $M_{z=1}$ rederives the correct discontinuity of the exact block in (3.2.12).

3.4.3 Fixing the four-point function

Repeating the steps done in the 4H case, we can fix the four-point function by combining holomorphic and anti-holomorphic terms. We write the four-point function as in (3.3.36) and determine the matrix C_{ij} by demanding the single-valuedness of the correlator around $z = 0$ and $z = 1$. Using the monodromy matrices $M_{z=0,1}$ in (3.4.23), we get

$$C = \begin{pmatrix} 1 & -\frac{i}{\sqrt{2}} e^{i\pi\gamma} \sqrt{\sin(2\pi\gamma)} \sec\left(\pi\left(\frac{1}{\epsilon} - 2\gamma\right)\right) \\ \frac{i}{\sqrt{2}} e^{-i\pi\gamma} \sqrt{\sin(2\pi\gamma)} \sec\left(\pi\left(\frac{1}{\epsilon} - 2\gamma\right)\right) & \sin\left(\frac{\pi}{\epsilon}\right) \sec\left(\pi\left(\frac{1}{\epsilon} - 2\gamma\right)\right) \end{pmatrix}. \quad (3.4.24)$$

We determine the normalization factor $g(\epsilon)$ in (3.3.36) by demanding the correct normalization of the identity block exchange in the s -channel as $z \rightarrow 0$. As before, this requires to consider $\epsilon > 1$, in which case the identity becomes the leading contribution in the correlator. Proceeding as before, we then have

$$g(\epsilon) = 1 + \sin(2\pi\gamma) \csc\left(2\pi\left(\frac{1}{\epsilon} - \gamma\right)\right). \quad (3.4.25)$$

In region R_0 , this matches (3.2.12), rederiving the OPE coefficient therein. For the full complex plane one must adjust C whenever we cross a branch cut or Stokes line, as discussed in Section 3.3.

We emphasize that we reconstructed the entire correlator starting just from the asymptotic series of the identity block in the s -channel.

3.4.4 Negative central charge

When $\epsilon < 0$ the conformal weight $h_{3,1}$ becomes positive and the identity operator provides the dominant contribution in the s -channel OPE. The situation is analogous to the 4H case, so we will be brief. The identity block is now a trans-series while the $(3,1)$ block is, up to z -independent factors, given by resummation of $\tilde{\mathcal{G}}$, as follows from (A.2.24). We have

$$\begin{aligned}\mathcal{F}_{1,1}^B(z; \epsilon) &= \mathcal{F}_{1,1}(z; \epsilon) + \frac{\sqrt{2} \sin(\frac{\pi}{\epsilon})}{\sin(2\pi(\gamma - \frac{1}{\epsilon}))} h(\epsilon) \mathcal{F}_{3,1}(z; \epsilon), \\ \mathcal{G}^B(z; \epsilon) &= -e^{-3i\pi\gamma} \left(\frac{2 \sin^2(\frac{\pi}{\epsilon})}{\sin(2\pi(\gamma - \frac{1}{\epsilon}))} \right) \frac{h(\epsilon)}{\sqrt{\sin(2\pi\gamma)}} \mathcal{F}_{3,1}(z; \epsilon),\end{aligned}\quad \epsilon < 0, \quad z \in R_0, \quad (3.4.26)$$

where $h(\epsilon)$ is defined in (3.4.17). Furthermore, the Stokes jumps change, becoming

$$S_+ = \begin{pmatrix} 1 & -ie^{3i\pi\gamma + \frac{i\pi}{\epsilon}} \sqrt{2 \sin(2\pi\gamma)} \\ 0 & e^{\frac{2i\pi}{\epsilon}} \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & 0 \\ -e^{-\frac{i\pi}{\epsilon} - 3i\pi\gamma} \sqrt{2 \sin(2\pi\gamma)} & e^{-\frac{2i\pi}{\epsilon}} \end{pmatrix}. \quad (3.4.27)$$

Here we have to be careful that not only the lines swap, but we also pick up additional phases due to (A.2.23).

As expected, (3.4.22) still applies and the monodromy matrix around $z = 2$ is again trivial. The forbidden singularity is resolved in the same way as in the $\epsilon > 0$ case. The monodromy matrices around $z = 0$ and $z = 1$ read

$$\begin{aligned}M_{z=0} &= \begin{pmatrix} ie^{-\frac{3i\pi}{\epsilon}} & 2e^{3i\pi\gamma} e^{-\frac{i\pi}{\epsilon}} \sqrt{2 \sin(2\pi\gamma)} \cos(\frac{\pi}{\epsilon}) \\ 0 & ie^{\frac{i\pi}{\epsilon}} \end{pmatrix}, \\ M_{z=1} &= \begin{pmatrix} e^{-2i\pi\gamma} & 0 \\ -e^{-3i\pi\gamma - \frac{i\pi}{\epsilon}} \sqrt{2 \sin(2\pi\gamma)} & -e^{2i\pi\gamma - \frac{2i\pi}{\epsilon}} \end{pmatrix}.\end{aligned}\quad (3.4.28)$$

We can fix the four-point function, as discussed in the previous cases, by starting from the ansatz (3.3.48). From the monodromy matrices (3.4.28) we can determine the matrix C entering (3.3.48). Like in the 4H correlator, for $\epsilon < 0$ the identity block dominates in the s -channel $z \rightarrow 0$ and this makes the determination of $g(\epsilon)$ trivial. Eventually we get

$$g(\epsilon) = 1, \quad C = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} e^{3i\pi\gamma} \sqrt{\sin(2\pi\gamma)} \csc(\frac{\pi}{\epsilon}) \\ \frac{1}{\sqrt{2}} e^{-3i\pi\gamma} \sqrt{\sin(2\pi\gamma)} \csc(\frac{\pi}{\epsilon}) & \csc(\frac{\pi}{\epsilon}) \cos(\pi(\frac{1}{\epsilon} - 2\gamma)) \end{pmatrix}, \quad \epsilon < 0. \quad (3.4.29)$$

We compare in Figure 3.7 $\mathcal{F}_{1,1}$ with its semi-classical approximation $\mathcal{F}_{1,1}^{\text{sc}}$ defined as in (3.3.51), for $\epsilon > 0$ (left panel) and $\epsilon < 0$ (right panel). For $\epsilon > 0$, $\mathcal{F}_{1,1} = \mathcal{F}_{1,1}^B$ for $z \notin (1, \infty)$ and $\mathcal{F}_{1,1}^{\text{sc}}$

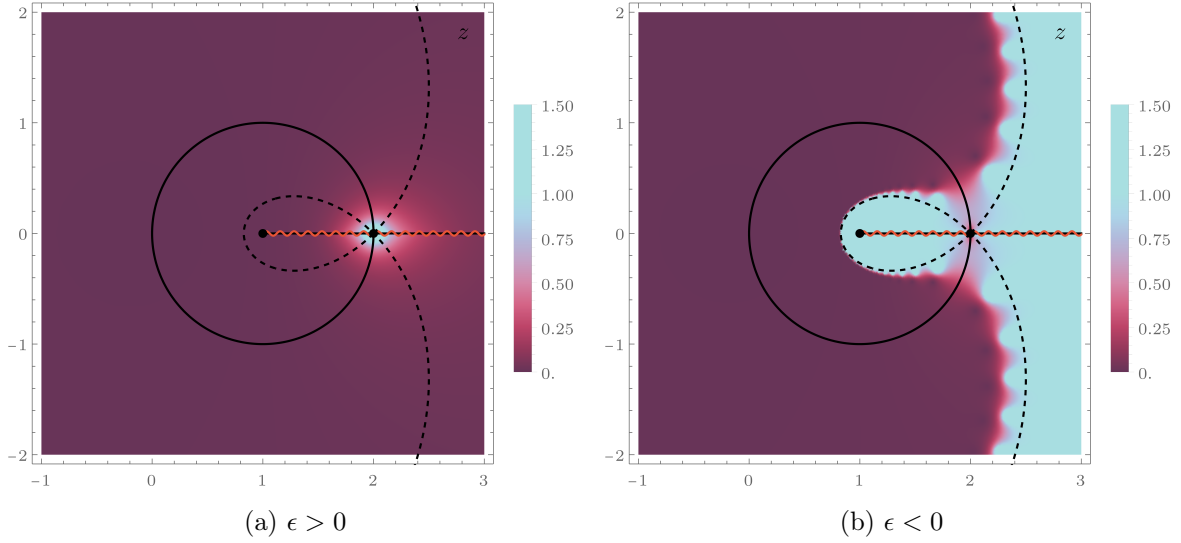


Figure 3.7: Density plots of $|1 - \mathcal{F}_{1,1}^{\text{sc}}/\mathcal{F}_{1,1}|$ as function of z . The black lines are the Stokes (solid) and anti-Stokes (dashed) lines. Wiggled lines are branch cuts of the semi-classical approximation emanating from $z = 1, 2$ (black dots). The density plot was generated with $\epsilon \simeq \pm 0.01447$ (corresponding to central charge $c \simeq 436$ and $c \simeq -393$, respectively).

provides a good approximation all over the complex plane, but a small region around the forbidden singularity. For $\epsilon < 0$, $\mathcal{F}_{1,1}$ is always a combination of $\mathcal{F}_{1,1}^B$ and $\mathcal{G}^B$. $\mathcal{F}_{1,1}^{\text{sc}}$ provides a good approximation in the purple region in panel (b) of Figure 3.7,¹⁴ until we cross anti-Stokes lines.

3.4.5 Unitary minimal models from a large c expansion

We have seen in the previous sections how we can determine the whole correlator of 4H and 2H2L starting from the asymptotic expansion of the identity block, both for $\epsilon > 0$ and $\epsilon < 0$. The result applies also for finite values of ϵ , and hence at finite central charge. In fact, starting from $\epsilon > 0$, we see that as ϵ increases, at some point b^2 turns negative and the conformal weights (3.2.4) turn positive. In particular, the values

$$\epsilon = \frac{2(p+1)}{p+3}, \quad p \geq 3, \quad (3.4.30)$$

correspond to the *unitary* minimal models $\mathcal{M}_{p+1,p}$ with

$$c = 1 - \frac{6}{p(p+1)}, \quad h_{r,s} = \frac{(pr - (p+1)s)^2 - 1}{4p(p+1)}. \quad (3.4.31)$$

Interestingly enough, the Borel summability of $\tilde{\mathcal{F}}_{1,1}$ extends all the way to these values. Consider for example the 4- σ correlator, with $h_\sigma = \bar{h}_\sigma = 1/16$, in the Ising model $p = 3$. The identity

¹⁴This purple region does not extend to infinity, but is enclosed by the anti-Stokes line which has a “limaçon” shape [158].

block exchanged in this correlator is known to be given by

$$\mathcal{F}_{1,1}^{\text{Ising}}(z) = \frac{1}{(z(1-z))^{\frac{1}{8}}} \sqrt{\frac{1+\sqrt{1-z}}{2}}. \quad (3.4.32)$$

We can match this formula in two ways. We can take the $\tilde{\mathcal{F}}_{1,1}$ series of the 4H case and resum it for $\epsilon = 4/3$ to get

$$\mathcal{F}_{1,1}^{\text{Ising}}(z) = \mathcal{F}_{1,1}^B\left(z; \epsilon = \frac{4}{3}\right), \quad z \in E_0 \quad (4\text{H}). \quad (3.4.33)$$

By tracking the Stokes jumps across the z -plane, the above formula can be extended to retrieve $\mathcal{F}_{1,1}^{\text{Ising}}(z)$ everywhere in terms of $\mathcal{F}_{1,1}^B$ and $\mathcal{G}^B$, using (3.3.27). Alternatively, we can consider the 2H2L correlator, where the light field has $\gamma = 3/8$, which corresponds to the conformal weight $h_L = 1/16$ at $c = 1/2$, and resum it for $\epsilon = 4/3$. Again we get

$$\mathcal{F}_{1,1}^{\text{Ising}}(z) = \mathcal{F}_{1,1}^B\left(z; \epsilon = \frac{4}{3}\right)_{\gamma=3/8}, \quad z \in R_0, \quad (2\text{H2L}). \quad (3.4.34)$$

As in the 4H approach, we can use our results to continue this formula into the full complex z -plane by incorporating the Stokes phenomena of 2H2L. For both 4H and 2H2L we can extend the outcome of the discussion about discovering $\mathcal{F}_{3,1}$, resumming its trans-series, and fixing the four-point correlator to the Ising model. The above analysis applies of course to correlators involving the fields $\phi_{2,1}$ in all $\mathcal{M}_{p+1,p}$ unitary minimal models.

From a 3d “dual” theory point of view, we are interpolating the 4- σ Ising correlator to a family of four-point correlators in holographic non-unitary CFTs, which would correspond to a pair of negative energy sources interacting with each other (4H) or to a small perturbation in the presence of a negative energy source (2H2L). Correlators of unitary models can then be seen as the “deep quantum gravity” regime of amplitudes in exotic 3d theories with negative energy sources.

3.5 Discussion

To conclude, we point out connections among our results and related topics. In particular, we highlight how to approach unitary theories. As a general remark, our analysis can be useful in the countless contexts involving large parameter expansions of ${}_2F_1$.

3.5.1 Related topics

AdS picture. In this paper we have discussed the systematics of the large c expansion of Virasoro blocks associated with the identity operator for a class of correlators involving degenerate operators. Due to the identification of the central charge with the 3d Newton constant through $c = 3\ell/2G_N$ [111], such an expansion is then mimicked by the small G_N ones in the semi-classical gravitational theory on AdS_3 . Depending on the nature of the external operators, e.g. how their conformal dimensions behave as c becomes large, we expect the

Virasoro block of the identity to encapsulate the contribution of the propagation of gravitons in empty AdS, in the defect/excess conical defect geometry or in the BTZ black hole background. For the first few orders in the large c expansion, this has been proven using the techniques of the geodesics Witten diagrams in [124, 130, 159–162] and while challenging to explicitly check, especially in the 4H case, we envision that our analysis can be a testing ground for further holographic explorations.¹⁵ In this context, it would be interesting to understand the role of the maps (3.3.5) and (3.4.2) in the holographic computations. An interesting observation is that the presence of forbidden singularities, reflected in the gravitational setup by singularities of the bulk to boundary propagators [125, 133, 163] is not exclusive to the 2H2L case but extends also to the 4H case, predicting that this should persist in the holographic counterpart and that they would be higher order singularities as probing larger and larger orders in the large c expansion. Our findings provide hints on characterizing the geometries dual to 2H2L and 4H correlators in the dual CFT. In particular, they enter in the set of saddle points that we need to sum over to reconstruct the gravitational partition function. This is generically a complicated task and we envision that our results can shed light on some aspects of it.

Chern-Simons description. Due to its natural split into a holomorphic and anti-holomorphic part, the $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ Chern-Simons description of AdS_3 gravity is particularly useful to directly compute Virasoro blocks in a $1/c$ expansion [164–166]. In particular, Wilson lines anchored at the boundary have been recognized to be the correct gauge-invariant observables. In this framework, one can construct Virasoro blocks in the $1/c$ expansion using Wilson lines networks [167] making order-by-order computations technically easier than AdS_3 Witten diagrams, provided one finds a good regularization prescription for the lines. As discussed in [167], this turns out to be quite a non-trivial task. It would be interesting to explore whether our all-order results in Section 3.4 might help determine such regularization procedure at higher order in $1/c$ for Wilson line correlators corresponding to a 2H2L configuration.

More general degenerate operators. The four-point function of the degenerate $\phi_{2,1}$ satisfies a differential equation which is hypergeometric, leading to the analytical results contained in this work. One might ask what can be concluded concerning resurgent structures for four-point functions of higher degenerates of the form $\phi_{r,1}$ appearing in different minimal models. Even though they satisfy ordinary differential equations of order r , we still expect an ansatz of the form (A.1.3) to be valid. In these cases, Stokes jumps are expected to mix multiple solutions of the underlying differential equation. This mixing pattern can be deduced from knowledge of the saddle actions $S(z)$ corresponding to each Virasoro block. Generalizing the discussion in Appendix A.1.3, it follows that $S'(z)$ solves a polynomial equation of order r , suggesting that the cases $r = 3, 4$ might be still treatable semi-analytically.

Higher dimensional CFTs. While the structure of CFTs defined in spacetime $d > 2$ is different from the $2d$ case, the explicit form of global conformal blocks still involves hypergeo-

¹⁵Even though the CFT that we consider is non-unitary, thus generically not suited for a straightforward holographic description, this identification should still hold.

metric functions in even spacetime dimensions. It would be interesting to see if the expansions that we found for generic hypergeometric functions are useful in some limiting cases, such as the large dimension expansion. A potential arena for such explorations could be the analytic bootstrap. In particular, the fact that the identity block, either global or Virasoro, is present when expanding a four-point function of (at least pairwise) identical operators in the s -channel lead to conclude that specific composite operators (dubbed double twist or double trace) need to be exchanged in the t -channel, and the correction to the conformal dimension of such operators for large spin is completely determined by the behaviour of the identity block, together with unitarity [67, 68, 168] (see [123, 125, 169] for studies in $d = 2$). There has been a plethora of results generalising, extending and complementing these findings (see for instance [23, 32]), and we envision that the manipulations of the hypergeometric functions that we present in this paper can be handy also in that context.

3.5.2 Unitary case: challenges and directions

Virasoro blocks associated with degenerate operators do not appear in unitary CFTs aside from minimal models. The most compelling question is whether there are general lessons from our analysis that extend to generic correlators and blocks in *unitary* $2d$ CFTs.

A partial list of key questions to answer are the following: does the Stokes phenomenon consistently occur as we move z over the complex plane? Is there a region in cross-ratio space (z) where the asymptotic series of the identity block reconstructs the full block or not? In other words, does the unitary case resemble the $\epsilon > 0$ or the $\epsilon < 0$ scenarios we discussed? Can we always resolve forbidden singularities with non-perturbative effects discovered through Stokes jumps? Can resurgence techniques provide predictions on the spectrum of unitary $2d$ CFTs at large central charge, such as in holographic CFTs? How would they compare with similar bounds found using the modular bootstrap [170]?¹⁶ Is our analysis generalizable to averages of product of correlators in CFT ensembles, e.g. those discussed in [172–174]?

The same Stokes phenomena which resolve the appearance of forbidden singularities are expected to play a role in the problem of information loss when we consider the correlator in Lorentzian signature. It would be extremely interesting to extend our study to this regime and possibly shed light on this issue. However, the analysis of the large-order behaviour in unitary cases is challenging. In [152], the authors analysed the leading asymptotic behaviour in the generic unitary case using Zamolodchikov’s recursion for Virasoro blocks with four light or four heavy operators in the regime $z \ll 1$. In these cases, the leading singularity in the Borel plane lies away from the positive real axis in this limit. With the insight from our analysis, we know that finite z is crucial to discern the non-perturbative effects.

A promising setup where we might be able to answer these questions is 2H2L unitary correlators. The Virasoro blocks entering such correlators have been shown to have an infinite number of forbidden singularities in the leading semi-classical limit, whose position is known analytically [123, 133], see (3.4.6). The behaviour of generic and unitary heavy-light Virasoro

¹⁶See [171] for the state-of-the-art bound.

blocks near forbidden singularities was investigated in the numerical work [136]. The exact block was computed using Zamolodchikov’s h -recursion at very high precision and compared with the semi-classical block at large central charge (an analogue of what we reported in Figure 3.5 and 3.7 for blocks involving degenerate operators). It was shown that the exact Virasoro block deviates drastically from its semi-classical approximation in a neighbourhood of the forbidden singularity, the former being regular there. It was speculated that this deviation could be explained by a web of Stokes and anti-Stokes lines emanating from the forbidden singularities (see also [137]).

A systematic $1/c$ expansion of heavy-light Virasoro blocks can be in principle obtained from a Zamolodchikov-like recursion relation using the semi-classical heavy-light block as the seed of the recursion [117, 118]. It then follows that we should be able to build upon the analysis of [136] to explore the resurgence structure of generic and unitary 2H2L correlators, by studying the large order behaviour of the asymptotic series of the Virasoro identity blocks and its Borel transform. A limitation of this approach is that we don’t have access to the full analytic form of each $1/c$ term, but only to a truncated power series in the cross-ratio z , thus requiring us to resort to numerical techniques. Nevertheless, preliminary investigations in this direction have shown evidence that forbidden singularities occur not only in the leading semi-classical term but in each order in the $1/c$ expansion. Moreover, by studying the large-order behaviour of the asymptotic series of the identity block, we have gathered numerical evidence that Stokes phenomenon occurs (at least in a neighbourhood of the forbidden singularity closest to $z = 0$). From a resurgence point of view, this would imply that the asymptotic series of the identity block cannot be Borel summable over the entire complex z -plane, and hence that non-perturbative terms, associated with heavy operator exchange, must be included in unitary heavy-light correlator. We aim to report and refine our findings in this direction in a separate publication.

Chapter 4

3d: conformal bootstrap and deconfined quantum criticality

This chapter applies the numerical conformal bootstrap to the CP^2 model, the simplest deconfined quantum critical point, and is organised as follows. Section 4.1 reviews deconfined quantum criticality, motivates the study of the CP^2 theory, and summarises our results. Section 4.2 introduces the CP^{N-1} model, its monopole operators and the large charge effective theory used to predict their scaling dimensions. Section 4.3 sets up the conformal bootstrap for the $U(1)$ sector and applies it to bound and determine the low-lying CFT data. The technical details are collected in four further sections of this chapter: the numerical bootstrap setup (Section 4.4), extended results for the CP^2 model (Section 4.5) and for the CP^3 and CP^4 models (Section 4.6), and a comparison of the bootstrap to the large charge expansion for the critical $O(2)$ model (Section 4.7). Section 4.8 discusses the results and future directions. This chapter is based on [2].

4.1 Introduction

Deconfined quantum critical points (DQCPs) describe the continuous phase transition between one phase that preserves a symmetry H but breaks another symmetry H' , and a second phase that preserves H' but breaks H [175, 176]. The phase transition is described by a conformal field theory with symmetry $H \times H'$. The simplest example of a DQCP is quantum electrodynamics in 2+1 dimensions (QED3) with $N > 1$ matter fields (either fermions or bosons), where $H = U(1)$ and $H' = SU(N)$, and the gauge fields are emergent at the critical point, hence deconfined. At large N , it can be shown that this theory flows to a CFT in the IR, and observables can be computed in a $1/N$ expansion [177, 178]. However, at finite N the theory is strongly coupled, which makes it challenging to compute observables or even determine if the theory flows to a CFT.

The simplest putative DQCP would have $N = 2$ bosons or fermions.¹ The conformal bootstrap was used in [179] to show that the $N = 2$ fermionic theory is likely not conformal.

¹By “simplest”, we mean the theory with the least degrees of freedom.

CP^2	Δ_1	Δ_2	Δ_3	Δ_4	Δ_0
Bootstrap	0.755*	1.841(1)	3.173(4)	4.65(9)	1.61(1)
Large N	0.755	1.81	3.10	4.59	–
Lattice [192]	0.71(4)	–	–	–	1.46(7)

Table 4.1: Comparison of scaling dimensions Δ_q of the lowest dimension scalar operators with $U(1)$ charge q in the CP^2 model, as determined from the bootstrap study here, the large N expansion for monopoles $q > 0$, and a lattice study for both monopoles and the lowest non-monopole singlet $q = 0$. The asterisk by Δ_1 for bootstrap means we put it in to determine the others, while the bootstrap errors come from extrapolation in the bootstrap truncation parameter Λ .

In particular, if the theory were conformal, then it would be expected to have an enhanced $O(4)$ symmetry [180–183], but the bootstrap puts bounds on the scaling dimension of the order parameter that exclude the lattice estimate [184] by a huge margin. Instead the theory has been recently argued to spontaneously break $SU(2) \times U(1) \rightarrow U(1)$ [185, 186]. The bootstrap similarly rules out the possibility that the $N = 2$ bosonic theory is critical [23, 179], i.e. that it is a CFT with one relevant $SU(2) \times U(1)$ singlet. The theory is believed to have an enhanced $SO(5)$ symmetry [187], but the lattice estimate for the order parameter is ruled out by the bootstrap if one assumes there is just one relevant $SU(2) \times U(1)$ singlet, i.e. no relevant $SO(5)$ singlets. Instead, the theory is believed to either be a tricritical CFT (i.e. two relevant singlets) [188, 189], or be weakly first order [190]. Since N must be even in the parity preserving fermionic case,² this leaves the $N = 3$ bosonic theory as the simplest possible DQCP with one relevant singlet.

QED3 with N bosons, also known as the CP^{N-1} model,³ has been extensively studied with lattice simulations, but there is no agreement on the values of scaling dimensions, or even for what value of N the theory flows to a CFT. A recent simulation of lattice Abelian-Higgs model suggested that the theory is conformal for $N = 10$ but not for $N = 4$ [191], which suggests it is also not conformal for $N = 3$. The theory is believed to be conformal for all $N \geq N_{\text{crit}}$, so if $N_{\text{crit}} = 4$, then $N = 3$ would not be conformal. A previous simulation of the Néel-VBS transition in quantum $SU(N)$ anti-ferromagnets, using the JQ model, suggested that the theory is conformal for $N = 3$ and higher values [192]. Another simulation with the JQ model claimed it was only conformal for $N = 4$ and higher [193].

In this chapter, we will show evidence that the CP^2 model flows to a critical point, and predict the scaling dimensions of low-lying operators. We will make use of the fact that the large N expansion of scaling dimensions Δ_q of the lowest dimension scalar operators with charge $q \in \mathbb{Z}$ under the $U(1)$ [194, 195], which are called monopole operators [196], has been shown to be extremely accurate for all values of N by comparison to lattice predictions for small values

²For odd N , one must have non-zero Chern-Simons coupling.

³This theory is also referred to as non-compact QED3 or $NCCP^{N-1}$ model in condensed matter literature, which refers to the fact that the theory has an explicit $U(1)$ symmetry that forbids monopoles from being added to the action.

Bootstrap	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$q = 1$	3.328(4)	3.21(1)	4.1(1)	5.42(3)
$q = 2$	4.83(3)	3.634(6)	5.64(9)	5.6(1)
$q = 3$	6.0(1)	4.893(4)	5.6(1)	6.82(2)
$q = 4$	–	6.1(2)	–	7.87(7)
$q = 0$	2	3.76(2)	4.328(2)	5.0(1)

Table 4.2: Scaling dimensions $\Delta_{q,\ell}$ of the lowest dimension operators with $U(1)$ charge $q \in \mathbb{Z}$ and spin $\ell > 0$ as determined from the bootstrap study here, with errors estimates coming from the extrapolation in the bootstrap truncation parameter Λ . Note that odd spin $q = 4$ operators exist, but do not appear in the correlators we bootstrap here. The $\ell = 1, q = 0$ operator is the conserved $U(1)$ current which has exact dimension $\Delta = 2$.

of N [193, 195, 197] and the duality to the critical $O(2)$ model for $N = 1$ [198–200]. This evidence is summarized in [188, Table II], except that the $N = 3$ value in Table II should be compared to [192] as shown in Table 4.1, since [193] did not find conformality for $N = 3$. Note that for $N = 2$, it was argued in [188] that the theory is a CFT with two relevant singlets, with bootstrap results that compare well to the large N expansion, as shown in Table I of that paper. Both the large N and bootstrap results were later confirmed by a lattice study [189]. For the duality with the $O(2)$ model, see instead [188, Table III].

We apply the numerical conformal bootstrap to $U(1)$ invariant CFTs in $3d$ whose only relevant operators are scalars with $q = 0, 1, 2$, as suggested for the CP^2 model from large N . This excludes the critical $O(2)$ model, which also has a relevant $q = 3$ operator [77].

The bootstrap rigorously bounds the space of allowed scaling dimensions of these operators, as well as the $q = 3, 4$ operators that appear in the operator product expansion of the $q = 0, 1, 2$ operators. Physical theories often appear approximately on the boundary of allowed regions. We emphasize that physical theories do not exactly appear on the boundary of an allowed region formed by bootstrapping a finite amount of correlators, because this would then imply that adding further correlators to the bootstrap could not change the bound even slightly. Nonetheless, physical theories are very near the boundary, as the CFT data read off from the approximate solution to crossing at the boundary matches independent estimates to good accuracy. This has been shown for many other models, such as the critical $O(N)$ models [27, 201, 202], $O(N) \times O(2)$ models [203], $N = 4$ fermionic QED3 [204, 205], $N = 2$ bosonic QED3 [188], bosonic QED3 for very large N [206], the 3-state Potts model [207, 208], the Gross-Neveu-Yukawa model [95, 96], and the $\mathcal{N} = 1$ Ising model [209–211].

In our case, by looking at the point on the boundary given by the large N value of Δ_1 and then minimizing the scaling dimension Δ_0 of the lowest singlet operator (which is not a monopole), we can read off the values of all the other scaling dimensions. As shown in Table 4.1, our results match the large N estimates of the scalar monopole scaling dimensions Δ_q for $q = 2, 3, 4$, as well as the lattice estimate in [192] for Δ_0 .

Large charge	$\ell = 0$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$q = 1$	0.750	2.482	3.199	3.912
$q = 2$	1.803	3.536	4.253	4.966
$q = 3$	3.093	4.825	5.543	6.256
$q = 4$	4.583	6.315	7.032	7.745

Table 4.3: Scaling dimensions $\Delta_{q,\ell}$ of the lowest dimension scalar operators with $U(1)$ charge $q \in \mathbb{Z}$ and spin ℓ as computed from the large q effective theory for general $\ell \neq 1$, with coefficients fixed by large N for $\ell = 0$. The red values for $\ell > q$ are where this small ℓ expansion is observed to be less accurate.

We also make predictions for the lowest spin ℓ monopole operators with scaling dimension $\Delta_{q,\ell}$ for $\ell \leq 4$, reported in Table 4.2. We can compare these values to the large charge q effective theory at small $\ell \neq 1$ [212, 213], whose coefficients were fixed using the large N expansion for $\ell = 0$ in [214]. The extrapolation is summarized in Table 4.3. We find that these values match our bootstrap predictions for $\ell \leq q$, which is presumably where this small ℓ expansion is accurate. Curiously, the exact same pattern is observed for the critical $O(2)$ model, where bootstrap results from [215] match the large q effective theory from [216] for $\ell \leq q$, as summarized in Section 4.7. The large charge expansion for the critical $O(2)$ model was previously compared against lattice predictions for $\ell = 0$ in [217].

4.2 The CP^{N-1} model

The Lagrangian of N complex scalar fields ϕ_i coupled to an Abelian gauge field A_μ in $3d$ is

$$\mathcal{L} = \sum_{i=1}^N (|(\partial_\mu - iA_\mu)\phi^i|^2 + m^2|\phi^i|^2) + u \left(\sum_{i=1}^N |\phi^i|^2 \right)^2 + \frac{F^2}{4e^2}, \quad (4.2.1)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. At large N , we can tune $m^2 = 0$ to get a critical theory in the IR. One can also tune both m^2 and u to zero to get a tricritical theory. We can then decouple the quartic term by introducing a Hubbard-Stratonovich field σ and replacing $u \left(\sum_{i=1}^N |\phi^i|^2 \right)^2$ by $\sigma \sum_{i=1}^N |\phi^i|^2 - \sigma^2/4u$, such that the functional integral over σ reproduces (4.2.1). Since the scaling dimension of σ is $\Delta_0 = 2$ at large N , in the IR we set $e, u \rightarrow \infty$ to get the conformally invariant action

$$\mathcal{L} = \sum_{i=1}^N (|(\partial_\mu - iA_\mu)\phi^i|^2 + \sigma|\phi^i|^2) \quad (4.2.2)$$

This is in the same universality class as the CP^{N-1} model, since shifting σ by one gives a non-linear sigma model with CP^{N-1} target space [218]. The theory has a $SU(N)$ flavor symmetry that rotates the ϕ_i , as well as a $U(1)$ topological symmetry whose current $\epsilon_{\mu\nu\rho} F^{\nu\rho}$ is conserved due to the Bianchi identity. Including discrete groups, the faithful global symmetry for $N > 2$ is $(SU(N)/\mathbb{Z}_N \times U(1)) \rtimes \mathbb{Z}_2^c$, where $\mathbb{Z}_2^c$ is charge conjugation and $\mathbb{Z}_N$ is the center of $SU(N)$.

We can construct local operators out of ϕ_i , A_μ , and σ that transform under $SU(N)$ but not $U(1)$. For instance the unique relevant singlet scalar operator is σ , whose scaling dimension is $\Delta_0 = 2$ at $N \rightarrow \infty$. The scaling dimension of these operators can be computed at large N using Feynman diagrams to get $\Delta \sim N^0$ and the sub-leading correction [177, 219–223], but this expansion is not very accurate for small N . The scaling dimensions were also computed using the $4 - \epsilon$ expansion [177, 224, 225], which is also not accurate for $\epsilon = 1$.

Monopole operators. We define monopole operators as local gauge invariant operators that transform under the $U(1)$ with charge $q \in \mathbb{Z}$. These operators are not built from fields in the action, but instead are defined as inserting magnetic flux $q = \frac{1}{2\pi} \int F$ [226].⁴ The lowest dimension monopoles are scalars and singlets under $SU(N)$.

Their scaling dimensions Δ_q are identified via the state-operator correspondence with the ground state energies on $S^2 \times \mathbb{R}$ with $2\pi q$ magnetic flux through the S^2 , which can be computed at large N using a saddle point expansion [194, 196] to get $\Delta_q \sim N$. This calculation was carried out to sub-leading order in [195],⁵ and the results were found to be extremely accurate even at small N . We list some values extrapolated to $N = 3$ in Table 4.1.

Large q effective theory. As shown in [212, 213], operators in any CFT with large charge q under a $U(1)$ global symmetry are described by an effective theory. This effective theory can be used to compute the scaling dimension $\Delta_{q,\ell}$ of the lowest dimension primary operator with charge q and spin $\ell \ll q^{\frac{1}{2}}$ as

$$\Delta_{q,\ell} = c_{\frac{3}{2}} q^{\frac{3}{2}} + c_{\frac{1}{2}} q^{\frac{1}{2}} - 0.0937 + \sqrt{\frac{\ell(\ell+1)}{2}} + O(q^{-\frac{1}{2}}) \quad (4.2.3)$$

except for $\ell = 1$, where the formula describes a conformal descendant (as can be seen from the fact that $\sqrt{\ell(\ell+1)/2} = 1$ for $\ell = 1$). Note that the q^0 terms are universal for any theory and any $\ell \ll q^{\frac{1}{2}}$. For the CP^{N-1} model, the lowest dimension monopoles are described by this effective theory. The large N results from [195], valid for $\ell = 0$ and finite q , were used in [214] to compute the theory-dependent coefficients $c_{\frac{3}{2}}$ and $c_{\frac{1}{2}}$ to sub-leading order in N (and verify the universal q^0 term), which for $N = 3$ give

$$c_{\frac{3}{2}} = 0.4983, \quad c_{\frac{1}{2}} = 0.3449. \quad (4.2.4)$$

We show the values of $\Delta_{q,\ell}$ from this expansion in Table 4.3, which for $\ell = 0$ match the finite q large N results in Table 4.1 to good accuracy.

4.3 Conformal bootstrap

We apply the numerical conformal bootstrap to just the $O(2) \cong U(1) \rtimes \mathbb{Z}_2^c$ sector of the theory, following the setup for $O(2)$ invariant systems in [77]. The generalities of the bootstrap algo-

⁴We normalize q as twice the value given in [188, 195].

⁵These values can be most conveniently extracted from [214], in particular equations 24 and Appendix A of that paper.

rithm are reviewed in Section 2.2, while additional details on this specific setup are postponed to Section 4.4. In this section we focus on the main ingredients and the strategy we used.

Four point functions of local scalar primary operators $\varphi_q(x)$ with charge q under a $U(1)$ global symmetry can be expanded in conformal blocks $g_{\Delta,\ell}^{\Delta_{12}^-, \Delta_{34}^-}(u, v)$ as

$$\begin{aligned} \langle \varphi_{q_1}(x_1) \varphi_{q_2}(x_2) \varphi_{q_3}(x_3) \varphi_{q_4}(x_4) \rangle &= \frac{x_{24}^{\Delta_{12}^-} x_{14}^{\Delta_{34}^-}}{x_{14}^{\Delta_{12}^-} x_{13}^{\Delta_{34}^-}} \\ &\times \frac{1}{x_{12}^{\Delta_{12}^+} x_{34}^{\Delta_{34}^+}} \sum_{\mathcal{O}} \lambda_{\varphi_{q_1} \varphi_{q_2} \mathcal{O}} \lambda_{\varphi_{q_3} \varphi_{q_4} \mathcal{O}} T_{\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \mathbf{r}_4}^{\mathbf{r}} g_{\Delta,\ell}^{\Delta_{12}^-, \Delta_{34}^-}(u, v), \end{aligned} \quad (4.3.1)$$

where $\Delta_{ij}^{\pm} \equiv \Delta_i \pm \Delta_j$, $x_{ij} \equiv x_i - x_j$, and we define the conformal cross ratios as u, v as in (2.2.6). The primary operators $\mathcal{O}$ that appear in both OPEs $\varphi_{q_1} \times \varphi_{q_2}$ and $\varphi_{q_3} \times \varphi_{q_4}$ have scaling dimension Δ , spin ℓ , and transform in an irreducible representation $\mathbf{r}$ of $O(2)$ that appears in both the tensor products $\mathbf{r}_1 \otimes \mathbf{r}_2$ and $\mathbf{r}_3 \otimes \mathbf{r}_4$. The global symmetry tensor structures are denoted by $T_{\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \mathbf{r}_4}^{\mathbf{r}}$, whose explicit expression can be found in [77]. Finally, $\lambda_{\varphi_i \varphi_j \mathcal{O}}$ are the OPE coefficients. In the following we denote by $\lambda_{ijk} \equiv \lambda_{\varphi_i \varphi_j \varphi_k}$ the OPE coefficients involving the “external” operators, whenever they are exchanged in the respective OPE. The LHS of (4.3.1) is invariant under swapping $\varphi_{q_1}(x_1)$ with $\varphi_{q_3}(x_3)$, but the block expansion on the RHS changes giving the crossing equations

$$\begin{aligned} 0 &= \sum_{\mathcal{O}} \lambda_{\varphi_{q_1} \varphi_{q_2} \mathcal{O}} \lambda_{\varphi_{q_3} \varphi_{q_4} \mathcal{O}} T_{\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \mathbf{r}_4}^{\mathbf{r}} v^{\frac{\Delta_{23}^+}{2}} g_{\Delta,\ell}^{\Delta_{12}^-, \Delta_{34}^-}(u, v) \\ &- \sum_{\mathcal{O}} \lambda_{\varphi_{q_3} \varphi_{q_2} \mathcal{O}} \lambda_{\varphi_{q_1} \varphi_{q_4} \mathcal{O}} T_{\mathbf{r}_3 \mathbf{r}_2 \mathbf{r}_1 \mathbf{r}_4}^{\mathbf{r}} u^{\frac{\Delta_{12}^+}{2}} g_{\Delta,\ell}^{\Delta_{32}^-, \Delta_{14}^-}(v, u). \end{aligned} \quad (4.3.2)$$

By projecting this equation into a basis for the tensor structures $T_{\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3 \mathbf{r}_4}^{\mathbf{r}}$, one can get a finite set of crossing equations, each a function of u, v . As we reviewed in Section 2.2, one can then truncate the infinite dimensional vector space of functions of u, v to the space of their derivatives at the crossing symmetric point up to a given order Λ . By imposing the reality of the OPE coefficients that follows from unitarity, one can then use this finite set of crossing equations to compute bounds on the allowed scaling dimensions and OPE coefficients that appear in (4.3.1), and these bounds monotonically shrink as Λ is increased. We use the **Navigator** algorithm to efficiently scan over the allowed region [78].

We first consider correlators of φ_0 and φ_1 for $\Lambda = 19$, which yields 7 crossing equations as given in [201]. We assume that these are the only relevant scalar operators with their $U(1)$ charges, and scan over the OPE coefficient ratio $\lambda_{000}/\lambda_{110}$ to further shrink the bound. The allowed region in the space of (Δ_0, Δ_1) is shown in Figure 4.1, which generalizes Figure 2 of [201] by extending to larger values of Δ_1 .⁶ Note that the critical $O(2)$ Wilson-Fisher model and the CP^{N-1} model for $N > 2$ (if critical) should appear in the allowed region, since they both have one relevant $q = 0, 1$ operators.⁷ In particular, while the critical $O(2)$ theory appears in a small

⁶In [201], Δ_1 and Δ_0 are denoted as Δ_ϕ and Δ_s , respectively.

⁷Recall that the CP^1 theory is believed to either have two $q = 0$ relevant operators or to have a weakly first order transition.

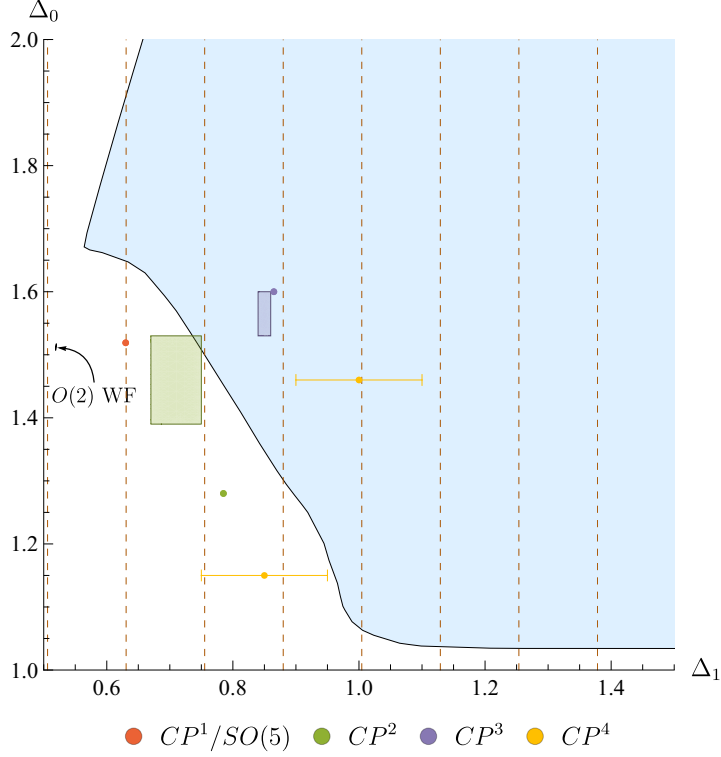


Figure 4.1: Allowed region for (Δ_1, Δ_0) in 3d CFTs with $O(2)$ global symmetry and only one relevant $q = 0, 1$ operators, computed with $\Lambda = 19$. These assumptions allow for the critical $O(2)$ Wilson-Fisher model, contained in the tiny island indicated by the arrow. They also allow for the CP^{N-1} model for $N > 2$. We show lattice results for $N = 3$ [192], for $N = 4$ [192, 193], and for $N = 5$ [227], where errors bars are shown where available, and otherwise are shown by colored dots. The dashed vertical lines correspond to the value for Δ_1 extrapolated from large N estimates [195] down to $N = 1, 2, \dots, 8$ from left to right. The red point denotes the scaling dimensions of the $SO(5)$ bootstrap [188] which is proposed to describe the tricritical $N = 2$ theory, and is ruled out by our assumption of just one relevant $U(1)$ singlet.

“island” as shown first in [201], the lattice and large N estimates for the CP^2 theory appear on the lower boundary of the “continent”, which motivates us to minimize Δ_0 to find this theory. Note that as Δ_1 gets bigger, the lower bound on Δ_0 goes to one, so with these assumptions the CP^{N-1} model for higher N cannot lie on this bound, as we expect $\Delta_0 \approx 2$ for large N . In Section 4.6, we discuss results for $N = 4, 5$, which show some agreement with the large N expansion for scalar monopole scaling dimensions.

We next consider correlators of φ_0, φ_1 , and φ_2 for $\Lambda = 19$, which yields 22 crossing equations as given in [77, 228]. We now assume that these are the only relevant operators, by imposing that all $q = 3, 4$ scalar operators are irrelevant. This excludes the critical $O(2)$ model, which has a relevant $q = 3$ operator [77]. The output of the bootstrap is an allowed region in the 6-dimensional space $(\Delta_0, \Delta_1, \Delta_2, \lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110})$. We look at the point on the boundary of this region given by inputting the value $\Delta_1 = 0.755$, as given by large N , and minimizing Δ_0 as suggested by the previous plot. In Figure 4.2 we look at a cross section of

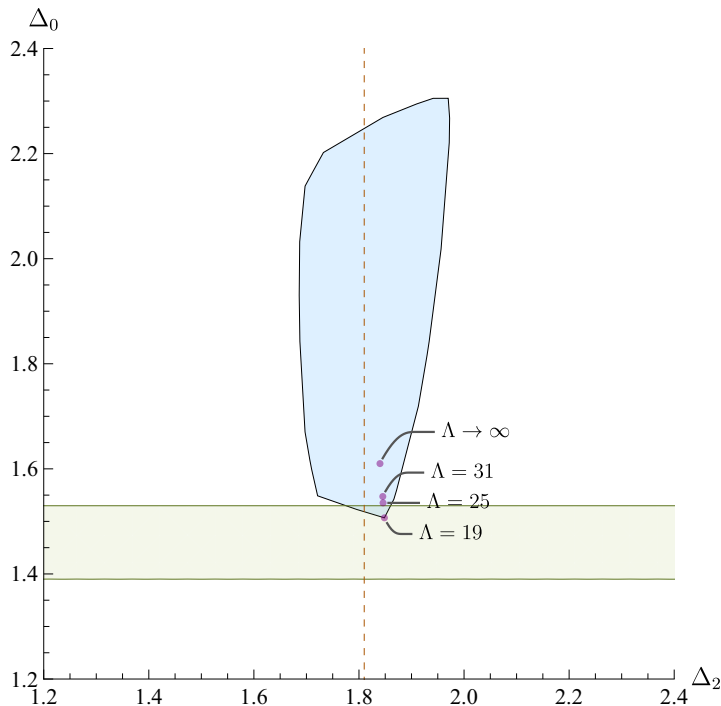


Figure 4.2: Allowed region for (Δ_2, Δ_0) in $3d$ CFTs with $O(2)$ symmetry and only one relevant $q = 0, 1, 2$ operators, with $\Delta_1 = 0.755$, and computed with $\Lambda = 19$. The green band is the lattice result for Δ_0 from [192], while the dashed vertical line is the large N value of Δ_2 extrapolated to $N = 3$. The purple dots are the result of Δ_0 minimization for some values of Λ , together with the $\Lambda \rightarrow \infty$ extrapolation.

this allowed region with $\Delta_1 = 0.755$ in the space of (Δ_0, Δ_2) , which shows that minimizing Δ_0 corresponds to a corner of this allowed region.

We can then read off all CFT data that appears in the approximate solution to crossing at the boundary of this allowed region. We report the values of the lowest dimension $q = 0, 2, 3, 4$ scalar operators in Table 4.1, and compare them to large N and lattice estimates. To obtain these values, we ran the bootstrap for $\Lambda = 19$ to $\Lambda = 31$ in steps of two, and then extrapolated to $\Lambda \rightarrow \infty$, so the error shown in our results comes from this extrapolation (see Section 4.5 for more details). We also report some of the lowest dimension operators with higher spin in Table 4.2. We do not report higher twist operators, since these are less accurate.

4.4 Numerical bootstrap details

We remind the reader that representations of $O(2) \cong U(1) \rtimes \mathbb{Z}_2$ can be induced from the 1 dimensional complex representations of $U(1)$, labeled by $q \in \mathbb{Z}$. The $q = 0$ representation induces the trivial representation $\mathbf{0}^+$ (singlet) and the sign representation $\mathbf{0}^-$ (odd under $\mathbb{Z}_2$), both of which are 1 dimensional. The $q > 0$ representation induces a 2 dimensional representation $\mathbf{q}$, isomorphic to $q \oplus (-q)$ as $U(1)$ representation and where $\mathbb{Z}_2$ acts by swapping the two subspaces. Our bootstrap setup involves all 4-point functions of scalar primary operators

in the $\mathbf{0}^+, \mathbf{1}, \mathbf{2}$ representations, which we denote by $\varphi_0, \varphi_1, \varphi_2$ respectively. Primary operators $\mathcal{O}_{\mathbf{r}, \Delta, \ell}$ exchanged in the conformal block decomposition (4.3.1) are in representations of $O(2)$ determined by the tensor product decomposition of the representations of the possible pairs of external operators. These are

$$\begin{aligned} \mathbf{0}^+ \otimes \mathbf{0}^+ &= \mathbf{0}^+, & \mathbf{1} \otimes \mathbf{1} &= \mathbf{0}^+ \oplus \mathbf{0}^- \oplus \mathbf{2}, & \mathbf{2} \otimes \mathbf{2} &= \mathbf{0}^+ \oplus \mathbf{0}^- \oplus \mathbf{4}, \\ \mathbf{0}^+ \otimes \mathbf{1} &= \mathbf{1}, & \mathbf{0}^+ \otimes \mathbf{2} &= \mathbf{2}, & \mathbf{1} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{3}. \end{aligned}$$

Moreover, permutation symmetry of 3-point function implies that in the $\varphi_0 \times \varphi_0$, $\varphi_1 \times \varphi_1$ and $\varphi_2 \times \varphi_2$ OPEs the primary operators $\mathcal{O}_{\mathbf{0}^+, \Delta, \ell}$, $\mathcal{O}_{\mathbf{2}, \Delta, \ell}$ and $\mathcal{O}_{\mathbf{4}, \Delta, \ell}$ are exchanged with even ℓ , while $\mathcal{O}_{\mathbf{0}^-, \Delta, \ell}$ are exchanged with odd ℓ . Instead, in the $\varphi_0 \times \varphi_1$, $\varphi_0 \times \varphi_2$ and $\varphi_1 \times \varphi_2$ OPEs the primary operators $\mathcal{O}_{\mathbf{1}, \Delta, \ell}$, $\mathcal{O}_{\mathbf{2}, \Delta, \ell}$ and $\mathcal{O}_{\mathbf{3}, \Delta, \ell}$ are exchanged with any ℓ . The “external” operators appear themselves in the OPEs considered: φ_0 appears in $\varphi_0 \times \varphi_0$, $\varphi_1 \times \varphi_1$ and $\varphi_2 \times \varphi_2$ with OPE coefficients λ_{000} , λ_{110} and λ_{220} respectively; φ_1 appears in $\varphi_0 \times \varphi_1$ with coefficient $\lambda_{101} = \lambda_{110}$; φ_2 appears in $\varphi_1 \times \varphi_1$ and $\varphi_0 \times \varphi_2$ with coefficients λ_{112} and $\lambda_{202} = \lambda_{220}$ respectively.

When sitting at points that saturate bootstrap bounds, i.e. when we minimize Δ_0 at fixed Δ_1 , it’s possible to obtain approximate solutions to the set of crossing equations using the Extremal Functional Method (EFM) [72, 229, 230]. The EFM gives a non-rigorous estimate of the low lying spectrum and OPE coefficients of operators exchanged in OPEs involved in the crossing equations. With our bootstrap setup we can thus access the following operators: $\mathcal{O}_{\mathbf{0}^+, \Delta, \ell}$ and $\mathcal{O}_{\mathbf{4}, \Delta, \ell}$ with ℓ even; $\mathcal{O}_{\mathbf{0}^-, \Delta, \ell}$ with ℓ odd; $\mathcal{O}_{\mathbf{1}, \Delta, \ell}$, $\mathcal{O}_{\mathbf{2}, \Delta, \ell}$ and $\mathcal{O}_{\mathbf{3}, \Delta, \ell}$ with any ℓ . The estimates from EFM are subject to a dependence on the dimension of the space of derivative functionals used in the bootstrap bound (parametrized by Λ) and typically we can obtain a reliable estimate only for the lowest twist operators. See Sections 4.5 and 4.6 for the Λ -dependence of our bootstrap estimates of scaling dimensions.

We used the software `autoboot` [228] to obtain the crossing equations (4.3.2) for correlators involving scalar operators φ_0, φ_1 , used for Figure 4.1, and for correlators involving $\varphi_0, \varphi_1, \varphi_2$, used for Figure 4.2 as well as for the `NavigatoR` computations. For the normalization convention of the OPE coefficients we refer to section 2 of the [228].

The numerical bootstrap computations of this chapter were carried out using the `simpleboot` package [231]. The computation of the conformal blocks was done using `scalar_blocks` [232], while the semi-definite optimization was done using `SDPB` [74]. The feasibility bounds for scaling dimensions in Figures 4.1 and 4.2 are obtained with a delaunay triangulation in scaling dimensions and by performing a scan over the OPE ratios of external operators using the cutting surface algorithm introduced in [77]. The constrained minimization of Δ_0 at fixed Δ_1 in the 5-dimensional space $(\Delta_0, \Delta_2, \lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110})$, reported in Tables 4.6, 4.8 and 4.10, was carried out with the `NavigatoR` method, using a modified BFGS algorithm introduced in [78]. Both the cutting surface and the BFGS algorithm are implemented in `simpleboot`.

The parameters used in the numerical computation of the conformal blocks are reported in Table 4.4 for the various values of the bootstrap parameter Λ we explored. There, S_Λ denotes

Λ	spin-ranges	poles	order	precision
19	S_{19}	14	56	1024
21	S_{21}	18	72	1024
23	S_{23}	18	72	1024
25	S_{25}	18	72	1024
27	S_{27}	20	80	1024
29	S_{29}	22	88	1024
31	S_{31}	24	96	1024

Table 4.4: Parameters of `scalar_blocks` used for the computation of the conformal blocks. See (4.4.1) for the definition of the spin ranges.

Parameter	Feasibility	Navigator
<code>dualityGapThreshold</code>	10^{-20}	10^{-30}
<code>primalErrorThreshold</code>	10^{-60}	10^{-30}
<code>dualErrorThreshold</code>	10^{-60}	10^{-30}
<code>initialMatrixScalePrimal</code>	10^{20}	10^{20}
<code>initialMatrixScaleDual</code>	10^{20}	10^{20}
<code>maxComplementarity</code>	10^{100}	10^{100}
<code>maxIterations</code>	1000	1000
<code>precision</code>	1024	1024
<code>detectPrimalFeasibleJump</code>	true	false
<code>detectDualFeasibleJump</code>	true	false
<code>findPrimalFeasible</code>	false	false
<code>findDualFeasible</code>	false	false

Table 4.5: Parameters of `SDPB` used for the semi-definite optimization.

the following sets of spins included in the bootstrap constraints:

$$\begin{aligned}
S_{19} &= \{0, \dots, 26\} \cup \{49, 52\}, \\
S_{23} &= \{0, \dots, 30\} \cup \{38, 49, 52\}, \\
S_{25} &= \{0, \dots, 30\} \cup \{38, 49, 52\}, \\
S_{27} &= \{0, \dots, 31\} \cup \{33, 34, 37, 38, 41, 42, 45, 46, 49, 50\}, \\
S_{29} &= \{0, \dots, 31\} \cup \{33, 34, 37, 38, 41, 42, 45, 46, 49, 50, 55, 56\}, \\
S_{31} &= \{0, \dots, 31\} \cup \{33, 34, 37, 38, 41, 42, 45, 46, 49, 50, 55, 56, 59, 60\}.
\end{aligned} \tag{4.4.1}$$

The parameters relevant for the semi-definite optimization are reported in Table 4.5, both for the feasibility bounds and the Navigator run. For the feasibility bounds, we assumed a bounding box on the OPE ratios for the cutting surface algorithm of $(\lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110}) \in [-10, 10] \times [-10, 10] \times [-10, 10]$.

For the extraction of the Δ_3 and Δ_4 scaling dimensions from the extremal functional, as well as for spinning operators, we used the `spectrum.py` package [233, 234].

CP^2	$\Lambda = 19$	$\Lambda = 21$	$\Lambda = 23$	$\Lambda = 25$	$\Lambda = 27$	$\Lambda = 29$	$\Lambda = 31$	$\Lambda \rightarrow \infty$
Δ_0	1.507	1.519	1.528	1.535	1.540	1.544	1.547	1.608
Δ_2	1.848	1.847	1.847	1.846	1.846	1.845	1.845	1.841
Δ_3	3.209	3.204	3.203	3.198	3.199	3.196	3.195	3.173
Δ_4	5.130	5.137	5.050	5.026	5.012	4.975	4.952	4.653
$\lambda_{000}/\lambda_{110}$	0.182	0.200	0.213	0.225	0.234	0.241	0.246	0.344
$\lambda_{220}/\lambda_{110}$	1.766	1.775	1.782	1.786	1.789	1.792	1.794	1.836
$\lambda_{112}/\lambda_{110}$	1.695	1.698	1.700	1.701	1.703	1.703	1.704	1.717

Table 4.6: Scaling dimensions and OPE ratios for the lowest dimensional scalars for $\Delta_1 = 0.755$ (corresponding to the CP^2 model) at various Λ values, together with the $\Lambda \rightarrow \infty$ fit. The values of Δ_0 are rigorous lower bounds. The values of Δ_2 and the OPE ratios correspond to the point on the boundary of the 5-dimensional allowed region $(\Delta_0, \Delta_2, \lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110})$ obtained by minimizing Δ_0 at fixed Δ_1 . The values of Δ_3 and Δ_4 are non-rigorous estimates from truncated crossing equations.

4.5 Extended results for the CP^2 model

In this section we describe in more detail the results obtained for the CP^2 model, which corresponds to studying the crossing equations (4.3.2) for $q = 0, 1, 2$ operators at $\Delta_1 = 0.755$, in accordance with the large N prediction. In Tables 4.1 and 4.2 we showed the estimates for scaling dimension of the lowest dimensional monopoles.

These estimates are obtained as follows. First we take the lowest dimension operator at multiple values of Λ , either as found using the EFM, or in the case of Δ_2 and the ratio of OPE coefficients, the values found through a rigorous minimization. Then we extrapolate our finite Λ results to infinite Λ assuming a linear fit in $1/\Lambda$. Linear convergence in $1/\Lambda$ has been previously observed in bootstrap studies for a wide range of observables (for sufficiently large Λ). Including a quadratic correction doesn't significantly improve the fit and does result in less physical behavior in the extrapolated regions. To illustrate the fitting procedure we show the fits used in extrapolation for the scalar monopole dimension in Figure 4.3. The data at finite values of Λ used for the extrapolations is also shown in Table 4.6. The full data used for all the presented extrapolations can be found in the auxiliary file `QED3_data.nb` of [2].

It should be noted that no rigorous errors can be assigned to the values extracted by the EFM. Instead we include a non-rigorous estimate of the error in the extrapolated value in the form of the variation of the extrapolated value when leaving up to 2 points out (this method of estimating the predictive value of a prediction based on finite set of points is known as *leave-p-out cross validation*, see for example [235, Ch. 7.11]).

For some operators in Table 4.2 we interpret not the lowest but the second lowest operator detected in the EFM to be the true physical lowest operator. This is motivated because these spectra contain an extra operator very close to the unitary bound (and thus the assumed gap) with a significantly smaller OPE then the following operator. Moreover, this difference becomes

larger as Λ increases. This spurious operator could be related to the *sharing effect* [215, 230]. Such an operator close to the unitary bound is also not expected to exist according to the large-charge expansion. The choice of fit and the alternative points that we could have chosen to fit are illustrated in Figure 4.4.

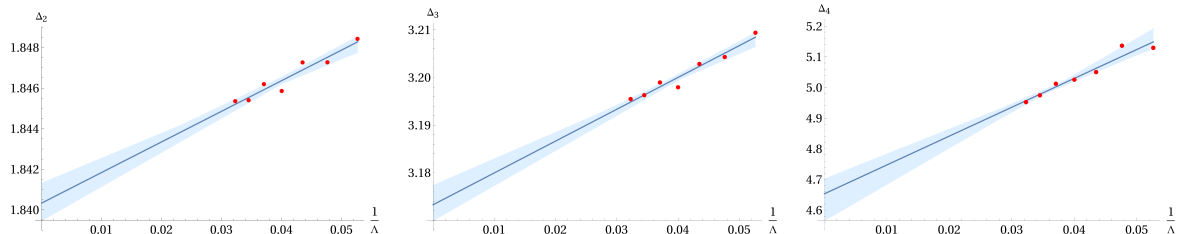


Figure 4.3: Linear least-squares fits in $1/\Lambda$ of the dimension of the lowest scalar monopoles in the CP^2 model. In the case of Δ_4 we ignore a “fake” operator at the imposed gap of 3. These fits are used to obtain the extrapolations at $\Lambda \rightarrow \infty$ shown in Table 4.1. A blue band indicate the range of fits that can be obtained by omitting up to 2 points. The red dots indicate the dimensions (obtained by EFM except for Δ_2 which is one of the variables in the search space).

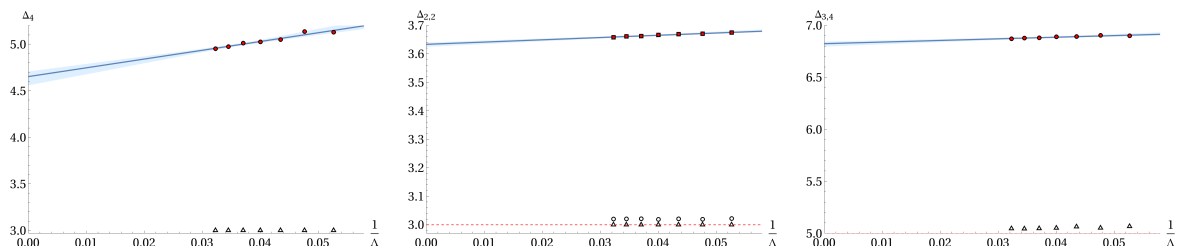


Figure 4.4: Linear least-squares fits in $1/\Lambda$ of the suspected physical lowest dimensional operators in the CP^2 model. These fits are used to obtain the extrapolations at $\Lambda \rightarrow \infty$ shown in Tables 4.1 and 4.2. The operator dimensions obtained by EFM are plotted as black triangles (lowest dimensional operator), circles (next-lowest) and squares (next-to-next-to-lowest). The red dots indicate the dimensions used for extrapolations presented in Tables 4.1 and 4.2. Rather than using the lowest dimensional operators from the EFM as we did for the other sectors, in these case we treat operators at the imposed gaps or close to it as *spurious*. This is supported by the fact that the OPE coefficients of these operators near the unitarity bound are significantly smaller than the OPE coefficients of the fitted operator and are getting smaller still as Λ increases.

4.6 Results for the CP^3 and CP^4 models.

In this appendix we describe the results obtained for CP^3 and CP^4 models by analogous bootstrap methods, i.e. by setting Δ_1 to 0.88 and 1.005 (as predicted by the large N prediction). In Tables 4.7 and 4.9 we show the prediction of the lowest operator dimensions for scalars with charge q in the CP^3 and CP^4 models, respectively, and compare these against large N and lattice estimates. The bootstrap estimate is obtained by extrapolating our finite Λ results to infinite Λ according to the linear fits shown in Figure 4.5 and 4.6, as described in Section 4.5. The data at finite values of Λ is included in Tables 4.8 and 4.10 for CP^3 and CP^4 , respectively.

CP^3	Δ_1	Δ_2	Δ_3	Δ_4	Δ_0
Bootstrap	0.880*	2.12(1)	3.65(4)	5.8(2)	1.43(2)
Large N	0.880	2.12	3.64	5.40	–
Lattice [192]	0.85(1)	–	–	–	1.57(4)
Lattice [193]	0.865	–	–	–	1.60

Table 4.7: Comparison of scaling dimensions Δ_q of the lowest dimension scalar operators with $U(1)$ charge q in the CP^3 model, as determined from the bootstrap study here, the large N expansion for monopoles $q > 0$, and lattice studies. The asterisk by Δ_1 for bootstrap means we put it in to determine the others, while the bootstrap errors come from extrapolation in the bootstrap truncation parameter Λ .

CP^3	$\Lambda = 19$	$\Lambda = 21$	$\Lambda = 23$	$\Lambda = 25$	$\Lambda = 27$	$\Lambda = 29$	$\Lambda \rightarrow \infty$
Δ_0	1.303	1.317	1.329	1.336	1.342	1.347	1.432
Δ_2	2.161	2.157	2.151	2.151	2.148	2.149	2.123
Δ_3	3.766	3.753	3.737	3.735	3.727	3.728	3.647
Δ_4	6.734	6.669	6.542	6.510	6.436	6.435	5.795
$\lambda_{000}/\lambda_{110}$	0.096	0.103	0.108	0.112	0.115	0.118	0.161
$\lambda_{220}/\lambda_{110}$	1.636	1.645	1.650	1.656	1.658	1.662	1.711
$\lambda_{112}/\lambda_{110}$	1.633	1.635	1.643	1.644	1.647	1.648	1.681

Table 4.8: Scaling dimensions and OPE ratios for the lowest dimensional scalars for $\Delta_1 = 0.88$ (corresponding to the CP^3 model) at various Λ values, together with the $\Lambda \rightarrow \infty$ fit. The values of Δ_0 are rigorous lower bounds. The values of Δ_2 and the OPE ratios correspond to the point on the boundary of the 5-dimensional allowed region $(\Delta_0, \Delta_2, \lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110})$ obtained by minimizing Δ_0 at fixed Δ_1 . The values of Δ_3 and Δ_4 are non-rigorous estimates from truncated crossing equations.

CP^4	Δ_1	Δ_2	Δ_3	Δ_4	Δ_0
Bootstrap	1.005*	2.58(2)	4.6(4)	3.5(4)	1.154(5)
Large N	1.005	2.43	4.18	6.21	–
Lattice rect. [227]	0.85(10)	–	–	–	1.15
Lattice honeyc. [227]	1.0(1)	–	–	–	1.46

Table 4.9: Comparison of scaling dimensions Δ_q of the lowest dimension scalar operators with $U(1)$ charge q in the CP^4 model, as determined from the bootstrap study here, the large N expansion for monopoles $q > 0$, and lattice studies (on rectangular and honeycomb lattices). The asterisk by Δ_1 for bootstrap means we put it in to determine the others, while the bootstrap errors come from extrapolation in the bootstrap truncation parameter Λ .

CP^4	$\Lambda = 19$	$\Lambda = 21$	$\Lambda = 23$	$\Lambda = 25$	$\Lambda = 27$	$\Lambda = 29$	$\Lambda = 31$	$\Lambda \rightarrow \infty$
Δ_0	1.063	1.071	1.080	1.086	1.091	1.094	1.098	1.154
Δ_2	2.534	2.552	2.539	2.547	2.545	2.554	2.553	2.576
Δ_3	–	3.331	–	4.575	4.558	4.588	4.569	4.589
Δ_4	–	4.129	10.348	3.787	3.746	3.758	3.731	3.534
$\lambda_{000}/\lambda_{110}$	–0.830	–0.918	–0.972	–1.051	–1.072	–1.131	–1.106	–1.615
$\lambda_{220}/\lambda_{110}$	1.567	1.561	1.560	1.557	1.563	1.568	1.569	1.571
$\lambda_{112}/\lambda_{110}$	2.892	2.979	3.003	3.065	3.058	3.099	3.065	3.396

Table 4.10: Scaling dimensions and OPE ratios for the lowest dimensional scalars for $\Delta_1 = 0.1005$ (corresponding to the CP^4 model) at various Λ values, together with the $\Lambda \rightarrow \infty$ fit. The values of Δ_0 are rigorous lower bounds. The values of Δ_2 and the OPE ratios correspond to the point on the boundary of the 5-dimensional allowed region ($\Delta_0, \Delta_2, \lambda_{000}/\lambda_{110}, \lambda_{220}/\lambda_{110}, \lambda_{112}/\lambda_{110}$) obtained by minimizing Δ_0 at fixed Δ_1 . The values of Δ_3 and Δ_4 are non-rigorous estimates from truncated crossing equations.

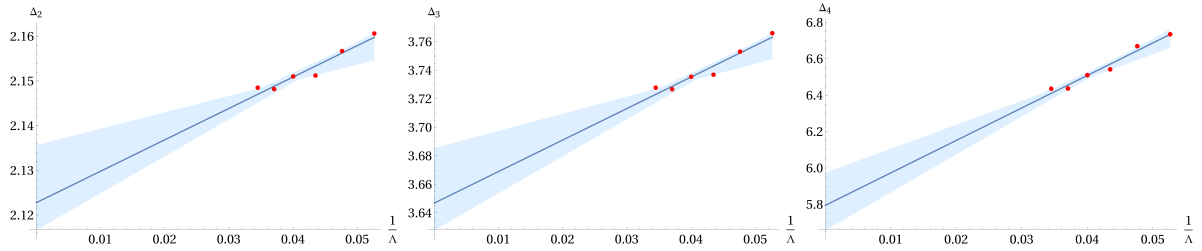


Figure 4.5: Linear least-squares fits in $1/\Lambda$ of the dimension of the lowest scalar monopoles in the CP^3 model (excluding “fake” operators at the imposed gap, when present). These fits are used to obtain the extrapolations at infinite Λ shown in Table 4.7. To give an estimate of the stability of the fit and its predictive accuracy a blue band indicate the range of fits that can be obtained by omitting up to 2 points. The red dots indicate the dimensions (obtained by EFM except for Δ_2 which is one of the variables in the search space).

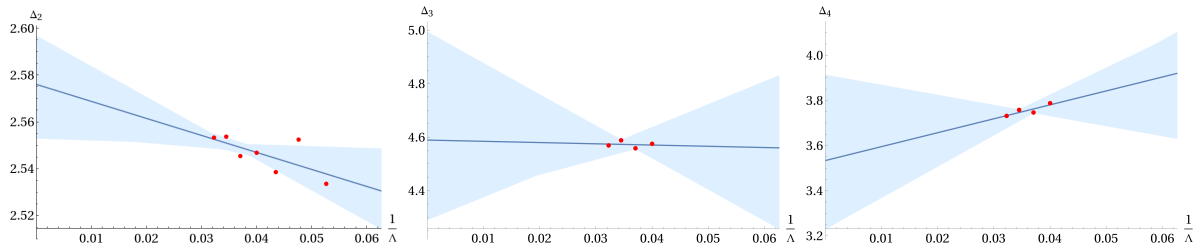


Figure 4.6: Linear least-squares fits in $1/\Lambda$ of the dimension of the lowest scalar monopoles in the CP^4 model (excluding “fake” operators at the imposed gap, when present). These fits are used to obtain the extrapolations at infinite Λ shown in Table 4.9. To give an estimate of the stability of the fit and its predictive accuracy a blue band indicate the range of fits that can be obtained by omitting up to 2 points. The red dots indicate the dimensions (obtained by EFM except for Δ_2 which is one of the variables in the search space). Some figures contain less points since some operators are not detected at for all values of Λ .

4.7 Comparison of bootstrap to large q for critical $O(2)$

In [216], the scaling dimensions of scalar operators in the critical $O(2)$ model were computed for $q \leq 16$ using lattice Monte Carlo. These values were fit to the general large q expansion in (4.2.3) for $\ell = 0$ to get the coefficients

$$c_{\frac{3}{2}} = 0.3371, \quad c_{\frac{1}{2}} = 0.2658. \quad (4.7.1)$$

We show the predictions from (4.2.3) with these value of the coefficients for $q = 1, 2, 3, 4$ and $\ell = 0, 2, 3, 4$ in Table 4.11. Note that just as for the CP^2 model, for some values of $\ell > q$ we observe violations of the unitarity bound, which suggests the expansion is not good for these small values of q .

The conformal bootstrap was used in [215] to compute scaling dimensions of operators in the critical $O(2)$, using the same set of correlators of $q = 0, 1, 2$ scalar operators considered in this chapter, except with assumptions on the spectrum suitable to the critical $O(2)$ model. In Table 4.12, we summarize the data from this bootstrap study for $q = 1, 2, 3, 4$ and $\ell \leq 4$, where recall that odd spins do not appear for $q = 4$ in the correlators considered in this bootstrap. Some of this data is known to more digits of precision than shown here, but for the purpose of the comparison to large q , we find that the 4 digits is more than enough. As one can see from comparing Tables 4.11 and 4.12, the data from each method matches for $\ell \leq q$.

Note that [215] did not give data for $q = 3$ odd ℓ or $q = \ell = 1$, so we reran those values ourselves.

Large charge	$\ell = 0$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$q = 1$	0.509	2.241	2.958	3.672
$q = 2$	1.236	2.968	3.685	4.399
$q = 3$	2.118	3.851	4.568	5.281
$q = 4$	3.135	4.867	5.584	6.297

Table 4.11: Scaling dimensions of the lowest dimension scalar operators in the critical $O(2)$ model as computed from the large q effective theory for general $\ell \neq 1$ with coefficients fixed by Monte Carlo for $\ell = 0$. The red values for $\ell > q$ are where this small ℓ expansion is observed to be less accurate.

Bootstrap	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$q = 1$	0.519	2.950	3.650	4.615	5.700
$q = 2$	1.236	5.800	3.015	5.766	5.030
$q = 3$	2.106	2.078	3.884	4.582	5.852
$q = 4$	3.115	–	4.893	–	6.730

Table 4.12: Scaling dimensions of the lowest dimension scalar operators in the critical $O(2)$ model as computed from the conformal bootstrap.

4.8 Discussion

In this chapter we bootstrapped the CP^2 model, which is the simplest DQCP. In particular, we showed that the point on the boundary of the allowed region of $U(1)$ CFTs in $3d$ given by inputting Δ_1 as computed from the large N expansion of the CP^{N-1} model extrapolated to $N = 3$, gives Δ_q for $q = 2, 3, 4$ that match their large N values, as well as Δ_0 that matches the lattice estimate from [192]. We also made predictions for $\Delta_{q,\ell}$ for $\ell > 0$, which match the large q effective theory as computed using the large N expansion in [214] for $\ell \leq q$.

It is striking that the large q expansion matches our bootstrap results for every $\ell \leq q$ that we computed, as strictly speaking this expansion should only be valid for $\ell \ll q^{\frac{1}{2}}$. A similar successful match was also observed for the critical $O(3)$ model in [236] for $\ell = 2$ and $q = 1, \dots, 6$. In Section 4.7, we compare the bootstrap results from [215] to the large q expansion for the critical $O(2)$ from [216], where in this case lattice simulations for $\ell = 0$ data were used to fix $c_{\frac{3}{2}}$ and $c_{\frac{1}{2}}$ in (4.2.3). We find the exact same pattern as observed here for CP^2 . This suggests the general large q effective theory works for a larger regime of ℓ than one might expect. Note that the large charge expansion for larger values of ℓ was worked out in [237], but these expansions do not match bootstrap data either for the critical $O(2)$ model or for our theory, which might be because these large q expansions are known to fewer orders than the small ℓ regime expansion considered here.

Looking ahead, we would like to access the $SU(3)$ sector of the CP^2 model, perhaps by performing a mixed correlator study between adjoints of $SU(3)$ as well as the monopoles considered in this chapter. Unfortunately, we found that the simplest bootstrap setup, which only assumes one relevant $q = 0, 1, 2$ and one adjoint, leads to weak bounds in the $SU(3)$ sector, which are far from the lattice estimates of lowest dimension adjoint operator in [192]. This is because in this setup, we have found the mixing between the adjoint correlator and the correlators in this chapter to be very weak. Indeed, it was known from earlier work that the $SU(N)$ adjoint bootstrap yields weak bounds for small N [206, 238]. It would be interesting to find well motivated physical assumptions that could overcome this problem.

It would also be nice to confirm our predictions for higher spin monopoles for general q by generalizing the large N and finite q calculation for lowest dimension monopoles in [194, 195] to non-lowest monopoles, as all the lowest monopoles in the CP^{N-1} theory have zero spin. This would require one to compute the energy of excited states on the cylinder. Such a calculation could also show evidence for the putative $SO(5)$ symmetry enhancement for the $N = 2$ theory [187–189].

We also expect that some of our predictions can be compared to lattice simulations in the near future. In particular, Δ_3 can be compared to the ω component of lattice simulations on the honeycomb lattice, while $\Delta_{2,2}$ could be compared to the ω component on the square lattice [239].

We would also like to generalize our study to larger N , even just in the $U(1)$ sector. Since the scaling dimensions of monopole operators grows linearly with N , it will require stronger constraints for CFTs with larger N to saturate the bound. Note that the only specification of N

into our $U(1)$ bootstrap was by inputting the value of Δ_1 as given from the large N expansion, so one could in principle bootstrap CP^{N-1} for larger N by inputting a larger value of Δ_1 . As noted above though, the lower bound on Δ_0 goes to one as Δ_1 increases, so the CP^{N-1} model cannot lie on this lower bound in that regime. Preliminary investigation suggests that increasing the gap above Δ_0 above three, which was what we assumed in this chapter, might address this problem, but we have no independent way of determining how large that gap should be.

Finally, it would be nice to bootstrap other $3d$ non-supersymmetric gauge theories,⁸ such as QED3 with non-zero Chern-Simons coupling, or QCD3. It would also be nice to upgrade the existing bounds for low N theories in this chapter and [188, 204, 205] into precise islands, so that the bootstrap can numerically solve these theories in the same sense that the critical $O(N)$ model was solved in [29, 30, 77, 201, 242].

⁸Precise bootstrap islands have been found for $3d$ supersymmetric gauge theories such as ABJM theory in [240, 241], but this required the additional input of supersymmetric localization constraints.

Chapter 5

$3d$ to $4d$: QFT in AdS and confinement

This chapter uses the numerical conformal bootstrap to study confinement of four-dimensional Yang–Mills theory in Anti-de Sitter space, through the boundary conformal correlators of its conserved $SU(N)$ currents, and is organised as follows. Section 5.1 introduces the boundary theories, lays out the possible confinement scenarios, and summarises our results. Section 5.2 examines the confinement-deconfinement transition in more detail, and Section 5.3 reviews the boundary spectrum of Yang–Mills in AdS, from the free limit to perturbation theory. Section 5.4 sets up the non-Abelian current bootstrap, which we apply to derive a universal bound on C_J in Section 5.5 and to constrain the scaling dimensions of the lightest scalars in Section 5.6. Section 5.7 collects our conclusions and outlook. Three appendices, gathered at the end of the thesis, discuss a bulk Higgsing toy model (Appendix B), the character techniques used to obtain the free-theory spectrum (Appendix C), and the details of the numerical bootstrap (Appendix D). This chapter is based on [3].

5.1 Introduction

A quantitative understanding of confinement and the emergence of a mass gap in $4d$ non-Abelian gauge theories remains one of the hardest and most enduring open problems in quantum field theory [243, 244]. Despite decades of effort, progress has been limited by the absence of small parameters away from the limit of a large number of colours, as well as by the intrinsically strong coupling that governs the infrared (IR) regime.

As initially proposed in [245, 246], the problem can be addressed from a new angle by studying pure Yang–Mills (YM) theory on a curved four-dimensional background, namely Euclidean Anti-de Sitter space (AdS_4). While this reformulation does not solve the issue of strong coupling in any obvious way, it introduces a natural infrared regulator without sacrificing spacetime symmetries. Excitations with wavelengths larger than the AdS radius L are gapped already at the classical level, and the running of the coupling $g_{\text{YM}}(\mu)$ effectively freezes below the scale $\mu \sim 1/L$, where curvature effect dominates over quantum fluctuations. Unlike a

finite box, however, AdS_4 preserves the full $SO(4,1)$ symmetry group, which coincides with the conformal group in $3d$ flat space.

As a result, asymptotically free gauge theories in AdS are controlled by a single dimensionless parameter, which may be taken as $g_{\text{YM}}(1/L)$ or, equivalently, $L\Lambda_{\text{YM}}$, where Λ_{YM} denotes the dynamically generated scale of the flat-space theory. The physics interpolates between two regimes.

- *Weak coupling* ($L\Lambda_{\text{YM}} \ll 1$): the theory can be described by the classical Yang–Mills action with perturbative corrections. The bulk spectrum consists of weakly interacting gluonic excitations with a mass gap of order $1/L$.
- *Strong coupling* ($L\Lambda_{\text{YM}} \gg 1$): the gauge interactions become non-perturbative before curvature effects set in, leading to confinement. Physical states are colour-singlet bound states with a mass gap of order Λ_{YM} .

Varying the dimensionless parameter $L\Lambda_{\text{YM}}$ interpolates between these limits and is expected to induce a confinement–deconfinement transition at a critical value of the radius $L_* \sim 1/\Lambda_{\text{YM}}$. Such a transition has been conjectured and explored in [246, 247], and more recently studied perturbatively in [248, 249]. Its existence is also required by consistency with the flat-space limit $L \rightarrow \infty$, where pure YM theory on $\mathbb{R}^4$ confines.

Last but not least, this framework allows us to use the powerful machinery of conformal bootstrap to investigate the boundary conformal field theory.¹ Since the formulation of the AdS/CFT correspondence [34–36], CFTs have been used to study theories in AdS, initially as a tool to understand gravity and string theory, but later even to study ordinary QFT in a static AdS background [37–40, 250, 251]. Among the tools that CFTs offer, one particularly promising one is the numerical conformal bootstrap that we reviewed in Section 2.2.

In this chapter, we make use of the numerical conformal bootstrap to study the correlation functions of conserved currents in the boundary CFT. The goal of this chapter is to elucidate the mechanism underlying the (de)confinement transition in AdS_4 and to investigate how the appropriate boundary conditions for the gauge fields evolve as one moves between the two regimes. In particular, we study the disappearance of the Dirichlet boundary condition, identified in [246] as the smoking gun for confinement, and we find non-perturbative constraints on the possible realisations of this phenomenon.

A family of boundary theories. The bulk gauge coupling, or equivalently

$$L\Lambda_{\text{YM}} \sim \exp\left(-\frac{8\pi^2}{\beta_0 g_{\text{YM}}^2}\right), \quad (5.1.1)$$

¹Through this chapter, we will use the expression “boundary CFT” to refer to the boundary theory. Formally, this is a set of correlation functions that behave like correlators of a conformally invariant theory, even though there is no local stress tensor among the boundary operators. In any case, the axioms of a conformal theory, such as unitarity, crossing, and the existence of a convergent OPE, are all we need to make use of bootstrap techniques.

corresponds to a one-parameter family of deformations of the boundary CFTs. As reviewed in the previous section, when we vary $L\Lambda_{\text{YM}}$, we interpolate between the two following regimes.

- **Small radius** ($L\Lambda_{\text{YM}} \ll 1$): in this regime, computations can be carried out perturbatively using the UV Lagrangian. In particular, one can use a classical analysis to understand the possible AdS-invariant boundary conditions for the bulk gauge field A_μ^a , which transforms in the adjoint representation of the gauge group G_{YM} . The standard possibilities are the Dirichlet (D) boundary condition (bc)

$$A_\mu^a(z, \vec{x}) \underset{z \rightarrow 0}{\sim} z J_\mu^a(\vec{x}), \quad (5.1.2)$$

and the Neumann (N) boundary condition

$$A_\mu^a(z, \vec{x}) \underset{z \rightarrow 0}{\sim} a_\mu^a(\vec{x}). \quad (5.1.3)$$

Here we are using Poincaré coordinates $(z, \vec{x})$ with $z > 0$, and μ denotes an index parallel to the boundary. The metric is taken to be

$$ds^2 = \frac{L^2}{z^2} (dz^2 + d\vec{x}^2). \quad (5.1.4)$$

Choosing Dirichlet as a bc, the theory has a G_{YM} global symmetry. This symmetry acts locally on the boundary, meaning that among the boundary operators there is an associated conserved current J_μ^a , with protected scaling dimension $\Delta = 2$ (i.e. $L\Lambda_{\text{YM}}$ -independent) in the adjoint representation of G_{YM} . In fact, in the limit $g_{\text{YM}} \rightarrow 0$, the boundary CFT approaches the Generalised Free Vector (GFV) theory built out of this conserved current. Choosing Neumann as a bc, G_{YM} is now gauged also at the boundary. In practice, this means that the boundary CFT has no continuous global symmetry.² In the limit $g_{\text{YM}} \rightarrow 0$, all the local boundary operators are gauge invariant combinations of a boundary field-strength operator $f_{\mu\nu}^a$. There is no boundary operator of protected scaling dimension. Note that in both Dirichlet and Neumann boundary condition, for $g_{\text{YM}} \rightarrow 0$, all Lorentz-scalar operators are irrelevant. In other words, there is no additional bc that we can define by perturbing Dirichlet or Neumann boundary conditions with a relevant operator. The boundary theories are stable as far as such perturbations are concerned.

- **Large radius** ($L\Lambda_{\text{YM}} \gg 1$): in this regime, the spectrum of boundary operators reproduces the spectrum of asymptotic scattering states of the theory in flat space (both single- and multi-particle states). Since a unique vacuum with a definite spectrum is expected to exist in flat space, there should be a single possibility as far as boundary conditions are

²Moreover, the only discrete global symmetry is charge conjugation as far as point-like operators are concerned. Since we restrict to point-like operators, our analysis is not sensitive to the global structure of the gauge group. Boundary operators can also arise from the line operators of the bulk theory, which carry information about the global structure of the gauge theory [252]. For instance, the bulk line operators can give rise to boundary line operators, or they can be open and end at boundary points. Some discrete global symmetry, whose precise form depends on the choice of global structure, can act on this broader class of operators, even with Neumann boundary condition.

concerned. Given that a mass gap is also expected in flat space, all scaling dimensions should grow linearly with L , $\Delta_i \sim M_i L$, where M_i are the masses of the corresponding asymptotic states in the flat space theory. Furthermore, due to the presumed confinement, there should be no colour symmetry G_{YM} labelling the states. More generally, glueball asymptotic states should not carry charges under any continuous global symmetry.

- **Intermediate regime** ($L\Lambda_{\text{YM}} \sim 1$) as the radius of AdS increases, we depart from the perturbative description, and we enter the strongly coupled regime. The arguments reviewed above suggest that the Dirichlet bc cannot be valid all the way to infinite L : something must occur to prevent this from happening. The confinement-deconfinement transition is then conjectured to happen at some critical radius L_* . For $L \lesssim L_*$, the Dirichlet bc is stable and the boundary CFT has a G_{YM} global symmetry, having in its spectrum a non-Abelian conserved current J_μ^a . For $L \gtrsim L_*$, the Dirichlet bc no longer exists as a stable bc and the continuous boundary global symmetry is lost.

Summary of confinement scenarios. Various scenarios for the disappearance of the Dirichlet bc have been proposed. They can be phrased in terms of features of the data of the boundary CFT as we approach L_* in a continuous way from below ($L \leq L_*$).

- **Decoupling:** the current J_μ^a decouples from the spectrum by becoming a null operator. In terms of CFT data, this means that the current two-point function coefficient $C_J = C_J(L\Lambda_{\text{YM}})$ satisfies $C_J \rightarrow 0$ as $L \rightarrow L_*$.
- **Merging:** a scalar S singlet under G_{YM} becomes marginal, triggering a merging and annihilation of the boundary CFT with another putative CFT. This is a generic mechanism discussed in [253–255], and considered in the context of QFT in AdS in [247, 256, 257]. In terms of CFT data this means that the dimension of S satisfies $\Delta_S = \Delta_S(L\Lambda_{\text{YM}}) \rightarrow 3$ as $L \rightarrow L_*$.
- **Higgsing:** a scalar $\mathcal{O}$ charged under G_{YM} becomes marginal, allowing for a new boundary condition for YM in AdS_4 where the boundary global symmetry is explicitly broken. In terms of CFT data this means that $\Delta_{\mathcal{O}} = \Delta_{\mathcal{O}}(L\Lambda_{\text{YM}}) \rightarrow 3$ as $L \rightarrow L_*$.

Additional details on these scenarios are discussed in Section 5.2.

The above scenarios have been analysed in [248] through the lens of perturbation theory, valid in the regime $L\Lambda_{\text{YM}} \ll 1$. On the one hand, this provides a first quantitative approach going beyond the qualitative explorations of [246] and a first evidence in favour of merging. On the other hand, the results obtained there were compatible with all the scenarios above and were based on a one-loop analysis. The bootstrap analysis presented in this chapter shares the same goal of a quantitative approach to the (de)confinement transition of YM in AdS, this time on a non-perturbative footing.

Results. The boundary CFT of YM in AdS_4 with Dirichlet bc contains a non-Abelian conserved current J_μ^a , in the adjoint representation of the gauge group. For definiteness, the gauge group will be taken to be $G_{\text{YM}} = \text{SU}(N)$ from now on. We set the θ -angle to zero, so the YM theory preserves parity and charge conjugation symmetry. The most important observables involved in the above scenarios are accessible in the operator product expansion of such a current, and hence in the four-point function $\langle JJJJ \rangle$. This motivates us to consider the setup of the non-Abelian current bootstrap developed in [101], and reviewed in Section 5.4.

As a first application, we considered a lower bound on the current central charge C_J . This quantity is normally defined as the normalisation of the current two-point function, in the specific choice of normalisation where J obeys the Ward identities of the global symmetry [258]. In the bootstrap setup, after the appropriate rescaling, the current central charge appears in the coefficient of the J exchange in the $J \times J$ OPE. This OPE contains two possible tensor structures, and the precise mixture is parametrised by a further coefficient α_{JJJ} [259]. A lower bound on C_J as a function of α_{JJJ} is reported in Figures 5.3 and 5.4 respectively for $\text{SU}(2)$ and $\text{SU}(3)$. It is strictly positive, preventing the current central charge from vanishing in any unitary CFT. A lower bound on C_J as a function of N is reported in Figure 5.6, which again shows that it is always strictly above zero. In this case, the bound is obtained by setting α_{JJJ} at the value α_{JJJ}^* , which minimises C_J .

The striking consequence of these bounds is that *the “decoupling” scenario discussed in the previous section cannot be reached continuously*: the boundary theory must violate some assumption (unitarity or conformal symmetry) before reaching the limit of vanishing C_J .

Even though a map between C_J and g_{YM} valid at strong coupling is not known, it is expected that, at least within some natural set of scheme choices for the definition of g_{YM} , C_J is monotonically decreasing as g_{YM} grows. This is indeed the case perturbatively. If this relation persists at strong coupling, our results show that the coupling cannot diverge, since $C_J > 0$.

Lower bounds on central charges have been on the numerical bootstrap menu since its inception [72, 260]. However, while the coefficient of the two-point function of the stress tensor, C_T , can be bounded using the four-point function of scalars, C_J only appears universally in the four-point function of the currents, whose global charge is fixed. It is also worth pointing out that, for both C_T and C_J , analytic lower bounds have been known in two dimensions for much longer—see e.g. [21]—where, in the standard $2d$ normalization, the central charges c and k obey $c \geq 1/2$ and $k \geq 1$.

Next, we search for evidence in favour of or against the “merging” and “Higgsing” scenarios. Our strategy is to show that, in YM-like theories, either a singlet scalar (for merging) or a charged scalar (for Higgsing) operator is allowed to cross marginality. Here we face a subtle problem, namely, defining what it means for a conformal theory to be YM-like – in particular, at strong coupling, where the transition is expected to happen. At this stage, we cannot precisely characterise what a CFT describing the boundary of YM in the bulk should look like. Our intuitive picture is that, at weak coupling, the CFT spectrum should be perturbatively close to that of a GFV. Then, as the bulk coupling increases, the spectrum will continuously

be deformed away from the GFV. This motivates us to study in detail the GFV spectrum. We perform such task using character techniques. Taking into account charge conjugation symmetries of operators is not entirely trivial and requires some more formalism, which we extensively develop in Appendix C.

The above intuition is verified in Figure 5.10, where it is shown that with extreme gap assumptions, the allowed region shrinks to an island around the GFV point. More realistic assumptions are obtained by parametrising the deformation away from the GFV spectrum schematically as follows:

$$|\Delta - \Delta_{\text{GFV}}| \leq \frac{\delta}{\ell^n}, \quad (5.1.5)$$

for the leading spin $\ell > 0$ operators in the spectrum – see Section 5.6.4 for precise details. We then explore allowed values for the dimensions of the lightest scalar operators in the $J \times J$ OPE.³ As long as $\delta \ll 1$, we are in the perturbative regime and we find that (as expected) no scalar operator is allowed to be relevant. As δ increases, the space of theories consistent with the spectral assumption (5.1.5) becomes larger, with YM at strong coupling being part of it. Figures 5.13 and 5.15 for $SU(2)$ and $SU(3)$ show that, as δ increases, a scalar singlet is allowed to cross marginality before scalar operators in any other non-singlet representation. We interpret this as *evidence in favour of the merging scenario*.

Similar evidence can be obtained by examining the extremal spectrum arising from the optimisation of various quantities. As shown in Figures 5.14 and 5.16, we obtained bounding boxes for the scaling dimensions of singlet and non-singlet scalars, as a function of δ . Although we don't have evidence that these extremal spectra are necessarily related to YM theory, among all CFTs consistent with the assumption (5.1.5), those that minimise the dimension of a non-singlet scalar operator necessarily contain a lighter singlet scalar in the spectrum, which crosses marginality first.

To summarise, our analysis shows that there exist two values δ_* and δ_{**} such that, for $\delta < \delta_*$, no scalar operator with a spectrum of the form (5.1.5) is allowed to be relevant. For $\delta_* < \delta < \delta_{**}$, only a scalar singlet can become marginal. That doesn't necessarily imply that this will be the case in YM, leading to the merging scenario, but only that this possibility can occur before anything else. This picture confirms the perturbative result of [248], which showed that the scalar singlet acquires the largest negative anomalous dimension at leading order. Finally, for $\delta > \delta_{**}$, non-scalar singlets are allowed to be relevant as well. Beyond this point, we are not able to make any conclusive statements. Nevertheless, it appears that the merging scenario is the most plausible.

5.2 The confinement-deconfinement transition

In this section, we review and clarify the proposed scenarios for the transition between the small-radius and large-radius phases of non-Abelian gauge theories in AdS.

³We assume that there is one such isolated operator below a certain threshold. See Section 5.6.4 for more details.

The puzzle is to find a way to connect these two regimes coherently at intermediate values of $L\Lambda_{\text{YM}}$. A necessary condition to connect them is that the Dirichlet bc should disappear at some non zero critical value of $L\Lambda_{\text{YM}}$. Such a critical value must exist because there is no scalar operator at $\Delta = 3$ in the GFV. On the other hand, the boundary condition Neumann could very well be the one that approaches continuously the single confined theory in the large radius limit.⁴ The latter possibility seems very natural, but it is not guaranteed to be the case. Let us outline some possible scenarios [246, 247] for the behaviour at intermediate $L\Lambda_{\text{YM}}$.

Decoupling. The coefficient C_J of the two-point function of the $SU(N)$ currents vanishes at a critical value of the radius. Because of the Cauchy-Schwarz inequality, the null state $|J_\mu^a\rangle$ must have vanishing overlap with any finite norm state in a unitary CFT. One may object that a state with the same quantum numbers still exists, and is created by the unit-normalised operator $J_\mu^a/\sqrt{C_J}$. In fact, this scenario involves a radical change in the spectrum: all the local operators that are charged under $SU(N)$ should be quotiented out. From the bulk point of view, this is an expected feature of a confining transition: all the multi-gluon states should disappear. From the point of view of the CFT data, the fate of these scaling operators is sealed by the three-point function $\langle J_\mu^a O^* O \rangle$, where O and O^* are in conjugate representations of the $\mathfrak{su}(N)$ algebra and have non-vanishing two-point function. This correlator is fixed by the $\mathfrak{su}(N)$ representation and is, in particular, of order one. Hence, the Cauchy-Schwarz inequality requires that the state created by acting with the two charged local operators at separated points becomes non-normalizable as $C_J \rightarrow 0$. While single operator insertions can be normalised to one, this statement, corresponding to the divergence of a four-point function, is physical. The origin of the divergence in the four-point function $\langle OO^* OO^* \rangle$ is, of course, visible in the OPE, and related to the discussion above: the conformal block decomposition contains the current, with an OPE coefficient proportional to $1/C_J$. Other blocks cannot compensate for the corresponding pole, as one can see, for instance, by evaluating the correlator in a reflection positive configuration.

The singlet sector of the theory is unaffected by these divergences and can continue to exist, at least on the plane, at smaller AdS radii. In the language of [246], this is a continuous transition. Nevertheless, crossing and unitarity do constrain the behaviour of the charged sector as C_J gets smaller, and one of the aims of this chapter is to show, via the conformal bootstrap, that the unit normalisation of the two-point function $\langle OO^* \rangle$ and the growth of the four-point function are in tension with each other.

Merging with transition to Neumann. One of the initially irrelevant scalar singlet operators of Dirichlet bc becomes marginal at some critical value $(L\Lambda_{\text{YM}})_M$, before any operator charged under G_{YM} . As discussed in [248, 257], this generically implies that the Dirichlet bc merges with a second G_{YM} -symmetric bc, called Dirichlet* (D^*), and for $L\Lambda_{\text{YM}} > (L\Lambda_{\text{YM}})_M$ both Dirichlet and D^* are *complex CFTs* [254, 255]. In this context, this means that they remain AdS-invariant boundary conditions (i.e. boundary conditions that preserve the whole isometry

⁴See [261] for a recent exploration in this direction from the point of view of flux tubes.

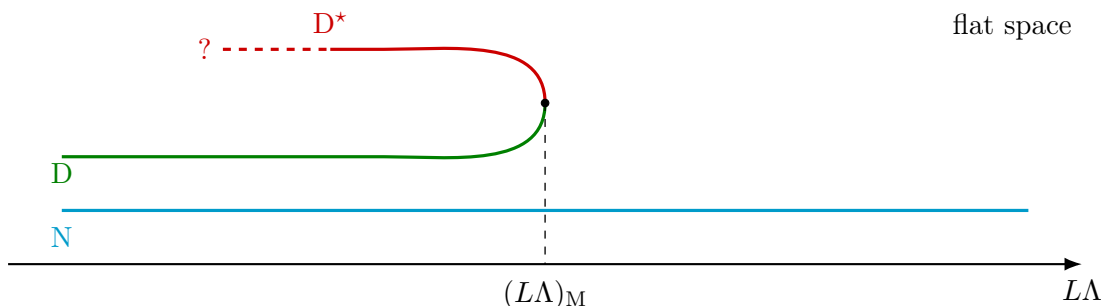


Figure 5.1: Sketch of the boundary conditions available in terms of real boundary CFTs as the coupling g_{YM} grows towards flat space. Here we illustrate the simplest possible scenario, namely merging with transition to Neumann. At some critical value, the Dirichlet boundary condition is assumed to merge with a hitherto unknown boundary condition dubbed D^* and annihilate into the complex plane. Thus, sufficiently close to flat space, one naively expects only the Neumann boundary condition to remain. Note that the vertical axis does not represent a definite quantity; it is merely used to juxtapose the different options.

group) only if one deforms the action by a complex coupling proportional to the operator that crossed marginality, thus breaking unitarity. The consequence of this for the physical theory, constrained to real values of the coupling, is that necessarily the boundary condition breaks the AdS isometries, i.e. the conformal group at the boundary. The breaking takes the form of a boundary RG, triggered by the running of the coupling associated with the singlet operator. In the deep IR of this RG flow, barring the exotic possibility of a boundary condition that preserves only scale transformations and not the full conformal group, the theory must settle in a new AdS-invariant boundary condition. The most natural possibility is that it settles in N , which then continuously extrapolates to the large radius boundary condition. This is the simplest possible scenario. It is illustrated in Figure 5.1.

Higgsing. One of the initially irrelevant Lorentz scalar operators of Dirichlet bc, charged under G_{YM} , becomes marginal at some critical value $(L\Lambda_{\text{YM}})_H$, before any singlet operator does. We expand more on this scenario because it was only mentioned in [247, 248] briefly. Let us denote the operator becoming marginal as $\mathcal{O}^I$, where I is an index in the (non-trivial) representation R of G_{YM} this operator belongs to. When this happens, we gain the possibility of defining a new boundary condition: we simply perturb the Dirichlet bc by the relevant coupling g_I

$$S_{\text{YM}} \rightarrow S_{\text{YM}} + \int_{z=0} d^3\vec{x} g_I \mathcal{O}^I(\vec{x}), \quad (5.2.1)$$

and let the boundary RG flow settle at a new AdS-invariant boundary condition.⁵ This RG flow is short when $L\Lambda_{\text{YM}}$ is larger than $(L\Lambda_{\text{YM}})_H$ but parametrically close to it. In this limit,

⁵In [247, 248], the operator $\mathcal{O}^I$ triggering Higgsing was assumed to be in the adjoint representation of G_{YM} . However, this does not have to be the case; any non-trivial representation could do the job. Clearly, point-like operators of the Dirichlet boundary condition can only be in representations obtained by products of the adjoint representation with itself. One reason why the adjoint representation is special is that, according to the perturbative calculations for $SU(N)$ YM in [248], it is the charged operator with the smallest (irrelevant) scaling dimension at one-loop order.

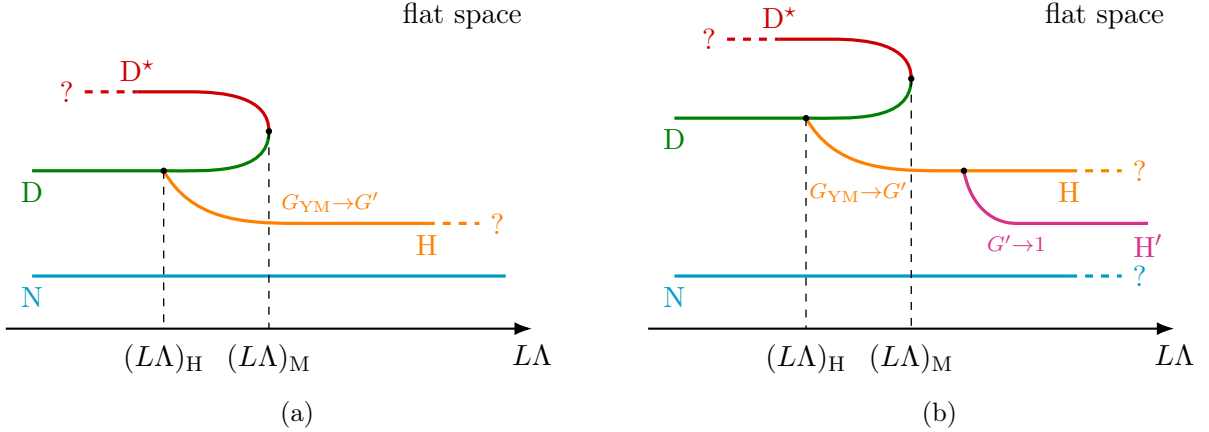


Figure 5.2: Two sketches of possible Higgsing scenarios. In both panels, at $(L\Lambda)_H < (L\Lambda)_M$ a charged operator under G_{YM} becomes relevant, guaranteeing the existence of a boundary condition with smaller global symmetry. This new boundary condition may itself contain other charged operators which eventually also become relevant, leading to a cascade of symmetry breaking until no symmetry is left. In panel (a), the Neumann bc extrapolates to flat space. However, since H emerges before the merging transition, the latter has one more possible endpoint. In panel (b), a completely Higgsed boundary condition H' , rather than N , extrapolates to flat space.

the spectrum of the new boundary condition can be obtained as a perturbation of Dirichlet bc. Since the perturbing operator is charged, the global symmetry of the new boundary condition is explicitly broken to a subgroup of G_{YM} . The current conservation is violated by the deformation as follows

$$\partial^\mu J_\mu^a \propto g_I^*(T^a)^I_J \mathcal{O}^J, \quad (5.2.2)$$

where $(T^a)^I_J$ denotes the G_{YM} generator in the representation R , and g_I^* is the value of the coupling at the IR fixed point of the boundary RG flow. We see that the broken currents are those associated with the generators for which $g_I^*(T^a)^I_J \neq 0$. From the point of view of the bulk, this breaking is interpreted as Higgsing of the gauge degrees of freedom. The resulting anomalous dimension of the current operator corresponds to a non-zero mass of the gluons. For this reason, we call this a Higgsing bc and denote it with H . To illustrate this characterisation of Higgsing in AdS in a simpler weakly-coupled example, in Appendix B we discuss the abelian Higgs model in AdS, in particular deriving equations analogous to (5.2.1) and (5.2.2).

It is important to stress that the emergence of this new Higgsing bc *does not make Dirichlet disappear*. Indeed, as depicted in Figure 5.2, while the two continuous families of CFTs defined by Dirichlet and H join at a point, they do so in a different way compared to the merging of Dirichlet and D^* discussed above. The two associated lines do not intersect smoothly, but rather cross and continue past the intersection point.⁶ This is allowed because the smooth merger

⁶Depending on the form of the leading non-linear terms in the β function of g_I , the line associated with Higgsing might exist as a unitary and AdS-invariant boundary condition on both sides of the intersection with D , or only on one side. The latter case is realised in the perturbative example discussed in the Appendix B. Note that in the discussion around (5.2.2) we are implicitly assuming that the sign of the leading non-linear term leads to a real IR fixed point parametrically close to the intersection point.

and annihilation can only happen between families of CFTs sharing the same symmetry, while Dirichlet and H, by definition, have different symmetries. Similar behaviours arise, for instance, between the free and interacting CFTs near Wilson-Fisher and $d = 2 + \epsilon$ fixed points, if one treats the spacetime dimension as a continuous parameter.⁷ This scenario recently appeared in [262], which defined the two families as “continuously but not analytically connected”.

One still needs to find a mechanism for Dirichlet to disappear. One possibility is that it still happens through a merger and annihilation, only at a larger value of the coupling $(L\Lambda_{\text{YM}})_{\text{M}} > (L\Lambda_{\text{YM}})_{\text{H}}$. In fact, rather than providing a full and clear picture of the connection between the regimes of small and large AdS, this scenario suggests that things could be far more involved than in the simple merger picture above. For instance, we also need to consider what happens to the H boundary condition in the limit of a large radius. Note that H will be charged under an unbroken subgroup G' of G_{YM} , where G' is the group that leaves invariant the deforming operator in (5.2.1). This implies that H shouldn't extrapolate continuously to the large radius boundary condition either. One possibility, illustrated in panel (a) of Figure 5.2, is that it is still Neumann that is connected to the flat space observables. Another possibility is that a further Higgsing takes place in H, breaking G' completely and giving rise to a fully Higgsed boundary condition H' with no continuous global symmetry. As illustrated in panel (b), H' could, in principle, be a valid alternative to Neumann as the boundary condition that is continuously connected to the flat space observables, compatibly with colour confinement and the mass gap.⁸

From the discussion above, it is clear that the more intricate Higgsing scenario can be ruled out if one can show that Dirichlet disappears *before* any charged scalar operator becomes relevant. This observation motivates part of our investigations below.

5.3 Boundary spectrum of YM in AdS

At zero coupling $g_{\text{YM}} = 0$, the $d = 3$ boundary CFT is given by a generalised Free Vector Theory (GFV): the Mean Field Theory constructed from J_μ^a , which is a spin $\ell = 1$ operator in the adjoint representation of $SU(N)$ with dimension $\Delta = d - 1 = 2$. As we turn on bulk interactions $g_{\text{YM}} \neq 0$, the spectrum of the boundary CFT becomes a deformation of the GFV, where all composite operators built out of J_μ^a acquire anomalous dimensions. The current itself and the vacuum do not obtain an anomalous dimension. We will assume that the spectrum of primaries in the boundary CFT of YM in AdS_4 is continuously connected to the spectrum of the GFV. Contrary to what typically happens for perturbative IR fixed points of flat space QFTs,

⁷Examples of continuous families of CFTs with a charged operator that crosses marginality can also be found in the context of local CFTs. For example, one can consider the $O(N)$ -symmetric scalar CFT in $3d$, where the role of $L\Lambda_{\text{YM}}$ is played by the number of fields N . In this family of theories, a rank-4 traceless symmetric operator approaches marginality for $N_{\text{H}} \gtrsim 2.99$ [242]. This implies that, for $N > N_{\text{H}}$, the $O(N)$ model is unstable with respect to this perturbation, which, however, needs to be turned on explicitly.

⁸A natural question is whether such a putative H' bc is in fact the same as N. Depending on the global structure of the theory, one might be able to use discrete symmetries and boundary operators originating from bulk line operators to distinguish between H' and N, see the comments in footnote 2. A comprehensive discussion of these global aspects and of bulk line operators goes beyond the scope of this chapter.

the bulk equations of motion are not expected to imply discontinuous changes in the boundary spectrum of primaries between the free and the interacting theory. From the boundary point of view, this is easy to see since, other than the current itself (which remains conserved), there are no operators located at the unitarity bound at zero coupling.

5.3.1 Symmetries

The classical Lagrangian of pure YM theory in AdS_4 with gauge group $SU(N)$ is

$$S_{\text{YM}} = -\frac{1}{2g_{\text{YM}}^2} \int dx \, \text{tr} [F_{MN} F^{MN}], \quad (5.3.1)$$

where the field strength for an $\mathfrak{su}(N)$ connection $A_M = A_M^a T^a$ is

$$F_{MN} = F_{MN}^a T^a, \quad F_{MN}^a = \nabla_M A_N^a - \nabla_N A_M^a + f^{abc} A_M^b A_N^c, \quad (5.3.2)$$

and g_{YM} is the YM coupling constant. In what follows, we will use capital Latin letters to denote AdS_4 indices and Greek letters to denote CFT_3 indexes. Lower case Latin letter will be reserved for $\mathfrak{su}(N)$ adjoint indices. Here $\{T^a, a = 1, \dots, N^2 - 1\}$ are anti-hermitian $\mathfrak{su}(N)$ generators with

$$\text{tr} [T^a T^b] = -\frac{1}{2} \delta^{ab}, \quad [T^a, T^b] = f^{abc} T^c, \quad (5.3.3)$$

where the trace is in the fundamental representation, and the structure constants f^{abc} are real. The integral in (5.3.1) is short-hand for the integral over AdS

$$\int dx \, f(x) \equiv \int_{\text{AdS}_4} d^4x \, \sqrt{g} f(x), \quad (5.3.4)$$

and ∇_M in (5.3.2) is the covariant derivative in AdS .

When Dirichlet bc (5.1.2) are imposed, boundary correlators of local operators are invariant under the faithful symmetry

$$G = G_{\text{space}} \times G_{\text{global}} = \left(SO(4, 1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}} \right) \times \left(PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}} \right), \quad (N \geq 3), \quad (5.3.5)$$

while for $N = 2$ we have

$$G = \left(SO(4, 1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}} \right) \times SO(3), \quad (N = 2). \quad (5.3.6)$$

Let us explain the origin of each factor.

- **Conformal symmetry:** $SO(4, 1)_0$ denotes the connected component of the Euclidean conformal group of $\mathbb{R}^3$. This coincides with the isometries of Euclidean AdS_4 that are connected to the identity. It is a well-known fact that boundary correlators of QFTs in rigid AdS space are invariant under this group.
- **Parity:** $\mathbb{Z}_2^{\mathcal{P}}$ denotes the $\mathbb{Z}_2$ group generated by the identity and parity transformations on the conformal boundary. In Poincaré coordinates $(z, \vec{x})$, where AdS_4 is Weyl equivalent

to $\mathbb{R}_+ \times \mathbb{R}^3$, a parity transformation is generated by a single coordinate reflection on $\mathbb{R}^3$, or equivalently by a total reflection $(z, \vec{x}) \rightarrow (z, -\vec{x})$. With the latter definition, the gauge connection transforms as $A^M(z, \vec{x}) \rightarrow \mathcal{P}^M_N A^N(z, -\vec{x})$ with $\mathcal{P}^z_z = 1, \mathcal{P}^\mu_\nu = -\delta^\mu_\nu$ and zero otherwise. The action (5.3.1) is manifestly invariant under this transformation. Parity does not commute with translations and special-conformal transformations, hence the product is semi-direct in (5.3.5).⁹ However, it still commutes with the maximally compact subgroup $SO(1,1) \times SO(3)$, generated by dilatation and Euclidean rotations. This makes it easy to take parity into account. Its action on a local primary operator with scaling dimension Δ and spin ℓ , independently of its $PSU(N) \rtimes \mathbb{Z}_2^C$ representation, is

$$\mathcal{O}_{\Delta,\ell}(\vec{x}) \xrightarrow{\mathcal{P}} p(-1)^\ell \mathcal{O}_{\Delta,\ell}(-\vec{x}), \quad (5.3.7)$$

where $p = \pm$ denotes the intrinsic parity of the operator.

- **$SU(N)$ symmetry:** even though $SU(N)$ transformations are gauged in the bulk of AdS, they are genuine symmetries of the Dirichlet boundary CFT. Local boundary operators are obtained as products of currents, in irreducible representations arising from the tensor product of adjoint representations. Since the adjoint representation has zero N -ality, being invariant under transformations in the centre of $SU(N)$, all boundary operators will also be so. Thus the faithful symmetry is $PSU(N) = SU(N)/\mathbb{Z}_N$. For $N = 2$, we have $PSU(2) = SO(3)$.
- **Charge conjugation:** $\mathbb{Z}_2^C$ is the $\mathbb{Z}_2$ group generated by the identity and charge conjugation $\mathcal{C}$. We define the action of $\mathcal{C}$ on the $\mathfrak{su}(N)$ connection as

$$A_M(x) = A_M^a(x) T^a \xrightarrow{\mathcal{C}} A_M(x)^* = \left(-C^{ab} A_M^b(x) \right) T^a, \quad (5.3.8)$$

where $*$ denotes complex conjugation, and we used that $(T^a)^* = -(T^a)^t$ for anti-hermitian generators, while also defining the matrix

$$(T^a)^t = C^{ab} T^b, \quad C^{ab} = -2 \operatorname{tr}[(T^a)^t T^b]. \quad (5.3.9)$$

Under this transformation the field strength (5.3.2) transforms as¹⁰

$$F_{MN}(x) = F_{MN}^a(x) T^a \xrightarrow{\mathcal{C}} F_{MN}(x)^* = \left(-C^{ab} F_{MN}^b(x) \right) T^a. \quad (5.3.10)$$

It is easy to check that this is a symmetry of (5.3.1). Since (5.3.8) does not commute with $SU(N)$ transformations, the product in (5.3.5) is semi-direct. For $N = 2$, complex conjugation can be realised by an $SU(2)$ transformation, i.e. there is g_0 such that $g^* = g_0 g g_0^{-1}$ for any $g \in SU(2)$, explaining why we don't have a $\mathbb{Z}_2^C$ factor in (5.3.6).

The boundary correlators inherit this symmetry. The action of $\mathcal{C}$ on a generic boundary primary operator $\mathcal{O}$ is

$$\mathcal{O}(\vec{x}) \xrightarrow{\mathcal{C}} \mathfrak{c} \mathcal{O}(\vec{x})^*, \quad (5.3.11)$$

⁹For example, a level one descendant picks up an additional -1 factor compared to its corresponding primary.

¹⁰The structure constants $f^{abc} = -2 \operatorname{tr}[[T^a, T^b] T^c]$ satisfy the following relation $C^{aa'} C^{bb'} C^{cc'} f^{a'b'c'} = -2 \operatorname{tr}[[(T^a)^t, (T^b)^t] (T^c)^t] = -f^{abc}$, which implies $f^{abc} A_M^b A_N^c \xrightarrow{\mathcal{C}} f^{abc} C^{bb'} C^{cc'} A_M^{b'} A_N^{c'} = -C^{aa'} f^{a'b'c'} A_M^{b'} A_N^{c'}$.

valid for any scaling dimension, spin, and $SU(N)$ representation. The factor $\mathfrak{c}$ is a possibly present intrinsic charge parity, which we will discuss shortly. Charge conjugation $\mathcal{C}$ acts on an irreducible representation R_λ with highest-weight Dynkin labels $\lambda = (\lambda_1, \dots, \lambda_{N-1})$ as

$$\lambda \xrightarrow{\mathcal{C}} \lambda^* = (\lambda_{N-1}, \dots, \lambda_1). \quad (5.3.12)$$

If $\lambda \neq \lambda^*$, irreducible representations of $PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}}$ are given by $R_\lambda^{\mathcal{C}} = R_\lambda \oplus R_{\lambda^*}$, where $\mathcal{C}$ acts by swapping the two subspaces. In this case, the constant $\mathfrak{c}$ in (5.3.11) is immaterial and can be taken to be one. For brevity, we will denote by λ_c the representation $R_\lambda^{\mathcal{C}}$. If $\lambda = \lambda^*$, the representation is self-conjugate, and in this case we can have a non-trivial intrinsic charge conjugation $\mathfrak{c} = \pm 1$ in (5.3.11). The corresponding representations of $PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}}$ are $R_\lambda^\pm$ and we denote them simply as $\lambda_\pm$.¹¹

Whenever we do not need to distinguish between the two cases, we will just write $\lambda_{(c,\pm)}$.

See Appendix C for further details on the action of parity and charge conjugation.

Irreducible representations of $G = G_{\text{space}} \times G_{\text{global}}$ of the boundary primary fields will then be characterised by the quantum numbers $(\Delta, \ell, p, \lambda_{(c,\pm)})$. For example, the local conserved $SU(3)$ current has

$$J_\mu^a : (2, 1, +, (1, 1)_-). \quad (5.3.13)$$

Details on how we fix the intrinsic sign under $\mathbb{Z}_2^{\mathcal{C}}$ are given below. For bootstrap applications, it will be of particular importance to know the spectrum of operators in the $J \times J$ OPE that are allowed by symmetries. This will be reviewed in Section 5.4, but let us anticipate here the first restriction imposed by the global symmetry to keep an eye out for the relevant representations in the following section.

We denote by **adj** the adjoint representation of $SU(N)$. For $N \geq 4$, the tensor product of two **adj** decomposes as

$$\mathbf{adj} \otimes \mathbf{adj} \cong \mathbf{S} \oplus 2 \mathbf{adj} \oplus \mathbf{S}\bar{\mathbf{S}} \oplus \mathbf{A}\bar{\mathbf{A}} \oplus \mathbf{S}\bar{\mathbf{A}} \oplus \mathbf{A}\bar{\mathbf{S}}, \quad (5.3.14)$$

where the Young diagrams and Dynkin labels of the above representations are reported in Table 5.1.¹² All these representations have vanishing N -ality and lift to representations of $PSU(N)$. For $N = 3$, the decomposition (5.3.14) still applies but the $\mathbf{A}\bar{\mathbf{A}}$ representation does not appear. For $N = 2$, we have

$$\mathbf{adj} \otimes \mathbf{adj} \cong \mathbf{S} \oplus \mathbf{adj} \oplus \mathbf{S}\bar{\mathbf{S}}, \quad (5.3.15)$$

where $\mathbf{S}$, **adj** and $\mathbf{S}\bar{\mathbf{S}}$ representations are identified with the singlet, the vector, and the traceless symmetric representations of $SO(3)$, respectively.

When taking into account the transformation property under charge conjugation, for $N \geq 3$ the two **adj** in the right-hand side of (5.3.14) have opposite charge conjugation parity and induce

¹¹These exhaust all the finite-dimensional unitary irreducible representations, see e.g. [263, Prop. 1.5].

¹²The $\mathbf{S}\bar{\mathbf{S}}$, $\mathbf{S}\bar{\mathbf{A}}$, $\mathbf{A}\bar{\mathbf{S}}$ and $\mathbf{A}\bar{\mathbf{A}}$ representations were denoted in [248] as R_1 , $\bar{R}_2$, R_2 and R_3 , respectively.

Name	Dynkin lables		Young diagram	Dimension
S	$(0, \dots, 0)$		$[0]$	1
adj	$(1, 0, \dots, 0, 1)$	$[(2) \text{ for } N = 2]$	$[N - 1, 1]$	$N^2 - 1$
S$\bar{S}$	$(2, 0, \dots, 0, 2)$	$[(4) \text{ for } N = 2]$	$[N - 1, N - 1, 1, 1]$	$\frac{1}{4}(N - 1)N^2(N + 3)$
S$\bar{A}$	$(2, 0, \dots, 0, 1, 0)$	$[(3, 0) \text{ for } N = 3]$	$[N - 2, 1, 1]$	$\frac{1}{4}(N^2 - 4)(N^2 - 1)$
A$\bar{S}$	$(0, 1, 0, \dots, 0, 2)$	$[(0, 3) \text{ for } N = 3]$	$[N - 1, N - 1, 2]$	$\frac{1}{4}(N^2 - 4)(N^2 - 1)$
A$\bar{A}$	$(0, 1, 0, \dots, 0, 1, 0)$	$[(0, 2, 0) \text{ for } N = 4]$	$[N - 2, 2]$	$\frac{1}{4}(N - 3)N^2(N + 1)$

Table 5.1: Irreducible $SU(N)$ representations appearing in the tensor product of two adjoint representations. Note that **A $\bar{A}$** does not appear for $N = 2, 3$, while **S $\bar{A}$** and **A $\bar{S}$** do not appear for $N = 2$. A Young diagram $[l_1, \dots, l_k]$ has l_i boxes in the i -th column.

two inequivalent representations of $PSU(N) \rtimes \mathbb{Z}_2^C$, denoted by **adj $_{\pm}$** . Moreover, the **S $\bar{A}$** and **A $\bar{S}$** representations are grouped into a representation denoted by **S $\bar{A}_c$** . The tensor product (5.3.14) is then “refined” as follows for two **adj $_{\pm}$** representations:

$$\mathbf{adj}_{\pm} \otimes \mathbf{adj}_{\pm} \cong \mathbf{S}_+ \oplus \mathbf{adj}_+ \oplus \mathbf{adj}_- \oplus \mathbf{S}\bar{\mathbf{S}}_+ \oplus \mathbf{A}\bar{\mathbf{A}}_+ \oplus \mathbf{S}\bar{\mathbf{A}}_c. \quad (5.3.16)$$

The signs under charge conjugation on the right-hand side can be determined using the characters of Appendix C.2.

We further split representations in (5.3.16) and (5.3.15) in terms of their behaviour under the swapping of the two tensor product factors, i.e. in the symmetric square (Sym^2) and alternating square (Alt^2) components. This is useful, since, as we will review in Section 5.4, their symmetry properties under swapping determine further selection rules in the $J \times J$ OPE. The decomposition for $N \geq 3$ reads¹³

$$\begin{aligned}
N \text{ odd : } \quad & \mathbf{R}_s = \text{Sym}^2(\mathbf{adj}_{\pm}) \cong \mathbf{S}_+ \oplus \mathbf{adj}_+ \oplus \mathbf{S}\bar{\mathbf{S}}_+ \oplus \mathbf{A}\bar{\mathbf{A}}_+, \\
& \mathbf{R}_a = \text{Alt}^2(\mathbf{adj}_{\pm}) \cong \mathbf{adj}_- \oplus \mathbf{S}\bar{\mathbf{A}}_c, \\
N \text{ even : } \quad & \mathbf{R}_s = \text{Sym}^2(\mathbf{adj}_{\pm}) \cong \mathbf{S}_+ \oplus \mathbf{adj}_- \oplus \mathbf{S}\bar{\mathbf{S}}_+ \oplus \mathbf{A}\bar{\mathbf{A}}_+, \\
& \mathbf{R}_a = \text{Alt}^2(\mathbf{adj}_{\pm}) \cong \mathbf{adj}_+ \oplus \mathbf{S}\bar{\mathbf{A}}_c,
\end{aligned} \quad (5.3.17)$$

and for $N = 2$

$$\begin{aligned}
N = 2 : \quad & \mathbf{R}_s = \text{Sym}^2(\mathbf{adj}) \cong \mathbf{S} \oplus \mathbf{S}\bar{\mathbf{S}}, \\
& \mathbf{R}_a = \text{Alt}^2(\mathbf{adj}) \cong \mathbf{adj}.
\end{aligned} \quad (5.3.18)$$

We observe, in particular, that **adj $_{-}$** (resp. **adj $_{+}$**) appears in $\mathbf{R}_a$ for N odd (reps. N even). For Yang-Mills in AdS we require $A_{\mu}^a \underset{z \rightarrow 0}{\sim} z J_{\mu}^a$ to transform with the same sign as $f^{abc} A_{\mu}^a A_{\nu}^c \underset{z \rightarrow 0}{\sim} z^2 f^{abc} J_{\mu}^a J_{\nu}^b$, in order for $F_{\mu\nu}^a$ to transform covariantly as in (5.3.10). Since $f^{abc} A_{\mu}^a A_{\nu}^c$ is precisely the projection in the alternating square adjoint subspace, this implies that A_{μ}^a , and thus J_{μ}^a , has to be identified with the **adj $_{-}$** representation of $PSU(N) \rtimes \mathbb{Z}_2^C$ for N odd and with **adj $_{+}$** for N even. This explains the origin of the “ $-$ ” sign in (5.3.13).

¹³Again, for $N = 3$ the representation **A $\bar{A}$** does not appear.

5.3.2 Free theory spectrum

The spectrum of the GFV generated from a non-Abelian $SU(N)$ conserved current can be obtained with character techniques, as was done in Appendix A of [248], and further refined in Appendix C for the purposes of the present chapter. In this section, we discuss some features of the spectrum that are relevant for the scenarios discussed in Section 5.1 and for the gap assumptions used in Sections 5.6.2 and 5.6.4. The spectrum of primary operators of the GFV is obtained by J_μ^a and products thereof, with appropriate insertions of derivatives and projection into irreducible representations of $SU(N)$. Moreover, one can contract these operators with the $\epsilon^{\mu\nu\rho}$ tensor to form operators that have intrinsic parity $p = -$.

We start by discussing the double trace operators of the form $[JJ]$: these are the lowest-dimensional operators, besides the identity and the conserved current itself. They are also the only operators for which anomalous dimensions have been computed, as we will review in Section 5.3.3. In terms of the twist

$$\Delta - \ell = \tau = 2 + m, \quad (5.3.19)$$

parity-even and parity-odd operators have even and odd integer values of $m \geq 0$, respectively. Consider first parity-even operators. They are of the form¹⁴

$$\begin{aligned} (\Delta, \ell, p) = (\ell + 4 + 2n, \ell, +) & : J_\mu^a \partial_{\mu_1} \cdots \partial_{\mu_\ell} \square^n J_\mu^b, & \ell \geq 0, n \geq 0, \\ (\Delta, \ell, p) = (\ell + 2 + 2n, \ell, +) & : J_\mu^a \partial_{\mu_1} \cdots \partial_{\mu_{\ell-2}} \square^n J_\nu^b, & \ell \geq 2, n \geq 0, \end{aligned} \quad (5.3.20)$$

and are in representations R_s for ℓ even and R_a for ℓ odd. The first Regge trajectory $m = 0$ starts from $\ell \geq 2$ and is non-degenerate, as are the operators with $\ell = 0$ and $\ell = 1$ for any $m \geq 2$. Operators with $\ell \geq 2$ and $m \geq 2$ are instead doubly degenerate.

Double trace parity-odd operators have the same schematic form (5.3.20), provided we replace one of the two currents with its dual $\tilde{J}^{\mu,a} = \epsilon^{\mu\nu\rho} \partial_\nu J_\rho^a$. We then have

$$\begin{aligned} (\Delta, \ell, p) = (\ell + 5 + 2n, \ell, -) & : \tilde{J}_\mu^a \partial_{\mu_1} \cdots \partial_{\mu_\ell} \square^n J_\mu^b, & \ell \geq 0, n \geq 0, \\ (\Delta, \ell, p) = (\ell + 3 + 2n, \ell, -) & : \tilde{J}_\mu^a \partial_{\mu_1} \cdots \partial_{\mu_{\ell-2}} \square^n J_\nu^b, & \ell \geq 2, n \geq 0. \end{aligned} \quad (5.3.21)$$

Operators with $\ell = 0$ and $\ell = 1$ are non-degenerate for any $m \geq 3$ and are respectively in representations R_s and R_a . Operators with $\ell \geq 2$ and any $m \geq 1$ are also non-degenerate, but they appear in both R_s and R_a representations.

Determining the spectrum (quantum numbers and degeneracies) of k -trace primary operators by explicit construction becomes increasingly hard as k increases, since one has to disentangle primaries from descendants and decompose into irreducible representations products of more and more adjoint currents. For this purpose, in Appendix C we use character techniques that allow us to determine the spectrum of the GFV systematically – including the intrinsic charge conjugation for self-conjugate representations – without the need to construct

¹⁴The position of the derivative insertions it to be understood as schematic, for example by $J_\mu^a \partial_{\mu_1} \cdots \partial_{\mu_{\ell-2}} J_\nu^b$ we mean that among all possible linear combinations of operators with 2 J -s and $\ell - 2$ derivatives, we have to select the ones that correspond to primaries.

the operators explicitly. We refer the reader to Tables C.1 and C.2 and to the ancillary file `GFV-spectrum.nb` of [3] for a systematic collection of the lowest lying GFV operators for the cases of $SU(2)$, $SU(3)$ and $SU(4)$.

For illustration purposes, we report below the schematic form of the lowest-dimensional triple trace operators appearing in the GFV spectrum. The first parity-even $[J^3]$ operators read

$$\begin{aligned}
(\Delta, \ell, p) = (7, 0, +) & : J_\mu^a J_\nu^b \partial^\mu J_\nu^c, \\
(\Delta, \ell, p) = (6, 1, +) & : J_\mu^a J_\mu^b J_\rho^c, \\
(\Delta, \ell, p) = (7, 2, +) & : J_\mu^a J_\nu^b \partial^\mu J_\rho^c, \\
(\Delta, \ell, p) = (3 + \ell, \ell, +) & : J_\mu^a J_\nu^b \partial_{\mu_1} \cdots \partial_{\mu_{\ell-3}} J_\nu^c, \quad \ell \geq 3.
\end{aligned} \tag{5.3.22}$$

While for the parity-odd sector, these are

$$\begin{aligned}
(\Delta, \ell, p) = (6, 0, -) & : \epsilon^{\mu\nu\rho} J_\mu^a J_\nu^b J_\rho^c, \\
(\Delta, \ell, p) = (7, 1, -) & : J_\mu^a J_\mu^b \tilde{J}_\rho^c, \\
(\Delta, \ell, p) = (6, 2, -) & : \epsilon^{\mu\nu\rho} J_\mu^a J_\nu^b J_\sigma^c, \\
(\Delta, \ell, p) = (4 + \ell, \ell, -) & : J_\mu^a J_\nu^b \partial_{\mu_1} \cdots \partial_{\mu_{\ell-3}} \tilde{J}_\nu^b, \quad \ell \geq 3.
\end{aligned} \tag{5.3.23}$$

These operators are generically in irreducible representations appearing in $\mathbf{adj}^{\otimes 3}$, which include both R_s and R_a representations in (5.3.17).

5.3.3 Perturbative results

In this subsection, we collect the known results about anomalous dimensions of double-trace operators of the form $[JJ]$ in the GFV spectrum. It should be emphasized that we will not make direct use of all of these in our numerical explorations. What we will mostly make use of, their scaling with spin, will inform spin-dependent gap assumptions. Nevertheless, we find it instructive for future reference to collect all such relevant results in one place. We define the anomalous dimensions $\vec{\gamma}$ as

$$\vec{\Delta}_{[JJ]_m} = 2 + \ell + m + \vec{\gamma}_{[JJ]_m}, \tag{5.3.24}$$

where $\vec{\gamma}$ takes into account the different irreducible representations which can appear. For $N \geq 4$, $\vec{\gamma} = (\gamma_{\mathbf{S}_+}, \gamma_{\mathbf{adj}_+}, \gamma_{\mathbf{S}\bar{\mathbf{S}}_+}, \gamma_{\mathbf{A}\bar{\mathbf{A}}_+}, \gamma_{\mathbf{adj}_-}, \gamma_{\mathbf{S}\bar{\mathbf{A}}_c})$.¹⁵ The two-point function of double trace operators with spin ℓ has the following schematic group-theoretical structure at NLO:¹⁶

$$\langle [JJ]^{a_1 b_1} [JJ]^{a_2 b_2} \rangle \big|_{1\text{-loop}} \propto \left(T \pm (-1)^\ell U \right)^{a_1 b_1 a_2 b_2}, \tag{5.3.25}$$

where

$$T^{a_1 b_1 a_2 b_2} = f^{a_1 a_2 e} f^{b_1 b_2 e}, \quad U^{a_1 b_1 a_2 b_2} = f^{a_1 b_2 e} f^{b_1 a_2 e}. \tag{5.3.26}$$

Under the exchange $a_1 \leftrightarrow b_1$ or $a_2 \leftrightarrow b_2$, $U \leftrightarrow T$, and $T+U$ and $T-U$ are respectively symmetric and antisymmetric. Depending on the spin ℓ , then, only symmetric (R_s) or antisymmetric (R_a)

¹⁵Recall from Section 5.3.2 that not always all representations appear for given quantum numbers of $[JJ]$. In particular, for parity even scalars, we can only have the representations $\mathbf{S}_+$, $\mathbf{adj}_+$, $\mathbf{S}\bar{\mathbf{S}}_+$, $\mathbf{A}\bar{\mathbf{A}}_+$.

¹⁶In terms of Witten diagrams in the bulk, NLO means two loop level.

representations can acquire a non-trivial anomalous dimension at NLO. The group theoretical coefficients $\vec{v}$ of anomalous dimensions for operators with the same spacetime quantum numbers but in different representations of $SU(N)$ can be easily obtained by projecting (5.3.25) on the various representations using the results in Section 5.3.4. We get

$$\vec{v}_s = (2N, N, -2, 2), \quad \vec{v}_a = (N, 0), \quad (5.3.27)$$

where $\vec{v} = \vec{v}_s \oplus \vec{v}_a$, and the subscripts s, a refer respectively to the representations R_s and R_a in (5.3.17). The same decomposition applies at leading order in a $1/\ell$ expansion [259].

The anomalous dimensions for the first three Regge trajectories $m = 0, 1, 2$ have been computed in [259, Eqs. 5.16–5.18] at leading order in a $1/\ell$ expansion, using the light-cone analytic bootstrap [67, 68]. The leading anomalous dimensions $\gamma_{[JJ]_m}$ for any m and at finite $\ell > 1$ have been computed in [264, Eqs. 4.6–4.8], at leading order in a $1/C_J$ (i.e. g_{YM}^2) expansion, using the Lorentzian inversion formula [168]. The decomposition into irreducible representations was not performed in [264], and the results are subject to an ambiguity in the overall normalisation factor.¹⁷ However, we can determine the proper factors by matching the results of [264] with the large ℓ ones of [259], restricting both results to the special case of Yang-Mills theory in rigid AdS. For completeness, we report here the final formulas:

$$\begin{aligned} \vec{\gamma}_{[JJ]_0} &= -\frac{g_{\text{YM}}^2}{4\pi^2} \vec{P}_\eta \left(\frac{\beta^4 - 4\beta^3 + 28\beta^2 - 48\beta + 32}{(\beta - 4)(\beta - 2)(\beta - 1)\beta(\beta + 2)} \right), \\ \vec{\gamma}_{[JJ]_{2n}}^{(1)} &= \frac{g_{\text{YM}}^2}{2\pi^2} \vec{P}_\eta \left(\psi \left(\frac{\beta}{2} - n - 2 \right) - \psi \left(\frac{\beta}{2} + n \right) + \frac{4}{\beta - 2n - 2} - \frac{4}{(\beta - 2n)(\beta - 2n + 2)} \right), \\ \vec{\gamma}_{[JJ]_{2n}}^{(2)} &= \frac{g_{\text{YM}}^2}{2\pi^2} \vec{P}_\eta \left(\psi \left(\frac{\beta}{2} - n \right) - \psi \left(\frac{\beta}{2} + n + 2 \right) + \frac{4}{\beta + 2n} + \frac{4}{(\beta + 2n - 4)(\beta + 2n - 2)} \right), \\ \vec{\gamma}_{[JJ]_{2n-1}}^{(1)} &= \frac{g_{\text{YM}}^2}{2\pi^2} \vec{Q}_\eta \left(\psi \left(\frac{\beta + 1}{2} - n \right) - \psi \left(\frac{\beta + 1}{2} + n \right) - \frac{8}{(\beta - 2n - 1)(\beta - 2n + 1)} \right), \\ \vec{\gamma}_{[JJ]_{2n-1}}^{(2)} &= \frac{g_{\text{YM}}^2}{2\pi^2} \vec{P}_\eta \left(\psi \left(\frac{\beta + 1}{2} - n \right) - \psi \left(\frac{\beta + 1}{2} + n \right) + \frac{8}{(\beta + 2n - 1)(\beta + 2n + 1)} \right), \end{aligned} \quad (5.3.28)$$

where $\psi(x)$ is the digamma function, $\beta = \Delta + \ell$, $\ell > 1$,

$$\vec{P}_\eta = \vec{v}_\eta \frac{1 + c_\eta(-1)^\ell}{2}, \quad \vec{Q}_\eta = \vec{v}_\eta \frac{1 - c_\eta(-1)^\ell}{2}, \quad c_{s,a} = \pm 1, \quad \eta = s, a, \quad (5.3.29)$$

and the superscripts (1) and (2) in $\gamma_{[JJ]_m}$ refer to the two degenerate operators which appear for $n > 0$. Note that for parity-odd operators, all representations appear for any given spin, in agreement with the discussion of the previous subsection.

More recently, the anomalous dimensions for scalars with $m = 2$ have been determined by Witten diagrams and by a broken conformal Ward identity [248]:

$$\vec{\gamma}_{[JJ]_2}(\ell = 0) = -\frac{11g_{\text{YM}}^2}{48\pi^2} \vec{v}_s. \quad (5.3.30)$$

¹⁷In addition to that, [264, Eq. 4.6] appears to have a typo. We thank Yue-Zhou Li for correspondence on this point.

The anomalous dimensions of all the double trace operators appearing in (5.3.28) are negative, with the exception of $\gamma_{\mathbf{S}\bar{\mathbf{S}}_+}$, which is positive, and of $\gamma_{\mathbf{S}\bar{\mathbf{A}}_c}$, which is vanishing at leading order, both in a $1/\ell$ and in a g_{YM}^2 expansion. The charged operator with the largest (negative) one-loop anomalous dimension is in the adjoint representation. We will focus our bootstrap analysis on the lightest scalar operators in the singlet, adjoint and $\mathbf{S}\bar{\mathbf{S}}$ representations. For the $SU(3)$ case, these are perturbatively the lightest double-trace scalars, among which the singlet has the most negative γ . For the $SU(2)$ case, we recall that perturbatively the lightest adjoint scalar is not a double-trace but rather of $[J^3]$ type with $\Delta = 7 + O(g_{\text{YM}}^2)$.

We also recall that the current central charge at LO in perturbation theory is [265]

$$C_J = \frac{C_J^0}{g_{\text{YM}}^2} \left(1 + O(g_{\text{YM}}^2)\right), \quad C_J^0 = \frac{2}{\pi^2}, \quad (5.3.31)$$

which shows explicitly that a large central charge corresponds to weak coupling.

5.3.4 Projectors to irreducible representations

Our task is to find the explicit decomposition into irreducible representations of a tensor O^{ab} , where $a, b = 1, \dots, N^2 - 1$ are adjoint indices of $SU(N)$. The decomposition can be made by decomposing or not decomposing the adjoint indices into fundamental ones. We choose not to split them, using the results of [266], as done in [248]. A generic tensor operator O^{ab} can be split into its symmetric and antisymmetric parts:

$$O^{ab} = O^{(ab)} + O^{[ab]} = O_s + O_a. \quad (5.3.32)$$

The two tensors have $M(M \pm 1)/2$ components, with $M = (N^2 - 1)$, respectively. They coincide with the projection of $\mathbf{adj} \otimes \mathbf{adj}$ in (5.3.14) into the symmetric square and alternating square components

$$\text{Sym}^2 \mathbf{adj} \cong \mathbf{S} \oplus \mathbf{adj}_s \oplus \mathbf{S}\bar{\mathbf{S}} \oplus \mathbf{A}\bar{\mathbf{A}}, \quad \text{Alt}^2 \mathbf{adj} \cong \mathbf{adj}_a \oplus \mathbf{S}\bar{\mathbf{A}} \oplus \mathbf{A}\bar{\mathbf{S}}. \quad (5.3.33)$$

It is easy to check that the dimensions agree between the left- and right-hand side. We write

$$O^{(ab)} = \sum_{\mathbf{r} \in \text{Sym}^2 \mathbf{adj}} P_{i_s, (cd)}^{(ab)} O^{(cd)}, \quad O^{[ab]} = \sum_{\mathbf{r} \in \text{Alt}^2 \mathbf{adj}} P_{\mathbf{r}, [cd]}^{[ab]} O^{[cd]}. \quad (5.3.34)$$

The projectors $P_{\mathbf{r}}$ and $P_{\mathbf{r}}$ can be determined by demanding the following conditions:

$$P_{\mathbf{r}, (ab)}^{(ab)} = \dim \mathbf{r}, \quad P_{\mathbf{r}, [ab]}^{[ab]} = \dim \mathbf{r}, \quad P_{\mathbf{r}} P_{\mathbf{s}} = \delta_{\mathbf{r}, \mathbf{s}} P_{\mathbf{s}}, \quad (5.3.35)$$

where we have omitted the indices on the last of the three equations, as well as completeness:

$$\sum_{\mathbf{r} \in \text{Sym}^2 \mathbf{adj}} P_{\mathbf{r}, (cd)}^{(ab)} = (\mathbb{1}_s)_{cd}^{ab}, \quad \sum_{\mathbf{r} \in \text{Alt}^2 \mathbf{adj}} P_{\mathbf{r}, [cd]}^{[ab]} = (\mathbb{1}_a)_{cd}^{ab}, \quad (5.3.36)$$

where

$$(\mathbb{1}_s)_{cd}^{ab} = \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \quad (\mathbb{1}_a)_{cd}^{ab} = \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \quad (5.3.37)$$

The conditions (5.3.35), (5.3.36), (5.3.37) are overconstraining. We only report here the final results:

$$\begin{aligned}
P_{\mathbf{S},cd}^{ab} &= \frac{1}{N^2-1} \delta_{ab} \delta_{cd}, \\
P_{\mathbf{adj}_s,cd}^{ab} &= \frac{N}{N^2-4} d_{abz} d_{cdz}, \\
P_{\mathbf{S}\bar{\mathbf{S}},cd}^{ab} &= \frac{1}{4} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) + \frac{1}{2(N+1)} \delta_{ab} \delta_{cd} - \frac{1}{4} (f_{adz} f_{bcz} + f_{acz} f_{bdz}) + \frac{N}{4(N+2)} d_{abz} d_{cdz}, \\
P_{\mathbf{A}\bar{\mathbf{A}},cd}^{ab} &= \frac{1}{4} (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) - \frac{1}{2(N-1)} \delta_{ab} \delta_{cd} + \frac{1}{4} (f_{adz} f_{bcz} + f_{acz} f_{bdz}) - \frac{N}{4(N-2)} d_{abz} d_{cdz},
\end{aligned} \tag{5.3.38}$$

for the symmetric tensors, and

$$\begin{aligned}
P_{\mathbf{adj}_a,cd}^{ab} &= \frac{1}{N} f_{abz} f_{cdz}, \\
P_{\mathbf{A}\bar{\mathbf{S}},cd}^{ab} &= \frac{1}{4} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) - \frac{1}{2N} f_{abz} f_{cdz} + \frac{i}{8} (f_{acz} d_{bdz} - f_{adz} d_{bcz} - f_{bcz} d_{adz} + f_{bdz} d_{acz}), \\
P_{\mathbf{S}\bar{\mathbf{A}},cd}^{ab} &= \frac{1}{4} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) - \frac{1}{2N} f_{abz} f_{cdz} - \frac{i}{8} (f_{acz} d_{bdz} - f_{adz} d_{bcz} - f_{bcz} d_{adz} + f_{bdz} d_{acz}),
\end{aligned} \tag{5.3.39}$$

for the antisymmetric ones, where

$$d^{abc} = -2i \operatorname{tr} [\{T^a, T^b\} T^c], \quad f^{abc} = -2 \operatorname{tr} [T^a, T^b] T^c, \tag{5.3.40}$$

in terms of the anti-hermitian generators in (5.3.3). The same projectors can also be used to write projectors in the representations in (5.3.17), which read

$$\begin{aligned}
P_{\mathbf{S}_+} &= P_{\mathbf{S}}, & P_{\mathbf{S}\bar{\mathbf{S}}_+} &= P_{\mathbf{S}\bar{\mathbf{S}}}, & P_{\mathbf{A}\bar{\mathbf{A}}_+} &= P_{\mathbf{A}\bar{\mathbf{A}}}, \\
P_{\mathbf{adj}_+} &= P_{\mathbf{adj}_s}, & P_{\mathbf{adj}_-} &= P_{\mathbf{adj}_a}, & & (N \text{ odd}), \\
P_{\mathbf{adj}_+} &= P_{\mathbf{adj}_a}, & P_{\mathbf{adj}_-} &= P_{\mathbf{adj}_s}, & & (N \text{ even}), \\
P_{\mathbf{S}\bar{\mathbf{A}}_c} &= P_{\mathbf{S}\bar{\mathbf{A}}} + P_{\mathbf{A}\bar{\mathbf{S}}}.
\end{aligned} \tag{5.3.41}$$

5.4 Bootstrap setup

The general framework of the numerical conformal bootstrap is reviewed in Section 2.2; here we review the setup for conserved currents developed in [99, 100] and extended to non-Abelian currents in [101]. In particular, we will consider a four-point function of non-Abelian conserved spin 1 currents J_μ^a in $3d$, transforming in the adjoint representation of $SU(N)$. The four-point function is decomposed in terms of conformal tensor structures $\mathbb{T}_I$ and global symmetry tensor structures $T_{\mathbf{r}}$:

$$\langle J^a(P_1, Z_1) J^b(P_2, Z_2) J^c(P_3, Z_3) J^d(P_4, Z_3) \rangle = \sum_I \sum_{\mathbf{r}} \mathbb{T}_I(P_i, Z_i) T_{\mathbf{r}}^{abcd} g_{\mathbf{r}}^I(z, \bar{z}), \tag{5.4.1}$$

while the ‘‘dynamical’’ content is encoded in the g -function. The coordinates (P_i, Z_i) are those of the embedding formalism of [59]. The sum over I spans the conformal tensor structures. We refer to Appendix B of [101] for the discussion on the conformal tensor structures and the

choice of linearly independent g -functions. The sum over $\mathbf{r}$ runs over the global symmetry singlets, whose number is n_{4c} , in which the correlator can be decomposed. Determining n_{4c} is straightforward if we consider $\mathbf{adj}^{\otimes 4} = (\mathbf{adj}^{\otimes 2}) \otimes (\mathbf{adj}^{\otimes 2})$ and use the decomposition (5.3.14), recalling that singlets can only arise in tensor products between one representation and its complex conjugate.

In [101], the factorisation of conformal and global symmetry of (5.3.5) was explicitly assumed, together with invariance under parity. Invariance under charge conjugation was not stated explicitly, but implied by [101, Eq. C.2], when assuming certain symmetry properties of the global symmetry tensor structures $T_{\mathbf{r}}^{abcd}$.¹⁸ At the level of $SU(N)$ -invariant tensors, i.e. without imposing charge conjugation symmetry, for $N \geq 4$ we have 1 singlet each from the $\mathbf{S}, \mathbf{S}\bar{\mathbf{S}}, \mathbf{A}\bar{\mathbf{A}}$; 4 singlets from $(2\mathbf{adj}) \otimes (2\mathbf{adj})$; 2 singlets from $\mathbf{S}\bar{\mathbf{A}} \otimes \mathbf{A}\bar{\mathbf{S}}$ and $\mathbf{A}\bar{\mathbf{S}} \otimes \mathbf{S}\bar{\mathbf{A}}$, for a total of $n_{4c} = 3 \times 1 + 4 + 2 = 9$. This number is reduced to $n_{4c} = 8$ for $N = 3$, and $n_{4c} = 3$ for $N = 2$. Independently of N , the two adjoint representations $\mathbf{adj}^{\otimes 2}$ have always opposite charge conjugation parity, while $\mathbf{S}\bar{\mathbf{A}}$ and $\bar{\mathbf{S}}\mathbf{A}$ combine into a unique irreducible representation of $PSU(N) \rtimes \mathbb{Z}_2^C$. So, under charge conjugation, the 9 singlets for $N \geq 4$ decompose as $n_{4c} = n_{4c}^+ + n_{4c}^-$, with $n_{4c}^+ = 6$ and $n_{4c}^- = 3$.¹⁹ Imposing charge conjugation symmetry reduces the number of singlets from n_{4c} to n_{4c}^+ , and their associated tensor structures $T_{\mathbf{r}}^{abcd}$ in (5.4.1) do indeed satisfy [101, Eq. C.2]. For the $N = 2$ case, this issue does not arise since $\mathbf{adj}^{\otimes 2}$ decomposes only into self-conjugate and degeneracy-free representations. We use symmetry tensor structures

$$T_{\mathbf{r}}^{abcd} = \frac{1}{\sqrt{\dim \mathbf{r}}} P_{\mathbf{r}}^{abcd}, \quad (5.4.2)$$

where $\mathbf{r}$ runs over the representation labels in (5.3.16) and $P_{\mathbf{r}}^{abcd}$ are the corresponding projectors, whose explicit form is given in Section 5.3.4.

Coming back to (5.4.1), each g -function can be decomposed into conformal blocks as

$$g_I^{\mathbf{r}}(z, \bar{z}) = \sum_{\mathcal{O}_{\Delta, \ell, p, \mathbf{r}}} \lambda_{JJ\mathcal{O}}^{(a)} \lambda_{JJ\mathcal{O}}^{(b)} g_{\Delta, \ell}^{(I, ab)}(z, \bar{z}), \quad (5.4.3)$$

where a, b run over the basis of three-point tensor structures and operators exchanged are labelled by spacetime parity $p = \pm 1$, spin ℓ , conformal dimension Δ and irreducible representation $\mathbf{r}$ of $PSU(N) \rtimes \mathbb{Z}_2^C$. In the above decomposition, it is assumed that operators (but the current) have a unit-normalised two-point function $\langle \mathcal{OO} \rangle$.

¹⁸We are grateful to Petr Kravchuk and Marten Reehorst for bringing this to our attention and for correspondence. The bootstrap equations for non-Abelian conserved currents without assuming charge conjugation invariance (in $4d$) will appear in [267].

¹⁹For $N = 3$ we have $n_{4c} = n_{4c}^+ + n_{4c}^-$, with $n_{4c}^+ = 5$ and $n_{4c}^- = 3$. This counting can be checked using the characters in Appendix C.2.

The contribution of the identity and of the current itself to the block decomposition requires a separate discussion. In embedding coordinates, we have in d dimensions (see e.g. [259])

$$\begin{aligned}\langle J^a(Z_1, P_1) J^b(Z_2, P_2) \rangle &= C_J \delta^{ab} \frac{H_{12}}{P_{12}^d}, \\ \langle J^a(Z_1, P_1) J^b(Z_2, P_2) J^c(Z_3, P_3) \rangle &= \frac{C_J}{S_d} f^{abc} \frac{1}{P_{12}^{d/2} P_{23}^{d/2} P_{13}^{d/2}} \\ &\quad \times \left((d - (d+2)\alpha_{JJJ}) V_1 V_2 V_3 - \alpha_{JJJ} (V_1 H_{23} + V_2 H_{13} + V_3 H_{23}) \right),\end{aligned}\tag{5.4.4}$$

where C_J is the current central charge, $S_d = 2\pi^{d/2}/\Gamma(d/2)$. The norm of the two-dimensional OPE vector $(\lambda_{JJJ}^{(1)}, \lambda_{JJJ}^{(2)})$ is fixed by Ward identities to be proportional to C_J/S_d , while α_{JJJ} is a parametrization for the remaining degrees of freedom. For reference, in $d = 3$, we have $\alpha_{JJJ} = 1/2$ for an $SU(N)$ conserved current of N free bosons in the fundamental representation of $SU(N)$ and $\alpha_{JJJ} = 1$ for free fermions [258]; for YM in AdS_4 at tree level we have $\alpha_{JJJ} = 3/4 + O(g_{\text{YM}}^2)$, see Appendix A of [264].

The contributions of the identity and the current to the block decomposition are then

$$\begin{aligned}g_I^{\mathbf{S}^+}(z, \bar{z}) &= C_J^2 g_{0,0}^{(I)}(z, \bar{z}) + \dots \\ g_I^{\mathbf{r}_J}(z, \bar{z}) &= C_J \theta^{(a)} \theta^{(b)} g_{2,1}^{(I,ab)}(z, \bar{z}) + \dots\end{aligned}\tag{5.4.5}$$

where $\mathbf{r}_J = \mathbf{adj}$ for $N = 2$, $\mathbf{r}_J = \mathbf{adj}_-$ for $N \geq 3$ odd and $\mathbf{r}_J = \mathbf{adj}_+$ for $N \geq 3$ even, and where we have defined the OPE vector

$$\theta = \begin{pmatrix} \theta^{(1)} \\ \theta^{(2)} \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} 5 - 3\alpha_{JJJ} \\ -\alpha_{JJJ} \end{pmatrix}.\tag{5.4.6}$$

The symmetry properties of three-point functions

$$\langle J^a(P_1, Z_1) J^b(P_2, Z_2) O_{\Delta, \ell, p, \mathbf{r}}^A(P_3, Z_3) \rangle = \lambda_{JJO}^{(i)} \mathbb{K}_{\ell, p}^{(i)} C_{\mathbf{r}}^{ab, A}\tag{5.4.7}$$

with respect to permutations of their operators, impose selection rules on the spin of operators exchanged in the OPE, which depend on the spacetime parity p . These are determined by the symmetry properties under operator exchange of the tensor structure $\mathbb{K}_{\ell, p}$, and those of the tensor structure $C_{\mathbf{r}}$ (i.e. under $a \leftrightarrow b$). The latter is related to the exchange symmetry of the four-point tensor structure $T_{\mathbf{r}}^{abcd}$, being proportional to $C_{\mathbf{r}}^{ab, A} C_{\mathbf{r}}^{cd, B} \delta_{AB}$. Moreover, there exist combinations of $(\ell, p, \mathbf{r})$ for which there are no compatible conformal symmetry tensor structures. Finally, further restrictions on OPE coefficients are imposed by the conservation of the non-Abelian current. The selection rules were worked out in [101] and are summarised in Table 5.2.

The choice of crossing conditions is described in [101, App. B] and consists of 4 bulk constraints and 1 point constraint, each of which is a system of constraints labelled by $\mathbf{r}$.

parity	rep	spin
+	R_s	ℓ even
+	R_a	ℓ odd
-	R_s	$\ell \neq 1$
-	R_a	$\ell \geq 1$

Table 5.2: Spin selection rules in $\langle JJJJ \rangle$ depending on parity and symmetry property under exchange.

The crossing equations are

$$\begin{aligned}
\mathcal{M}_s^r g_{[0,1,1,0]}^s(z, \bar{z}) &= g_{[1,1,0,0]}^r(1-z, 1-\bar{z}), \\
\mathcal{M}_s^r g_{[0,0,0,0]}^s(z, \bar{z}) &= g_{[0,0,0,0]}^r(1-z, 1-\bar{z}), \\
\mathcal{M}_s^r g_{[1,0,1,0]}^s(z, \bar{z}) &= g_{[1,0,1,0]}^r(1-z, 1-\bar{z}), \\
\mathcal{M}_s^r g_{[1,1,1,1]}^s(z, \bar{z}) &= g_{[1,1,1,1]}^r(1-z, 1-\bar{z}), \\
\mathcal{M}_s^r g_{[-1,-1,1,1]}^s(1/2, 1/2) &= g_{[1,-1,-1,1]}^r(1/2, 1/2),
\end{aligned} \tag{5.4.8}$$

where $I = [h_1, h_2, h_3, h_4]$ are convenient labels for the conformal tensor structures (see [268]), and where $\mathcal{M}$ is the crossing matrix (see [101, Eq. C.4–C.5] for explicit expression)²⁰

$$\mathcal{M}_s^r = \frac{\sum_{a,b,c,d} T_{\mathbf{r}}^{cbad} T_{\mathbf{s}}^{abcd}}{\sum_{a,b,c,d} T_{\mathbf{r}}^{cbad} T_{\mathbf{r}}^{cbad}}. \tag{5.4.9}$$

Each equation is then differentiated with respect to $(z, \bar{z})$ at $z = \bar{z} = 1/2$ obtaining

$$\mathcal{M}_s^r \partial_z^m \partial_{\bar{z}}^n g_{[0,1,1,0]}^s(1/2, 1/2) - (-1)^{n+m} \partial_z^m \partial_{\bar{z}}^n g_{[1,1,0,0]}^r(1/2, 1/2) = 0, \tag{5.4.10}$$

$$(\mathcal{M}_s^r - (-1)^{n+m} \delta_s^r) \partial_z^m \partial_{\bar{z}}^n g_{[0,0,0,0]}^s(1/2, 1/2) = 0, \tag{5.4.11}$$

$$(\mathcal{M}_s^r - (-1)^{n+m} \delta_s^r) \partial_z^m \partial_{\bar{z}}^n g_{[1,0,1,0]}^s(1/2, 1/2) = 0, \tag{5.4.12}$$

$$(\mathcal{M}_s^r - (-1)^{n+m} \delta_s^r) \partial_z^m \partial_{\bar{z}}^n g_{[1,1,1,1]}^s(1/2, 1/2) = 0, \tag{5.4.13}$$

$$\mathcal{M}_s^r g_{[0,1,1,0]}^s(1/2, 1/2) - g_{[1,-1,-1,1]}^r(1/2, 1/2) = 0. \tag{5.4.14}$$

Note that the matrix $\mathcal{M}_s^r - (-1)^{n+m} \delta_s^r$ does not have maximal rank. In particular, when $n+m$ is even, the rank is equal to the number of anti-symmetric irreps r_a and when $n+m$ is odd, the rank is equal to the number of symmetric irreps r_s , see (5.3.17) and (5.4.16) below.

All the crossing constraints can be put together into the following sum rule

$$\sum_{\mathcal{O}_{\Delta,\ell,p,\mathbf{r}}} \lambda_{JJ\mathcal{O}}^T \vec{\mathcal{V}}_{\Delta,\ell,p,\mathbf{r}} \lambda_{JJ\mathcal{O}} = 0, \tag{5.4.15}$$

where the “crossing vector” $\vec{\mathcal{V}}$ is a vector of matrices built from derivatives of conformal blocks. Each component is a matrix (possibly 1×1) contracted with a vector of OPE coefficients

²⁰The crossing matrix for $SU(2)$ is the same as the crossing matrix for $SO(3)$.

$\lambda_{JJ\mathcal{O}} = (\lambda_{JJ\mathcal{O}}^{(I)})$. The length of these crossing vectors corresponds to the number of non-trivial equations included in (5.4.10) through (5.4.14), times the number of derivatives included ($n + m \leq \Lambda$, $n \leq m$). In particular

- (5.4.10) gives $(r_s + r_a) \times (d_{\text{even}} + d_{\text{odd}})$ equations
- (5.4.11), (5.4.12) and (5.4.13) give each $r_s \times d_{\text{odd}} + r_a \times d_{\text{even}}$ equations
- (5.4.14) gives $r_s + r_a$ equations

where

$$r_s = \begin{cases} 2 & N = 2 \\ 3 & N \geq 3 \end{cases}, \quad r_a = \begin{cases} 1 & N = 2 \\ 2 & N = 3 \\ 3 & N \geq 4 \end{cases}, \quad (5.4.16)$$

$$d_{\text{even}} = \frac{1}{2} \left(\left\lfloor \frac{\Lambda}{2} \right\rfloor + 1 \right) \left(\left\lfloor \frac{\Lambda}{2} \right\rfloor + 2 \right), \quad d_{\text{odd}} = \frac{1}{2} \left\lfloor \frac{\Lambda+1}{2} \right\rfloor \left(\left\lfloor \frac{\Lambda+1}{2} \right\rfloor + 1 \right).$$

Thus, we find a total of $(d_{\text{even}} + 4d_{\text{odd}} + 1)r_s + (4d_{\text{even}} + d_{\text{odd}} + 1)r_a$ components.

In (5.4.15), two particular contributions will play a role in our bootstrap bounds: the identity $\mathbb{1}$ and the current itself J . The contribution of these two particular operators to the block decomposition is given in (5.4.5). We single out these terms in the sum rule (5.4.15) and divide by the identity OPE coefficient to obtain

$$\vec{\mathcal{V}}_{\mathbb{1}} + \frac{1}{C_J} \theta^\top \vec{\mathcal{V}}_J \theta + \sum_{\mathcal{O}_{\Delta, \ell, p, \mathbf{r}}} \frac{1}{C_J^2} \lambda_{JJ\mathcal{O}}^\top \vec{\mathcal{V}}_{\Delta, \ell, p, \mathbf{r}} \lambda_{JJ\mathcal{O}} = 0. \quad (5.4.17)$$

As usual in the numerical conformal bootstrap, each component of the crossing vector can be approximated arbitrarily well by rational functions in the scaling dimension Δ . More precisely,

$$\vec{\mathcal{V}}_{\Delta, \ell, p, \mathbf{r}} \simeq \chi_\ell(\Delta) \vec{M}_{\ell, p, \mathbf{r}}(\Delta), \quad (5.4.18)$$

where $\chi_\ell(\Delta) = (4r_*)^\Delta / \prod_i (\Delta - p_i) \geq 0$ for Δ above the unitarity bound, with $r_* = 3 - 2\sqrt{2}$ is the modulus of the radial coordinate [65] at the crossing symmetric point, and the $\vec{M}_{\ell, p, \mathbf{r}}(\Delta)$ are vectors of polynomial matrices in Δ .

In the Semi-Definite Programs (SDPs) that we will formulate, the action of functionals on $\vec{\mathcal{V}}$ translates into acting on the vector of polynomial matrices $\vec{M}$. Since $\chi_\ell \geq 0$ in unitary theories, the conditions of positivity on $\vec{\mathcal{V}}$ are equivalent to positivity conditions on $\vec{M}$.

Notation. We denote as s , a and $s\bar{s}$ the lowest dimensional parity-even scalars $(\ell, p) = (0, +)$ that appear in the $\mathbf{S}_+$, $\mathbf{adj}_+$ and $\mathbf{S}\bar{\mathbf{S}}_+$ representations respectively. We use primes to indicate sub-leading operators in a given selection sector, determined by spin, spacetime parity and global symmetry representation. As an exception, we denote by $T'_{\mu\nu}$ the first parity-even,

singlet, $\ell = 2$ operator (recall that the stress tensor is not present in the boundary CFT of a QFT in rigid AdS). The unitarity bound is denoted by

$$\Delta_{\text{UB}}(\ell) = \begin{cases} \frac{d-2}{2} & \ell = 0, \\ \ell + d - 2 & \ell \geq 1, \end{cases} \quad (d = 3). \quad (5.4.19)$$

5.5 Bounds on C_J

In order to address the decoupling scenario discussed earlier, it will be useful to set up an SDP suited for OPE coefficient extremisation. This is because if we can bound C_J to be strictly above zero, the two-point function of J will be non-zero for any choice of the AdS radius L , ruling out the decoupling scenario. In particular, we look for the optimal functional α satisfying:

$$\begin{aligned} \text{objective:} & \quad \alpha(\vec{\mathcal{V}}_1) \\ \text{normalisation:} & \quad \alpha(\theta^\top \vec{\mathcal{V}}_J \theta) \\ \text{conditions:} & \quad \begin{aligned} \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) &\succeq 0 & \Delta &\geq \Delta_{\text{UB}}(\ell) & \mathbf{r} \in \mathbf{R}_s & \ell = 0, 2, 4, \dots \\ \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) &\succeq 0 & \Delta &\geq \Delta_{\text{UB}}(\ell) & \mathbf{r} \in \mathbf{R}_a & \ell = 1, 3, 5, \dots \\ \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) &\succeq 0 & \Delta &\geq \Delta_{\text{UB}}(\ell) & \mathbf{r} \in \mathbf{R}_s & \ell = 0, 2, 3, \dots \\ \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) &\succeq 0 & \Delta &\geq \Delta_{\text{UB}}(\ell) & \mathbf{r} \in \mathbf{R}_a & \ell = 1, 2, 3, \dots \end{aligned} \end{aligned} \quad (5.5.1)$$

This SDP is parametric in θ , namely in α_{JJJ} , see (5.4.6). The optimal solution minimises the objective. Let $P = \max \alpha(\vec{\mathcal{V}}_1)$ be the optimal objective for a given value of α_{JJJ} . Acting with a functional satisfying the SDP constraints on the crossing equation (5.4.17), we have

$$\frac{1}{C_J} = -\alpha(\mathcal{V}_1) - \sum_{\mathcal{O}_{\Delta,\ell,p,\mathbf{r}}} \frac{1}{C_J^2} \lambda_{JJ\mathcal{O}}^\top \alpha(\vec{\mathcal{V}}_{\Delta,\ell,p,\mathbf{r}}) \lambda_{JJ\mathcal{O}} \leq -\alpha(\mathcal{V}_1), \quad (5.5.2)$$

and thus

$$C_J \geq \max \left(-\frac{1}{\alpha(\vec{\mathcal{V}}_1)} \right) = -\frac{1}{P}. \quad (5.5.3)$$

We recall that $\alpha(\vec{\mathcal{V}}_1) < 0$ is required for the SDP (5.5.1) to be feasible.

Hence, for each value of α_{JJJ} we obtain a lower bound on C_J . A priori, the problem could be unbounded and there could exist values of α_{JJJ} for which $P \rightarrow -\infty$. This happens in our setup for all α_{JJJ} at low derivative order, e.g. at $\Lambda = 4$, and is known to generically happen when bounding OPE coefficients from the bootstrap of scalar four-point functions for large values of the external operator dimension. Fortunately, the problem is bounded in this case, and we establish a non-trivial rigorous lower bound on the current central charge C_J . This is analogous to the lower bounds on the stress tensor central charge C_T that have been obtained from the bootstrap of four stress tensors [104] or four abelian conserved currents [99].

We normalise our numerical bounds by the current central charge in a theory of a Free Boson (FB) in the fundamental of $SU(N)$, denoted by C_J^{FB} (see Appendix D.1 for further details). In the plots of the lower bounds as a function of α_{JJJ} , we add for reference blue dots

corresponding to this theory, $(\alpha_{JJJ}, C_J/C_J^{\text{FB}}) = (1/2, 1)$, and to the theory of a free fermion in the fundamental of $SU(N)$, $(C_J^{\text{FF}}/C_J^{\text{FB}}, \alpha_{JJJ}) = (1, 2\sqrt{2})$ [258].

The $SU(2)$ bound is reported in Figure 5.3 with $\Lambda = 11, 19, 23$, and for $SU(3)$ in Figure 5.4, with $\Lambda = 11, 19$. As no rigorous analytic bound is known on the variable α_{JJJ} ,²¹ we also present in Figure 5.5 the same $\Lambda = 23$ bound in terms of the compact variable $\chi = \frac{2}{\pi} \arctan(\alpha_{JJJ} - \frac{3}{4})$, to show that the bound monotonically increases as $\alpha_{JJJ} \rightarrow \pm\infty$, i.e. $\chi \rightarrow \pm 1$.

Note that for $SU(2)$ the SDP is numerically cheaper than the one in (5.5.1), since fewer representations are exchanged in the tensor product of two adjoints (5.3.18), resulting in fewer sum rules obtained from a given crossing equation.

Finally, we have obtained the absolute lower bound on C_J for higher rank $SU(N)$ groups at $\Lambda = 19$. In this case, we have not mapped the entire lower bound as a function of α_{JJJ} , but used instead the Navigator method [78] to minimize the objective $\max \alpha(\mathcal{V}_1)$ (inversely proportional to the C_J lower bound) as a function of α_{JJJ} . A plot of $\min C_J$ as a function of N is reported in Figure 5.6, together with the value of α_{JJJ}^* for which the minimum is attained.

As shown in these figures, the values of the minima and the corresponding α_{JJJ} seem to converge to a fixed value in the large N limit. This mirrors the behaviour of C_J in theories such as the $O(N)$ vector model and Gross–Neveu–Yukawa, as observed in [271]. By contrast, it differs from the expected behaviour of the central charge C_T [258], which instead scales linearly with N .

The assumptions listed in (5.5.1) hold in any unitary theory, making our bounds applicable to any CFT with global $SU(N)$ symmetry. In this chapter, however, we focus on the dual description of YM in AdS, which is a non-local CFT without a stress tensor. In principle, the bounds presented in this section could be strengthened by removing $T_{\mu\nu}$ from the spectrum and imposing a gap in the spin-2, parity-even singlet sector. The challenge is identifying the appropriate gap: when $g_{\text{YM}} \rightarrow 0$, the first operator is (schematically) $J_\mu J_\nu$ with dimension $\Delta = 4$, but as the coupling grows, this operator acquires an anomalous dimension. Since the minimum of C_J should correspond to a strong-coupling regime, this anomalous dimension could be $O(1)$, making the analysis highly sensitive to the assumed gap. We leave this investigation to future work, since it would not alter the main conclusion: the decoupling scenario is ruled out.

5.6 Bounds on scaling dimensions of scalars

Having rigorously excluded the decoupling scenario in the previous subsection, we now turn to the merging–annihilation and Higgsing scenarios. Our strategy is to impose constraints on operator dimensions. In both cases, the key question is which operators can cross marginality, and at what point this occurs. We start in Section 5.6.1 by obtaining general exclusion bounds

²¹This is to contrast with the parameters of the OPE ratios entering $\langle TTT \rangle$ and $\langle JJJ \rangle$, which are known to satisfy the “conformal collider bounds” [269, 270]. It is known that $1/2 \leq \alpha_{JJJ} \leq 1$ if and only if the binding energy from non-Abelian gauge interactions for well-separated neutral 2-particle states is attractive, see [259, Eq. 4.23], where in our conventions $\lambda_{JJJ} = \frac{C_J}{4\pi} \alpha_{JJJ}$.

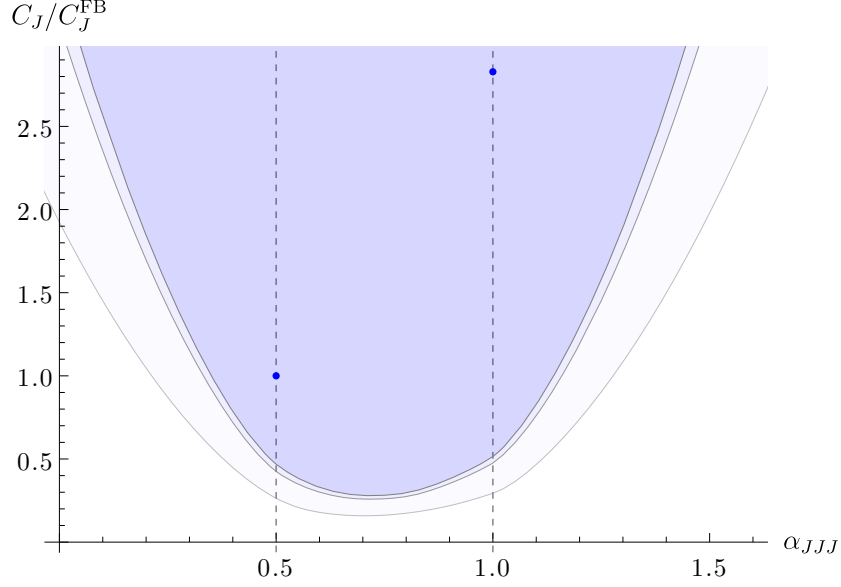


Figure 5.3: Lower bound on C_J in the case of $SU(2)$ as a function of α_{JJJ} . The shaded area is allowed, darker colour corresponding to higher Λ , we have $\Lambda = 11, 19$ and 23 respectively. The dashed vertical lines and the blue dots at $\alpha_{JJJ} = 1/2, 1$ denote the free boson and free fermion theory, respectively. We see that $C_J = 0$ is excluded for any choice of α_{JJJ} .

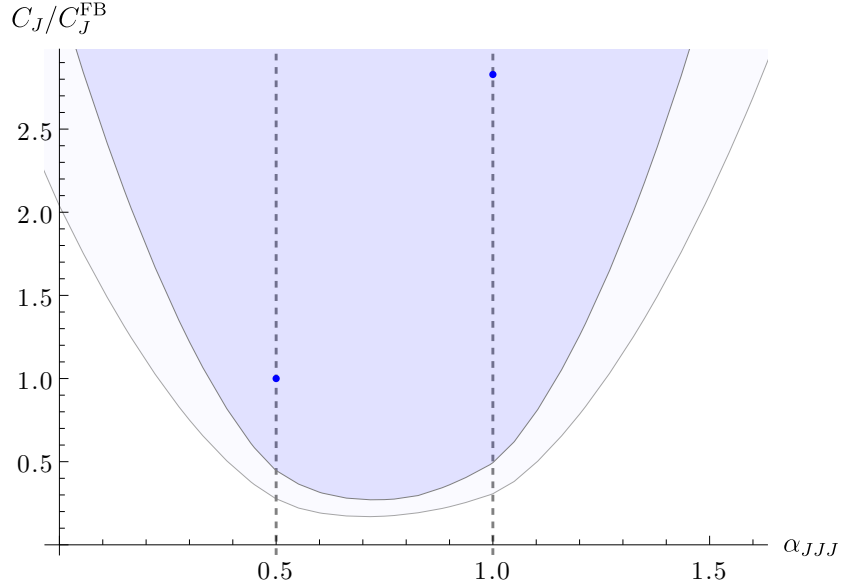


Figure 5.4: Lower bound on C_J in the case of $SU(3)$ as a function of α_{JJJ} . The shaded area is allowed, the darker area corresponding to $\Lambda = 19$ (and the rest to $\Lambda = 11$). The dashed vertical lines and the blue dots at $\alpha_{JJJ} = 1/2, 1$ denote the free boson and free fermion theory, respectively. As in the $SU(2)$ case, $C_J = 0$ is excluded for any choice of α_{JJJ} .

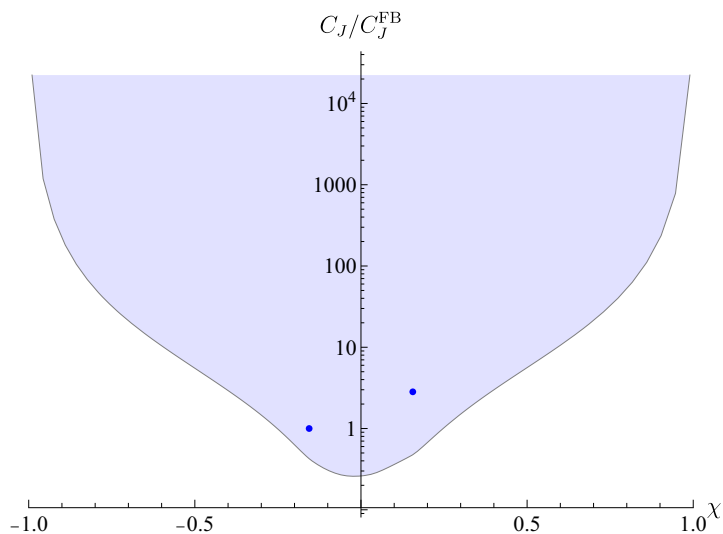


Figure 5.5: Lower bound on C_J in the case of $SU(2)$ as a function of a compact parametrization of the OPE ratio $\chi \in [-1, 1]$. The shaded area is allowed and corresponds to the $\Lambda = 19$ computation of Figure 5.3. The vertical axis is on a logarithmic scale.

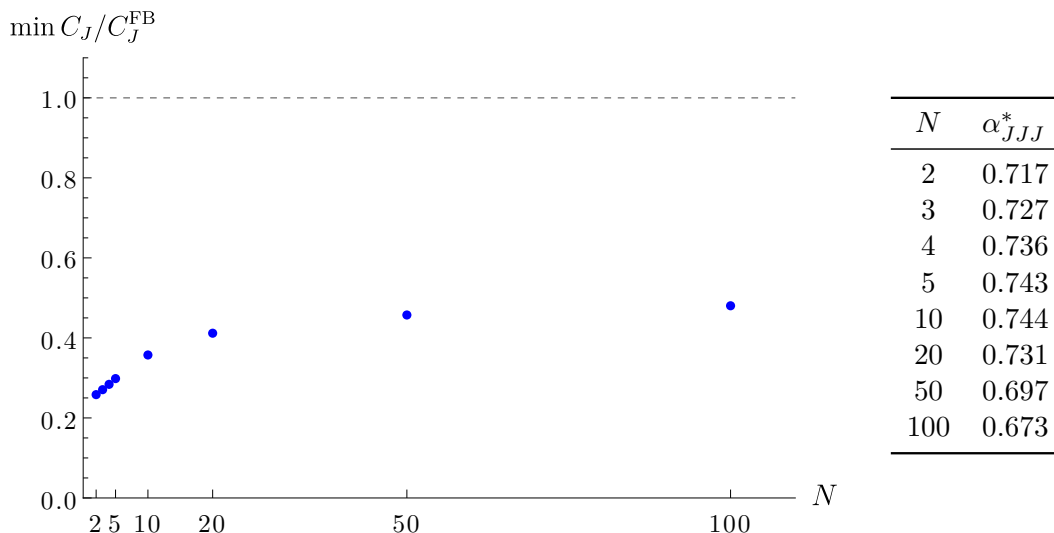


Figure 5.6: Absolute lower bound of C_J as a function of N for $SU(N)$ non-Abelian conserved currents. The bound is obtained by scanning over α_{JJJ} . The minimum is reached for α_{JJJ}^* , whose value is reported on the table to the right of the plot. The bound was obtained at $\Lambda = 19$.

assuming nothing but the unitarity bound, and mild twist gaps used for the stability of the numerical algorithm. These bounds are clearly generic, as we have not yet supplied any input that would single out specific theories. Thus, in order to get more constraining results, we must direct the bootstrap to focus on theories that arise as boundary limits of YM in AdS, rather than on generic $3d$ CFTs with a conserved current. We implement this refinement in two steps. First, in Sections 5.6.2 and 5.6.3 we inject information about the theory in the form of gaps on a handful of sectors and on sub-leading operators after those we are trying to bound. Second, in Section 5.6.4, we probe the “flow” itself by first demanding that leading operators sit at the

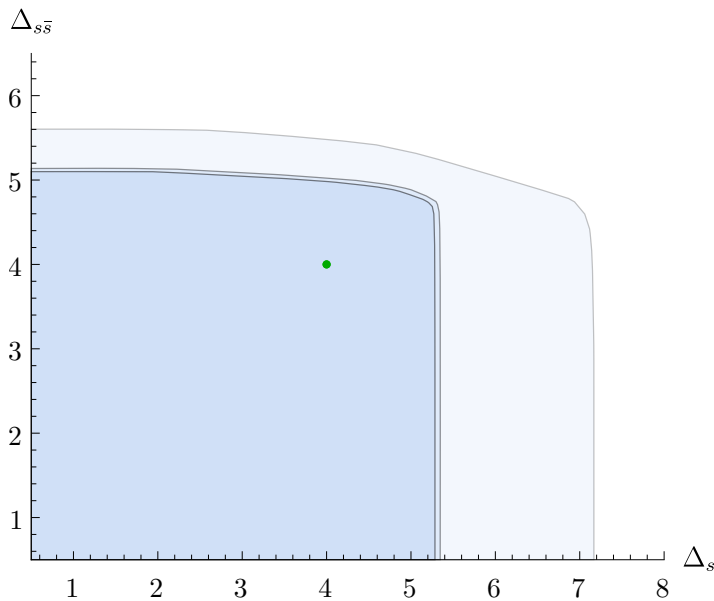


Figure 5.7: Gap maximisation bound for $(\Delta_s, \Delta_{s\bar{s}})$ in the case of $SU(2)$. The shaded area is allowed, darker colour means higher $\Lambda = 11, 19, 23$. The green dot at $(\Delta_s, \Delta_{s\bar{s}}) = (4, 4)$ denotes the generalised free vector current theory. The precise spectrum assumptions are reported in (D.2.2).

values predicted by the GFV, and then relaxing this assumption progressively in terms of a parameter δ . This parameter provides an indirect probe for L .

Since the numerics involved in this section are quite expensive, we consider only $SU(2)$ and $SU(3)$ theories. The precise Semi-Definite Programs (SDP) used to obtain the bounds of this section are deferred to Appendix D. Before proceeding to specific plots we would like to emphasize that while an extrapolation to $\Lambda \rightarrow \infty$ would be desirable, and perhaps provide a clearer physical picture, the values of Λ for which we have run do not yet seem to exhibit linear convergence, preventing us from doing so. While higher values of Λ would allow us to improve in this direction, we found that the numerics became prohibitively expensive.

5.6.1 Upper bounds

We start by providing upper bounds on the scaling dimensions of the leading parity-even scalar operators. These are s , a , and $s\bar{s}$ for $SU(3)$, while for $SU(2)$ we have only s and $s\bar{s}$.²² In Figure 5.7 we report the allowed region in the $(\Delta_s, \Delta_{s\bar{s}})$ plane for $SU(2)$, and in Figure 5.8 in $(\Delta_s, \Delta_a, \Delta_{s\bar{s}})$ space for $SU(3)$. In Figure 5.9 we provide two-dimensional projections of the bounds in Figure 5.8.

These bounds assume only unitarity and apply to any CFT with conserved $SU(2)$ or $SU(3)$ conserved currents. Note that since a generalised free current is a consistent solution to crossing, the points $(\Delta_s, \Delta_{s\bar{s}}) = (4, 4)$ for $SU(2)$ and $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}) = (4, 4, 4)$ for $SU(3)$ are

²²As discussed in Section 5.3.2 and shown in Table C.1, the first adjoint scalar in the $SU(2)$ GFV has dimension $\Delta = 7$, which makes this charged scalar sector less relevant for the Higgsing scenario.

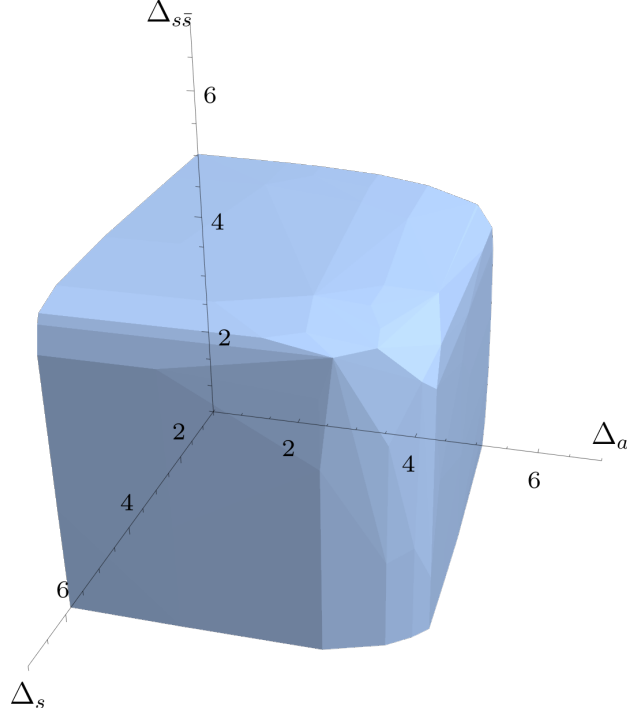


Figure 5.8: Gap maximisation bound for $(\Delta_s, \Delta_a, \Delta_{s\bar{s}})$ in the case of $SU(3)$ for $\Lambda = 23$. The shaded coloured region is allowed. The precise spectrum assumptions are reported in (D.2.3).

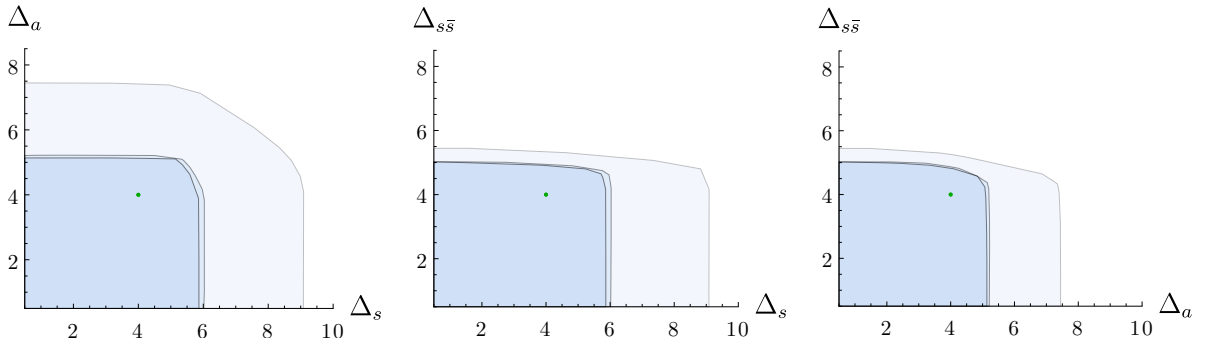


Figure 5.9: Projections of the gap maximisation bound in Figure 5.8 for $(\Delta_s, \Delta_a, \Delta_{s\bar{s}})$ in the case of $SU(3)$, to the three planes (Δ_s, Δ_a) , $(\Delta_s, \Delta_{s\bar{s}})$ and $(\Delta_a, \Delta_{s\bar{s}})$ respectively. The shaded area is allowed; darker colours indicate higher values of $\Lambda = 11, 19, 23$. The green dot at $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}) = (4, 4, 4)$ denotes the generalised free vector current theory. The precise spectrum assumptions are reported in (D.2.3).

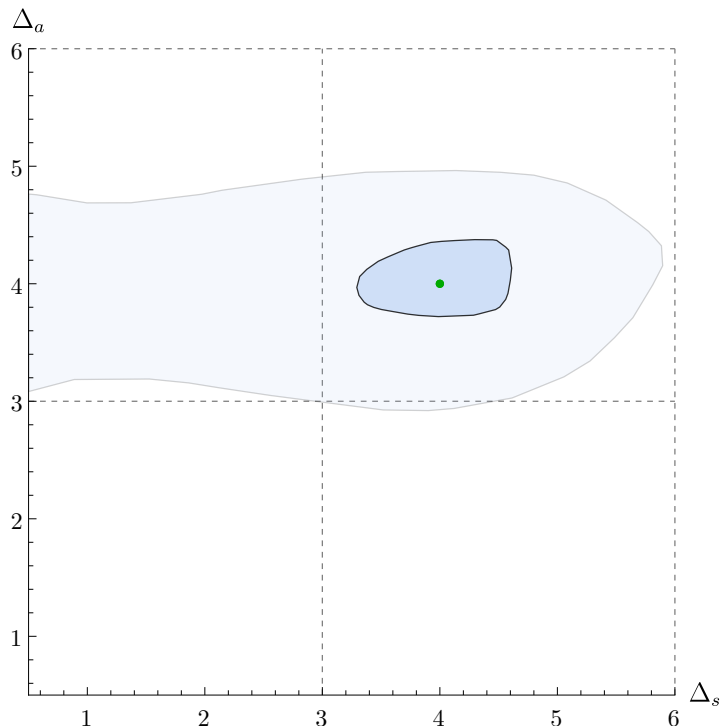


Figure 5.10: Allowed region for (Δ_s, Δ_a) in the case of $SU(3)$ assuming $\Delta_{s'} \geq 6$, $\Delta_{a'} \geq 6$ and $\Delta_{T'_{\mu\nu}} \geq 4$ (see (D.3.1)). The shaded areas correspond to $\Lambda = 11$ (lighter) and $\Lambda = 19$ (darker). The green dot $(\Delta_s, \Delta_a) = (4, 4)$ corresponds to the GFV. The dashed lines denote either marginality or the value of the gap imposed on the sub-leading scalar operators.

always allowed. Moreover, since these are gap maximisation bounds, any point in the square $(\Delta_s, \Delta_{s\bar{s}}) \in [1/2, 4] \times [1/2, 4]$ or in the box $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}) \in [1/2, 4] \times [1/2, 4] \times [1/2, 4]$ is also allowed. As expected, we cannot infer much about YM from these bounds, given that all the scalars we considered are allowed to be both relevant and irrelevant.

5.6.2 Allowed region with strong assumptions

Before considering more physically motivated gaps, we study what kind of constraints arise for Δ_s and Δ_a when imposing very aggressive gaps, which essentially correspond to the spectral gaps of the UV GFV. In particular, we take s and a to be isolated in the spectrum and we impose $\Delta_{s'} \geq 6$, $\Delta_{a'} \geq 6$ and $\Delta_{T'_{\mu\nu}} \geq 4$.²³ The result, reported in Figure 5.10 for $SU(3)$, is an isolated island of allowed parameter space around the generalised free vector (GFV) theory. The observation that the entire allowed region can collapse into an island centred around the UV of the RG flow, provided sufficiently strong assumptions are made, is encouraging and motivated the methodology presented in Section 5.6.4. As one might expect, the resulting bound already rules out both the Higgsing and merging scenarios. It is noteworthy that this occurs already for $\Lambda = 19$. If we believe that at least one of these scenarios must be realised, then at least one

²³Recall that the boundary CFT does not have a stress tensor, and $T'_{\mu\nu}$ denotes the lightest spin 2 parity-even operator.

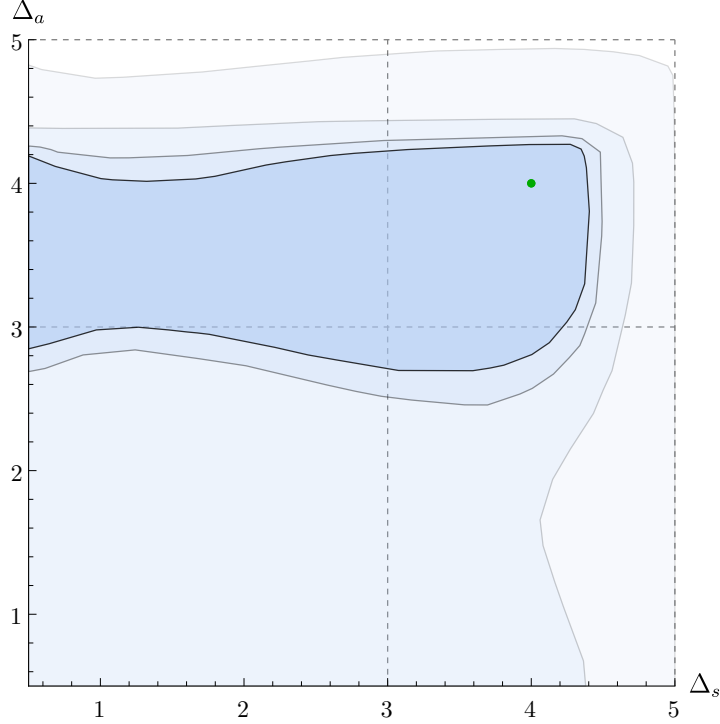


Figure 5.11: Allowed region for (Δ_s, Δ_a) in the case of $SU(3)$ with the gaps from (D.4.1). The shaded areas (from lighter to darker) correspond to $\Lambda = 11, 15, 19, 23$. The green dot $(\Delta_s, \Delta_a) = (4, 4)$ denotes the GFV. The dashed lines denote either marginality or the value of the gap imposed on the sub-leading operators.

of our assumptions cannot hold across the full flow. In more precise terms, at least one among the operators s' , a' , and $T'_{\mu\nu}$ must acquire a negative anomalous dimension.

5.6.3 Allowed region with moderate assumptions

We now take a middle route between Sections 5.6.1 and 5.6.2, assuming gaps inspired by the GFV, but at the same time allowing operators to obtain anomalous dimensions. In particular, we assume a handful of gaps $\Delta \geq \Delta_{\ell,p,\mathbf{r}}^{\text{gap}}$ in sectors $(\ell, p, \mathbf{r})$ with small spin ℓ , which lie between the unitarity bound and spectral gaps of the GFV. The precise gap assumptions are reported in Table D.1. We impose that s and a , the lightest singlet and adjoint scalars, are isolated in the spectrum, and $\Delta_{s'}, \Delta_{a'} \geq 5$ as gaps on the sub-leading operators. The allowed region in (Δ_s, Δ_a) space, which has been calculated for $\Lambda = 11, 15, 19$ and 23 , is given in Figure 5.11. The SDP used to obtain this allowed region is reported in (D.4.1). While in this case we do not observe the creation of an island, there seems to be a trend with increasing constraining power, measured by Λ , towards pushing Δ_a above the value of $\Delta_a = 3$. Figure 5.11 provides the first faint evidence in favour of the merging scenario, which will be strengthened by the results of next section.

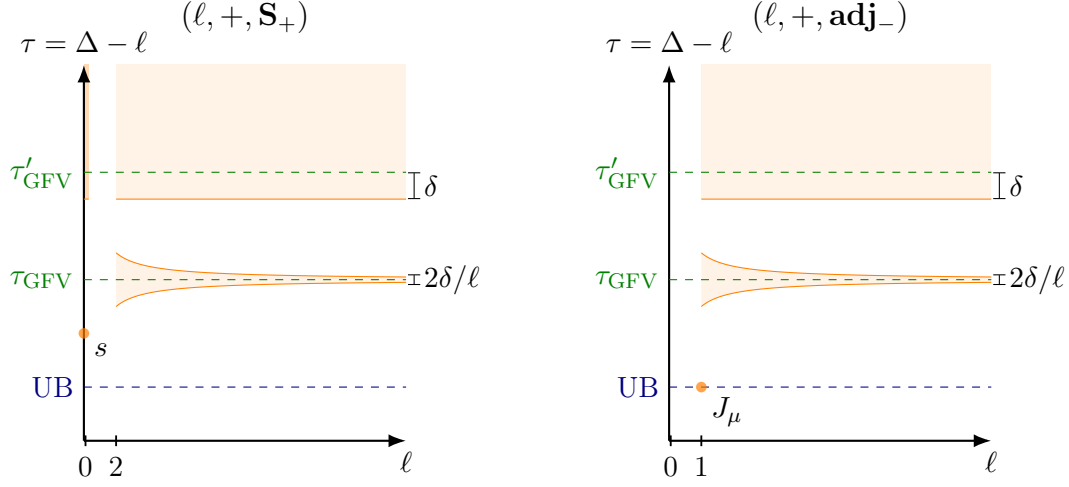


Figure 5.12: Schematic representation of the δ -gaps for two examples, the $(\ell, +, \mathbf{S}_+)$ sector (on the left) and the $(\ell, +, \mathbf{adj}_-)$ sector (on the right). The dots correspond to isolated operators, while bands correspond to intervals or gaps. We represented the spin ℓ as a continuum (except for $\ell = 0$ in the left panel, which has to be treated separately) even though the two sectors only contain even and odd integer spins, respectively. Note that the twist of the leading and sub-leading operators in the GFV (the dashed green lines) is not constant at small spin, so the vertical position in the picture is just schematic in this case.

5.6.4 Tracing the flow

We now implement the idea of “following” the RG flow of YM in AdS, alluded to in the previous subsections, starting from the GFV and flowing to what is (potentially) the transition point.²⁴ Recall that as long as the Dirichlet bc exists in the form of a unitary and stable boundary CFT, the boundary limit of YM in AdS is described by this one-parameter family of CFTs. The parameter of the flow is g_{YM} (or $L\Lambda_{\text{YM}}$) and continuously connects to the GFV at $g_{\text{YM}} = 0$. The above idea is implemented by assuming a spectrum for operators exchanged in the OPE $J \times J$ which is parametrically close to that of the GFV. This parametric proximity is quantified in terms of a parameter that we call δ . At $\delta = 0$ the spectrum of the first few operators appearing in $J \times J$ is forced to be that of the GFV (see precise details below). As we relax δ to some non-zero value, the scaling dimensions of operators are allowed to live in an interval centred around the GFV value, whose precise size depends on the chosen value of δ . This, then, allows the operators to attain an anomalous dimension, the permissible size of which depends on δ . It is in this sense that δ behaves as a probe for the RG flow. At a technical level, this type of spectrum is implemented in the bootstrap using what is now referred to as “interval positivity”, introduced in [274, Sec. 5.4].

The dependence on the parameter δ of our gaps is inspired by the universal light-cone bootstrap result stating that anomalous dimensions of spin ℓ $[JJ]$ operators in $3d$ scale as $1/\ell$ at large spin [67, 68]. As discussed in Section 5.3.2, the first operator in each $(\ell, p, \mathbf{r})$ sector

²⁴In addition to the analysis presented in the preceding sections, the strategy and gap assumptions adopted in this section were partially inspired by [272] and the hybrid bootstrap study of [273].

is of the type $[JJ]$, with the exception of the identity $\mathbf{1}$ and the current itself J , which have fixed dimensions. Moreover, thanks to the result reviewed in Section 5.3.3, we know that the $(\ell, p, \mathbf{S}\bar{\mathbf{A}}_c)$ sector anomalous dimensions scale as $1/\ell^a$ with $a \geq 2$. The precise set of gap assumptions for the case of $SU(3)$ is as follows:

- In each $(\ell, p, \mathbf{r})$ sector exchanged in $\langle JJJJ \rangle$, except for $\ell = 0$ or $\mathbf{S}\bar{\mathbf{A}}_c$, we assume that
 - the first operator can be in an interval $[\Delta_{\text{GFV}} - \delta/\ell, \Delta_{\text{GFV}} + \delta/\ell]$ of width $2\delta/\ell$ centered around the first operator in the GFV spectrum Δ_{GFV} in the $(\ell, p, \mathbf{r})$ sector;
 - the second operator can have dimension $\Delta \geq \Delta'_{\text{GFV}} - \delta$, where Δ'_{GFV} is the dimension of the sub-leading operator in the $(\ell, p, \mathbf{r})$ of the GFV spectrum.²⁵
- In each $(\ell, p, \mathbf{S}\bar{\mathbf{A}}_c)$ sector we assume the first operator can be in an interval $[\Delta_{\text{GFV}} - \delta/\ell^2, \Delta_{\text{GFV}} + \delta/\ell^2]$, while the second operator can have dimension $\Delta \geq \Delta'_{\text{GFV}} - \delta$
- In the $(0, -, \mathbf{r})$ sector we assume that the first operator can be in an interval $[\Delta_{\text{GFV}} - \delta, \Delta_{\text{GFV}} + \delta]$, while the second operator can have dimensions $\Delta \geq \Delta'_{\text{GFV}} - \delta$
- In the $(0, +, \mathbf{S}_+)$, $(0, +, \mathbf{adj}_+)$ and $(0, +, \mathbf{S}\bar{\mathbf{S}}_+)$ sectors we assume first an isolated operator of dimension Δ_s , Δ_a and $\Delta_{s\bar{s}}$ respectively, and a second operator that can have dimension $\Delta \geq \Delta'_{\text{GFV}} - \delta = 6 - \delta$
- The current itself is exchanged in the $(1, +, \mathbf{adj}_-)$ sector at $\Delta = 2$ with OPE ratio parametrised by α_{JJJ} as described in (5.4.6).

We will refer to this set of gap assumptions as δ -gaps.²⁶

The gap assumptions depend on 5 parameters $(\delta, \Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$: we will choose an array of values for δ and scan over the remaining ones. For this, we will use the Navigator method [78] to explore the space of allowed scaling dimensions $(\Delta_s, \Delta_a, \Delta_{s\bar{s}})$ as well as the OPE ratio α_{JJJ} . For a fixed value of δ , the allowed region in $(\Delta_s, \Delta_s, \Delta_{s\bar{s}})$ has the interpretation of containing the one-parameter family of conformal data of the Dirichlet boundary CFT for all values of the coupling $0 \leq g_{\text{YM}} \leq g_{\text{YM}}(\delta)$,²⁷ where we have defined $g_{\text{YM}}(\delta)$ to be the maximum value of g_{YM} that can be probed with our gap assumptions at a given δ . By definition, the allowed region with δ -gaps $\delta = \delta_1$ contains the region with $\delta = \delta_2 < \delta_1$. We will not map the allowed region in $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$ space to its full extent, as this would be numerically rather

²⁵As reviewed in Section 5.3.3, the sub-leading operator is typically a “ J^3 ” operator for which the scaling of anomalous dimensions in $1/\ell$ is not known—but see [275–277] for recent work on this topic. Determining the fall-off of such sub-leading operators at large spin would allow us to improve our numerical results in the future.

²⁶It is not uncommon for scalar operators to have larger anomalous dimensions than spinning ones. With this set of gap assumptions, the anomalous dimension of the lightest parity-odd scalar and the lightest $\ell = 1$ operator (except for the current) are allowed to be of the same magnitude (while a $\sim 1/\ell$ factor effectively suppresses anomalous dimensions of the higher spin operators). Nevertheless, we have tested that our bounds are not so sensitive to the size of the interval assumption on the lightest parity-odd scalar. For example, at $\delta = 0.2$, enlarging the window to $[\Delta_{\text{GFV}} - 2\delta, \Delta_{\text{GFV}} + 2\delta]$ affects the bound by $\sim 0.1\%$.

²⁷If all anomalous dimensions are monotonic for $g_{\text{YM}} \leq g_{\text{YM}}(\delta)$.

δ	$(\min \Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ})$	$(\Delta_s, \min \Delta_{s\bar{s}}, \alpha_{JJJ})$
0	(3.970, 3.999, 0.867)	(3.989, 3.987, $-$)
0.1	(3.780, 4.101, 0.708)	(3.971, 3.931, $-$)
0.3	(3.521, 4.180, 0.713)	(3.883, 3.851, $-$)
0.5	(3.227, 4.229, 0.724)	(3.746, 3.754, $-$)
0.7	(2.865, 4.135, 0.730)	(3.504, 3.611, $-$)
1	(1.956, 3.421, 0.771)	(2.698, 3.172, $-$)

Table 5.3: Results from the minimization of either Δ_s or $\Delta_{s\bar{s}}$ for different δ -gaps for $SU(2)$ at $\Lambda = 19$. In the $\Delta_{s\bar{s}}$ minimisation, the α_{JJJ} direction has a flat gradient for most of the minimization (i.e. smaller than the duality gap); the value of α_{JJJ} is thus not meaningful and is not reported.

expensive. Instead, we will maximise and minimise scaling dimensions, which will provide a “bounding box” of the actual allowed region.

In the case of $SU(2)$ we adopt analogous gap assumptions, with the exception that the $\mathbf{S}\bar{\mathbf{A}}_c$ representation is not present in this case, and the search space is just $(\Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ})$.

The strategy, then, is to look at how the allowed region changes as we increase δ . We will consider as evidence in favour of the “merger and annihilation” scenario the fact that $\Delta_s \leq 3$ is allowed in this CFT space for a smaller value of δ with respect to the value of δ for which either Δ_a or $\Delta_{s\bar{s}}$ are allowed to be below 3. This is non-rigorous evidence, since we are not guaranteed that the boundary CFT of YM in AdS at $g_{\text{YM}} = g_{\text{YM}}(\delta)$ extremizes (or is close to extremizing) Δ_s in this CFT space.

Recall that, in the case of $SU(2)$, the leading order anomalous dimension of the lightest s and $s\bar{s}$ operators are respectively negative and positive, as we reviewed in Section 5.3.3, while the lightest adjoint scalar has dimension $\Delta = 7 + O(g_{\text{YM}}^2)$, as we reviewed in Section 5.3.2. Thus, the Higgsing scenario is “perturbatively” disfavoured in this case. In the case of $SU(3)$, instead, the a operator is much lighter in the GFV ($\Delta = 4$) and has a negative anomalous dimension at leading order, which makes Higgsing a more likely possibility. We have thus decided to invest more computational resources in the case of $SU(3)$, i.e. performing the present computation for a denser array of δ values.

$SU(2)$ case

Let us start with the $SU(2)$ case. In Table 5.3 we report the results from the minimization of Δ_s and $\Delta_{s\bar{s}}$ in $(\Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ})$ space for different values of δ that we have explored. The bootstrap computation is done at $\Lambda = 19$. A plot of $\min \Delta_s$ and $\min \Delta_{s\bar{s}}$ as a function of δ is reported in Figure 5.13, where we observe that the singlet is allowed to cross the marginality condition for $\delta_* \simeq 0.6$, while the $s\bar{s}$ operator cannot be relevant until at least $\delta = 1$, and thus we expect $\delta_{**} \gtrsim 1$ in the notation used in Section 5.1. While it would certainly be instructive to add the perturbative results in Figure 5.13 in order to assess the proximity of the bounds to

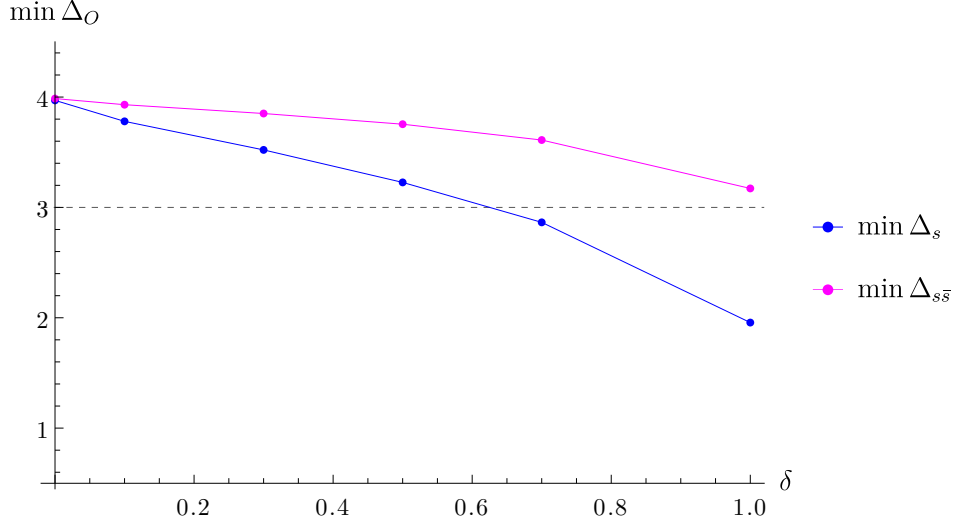


Figure 5.13: Comparison of $\min \Delta_s$ and $\min \Delta_{s\bar{s}}$ for different δ -gaps in the case of $SU(2)$ at $\Lambda = 19$.

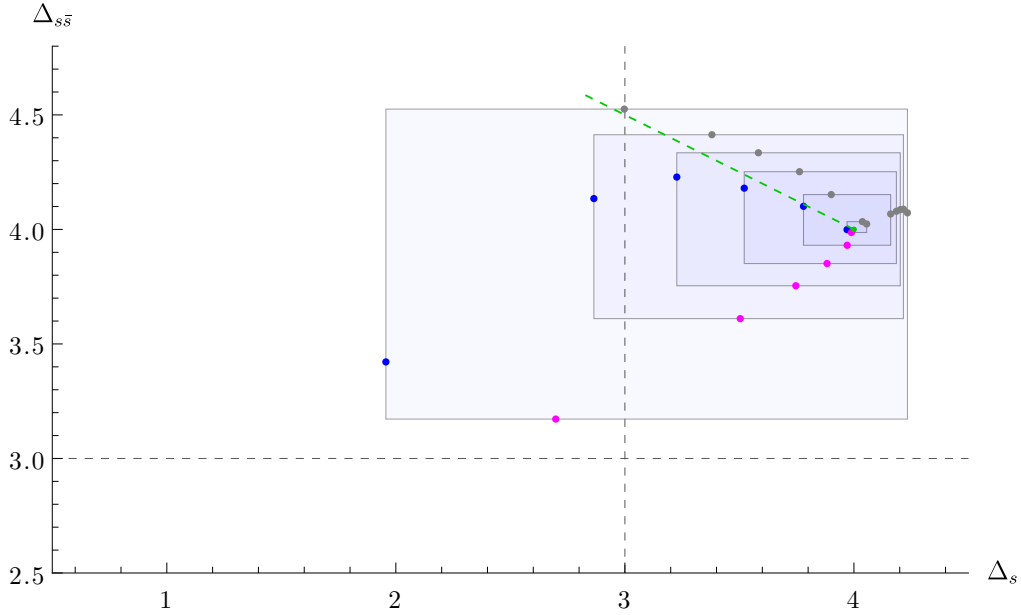


Figure 5.14: Projection in the $(\Delta_s, \Delta_{s\bar{s}})$ plane of the allowed region in $(\Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ})$ space for different δ -gaps for the case of $SU(2)$ at $\Lambda = 19$. Shaded areas, from darker to lighter, correspond to $\delta = 0, 0.1, 0.03, 0.5, 0.7, 1$. Note that this overestimates the size of the actual allowed region since it's obtained by only minimising/maximising Δ_s and $\Delta_{s\bar{s}}$. The blue and magenta dots denote the endpoints of the minimisation of Δ_s and $\Delta_{s\bar{s}}$, respectively. The grey dots denote the endpoints of the maximisation procedure. The dashed green line corresponds to the anomalous dimension (5.3.30). The dashed grey lines denote the marginality condition.

them, especially at weak coupling, the parameter δ cannot be directly determined as a function of the AdS radius L in order to do so.

In Figure 5.14 we draw bounding boxes for the allowed region in $(\Delta_s, \Delta_{s\bar{s}})$ space for different values of δ , as reported in Table 5.3. In this case we have eliminated the parameter δ from our plot by plotting each dimension as a function of the other, thus allowing us to display the perturbative results. For a given δ , the actual allowed region for the scaling dimensions is a two-dimensional island-like shape that is tangent to the boxes at the corresponding dots. For $\delta = 0$, the allowed region is essentially a small island centred at the GFV values $(\Delta_s, \Delta_{s\bar{s}}) = (4, 4)$. As δ increases, the island enlarges asymmetrically, allowing the singlet to cross marginality while the charged $s\bar{s}$ scalar is still bounded to be irrelevant.

The SDP used to obtain Figure 5.13 and 5.14 is reported in (D.5.5). The SDP is parametric in $(\Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ}, \delta)$. For a fixed value of δ , we used the Navigator method to minimise/maximise each direction in scaling dimensions, keeping the remaining parameters free. Namely, we find the minimal and maximal values of either Δ_s or $\Delta_{s\bar{s}}$ for which the navigator function $\mathcal{N}(\Delta_s, \Delta_{s\bar{s}}, \alpha_{JJJ}) = \max \alpha(\vec{\mathcal{V}}_1)$ is negative.

SU(3) case

We now proceed to $SU(3)$. The results from the minimization of Δ_s , Δ_a and $\Delta_{s\bar{s}}$ in $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$ space are reported in Table 5.4 for different values of δ that we have explored. The bootstrap computation is done at $\Lambda = 19$. A plot of $\min \Delta_s$, $\min \Delta_a$ and $\min \Delta_{s\bar{s}}$ as a function of δ is reported in Figure 5.15, where we observe that the singlet is allowed to cross marginality for $\delta_* \simeq 0.5$, the adjoint for $\delta_{**} \simeq 0.9$, while the $s\bar{s}$ operator cannot be relevant until at least $\delta = 1$. Similarly to the case of Figure 5.13, discussed previously, we omit the perturbative results in Figure 5.15.

In panel (a) of Figure 5.16 we draw bounding boxes for the allowed region in the (Δ_s, Δ_a) space for different values of δ , as reported in Table 5.4. For a given δ , the actual allowed region for the scaling dimensions is a three-dimensional island-like shape, whose two-dimensional projection is tangent to the boxes at the corresponding dots. For $\delta = 0$, the allowed region is essentially a small island centred at the GFV values $(\Delta_s, \Delta_a) = (4, 4)$. As δ increases, the island enlarges asymmetrically, allowing the lightest singlet to cross marginality while the lightest adjoint is still bounded to be irrelevant. For $\delta \geq \delta_{**} \simeq 0.9$, the a scalar is allowed to be relevant. When $\delta = 1$, Δ_s is permitted to lie at the unitarity bound, and the allowed region is roughly comparable to the peninsula in Figure 5.11. In panel (b) of Figure 5.16 we consider instead the projection to the $(\Delta_s, \Delta_{s\bar{s}})$ plane.

The SDP used to obtain Figures 5.15 and 5.16 is reported in (D.5.1). This SDP is parametric in $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ}, \delta)$. For a fixed value of δ , we used the Navigator method to minimise/maximise each scaling dimension, keeping the remaining parameters free. Namely, find the minimal and maximal values of either Δ_s , Δ_a or $\Delta_{s\bar{s}}$ for which the navigator function $\mathcal{N}(\Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ}) = \max \alpha(\vec{\mathcal{V}}_1)$ is negative.

To summarise, for both $SU(2)$ and $SU(3)$, the singlet operator is permitted to cross marginality at somewhat small values of δ , roughly $\delta_* \sim 1/2$. As noted in the introduction, this does not guarantee that it will actually do so. By contrast, charged scalars require values

δ	$(\min \Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$	$(\Delta_s, \min \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$	$(\Delta_s, \Delta_a, \min \Delta_{s\bar{s}}, \alpha_{JJJ})$
0	(3.943, 3.984, 3.997, 0.913)	(3.963, 3.975, 3.998, 0.912)	(3.969, 3.994, 3.986, -)
0.0125	(3.881, 3.972, 4.014, 0.840)	(3.969, 3.944, 4.021, 0.820)	(3.957, 4.014, 3.974, -)
0.05	(3.775, 3.945, 4.034, 0.790)	(3.969, 3.899, 4.064, 0.784)	(3.910, 4.029, 3.956, -)
0.1	(3.664, 3.925, 4.065, 0.778)	(3.918, 3.848, 4.093, 0.758)	(3.876, 4.033, 3.937, -)
0.2	(3.491, 3.892, 4.112, 0.756)	(3.807, 3.765, 4.127, 0.731)	(3.820, 4.016, 3.903, -)
0.3	(3.320, 3.847, 4.135, 0.751)	(3.696, 3.687, 4.156, 0.726)	(3.742, 3.992, 3.865, -)
0.4	(3.139, 3.796, 4.149, 0.752)	(3.586, 3.604, 4.183, 0.729)	(3.670, 3.958, 3.825, -)
0.5	(2.944, 3.676, 4.152, 0.757)	(3.448, 3.513, 4.206, 0.731)	(3.518, 3.938, 3.779, -)
0.6	(2.728, 3.600, 4.116, 0.760)	(3.308, 3.413, 4.227, 0.733)	(3.318, 3.912, 3.724, -)
0.7	(2.471, 3.536, 4.028, 0.765)	(3.172, 3.300, 4.245, 0.734)	(3.116, 3.885, 3.658, -)
0.8	(2.121, 3.467, 3.891, 0.775)	(3.055, 3.172, 4.259, 0.736)	(2.852, 3.856, 3.575, -)
0.9	(1.177, 3.507, 3.655, 0.831)	(2.947, 3.023, 4.268, 0.738)	(2.502, 3.808, 3.471, -)
1	(0.5000, -, -, -)	(2.844, 2.846, 4.265, 0.742)	(2.018, 3.731, 3.337, -)

Table 5.4: Results from the minimization of either Δ_s , Δ_a or $\Delta_{s\bar{s}}$ for different δ -gaps for $SU(3)$ at $\Lambda = 19$. In some cases, the gradient in one or more directions is flat for most of the minimisation (i.e. below the duality gap) and thus the value in that direction is not reported.

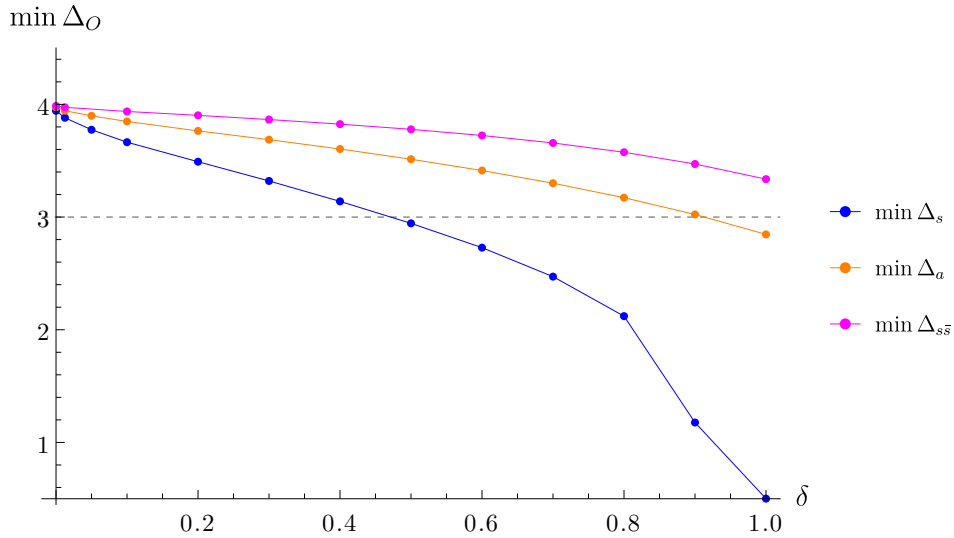
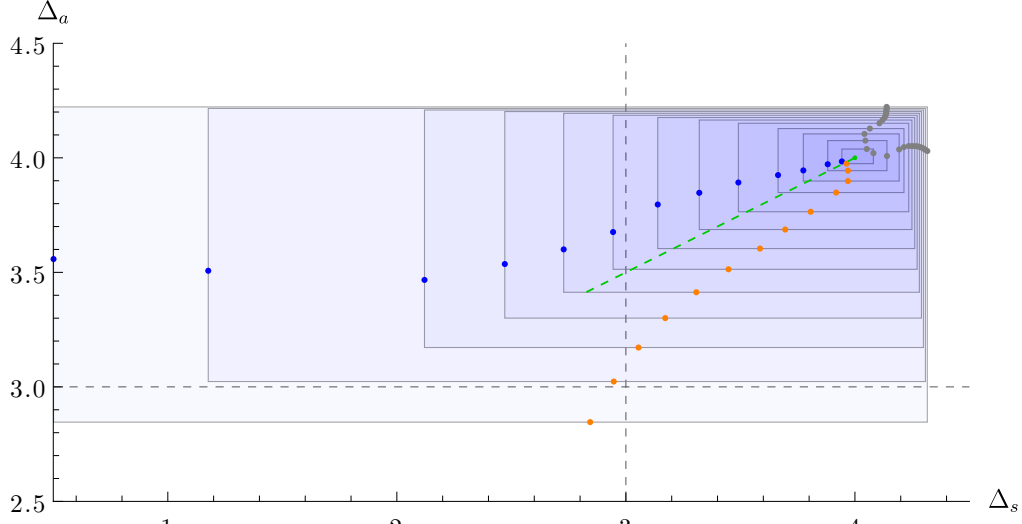
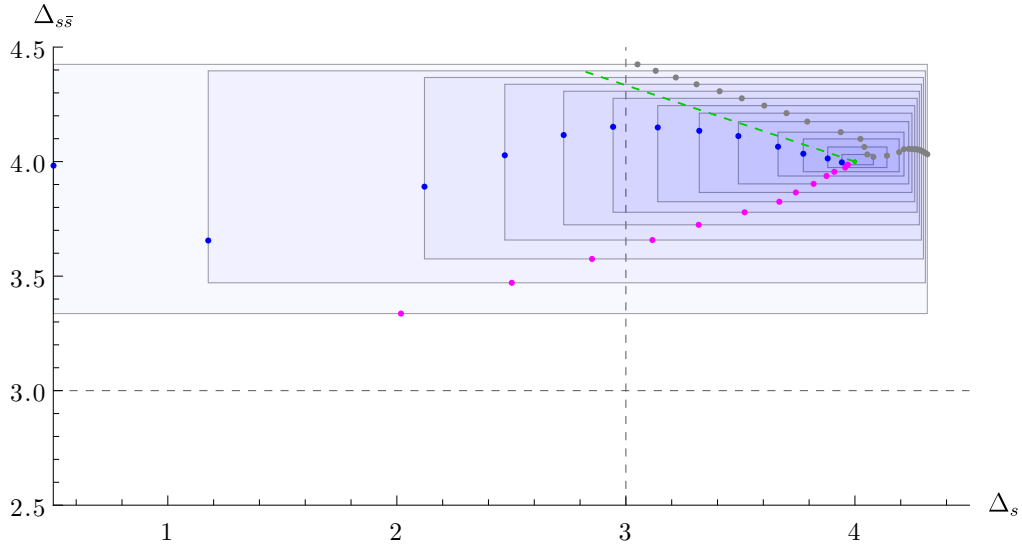


Figure 5.15: Comparison of $\min \Delta_s$, $\min \Delta_a$ and $\min \Delta_{s\bar{s}}$ for different values of the δ parameter in the case of $SU(3)$ at $\Lambda = 19$.



(a)



(b)

Figure 5.16: Projection to the (Δ_s, Δ_a) and $(\Delta_s, \Delta_{s\bar{s}})$ planes in panels (a) and (b), respectively, of the allowed region in $(\Delta_s, \Delta_a, \Delta_{s\bar{s}}, \alpha_{JJJ})$ space for different δ -gaps in the case of $SU(3)$ at $\Lambda = 19$. Shaded areas, from darker to lighter, correspond to $\delta = 0.0125, 0.05, 0.1, 0.2, \dots, 1$. Note that this overestimates the actual size of the allowed region since it is obtained by only extremising Δ_s, Δ_a and $\Delta_{s\bar{s}}$. The blue, orange and magenta dots denote the endpoints of the minimisation of Δ_s, Δ_a and $\Delta_{s\bar{s}}$, respectively. The grey dots denote the endpoints of the maximisation procedure. The dashed green line corresponds to the anomalous dimension (5.3.30). The dashed grey lines denote the marginality condition.

of δ that are nearly twice as large. This indicates that there exist solutions to crossing in which a scalar singlet becomes marginal while the spectrum remains GFV (or YM)-like. In contrast, theories with a marginal charged operator lie much farther from such a regime. Our analysis at least suggests that a merger-annihilation scenario remains feasible even with a spectrum resembling the UV one, while Higgsing would necessitate more substantial corrections across a larger portion of the spectrum.

We envision that pushing the numerics to higher Λ would make the distinction between the two scenarios even more pronounced,²⁸ and possibly start showing hints of a special point in CFT space in the form of a kink or a feature of some form.

5.7 Discussion

In this chapter, we have studied the constraints imposed by unitarity and crossing on the correlator of four non-Abelian conserved currents in three dimensions. Our spectral assumptions are compatible with the conformal theory induced on the boundary of AdS_4 by pure Yang-Mills theory with $SU(N)$ gauge group and Dirichlet boundary condition. Since the boundary CFT has a global symmetry, it is incompatible with colour confinement and should cease to exist in the flat space limit. Conversely, rigorously showing that no boundary condition with a conserved $SU(N)$ current survives in this limit would amount to solving the confinement problem. To this end, we tested the three currently available scenarios for confinement in AdS: (a) decoupling, (b) merging and annihilation, (c) Higgsing. We were able to exclude scenario (a) by showing that the two-point function of currents $\langle JJ \rangle$ is bounded to be strictly positive, hence J cannot become null and decouple from the theory. We have not been able to sharply rule out scenarios (b) or (c), but our constraints strongly favour (b), since a scalar singlet operator is allowed to become marginal long before any charged operator does.

Given that our results depend sensitively on the gaps imposed in the operator spectrum, in this chapter, we erred on the side of caution, and our assumptions simply parametrise any continuous deformation of free gluons in AdS. While this makes our results fully rigorous and of interest beyond the context of QFT in AdS, better isolating Yang-Mills would be necessary to discriminate between scenarios (b) and (c), or at least to prove that the Dirichlet boundary condition does cease to exist beyond a finite radius.

The bootstrap excels at isolating theories with large gaps in the spectrum. While the requirement may be ambiguous, the prototypical example is the $3d$ Ising model. The interacting equations of motion of the Wilson-Fisher fixed point [278] justify the presence of a single relevant scalar in the $\mathbb{Z}_2$ odd sector, in contrast to the generalised free theory of a field with the same dimension. The role of the equations of motion is, in general, to render some operator redundant. These redundancy channels [279] are sometimes able to isolate special solutions to crossing from a discrete or continuous family with otherwise similar spectra. As remarked in Section 5.2, a similar mechanism is at work in the Higgsing scenario, where the current multiplet recombines

²⁸Since these are lower bounds, when increasing Λ , the value of δ where a scalar is allowed to cross marginality can only increase (or remain constant).

with a charged operator. It would be interesting to explore whether the related redundancy channels can help isolate (or rule out) the corresponding boundary CFT.

As for the Dirichlet boundary condition, its spectrum is a continuous deformation of a Generalised Free Vector, with no operator recombining perturbatively close to the free point. Therefore, redundancy channels are absent and cannot distinguish the interacting theory from free gluons or from a different deformation thereof. Our spectral assumptions do exclude generic $SU(N)$ gauge theories with matter, and in particular, crucially, QCD in the conformal window, whose boundary spectrum includes a protected displacement operator [249, 280]. To do better, one option might be to inject the presence of AdS directly by including bulk operators in the bootstrap. A system of correlators, amenable to the bootstrap, involving bulk and boundary operators was proposed in [281] for AdS_2 – see also [282, 283] for related work. One of its advantages is a sum rule for the bulk UV central charge: a four-dimensional analogue would allow us to exclude any additional UV degree of freedom beyond the gluons.

Other avenues to isolate Yang-Mills theory include the study of Wilson lines ending on the boundary, or, moving away from Dirichlet and onto Neumann boundary conditions, Wilson lines fully embedded in the three-dimensional boundary, as recently studied in [261]. This would lead us into the territory of the bootstrap in defect CFT.

A related weakness of our setup is that the value of g_{YM} (or equivalently $L\Lambda_{\text{YM}}$) was not a direct input. While C_J acts naturally as a physical parametrisation of the bulk coupling, and its value can be injected into the bootstrap, a way to parametrise gap assumptions in terms of C_J would still be needed. Of course, one can put an upper bound on the leading scaling dimension in each symmetry sector as a function of C_J , and simply assume that C_J is monotonically decreasing with $L\Lambda_{\text{YM}}$. While we have explored this strategy, it unfortunately does not discriminate between merging and Higgsing in a conclusive way, since either the singlet or a charged operator crosses marginality first, depending on the value of α_{JJJ} .

In Section 5.6.4, we instead used parametric gap assumptions as an indirect proxy for the RG scale. However, this method can only bound $L\Lambda_{\text{YM}}$ from above, and for instance does not distinguish the interacting theory from the free UV limit, as is visible in the plots of that section. Tracking the value of an exactly marginal coupling through the bootstrap bound is not without precedent: in $\mathcal{N} = 4$ SYM [284, 285] and $\mathcal{N} = 2$ SQCD [286], this was done with the help of constraints coming from localisation. A different idea, tailored for QFT in AdS, but so far only implemented in AdS_2 , is to combine perturbation theory with the rigid structure of the OPE to derive a set of coupled differential equations which determines the dependence of the CFT data on the AdS radius [287]. An approach closer to the one we adopted in this chapter would be, in the spirit of [273], to allow the leading operator in each sector to lie in an interval centred around the dimensions predicted by the Lorentzian Inversion Formula [168], which were computed to first order in the bulk coupling in [264]. Any method that permits injecting information about the coupling, or (part of) the spectrum of the interacting theory, would also alleviate the need to scan over δ , thus making the problem much more numerically tractable.

Beyond the effort to isolate the boundary CFT corresponding to the Dirichlet boundary condition, it would be interesting to formulate bootstrap assumptions which place the putative $D^\star$ boundary condition in the allowed region. An optimistic possibility is that the corresponding bound would reveal the presence of two kinks, associated with Dirichlet and $D^\star$, which approach each other, merge and disappear. Hints of such a phenomenon have been observed in [203, 208]. Such a finding would go a long way towards a conclusive answer for the nature of the deconfinement transition. However, at the moment, too little is known about the nature of $D^\star$ to turn this hope into a concrete strategy. For instance, it could be that $D^\star$ does not exist as a unitary boundary condition at weak coupling, but only appears at $g_{\text{YM}} \sim 1$, before disappearing again together with D . Analytical studies of the possible boundary conditions at weak coupling, or explicit constructions of candidates for $D^\star$, can help clarify what to hope for in future bootstrap analyses.

Chapter 6

$4d$: S -matrix bootstrap for non-identical scalars

This chapter uses the primal S -matrix bootstrap to constrain the two-to-two scattering of two non-identical scalar particles in four dimensions, and is organised as follows. Section 6.1 motivates the problem and summarises our results. Section 6.2 sets up the kinematics, crossing relations and unitarity constraints of the amplitudes. Section 6.3 analyses their analytic structure, in particular the pseudo-physical regions and the anomalous thresholds. Section 6.4 defines the observables that we bound, and Section 6.5 constrains them numerically. Section 6.6 shows how additional assumptions on the pseudo-physical region can improve the bounds. Section 6.7 collects our conclusions. Three appendices, gathered at the end of the thesis, discuss the analyticity of the partial-wave expansions (Appendix E), the dispersion relations and positivity bounds (Appendix F), and the details of the numerical setup (Appendix G). This chapter is based on [4].

6.1 Introduction

The S -matrix bootstrap is a collection of methods for studying scattering amplitudes in quantum field theory in a model-independent way, without committing to a particular action. Its basic input consists of Lorentz invariance, crossing symmetry, causality/analyticity and unitarity, and most applications have focused on two-to-two scattering amplitudes. See [10, 288] for classic references. For a modern overview of the program, see the white paper [44]. One may think of amplitudes as forming an infinite-dimensional space parametrised by dimensionless observables. The general principles of QFT carve out the allowed region in this space. The bootstrap aims both to bound this region and to construct amplitudes that lie on its boundary. The first non-perturbative bounds of this kind were obtained by Lopez and Mennessier [289, 290], while a simple modern implementation was introduced in [42]. The general framework, together with the primal formulation for the elastic scattering of identical scalars that we generalise in this chapter, is reviewed in Section 2.3.

For present purposes, it is useful to organise S -matrix bootstrap methods into two broad classes, called *primal* and *dual*. Dual methods rule out inconsistent amplitudes, whereas primal methods “rule in” amplitudes by constructing explicit solutions. Dual approaches include [45–47, 289, 290], while primal approaches include [41, 42, 47, 291]. The two viewpoints are complementary: dual methods offer more direct control over the rigour of exclusion bounds, while primal methods are convenient for constructing amplitudes in practice. In this chapter we use the primal method of [42]. Bootstrap methods based on neural networks were proposed in [292, 293], while other constructive approaches were developed in [294, 295].

The S -matrix bootstrap is particularly powerful in $d = 2$, where many bounds are saturated by integrable models [41], further developments include [43, 296–302]. In higher dimensions, bounds on identical neutral massive particles were obtained in [42, 289, 290], with further results in $d = 4$ [106, 303, 304] and in general dimension [106, 305]. Most of these works concern massive scalars. Spinning particles in $d = 4$ were treated in [306], see also [307, 308]. In particular, bounds on identical massive Majorana fermions were obtained in [306]. For massless particles, universal soft expansions, which follows from the shift symmetry at low energy, must also be imposed. A practical implementation was introduced in [309], with further developments in [310, 311]; bounds on photons were obtained in [312]. Other applications include QCD flux tubes [313–315], the QCD world-sheet axion [316], glueballs [317], the c -anomaly in $d = 2$ and the a -anomaly in $d = 4$ [318, 319], Froissart growth [320], and scattering amplitudes in string theory and M-theory [321, 322].

Under the standard assumptions of analyticity and Froissart boundedness [12], scattering amplitudes obey dispersion relations. Combining these relations with positivity, which captures a restricted subset of unitarity, gives non-perturbative bounds on the space of observables. These dispersive methods should be viewed as dual S -matrix bootstrap methods: they generally give weaker bounds because they do not use the full unitarity constraints, but are much easier to implement in practice. Modern formulations were developed in [110, 323, 324], with further developments in [325–330]. Many applications concern massless amplitudes, where one is often forced to assume a tree-level-like analytic structure. Examples include string-like amplitudes [331–338] and pion amplitudes in the $N_c \rightarrow \infty$ limit of QCD [339–345].¹ For massive amplitudes, dispersion relations were studied systematically for identical scalars in [106] and have recently been applied to spin-one particles [365] and particles of unequal masses [366].

What is this chapter about? In this chapter we are particularly interested in applications of the non-perturbative S -matrix bootstrap to low-energy QCD. Theoretical bounds on pion-pion scattering were obtained in the pioneering work of [367]. By injecting experimental data into the S -matrix bootstrap setup one can isolate the pion-pion amplitude and study the resonances it contains, see [368].²

¹Dispersive methods are also powerful for bounding effective field theories. For early works see [346, 347]; further developments include [348–350], and a review is given in [351]. More recent works include [352–364].

²An interesting new direction is to combine the S -matrix and form-factor bootstrap in the study of pion-pion scattering; see [369–371].

A natural next step is to study processes involving different hadrons, in particular pion-kaon scattering, $\pi K \rightarrow \pi K$, and pion-nucleon scattering, $\pi N \rightarrow \pi N$. Both processes involve two non-identical particles. Apart from the $d = 2$ studies of [43, 45], the S -matrix bootstrap literature has so far focused almost entirely on the scattering of identical particles. The reason is that scattering of non-identical particles brings new analytic structures and a larger system of amplitudes and unitarity constraints.

The main new issue concerns analyticity. For elastic scattering of the lightest particle in the spectrum with itself, one can adopt the standard assumption of maximal analyticity: the amplitude on the principal sheet is analytic for generic complex values of s and t , apart from physical-threshold cuts and possible bound-state poles. For scattering involving heavier or non-identical particles, this analytic structure is no longer sufficient. In particular, some amplitudes can develop a branch cut whose branch point lies below the physical threshold of the corresponding process. The segment of this cut below the physical threshold is called the *pseudo-physical* region. So-called anomalous thresholds can also occur. A bootstrap for non-identical particles therefore requires a generalisation of maximal analyticity that accounts for these additional singularities.

The goal of this chapter is to initiate the study of non-identical particles in $d = 4$ spacetime dimensions in the simplest possible setting. We consider two real scalar particles, denoted by A and B , and impose no continuous global symmetry. We denote their masses by m_A and m_B , take $m_A \leq m_B$ without loss of generality, and use the notation

$$m \equiv m_A, \quad M \equiv m_B, \quad \mu \equiv \frac{M}{m} \geq 1. \quad (6.1.1)$$

Without further assumptions, one would in principle have to consider the full set of two-to-two processes

$$\{ij \rightarrow kl\}, \quad (6.1.2)$$

where each of i, j, k, l can be either A or B . We simplify this setup by imposing a $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry under which A and B carry charges $(-, +)$ and $(+, -)$, respectively. This symmetry has four useful consequences. First, taking A and B to be the lightest states in their charge sectors makes both particles absolutely stable for any value of μ . Second, it forbids all cubic couplings involving A and B and therefore removes one-particle exchange poles due to A or B . Third, up to relabelling it reduces the full set (6.1.2) to

$$AA \rightarrow AA, \quad BB \rightarrow BB, \quad AB \rightarrow AB, \quad AA \rightarrow BB. \quad (6.1.3)$$

Fourth, it excludes anomalous thresholds generated by triangular and effective-triangular Landau diagrams. We further assume that the amplitudes contain no one-particle poles due to additional states and no anomalous thresholds of any other origin.

Rather than studying all the amplitudes in (6.1.3), in this chapter we restrict our attention to the following subset of amplitudes

$$AB \rightarrow AB, \quad AA \rightarrow BB. \quad (6.1.4)$$

These two processes are related by crossing symmetry. The reduced setup (6.1.4) is necessarily less powerful than the full system (6.1.3) and is expected to give weaker bounds on observables. Its advantage is that it provides the minimal crossing-closed system on which non-trivial unitarity constraints can be imposed. Since the $\pi K \rightarrow \pi K$ and $\pi N \rightarrow \pi N$ systems are more complicated, it is useful to understand first whether this minimal setup already leads to non-trivial bounds. We show that it does.

Summary of the results. In Section 6.5 we construct bounds on the couplings $\lambda_{0,0}$, $\lambda_{2,0}$, and $\lambda_{2,1}$, and on the scalar scattering length a_0 . We focus on three values of the mass ratio:

$$\mu = 1, \quad \mu = 1.5, \quad \mu = 3.554. \quad (6.1.5)$$

The first value describes two distinct particles with equal masses. The other two are representative unequal-mass cases, with $\mu = 3.554$ approximating the ratio of the physical kaon and pion masses in the isospin limit.

Let us first discuss the unequal-mass case, $\mu > 1$. We find that $\lambda_{0,0}$ is unbounded from both sides, as illustrated in Figure 6.4. The coupling $\lambda_{2,0}$ obeys the lower bound $\lambda_{2,0} \geq 0$, but has no upper bound. The coupling $\lambda_{2,1}$ has a non-trivial absolute lower bound, shown in Figure 6.6, but no upper bound.

At fixed $\lambda_{2,0}$, we obtain a convergent lower bound on $\lambda_{2,1}$ that is stronger than the positivity bound (6.4.7); see Figure 6.7. We find no evidence for convergent absolute upper or lower bounds on the scalar scattering length a_0 . At fixed $\lambda_{2,0}$, however, a_0 has a convergent lower bound, shown in Figure 6.9.

The case $\mu = 1$ is qualitatively different. We find that all three couplings admit both upper and lower bounds. We obtain compact allowed regions in the $(\lambda_{0,0}, \lambda_{2,0})$, $(\lambda_{0,0}, \lambda_{2,1})$, and $(\lambda_{2,0}, \lambda_{2,1})$ planes; see Figures 6.10, 6.11, and 6.12. These regions are substantially larger than the corresponding regions for identical scalar particles, as shown in Figure 6.13.

In Section 6.6 we investigate whether the unbounded directions found for $\mu > 1$ can be removed by adding information about the pseudo-physical region. As a proof of concept, we consider $\mu = 1.5$ and restrict the discontinuity in the pseudo-physical region to a scalar Muskhelishvili-Omnès function. With this additional assumption, we obtain two-sided bounds on all three couplings and a convergent lower bound on a_0 ; see Figures 6.15–6.18. In a realistic application, instead of imposing the Muskhelishvili-Omnès function, one would use the experimental information on the amplitudes entering the extended-unitarity relation.

6.2 Scattering of unequal particles

In this section we summarise the main ingredients needed to describe the scattering of two non-identical particles and introduce the notation used throughout the chapter. In Section 6.2.1 we define the two-to-two scattering amplitudes. In Section 6.2.2 we determine the physical ranges of the Mandelstam invariants for the processes of interest. Finally, in Sections 6.2.3

and 6.2.4 we define the partial-wave amplitudes and formulate the corresponding unitarity constraints.³

6.2.1 Scattering amplitudes

In this chapter we work in $d = 4$ spacetime dimensions with the mostly-plus metric. We study the scattering processes $ij \rightarrow kl$, where each of i, j, k, l can be either particle A or B .

We assume the $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry under which A and B carry charges $(-, +)$ and $(+, -)$, respectively. This makes the particles absolutely stable, reduces the set of amplitudes to the four given in (6.1.3), and forbids simple poles due to self-interactions.

We define the interacting part of the scattering amplitude by

$$(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \times T_{ij \rightarrow kl}(s, t, u) \equiv \langle kl | \mathbb{T} | ij \rangle, \quad (6.2.1)$$

where $\mathbb{S} = \mathbb{I} + i\mathbb{T}$ is the unitary scattering operator, i, j, k , and l are the particles with four-momenta p_1, p_2, p_3 and p_4 . All these momenta are on shell, for instance $p_1^2 = -m_i^2$, where m_i is the mass of particle i . The Mandelstam variables s, t and u for the process $ij \rightarrow kl$ are defined as usual

$$s \equiv -(p_1 + p_2)^2, \quad t \equiv -(p_1 - p_3)^2, \quad u \equiv -(p_1 - p_4)^2, \quad (6.2.2)$$

and satisfy the relation

$$s + t + u = m_i^2 + m_j^2 + m_k^2 + m_l^2. \quad (6.2.3)$$

An important comment on the notation is in order. In this chapter we use the Mandelstam variables (s, t, u) to describe the s -, t -, and u -channels of *each* scattering process. For processes related by crossing symmetry, such as $AB \rightarrow AB$ and $AA \rightarrow BB$ related by $s \leftrightarrow t$, this implies that the variable s used in $AB \rightarrow AB$ is not the same as s in $AA \rightarrow BB$. This notation has to be contrasted with the one often used in the literature for pion-kaon scattering (see e.g. [372]), where s refers exclusively to the s -channel of $AB \rightarrow AB$, and hence the s -channel of $AA \rightarrow BB$ is denoted by t .

6.2.2 Kinematics

We have formally defined the interacting parts of the scattering amplitudes in (6.2.1). In this chapter we focus only on the processes (6.1.4), namely

$$AB \rightarrow AB \quad \text{and} \quad AA \rightarrow BB. \quad (6.2.4)$$

The scattering amplitudes describing the processes (6.2.4) are related by crossing, namely

$$T_{AB \rightarrow AB}(s, t, u) = T_{AB \rightarrow AB}(u, t, s) = T_{AA \rightarrow BB}(t, s, u). \quad (6.2.5)$$

³Kinematics and unitarity of scattering amplitudes of two unequal scalars were also spelled out in [319], which, however, made further special assumptions on the particles involved in the scattering processes.

In what follows we will carefully discuss kinematics of the processes (6.2.4). To write the equations below, we use the following notation

$$\Sigma \equiv 2m^2 + 2M^2, \quad (6.2.6)$$

together with

$$m_{\pm} \equiv M \pm m. \quad (6.2.7)$$

According to (6.2.3) for both sets of Mandelstam variables we have

$$s + t + u = \Sigma. \quad (6.2.8)$$

Amplitude $AB \rightarrow AB$. The Mandelstam variables in this channel are related to the particle kinematics as

$$\begin{cases} s = \left(\sqrt{m^2 + \mathbf{p}^2} + \sqrt{M^2 + \mathbf{p}^2} \right)^2 \\ t = -2\mathbf{p}^2(1 - z) \\ u = \Sigma - s - t \end{cases} \quad (6.2.9)$$

In the centre of mass (COM) frame the Mandelstam invariants can be parametrized in terms of the magnitude of the COM 3-momentum $\mathbf{p} \equiv |\vec{p}|$ and a scattering angle $z = \cos \theta$. This can be expressed in terms of the Mandelstam variables as

$$z = \cos \theta = 1 + \frac{2st}{(s - m^2 - M^2)^2 - 4m^2M^2} = 1 + \frac{2st}{(s - m_+^2)(s - m_-^2)}. \quad (6.2.10)$$

Imposing $\mathbf{p} \geq 0$ and $z \in [-1, 1]$ in the previous formulas, we find that the physical domain of the Mandelstam variables is contained within the region

$$s \geq m_+^2, \quad t \leq 0, \quad u \leq m_-^2. \quad (6.2.11)$$

It is sometimes also useful to write

$$t = -\frac{(s - m_+^2)(s - m_-^2)}{2s}(1 - z), \quad u = \Sigma - s - t. \quad (6.2.12)$$

This expresses the variables t and u as functions of s and z . Applying (6.2.11) to these expressions we immediately obtain the physical ranges of t and u at fixed s ,

$$-\frac{(s - m_+^2)(s - m_-^2)}{s} \leq t \leq 0, \quad \Sigma - s \leq u \leq \frac{(m_+ m_-)^2}{s}. \quad (6.2.13)$$

Amplitude $AA \rightarrow BB$. The relation between particle kinematics and Mandelstam variables are now given by

$$\begin{cases} s = 4(M^2 + \mathbf{q}^2) \\ t = m^2 - M^2 - 2\mathbf{q}^2 + 2\mathbf{q}\sqrt{M^2 - m^2 + \mathbf{q}^2}z \\ u = \Sigma - s - t \end{cases} \quad (6.2.14)$$

where $\mathbf{q}$ is the outgoing COM 3-momentum. We remind that these Mandelstam variables are different from the ones in (6.2.9). The scattering angle can be expressed in terms of the Mandelstam variables as

$$z = \cos \theta = \frac{2t + s - \Sigma}{\sqrt{(s - 4m^2)(s - 4M^2)}}. \quad (6.2.15)$$

For the process to be physical, both the incoming AA momentum and the outgoing BB momentum must be real. Since $M \geq m$, the physical domain for Mandelstam variables is contained within the region

$$s \geq 4M^2, \quad t \leq 0, \quad u \leq 0. \quad (6.2.16)$$

It is sometimes also useful to write

$$\begin{aligned} t(s, z) &= \frac{\Sigma - s}{2} + \frac{1}{2} \sqrt{(s - 4m^2)(s - 4M^2)} z, \\ u(s, z) &= \frac{\Sigma - s}{2} - \frac{1}{2} \sqrt{(s - 4m^2)(s - 4M^2)} z. \end{aligned} \quad (6.2.17)$$

These express the variables t and u as functions of s and z . Substituting (6.2.16) we immediately obtain the physical ranges of t and u at fixed s

$$B_-(s) \leq t \leq B_+(s), \quad B_-(s) \leq u \leq B_+(s), \quad (6.2.18)$$

where the boundary functions $B_{\pm}$ are defined as

$$B_{\pm}(s) \equiv \frac{\Sigma - s}{2} \pm \frac{1}{2} \sqrt{(s - 4m^2)(s - 4M^2)}. \quad (6.2.19)$$

In particular, the physical domain in t is a convex band contained between $t = -s$ and $t = 0$.

6.2.3 Partial amplitudes

Consider the interacting part of the scattering amplitude (6.2.1) describing the process $ij \rightarrow kl$. In this section we define the associated partial amplitudes. This mirrors the identical-scalar partial-wave decomposition reviewed in Section 2.3, here applied to the several coupled processes.

Let us define the following angular momentum projections of the scattering amplitude (6.2.1)

$$f_{ij \rightarrow kl}^{\ell}(s) \equiv \int_{-1}^1 dz P_{\ell}(z) T_{ij \rightarrow kl}(s, t(s, z), u(s, z)). \quad (6.2.20)$$

Here $\ell = 0, 1, 2, 3, \dots$ is the angular momentum, $P_{\ell}(z)$ are the standard Legendre polynomials and z is the cosine of the scattering angle of the process $ij \rightarrow kl$. The functions $t(s, z)$ and $u(s, z)$ depend on the kinematics of the specific process. For $AB \rightarrow AB$ and $AA \rightarrow BB$ these are respectively given by (6.2.12) and (6.2.17). One can invert the relation (6.2.20), getting

$$T_{ij \rightarrow kl}(s, t, u) = \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(z(s, t)) f_{ij \rightarrow kl}^{\ell}(s), \quad (6.2.21)$$

which is valid a priori only in the physical domain of the Mandelstam variables.

The interacting part of the partial amplitude is defined as

$$T_{ij \rightarrow kl}^\ell(s) = \mathcal{N}_{ij \rightarrow kl}(s) f_{ij \rightarrow kl}^\ell(s). \quad (6.2.22)$$

Here $\mathcal{N}_{ij \rightarrow kl}(s)$ is a simple kinematic factor. In this chapter we use the convention of [306]. In these conventions we obtain

$$\begin{aligned} s \geq 4m^2 : \quad \mathcal{N}_{AA \rightarrow AA}(s) &= \frac{1}{32\pi} \sqrt{1 - \frac{4m^2}{s}}, \\ s \geq 4M^2 : \quad \mathcal{N}_{BB \rightarrow BB}(s) &= \frac{1}{32\pi} \sqrt{1 - \frac{4M^2}{s}}, \\ s \geq m_+^2 : \quad \mathcal{N}_{AB \rightarrow AB}(s) &= \frac{1}{16\pi} \frac{\sqrt{(s - m_+^2)(s - m_-^2)}}{s}, \\ s \geq 4M^2 : \quad \mathcal{N}_{AA \rightarrow BB}(s) &= \frac{1}{32\pi} \frac{(s - 4m^2)^{1/4} (s - 4M^2)^{1/4}}{\sqrt{s}}. \end{aligned} \quad (6.2.23)$$

Using (6.2.22) we can also write the full partial amplitude as

$$S_{ij \rightarrow kl}^\ell(s) = \delta_{(ij), (kl)} + iT_{ij \rightarrow kl}^\ell(s). \quad (6.2.24)$$

Here we have decided to present our results also for the processes $AA \rightarrow AA$ and $BB \rightarrow BB$. This will be useful for the next section.

The angular momentum ℓ in (6.2.20) and (6.2.22) satisfies some selection rules. By Bose symmetry, ℓ is an even integer for the $AA \rightarrow AA$, $BB \rightarrow BB$ and $AA \rightarrow BB$ processes, while it can be both even and odd for the $AB \rightarrow AB$ process. In the presence of identical particles it is standard to use conventions which involve a relative factor of 1/2 in the normalisations (6.2.23).

6.2.4 Unitarity constraints

Let us now write explicitly the unitarity constraints for our two-to-two scattering processes. For completeness, we present the constraints for all four processes in (6.1.3). Unitarity couples the amplitudes $AA \rightarrow AA$, $BB \rightarrow BB$, and $AA \rightarrow BB$ in the neutral sector, while $AB \rightarrow AB$ belongs to a separate charge sector. It will then be straightforward to reduce these constraints to the two processes in (6.1.4). The unitarity constraints are most easily formulated in terms of the partial-wave amplitudes defined in the previous subsection. In what follows, we use the definitions (6.2.24) and (6.2.22).

Positivity of the Hilbert-space norm implies that the Gram matrix formed by the asymptotic in- and out-states, projected to a definite angular momentum, is positive semidefinite, as explained in [306].

On the subspace spanned by two-particle irreducible in/out states $|AA\rangle, |BB\rangle, |AB\rangle$, unitarity generalises the single-amplitude positivity of Section 2.3 to positive semi-definiteness of

the matrix

$$\begin{pmatrix} \delta_{a,b} & S_{a \rightarrow b}^\ell(s)^* \\ S_{a \rightarrow b}^\ell(s) & \delta_{a,b} \end{pmatrix} \succeq 0, \quad (6.2.25)$$

for physical energies and angular momenta, and where $a, b \in \{(AA), (BB), (AB)\}$ and $S_{a \rightarrow b}^\ell(s)$ as in (6.2.24).

We now summarize the resulting unitarity constraints in the two charge sectors.

- For $4m^2 \leq s \leq 4M^2$ and $\ell = 0, 2, 4, \dots$ we have

$$\begin{pmatrix} 1 & 1 - iT_{AA \rightarrow AA}^\ell(s)^* \\ 1 + iT_{AA \rightarrow AA}^\ell(s) & 1 \end{pmatrix} \succeq 0. \quad (6.2.26)$$

- For $s \geq 4M^2$ and $\ell = 0, 2, 4, \dots$ we have

$$\begin{pmatrix} 1 & 0 & 1 - iT_{AA \rightarrow AA}^\ell(s)^* & -iT_{AA \rightarrow BB}^\ell(s)^* \\ 0 & 1 & -iT_{AA \rightarrow BB}^\ell(s)^* & 1 - iT_{BB \rightarrow BB}^\ell(s)^* \\ 1 + iT_{AA \rightarrow AA}^\ell(s) & iT_{AA \rightarrow BB}^\ell(s) & 1 & 0 \\ iT_{AA \rightarrow BB}^\ell(s) & 1 + iT_{BB \rightarrow BB}^\ell(s) & 0 & 1 \end{pmatrix} \succeq 0. \quad (6.2.27)$$

- For $s \geq m_+^2$ and $\ell = 0, 1, 2, \dots$ we have

$$\begin{pmatrix} 1 & 1 - iT_{AB \rightarrow AB}^\ell(s)^* \\ 1 + iT_{AB \rightarrow AB}^\ell(s) & 1 \end{pmatrix} \succeq 0. \quad (6.2.28)$$

Neglecting the processes $AA \rightarrow AA$ and $BB \rightarrow BB$, we see from (6.2.26) and (6.2.27) that the following sub-set of unitarity constraints holds:

$$\begin{pmatrix} 1 & -iT_{AA \rightarrow BB}^\ell(s)^* \\ iT_{AA \rightarrow BB}^\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4M^2, \quad \text{and} \quad \ell = 0, 2, 4, \dots \quad (6.2.29)$$

The relations (6.2.28) and (6.2.29) are the unitarity constraints that we will use in practice in our numerical studies in Section 6.5.

6.3 Analytic structure of the amplitudes

In this section we follow the viewpoint advocated by Gribov [373]. To identify the singularities that may occur in a scattering amplitude, we consider all Feynman-graph topologies compatible with the assumed spectrum and symmetries. For this purpose, one keeps only the scalar propagator denominators and replaces numerator factors, spin structures, and regular vertex functions by constants. We will call the resulting reduced Feynman graphs *Landau diagrams*.⁴

⁴Alternatively, Landau diagrams can be understood as graphs generated by iterating analytically continued unitarity; see, for example, [107, 374]. In this construction, the vertices represent exact scattering subamplitudes, while the internal lines represent on-shell intermediate states integrated over phase space. For the purpose of locating candidate singularities, the subamplitudes can be replaced by constants, reducing the analysis to the Landau equations of the corresponding denominator graph. These denominator graphs are precisely the Landau diagrams used in this section.

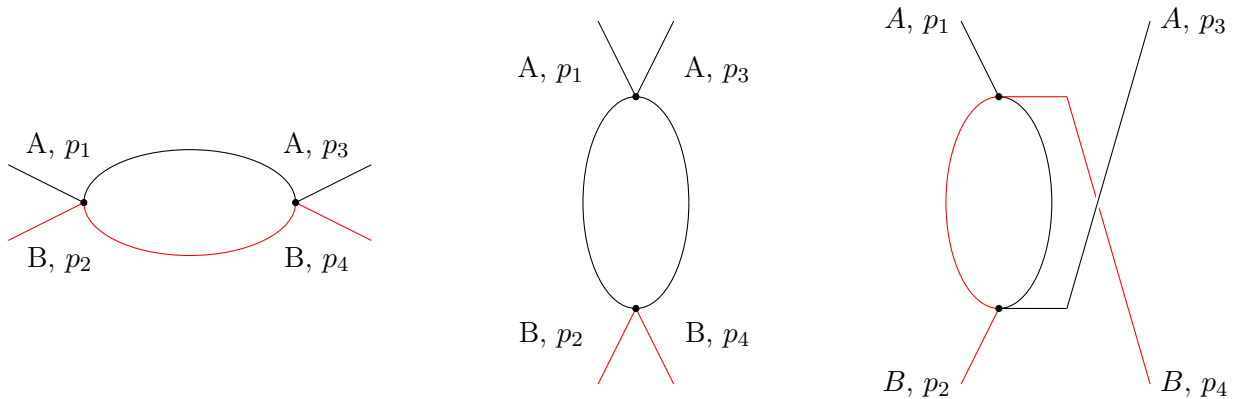


Figure 6.1: Bubble Landau diagrams associated with the leading two-particle singularities of the $AB \rightarrow AB$ amplitude. From left to right, the diagrams contain an AB intermediate state in the s -channel, an AA intermediate state in the t -channel, and an AB intermediate state in the u -channel. Black and red lines represent particles A and B , respectively.

Their Landau equations [375]⁵ determine candidate singularities in the external Mandelstam invariants. Although this analysis originates in perturbation theory, we assume, following Gribov, that the same singularities can occur in the exact amplitude. Special zeros or dynamical cancellations may nevertheless remove a candidate singularity.

We now apply this machinery to the amplitude $AB \rightarrow AB$. The simplest relevant Landau diagrams are the two-particle bubble graphs shown in Figure 6.1. They describe the leading singularities in the s -, t -, and u -channels. Their locations follow directly from the corresponding Landau equations.

Physical and pseudo-physical thresholds. For the first diagram in Figure 6.1, the internal particles A and B can simultaneously go on shell at $s = m_+^2$. This is the leading physical threshold in the s -channel. The corresponding branch cut extends along $s \geq m_+^2$.

The second diagram contains two internal particles A in the t -channel. Its Landau equations give a branch point at $t = 4m^2$, and the corresponding cut begins at $t \geq 4m^2$. There is also another t -channel bubble diagram, not shown in Figure 6.1, whose two internal lines are particles B . It gives an additional branch point at $t = 4M^2$. We refer to the interval

$$4m^2 \leq t < 4M^2 \quad (6.3.1)$$

as the *pseudo-physical region* and to the point $4m^2$ as the *pseudo-physical threshold*. In the special case $M = m$, the pseudo-physical region collapses to a point and effectively disappears.

By crossing symmetry, the analytic structure of the $AB \rightarrow AB$ amplitude in the t -channel coincides with that of the $AA \rightarrow BB$ amplitude in the s -channel. The pseudo-physical region of the latter process is therefore $4m^2 \leq s < 4M^2$. The most important feature of the corresponding

⁵For recent works using Landau equations see [376–386]. For nonperturbative applications see [374, 387].

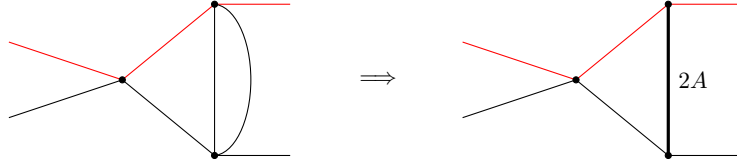


Figure 6.2: A Landau diagram containing four internal lines (left) and its reduced triangular form (right).

cut is that its discontinuity is not directly constrained by unitarity. Indeed, the unitarity condition (6.2.29) for the $AA \rightarrow BB$ process applies only in the physical region $s \geq 4M^2$.

An analogous discussion applies to the amplitudes $AA \rightarrow AA$ and $BB \rightarrow BB$. The former has only physical thresholds, while the latter has both a physical and a pseudo-physical threshold. We summarize the relevant two-particle thresholds in Table 6.1.

amplitude	physical s -region	two-particle branch points	pseudo-physical region
$AA \rightarrow AA$	$s \geq 4m^2$	$4m^2, 4M^2$	none
$BB \rightarrow BB$	$s \geq 4M^2$	$4m^2, 4M^2$	$4m^2 \leq s < 4M^2$
$AA \rightarrow BB$	$s \geq 4M^2$	$4m^2, 4M^2$	$4m^2 \leq s < 4M^2$
$AB \rightarrow AB$	$s \geq (m + M)^2$	$(m + M)^2$	none

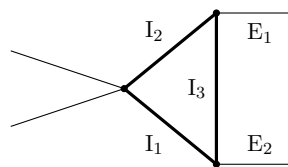
Table 6.1: Summary of the leading two-particle branch points and pseudo-physical regions for the two-to-two scattering amplitudes of particles A and B .

Anomalous thresholds. An anomalous threshold is a Landau singularity that does not coincide with an ordinary physical or pseudo-physical threshold. The simplest Landau topology that can generate such a singularity is a triangle [375].

In our setup, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry forbids all cubic vertices involving the single-particle states A and B . Triangular diagrams whose internal lines are all single-particle propagators are therefore absent. An effective triangle can nevertheless arise from a diagram with a multi-particle subgraph. A simple example is shown in the left panel of Figure 6.2.

The two parallel black lines in this diagram form an AA bubble. At the AA threshold, the two particles have the same four-momentum. At the level of the Landau equations, the bubble is then equivalent to a single effective internal line of mass $2m$. Replacing the AA bubble by this effective line gives the reduced triangular diagram shown in the right panel of Figure 6.2.

To analyse this possibility, consider a general effective triangle:



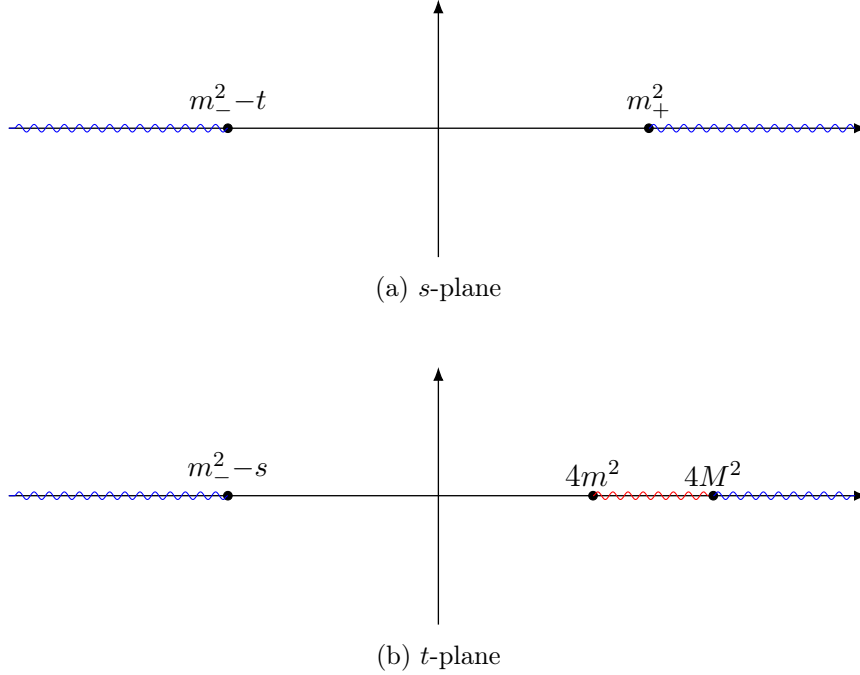


Figure 6.3: Analytic structure of the $AB \rightarrow AB$ amplitude in the complex s -plane at fixed t and in the complex t -plane at fixed s .

The external lines E_1 and E_2 represent particles A or B . The internal lines I_1 , I_2 , and I_3 may represent either single-particle or multi-particle states with the appropriate $\mathbb{Z}_2 \times \mathbb{Z}_2$ charges. We denote by m_{I_a} the corresponding threshold mass.

A necessary, but not sufficient, condition for this diagram to generate an anomalous threshold on the principal sheet is

$$m_{E_1}^2 > m_{I_2}^2 + m_{I_3}^2 \quad \text{or} \quad m_{E_2}^2 > m_{I_1}^2 + m_{I_3}^2. \quad (6.3.2)$$

Neither condition can be satisfied in our setup. Indeed, charge conservation at each vertex fixes the possible charge sectors of the adjacent internal states. For an external particle A , the lightest allowed pairs are A together with an AA state, or B together with an AB state. For an external particle B , the lightest allowed pairs are B together with an AA state, or A together with an AB state. In every case, the sum of the squared internal masses is strictly larger than the squared mass of the corresponding external particle. Thus, no anomalous threshold can arise from an elementary or effective triangular Landau diagram of the form considered above.

More complicated Landau diagrams could in principle generate other anomalous thresholds. A complete classification of all such topologies is beyond the scope of this chapter. We therefore assume that no anomalous thresholds arise from more complicated Landau diagrams.

Assumed analytic structure. The resulting principal-sheet analytic structure assumed for the amplitude $AB \rightarrow AB$ is summarized in Figure 6.3. This is the main conclusion of the present section. It provides the analytic input for the numerical bootstrap of Section 6.5.

6.4 Observables and the space of amplitudes

In this short section we define the observables associated with the amplitude $T_{AB \rightarrow AB}(s, t, u)$. These observables are dimensionless quantities constructed directly from the amplitude. A complete set of them can be viewed as coordinates on the infinite-dimensional space of scattering amplitudes: each amplitude corresponds to a point in this space, specified by the values of all its observables. In practice, we consider only a finite subset of observables. In the next section we construct bounds on various sub-sets of observables.

We consider two classes of observables. The first class consists of couplings, defined by derivatives of the amplitude at a fixed kinematic point. The second class consists of threshold parameters, called scattering lengths and effective ranges.

Couplings. Non-perturbative couplings are defined as derivatives of the amplitude at a fixed kinematical configuration, generalising the identical-scalar definition of Section 2.3. We define

$$\lambda_{k,l} \equiv \frac{m^{2k+2l}}{k!l!} \frac{\partial^k}{\partial s^k} \frac{\partial^l}{\partial t^l} T_{AB \rightarrow AB}(s_0, t_0, u_0), \quad (6.4.1)$$

with

$$s_0 = u_0 \equiv \frac{m^2}{3} + M^2, \quad t_0 \equiv \Sigma - s_0 - u_0 = \frac{4}{3}m^2. \quad (6.4.2)$$

This choice is such that the couplings are real (thanks to real-analyticity), they respect the $s-u$ crossing symmetry of the amplitude, and they coincide with the couplings defined in [106] when $m = M$. This definition is also motivated by the dispersion relations and positivity bounds discussed in Appendix F. The mass-dependent prefactor is chosen so that the couplings are dimensionless. Not all the couplings in (6.4.1) are independent or non-trivial because of $s-u$ crossing. For example we have

$$\lambda_{2k+1,0} = 0, \quad (k \geq 0), \quad \lambda_{1,1} = \lambda_{2,0}, \quad \lambda_{1,2} = \lambda_{2,1}, \quad \lambda_{3,1} = 2\lambda_{4,0}, \quad \lambda_{1,3} = \lambda_{2,2} - \lambda_{4,0}. \quad (6.4.3)$$

We will be in particular interested in the first three couplings, namely

$$\lambda_{0,0}, \quad \lambda_{2,0}, \quad \lambda_{2,1}. \quad (6.4.4)$$

The first one, $\lambda_{0,0}$, is just the value of the amplitude at the point (s_0, t_0, u_0) . The remaining two admit a dispersive representation⁶

$$\begin{aligned} \lambda_{2,0} &= \frac{m^4}{\pi} \int_{m_+^2}^{\infty} ds \frac{2 \operatorname{Im} T(s, t_0)}{(s - s_0)^3}, \\ \lambda_{2,1} &= \frac{m^6}{\pi} \int_{m_+^2}^{\infty} ds \left[\frac{2 \partial_t \operatorname{Im} T(s, t_0)}{(s - s_0)^3} - \frac{3 \operatorname{Im} T(s, t_0)}{(s - s_0)^4} \right], \end{aligned} \quad (6.4.5)$$

where we have defined

$$T(s, t) \equiv T_{AB \rightarrow AB}(s, t, \Sigma - s - t), \quad (6.4.6)$$

⁶If a stronger bound than the Froissart bound is assumed, then $\lambda_{0,0}$ also admits a dispersive representation.

to lighten the notation. Finally, the dispersion relations and positivity of the $AB \rightarrow AB$ amplitude imply the following bounds

$$\lambda_{2,0} \geq 0, \quad \lambda_{2,1} \geq -\frac{9}{4(1+3\mu)}\lambda_{2,0}. \quad (6.4.7)$$

We refer to Appendix F for the derivation and further details.

Threshold parameters. Scattering lengths and effective ranges are natural observables that can be extracted from the threshold expansion of the partial amplitudes. For the $AB \rightarrow AB$ process we define the threshold parameters a_ℓ and b_ℓ as

$$\text{Re } T_{AB \rightarrow AB}^\ell(s) = 2(\mathbf{p}/m)^{2\ell+1} \left(a_\ell + b_\ell (\mathbf{p}/m)^2 + O((\mathbf{p}/m)^4) \right). \quad (6.4.8)$$

where $\mathbf{p}$ is the modulus of the COM 3-momentum, related to s as in (6.2.9). The observable a_ℓ is sometimes referred to directly as the “scattering length” expressed in units of m .

We recall that in our conventions the unitary S -matrix is $S_{AB \rightarrow AB}^\ell(s) = 1 + iT_{AB \rightarrow AB}^\ell(s)$, i.e. $|S_{AB \rightarrow AB}^\ell(s)|^2 \leq 1$. Defining phase shifts as

$$S_{AB \rightarrow AB}^\ell(s) = e^{2i\delta_\ell(s)}, \quad (6.4.9)$$

we have the following commonly-written threshold expansion for the scalar phase shift

$$(\mathbf{p}/m) \cot \delta_0(s) = \frac{1}{a_0} + O((\mathbf{p}/m)^2). \quad (6.4.10)$$

For scattering of identical particles, the scattering lengths a_ℓ with $\ell \geq 2$ admit a dispersive representation which can be obtained by expanding at threshold the Froissart-Gribov formula. This implies that $a_\ell \geq 0$ for $\ell \geq 2$, see for example section 2.5 in [107]. For unequal particles, dispersive representations exists for $\ell \geq 2$ using again the Froissart-Gribov formula, but no simple bounds can be derived because of absence of sign-definiteness.

Still, in the case of $\pi K \rightarrow \pi K$ and $\pi N \rightarrow \pi N$ scattering, relative analytic bounds between scattering lengths on fixed isospin channels have been obtained [388]. Scalar scattering lengths are in general unbounded from above because of the possible presence of resonances close to the physical region, see [42] for a discussion. Several analytic lower bounds on the scalar scattering length have been obtained for equal particles [388],⁷ but we are not aware of generalisations to unequal particles.

6.5 Numerical results

In this section we investigate non-perturbative bounds on quantities that can be extracted from the $AB \rightarrow AB$ scattering amplitude. The bounds apply to any massive quantum field theory in $4d$ that contains the particles A and B in its spectrum.

⁷Numerical lower bounds using primal S -matrix bootstrap have been obtained in [42]. See also [367] for bounds on scattering lengths in pion scattering.

6.5.1 Primal S -matrix bootstrap

We obtain numerical bounds using the primal S -matrix bootstrap introduced in [42] and reviewed for identical scalars in Section 2.3. As a first step we parameterise the most general scattering amplitude for the $AB \rightarrow AB$ process that is compatible with the analytic structure described in Section 6.3. We define the ρ -variable

$$\rho(z, z_{\text{th}}, z_0) = \frac{\sqrt{z_{\text{th}} - z_0} - \sqrt{z_{\text{th}} - z}}{\sqrt{z_{\text{th}} - z_0} + \sqrt{z_{\text{th}} - z}}. \quad (6.5.1)$$

This function maps the cut complex plane $\mathbb{C} \setminus [z_{\text{th}}, +\infty]$ to the unit disk, sending the cut to the boundary of the disk, and has the further property that $\rho(z_0, z_{\text{th}}, z_0) = 0$. In order to accommodate the branch cut structure in Figure 6.3 we define

$$\rho_+(z) = \rho(z, m_+^2, s_0), \quad \rho_m(z) = \rho(z, 4m^2, \Sigma - 2s_0). \quad (6.5.2)$$

For s_0 we make the following choice

$$s_0 = \frac{m^2}{3} + M^2. \quad (6.5.3)$$

We can represent the interacting part of the scattering amplitude for the $AB \rightarrow AB$ process as a power series expansion in the variables (6.5.2) as follows

$$T_{AB \rightarrow AB}(s, t, u) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \alpha_{a,b,c} (\rho_+(s))^a (\rho_m(t))^b (\rho_+(u))^c, \quad (6.5.4)$$

where $\alpha_{a,b,c}$ are real parameters. The ansatz (6.5.4) contains several redundancies because of the relation $s + t + u = \Sigma$. In this chapter we reduce the redundancies by further imposing⁸

$$\alpha_{a,b,c} = 0 \quad \text{if } abc \neq 0. \quad (6.5.5)$$

The crossing relations (6.2.5) constrain the ansatz. The $s \leftrightarrow u$ relation imposes a further constraint on the coefficients of the ansatz (6.5.4), which have to satisfy

$$\alpha_{a,b,c} = \alpha_{c,b,a}. \quad (6.5.6)$$

The $s \leftrightarrow t$ relation implies that $T_{AA \rightarrow BB}$ is not an independent amplitude and can be written as

$$T_{AA \rightarrow BB}(s, t, u) = \sum_{a,b,c} \alpha_{a,b,c} (\rho_+(t))^a (\rho_m(s))^b (\rho_+(u))^c. \quad (6.5.7)$$

The ansatz (6.5.7) is $t - u$ symmetric thanks to (6.5.6).

Note that we have chosen the variable ρ_m so that the amplitude can accommodate for a non-analyticity in the pseudo-physical region discussed in Section 6.3. One could include also a ρ -variable with a cut at $z_{\text{th}} = 4M^2$ to capture an additional discontinuity at the physical

⁸This condition is slightly stronger than what follows just from the constraints on the Mandelstam variables, see [42, App. C] for a discussion.

threshold. However, this would further increase the size of the ansatz and would probably be redundant to the current ansatz in absence of additional inputs.

In the case where we relax the symmetries on the A and B particles to allow for 3-point interactions, or if we allow for additional bound states, the ansatz has to be modified to include simple poles. This possibility will not be considered in this chapter, and is left for future exploration.

The ansatz (6.5.4) has an infinite number of terms. In order to obtain numerical bounds, it is crucial to perform a truncation such that

$$a + b + c \leq N_{\max}, \quad (6.5.8)$$

where $N_{\max}$ is a cut-off parameter, and explore convergence of the bounds as $N_{\max} \rightarrow +\infty$. In practice, the computational cost increases rapidly with the cut-off and in this chapter we will work with $N_{\max} \leq 26$.

Once we have truncated the sums we can collect the coefficients $\alpha_{a,b,c}$ in a finite-dimensional vector $\vec{\alpha}$, and write the ansatz (6.5.4) and (6.5.7) as

$$T_{AB \rightarrow AB}(s, t, u) = \vec{\alpha} \cdot \vec{X}(s, t, u), \quad T_{AA \rightarrow BB}(s, t, u) = \vec{\alpha} \cdot \vec{X}(t, s, u), \quad (6.5.9)$$

where

$$X_{a,b,c}(s, t, u) = (\rho_+(s))^a (\rho_m(t))^b (\rho_+(u))^c. \quad (6.5.10)$$

The observables introduced in Section 6.4 can be extracted from (6.5.9) and are linear in the coefficients $\alpha_{a,b,c}$. The numerical strategy is to maximise or minimise such observables, subject to the unitarity constraints discussed in Section 6.2.4, which can be cast as a semi-definite optimisation problem (SDP). To impose the unitarity constraint on the ansatz we compute the partial wave projection

$$T_{AB \rightarrow AB}^\ell(s) = \vec{\alpha} \cdot \vec{X}_{AB \rightarrow AB}^\ell(s), \quad T_{AA \rightarrow BB}^\ell(s) = \vec{\alpha} \cdot \vec{X}_{AA \rightarrow BB}^\ell(s) \quad (6.5.11)$$

where

$$\begin{aligned} \vec{X}_{AB \rightarrow AB}^\ell(s) &= \mathcal{N}_{AB \rightarrow AB}(s) \int_{-1}^1 dz P_\ell(z) \vec{X}(s, t(s, z), u(s, z)), \\ \vec{X}_{AA \rightarrow BB}^\ell(s) &= \mathcal{N}_{AA \rightarrow BB}(s) \int_{-1}^1 dz P_\ell(z) \vec{X}(t(s, z), s, u(s, z)). \end{aligned} \quad (6.5.12)$$

See equations (6.2.20), (6.2.22), (6.2.23) for the definitions of the partial wave projection, as well as (6.2.12) and (6.2.17) for the expression of $t(s, z)$ and $u(s, z)$ in the appropriate channel.

Unitarity of the partial amplitudes, equations (6.2.28) and (6.2.29), can then be formulated as positive-semidefinite conditions on the following matrices, linear in the coefficients α :

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \vec{\alpha} \cdot \begin{pmatrix} 0 & -i\vec{X}_{AB \rightarrow AB}^\ell(s)^* \\ i\vec{X}_{AB \rightarrow AB}^\ell(s) & 0 \end{pmatrix} &\succeq 0, \quad s \geq m_+^2, \quad \ell = 0, 1, 2, \dots, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \vec{\alpha} \cdot \begin{pmatrix} 0 & -i\vec{X}_{AA \rightarrow BB}^\ell(s)^* \\ i\vec{X}_{AA \rightarrow BB}^\ell(s) & 0 \end{pmatrix} &\succeq 0, \quad s \geq 4M^2, \quad \ell = 0, 2, 4, \dots, \end{aligned} \quad (6.5.13)$$

where $*$ denotes complex conjugation. In addition, two further truncations are needed to formulate an SDP. We put a cut-off in the spin $\ell \leq L_{\max}$ and sample the physical regions in s at n_{pts} points. See Appendix G for further details. Again, convergence of the bounds in $L_{\max}$ and n_{pts} has to be carefully determined.

Finally, a bound on an observable $\vec{\alpha} \cdot \vec{b}$ linear in the α 's, at finite $(N_{\max}, L_{\max}, n_{\text{pts}})$, can be obtained from the following SDP:

$$\begin{aligned} \max \quad & \vec{\alpha} \cdot \vec{b} \\ \text{such that} \quad & M_{0,j} + \vec{\alpha} \cdot \vec{M}_j \succeq 0, \end{aligned} \tag{6.5.14}$$

where $\vec{M}_j$ and $M_{0,j}$ are 2×2 matrices. More concretely j is a multi-index labelling all the positivity conditions in (6.5.13): for $j = (AB \rightarrow AB, \ell, s)$, $M_{0,j}$ denotes the $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ matrix and $\vec{M}_j$ the vector of matrices constructed out of $\vec{X}_{AB \rightarrow AB}^\ell(s)$; for $j = (AA \rightarrow BB, \ell, t)$, $M_{0,j}$ is the identity matrix and $\vec{M}_j$ the vector of matrices constructed out of $\vec{X}_{AA \rightarrow BB}^\ell(s)$. In this chapter we used SDPB [73, 74] to solve these optimisation problems, see Appendix G for further details.

6.5.2 Bounds on couplings

In this section we report the bounds that have been obtained on the non-perturbative couplings $\lambda_{k,l}$ defined in Section 6.4. As already mentioned in Section 6.3, we have to make an important distinction between the special case of equal masses and the generic case of unequal masses. The former is postponed to Section 6.5.4. Here we discuss the generic case of mass ratio $\mu > 1$. In practice, due to the numerical complexity of the problem, we have decided to study two representative cases

$$\mu = 1.5, 3.554. \tag{6.5.15}$$

We believe that the patterns observed in these cases will hold for any $\mu > 1$, and indeed we have obtained partial confirmation of this also for other values of μ , but large-scale numerical optimisations have been carried out only in the aforementioned cases. The value $\mu = 3.554$ might seem arbitrary at first sight, but it corresponds to the mass ratio of pions and kaons in the isospin limit,⁹ as we aim to investigate S -matrix bootstrap bounds on $\pi K \rightarrow \pi K$ scattering in the future.

Using the definition of the non-perturbative couplings (6.4.1) in the ansatz (6.5.4), we can express the $\lambda_{k,l}$ as finite linear combinations of the $\alpha_{a,b,c}$. These are finite since we have chosen the zero of the ρ -variables to coincide with the point (s_0, t_0, s_0) at which the $\lambda_{k,l}$ are defined.

⁹More precisely, we used $m_\pi = m_{\pi^\pm} = 139.57 \text{ MeV}$ and $m_K = (m_{K^0} + m_{K^+})/2 = 496 \text{ MeV}$ [372], which gives $\mu \approx 3.554$.

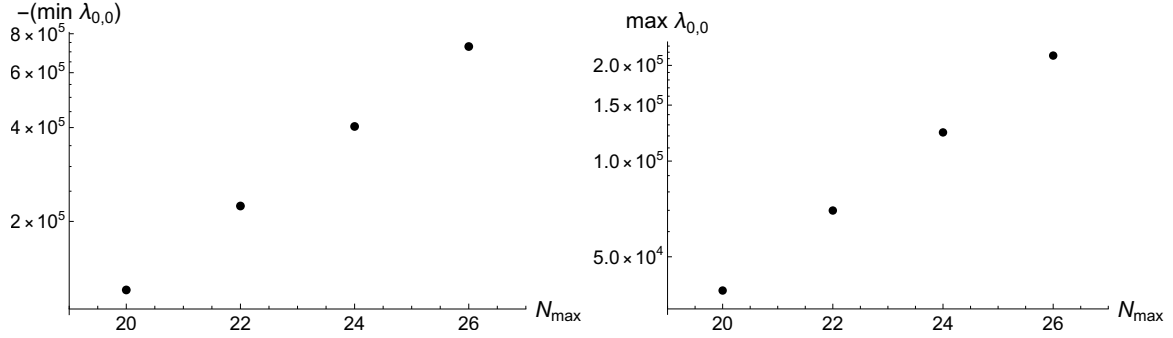


Figure 6.4: Minimisation and maximisation of $\lambda_{0,0}$ as a function of $N_{\max}$ for the $\mu = 1.5$ case. The bound is obtained at $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The vertical axis is in logarithmic scale.

For example we have

$$\begin{aligned}
\lambda_{0,0} &= \alpha_{0,0,0}, \\
\lambda_{2,0} &= \frac{9}{16(1+3\mu)^2} \left(\alpha_{0,0,1} + \frac{1}{2}\alpha_{0,0,2} - \frac{1}{4}\alpha_{1,0,1} \right), \\
\lambda_{2,1} &= \frac{27}{128(1+3\mu)^2} \left(\frac{1}{4}(1+3\mu)\alpha_{0,1,1} + \frac{1}{8}(1+3\mu)\alpha_{0,1,2} - \frac{15}{4}\alpha_{0,0,1} - 3\alpha_{0,0,2} - \frac{3}{4}\alpha_{0,0,3} + \frac{1}{2}\alpha_{1,0,1} + \frac{1}{4}\alpha_{1,0,2} \right).
\end{aligned} \tag{6.5.16}$$

As anticipated in Section 6.1, in contrast to the case of scattering of the lightest particle, we find several unbounded directions in the space of non-perturbative couplings when imposing unitarity of the $AB \rightarrow AB$ and $AA \rightarrow BB$ amplitudes. We believe this is mostly due to the absence of unitarity constraints on the pseudo-physical region where the ansatz 6.5.4 has an explicit discontinuity.¹⁰ Perhaps the most dramatic effect is that the value of the amplitude at the symmetric point, $\lambda_{0,0}$, is unbounded. This is illustrated in Figure 6.4: both the upper and lower bounds grow exponentially in $N_{\max}$.

As a next step, we consider instead the couplings $\lambda_{2,0}$ and $\lambda_{2,1}$. These are likely to be on a better footing since they can be bounded analytically using dispersion relations in the $AB \rightarrow AB$ channel and positivity. See Section 6.4 and equation (6.4.7).

For $\lambda_{2,0}$ we obtain an absolute lower bound compatible with zero, up to the precision we used, for any $N_{\max}$. This is consistent with (6.4.7) and the fact that in the free theory $\lambda_{2,0} = 0$. Unfortunately, we do not find any absolute upper bound: at finite $N_{\max}$ it grows exponentially, as shown in Figure 6.5a.

For $\lambda_{2,1}$ we are able to obtain a non-trivial absolute lower bound. Performing an extrapolation in $1/N_{\max}$ with a quadratic model we obtain our best estimate

$$\begin{aligned}
\mu = 1.5 & : \quad \lambda_{2,1} \geq -1.289(3), \\
\mu = 3.554 & : \quad \lambda_{2,1} \geq -0.194(1),
\end{aligned} \tag{6.5.17}$$

The error is a non-rigorous estimate based on the leave-one-out uncertainty. We report in Figure 6.6 the lower bound at finite $N_{\max} = 16, 18, 20, 22, 24, 26$, plotted as a function of

¹⁰At finite $N_{\max}$ the semi-definite problem is always bounded. As we increase the size of the ansatz, the “freedom” in the pseudo-physical region can make a coupling larger and larger, so that the bound as a function of $N_{\max}$ shows no sign of convergence.

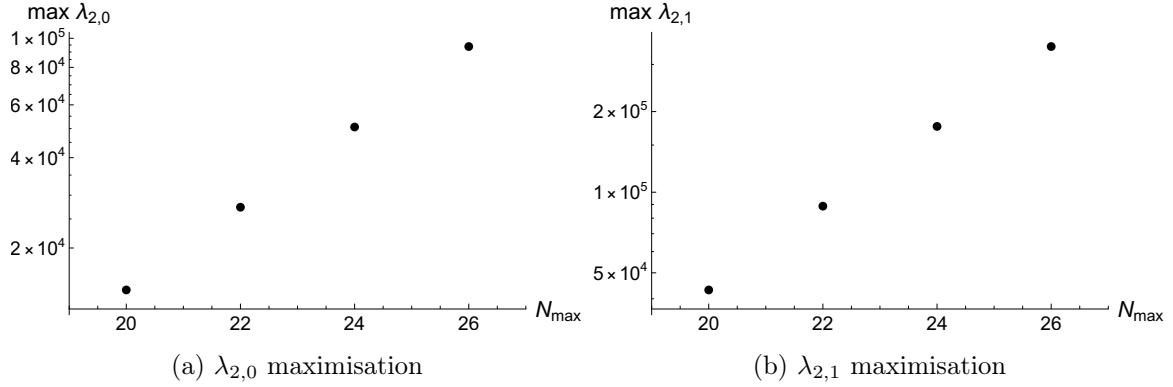


Figure 6.5: Maximisation of $\lambda_{2,0}$ and $\lambda_{2,1}$ as a function of $N_{\max}$ for the $\mu = 1.5$ case. The bound is obtained at $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The vertical axis is in logarithmic scale.

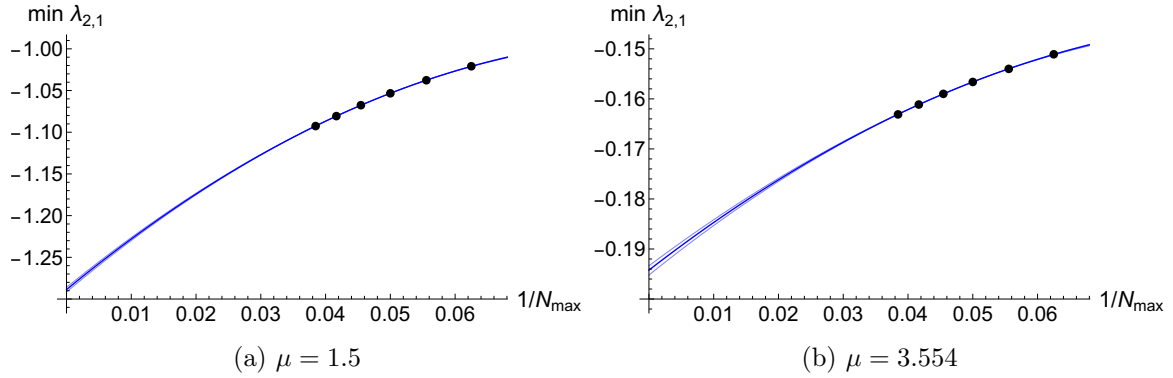
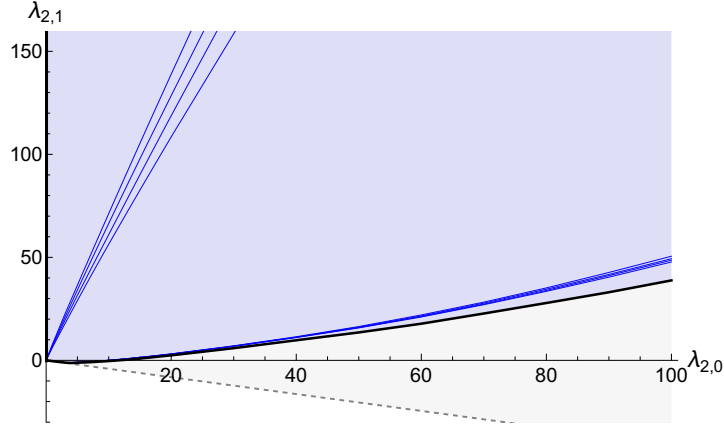


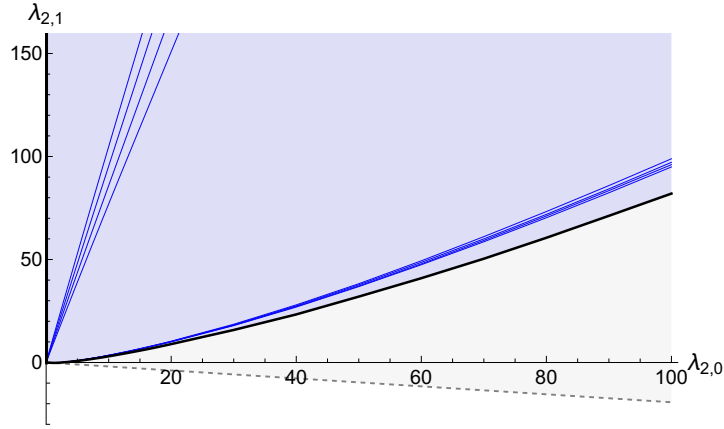
Figure 6.6: Minimisation of $\lambda_{2,1}$ as a function of $1/N_{\max}$ (black dots) for mass ratios $\mu = 1.5$ and $\mu = 3.554$. The blue line is a fit with a model $a + b/N_{\max} + c/N_{\max}^2$. The error band denotes the estimate of uncertainty based on a leave-one-out procedure.

$1/N_{\max}$ for both mass ratios, together with a quadratic fit. Figure 6.5b shows instead how the absolute upper bound grows exponentially in $N_{\max}$.

We can further improve on the positivity bound (6.4.7) by fixing a value of $\lambda_{2,0}$ and minimising/maximising $\lambda_{2,1}$ subject to non-linear unitarity. This again can be cast into an SDP of the form (6.5.14). The result is shown in Figure 6.7a and in Figure 6.7b for $\mu = 1.5$ and $\mu = 3.554$, respectively. We observe convergence for the lower bound, while the upper bound appears to be growing linearly with $N_{\max}$, at least for the range of values we have explored. The bound extrapolated at $N_{\max} \rightarrow \infty$ shows a significant improvement compared to the positivity bound, especially for large enough $\lambda_{2,0}$. For small $\lambda_{2,0}$ our bound approaches the positivity bound, see Figure 6.8; this is expected since non-linear unitarity reduces to positivity in the perturbative limit. The bound can of course also be read as upper and lower bounds on $\lambda_{2,0}$ at fixed $\lambda_{2,1}$; for $\lambda_{2,1} \geq 0$ the lower bound reduces to $\lambda_{2,0} \geq 0$ but we still have a non-trivial upper bound, as we can see in Figure 6.7; for $(\min \lambda_{2,1}) \leq \lambda_{2,1} \leq 0$, we obtain both non-trivial upper and lower bounds, as shown more clearly in Figure 6.8.



(a) $\mu = 1.5$



(b) $\mu = 3.554$

Figure 6.7: Minimisation and maximisation of $\lambda_{2,1}$ at fixed $\lambda_{2,0}$ for $\mu = 1.5$ and $\mu = 3.554$. The blue lines denote the boundary of the “allowed” primal region, of increasing size corresponding to increasing values of $N_{\max} = 20, 22, 24, 26$. The gray area bounded by the vertical axis and the dashed line denotes the positivity bound (6.4.7). The black line enclosing the blue area denotes the $N_{\max} \rightarrow \infty$ extrapolation of the bound, which converges to a finite value only for the lower bound. The upper bound at fixed $\lambda_{2,0}$ is observed to grow linearly in $N_{\max}$. All bounds use $n_{\text{pts}} = 200$ and $L_{\max} = 60$, except for the $\mu = 3.554$, $N_{\max} = 26$ case, for which $L_{\max} = 70$.

6.5.3 Bounds on scalar scattering length

In this section we explore bounds on the threshold expansion parameters (6.4.8), for small ℓ , focusing in particular on $\ell = 0$. We consider the generic case $\mu > 1$ and we report bounds for the representative values of mass ratios in (6.5.15).

The threshold expansion parameters can again be expressed in terms of linear combinations of the $\alpha_{a,b,c}$ parameters. Performing a threshold expansion of the ρ -variables (6.5.2) in the ansatz (6.5.4), in the COM 3-momentum $\mathbf{p}$ at fixed scattering angle $z = \cos \theta$, we find

$$T_{AB \rightarrow AB}(s, t, u) = \sum_{m=0}^{+\infty} \sum_{n=0}^{\lfloor m/2 \rfloor} A_{m,n} \mathbf{p}^m z^n, \quad (6.5.18)$$

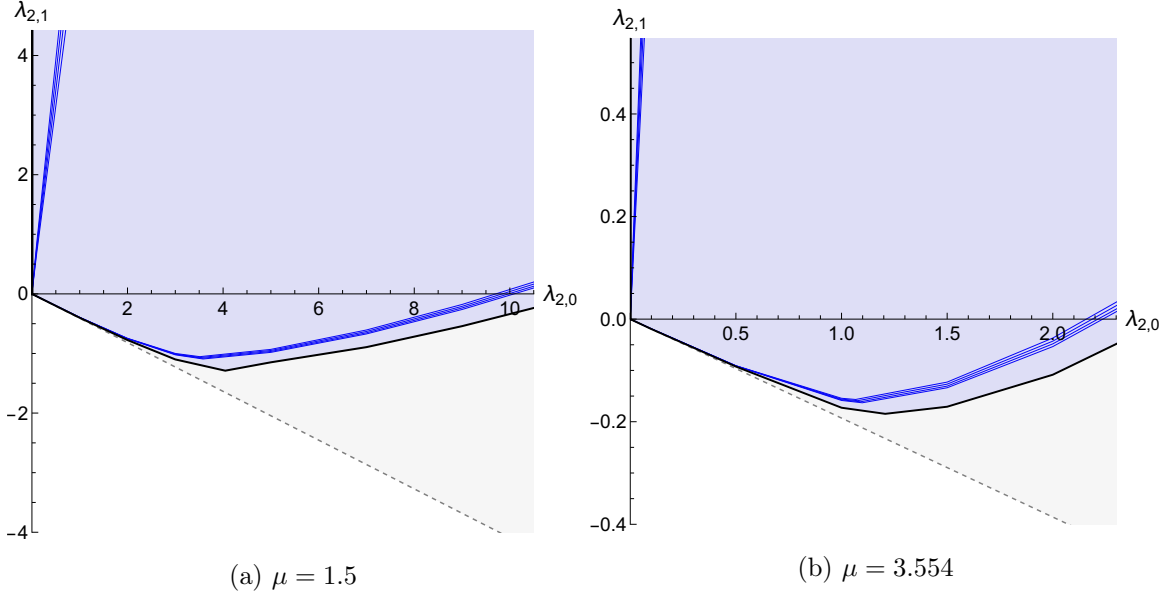


Figure 6.8: Zoom on small values of $\lambda_{2,0}$ in Figures 6.7a and 6.7b. The bound agrees with and asymptotes to the positivity bound as $\lambda_{2,0} \rightarrow 0$.

where, for example,

$$A_{0,0} = \sum_{a,b,c} \alpha_{a,b,c} (2\sqrt{6} - 5)^b \left(\frac{1 + 9\mu - 2\sqrt{6}\sqrt{\mu(3\mu+1)}}{1 - 3\mu} \right)^c. \quad (6.5.19)$$

Note that, by construction, the $A_{m,n}$ with m even are real, while the $A_{m,n}$ with m odd are imaginary. Performing the partial wave projection of the expansion (6.5.18), we obtain the relation between the coefficients $A_{m,n}$ and the threshold expansion parameters. We find

$$a_\ell = \gamma_\ell A_{2\ell,\ell}, \quad b_\ell = \gamma_\ell \left(A_{2\ell+2,\ell} - \frac{1}{2\mu} A_{2\ell,\ell} \right), \quad (6.5.20)$$

with

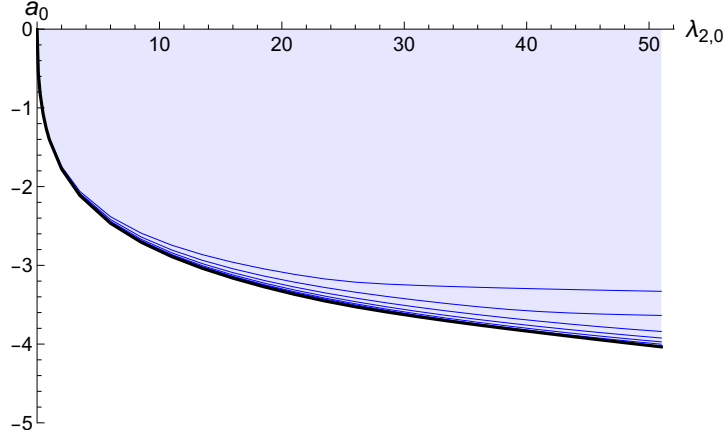
$$\gamma_\ell = \frac{(2\ell-1)!!}{(2\ell+1)\ell!} \frac{1}{8\pi(1+\mu)}. \quad (6.5.21)$$

In particular the scalar scattering length is

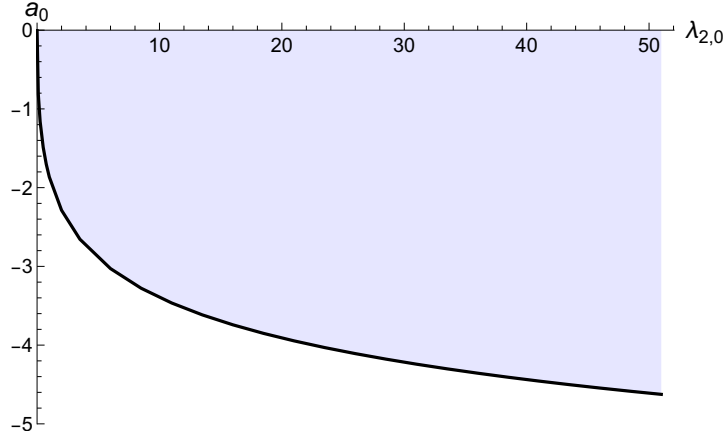
$$a_0 = \frac{A_{0,0}}{8\pi(1+\mu)}. \quad (6.5.22)$$

In contrast to the couplings $\lambda_{k,l}$, the threshold expansion involves all terms of the ansatz (6.5.4) and thus depends on the $N_{\max}$ truncation.

Starting with the scalar scattering length a_0 , we have not found evidence of convergent upper and lower bounds. The absolute lower bound appears to be growing linearly in $N_{\max}$, while the absolute upper bound diverges exponentially in $N_{\max}$. The absence of an upper bound is expected and was already observed in [42] for the case of identical particles. The absence of a lower bound is an unwelcome feature, similar to what happens for some couplings $\lambda_{k,l}$.



(a) $\mu = 1.5$



(b) $\mu = 3.554$

Figure 6.9: Minimisation of a_0 at fixed $\lambda_{2,0}$ for $\mu = 1.5$ and $\mu = 3.554$. The blue lines denote the boundary of the “allowed” primal region, of increasing size corresponding to increasing values of $N_{\max} = 16, 18, 20, 22, 24, 26$ for $\mu = 1.5$ and $N_{\max} = 20, 22, 24, 26$ for $\mu = 3.554$ (the bounds essentially overlap on this scale). The black line enclosing the blue area denotes the $N_{\max} \rightarrow \infty$ extrapolation of the lower bound. All bounds use $n_{\text{pts}} = 200$ and $L_{\max} = 60$, except for the $\mu = 3.554$, $N_{\max} = 26$ case, for which $L_{\max} = 70$.

We have also explored absolute upper and lower bounds on a_1 , a_2 and b_0 and found similar results. None of the bounds appear to be convergent in $N_{\max}$. Finally, we have obtained bounds on a_0 for a fixed value of $\lambda_{2,0}$. In this case, perhaps surprisingly, we obtain a lower bound that converges exponentially in $N_{\max}$. This is reported in Figure 6.9a and Figure 6.9b for $\mu = 1.5$ and $\mu = 3.554$, respectively, together with an $N_{\max} \rightarrow \infty$ extrapolation with a model $a + b \exp(-cN_{\max})$.¹¹

¹¹We find $c \approx 0.3$ for $\mu = 1.5$ and $c \approx 0.7$ for $\mu = 3.554$.

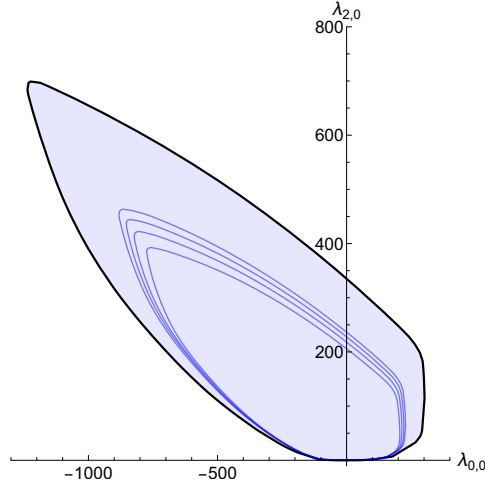


Figure 6.10: Bound in the space of $(\lambda_{0,0}, \lambda_{2,0})$ couplings for $\mu = 1$. The blue lines denote the boundary “allowed” primal region for increasing values of $N_{\max} = 20, 22, 24, 26$. The black line enclosing the blue region denotes the $N_{\max} \rightarrow \infty$ extrapolation. The bound is obtained with $L_{\max} = 46$ and $n_{\text{pts}} = 200$.

6.5.4 Bounds on couplings for equal masses

In this section we report bounds on the non-perturbative couplings $\lambda_{k,l}$ for the case where the particles A and B have the same mass, namely for $\mu = 1$. As discussed in Section 6.3 this special case is on a much better footing since the unitarity constraints bound all discontinuities of the ansatz. Indeed we find that now all the couplings $\lambda_{0,0}$, $\lambda_{2,0}$ and $\lambda_{2,1}$ admit upper and lower bounds that converge as $N_{\max} \rightarrow \infty$, analogously to what happens for the scattering of identical particles [106]. For each pair of couplings $(\lambda_{0,0}, \lambda_{2,0})$, $(\lambda_{0,0}, \lambda_{2,1})$ and $(\lambda_{2,0}, \lambda_{2,1})$, we carve out the allowed region in the space of couplings at finite $N_{\max}$,¹² and extrapolate to $N_{\max} \rightarrow \infty$ with a linear model $a + b/N_{\max}$. The results are reported in Figures 6.10, 6.11 and 6.12, respectively. The bound in the $(\lambda_{2,0}, \lambda_{2,1})$ space is again compatible with the positivity bound (6.4.7).

For completeness and comparison with (6.5.17), we also report our estimate for the lower bound on $\lambda_{2,1}$ extrapolated from a linear fit in $1/N_{\max}$

$$\mu = 1 \quad : \quad \lambda_{2,1} \geq -2.98(2). \quad (6.5.23)$$

Comparison with identical scalars. The scattering of identical scalars can be seen as a special case of the $AB \rightarrow AB$ and $AA \rightarrow BB$ system when $\mu = 1$. Bounds on the non-perturbative couplings $\lambda_{k,l}$ have been obtained for identical scalars in [106] and it is useful to compare with them. A priori, one could wonder if the bounds presented in this section are

¹²In practice we maximise the objective $p_\theta = \cos(\theta)v_1 + \sin(\theta)v_2 = \vec{b} \cdot \vec{a}$ for each pair $v_1, v_2 \in \{\lambda_{0,0}, \lambda_{2,0}, \lambda_{2,1}\}$ [297, 304]. Maximising p_θ selects the boundary point of the (v_1, v_2) region whose outward normal points along $(\cos \theta, \sin \theta)$; we read its coordinates (v_1, v_2) from the optimal solution of the SDP (6.5.14). Scanning θ then traces out the boundary. We sample the angle θ by performing a dynamic bisection based on arc-length criteria.

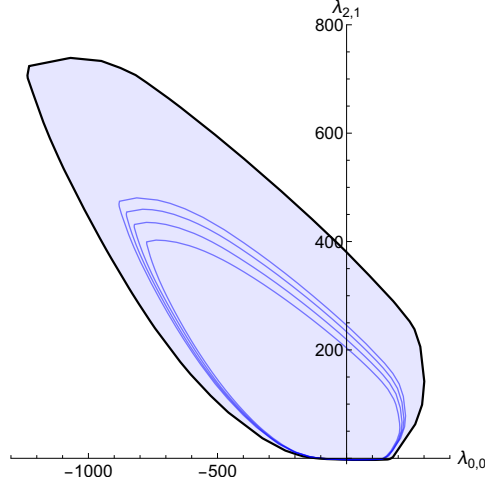


Figure 6.11: Bound in the space of $(\lambda_{0,0}, \lambda_{2,1})$ couplings for $\mu = 1$. The blue lines denote the boundary “allowed” primal region for increasing values of $N_{\max} = 20, 22, 24, 26$. The black line enclosing the blue region denotes the $N_{\max} \rightarrow \infty$ extrapolation. The bound is obtained with $L_{\max} = 46$ and $n_{\text{pts}} = 200$.

saturated by scattering amplitudes of identical scalars. We will see that this is not the case, generically.

In order to make a fair comparison between the bounds, we need to spell out schematically the SDPs we are solving in the two cases. For the case of non-identical particles we have two amplitudes $T_{AB \rightarrow AB}$ and $T_{AA \rightarrow BB}$ which are written in terms of a single scalar function F

$$F(s, t, u) = T_{AB \rightarrow AB}(s, t, u) = T_{AA \rightarrow BB}(t, s, u), \quad (6.5.24)$$

which is $s - u$ symmetric and for $m = M$ we solve the following SDP

$$\begin{aligned} & \max \quad \partial_s^k \partial_t^l F(s_0, t_0, u_0), \\ & \text{s.t.} \quad \begin{pmatrix} 1 & 1 - iF_\ell(s)^* \\ 1 + iF_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad \ell = 0, 1, 2, \dots, \\ & \quad \begin{pmatrix} 1 & -i\tilde{F}_\ell(s)^* \\ i\tilde{F}_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad \ell = 0, 2, 4, \dots, \end{aligned} \quad (6.5.25)$$

The partial amplitudes are

$$\begin{aligned} F_\ell(s) &= \mathcal{N}(s) \int_{-1}^1 dz P_\ell(z) F(s, t(s, z), u(s, z)), \\ \tilde{F}_\ell(s) &= \frac{1}{2} \mathcal{N}(s) \int_{-1}^1 dz P_\ell(z) F(t(s, z), s, u(s, z)), \end{aligned} \quad (6.5.26)$$

and

$$\mathcal{N}(s) = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{s}}, \quad (6.5.27)$$

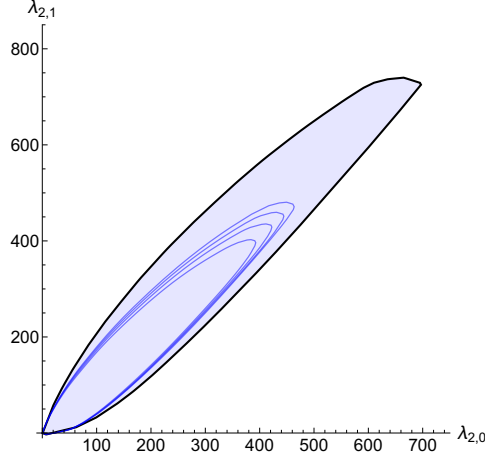


Figure 6.12: Bound in the space of $(\lambda_{2,0}, \lambda_{2,1})$ couplings for $\mu = 1$. The blue lines denote the boundary “allowed” primal region for increasing values of $N_{\max} = 20, 22, 24, 26$. The black line enclosing the blue region denotes the $N_{\max} \rightarrow \infty$ extrapolation. The bound is obtained with $L_{\max} = 46$ and $n_{\text{pts}} = 200$.

as can be read off (6.2.23) setting $M = m$. For the case of identical particles we have a single amplitude $T_{AA \rightarrow AA}(s, t, u) \equiv G(s, t, u)$ which is fully crossing symmetric and we solve the following SDP

$$\begin{aligned} & \max \quad \partial_s^k \partial_t^l G(s_0, t_0, u_0), \\ & \text{s.t.} \quad \begin{pmatrix} 1 & 1 - iG_\ell(s)^* \\ 1 + iG_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad \ell = 0, 2, 4, \dots, \end{aligned} \quad (6.5.28)$$

where

$$G_\ell(s) = \frac{1}{2} \mathcal{N}(s) \int_{-1}^1 dz P_\ell(z) G(s, t(s, z), u(s, z)), \quad (6.5.29)$$

with $\mathcal{N}(s)$ as in (6.5.27).

Let us now take $F(s, t, u) = \frac{1}{2} G(s, t, u)$ in the SDP (6.5.25). This restricts the SDP to the space of amplitudes that are also $s - t$ symmetric. The partial wave projections become in this case

$$F_\ell(s) = G_\ell(s), \quad \tilde{F}_\ell(s) = \frac{1}{2} G_\ell(s), \quad (6.5.30)$$

thanks to $s - t$ symmetry of G . The semi-definite constraint for $\tilde{F}_\ell(s)$

$$\begin{pmatrix} 1 & -\frac{i}{2} G_\ell(s)^* \\ \frac{i}{2} G_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2 \quad \ell = 0, 2, 4, \dots, \quad (6.5.31)$$

is implied by the semi-definite constraint for $F_\ell(s)$ ¹³

$$\begin{pmatrix} 1 & 1 - iG_\ell(s)^* \\ 1 + iG_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad \ell = 0, 1, 2, \dots \quad (6.5.32)$$

¹³Setting $G_\ell(s) = x + iy$, the constraint (6.5.32), which reads $x^2 + (y - 1)^2 \leq 1$, implies (6.5.31), which reads $x^2 + y^2 \leq 4$ in these variables.

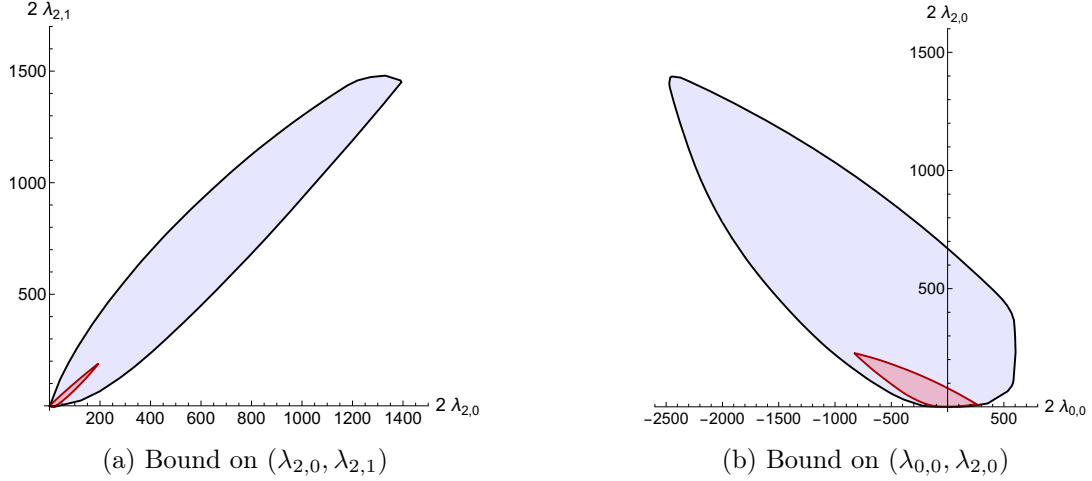


Figure 6.13: Comparison between the bounds on the couplings $\lambda_{0,0}, \lambda_{2,0}, \lambda_{2,1}$ for non-identical particles (blue region) at $\mu = 1$ and the bounds on the same couplings for identical particles (red region), up to a factor of 2, see (6.5.34). We report the bounds extrapolated at $N_{\max} \rightarrow \infty$. The bounds for identical particles are taken from [106].

Moreover, (6.5.32) is trivial for odd ℓ since the partial wave projection is non-trivial only for even ℓ thanks to $t - u$ symmetry and the fact that $t(s, -z) = u(s, z)$.

We conclude that the SDP (6.5.25) restricted to $s - t - u$ symmetric functions $G(s, t, u)$ is equivalent to the following SDP

$$\begin{aligned} \max \quad & \frac{1}{2} \partial_s^k \partial_t^l G(s_0, t_0, u_0), \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 - iG_\ell(s)^* \\ 1 + iG_\ell(s) & 1 \end{pmatrix} \succeq 0, \quad s \geq 4m^2, \quad \ell = 0, 2, 4, \dots \end{aligned} \quad (6.5.33)$$

We see that the only difference w.r.t. the SDP (6.5.28) is a factor of $1/2$ in the objective. If $\bar{\lambda}_{k,l}$ denotes the upper bound on the non-perturbative coupling, we then conclude that

$$\bar{\lambda}_{k,l}^{(\text{id})} \leq 2\bar{\lambda}_{k,l}^{(\text{non-id})} \quad (\mu = 1). \quad (6.5.34)$$

In Figure 6.13 we show the comparison between our bounds on $(\lambda_{0,0}, \lambda_{2,0})$ and $(\lambda_{2,0}, \lambda_{2,1})$ for the $\mu = 1$ case and the bounds on the same observables taken from [106].

6.6 Taming the pseudo-physical region

In this section we present a very simple toy model that shows how additional assumptions on the pseudo-physical region can cure unbounded directions in the primal S -matrix bootstrap. The idea is twofold: first we separate the pseudo-physical region $4m^2 \leq s \leq 4M^2$ from the physical region $s \geq 4M^2$; second we assume that the physics in the pseudo-physical region is completely dominated by a scalar resonance. Of course the latter is a very crude assumption (and unrealistic

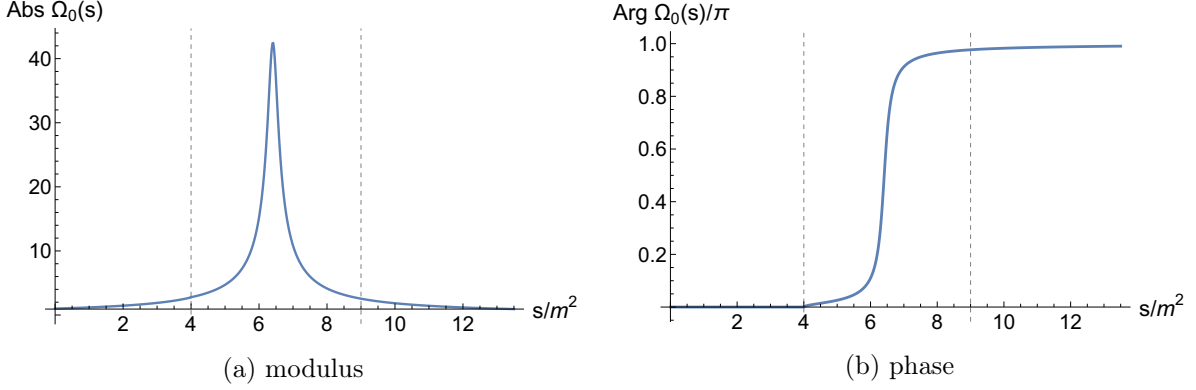


Figure 6.14: Plot of the absolute value and phase of the Omnès function parametrization (6.6.6). The dashed vertical lines denote the pseudo-physical threshold $4m^2$ and the physical threshold $4M^2$. The plot is done for $\mu = 3/2$, $m_0^2 = 13/2$ and $g = 1/2$.

for most physical scatterings), but the aim is to show how far physical assumptions on the pseudo-physical behaviour can get us.

The parametrization of the scalar resonance is done using the so-called Muskhelishvili-Omnès function [389, 390]. Analytically continued unitarity for the $AA \rightarrow BB$ process¹⁴

$$2 \operatorname{Im} f_{AA \rightarrow BB}^\ell(s) = \mathcal{N}_{AA \rightarrow AA}(s) f_{AA \rightarrow AA}^\ell(s) f_{AA \rightarrow BB}^\ell(s)^*, \quad 4m^2 \leq s \leq \min\{16m^2, 4M^2\}, \quad (6.6.1)$$

implies the Watson's theorem, which states that the phase of the $AA \rightarrow BB$ amplitude coincides with the phase shift of the $AA \rightarrow AA$ amplitude in the pseudo-physical region.¹⁵ In formulas, if $\delta_\ell(s)$ is the phase shift of the $AA \rightarrow AA$ scattering

$$S_{AA \rightarrow AA}^\ell(s) = 1 + i \mathcal{N}_{AA \rightarrow AA}(s) f_{AA \rightarrow AA}^\ell(s) = e^{2i\delta_\ell(s)}, \quad 4m^2 \leq s \leq 4M^2, \quad (6.6.2)$$

then

$$f_{AA \rightarrow BB}^\ell(s) = \left| f_{AA \rightarrow BB}^\ell(s) \right| e^{i\delta_\ell(s)}, \quad 4m^2 \leq s \leq 4M^2. \quad (6.6.3)$$

The modulus is naively unconstrained, but it can be reconstructed from its phase, up to a polynomial term, using a dispersion relation:

$$f_{AA \rightarrow BB}^\ell(s) = P^\ell(s) \Omega^\ell(s), \quad (6.6.4)$$

where

$$\Omega^\ell(s) = \exp\left(\frac{s}{\pi} \int_{4m^2}^{+\infty} ds' \frac{\delta_\ell(s')}{s'(s' - s)}\right) \quad (6.6.5)$$

is called the Muskhelishvili-Omnès function, and $P^\ell(s)$ is a polynomial in s , whose degree is associated to the behaviour of the amplitude and the phase shift at infinity. We have taken the upper extreme of integration in (6.6.5) to be ∞ rather than $4M^2$, for simplicity. The distinction is immaterial since, for $s > 4M^2$, the terms coming from the ρ -ansatz also contribute to the phase of the amplitude, see below.

¹⁴Note that $f_{AA \rightarrow BB}^\ell(s)^*$ has to be intended as $f_{AA \rightarrow BB}^\ell(s - i\epsilon)$ to perform analytic continuation.

¹⁵This applies for $M < 2m$, while for $M > 2m$ it holds only if inelasticities are negligible.

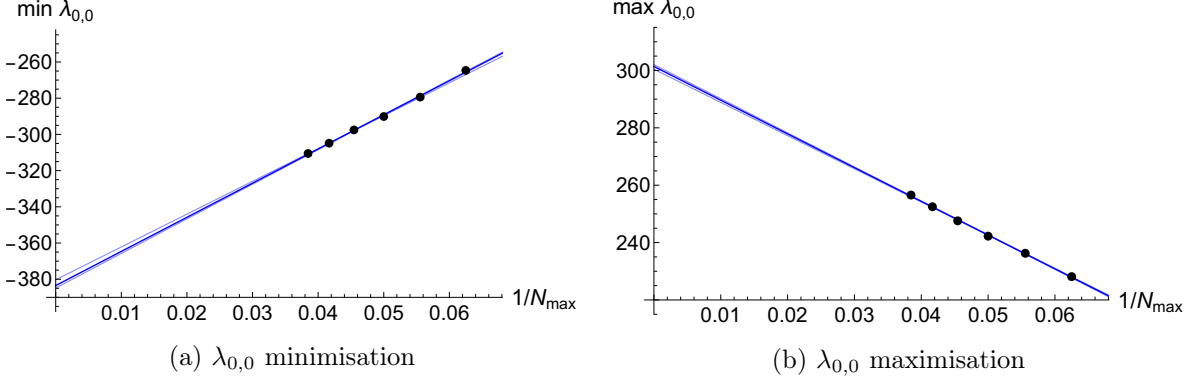


Figure 6.15: Minimisation and maximisation of $\lambda_{0,0}$ as a function of $1/N_{\max}$ using the Omnès function ansatz. The bound is obtained for $\mu = 3/2$, $m_0^2 = 13/2$ and $g = 1/2$, and using $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The blue line is a fit with a model $a + b/N_{\max}$. The error band denotes the estimate of the uncertainty based on a leave-one-out procedure.

Assuming that the $AA \rightarrow AA$ process is dominated by the scalar partial wave, and this is dominated by a resonance, we show in Section 6.6.1 that the corresponding Omnès function for $\ell = 0$ can be written as

$$\Omega^0(s) = \frac{m_0^2}{s - m_0^2 + g^2 C(s)}, \quad (6.6.6)$$

where m_0 and g are free parameters governing the mass and the width of the resonance, while $C(s)$ is the Chew-Mandelstam function [391]

$$C(s) = \frac{s}{\pi} \int_{4m^2}^{\infty} ds' \frac{r(s')}{s'(s' - s)} = \frac{1}{\pi} \left(2 + r(s) \log \frac{r(s) - 1}{r(s) + 1} \right), \quad r(s) = \sqrt{1 - \frac{4m^2}{s}}. \quad (6.6.7)$$

We refer the reader to Section 6.6.1 for a derivation of (6.6.6) and its relation to (6.6.5) when δ_0 is dominated by a resonance, and for the analytic properties of the function (6.6.6).

In Figure 6.14 we report the plot of our parametrization of the Omnès function which presents the typical resonance behaviour.

For the primal S -matrix bootstrap, we then propose to modify the ansatz in (6.5.4) and (6.5.7) as follows

$$T_{AB \rightarrow AB}(s, t, u) = \alpha_0 \Omega^0(t) + \sum_{a,b,c} \alpha_{a,b,c} (\rho_+(s))^a (\rho_M(t))^b (\rho_+(u))^c, \quad (6.6.8)$$

$$T_{AA \rightarrow BB}(s, t, u) = \alpha_0 \Omega^0(s) + \sum_{a,b,c} \alpha_{a,b,c} (\rho_+(t))^a (\rho_M(s))^b (\rho_+(u))^c, \quad (6.6.9)$$

where, for simplicity, we assume that $P^{\ell=0}(s)$ is just a constant, the parameter α_0 , and $\rho_3(z)$ is a ρ -variable with a branch-cut starting at the *physical* threshold

$$\rho_M(z) = \rho(z, 4M^2, \Sigma - 2s_0). \quad (6.6.10)$$

The ansatz (6.6.9) is constructed in such a way that the discontinuity of the partial wave projection $f_{AA \rightarrow BB}^{\ell}(s)$ in the pseudo-physical region is given by the Omnès function $\Omega^0(s)$ only

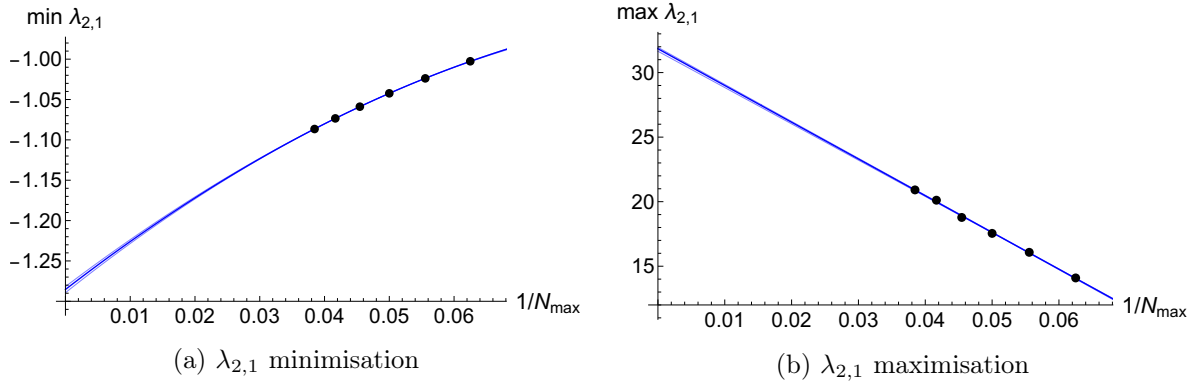


Figure 6.16: Minimisation and maximisation of $\lambda_{2,1}$ as a function of $1/N_{\max}$ using the Omnès function ansatz. The bound is obtained for $\mu = 3/2$, $m_0^2 = 13/2$ and $g = 1/2$, and using $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The blue line is a fit with a model $a + b/N_{\max}$ for the maximisation and a quadratic model $a + b/N_{\max} + c/N_{\max}^2$ for the minimisation. The error band denotes the estimate of the uncertainty based on a leave-one-out procedure.

for $\ell = 0$ and it is trivial for $\ell > 0$

$$\text{Im } f_{AA \rightarrow BB}^\ell(s) = \delta_{\ell,0} \alpha_0 \text{Im } \Omega^0(s), \quad 4m^2 \leq s < 4M^2. \quad (6.6.11)$$

The ansatz (6.6.8) is simply fixed by $s - t$ crossing. This is not exactly what we would like from Watson's theorem (6.6.3), since that would require the phase of $f_{AA \rightarrow BB}^0$ to match the phase of Ω^0 . However, given an explicit solution to an optimisation problem, we can check a posteriori that the Omnès function term is dominant w.r.t. the pure ρ -ansatz part in the pseudo-physical region, and that approximately

$$\arg\left(f_{AA \rightarrow BB}^\ell(s)\right)\Big|_{\text{optimal}} \approx \delta_{\ell,0} \arg \Omega^0(s), \quad 4m^2 \leq s < 4M^2. \quad (6.6.12)$$

Adopting now this modified ansatz, we explore bounds on the non-perturbative couplings $\lambda_{k,l}$ using the same strategy discussed in Section 6.5.1. In particular we present upper and lower bounds on $\lambda_{0,0}$, namely the value of the amplitude at a point, for different values of $N_{\max}$. The problem now depends not only on the mass ratio μ , but also on the parameters m_0 and g of the Omnès function (6.6.6). We have not explored the full range of those parameters, but rather have chosen just a sample value since the purpose of this section is a proof of concept. We have taken

$$\mu = \frac{3}{2}, \quad m_0^2 = \frac{13}{2}, \quad g = \frac{1}{2}. \quad (6.6.13)$$

The value of m_0^2 is chosen to be the mid-point of the pseudo-physical region, while g is an arbitrary value that satisfies the constraint (6.6.26).

In Figure 6.15 we report upper and lower bounds on $\lambda_{0,0}$ as a function of $N_{\max}$. Nicely enough, we observe now good linear convergence of the bound in $1/N_{\max}$. This is to be contrasted with Figure 6.4 which showed exponential growth of the bound with $N_{\max}$. This shows that our crude assumption on the pseudo-physical region is strong enough to allow us to recover a bound on the amplitude at one point.

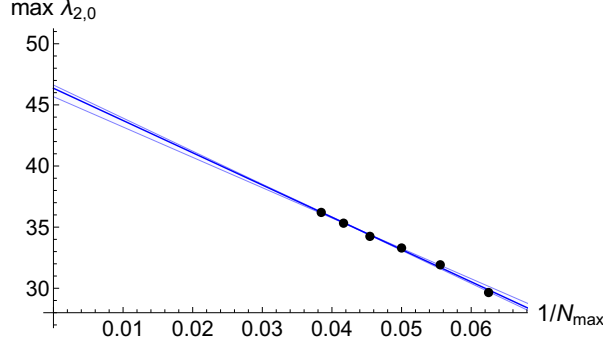


Figure 6.17: Maximisation of $\lambda_{2,0}$ as a function of $1/N_{\max}$ using the Omnès function ansatz. The bound is obtained for $\mu = 3/2$, $m_0^2 = 13/2$ and $g = 1/2$, and using $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The blue line is a fit with a model $a + b/N_{\max}$. The error band denotes the estimate of the uncertainty based on a leave-one-out procedure.

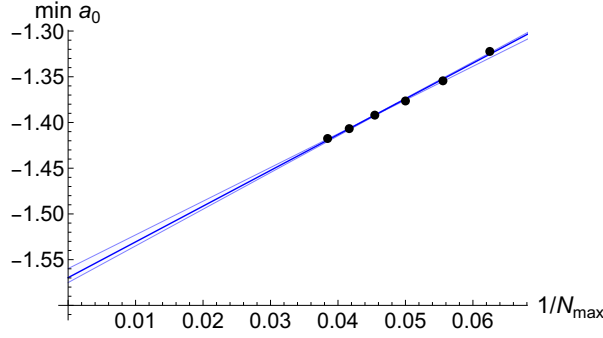


Figure 6.18: Minimisation of a_0 as a function of $1/N_{\max}$ using the Omnès function ansatz. The bound is obtained for $\mu = 3/2$, $m_0^2 = 13/2$ and $g = 1/2$, and using $L_{\max} = 60$ and $n_{\text{pts}} = 200$. The blue line is a fit with a model $a + b/N_{\max}$. The error band denotes the estimate of the uncertainty based on a leave-one-out procedure.

Similar convergent bounds are now also found for minimisation and maximisation of $\lambda_{2,1}$, as reported in Figure 6.16, as well as for maximisation of $\lambda_{2,0}$, shown in Figure 6.17. Finally, a convergent lower bound is found also for the scalar scattering length a_0 and is reported in Figure 6.18.

For all these extremisation procedures, we always find the Omnès part of the ansatz (6.6.6) to be dominant in the scalar partial amplitude $f_{AA \rightarrow BB}^\ell(s)$ on the pseudo-physical region. An example plot is shown in Figure 6.19 for the extremal amplitude obtained by minimising a_0 at $N_{\max} = 26$.

6.6.1 Details on Muskhelishvili-Omnès function

Let us give some further details on the origin of (6.6.6), its analytic properties, and how this is related to the definition of Muskhelishvili-Omnès function in (6.6.5). Let $f \equiv f_{AA \rightarrow AA}^{\ell=0}$ be the scalar partial amplitude for $AA \rightarrow AA$ scattering. In the elastic region, unitarity requires

$$2\text{Im } f = r(s)|f^2| \longrightarrow \text{Im } f^{-1} = -\frac{r(s)}{2}, \quad 4m^2 \leq s \leq \min(4M^2, 16m^2), \quad (6.6.14)$$

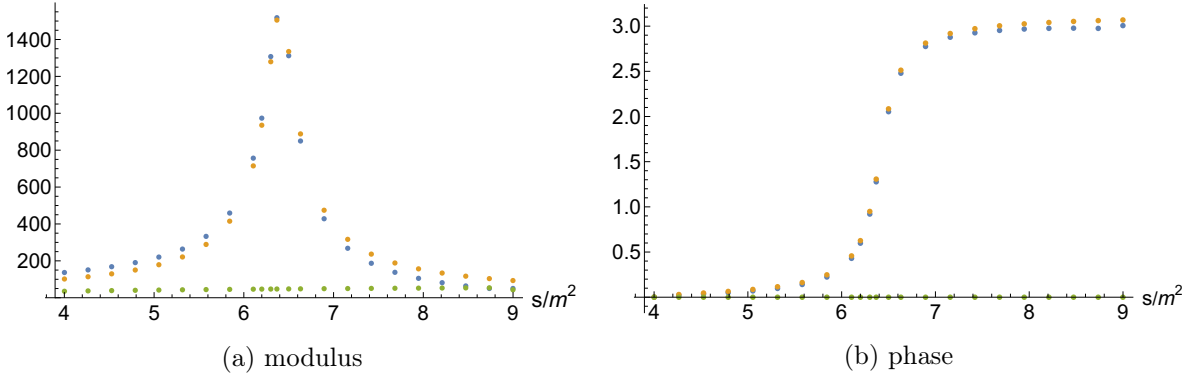


Figure 6.19: Scalar partial amplitude in the pseudo-physical region for the $\max \lambda_{0,0}$ problem at $N_{\max} = 26$ using the Omnès function ansatz. The blue dots denote extremal scalar partial amplitude $f_{AA \rightarrow BB}^0(s)$, the orange dots denote the Omnès function contribution $\alpha_0 \Omega^0(s)$, while the green dots denote the ρ -variables part of the ansatz. This shows that the Omnès function is dominant and that (6.6.12) holds.

where $r(s)$ is defined in (6.6.7). Consider now the Chew-Mandelstam function $C(s)$ entering (6.6.7). Using $\lim_{\epsilon \rightarrow 0} 1/(x \pm i\epsilon) = P(1/x) \mp i\pi\delta(x)$ in the integral form, one easily finds

$$C(s \pm i\epsilon) = \text{P.V.} C(s) \pm ir(s) \longrightarrow \text{Im } C(s) = r(s) \quad \text{for } s > 4m^2, \quad (6.6.15)$$

where P.V. stands for the principal value of the integral. The combination $K^{-1} \equiv 2/f + C$ is then an arbitrary function, analytic on the cut. We can then express the partial wave as

$$f = \frac{1}{2} \frac{1}{K^{-1} - C}. \quad (6.6.16)$$

Assuming that f is dominated by a resonance, a minimal choice for K^{-1} reads

$$K^{-1} = \frac{m_0^2 - s}{g^2}, \quad (6.6.17)$$

where m_0 and g^2 are related to the mass, width, and coupling of the resonance.¹⁶ Up to an overall constant, the resulting partial wave f is proportional to the Omnes function (6.6.6). The phase shift δ^0 can be determined either from f , or equivalently from the argument of Ω^0 , using (6.6.15) and (6.6.6). We get in both cases

$$\delta^0(s) = -\arctan \frac{g^2 r(s)}{s - m_0^2 + g^2 \text{P.V.} C(s)}. \quad (6.6.18)$$

The phase shift δ^0 is of the form expected for a resonance. For a thin resonance, $\delta^0(s)$ approaches a Breit-Wigner form, where m_0^2 and $m_0\Gamma \approx g^2 r(m_0^2)$ are approximately the mass and the width of the resonance, respectively.¹⁷ Note that $\delta^0(s)$ has the correct square-root behaviour close to

¹⁶Alternative choices for K^{-1} may be adopted when additional features of the amplitude need to be incorporated, such as the Adler zeros that occur, for example, in pion-pion scattering.

¹⁷The formalism applies also for broad resonances. In this case, m_0 and $g^2/(4m_0^2)$ substantially deviate from the mass and width of the resonance, which are properly determined by looking at the pole of Ω^0 in the second Riemann sheet of the s -complex plane, see below.

the threshold $s = 4m^2$:

$$\delta_0(s) = \frac{g^2\pi}{-4m\left(g^2 - 2m^2\pi + \frac{m_0^2\pi}{2}\right)}\sqrt{s - 4m^2} + \mathcal{O}(s - 4m^2)^{\frac{3}{2}}, \quad (6.6.19)$$

and monotonically approaches π as $s \rightarrow \infty$. We now show that Ω_0 in (6.6.6) can be written as in the defining form (6.6.5), with δ^0 as in (6.6.18). Define

$$L \equiv \log \Omega_0 = \log m_{\text{res}}^2 - \log D(s), \quad D(s) \equiv s - m_0^2 + g^2 C(s), \quad (6.6.20)$$

so that

$$L(s + i\epsilon) - L(s - \epsilon) = 2i\delta^0(s). \quad (6.6.21)$$

Write then a dispersion relation starting from an analytic point in the s -plane,

$$\frac{L(s)}{s} = \frac{1}{2i\pi} \oint_{\gamma} \frac{L(z)dz}{z(z-s)}, \quad (6.6.22)$$

where γ is a small circle around the point s . Blowing the contour, we get

$$L(s) = \frac{s}{2i\pi} \int_{4m^2}^{\infty} \frac{\text{Disc } L(z)dz}{z(z-s)} = \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{\delta_0(s')ds'}{s'(s'-s)}. \quad (6.6.23)$$

By exponentiating, we then reproduce (6.6.5).

We can finally verify that Ω^0 cannot have poles in the principal sheet, but it can in the second sheet, as expected for a resonance appearing in an elastic amplitude. Away from the branch-cut, the denominator in (6.6.6) can not have a zero for complex s , since its imaginary part never vanishes. Indeed, simple algebra gives

$$\text{Im } D^{\text{I}}(s) = \text{Im } s \left(1 + g^2 \int_{4m^2}^{\infty} \frac{\rho(s')ds'}{|s' - s|^2} \right), \quad (6.6.24)$$

where I indicates the principal sheet. The term in square bracket is positive definite and hence, for $\text{Im } s \neq 0$, $\text{Im } D^{\text{I}}$ cannot vanish. For real $s < 4m^2$, we should check whether $D(s)$ can give rise to zeros, namely to stable bound states. It is easy to see that $D^{\text{I}}(s)$ is monotonically increasing from $-\infty$ for $s \rightarrow -\infty$ to some value as $s \rightarrow 4m^2$. If $D^{\text{I}}(s = 4m^2) < 0$, no zeros can appear in the whole range $s \in (-\infty, 4m^2)$. We have

$$D^{\text{I}}(s = 4m^2) = 4m^2 - m_0^2 + \frac{2g^2}{\pi}, \quad (6.6.25)$$

so demanding the absence of poles requires

$$g^2 < \frac{\pi}{2}(m_0^2 - 4m^2). \quad (6.6.26)$$

In the second Riemann sheet II, we have

$$D^{\text{II}} = D^{\text{I}} + 2ig^2 r(s), \quad (6.6.27)$$

and now a complex zero will appear at s^* : $D^{\text{II}}(s^*) = 0$. For simplicity, let us work out the case for a thin resonance, with $s^* = m_{\text{res}}^2 - im_{\text{res}}\Gamma$, with $\Gamma \ll m_{\text{res}}$. Expanding at linear order in Γ and g^2 , and recalling that $C(s) \approx \text{P.V.} C(m_{\text{res}}^2) - ig^2 r(m_{\text{res}}^2)$, we have

$$D^{\text{II}}(s) \approx m_{\text{res}}^2 - im_{\text{res}}\Gamma - m_0^2 + g^2 C_0(m_{\text{res}}^2) + ig^2 r(m_{\text{res}}^2) = 0, \quad (6.6.28)$$

which gives, up to order g^2 :

$$m_{\text{res}}^2 \approx m_0^2 - g^2 C_0(m_0^2), \quad \Gamma \approx \frac{g^2 r(m_0^2)}{m_0}. \quad (6.6.29)$$

6.7 Discussion

We explored the constraints on the space of scattering amplitudes of two unequal scalar massive particles in $4d$, denoted by A and B . A $\mathbb{Z}_2 \times \mathbb{Z}_2$ global symmetry was assumed, ensuring the stability of the particles, forbidding triangular anomalous thresholds, and slightly simplifying our setup. We used the primal S -matrix bootstrap to obtain non-perturbative bounds on several observables that can be extracted from the scattering amplitude for the $AB \rightarrow AB$ process. This is not a trivial generalization of the identical scalar S -matrix bootstrap because of the unavoidable presence of another source of non-analyticity, the pseudo-physical regions. We found that for $M > m$ there exist observables that admit one-sided bounds or admit a bound once another observable is fixed. Upper and lower bounds on all the observables are recovered when the behaviour of the amplitude is controlled in the pseudo-physical region by some assumption, or for $M = m$, where this region collapses to a point.

There are several interesting directions for further explorations.

Role of the pseudo-physical region. Singularities in the pseudo-physical region present a challenge to the bootstrap since they are not bounded by unitarity, which strictly applies to the physical region, and played a prominent role in this chapter. It would be interesting to better understand their role and predict which observables can be bounded and how. The naive expectation is that maximisation or minimisation of an observable is unbounded in the $N_{\text{max}} \rightarrow \infty$ limit. This phenomenon can be understood in simple toy models of analytic functions of one variable,¹⁸ but the extension to the actual scattering in $4d$ is not straightforward because of the complication of two complex variables, crossing symmetry and the partial wave projection. In fact, the key result of this chapter is that some observables can be bounded.

¹⁸Consider the space of real analytic functions $f(z)$ on $\mathbb{C} \setminus [1, +\infty)$ such that $f(z) \sim 1/z$ as $z \rightarrow \infty$. Let $x_0 \in [1, +\infty)$ and consider the problem of maximising $f(0)$ subject to $|f(x)| \leq 1$ for $x \geq x_0$. If $x_0 = 1$ the maximum modulus principle gives $f(0) = 1$, with the optimal function being $f(z) = 1$. As soon as $x_0 > 1$ the problem becomes unbounded. This can be most easily seen by writing the dispersion relation

$$f(z) = \int_1^\infty \frac{dx}{\pi} \frac{\text{Im } f(x)}{x - z} \quad \Rightarrow \quad f(0) = \int_1^{x_0} \frac{dx}{\pi} \frac{\text{Im } f(x)}{x} + \int_{x_0}^\infty \frac{dx}{\pi} \frac{\text{Im } f(x)}{x}.$$

The second term is bounded by $|\text{Im } f(x)| \leq 1$ and by the fall off at infinity. The first term is instead unbounded, so that $\max f(0) = +\infty$. This can also be seen numerically by solving the primal semi-definite optimisation problem by taking an ansatz $f(z) = \sum_{n=0}^{n_{\text{max}}} a_n \rho(z, 1, 0)^n$, with the ρ -variable in (6.5.1). One finds that a bound on $f(0)$ does not converge in n_{max} . A similar argument was presented in [43, App. E].

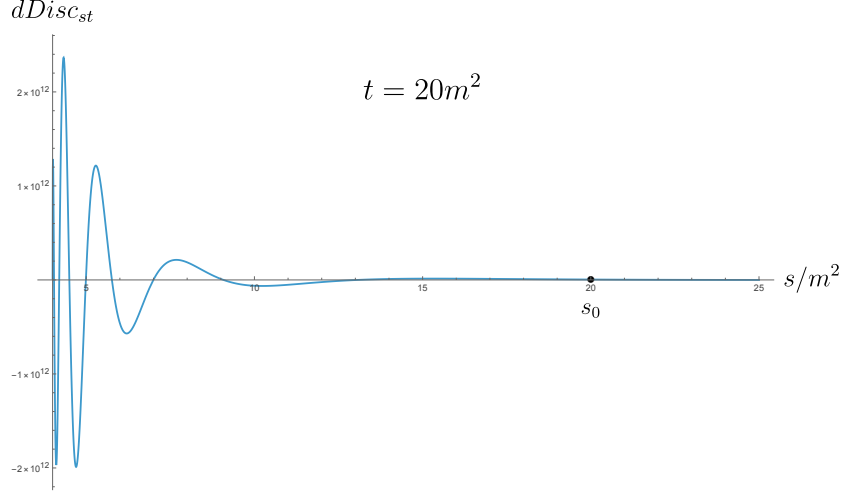


Figure 6.20: The value of $d\text{Disc}_{st}$ as a function of $s > s_{\text{th}}$ for the extremal $AB \rightarrow AB$ amplitude obtained by maximising $\lambda_{2,0}$ at fixed t , $\mu = 1$. We have $N_{\text{max}} = 26$, $L_{\text{max}} = 46$.

Double discontinuity. The expansion of the amplitude (6.5.4) was motivated by its analytic structure, but it actually fails to reproduce the correct $d\text{Disc}$ of the amplitude. Indeed, while the actual $d\text{Disc}$ of the $AB \rightarrow AB$ amplitude for physical values of s is non-vanishing for $t > t^{\text{LL}}(s) > 4m^2$, with t^{LL} determined in (E.3.4), amplitudes written as in (6.5.4) have a non-vanishing $d\text{Disc}$ starting directly from $t > 4m^2$. Its value is computed noting that, for $s > m_+^2$ and $t > 4m^2$, $\rho_+(s)$ and $\rho_m(t)$ are pure phases $\rho_+(s) = e^{i\theta_s(s)}$, $\rho_m(t) = e^{i\theta_t(t)}$ and we get

$$d\text{Disc}_{st} T(s, t) = \sum_{a,b} \alpha_{a,b,0} \sin(a\theta_s(s)) \sin(b\theta_t(t)), \quad AB \rightarrow AB. \quad (6.7.1)$$

At first sight this might suggest that the amplitudes (6.5.4) are unphysical. Very preliminary investigations indicate, however, that the observables we consider are very weakly sensitive to the double discontinuity. A possible explanation of this welcome feature is the following. The coefficients $\alpha_{a,b,0}$ for the allowed amplitudes are such that $d\text{Disc}_{st}$ oscillates rapidly for $t < t^{\text{LL}}$, effectively averaging it to zero, and then changes behaviour for $t > t^{\text{LL}}$, when it is supposed to be non-vanishing. For illustration, we show in Figure 6.20 the extremal amplitude obtained by maximising $\lambda_{2,0}$ at fixed $t = 20m^2$ for $\mu = 1$. For this value of t , $d\text{Disc}_{st}$ should identically vanish in the interval $s \in [4m^2, s_0]$, with $s_0 = 20m^2$. The figure shows that $d\text{Disc}_{st}$ oscillates in the region where it should vanish, and approaches a monotonic behaviour when s approaches s_0 .¹⁹ The same behaviour is found in other cases. It would be interesting to undertake a systematic analysis of this kind and understand to what extent and for which observables the ρ -expansion is just fine, although it does not reproduce the correct analytic behaviour of the amplitude.²⁰

¹⁹For $s > s_0$, $d\text{Disc}_{st}$ is non-zero. This is not evident from the figure at the scales reported.

²⁰As far as we are aware, this question has not been settled even for the much studied case of identical scalar scattering.

Full system. An obvious generalisation of the analysis in this chapter would be to consider the full system of two-to-two scattering amplitudes of the A and B particles (6.1.3). Bounds from this mixed system will be at least as strong as the ones obtained in this chapter, if not stronger. We believe that the ordinary unitarity constraints in (6.2.29) and (6.2.27) would still be too weak to obtain two-sided bounds on all observables. It is possible that analytically continued unitarity makes the problem bounded. The intuition behind this is that such form of extended unitarity implies

$$\left| \text{Im } f_{AA \rightarrow BB}^\ell(s) \right|^2 \leq \text{Im } f_{AA \rightarrow AA}^\ell(s) \text{Im } f_{BB \rightarrow BB}^\ell(s), \quad (6.7.2)$$

also for the pseudo-physical region $4m^2 \leq s \leq 4M^2$. Indeed, in $2d$ extended unitarity was shown to bound certain observables extracted from the $AB \rightarrow AB$ amplitude [43]. Whether this persists in $4d$ remains to be seen. A natural concern is that both $AA \rightarrow BB$ and $BB \rightarrow BB$ have a discontinuity in the pseudo-physical region, so that (6.7.2) only imposes a relative constraint between the two, leaving the amplitudes free to behave wildly in the pseudo-physical region and rendering some observables unbounded. Extended unitarity does, however, force the discontinuity in the pseudo-physical region of the $BB \rightarrow BB$ process to be sign-definite, which may tame the optimisation problem.²¹

Phenomenological applications. Looking ahead to phenomenological applications, it would be interesting to extend this setup to the scattering of different hadrons in low-energy QCD. So far the applications of the primal S -matrix bootstrap in this spirit have been restricted to pion-pion scattering. Natural targets are pion-kaon and pion-nucleon scattering. This was, at its core, the main goal that motivated this largely theoretical chapter. We would need to dress our setup with global symmetries (which can be approximately taken to be $SU(2) \times U(1)$) and the helicity structure of the nucleons.²² We believe non-analyticities in the pseudo-physical region will still represent the main technical complication. In addition to that, a crucial non-trivial point would be to identify both inputs from QCD and a good set of observables that make the problem bounded and, importantly, the QCD scattering amplitude close to be extremal. In this respect, restricting to the amplitudes in (6.1.4) has a further potentially useful by-product. Extremal amplitudes obtained in the primal bootstrap are often elastic, i.e. saturate unitarity conditions. In studying the full system, the first particle-production threshold is at $s = 16m^2$ and comes from $4A$ particles in the $AA \rightarrow AA$ process. By contrast, the lightest multi-particle state in the $AB \rightarrow AB$ process is $3A + B$, with threshold $s = (3m + M)^2 = m^2(3 + \mu)^2$, parametrically larger than $16m^2$ for $\mu \gg 1$. In the absence of additional lighter channels, the $AB \rightarrow AB$ amplitude therefore remains elastic over a much wider range of squared energies. Extremal amplitudes obtained from the reduced system (6.1.4) may consequently be closer to physical amplitudes.

²¹Instead, the discontinuity in the pseudo-physical region of $AA \rightarrow BB$ can be of any sign. We have indeed observed that even for the optimisation problems of Section 6.5 that admit a convergent bound, the extremal amplitudes oscillate wildly in the pseudo-physical region.

²²The analysis of $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow N\bar{N}$ amplitude with dispersion relations was already carried out in the seminal work of Mandelstam [11].

Both $\pi K \rightarrow \pi K$ and $\pi N \rightarrow \pi N$ scatterings have been extensively studied using first-principles approaches such as the Roy–Steiner equations [392–394] (see e.g. [372, 395, 396]) and it would be interesting to see how the S -matrix bootstrap can compare to that. A lot of experimental data about these processes has been available for a long time, making this an exciting prospect for the bootstrap.

Appendix A

Hypergeometric gymnastics

A.1 Asymptotic expansion of ${}_2F_1$

The theory of hypergeometric functions is a well-known subject in mathematics, see e.g. section 15 of [397] for a collection of results. Since the early work of Watson [398] several asymptotic expressions of ${}_2F_1$ when different combinations of parameters are large have been discussed in the literature, see e.g. [399–401], and [397, 402] for a collection of results.¹ In these works the coefficients of the resulting asymptotic expansion are either given in implicit form or only the first few terms of the expansion are explicitly reported. To the best of our knowledge, a completely explicit form of the coefficients of the whole asymptotic expansion of the ${}_2F_1$ entering (3.2.9) and (3.2.10) is missing. This appendix aims to close this gap and discuss some generalizations.

The hypergeometric function ${}_2F_1(a, b, c; z)$ is one of the solutions of the second order differential equation

$$z(1-z)f''(z) + (c - (a+b+1)z)f'(z) - abf(z) = 0. \quad (\text{A.1.1})$$

This equation has three regular-singular points at $z = 0, 1, \infty$. The solution ${}_2F_1(a, b, c; z)$ is analytic at $z = 0$ in the principal sheet, and has a branch cut singularity at $z = 1$, with a branch cut over the real positive axis from $z = 1$ up to infinity. Its series expansion in z , known as the Gauss series, is convergent in the open disc $|z| < 1$. We are here interested in an expansion for large parameters a , b and c .

The general problem is to determine, at least in some domain $z \in D \subseteq \mathbb{C}$, the asymptotic series in $\lambda \rightarrow +\infty$ of a ${}_2F_1(a, b, c; z)$ where the parameters scale as

$$a = a_0 + a_1\lambda, \quad b = b_0 + b_1\lambda, \quad c = c_0 + c_1\lambda. \quad (\text{A.1.2})$$

When $c_1 \neq 0$ the series is expected to take the semi-classical form

$${}_2F_1(a_0 + a_1\lambda, b_0 + b_1\lambda, c_0 + c_1\lambda; z) \sim e^{-\lambda S(z)} A(z) \left(1 + \sum_{n=1}^{+\infty} f_n(z) \lambda^{-n} \right), \quad z \in D. \quad (\text{A.1.3})$$

¹See also [403] for a WKB-like approach. This is useful to quickly get the Stokes lines associated with the resulting asymptotic series (see appendix A.1.2), but we did not find it convenient to get the whole explicit asymptotic series.

The case $c_1 = 0$ will not be considered. This case could be in principle worked out with a similar approach to what follows, but needs a slight modification of the semi-classical ansatz.

This appendix is quite articulated and is then useful to summarize its structure. We discuss in sections A.1.1 and A.1.2 how to determine the domain $z \in D \subseteq \mathbb{C}$ where the asymptotic expansion (A.1.3) is Borel resummable and where (A.1.3) is a good approximation of the function. This requires to determine the so-called Stokes and anti-Stokes lines associated to the differential equation (A.1.1), which are governed by the factor $S(z)$ in (A.1.3). The explicit form of the factors of S and A is derived in section A.1.3. The remaining sections are dedicated to the derivation of the coefficients f_n in (A.1.3). It is hard to determine a closed analytic form of f_n when all parameters scale with λ . The case where

$$a_1 = 0, \quad b_1 = 0, \quad c_1 \neq 0, \quad (\text{A.1.4})$$

is instead particularly simple and will be discussed first in section A.1.4. Fortunately enough, the hypergeometric functions with special parameters in (3.2.9) and (3.2.10) entering our $2d$ CFT problem can be reduced to this case thanks to hypergeometric identities that we discuss in sections A.1.5 and A.1.6. As a by-product, we present in section A.1.7 a sketch of the derivation and a closed formula for the coefficients f_n in the case

$$a_1 = 0, \quad b_1 \neq 0, \quad c_1 \neq 0. \quad (\text{A.1.5})$$

For the convenience of the reader, we summarize all the hypergeometric expansions we found in appendix A.3.

A.1.1 Determining Stokes lines I: Steepest descent contours

Whenever a function can be recast into a definite integral form, a steepest descent contour analysis can be used to develop its asymptotic expansions, classify the possible non-perturbative corrections and reveal possible Stokes lines. See e.g. [404] for a detailed review of the method applied to the Airy function. Hypergeometric functions admit such a representation:

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)} \mathcal{I}(a, b, c, z), \quad \mathcal{I}(a, b, c, z) = \int_0^1 dx x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a}, \quad (\text{A.1.6})$$

where the integral converges for $\text{Re}(b) > 0$, $\text{Re}(b) < \text{Re}(c)$ and $z \in \mathbb{C} \setminus [1, +\infty)$. The integral can be recast in exponential form as follows:

$$\begin{aligned} \mathcal{I}(\lambda, z) &= \int_0^1 dx x^{b_0-1} (1-x)^{c_0-b_0-1} (1-zx)^{-a_0} e^{-\lambda S(x, z)}, \\ S(x, z) &= -\log \left[x^{b_1} (1-x)^{c_1-b_1} (1-zx)^{-a_1} \right]. \end{aligned} \quad (\text{A.1.7})$$

The original integration contour $x \in [0, 1]$ can be freely deformed in the complex x -plane, as long as no integrand singularities are crossed. In this form, the $\lambda \rightarrow \infty$ expansion takes the form of a saddle-point expansion. The saddles of $S(x, z)$ are at

$$x_{\pm} = \frac{(a_1 - b_1)z - c_1 \pm \sqrt{[(a_1 - b_1)z - c_1]^2 + 4b_1(a_1 - c_1)z}}{2(a_1 - c_1)z}. \quad (\text{A.1.8})$$

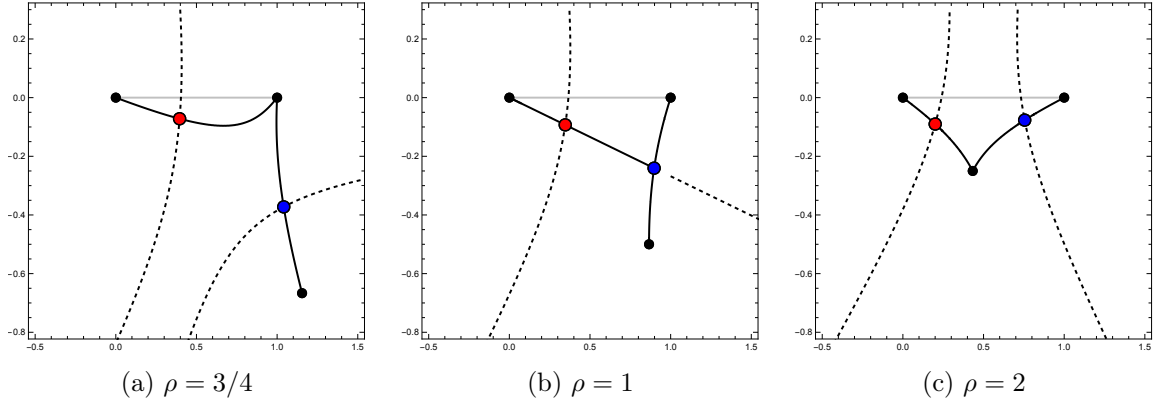


Figure A.1: Thimble decomposition for $\mathcal{I}(\lambda, z)$ in (A.1.7) in the complex x -plane with $a_1 = -1$, $b_1 = 2$, $c_1 = 2$. We take $\lambda = 100$ and $z = \rho e^{i\pi/6}$ with different values of ρ , corresponding to three points in the trajectory crossing the Stokes line depicted in orange in figure 3.1, with $\rho = 1$ being on the line itself. Black dots correspond to the singularities of $S(x, z)$ at $x = 0, 1, 1/z$ while the red and blue dots are the saddles x_- and x_+ respectively. Solid and dashed lines are the steepest descent contours and steepest ascent contours respectively. The grey solid line is the original contour of integration. In A.1a, only the saddle point x_- contributes. In A.1b, the thimble decomposition is degenerate. In A.1c, both saddles x_- and x_+ contribute.

From the saddle $x_{\pm}$ emanate steepest descent contours $\mathcal{C}_{\pm}$ where $\text{Re}(S)$ monotonically decreases moving away from the saddle. Not all saddles can contribute simultaneously to the asymptotic integral $\mathcal{I}(\lambda, z)$ at given values of z (for simplicity we assume $z \in \mathbb{C}$ to be the only parameter we vary, with a_1, b_1, c_1 kept fixed). Which saddle contributes is determined by how the original contour of integration can be decomposed into a sum of the steepest descent contours $\mathcal{C}_{\pm}$ (Lefschetz thimbles). An example of thimble decomposition as z varies is shown in figure A.1.

Since the imaginary part of the action remains constant on steepest descent contours, this decomposition becomes degenerate when $\mathcal{C}_{\pm}$ intersects, or equivalently:

$$\text{Im}\left(\lambda(S_- - S_+)\right) = 0, \quad (\text{A.1.9})$$

where $S_{\pm}(z) = S(x_{\pm}(z), z)$. This is the condition which determines where Stokes lines are in the z -plane. We can write this condition more explicitly noticing that

$$\frac{d}{dz}(S_- - S_+) = \sqrt{Q_0(z)}, \quad Q_0 = \frac{z^2(a_1 - b_1)^2 + 2z(2a_1b_1 - c_1(a_1 + b_1)) + c_1^2}{4z^2(1 - z)^2}. \quad (\text{A.1.10})$$

The function Q_0 vanishes at $z_{\pm}$, the two roots of its numerator:

$$z_{\pm} = \frac{c_1(a_1 + b_1) - 2a_1b_1 \pm 2\sqrt{a_1b_1(a_1 - c_1)(b_1 - c_1)}}{(a_1 - b_1)^2}. \quad (\text{A.1.11})$$

Noticing that $(S_- - S_+)(z_{\pm}) = 0$ we can integrate the above equation and obtain the following condition for $z \in \mathbb{C}$ to be a Stokes point for generic $\lambda \in \mathbb{C}$:

$$\text{Im}\left\{\lambda \int_{z_{\pm}}^z dw \sqrt{Q_0(w)}\right\} = 0, \quad (\text{Stokes lines}). \quad (\text{A.1.12})$$

For the specific cases of identity blocks in (3.2.11) and (3.2.12), the above condition gives the Stokes line depicted in figure 3.1 and figure 3.2 respectively. Anti-Stokes lines are instead defined as

$$\operatorname{Re}\left\{\lambda \int_{z_{\pm}}^z dw \sqrt{Q_0(w)}\right\} = 0, \quad (\text{anti-Stokes lines}). \quad (\text{A.1.13})$$

Stokes lines are lines at which non-perturbative effects “appear” and anti-Stokes lines are the lines at which non-perturbative effects become of the same order as perturbative effects.

A.1.2 Determining Stokes lines II: exact WKB

The large parameter expansion of the hypergeometric function and its relation to exact WKB² has been worked out in a series of papers, see e.g. [406].³

The hypergeometric differential equation (A.1.1) can be rewritten in a Schrödinger-like form by defining a new function g as

$${}_2F_1(a, b, c, z) = z^{-\frac{c}{2}}(1-z)^{-\frac{1}{2}(a+b-c+1)}g(z), \quad (\text{A.1.14})$$

such that the linear derivatives terms in (A.1.1) cancel. We then get

$$-g'' + Q(z)g = 0, \quad (\text{A.1.15})$$

where

$$Q(z) = \frac{z^2((a-b)^2 - 1) + 2z(2ab - c(a+b-1)) + c(c-2)}{4z^2(1-z)^2} \quad (\text{A.1.16})$$

is a “potential”. If we take the limit of large $a, b, c \sim \lambda \rightarrow \infty$, as in (A.1.2) then (A.1.15) turns into a Schrödinger equation containing classical and quantum (one and two-loop) potentials, where $1/\lambda$ is identified with $\hbar$. Then one finds

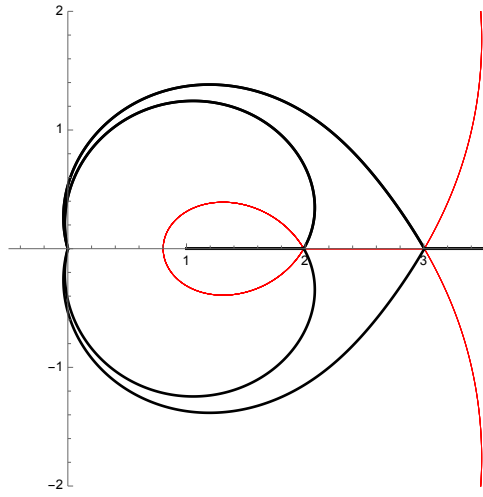
$$-\lambda^{-2}g'' + \left(Q_0(z) + \lambda^{-1}Q_1(z) + \lambda^{-2}Q_2(z)\right)g = 0, \quad (\text{A.1.17})$$

where the leading potential Q_0 is given in (A.1.10).

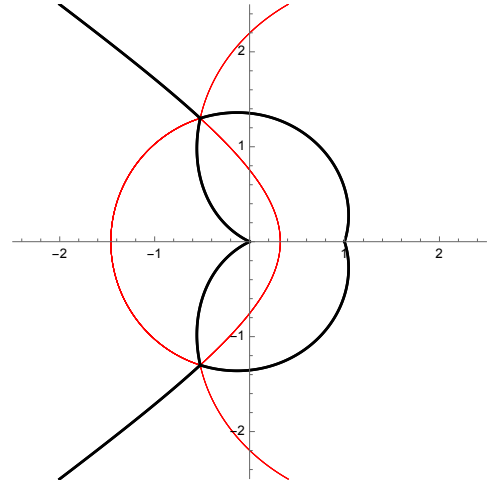
It is well-known that Stokes lines emanate from the zeros $z_{\pm}$ of Q_0 , which are turning points of the potential Q_0 . For each $z_{\pm}$, Stokes lines and anti-Stokes lines are given by (A.1.12) and (A.1.13). The Stokes lines of simple zeros are locally those of the Airy function, with three Stokes lines emanating from the point with a $2\pi/3$ angle between them. When the z_+ and z_- collide to form a double zero, four Stokes lines emanate from the point with a $\pi/2$ angle between them. For non-singular Q_0 , Stokes lines emanating from turning points can go to infinity or end at another turning point. A Stokes line ending at another turning point can be resolved by assigning a small phase to the parameter λ . If Q_0 has singularities, Stokes lines emanating from turning points can also end at the singular points. The potential (A.1.10) has two order-two poles at $z = 0$ and $z = 1$. The way Stokes lines approach a singular order-two pole depends on

²See e.g. [405] for a comprehensive math-oriented or [156] for a more basic and physics-oriented introductions to exact WKB.

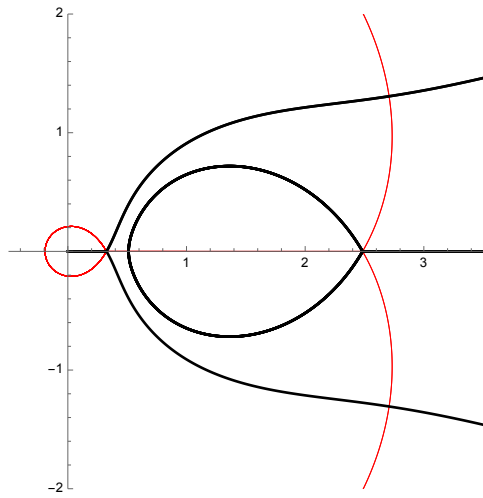
³We thank Kohei Iwaki for drawing our attention to these papers.



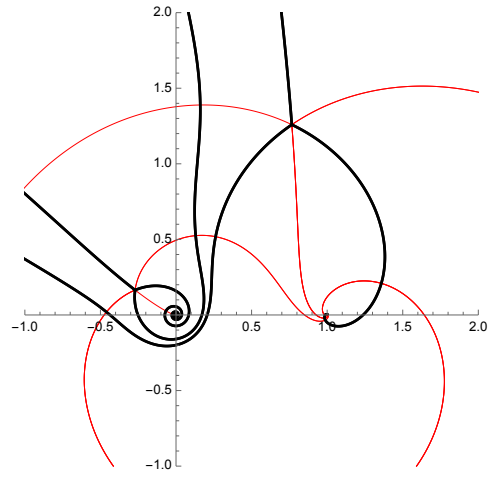
(a) $a_1 = 1$, $b_1 = 0.018$, $c_1 = 2.4$



(b) $a_1 = 1.2$, $b_1 = 2.2$, $c_1 = 1.4$



$a_1 = -0.95 - 0.5i$,
(c) $b_1 = 0.05 - 0.5i$,
 $c_1 = -0.9$



$a_1 = -1 + 0.9i$,
(d) $b_1 = 0.03 + 0.8i$,
 $c_1 = -0.1 + 0.7i$

Figure A.2: Four examples of Stokes (black) and anti-Stokes (red) lines for potentials of the form (A.1.10). While it is not visible in the figures, anti-Stokes lines in these examples are closed.

the value of the residue of $\sqrt{Q_0}$ [405]. The Stokes lines approach the singular point radially if the residue is real, form closed trajectories around the singular point if it is purely imaginary, and spiral around it for a generic complex residue. For illustration we report in figure A.2 the Stokes and anti-Stokes lines as given by (A.1.12), (A.1.13) for a sample of choices of parameters (a_1, b_1, c_1) .

A.1.3 Saddle and prefactor

The form of the two factors $S_{\pm}$ (from now on loosely dubbed “saddles”) can easily be derived from the results in sections A.1.1 or A.1.2. From now on we consider for simplicity only the case of *real* parameters and focus on the solution which is regular as $z \rightarrow 0$ and corresponds to ${}_2F_1(a, b, c; z)$, the other solution corresponding instead to the solution $z^{1-c} {}_2F_1(a - c + 1, b - c + 1, 2 - c; z)$. For $c_1 > 0$ the value of $S(z)$ for ${}_2F_1(a, b, c; z)$ is⁴

$$S(z) = \log \left[\frac{((a_1 - b_1)z + c_1 - 2a_1 + 2u(z))^{a_1+b_1-c_1} ((a_1 - b_1)z + c_1 + 2u(z))^{c_1}}{((a_1 - b_1)^2 z + 2a_1 b_1 - c_1(a_1 + b_1) + 2(a_1 - b_1)u(z))^{b_1}} \right] - S_0, \\ S_0 = \log \left[2^{a_1} c_1^{c_1} (b_1(a_1 - c_1))^{-b_1} (c_1 - a_1)^{a_1+b_1-c_1} \right], \quad (\text{A.1.18})$$

with S_0 fixed by the boundary condition $S(0) = 0$, and

$$u(z) = \sqrt{\frac{c_1^2}{4} + \left(a_1 b_1 - \frac{1}{2} c_1 (a_1 + b_1) \right) z + \frac{(a_1 - b_1)^2}{4} z^2}. \quad (\text{A.1.19})$$

At next-to-leading order in λ we can determine the factor $A(z)$, given by

$$A(z) = A_0 \frac{(1-z)^{c_0-a_0-b_0}}{\sqrt{u(z)}} \left((a_1 - b_1)^2 z + 2a_1 b_1 - c_1(a_1 + b_1) + 2(a_1 - b_1)u(z) \right)^{\frac{b_0-a_0}{2}} \\ \times \left((a_1^2 + b_1^2 - c_1(a_1 + b_1))z + 2a_1 b_1 + c_1^2 - c_1(a_1 + b_1) + 2(a_1 + b_1 - c_1)u(z) \right)^{\frac{a_0+b_0-c_0}{2}} \\ \times \left((2a_1 b_1 - c_1(a_1 + b_1))z + c_1^2 + 2c_1 u(z) \right)^{\frac{1-c_0}{2}}, \\ A_0 = 2^{c_0-b_0-1} c_1^{c_0-\frac{1}{2}} a_1^{\frac{c_0-a_0-b_0}{2}} b_1^{b_0-\frac{c_0}{2}} (a_1 - c_1)^{\frac{a_0-b_0}{2}}, \quad (\text{A.1.20})$$

with A_0 fixed by the boundary condition $A(0) = 1$.

When $a_1 = 0$ the previous formulas appear singular but the solutions are still well-defined and dramatically simplify:

$$S(z) = 0, \quad A(z) = \left(1 - \frac{b_1 z}{c_1} \right)^{-a_0}. \quad (\text{A.1.21})$$

When both $a_1 = 0$ and $b_1 = 0$, both the saddle and the prefactor become trivial:

$$S(z) = 0, \quad A(z) = 1. \quad (\text{A.1.22})$$

⁴Without loss of generality, we can assume $a_1 \leq b_1$ since the problem is symmetric in (a, b) exchange.

A.1.4 Expansion of ${}_2F_1(a, b, c + \lambda; z)$

In this section we provide a formula for the coefficients f_n of the asymptotic expansion (A.1.3) in the case where only the third parameter scales with λ , namely $a_1 = b_1 = 0$ in (A.1.2). Without loss of generality, we can set $c_1 = 1$. To lighten the notation we will drop the subscript “0” from the coefficients in (A.1.3).

An implicit form of this expansion is known, see e.g. section 15.12(iii) of [397], but the coefficients (A.1.3) are not given in a closed form, but rather in terms of a generating function.

An explicit form of the coefficients can be obtained instead by manipulating the Gauss series and expanding the Pochhammer symbols. To this end, we can use the identity

$$\frac{1}{(\lambda)_k} = \sum_{n=k}^{+\infty} (-1)^{n+k} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \lambda^{-n}, \quad |\lambda| > k-1, \quad k \geq 1, \quad (\text{A.1.23})$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the Stirling numbers of the second kind, or rather the following generalization

$$\frac{1}{(c+\lambda)_k} = \sum_{n=k}^{+\infty} (-1)^{n+k} S_2 \left(\begin{matrix} n-1 \\ k-1 \end{matrix}; c \right) \lambda^{-n}, \quad |\lambda| > \max_{0 \leq p \leq k-1} |c+p|, \quad k \geq 1, \quad (\text{A.1.24})$$

where we have introduced the Stirling polynomials of the second kind

$$S_2 \left(\begin{matrix} n \\ k \end{matrix}; x \right) = \sum_{m=k}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{m} x^{n-m}. \quad (\text{A.1.25})$$

Plugging (A.1.24) into the Gauss series gives

$${}_2F_1(a, b, c + \lambda; z) \sim 1 + \sum_{k=1}^{+\infty} \sum_{n=k}^{+\infty} \frac{(a)_k (b)_k}{k!} z^k (-1)^{n+k} S_2 \left(\begin{matrix} n-1 \\ k-1 \end{matrix}; c \right) \lambda^{-n}. \quad (\text{A.1.26})$$

Exchanging the order of the two sums, we get, as $\lambda \rightarrow +\infty$:

$${}_2F_1(a, b, c + \lambda; z) \sim 1 + \sum_{n=1}^{+\infty} f_n(a, b, c; z) \lambda^{-n}, \quad z \in \mathbb{C} \setminus [1, +\infty), \quad (\text{A.1.27})$$

where

$$f_n(a, b, c; z) = \sum_{k=1}^n (-1)^{n+k} S_2 \left(\begin{matrix} n-1 \\ k-1 \end{matrix}; c \right) \frac{(a)_k (b)_k}{k!} z^k. \quad (\text{A.1.28})$$

The sums in (A.1.26) are both convergent, but the exchange of the two is merely formal and makes the sum over n in (A.1.27) asymptotic. In fact, applying the identity (A.1.24) for all $k \in \mathbb{N}$ would imply a zero radius of convergence in $1/\lambda$. The matching of the expansions (A.1.27) and the known one, e.g. equation 15.12.3 of [397], can be numerically verified to any order. Note that this asymptotic series actually holds in a much larger domain in z than the Gauss series we started from. This explicit form of the expansion in λ is particularly useful since each term of the asymptotic series is a degree n polynomial in z with coefficients known in closed form.

This will make it feasible to carry out the resurgence analysis of such asymptotic series in appendix A.2.

In the case where $c = 0$, the Stirling polynomials reduce to ordinary Stirling numbers and thus we have the following simpler asymptotic expansion as $\lambda \rightarrow +\infty$:

$${}_2F_1(a, b, \lambda; z) \sim 1 + \sum_{n=1}^{+\infty} f_n(a, b; z) \lambda^{-n}, \quad f_n(a, b; z) = \sum_{k=1}^n (-1)^{n+k} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \frac{(a)_k (b)_k}{k!} z^k. \quad (\text{A.1.29})$$

The formula holds for $z \in \mathbb{C} \setminus [1, +\infty)$ as can be deduced by analyzing the web of Stokes and anti-Stokes lines that can be obtained as in A.1.2. This asymptotic expansion also holds for $\lambda \rightarrow -\infty$, in the domain $\{z \in \mathbb{C} \setminus (-\infty, 0] : \text{Re}(z) < 1/2\}$, as can be understood by studying the Stokes and anti-Stokes lines of the associated differential equation studied in A.1.2. See appendix A.2.2 for further discussion.

A.1.5 Reduction of ${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z)$

The asymptotic expansion as $\lambda \rightarrow +\infty$ of ${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z)$, corresponding to the ${}_2F_1$ entering the identity block in the 4H correlator (3.2.9), can be reduced to the case discussed in section A.1.4 with a non-trivial change of variable.

The saddle $S(z)$ and the prefactor $A(z)$ are obtained from (A.1.18) and (A.1.20) respectively, with $a_0 = 0$, $a_1 = -1$, $b_0 = 1$, $b_1 = 1$, $c_0 = 2$ and $c_1 = 2$.

The idea is to solve the following auxiliary problem. ${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z)$ is a solution of the hypergeometric differential equation (A.1.1) with $a = -\lambda$, $b = 1 + \lambda$, $c = 2 + 2\lambda$. We factor out the leading asymptotic and define

$$g(w(z)) = \frac{f(z)}{e^{-\lambda S(z)} A(z)}, \quad (\text{A.1.30})$$

where w is a map to be determined. The function g satisfies a second-order differential equation which can be put in the the form

$$w(1-w)\partial_w^2 g + H_1(z)\partial_w g - H_0(z)g = 0, \quad (\text{A.1.31})$$

where H_1, H_0 are functions of z and the map $w(z)$, given by

$$H_1(z) = \frac{w(z)(1-w(z))}{z(1-z)w'(z)^2} \left[2\lambda u(z)w'(z) + \left(\frac{1}{2} - z + \frac{1-2z}{2u(z)^2} + u(z) \right) w'(z) + z(1-z)w''(z) \right],$$

$$H_0(z) = \frac{15}{16} \frac{w(z)(1-w(z))}{u(z)^4 w'(z)^2}, \quad (\text{A.1.32})$$

with

$$u(z) = \sqrt{1-z+z^2}. \quad (\text{A.1.33})$$

The goal is to find a map $w(z)$ such that (A.1.31) is again a hypergeometric differential equation where now only the “third” parameter scales with λ (namely $a_1 = b_1 = 0$ and $c_1 \neq 0$ in (A.1.2)).

More precisely we have to find $w(z)$ and constants (A, B, C_0, C_1) such that

$$H_1(z) = C_0 + C_1\lambda - (A + B + 1)w(z), \quad (\text{A.1.34})$$

$$H_0(z) = AB. \quad (\text{A.1.35})$$

It is highly non-trivial that this map exists. If a solution exists, we will have derived the identity

$${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z) = e^{-\lambda S(z)} A(z) {}_2F_1(A, B, C_0 + C_1\lambda; w(z)), \quad z \in D, \quad (\text{A.1.36})$$

at least in some domain $D \subseteq \mathbb{C}$ where the asymptotic ansatz holds. We have also assumed that the map w behaves as $w(z) \sim \alpha z + O(z^2)$ for small z , so that we can correctly match the behaviour as $z \rightarrow 0$ of the ${}_2F_1$ on both sides. Otherwise one needs to pick another appropriate solution of the hypergeometric differential equation.

The problem is in principle overconstrained and the existence of a solution is related to the particular way in which the parameters in the ${}_2F_1$ scale with λ . Remarkably, for the case at hand a map $w(z)$ does exist and its form can be found in the following way. Isolating from (A.1.34) the term proportional to λ gives the following first order differential equation for $w(z)$:

$$w'(z) = \frac{2u(z)}{C_1 z(1-z)} w(z)(1-w(z)). \quad (\text{A.1.37})$$

We don't solve this equation directly but rather use this relation in H_0 to remove the $w'(z)$ term. Then the constraint (A.1.35) reduces to a simple second-degree polynomial equation in w

$$w(z)^2 - w(z) + \frac{15}{64} \frac{C_1^2}{AB} \frac{z^2(1-z)^2}{u(z)^6} = 0 \quad \Rightarrow \quad w_{\pm}(z) = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{15}{16} \frac{C_1^2}{AB} \frac{z^2(1-z)^2}{u(z)^6}} \right). \quad (\text{A.1.38})$$

Plugging these solutions into the differential equation (A.1.37) and demanding that the constraint is solved for all z implies⁵

$$w_{\pm} \quad : \quad AB = \frac{5}{36}, \quad C_1 = \mp 1. \quad (\text{A.1.39})$$

Finally, plugging the relations (A.1.37) and (A.1.38) in (A.1.34) results in a simple equation:

$$(A + B - 1)w(z) + 1 - C_0 + \frac{1}{2}C_1 = 0 \quad \Rightarrow \quad A + B = 1, \quad C_0 = 1 + \frac{1}{2}C_1. \quad (\text{A.1.40})$$

We have therefore found two solutions to the constraints (A.1.34) and (A.1.35). Demanding that $w(z) \sim \alpha z + O(z^2)$ for small z selects the solution⁶

$$r(z) = w_-(z) = \frac{1}{2} + \frac{(z+1)(2z-1)(2-z)}{4u(z)^3}, \quad (A, B) = \left(\frac{1}{6}, \frac{5}{6} \right), \quad C_0 = \frac{3}{2}, \quad C_1 = 1. \quad (\text{A.1.41})$$

⁵Note that while the solutions $w_{\pm}$ in (A.1.38) depend on C_1^2 , the constraint (A.1.37) depends on C_1 . For this reason, picking either w_+ or w_- requires a different sign of C_1 to solve the constraint.

⁶The other solution is given by $w_+(z) = 1 - r(z)$, with $C_1 = -1$, $C_0 = 1/2$, and the same values of A and B . In both cases the solutions are symmetric under (A, B) exchange.

We have thus established the identity

$${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda) = e^{-\lambda S(z)} A(z) g(r(z)), \quad g(r) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, \frac{3}{2} + \lambda; r\right), \quad z \in D_0. \quad (\text{A.1.42})$$

Note that the saddle and the prefactor can be written more conveniently in terms of $r(z)$ as

$$S(z) = \frac{1}{2} \log\left(\frac{27}{16} \frac{z^2}{(1-z)^2} \frac{1-r(z)}{r(z)}\right), \quad A(z) = \frac{e^{-\frac{1}{2}S(z)}}{\sqrt{u(z)}}. \quad (\text{A.1.43})$$

Comments on the analytic structure Let us discuss in more detail the analytic structure stemming from (A.1.42), and specify in particular the domain D_0 in which it is valid. We restrict our analysis to the principal sheet in the z -plane. We have possible branching points where $u(z)$ vanishes, at $z = z_{\pm} = \exp(i\pi/3)$, branching points of the logarithm in S , and branching points of $g(r)$ when $r > 1$. We report in figure A.3 the branching points and branch cuts which result in the complex z -plane. From $z = 1$, where $r = 1$, the two red branch cuts represent the points where r is real and greater than one, reaching infinity at $z_{\pm}$. The two blue branch cuts are instead square root branch cuts of the function $u(z)$, which induce a transformation $r \rightarrow 1 - r$. Despite the two branch cuts looking quite different, they act equally on the whole function, implying that $z_{\pm}$ are actually analytic regular points, as expected from the left-hand side of (A.1.42). Finally, we have the orange branch cut from $z = 1$ up to infinity, which is invisible in the principal sheet of $g(r)$, but is a branch cut of its continuation to the nearby Riemann sheets. The z -plane is then split into three regions, denoted D_p in the figure, $p = 0, \pm$. One can verify that over the whole principal sheet in the z -plane, we have

$${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z) = e^{-\lambda S(z)} A(z) g^{(p)}(r(z)), \quad z \in D_p, \quad (\text{A.1.44})$$

where $g^{(0)}(r) = g(r)$ represents the ${}_2F_1$ on the right-hand side of (A.1.42) on the principal Riemann sheet in the r -plane, while

$$g^{(\pm)}(r) = g(r) \pm \text{Disc}[g(r)], \quad \text{Disc}[g(r)] = i \left(\frac{r}{r-1}\right)^{-\frac{1}{2}-\lambda} g(1-r), \quad (\text{A.1.45})$$

is the function g in the two Riemann sheets adjacent to the principal one. One can see from (A.1.45) the appearance of the orange branch cut for $g^{(\pm)}$, which eventually maps to the ordinary branch cut of the ${}_2F_1$ in z in the left-hand side of (A.1.44). Region D_0 and (A.1.45) reappear in the resurgence analysis of the asymptotic series, as seen in the main text around (3.3.19).

A.1.6 Reduction of ${}_2F_1(2\gamma, 1 + \lambda, 2 + 2\lambda; z)$

The asymptotic expansion as $\lambda \rightarrow +\infty$ of ${}_2F_1(2\gamma, 1 + \lambda, 2 + 2\lambda; z)$, the ${}_2F_1$ entering the identity block in the 2H2L correlator (3.2.10), can also be reduced to the case discussed in section A.1.4 with a non-trivial change of variable. The procedure is identical to what we described in section A.1.5 so we will be more brief.

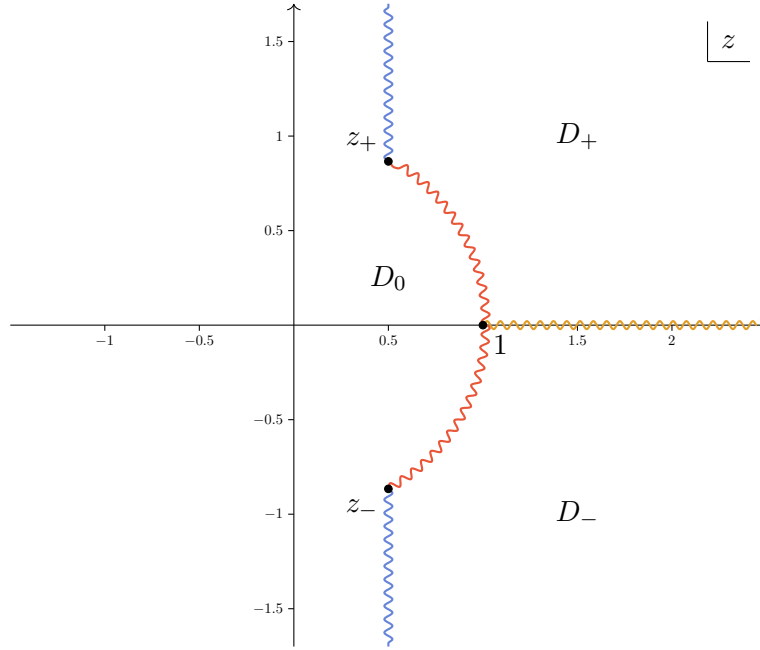


Figure A.3: Branch cuts of the hypergeometric function when written using the variable $r(z)$. The points $z_{\pm}$ turn out to be analytic points and we are only left with the discontinuity at $z = 1$ (orange branch cut).

The saddle and prefactor can be read from (A.1.21) with $a_0 = 2\gamma$, $b_1 = 1$ and $c_1 = 2$. The functions H_0, H_1 can be computed straightforwardly and will not be reported. Isolating from the H_1 equation the term proportional to λ gives the following first order differential equation for $w(z)$:

$$w'(z) = \frac{(2-z)}{C_1 z(1-z)} w(z)(1-w(z)). \quad (\text{A.1.46})$$

Using this relation into the constraint (A.1.35), we reduce to an algebraic equation for w :

$$w(z)^2 - w(z) + 2\gamma(2\gamma+1) \frac{C_1^2}{AB} \frac{z^2(1-z)}{(2-z)^4} = 0 \quad \Rightarrow \quad w_{\pm}(z) = \frac{1}{2} \left(1 \pm \sqrt{1 - 8\gamma(2\gamma+1) \frac{C_1^2}{AB} \frac{z^2(1-z)}{(2-z)^4}} \right). \quad (\text{A.1.47})$$

Plugging these solutions into the differential equation (A.1.46) and asking for the constraint to be solved for all z and generic γ implies⁵

$$w_{\pm} \quad : \quad AB = \frac{\gamma(2\gamma+1)}{2}, \quad C_1 = \mp 1. \quad (\text{A.1.48})$$

Finally, we can use the relation (A.1.46) and (A.1.47) in (A.1.34) to fix A , B , C_0 and C_1 . Demanding that $w(z) \sim \alpha z + O(z^2)$ for small z selects the solution⁷

$$v(z) = w_-(z) = \frac{z^2}{(2-z)^2}, \quad (A, B) = \left(\gamma, \gamma + \frac{1}{2} \right), \quad C_0 = \frac{3}{2}, \quad C_1 = 1. \quad (\text{A.1.49})$$

⁷The other solution is given by $w_+(z) = 1 - v(z)$, with $C_1 = -1$, $C_0 = 2\gamma$, and the same values of A and B .

We have thus established the identity

$${}_2F_1(2\gamma, 1 + \lambda, 2 + 2\lambda) = A(z)g(v(z)), \quad g(v) = {}_2F_1\left(\gamma, \frac{1}{2} + \gamma, \frac{3}{2} + \lambda; v\right), \quad z \in D, \quad (\text{A.1.50})$$

where

$$A(z) = \left(1 - \frac{z}{2}\right)^{-2\gamma}. \quad (\text{A.1.51})$$

In this case, the analytic structure is much simpler than the one discussed in section A.1.5. The region where (A.1.50) holds is the entire cut complex plane $D = \mathbb{C} \setminus [1, +\infty)$.

Note that the relation (A.1.50) can also be obtained using identities between hypergeometric functions, see e.g. equation 15.8.13 of [397].

A.1.7 Expansion of ${}_2F_1(a, b_0 + b_1\lambda, c + \lambda; z)$

In this section, we provide a formula for the coefficients f_n of the asymptotic expansion (A.1.3) in the case where $a_1 = 0$ in (A.1.2) while $b_1, c_1 \neq 0$. Without loss of generality, we can set $c_1 = 1$. To lighten the notation we will drop the subscript “0” from the coefficients in a_0 and c_0 . This generalization is not used in the main text.

In the case where a, b_0, b_1, c are generic, the strategy devised in section A.1.5 does not work, namely, the map $w(z)$ does not exist unless the parameters satisfy particular relations.

Nevertheless, it’s possible to plug the asymptotic ansatz (the saddle and prefactor are given in (A.1.21))

$$f(z) = {}_2F_1(a, b_0 + b_1\lambda, c + \lambda; z) \sim (1 - b_1z)^{-a} \left(1 + \sum_{n=1}^{+\infty} f_n(z)\lambda^{-n}\right), \quad (\text{A.1.52})$$

into the differential equation (A.1.1) to obtain a differential recursion formula for the coefficients f_n . The recursion gives $f'_n(z)$ in terms of $f_{n-1}(z)$, $f'_{n-1}(z)$ and $f''_{n-1}(z)$. Starting from the seed $f_0(z) = 1$ and imposing the boundary condition $f_n(0) = 0$ it’s straightforward to obtain the first few coefficients of the asymptotic series. It’s easy to realize (and prove inductively) that the coefficient f_n is always a polynomial of degree $2n$ in the variable

$$w(z) = \frac{z}{1 - b_1z}. \quad (\text{A.1.53})$$

Thus, using the ansatz $f_n(z) = \sum_{l=0}^{2n} d_{n,l}w(z)^l$ into the recursion formula for f_n we can obtain an algebraic recursion formula for the coefficients $d_{n,k}$.

The explicit form of the coefficients for all n, k is difficult to obtain from the recursion, but we can leverage the knowledge we obtained so far to compute the coefficients $d_{n,k}$ by directly expanding Pochhammers in the Gauss series. Given the previous discussion, it’s clear that a better starting point is obtained after absorbing the prefactor $(1 - b_1z)^{-a}$ and changing variable $z \rightarrow z(w) = w/(1 + b_1w)$:

$$g(w) = (1 + b_1w)^{-a} f\left(\frac{w}{1 + b_1w}\right) = \sum_{l=0}^{+\infty} \sum_{k=0}^l \binom{l}{k} \frac{(b_0 + b_1\lambda)_k}{(c + \lambda)_m} (-b_1)^{l-k} \frac{(a)_l}{l!} w^l. \quad (\text{A.1.54})$$

The Pochhammer in the denominator can be expanded in terms of Stirling polynomials of the second kind as in (A.1.24), while the one in the numerator in terms of Stirling polynomials of the first kind

$$(b_0 + b_1\lambda)_n = \sum_{k=0}^n S_1\left(\begin{matrix} n \\ k \end{matrix}; b_0\right) (b_1\lambda)^k, \quad (n \geq 0), \quad S_1\left(\begin{matrix} n \\ k \end{matrix}; x\right) = \sum_{m=k}^n \left[\begin{matrix} n \\ m \end{matrix}\right] \binom{m}{k} x^{m-k}, \quad (\text{A.1.55})$$

and $\left[\begin{matrix} n \\ k \end{matrix}\right]$ are *unsigned* Stirling numbers of the first kind. This produces an expression for g with four nested summations. After performing several sum exchanges and index re-labeling, we can collect the inverse powers of λ and obtain the asymptotic expansion of $g(w)$ as $\lambda \rightarrow +\infty$:

$$g(w) \sim 1 + \sum_{n=1}^{+\infty} g_n(w) \lambda^{-n}. \quad (\text{A.1.56})$$

The coefficients g_n are then explicitly written as polynomials in w of degree $2n$. We omit the details of the computation. We find:

$$g_n(w) = \sum_{l=1}^{2n} d_{n,l}(a, b_0, b_1, c) w^l, \quad (\text{A.1.57})$$

$$d_{n,l}(a, b_0, b_1, c) = \frac{(a)_l}{l!} \sum_{r=1}^l \sum_{k=0}^{\min\{n,r\}} (-1)^{n+l+r-k} \binom{l}{r} S_2\left(\begin{matrix} r+n-k-1 \\ r-1 \end{matrix}; c\right) S_1\left(\begin{matrix} r \\ r-k \end{matrix}; b_0\right) b_1^{l-k}.$$

Combining (A.1.57) with (A.1.52), where $f_n(z) = g_n(z/(1-b_1z))$, provides the full asymptotic expansion of ${}_2F_1(a, b_0 + b_1\lambda, c + \lambda; z)$ as $\lambda \rightarrow +\infty$.

A.2 Resurgence of ${}_2F_1$

In this section we study the resurgence properties of the asymptotic series (A.1.29), using the framework and notation reviewed in Section 2.1.

The asymptotic series (A.1.29) is the solution of a second-order differential equation and it has simple resurgent properties involving just two series, the original one and another obtained using (2.1.6). The presence of just two asymptotic series is related to the second order of the hypergeometric differential equation (A.1.1) which admits two independent solutions. The key point of the analysis that follows is to see how the resurgence properties of

$$\tilde{F}_1(\epsilon; w) \sim {}_2F_1\left(a, b, \frac{1}{\epsilon}; w\right), \quad (\text{A.2.1})$$

vary as we vary w . We find it more convenient for the resurgence analysis to work in terms of

$$\epsilon = \frac{1}{\lambda}, \quad (\text{A.2.2})$$

so that the asymptotic limit is $\epsilon \rightarrow 0^\pm$ instead of $\lambda \rightarrow \pm\infty$. We take ϵ real and discuss the cases $\epsilon > 0$ and $\epsilon < 0$ separately.

A.2.1 Stokes jumps for large positive parameter

We determine the Borel function $\widehat{F}_1$ associated to the $\widetilde{F}_1$ series. Using (2.1.3) and (A.1.29) we have

$$\begin{aligned}\widehat{F}_1(w; t) &= \sum_{n=1}^{+\infty} \sum_{k=1}^n (-1)^{n+k} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \frac{(a)_k (b)_k}{k!} w^k \frac{t^{n-1}}{(n-1)!} = \sum_{k=1}^{+\infty} \frac{(a)_k (b)_k}{k!} (-w)^k \sum_{n=k}^{+\infty} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \frac{-(-t)^{n-1}}{(n-1)!} \\ &= abw \sum_{k=0}^{+\infty} \frac{(a+1)_k (b+1)_k}{(k+1)!} \frac{(w(1-e^{-t}))^k}{k!} = abw {}_2F_1(a+1, b+1, 2; u),\end{aligned}\tag{A.2.3}$$

where

$$u = w(1 - e^{-t}),\tag{A.2.4}$$

and from the first to the second row of (A.2.3) we have used the relation

$$\sum_{n=k}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}.\tag{A.2.5}$$

We can determine the singularities of $\widehat{F}_1$ in the Borel t -plane analytically as a function of $w \in \mathbb{C}$. In a generic Riemann sheet ${}_2F_1(A, B, C; u)$ is singular for $u = 0, 1, \infty$. We get a rich spectrum of singularities:

$$\begin{aligned}u = 0 &\implies w = 0, \forall t, \text{ or } t = 2i\pi k, k \in \mathbb{Z}_{\neq 0}, \forall w, \\ u = 1 &\implies t = t_k(w) = \log\left(\frac{w}{w-1}\right) + 2i\pi k, \quad k \in \mathbb{Z}, \\ u = \infty &\implies w = \infty, \forall t, \text{ or } t = -\infty, \forall w.\end{aligned}\tag{A.2.6}$$

The most interesting ones are those which depend on w , in the second-row of (A.2.6), because as we vary $w \in \mathbb{C}$ they can cross the positive real axis and lead to the Stokes discontinuity (2.1.6). When $w \notin [1, +\infty)$, the positive real line in the Borel t -plane is free of singularities. In this case $\widetilde{F}_1$ is Borel summable and $F_1^B = s_0(\widetilde{F}_1)$ coincides with the original function:⁸

$$s_0(\widetilde{F}_1)(w; \epsilon) = 1 + abw \int_0^{+\infty} dt e^{-\frac{t}{\epsilon}} {}_2F_1(a+1, b+1, 2; u)\tag{A.2.7}$$

$$\begin{aligned}&\stackrel{s=1-e^{-t}}{=} 1 + abw \int_0^1 ds (1-s)^{\frac{1}{\epsilon}-1} {}_2F_1(a+1, b+1, 2; ws) \\ &= {}_2F_1\left(a, b, \frac{1}{\epsilon}; w\right) = F_1(w; \epsilon).\end{aligned}\tag{A.2.8}$$

When $w \in (1, +\infty)$, the singularity $t_0(w)$ in the second row of (A.2.6) lie on the positive real axis of the Borel t -plane and gives rise to a Stokes discontinuity. This is easily computed. We have

$$\begin{aligned}(s_{0+} - s_{0-})[\widetilde{F}_1] &= \lim_{\delta \rightarrow 0^+} \left(\int_0^{e^{i\delta}\infty} - \int_0^{e^{-i\delta}\infty} \right) dt e^{-\frac{t}{\epsilon}} \widehat{F}_1(w; t) = \int_{\mathcal{H}} dt e^{-\frac{t}{\epsilon}} \widehat{F}_1(w; t) \\ &= abw \int_{t_0(w)}^{+\infty} dt e^{-\frac{t}{\epsilon}} \text{Disc}[_2F_1(a+1, b+1, 2; u)],\end{aligned}\tag{A.2.9}$$

⁸This is expected, because F_1 is analytic in the appropriate disc in the $1/\epsilon$ plane.

where $\mathcal{H}$ is a Hankel contour starting around $t_0(w)$ and extending over the positive real axis. The contour integral picks up the discontinuity of the hypergeometric ${}_2F_1(A, B, C; u)$ across the branch cut starting at $u = 1$. For reference the explicit formula is

$$\begin{aligned} \text{Disc}[_2F_1(A, B, C; u)] &= 2\pi i \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)\Gamma(C-A-B+1)} \\ &\times (u-1)^{C-A-B} {}_2F_1(C-A, C-B, C-A-B+1; 1-u), \quad u \in (1, +\infty). \end{aligned} \quad (\text{A.2.10})$$

Applying this formula to (A.2.9) and shifting $t \rightarrow t_0(w) + t$ gives the integral

$$\begin{aligned} (s_{0+} - s_{0-})(\tilde{F}_1) &= \frac{2\pi i}{\Gamma(a)\Gamma(b)\Gamma(1-a-b)} \left(\frac{w}{w-1} \right)^{-\frac{1}{\epsilon}} w(w-1)^{-a-b} \\ &\times \int_0^{+\infty} dt e^{-\frac{t}{\epsilon}} (1-e^{-t})^{-a-b} {}_2F_1(1-a, 1-b, 1-a-b, (1-w)(1-e^{-t})), \end{aligned} \quad (\text{A.2.11})$$

leading to

$$(s_{0+} - s_{0-})(\tilde{F}_1) = \frac{2\pi i \Gamma(\frac{1}{\epsilon}) w^{-\frac{1}{\epsilon}+1} (w-1)^{\frac{1}{\epsilon}-a-b}}{\Gamma(a)\Gamma(b)\Gamma(1-a-b+\frac{1}{\epsilon})} {}_2F_1\left(1-a, 1-b, 1-a-b+\frac{1}{\epsilon}, 1-w\right). \quad (\text{A.2.12})$$

Note that while (A.2.12) applies also when $a+b=1$, its derivation needs to be adjusted. In such cases, (A.2.10) has to be adapted for $A+B=C+1$ and the singularity of $\hat{F}_1$ changes nature from a rational branch cut to a pole and a logarithmic cut. We rewrite (A.2.12) as

$$(s_{0+} - s_{0-})(\tilde{F}_1) = i\eta F_2(\epsilon; w), \quad w \in (1, +\infty), \quad (\text{A.2.13})$$

where

$$F_2(w; \epsilon) = \frac{1}{\eta} e^{-\frac{1}{\epsilon} S(w)} A(w) \mathcal{C}(\epsilon) {}_2F_1\left(1-a, 1-b, 1-a-b+\frac{1}{\epsilon}, 1-w\right), \quad (\text{A.2.14})$$

$$S(w) = \log\left(\frac{w}{w-1}\right), \quad A(w) = \frac{w}{(w-1)^{a+b}}, \quad \mathcal{C}(\epsilon) = \frac{2\pi\Gamma(\frac{1}{\epsilon})}{\Gamma(a)\Gamma(b)\Gamma(1-a-b+\frac{1}{\epsilon})}, \quad (\text{A.2.15})$$

and we have multiplied and divided by the factor

$$\eta = 2\sqrt{\sin(\pi a)\sin(\pi b)}, \quad (\text{A.2.16})$$

to simplify some of the formulas which will follow. Note that F_2 in the right-hand side of (A.2.13) is proportional to

$$(1-w)^{\frac{1}{\epsilon}-a-b} {}_2F_1\left(\frac{1}{\epsilon}-a, \frac{1}{\epsilon}-b, \frac{1}{\epsilon}+1-a-b, 1-w\right), \quad (\text{A.2.17})$$

which is one of the two independent solutions of (A.1.1) when expanded around $w = 1$. Using Kummer's connection formulas, see e.g. equation 15.10.21 of [397], the Stokes discontinuity (A.2.13) can be rewritten as a linear combination of the two independent solutions

around $w = 0$, ${}_2F_1(a, b, 1/\epsilon; w)$ and $w^{1-1/\epsilon} {}_2F_1(1+a-1/\epsilon, 1+b-1/\epsilon, 2-1/\epsilon; w)$. From the Stokes jump of the asymptotic series of one solution, we have then “discovered” the other one. The formulation in terms of the hypergeometric (A.2.14) is useful, as we have a closed-form expression for the asymptotic series of ${}_2F_1$ when one parameter is large.

We now compute the Borel transform $\widehat{F}_2(t)$, determine its Stokes discontinuities, and show that they give back F_1 . The asymptotic series $\widetilde{F}_2$ in ϵ is the product of the ones coming from $\mathcal{C}(\epsilon)$ and ${}_2F_1$.⁹ Its explicit expression is not needed, because we can immediately deduce $\widehat{F}_2$ from the integrand appearing in (A.2.11). We have

$$\widehat{F}_2(w; t) = e^{-\frac{1}{\epsilon}S(w)} \frac{A(w)}{\eta} \frac{2\pi(1-e^{-t})^{-a-b}}{\Gamma(a)\Gamma(b)\Gamma(1-a-b)} {}_2F_1(1-a, 1-b; 1-a-b; (1-w)(1-e^{-t})). \quad (\text{A.2.18})$$

The singularities of $\widehat{F}_2$ in the Borel t -plane are identical to the ones of $\widehat{F}_1$ in (A.2.6), provided $w \rightarrow 1-w$. As for $\widehat{F}_1$, the most interesting ones are those which depend on w . When $w \in \mathbb{C} \setminus (-\infty, 0]$, the positive real line in the Borel t -plane is free of singularities. In this case, $\widetilde{F}_2$ is Borel summable and we recover the original function F_2 . When $w \in (-\infty, 0)$, instead, the singularities $t_0(1-w) = \log(\frac{w-1}{w})$ lie on the positive real axis of the Borel t -plane. The computation of the Stokes discontinuity is very similar to the one in (A.2.11) and will not be reported. We have

$$(s_{0+} - s_{0-})(\widetilde{F}_2) = i\eta A(w)A'(w)F_1, \quad w \in (-\infty, 0), \quad (\text{A.2.19})$$

where

$$A'(w) = -\frac{(1-w)^{a+b}}{w}. \quad (\text{A.2.20})$$

The factor of A in (A.2.15) has a branch cut on the line $(-\infty, 1)$ and thus the product AA' is ambiguous for $w \in (-\infty, 0)$. The ambiguity amounts to the choice of the Riemann sheet of $w \in (-\infty, 0)$. We take $w-1 = \exp(i\pi)(1-w)$ in the principal sheet of A , so that

$$A(w)A'(w) = -e^{-i\pi(a+b)}. \quad (\text{A.2.21})$$

Summarizing, we have the following relations:

$$\left\{ \begin{array}{ll} s_0(\widetilde{F}_1) = F_1, & w \in \mathbb{C} \setminus [1, +\infty), \\ s_0(\widetilde{F}_2) = F_2, & w \in \mathbb{C} \setminus (-\infty, 0], \\ (s_{0+} - s_{0-})(\widetilde{F}_1) = i\eta s_0(\widetilde{F}_2), & w \in (1, +\infty), \\ (s_{0+} - s_{0-})(\widetilde{F}_2) = -ie^{-i\pi(a+b)}\eta s_0(\widetilde{F}_1), & w \in (-\infty, 0). \end{array} \right. \quad (\text{A.2.22})$$

The structure of Stokes lines in the w -plane is displayed in figure A.4. The lines emanate from the “singular” points $w = 0, 1$ and go to infinity along the real axis. Out-going arrows from the singular points denote where $\widetilde{F}_1$ jumps while in-going arrows denote where $\widetilde{F}_2$ jumps. In the main text, these “fundamental” Stokes lines are mapped into the Stokes lines in the z -plane of figures 3.3 and 3.6 through the maps $w = r(z)$ and $w = v(z)$, respectively.

⁹See (A.1.28) for the asymptotic series of the ${}_2F_1$ part, and equation 5.11.13 of [397] for $\mathcal{C}(\epsilon)$.

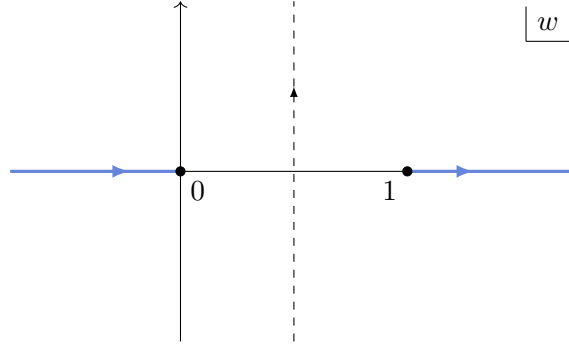


Figure A.4: Stokes lines (blue) and anti-Stokes lines (dashed) in the w -plane for the $(\tilde{F}_1, \tilde{F}_2)$ system. $w = 0, 1$ are singular points. Outgoing arrows denote lines where $\tilde{F}_1$ has a Stokes jump, while ongoing denote lines where $\tilde{F}_2$ jumps.

A.2.2 Stokes jumps for large negative parameter

For $\epsilon < 0$ the relevant Stokes jumps for $\tilde{F}_1$ or $\tilde{F}_2$ happen when $t_k(w)$ or $t_k(1-w)$ (defined in (A.2.6)), respectively, cross the *negative* real axis. It is simple to see that this happens precisely when one of the other is real and positive. Thus, the Stokes lines of $\tilde{F}_{1,2}$ swap, $\tilde{F}_1$ is not Borel summable for $w \in (-\infty, 0)$ and $\tilde{F}_2$ is not Borel summable for $w \in (1, \infty)$.

Since the asymptotic series associated with the singularity does not change with either the sign of ϵ or w , the Stokes jump itself is the same. However, due to the analytic continuation in the branch cuts of the pre-factors, there is a small phase difference. Summarily, we can write the Stokes jumps for $\epsilon < 0$ as

$$\begin{aligned} (s_{\pi+} - s_{\pi-})(\tilde{F}_1) &= ie^{2\pi i(a+b)} \eta s_{\pi}(\tilde{F}_2), & w \in (-\infty, 0), \\ (s_{\pi+} - s_{\pi-})(\tilde{F}_2) &= -ie^{-\pi i(a+b)} \eta s_{\pi}(\tilde{F}_1), & w \in (1, +\infty). \end{aligned} \quad (\text{A.2.23})$$

Note that the resummations of the asymptotic series differ substantially from the $\epsilon > 0$ case. This is to be expected since we cross multiple Stokes phenomena as we analytically continue from the positive to the negative real axis. However, starting from the exact Borel transforms we can calculate the resummation with a change of variables analogous to (A.2.7), with the ray along the negative real axis instead. We find

$$\begin{aligned} s_{\pi}(\tilde{F}_1) &= {}_2F_1\left(a, b; \frac{1}{\epsilon}; w\right) + \mathcal{D}(\epsilon) w^{1-\frac{1}{\epsilon}} {}_2F_1\left(1+a-\frac{1}{\epsilon}, 1+b-\frac{1}{\epsilon}; 2-\frac{1}{\epsilon}; w\right), & w \in \mathbb{C} \setminus (-\infty, 0] \\ s_{\pi}(\tilde{F}_2) &= -\mathcal{D}'(\epsilon) \left(\frac{w}{w-1}\right)^{-\frac{1}{\epsilon}} \frac{w e^{-i\pi(a+b)}}{(w-1)^{a+b}} {}_2F_1\left(1-a, 1-b; 2-\frac{1}{\epsilon}; w\right), & w \in \mathbb{C} \setminus [1, +\infty) \end{aligned} \quad (\text{A.2.24})$$

where

$$\begin{aligned} \mathcal{D}(\epsilon) &= \frac{\Gamma(\frac{1}{\epsilon}-1)\Gamma(a-\frac{1}{\epsilon}+1)\Gamma(b-\frac{1}{\epsilon}+1)}{\Gamma(a)\Gamma(b)\Gamma(1-\frac{1}{\epsilon})}, \\ \mathcal{D}'(\epsilon) &= \frac{\pi}{\sqrt{\sin(\pi a)\sin(\pi b)}} \frac{\Gamma(a-\frac{1}{\epsilon}+1)\Gamma(b-\frac{1}{\epsilon}+1)}{\Gamma(a)\Gamma(b)\Gamma(2-\frac{1}{\epsilon})\Gamma(1-\frac{1}{\epsilon})}. \end{aligned} \quad (\text{A.2.25})$$

For $\epsilon < 0$ we see that $s_\pi(\tilde{F}_1) \neq F_1$, but rather it equals a specific linear combination of the two independent solutions of the hypergeometric equation (A.1.1). This combination is the only representative in the class of functions with asymptotic series $\tilde{F}_1$ which has the correct analytic behaviour in $1/\epsilon$ in an open disc D_R of radius R with centre at $-\epsilon = R$. Up to a numerical factor, the resummation of $\tilde{F}_2$ in (A.2.24) equals $w^{1-1/\epsilon} {}_2F_1(1+a-1/\epsilon, 1+b-1/\epsilon, 2-1/\epsilon; w)$, the “other” solution around $w = 0$.

A.2.3 Large-order behaviour

The large order behaviour of the coefficients f_n in (A.1.29) can be extracted from the Borel plane analysis, as pointed out in (2.1.7) and (2.1.8). Since we have access to the exact Borel transform in (A.2.3), we can in principle extract the full large order behaviour. To do so, we inspect the rational branch cuts, which are of the form¹⁰

$$\hat{F}(t) = \text{holomorphic part} + \frac{\pi}{\sin(\pi\beta)} (S-t)^{-\beta} \sum_{k \geq 0} \frac{c_k^{(S)}}{\Gamma(n+1-\beta)} (t-S)^k + \dots \quad (\text{A.2.26})$$

These singularities then show up in the coefficients f_n of $\tilde{f}$ as

$$f_n \sim \sum_{k \geq 0} c_k^{(S)} \Gamma((n-k)-\beta) S^{-(n-k)+\beta}. \quad (\text{A.2.27})$$

For real power series $\tilde{f}$, the leading contribution is given by either a single real singularity or a pair of complex conjugate singularities. The latter case can be simplified to

$$f_n \sim \sum_{k \geq 0} 2 \cos((n-k-\beta)\phi - \alpha_k) \Gamma((n-k)-\beta) \rho^{-(n-k)+\beta} |c_k^{(S)}|, \quad (\text{A.2.28})$$

where $S = \rho e^{i\phi}$, $c_k^{(S)} = |c_k^{(S)}| e^{i\alpha_k}$.

The leading large-order behaviour is then determined by the largest term in the sum appearing in (A.2.28), generally given by the $k = 0$ term of the singularity with the smallest ρ (closest to the origin). Due to the many, w -dependent, singularities this leading behaviour depends on w . For simplicity, we focus on w real. We have three regimes for w real:

$$f_n(a, b, w) \approx \begin{cases} \frac{w \left((w-1) \log \frac{w}{w-1} \right)^{-a-b}}{\Gamma(a)\Gamma(b)} \Gamma(n+a+b-1) \left(\log \frac{w}{w-1} \right)^{1-n}, & w < 0 \vee w > \frac{1}{1-e^{-2\pi}}, \\ \frac{w(1-w)^{-a-b}}{\Gamma(a)\Gamma(b)} \Gamma(n+a+b-1) \left| \log \frac{w}{w-1} \right|^{1-n-a-b} \\ \quad \times 2 \cos \left((n+a+b-1) \arg \left(\log \frac{w}{w-1} \right) - \pi(a+b) \right), & 0 < w < \frac{1}{1+e^{-\sqrt{3}\pi}}, \\ 2^{2-n} \pi^{-n-1} \sin(\pi a) \sin(\pi b) \Gamma(n) \sin \left(\pi(a+b) + \frac{\pi n}{2} \right), & \frac{1}{1+e^{-\sqrt{3}\pi}} < w < \frac{1}{1-e^{-2\pi}}. \end{cases} \quad (\text{A.2.29})$$

¹⁰The relation (A.2.26) is easy to generalize for integer β for which it corresponds to pole singularities (if $\beta = 0$) and logarithmic branch cuts ($\beta \geq 0$).

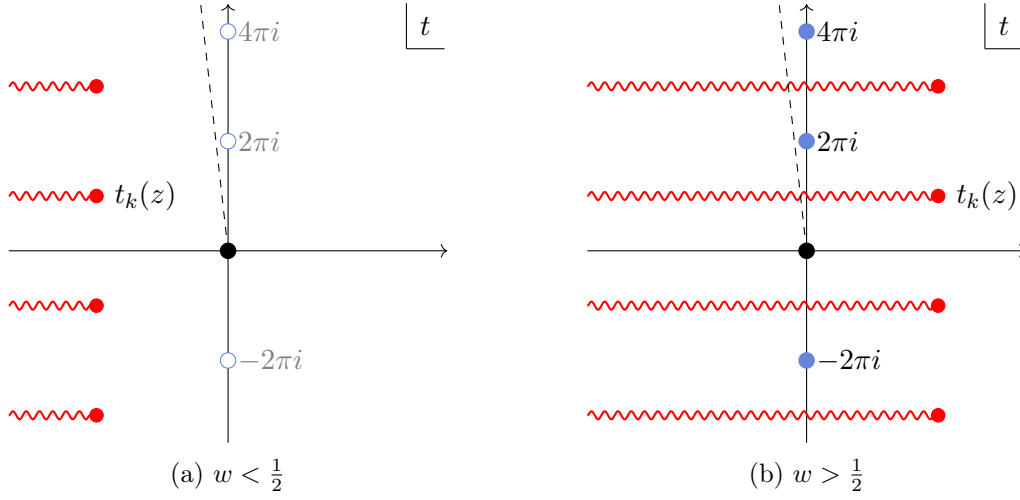


Figure A.5: Higher order Stokes phenomenon in the Borel transform (A.2.3). The red dots are the singularities t_k defined in (A.2.6), with their respective branch cuts drawn as wavy lines. The branch cuts are placed along the natural choice for ${}_2F_1$ after the exponential map u . The singularities at non-zero multiples of $2\pi i$ are in the non-principal sheet of the hypergeometric function and are only reached by the ray of Borel summation if the ray crosses a branch. This happens for $w > \frac{1}{2}$ (right) but not for $w < \frac{1}{2}$ (left). We colour these singularities as grey when they are invisible and as blue otherwise. The dashed line represents the ray of (2.1.5) for an angle close to $\theta = \pi/2$.

In the first and last cases, there is a single closest singularity at t_0 . In the middle case, there are two complex singularities which dominate, corresponding to $t_{0,1} = \log\left(\frac{w}{1-w}\right) \pm \pi i$. Lastly, when w is very close to 1, $\left|\log\frac{w}{w-1}\right|$ becomes large and the closest singularities are those at $t = \pm 2\pi i$.

One could be surprised that the singularities at $t = \pm 2\pi i$ do not also dominate the large order behaviour for w close to 0 as $|t_0| \gg 1$. In fact, they do not contribute at all due to a so-called higher-order Stokes phenomenon [407]. Higher-order Stokes phenomena happen when two singularities in the Borel plane become collinear with the origin. Then the branch cut emanating from the closer singularity affects whether rays emanating at the origin reach the further singularity. The large-order behaviour is sensible to singularities which intersect the rays on which the Borel summation is performed for any angle, see (2.1.7). In the case of aligned singularities, the closest one can eclipse all others, preventing them from showing up in the large-order behaviour.

In this case, the points $2\pi i k$ are not singularities of the hypergeometric function in (A.2.3) in the principal sheet, but they are on other ones. When $\text{Re}(w) < 1/2$, the branch cuts starting at t_k are such that the rays emanating from the origin remain in the principal sheet of the hypergeometric function along the imaginary axis. At $\text{Re}(w) = 1/2$ the singularities at t_k align with those $2\pi i k$, leading to a higher order Stokes phenomenon. Thus, if $\text{Re}(w) > 1/2$, a ray emanating from the origin crosses the branch cut and reaches $t = \pm 2\pi i$ in the other sheet, where the hypergeometric function is singular. This is illustrated in figure A.5. Thus, the imaginary singularities only contribute to the large order behaviour when $\text{Re}(w) > 1/2$.

A.3 Summary of asymptotic expansions

We define *Stirling polynomials* of the first and second kind, respectively, as

$$S_1\left(\begin{matrix} n \\ k \end{matrix}; x\right) = \sum_{m=k}^n \left[\begin{matrix} n \\ m \end{matrix}\right] \binom{m}{k} x^{m-k}, \quad n \geq 0, \quad 0 \leq k \leq n, \quad (\text{A.3.1})$$

$$S_2\left(\begin{matrix} n \\ k \end{matrix}; x\right) = \sum_{m=k}^n \left\{\begin{matrix} m \\ k \end{matrix}\right\} \binom{n}{m} x^{n-m}, \quad n \geq 0, \quad 0 \leq k \leq n, \quad (\text{A.3.2})$$

where $\left[\begin{matrix} n \\ m \end{matrix}\right]$ are *unsigned* Stirling numbers of the first kind and $\left\{\begin{matrix} m \\ k \end{matrix}\right\}$ are Stirling numbers of the second kind. When $x = 0$, the Stirling polynomials reduce to the corresponding Stirling numbers.

Large b and c As $\lambda \rightarrow +\infty$, we have

$${}_2F_1(a, b_0 + b_1\lambda, c + \lambda; z) \sim (1 - b_1 z)^{-a} \left(1 + \sum_{n=1}^{+\infty} g_n\left(a, b_0, b_1, c; \frac{z}{1 - b_1 z}\right) \lambda^{-n} \right), \quad (\text{A.3.3})$$

$$g_n(a, b_0, b_1, c; w) = \sum_{l=1}^{2n} d_{n,l}(a, b_0, b_1, c) w^l,$$

$$d_{n,l}(a, b_0, b_1, c) = \frac{(a)_l}{l!} \sum_{r=1}^l \sum_{k=0}^{\min\{n,r\}} (-1)^{n+l+r-k} \binom{l}{r} S_2\left(\begin{matrix} r+n-k-1 \\ r-1 \end{matrix}; c\right) S_1\left(\begin{matrix} r \\ r-k \end{matrix}; b_0\right) b_1^{l-k}.$$

We have not investigated in detail the domain $z \in D \subseteq \mathbb{C}$ and parameters $a, b_0, b_1, c \in \mathbb{C}$ where this asymptotic approximation is valid. This domain could be deduced by studying the web of Stokes and anti-Stokes lines associated with the differential equation. It would be interesting to fill this gap.

Large c As $\lambda \rightarrow +\infty$, we have

$${}_2F_1(a, b, c + \lambda; z) \sim 1 + \sum_{n=1}^{+\infty} f_n(a, b, c; z) \lambda^{-n}, \quad z \in \mathbb{C} \setminus [1, +\infty), \quad (\text{A.3.4})$$

$$f_n(a, b, c; z) = \sum_{k=1}^n (-1)^{n+k} S_2\left(\begin{matrix} n-1 \\ k-1 \end{matrix}; c\right) \frac{(a)_k (b)_k}{k!} z^k.$$

This can also be obtained as a special case of (A.3.3) with $b_1 = 0$. As $\lambda \rightarrow -\infty$, the asymptotic form is the same but the domain is restricted to $\{z \in \mathbb{C} \setminus (-\infty, 0] : \operatorname{Re}(z) < 1/2\}$.

Special cases Large parameter asymptotic expansion of ${}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z)$ can be reduced to special cases of the asymptotic expansion (A.3.4) after a change of variable. As

$\lambda \rightarrow +\infty$, we have

$$\begin{aligned}
{}_2F_1(-\lambda, 1 + \lambda, 2 + 2\lambda; z) &\sim e^{-\lambda S(z)} A(z) \left(1 + \sum_{n=1}^{+\infty} f_n(r(z)) \lambda^{-n} \right), \\
f_n(r) &= \sum_{k=1}^n (-1)^{n+k} S_2 \left(\begin{matrix} n-1 \\ k-1 \end{matrix}; \frac{3}{2} \right) \frac{(\frac{1}{6})_k (\frac{5}{6})_k}{k!} r^k,
\end{aligned} \tag{A.3.5}$$

where

$$S(z) = \frac{1}{2} \log \left(\frac{27}{16} \frac{z^2}{(1-z)^2} \frac{1-r(z)}{r(z)} \right), \quad A(z) = \frac{e^{-\frac{1}{2}S(z)}}{(1-z+z^2)^{\frac{1}{4}}}, \tag{A.3.6}$$

and

$$r(z) = \frac{1}{2} + \frac{(z+1)(2z-1)(2-z)}{4(1-z+z^2)^{\frac{3}{2}}}. \tag{A.3.7}$$

Similarly, large parameter asymptotic expansion of ${}_2F_1(2\gamma, 1 + \lambda, 2 + 2\lambda)$ can be reduced to the asymptotic expansion (A.3.4). As $\lambda \rightarrow +\infty$, we have

$$\begin{aligned}
{}_2F_1(2\gamma, 1 + \lambda, 2 + 2\lambda; z) &\sim \left(1 - \frac{z}{2} \right)^{-2\gamma} \left(1 + \sum_{n=1}^{+\infty} f_n \left(\gamma; \frac{z^2}{(2-z)^2} \right) \lambda^{-n} \right), \\
f_n(\gamma; w) &= \sum_{k=1}^n (-1)^{n+k} S_2 \left(\begin{matrix} n-1 \\ k-1 \end{matrix}; \frac{3}{2} \right) \frac{(\gamma)_k (\gamma + 1/2)_k}{k!} w^k.
\end{aligned} \tag{A.3.8}$$

See the discussion at the end of section 3.3.4 and 3.4.4 and the related figure 3.5 and 3.7, respectively, for a discussion on the domain in z of validity of (A.3.8), also in the case $\lambda \rightarrow -\infty$.

Appendix B

Higgsing transition in the Abelian Higgs model

In this appendix we illustrate this characterisation of Higgsing in AdS in a simpler weakly-coupled example, the Abelian Higgs model. We consider first the theory of a complex scalar field Φ in AdS, and later couple it to a dynamical gauge field.

We take the following action

$$S = \int dx \left(|\nabla_M \Phi|^2 + m^2 |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4 \right), \quad (\text{B.0.1})$$

where the integration is over AdS as in (5.3.4), and we use M to denote an index in the tangent space. This action has two real parameters m^2 and λ , the latter assumed to be strictly positive to ensure global stability. The action is invariant under a $U(1)$ symmetry under which Φ has charge 1, namely the transformation is $\Phi \rightarrow e^{i\alpha} \Phi$ with parameter $\alpha \sim \alpha + 2\pi$.

As long as we can treat λ as a small interaction, meaning that the classical analysis of the potential is reliable, we have the following vacuum structure for the theory as a function of m^2 :

- For $L^2 m^2 \geq L^2 m_{\text{BF}}^2 = -\frac{d^2}{4}$ [408] there is a stable vacuum in the origin $\Phi = 0$. Note that this vacuum is a global minimum of the potential for $m^2 \geq 0$ but becomes a local maximum for $m_{\text{BF}}^2 \leq m^2 < 0$. The fluctuations around this vacuum are simply given by the weakly interacting scalar field Φ of mass m^2 , which admits a $U(1)$ invariant boundary condition: $\Phi(z, \vec{x}) \underset{z \rightarrow 0}{\sim} z^\Delta \hat{\Phi}(\vec{x})$, with $\Delta(\Delta - d) = L^2 m^2$.

In the range $L^2 m_{\text{BF}}^2 \leq L^2 m^2 \leq L^2 m_{\text{BF}}^2 + 1$ there are actually two choices of $U(1)$ invariant boundary conditions, because both solutions Δ_- and $\Delta_+ = d - \Delta_- \geq \Delta_-$ to the quadratic equation for Δ are above the unitarity bound.

The boundary conformal theory is understood as a Generalised Free Scalar theory of the operator $\hat{\Phi}(\vec{x})$, which has scaling dimension Δ and charge 1 under the $U(1)$ symmetry, perturbed by the small interaction λ . Note that from the point of view of the boundary

theory, the $U(1)$ symmetry is non-locally realised. It is inherited from the bulk symmetry, but it does not have an associated local conserved current operator.

- For $m^2 < 0$, there is a circle of vacua outside of the origin at the local minimum of the potential, namely $|\Phi| = v \equiv \sqrt{-\lambda/m^2}$. Expanding around one of the possible expectation values on the circle, say $ve^{i\theta}$, with $\theta \sim \theta + 2\pi$, we write

$$\Phi(z, \vec{x}) = \left(v + \frac{\rho(z, \vec{x})}{\sqrt{2}} \right) e^{i\left(\theta + \frac{\pi(z, \vec{x})}{\sqrt{2}v}\right)}. \quad (\text{B.0.2})$$

The fluctuations are the radial mode ρ of mass $m_\rho^2 = 2(-m^2) \geq 0$ and the massless Goldstone boson $\pi(z, \vec{x}) \sim \pi(z, \vec{x}) + 2\pi \sqrt{2}v$. The boundary conditions of these modes are $\rho(z, \vec{x}) \underset{z \rightarrow 0}{\sim} z^{\Delta_\rho} \hat{\rho}(\vec{x})$, with $\Delta_\rho(\Delta_\rho - d) = L^2 m_\rho^2$, and $\pi(z, \vec{x}) \underset{z \rightarrow 0}{\sim} z^d \hat{\pi}(\vec{x})$. The resulting boundary condition for Φ is not invariant under a $U(1)$ transformation, because it approaches the constant expectation value $ve^{i\theta}$ at the boundary.

A boundary condition that breaks a bulk symmetry is the AdS version of spontaneous symmetry breaking. The boundary theory contains the operator $\hat{\rho}$ of dimension Δ_ρ and the operator $\hat{\pi}$ of exactly marginal dimension d . The latter dimension is a consequence of the bulk symmetry breaking [247] and therefore does not receive corrections in λ . The operator $\hat{\pi}$ is referred to as “tilt operator”, borrowing the terminology used in the context of defect and boundary CFT [409]. Deforming the action by a boundary term proportional to the tilt induces a shift in the constant θ in the boundary condition of Φ , i.e. it moves the theory to a nearby point in the circle of vacua. This means that the space of bulk vacua parametrised by θ corresponds to a continuous family of boundary conformal theories, parametrised by the exactly marginal coupling associated to the tilt [410].

It is useful to repackage the above information by listing the various options for the boundary conditions as a function of m^2 , keeping $\lambda > 0$ fixed and small as before.

- For $L^2 m^2 \geq 0$, the only available boundary condition is the $U(1)$ preserving boundary condition with scaling dimension Δ_+ . The choice Δ_- is AdS invariant but leads to a “real non-unitary” theory.
- For $L^2 m_{\text{BF}}^2 + 1 \leq L^2 m^2 < 0$, in addition to the option above that keeps existing, a continuous family (circle) of boundary conditions that breaks the $U(1)$ symmetry becomes available. Note that this additional continuous set of boundary theories starts existing precisely when the lightest $U(1)$ charged operator $\hat{\Phi}$ in the boundary spectrum of the $U(1)$ preserving boundary condition crosses marginality, i.e. when we cross $m^2 = 0$. Indeed, the new $U(1)$ breaking boundary condition can also be defined as the endpoint of the boundary RG flow obtained by perturbing the $U(1)$ preserving one with the relevant operator $\hat{\Phi}$. The phase of the complex coupling associated with this deformation determines which vacuum is selected.
- For $L^2 m_{\text{BF}}^2 \leq L^2 m^2 < L^2 m_{\text{BF}}^2 + 1$, in addition to the two options above, there is one more $U(1)$ preserving boundary condition, the one with dimension Δ_- , which becomes

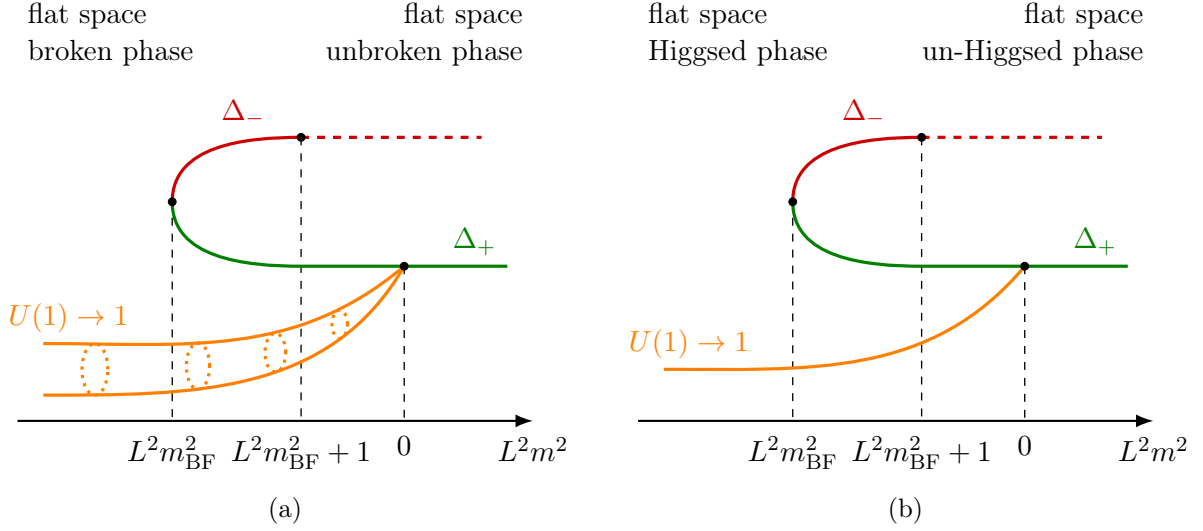


Figure B.1: Illustration of the possible boundary conditions as a function of $L^2 m^2$. Panel (a) refers to the theory of the complex scalar: at $m^2 = 0$, the boundary charged operator $\hat{\Phi}$ of the Δ_+ boundary condition crosses marginality, and the theory develops a circle of symmetry-breaking boundary conditions. The symmetry-preserving boundary condition keeps existing past that point, until it annihilates with the Δ_- boundary condition. Upon gauging, as illustrated in panel (b), the circle of symmetry-breaking boundary conditions gets replaced by a single Higgsed boundary condition, in which the conservation of the boundary $U(1)$ current is violated. Note that the vertical axis does not represent a definite quantity; it is merely used to juxtapose the different options.

unitary in this range of m^2 . Both the Δ_+ and the Δ_- can be connected to the symmetry-breaking boundary condition by an RG flow triggered by the relevant charged operator $\hat{\Phi}$. Moreover, they can be connected by an RG flow triggered by the singlet operator $|\hat{\Phi}|^2$, starting from the Δ_- boundary condition where this operator is relevant, and ending at the Δ_+ boundary condition where it is irrelevant.

- For $L^2 m^2 < L^2 m_{\text{BF}}^2$ only the continuous family of symmetry-breaking boundary conditions is available. Indeed, as we approach the limit $m^2 \rightarrow m_{\text{BF}}^2$ from above, the two available symmetry-preserving boundary conditions, with scaling dimensions Δ_- and Δ_+ , merge and annihilate with each other [253, 256]. The singlet operator $|\hat{\Phi}|^2$ is marginal at the merging point. Moving to more negative values of m^2 , these boundary conditions can only exist as a pair of conjugate complex conformal theories, or as real boundary conditions that break AdS isometries.

These boundary conditions are illustrated in panel (a) of Figure B.1.

We now couple Φ to a dynamical $U(1)$ gauge field A_M , i.e. a Maxwell field, by promoting the derivative to a gauge covariant one $\nabla_M \Phi \rightarrow D_M \Phi \equiv (\nabla_M + iA_M)\Phi$ and adding to the action the kinetic term for the gauge field

$$S \rightarrow S|_{\nabla \Phi \rightarrow D \Phi} + \int dx \frac{1}{4e^2} F^{MN} F_{MN}, \quad (\text{B.0.3})$$

where e^2 is the gauge coupling and $F_{MN} = \nabla_M A_N - \nabla_N A_M$. In the interest of illustrating the Higgsing transition in AdS in this simpler example, let us focus on the Dirichlet boundary condition for the gauge field. Neglecting for a moment the interaction with matter, this boundary condition gives a $U(1)$ conserved current operator j^μ of protected dimension $d - 1$.

In the range $m^2 \geq m_{\text{BF}}^2$, by picking a $U(1)$ preserving boundary condition for the matter fields, the interaction with matter preserves the current conservation and forces us to identify the $U(1)$ symmetry associated to the boundary current with the $U(1)$ symmetry acting on the operators from the matter sector. Therefore, the $U(1)$ symmetry under which $\hat{\Phi}$ is charged has become locally realised on the boundary as a consequence of the bulk gauging.

Let us now ask what happens in the range $m^2 < 0$ when we pick a symmetry-breaking boundary condition for the matter field. In the bulk, one can repeat the treatment familiar from flat space, by expanding the complex scalar field around its expectation value as in (B.0.2), and reach the same conclusion that the gauge field gains a positive mass-squared $m_A^2 = 2e^2 v^2$. This means that the current operator stops being conserved and it gains some positive anomalous dimension.

Let us explain this phenomenon from the boundary point of view. As we already observed above in the discussion of the theory before gauging, the symmetry-breaking boundary condition becomes available precisely as we cross $m^2 = 0$, i.e. when the lightest charged operator $\hat{\Phi}$ of the symmetry-preserving boundary condition crosses marginality. We also remarked above that the symmetry-breaking boundary condition can be reached, starting from the symmetry-preserving one, by perturbing with the relevant operator $\hat{\Phi}$. This is analogous to the deformation (5.2.1) considered in the case of Yang-Mills. As we turn on this deformation, since it is charged, the boundary current stops being conserved

$$\partial_\mu j^\mu \propto e^{-i\theta} \hat{\Phi} + e^{i\theta} \hat{\Phi}^*, \quad (\text{B.0.4})$$

where θ is the phase of the complex boundary coupling. The analogous equation for Yang-Mills is (5.2.2). This means that, as we flow to the IR and reach the new AdS invariant boundary condition, the operator j^μ is no longer conserved and can have a generic dimension. This boundary RG flow can be followed reliably if m^2 is negative but small.

A complementary way to reach the Higgsed boundary condition is to start from the matter sector with a symmetry-breaking boundary condition and turn on a small gauge coupling e , treating it perturbatively. Doing so, one finds at leading order $\partial_\mu j^\mu \propto evL \hat{\pi}$, expressing that there is a recombination between the conserved current operator and the tilt operator. This shows again that the current stops being conserved and gets a non-zero anomalous dimension. This relation is also useful to clarify that, since the tilt operator $\hat{\pi}$ is replaced in the spectrum by a descendant of the current, it cannot be used any more to give rise to a continuous family of boundary conditions: the circle of boundary theories collapses to a single point after gauging, see panel (b) of Figure B.1.

Putting things together, we see that this example is a simple realisation of the Higgsing transition as we described for YM in AdS in Section 5.2. The starting point is a family of

symmetry-preserving boundary conditions, which has only irrelevant charged operators (Δ_+ boundary condition for $m^2 > 0$ here, Dirichlet boundary condition at small coupling in Section 5.2). At some point, as we move along this family (decreasing m^2 here, increasing g_{YM} in Section 5.2) a charged boundary operator crosses marginality (in the current example at $m^2 = 0$, in Section 5.2 possibly at $(L\Lambda_{\text{YM}})_\text{H}$). When this happens, a new line of Higgsed boundary conditions emerges, in which the current stops being conserved. We see clearly in the example discussed here that the symmetry-preserving boundary condition keeps existing past the point at which the Higgsed boundary condition emerges. When it stops existing, this is actually due to a merger and annihilation with a boundary condition with the same symmetry (Δ_+ boundary condition here, D^* in Section 5.2).

Appendix C

Spectrum of $SU(N)$ GFV

In this appendix, we compute the low-lying spectrum of primary operators in the Mean Field Theory generated by a non-Abelian $SU(N)$ conserved current in $3d$. We do this by using character techniques introduced in [411, 412]. This section provides an extension of the results presented in Appendix A of [248]. In particular, we distinguish between parity-even and parity-odd sectors and show how to identify the intrinsic charge under charge conjugation for self-conjugate representations. The latter identification requires a somewhat more sophisticated formalism, which will be discussed in some detail. In Section C.1 we discuss characters of the $3d$ conformal group in the presence of parity. In Section C.2 we discuss $SU(N)$ characters in the presence of charge conjugation. In Section C.3, we finally review the general technique for obtaining the spectrum. At the end of the appendix, in Table C.1 and C.2 we explicitly report the spectrum of the first primary operators for $SU(2)$ and $SU(3)$.

C.1 Characters of $SO(4, 1)_0 \rtimes \mathbb{Z}_2^P$

The orthogonal group $O(d)$ has two disconnected components $O(d)_+$ and $O(d)_-$, the one connected to the identity being $SO(d)$. The other component can be realised as $O(d)_- = \{RP : R \in SO(d)\}$ where P is an orthogonal matrix with $\det P = -1$. As a group, we have $O(d) \cong SO(d) \times \mathbb{Z}_2$ for d odd and $O(d) \cong SO(d) \rtimes \mathbb{Z}_2$ for d even, where $\mathbb{Z}_2 = \{1, P\}$. The choice of P is arbitrary and gives rise to different realisations of $O(d)$, which are all isomorphic.¹ A commonly used convention is to pick $P = P_i$ to be a reflection of the i -th coordinate in $\mathbb{R}^d$, since this chapters for any $O(d)$.² This is, however, not the most convenient choice for the present discussion, since, even in odd d , such P_i -s would not commute with all the $SO(d)$ generators. In $d = 3$ (or in general for odd d), it is instead more convenient to choose a reflection of all the

¹This is not the case for $Pin(d)$, the universal covering group of $O(n)$, which is known to have two inequivalent choices, $Pin_{\pm}(d)$. Primary operators of the boundary theory considered here are, however, all bosonic and only in tensor representations; therefore, the two actions are equivalent.

²This convention is also useful when making contact with parity and time-reversal symmetry in Lorentzian QFTs.

coordinates as a representative

$$\mathcal{P} = P_1 P_2 P_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\mathbb{1}. \quad (\text{C.1.1})$$

This is related to P_i by $P_i = e^{i\pi M_i} \mathcal{P}$, where M_i is the generator of rotations in the i -th direction of $\mathbb{R}^3$.

We define primary operators of the $SO(4,1)_0$ conformal group to transform under parity as in (5.3.7). Irreducible representations of $SO(4,1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}}$ are labelled by (Δ, ℓ, p) , where Δ is the scaling dimension, ℓ is the $SO(3)$ spin, and $p = \pm$ is the intrinsic parity. Let us denote by $g \in \{\mathbb{1}, \mathcal{P}\}$ the two elements of $\mathbb{Z}_2^{\mathcal{P}}$. We define the character of “long” representations of $SO(4,1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}}$ as

$$\chi_{\Delta, \ell, p}(q, x, g) = \text{tr}[q^D x^L g], \quad (\text{C.1.2})$$

where the trace is over the (Δ, ℓ, p) representation, D is the generator of dilations and L is the Cartan generator of $SO(3)$, q and x being the corresponding fugacities. These characters are given explicitly by

$$\begin{aligned} \chi_{\Delta, \ell, p}(q, x, \mathbb{1}) &= \text{tr}[q^D x^L \mathbb{1}] = \frac{q^\Delta \chi_\ell(x)}{(1-q)(1-qx)(1-q/x)}, \\ \chi_{\Delta, \ell, p}(q, x, \mathcal{P}) &= \text{tr}[q^D x^L \mathcal{P}] = \frac{p(-1)^\ell q^\Delta \chi_\ell(x)}{(1+q)(1+qx)(1+q/x)}, \end{aligned} \quad (\text{C.1.3})$$

where $\chi_\ell(x) = \sum_{m=-\ell}^{\ell} x^m$ is the $SO(3)$ character for a spin ℓ representation. When $g = \mathbb{1}$, the character in (C.1.3) is the ordinary conformal character for a primary operator of spin ℓ in a 3d CFT. When $g = \mathcal{P}$, we obtain a “twisted” character. Its form is easily derived from the untwisted character by recalling that D and the $SO(3)$ subgroup commute with $\mathcal{P}$, so the contribution to the trace of the $SO(3)$ highest-weight sub-module $|\Delta, \ell, p\rangle$ is $p(-1)^\ell q^\Delta \chi_\ell(x)$. The contributions of the descendants $P_{\mu_1} \cdots P_{\mu_k} |\Delta, \ell, p\rangle$ picks a $(-1)^k$ factor compared to the “untwisted” case since $\mathcal{P} P_\mu = -P_\mu \mathcal{P}$. This explains the form of $\chi_{\Delta, \ell, p}(q, x, \mathcal{P})$ in (C.1.3).

Characters of “short” representations for which Δ saturates the unitarity bound, i.e. the identity, the free scalar, and conserved spin currents, are given by

$$\begin{aligned} \chi_{0,0,p}^{(\text{short})}(q, x, g) &= 1, \\ \chi_{1/2,0,p}^{(\text{short})}(q, x, g) &= \chi_{1/2,0,p}(q, x, g) - \chi_{5/2,0,p}(q, x, g), \\ \chi_{\ell+1,\ell,p}^{(\text{short})}(q, x, g) &= \chi_{\ell+1,\ell,p}(q, x, g) - \chi_{\ell+2,\ell-1,p}(q, x, g). \end{aligned} \quad (\text{C.1.4})$$

C.2 Characters of $PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}}$

The determination of the characters of $\widetilde{PSU}(N) = PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}}$, where $\mathbb{Z}_2^{\mathcal{C}} = \{1, \mathcal{C}\}$, is more involved than that of $SO(4,1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}}$ because the action of $\mathcal{C}$ on $SU(N)$ cannot be “trivialized” as in the case for parity. The action of $\mathbb{Z}_2^{\mathcal{C}}$ on $PSU(N)$ is given by a map from $\mathbb{Z}_2$

to the automorphism group of $PSU(N)$, $\text{Out}(PSU(N))$, isomorphic to $\mathbb{Z}_2$. The map $\varphi: \mathbb{Z}_2 \rightarrow \text{Out}(PSU(N))$ is given by $\varphi_{\mathbb{1}}(g) = g$, $\varphi_{\mathcal{C}}(g) = g^*$, with $g \in PSU(N)$ and $*$ denotes complex conjugation. We consider here $N \geq 3$, since $\text{Out}(PSU(2))$ is trivial.

The action of charge conjugation on primary fields is given by (5.3.11). When the primary operator $\mathcal{O}$ is in a non-self-conjugate representation $\lambda \neq \lambda^*$, the intrinsic charge conjugation phase $\mathfrak{c}$ in (5.3.11) can be set to one by a unitary transformation. In contrast, if $\lambda^* = \lambda$, we can induce two inequivalent representations $\lambda_{\pm}$ which are isomorphic to λ as $PSU(N)$ representations and where $\mathcal{C}$ acts as $\pm C_{\lambda}$, where C_{λ} is the intertwiner between the representation $R_{\lambda}(g)$, $g \in PSU(N)$, and the twisted representation $\tilde{R}_{\lambda}(g) = R_{\lambda}(\varphi(g))$ (i.e. if $\lambda^* = \lambda$, then there exists C_{λ} such that $\tilde{R}_{\lambda}(g) = C_{\lambda} R_{\lambda}(g) C_{\lambda}^{-1}$). For example, for the adjoint representation charge conjugation may be represented by $(C_{\lambda})^{ab} = (-1)^{N+1} C^{ab}$, with C^{ab} as given in (5.3.9), in the $(1, 0, \dots, 0, 1)_+$ representation.³

Complex conjugation acts on anti-hermitian Cartan generators as $H_i \rightarrow -H_i$, thus mapping weights to their opposite: $\mu \rightarrow -\mu$. The highest-weight state is then not mapped to itself but to the lowest-weight state. It also does not preserve the canonical decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, but swaps positive and negative roots: $\mathfrak{g}_+ \leftrightarrow \mathfrak{g}_-$. For these reasons, as for parity, it is convenient for the construction of characters to use another definition of charge conjugation. This is the natural definition at the level of the Lie algebra, and corresponds to the symmetry of the Dynkin diagram of A_{N-1} , reflecting the simple roots $\alpha_i \rightarrow \alpha_{N-i}$, with $i = 1, \dots, N-1$. We then have $\varphi(H_i) = H_{N-i}$ and on the step operators of the simple root $\varphi(E_{\pm\alpha_i}) = E_{\pm\alpha_{N-i}}$. The action on any other step operator is determined by $\varphi([x, y]) = [\varphi(x), \varphi(y)]$. The action on any weight of the Lie algebra follows, and is reported in (5.3.12) in the main text.

Note that the group extension of $\text{Out}(SU(N))$ by $SU(N)$ is unique for odd N , while for even N there are two inequivalent extensions, depending on the representative chosen for $\text{Out}(SU(N))$, see the discussion e.g. in [414, 415]. In contrast, for $PSU(N)$ there is a unique extension for any N , which is isomorphic to the one induced by complex conjugation or, equivalently, by the Lie algebra automorphism φ that swaps the simple roots.

The group $\widetilde{PSU}(N)$ has two disconnected components. We can define the character of an irreducible representation $\lambda_{(c, \pm)}$ by taking the trace of $(e^h, \mathbb{1}) \in \widetilde{PSU}(N)$ or $(e^h, \mathcal{C}) \in \widetilde{PSU}(N)$ over the representation, with h in the Cartan subalgebra of $\mathfrak{su}(N)$.

For λ_c representations, $(e^h, \mathbb{1})$ elements are represented as the direct sum $R_{\lambda}(e^h) \oplus R_{\lambda^*}(e^h)$, while the elements $(e^h, \mathcal{C})$ are off-diagonal matrices. We thus have

$$\begin{aligned}\chi_{\lambda_c}(e^h, \mathbb{1}) &= \chi_{\lambda}(e^h) + \chi_{\lambda^*}(e^h), \\ \chi_{\lambda_c}(e^h, \mathcal{C}) &= 0,\end{aligned}\tag{C.2.1}$$

where $\chi_{\lambda}(e^h) = \text{tr } R_{\lambda}(e^h)$ is the usual character of $SU(N)$ representations, see later (C.2.6) and (C.2.7).

³The conventional choice for the $(-1)^{N+1}$ factor reflects the conventions used for the outer automorphism used in [413].

For $\lambda_{\pm}$ representations, $(e^h, \mathbb{1})$ elements are represented as $R_{\lambda}(e^h)$, while $(e^h, \mathcal{C})$ are represented by $\pm C_{\lambda} R_{\lambda}(e^h)$. Hence

$$\begin{aligned}\chi_{\lambda_{\pm}}(e^h, \mathbb{1}) &= \chi_{\lambda}(e^h), \\ \chi_{\lambda_{\pm}}(e^h, \mathcal{C}) &= \pm \operatorname{tr} [C_{\lambda} R_{\lambda}(e^h)] = \pm \chi_{\lambda}^{\varphi}(e^h).\end{aligned}\tag{C.2.2}$$

The insertion of C_{λ} in the trace defines a so-called “twisted” or “twining” character χ_{λ}^{φ} . The computation of this character requires an in-depth analysis of how the outer automorphisms of the $\mathfrak{su}(N)$ algebra act on the irreducible representations, and was worked out in [413] for general outer automorphisms of Kac-Moody Lie algebras.⁴ The key result of [413] that we need here is that the twining character χ^{φ} for a Lie algebra $\mathfrak{g}$ is identical to the ordinary character $\check{\chi}$ of the “orbit Lie algebra” $\check{\mathfrak{g}}$:

$$\chi_{\lambda}^{\varphi}(e^h) = \check{\chi}_{\check{\lambda}}(e^{\check{h}}).\tag{C.2.3}$$

Note that this relation holds for characters of infinite-dimensional Verma modules as well as for finite-dimensional representations (see the Example later, where the identity can be easily worked out for $\mathfrak{su}(3)$ Verma modules).

In the case of $\mathfrak{g} = A_{N-1} = \mathfrak{su}(N)$ Lie algebra, the orbit Lie algebra is

$$\check{\mathfrak{g}} = \begin{cases} C_n = \mathfrak{sp}(2n), & N = 2n + 1, \\ B_{n+1} = \mathfrak{so}(2n + 3), & N = 2n + 2. \end{cases}\tag{C.2.4}$$

The difference between even and odd N arises from the fact that for odd N the action of φ is free, while it is not for even N , as one simple root – the one in the middle of the Dynkin diagram – is left invariant.

The Dynkin labels λ and $\check{\lambda}$, and the Cartans h and $\check{h}$ appearing in (C.2.3) are related as follows:

$$\begin{aligned}N = 2n + 1 : \quad & \lambda = (\lambda_1, \dots, \lambda_n, \lambda_n, \dots, \lambda_1), \\ & h = x_1 H_1 + \dots + x_{2n} H_{2n}, \\ & \check{\lambda} = (\lambda_1, \dots, \lambda_n), \\ & \check{h} = (x_1 + x_{2n}) \check{H}_1 + \dots + (x_n + x_{n+1}) \check{H}_n; \\ N = 2n + 2 : \quad & \lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \lambda_n, \dots, \lambda_1), \\ & h = x_1 H_1 + \dots + x_{2n+1} H_{2n+1}, \\ & \check{\lambda} = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}), \\ & \check{h} = (x_1 + x_{2n}) \check{H}_1 + \dots + (x_n + x_{n+2}) \check{H}_n + x_{n+1} \check{H}_{n+1};\end{aligned}\tag{C.2.5}$$

where H_i and $\check{H}_i$ are the Cartan generators of $\mathfrak{g}$ and $\check{\mathfrak{g}}$ in the Cartan-Weyl basis, respectively.

⁴See also [416, 417] for some recent literature that uses these characters.

The characters for any representation of the classical Lie algebras A_n, B_n, C_n are well-known and can be written by a ratio of determinants as follows:

$$\chi_\lambda(y) = \frac{\det M(p, y)}{\det M(0, y)}, \quad (\text{C.2.6})$$

where (see e.g. [418])

$$\begin{aligned} A_n : \quad & M(p, y)_{ij} = y_j^{p_i+n+1-i}, \quad i, j = 1, \dots, n+1, \\ & p = (\lambda_1 + \dots + \lambda_n, \lambda_2 + \dots + \lambda_n, \dots, \lambda_n, 0), \\ & y_{n+1} = 1/(y_1 \dots y_n); \\ B_n : \quad & M(p, y)_{ij} = y_j^{p_i+n+\frac{1}{2}-i} - y_j^{-(p_i+n+\frac{1}{2}-i)}, \quad i, j = 1, \dots, n, \\ & p = (\lambda_1 + \dots + \lambda_{n-1} + \lambda_n/2, \lambda_2 + \dots + \lambda_{n-1} + \lambda_n/2, \lambda_n/2); \\ C_n : \quad & M(p, y)_{ij} = y_j^{p_i+n+1-i} - y_j^{-(p_i+n+1-i)}, \quad i, j = 1, \dots, n, \\ & p = (\lambda_1 + \dots + \lambda_n, \lambda_2 + \dots + \lambda_n, \dots, \lambda_n). \end{aligned} \quad (\text{C.2.7})$$

Here $e^h = y_1^{L_1} \dots y_n^{L_n}$ are the fugacities in a orthonormal basis, and p_i are the weights of the highest-weight state in this basis: $L_i|\lambda\rangle = p_i|\lambda\rangle$. These are related to the fugacities in the Cartan-Weyl basis $e^h = z_1^{H_1} \dots z_n^{H_n}$, $z_i = e^{x_i}$, where $H_i|\lambda\rangle = \lambda_i|\lambda\rangle$, by

$$\begin{aligned} A_n : \quad & y_1 = z_1, \quad y_i = \frac{z_i}{z_{i-1}}, \quad (i = 2, \dots, n) \\ B_n : \quad & y_1 = z_1, \quad y_i = \frac{z_i}{z_{i-1}}, \quad (i = 2, \dots, n-1), \quad y_n = \frac{z_n^2}{z_{n-1}}, \\ C_n : \quad & y_1 = z_1, \quad y_i = \frac{z_i}{z_{i-1}}, \quad (i = 2, \dots, n), \end{aligned} \quad (\text{C.2.8})$$

Summarising, we have the following character formulas

$$\begin{aligned} \chi_{\lambda_c}(z, \mathbb{1}) &= \chi_\lambda(z) + \chi_{\lambda^*}(z), \\ \chi_{\lambda_c}(z, \mathcal{C}) &= 0, \end{aligned} \quad (\text{C.2.9})$$

and

$$\begin{aligned} \chi_{\lambda_\pm}(z, \mathbb{1}) &= \chi_\lambda(z), \\ \chi_{\lambda_\pm}(z, \mathcal{C}) &= \begin{cases} \pm \chi_\lambda^{(C_n)}(\check{z}), & N = 2n+1, \\ \pm \chi_\lambda^{(B_{n+1})}(\check{z}), & N = 2n+2, \end{cases} \end{aligned} \quad (\text{C.2.10})$$

where

$$\begin{aligned} N = 2n+1 : \quad & \check{z} = (z_1 z_{2n}, z_2 z_{2n-1}, \dots, z_n z_{n+1}), \\ N = 2n+2 : \quad & \check{z} = (z_1 z_{2n+1}, z_2 z_{2n}, \dots, z_n z_{n+2}, z_{n+1}). \end{aligned} \quad (\text{C.2.11})$$

The intertwiner C_λ can be explicitly constructed by the action on the universal enveloping algebra $U(\mathfrak{g})$, and thus on any representation, by the Poincaré-Birkhoff-Witt (PBW) theorem.

Namely, we recall that given a basis $\{F_1, \dots, F_r\}$ for $\mathfrak{g}_-$, the Verma module V_λ of highest-weight state v_λ is spanned by the ordered monomials in $U(\mathfrak{g}_-)$

$$V_\lambda \cong \text{Span}\{F_1^{n_1} \cdots F_r^{n_r} v_\lambda : H_i v_\lambda = \lambda_i v_\lambda, U(\mathfrak{g}_+) v_\lambda = 0\}. \quad (\text{C.2.12})$$

The intertwiner in the PBW basis is then

$$C_\lambda F_1^{n_1} \cdots F_r^{n_r} v_\lambda = \varphi(F_1)^{n_1} \cdots \varphi(F_r)^{n_r} v_\lambda, \quad (\text{C.2.13})$$

Since φ is an algebra automorphism, this can always be rewritten as a linear combination of elements of a basis of $U(\mathfrak{g}_-)$. A similar construction applies to finite-dimensional irreducible representations, which are quotients of Verma modules.

Example. We consider the case of infinite-dimensional Verma modules of $SU(3)$. The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (\text{C.2.14})$$

The simple roots are $\{\alpha_1, \alpha_2\}$, the positive roots are $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. In the basis of fundamental weights $\{\omega_1, \omega_2\}$ we have $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2$ and $\alpha_1 + \alpha_2 = \omega_1 + \omega_2$. The Cartan-Weyl basis of $\mathfrak{su}(3)$ is

$$\begin{aligned} \{H_1, H_2\}, \quad \{E_1 = E_{\alpha_1}, E_2 = E_{\alpha_2}, E_3 = E_{\alpha_1 + \alpha_2} = [E_{\alpha_1}, E_{\alpha_2}]\}, \\ \{F_1 = E_{-\alpha_1}, F_2 = E_{-\alpha_2}, F_3 = E_{-\alpha_1 - \alpha_2} = -[E_{-\alpha_2}, E_{-\alpha_1}]\}, \end{aligned} \quad (\text{C.2.15})$$

which span $\mathfrak{h}$, $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively. The outer automorphism acts by swapping the two simple roots $\alpha_1 \leftrightarrow \alpha_2$. On the Cartan-Weyl basis

$$\begin{aligned} \varphi(H_1) &= H_2, & \varphi(H_2) &= H_1, \\ \varphi(E_1) &= E_2, & \varphi(E_2) &= E_1, & \varphi(E_3) &= -E_3, \\ \varphi(F_1) &= F_2, & \varphi(F_2) &= F_1, & \varphi(F_3) &= -F_3. \end{aligned} \quad (\text{C.2.16})$$

Let v_λ be a highest-weight state with Dynkin labels (λ_1, λ_2) . The PBW basis of the Verma module is given by $F_1^{n_1} F_2^{n_2} F_3^{n_3} v_\lambda$ with n_i integers. The group element e^h , with h in the Cartan sub-algebra $\mathfrak{h}$, acts diagonally as

$$e^h F_1^{n_1} F_2^{n_2} F_3^{n_3} v_\lambda = e^{\mu(h)} F_1^{n_1} F_2^{n_2} F_3^{n_3} v_\lambda \quad \mu = \lambda - n_1 \alpha_1 - n_2 \alpha_2 - n_3 (\alpha_1 + \alpha_2), \quad (\text{C.2.17})$$

Recall that if $h = x_1 H_1 + x_2 H_2$ and $\mu = \mu_1 \omega_1 + \mu_2 \omega_2$, then $\mu(h) = x_1 \mu_1 + x_2 \mu_2$. From this, we recover the well-known expression for Verma module characters

$$\mathcal{V}_\lambda(e^h) = \sum_{n_1, n_2, n_3=0}^{+\infty} e^{\lambda(h)} e^{-n_1 \alpha_1(h)} e^{-n_2 \alpha_2(h)} e^{-n_3 (\alpha_1 + \alpha_2)(h)} = e^{\lambda(h)} \prod_{\alpha \in \Delta_+} \frac{1}{1 - e^{-\alpha(h)}}. \quad (\text{C.2.18})$$

For a symmetric weight $\lambda = \lambda_1 \omega_1 + \lambda_1 \omega_2$, the action of C_λ on the Verma module is given in (C.2.13) and in this case reads

$$C_\lambda F_1^{n_1} F_2^{n_2} F_3^{n_3} v_\lambda = (-1)^{n_3} F_1^{n_2} F_2^{n_1} F_3^{n_3} v_\lambda + \sum_{k=0}^{\min\{n_1, n_2\}} c_k F_1^{n_2-k} F_2^{n_1-k} F_3^{n_3+k} v_\lambda, \quad (\text{C.2.19})$$

for some coefficients c_k that are determined by the commutation relations of the $\mathfrak{su}(3)$ algebra, but whose precise form is not important here. Thus, only the states with $n_1 = n_2$ can contribute to the twisted character, and those contribute with $(-1)^{n_3} e^{\mu(h)}$. We then have

$$\mathcal{V}_\lambda^\varphi(e^h) = \text{tr} \left[C_\lambda R_\lambda(e^h) \right] = \sum_{n_1, n_3=0}^{+\infty} (-1)^{n_3} e^{\lambda(h)} e^{-n_1(\alpha_1+\alpha_2)(h)} e^{-n_3(\alpha_1+\alpha_2)(h)} = \frac{e^{\lambda(h)}}{1 - e^{-(2\alpha_1+2\alpha_2)(h)}}. \quad (\text{C.2.20})$$

Writing $h = x_1 H_1 + x_2 H_2$ and the weights in the fundamental basis, this is

$$\mathcal{V}_\lambda^\varphi(e^h) = \frac{e^{(x_1+x_2)\lambda_1}}{1 - e^{-2(x_1+x_2)}}, \quad (\text{C.2.21})$$

which coincides with the character of the $C_1 = \mathfrak{sp}(2) = \mathfrak{su}(2)$ Verma module

$$\check{\mathcal{V}}_{\check{\lambda}}(e^{\check{h}}) = \text{tr} \left[R_{\check{\lambda}}(e^{\check{h}}) \right] = \frac{e^{\check{\lambda}(\check{h})}}{1 - e^{-\check{\alpha}(\check{h})}} = \frac{e^{\check{x}_1 \check{\lambda}_1}}{1 - e^{-2\check{x}_1}}, \quad \check{\lambda} = \check{\lambda}_1 \check{\omega}_1, \quad \check{h} = \check{x}_1 \check{H}_1, \quad (\text{C.2.22})$$

upon identifying $\check{\lambda}_1 = \lambda_1$ and $\check{x}_1 = x_1 + x_2$.

C.3 Technique

In Section C.1 and C.2 we presented the characters of the $3d$ conformal group in the presence of parity $SO(4,1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}}$ and of $PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}}$, respectively. Characters for representations of (5.3.5) can be easily written down with the knowledge of these two ingredients. For a representation $(\Delta, \ell, p, \lambda_{(c,\pm)})$ these are

$$\chi_{\Delta, \ell, p, \lambda_{(c,\pm)}}(q, x, y, g_{\mathcal{P}}, g_{\mathcal{C}}) = \begin{cases} \chi_{\Delta, \ell, p}^{(\text{short})}(q, x, g_{\mathcal{P}}) \chi_{\lambda_{(c,\pm)}}(y, g_{\mathcal{C}}) & \text{if } (\Delta, \ell) = (0, 0), (1/2, 0), (\ell + 1, \ell), \\ \chi_{\Delta, \ell, p}(q, x, g_{\mathcal{P}}) \chi_{\lambda_{(c,\pm)}}(y, g_{\mathcal{C}}) & \text{otherwise,} \end{cases} \quad (\text{C.3.1})$$

where $g_{\mathcal{P}} \in \{1, \mathcal{P}\}$, $g_{\mathcal{C}} \in \{1, \mathcal{C}\}$, $\chi_{\Delta, \ell, p}$ and $\chi_{\Delta, \ell, p}^{(\text{short})}$ are given by (C.1.3) and (C.1.4), and $\chi_{\lambda_{(c,\pm)}}$ is given by (C.2.9). We denote by (q, x) the fugacities for the $SO(4,1)_0$ Cartan generators, while $y = (y_1, \dots, y_{N-1})$ are the fugacities for the $\mathfrak{su}(N)$ Cartan subalgebra in the orthonormal basis (C.2.8).

The generating function for the spectrum of a free bosonic field in a representation R of a group G is given by the generating function of symmetric products

$$Z_R(\eta, g) = \sum_{n=0}^{+\infty} \eta^n \chi_{\text{Sym}^n R}(g) = \frac{1}{\det(1 - \eta R(g))} = \exp \left(\sum_{n=1}^{+\infty} \frac{\eta^n}{n} \chi_R(g^n) \right), \quad g \in G, \quad (\text{C.3.2})$$

where η is a fugacity for the particle number. In the GFV case we use $R = (2, 1, +, \mathbf{adj}_{\pm}) \equiv J$ for N even/odd, and

$$g = \left((q^D x^L, g_{\mathcal{P}}), (z^H, g_{\mathcal{C}}) \right) \in G = \left(SO(4,1)_0 \rtimes \mathbb{Z}_2^{\mathcal{P}} \right) \times \left(PSU(N) \rtimes \mathbb{Z}_2^{\mathcal{C}} \right), \quad (\text{C.3.3})$$

where $z^H = \prod_{i=1}^{N-1} z_i^{H_i}$. Taking powers of g requires some care because of the semi-direct product. $\mathcal{P}$ commutes with D and L , so $(q^D x^L, g_{\mathcal{P}})^n = (q^{nD} x^{nL}, g_{\mathcal{P}}^n)$, while for $(z^H, g_{\mathcal{C}})^n$ we have to distinguish between n even and n odd when $g_{\mathcal{C}} = \mathcal{C}$. The key identity is

$$(z^H, \mathcal{C})^2 = (z^H z^{\varphi(H)}, \mathbb{1}) \equiv (\tilde{z}^H, \mathbb{1}), \quad (\text{C.3.4})$$

where

$$\begin{aligned} N = 2n + 1 : \quad \tilde{z} &= (z_1 z_{2n}, z_2 z_{2n-1}, \dots, z_{2n} z_1), \\ N = 2n + 2 : \quad \tilde{z} &= (z_1 z_{2n}, z_2 z_{2n-1}, \dots, z_{n+1}^2, \dots, z_{2n} z_1). \end{aligned} \quad (\text{C.3.5})$$

We then have

$$\begin{aligned} (z^H, \mathcal{C})^{2m} &= (\tilde{z}^{mH}, \mathbb{1}), \\ (z^H, \mathcal{C})^{2m+1} &= (z^H \tilde{z}^{mH}, \mathcal{C}), \end{aligned} \quad (\text{C.3.6})$$

which can be mapped back to the y -fugacities using (C.2.8). We can now explicitly determine the form of the characters $\chi_J(g^n)$ appearing in (C.3.2) in terms of the character of a single current J , which will be denoted by $z_J(q, x, y, g_{\mathcal{P}}, g_{\mathcal{C}})$. To avoid cluttered notation, let us split the element g in (C.3.3) into four pieces, and define

$$\begin{aligned} g_{++} &= ((q^D x^L, \mathbb{1}), (z^H, \mathbb{1})), & g_{+-} &= ((q^D x^L, \mathbb{1}), (z^H, \mathcal{C})), \\ g_{-+} &= ((q^D x^L, \mathcal{P}), (z^H, \mathbb{1})), & g_{--} &= ((q^D x^L, \mathcal{P}), (z^H, \mathcal{C})). \end{aligned} \quad (\text{C.3.7})$$

We then have

$$\begin{aligned} \chi_J(g_{++}^n) &= z_J(q^n, x^n, y^n, \mathbb{1}, \mathbb{1}), \\ \chi_J(g_{-+}^{2m}) &= z_J(q^{2m}, x^{2m}, y^{2m}, \mathbb{1}, \mathbb{1}), & \chi_J(g_{-+}^{2m+1}) &= z_J(q^{2m+1}, x^{2m+1}, y^{2m+1}, \mathcal{P}, \mathbb{1}), \\ \chi_J(g_{+-}^{2m}) &= z_J(q^{2m}, x^{2m}, \tilde{y}^m, \mathbb{1}, \mathbb{1}), & \chi_J(g_{+-}^{2m+1}) &= z_J(q^{2m+1}, x^{2m+1}, y \tilde{y}^m, \mathbb{1}, \mathcal{C}), \\ \chi_J(g_{--}^{2m}) &= z_J(q^{2m}, x^{2m}, \tilde{y}^m, \mathbb{1}, \mathbb{1}), & \chi_J(g_{--}^{2m+1}) &= z_J(q^{2m+1}, x^{2m+1}, y \tilde{y}^m, \mathcal{P}, \mathcal{C}), \end{aligned} \quad (\text{C.3.8})$$

where $y^n = (y_1^n, \dots, y_{N-1}^n)$, and similarly for $\tilde{y}^m$ and $y \tilde{y}^m$.

We then define a vector for generating functions on each connected component of the group

$$\vec{Z}_J(\eta, q, x, y) = \begin{pmatrix} Z_J(\eta, q, x, y, \mathbb{1}, \mathbb{1}) \\ Z_J(\eta, q, x, y, \mathcal{P}, \mathbb{1}) \\ Z_J(\eta, q, x, y, \mathbb{1}, \mathcal{C}) \\ Z_J(\eta, q, x, y, \mathcal{P}, \mathcal{C}) \end{pmatrix}. \quad (\text{C.3.9})$$

Each entry is computed by combining (C.3.2) with (C.3.8):

$$\begin{aligned} Z_J(\eta, q, x, y, \mathbb{1}, \mathbb{1}) &= \exp \left(\sum_{n=1}^{+\infty} \frac{\eta^n}{n} z_J(q^n, x^n, y^n, \mathbb{1}, \mathbb{1}) \right), \\ Z_J(\eta, q, x, y, \mathcal{P}, \mathbb{1}) &= \exp \left(\sum_{m=0}^{+\infty} \frac{\eta^{2m+1}}{2m+1} z_J(q^{2m+1}, x^{2m+1}, y^{2m+1}, \mathcal{P}, \mathbb{1}) + \sum_{m=2}^{+\infty} \frac{\eta^{2m}}{2m} z_J(q^{2m}, x^{2m}, y^{2m}, \mathbb{1}, \mathbb{1}) \right), \\ Z_J(\eta, q, x, y, \mathbb{1}, \mathcal{C}) &= \exp \left(\sum_{m=0}^{+\infty} \frac{\eta^{2m+1}}{2m+1} z_J(q^{2m+1}, x^{2m+1}, y \tilde{y}^m, \mathbb{1}, \mathcal{C}) + \sum_{m=2}^{+\infty} \frac{\eta^{2m}}{2m} z_J(q^{2m}, x^{2m}, \tilde{y}^m, \mathbb{1}, \mathbb{1}) \right), \\ Z_J(\eta, q, x, y, \mathcal{P}, \mathcal{C}) &= \exp \left(\sum_{m=0}^{+\infty} \frac{\eta^{2m+1}}{2m+1} z_J(q^{2m+1}, x^{2m+1}, y \tilde{y}^m, \mathcal{P}, \mathcal{C}) + \sum_{m=2}^{+\infty} \frac{\eta^{2m}}{2m} z_J(q^{2m}, x^{2m}, \tilde{y}^m, \mathbb{1}, \mathbb{1}) \right). \end{aligned} \quad (\text{C.3.10})$$

We decompose (C.3.9) as a sum of characters of irreducible representations,

$$\vec{Z}_J(\eta, q, x, y) = \sum_{R,n} d_{R,n} \eta^n \vec{\chi}_R(q, x, y), \quad \vec{\chi}_R(q, x, y) = \begin{pmatrix} \chi_R(q, x, y, \mathbb{1}, \mathbb{1}) \\ \chi_R(q, x, y, \mathcal{P}, \mathbb{1}) \\ \chi_R(q, x, y, \mathbb{1}, \mathcal{C}) \\ \chi_R(q, x, y, \mathcal{P}, \mathcal{C}) \end{pmatrix}, \quad (\text{C.3.11})$$

where $R = (\Delta, \ell, p, \lambda_{(c,\pm)})$ and $d_{R,n}$ denotes the degeneracy of a primary operator in the representation R built out of n currents J (i.e. the degree of the symmetric product).

To perform this decomposition, we work recursively in a formal power series in q up to a given cut-off in scaling dimension. Given the spectrum S_K up to $\Delta = K$ we consider the difference

$$\vec{Z}_J(\eta, q, x, y) - \sum_{(R,n) \in S_K} d_{R,n} \eta^n \vec{\chi}_R(q, x, y) = \vec{f}_{K+1}(\eta, x, y) q^{K+1} + O(q^{N+2}). \quad (\text{C.3.12})$$

The function $\vec{f}_{K+1}(\eta, x, y) q^{K+1}$ is decomposed in characters by matching the leading term in the $\eta, q, x, y_1, \dots, y_{N-1} \rightarrow 0$ limit with the behaviour of the character (C.3.1) in the same limit

$$\begin{aligned} \vec{\chi}_{\Delta, \ell, p, \lambda_{\pm}}(q, x, y) &\sim \frac{q^{\Delta}}{x^{\ell} y_1^{\lambda_1 + \dots + \lambda_N} y_2^{\lambda_1 + \dots + \lambda_{N-2}} \dots y_{N-1}^{\lambda_1}} \begin{pmatrix} 1 \\ p(-1)^{\ell} \\ \pm 1 \\ \pm p(-1)^{\ell} \end{pmatrix}, \\ \vec{\chi}_{\Delta, \ell, p, \lambda_c}(q, x, y) &\sim \frac{q^{\Delta}}{x^{\ell} y_1^{\lambda_1 + \dots + \lambda_N} y_2^{\lambda_1 + \dots + \lambda_{N-2}} \dots y_{N-1}^{\lambda_1}} \begin{pmatrix} 1 \\ p(-1)^{\ell} \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (\text{C.3.13})$$

and recursively adding this contribution to S_K . In practice, there can be degeneracies between operators with the same $(\Delta = K+1, \ell, \lambda, n)$ but different behaviour under parity and charge conjugation. To disentangle the contributions, we use the leading behaviour of the first component of $\vec{f}_{K+1}(\eta, x, y) q^{K+1}$ to fix the $(\Delta, \ell, \lambda, n)$ quantum numbers. We then consider the coefficient associated with these quantum numbers in the full vector, i.e.

$$\vec{f}_{K+1}(\eta, x, y) q^{K+1} \sim \frac{\eta^n q^{K+1}}{x^{\ell} y_1^{\lambda_1 + \dots + \lambda_N} \dots y_{N-1}^{\lambda_1}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}. \quad (\text{C.3.14})$$

If $\lambda \neq \lambda^*$, we have $a_3 = a_4 = 0$ and we solve the system

$$\begin{cases} a_1 = d_+ + d_- , \\ a_2 = (-1)^{\ell} (d_+ - d_-) , \end{cases} \quad (\text{C.3.15})$$

p	ℓ	λ_1	Δ		
+	0	0	0	4	6
+	0	2	7	9	10
+	0	4	4	6	7
+	0	6	9	10	
+	0	8	8	10	
+	1	2	2	5	6
+	2	0	4	6	7
+	2	4	4	6	7
+	3	2	5	6	7
+	4	0	6	8	9
+	4	4	6	7	8
+	5	2	7	8	9
+	6	0	8	9	10
+	6	4	8	9	10
+	7	2	9	10	
+	8	0	10		
+	8	4	10		

p	ℓ	λ_1	Δ		
-	0	0	5	6	7
-	0	4	5	7	8
-	1	2	4	6	7
-	2	0	5	7	8
-	2	2	5	6	7
-	2	4	5	6	7
-	3	0	6	7	8
-	3	2	6	7	8
-	3	4	6	7	8
-	4	0	7	8	9
-	4	2	7	8	9
-	4	4	7	8	9
-	5	0	8	9	10
-	5	2	8	9	10
-	5	4	8	9	10
-	6	0	9	10	
-	6	2	9	10	
-	6	4	9	10	
-	7	0	10		
-	7	2	10		
-	7	4	10		

Table C.1: $SU(2)$ GFV spectrum up to $\Delta = 10$, showing only sectors exchanged in $\langle JJJJ \rangle$. In the parity-even scalar sector, non-exchanged sectors are highlighted in orange.

where $d_{\pm} = d_{K+1,\ell,\pm,\lambda_c,n}$. If $\lambda = \lambda^*$ we solve

$$\begin{cases} a_1 = d_{+,+} + d_{+,-} + d_{-,+} + d_{-,-}, \\ a_2 = (-1)^\ell (d_{+,+} + d_{+,-} - d_{-,+} - d_{-,-}), \\ a_3 = d_{+,+} - d_{+,-} + d_{-,+} - d_{-,-}, \\ a_4 = (-1)^\ell (d_{+,+} - d_{+,-} - d_{-,+} + d_{-,-}), \end{cases} \quad (\text{C.3.16})$$

where $d_{\pm,\pm} = d_{K+1,\ell,\pm,\lambda_\pm,n}$. A consistent solution requires all the degeneracies to be non-negative integers. When $\vec{f}_{K+1}$ vanishes, the new set S'_K will be S_{K+1} , namely we will have determined the spectrum at level $\Delta = K+1$. This strategy can be implemented algorithmically, e.g. in **Mathematica**, for a fixed value of N .

In the $N = 2$ case, the computation is much simpler since we replace $PSU(N) \rtimes \mathbb{Z}_2^C \rightarrow SO(3)$. The group still has two disconnected components due to the action of parity, resulting in a two-dimensional vector of generating functions. The previous discussion can be easily adapted to this simpler case.

We report in Tables C.1 and C.2 the spectrum of the first primary operators in the $SU(2)$ and $SU(3)$ GFV up to $\Delta = 10$, and up to $\ell = 5$ for $SU(3)$. For each parity, spin, and $PSU(N)$ representation, we report the scaling dimensions of up to the first 3 operators. For self-conjugate representations in $SU(3)$, the intrinsic charge conjugation is also reported. For the specific $(p, \ell) = (+, 0)$ sector, and only for this one, we show all operators, highlighting in

p	ℓ	$(\lambda_1, \lambda_2)_{(c,\pm)}$	Δ		
+	0	$(0,0)_+$	0	4	6
+	0	$(0,0)_-$	9	10	
+	0	$(1,1)_+$	4	6	7
+	0	$(1,1)_-$	7	9	10
+	0	$(3,0)_c$	7	8	9
+	0	$(2,2)_+$	4	6	7
+	0	$(2,2)_-$	7	8	9
+	0	$(4,1)_c$	7	8	9
+	0	$(3,3)_+$	8	10	
+	0	$(3,3)_-$	9	10	
+	0	$(6,0)_c$	8	10	
+	0	$(5,2)_c$	10		
+	0	$(4,4)_+$	8	10	
+	1	$(1,1)_-$	2	5	6
+	1	$(3,0)_c$	5	6	7
+	2	$(0,0)_+$	4	6	7
+	2	$(1,1)_+$	4	6	7
+	2	$(2,2)_+$	4	6	7
+	3	$(1,1)_-$	5	6	7
+	3	$(3,0)_c$	5	6	7
+	4	$(0,0)_+$	6	8	9
+	4	$(1,1)_+$	6	7	8
+	4	$(2,2)_+$	6	7	8
+	5	$(1,1)_-$	7	8	9
+	5	$(3,0)_c$	7	8	9

p	ℓ	$(\lambda_1, \lambda_2)_{(c,\pm)}$	Δ		
-	0	$(0,0)_+$	5	6	7
-	0	$(1,1)_+$	5	6	7
-	0	$(2,2)_+$	5	6	7
-	1	$(1,1)_-$	4	6	7
-	1	$(3,0)_c$	4	6	7
-	2	$(0,0)_+$	5	7	8
-	2	$(1,1)_+$	5	6	7
-	2	$(1,1)_-$	5	6	7
-	2	$(3,0)_c$	5	6	7
-	2	$(2,2)_+$	5	6	7
-	3	$(0,0)_+$	6	7	8
-	3	$(1,1)_+$	6	7	8
-	3	$(1,1)_-$	6	7	8
-	3	$(3,0)_c$	6	7	8
-	3	$(2,2)_+$	6	7	8
-	4	$(0,0)_+$	7	8	9
-	4	$(1,1)_+$	7	8	9
-	4	$(1,1)_-$	7	8	9
-	4	$(3,0)_c$	7	8	9
-	4	$(2,2)_+$	7	8	9
-	5	$(0,0)_+$	8	9	10
-	5	$(1,1)_+$	8	9	10
-	5	$(1,1)_-$	8	9	10
-	5	$(3,0)_c$	8	9	10
-	5	$(2,2)_+$	8	9	10

Table C.2: $SU(3)$ GFV spectrum up to $\Delta = 10$ and $\ell = 5$, showing only sectors exchanged in $\langle JJJJ \rangle$. In the parity-even scalar sector, non-exchanged sectors are highlighted in orange.

orange the ones that cannot be exchanged in the correlator $\langle JJJJ \rangle$ in general. These include operators in representations different from R_s and R_a in (5.3.17) and (5.3.18), and those in R_s and R_a which do not satisfy the selection rules reported in Table 5.2. The purpose is to show that the lightest scalars are exchanged in our bootstrap setup. In all other sectors, we display only the operators exchanged in the $\langle JJJJ \rangle$ correlator. We omit the label n denoting the number of constituent currents. For $SU(2)$, $SU(3)$ and $SU(4)$ cases, the full spectrum up to $\Delta = 10$ ($\Delta = 8$ for $SU(4)$), including degeneracies, is available in the ancillary file `GFV-spectrum.nb` of [3].

Appendix D

Details on non-Abelian currents bootstrap

This chapter provides technical details on the numerical bootstrap setup used in Chapter 5.

D.1 C_J normalisation

The relation between the current central charge in the theory of a Free Boson (FB) in the fundamental of $SU(N)$, C_J^{FB} , and the optimal objective of the SDP (5.5.1) is not completely straightforward to obtain. It requires matching the normalisation conventions in (5.4.4) with those of the conformal blocks in `blocks_3d` [419] used in the numerics.

We have thus adopted a pragmatic approach. We can focus the SDP (5.5.1) on FB theory by fixing $\alpha_{JJJ} = 1/2$ and inputting strong assumptions on the spectrum, i.e. imposing that the lightest 10 operators in each sector $(\ell, p, \mathbf{r})$ are isolated and have dimensions as in the FB theory. The primary spectrum of the FB theory can be computed at low scaling dimensions by adapting the character technique described in Appendix C, and then inferring integer gaps between the heavier operators. Then we computed upper and lower bounds on C_J . The result was an interval of values that allowed us to determine C_J^{FB} within a precision of 10^{-3} . We repeated it for several results of N .

Our findings are consistent with the normalisation-independent relation

$$\frac{C_J}{C_J^{\text{FB}}} \geq -\frac{4N}{\sqrt{N^2-1}} \frac{1}{P}, \quad (\text{D.1.1})$$

whose N -dependence can be inferred by comparing (5.4.4) with the symmetry tensor structures (5.4.2) used in (5.4.1).

D.2 Gap maximisation bounds

In what follows we denote by $\vec{0}$ a vector of identical zero components of length

$$\dim(\vec{0}) = (d_{\text{even}} + 4d_{\text{odd}} + 1)r_s + (4d_{\text{even}} + d_{\text{odd}} + 1)r_a, \quad (\text{D.2.1})$$

see (5.4.16) for the definition.

The SDP used to obtain the $SU(2)$ bounds is

$$\begin{aligned}
\text{objective: } & \alpha(\vec{0}) = 0 \\
\text{normalisation: } & \alpha(\vec{\mathcal{V}}_{\mathbb{1}}) \\
\text{conditions: } & \alpha(\vec{\mathcal{V}}_{\Delta,0,+,\mathbf{s}}) \succeq 0 & \Delta \geq \Delta_s \\
& \alpha(\vec{\mathcal{V}}_{\Delta,0,+,\mathbf{s}\bar{\mathbf{s}}}) \succeq 0 & \Delta \geq \Delta_{s\bar{s}} \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 2, 4, 6, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_J) \succeq 0
\end{aligned} \tag{D.2.2}$$

where we imposed a small twist gap $\tau = 10^{-6}$, which improves numerical stability but does not have a noticeable impact on the bounds.

The SDP used to obtain the $SU(3)$ bounds is

$$\begin{aligned}
\text{objective: } & \alpha(\vec{0}) = 0 \\
\text{normalisation: } & \alpha(\vec{\mathcal{V}}_{\mathbb{1}}) \\
\text{conditions: } & \alpha(\vec{\mathcal{V}}_{\Delta,0,+,\mathbf{s}_+}) \succeq 0 & \Delta \geq \Delta_s \\
& \alpha(\vec{\mathcal{V}}_{\Delta,+,0,\mathbf{adj}_+}) \succeq 0 & \Delta \geq \Delta_a \\
& \alpha(\vec{\mathcal{V}}_{\Delta,0,+,\mathbf{s}\bar{\mathbf{s}}_+}) \succeq 0 & \Delta \geq \Delta_{s\bar{s}} \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 2, 4, 6, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 & \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_J) \succeq 0
\end{aligned} \tag{D.2.3}$$

where $\tau = 10^{-6}$.

D.3 Allowed region with strong assumptions

The bound was obtained using the following SDP

$$\begin{aligned}
\text{objective:} \quad & \alpha(\vec{0}) = 0 \\
\text{normalisation:} \quad & \alpha(\vec{\mathcal{V}}_{\mathbb{1}}) \\
\text{conditions:} \quad & \alpha(\vec{\mathcal{V}}_{\Delta_s,0,+, \mathbf{s}_+}) \succeq 0 \\
& \alpha(\vec{\mathcal{V}}_{\Delta_a,+,0, \mathbf{adj}_+}) \succeq 0 \\
& \alpha(\vec{\mathcal{V}}_{\Delta,0,+, \mathbf{s}_+}) \succeq 0 \quad \Delta \geq 6 \\
& \alpha(\vec{\mathcal{V}}_{\Delta,+,0, \mathbf{adj}_+}) \succeq 0 \quad \Delta \geq 6 \\
& \alpha(\vec{\mathcal{V}}_{\Delta,2,+, \mathbf{s}_+}) \succeq 0 \quad \Delta \geq 4 \\
& \alpha(\vec{\mathcal{V}}_{\Delta,0,+, \mathbf{s}\bar{\mathbf{s}}_+}) \succeq 0 \quad \Delta \geq \Delta_{\text{UB}}(0) \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 4, 6, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\text{UB}}(\ell) + \tau \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_J) \succeq 0
\end{aligned} \tag{D.3.1}$$

where $\tau = 10^{-6}$.

D.4 Allowed region with moderate assumptions

The bound was obtained using the following SDP

$$\begin{aligned}
\text{objective:} \quad & \alpha(\vec{0}) = 0 \\
\text{normalisation:} \quad & \alpha(\vec{\mathcal{V}}_{\mathbb{1}}) \\
\text{conditions:} \quad & \alpha(\vec{\mathcal{V}}_{\Delta_s,0,+, \mathbf{s}_+}) \succeq 0 \\
& \alpha(\vec{\mathcal{V}}_{\Delta_a,+,0, \mathbf{adj}_+}) \succeq 0 \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\ell,+, \mathbf{r}}^{\text{gap}} \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 4, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\ell,+, \mathbf{r}}^{\text{gap}} \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\ell,-, \mathbf{r}}^{\text{gap}} \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3, \dots \\
& \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \geq \Delta_{\ell,-, \mathbf{r}}^{\text{gap}} \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3 \dots \\
& \alpha(\vec{\mathcal{V}}_J) \succeq 0
\end{aligned} \tag{D.4.1}$$

with $\Delta_{\ell,p, \mathbf{r}}^{\text{gap}}$ given in Table D.1.

parity	rep	spin	$\Delta_{\ell,p,\mathbf{r}}^{\text{gap}}$	parity	rep	spin	$\Delta_{\ell,p,\mathbf{r}}^{\text{gap}}$
+	$\mathbf{S}_+, \mathbf{adj}_+$	$\ell = 0$	5	−	$\mathbf{R}_s$	$\ell = 0$	3
+	$\mathbf{S}\bar{\mathbf{S}}_+$	$\ell = 0$	3	−	$\mathbf{R}_s$	$\ell = 2$	4
+	$\mathbf{R}_s$	$\ell = 2$	3.5	−	$\mathbf{R}_s$	$\ell = 3$	5
+	$\mathbf{R}_s$	$\ell = 4$	4.5	−	$\mathbf{R}_s$	$\ell = 4$	6
+	$\mathbf{R}_s$	$\ell \geq 6$	$\Delta_{\text{UB}}(\ell) + \tau$	−	$\mathbf{R}_s$	$\ell \geq 5$	$\Delta_{\text{UB}}(\ell) + \tau$
+	$\mathbf{R}_a$	$\ell = 1$	4	−	$\mathbf{R}_a$	$\ell = 1$	3
+	$\mathbf{R}_a$	$\ell = 3$	4.5	−	$\mathbf{R}_a$	$\ell = 2$	4
+	$\mathbf{R}_a$	$\ell = 5$	6.5	−	$\mathbf{R}_a$	$\ell = 3$	5
+	$\mathbf{R}_a$	$\ell \geq 7$	$\Delta_{\text{UB}}(\ell) + \tau$	−	$\mathbf{R}_a$	$\ell = 4$	6
				−	$\mathbf{R}_a$	$\ell \geq 5$	$\Delta_{\text{UB}}(\ell) + \tau$

Table D.1: Gaps imposed in the SDP (D.4.1). Here $\tau = 10^{-6}$.

D.5 Allowed region with parametric assumptions

The bounds in the case of $SU(3)$ are obtained with the following SDP

$$\begin{aligned}
\text{objective:} & \quad \alpha(\vec{\mathcal{V}}_1) \\
\text{normalisation:} & \quad \alpha(\vec{\Sigma}) \\
\text{conditions:} & \quad \alpha(\vec{\mathcal{V}}_{\Delta_s,0,+, \mathbf{S}_+}) \succeq 0 \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta_a,0,+, \mathbf{adj}_+}) \succeq 0 \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta_{s\bar{s}},0,+, \mathbf{S}\bar{\mathbf{S}}_+}) \succeq 0 \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,+, \mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 4, \dots \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+, \mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,+, \mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,-, \mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3, \dots \\
& \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-, \mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,-, \mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3, \dots \\
& \quad \alpha(\theta^T \vec{\mathcal{V}}_J \theta) \succeq 0
\end{aligned} \tag{D.5.1}$$

The normalisation used is the so-called “ Σ -navigator”, see [78, Sec. 2.1.2]. The gap assumptions are defined as

$$\begin{aligned}
\Delta_{0,+, \mathbf{r}}(\delta) &= [\Delta'_{0,+, \mathbf{r}} - \delta, \infty), \\
\Delta_{0,-, \mathbf{r}}(\delta) &= [\Delta_{0,-, \mathbf{r}} - \delta, \Delta_{0,-, \mathbf{r}} - \delta] \cup [\Delta'_{0,-, \mathbf{r}} - \delta, \infty), \\
\Delta_{\ell,p, \mathbf{r}}(\delta) &= \left[\Delta_{\ell,p, \mathbf{r}} - \frac{\delta}{\ell}, \Delta_{\ell,p, \mathbf{r}} + \frac{\delta}{\ell} \right] \cup [\Delta'_{\ell,p, \mathbf{r}} - \delta, \infty), \quad \ell \geq 1, \mathbf{r} \neq \mathbf{S}\bar{\mathbf{A}}, \\
\Delta_{\ell,p, \mathbf{S}\bar{\mathbf{A}}_c}(\delta) &= \left[\Delta_{\ell,p, \mathbf{S}\bar{\mathbf{A}}_c} - \frac{\delta}{\ell^2}, \Delta_{\mathbf{S}\bar{\mathbf{A}}_c, p, \ell} + \frac{\delta}{\ell^2} \right] \cup [\Delta'_{\ell,p, \mathbf{S}\bar{\mathbf{A}}_c} - \delta, \infty), \quad \ell \geq 1,
\end{aligned} \tag{D.5.2}$$

where $\Delta_{\ell,p,\mathbf{r}}$ and $\Delta'_{\ell,p,\mathbf{r}}$ are the dimensions of the leading and sub-leading operators in the GFV in the $(\ell, p, \mathbf{r})$ sector.¹ These are

$$\begin{aligned} \Delta_{\ell,+,\mathbf{r}} &= 2 + \ell, & \ell > 4, \\ \Delta'_{\ell,+,\mathbf{r}} &= 3 + \ell, & \ell > 4, \\ \Delta_{\ell,-,\mathbf{r}} &= 3 + \ell, & \ell > 2, \\ \Delta'_{\ell,-,\mathbf{r}} &= 4 + \ell, & \ell > 2, \end{aligned} \tag{D.5.3}$$

as discussed in Section (5.3.2). For $\ell \leq 4$, in the case of $SU(3)$ we have

$\Delta_{\ell,+,\mathbf{r}} = 4$	$\ell = 0, 2$ and $\mathbf{r} \in \mathbf{R}_s$	$\Delta_{0,-,\mathbf{r}} = 5$	$\mathbf{r} \in \mathbf{R}_s$
$\Delta'_{\ell,+,\mathbf{r}} = 6$	$\ell = 0, 2$ and $\mathbf{r} \in \mathbf{R}_s$	$\Delta'_{0,-,\mathbf{r}} = 6$	$\mathbf{r} \in \mathbf{R}_s$
$\Delta_{\ell,+,\mathbf{r}} = 5$	$\ell = 1, 3$ and $\mathbf{r} \in \mathbf{R}_a$	$\Delta_{1,-,\mathbf{r}} = 4$	$\mathbf{r} \in \mathbf{R}_a$
$\Delta'_{\ell,+,\mathbf{r}} = 6$	$\ell = 1, 3$ and $\mathbf{r} \in \mathbf{R}_a$	$\Delta'_{1,-,\mathbf{r}} = 6$	$\mathbf{r} \in \mathbf{R}_a$
$\Delta_{4,+,\mathbf{r}} = 6$	$\mathbf{r} \in \mathbf{R}_s$	$\Delta'_{2,-,\mathbf{r}} = 5$	$\mathbf{r} \in \mathbf{R}_s, \mathbf{R}_a$
$\Delta'_{4,+,\mathbf{r}} = 7$	$\mathbf{r} \in \{\mathbf{adj}_+, \mathbf{S}\bar{\mathbf{S}}_+\}$	$\Delta'_{2,-,\mathbf{r}} = 6$	$\mathbf{r} \in \{\mathbf{adj}_+, \mathbf{S}\bar{\mathbf{S}}_+\}, \mathbf{R}_a$
$\Delta'_{4,+,\mathbf{S}_+} = 8$		$\Delta_{2,-,\mathbf{S}_+} = 7$	

(D.5.4)

See Table C.2 and the ancillary file `GFV-spectrum.nb` of [3].

In the $SU(2)$ case, we have a similar SDP

$$\begin{aligned} \text{objective:} & \quad \alpha(\vec{\mathcal{V}}_1) \\ \text{normalisation:} & \quad \alpha(\vec{\Sigma}) \\ \text{conditions:} & \quad \alpha(\vec{\mathcal{V}}_{\Delta_s,0,+,\mathbf{S}}) \succeq 0 \\ & \quad \alpha(\vec{\mathcal{V}}_{\Delta_{s\bar{s}},0,+,\mathbf{S}\bar{\mathbf{S}}}) \succeq 0 \\ & \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,+,\mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 4, \dots \\ & \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,+,\mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,+,\mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 3, 5, \dots \\ & \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,-,\mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_s \quad \ell = 0, 2, 3, \dots \\ & \quad \alpha(\vec{\mathcal{V}}_{\Delta,\ell,-,\mathbf{r}}) \succeq 0 \quad \Delta \in \Delta_{\ell,-,\mathbf{r}}(\delta) \quad \mathbf{r} \in \mathbf{R}_a \quad \ell = 1, 2, 3 \dots \\ & \quad \alpha(\theta^T \vec{\mathcal{V}}_J \theta) \succeq 0 \end{aligned} \tag{D.5.5}$$

where $\Delta_{\ell,p,\mathbf{r}}$ are as in (D.5.3) and (D.5.4), except for the absence of the $\mathbf{adj}_+$ and $\mathbf{S}\bar{\mathbf{A}}_c$ representations, and the following change in the $(\ell, p) = (4, +)$ sector

$$\begin{aligned} \Delta_{4,+,\mathbf{r}} &= 6, & \mathbf{r} \in \mathbf{R}_s, \\ \Delta'_{4,+,\mathbf{r}} &= 8, & \mathbf{r} \in \mathbf{R}_s. \end{aligned} \tag{D.5.6}$$

¹Ignoring the identity and the current J itself.

Λ	11	15	19	23
spins	S_{11}	S_{15}	S_{19}	S_{23}
keptPoleOrder	11	14	14	18
order	30	56	56	72

Table D.2: Parameters used for `blocks_3d`, with S_Λ given in (D.6.1).

D.6 Details on the numerical implementation

The numerical bootstrap computations of this chapter were carried out using the `simpleboot` package [231], based on the code used in [101]. The computation of the conformal blocks was done using `blocks_3d` [419]. The semi-definite optimisation was done using SDPB [73, 74].

For the computations of Sections 5.5 and 5.6.4, we used a slightly modified version of SDPB v3 which saves a “middle checkpoint” at a finite value of the μ parameter and recovers from a “stalling” phase by aggressively increasing the β parameter (see [73, Sec. 2.4] for the definition of these parameters). This modified version was previously implemented for SDPB v2.4 by Ning Su, available at https://github.com/suning-git/sdpb/tree/sdpb2.4.0_midck_stallingrecover, and adapted to SDPB v3 by one of the authors for the present chapter, and is made available at <https://gitlab.com/apiazza134/sdpb-midck-stallingrecover>.

In our experience, SDPB v3 provided a significant speed-up in bootstrap computations compared to SDPB v2.x, offering a much better scalability across multiple nodes. As a benchmark, a single SDPB step for a computation of Section 5.6.4 took on average 40s with v3 and 300s with v2.4 (using 6 nodes, 40 cpus and 40GB each, on the SISSA Ulysses cluster). On the other hand, when SDPB is run in optimisation mode, as needed for Navigator computations, it is usually impossible to “hot start” a new optimisation from a previous checkpoint. The “middle checkpoint” and “stalling recover” version of SDPB v2.4 was designed to overcome this problem. We would thus like to stress the importance of using this modified version of SDPB v3 for the computations presented in Section 5.6.4.

The parameters used for the conformal blocks computations are reported in Table D.2, where S_Λ denotes the set of spins included in the positivity conditions:

$$\begin{aligned}
S_{11} &= \{0, \dots, 24\} \cup \{49, 50\}, \\
S_{15} &= \{0, \dots, 26\} \cup \{49, 50\}, \\
S_{19} &= \{0, \dots, 26\} \cup \{49, 50\}, \\
S_{23} &= \{0, \dots, 26\} \cup \{29, 30, 33, 34, 37, 38, 41, 42, 45, 46, 49, 50\}.
\end{aligned}
\tag{D.6.1}$$

Appendix E

Analyticity of partial-wave expansions

We report in this appendix some details which aim at clarifying the analyticity properties of partial wave expansions. We first explain why and how ellipses arise in complete generality. We then discuss the small and large Lehmann ellipsis for the cases of interest in Chapter 6, the $AB \rightarrow AB$ and $AA \rightarrow BB$ scattering amplitudes. Finally, we show how the pseudo-physical region is responsible for the appearance of a circular cut in the partial waves for $AB \rightarrow AB$ scattering.

E.1 The origin of Lehmann ellipses

The origin of ellipses is a mathematical fact, entirely determined by the behaviour of the partial wave expansion at large spin. Any function $f(x) \in L^2$ on the interval $x \in [-1, 1]$ can be expanded in partial waves as

$$f(x) = \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}(x), \quad (\text{E.1.1})$$

where

$$a_{\ell} = \frac{1+2\ell}{2} \int_{-1}^1 dx f(x) P_{\ell}(x), \quad (\text{E.1.2})$$

and $P_{\ell}(x)$ are the Legendre polynomials. On $x \in [-1, 1]$, the expansion (E.1.1) is convergent. Consider now the analytic continuation of f over some domain and let us determine in which region $z \in E \supset [-1, 1]$ the partial wave expansion (E.1.1) applies. Evidently, this requires to know the asymptotic behaviour of $P_{\ell}(z)$ for $\ell \gg 1$. An easy way to determine it is to use a saddle point approximation starting from the integral representation

$$P_{\ell}(z) = \frac{1}{\pi} \int_0^{\pi} d\alpha \left(z + \sqrt{z^2 - 1} \cos \alpha \right)^{\ell}, \quad (\text{E.1.3})$$

valid for any integer ℓ . For $z \notin [-1, 1]$, the only saddle contributing to the integral is the one at $\alpha = 0$. Straightforward saddle-point methods give

$$P_\ell(z) \approx \frac{\Phi(z)^{\ell+\frac{1}{2}}}{\sqrt{2\pi\ell}(z^2-1)^{\frac{1}{4}}} \left(1 + \mathcal{O}(\ell^{-1})\right), \quad \ell \gg 1, \quad z \notin [-1, 1], \quad (\text{E.1.4})$$

where

$$\Phi(z) \equiv z + \sqrt{z^2 - 1}. \quad (\text{E.1.5})$$

When $z \in [-1, 1]$, the relation (E.1.3) still applies, with $z + \sqrt{z^2 - 1} \cos \alpha \rightarrow z + i\sqrt{1 - z^2} \cos \alpha$. The function Φ becomes a phase:

$$\Phi(z) = e^{i\theta}, \quad z = \cos \theta, \quad (\text{E.1.6})$$

and now both saddle points at $\alpha = 0$ and at $\alpha = \pi$ contribute. Paying attention to rotate the contour to reach the steepest-descent trajectory, eventually we get

$$P_\ell(z = \cos \theta) \approx \sqrt{\frac{2}{\pi\ell \sin \theta}} \cos \left[\left(\ell + \frac{1}{2}\right)\theta - \frac{\pi}{4} \right] \left(1 + \mathcal{O}(\ell^{-1})\right), \quad \ell \gg 1, \quad z \in [-1, 1]. \quad (\text{E.1.7})$$

Coming back to our analytic continuation, for large ℓ , we have

$$f(z) \sim \sum_{\ell} a_{\ell} \Phi(z)^{\ell}. \quad (\text{E.1.8})$$

In order to map this expression to an ordinary Taylor expansion, we perform the conformal map

$$w = \Phi(z) \longrightarrow z = \frac{1}{2} \left(w + \frac{1}{w} \right). \quad (\text{E.1.9})$$

The domain of convergence of (E.1.8) is determined by the behaviour of a_{ℓ} or, equivalently, from the position of the singularity of f closest to the origin. Let z_s be such singularity. Then the domain of convergence is given by a disk in the w -plane of radius

$$R = |w(z_s)|. \quad (\text{E.1.10})$$

This disk is mapped to an ellipse in the z -plane. Indeed, if $w = Re^{i\phi}$, we have

$$z = x + iy = a \cos \phi + ib \sin \phi \longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (\text{E.1.11})$$

with

$$a = \frac{1}{2}(R + R^{-1}), \quad b = \frac{1}{2}(R - R^{-1}). \quad (\text{E.1.12})$$

So, the partial wave expansion converges uniformly in the z -plane within an ellipse E with foci at $(-1, 0)$ and $(1, 0)$, and major and minor axis given by $2a$ and $2b$, respectively. See Figure E.1 for an illustration. We also have that

$$a_{\ell} \sim w(z_s)^{-\ell}, \quad \ell \gg 1. \quad (\text{E.1.13})$$

For real z_s , the semi-major axis of the ellipse in the z -plane corresponds directly to $z_s = a$. When $z > 1$, we can set $z = \cosh \alpha$, so that $\Phi(z(\alpha)) = e^{\alpha}$, and $P_{\ell \gg 1} \approx e^{\ell \alpha}$.

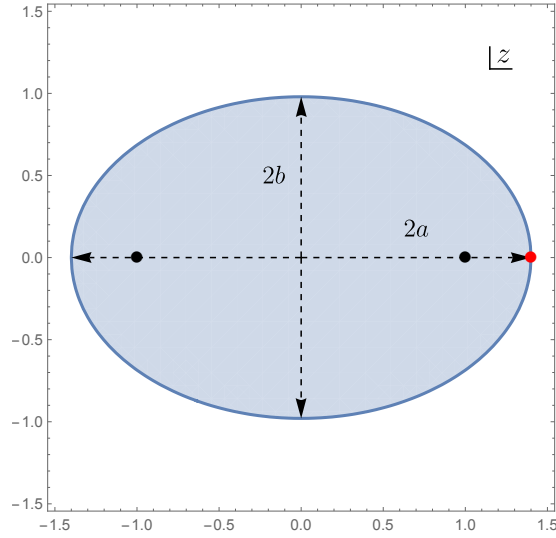


Figure E.1: Ellipse of convergence of the partial wave expansion in the z -plane for a function with a singularity at z_s (red bullett). The two black bullets denote the two foci at $(-1, 0)$ and $(1, 0)$. We have taken $z_s = 1.4$ in the picture.

E.2 Small Lehmann ellipses

In the context of scattering amplitudes, the function $f(z)$ introduced in Section E.1 can be identified with the amplitude $T(s, t(s, z), u(s, z))$ at fixed s , with $t(s, z)$ as given in (6.2.12) for $AB \rightarrow AB$ and (6.2.15) for $AA \rightarrow BB$. The region of convergence of the partial wave expansion of T determines what is called the small Lehmann ellipse (SL). As we have seen, the size of the semi-major axis of the ellipse is determined by the singularity closest to the origin in the z -plane, z_s^{SL} , at given $s > s_{\text{th}}$. For scattering amplitudes, there are two natural candidates, namely the first singularities in the t and u channel, respectively, given by $t = t_{\text{th}}$ and $u = u_{\text{th}}$. They respectively give $z_s(t_{\text{th}}) > 1$ and $z_s(u_{\text{th}}) < -1$. For identical particles, $t - u$ crossing implies $z_s^{\text{SL}} = |z_s(t_{\text{th}})| = |z_s(u_{\text{th}})|$, while for unequal particles one has

$$z_s^{\text{SL}} = \min(|z(t = t_{\text{th}}, s)|, |z(u = u_{\text{th}}, s)|). \quad (\text{E.2.1})$$

For the $AB \rightarrow AB$ scattering amplitude, $t_{\text{th}} = 4m^2$, $u_{\text{th}} = m_+^2$, and it is easily verified that, for any $s > s_{\text{th}} = m_+^2$, the singularity closest to the origin is the one given by $t = 4m^2$. Using (6.2.10), we find

$$z_s^{\text{SL}} = z(t = t_{\text{th}}, s) = 1 + \frac{8sm^2}{(s - m_+^2)(s - m_-^2)}, \quad AB \rightarrow AB. \quad (\text{E.2.2})$$

For $AA \rightarrow BB$ scattering, $t - u$ crossing fixes $u_{\text{th}} = t_{\text{th}} = m_+^2$, and from (6.2.15) we get

$$z_s^{\text{SL}} = \frac{4mM + s}{\sqrt{(s - 4m^2)(s - 4M^2)}}, \quad AA \rightarrow BB. \quad (\text{E.2.3})$$

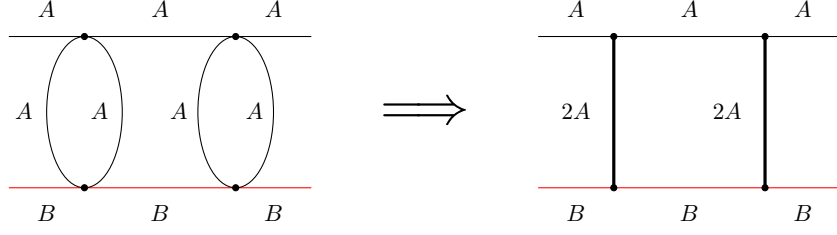


Figure E.2: Example of leading Landau diagram contributing to the double discontinuity of the $AB \rightarrow AB$ amplitude (left) and its reduced box diagram (right). Like in Figure 6.2, the thick vertical lines represent effective propagators for a state of $2A$ particles at threshold.

E.3 Double discontinuities and large Lehmann ellipses

When the function f in Section E.1 is identified with the s -discontinuity of the amplitude, the convergence of the partial wave expansion defines the so called large Lehmann ellipse (LL). Its boundary coincides with the appearance of double discontinuities in the amplitude. We define the double $s - t$ discontinuity of an amplitude $T(s, t, u)$ as

$$\begin{aligned} \text{dDisc}_{st} T &\equiv -\frac{1}{4} \lim_{\epsilon \rightarrow 0} \left(T(s + i\epsilon, t + i\epsilon) - T(s - i\epsilon, t + i\epsilon) - T(s + i\epsilon, t - i\epsilon) + T(s - i\epsilon, t - i\epsilon) \right) \\ &= \text{Disc}_t \text{Disc}_s T(s, t) = \text{Disc}_s \text{Disc}_t T(s, t) = \text{dDisc}_{ts} T, \end{aligned} \quad (\text{E.3.1})$$

where $T(s, t) = T(s, t, u(s, t))$. Similarly the double $s - u$ discontinuity is defined as

$$\begin{aligned} \text{dDisc}_{su} \tilde{T} &\equiv -\frac{1}{4} \lim_{\epsilon \rightarrow 0} \left(\tilde{T}(s + i\epsilon, u + i\epsilon) - \tilde{T}(s - i\epsilon, u + i\epsilon) - \tilde{T}(s + i\epsilon, u - i\epsilon) + \tilde{T}(s - i\epsilon, u - i\epsilon) \right) \\ &= \text{Disc}_u \text{Disc}_s \tilde{T}(s, u) = \text{Disc}_s \text{Disc}_u \tilde{T}(s, u) = \text{dDisc}_{us} \tilde{T}, \end{aligned} \quad (\text{E.3.2})$$

where $\tilde{T}(s, u) = T(s, t(s, u), u)$. We also have a double $t - u$ discontinuity dDisc_{tu} , but this vanishes for $s > s_{\text{th}}$ and will not be considered in what follows.

The onset of the double discontinuity can be detected by considering Landau diagrams, see e.g. [372] for the specific case of pion-kaon scattering. Similarly to the analysis of triangular diagrams made in Section 6.3, the leading Landau diagrams associated to dDisc can be reduced to effective box diagrams, as illustrated in Figure E.2. Like for the small Lehmann ellipse, in absence of $t - u$ crossing symmetry ($AB \rightarrow AB$), we should check if the singularity closest to the origin in the z -plane, z_s^{LL} , is given by dDisc_{st} or dDisc_{su} . For any $s > m_+^2$, one finds that dDisc_{st} dominates and eventually

$$z_s^{\text{LL}} = 1 + \frac{2s t^{\text{LL}}(s)}{(s - m_+^2)(s - m_-^2)}, \quad AB \rightarrow AB, \quad (\text{E.3.3})$$

with

$$t^{\text{LL}} = \min(t_1, t_2) = \begin{cases} t_1 & s \leq s_0 \\ t_2 & s \geq s_0 \end{cases}. \quad (\text{E.3.4})$$

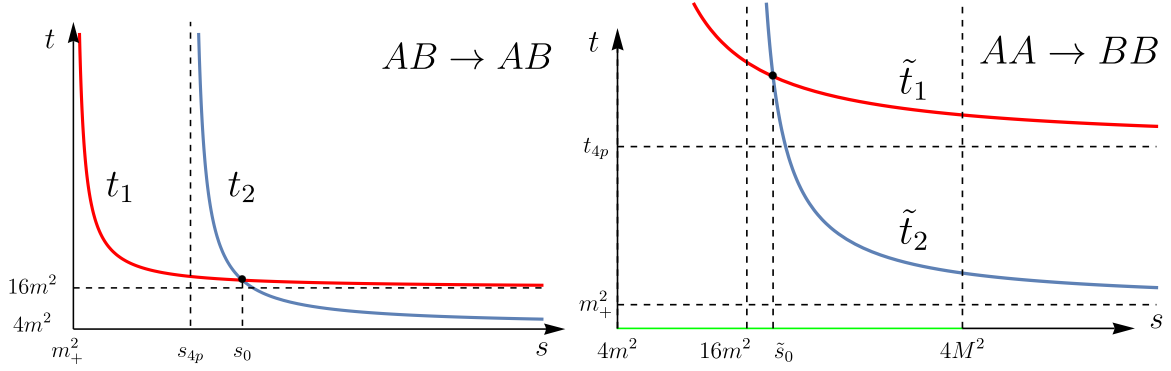


Figure E.3: Onset of the double discontinuity in the real (s, t) plane. (left) The two curves intersect at s_0 , reported in (E.3.6). The point $s_{4p} = (3m + M)^2$ is the threshold for 4-particle production. (right) The two curves intersect at $\tilde{s}_0$, reported in (E.3.9), while $t_{4p} = s_{4p}$. The green line represents the pseudo-physical region. The plots shown correspond to $M = 3m$.

In (E.3.4),

$$\begin{aligned} t_1(s) &= 16m^2 + \frac{64m^4 s}{(s - m_+^2)(s - m_-^2)}, & s > m_+^2, \\ t_2(s) &= 4m^2 + \frac{32m^3(M + m)}{s - (M + 3m)^2}, & s > s_1 = (3m + M)^2, \end{aligned} \quad (\text{E.3.5})$$

are the leading Karplus curves, and

$$s_0 = M^2 + 4Mm + 5m^2 + 2m\sqrt{5M^2 + 12Mm + 8m^2} > s_1. \quad (\text{E.3.6})$$

As expected, for $M = m$, (E.3.5) reduces to the leading Karplus curves for identical particles, see e.g. [107].

For $AA \rightarrow BB$ scattering, $\text{dDisc}_{st} = \text{dDisc}_{su}$. The leading singularity z_s^{LL} is detected by considering the same Landau diagrams relevant for $AB \rightarrow AB$ scattering, but with $s \leftrightarrow t$. The final result is that, for $s > 4M^2$,

$$z_s^{\text{LL}} = \frac{2t^{\text{LL}}(s) + s - 2m^2 - 2M^2}{\sqrt{(s - 4m^2)(s - 4M^2)}}, \quad AA \rightarrow BB, \quad (\text{E.3.7})$$

where

$$t^{\text{LL}} = \min(\tilde{t}_1, \tilde{t}_2) = \begin{cases} \tilde{t}_1 & s \leq \tilde{s}_0 \\ \tilde{t}_2 & s \geq \tilde{s}_0 \end{cases}, \quad (\text{E.3.8})$$

with

$$\begin{aligned} \tilde{t}_1(s) &= (3m + M)^2 + \frac{32m^3(M + m)}{s - 4m^2}, & s > 4m^2, \\ \tilde{t}_2(s) &= \frac{16m^4 + M^2 s + m^2(s - 16M^2) + 2m\sqrt{s(sM^2 - 16m^2M^2 + 16m^4)}}{s - 16m^2}, & s > 16m^2, \\ \tilde{s}_0 &= \frac{4m^2}{m + M} \left(3m + 2M + \sqrt{5M^2 + 12Mm + 8m^2} \right) > 16m^2. \end{aligned} \quad (\text{E.3.9})$$

The leading Karplus curves for identical particles are again reproduced for $M = m$, as expected.

We report in Figure E.3 the leading Karplus curves beyond which dDisc_{st} is non-vanishing for both $AB \rightarrow AB$ and $AA \rightarrow BB$ scattering amplitudes. The picture shows the case $M = 3m > 2m$. While the detailed form of the curves depends on μ , their qualitative behaviour applies for any μ . In contrast, the higher Karplus curves depend sensitively on μ .

E.4 Circular cut of the $AB \rightarrow AB$ partial waves

One of the consequences of the pseudo-physical region is to give rise to a circular branch-cut singularity in the complex s -plane for the partial waves $f_{AB \rightarrow AB}^\ell(s)$. The emergence of this cut can be seen from (6.2.10), rewritten as

$$t = \frac{z-1}{2s}(s-m_+^2)(s-m_-^2). \quad (\text{E.4.1})$$

Partial waves are obtained by integrating over $z \in [-1, 1]$, namely from $t(-1)$ to $t(1) = 0$. As s is varied, it can happen that the integration over z hits a value of t where the amplitude is non-analytic. The first singularity occurs at $t_{\text{th}} = 4m^2$. Since t_{th} is real, we can first determine for which complex values of s , $t(-1)$ is real. Setting $s = r \exp(i\omega)$ and demanding $\text{Im } z(-1) = 0$ gives

$$\sin \omega \left(r - \frac{(M^2 - m^2)^2}{r} \right) = 0. \quad (\text{E.4.2})$$

This is satisfied either for $\omega = 0, \pi$, corresponding to real s , or $r = M^2 - m^2$ and any ω . The second option corresponds to the circular cut. Along it, we have

$$t(z) = (z-1) \left[(M^2 - m^2) \cos \omega - (M^2 + m^2) \right]. \quad (\text{E.4.3})$$

At $z = -1$, as ω varies from 0 to π , $t(-1)$ ranges from $4m^2$ to $4M^2$, precisely the pseudo-physical region, signalling that indeed the integration over $z \in [-1, 1]$ hits the singularity along this circle. So, partial waves $f_{AB \rightarrow AB}^\ell(s)$, in addition to the usual physical branch-cuts over the real axis, exhibit non-analyticities along a circular cut encircling the origin with radius

$$R_{\text{cut}} = M^2 - m^2. \quad (\text{E.4.4})$$

For equal particles the circle shrinks to the point $s = 0$. In our numerical study, partial waves are evaluated for physical values of s , so away from this circular cut. However, in arguments based on dispersion relations, or for extrapolating the form of extremal partial waves in the complex s -plane, care has to be paid to it.

Appendix F

Dispersion relations and positivity bounds

In this appendix we derive dispersive representations for some of the non-perturbative couplings $\lambda_{k,l}$ defined in Section 6.4 and obtain bounds using positivity techniques. In this context we follow the strategy outlined in [110], properly generalised for unequal particles. In contrast to [110], however, we do not rely on a Lagrangian description for the definition of the couplings, but rather use the non-perturbative definition (6.4.1), as done in [106]. The bounds so obtained are then exact and not restricted to tree-level amplitudes. Positivity bounds on scattering of non-identical scalars have also been explored in [366], but considering the full system of two-to-two scattering amplitudes and restricting to particular ranges of mass ratios. We will instead present bounds that use only the $AB \rightarrow AB$ amplitude but are valid for any mass ratio.

Dispersion relation for $AB \rightarrow AB$. We are interested in writing fixed- t dispersion relations in the complex s -plane for the amplitude $T(s, t)$ in (6.4.6). The starting point is the Cauchy's integral with two subtractions

$$\oint_{\gamma} \frac{ds'}{2\pi i} \frac{T(s', t)}{\prod_{i=1}^3 (s' - z_i)} = 0, \quad (\text{F.0.1})$$

where γ is an arbitrary closed contour around a generic complex point in the s -plane in Figure 6.3a, where T is analytic, and away from the subtraction points z_i , that we will fix momentarily. Blowing up the contour, we can neglect the arc at infinity thanks to the Froissart bound and the two subtractions. We then pick up the residues at $s' = z_i$ as well as the Hankel contours on the branch cuts. We obtain

$$\sum_{i=1}^3 \text{Res}_{s'=z_i} \frac{T(s', t)}{\prod_{j=1}^3 (s' - z_j)} = \int_{-\infty}^{m_-^2 - t} \frac{ds'}{2\pi i} K(s') \text{Disc}_s T(s', t) + \int_{m_+^2}^{+\infty} \frac{ds'}{2\pi i} K(s') \text{Disc}_s T(s', t), \quad (\text{F.0.2})$$

where

$$\text{Disc}_s T(s, t) = \lim_{\epsilon \rightarrow 0} (T(s + i\epsilon, t) - T(s - i\epsilon, t)) = 2i \text{Im } T(s, t) \quad (\text{F.0.3})$$

by real analyticity and K is the kernel

$$K(s') = \frac{1}{(s' - z_1)(s' - z_2)(s' - z_3)}. \quad (\text{F.0.4})$$

Using $s - u$ crossing, $T(s, t) = T(\Sigma - s - t, t)$, we can combine the first integral with the second one after changing variable $s' \rightarrow \Sigma - s' - t$. After simple manipulations, we get

$$\sum_{i=1}^3 \text{Res}_{s'=z_i} \frac{T(s', t)}{\prod_{j=1}^3 (s' - z_j)} = \int_{m_+^2}^{+\infty} \frac{ds'}{\pi} (K(s') - K(\Sigma - s' - t)) \text{Im} T(s', t). \quad (\text{F.0.5})$$

Dispersion relation for $AA \rightarrow BB$. We apply the same logic of the previous dispersion relation to the $AA \rightarrow BB$ amplitude

$$G(s, t) \equiv T_{AA \rightarrow BB}(s, t, \Sigma - s - t) = T(t, s). \quad (\text{F.0.6})$$

We write a fixed- t Cauchy's integral with two subtractions in the complex s -plane (which is the same as the complex t -plane for $T(s, t)$ in Figure 6.3b). We land on an identity similar to (F.0.2)

$$\sum_{i=1}^3 \text{Res}_{s'=w_i} \frac{G(s', t)}{\prod_{j=1}^3 (s' - w_j)} = \int_{-\infty}^{m_-^2 - t} \frac{ds'}{2\pi i} \tilde{K}(s') \text{Disc}_s G(s', t) + \int_{4m^2}^{+\infty} \frac{ds'}{2\pi i} \tilde{K}(s') \text{Disc}_s G(s', t) \quad (\text{F.0.7})$$

where $\tilde{K}(s')$ is the same as in (F.0.4) with $z_i \rightarrow w_i$. The discontinuity on the left-hand cut of the $AA \rightarrow BB$ amplitude can be mapped to the discontinuity on the right-hand cut of the $AB \rightarrow AB$ amplitude by combining $t - u$ and $s - t$ crossing

$$\text{Disc}_s G(s, t) = -\text{Disc}_s T(\Sigma - s - t, t), \quad s \geq 4m^2. \quad (\text{F.0.8})$$

After a change of variable $s' \rightarrow \Sigma - s' - t$ in the first integral we obtain

$$\sum_{i=1}^3 \text{Res}_{s'=w_i} \frac{G(s', t)}{\prod_{j=1}^3 (s' - w_j)} = \int_{4m^2}^{+\infty} \frac{ds'}{\pi} \tilde{K}(s') \text{Im} G(s', t) - \int_{m_+^2}^{+\infty} \frac{ds'}{\pi} \tilde{K}(\Sigma - s' - t) \text{Im} T(s', t). \quad (\text{F.0.9})$$

By $s - t$ crossing this can be equivalently written as a fixed- s dispersion relation for $T(s, t)$

$$\sum_{i=1}^3 \text{Res}_{t'=w_i} \frac{T(s, t')}{\prod_{j=1}^3 (t' - w_j)} = \int_{4m^2}^{+\infty} \frac{dt'}{\pi} \tilde{K}(t') \text{Im} T(s, t') - \int_{m_+^2}^{+\infty} \frac{ds'}{\pi} \tilde{K}(\Sigma - s' - s) \text{Im} T(s', s). \quad (\text{F.0.10})$$

Dispersive formulas for the couplings. By appropriately choosing $z_i = z_i(s, t)$ and $w_i = w_i(s, t)$ to be linear functions and by Taylor expanding (F.0.5) and (F.0.10) around the point (s_0, t_0) , we can obtain dispersive representations for the couplings $\lambda_{k,l}$ defined in (6.4.1). When taking the residues and Taylor expanding at $(s, t) = (s_0, t_0)$ the left-hand side of (F.0.5), we generate derivatives of $T(s, t)$ at the point $(s, t) = (z_i(s_0, t_0), t_0)$. Similarly, taking residues and Taylor expanding the left-hand side of (F.0.10), we generate derivatives of $T(s, t)$ at the

point $(s, t) = (s_0, w_i(s_0, t_0))$. In order to extract the couplings $\lambda_{k,l}$ we then want to choose $z_i(s_0, t_0) = s_0$ and $w_i(s_0, t_0) = t_0$. A convenient choice, that does not produce accidental cancellations and respects $s - t$ symmetry, is the following

$$\begin{cases} z_1 = s_0 \\ z_2 = s_0 + \hat{s} + \hat{t} = s + t - t_0 \\ z_3 = s_0 + \hat{s} - \hat{t} = s - t + t_0 \end{cases} \quad \begin{cases} w_1 = t_0 \\ w_2 = t_0 + \hat{s} + \hat{t} = s + t - s_0 \\ w_3 = t_0 - \hat{s} + \hat{t} = t - s + s_0 \end{cases} \quad (\text{F.0.11})$$

where we have defined shifted Mandelstam variables

$$\hat{s} = s - s_0, \quad \hat{t} = t - t_0. \quad (\text{F.0.12})$$

Let us motivate the choice for s_0 in (6.4.2). For the dispersion relations to be valid, the points z_i and w_i must be away from branch cut singularities; moreover we want the branch cuts in Figure 6.3 to not overlap. Working locally around $(s, t) \sim (s_0, t_0)$ the conditions are

$$m_-^2 - t_0 < s_0 < m_+^2, \quad m_-^2 - s_0 < t_0 < 4m^2. \quad (\text{F.0.13})$$

Fixing $t_0 = \Sigma - 2s_0$ to preserve $s - u$ symmetry, and using $m \leq M$ in the previous constraint, we obtain

$$M^2 - m^2 < s_0 < m_+^2. \quad (\text{F.0.14})$$

Looking for an expression for s_0 linear in m^2 and M^2 , that reduces to crossing symmetric point $4m^2/3$ in the case of equal masses, fixes $s_0 = 4m^2/3 + b(M^2 - m^2)$. The constraint (F.0.14) is then solved for any value of the masses by $b = 1$, which reduces s_0 to (6.4.2).

Let us now go back to (F.0.5) and (F.0.10), plug in the subtraction point (F.0.11) and expand in $(\hat{s}, \hat{t})$ the left-hand side (LHS). At first order in the expansion, we find

$$\text{LHS (F.0.5)} = \lambda_{2,0} + \lambda_{2,1}\hat{t} + 3\lambda_{4,0}\hat{s}^2 + (\lambda_{2,2} + \lambda_{4,0})\hat{t}^2 + 4\lambda_{4,0}\hat{s}\hat{t} + \dots$$

$$\text{LHS (F.0.10)} = \lambda_{0,2} + \lambda_{2,1}\hat{s} + 2\lambda_{0,3}\hat{t} + (\lambda_{0,4} + \lambda_{2,2})\hat{s}^2 + 3\lambda_{0,4}\hat{t}^2 + (2\lambda_{2,2} - 2\lambda_{4,0})\hat{s}\hat{t} + \dots \quad (\text{F.0.15})$$

Note that we have set $m = 1$ in the previous formulas. In writing this expansion we have already accounted for the constraints from $s - u$ crossing symmetry on the couplings $\lambda_{k,l}$.

Performing now the same expansion on the right-hand side (RHS) of (F.0.5) and (F.0.10) we obtain

$$\begin{aligned} \text{RHS (F.0.5)} &= \int_{m_+^2}^{\infty} \frac{ds'}{\pi} \left[\frac{2 \text{Im } T(s', t_0)}{(s' - s_0)^3} + \left(\frac{2\partial_t \text{Im } T(s', t_0)}{(s' - s_0)^3} - \frac{3 \text{Im } T(s', t_0)}{(s' - s_0)^4} \right) \hat{t} + \dots \right], \\ \text{RHS (F.0.10)} &= \int_{4m^2}^{\infty} \frac{dt'}{\pi} \left[\frac{\text{Im } T(s_0, t')}{(t' - t_0)^3} + \frac{\partial_s \text{Im } T(s_0, t')}{(t' - t_0)^3} \hat{s} + \frac{2 \text{Im } T(s_0, t')}{(t' - t_0)^4} \hat{t} + \dots \right] \\ &\quad + \int_{m_+^2}^{\infty} \frac{ds'}{\pi} \left[\frac{\text{Im } T(s', s_0)}{(s' - s_0)} + \left(\frac{\partial_t \text{Im } T(s', s_0)}{(s' - s_0)^3} - \frac{3 \text{Im } T(s', s_0)}{(s' - s_0)^4} \right) \hat{s} - \frac{2 \text{Im } T(s', s_0)}{(s' - s_0)^4} \hat{t} + \dots \right]. \end{aligned} \quad (\text{F.0.16})$$

For simplicity, we have reported only the first order of the expansion. We remark that in these formulas it is important to use the relation $\Sigma = 2s_0 + t_0$ to obtain simple expressions.

By matching the two sides we find dispersive representations for the couplings. For example, at first order in $(\hat{s}, \hat{t})$, we obtain

$$\begin{aligned}
\lambda_{2,0} &= \int_{m_+^2}^{\infty} \frac{ds}{\pi} \frac{2 \operatorname{Im} T(s, t_0)}{(s - s_0)^3}, \\
\lambda_{0,2} &= \int_{4m^2}^{\infty} \frac{dt}{\pi} \frac{\operatorname{Im} T(s_0, t)}{(t - t_0)^3} - \int_{m_+^2}^{\infty} \frac{ds}{\pi} \frac{\operatorname{Im} T(s, s_0)}{(s - s_0)^3}, \\
\lambda_{0,3} &= \int_{4m^2}^{\infty} \frac{dt}{\pi} \frac{\operatorname{Im} T(s_0, t)}{(t - t_0)^4} - \int_{m_+^2}^{\infty} \frac{ds}{\pi} \frac{\operatorname{Im} T(s, s_0)}{(s - s_0)^4}, \\
\lambda_{2,1} &= \int_{m_+^2}^{\infty} \frac{ds}{\pi} \left(\frac{2\partial_t \operatorname{Im} T(s, t_0)}{(s - s_0)^3} - \frac{3 \operatorname{Im} T(s, t_0)}{(s - s_0)^4} \right) \\
&= \int_{4m^2}^{\infty} \frac{dt}{\pi} \frac{\partial_s \operatorname{Im} T(s_0, t)}{(t - t_0)^3} + \int_{m_+^2}^{\infty} \frac{ds}{\pi} \left(\frac{\partial_t \operatorname{Im} T(s, s_0)}{(s - s_0)^3} - \frac{3 \operatorname{Im} T(s, s_0)}{(s - s_0)^4} \right).
\end{aligned} \tag{F.0.17}$$

We have dropped the primes for readability. The expansion can be carried out systematically at higher order but the expressions become increasingly more complicated. An exception is given by $\lambda_{2k,0}$ and $\lambda_{0,l}$ for which we find

$$\begin{aligned}
\lambda_{2k,0} &= \int_{m_+^2}^{\infty} \frac{ds}{\pi} \frac{2 \operatorname{Im} T(s, t_0)}{(s - s_0)^{2k+1}}, & k \geq 1, \\
\lambda_{0,l} &= \int_{4m^2}^{\infty} \frac{dt}{\pi} \frac{\operatorname{Im} T(s_0, t)}{(t - t_0)^{l+1}} - \int_{m_+^2}^{\infty} \frac{ds}{\pi} \frac{\operatorname{Im} T(s, s_0)}{(s - s_0)^{l+1}}, & l \geq 2.
\end{aligned} \tag{F.0.18}$$

We remark that, when matching powers of $\hat{s}$ and $\hat{t}$ on each dispersion relation and solving for the couplings, we find the linear system to be of maximal rank (at least up to order 8 in the expansion). This is in contrast to the case of dispersion relations for identical scalars, where full crossing symmetry leads to an over-determined system and so-called “null constraints” on dispersive integrals of $\operatorname{Im} T(s, t)$ itself. We do find, however, that some couplings appear in both dispersion relations, like $\lambda_{2,1}$ in the previous formulas. In this case this seems to give a non-trivial dispersive constrain on $\operatorname{Im} T(s, t)$ which we have not explored.

Positivity bounds. As a consequence of unitarity (6.2.28), the imaginary part of the $AB \rightarrow AB$ partial amplitude is positive

$$\operatorname{Im} f_{AB \rightarrow AB}^{\ell}(s) \geq 0, \quad s \geq m_+^2, \quad \ell = 0, 1, 2, \dots \tag{F.0.19}$$

This property can be used to obtain bounds on the couplings $\lambda_{k,l}$ that admit dispersive representations derived from (F.0.5). We plug in (F.0.17) the partial wave decomposition (6.2.21) of $T(s, t) = T_{AB \rightarrow AB}(s, t, u)$ and cast the dispersive representation as an average over a positive measure

$$\langle X_{\ell}(s) \rangle = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \int_{m_+^2}^{\infty} \frac{ds}{\pi} \operatorname{Im} f_{AB \rightarrow AB}^{\ell}(s) X_{\ell}(s). \tag{F.0.20}$$

For example, for the $\lambda_{2,0}$ and $\lambda_{2,1}$ couplings in (F.0.17) we get

$$\lambda_{2,0} = \left\langle \frac{2P_\ell(z(s, t_0))}{(s - s_0)^3} \right\rangle, \quad (\text{F.0.21})$$

$$\lambda_{2,1} = \left\langle \frac{4sP'_\ell(z(s, t_0))}{(s - s_0)^3(s - m_+^2)(s - m_-^2)} \right\rangle - \left\langle \frac{3P_\ell(z(s, t_0))}{(s - s_0)^4} \right\rangle, \quad (\text{F.0.22})$$

with $z(s, t)$ given in (6.2.10) and $P'_\ell(z) = \partial_z P_\ell(z)$. Note that $s \geq m_+^2 \geq s_0$ and $z(s, t_0) > 1$ in the integration region of the average (F.0.20). The Legendre polynomials satisfy $P_\ell(z) \geq 0$ for $z \geq 1$ and therefore the averaged object in (F.0.21) is positive. By positivity of the measure we conclude that

$$\lambda_{2,0} \geq 0. \quad (\text{F.0.23})$$

Similarly, the two terms in (F.0.22) are separately positive, but we cannot derive an absolute bound because of the minus sign. We can notice, however, that $1/(s - s_0)$ is monotonically decreasing for $s \geq m_+^2$ and reaches its maximum at $s = m_+^2$, where $1/(m_+^2 - s_0) = 3/(2 + 6\mu)$. Using this inequality in (F.0.22) we conclude

$$\lambda_{2,1} \geq - \left\langle \frac{3P_\ell(z(s, t_0))}{(s - s_0)^4} \right\rangle \geq - \frac{9}{4(1 + 3\mu)} \left\langle \frac{2P_\ell(z(s, t_0))}{(s - s_0)^3} \right\rangle = - \frac{9}{4(1 + 3\mu)} \lambda_{2,0}, \quad (\text{F.0.24})$$

Bounds on higher order couplings can be systematically obtained using semi-definite optimization as described in [106], which shows that the analytic bound (F.0.24) is optimal. In contrast to the case of identical scalars, in the absence of null constraints, we do not find any numerical upper bound on the ratio $\lambda_{2,1}/\lambda_{2,0}$.

Appendix G

Details on primal S -matrix bootstrap

In this appendix we give further details on the numerical setup used to obtain the bounds presented in Chapter 6. We first comment on how our bounds depend on the truncations in energy and spin, controlled by the parameters n_{pts} and L_{max} . We then explain how a judicious rescaling of the SDP (6.5.14) makes the numerical optimisation more stable and faster. Next, we analyse the relative importance of the various positivity conditions imposed, revealing some patterns that perhaps deserve a better understanding. Finally, we list the parameters used for the semi-definite optimisation.

Sampling convergence. The semi-definite programs that we solve depend on the parameter n_{pts} that controls the number of values of s where we impose the unitarity constraints (6.5.13). As we increase n_{pts} the bound can only get stronger. In practice we use a Chebyshev distribution for the angular variable $\varphi(s)$ defined as $e^{i\varphi(s)} = \rho_+(s)$ for the $AB \rightarrow AB$ unitarity and $e^{i\varphi(s)} = \rho(s, 4M^2, t_0)$ for the $AA \rightarrow BB$ unitarity. See (6.5.1) and (6.5.2) for the definitions of the ρ -variables. In both cases the physical regions $[m_+^2, +\infty]$ and $[4M^2, +\infty]$ are mapped to the interval $[0, \pi]$ in φ , where we pick a grid

$$\varphi_k = \frac{\pi}{2} + \frac{\pi}{2} \cos\left(\frac{k + 1/2}{n_{\text{pts}}} \pi\right), \quad k = 1, \dots, n_{\text{pts}}. \quad (\text{G.0.1})$$

For a representative set of observables, we have checked the dependence of our bounds on n_{pts} by solving the SDP at $n_{\text{pts}} = 100, 150, 200, 250$. We have found non-trivial dependence on this truncation up to 200, but little change between 200 and 250 (below 1%). We have then settled on fixing $n_{\text{pts}} = 200$ for all the numerical optimisations.

Spin convergence and subtracted positivity. The semi-definite programs that we solve depend on the parameter L_{max} which sets the maximum spin where we impose the unitarity constraints (6.5.13). As we increase L_{max} the bound can only get stronger. Our setup turned out to be quite sensitive to this parameter, which we had to push to much higher values than

what has been commonly used for the primal S -matrix bootstrap of identical particles. The rule of thumb is that one needs to increase $L_{\max}$ as the size of the ansatz $N_{\max}$ increases. In our case we also observed that we had to increase $L_{\max}$ as we increase the mass ratio μ . In practice, we have adopted the strategy of fixing $L_{\max}$ to be a value such that the bound on a given observable at $N_{\max} = 26$ is stable. In order to accelerate the convergence of the bound in $L_{\max}$ we have imposed the *subtracted positivity* constraints introduced in [303].¹ Subtracted positivity is the statement that

$$\begin{aligned} \operatorname{Im} f_{AB \rightarrow AB}(s, t, u) - \sum_{\ell \leq L_{\max}} \frac{2\ell + 1}{2} P_{\ell}(z(s, t)) \operatorname{Im} f_{AB \rightarrow AB}^{\ell}(s) \\ = \sum_{\ell > L_{\max}} \frac{2\ell + 1}{2} P_{\ell}(z(s, t)) \operatorname{Im} f_{AB \rightarrow AB}^{\ell}(s) \geq 0, \end{aligned} \quad (\text{G.0.2})$$

for $s \geq m_+^2$ and $t \leq 4m^2$ and with $z(s, t)$ given in (6.2.10). Indeed in this kinematic regime the partial wave decomposition converges, as recalled in Appendix E, the partial amplitudes $f_{AB \rightarrow AB}^{\ell}(s)$ have positive imaginary part as a consequence of (6.2.28), and the Legendre polynomials $P_{\ell}(z)$ are positive since $z(s, t) \geq 1$. This condition can again be formulated in the form (6.5.14). We have imposed this constraint on a grid in (s, t) , by sampling s as before and picking 10 values of t_n in the interval $[0, 4m^2]$ accumulating at the endpoint

$$t_n \in \{0, 0.95, 1.9, 2.85, 3.8, 3.9, 3.99, 3.999, 3.9999, 3.99999\}. \quad (\text{G.0.3})$$

Here we report an example check of spin convergence for the minimisation of $\lambda_{2,1}$ at $N_{\max} = 26$, imposing the subtracted positivity constraints.

$L_{\max}$	30	40	50	60	70
$\min \lambda_{2,1}$	-10.063271	-1.144910	-1.092609	-1.092587	-1.092583

A common pattern we observe is that in the presence of the subtracted positivity constraint the bound gets very stable after a threshold $L_{\max}$, while in its absence it changes quite smoothly in the above range but extrapolates to the same value. Subtracted positivity is then useful to avoid the need for a double extrapolation in $L_{\max}$ and $N_{\max}$.

Integrals precision. The partial wave projection of the ρ -ansatz (6.5.12) is computed by numerical integration with `Mathematica`. This requires working at quite high precision because the integrals for s close to the threshold and large ℓ are exponentially suppressed as in (6.4.8). For large n_{pts} and large $L_{\max}$, the smallest partial wave projection scales roughly as $\sim n_{\text{pts}}^{-4L_{\max}-2}$. This means that numerical integration for points close to threshold has to be performed at very high `WorkingPrecision` and it is essentially insensitive to `PrecisionGoal`. Away from threshold, instead, the precision of the numerical integration is controlled by `PrecisionGoal`.

¹We thank Andrea Guerrieri for suggesting this.

In practice we have used the following

```
NIntegrate[...,"WorkingPrecision" -> 900, "MaxRecursion" -> 150,
  "PrecisionGoal" -> 200, "AccuracyGoal" -> 200,
  "Method" -> {"GlobalAdaptive", Method -> "ClenshawCurtisRule"}]
```

which guarantees at least 200 correct digits for the smallest integrals that we faced.

SDP rescaling. The suppression of the partial wave projections at threshold (6.4.8) also leads to ill-conditioning of the SDP (6.5.14) in its naive formulation. Unitarity of the $AB \rightarrow AB$ amplitude (6.2.28) is imposed in the numerics as positive semi-definiteness of the real matrix which takes the form

$$\begin{pmatrix} 2 - I_j & R_j \\ R_j & I_j \end{pmatrix} \succeq 0, \quad j = (AB \rightarrow AB, \ell, s), \quad (\text{G.0.4})$$

where

$$R_j = \vec{\alpha} \cdot \text{Re} \left[\vec{X}_{AB \rightarrow AB}^\ell(s) \right] \quad I_j = \vec{\alpha} \cdot \text{Im} \left[\vec{X}_{AB \rightarrow AB}^\ell(s) \right] \quad (\text{G.0.5})$$

This is equivalent to the first constraint in (6.5.13) for hermitian matrices. Close to threshold, both the real and the imaginary parts of $\vec{X}_{AB \rightarrow AB}^\ell(s)$ are exponentially suppressed in the COM momentum $\mathbf{p}$, implying that for generic $\vec{\alpha}$ the matrix in (G.0.4) has an eigenvalue which is extremely close to zero. Conditions in the SDP with eigenvalues very close to zero are problematic for the solver SDPB and can lead to instabilities where the algorithm never manages to achieve a *dual jump*, unless the error thresholds are set extremely small, roughly below the size of the smallest eigenvalue involved, which, in turn, also requires working at extremely high precision.

A useful trick to overcome this difficulty is to perform a diagonal rescaling of the SDP conditions. Defining

$$D_j = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{d_j} \end{pmatrix} \quad d_j = \left\| \text{Im} \vec{X}_{AB \rightarrow AB}^\ell(s) \right\|_\infty \quad (\text{G.0.6})$$

where $\|\vec{x}\|_\infty = \max_i |x_i|$ is the supremum norm, we construct the following equivalent SDP constraint

$$D_j^t \begin{pmatrix} 2 - I_j & R_j \\ R_j & I_j \end{pmatrix} D_j = \begin{pmatrix} 2 - I_j & R_j/\sqrt{d_j} \\ R_j/\sqrt{d_j} & I_j/d_j \end{pmatrix} \succeq 0 \quad (\text{G.0.7})$$

The matrix D_j is invertible, so the congruence transformation preserves the cone of positive semi-definite matrices and the rescaled constraint is equivalent to (G.0.4). An important observation is that, at fixed ℓ and s , both the real and the imaginary parts of the components of $\vec{X}_{AB \rightarrow AB}^\ell(s)$ are roughly of the same size and do not span orders of magnitude. This implies then that the new positive semi-definite condition no longer has almost-zero eigenvalues, since I_j/d_j is now made of roughly order-one numbers.

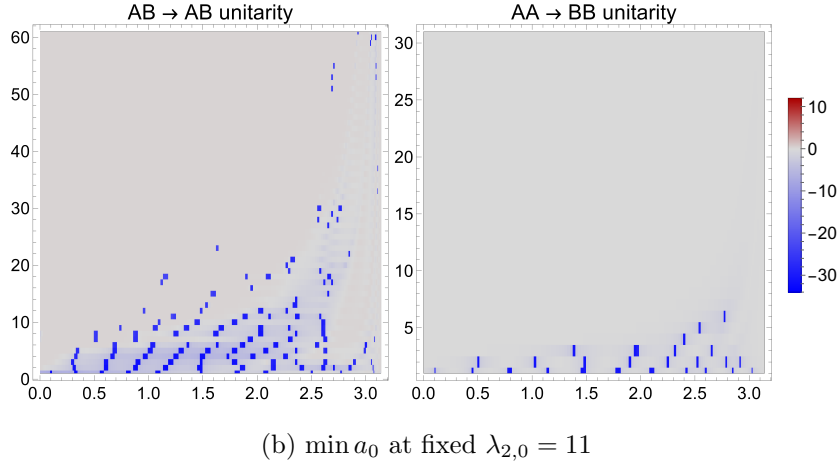
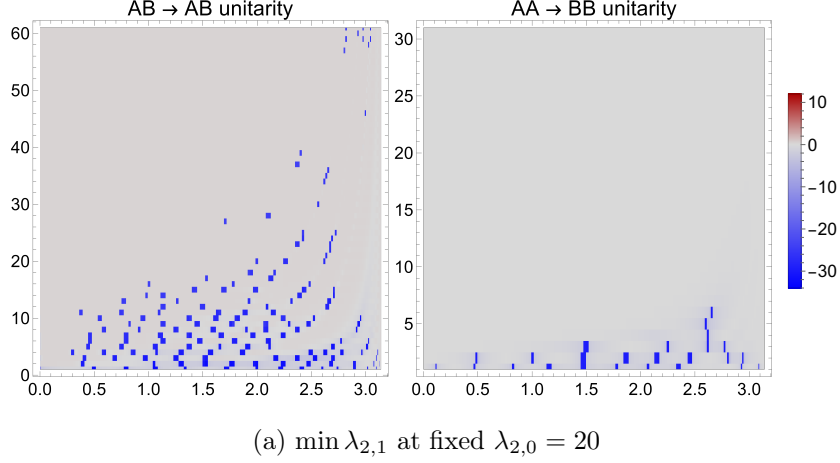


Figure G.1: Density plot of $\log \lambda_j$, the logarithm of the minimum eigenvalue of the condition matrix $M_j(\alpha^*)$ at the optimal solution of two optimisation problems. The index j is expanded into a two-dimensional grid (φ_k, ℓ) for the unitarity conditions. A blue colour denotes the conditions that have the most impact on the bound, while conditions with $\lambda_j > 1$ can be removed from the SDP with little change in the optimal objective.

In practice we apply this rescaling to the SDP not only close to threshold but for all the (ℓ, s) that we sample. This has the additional benefit that the SDP constraints are all roughly of the same size, independently of ℓ and s . After performing this rescaling we observe that the SDPB solver always has both primal and dual jumps, that the internal precision needed can be dramatically reduced, and that the internal convergence of the SDP algorithm is much faster.

Which conditions are more important? Let $\vec{\alpha}^*$ be the optimal solution to the SDP (6.5.14) and let

$$M_j(\alpha^*) = M_{0,j} + \vec{\alpha}^* \cdot \vec{M}_j. \quad (\text{G.0.8})$$

We recall that j is a multi-index labelling all the positive-semi-definite constraints that we compose indexed by $j = (\text{channel}, \ell, \varphi_k)$ (here we trade s for φ_k in (G.0.1)). A useful measure

of how much a given constraint contributes to the optimal bound is given by the minimal eigenvalue of the matrix $\lambda_j = \lambda_{\min}(M_j(\alpha^*))$. Typically, the condition j has an important impact on the bound if $\lambda_j < 1$ and the conditions with the smallest eigenvalue are the most important, in the sense that removing them can alter significantly the value of the optimal objective.² In Figure G.1 we report a plot of such quantity for a sample of optimisation problems. The index j is expanded into a two-dimensional grid: for the $AB \rightarrow AB$ unitarity conditions this is (φ_k, ℓ) , with $k = 1, \dots, n_{\text{pts}}$ and $\ell = 0, \dots, L_{\text{max}}$; for the $AA \rightarrow BB$ unitarity conditions this is (φ_k, ℓ) with k as before and $\ell = 0, 2, \dots, L_{\text{max}}$. We observe that the bound on this specific observable is most sensitive to the $AB \rightarrow AB$ unitarity conditions for $\ell \lesssim 20$ and for a wide range of energies, and also to the high energy and high spin region. The bound is also sensitive to $AA \rightarrow BB$ unitarity conditions with small $\ell \lesssim 5$.

It would be interesting to understand better this pattern, which is shared also by other observables. In particular, it would be interesting to understand why it is not so sensitive to low-energy and high-spin for $AB \rightarrow AB$ and not sensitive to high spin, for any energy, for $AA \rightarrow BB$. Figure G.1 suggests that the grid used in (φ_k, ℓ) is probably not the most efficient choice. A smarter, and perhaps adaptive, sampling scheme in s and ℓ could be adopted to speed up the numerics in the future.

SDP solver. The semi-definite optimisation problems considered in this chapter were solved using the SDP solver SDPB [73, 74]. On certain occasions, especially before applying the SDP rescaling described above, we also used a modified version to overcome some stalling problems, available at <https://gitlab.com/apiazza134/sdpb-midck-stallingrecover>. The parameters we used for the solver are the default ones, in particular a `dualityGap` of 10^{-30} , and a binary `precision` of 655.

²The minimum value of λ_j is of the same size as the duality gap used in the SDP solver.

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