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Growth of Sobolev norms in quasi-integrable quantum systems

Croissance des normes de Sobolev dans les systèmes quantiques quasi-intégrables

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Abstract

We prove an abstract result giving a $\langle t\rangle^{\varepsilon}$ upper bound on the growth of the Sobolev norms of a time-dependent Schrödinger equation of the form $\mathrm{i}\dot{\psi} = H_0\psi + V(t)\psi$. Here H_0 is assumed to be the Hamiltonian of a steep quantum integrable system and to be a pseudodifferential operator of order $\mathtt{d} > 1$; $V(t)$ is a time-dependent family of pseudodifferential operators, unbounded, but of order $\mathtt{b} < \mathtt{d}$. The abstract theorem is then applied to perturbations of the quantum anharmonic oscillators in dimension 2 and to perturbations of the Laplacian on a manifold with integrable geodesic flow, and in particular Zoll manifolds, rotation-invariant surfaces and Lie groups. The proof is based on a quantum version of the proof of the classical Nekhoroshev theorem.

Résumé

Nous prouvons un résultat abstrait donnant une majoration de la forme $\langle t\rangle^{\varepsilon}$ pour la croissance des normes de Sobolev d'une équation de Schrödinger dépendant du temps de la forme $\mathrm{i}\dot{\psi} = H_0\psi + V(t)\psi$. On suppose que H_0 est l'hamiltonien d'un système quantique intégrable escarpé (steep) et qu'il est un opérateur pseudo-différentiel d'ordre $\mathtt{d} > 1$; $V(t)$ est une famille dépendant du temps d'opérateurs pseudo-différentiels, non bornés, mais d'ordre $\mathtt{b} < \mathtt{d}$. Le théorème abstrait est ensuite appliqué aux perturbations des oscillateurs quantiques anharmoniques en dimension 2 et aux perturbations du laplacien sur une

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variété avec flot géodésique intégrable, et en particulier sur les variétés de Zoll, les surfaces invariantes par rotation et les groupes de Lie. La démonstration repose sur une version quantique de la preuve du théorème de Nekhoroshev classique.

Keywords: Schrödinger operator, normal form, Nekhoroshev theorem, pseudo differential operators

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Contents

1 Introduction

In this paper we prove an abstract theorem giving a $\langle t\rangle^{\varepsilon}$ upper bound for the Sobolev norms of the solutions of an abstract Schrödinger equation of the form

$$\mathrm{i}\dot{\psi} = (H_0 + V(t))\psi \ , \quad \psi \in \mathcal{H} \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space, and H_0 is the Hamiltonian of a quantum system which is globally integrable in a sense defined below and steep; H_0 is also assumed to be a pseudodifferential operator of order $\mathtt{d} > 1$, and $V(t)$ is a smooth time-dependent family of self-adjoint pseudodifferential operators of order $\mathtt{b} < \mathtt{d}$. In this sense V can be considered a perturbation of H_0 and the system $H_0 + V(t)$ can be called *quasi-integrable*.

The novelty of this result is twofold: it extends the class of unperturbed systems for which upper bounds of the Sobolev norms can be obtained, and it allows to treat the case of unbounded perturbations. We emphasize that the class of unperturbed systems treated here strictly contains *all* the systems of order $\mathtt{d} > 1$ for which estimates on the growth of Sobolev norms have been obtained; it also contains new systems, like the quantum 2-d anharmonic oscillator and Lie groups.

We now describe more in detail our result. First, following [BGMR21] (see also [Fis15]) we introduce some abstract algebras of linear operators in $\mathcal{H}$ that enjoy the properties typical of pseudodifferential operators. We actually call the elements of these algebras pseudodifferential operators.

Then we give the definition of globally integrable quantum system. The idea is to introduce some operators A_j, $j = 1, ..., d$ that are the quantum analogue of the classical action variables and to consider quantum systems with Hamiltonian $H_0 = h_0(A_1, ..., A_d)$, namely Hamiltonians which are functions of the quantum action variables. In turn, the quantum action variables are defined to be d commuting self-adjoint pseudodifferential operators of order 1, whose joint spectrum is contained in $\mathbb{Z}^d + \kappa$, with some $\kappa \in \mathbb{R}^d$. The motivation of this definition rests in the classical results by Duistermaat–Guillemin [DG75], Colin de Verdière [CdV80], and Helffer–Robert [HR82], which ensure that, under suitable assumptions, the quantization of a classical action variable is a perturbation of a pseudodifferential operator with spectrum contained in $\mathbb{Z} + \kappa$ with $\kappa \in \mathbb{R}$. We point out that our definition, which does not involve directly the action variables of the classical system, is quite flexible, since it applies also to systems whose classical action variables are poorly known and to quantization of some superintegrable systems [Nek72, Fas05] in which the Hamiltonian is independent of some of the action variables. Finally, we assume that the function h_0 is homogeneous of degree $\mathtt{d} > 1$ and fulfills the steepness assumption of the classical Nekhoroshev's theorem (see Definition 2.8 below). Concerning the perturbation V, we assume that it is a pseudodifferential operator of order $\mathtt{b} < \mathtt{d}$. In the case $\mathtt{b} = \mathtt{d}$, V cannot be considered as a perturbation of H_0. For this reason, we do not think it can be treated within our framework.

We come now to the applications of the abstract result. The first application is to perturbations of a 2-dimensional quantum anharmonic oscillator with Hamiltonian

$$H_0 = -\frac{\Delta}{2} + \frac{\|x\|^{2\ell}}{2\ell} \ , \quad x \in \mathbb{R}^2 \ , \quad \ell \in \mathbb{N} \ , \quad \ell \geq 2 \ , \tag{1.2}$$

for which we prove the $\langle t \rangle^{\varepsilon}$ upper bound on the growth of Sobolev norms. We recall that for anharmonic oscillators the situation was clear only for the 1-d case [BGMR21], while for the higher dimensional case only an estimate by $\langle t \rangle^{s}$ for the $\mathcal{H}^s$ norm was known for the case of bounded perturbations (see [MR17]), while, for unbounded perturbations, an upper bound of the form $\langle t \rangle^{s\alpha_s}$, with an exponent $\alpha_s > 1$ which diverges as $\mathtt{b} \to \mathtt{d}$, was proven in [MR17].[1]. A $\langle t \rangle^{\varepsilon}$ estimate was out of reach for the 2d case with previous methods. The second application is to Schrödinger equations on compact manifolds with globally integrable geodesic flow, namely manifolds in which the Hamiltonian of the geodesic flow is integrable, and furthermore the action variables are globally defined and the Hamiltonian is steep. In this paper, in order to be determined, we present some specific examples. First (i) we recover the known results for tori [Bou99, Del10, BM19, BLM22a] and Zoll Manifolds [BGMR21], then (ii) we consider rotation invariant surfaces (following [CdV80, Del10]) and construct the operators A_j by quantizing the classical action variables. The novelty of the result we get for rotation invariant surfaces is that we can deal with *unbounded* perturbations of the Laplacian. Another example (iii) is that of the Schrödinger equation on a Lie group (see [BP11, BCP15] for a KAM type result): it is known that the geodesic flow on a compact Lie group is integrable ([Mis82, Bol04]), but very little is known on the action angle variables. For this reason we work directly at a quantum level. To this end we use the intrinsic pseudodifferential calculus on Lie groups developed in [Fis15, RT09] and construct directly the quantum actions in the case of compact, simply connected Lie groups. Here the lattice $\mathbb{Z}^d$ is essentially the lattice of the dominant weights of the irreducible representations of the Lie group. The result controlling growth of Sobolev norms for solutions of the Schrödinger equation on Lie groups is new.

We come now to a description of the proof of the abstract result (namely Theorem 2.10). The present paper is a direct continuation of the works [BLM20, BLM22b, BLM22a, BLR22], and is based on the quantization of the proof of the classical Nekhoroshev theorem. Here in particular we develop an abstract version of the proof, which is based just on the use of the lattice of the joint spectrum of the actions.

Precisely, the proof consists of two steps: first one uses the methods of normal form to construct iteratively a family of unitary time-dependent operators conjugating the

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[1] Actually the result of [MR17] applies to very general systems and does not rely on a pseudo-differential setting, but typically only allows to prove estimates with the exponent of $\langle t \rangle$ which depends on s.

original Hamiltonian (1.1) to a Hamiltonian of the form

$$H_0 + Z(t) + R(t) \tag{1.3}$$

with $Z(t)$ which is a "normal form" operator (see Definition 4.4 below) and $R(t)$ a smoothing operator, which plays the role of a remainder. This was done in [BLM22b, BLM22a] for the Schrödinger operator on $\mathbb{T}^d$ by quantizing the classical normal form procedure. We remark that the use of pseudo-differential calculus is what allows in [BLM22b, BLM22a] to deal with unbounded perturbations. The second step of the proof in [BLM22b, BLM22a] consists in analyzing the structure of the normal form operator, namely in studying the way it couples different Fourier modes depending on the resonance relations fulfilled by the frequencies of the classical system. This is essentially a quantum version of the geometric construction of Nekhoroshev theorem (see e.g. [Nek77, Nek79, GCB16, BL20] for the classical construction). As a result we obtain that $Z(t)$ has a block diagonal structure with dyadic blocks, so that for the dynamics of $H_0 + Z(t)$ the Sobolev norms remain bounded forever. The addition of the remainder $R(t)$ is the responsible for the $\langle t\rangle^{\varepsilon}$ estimate on the growth.

To realize an abstract version of the above construction, one has to develop several new tools. A major difference with respect to the works [BLM20, BLM22b, BLM22a, BLR22] is that we use here directly pseudo-differential operators, never dealing with the corresponding symbols. This enables to work only with the quantum actions, instead of the classical ones. In turn, the quantum actions are used to define a Fourier expansion of pseudo-differential operators, in which the Fourier coefficients are labeled by the points of the lattice of their joint spectrum. This is the main new technical ingredient of the present paper.

We point out that the geometric part of the proof is more complicated here than in [BLM22b, BLM22a] because we deal here with the steep case, while in [BLM22b, BLM22a] only the convex case was treated.

We conclude this introduction by adding some further comments on the connection of the present result with previous ones. Our main references is the paper [Bou99] (see also [Del10, BM19]), in which Bourgain deals with the Schrödinger equation on the torus. Bourgain's approach is based on a dyadic decomposition of $\mathbb{Z}^d$ which is almost invariant for H and only allows to deal with *bounded* perturbations of $-\Delta$ *on tori*. Bourgain's result was extended to the case of unbounded perturbations in [BLM22a], which in turn is the starting point of the present work. The main new point of [BLM22a] is that it gives a decomposition of $\mathbb{Z}^d$ analogous to Bourgain's one, but based on the resonance properties of the frequencies of the underlying classical system. Such resonance properties admit a natural generalization to more general systems, and this is what is developed in the present paper.

The other direct reference for our work is [BGMR21] (which in turn is closely related to [Bam18, Bam17, BM18, MR17, BGMR18]). In [BGMR21] essentially two cases were considered: systems with periodic classical flows (1-d systems and Zoll manifolds) and systems of degree $\mathtt{d} = 1$ (half wave equation in dimension 1 and harmonic oscillators). The main novelty of the present approach is that we are able to deal with the case of quantization of "general" classical integrable systems, in which the flow is either periodic or quasiperiodic depending on the initial datum. However we restrict here to the case $\mathtt{d} > 1$.

Concerning the case $\mathtt{d} = 1$, we remark that known upper bounds on the growth of Sobolev norms just pertain the case of a perturbation $V(t)$ which depends quasiperiodically on time, with frequencies which are nonresonant with the asymptotic of the gaps between eigenvalues of the unperturbed operator H_0. Still for the case $\mathtt{d} = 1$ some interesting counterexamples, showing that growth of Sobolev norms actually occurs, and furthermore in the resonant case is a quite general phenomenon, have been obtained [Del14, BGMR21, Mas19, Mas21, Mas23, Tho21]. We plan to investigate possible extensions of our method to the case $\mathtt{d} = 1$ in the future.

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Part I

Statements

2 Main results

2.1 Abstract pseudodifferential operators

Let $\mathcal{H}$ be a Hilbert space and K_0 a self-adjoint positive operator with compact inverse.

We define a scale of Hilbert spaces by $\mathcal{H}^r := \text{dom}(K_0^r)$ (the domain of the operator K_0^r) if $r \geq 0$, and $\mathcal{H}^r := (\mathcal{H}^{-r})'$ (the dual space w.r.t. the scalar product of $\mathcal{H}$) if $r < 0$. We denote by $\mathcal{H}^{-\infty} = \bigcup_{r\in\mathbb{R}} \mathcal{H}^r$ and $\mathcal{H}^{+\infty} = \bigcap_{r\in\mathbb{R}} \mathcal{H}^r$. We endow $\mathcal{H}^r$ with the natural norm $\|\psi\|_r := \|(K_0)^r\psi\|_0$, where $\|\cdot\|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$. From the spectral decomposition of K_0 it follows that for any $m \in \mathbb{R}$, $\mathcal{H}^{+\infty}$ is a dense linear subspace of $\mathcal{H}^m$. In the following, when we say that an operator is self-adjoint or unitary, we always mean that it is self-adjoint or unitary with respect to the scalar product in $\mathcal{H}$.

We denote by $\mathcal{B}(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ the space of bounded linear operators from $\mathcal{H}^{s_1}$ to $\mathcal{H}^{s_2}$. As usual in the framework of pseudodifferential calculus, we will identify two operators $A \in \mathcal{B}(\mathcal{H}^{s_1};\mathcal{H}^{s_2})$ and $A' \in \mathcal{B}(\mathcal{H}^{s_1'};\mathcal{H}^{s_2'})$ with $s_1' > s_1$ if $A\big|_{\mathcal{H}^{s_1'}} = A'$. Correspondingly we will write $A \in \mathcal{B}(\mathcal{H}^{s_1};\mathcal{H}^{s_2}) \cap \mathcal{B}(\mathcal{H}^{s_1'};\mathcal{H}^{s_2'})$

We will also consider the space $\bigcap_{s\in\mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$ which is a Fréchet space when endowed with the semi-norms $\|\cdot\|_{\mathcal{B}(\mathcal{H}^s,\mathcal{H}^{s-m})}$.

Definition 2.1. *We will say that F is of* order m *if $F \in \bigcap_{s\in\mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$. In the case of negative $m = -N$, we will also say that F is N*-smoothing.

Definition 2.2. *If $\mathcal{F}$ is a Fréchet (or a Banach) space, we denote by $C_b^k\left(\mathbb{R}^d;\mathcal{F}\right)$ the space of the functions $F \in C^k\left(\mathbb{R}^d;\mathcal{F}\right)$, such that all the seminorms of d^jF are bounded uniformly over $\mathbb{R}^d$ for all $j \leq k$. If this is true for all k we write $F \in C_b^\infty\left(\mathbb{R}^d;\mathcal{F}\right)$.*

Given $0 \leq \rho_0 < 1$, we introduce a family of algebras $\mathcal{A}_\rho$ with $\rho \in (\rho_0, 1]$ of operators encoding the fundamental properties of pseudodifferential operators.

For $m \in \mathbb{R}$ and $\rho \in (\rho_0, 1]$ let $\mathcal{A}_\rho^m$ be a linear subspace of $\bigcap_{s\in\mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$ and define $\mathcal{A}_\rho := \bigcup_{m\in\mathbb{R}} \mathcal{A}_\rho^m$.

Assumption I.

i. For each $m \in \mathbb{R}$, one has $K_0^m \in \mathcal{A}_1^m$.

ii. For each $m \in \mathbb{R}$ and $\rho \in (\rho_0, 1]$, $\mathcal{A}_\rho^m$ is a Fréchet space for a family of semi-norms $\{\wp_{\rho,j}^m\}_{j\geq 1}$ such that the embedding $\mathcal{A}_\rho^m \hookrightarrow \bigcap_{s\in\mathbb{R}} \mathcal{B}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is continuous.

If $m'<m$ then $\mathcal{A}_\rho^{m'} \hookrightarrow \mathcal{A}_\rho^m$ with a continuous embedding.
If $\rho_1 < \rho_2$, then $\mathcal{A}_{\rho_2}^m \hookrightarrow \mathcal{A}_{\rho_1}^m$ with continuous embedding.

iii. $\forall m, n \in \mathbb{R}$: if $F \in \mathcal{A}_\rho^m$ and $G \in \mathcal{A}_\rho^n$ then $FG \in \mathcal{A}_\rho^{m+n}$ and the map $(F, G) \mapsto FG$ is continuous from $\mathcal{A}_\rho^m \times \mathcal{A}_\rho^n$ into $\mathcal{A}_\rho^{m+n}$.

iv. If $F \in \mathcal{A}_\rho^m$ and $G \in \mathcal{A}_\rho^n$ then the commutator $[F, G] \in \mathcal{A}_\rho^{m+n-\rho}$ and the map $(F, G) \mapsto [F, G]$ is continuous from $\mathcal{A}_\rho^m \times \mathcal{A}_\rho^n$ into $\mathcal{A}_\rho^{m+n-\rho}$.

v. $\mathcal{A}_\rho$ is closed under perturbations by smoothing operators: let $F : \mathcal{H}^{+\infty} \to \mathcal{H}^{-\infty}$ be a linear map. If there exists $m \in \mathbb{R}$ such that for every $N > 0$ we have a decomposition $F = F^{(N)} + S^{(N)}$, with $F^{(N)} \in \mathcal{A}_\rho^m$ and $S^{(N)}$ is N-smoothing, then $F \in \mathcal{A}_\rho^m$.

vi. If $F \in \mathcal{A}_\rho^m$, then also the adjoint operator $F^* \in \mathcal{A}_\rho^m$.

vii. For any $F \in \mathcal{A}_1^m$, any $G \in \mathcal{A}_1^1$, the map $\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}tG} F \mathrm{e}^{-\mathrm{i}tG} \in C_b^0(\mathbb{R}, \mathcal{A}_1^m)$.

Remark 2.3. *Property I.iv is the one which makes the algebras* $\mathcal{A}_\rho$ *different for different* ρ.
Property I.vii is an abstract version of the Egorov theorem.

2.2 Globally integrable quantum systems

The idea is to define a quantum integrable system as a system whose Hamiltonian operator can be written as a function of some *action operators*. The action operators are d self-adjoint pairwise commuting operators (in the sense that their projection-valued measures commute) $A_1, ..., A_d$, fulfilling the following Assumption A.

Definition 2.4. *Let* $m \in \mathbb{R}$ *and let* $0 < \varsigma \leq 1$ *be a parameter; a function* $f \in C^\infty(\mathbb{R}^d)$ *is said to be a symbol of class* S_ς^m *if* $\forall \alpha \in \mathbb{N}^d$ *one has*

$$\sup_{a \in \mathbb{R}^d} \frac{|\partial^\alpha f(a)|}{\langle a \rangle^{m-\varsigma|\alpha|}} < \infty \tag{2.1}$$

where $\langle a \rangle := \sqrt{1 + \sum_j a_j^2}$.

The quantities at l.h.s. of (2.1) form a family of seminorms for S_ς^m.

Assumption A.

i. For $j = 1, ..., d$, the operators A_j fulfill $A_j \in \mathcal{A}_1^1$.

ii. $\exists c_1 > 0$ s.t. $c_1 K_0^2 < \mathbf{1} + \sum_{j=0}^{d} A_j^2$.

iii. There exist a convex closed cone $\mathcal{C} \subseteq \mathbb{R}^d$ and a vector $\kappa = (\kappa_1, ..., \kappa_d) \in \mathbb{R}^d$, such that the joint spectrum Λ of the A_j's fulfills

$$\Lambda \subset (\mathbb{Z}^d + \kappa) \cap \mathcal{C} \ . \tag{2.2}$$

iv. There exist $\varsigma_0 \in [0, 1)$ and an increasing continuous function $(\varsigma_0, 1] \ni \varsigma \mapsto \rho(\varsigma) \in (\rho_0, 1]$, with $\rho(1) = 1$, s.t., if $f \in S^m_\varsigma$, then $f(A_1, ..., A_d) \in \mathcal{A}^m_{\rho(\varsigma)}$. Furthermore its seminorms depend only on the seminorms of f and on the seminorms of the A_j's.

Remark 2.5. *By A.ii, the operator* $\langle A\rangle^2 := \mathbf{1} + \sum_{j=0}^{d} A_j^2$ *has compact inverse, therefore it has pure point spectrum and there exists a basis of* $\mathcal{H}$ *formed by eigenfunctions of* $\langle A\rangle^2$*. Since the operators* A_j *pairwise commute and commute with* $\langle A\rangle^2$*, there exists also a basis* $\{\psi_L\}$ *of* $\mathcal{H}$ *formed by eigenfunctions common to all these operators. Then the joint spectrum* Λ *of the operators* A_j *is defined as the set of the* $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ *s.t. there exists* L *with*

$$A_j \psi_L = a_j \psi_L \ , \quad \forall j = 1, ..., d \ . \tag{2.3}$$

We are now ready to give our definition of a globally integrable quantum system:

Definition 2.6 (Globally integrable quantum system)**.** *We say that* H_0 *is a* globally integrable quantum system *if there exists* $h_0 \in C^\infty(\mathbb{R}^d; \mathbb{R})$ *such that*

$$H_0 = h_0(A_1, \ldots, A_d) \, , \tag{2.4}$$

where $A_1, \ldots, A_d$ *satisfy Assumption A and the function* (2.4) *is spectrally defined.*

2.3 The statement

To state our assumptions on the function h_0 defining H_0 we still need a couple of definitions. First we recall that a function $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is said to be homogeneous of degree m if $f(\lambda a) = \lambda^m f(a)$, $\forall \lambda > 0$.

Homogeneous functions are typically singular at the origin, but, in the context of pseudodifferential operators, the behavior of functions in a neighborhood of the origin is not important. This is captured by the next definition.

Definition 2.7. *A function* $f \in C^\infty(\mathbb{R}^d)$ *will be said to be homogeneous of degree* m *at infinity if it fulfills* $f(\lambda a) = \lambda^m f(a)$, $\forall \lambda > 1$ *and* $\forall a \in \mathbb{R}^d \setminus B_{1/4}$*, where* B_r *is the ball of radius* r *centered at the origin.*

We recall, from [GCB16], the definition of steepness:

Definition 2.8 (Steepness)**.** *Let $\mathcal{U} \subset \mathbb{R}^d$ be a bounded connected open set with nonempty interior. A function $h_0 \in C^1(\mathcal{U})$, is said to be steep in $\mathcal{U}$ with steepness radius $\mathtt{r}$, steepness indices $\alpha_1 \geq 1, \dots, \alpha_{d-1} \geq 1$ and (strictly positive) steepness coefficients $\mathtt{B}_1, ..., \mathtt{B}_{d-1}$, if its gradient $\omega_i(a) := \frac{\partial h_0}{\partial a_i}(a)$ fulfills: $\inf_{a \in \mathcal{U}} \|\omega(a)\| > 0$ and for any $a \in \mathcal{U}$ and for any s dimensional linear subspace $M \subset \mathbb{R}^d$ orthogonal to $\omega(a)$, one has*

$$\max_{0 \leq \eta \leq \xi} \min_{u \in M : \|u\|=1} \|\Pi_M \omega(a + \eta u)\| \geq \mathtt{B}_s \xi^{\alpha_s} \quad \forall \xi \in (0, \mathtt{r}] \ , \tag{2.5}$$

where Π_M is the orthogonal projector on M; the quantities u and η are also subject to the limitation $a + \eta u \in \mathcal{U}$.

Remark 2.9. *It is well known that steepness is generic. Examples of steep functions are given by functions which are convex or quasiconvex. In the applications we will verify steepness by verifying an equivalent condition due to Niederman [Nie06] (see Theorem B.1 below).*

On H_0 we assume:

Assumption H.

i. H_0 is the Hamiltonian of a globally integrable quantum system and the function h_0 of (2.4) is homogeneous of degree $\mathtt{d} > 1$ at infinity.

ii. There exists an open set $\mathcal{U} \subset \mathbb{R}^d$, s.t. $\mathcal{U} \supset \overline{(B_2 \setminus B_{1/2}) \cap \mathcal{C}}$, with the property that h_0 is steep on $\mathcal{U}$.

Theorem 2.10. *Let $H = H(t)$ be of the form*

$$H(t) := H_0 + V(t) \tag{2.6}$$

with H_0 the Hamiltonian of a globally integrable quantum system. Assume that Assumption H holds and that $V(\cdot) \in C^\infty_b(\mathbb{R}; \mathcal{A}^{\mathtt{b}}_1)$ is a family of self-adjoint operators. Assume $\mathtt{b} < \mathtt{d}$; then for any $s \geq 0$ and for any initial datum $\psi \in \mathcal{H}^s$ there exists a unique global solution $\psi(t) := \mathcal{U}(t, \tau)\psi \in \mathcal{H}^s$ of the initial value problem

$$\mathrm{i}\partial_t \psi(t) = H(t)\psi(t) \,, \quad \psi(\tau) = \psi \,, \tag{2.7}$$

furthermore, for any $s > 0$ and $\varepsilon > 0$ there exists a positive constant $K_{s,\varepsilon}$ such that for any $\psi \in \mathcal{H}^s$

$$\|\mathcal{U}(t, \tau)\psi\|_s \leq K_{s,\varepsilon} \langle t - \tau \rangle^\varepsilon \|\psi\|_s \ , \quad \forall t, \tau \in \mathbb{R} \,. \tag{2.8}$$

3 Applications

3.1 Application 1: the anharmonic oscillator in dimension 2

We define $\mathcal{H} := L^2(\mathbb{R}^2)$ and, for $\ell \in \mathbb{N}$, $\ell \geq 2$, consider the Hamiltonian of the quantum anharmonic oscillator

$$H_0 := -\frac{\Delta}{2} + \frac{\|x\|^{2\ell}}{2\ell} \ , \quad x \in \mathbb{R}^2 \ . \tag{3.1}$$

In order to define the scale of Hilbert spaces $\mathcal{H}^s$, we define $K_0 := (\mathbf{1} + H_0)^{\frac{\ell+1}{2\ell}}$, whose principal symbol is

$$\mathtt{k}_0(x,\xi) := \Big(1 + \frac{\|x\|^{2\ell}}{2\ell} + \frac{\|\xi\|^2}{2}\Big)^{\frac{\ell+1}{2\ell}} \ . \tag{3.2}$$

For $\rho \in \big(\frac{\ell-1}{\ell+1}, 1\big]$, define

$$\delta_1 := \frac{1}{2}\left(\rho - \frac{\ell-1}{\ell+1}\right) \ , \quad \delta_2 := \frac{1}{2}\left(\rho + \frac{\ell-1}{\ell+1}\right) \ . \tag{3.3}$$

Definition 3.1. *Given $f \in C^\infty(\mathbb{R}^4)$, we will write $f \in S^m_{AN,\rho}$ if $\forall \alpha, \beta \in \mathbb{N}^2$, there exists $C_{\alpha,\beta} > 0$ s.t.*

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)| \leq C_{\alpha,\beta} \ \left(\mathtt{k}_0(x,\xi)\right)^{m-\delta_1|\alpha|-\delta_2|\beta|} \quad \forall (x,\xi) \in \mathbb{R}^4 \ , \tag{3.4}$$

with δ_1, δ_2 given by (3.3). We will say that an operator F is a pseudodifferential operator of class $\mathcal{A}^m_\rho$ if there exists a symbol $f \in S^m_{AN,\rho}$ s.t. F is the Weyl quantization of f.

Remark 3.2. *When $\rho = 1$ the class of symbol $\mathcal{A}^m_1$ reduces to the standard classes used to study the anharmonic oscillator (see e.g. [HR82]). The case with $\rho < 1$ was studied in [BLR22].*

The properties I are immediate consequences of standard pseudodifferential calculus in $\mathbb{R}^4$. Following [CdV80, Cha83, BLR22], the operators A_1, A_2 will be constructed in Subsection 6.1 by quantizing the classical actions. Assumptions A and H will be verified in Subsection 6.1, so that we have the following:

Theorem 3.3. *Consider the Schrödinger equation (2.7) with H_0 given by (3.1) and $V(.) \in C^\infty_b(\mathbb{R}; \mathcal{A}^{\mathtt{b}}_1)$, with $\mathtt{b} < \frac{2\ell}{\ell+1}$, then the corresponding evolution operator fulfills (2.8).*

3.2 Application 2: manifolds with globally integrable geodesic flow

Let (M, g) be a compact n-dimensional Riemannian manifold without boundary; to fit our scheme we define $\mathcal{H} := L^2(M)$, $K_0 := \sqrt{\mathbf{1} - \Delta_g}$, with Δ_g the negative Laplace-Beltrami operator relative to the metric g, so that $\mathcal{H}^s$ coincides with the classical Sobolev space H^s. In this case, the pseudodifferential operators are the standard ones defined by Hörmander. Precisely we give the following definition.

Definition 3.4. *A function $f \in C^\infty(T^*M)$ is said to be a symbol of class $S^m_{H,\varrho}$, if, when written in any canonical coordinate system (in the sense of T^*M) it fulfills*

$$\left|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)\right| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\varrho|\beta|+(1-\varrho)|\alpha|} \ , \quad \forall \alpha, \beta \in \mathbb{N}^n \ , \quad \forall (x,\xi) \in T^*M \ . \tag{3.5}$$

Definition 3.5. *We say that $F \in \mathcal{A}^m_\rho$, if it is a pseudodifferential operator (in the sense of Hörmander [Hör85]) with Weyl symbol of class $S^m_{H,\varrho}$, with $\varrho = \dfrac{\rho+1}{2}$.*

Then Assumption I holds. In particular, the commutator between two pseudodifferential operators gains ρ with respect to their product. Furthermore, Assumption A.iv with $\rho(\varsigma) = 2\varsigma - 1$ follows from functional calculus. We are now going to study some specific manifolds M. The applications are dealt with in different subsections, since the construction is different in each specific case. The result will always be that the solution of the Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = (-\Delta_g + V(t))\psi \ , \quad \psi \in H^s(M) \tag{3.6}$$

with $V(.) \in C^\infty_b(\mathbb{R}; \mathcal{A}^{\mathtt{b}}_1)$, $\mathtt{b} < 2$, fulfills the estimate (2.8).

3.2.1 Flat tori

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be a basis of $\mathbb{R}^n$ and let $\Gamma : \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, be the maximal lattice that they generate. Define $M \equiv \mathbb{T}_\Gamma := \mathbb{R}^n/\Gamma$. By introducing in $\mathbb{T}_\Gamma$ the basis of the vectors $\mathbf{e}_i$, the Laplacian is transformed in the operator $H_0 := \sum_{k,l} g^{kl}(-\mathrm{i}\partial_k)(-\mathrm{i}\partial_l)$, with g^{kl} the inverse matrix of $g_{jk} := \mathbf{e}_j \cdot \mathbf{e}_k$. In this case one has $A_j := -\mathrm{i}\partial_j$, $j = 1, ..., n = d$ and $h_0(\xi) := \sum_{kl} g^{kl}\xi_l\xi_k$, which is convex and thus steep. So Theorem 2.10 applies and we get the estimate (2.8). This result was already obtained in [BLM22a], which improved the results [Bou99, Del10, BM19].

3.2.2 Zoll manifolds

We recall that a Zoll manifold is a compact manifold s.t. all its geodesics are closed; the typical example of a Zoll manifold is a sphere. By Theorem 1 of [CdV79], there exists a pseudodifferential operator Q of order -1, commuting with $-\Delta_g$, s.t. $\mathrm{spec}(\sqrt{-\Delta_g}+Q) \subset \mathbb{N}+\kappa$, with $\kappa \geq 0$. We put $A := \sqrt{-\Delta_g}+Q$ and $h_0(a) := a^2$. We remark that in this case one has $d = 1$. We thus get that the solutions of (3.6) on a Zoll manifold fulfill (2.8). We recall that this result was already obtained in [BGMR21].

3.2.3 Rotation-invariant surfaces

Consider a real *analytic* function $f : \mathbb{R}^3 \to \mathbb{R}$ invariant by rotations around the z axis, and assume it is a submersion at $f(x, y, z) = 1$. Denote by M the level surface $f(x, y, z) = 1$, suppose that M is diffeomorphic to $\mathbb{S}^2$ and endow it by the natural metric g induced by the euclidean metric of $\mathbb{R}^3$, then M has integrable geodesic flow. Following [CdV80], we introduce suitable coordinates in M as follows: let N and S be the north and the south poles (intersection of M with the z axis) and denote by $\theta \in [0, L]$ the curvilinear abscissa along the geodesic given by the intersection of M with the xz plane; we orient it as going from N to S and consider also the cylindrical coordinates (r, ϕ, z) of $\mathbb{R}^3$: we will use the coordinates $(\theta, \phi) \in (0, L) \times (0, 2\pi)$ as coordinates in M. Using such coordinates, one can write the equation of M by expressing the cylindrical coordinates of a point in $\mathbb{R}^3$ as a function of (θ, ϕ) getting

$$M = \left\{(r(\theta), \phi, z(\theta)) , \quad (\theta, \phi) \in (0, L) \times \mathbb{T}^1\right\} .$$

Since θ is a geodesic parameter, the metric takes the form $g = r^2(\theta)d\phi^2 + d\theta^2$. We assume that the function $r(\theta)$ has only one critical point $\theta_0 \in (0, L)$. Furthermore, we need to ensure steepness. To this aim, consider the following Taylor expansion at $\theta = \theta_0$:

$$\frac{1}{2r^2(\theta)} = \beta_0 + \frac{1}{2}\beta_2(\theta-\theta_0)^2 + \frac{1}{3!}\beta_3(\theta-\theta_0)^3 + \frac{1}{4!}\beta_4(\theta-\theta_0)^4 + O(|\theta-\theta_0|^5) . \tag{3.7}$$

Theorem 3.6. *Consider the Schrödinger equation* (3.6) *with* $V(.) \in C_b^\infty(\mathbb{R}; \mathcal{A}_1^{\mathtt{b}})$*, with* $\mathtt{b} < 2$*. Assume also that* $\beta_2 \neq 0$*, and that*

$$\beta_0 \frac{-5\beta_3^2 + 3\beta_2\beta_4}{24\beta_2^2} - \beta_2 \neq 0 , \tag{3.8}$$

then Assumption H holds.

This theorem will be proved in Subsection 6.2. In this example, the actions were actually constructed in [CdV80] by quantizing the classical action variables.

Remark 3.7. *For the case of* bounded potentials*, this theorem was proved in [Del10] where the condition* (3.8) *was not required.*

3.2.4 Compact, simply connected Lie groups

Let $M \equiv G$ be a simply connected compact Lie group endowed with the bi-invariant metric g. To apply Theorem 2.10 to equation (3.6) on G we use the intrinsic formulation of pseudodifferential calculus on Lie groups, developed in [RT09, Fis15]. In particular this will be needed to construct the quantum actions A_j and to verify their properties. We remark that Lie groups which are simply connected and compact are given by $\mathrm{SU}(n)$ with $n \geq 2$, $\mathrm{Sp}(n)$ with $n \geq 3$, $\mathrm{Spin}(n)$ with $n \geq 7$, and G_2, F_4, E_6, E_7, E_8 and their direct products. Our result also extends to direct products of compact, simply connected Lie groups with tori of any dimension. An extension to more general compact Lie groups and homogeneous spaces can also be obtained; the details are left for a future work.

The starting point of the construction is the fact that in the intrinsic Fourier calculus in Lie groups, the Fourier coefficients of a smooth function are labeled by the irreducible unitary representations of the group and each Fourier coefficient is a unitary operator in the representation space.

More precisely, denote by $\widehat{G}$ the set of unitary irreducible representations of G modulo unitary equivalence and by $\mathrm{Rep}(G)$ the set of unitary representations of G (still modulo equivalence). Given $\xi \in \mathrm{Rep}(G)$, denote by $\mathcal{H}_\xi$ the corresponding representation space; then the Fourier coefficients of a function $\psi : G \to \mathbb{R}$ are a sequence $\{\hat{\psi}_\xi \mid \xi \in \widehat{G}\}$ with $\hat{\psi}_\xi \in \mathcal{B}(\mathcal{H}_\xi)$.

There is a way of defining symbols of pseudodifferential operators as maps σ,

$$G \times \mathrm{Rep}(G) \ni (x, \xi) \mapsto \sigma(x, \xi) \in \mathcal{B}(\mathcal{H}_\xi) \ , \tag{3.9}$$

with suitable properties (see Definition 6.3 below for the precise definition taken from [Fis15]). Actually a symbol is usually defined by its action on $\widehat{G}$ and extended to $\mathrm{Rep}(G)$ by direct sum.

To define the actions we need a further step in the theory of Lie groups: to a representation $\xi \in \widehat{G}$, one associates its highest weight $\mathtt{w}_\xi$, and it turns out (see e.g. [FH13]) that there is a 1-1 correspondence between the elements of $\widehat{G}$ and the elements of the cone $\Lambda^+(G)$ of dominant weights, defined by

$$\Lambda^+(G) = \left\{ \mathtt{w} \in \mathbb{R}^d \;\middle|\; \mathtt{w} = \mathtt{w}^1 \mathtt{f}_1 + \cdots + \mathtt{w}^d \mathtt{f}_d \,, \quad \mathtt{w}^j \in \mathbb{N} \ , \ \forall j = 1, \ldots, d \right\} \ , \tag{3.10}$$

where $\mathtt{f}_1, \ldots, \mathtt{f}_d \in \mathbb{R}^d$ are the *fundamental weights* of G. In the following we also denote $\underline{\mathtt{f}} := \sum_{j=1}^d \mathtt{f}_j \in \mathbb{R}^d$ and, given a dominant weight $\mathtt{w}$, we denote by $\mathtt{w}^j$ its components on the basis $\mathtt{f}_j$, namely the numbers such that $\mathtt{w} = \sum_{j=1}^d \mathtt{w}^j \mathtt{f}_j$. With this notation, the Laplacian $-\Delta_g$ acts in Fourier space as follows:

$$\widehat{(-\Delta_g \phi)}_\xi = (\|\mathtt{w}_\xi + \underline{\mathtt{f}}\|^2 - \|\underline{\mathtt{f}}\|^2) \hat{\phi}_\xi \ , \tag{3.11}$$

and its symbol is given by

$$\sigma_{-\Delta_g}(\xi) = (\|\mathtt{w}_\xi + \underline{\mathtt{f}}\|^2 - \|\underline{\mathtt{f}}\|^2)\mathbf{1}_{\mathcal{H}_\xi} \ .$$

We are now ready to define the quantum actions $A_1, \ldots, A_d$ as the operators acting in Fourier space as follows:

$$\widehat{(A_j\phi)}_\xi := (\mathtt{w}^j_\xi + 1)\hat{\phi}_\xi \ , \tag{3.12}$$

whose symbol is given by

$$\sigma_{A_j}(\xi) = \left(\mathtt{w}^j_\xi + 1\right) \mathbf{1}_{\mathcal{H}_\xi} \ . \tag{3.13}$$

By direct computation one can see that the operators A_j commute, that their joint spectrum is $\Lambda = \mathbb{N}^d + \kappa \ , \quad \kappa = (1, ..., 1)$, and that

$$-\Delta_g = \sum_{i,j=1}^{d} A_i A_j \, \mathtt{f}_i \cdot \mathtt{f}_j - \|\underline{\mathtt{f}}\|^2 \ , \tag{3.14}$$

so that we can define

$$h_0(A) := \sum_{i,j=1}^{n} A_i A_j \, \mathtt{f}_i \cdot \mathtt{f}_j \ . \tag{3.15}$$

Note that h_0 is homogeneous of degree 2, convex and thus steep. In Section 6.3 we will prove that the A_j's are pseudodifferential operators, so that we can apply Theorem 2.10 and deduce that the estimate (2.8) holds for the solutions of the equation (3.6) on a compact, simply connected Lie group.

Part II

Proofs

4 Analytic part

We start by fixing some notations and definitions that will be used in the rest of the paper, then we will state and prove the normal form Lemma.

Given two real valued functions f and g, sometimes we will use the notation $f \lesssim g$ to mean that there exists a constant $C > 0$, independent of all the relevant quantities, such that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we will write $f \simeq g$.

We recall that we denote

$$\omega(a) := \frac{\partial h_0}{\partial a}(a) \ ,$$

which is homogeneous at infinity of degree

$$\mathtt{M} := \mathtt{d} - 1 . \tag{4.1}$$

Furthermore, given δ such that $\max\{0, \varsigma_0 + \mathtt{M} - 1\} < \delta < \mathtt{M}$, where ς_0 is the quantity defined in Assumption A.iv, we set

$$\varsigma := 1 - (\mathtt{M} - \delta) \ , \qquad \rho := \rho(\varsigma) \, , \tag{4.2}$$

where $\rho(\cdot)$ is the function defined in Assumption A.iv.

Definition 4.1. *Given a joint eigenvalue $a = (a_1, \dots, a_d) \in \Lambda$ of the A_j's, we consider the corresponding joint eigenspace, namely the space $\Sigma_a \subset \mathrm{dom}(K_0)$ with the property that*

$$\psi \in \Sigma_a \qquad \Longleftrightarrow \qquad A_j \psi = a_j \psi \ , \quad \forall j = 1, ..., d \ . \tag{4.3}$$

The orthogonal projector on Σ_a will be denoted by Π_a.

Remark 4.2. *Given $\psi \in \mathcal{H}$, one can consider its spectral decomposition, namely*

$$\psi = \sum_{a \in \Lambda} \Pi_a \psi \ , \tag{4.4}$$

then by Assumption A.ii $\forall s \in \mathbb{N}$ there exist $c_{1,s}, c_{2,s} > 0$ such that one has that

$$c_{1,s} \left\| \psi \right\|_s^2 \leq \sum_{a \in \Lambda} \langle a \rangle^{2s} \| \Pi_a \psi \|_0^2 \leq c_{2,s} \left\| \psi \right\|_s^2 \ . \tag{4.5}$$

Given $\mu \in (0,1)$ and $\mathtt{R} > 1$ (typically $\mu \ll 1$ and $\mathtt{R} \gg 1$) , we give the following definitions:

Definition 4.3. *We say that a point $a \in \Lambda$ is resonant with $k \in \mathbb{Z}^d \setminus \{0\}$ if $\|a\| \geq \mathtt{R}$ and*

$$|\omega(a) \cdot k| \leq \|a\|^{\delta} \|k\| \quad \text{and} \quad \|k\| \leq \|a\|^{\mu} \, . \tag{4.6}$$

If $a \in \Lambda$ does not satisfy (4.6), we say that a is nonresonant with k.

Definition 4.4 (Normal form)**.** *We say that an operator $Z \in \mathcal{A}^m_\rho$ is* in normal form *if*

$$\exists \psi \in \mathcal{H} \ s.t. \ \langle \Pi_a \psi ; Z \Pi_b \psi \rangle \neq 0 \tag{4.7}$$

implies that either a is resonant with $b - a$, or b is resonant with $b - a$.

Definition 4.5. *We say that a family of unitary operators $U(t)$, conjugates H to H^+, if, when $\psi(t) = U(t)\phi(t)$, one has*

$$\mathrm{i}\dot{\psi}(t) = H(t)\psi(t) \qquad \Longleftrightarrow \qquad \mathrm{i}\dot{\phi}(t) = H^+(t)\phi(t) \ . \tag{4.8}$$

We are going to prove the following normal form theorem

Theorem 4.6 (Normal form lemma)**.** *Let H be as in equation (2.6), with $V \in C^\infty_b(\mathbb{R}; \mathcal{A}^{\mathtt{b}}_1)$, $\mathtt{b} < \mathtt{d}$, and assume that $V(t)$ is a family of self-adjoint operators. There exists $0 < \delta_* < \mathtt{M}$ such that, if $\delta_* < \delta < \mathtt{M}$, and*

$$\mathfrak{a} := \min\{2\rho + \delta - \mathtt{d}; \rho + \delta - \mathtt{b}; \delta\} > 0 \ , \tag{4.9}$$

then for any $N \in \mathbb{N}$, for any $\mu, \mathtt{R}$ as above there exists a time-dependent family of unitary maps $U_N(t)$ which conjugates H to

$$H^{(N)} := H_0 + Z_N(t) + R^{(N)}(t) \, , \tag{4.10}$$

and the following properties hold

1. *$Z_N \in C^\infty_b\left(\mathbb{R}; \mathcal{A}^{\mathtt{b}}_\rho\right)$ is a family of self-adjoint operators in normal form;*
2. *$R^{(N)} \in C^\infty_b\left(\mathbb{R}; \mathcal{A}^{\mathtt{b}-\mathfrak{a}N}_\rho\right)$, is a family of self-adjoint operators;*
3. *For any $s \geq 0$, $U_N, \ U_N^{-1} \in L^\infty(\mathbb{R}; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^s))$.*

The rest of this section is devoted to the proof of this theorem. Actually this is the generalization to the abstract setting of theorems proven in [BLM22b, BLR22] so we only present in detail the points of the proofs different from those of [BLM22b, BLR22].

The conjugating maps $U_N(t)$ that one looks for are compositions of maps of the form $e^{-\mathrm{i}G(t)}$, with $G(.) \in C^\infty_b(\mathbb{R}; \mathcal{A}^\eta_\rho)$ a family of self-adjoint pseudodifferential operators with $\eta < \rho$ and for each fixed t, $e^{-\mathrm{i}G(t)}$ is the complex exponential of the operator $-\mathrm{i}G(t)$, as defined through functional calculus for self-adjoint operators. The detailed study of the properties of $e^{-\mathrm{i}G(t)}$ was done in a context very similar to the present one in [BGMR21], to which we refer for more details.

Lemma 4.7. *Let $G \in C^\infty_b(\mathbb{R}; \mathcal{A}^\eta_\rho)$, $\eta < \rho$ and $H \in C^\infty_b(\mathbb{R}; \mathcal{A}^m_\rho)$, be families of self-adjoint operators; then $e^{-\mathrm{i}G(t)}$ conjugates H to H^+ given by*

$$H^+ = H - \mathrm{i}[H, G] + \frac{1}{2}[[H; G]; G] + C^\infty_b(\mathbb{R}; \mathcal{A}^{m+3(\eta-\rho)}_\rho) + C^\infty_b(\mathbb{R}; \mathcal{A}^\eta_\rho) \ . \tag{4.11}$$

Proof. By Lemma 3.1 of [BGMR21], one obtains

$$H^+ = e^{\mathrm{i}G(t)} H e^{-\mathrm{i}G(t)} - \int_0^1 e^{\mathrm{i}sG(t)} \partial_t G e^{-\mathrm{i}sG(t)} \, ds \, . \tag{4.12}$$

Then one applies Lemma 3.2 of [BGMR21] with $M = 2$ to the first summand of (4.12) and Lemma 3.2 of [BGMR21] with $M = 1$ to the second summand of (4.12). □

We use this formula to compute the structure of the transformed Hamiltonian. To this end remark that, since $H_0 \in \mathcal{A}_1^{\mathtt{d}}$ (by A.iv and H.i) and $V \in C_b^\infty(\mathbb{R}; \mathcal{A}_1^{\mathtt{b}})$, taking $G \in C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^\eta)$ with $0 < \rho - \eta < (\mathtt{d} - \mathtt{b})/2$ and $\eta < \mathtt{b}$ (which will be our case), one gets that H^+ has the structure

$$H^+ = H_0 - \mathrm{i}[H_0, G] + V + C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^{\mathtt{d}+2(\eta-\rho)}) + C_b^\infty(\mathbb{R}; \mathcal{A}_\rho^\eta) \ , \tag{4.13}$$

with $\max\{\mathtt{d} + 2(\eta - \rho),\ \eta\} < \mathtt{b}$. Then the idea is to determine G which solves the so called co-homological equation, namely

$$-\mathrm{i}[H_0, G] + V = Z + \textit{lower order terms} \tag{4.14}$$

with Z in normal form and then to iterate the construction. The solution of (4.14) is the main issue of this section and will be done by developing an abstract version of the normal form theory of [BLM20, BLM22b, BLR22]. This requires some work that will be done in the next subsections.

4.1 The Fourier expansion

Here we extend the theory of Sect. 3.3 of [BGMR21] (see also [Bam96, BL20]) to the case where H_0 is a globally integrable quantum system. The idea is that the conjugation of operators with the unitary groups generated by the quantum actions defines a group action of the torus $\mathbb{T}^d$ on the space of pseudo-differential operators. Such a group action is used to define a Fourier expansion of pseudodifferential operators and to develop in a corresponding way normal form theory. A delicate point consists in solving the co-homological equation (Eq. (4.14)), and this is done in Subsection 4.2. We start by giving the following definition.

Definition 4.8. *Let $F \in \mathcal{A}_\rho^m$ with $\rho \in (\rho_0, 1]$, then, for $k \in \mathbb{Z}^d$, we define its k-th Fourier coefficient to be:*

$$\hat{F}_k := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{\mathrm{i}\varphi \cdot A} F e^{-\mathrm{i}\varphi \cdot A} e^{-\mathrm{i}k \cdot \varphi} d\varphi \, . \tag{4.15}$$

In the following we will use the notation

$$F(\varphi) := e^{\mathrm{i}\varphi \cdot A} F e^{-\mathrm{i}\varphi \cdot A} \ . \tag{4.16}$$

Remark 4.9. *By the formula*

$$\frac{d}{d\varphi_j} \left(e^{\mathrm{i}\varphi \cdot A}\, F\, e^{-\mathrm{i}\varphi \cdot A} \right) = e^{\mathrm{i}\varphi \cdot A} \, \left(-\mathrm{i}[F; A_j] \right) \, e^{-\mathrm{i}\varphi \cdot A}$$

Assumption I.vii implies that, for $F \in \mathcal{A}_1^m$, the right-hand side is in $C_b^0(\mathbb{T}^d, \mathcal{A}_1^m)$, so that, by iteration, the map $\varphi \mapsto F(\varphi) = e^{\mathrm{i}\varphi \cdot A}\, F\, e^{-\mathrm{i}\varphi \cdot A} \in C_b^\infty(\mathbb{T}^d, \mathcal{A}_1^m)$.

Remark 4.10. *If F is selfadjoint then, $\forall k \in \mathbb{Z}^d$, one has $(\hat{F}_k)^* = \hat{F}_{-k}$.*

Lemma 4.11. *Let $F \in \mathcal{A}^m_1$, then for any j and $N \in \mathbb{N}$ there exist $C > 0$ and J, independent of F, such that*

$$\wp^m_{1,j}\left(\hat{F}_k\right) \le C \frac{\wp^m_{1,J}(F)}{\langle k\rangle^N} \quad \forall k \in \mathbb{Z}^d. \tag{4.17}$$

It follows that the series

$$F(\varphi) = \sum_{k\in\mathbb{Z}^d} \hat{F}_k e^{\mathrm{i}k\cdot\varphi} \tag{4.18}$$

is convergent.

The proof is an immediate consequence of Remark 4.9.

Definition 4.12. *For $m \in \mathbb{R}$, $\rho \in (\rho_0, 1]$, the set of the operators $F \in \mathcal{A}^m_\rho$, s.t. $\forall j$, and $\forall N \in \mathbb{N}$*

$$\wp^m_{\rho,j,N}(F) := \sum_{k\in\mathbb{Z}^d} \langle k\rangle^N \wp^m_{\rho,j}\left(\hat{F}_k\right) < \infty\,, \tag{4.19}$$

will be denoted by $\mathcal{S}^m_\rho$. This is a Fréchet space with the family of seminorms (4.19).

Remark 4.13. *By Lemma 4.11 one has $\mathcal{A}^m_1 \hookrightarrow \mathcal{S}^m_1$ continuously.*

By proceeding as in the proof of Lemma 5.16 of [BLR22] one gets

Lemma 4.14. *Let $F \in \mathcal{S}^m_\rho$ and $G \in \mathcal{S}^{m'}_\rho$. Then $FG \in \mathcal{S}^{m+m'}_\rho$, $[F; G] \in \mathcal{S}^{m+m'-\rho}_\rho$ and $\forall N \in \mathbb{N}$, $\forall j$, $\exists J, C$ s.t. one has*

$$\wp^{m+m'}_{\rho,j,N}(FG) \le C \wp^m_{\rho,J,N}(F)\, \wp^{m'}_{\rho,J,N}(G)\,, \tag{4.20}$$

$$\wp^{m+m'-\rho}_{\rho,j,N}([F; G]) \le C \wp^m_{\rho,J,N}(F)\, \wp^{m'}_{\rho,J,N}(G)\,. \tag{4.21}$$

Remark 4.15. *By deriving $F(\varphi)$ with respect to φ_j one gets*

$$\frac{\partial F}{\partial \varphi_j}(\varphi) = \sum_k \mathrm{i}k_j \hat{F}_k e^{\mathrm{i}k\cdot\varphi} = -\mathrm{i}[F(\varphi); A_j]\,,$$

which implies

$$\sum_k \mathrm{i}k_j \hat{F}_k = -\mathrm{i}[F; A_j]\,. \tag{4.22}$$

That's why this Fourier expansion is useful for the solution of the cohomological equation.

The main property relating the lattice Λ and the Fourier expansion is given by the following lemma

Lemma 4.16. *Let $F \in \mathcal{A}^m_\rho$ for some m. For any $a, b \in \Lambda$ and $k \in \mathbb{Z}^d$, if ψ_a is an eigenfunction corresponding to a and ψ_b is an eigenfunction corresponding to b, one has*

$$\langle \Pi_b\psi; \hat{F}_k \Pi_a\psi\rangle = \delta_k^{a-b}\langle \Pi_b\psi; F\Pi_a\psi\rangle \ . \tag{4.23}$$

Proof. Just compute

$$\begin{aligned}\langle \Pi_b\psi; \hat{F}_k \Pi_a\psi\rangle &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\langle \Pi_b\psi; e^{\mathrm{i}\varphi\cdot A}Fe^{-\mathrm{i}\varphi\cdot A}\Pi_a\psi\rangle e^{-\mathrm{i}\varphi\cdot k}d\varphi\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\langle e^{-\mathrm{i}\varphi\cdot A}\Pi_b\psi; Fe^{-\mathrm{i}\varphi\cdot A}\Pi_a\psi\rangle e^{-\mathrm{i}\varphi\cdot k}d\varphi\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\langle e^{-\mathrm{i}\varphi\cdot b}\Pi_b\psi; Fe^{-\mathrm{i}\varphi\cdot a}\Pi_a\psi\rangle e^{-\mathrm{i}\varphi\cdot k}d\varphi\\ &= \langle \Pi_b\psi; F\Pi_a\psi\rangle\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}e^{-\mathrm{i}\varphi\cdot(k+a-b)}d\varphi\,.\end{aligned}$$

□

4.2 Solution of the cohomological equation

In this subsection we are going to prove the following lemma.

Lemma 4.17. *There exists $\delta_* < \mathtt{M}$ s.t. for all $\delta_* < \delta < \mathtt{M}$, the following holds true: define $\varsigma := \varsigma(\delta)$ and $\rho = \rho(\varsigma(\delta))$ according to (4.2), then $\forall F \in \mathcal{S}^m_\rho$ self-adjoint, there exist self-adjoint operators $G \in \mathcal{S}^{m-\delta}_\rho$, $Z \in \mathcal{S}^m_\rho$, with Z in normal form, s.t.*

$$-\mathrm{i}[H_0; G] + F - Z \in \mathcal{S}^{m-(2\rho+\delta-\mathtt{d})}_\rho + \mathcal{A}^{-\infty}_\rho \ , \tag{4.24}$$

furthermore, for $\delta_* < \delta < \mathtt{M}$ one has $2\rho + \delta - \mathtt{d} > 0$.

First, following [BLR22], we split the perturbation F in a resonant, a nonresonant and a smoothing part. This will be done with the help of suitable pseudodifferential cutoffs.

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a symmetric cutoff function which is equal to 1 in $[-\frac{1}{2}, \frac{1}{2}]$ and has support in $[-1, 1]$, and given $\mathtt{R} > 0$, define

$$\chi^{(\mathtt{R})}(t) := \chi(\mathtt{R}^{-1}\|t\|) \ , \quad t \in \mathbb{R}^d \,,$$

which is of class C^∞ notwithstanding the singularity of $\|t\|$ at $t = 0$, since χ is constantly equal to 1 in a neighborhood on 0.

With its help we define, for $k \in \mathbb{Z}^d \setminus \{0\}$,

$$\tilde{\chi}_k(a) := \chi\left(\frac{\|k\|}{\|a\|^\mu}\right) \ , \quad \chi_k(a) := \chi\left(\frac{\omega(a)\cdot k}{\|a\|^\delta\|k\|}\right) \ , \tag{4.25}$$

$$d_k(a) := \frac{1}{\omega(a)\cdot k}\left(1 - \chi\left(\frac{\omega(a)\cdot k}{\|a\|^\delta\|k\|}\right)\right) \ . \tag{4.26}$$

We also put

$$\chi_0(a) := 1\,, \quad \tilde{\chi}_0(a) := 1\,, \tag{4.27}$$

$$\chi_k^T(a) := (1-\chi^{(\mathtt{R})}(a))\chi_k(a)\,, \quad d_k^T(a) := (1-\chi^{(\mathtt{R})}(a))d_k(a)\,.$$

By the following lemma the above functions are symbols

Lemma 4.18. *$\forall k \in \mathbb{Z}^d \setminus \{0\}$ one has*

$$\begin{gathered} \chi_k^T,\ (1-\chi^{(\mathtt{R})})(1-\chi_k),\ \tilde{\chi}_k \in S_\varsigma^0\,, \\ d_k^T \in S_\varsigma^{-\delta}\,, \end{gathered}$$

with seminorms uniformly bounded in k.

We omit the proof, which is a variant of the proofs of Lemmas 6.3, 6.4, 6.6 of [BLR22].

Given $F \in \mathcal{S}_\rho^m$ self-adjoint, we use the above functions to decompose F:

$$F_0^{(\mathrm{res})} := \sum_{k\in\mathbb{Z}^d} \chi_k^T(A)\tilde{\chi}_k(A)\hat{F}_k\ , \tag{4.28}$$

$$F_0^{(\mathrm{nr})} := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1-\chi^{(\mathtt{R})}(A))(1-\chi_k(A))\tilde{\chi}_k(A)\hat{F}_k\ , \tag{4.29}$$

$$F_0^{(\mathrm{S})} := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1-\chi^{(\mathtt{R})}(A))(1-\tilde{\chi}_k(A))\hat{F}_k + \chi^{(\mathtt{R})}(A)F\ , \tag{4.30}$$

and

$$F^{(\mathrm{res})} := \frac{F_0^{(\mathrm{res})} + (F_0^{(\mathrm{res})})^*}{2}\ , \quad F^{(\mathrm{nr})} := \frac{F_0^{(\mathrm{nr})} + (F_0^{(\mathrm{nr})})^*}{2}\ , \tag{4.31}$$

$$F^{(\mathrm{S})} := \frac{F_0^{(\mathrm{S})} + (F_0^{(\mathrm{S})})^*}{2}\ , \tag{4.32}$$

so that each one of the operators is self-adjoint and one has $F := (F+F^*)/2$, and therefore $F = F^{(\mathrm{nr})} + F^{(\mathrm{res})} + F^{(\mathrm{S})}$.

Remark 4.19. *Let ς and ρ be defined as in (4.2). If $F \in \mathcal{S}_\rho^m$ for some $m \in \mathbb{R}$, then $F^{(\mathrm{res})}, F^{(\mathrm{nr})} \in \mathcal{S}_\rho^m$.*

Remark 4.20. *By Lemma 4.14 and Remark 4.10 one has that $F^{(\mathrm{nr})} = F_0^{(\mathrm{nr})} + \mathcal{S}_\rho^{m-\rho}$.*

Concerning $F^{(\mathrm{S})}$, we have the following lemma.

Lemma 4.21. *Assume* $F \in \mathcal{S}^m_\rho$*, then* $F^{(\mathrm{S})} \in \mathcal{A}^{-\infty}_\rho$*.*

Proof. First remark that the statement is obviously true for the second term of (4.30). Consider now the first term of (4.30). We are going to prove that $\forall s$ it is smoothing of order $s-m$, from which the result follows. First consider the case $m = 0$; we prove that $F^{(\mathrm{S})}_0$ maps $\mathcal{H}^0$ to $\mathcal{H}^s$ for all $s \in \mathbb{N}$. Using the spectral decomposition of the operators A, we have

$$
\begin{aligned}
F^{(\mathrm{S})}_0 \psi &= \sum_{a,k} (1 - \chi^{(\mathtt{R})}(A))(1 - \tilde{\chi}_k(A)) \Pi_a \hat{F}_k \psi \\
&= \sum_{a,k} (1 - \chi^{(\mathtt{R})}(a))(1 - \tilde{\chi}_k(a)) \Pi_a \hat{F}_k \psi \ ,
\end{aligned}
$$

so that, using that $(1 - \tilde{\chi}_k(A))$ is different from zero only if $\|a\|^\mu < \|k\|$ and applying Remark 4.2, one has

$$
\begin{aligned}
\left\| F^{(\mathrm{S})}_0 \psi \right\|^2_s &\leq c_1^{-s} \sum_a \langle a \rangle^{2s} \left\| \sum_k (1 - \chi^{(\mathtt{R})}(a))(1 - \tilde{\chi}_k(a)) \Pi_a \hat{F}_k \psi \right\|^2_0 \\
&\leq c_1^{-s} \sum_a \langle a \rangle^{2s} \Big(\sum_{\|k\| > \|a\|^\mu} \left\| \Pi_a \hat{F}_k \psi \right\|_0 \Big)^2 \\
&\leq c_1^{-s} \sum_a \langle a \rangle^{2s} \Big(\sum_{\|k\| > \|a\|^\mu} \left\| \hat{F}_k \right\|_{\mathcal{B}(\mathcal{H}^0;\mathcal{H}^0)} \|\psi\|_0 \Big)^2 \\
&\leq c_1^{-s} C \sum_a \langle a \rangle^{2s} \Big(\sum_{\|k\| > \|a\|^\mu} \frac{\langle k \rangle^N \wp^0_{\rho,j}(\hat{F}_k)}{\langle k \rangle^N} \|\psi\|_0 \Big)^2 \\
&\leq c_1^{-s} C \sum_a \langle a \rangle^{2s} \frac{1}{\|a\|^{2\mu N}} \Big(\sum_k \langle k \rangle^N \wp^0_{\rho,j}(\hat{F}_k) \Big)^2 \|\psi\|^2_0 \ .
\end{aligned}
$$

Taking $N > \frac{s+d}{\mu}$, one gets that the above quantity is bounded by a constant times $\left(\wp^0_{\rho,j,N}(F)\right)^2 \|\psi\|^2_0$, which proves the statement in the considered case. The general case $m \neq 0$ and $\psi \in \mathcal{H}^{s'}$ is easily reduced to the previous one by considering the operator $\langle A \rangle^{-m+s'} F \langle A \rangle^{-s'}$. □

Lemma 4.22. $F^{(\mathrm{res})}$ *is in normal form.*

Proof. One has

$$
F^{(\mathrm{res})} := \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \left[(1 - \chi^{(\mathtt{R})}(A)) \chi_k(A) \tilde{\chi}_k(A) \hat{F}_k + \hat{F}_{-k} (1 - \chi^{(\mathtt{R})}(A)) \chi_k(A) \tilde{\chi}_k(A) \right] \ .
$$

Consider first the first term in the sum. By Lemma 4.16 and by the definition of the cutoffs one has

$$\langle\psi_b; (1-\chi^{(\mathtt{R})}(A))\chi_k(A)\tilde{\chi}_k(A)\hat{F}_k\psi_a\rangle = \langle(1-\chi^{(\mathtt{R})}(A))\chi_k(A)\tilde{\chi}_k(A)\psi_b; \hat{F}_k\psi_a\rangle$$
$$= (1-\chi^{(\mathtt{R})}(b))\chi_k(b)\tilde{\chi}_k(b)\langle\psi_b; \hat{F}_k\psi_a\rangle = (1-\chi^{(\mathtt{R})}(b))\chi_k(b)\tilde{\chi}_k(b)\delta_b^{a+k}\langle\psi_b; F\psi_a\rangle \ ,$$

which means that this term does not vanish only when b is resonant with $k = b-a$, so this term is in normal form. An equal computation shows that the second term is different from zero only if a is resonant with $k = b-a$, so that the lemma is proven. □

Proof of Lemma 4.17. First, we remark that for δ_* sufficiently close to $\mathtt{M} = \mathtt{d} - 1$ one has that $\varsigma := 1 - (\mathtt{M} - \delta)$ is close to 1. Consequently, since the function $\rho(\cdot)$ defined in Assumption A.iv is continuous, also $\rho(\varsigma)$ is close to 1, and one gets

$$2\rho + \delta - \mathtt{d} = 2\rho + \varsigma - 2 > 0\,.$$

Then we put $Z := F^{(\mathrm{res})}$ and we include $F^{(\mathrm{S})}$ in the remainder term $\mathcal{A}_\rho^{-\infty}$. Then we define

$$G_0 := \sum_{k\neq 0} d_k^T(A)\hat{F}_k\,, \quad G := \frac{G_0 + G_0^*}{2} = G_0 + \mathcal{S}_\rho^{m-\delta-\rho} \ , \tag{4.33}$$

so that G is self-adjoint.

Since by Lemma 4.18 and Assumption A.iv, $d_k^T(A) \in \mathcal{A}_\rho^{-\delta}$, one has $G_0 \in \mathcal{S}_\rho^{m-\delta}$ and also $G \in \mathcal{S}_\rho^{m-\delta}$ with seminorms bounded by the seminorms of F.

We verify now that such a G solves the cohomological equation. By the generalized commutator lemma (Theorem A.1) and Remark 4.15 one has

$$-\mathrm{i}[H_0; G] = -\mathrm{i}[H_0; G_0] + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho} = \sum_{j=1}^{d} -\mathrm{i}\frac{\partial h_0}{\partial a_j}(A)[A_j; G_0] + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho}$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{j=1}^{d} \mathrm{i}\omega_j(A)k_j\hat{G}_{0,k} + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho} = \sum_{k\in\mathbb{Z}^d}(1-\chi^{(\mathtt{R})}(A))(1-\chi_k(A))\hat{F}_k + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho}$$
$$= F_0^{(\mathrm{nr})} + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho} = F^{(\mathrm{nr})} + \mathcal{S}_\rho^{m-\delta+\mathtt{d}-2\rho} \ ,$$

since $F_0^{(\mathrm{nr})} - F^{(\mathrm{nr})} \in \mathcal{S}_\rho^{m-\rho}$ and $\rho > \rho + (\rho - 1) + (\delta - M) = 2\rho + \delta - \mathtt{d}$. □

4.3 End of the proof of Theorem 4.6

We are now going to prove the following iterative lemma, which immediately yields Theorem 4.6

Lemma 4.23. *Let H be as in equation (2.6), with $V \in C^\infty_b(\mathbb{R}; \mathcal{S}^{\mathtt{b}}_\rho)$, $\mathtt{b} < \mathtt{d}$, a family of self-adjoint operators. There exists $0 < \delta_* < \mathtt{M}$ such that, if $\delta_* < \delta < \mathtt{M}$, then $\mathfrak{a}$ defined as in (4.9) satisfies $\mathfrak{a} > 0$ and the following holds. For any $\mu, \mathtt{R}$ and $\forall n \geq 0$ there exists a time-dependent family $U_n(t)$ of unitary maps conjugating the operator H of (2.6) to*

$$H_n(t) = H_0 + Z_n(t) + R_n(t) + \widetilde{R}_n(t)\,, \tag{4.34}$$

where:

1. *$Z_n \in C^\infty_b(\mathbb{R}; \mathcal{S}^{\mathtt{b}}_\rho)$ is a family of self-adjoint operators in normal form*
2. *$R_n \in C^\infty_b(\mathbb{R}; \mathcal{S}^{\mathtt{b}-n\mathfrak{a}}_\rho)$ and $R_n(t)$ is a family of self-adjoint operators*
3. *$\widetilde{R}_n \in C^\infty(\mathbb{R}; \mathcal{A}^{-\infty}_\rho)$ and $\widetilde{R}_n(t)$ is a family of self-adjoint operators*
4. *For any $s \geq 0$, $U_N,\ U_N^{-1} \in L^\infty(\mathbb{R}; \mathcal{B}(\mathcal{H}^s; \mathcal{H}^s))$.*

Proof. First we observe that there exists $\delta_* = \delta_*(\mathtt{b}) > 0$ such that, if $\delta_* < \delta < \mathtt{M}$, then $\mathfrak{a} > 0$. We prove the theorem by induction. In the case $n = 0$, the claim is trivially true taking $U_0(t) = \mathbf{1}$, $Z_0 = 0$, $R_0 = V$ and $\widetilde{R}_0 = 0$.

We consider now the case $n > 0$. Denote $m := \mathtt{b} - n\mathfrak{a}$; we determine $G_{n+1} \in \mathcal{S}^\eta_\rho$, $\eta = \mathtt{b} - n\mathfrak{a} - \delta$, according to Lemma 4.17 with F replaced by R_n. Up to increasing again the value of δ_*, and using again Assumption A.iv, one has that

$$\eta = \mathtt{b} - n\mathfrak{a} - \delta \leq \mathtt{b} - \delta = -(\mathtt{d} - \mathtt{b}) + (\mathtt{M} - \delta) + 1 < \rho\,.$$

Then one uses $e^{\mathrm{i}G_{n+1}}$ to conjugate H_n to H^+ given by (see Lemma 4.7 with m replaced by $\mathtt{d}$)

$$\begin{aligned} H^+ &= H_n - \mathrm{i}\,[H_n; G_{n+1}] + \frac{1}{2}[[H_n; G_{n+1}]; G_{n+1}] + \mathcal{S}^{\mathtt{d}+3(\eta-\rho)}_\rho + \mathcal{S}^\eta_\rho + \mathcal{A}^{-\infty}_\rho \\ &= H_n - \mathrm{i}\,[H_0; G_{n+1}] + \frac{1}{2}[[H_0; G_{n+1}]; G_{n+1}] + \mathcal{S}^{\mathtt{b}+\eta-\rho}_\rho + \mathcal{S}^{\mathtt{d}+3(\eta-\rho)}_\rho + \mathcal{S}^\eta_\rho + \mathcal{A}^{-\infty}_\rho \\ &= H_0 + Z_n + R_n^{(\mathrm{res})} + \mathcal{S}^{m-(2\rho+\delta-\mathtt{d})}_\rho + \mathcal{S}^{\mathtt{b}+\eta-\rho}_\rho + \mathcal{S}^{\mathtt{d}+3(\eta-\rho)}_\rho + \mathcal{S}^\eta_\rho + \mathcal{A}^{-\infty}_\rho\ , \end{aligned} \tag{4.35}$$

where we go from the first to the second line by inserting the expression of H_n and from the second to the third line using Lemma 4.17.

Writing explicitly the different exponents of the classes of the remainder terms in the last line of (4.35), we get that they are given by

$$\begin{aligned} e_1 &:= \mathtt{b} - n\mathfrak{a} - (2\rho + \delta - \mathtt{d}) = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_1\ , \quad \mathfrak{a}_1 := 2\rho + \delta - \mathtt{d} \\ e_2 &:= \mathtt{b} + \mathtt{b} - n\mathfrak{a} - \delta - \rho = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_2\ , \quad \mathfrak{a}_2 := \delta + \rho - \mathtt{b}\ , \\ e_3 &:= \mathtt{d} + 3(\mathtt{b} - n\mathfrak{a} - \delta - \rho) = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_3\ , \quad \mathfrak{a}_3 := -2(b - n\mathfrak{a}) - \mathtt{d} + 3\delta + 3\rho\ , \\ e_4 &:= \mathtt{b} - n\mathfrak{a} - \delta = \mathtt{b} - n\mathfrak{a} - \mathfrak{a}_4\ , \quad \mathfrak{a}_4 := \delta\ . \end{aligned}$$

Noting that $\mathfrak{a}_3 \geq \mathfrak{a}_2$ and choosing the smallest value of $\mathfrak{a}$, the conclusion follows. □

5 Geometric part and conclusion of the proof

Definition 5.1. *Given a subset* $E \subset \Lambda$ *we define* $\Pi_E := \sum_{a\in E} \Pi_a$.

In the present section we prove that an operator in normal form leaves invariant a dyadic partition, precisely we prove the following theorem.

Theorem 5.2. *Suppose that* H_0 *satisfies Assumption H, then there exists a partition* $\{W_\ell\}_{\ell\in\mathbb{N}}$ *of* Λ *with the following properties.*

(1) The sets W_ℓ *are finite and dyadic, namely*

$$\max\{\|a\| \ : \ a \in W_\ell\} \leq 2 \min\{\|a\| \ : \ a \in W_\ell\} \quad \forall \ell \, . \tag{5.1}$$

(2) The sets W_ℓ *are invariant for any operator* Z *in normal form, namely*

$$[\Pi_{W_\ell}, Z] = 0 \ , \quad \forall \ell \in \mathbb{N} \, . \tag{5.2}$$

This is the heart of the proof, indeed, by exploiting this Theorem the proof of Theorem 2.10 is concluded following exactly the procedure of Sect. 5 of [BLM22a]. Let us summarize the strategy: given $A(t)$ a family of time dependent self-adjoint operators, let $\mathcal{U}_A(t,\tau)$ be the time dependent unitary flow associated to A. First one proves that, due to the dyadic property of the blocks, $\|\mathcal{U}_{H_0+Z_N}(t,\tau)\psi\|_s \leq K_s\|\psi\|_s$ for all times t,τ. This is Lemma 5.1 of [BLM22a]. Then, by Duhamel formula, one obtains $\|\mathcal{U}_{H^{(N)}}(t,\tau)\psi\|_s \leq K_{s,N}\langle t-\tau\rangle\|\psi\|_s$ for all t,τ (see Proposition 5.2 of [BLM22a]) and by interpolation one gets (2.8) with $\mathcal{U} = \mathcal{U}_{H^{(N)}}$. Finally Theorem 2.10 follows using boundedness in $\mathcal{H}^s$ of the unitary maps U_N of Theorem 4.6.

The construction of the sets W_ℓ is a generalization to the abstract steep case of that done in [BLM22b, BLM22a] for quadratic H_0. It represents a quantum counterpart of the geometric decomposition at the basis of the proof of Nekhoroshev theorem. The key point to have in mind is that an operator in normal form only connects points a and b of the lattice Λ s.t. $b-a$ is either resonant with a or resonant with b (see Lemma 4.22 and its proof).

Remark 5.3. *The whole construction that will be developed in this section is based only on the structural properties of* H_0*. In particular Theorem 5.2 holds also for any linear self-adjoint operator* Z *which satisfies* (4.7) *and not only for pseudodifferential operators.*

We recall the following:

Definition 5.4. *A subgroup* M *of* $\mathbb{Z}^d$ *is said to be pure*[2] *if* $\mathrm{span}_{\mathbb{R}}\{M\} \cap \mathbb{Z}^d = M$.

---

[2] This notion is sometimes referred to in the literature on Hamiltonian systems as *resonance module*, see for instance [Gio03, BL20].

5.1 The invariant partition: definitions and statement

The construction depends on the parameters δ, μ (controlling the notion of resonant point, see Definition 4.3) and on some further parameters. In order to get the result they have to fulfill some relations that we anticipate here. We assume

$$\alpha d(d-1)\mu < \mathtt{M} - \delta \,, \quad \alpha := \alpha_1 \cdots \alpha_{d-1} \,, \tag{5.3}$$

where the steepness indexes $\alpha_1, \ldots, \alpha_{d-1}$ have been introduced in Definition 2.8. We also take positive parameters $1 = \mathtt{C}_1 < \mathtt{C}_2 \cdots < \mathtt{C}_d$, $1 = \mathtt{D}_1 < \mathtt{D}_2 \cdots < \mathtt{D}_d$ and define

$$\gamma_1 := \mathtt{M} - \delta \,, \quad \gamma_{s+1} := \frac{\gamma_s - s\mu}{\alpha_s} \quad \forall s = 1, \ldots, d-1 \,. \tag{5.4}$$

Remark 5.5. *Equation* (5.3) *ensures that the parameters $\gamma_1, \ldots, \gamma_d$ are positive, with $\gamma_1 > \cdots > \gamma_d$.*

In the following, neighborhoods of the origin will not be relevant, so we will only consider the set

$$\|a\| \geq \frac{1}{2} \,. \tag{5.5}$$

We fix now an open convex cone $\mathcal{C}_e$ s.t. $\mathcal{C}_e \supset \overline{\mathcal{C}} \setminus \{0\}$ and $\mathcal{C}_e \setminus \{0\}$ is contained in the cone generated by $\mathcal{U}$, namely

$$\mathcal{C}_e \setminus \{0\} \subseteq \{a \in \mathbb{R}^d \mid \exists \lambda \in \mathbb{R}^+, u \in \mathcal{U} \text{ s.t. } a = \lambda u\} \,,$$

with $\mathcal{U}$ as in Assumption H.ii.

By the steepness assumption, on the ball $\|a\| = 1$, $\omega(a)$ is bounded from below and from above and thus, by homogeneity, $\exists K > 0$ such that

$$\|a\|^{\mathtt{M}} K^{-1} \leq \|\omega(a)\| \leq K\|a\|^{\mathtt{M}} \,, \quad \forall a \in \mathbb{R}^d \cap \mathcal{C}_e \,. \tag{5.6}$$

Remark 5.6. *By homogeneity, the steepness condition* (2.5) *implies that $\forall r \geq 1$ and $\forall a \in \left(B_{2r} \setminus B_{r/2}\right) \cap \mathcal{C}_e$, one has*

$$\max_{0 \leq \eta \leq \xi} \min_{u \in M : \|u\|=1} \|\Pi_M \omega(a + \eta u)\| \geq \mathtt{B}_s r^{\mathtt{M}-\alpha_s} \xi^{\alpha_s} \quad \forall \xi \in (0, r\mathtt{r}] \,, \tag{5.7}$$

This is the condition we will use in the proof.

Keeping in mind Definition 4.4 of normal form, one has that, if Z is in normal form, then

$$\exists \psi \quad \text{s.t. } \langle \Pi_a \psi; Z \Pi_{a+k} \psi \rangle \neq 0 \tag{5.8}$$

only if either a is resonant with k or $a + k$ is resonant with k according to Definition 4.3.

We start by identifying the points $a \in \Lambda$ which are in resonance with the vectors of a one dimensional pure subgroup M. They are the points such that (5.8) holds for some $k \in M$, namely the points that a normal form operator moves in the directions of M. For this reason, we give the following definition:

Definition 5.7. *Given M a pure subgroup of dimension 1 and $k \in M$, we define for $\sigma \in \{0, 1\}$*

$$\mathcal{Z}_k^\sigma := \{a \in \Lambda \mid a + k \in \Lambda \ \ \textit{and} \ \ a + \sigma k \ \ \textit{is resonant with } k\} \tag{5.9}$$

and

$$\mathcal{Z}_M^{(1)} := \bigcup_{k \in M} \left(\mathcal{Z}_k^0 \cup \mathcal{Z}_k^1\right) . \tag{5.10}$$

Remark 5.8. *Points $a \in \mathcal{Z}_k^0$ fulfill in particular $|\omega(a) \cdot k| \leq \|k\| \|a\|^\delta$, whereas points $a \in \mathcal{Z}_k^1$ fulfill $|\omega(a + k) \cdot k| \leq \|k\| \|a + k\|^\delta$.*

We define now the nonresonant points as the points in Λ that belong to none of the resonant zones $\mathcal{Z}_M^{(1)}$.

Definition 5.9 (Nonresonant set)**.** *We define*

$$\begin{aligned} \Omega_0 &:= \{a \in \Lambda \mid a \textit{ is nonresonant with } b - a \quad \forall b \in \Lambda\} , \\ \Omega_1 &:= \{a \in \Lambda \mid b \textit{ is nonresonant with } b - a \quad \forall b \in \Lambda\} . \end{aligned}$$

The set $\Omega = \Omega_0 \cap \Omega_1$. is called nonresonant set.

Remark 5.10. *One has*

$$\Omega_\sigma := \Lambda \setminus \left(\bigcup_{k \in \mathbb{Z}^d} \mathcal{Z}_k^\sigma \right) . \tag{5.11}$$

We come to multiple resonances.

Definition 5.11. *Let $j = 1, \dots, d$. Given $a \in \Lambda$ and $k \in \mathbb{Z}^d$, we say that a* is resonant with k at order j *if*

$$\|a\| \geq \mathtt{R} \quad \wedge \quad \|k\| \leq \mathtt{D}_j \|a\|^\mu \tag{5.12}$$

and

$$|\omega(a) \cdot k| \leq \mathtt{C}_j \|k\| \|a\|^{\mathtt{M} - \gamma_j} . \tag{5.13}$$

When $j = 1$, since $\mathtt{C}_1 = \mathtt{D}_1 = 1$ and $M - \gamma_1 = \delta$, one recovers the Definition 4.3.

Definition 5.12 (Resonant zones)**.** *Let M be a pure subgroup of $\mathbb{Z}^d$ of dimension s.*

(i) If $s = 0$, namely $M = \{0\}$, we set $\mathcal{Z}_M^{(0)} = \mathcal{Z}_{\{0\}}^{(0)} := \Omega$.

(ii) If $s \geq 1$, for any set of linearly independent vectors $\{k_1, \ldots, k_s\}$ in M, we define, for $\sigma = 0, 1$

$$\begin{aligned} &\mathcal{Z}^{\sigma}_{k_1,\ldots,k_s} = \{a \in \Lambda \mid \\ &(a + k_1 \in \Lambda) \quad \wedge \quad (a + \sigma k_1 \text{ is resonant with } k_j \text{ at order } j \quad \forall j = 1, \ldots, s)\}\,, \end{aligned}$$

and

$$\mathcal{Z}_M^{(s)} := \bigcup_{\substack{k_1,\ldots,k_s \\ \text{lin. ind. in } M}} \left(\mathcal{Z}^0_{k_1,\ldots,k_s} \cup \mathcal{Z}^1_{k_1,\ldots,k_s} \right) . \tag{5.14}$$

The sets $\mathcal{Z}_M^{(s)}$ are called resonant zones.

We point out that, if $s = 1$, we recover Definition 5.7. Strictly speaking the points in $\mathcal{Z}_{\{0\}}^{(0)}$ are non resonant, however, for compactness of language it is convenient to call them 0-resonant.
The sets $\mathcal{Z}_M^{(s)}$ contain lattice points a which are in resonance with *at least* s linearly independent vectors in M.

Lemma 5.13. *Fix $r, s \in \{1, \ldots, d\}$ with $1 \leq r < s$, then for any M with dim $M = s$, one has*

$$\mathcal{Z}_M^{(s)} \subseteq \bigcup_{\substack{M' \subset M \\ dim. M' = r}} \mathcal{Z}^{(r)}_{M'} .$$

The result follows from the very definition of the resonant zones, the details are left to the reader.

We start the study of the properties of the resonant and nonresonant zones. First, one has that, by the first of (5.12) points of Λ with a small norm are considered as nonresonant. A quantitative statement is the content of the next lemma.

Lemma 5.14. *Assume $\mathtt{R}$ large enough, then*

$$B_{\mathtt{R}/2} \subset \Omega .$$

Proof. Given $a \in \Lambda$, suppose that there exist $j = 1, \ldots, d$, $\sigma \in \{0, 1\}$ and $k \in \mathbb{Z}^d$ such that $a + k \in \Lambda$ and $a + \sigma k$ is resonant with k at order j. When $\sigma = 0$, (5.12) gives $\|a\| \geq \mathtt{R}$, therefore $\Omega_0 \supset B_{\mathtt{R}}$. On the other hand, for $\sigma = 1$, by (5.12) with a replaced by $a + k$ one has $\|a + k\| \geq \mathtt{R}$, and

$$\|a\| \geq \|a + k\| - \|k\| \geq \|a + k\| - \mathtt{D}_j \|a + k\|^{\mu} ,$$

but the right-hand side turns out to be larger than $\mathtt{R}/2$, provided $\|a + k\|$ is large enough. This gives $\Omega_1 \supset B_{\frac{\mathtt{R}}{2}}$, and therefore $\Omega = \Omega_0 \cap \Omega_1 \supset B_{\frac{\mathtt{R}}{2}}$. $\square$

Remark 5.15. *Since the resonant zones $\mathcal{Z}_M^{(s)}$ with $s>0$ and Ω are disjoint sets, by Lemma 5.14 one has that $a\in\mathcal{Z}_M^{(s)}$ for some M implies $\|a\|\geq\frac{\mathtt{R}}{2}$.*

In the next lemma, whose proof is deferred to Subsection 5.2, we claim that the "completely resonant zone" is empty:

Lemma 5.16. *Provided $\mathtt{R}$ is large enough, the resonant zone $\mathcal{Z}_{\mathbb{Z}^d}^{(d)}$ is empty.*

We now introduce an equivalence relation, which encodes the fact that two sites $a,b\in\Lambda$ can be connected one each other by an operator in normal form only if they are not too distant. The equivalence classes play in the present construction the same role that the different connected components of the resonant zones play in classical Nekhoroshev theorem.

Definition 5.17. *On $\mathcal{Z}_M^{(s)}$ we define the following pre-equivalence relation: $a\sim' b$ if $a-b\in M$ and*

$$\|a-b\|\leq\max\{\|a\|^{\mu},\,\|b\|^{\mu}\}\,. \tag{5.15}$$

We then complete such a pre-equivalence relation to an equivalence relation setting $a\sim b$ if there exists a finite sequence $\{a_j\}_{j=1}^N$ such that $a_1=a$, $a_N=b$ and $a_j\sim' a_{j+1}$ for any $j=1,\dots N-1$.

Definition 5.18. *For any pure subgroup M of dimension s there could be several equivalence classes that will be labeled by an index $j=1,...,j_M$, with $j_M\leq\infty$, we will denote $\mathcal{J}_M:=\{1,...,j_M\}$ and by $A_{M,j}^{(s)}$ the j-th equivalence classes with respect to the equivalence relation $\sim$.*

Remark 5.19. *In the case $M=\{0\}$, the condition $a-b\in M$ implies that each equivalence class $A_{\{0\},j}^{(0)}$ contains only one element of $\mathcal{Z}_{\{0\}}^{(0)}$, and $\mathcal{J}_M=\sharp\mathcal{Z}_{\{0\}}^{(0)}$.*

The following lemma encodes the main properties of the equivalence classes. In particular, Items 1. and 3. ensure that points belonging to the same equivalence class are connected by a vector in M and are not too far each other.

Lemma 5.20. *The equivalence classes $\{A_{M,j}^{(s)}\}_{j\in\mathcal{J}_M}$ fulfill the following properties:*

1. *$a,b\in A_{M,j}^{(s)}\Rightarrow a-b\in M$*
2. *If $a\in A_{M,j_1}^{(s)}$ and $b\in A_{M,j_2}^{(s)}$ with $j_1\neq j_2$, then either $\|a-b\|>\max\{\|a\|^{\mu},\,\|b\|^{\mu}\}$, or $a-b\notin M$.*

Furthermore, provided $\mathtt{R}$ is large enough, there exists a positive constant C, depending only on γ_s, $\mathtt{C}_s$ and $\mathtt{D}_s$, such that the following holds:

3. *$a,b\in A_{M,j}^{(s)}\Rightarrow\|a-b\|\leq C\|a\|^{1-\gamma_{s+1}}$.*

Since the proof of Item 3. is nontrivial, we postpone the proof of this lemma to Subsection 5.3.

Clearly the regions $\mathcal{Z}_M^{(s)}$ are not reciprocally disjoint. Following the construction of [BLM22b, GCB16, BL20], we identify now sets of points $a \in \Lambda$ which admit *exactly* s linearly independent resonance relations.

Definition 5.21 (Resonant blocks)**.**

1. *(Nonresonant blocks $B_{M,j}^{(s)}$): If $M = \{0\}$,*
$$B_{\{0\},j}^{(0)} := A_{\{0\},j}^{(0)} \quad , \forall j \in \mathcal{J}_{\{0\}}$$
2. *(s-resonant blocks): Given a pure subgroup $M \subset \mathbb{Z}^d$ of dimension $s \in \{1, \dots, d-1\}$, we define*
$$B_{M,j}^{(s)} := A_{M,j}^{(s)} \backslash \left\{ \bigcup_{M' s.t. \dim M' = s+1} \mathcal{Z}_{M'}^{(s+1)} \right\}, \quad j \in \mathcal{J}_M \ ,$$

where $\mathcal{J}_M$ and $\{A_{M,j}^{(s)}\}_{j \in \mathcal{J}_M}$ are the sets whose existence is ensured in Lemma 5.20.

Remark 5.22. *The resonant blocks form a covering of Λ.*

As we will prove in the following section, $\forall s$ there exists a suitable choice of the constants $\mathtt{C}_s$, $\mathtt{D}_s$ such that the blocks $B_{M,j}^{(s)}$ are reciprocally disjoint. However, as in the classical case, they are not invariant under the action of a normal form operator, so we define the quantum extended blocks, which instead will be invariant. First, given two sets A and B, we define:
$$A + B := \{a + b \mid a \in A\,,\ b \in B\} \ .$$

Definition 5.23 (Extended blocks $E_{M,j}^{(s)}$)**.**

1. *$E_{\{0\},j}^{(0)} := B_{\{0\},j}^{(0)} \equiv A_{\{0\},j}^{(0)}$ $j \in \mathcal{J}_{\{0\}}$*
2. *Given a pure subgroup M of dimension 1, $\forall j \in \mathcal{J}_M$ we define*
$$E_{M,j}^{(1)} := \left\{ B_{M,j}^{(1)} + M \right\} \cap A_{M,j}^{(1)}\,, \quad E^{(1)} := \bigcup_{M \text{ of dim.} 1,\, j \in \mathcal{J}_M} E_{M,j}^{(1)} \,.$$
3. *Given a pure subgroup M of dimension s, with $2 \leq s < d$, for any $j \in \mathcal{J}_M$ we define*
$$E_{M,j}^{(s)} := \left\{ B_{M,j}^{(s)} + M \right\} \cap A_{M,j}^{(s)} \cap \bigcap_{k=1}^{s-1} \left(E^{(s-k)}\right)^c\,, \quad E^{(s)} := \bigcup_{M \text{ of dim.} s\,,\, j \in \mathcal{J}_M} E_{M,j}^{(s)}\,,$$
where, given $E \subseteq \Lambda$, $E^c := \Lambda \setminus E$.

Remark 5.24. *The extended blocks* $\{E^{(s)}_{M,j}\}$ *are still a covering of* Λ*.*

The following theorem is the main result of the present section, and it is the heart of the geometric part of the proof. Its proof is postponed to Subsection 5.4:

Theorem 5.25. *There exists a choice of the parameters* $\mathtt{C}_1, \ldots, \mathtt{C}_d$*,* $\mathtt{D}_1, \ldots, \mathtt{D}_d$ *and* $\mathtt{R}$ *such that the blocks* $\{E^{(s)}_{M,j}\}_{M\subset\mathbb{Z}^d,\, j\in\mathcal{J}_M}$ *are a partition of* Λ*, which is dyadic and is left invariant by operators* Z *which are in normal form.*

5.2 Proof of Lemma 5.16

As in the proof of the classical Nekhoroshev theorem (see also [BLM22b]), the following Lemma from [Gio03], to which we refer for the proof, plays a fundamental role.

Lemma 5.26 (Lemma 5.7 of [Gio03])**.** *Let* $s \in \{1, \ldots, d\}$ *and let* $\{u_1, \ldots u_s\}$ *be linearly independent vectors in* $\mathbb{R}^d$*. Let* $w \in \operatorname{span}\{u_1, \ldots u_s\}$ *be any vector. If* α*,* N *are such that*

$$\|u_j\| \le N \quad \forall j = 1, \ldots s\,,$$
$$|w\cdot u_j| \le \alpha \quad \forall j = 1, \ldots s\,,$$

then

$$\|w\| \le \frac{sN^{s-1}\alpha}{\mathrm{Vol}\{u_1\,|\cdots|u_s\}}\,.$$

Proof of Lemma 5.16. Assume that $\mathcal{Z}^{(d)}_{\mathbb{Z}^d}$ is not empty and take $a \in \mathcal{Z}^{(d)}_{\mathbb{Z}^d}$. First, by Remark 5.15, one has $\|a\| \ge \mathtt{R}/2$. Furthermore there exists $\sigma \in \{0,1\}$ and $\{k_1, \ldots, k_d\} \subset \mathbb{Z}^d$ linear independent vectors such that $\forall j$

$$\begin{aligned} \|k_j\| &\le \mathtt{D}_j\|a+\sigma k_1\|^{\mu} \le \mathtt{D}_d\|a+\sigma k_1\|^{\mu}\,,\\ |\omega(a+\sigma k_1)\cdot k_j| &\le \mathtt{C}_j\|k_j\|\|a+\sigma k_1\|^{\mathtt{M}-\gamma_j} \le \mathtt{C}_d\mathtt{D}_d\|a+\sigma k_1\|^{\mathtt{M}-\gamma_d+\mu}\,. \end{aligned} \tag{5.16}$$

Consider first the case $\sigma = 0$. By the second of (5.16), using Lemma 5.26 we deduce

$$\|\omega(a)\| \le d(\mathtt{D}_d)^d\mathtt{C}_d\|a\|^{\mathtt{M}-\gamma_d+\mu d}\,.$$

Then by Equation (5.6), one has

$$K^{-1}\|a\|^{\mathtt{M}} \le \|\omega(a)\| \le d(\mathtt{D}_d)^d\mathtt{C}_d\|a\|^{\mathtt{M}-\gamma_d+d\mu}\,. \tag{5.17}$$

Recalling that $\mathtt{M}-\gamma_d+d\mu < \mathtt{M}$, (5.17) implies $\|a\| \le \mathtt{R}_0$ for some positive $\mathtt{R}_0$, but if $\mathtt{R}/2 > \mathtt{R}_0$ this is in contradiction with the fact that $\|a\| \ge \frac{\mathtt{R}}{2}$.
If instead $\sigma = 1$, in order to eliminate the presence of the vectors k_1 in estimates (5.16), we apply Lemma C.3 of the appendix, with $k = k_1$, $h = k_j$, $a = b$, $l = 0$, and then conclude the proof as in the case $\sigma = 0$. $\square$

5.3 Proof of Lemma 5.20

Items 1 and 2 of Lemma 5.20 are easily verified. Due to Remark 5.19, Item 3 also immediately follows in the case $M = \{0\}$. We now tackle the case $M \neq \{0\}$; this is the heart of the proof of Theorem 5.2, in particular it is where steepness comes into play.

In the remaining part of the present subsection, *we will restrict to the case where M is a proper nonzero pure subgroup of $\mathbb{Z}^d$*, namely $1 \le \dim M \le d-1$.

Lemma 5.27. *If $a \in \mathcal{Z}_M^{(s)}$, then there exists a positive constant K depending only on $d, \mu, \gamma_s, \mathtt{C}_s, \mathtt{D}_s$, such that*

$$\|\Pi_M \omega(a)\| \le K \|a\|^{\mathtt{M}-\gamma_s+s\mu} . \tag{5.18}$$

Proof. One argues as in the proof of Lemma 5.16. By Definition 5.12 of $\mathcal{Z}_M^{(s)}$, if $a \in \mathcal{Z}_M^{(s)}$ there exist $k_1, \dots, k_s$ and $\sigma \in \{0,1\}$ s.t. for all $j = 1, \dots, s$

$$|\omega(a+\sigma k_1)\cdot k_j| \le \mathtt{C}_s \|k_j\| \|a+\sigma k_1\|^{\mathtt{M}-\gamma_s} , \quad \|k_j\| \le D_s \|a+\sigma k_1\|^{\mu} . \tag{5.19}$$

Then if $\sigma = 0$, one observes that $|(\Pi_M\omega(a))\cdot k_j| = |\omega(a)\cdot k_j|$ for all j, and applies Lemma 5.26 with $w = \Pi_M\omega(a)$ and $u_j = k_j$ to deduce $\|\Pi_M\omega(a)\| \le s(\mathtt{D}_s)^{s-1}\mathtt{C}_s\|a\|^{\mathtt{M}-\gamma_s+\mu s}$. If instead $\sigma = 1$, one applies Lemma C.3 with $k = k_1$, $h = k_j$ $a = b$, and $l = 0$, to obtain estimates of the form (5.19) involving only a instead of $a+\sigma k_j$, and then applies Lemma 5.26 as in the case $\sigma = 0$. □

In order to be able to use steepness, one has to ensure that, given two points belonging to the same equivalence class, there always exists a continuous curve joining them such that, for any point in its support, a property analogous to (5.18) holds:

Lemma 5.28 (Interpolation)**.** *For any $s = 1, \dots, d-1$, there exists a positive constant C depending only on $d, \mu, \gamma_s, \mathtt{C}_s, \mathtt{D}_s$, such that the following holds. For any M, j and for any $a, b \in A_{M,j}^{(s)}$, there exists a curve $\gamma : [0,1] \to (\{a\} + \operatorname{span}_{\mathbb{R}} M) \cap \mathcal{C}$ such that*

$$\gamma(0) = a , \quad \gamma(1) = b , \tag{5.20}$$

and

$$\|\Pi_M\omega(\gamma(t))\| \le C \langle \gamma(t)\rangle^{\mathtt{M}-\gamma_s+s\mu} \quad \forall t \in [0,1] . \tag{5.21}$$

Proof. Suppose first that a and b are such that $a \sim' b$, and define γ by $\gamma(t) = a + t(b-a) \subset \mathcal{C}$. Furthermore, by Remark C.2 there exists a positive constant C such that

$$C^{-1}\|a\| < \|a + t(b-a)\| \le C\|a\| \quad \forall t \in [0,\, 1] , \tag{5.22}$$

so that (by the $(\mathtt{M}-1)$-homogeneity of $\partial\omega/\partial a$):

$$\begin{aligned} \|\Pi_M\omega(a+t(b-a))\| &\le \|\Pi_M\omega(a)\| + \left\| \int_0^1 \frac{\partial \omega}{\partial a}(a + t_1 t(b-a)) t(b-a)\, dt_1 \right\| \\ &\lesssim \|a\|^{\mathtt{M}-\gamma_s+s\mu} + \|a\|^{\mathtt{M}-1+\mu} \lesssim \|a\|^{\mathtt{M}-\gamma_s+s\mu} , \end{aligned}$$

where we used Lemma 5.27. Then, using again (5.22), one also obtains

$$\|\Pi_M\omega(a+t(b-a))\| \lesssim \|a+t(b-a)\|^{\mathtt{M}-\gamma_s+s\mu} \quad \forall t\,.$$

The general case $a \sim b$ follows by exploiting the previous result. □

Roughly speaking, the idea in order to prove Item 3 of Lemma 5.20 is that one would like to consider the curve joining a and b and to exploit steepness in order to deduce that, if by contradiction Item 3 of Lemma 5.20 does not hold true, then (5.21) is violated. This is obtained essentially as in the classical case. To this end, we first need a few technical preparation Lemmas.

Definition 5.29. *For any $a \in \mathcal{Z}_M^{(s)}$, define $\omega^\perp(a)$ as the $d-1$ dimensional subspace of $\mathbb{R}^d$ orthogonal to $\omega(a)$, and*

$$M_a := \Pi_{\omega^\perp(a)}\mathrm{span}_{\mathbb{R}}M\,. \tag{5.23}$$

The following lemma ensures that in an appropriate sense M_a is close to M.

Lemma 5.30. *Let M be a pure subgroup of dimension s, $1 \le s < d$. There exists positive constants C and $\mathtt{R}_0$, depending only on γ_s, μ, $\mathtt{C}_s$, $\mathtt{D}_s$, such that, if $\mathtt{R} > \mathtt{R}_0$, then for any $a \in \mathcal{Z}_M^{(s)}$,*

$$\|\Pi_{M_a} - \Pi_M\| \le C\|a\|^{-\gamma_s+s\mu}\,. \tag{5.24}$$

Proof. By Lemma 5.27, there exists a positive constant C_1, depending only on $\mathtt{C}_s, \mathtt{D}_s, \gamma_s, \mu$, such that for any $a \in \mathcal{Z}_M^{(s)}$ we have

$$\|\Pi_M\omega(a)\| \le C_1\|a\|^{\mathtt{M}-\gamma_s+s\mu}\,.$$

Then Lemma C.4 of the appendix and Equation (5.6) give

$$\|\Pi_{M_a} - \Pi_M\| \le 9C_1\|a\|^{\mathtt{M}-\gamma_s+s\mu}\|\omega(a)\|^{-1} \le 9CC_1\|a\|^{-\gamma_s+s\mu} \tag{5.25}$$

(with C the constant in eq. (5.6)), provided $\varepsilon := C_1\|a\|^{\mathtt{M}-\gamma_s+s\mu} \le \frac{1}{2}\|\omega(a)\|$, but this inequality holds provided $\mathtt{R}$ is large enough. □

Lemma 5.31. *Given a pure subgroup M of dimension s, let $a \in \mathcal{Z}_M^{(s)}$ and suppose $u \in \mathrm{span}_{\mathbb{R}}M$ and $a+u \in \mathcal{C}_e$ is such that $\exists C, \tau$ s.t.*

$$\|\Pi_M\omega(a+u)\| \le C\|a+u\|^{\mathtt{M}-\gamma_s+s\mu}\,, \quad \|u\| \le C\|a\|^{1-\tau}\,. \tag{5.26}$$

Assume that

$$a + \Pi_{M_a}u \in \mathcal{C}_e\ , \tag{5.27}$$

then there exists positive constants C^+ and $\mathtt{R}_0$, both depending only on $\gamma_s, \mu, \mathtt{C}_s, \mathtt{D}_s, C$ and τ, such that if $\mathtt{R} > \mathtt{R}_0$ one has

$$\|\Pi_{M_a}\omega\,(a+\Pi_{M_a}u)\| \le C^+\|a\|^{\mathtt{M}-\gamma_s+s\mu}\,. \tag{5.28}$$

Proof. One has

$$\Pi_{M_a}\omega(a+\Pi_{M_a}u) = \Pi_M\omega(a+u) + (\Pi_{M_a}-\Pi_M)\,\omega(a+\Pi_{M_a}u) \tag{5.29}$$
$$+\,\Pi_M\left(\omega(a+\Pi_{M_a}u)-\omega(a+u)\right). \tag{5.30}$$

By (5.26), one has

$$\|\Pi_M\omega(a+u)\| \lesssim \langle a+u\rangle^{\mathtt{M}-\gamma_s+s\mu} \tag{5.31}$$
$$\lesssim \|a\|^{\mathtt{M}-\gamma_s+s\mu} + \|u\|^{\mathtt{M}-\gamma_s+s\mu} \lesssim \|a\|^{\mathtt{M}-\gamma_s+s\mu}, \tag{5.32}$$

since $1-\tau<1$.

The second term at right-hand side of Eq. (5.29) is estimated using again homogeneity and Eq. (5.24). We come to (5.30). Recalling that $u\in\mathrm{span}_{\mathbb{R}}M$, one has

$$\|\Pi_M\left(\omega(a+\Pi_{M_a}u)-\omega(a+u)\right)\| = \|\Pi_M\left(\omega(a+\Pi_{M_a}u)-\omega(a+\Pi_M u)\right)\|$$
$$\leq \left\|\int_0^1 \frac{\partial\omega}{\partial a}\left(a+t(\Pi_{M_a}-\Pi_M)u)\right)\,dt\right\| \|(\Pi_{M_a}-\Pi_M)\,u\|\,. \tag{5.33}$$

Using again Lemma 5.30, one concludes the proof. ☐

We use now steepness in order to prove the following lemma, which concludes the proof of Lemma 5.20.

Lemma 5.32. *There exist positive constants* $\overline{C}$ *and* $\mathtt{R}_0$*, depending on* $\gamma_s,\mu,\mathtt{C}_s,\mathtt{D}_s,\mathtt{r}$ *only, such that if* $\mathtt{R}\geq\mathtt{R}_0$, $\forall a\in A^{(s)}_{M,j}$ *one has*

$$\|a-b\|\leq\overline{C}\|a\|^{1-\gamma_{s+1}} \quad \forall b\in A^{(s)}_{M,j}\,. \tag{5.34}$$

Proof. We prove that there exist large constants $\overline{R}$ and $\overline{C}>0$, depending on $\mathtt{r},\gamma_s,\mu_s,\mathtt{D}_s,\mathtt{C}_s$ only, such that if there exists a couple of points $a,b\in A^{(s)}_{M,j}$ satisfying $\|a\|\geq\overline{R}$ and

$$\|a-b\|\geq\overline{C}\|a\|^{1-\frac{(\gamma_s-s\mu)}{\alpha_s}}, \tag{5.35}$$

one gets a contradiction. Then the result will follow taking $\mathtt{R}\geq 2\overline{R}$ and using Remark 5.15.

Let us fix a and b in some $A^{(s)}_{M,j}$ and suppose that (5.35) holds. Consider the curve $\gamma_{a,b}$ joining a and b constructed in Lemma 5.28; then there exists $t^*>0$ such that

$$\|\gamma_{a,b}(t^*)-a\| = \overline{C}\|a\|^{1-\frac{(\gamma_s-s\mu)}{\alpha_s}}, \tag{5.36}$$
$$\|\gamma_{a,b}(t)-a\| < \overline{C}\|a\|^{1-\frac{(\gamma_s-s\mu)}{\alpha_s}} \quad \forall t\in[0,t^*)\,. \tag{5.37}$$

By construction of $\gamma_{a,b}$ one has

$$\gamma_{a,b}(t){=}a + \Pi_M(\gamma_{a,b}(t) - a) \in \mathcal{C} \quad \forall t \in [0, t^*) \, .$$

Denote $u(t) := \Pi_{M_a}(\gamma_{a,b}(t) - a)$ and note that, by Lemma 5.30 and (5.37), if $\|a\|$ is large enough we have

$$\|u(t)\| \le 2\overline{C}\|a\|^{1-\frac{(\gamma_s - s\mu)}{\alpha_s}} \, . \tag{5.38}$$

This, taking $\|a\|$ large enough and using that $1-\frac{(\gamma_s-s\mu)}{\alpha_s} < 1$, implies that $\tilde{u} := \gamma_{ab}(t) - a$ satisfies $a + \Pi_{M_a}\tilde{u} = a + u(t) \in \mathcal{C}_e$. Therefore, from (5.21) and applying Lemma 5.31 with u replaced by $\tilde{u}$, $\exists C^+$, depending only on $\gamma_s, \mu, \mathtt{C}_s, \mathtt{D}_s$, such that

$$\|\Pi_{M_a}\omega(a + \Pi_{M_a}(\gamma_{ab}(t) - a))\| \le C^+ \|a\|^{\mathtt{M}-\gamma_s + s\mu} \, . \tag{5.39}$$

We are now going to use steepness in the form (5.7). Note that, by Lemma 5.30 and (5.36), if $\|a\|$ is large enough we also have

$$\frac{\overline{C}}{2}\|a\|^{1-\frac{(\gamma_s - s\mu)}{\alpha_s}} \le \|u(t^*)\| \le 2\overline{C}\|a\|^{1-\frac{(\gamma_s - s\mu)}{\alpha_s}} \, . \tag{5.40}$$

Take then $r = \|a\|$ and $\xi := \|u(t^*)\|$ and, for $\eta \in [0, \xi]$ let t_η be the smallest time in $[0, t^*]$ such that $\|u(t_\eta)\| = \eta$; by (5.40) one has $\xi < \mathtt{r}\,\|a\|$ (with $\mathtt{r}$ the quantity in Eq. (2.5)). Let $\bar{t}$ be the point realizing the maximum on $[0, \xi]$ of the quantity $\|\Pi_{M_a}\omega(a + u(t_\eta))\|$, then steepness ensures that

$$\begin{aligned} \|\Pi_{M_a}\omega(\gamma_{ab}(\bar{t}))\| = \|\Pi_{M_a}\omega(a + u(\bar{t}))\| &= \max_{\eta \in [0,\xi]} \|\Pi_{M_a}\omega(a + u(t_\eta))\| \\ &\ge \mathtt{B}_s \, \|a\|^{\mathtt{M}-\alpha_s} \, \xi^{\alpha_s} \ge \mathtt{B}_s (2^{-1}\overline{C})^{\alpha_s} \|a\|^{\mathtt{M}-\gamma_s + s\mu} \, . \end{aligned} \tag{5.41}$$

But, taking $\overline{C}$ large enough, this contradicts Eq. (5.39), and so the conclusion follows. □

5.4 Proof of Theorem 5.25

This subsection follows very closely the proof given in Subsection 5.1 of [BLM22b] for the convex case. We prove in detail only the lemmas with a new proof and we make reference to [BLM22b] for the others.

The next two lemmas ensure that, if the parameters $\mathtt{C}_j, \mathtt{D}_j$ are suitably chosen, an extended block $E^{(s)}_{M,j}$ is separated from every resonant zone associated to a lower dimensional subgroup M' which is not contained in M. The mechanism is that a point in $E^{(s)}_{M,j}$ which is close to the resonant zone associated to a different subgroup M' would gain one more resonance relation, and this would create a contradiction with Definition 5.21.

Lemma 5.33 (Nonoverlapping of resonances)**.** *For all* $s = 1, \dots d-1$ *there exist positive constants* $\bar{\mathtt{R}}$*,* $\bar{\mathtt{C}}_{s+1}$ *and* $\bar{\mathtt{D}}_{s+1}$*, depending only on* $d, \mathtt{C}_s, \mathtt{D}_s, \mu, \gamma_s$ *, such that the following holds: suppose that* M *and* M' *are two distinct pure subgroups of respective dimensions* s *and* s' *with* $s' \leq s$ *and* $M' \nsubseteq M$*. If*

$$\mathtt{C}_{s+1} > \bar{\mathtt{C}}_{s+1}\,, \quad \mathtt{D}_{s+1} > \bar{\mathtt{D}}_{s+1}\,, \quad \mathtt{R} > \bar{\mathtt{R}}\,,$$

then

$$E^{(s)}_{M,j} \cap \mathcal{Z}^{(s')}_{M'} = \emptyset \quad \forall j \in \mathcal{J}_M\,.$$

For the proof see Lemma 5.6 of [BLM22b]

Lemma 5.34 (Separation of resonances)**.** *There exist positive constants* $\bar{\mathtt{R}}$*,* $\tilde{\mathtt{C}}_{s+1}$ *and* $\tilde{\mathtt{D}}_{s+1}$ *depending only on* $d, \mu, \gamma_s, \mathtt{C}_s, \mathtt{D}_s$ *such that, if*

$$\mathtt{C}_{s+1} > \tilde{\mathtt{C}}_{s+1}\,, \quad \mathtt{D}_{s+1} > \tilde{\mathtt{D}}_{s+1}, \quad \mathtt{R} > \bar{\mathtt{R}}\,,$$

then the following holds true. Let $a \in E^{(s)}_{M,j}$ *for some* M *of dimension* $s = 1, \dots, d-1$ *and some* $j \in \mathcal{J}_M$*, and let* $k' \in \mathbb{Z}^d$ *be such that*

$$\|k'\| \leq \|a + \sigma k'\|^{\mu}\,,$$

for some $\sigma \in \{0,1\}$*. Then* $\forall M' \not\subset M$ *s. t.* $s' := \dim M' \leq s$ *one has*

$$a + k' \notin \mathcal{Z}^{(s')}_{M'}\,.$$

For the proof see Lemma 5.7 of [BLM22b]

As a consequence of Lemma 5.33, one has the following lemma, whose proof is a small variant of the proof of Theorem 5.8 of [BLM22b].

Lemma 5.35. *If the constants* $\mathtt{R}$*,* $\mathtt{C}_1, \dots, \mathtt{C}_d$*,* $\mathtt{D}_1, \dots, \mathtt{D}_d$ *are chosen as in Lemma 5.33, then the extended blocks* $\{E^{(s)}_{M,j}\}_{M \subset \mathbb{Z}^d, j \in \mathcal{J}_M}$ *are a partition of* Λ*.*

Then we have the following lemma, which is also a variant of Theorem 5.10 of [BLM22b], so we will omit its proof.

Lemma 5.36. *If the constants* $\mathtt{C}_1, \dots, \mathtt{C}_d$*,* $\mathtt{D}_1, \dots, \mathtt{D}_d$ *and* $\mathtt{R}$ *are chosen as in Lemma 5.34, and* Z *is an operator in normal form, then one has*

$$[\Pi_{E^{(s)}_{M,j}}, Z] = 0 \quad \forall M \subset \mathbb{Z}^d,\ \forall j \in \mathcal{J}_M\,.$$

We are now ready to prove Theorem 5.25:

Proof of Theorem 5.25. By Lemma 5.35, the blocks $\{E^{(s)}_{M,j}\}$ are a partition of Λ, and by Lemma 5.36 they are left invariant by any normal form operator Z. It remains to prove that they are dyadic. This follows from Item 3 of Lemma 5.20: let a and b respectively be the points of minimum and maximum norm in $E^{(s)}_{M,j}$. Then Lemma 5.20 implies

$$\|b\| \leq \|a\| + \|b-a\| \leq \|a\| + C\|a\|^{1-\gamma_{s+1}} \leq 2\|a\| ,$$

where in the last passage we have used that $1-\gamma_{s+1} < 1$, that $\|a\| \geq \mathtt{R}/2$ by Remark 5.15, and we have possibly increased the value of $\mathtt{R}$. $\square$

6 Proof of the results on the applications

6.1 Proof of the results on the anharmonic oscillator

First we recall the properties of the action variables for the classical Hamiltonian system

$$h_0(x,\xi) = \frac{\|\xi\|^2}{2} + \frac{\|x\|^{2\ell}}{2\ell} , \quad x \in \mathbb{R}^2, \ \xi \in \mathbb{R}^2 , \tag{6.1}$$

which were studied in [BLR22]. The first action is defined to be the angular momentum $a_2(x,\xi) := x_1\xi_2 - x_2\xi_1$. Following [BLR22], in order to define the action a_1, consider the polar coordinates (r,θ) in $\mathbb{R}^2$, and the effective potential $V^*_L(r) := \frac{L^2}{2r^2} + \frac{r^{2\ell}}{2\ell}$. For $L \neq 0$ and $E > \min_r V^*_L(r)$, we preliminary define

$$a_r = a_r(E,L) := \frac{\sqrt{2}}{\pi} \int_{r_m}^{r_M} \sqrt{E - V^*_L(r)} dr , \tag{6.2}$$

where $0 < r_m < r_M$ are the two solutions of the equation $E - V^*_L(r) = 0$.

The cone $\mathcal{C}$ is defined by

$$\mathcal{C} := \left\{ a \in \mathbb{R}^2 \ ; \ a_1 \geq 0 \ \text{ if } a_2 \geq 0 , \quad a_1 \geq |a_2| \ \text{ if } \ a_2 < 0 \right\} . \tag{6.3}$$

The following lemma was proved in [BLR22].

Lemma 6.1 (Lemma 4.5 of [BLR22])**.** *The function*

$$a_1(E,L) := \begin{cases} a_r(E,L) & for \quad L > 0 \\ a_r(E,L) - L & for \quad L < 0 \end{cases} \tag{6.4}$$

has the following properties:

(1) it extends to a complex analytic function of L and E defined in a complex neighbourhood of the set

$$\left\{(E, L) \in \mathbb{R}^2 \ : \ |L| < \left(\frac{2\ell}{\ell+1}E\right)^{\frac{\ell+1}{2\ell}} \quad , \quad E > 0\right\} \ ; \tag{6.5}$$

(2) the map $E \mapsto a_1(E, a_2)$ admits an inverse $E = h_0(a_1, a_2)$ which is analytic in the interior of $\mathcal{C}$. Furthermore it is homogeneous of degree $\frac{2\ell}{\ell+1}$ as a function of (a_1, a_2).

(3) the function $a_1(x, \xi) := a_1(h_0(x, \xi), a_2(x, \xi))$ is quasihomogeneous of degree $\ell + 1$, namely

$$a_1(\lambda x, \lambda^l \xi) = \lambda^{\ell+1} a_1(x, \xi) \ , \quad \forall \lambda > 0 \ .$$

(4) There exist positive constants C_1, C_2 s.t.

$$C_1 \langle a \rangle \leq \mathtt{k}_0 \leq C_2 \langle a \rangle \ ,$$

with $\mathtt{k}_0$ defined in (3.2).

Still one has to show that h_0 extends to a complex neighborhood of $\mathcal{C}$, that it is steep on such an extended domain and finally to prove the existence of the quantum actions.

To prove analyticity remark that a point of the boundary of the cone corresponds to a circular orbit of the particle, which in turn is a minimum of the effective 1-d Hamiltonian $h^*(p_r, r) := \frac{p_r^2}{2} + V_L^*(r)$. Let $r_c = r_c(L)$ be the radius of such a circular orbit and denote $\tilde{r} := r - r_c$. By Vey theorem [Vey78], the Birkhoff normal form of h^* is convergent in a complex neighborhood of the circular orbit, which means that there exists a canonical transformation which conjugates h^* to a function

$$\tilde{h}^* \left(\frac{p_r^2 + \tilde{r}^2}{2}\right)$$

analytic in a neighborhood of the origin. Keeping track of the the dependence on L, one easily verifies that $\tilde{h}^*$ is analytic in a neighbourhood of any point of the boundary of the cone.

Then steepness is obtained by applying Theorem B.2 of the appendix. According to Theorem B.2 it is enough to find a point where the Arnold determinant (defined in Eq. (B.1)) does not vanish. Such a point is an arbitrary point of the boundary of the cone: it was proved in Lemma 7 of [BFS18] that the Arnold determinant vanishes identically only when either $V(r) = -\frac{k}{r}$ or $V(r) = \frac{1}{2}kr^2$, which are not the case of the anharmonic oscillator.

To conclude the proof we have then to prove the existence of the quantum actions. This is granted by the following result by Colin de Verdière (see Theorem 3.1 and Theorem 3.2 of [CdV80]).

Theorem 6.2. *There exist two commuting pseudodifferential operators* $A_j \in \mathcal{A}^1_1$ *satisfying Assumption A. In particular there exists* $\kappa \in \left(\frac{\mathbb{Z}}{4}\right)^2$ *s.t their joint spectrum is contained in* $\mathcal{C} \cap (\mathbb{Z}^2 + \kappa)$ *and there exists a symbol* $h \in S^{\frac{2\ell}{\ell+1}}_{AN,1}$ *s.t.*

$$H_0 = h(A) \ . \tag{6.6}$$

Furthermore one has an asymptotic expansion

$$h = h_0 + l.o.t.$$

with h_0 *the classical Hamiltonian of the anharmonic oscillator written in terms of the action variables.*

Finally we have to regularize the Hamiltonian at the origin, but this is simply done by modifying it by the addition of a function with compact support, whose quantization is a smoothing operator.

6.2 Proof of the results on rotation-invariant surfaces

First we recall the Hamiltonian of the classical geodesic flow in the coordinates (ϕ, θ) namely

$$h_0 = \frac{p_\theta^2}{2} + \frac{p_\phi^2}{2r(\theta)^2} \ . \tag{6.7}$$

Again we put $a_2 := p_\phi$ and, for $p_\phi \neq 0$ we define

$$a_1 = a_1(E, p_\phi) := \frac{1}{\pi} \int_{\theta_m}^{\theta_M} \sqrt{E - \frac{p_\phi^2}{r(\theta)^2}} d\theta + |p_\phi| \ , \tag{6.8}$$

where θ_m and θ_M are the solutions of $E - \frac{p_\phi^2}{r(\theta)^2} = 0$. The cone $\mathcal{C}$ is defined by

$$\mathcal{C} := \left\{ (a_1, a_2) \in \mathbb{R}^2 \ : \ a_1 \geq |a_2| \geq 0 \right\} \ ,$$

and Colin de Verdière [CdV80] proved the analogue of Theorem 6.1 for this case. The proof can then be concluded exactly in the same way as for the anharmonic oscillator. An explicit computation shows that the conditions for the nonvanishing of the Arnold determinant at the boundary of the cone are $\beta_2 \neq 0$ and (3.8).

6.3 Proof of results on Lie groups

In order to prove that the quantum actions $A_1, \dots, A_d$ defined in (3.12) satisfy Assumptions A, in this section we use the intrinsic formulation of pseudodifferential calculus introduced in [Fis15, RT09]. We keep the notations of Subsection 3.2.4, and for any $\xi \in \widehat{G}$, we denote by

$$\lambda_\xi := \|\mathtt{w}_\xi + \underline{\mathtt{f}}\|^2 - \|\underline{\mathtt{f}}\|^2 \tag{6.9}$$

the corresponding eigenvalue of the Laplace-Beltrami operator $-\Delta_g$.

Following [Fis15], we consider the following class of symbols:

Definition 6.3. *Given $m \in \mathbb{R}$ and $0 \leq \delta_1 \leq \delta_2 \leq 1$, we say that a map $\sigma : G \times \mathrm{Rep}(G) \ni (x, \xi) \mapsto \sigma(x, \xi) \in \mathcal{B}(\mathcal{H}_\xi)$ is a symbol of order m, and we write $\sigma \in S^m_{F,\delta_1,\delta_2}$, if for any $\xi \in \widehat{G}$ the map $x \mapsto \sigma(x, \xi)$ is smooth and $\forall \alpha, \beta \in \mathbb{N}$ and for any smooth differential operator D^α_x of order α and any $\tau = (\tau_1, \dots, \tau_\alpha)$ with $\tau_i \in \widehat{G}$, there exists $C > 0$ such that*

$$\|D^\alpha_x \Delta^\beta_\tau \sigma(x, \xi)\|_{\mathcal{B}(\mathcal{H}_\xi^{\otimes\tau})} \leq C(1 + \lambda_\xi)^{\frac{m - \delta_2\beta + \delta_1\alpha}{2}} , \tag{6.10}$$

where $\Delta^\beta_\tau := \Delta_{\tau_1} \cdots \Delta_{\tau_\beta}$,

$$\Delta_{\tau_i}\sigma(x, \xi) := \sigma(x, \tau_i \otimes \xi) - \sigma(x, \mathbf{1}_{\mathcal{H}_{\tau_i}} \otimes \xi) , \tag{6.11}$$

and $\| \cdot \|_{\mathcal{B}(\mathcal{H}_\xi^{\otimes\tau})}$ is the operatorial norm on $\mathcal{H}_{\tau_1} \otimes \cdots \otimes \mathcal{H}_{\tau_\beta} \otimes \mathcal{H}_\xi$.

With the above definition, one defines the quantization $\mathrm{Op}(\sigma)$ of a symbol σ as

$$\mathrm{Op}(\sigma)\psi(x) = \sum_{\xi \in \widehat{G}} d_\xi \mathrm{Tr}\left(\xi(x)\sigma(x, \xi)\hat{\psi}_\xi\right) , \tag{6.12}$$

where $d_\xi = \mathrm{dim}\mathcal{H}_\xi$ and $\hat{\psi}_\xi$ is the ξ-th Fourier coefficient of ψ. If there exists $\sigma \in S^m_{F,\delta_1,\delta_2}$ such that $\Sigma = \mathrm{Op}(\sigma)$, we write $\Sigma \in \mathrm{OPS}^m_{F,\delta_1,\delta_2}$.

Remark 6.4. *Assume that the symbol σ does not depend on x and that it has the form $\sigma(\xi) = m(\xi)\mathbf{1}_{\mathcal{H}_\xi}$ for some function m. Then $Op(\sigma)$ defined as in (6.12) acts as a Fourier multiplier, which multiplies each frequency $\hat{\psi}_\xi$ by the factor $m(\xi)$. This is for instance the case of the Laplace-Beltrami operator $-\Delta_g$ and of the quantum actions $A_1, \dots, A_d$.*

The remarkable fact is that, as proved in [Fis15], the pseudodifferential calculus constructed in this way is equivalent to the pseudodifferential calculus constructed considering G as a manifold. Precisely the following result was proven in [Fis15]:

Theorem 6.5. *Let $\delta_2 = 1 - \delta_1 =: \varrho$; then the class $\mathrm{OPS}^m_{F,\delta_1,\delta_2}$ coincides with the class of pseudodifferential operators in the sense of Hörmander with symbol $S^m_{H,\varrho}$ on G.*

By Theorem 6.5, the proof that the operators $A_1, \dots, A_d$ defined in (3.12) are actually pseudodifferential operators of order 1 reduces to the following:

Lemma 6.6. *For* $j = 1, \dots, d$ *let* σ_{A_j} *be defined as in* (3.13), *then* $\sigma_{A_j} \in S^1_{F,1,0}$.

Proof. According to Section 3.2 of [Fis15], since the symbols σ_{A_j} are independent of x, it is sufficient to prove that for any $\beta \in \mathbb{N}$ and $\Delta_\tau^\beta = \Delta_{\tau_1} \cdots \Delta_{\tau_\beta}$ there exists $C = C_\tau > 0$ such that

$$\|\Delta_\tau^\beta \sigma_{A_j}(\xi)\|_{\mathcal{B}(\mathcal{H}^{\otimes \tau})} \leq C_\tau (1 + \lambda_\xi)^{\frac{1-\beta}{2}} \quad \forall \xi \in \widehat{G} \,. \tag{6.13}$$

If $\beta = 0$, one has

$$\|\sigma_{A_j}(\xi)\|_{\mathcal{B}(\mathcal{H}^{\otimes \tau})} = \max_{\xi \in \widehat{G}} \{\mathtt{w}_\xi^j + 1\} \lesssim (1 + \lambda_\xi)^{\frac{1}{2}} \,. \tag{6.14}$$

Consider now $\beta = 1$: then one has to estimate $\forall \tau \in \widehat{G}$

$$\|\Delta_\tau \sigma_{A_j}(\xi)\|_{\mathcal{B}(\mathcal{H}_\tau \otimes \mathcal{H}_\xi)} = \|\sigma_{A_j}(\tau \otimes \xi) - \sigma_{A_j}(\mathbf{1}_{\mathcal{H}_\tau} \otimes \xi)\|_{\mathcal{B}(\mathcal{H}_\tau \otimes \mathcal{H}_\xi)} \,.$$

Now (see for instance [FH13], Exercise 25.33), for any $\tau, \xi \in \widehat{G}$, if $\mathtt{w}_\xi$ is the highest weight of ξ, then the representation $\tau \otimes \xi$ is isomorphic to a finite direct sum of representations ζ_μ with highest weight $\mathtt{w}_\xi + \mu$, for any weight μ of the representation τ. Thus one has

$$\begin{aligned} \Delta_\tau \sigma_{A_j}(\xi) &= \bigoplus_{\mu \in \text{ weight of } \tau} \left((\mathtt{w}_\xi + \mu)^j + 1 - \left(\mathtt{w}_\xi^j + 1\right)\right) \mathbf{1}_{\mathcal{H}_{\zeta_\mu}} \\ &= \bigoplus_{\mu \in \text{ weight of } \tau} \mu^j \, \mathbf{1}_{\mathcal{H}_{\zeta_\mu}} \,. \end{aligned} \tag{6.15}$$

This implies

$$\|\Delta_\tau \sigma_{A_j}(\xi)\|_{\mathcal{B}(\mathcal{H}_\tau \otimes \mathcal{H}_\xi)} = \max_{\mu \in \text{ weight of } \tau} |\mu^j| =: C_\tau \,,$$

which gives (6.13) with $\beta = 1$. Finally, if $\beta \geq 2$, by (6.15) one gets that $\Delta_\tau^\beta \sigma_{A_j} = 0\,$, thus (6.13) is trivially verified. □

A The generalized commutator lemma

In this section we apply the commutator lemma proved in [Ras12] to our pseudodifferential setting.

Given a multiindex $j = (j_1, ..., j_d)$, and d selfadjoint pairwise commuting operators $A_1, ..., A_d$, we will denote

$$\mathrm{Ad}_A^j := \mathrm{Ad}_{A_1}^{j_1} \ldots \mathrm{Ad}_{A_d}^{j_d} \ .$$

Then the following theorem holds:

Theorem A.1 (Theorem 3 of [Ras12])**.** *Let $B \in \mathcal{B}(\mathcal{H})$ be such that $\mathrm{Ad}_A^j B \in \mathcal{B}(\mathcal{H})$ for all multiindexes j with $|j| \leq n_0$. Let $f \in S_1^m$, then, for all positive t_1, t_2 and all integers n fulfilling*

$$t_1 + t_2 + m < n + 1 \ , \quad n + 1 \leq n_0$$

one has

$$[B; f(A)] = \sum_{|j|=1}^{n} \frac{1}{j!} \partial^j f(A)\, \mathrm{Ad}_A^j B + R_n(A, B)\, . \tag{A.1}$$

Furthermore there exists a positive constant C (which depends on n), independent of A and B, s.t.

$$\| R_n(A, B) \|_{\mathcal{B}(\mathcal{H}^{-t_2}, \mathcal{H}^{t_1})} \leq C \sum_{|j| \leq n+1} \left\| \mathrm{Ad}_A^j B \right\|_{\mathcal{B}(\mathcal{H})} \ .$$

Corollary A.2. *Let $B \in \mathcal{A}_\rho^{m'}$, let f be as above and let $A_1, ..., A_d$ be quantum actions as in Assumption A; let N and n fulfill $N + m < \rho(n+1)$ then equation* (A.1) *holds. Furthermore there exists J and for any $s \in \mathbb{R}$ there exists a constant $C_{s,N}$ s.t. the following estimate holds*

$$\| R_n(A, B) \|_{\mathcal{B}(\mathcal{H}^s; \mathcal{H}^{s+N-m'})} \leq C_{s,N} \wp_{\rho,J}^m (B) \ . \tag{A.2}$$

To get the corollary, just apply Theorem A.1 to the operator $\langle A \rangle^{-m'-n(1-\rho)+s} B \langle A \rangle^{-s}$ and take $t_2 = 0$, $t_1 = N$. □

B Steepness in 2 degrees of freedom

We will use here a condition equivalent to steepness introduced in [Nie06].

Theorem B.1 (Niederman)**.** *Let h be a function real analytic in an open set $\mathcal{U} \subset \mathbb{R}^d$. Then h is steep on any compact set $\Sigma \subset \mathcal{U}$ if and only if h has no critical points in $\mathcal{U}$, and its restriction $h\big|_M$ to any affine subspace $M \subset \mathbb{R}^d$ admits only isolated critical points.*

If $d = 2$, then, by homogeneity, one gets a condition easy to verify. To state the corresponding result, we recall the definition of the Arnold determinant $\mathcal{D}$:

$$\mathcal{D} := \det \begin{bmatrix} \frac{\partial^2 h}{\partial a^2} & \left(\frac{\partial h}{\partial a}\right)^t \\ \frac{\partial h}{\partial a} & 0 \end{bmatrix} \ . \tag{B.1}$$

Theorem B.2. *Let $\mathcal{C}_e \subset \mathbb{R}^2$ be an open convex cone, and let $h : \mathcal{C}_e \to \mathbb{R}$ be a real analytic function homogeneous of degree $\mathtt{d}$. Assume that there exists $\bar{a} \in \mathcal{C}_e$ s.t. $\mathcal{D}(\bar{a}) \neq 0$, and that $h(a) > 0\ \forall a \in \mathcal{C}_e$, then h is steep in any compact subset of $\mathcal{C}_e$.*

Proof. Step 1. We prove that h has no critical points in $\mathcal{C}_e$. Indeed, if $\bar{a}$ is a critical point of h then, $\forall t > 0$, also $t\bar{a}$ is a critical point of h. It follows that h is constant along the line $t\bar{a}$, $t \in \mathbb{R}^+$, but one has $h(t\bar{a}) = t^{\mathtt{d}} h(\bar{a})$ which is not constant, since h is positive.

Step 2. We show that the set where $\mathcal{D}(a)$ vanish is composed of radial lines. Assume by contradiction that it vanishes on a curve intersecting transversally (topologically) a radial line Δ, then, since $\mathcal{D}$ is a homogeneous function, it vanishes on a whole open cone containing Δ, thus by analyticity it is identically zero, which contradicts the assumptions.

Step 3. The restriction of h to any non radial straight line Δ is a nontrivial analytic function. Indeed, by step 2, there exists at least one point $a_0 \in \Delta$ with $\mathcal{D}(a_0) \neq 0$, and recall that $\mathcal{D}(a_0) \neq 0$ implies that h is quasiconvex at a_0. Write Δ as the set of the points of the form $a = tn + a_0$, with n a unitary vector, one has

$$h(tn + a_0) = h(a_0) + t dh(a_0)n + \frac{1}{2} d^2 h(a_0)(n, n)t^2 + ...$$

but, if $dh(a_0)n \neq 0$, then this is a nontrivial function of t, while, if $dh(a_0)n = 0$ then by quasiconvexity one has $d^2h(a_0)(n, n) \neq 0$, so also in this case this is a nontrivial function of t.

Step 4. The restriction of h to any radial line is a nontrivial analytic function. One has

$$h(ta_0) = t^{\mathtt{d}} h(a_0) \ ,$$

which is a nontrivial function of t, since $h(a_0) \neq 0$. ☐

C Further technical lemmas

In this appendix we collect a few technical results that we use to prove Theorem 5.25.

Remark C.1. *For $x > 0$, $y > 1$, $C > 0$ and $0 < a < 1$, one has*

$$x < C\,(y + x^a) \quad \Longrightarrow \quad \exists C' > 0 \ s.t. \ x < C' y \, .$$

Lemma C.2. *If $a, b \in \mathbb{R}^2$, $C > 0$ and $0 < \mu < 1$ are such that $\|a\|, \|b\| \geq 1$,*

$$\|a - b\| \leq C \|b\|^{\mu} \, ,$$

then one has

$$\begin{aligned} \|a - b\| \leq C\|b\|^{\mu} \lesssim_{\mu} C\|a\|^{\mu} + C\|a - b\|^{\mu} \, , \\ \|a - b\| \lesssim_{C,\mu} \|a\|^{\mu} \, . \end{aligned}$$

The proof is a simple computation and is omitted.

Lemma C.3. *Let $a, b \in \Lambda$ and $k, h, l \in \mathbb{Z}^d$ be such that $a+k, b+l \neq 0$. Suppose that there exist positive constants C, D, R, γ, μ and δ such that $\mu < 1-\delta$ and*

$$\begin{gathered} |\omega(a+k)\cdot h| \leq C\|a+k\|^{\mathtt{M}-\gamma}\|h\|\,, \\ \|a+k\| \geq R\,, \quad \|h\| \leq D\|a+k\|^{\mu}\,, \quad \|k\| \leq D\|a+k\|^{\mu}\,, \end{gathered} \tag{C.1}$$

$$\|b-a\| \leq D\|a\|^{1-\delta}\,, \quad \|l\| \leq D\|b+l\|^{\mu}\,. \tag{C.2}$$

Then there exist positive constants C^+, D^+, depending only on $C, D, \mathtt{M}, \gamma, \mu, \delta$, such that

$$|\omega(b+l)\cdot h| \leq C^+\|b+l\|^{\mathtt{M}-\min\{\gamma,\delta\}}\|h\|\,, \quad \|h\| \leq D^+\|b+l\|^{\mu}\,. \tag{C.3}$$

Proof. The second of (C.3) is simply obtained by triangle inequality and exploiting $\mu < 1$ and $1-\delta < 1$. To obtain the first one write

$$|\omega(b+l)\cdot h| \leq |\omega(a+k)\cdot h| + \int_0^1 \left|\left(\frac{\partial\omega(a+k+t\eta)}{\partial a}\eta\right)\cdot h\right| dt\,, \tag{C.4}$$

with $\eta := b+l-(a+k)$. Due to homogeneity of the function ω, this also implies

$$|\omega(b+l)\cdot h| \leq |\omega(a+k)\cdot h| + K\int_0^1 \|a+k+t\eta\|^{\mathtt{M}-1}\,dt\,\|\eta\|\|h\|\,, \tag{C.5}$$

with

$$K := \sup_{\|\hat{a}\|=1}\left\{\left\|\frac{\partial\omega(\hat{a})}{\partial a}\right\|\right\}.$$

Then the proof is concluded using triangular inequality. ☐

Lemma C.4. *Let $\omega_* \in \mathbb{R}^d$, $\varepsilon > 0$ and $M \subset \mathbb{R}^d$ be such that*

$$\|\Pi_M\omega_*\| \leq \varepsilon\,. \tag{C.6}$$

Define $M' := \Pi_{\omega_*^\perp}M$. If $\varepsilon \leq \frac{1}{2}\|\omega_*\|$, then one has

$$\|\Pi_{M'} - \Pi_M\| \leq 9\varepsilon\|\omega_*\|^{-1}\,. \tag{C.7}$$

Proof. Define $w = \Pi_M\omega_*$ and $v = \Pi_{M^\perp}\omega_*$, then one has

$$\omega_* = v + w\,, \quad v \in M^\perp\,, \quad w \in M\,, \tag{C.8}$$

with

$$\|\omega_* - v\| \leq \varepsilon\,. \tag{C.9}$$

Denote $M \oplus v := M \oplus \operatorname{span} v$; we first prove that

$$M \oplus v = M' \oplus \omega_* \, . \tag{C.10}$$

By the very definition of M' and v, one has

$$M' \subset M \oplus v \implies M' \oplus \omega_* \subset M \oplus \omega_* = M \oplus v \ .$$

To prove the opposite inclusion, remark that for any $m \in M$, one has $\Pi_{M'} m = \Pi_{\omega_*^\perp} m$

$$m = \Pi_{\omega_*} m + \Pi_{\omega_*^\perp} m = \Pi_{\omega_*} m + \Pi_{M'} m \in M' \oplus \operatorname{span}\{\omega_*\} \, ,$$

so we have

$$M \subset M' \oplus \omega_* \implies M' \oplus \omega_* \supset M \oplus \omega_* = M \oplus v \ .$$

Since $M \perp v = M' \perp \omega_*$, one has that, for any $u \in \mathbb{R}^d$

$$\Pi_M u = \Pi_{M \perp v} u - \frac{v \cdot u}{\|v\|^2} v \, , \quad \Pi_{M'} u = \Pi_{M' \perp \omega_*} - \frac{\omega_* \cdot u}{\|\omega_*\|^2} \omega_* \, . \tag{C.11}$$

thus

$$\|\Pi_M u - \Pi_{M'} u\| = \left\| \frac{v \cdot u}{\|v\|^2} v - \frac{\omega_* \cdot u}{\|\omega_*\|^2} \omega_* \right\| \, ,$$

from which the conclusion follows. □

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