

Mathematics Area — PhD course in
Geometry and Mathematical Physics

Modularity and Quantum Invariants of 3-Manifolds

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Abstract

Seen as complex-valued functions on rational numbers, quantum invariants of links and 3-manifolds have subtle analytic and algebraic features which seem to indicate that these topological invariants contain some information about the Riemannian structure of the underlying manifold. An example of this behavior is the quantum modularity conjecture, which is an asymptotic statement about the relation between quantum invariants whose arguments are related by a modular transformation.

In this thesis, we formulate a strong version of the quantum modularity conjecture for the $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev invariant of a geometric 3-manifold. In this formulation, the quantum modularity conjecture states that for a given geometric manifold (i) the $k \rightarrow \infty$ asymptotics of the WRT invariant $\tau(k)$ contains a sum over $\mathrm{SL}(2, \mathbb{C})$ flat connections on the manifold, (ii) there is a distinguished flat connection which appears with a factor of the WRT invariant itself, its argument having undergone a modular S -transformation, $\tau(-1/k)$, and (iii) this distinguished flat connection is determined by the Thurston geometry of the manifold. This generalizes the canonical $\mathrm{SL}(2, \mathbb{C})$ flat connection on hyperbolic manifolds to other Thurston geometries. We also prove that the conjecture holds for all Seifert homology spheres with three exceptional fibers, an infinite family of rational homology spheres, and some other examples.

Since the WRT invariant at a generic root of unity is morally the partition function of Chern–Simons theory at a fractional level, we also interpret this modular behavior in terms of the asymptotics of the contour path integral of analytically continued Chern–Simons theory.

Finally, we discuss an ongoing work about the unification of radial limits of the q -series valued $\widehat{Z}$ -invariants into a single element of an appropriately completed ring of polynomials $\mathbb{Z}[q]$. Existence of such a unification for integer homology spheres, and support from our numerical experiments indicate that such a unification might exist more generally.

“I do not care what comes after; I have seen the dragons on the wind of morning.”
— Ursula K. le Guin, *The Farthest Shore*.

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CHAPTER 1

Introduction

Quantum invariants of 3-manifolds, which are the subject of this thesis, belong to a relatively young branch of mathematics called quantum topology. Since they first appeared in the late 1980s [64, 56, 1], quantum invariants have provided topological invariants with the fresh perspective of topological quantum field theory, revealed deep connections with classical geometry (volume conjecture [33]), and sparked interest in entirely new kinds of mathematical objects (quantum modular forms [67], Habiro ring [25]).

One aim of this thesis is to understand the geometric information contained in the Witten–Reshetikhin–Turaev invariant of a 3-manifold—a purely topological object. This connection between geometry and topology is made manifest, by looking at the “small coupling” asymptotics of the WRT invariant and observing the phenomenon of quantum modularity. We give an elementary explanation of this quantum modular behavior later in this introduction.

An object related closely to quantum modularity, and one that is unique to quantum topology, is a certain completion of the ring of polynomials with integer coefficients, called the Habiro ring. It is inside the Habiro ring that the colored Jones polynomial of a knot finds its most natural home. In the final section of this chapter, we introduce some preliminary results from an ongoing work on the unification of radial limits of $\widehat{Z}$ -invariants into an element of the Habiro ring.

This chapter is dedicated to an opinionated tour of quantum topology for a reader unfamiliar with the subject. At the first two stops of this journey, we recall relevant facts about 3-manifold geometries, Chern–Simons theory, and quantum invariants necessary to motivate the statement of the quantum modularity conjecture in *Modularity*. The final stop is *Unification*, where we introduce unified invariants, connect them back to quantum modularity, and propose a plausible unification for the radial limits of $\widehat{Z}$ -invariants.

Geometry

Riemannian structures in topology are expected to be tied closely together in three dimensions due to Thurston’s geometrization [60]. Thurston’s geometrization is a weaker

statement than the uniformization of Riemann surfaces—not every 3-manifold is geometric, but any 3-manifold admits a decomposition into geometric pieces by cutting along 2-spheres and 2-tori. Each of these geometric pieces has the structure of a quotient M/Γ , where M is one of the eight Thurston geometries

$$S^3, \quad \mathbb{R}^3, \quad \mathbb{H}^3, \quad S^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \text{Nil}_3, \quad \text{Sol}_3, \quad \widetilde{\text{SL}(2, \mathbb{R})}, \quad (1.1)$$

and $\Gamma \subset \text{Isom } M$ is a discrete subgroup of isometries, acting freely and properly discontinuously on M . The geometric manifold $X \cong M/\Gamma$ inherits a Riemannian metric from M , and we say that X has a geometric structure modeled on M (see § 3.2 for an overview of the model geometries).

In the language of geometrization, the uniformization theorem says that every surface is geometric and has a geometric structure modeled on one of S^2 , $\mathbb{R}^2$ or $\mathbb{H}^2$. Moreover, which geometric structure a surface admits is determined completely by its Euler characteristic—a topological invariant.

For hyperbolic 3-manifolds, this connection between topology and Riemann structures is even stronger because the Mostow rigidity theorem implies that there is no moduli space of hyperbolic metrics—if it exists, the hyperbolic metric is unique [42, Chapter 13]. This means that the volume of a hyperbolic manifold is a topological invariant.

Euler characteristic	$\chi(S) > 0$	$\chi(S) = 0$	$\chi(S) < 0$
Geometric structure	S^2	$\mathbb{R}^2$	$\mathbb{H}^2$
Examples	$S^2, \mathbb{RP}^2$	$\mathbb{T}^2$, Klein bottle	Surfaces of genus ≥ 2

For complements of hyperbolic knots, this connection is captured by Kashaev’s volume conjecture [33]. Roughly speaking, the volume conjecture states that the “small coupling” asymptotics of the Jones polynomial are dominated by the hyperbolic volume of the knot complement. More precisely the conjecture states [44, 45],

$$\lim_{r \rightarrow \infty} \frac{2\pi}{r} \log J_r(K; e^{2\pi i/r}) \simeq \text{Vol}(S^3 \setminus K). \quad (1.2)$$

For hyperbolic 3-manifolds, there is a version of the volume conjecture due to Chen and Yang [8], which states that the “small coupling” asymptotics of the $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev (WRT) invariant is dominated by its complexified volume. For odd r , we have the following asymptotic statement

$$\lim_{r \rightarrow \infty} \frac{2\pi i}{r + 1/2} \log \tau_{r+1/2}(X) \simeq \text{Vol}_{\mathbb{C}}(X), \quad (1.3)$$

where $\text{Vol}_{\mathbb{C}}(X) := -4\pi^2 \text{CS}[A_*]$ for a canonical geometric $\text{SL}(2, \mathbb{C})$ flat connection on X determined by the hyperbolic structure. We explain why we consider the “small

coupling” asymptotics in the first place, and what the symbol $\text{CS}[A_*]$ means in the next section.

We emphasize that the Jones polynomial and the WRT invariant are topological invariants, and *a priori* there is no reason why their asymptotics should know about the hyperbolic structure.

Physics

The WRT invariant is a mathematical formalization of the Chern–Simons partition function which is defined heuristically by a path integral. We recall that Chern–Simons theory is a three dimensional topological gauge theory whose definition requires a compact connected simple Lie group G , and an inverse coupling constant k . The classical action of the theory is given by the integral of the Chern–Simons 3-form,

$$\text{CS}[A] = \frac{1}{8\pi^2} \int_X \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (1.4)$$

and the partition function of the quantum theory is given by the path integral

$$Z_k^G(X) = \int_{\mathcal{U}_G} \mathcal{D}A e^{2\pi i k \text{CS}[A]}. \quad (1.5)$$

Up to some constants and choice of normalization, we have $\tau_r(X) = Z_r^{\text{SU}(2)}(X)$. In what follows, we let $G = \text{SU}(2)$ be understood.

Gauge invariance of the path integral forces k to be an integer, but there is a way to analytically continue to arbitrary complex k by complexifying the gauge group $\text{SU}(2) \rightarrow \text{SL}(2, \mathbb{C})$ and integrating over an appropriate middle-dimensional contour in $\mathcal{U}_{\text{SL}(2, \mathbb{C})}$ (see § 4.3 for more details),

$$Z_k(X) = \int_{\Gamma \subset \mathcal{U}_{\text{SL}(2, \mathbb{C})}} \mathcal{D}A e^{2\pi i k \text{CS}[A]}. \quad (1.6)$$

A standard way to better understand path integrals of this type is to look at the small coupling limit $k \rightarrow \infty$. If we pretend that the path integral is finite dimensional, we can compute a saddle point expansion around stationary points of CS, which are $\text{SL}(2, \mathbb{C})$ flat connections. We let $k = r/s \in \mathbb{Q}$ and for $r \rightarrow \infty$ we have

$$Z_{r/s}(X) = \sum_{\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(X, \text{SL}(2, \mathbb{C})))} P_\alpha(e^{-2\pi i r/s}) e^{2\pi i \frac{s}{r} \text{CS}[A_\alpha]} Z_\alpha^{\text{pert}}(r/s), \quad (1.7)$$

where the sum is over $\text{SL}(2, \mathbb{C})$ flat connections, P_α are polynomials in $e^{-2\pi i r/s}$, and Z_α^{pert} are asymptotic series in s/r . Recall that the moduli space of flat connections is defined

as

$$m_{\text{flat}}(X, G) := \text{Hom}(\pi_1(X), G)/G, \quad (1.8)$$

where G acts by conjugation.

Comparing with the volume conjecture, we see that Chen and Yang's statement concerns the case $s = 2$, and says that there is a special connection A_* , compared to which, all other contributions $e^{2\pi i \frac{s}{r} \text{CS}[A]}$ are exponentially suppressed. Moreover, this special flat connection specified by the hyperbolic structure of X as follows. Note that $X \cong \mathbb{H}^2/\Gamma$ for some $\Gamma \subset \text{Isom}_+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$. The canonical geometric flat connection A_* is determined by the lift of the inclusion $\pi_1(X) \cong \Gamma \hookrightarrow \text{PSL}(2, \mathbb{C})$ to a map $\pi_1(X) \rightarrow \text{SL}(2, \mathbb{C})$.

There is a refinement of the Chen–Yang volume conjecture to arbitrary s , and it is called the quantum modularity conjecture.

Modularity

The quantum modularity conjecture for closed hyperbolic manifolds as formulated by Wheeler [63], makes the following claim about the $r \rightarrow \infty$ asymptotics of the WRT invariant

$$\tau_{r/s}(X) \simeq e^{2\pi i \frac{r}{s} \text{CS}[A_*]} (1 - e^{-2\pi i s/r}) \tau_{-s/r}(X) I(s/r) + o(e^{2\pi i \frac{r}{s} \text{CS}[A_*]}), \quad (1.9)$$

where $I(s/r)$ is an asymptotic series in s/r . Comparing with (1.7), we see that contributions from flat connections other than the geometric one are exponentially suppressed, and moreover the polynomial P_α , attached to the canonical flat connection is equal to the WRT invariant itself, its level having undergone a modular S -transformation $r/s \mapsto -s/r$, i.e., $P_{\alpha_*}(e^{-2\pi i r/s}) = (1 - e^{2\pi i s/r}) \tau_{-s/r}(X)$.

In [53], my supervisor and I formulated the following version of the quantum modularity conjecture for geometric integer homology spheres. To make expressions look nicer, we use the notation $\tau_X(\xi)$ for the WRT invariant that we denoted above by $\tau_{r/s}(X)$, where $\xi = e^{2\pi i s/r}$.

Conjecture 1.1. *Let $X \cong M/\Gamma$ be a geometric integer homology sphere with model geometry M , and let $\xi = e^{2\pi i s/r}$ be a primitive r -th root of unity for some $r \in 2\mathbb{Z}_{\geq 1} + 1$ and $s \in 4\mathbb{Z} + 1$.*

1. *As $r \rightarrow \infty$ along integers coprime with s , the WRT invariant has the following asymptotic expansion*

$$\tau_X(\xi) \simeq \sum_{\alpha \in \pi_0(m_{\text{flat}}(X, \text{SL}(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[A_\alpha]} P_\alpha(\xi) I_\alpha\left(\frac{s}{r}\right), \quad (1.10)$$

where $\tilde{\xi} = e^{-2\pi i r/s}$, $P_\alpha(\tilde{\xi}) \in \mathbb{Z}[\tilde{\xi}]$, and $I_\alpha(s/r) \in (s/r)^{\delta_\alpha/2} \mathbb{C}[[s/r]]$ for some $\delta_\alpha \in \mathbb{Z}$.

2. There is a geometric flat connection α_* specified by the map $\pi_1(X) \cong \Gamma \rightarrow \mathrm{SL}(2, \mathbb{C})$, which is a lift of the composition of the canonical inclusion $\Gamma \hookrightarrow \mathrm{Isom}_+ M$ with a certain natural map $\mathrm{Isom}_+ M \rightarrow \mathrm{PSL}(2, \mathbb{C})$, and we have

$$P_{\alpha_*}(\xi) = \xi^\delta (1 - \xi) \tau_X(\xi) - \chi_{S^3}(M) \quad (1.11)$$

where δ is an integer and the function $\chi_{S^3}(M) = 1$ if $M = S^3$ and zero otherwise.

A few remarks are in order.

Remark 1.1. The restriction $s = 1 \bmod 4$ in the conjecture is only stated to fix the sign of a Gauss sum. We can formulate a similar statement for $s = 3 \bmod 4$, at the cost of a phase factor.

Remark 1.2. The statement $P_{\alpha_*}(\xi) = \xi^\delta (1 - \xi) \tau_X(\xi)$ implies that $(1 - \xi) \tau_X(\xi) \in \mathbb{Z}[\xi]$. This is consistent with the integrality of the WRT invariant $\tau_X(\xi) \in \mathbb{Z}[\xi]$, which follows from the existence of a universal invariant for integer homology spheres in the Habiro ring, whose evaluation at a root of unity gives the WRT invariant [25, Theorem 4.4].

Remark 1.3. The main novelty here is the identification of a geometric flat connection for non-hyperbolic geometries. In Chapter 7, we give candidates for the map $\mathrm{Isom}_+ M \rightarrow \mathrm{PSL}(2, \mathbb{C})$ for every model geometry (§ 7.1.2), prove that the conjecture holds for all Brieskorn homology spheres (Theorem 7.1), formulate a version of the conjecture for $\mathbb{Z}/2\mathbb{Z}$ -homology spheres (Conjecture 7.1), and work out some rational homology sphere examples where the more general conjecture holds (§ 7.3).

Remark 1.4. Kashaev's volume conjecture for hyperbolic knot complements can also be further refined to a statement about quantum modularity [19].

Unification

A prototypical example of an element of the Habiro ring is the Kontsevich–Zagier function [66]

$$F(q) = \sum_{n=0}^{\infty} \left(\prod_{j=1}^n (1 - q^j) \right), \quad (1.12)$$

defined only when q is a root of unity. Despite not being a function on any open subset of the complex plane, or even a formal q -series, $F(q)$ has “good” behavior under modular transformations as a function on the rationals. In particular, the function $\phi(x) = q^{1/24} F(q)$ under a modular T -transformation acquires a phase $\phi(x+1) = e^{\pi i/12} \phi(x)$, and its failure to be modular under an S -transformation

$$\phi(x) \pm i^{1/2} |x|^{3/2} \phi(-1/x) = g(x) \quad (1.13)$$

where $q = e^{2\pi i x}$

extends to a real analytic function on the reals.

Such formulas are proved by relating $F(q)$ to the radial limit of the Eichler integral of a weight-1/2 modular form—in this case the Dedekind eta function. This kind of argument is the same one that appears on the way to the proof of the quantum modularity conjecture for Brieskorn homology spheres. This is not a surprise because a theorem due to Habiro [25, Theorems 4.1 and 4.4] says that the WRT invariant of integer homology spheres can be unified into an expression of the type (1.12)

$$\tau_X(q) = \sum_{n=0}^{\infty} a_n(q) \left(\prod_{j=1}^n (1 - q^j) \right), \quad (1.14)$$

where $a_n(q) \in \mathbb{Z}[q]$. Like the function $F(q)$, the above expression is defined only at roots of unity, and its evaluation at a root of unity would give the WRT invariant.

Recent years have seen a lot of interest in $\widehat{Z}$ -invariants [23, 21, 22], which are q -series valued invariants of 3-manifolds, labeled by $\text{Spin}^{\mathbb{C}}$ -structures [20, § 4.2]. These formal power series can variously be understood as a kind of quantum group invariant, a non-perturbative completion of complex Chern–Simons theory or BPS state counts of a certain supersymmetric quantum field theory. The relevant facts for our purposes are that (i) the q -series $\widehat{Z}_b(q)$ converges inside the unit disk $|q| < 1$, and its radial limit at roots of unity is related to well-defined quantum invariants of 3-manifolds, and (ii) the q -series are known to have quantum modular behavior [9, 6, 11, 10].

Motivated by its radial limits and quantum modularity, we can ask if there is an element of the Habiro ring $\widehat{\mathcal{W}}_b(q)$ with an expansion like (1.14), such that

$$\text{ev}_{\xi} \widehat{\mathcal{W}}_b(q) = \lim_{q \rightarrow \xi} \widehat{Z}(q). \quad (1.15)$$

We discuss the answer to this question in Chapter 8.

Outline

Like this introduction, the thesis is divided roughly into four parts.

Chapters 2 and 3 introduce the necessary background in the topology and geometry of 3-manifolds. Highlights from these chapters include the sections on the category of tangles (§ 2.3) and Thurston’s geometrization (§ 3.2).

Chapters 4 and 5 discuss the necessary background in Chern–Simons theory and quantum invariants. Highlights from this part include the asymptotic expansion of the Chern–Simons partition function when the level is analytically continued to a rational number (§ 4.3), and the definition of the WRT invariant (§ 5.2).

The next two chapters are the heart of the thesis. Chapter 6 introduces modular forms and Eichler integrals of theta functions, which are the main technical tools used to prove that Conjecture 1.1 holds for Brieskorn homology spheres (Theorem 7.1). Chapter 7 is based on joint work with my supervisor [53] and contains the most general version of the quantum modularity conjecture (Conjecture 7.1), candidates for geometric flat connection for every Thurston geometry (§ 7.1.2), Theorem 7.1 and its proof, and evidence that the conjecture holds in case of a number of rational homology sphere examples, including the infinite family $S^2(0; p, -2p - 1, -2p - 1)$ for $p \in \mathbb{Z}_{\geq 2}$.

Chapter 8 introduces the Habiro ring, $\widehat{Z}$ -invariants, strange identities and a conjectural unification of $\widehat{Z}$ -invariants into elements of Habiro ring, at least when the manifold is a knot surgery.

Appendix A collects some formulas and tricks for evaluating quadratic Gauss sums. We also use these Gauss sums to refine the radial limit conjecture for $\widehat{Z}$ -invariants (§ A.1). Appendix B contains a long proof of an integrality result for radial limits of a certain basis of false theta functions.

CHAPTER 2

Knots, Links, and Tangles

For the sake of self-containedness, we start by reviewing some preliminary knot theory— isotopies, Reidemeister moves, orientations, and framings—followed by a discussion of knot polynomials by their skein relations. The largest part of this chapter, however, is § 2.3 which describes the category of oriented and framed tangles, and explains how this category, OFrTang , is a model for the definition of a ribbon category, which sits at the heart of this subject.

2.1 Preliminaries

Knots are smooth embeddings of the circle into a 3-manifold. Let X be a 3-manifold; a *link* in X is a smooth embedding of a finite disjoint union of circles $K : S^1 \sqcup \cdots \sqcup S^1 \rightarrow X$. Each image $K(S^1)$ is called a component of the link K . We also use the notation $K = K_1 \sqcup \cdots \sqcup K_V$, where the integer V is the number of link components, and for each j , we have the embedding $K_j : S^1 \rightarrow X$ such that $K_j(S^1) \cap K_k(S^1) = \emptyset$ if $j \neq k$. For a good introduction to knot theory, we refer the reader to [34] and [54].

Remark 2.1. We will engage in a common abuse of notation and use the letter K to refer to both the embedding and its image in X interchangeably.

Remark 2.2. We will be a little cavalier about the category we are working in because the topological, piecewise-linear, and smooth categories are all equivalent in three-dimensions.

The appropriate notion of equivalence for knots and links is that of isotopy. Let K and L be links in a 3-manifold X . An *ambient isotopy* is a continuous family of homeomorphisms $f : [0, 1] \times X \rightarrow X$ such that $f_0 = \text{id}_X$ and $L = f_1 \circ K$. If such an isotopy exists, then we say that the links K and L are *ambient isotopic*. We will often drop the adjective “ambient” because these are the only isotopies we will encounter in this thesis.

Remark 2.3. Enumeration of link components, $K_1, \dots, K_V$ is completely superficial because differently enumerated links are identified under an isotopy.

Remark 2.4. Since $S^3 \cong \mathbb{R}^3 \cup \infty$, a knot in $\mathbb{R}^3$ can always be thought of as a knot in S^3 . Conversely, a knot in S^3 can always be thought of as a knot in $\mathbb{R}^3$ because we can always deform a knot in S^3 away from the point at infinity by ambient isotopy.

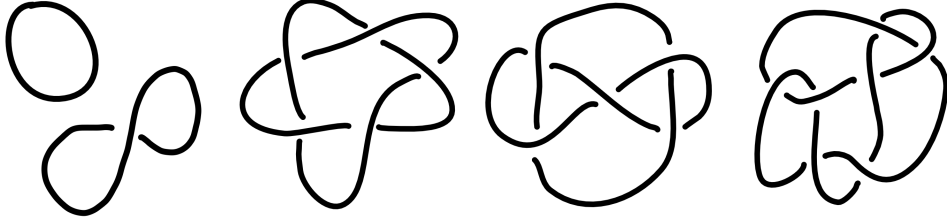


Figure 2.1: Some knot diagrams.

Knots and links in $\mathbb{R}^3$ (or S^3 , see Remark 2.4) are represented by diagrams in the plane. A knot diagram is the image of the knot under a projection of $\mathbb{R}^3$ onto a plane such that all self-intersections are isolated and consist of exactly two points. Self-intersections are resolved in a diagram by drawing over and under crossings (see Figure 2.1).

A cornerstone of knot theory is the *Reidemeister theorem* (see [54, § 1], which says that two diagrams belonging to the same isotopy class of a link are related by a finite sequence of the so-called *Reidemeister moves* (Figure 2.2).

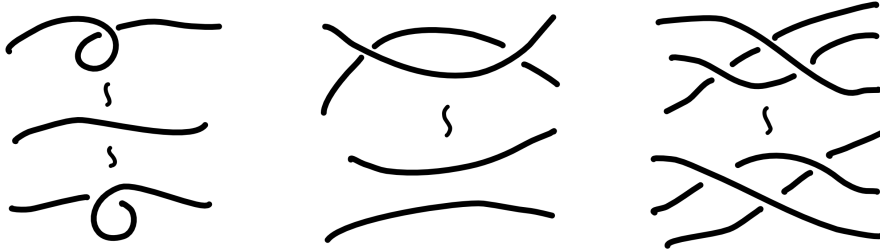


Figure 2.2: The three Reidemeister moves R_1 , R_2 , and R_3 .

Knots and links can carry extra structure. When this is the case, we require that the ambient isotopy preserve this structure. If a link is oriented, for example, we speak of ambient isotopies of oriented links, and the Reidemeister moves in Figure 2.1 have to be augmented to keep track of orientations.

One rudimentary isotopy invariant of oriented links is the *linking number*. For distinct components $K_i \neq K_j$ of an oriented link, it is defined as half the signed sum of crossings of K_i and K_j ,

$$\text{lk}(K_i, K_j) = \frac{1}{2} \sum_{p \in K_i \cap K_j} \epsilon(p), \quad (2.1)$$

where $\epsilon(p) = \pm 1$ for crossings of type $K_{\pm}$ respectively (see Figure 2.1). Note that the linking number changes sign if we flip the orientation on one of the components, and remains unchanged if both orientations are flipped simultaneously.

Remark 2.5. There is also an integral formula for the linking number due to Gauss

$$\text{lk}(K_i, K_j) = \frac{1}{4\pi} \int_{K_i} \int_{K_j} dx_i^\nu dx_j^\mu \epsilon_{\mu\nu\lambda} \frac{(x_i - x_j)^\lambda}{|x_i - x_j|^3}. \quad (2.2)$$

Such integrals also occur in the calculation of expectation values of Wilson lines in Chern–Simons theory.

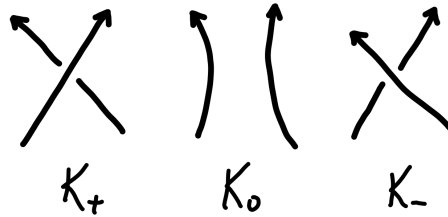


Figure 2.3: Convention for positive and negative crossings.

A link is *framed* if each of its components carries a homotopy class of non-singular normal vector field. For each link component, the homotopy class of this vector field is determined by the linking number of K_i and K'_i , where $K'_i \subset X \setminus K_i$ is obtained by pushing K_i a small distance along the normal vector field. The framing of a link component is alternatively called its *selflinking number*.

Remark 2.6. To compute the linking number, $\text{lk}(K_i, K'_i)$, we orient both K_i and K'_i in the same direction. The linking number is the same as long as both components are oriented in the same direction.

Remark 2.7. Framed knots are important because they describe how a 2-handle, $B^2 \times B^2$, should be glued to the boundary of the 4-ball, B^4 . It is a fact that every closed 3-manifold can be constructed from such gluing operations. See § 3.1 for details.

Remark 2.8. When $X = \mathbb{R}^3$ (or S^3), we can always pick a framing such that the self linking number is zero. But this is not always possible when X is a generic 3-manifold.

When we draw diagrams of framed links, we will denote the framing by writing the self linking number next to each component. When the self linking number is omitted, we shall assume that the knot is given the *blackboard framing*. A blackboard framed knot has its normal vector field pointed outward tangent to the blackboard (or a different drawing surface) at each point.

For blackboard framed diagrams, the first Reidemeister move is no longer an isotopy of framed links. We replace it by the following modified version which lets us undo twists.



Figure 2.4: The modified first Reidemeister move for blackboard framed link diagrams.

2.2 Polynomial invariants

Before we describe the hefty Reshetikhin–Turaev machinery for constructing isotopy invariants of knots and links, we recall the definition of some knot polynomials in terms of skein relations. The oldest and most classical of these is the Alexander polynomial, valued in $\mathbb{Z}[t^{\pm 1/2}]$ and defined by the skein relation (see [54, Proposition 2.8.3])

$$\Delta_{K_+} - \Delta_{K_-} = (t^{1/2} - t^{-1/2})\Delta_{K_0}, \quad (2.3)$$

and a normalization $\Delta_U = 1$, where U is the unknot.

The Jones polynomial, discovered in the 1980s [31, 32] satisfies the skein relation

$$q^2 V_{K_+} - q^{-2} V_{K_-} = (q - q^{-1})V_{K_0}. \quad (2.4)$$

To compute the Jones polynomial from this relation, one would need a choice of normalization—its value on the unknot, V_U . One convenient choice is to set $V_U = 1$. Note that despite the very similar looking skein relations, unlike the Alexander polynomial, the Jones polynomial can distinguish mirror images of achiral knots.

Finally, we have the two-variable HOMFLY-PT polynomial, valued in $\mathbb{Z}[a^{\pm}, q^{\pm}]$ and defined by the skein relation [18, 24]

$$a P_{K_+} - a^{-1} P_{K_-} = (q - q^{-1})P_{K_0}, \quad (2.5)$$

and a normalization $P_U = 1$. A curious thing about the HOMFLY-PT skein relation is that it subsumes both the Alexander polynomial and the Jones polynomial. Indeed $\Delta_K = P_K|_{a=1}$ and $V_K = P_K|_{a=q^2}$.

We also have the so-called $\mathfrak{sl}_N$ link polynomials which are valued in $\mathbb{Z}[q^{\pm 1}]$ and can be understood as another specialization of the HOMFLY-PT polynomial given by $V_K^{\mathfrak{sl}_N} = P_K|_{a=q^N}$. More fundamentally, these polynomials can be understood as the quantum link invariant associated to the quantum group $U_q \mathfrak{sl}_N$, when each component of the link is colored by the vector representation of $\mathfrak{sl}_N$. We discuss colored links again at the end of the following section.

2.3 The category of tangles

A tangle is an embedding of intervals, $I = [0, 1]$, and circles, S^1 , into $\mathbb{R}^2 \times I$ with the conditions that (i) the boundaries of intervals are incident on $\mathbb{R} \times 0 \times \{0, 1\}$, and (ii) no other part of the tangle intersects $\mathbb{R}^2 \times \{0, 1\}$. For non-negative integers k and l , a (k, l) -tangle intersects $\mathbb{R}^2 \times 0$ in k points, and $\mathbb{R}^2 \times 1$ in l points. For precise definitions of objects discussed in this section, we refer the reader to [61, Chapter I].

Remark 2.9. Like knots and links, we consider tangles upto ambient isotopy.

Remark 2.10. Knots and links are $(0, 0)$ -tangles, and a braid on n strands is an (n, n) -tangle.

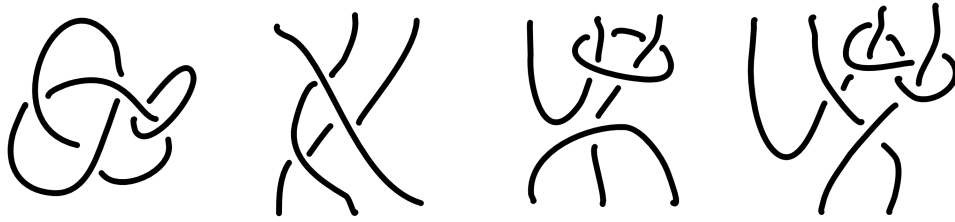


Figure 2.5: A $(0, 0)$ -tangle, a braid on three strands, a $(3, 3)$ -tangle that is not a braid, and a $(2, 4)$ -tangle.

Remark 2.11. Boundary of every (k, l) -tangle, T , admits a decomposition into *inputs*, $\partial_- T := \partial T \cap \mathbb{R} \times 0 \times 0$, and *outputs*, $\partial_+ T := \partial T \cap \mathbb{R} \times 0 \times 1$. We have $\partial T = \partial_- T \sqcup \partial_+ T$ and $|\partial_- T| = k$ and $|\partial_+ T| = l$.

Remark 2.12. Points in $\partial_- T$ and $\partial_+ T$ can be ordered from left to right (in ascending order of their x -coordinate).

We can now describe the category of tangles, Tang . Objects of this category are non-negative integers, and for any objects k and l , a morphism $k \rightarrow l$ is a (k, l) -tangle. Morphisms $T : k \rightarrow l$ and $U : l \rightarrow m$ are composed by gluing l outputs of T to the l inputs of U in order, and scaling the tangle to $\mathbb{R}^2 \times I$. The identity morphism $k \rightarrow k$ is the (k, k) -tangle with k parallel strands.

The motivation behind this categorical description of tangles is to build an algebraic description of knot theory in $\mathbb{R}^3$. In what follows, we shall describe a category with a richer structure—of oriented and framed tangles—which is a model for *ribbon categories*, and is a building block of three-dimensional topological quantum field theories.

Let OFrTang be the category of oriented framed tangles. Objects of this category are ordered sets of signs, $\mathcal{Ob} = \{u = (u_1, \dots, u_k) \in \{\pm 1\}^k \mid k \in \mathbb{Z}_{\geq 0}\}$. Morphisms, $T : u \rightarrow v$, are regular isotopy classes of tangles whose components carry orientations and framings. Orientations are such that the j -th strand incident on $\partial_- T$ (or $\partial_+ T$) points downwards if $u_j = +1$ (or $v_j = +1$ resp.) and upwards if $u_j = -1$ (or $v_j = -1$).

resp.). See Figure 2.6 for examples. As in Tang , morphisms are composed by gluing and rescaling tangles, and identity morphisms are appropriately oriented, 0-framed parallel strands.

Remark 2.13. If a framing is not indicated, we shall assume that framed tangle diagrams are blackboard framed.

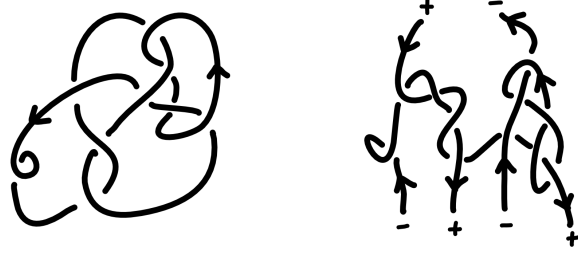


Figure 2.6: Morphisms in OFrTang .

Remark 2.14. Components of a framed tangle can also be thought of as ribbons, homeomorphic to $I \times I$ or $S^1 \times I$, embedded in $\mathbb{R}^2 \times I$ such that their cores, $I \times 1/2$ or $S^1 \times 1/2$, coincide with the tangle components and their surfaces are tangent to normal vector fields of the framing.

Remark 2.15. For brevity, we shall refer to OFrTang as the category of ribbon tangles.

OFrTang has a natural *monoidal structure*, i.e., there is a way to tensor objects and morphisms of the category. On objects, the tensor product is defined as disjoint union, for $u, v \in \mathcal{O}\mathcal{B}$, we have $u \otimes v := (u_1, \dots, u_k, v_1, \dots, v_k)$. For morphisms, $T : u \rightarrow v, T' : u' \rightarrow v'$, the tensor product $T \otimes T' : u \otimes u' \rightarrow v \otimes v'$ is defined as the tangle diagram obtained by adjoining T' to the right of T . The empty set in $\mathcal{O}\mathcal{B}$ is the monoidal unit, $\mathbb{1}$.

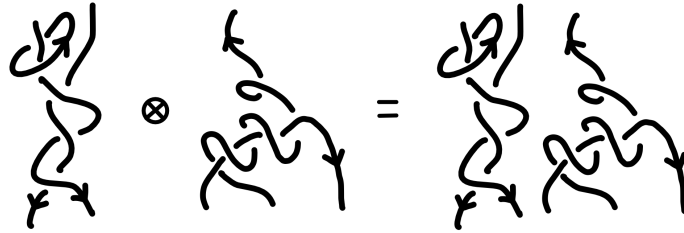


Figure 2.7: Tensor product of tangles.

OFrTang also has a braiding. In a monoidal category, *braiding* is defined as a natural family of isomorphisms $c = \{c_{u,v} : u \otimes v \rightarrow v \otimes u \mid u, v \in \mathcal{O}\mathcal{B}\}$ such that

$$c_{u,v \otimes w} = (\text{id}_v \otimes c_{u,w}) \circ (c_{u,v} \otimes \text{id}_w), \quad (2.6)$$

$$c_{u \otimes v, w} = (c_{u,w} \otimes \text{id}_v) \circ (\text{id}_u \otimes c_{v,w}), \quad (2.7)$$

for all objects u, v, w . Naturality of c means that for any $T : u \rightarrow u'$ and $U : v \rightarrow v'$, we have $(U \otimes T) \circ c_{u,v} = c_{u',v'} \circ (T \otimes U)$.

Remark 2.16. A monoidal category with a braiding is called a *braided monoidal category*.

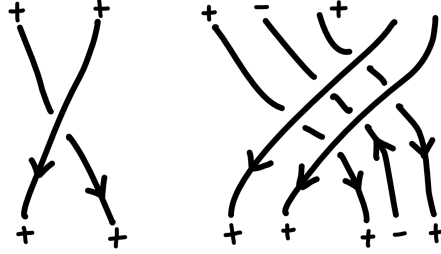


Figure 2.8: Braiding in OFrTang.

In the category of tangles, the braiding is given by the braiding of strands (see Figure 2.8). That this braiding satisfies Equations 2.6–2.7 and the naturality condition, follows from regular isotopy. Indeed, it can be shown that the braiding in any monoidal category satisfies the Yang–Baxter identity,

$$\begin{aligned} &(\text{id}_m \otimes c_{k,l}) \circ (c_{k,m} \otimes \text{id}_l) \circ (\text{id}_k \otimes c_{l,m}) \\ &= (c_{l,m} \otimes \text{id}_k) \circ (\text{id}_l \otimes c_{k,m}) \circ (c_{k,l} \otimes \text{id}_m), \end{aligned} \quad (2.8)$$

for any objects k, l , and m —which is just an algebraic version of the third Reidemeister move (see Figure 2.9).

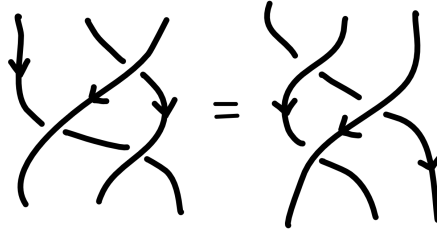


Figure 2.9: Yang–Baxter identity in OFrTang for $k = l = m = (+1)$.

OFrTang has a twist. In a braided monoidal category, a *twist* is defined as a natural family of isomorphisms $\theta = \{\theta_u : u \rightarrow u \mid u \in \mathcal{O}\mathfrak{b}\}$ that are compatible with the braiding in the following sense,

$$\theta_{u \otimes v} = c_{v,u} \circ c_{u,v} \circ (\theta_u \otimes \theta_v), \quad (2.9)$$

for all objects u, v . Naturality of θ means that for any $T : u \rightarrow v$, we have $\theta_v \circ T = T \circ \theta_u$. In OFrTang, the twist is given by twisting of strands (see Figure 2.10). Like the braiding, compatibility and naturality requirements for this twist follow from regular isotopy.

OFrTang has a compatible duality. A *duality* in a monoidal category is (i) an involution on its objects, i.e., for every object u , there is an object u^* , and (ii) morphisms $\text{ev}_u : u^* \otimes u \rightarrow \mathbb{1}$

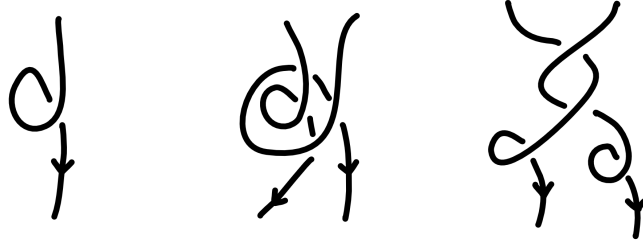


Figure 2.10: Twist in OFrTang.

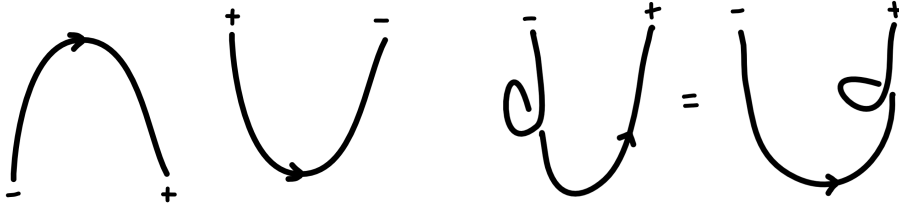


Figure 2.11: Evaluation and coevaluation maps in OFrTang and compatibility with the twist.

and $\text{coev}_u : \mathbb{1} \rightarrow u \otimes u^*$ such that

$$(\text{id}_u \otimes \text{ev}_u) \circ (\text{coev}_u \otimes \text{id}_u) = \text{id}_u \text{ and } (\text{ev}_u \otimes \text{id}_{u^*}) \circ (\text{id}_{u^*} \otimes \text{coev}_u) = \text{id}_{u^*}. \quad (2.10)$$

In OFrTang, the involution $u \rightarrow u^*$ is given by flipping all signs, and the maps coev_u and ev_u are given by cups and caps respectively (see Figure 2.11). If the monoidal category has a braiding and a twist, we require the duality to be compatible in the following way,

$$(\theta_u \otimes \text{id}_{u^*}) \circ \text{coev}_u = (\text{id}_u \otimes \theta_{u^*}) \circ \text{coev}_u. \quad (2.11)$$

Remark 2.17. A braided monoidal category with a compatible twist and duality is called a *ribbon category*.

Example 2.1. OFrTang is a ribbon category.

Example 2.2. Tang is also a ribbon category, but we chose to introduce OFrTang because in Tang, the duality involution on objects acts trivially, $k^* = k$, and the twist isomorphisms are given by the identity morphisms.

Example 2.3. The category $\text{FinVect}_{\mathbb{F}}$ of finite-dimensional vector spaces over a field $\mathbb{F}$ is also a ribbon category with the braiding, $c_{U,V} : U \otimes V \rightarrow V \otimes U$ given by a flip $u \otimes v \mapsto v \otimes u$ for $u \in U$ and $v \in V$ and extended by linearity, twist given by the identity map, $\theta_U = \text{id}_U$, and duality inherited from its vector space structure.

Remark 2.18. In the definition of a ribbon category, we did not require $(u^*)^* = u$ in the definition of the duality, but compatibility with the twist in a ribbon category makes

$u^{**} = (u^*)^*$ canonically isomorphic to u . Furthermore, the monoidal unit is canonically isomorphic to its own dual, $\mathbb{1}^* \cong \mathbb{1}$ [61, Chapter I § 2.6].

2.3.1 Colors

As a final step in the union of the geometry of tangles and the algebra of ribbon categories, we introduce the notion of *colored tangles*. Given a category, $\mathcal{C}$, a $\mathcal{C}$ -colored (oriented, framed) tangle is a tangle whose components are assigned (or colored with) objects of $\mathcal{C}$. An assignment of objects of $\mathcal{C}$ to components of a tangle is called a $\mathcal{C}$ -coloring.

For a ribbon category, $\mathcal{V}$, we define the category of $\mathcal{V}$ -colored oriented and framed tangles, $\text{OFrTang}_{\mathcal{V}}$. Objects of this category are finite sequences of pairs,

$$\eta = ((U_1, u_1), \dots, (U_k, u_k)), \quad (2.12)$$

where U_j are objects of $\mathcal{V}$ and $u_j \in \{\pm 1\}$ for every $j = 1, \dots, k$, and $k \in \mathbb{Z}_{\geq 0}$. A morphism $T : \eta \rightarrow \nu$ is a $\mathcal{V}$ -colored oriented and framed tangle such that the colors and orientations of $\partial_- T$ are the same as $\eta = ((U_1, u_1), \dots, (U_k, u_k))$ and the colors and orientations of $\partial_+ T$ are the same as $\nu = ((V_1, v_1), \dots, (V_l, v_l))$. As in OFrTang , the orientations of tangle components are such that they point downwards if $u_j, v_j = +1$ and upwards if $u_j, v_j = -1$.

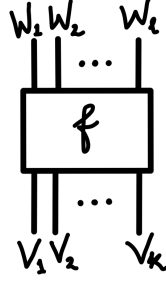
Remark 2.19. A *post-hoc* motivation for the definition of colored tangles comes from Chern–Simons theory, in which ribbon graphs are extended operators colored (or charged) with objects of $\text{Rep } G$, the category of representations of the gauge group. See also Remark 4.10.

2.3.2 Ribbon graphs

OFrTang is a subcategory of the category of ribbon graphs, Rib . Rib has the same objects as OFrTang but its morphisms are ribbon graphs— isotopy classes of oriented compact surfaces in $\mathbb{R}^2 \times I$ that can be decomposed into bands, annuli, and coupons such that only bases of bands intersect $\mathbb{R}^2 \times \{0, 1\}$.

Annuli are homeomorphic to $S^1 \rightarrow I$ and embedded in $\mathbb{R}^2 \times (0, 1)$. Bands are homeomorphic to $I \times I$ and have distinguished edges, $\{0, 1\} \times I$ called bases. Bases of bands meet the boundary $\mathbb{R} \times \{0, 1\}$ on $\mathbb{R} \times 0 \times \{0, 1\}$ or on the base of a coupon. Coupons are homeomorphic to $I \times I$ with distinguished edges, $\{0, 1\} \times I$, also called bases, embedded in $\mathbb{R}^2 \times (0, 1)$ such that it lies parallel to the xz -plane and its top base $1 \times I$ has a larger z -coordinate than its bottom base $0 \times I$.

Remark 2.20. A ribbon graph is said to be directed if the cores of bands, $S^1 \times 1/2$, and cores of annuli, $I \times 1/2$, are oriented. In what follows, all ribbon graphs will be assumed to be directed.

Figure 2.12: An elementary coupon in $\text{Rib}_\mathcal{V}$.

Like tangles, a (k, l) -ribbon graph, Γ , has k inputs, i.e., $|\partial_- \Gamma| = |\Gamma \cap \mathbb{R}^2 \times 0| = k$, and l outputs, i.e., $|\partial_+ \Gamma| = |\Gamma \cap \mathbb{R}^2 \times 1| = l$. And like tangles, we can order the points in $\partial_\pm \Gamma$ from left to right.

Morphisms of Rib are directed ribbon graphs. The ribbon category structure of OFrTang has a straightforward extension to the category of ribbon graphs— Rib is a ribbon category.

Given a ribbon category, $\mathcal{V}$, we can also define the category of $\mathcal{V}$ -colored ribbon graphs, $\text{Rib}_\mathcal{V}$, which has the same objects as $\text{OFrTang}_\mathcal{V}$, but whose morphisms are $\mathcal{V}$ -colored ribbon graphs. Bands and annuli of a $\mathcal{V}$ -colored ribbon graph are colored by objects of $\mathcal{V}$. An *elementary coupon* as in Figure 2.12 is colored by a morphism,

$$f : V_1^{v_1} \otimes \cdots \otimes V_k^{v_k} \rightarrow W_1^{w_1} \otimes \cdots \otimes W_l^{w_l}, \quad (2.13)$$

of $\mathcal{V}$, where $V_j^1 = V_j$ ($W_j^1 = W_j$ resp.) and $V_j^{-1} = V_j^*$ ($W_j^{-1} = W_j^*$ resp.).

CHAPTER 3

Manifolds

In this chapter, we cover the two pillars of 3-manifold topology: (i) a classical theorem of Lickorish and Wallace that reduces the homeomorphism classification problem of 3-manifolds to the study of framed links in the Euclidean three-space, and (ii) the geometrization conjecture (proved by Perelman) which says that every 3-manifold can be decomposed into geometric pieces, and every geometric piece has a Riemann structure modeled on one of the eight Thurston geometries.

We start with some examples.

Example 3.1 (Closed 3-manifolds). $S^1 \times S^2$, $\mathbb{T}^3 = S^1 \times S^1 \times S^1$, $S^1 \times S_g$ where S_g is a surface of genus $g \geq 2$.

Example 3.2 (Lens spaces). Let a, b be coprime integers and define a fixed point free action of $\mathbb{Z}_a$ on the 3-sphere by

$$S^3 = \{(w, z) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1\} \subset \mathbb{C}^2, \quad (3.1)$$

as $n \cdot (w, z) = (e^{2\pi i n/a} w, e^{2\pi i b n/a} z)$ for any $n \in \mathbb{Z}_a$ and $(w, z) \in S^3$. The quotient $S^3/\mathbb{Z}_a$ is the lens space $L(a, b)$.

Example 3.3 (Prism manifolds). Let S^3 be the group of unit quaternions and consider its subgroup D_{4m}^* generated by

$$\{e^{\pi \mathbf{i} a/m}, e^{\pi \mathbf{i} a/m} \mathbf{j}\}_{a=1, \dots, 2m}. \quad (3.2)$$

This group is isomorphic to the binary dihedral group. It acts on S^3 by multiplication and the quotient S^3/D_{4m}^* is the prism manifold $\mathcal{D}_m$.

Example 3.4 (Poincaré homology sphere). Let S^3 be the group of unit quaternions and consider its subgroup I_{120}^* generated by

$$s = \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \quad \text{and} \quad t = \frac{1}{2}(\varphi + \varphi^{-1} \mathbf{i} + \mathbf{j}), \quad (3.3)$$

where $\varphi = (1 + \sqrt{5})/2$. This group is isomorphic to the binary icosahedral group because the generators satisfy $(st)^2 = s^3 = t^5$. It acts on S^3 by multiplication and the quotient S^3/I_{120}^* is one possible description of the Poincaré homology sphere [38].

Remark 3.1. Recall that quaternions are generated by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as an algebra over $\mathbb{R}$ subject to relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Any quaternion can be written as $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ for some $w, x, y, z \in \mathbb{R}$, and its norm squared is given by $|q| = w^2 + x^2 + y^2 + z^2$.

3.1 Surgery along framed links

For our purposes, surgery will be an operation in which we start with a closed 4-ball, B^4 , with a framed link, K , in its bounding 3-sphere, $\partial B^4 \cong S^3$, and attach handles, $B^2 \times B^2$, to the boundary in ways specified by the framed link to get a new compact 4-manifold, W_K . The boundary of W_K is a 3-manifold and we say that it is the result of surgery on S^3 along K . See [37, Chapter I] for more details.

Let K be a framed link in the boundary S^3 with V components $K_1, \dots, K_V$. For every $j = 1, \dots, V$, let a_j be an integer that denotes the framing of K_j , and let $U_j \cong S^1 \times D^2$ be a solid torus in S^3 such that (i) its core, $S^1 \times 0$, coincides with the knot K_j , and (ii) U_j and U_k do not intersect if $j \neq k$. We attach a 2-handle $B^2 \times B^2$ along every U_j by the embedding $\phi_j : \partial D^2 \times D^2 \cong S^1 \times D^2 \hookrightarrow S^3 \cong \partial B^4$, such that the image, $\phi_j(\partial D^2 \times D^2)$, is identified with U_j , and $\phi_j(\partial D^2 \times (0, 1))$ goes a_j times around K_j . The resulting 4-manifold is defined inductively as

$$W_K = W_V \quad \text{where} \quad W_j := W_{j-1} \cup_{\phi_j} B^2 \times B^2 \quad \text{and} \quad W_0 = B^4. \quad (3.4)$$

Note that the order in which the handles are attached is not important because the tubular neighborhoods, U_j , are mutually disjoint.

Remark 3.2. Surgery along the empty link is defined to be $W_\emptyset := B^4$.

Example 3.5 (Surgery along ± 1 -framed unknot). $S_{\pm 1}^3(U) = S^3$.

Example 3.6 (Surgery along a 0-framed unknot). $S_0^3(U) = S^1 \times S^2$.

The following classical theorem states that every closed 3-manifold can be realized as ∂W_K for some framed link K .

Theorem 3.1 (Lickorish [41] and Wallace [62]). *Any closed connected 3-manifold (up to a degree one homeomorphism) can be obtained by surgery on S^3 along a certain framed link.*

Remark 3.3. We shall use the notation $S_a^3(K)$ for the resulting 3-manifold ∂W_K , where $a = (a_1, \dots, a_V)$. Sometimes the framings, a_j , are referred to as the *surgery coefficients*. See also § 3.1.2.

Remark 3.4. Every orientable 3-manifold is spin and therefore parallizable (see [37, Chapter VII]). In what follows, it will be relevant to keep track of the *framing* of a 3-manifold, which refers to the homotopy class of trivializations of its tangent bundle.

We define the *signature* of a link, $\sigma(K)$, to be the signature of the intersection form on $H_2(W_K, \mathbb{Z})$. If the link components are oriented, then the Seifert surfaces bounded by link components and the core of the associated handles form a basis of $H_2(W_K, \mathbb{Z})$. The basis element corresponding to the link component, K_j , is also called K_j . In this basis, the intersection form coincides with the linking matrix of K , i.e. [36, § 1],

$$K_j \cap_{W_K} K_k = \begin{cases} a_j & \text{if } j = k \\ \text{lk}(K_j, K_k) & \text{otherwise.} \end{cases} \quad (3.5)$$

Remark 3.5. Choosing to represent a 3-manifold, $X = \partial W_K$ as surgery along a link, implicitly picks a 2-framing on X that is induced by the bounding 4-manifold W_K . Introduced by Atiyah [4], the 2-framing is a weaker version of the framing, and is defined as a homotopy class of trivializations of $TX \oplus TX$ as a $\text{Spin}(6)$ -bundle. If surgeries along framed links K and L result in homeomorphic manifolds, then their induced 2-framings coincide if and only if they have the same signature, $\sigma(K) = \sigma(L)$.

Remark 3.6. A calculation with the Mayer–Vietoris sequence shows that the first homology group, $H_1(X, \mathbb{Z})$, is isomorphic to the cokernel of the linking matrix.

3.1.1 Kirby calculus

While every closed 3-manifold has a presentation as surgery along some framed link, this presentation is highly non-unique. We can define a relation $K \sim L$ if the manifolds resulting from surgeries along K and L are homeomorphic. This is an equivalence relation and the following deep theorem due to Kirby, and later refined by Fenn and Rourke, states that the moves FR_0 and FR_1 generate this relation.

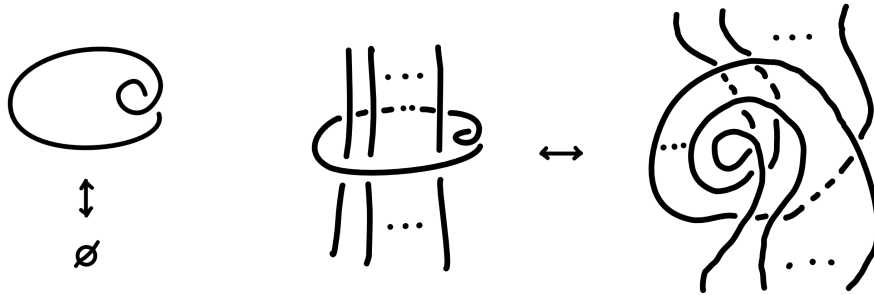


Figure 3.1: Fenn–Rourke moves FR_0 and FR_1 .

Theorem 3.2 (Kirby [35], Fenn–Rourke [17]). *Framed links L and K in S^3 are related by moves FR_0 and FR_1 if and only if the manifolds resulting from surgeries along L and K are homeomorphic.*

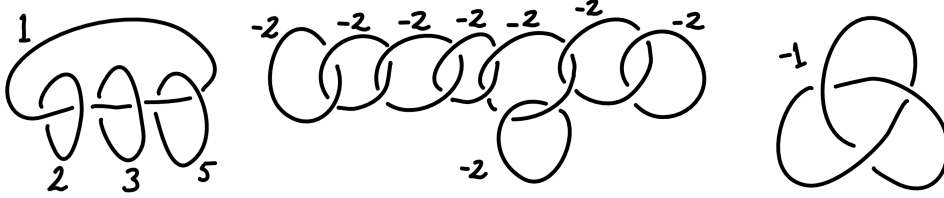


Figure 3.2: Surgery along all of the above framed links results in manifolds homeomorphic to the Poincaré homology sphere (see [38]).

Remark 3.7. We can generalize the above statement to the case of manifolds embedded with framed links. Let X_1, X_2 be manifolds that result from surgery on S^3 along framed links K_1, K_2 respectively; and let $L_1 \subset S^3 \setminus K_1, L_2 \subset S^3 \setminus K_2$ be framed links disjoint from K_1, K_2 respectively. L_1 is a framed link embedded in X_1 , and L_2 is a framed link embedded in X_2 . There is a homeomorphism between the pairs (X_1, L_1) and (X_2, L_2) if and only if the framed links $K_1 \cup L_1$ and $K_2 \cup L_2$ are related by a series of FR moves (see [61, Chapter II § 2.3]).

Remark 3.8. In the preceding remark, we can even let L_1 and L_2 be ribbon graphs, as long as the FR moves are being performed away from the coupons.

Remark 3.9. One possible way to invent 3-manifold invariants—or invariants of framed links in generic manifolds—is to start from framed links in S^3 and defining quantities that are invariant under FR moves. Such quantities will be homeomorphism invariants by Theorem 3.2. Homeomorphism invariance of the WRT invariant is proven in this manner.

3.1.2 Dehn surgery

A three-dimensional way to understand surgery along links is to think of it as cut-and-paste operations on solid tori in S^3 . Surgery data can alternatively be given as an link, K , in S^3 with V components, and pairs of integers $(a_1, b_1), \dots, (a_V, b_V)$ —one pair associated to each component. As before, we set $X_0 = S^3$, define X_j ’s inductively as described below, and define the resulting manifold as $X := X_V$ (see [57, Chapter 9]).

At the j -th step, we perform (a_j, b_j) -surgery along link component K_j , which consists of the following two steps.

1. Drilling: Let U_j be a tubular neighborhood of the knot K_j , and define $\mathring{X}_j := X_{j-1} \setminus U_j$, a 3-manifold with a torus boundary.

2. Filling: $X_j := \dot{X}_j \cup_{\phi_j} S^1 \times D^2$, where the gluing map $\phi_j : S^1 \times \partial D^2 \cong \mathbb{T}^2 \rightarrow \mathbb{T}^2 \cong \partial \dot{X}_j$ is a homeomorphism specified by the pair of integers (a_j, b_j) (explained below). Since we are only concerned with the homeomorphism type of X_j , we can consider the map ϕ_j upto isotopy and think of it as an element of the mapping class group of the torus.

We can pick the following generators for the first homology group of the boundary torus of $\dot{X}_j$: (i) a cycle l_j , called the longitude, that lies on the boundary, runs along K_j , and has linking number zero with the knot, and (ii) a cycle m_j , called the meridian, that lies on the boundary, bounds the unit disk of solid torus U_j , and has linking number 1 with K_j . So that $H_1(\partial \dot{X}_j, \mathbb{Z}) = \langle l_j, m_j \rangle_{\mathbb{Z}} \cong \mathbb{Z} \oplus \mathbb{Z}$. Similarly, we can pick the longitude and meridian generators for $H_1(S^1 \times \partial D^2, \mathbb{Z})$, if we replace K_j in the preceeding sentences with the core, $S^1 \times 0$, of the solid torus.

The isotopy class of ϕ_j is fixed by the map it induces on the first homology group, $\phi_{j,*} : H_1(S^1 \times \partial D^2, \mathbb{Z}) \rightarrow H_1(\partial \dot{X}_j, \mathbb{Z})$. We require that this induced map sends the meridian of the solid torus in $H_1(S^1 \times \partial D^2, \mathbb{Z})$ to $a_j l_j + b_j m_j$ in $H_1(\partial \dot{X}_j, \mathbb{Z})$. With the longitudes and meridians as generators, the matrix of $\phi_{j,*}$ as an isomorphism $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is given by

$$\phi_{j,*} = \begin{bmatrix} x_j & a_j \\ y_j & b_j \end{bmatrix}, \quad (3.6)$$

and the unknowns x_j, y_j are fixed by the requirement that the matrix is invertible over the integers, $x_j b_j - y_j a_j = \pm 1$.

Remark 3.10. The isotopy class of the gluing map, ϕ_j , can be understood as an element of the mapping class group, $\text{PSL}(2, \mathbb{Z})$, of the 2-torus.

Remark 3.11. The requirement $x_j b_j - y_j a_j = \pm 1$ implies that a_j and b_j are coprime and that an overall sign is irrelevant: $(-a_j, -b_j)$ surgery along K_j would result in the same manifold as (a_j, b_j) surgery along K_j .

Remark 3.12. The ratio $b_j/a_j \in \mathbb{Q} \cup \infty$ is called the *surgery coefficient* for K_j . We say that the manifold X is the result of *Dehn surgery* on S^3 along the link K , with surgery coefficients $\{b_j/a_j\}_{j=1,\dots,V}$.

Remark 3.13. Any 3-manifold X with a torus boundary, $\partial X = \mathbb{T}^2$, can be Dehn filled by gluing in a solid torus by an attaching map $\phi : S^1 \times \partial D^2 \cong \mathbb{T}^2 \rightarrow \mathbb{T}^2 \cong \partial X$, specified by some surgery coefficient b/a .

Example 3.7. Lens spaces, $L(a, b)$, can be realized as a/b -surgery on the unknot in S^3 .

3.2 Geometrization of 3-manifolds

Introduced by Thurston in 1982 [60] and proved by Perelman in 2003 [50, 52, 51], the geometrization conjecture states that any compact 3-manifold can be cut along 2-spheres

and 2-tori such that each piece locally has one of the following eight Thurston geometries

$$S^3, \quad \mathbb{R}^3, \quad \mathbb{H}^3, \quad S^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \text{Nil}_3, \quad \text{Sol}_3, \quad \widetilde{\text{SL}(2, \mathbb{R})}. \quad (3.7)$$

Each of these spaces is a Riemannian 3-manifold that admits a natural metric and is called a *model geometry*. Isotropic manifolds S^3 , $\mathbb{R}^3$ and $\mathbb{H}^3$ admit the usual constant curvature metrics; product manifolds $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ admit the product metric; and the Lie groups Nil_3 , Sol_3 and $\widetilde{\text{SL}(2, \mathbb{R})}$ admit metrics invariant under left multiplications (see [59, 42]).

The analogous statement for surfaces is the uniformization theorem, which states that any closed surface admits a regular covering by one of the three constant curvature model geometries S^2 , $\mathbb{R}^2$ or $\mathbb{H}^2$. Any surface covered by S^2 , $\mathbb{R}^2$ or $\mathbb{H}^2$ inherits a metric of constant positive, zero, or negative curvature respectively, which makes the covering map a local isometry. Which constant curvature metric a surface admits is determined completely by its Euler characteristic.

Euler characteristic	$\chi(S) > 0$	$\chi(S) = 0$	$\chi(S) < 0$
Geometric structure	S^2	$\mathbb{R}^2$	$\mathbb{H}^2$
Examples	$S^2, \mathbb{RP}^2$	$\mathbb{T}^2$, Klein bottle	Surfaces of genus ≥ 2

Let M be one of the eight Thurston geometries. We say that a Riemannian 3-manifold X has a *geometric structure modelled on M* , if X is locally isometric to M . If X is complete, then $X \cong M/\Gamma$ for some discrete subgroup $\Gamma \subset \text{Isom}(M)$ of isometries acting freely on M , and the covering $M \rightarrow M/\Gamma \cong X$ is a local isometry. A 3-manifold with (possibly empty) torus boundary is *geometric* if its interior has a finite volume complete geometric structure modelled on some M . In particular, a closed 3-manifold X is geometric if and only if—for some Thurston geometry M and discrete subgroup $\text{Isom}(M)$ —we have $X \cong M/\Gamma$.

Although the geometries Nil_3 , Sol_3 , and $\widetilde{\text{SL}(2, \mathbb{R})}$ look mysterious, they are just twisted products. For example, Nil_3 is the Heisenberg group of 3×3 upper triangular matrices with 1's on the diagonal,

$$\text{Nil}_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}, \quad (3.8)$$

and fits in the short exact sequence $0 \rightarrow \mathbb{R} \hookrightarrow \text{Nil}_3 \twoheadrightarrow \mathbb{R}^2 \rightarrow 0$. The inclusion $\mathbb{R} \hookrightarrow \text{Nil}_3$ sends a real number a to the matrix in (3.8) with $x = y = 0$ and $z = a$, and the projection $\text{Nil}_3 \twoheadrightarrow \mathbb{R}^2$ sends the matrix in (3.8) to $(x, y) \in \mathbb{R}^2$. We see that Nil_3 is an $\mathbb{R}$ -fibration over $\mathbb{R}^2$.

Sol_3 is the semidirect product $\mathbb{R} \rtimes \mathbb{R}^2$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by $t \cdot (x, y) = (e^t x, e^{-t} y)$, and can be thought of as an $\mathbb{R}^2$ bundle over $\mathbb{R}$.

$\widetilde{\text{SL}(2, \mathbb{R})}$ is the universal cover of the three-dimensional Lie groups $\text{SL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{R})$. The group $\text{PSL}(2, \mathbb{R})$ is the group of isometries of the hyperbolic plane and can be identified with $U\mathbb{H}^2$ —the unit tangent bundle of $\mathbb{H}^2$. The bundle $U\mathbb{H}^2$ is an S^1 -fibration over $\mathbb{H}^2$, so we can identify the universal cover $\widetilde{\text{SL}(2, \mathbb{R})}$ with an $\mathbb{R}$ -fibration over $\mathbb{H}^2$. We note that this fibration is topologically trivial, but not isometric to $\mathbb{H}^2 \times \mathbb{R}$.

For more details on Thurston geometrization and these model geometries, we refer the reader to the review [59] and the book [42, Chapter 12].

3.2.1 Seifert manifolds

We end this chapter by describing a family of closed 3-manifolds that populate six of the eight Thurston geometries. Indeed, every geometric 3-manifold—except those modelled on $\mathbb{H}^3$ and Sol_3 —is a Seifert manifold. A Seifert manifold with m exceptional fibers is a closed 3-manifold built out of the following set of data: an S^1 fibration over a surface S with Euler number b , and m pairs of coprime integers $(p_1, q_1), \dots, (p_m, q_m)$. The pairs of integers describe the m exceptional fibers of the fibration. The boundary of a tubular neighborhood of each fiber is a 2-torus with basis 1-cycles in homology given by the boundary of the disk, and a fiber at the boundary. The tubular neighborhood of the j -th exceptional fiber is a solid torus whose (p_j, q_j) boundary cycle is contractible.

Notice that the pair (p_j, q_j) is determined only up to an overall sign, so we can always choose $p_j > 0$ and denote the Seifert manifold as manifold $S(b; p_1/q_1, \dots, p_m/q_m)$. It is a fact that Seifert manifolds $S(b; p_1/q_1, \dots, p_m/q_m)$ and $S'(b'; p'_1/q'_1, \dots, p'_m/q'_m)$ are orientation preserving homeomorphic if and only if $S = S', e = e'$, up to reordering $p_j = p'_j$ and $q_j = q'_j \bmod p_j$ for every j [59, § 3]. In light of this result, Seifert invariants are sometimes normalized so that $b = 0$.

Two important invariants of a Seifert fibration are its Euler number

$$e = -b + \sum_{j=1}^m \frac{q_j}{p_j} \in \mathbb{Q}, \quad (3.9)$$

and the orbifold Euler characteristic of the base M/S^1 ,

$$\chi = \chi(S) - \sum_{j=1}^m \left(1 - \frac{1}{p_j}\right) \in \mathbb{Q}. \quad (3.10)$$

For six of the eight Thurston geometries, all closed manifolds are Seifert fibrations. Conversely, given a Seifert manifold, which model geometry it admits is completely determined by the two invariants e and χ according to the following table [59, Table 4.1].

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil_3	$\text{SL}(2, \mathbb{R})$

In this thesis, we will only consider Seifert fibrations over oriented bases. This, in particular, implies that the condition $\Gamma \subset \text{Isom}_0(N)$ holds.

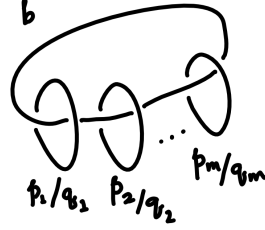


Figure 3.3: Surgery presentation of the manifold $S^2(b; p_1/q_1, \dots, p_m/q_m)$.

When the Seifert manifold is fibered over S^2 , it has an alternate description as Dehn surgery on the link in Figure 3.3 in S^3 . Furthermore, the first integral homology group of $M = S^2(b; p_1/q_1, \dots, p_m/q_m)$ is isomorphic to the cokernel of the matrix

$$B = \begin{bmatrix} b & 1 & \cdots & 1 & 1 \\ q_1 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{m-1} & 0 & \cdots & p_{m-1} & 0 \\ q_m & 0 & \cdots & 0 & p_m \end{bmatrix}, \quad (3.11)$$

i.e., $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{m+1}/B\mathbb{Z}^{m+1}$ [58, § 1.1.4]. Since $\det B = e p_1 p_2 \cdots p_m$, when M is a rational homology sphere we have $|H_1(M, \mathbb{Z})| = |e p_1 p_2 \cdots p_m|$. In particular, when M is an integer homology sphere,

$$p_1 p_2 \cdots p_m \left(-b + \sum_{j=1}^m \frac{q_j}{p_j} \right) = \pm 1 \quad (3.12)$$

which implies that (i) p_j 's are mutually coprime (ii) q_j 's are uniquely determined modulo p_j for every j , and therefore (iii) b is also determined. Conversely, given m mutually coprime integers $p_1, \dots, p_m$ such that $p_j \geq 2$, there is a unique Seifert integer homology sphere with m exceptional fibers, up to orientation reversal [58, § 1.1.4]. We shall pick an orientation such that $e > 0$ and denote this manifold as $\Sigma(p_1, \dots, p_m)$.

Remark 3.14. With the exception of Poincaré homology sphere, $\Sigma(2, 3, 5)$, for which $\chi > 0$, all other Seifert integer homology spheres $\Sigma(p_1, \dots, p_m)$ have $\chi < 0$ and therefore have geometry modeled on $\text{SL}(2, \mathbb{R})$.

CHAPTER 4

Chern–Simons theory

Let X be the three-dimensional spacetime manifold that is closed and oriented, G be a compact connected simple Lie group, and consider a principal G -bundle over X . When the *gauge group*, G , is simply connected—as will be the case in this thesis—the principal G -bundle is necessarily trivial and we can consider a theory for the *gauge field*, $A \in \Omega^1(X) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra associated to G , whose action is given by the integral of the Chern–Simons 3-form

$$\text{CS}[A] = \frac{1}{8\pi^2} \int_X \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (4.1)$$

The gauge field can be understood to be a 1-form representing a connection on the principal G -bundle, whose *field strength* or curvature is given by $F_A = dA + A \wedge A$.

Remark 4.1. Exterior derivative of the Chern–Simons 3-form is proportional to the second Chern character,

$$d \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = \text{tr} (F_A \wedge F_A). \quad (4.2)$$

Remark 4.2. Equations of motion for the classical theory are given by stationary points of the action, i.e., $\delta_A \text{CS}[A] = F_A = 0$, which means that classical solutions of this field theory are *flat connections*.

Since this is a gauge theory, field configurations that are related by a *gauge transformation* represent identical physical situations. If the action is to describe an actual physical theory, it should descend to the quotient, $\mathcal{A}_G/\mathcal{G}$, of the affine space of G -connections on the manifold, $\mathcal{A}_G$, by the group of gauge transformations $\mathcal{G}$. However, this does not happen because under a gauge transformation $g : X \rightarrow G$, the gauge field transforms as $A \rightarrow gAg^{-1} + gd(g^{-1})$ and the Chern–Simons action transforms as $\text{CS}[A] \rightarrow \text{CS}[A] + [g]$ as a result, where $[g]$ is an integer representing the homotopy class of the gauge transformation.

Even though the classical theory looks ill-defined and trivial, all hope is not lost because a consistent quantum theory requires only that the exponential, $\exp(2\pi i \text{CS}[A])$, be gauge invariant. We can write down the partition function of the quantum theory as a path integral,

$$Z_k^G(X) = \int_{\mathcal{A}_G/\mathcal{G}} \mathcal{D}A e^{2\pi i k \text{CS}[A]}, \quad (4.3)$$

where we have introduced a numerical parameter called the *level*, k , which is like an inverse coupling constant. Moreover, the quantum theory is far from trivial—one of the most celebrated results of modern mathematical physics is that the expectation values of Wilson lines in Chern–Simons theory equal Jones polynomials of corresponding knots and links embedded in X . We shall explain how this works in § 4.2.

Remark 4.3. Gauge invariance forces the level to be an integer.

Remark 4.4. For every level, k , and every compact simply connected simple Lie group, G , and the Chern–Simons partition function, $Z_k^G(X)$, should be a topological invariant of the manifold, X , because the path integral does not refer to the spacetime metric. Compare the Yang–Mills action,

$$\text{YM}[A] = \frac{1}{2} \int_X \text{tr} (E_A \wedge *E_A), \quad (4.4)$$

where a metric is implicit in the Hodge star operator.

4.1 Path integral and the framing anomaly

One way to get a better understanding of the Chern–Simons partition function is to compute the path integral in the small coupling, $k \rightarrow \infty$, limit. We can make a saddle point approximation and expand around the stationary point of the action,

$$Z_k^G(X) = \sum_{\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(X, G))} \int_{\mathcal{A}_G/\mathcal{G}} \mathcal{D}B e^{2\pi i k \text{CS}[A_\alpha + B]} \quad (4.5)$$

$$= \sum_{\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(X, G))} e^{2\pi i k \text{CS}[A_\alpha]} \int_{\mathcal{A}_G/\mathcal{G}} \mathcal{D}B \exp \left(\frac{ik}{4\pi} \int_X \text{tr} (B \wedge D_\alpha B) \right) \quad (4.6)$$

where $\mathcal{M}_{\text{flat}}(X, G)$ is the moduli space of flat G -connections on X , A_α is a $\mathfrak{g}$ -valued one-form representing the flat connection, and D_α is a covariant derivative with the background gauge field A_α . Formally, the integral over B looks like a Gaussian integral, but because B is a gauge field, we have to pick a metric, add a gauge fixing term, and introduce Faddeev–Popov ghosts to do the integral. As usual, the answer to such a computation is an appropriately regulated ratio of determinants,

$$\int_{\mathcal{A}_G/\mathcal{G}} \mathcal{D}B \exp \left(\frac{ik}{4\pi} \int_X \text{tr} (B \wedge D_\alpha B) \right) = \frac{\det \Delta_\alpha}{\sqrt{\det L_\alpha^-}}, \quad (4.7)$$

where $\Delta_\alpha = D_\alpha^2$ is the Laplacian, and L_α^- is the restriction of $L_\alpha = *D_\alpha + D_\alpha*$ to forms of odd degree (see [64, § 2] for details). The absolute value of the ratio of determinants is an invariant

$$T_\alpha(X) = \left| \frac{\det \Delta_\alpha}{\sqrt{\det L_\alpha^-}} \right|, \quad (4.8)$$

called the Ray–Singer analytic torsion of A_α , while its phase is more subtle. This is not surprising because both the Laplacian and L_α^- depend on a choice of metric.

Remark 4.5. The determinant of the Laplacian is a positive real number, so the only contribution to the phase is $\sqrt{\det L_\alpha^-}$.

The phase of $\sqrt{\det L_\alpha^-}$ is proportional to the η -invariant of the differential operator L_α^- , but this is not a topological invariant. More precisely,

$$\frac{\det \Delta_\alpha}{\sqrt{\det L_\alpha^-}} = T_\alpha(X) \exp\left(\frac{\pi i}{2} \eta(A_\alpha)\right) = T_\alpha(X) \exp\left(\frac{\pi i}{2} \eta(0) + 2\pi i \check{c}_\mathfrak{g} \text{CS}[A_\alpha]\right), \quad (4.9)$$

where $\eta(A_\alpha)$ is the η -invariant of L_α^- , $\eta(0)$ is the η -invariant for the trivial flat connection, and $\check{c}_\mathfrak{g}$ is the dual Coxeter number of the semisimple Lie algebra $\mathfrak{g}$ (see also [2]). With all of this, we have

$$Z_k^G(X) = e^{\pi i \eta(0)/2} \sum_{\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(X, G))} e^{2\pi i (k + \check{c}_\mathfrak{g}) \text{CS}[A_\alpha]} T_\alpha(X). \quad (4.10)$$

Remark 4.6. We pause to note that the Chern–Simons level $k \rightarrow k + \check{c}_\mathfrak{g}$ has renormalized.

The phase $e^{\pi i \eta(0)/2}$ is the only piece left that depends on a choice of metric, but we can add a counterterm to the action to compensate. First we note that the eta invariant $\eta(0)$ for the trivial flat connection is proportional to the eta invariant of L_- coupled only to the Levi–Civita connection, and the constant of proportionality is the dimension of the gauge group G . In symbols, $\eta(0) = d_G \eta_{\text{grav}}$. The appropriate counterterm to add to the action is

$$\text{CS}[\omega] = \frac{1}{8\pi^2} \int_X \text{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \quad (4.11)$$

where ω is the Levi–Civita connection on the spin bundle compatible with the chosen metric, because the combination

$$\frac{1}{2} \eta_{\text{grav}} + \frac{1}{12} \text{CS}[\omega] \quad (4.12)$$

does not depend on the metric. Note that the integral in $\text{CS}[\omega]$ depends on a choice of trivialization of its spin bundle, i.e., a framing (see Remarks 3.4 and 3.5). Like the usual Chern–Simons function under a gauge transformation, a change in framing causes a shift $\text{CS}[\omega] \rightarrow \text{CS}[\omega] + s$ for some integer s . For the Chern–Simons action with the added

counterterm $d_G \text{CS}[\omega]/24$, we have

$$Z_k^G(X) = \exp\left(\pi i d_G \left[\frac{\eta_{\text{grav}}}{2} + \frac{\text{CS}[\omega]}{24} \right]\right) \sum_{\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(X, G))} e^{2\pi i (k + \epsilon_0) \text{CS}[A_\alpha]} T_\alpha(X), \quad (4.13)$$

and we see that the partition function $Z_k^G(X)$ is a numerical topological invariant of framed 3-manifolds.

Remark 4.7. The sum in the saddle point approximation of the Chern–Simons path integral is over connected components of $\mathcal{M}_{\text{flat}}(X, G)$. We note that if these connected components are not isolated points, there would be additional integrals over that run over the moduli space, and computation of the path integral would need more care.

4.2 Link invariants

Wilson loops are extended observables in gauge theory supported on curves in spacetime, charged by representations of the gauge group. For a closed curve in spacetime, given by an embedding $K : S^1 \hookrightarrow X$, the associated Wilson loop operator carrying charge R is defined as

$$W_R(A, K) = \text{tr}_R \mathcal{P} \exp \int_K A, \quad (4.14)$$

where R is a representation of the gauge group, $\mathcal{P}$ is the path ordering operator, and A is the connection 1-form. The right-hand side is just the trace—with respect to the representation R —of the holonomy around loop K with respect to the connection A . This definition is gauge invariant, and therefore the Wilson loop descends to a well-defined function on $\mathcal{A}_G/\mathcal{G}$.

For non-intersecting closed curves $\{K_j : S^1 \hookrightarrow X\}_{j=1, \dots, V}$ charged with representations $R_1, \dots, R_V$ respectively, the expectation value of the associated Wilson loop operator is given by

$$Z_k^G(X, K) = \int_{\mathcal{A}_G/\mathcal{G}} \mathcal{D}A e^{2\pi i k \text{CS}[A]} \prod_{j=1}^n W_{R_j}(A, K_j), \quad (4.15)$$

where K stands for $K_1 \sqcup \dots \sqcup K_N$. Since the path integral on the right-hand side (1) does not refer to a spacetime metric and (2) holonomies only depend on the homotopy class of Wilson loops, the complex number $Z_k^G(X, K)$ should be an isotopy invariant of the link, K , embedded in X . There is a caveat, however, because when we try to actually compute the expectation value, we run into integrals of the following familiar type (compare (2.2), see also [64])

$$\sum_{i,j} \int_{K_i} dx_i^\mu \int_{K_j} dx_j^\nu \epsilon_{\mu\nu\lambda} \frac{(x_i - x_j)^\lambda}{|x_i - x_j|^3}. \quad (4.16)$$

When $i \neq j$, the above integral is the Gauss linking integral and computes the linking number of knots K_i and K_j , but the integral is ill-defined when $i = j$ and we need an extra piece of data to make sense of it—the self linking number or the framing.

The framing of a knot is the choice of a homotopy type of a normal vector field. With this piece of data, the $i = j$ integral can be regulated by pushing the knot a small distance along this normal vector field to get a second knot K'_i , and computing the integral along K_i and K'_i . While there is no canonical choice of framing for a knot embedded in a generic 3-manifold, when $X = S^3$ (or $\mathbb{R}^3$), all framings can be chosen so that the linking number of K_i and K'_i is zero.

Remark 4.8. Just as the partition function $Z_k^G(X)$ is not a diffeomorphism invariant of the manifold X , but an invariant of a *framed manifold*—i.e. the manifold with a choice of trivialization of its tangent bundle—the expectation value $Z_k^G(X, K)$ is not an isotopy invariant of the link K embedded in X , but an invariant of a *framed link*.

Remark 4.9. Framed knots in S^3 can also be thought of as ribbons by imagining the surface bound between the original knot and the knot pushed-off along the normal vector field.

Remark 4.10. The charged Wilson loop can serve as justification for why one might consider the notion of *colored* ribbon graphs natural. Using the term “color” for a group representation goes back to Murray Gell-Mann in the early 1970s and the genesis of quantum chromodynamics.

The Jones polynomial, $V_K(q)$, is an isotopy invariant of links in S^3 (or $\mathbb{R}^3$) that was discovered by considering certain representations of the braid group [31, 32]. The link invariant described above is a vast generalization of the Jones polynomial, if $G = \text{SU}(2)$ and all components of the link, K , are 0-framed and colored by the two-dimensional vector representation,

$$V_K(q = e^{2\pi i/k}) = Z_{k-2}^{\text{SU}(2)}(S^3, K) \quad \text{for } k = 3, 4, 5, \dots \quad (4.17)$$

Remark 4.11. The group $\text{SU}(2)$ has a unique (isomorphism class of) irreducible representation in each dimension. In dimension one, it is the trivial representation; in dimension two, it is the vector representation; in dimension three it is the adjoint representation; and so on.

Remark 4.12. Consider $G = \text{SU}(2)$ and color the K_j -th component of the link K by the n_j -dimensional irreducible representation of $\text{SU}(2)$. We denote the corresponding link invariant by

$$J_{n_1, \dots, n_V}(K; q = e^{2\pi i/k}) = Z_{k-2}^{\text{SU}(2)}(S^3, K) \quad \text{for } k = 3, 4, 5, \dots, \quad (4.18)$$

and call it the *colored Jones polynomial*.

4.3 Analytic continuation to non-integer levels

As we have stated elsewhere, a major focus of this thesis is to understand the modular behavior of the Reshetikhin–Turaev invariant at generic roots of unity. Since the Reshetikhin–Turaev invariant at a generic root of unity is morally equal to the Chern–Simons partition function at a rational level, we sketch how one might analytically continue the path integral in its level (see [65] for details).

If we let the level k take non-integer values, the integrand of the path integral $e^{2\pi i k \text{CS}[A]}$ no longer descends to a function on $\mathcal{U}_G = \mathcal{A}_G/\mathcal{G}$, because it picks up a phase $e^{2\pi i k}$ under a large gauge transformation. We can instead try to integrate over the universal cover, $\tilde{\mathcal{U}}_G = \mathcal{A}_G/\mathcal{G}_0$, the space of G -connections modulo gauge transformations connected to the identity, where $e^{2\pi i k \text{CS}[A]}$ is well-defined. This does not work. To see why, note that under a deck transformation of the covering $\tilde{\mathcal{U}}_G \rightarrow \mathcal{U}_G$, we have

$$\int_{\tilde{\mathcal{U}}_G} \mathcal{D}A e^{2\pi i k \text{CS}[A]} = e^{2\pi i k} \int_{\tilde{\mathcal{U}}_G} \mathcal{D}A e^{2\pi i k \text{CS}[A]}. \quad (4.19)$$

If k is not an integer, the path integral is forced to be zero.

To fix this, we complexify the gauge field—upgrade the G -connection A to a $G_{\mathbb{C}}$ -connection, replace the integration measure by the holomorphic volume form on $\tilde{\mathcal{U}}_{G_{\mathbb{C}}}$, and integrate over a middle-dimensional contour Γ in $\tilde{\mathcal{U}}_{G_{\mathbb{C}}}$,

$$Z_{k,\Gamma}^{G_{\mathbb{C}}}(X) = \int_{\Gamma \subset \tilde{\mathcal{U}}_{G_{\mathbb{C}}}} \mathcal{D}A e^{2\pi i k \text{CS}[A]}. \quad (4.20)$$

The integration cycle Γ is fixed by lifting the real cycle $\Gamma_G = \mathcal{U}$ in $\mathcal{U}_{\mathbb{C}}$ to a non-closed cycle in the universal cover $\tilde{\mathcal{U}}_{\mathbb{C}}$, and closing it by following downward Morse flow from its boundaries with respect to the Morse function $\text{Re } 2\pi i k \text{CS}[A]$. Fixing the integration contour according to this prescription ensures that $Z_{k,\Gamma}^{G_{\mathbb{C}}}(X) = Z_k^G(X)$ when $k \in \mathbb{Z}$.

Remark 4.13. Going forward, we shall assume the above choice of the integration contour and stop decorating the partition function with Γ .

Next, we would like to decompose the integration cycle into Lefschetz thimbles. Heuristically, a Lefschetz thimble is a middle-dimensional integration cycle attached to critical points of a Morse function and obtained as a union of downward flow trajectories with respect to the Morse function. Moreover, since Lefschetz thimbles form a basis of the appropriate middle-dimensional relative homology group, for the integration contour Γ described above, we have [21, § 2]

$$\Gamma = \sum_{(\alpha,n) \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}})) \times \mathbb{Z}} n_{(\alpha,n)} \Gamma_{(\alpha,n)}, \quad (4.21)$$

where the sum runs over critical submanifolds of $\mathcal{U}_{G_{\mathbb{C}}}$ with respect to the Morse function $\operatorname{Re} 2\pi i k \operatorname{CS}[A]$. In terms of these Lefschetz thimbles, we have

$$Z_k^{G_{\mathbb{C}}}(X) = \sum_{(\alpha, n) \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}})) \times \mathbb{Z}} n_{(\alpha, n)} \int_{\Gamma_{(\alpha, n)}} \mathcal{D}A e^{2\pi i k \operatorname{CS}[A]}. \quad (4.22)$$

As we are ultimately interested in an asymptotic statement about the WRT invariant, it makes sense to look at the small coupling expansion, $k \rightarrow \infty$. For the Lefschetz thimble integrals, we have

$$\int_{\Gamma_{(\alpha, n)}} \mathcal{D}A e^{2\pi i k \operatorname{CS}[A]} = e^{2\pi i k \operatorname{CS}[A_{(\alpha, n)}]} Z_{\alpha}^{\text{pert}}(k), \quad (4.23)$$

where $Z_{\alpha}^{\text{pert}}(k)$ is an asymptotic series in $1/k$ that only depends on $\alpha \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}}))$ and not on its lift [21, Appendix A]. Note that for a given $\alpha \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}}))$, we have

$$\operatorname{CS}[A_{(\alpha, n)}] - \operatorname{CS}[A_{(\alpha, m)}] \in \mathbb{Z}, \quad (4.24)$$

so that if we pick a particular lift of the Chern–Simons value, we can write

$$Z_k^{G_{\mathbb{C}}}(X) = \sum_{\alpha \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}}))} \left(\sum_{n \in \mathbb{Z}} n_{(\alpha, n)} e^{-2\pi i k n} \right) e^{2\pi i k \operatorname{CS}[A_{\alpha}]} Z_{\alpha}^{\text{pert}}(k) \quad (4.25)$$

$$= \sum_{\alpha \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}}))} P_{\alpha}(e^{-2\pi i k}) e^{2\pi i k \operatorname{CS}[A_{\alpha}]} Z_{\alpha}^{\text{pert}}(k), \quad (4.26)$$

where P_{α} is a power series in $e^{-2\pi i k}$ with integer coefficients.

When the level $k = r/s$ is a rational number with a fixed denominator, s , it suffices to lift the integrand of the path integral to the s -fold cover $\mathcal{U}_G^s = \mathcal{A}_G/\mathbb{G}_s$, the space of G -connections modulo gauge transformations whose degree is a multiple of s . However, for the same reason as before, this is not enough and we must complexify the gauge field. The path integral becomes

$$Z_{k=r/s}^{G_{\mathbb{C}}}(X) = \int_{\Gamma \subset \mathcal{U}_{G_{\mathbb{C}}}^s} \mathcal{D}A e^{2\pi i k \operatorname{CS}[A]}, \quad (4.27)$$

where Γ is an appropriately closed lift of the real cycle $\Gamma_G \subset \mathcal{U}_{G_{\mathbb{C}}}$ to the cover $\mathcal{U}_{G_{\mathbb{C}}}^s$. Since we are in an s -fold cover, there are many fewer Lefschetz thimbles,

$$\Gamma = \sum_{(\alpha, n) \in \pi_0(m_{\text{flat}}(X, G_{\mathbb{C}})) \times \mathbb{Z}/s\mathbb{Z}} n_{(\alpha, n)} \Gamma_{(\alpha, n)}. \quad (4.28)$$

If we consider the asymptotic expansion $r \rightarrow \infty$ and pick a lift of the Chern–Simons value, we get

$$Z_{k=r/s}^{G_{\mathbb{C}}}(X) = \sum_{\alpha \in \pi_0(M_{\text{flat}}(X, G_{\mathbb{C}}))} P_{\alpha}(e^{-2\pi i k}) e^{2\pi i k \text{CS}[A_{\alpha}]} Z_{\alpha}^{\text{pert}}(k), \quad (4.29)$$

where

$$P_{\alpha}(\xi) = \sum_{n \in \mathbb{Z}/s\mathbb{Z}} n_{(\alpha, n)} \xi^n \quad (4.30)$$

is a polynomial in the root of unity $\xi = e^{-2\pi i r/s}$, whose coefficients are the coefficients of the Lefschetz thimble decomposition of the contour Γ .

Remark 4.14. The quantum modularity conjecture is a statement about complex $\text{SU}(2)$ Chern–Simons theory which states that there is a distinguished flat connection $\alpha_* \in M_{\text{flat}}(X, \text{SL}(2, \mathbb{C}))$ which encodes information about the Thurston geometry of X , such that $P_{\alpha_*}(e^{-2\pi i k}) = Z_{-1/k}(X)$.

CHAPTER 5

Reshetikhin–Turaev construction

Heuristic arguments involving the path integral in the previous chapter indicated that there should exist numerical invariants of knots and closed 3-manifolds (and indeed a topological quantum field theory) that coincides with the Jones polynomial for links in S^3 . The algebraic construction of Reshetikhin and Turaev [56, 55] associates such numerical invariants (and indeed an extended TQFT) to any modular category. Moreover, when this modular category is that of certain finite dimensional representations of the small quantum group $\tilde{U}_\zeta \mathfrak{sl}_2$ for a root of unity $\zeta = e^{2\pi i/k}$, this construction gives mathematical meaning to Chern–Simons theory with $G = \mathrm{SU}(2)$ at level k , and we call the associated invariant the $\mathfrak{sl}_2$ Witten–Reshetikhin–Turaev invariant.

In the following sections, we sketch how this construction works in the general setting, and in § 5.3.2 we describe the relevant modular category for the $\mathfrak{sl}_2$ WRT invariant.

5.1 Link invariants

Having motivated the definition of ribbon categories in § 2.3 as an algebraic structure that encodes the combinatorics of Reidemeister moves, it should not be hard to see that, given a ribbon category, the following theorem gives a prescription for producing isotopy invariants of tangles (see [61, Chapter II §§ 2–4]).

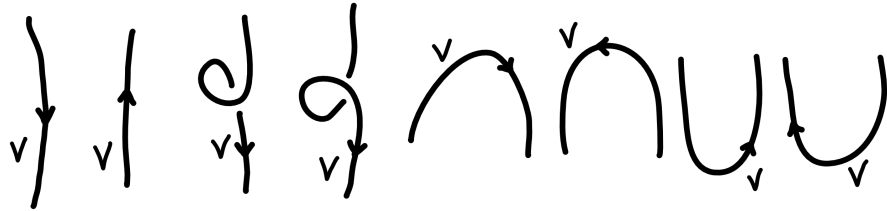


Figure 5.1: Morphisms $\downarrow_V, \uparrow_V, \phi_V, \phi'_V, \curvearrowright_V, \curvearrowleft_V, \smile_V, \frown_V$ in the category $\mathrm{OFrTang}_V$.

Theorem 5.1 (Reshetikhin–Turaev [56]). *For any ribbon category, $\mathcal{V}$, there is a unique covariant monoidal functor, $F : \mathrm{OFrTang}_V \rightarrow \mathcal{V}$, such that for every $U, V \in \mathcal{OB} \mathcal{V}$,*

1. $F((V, +1)) = V$ and $F((V, -1)) = V^*$,
2. $F(X_{U,V}^+) = c_{U,V}$, $F(\varphi_V) = \theta_V$, $F(\smile_V) = \text{coev}_V$, and $F(\frown_V) = \text{ev}_V$.

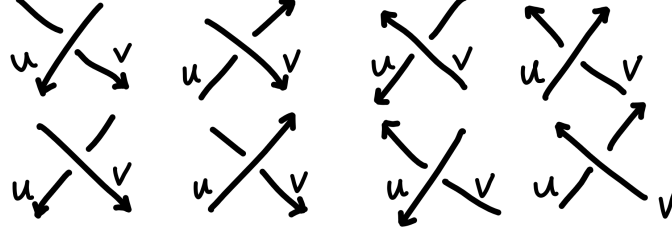


Figure 5.2: Crossing morphisms $X_{U,V}^\pm, Y_{U,V}^\pm, Z_{U,V}^\pm, T_{U,V}^\pm$ in the category OFrTang_V .

Isotopy invariance of F is almost a tautology because isotopic ribbon tangles represent the same morphism in OFrTang . Conversely, the theorem also says that morphisms in a ribbon category are isotopy invariant. See [61, Chapter I §§ 2.6–2.8] for some beautiful arguments that leverage this observation.

Remark 5.1. The functor F is called the Reshetikhin–Turaev functor, and $F(T)$ is called an operator invariant of the tangle, T .

Remark 5.2. Action of F on other objects and morphisms of OFrTang_V , follows from its functoriality and the fact that it respects tensor products and duality. For example, if $\eta = ((V_1, v_1), \dots, (V_k, v_k))$ is an object in OFrTang_V , then

$$F(\eta) = F((V_1, v_1) \otimes \dots \otimes (V_k, v_k)) = V_1^{v_1} \otimes \dots \otimes V_k^{v_k}, \quad (5.1)$$

where $V_j^{+1} = V_j$ and $V_j^{-1} = V_j^*$. It similarly follows that $F(X_{U,V}^-) = c_{V^*,U}^{-1}$ and $F(\varphi'_V) = \theta_V^{-1}$. And for the other crossings, we have

$$F(Y_{U,V}^+) = c_{V^*,U^*}^{-1}, \quad F(Z_{U,V}^+) = c_{V^*,U}^{-1}, \quad F(T_{U,V}^+) = c_{U^*,V^*}, \quad (5.2)$$

$$F(Y_{U,V}^-) = c_{U,V^*}, \quad F(Z_{U,V}^-) = c_{U,V^*}, \quad F(T_{U,V}^-) = c_{V^*,U^*}^{-1}. \quad (5.3)$$

We can also compute

$$F(\smile_V) = F(\varphi_V \otimes \uparrow_V \circ Y_{V,V}^- \circ \smile_V) = \theta_V \otimes \text{id}_{V^*} \circ c_{V^*,V} \circ \text{coev}_V, \quad (5.4)$$

$$F(\frown_V) = F(\frown_V \circ Z_{V,V}^- \circ \varphi_V \otimes \uparrow_V) = \text{ev}_V \circ c_{V,V^*} \circ \theta_V \otimes \text{id}_{V^*} \quad (5.5)$$

for cups and caps oriented the other way.

Remark 5.3. Since oriented framed links in S^3 can be deformed into ribbon $(0, 0)$ -tangles, the Reshetikhin–Turaev functor gives a numerical invariant, $F(K) \in \text{End } \mathbb{1}_V$, of the oriented framed link K for every V -coloring of K .

Example 5.1 (Invariant for the unknot). A p -framed unknot colored V is represented as the morphism $\smile_V \circ \theta_V^p \otimes \text{id}_{V^*} \circ \frown_V$ in OFrTang_V . We can use (5.5) to write the invariant

as

$$F(U_V^p) = \text{ev}_V \circ c_{V,V^*} \circ \theta_V^{p+1} \otimes \text{id}_{V^*} \circ \text{coev}_V = \text{tr } \theta_V^p, \quad (5.6)$$

where the trace of a morphism in a ribbon category is defined in the next section.

5.1.1 Trace in ribbon categories

There is a natural trace of morphisms, $T : \eta \rightarrow \eta$, in $\text{OFrTang}_\mathcal{V}$ given by closure of tangles. We call the closure a trace because it has the following trace-like properties:

1. for any tangles $S : u \rightarrow v$ and $T : v \rightarrow u$, we have $\text{tr}(S \circ T) = \text{tr}(T \circ S)$, and
2. for any tangles $S : u \rightarrow u$ and $T : v \rightarrow v$, we have $\text{tr}(S \otimes T) = \text{tr } S \otimes \text{tr } T$.

Consider the morphism $T : (V, +) \rightarrow (V, +)$ for simplicity; its closure is the colored $(0, 0)$ -tangle, $\text{tr } T = \smile_V \circ T \otimes \uparrow_V \circ \smile_V$. Writing $\smile_V$ using the more elementary building blocks of $\text{OFrTang}_\mathcal{V}$ we have

$$\text{tr } T = \smile_V \circ X_{V,V^*}^+ \circ (\varphi_V \circ T \otimes \uparrow_V) \circ \smile_V, \quad (5.7)$$

which motivates how trace should be defined in a general ribbon category. Given a morphism $f : V \rightarrow V$ in a ribbon category $\mathcal{V}$, its *trace* is defined as

$$\text{tr } f := \text{ev}_V \circ c_{V,V^*} \circ (\theta_V \circ f \otimes \text{id}_{V^*}) \circ \text{coev}_V \in \text{End } \mathbb{1}_V. \quad (5.8)$$

Note that this definition is just the image of (5.7) under the Reshetikhin–Turaev functor.

Remark 5.4. For any object V in a ribbon category, its *dimension* is defined as the trace of the identity map, $\dim V := \text{tr } \text{id}_V$.

Remark 5.5. The trace takes values in $\mathbb{K}_\mathcal{V} = \text{End } \mathbb{1}_\mathcal{V}$, which has the structure of a commutative semigroup, with the semigroup operation given by composition of morphisms. The semigroup $\mathbb{K}_\mathcal{V}$ is commutative because for any $k \in \mathbb{K}_\mathcal{V}$, we have $k = k \otimes \text{id}_{\mathbb{1}_\mathcal{V}} = \text{id}_{\mathbb{1}_\mathcal{V}} \otimes k$ and

$$k \circ k' = (k \otimes \text{id}_{\mathbb{1}_\mathcal{V}}) \circ (\text{id}_{\mathbb{1}_\mathcal{V}} \otimes k') = (\text{id}_{\mathbb{1}_\mathcal{V}} \otimes k) \circ (k' \otimes \text{id}_{\mathbb{1}_\mathcal{V}}) = k' \circ k \quad (5.9)$$

for any $k, k' \in \mathbb{K}_\mathcal{V}$.

In what follows, we would want the trace to take values in a commutative ring, instead of a semigroup. This occurs when $\mathcal{V}$ has the structure of an Ab-category, i.e., for any two objects U, V of $\mathcal{V}$, the set of morphisms $\text{Hom}(U, V)$ is an (additive) abelian group such that composition and tensor product of morphisms is bilinear with respect to the group operation.

Remark 5.6. In a (ribbon) Ab-category $\mathcal{V}$, the semigroup of endomorphisms of the monoidal unit $\mathbb{K}_\mathcal{V} = \text{End } \mathbb{1}_\mathcal{V}$, is promoted to a commutative ring—with the ring mul-

tiplication given by the composition of morphisms and the ring addition given by the addition of morphisms—and is called the *ground ring* of $\mathcal{V}$.

5.2 Manifold invariant

Extending the link invariant, F , to an invariant of 3-manifolds involves imposing some extra structure in the ribbon Ab-category $\mathcal{V}$, that makes it into a modular category. Unlike the category of tangles and ribbon categories, the author is unfortunately not aware of any geometrical considerations that directly motivate the definition of a modular category. The motivation comes from representation theory—the prototypical example one should think of when encountering definitions of simple objects and domination, is the category of finite dimensional representations of finite groups, and (the Drinfeld double of) their group algebras (see [61, Chapter II]).

Simple objects in a monoidal Ab-category are meant to act like irreducible representations. An object V is said to be a *simple object* if the only morphisms, $V \rightarrow V$, are the identity and scalar multiples of the identity, i.e., $\text{End}(V) = \{\kappa \otimes \text{id}_V : V \rightarrow V \mid \kappa \in \mathbb{K}\}$, where the scalars are elements of the ground ring of the category. Note that this is just a restatement of Schur’s lemma for irreducible representations over an algebraically closed field.

Remark 5.7. Equivalently, V is a simple object if $\text{End } V$ is generated over $\mathbb{K}$ by id_V .

Remark 5.8. For an algebra, A , over a commutative ring, $\mathbb{K}$, an A -module is said to be simple if the only non-trivial A -linear endomorphisms are the scalar multiples of the identity. An irreducible A -module, on the other hand, has no non-trivial A -submodules. An irreducible A -module is simple if and only if $\mathbb{K}$ is an algebraically closed field.

In the representation theory of finite groups (over a field of characteristic zero), any finite dimensional representation is equivalent to the direct sum of irreducible representations. An object in a monoidal Ab-category, V , is *dominated* by the family of objects, $\{V_i\}_{i \in I}$, if there exist a finite set, $J \subset I$, and families of morphisms, $f_j : V_j \rightarrow V$ and $g_j : V \rightarrow V_j$ for every $j \in J$, such that the identity map has the decomposition $\text{id}_V = \sum_{j \in J} f_j \circ g_j$.

Remark 5.9. Equivalently, V is dominated by $\{V_i\}_{i \in I}$ if the image of the pairing, $\text{Hom}(V, V_i) \otimes \text{Hom}(V_i, V) \ni g \otimes f \mapsto f \circ g \in \text{End } V$, generates $\text{End } V$ additively.

Remark 5.10. Note that the decomposition of the identity map is reminiscent of the universal property of direct sums. Indeed, if the category admits direct sums, then the decomposition of the identity map is equivalent to the requirement that there exist some object W such that $V \oplus W = \bigoplus_{j \in J} V_j$ for some finite subset $J \subset I$.

A *modular category* is a ribbon Ab-category, $\mathcal{V}$, along with a finite family of simple objects, $\{V_i\}_{i \in I}$, such that

1. there is an element $0 \in I$ such that $V_0 = \mathbb{1}_{\mathcal{V}}$ is the monoidal unit,

2. for every $i \in I$, there is an $i^* \in I$ such that $V_i^* = V_{i^*}$,
3. every object of $\mathcal{V}$ is dominated by the family, $\{V_i\}_{i \in I}$, and
4. the matrix $S_{ij} := \text{tr}(c_{V_j, V_i} \circ c_{V_i, V_j})$ is invertible over the ground ring, $\mathbb{K}_{\mathcal{V}}$.

Remark 5.11. The matrix S is symmetric and $S_{0i} = S_{i0} = \text{tr}(\text{id}_{V_i}) = \dim V_i$ for every $i \in I$. Invertibility of S implies that $\det S$ is invertible in $\mathbb{K}_{\mathcal{V}}$.

Remark 5.12. Objects V_i and V_j cannot be isomorphic if $i \neq j$. For if $V_i \cong V_j$, we would have $S_{ik} = S_{jk}$ for every $k \in I$ and therefore $\det S = 0$, making S degenerate. In particular, for every $i \in I$, there is a unique i^* such that $V_i^* = V_{i^*}$, and $0^* = 0$ because the monoidal unit is its own dual.

Remark 5.13. The dimension $\dim V_i$ is invertible over $\mathbb{K}_{\mathcal{V}}$ for every $i \in I$ [61, Chapter II § 1.4].

Remark 5.14. We will write $\dim i$ instead of $\dim V_i$ for brevity.

In addition to the modular category itself, the 3-manifold invariant also depends on a scalar called the rank of the modular category. A scalar, $\mathfrak{D} \in \mathbb{K}_{\mathcal{V}}$, is called the *rank* of $\mathcal{V}$ if

$$\mathfrak{D}^2 = \sum_{i \in I} (\dim i)^2. \quad (5.10)$$

Note that the rank may not exist, and may not be unique even if it does.

Remark 5.15. If such a $\mathfrak{D}$ does not exist, we can add it by hand by augmenting the ground ring, $\tilde{\mathbb{K}}_{\mathcal{V}} := \mathbb{K}_{\mathcal{V}}[x]/(x^2 - \mathfrak{D}^2)$ —and passing to the category $\tilde{\mathcal{V}}$ that has the same objects as $\mathcal{V}$ but more morphisms

$$\text{Hom}_{\tilde{\mathcal{V}}}(U, V) = \tilde{\mathbb{K}}_{\mathcal{V}} \otimes_{\mathbb{K}_{\mathcal{V}}} \text{Hom}_{\mathcal{V}}(U, V). \quad (5.11)$$

The bigger category $\tilde{\mathcal{V}}$ is modular with the same family of simple objects, and $x \in \tilde{\mathbb{K}}_{\mathcal{V}}$ is a rank of $\tilde{\mathcal{V}}$.

For every modular category $\mathcal{V}$, and a choice of rank $\mathfrak{D}$, there is a $\mathbb{K}_{\mathcal{V}}$ -valued homeomorphism invariant of 3-manifolds. This invariant is defined as follows.

Let K be an oriented framed link in S^3 , and let $\text{col}(K)$ be the set of maps $\lambda : \{K_1, \dots, K_V\} \rightarrow I$, denoting all possible colorings of K by simple objects of the modular category. The colored oriented framed link K_{λ} is obtained from K by coloring its j -th component, K_j , with $V_{\lambda(K_j)}$. For any oriented framed link, K , in S^3 , define

$$\{K\} := \sum_{\lambda \in \text{col}(K)} \left(\prod_{j=1}^V \dim \lambda(K_j) \right) F(K_{\lambda}) \in \mathbb{K}_{\mathcal{V}}, \quad (5.12)$$

where $F(K_{\lambda})$ is the $\mathbb{K}_{\mathcal{V}}$ -valued invariant of the colored ribbon $(0, 0)$ -tangle, K_{λ} .

Remark 5.16. The quantity, $\{K\}$, does not depend on the enumeration of the components of K or on its choice of orientation [61, Lemma 2.2.1 in Chapter II].

Remark 5.17. If we denote the 0-framed unknot by U , then the rank is formally a square root of $\{U\}$ because $F(U_{V_i}) = \dim i$. See also Example 5.4.

Let $\mathfrak{D}_\pm := \{U^{\pm 1}\}$ where $U^{\pm 1}$ is the ± 1 -framed unknot, and let X be the closed oriented 3-manifold obtained from surgery along a framed link K in S^3 . We define

$$\tau_V(X) := \frac{\{K\}}{\mathfrak{D}^{b_0} \mathfrak{D}_+^{b_+} \mathfrak{D}_-^{b_-}}, \quad (5.13)$$

where b_0 , b_+ , and b_- are the number of zero, positive, and negative eigenvalues of the linking matrix of K respectively.

Theorem 5.2 (Reshetikhin–Turaev [55]). $\tau_V(X)$ is a topological invariant of the manifold X .

Remark 5.18. The 3-manifold invariant can be extended to a homeomorphism invariant of colored framed links embedded in 3-manifolds—denoted as $\tau_V(X, K)$ for a framed link K in the 3-manifold X .

Remark 5.19. The numerical invariant $\tau_V(X)$ is called a quantum invariant of X and satisfies $\tau_V(X \# Y) = \tau_V(X) \tau_V(Y)$. Similarly, the link invariant satisfies $\tau_V(X \# Y, L \sqcup K) = \tau_V(X, L) \tau_V(Y, K)$.

Example 5.2 (Invariant for the 3-sphere). If we think of S^3 as the result of ± 1 -surgery on the unknot, we can easily compute, $\tau_V(S^3) = 1$.

Example 5.3 (Invariant for $S^1 \times S^2$). If we think of $S^1 \times S^2$ as the result of 0-surgery on the unknot, we can easily compute, $\tau_V(S^1 \times S^2) = \mathfrak{D}$.

Example 5.4 (Invariant for Lens spaces $L(p, 1)$). The Lens space $L(p, 1)$ can be realized as p -surgery on the unknot. Since V_i 's are simple objects, the twist should be a scalar multiple of the identity, i.e., $\theta_{V_i} = v_i \text{id}_{V_i}$ for some $v_i \in \mathbb{K}_V$ for every $i \in I$. Therefore the invariant of a p -framed unknot colored V_i is $F(U_{V_i}^p) = \text{tr } \theta_{V_i}^p = v_i^p \dim i$, and the manifold invariant for p -surgery on the unknot is

$$\tau_V(L(p, 1)) = \mathfrak{D}_{\sigma(p)}^{-1} \sum_{i \in I} v_i^p (\dim i)^2, \quad (5.14)$$

where $\sigma(p)$ is the sign of p .

5.2.1 An $\text{SL}(2, \mathbb{Z})$ representation from modular categories

In § 3.1.2 we encountered $\text{PSL}(2, \mathbb{Z})$ as the mapping class group of the 2-torus, and since a modular category is the algebraic foundation of three-dimensional TQFTs, it is not

surprising that there is a (projective) $\mathrm{SL}(2, \mathbb{Z})$ representation associated to every modular category.

Let $(\mathcal{V}, \{V_i\}_{i \in I})$ be a modular category with a choice of rank $\mathfrak{D}$. Recall that since V_i 's are simple modules, the twist isomorphism, $\theta_{V_i} : V_i \rightarrow V_i$, will be a scalar multiple of the identity. Let v_i be this scalar for V_i for every $i \in I$. We define $T_{\mathcal{V}}$ to be the diagonal matrix whose (i, j) -th entry is $\delta_{i,j} v_j$, and $S_{\mathcal{V}}$ to be the square matrix whose (i, j) -th entry is $\mathrm{tr}(c_{V_i, V_j} \circ c_{V_j, V_i})$. The associated representation is $|I|$ -dimensional over $\mathbb{K}_{\mathcal{V}}$, and is given by $t \mapsto T_{\mathcal{V}}^{-1}$ and $s \mapsto \mathfrak{D}^{-1} S_{\mathcal{V}}$ [61, Chapter II § 3.6]. See also [61, Chapter IV § 5] for the relation between projectivity of this representation and anomalies in the associated TQFT.

5.3 Invariants for $\mathfrak{sl}_2$

The largest class of examples of braided monoidal categories, ribbon categories and modular categories comes from the category of representations of quasitriangular bialgebras, ribbon Hopf algebras and modular Hopf algebras respectively.

Roughly speaking, the axioms in the definition of these unital associative algebras is motivated by the structure we would like its category of representations to have. A bialgebra, for example, has a comultiplication, which lets us define tensor product of its representations. The definition of a Hopf algebra has an antihomomorphism called the antipode, which lets us define duals of representations; a quasitriangular Hopf algebra has an R -matrix, which lets us define a braiding; a ribbon Hopf algebra has a universal twist, which lets us define a twist; and a modular Hopf algebra has a finite family of simple representations, which are the simple objects of the associated modular category.

Associative Algebra	Category of Representations
bialgebra	monoidal category
Hopf algebra	rigid monoidal category
quasitriangular Hopf algebra	rigid braided monoidal category
ribbon Hopf algebra	ribbon category
modular Hopf algebra	(quasi)modular category

Quantum groups are quasitriangular Hopf algebras that are q -deformations of the universal enveloping algebras of Lie algebras. These quasitriangular Hopf algebras admit a universal twist, which makes them a ribbon Hopf algebra. And if their representation theory is well understood, it is possible to pick a family of simple representations, $\{V_i\}_{i \in I}$, such that (the semisimplification of) the subcategory of its representation category whose objects are dominated by $\{V_i\}_{i \in I}$, is a modular category. In the following sections, we shall sketch this story for the q -deformation of $\mathfrak{sl}_2$ (see [49, Chapters 4 and 8]).

5.3.1 Quantum $\mathfrak{sl}_2$ as a ribbon Hopf algebra

Let $q = e^h$ be a formal parameter. The quantum group $U_q \mathfrak{sl}_2$ is generated by E, F, K , and K^{-1} over $\mathbb{C}[[h]]$ subject to the relations

$$\begin{aligned} KE &= q^2 EK, & KF &= q^{-2} FK \\ KK^{-1} &= K^{-1}K = 1, & EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned} \quad (5.15)$$

Remark 5.20. If we let H be such that $K = q^H = e^{hH}$ formally, and take the classical limit $h \rightarrow 0$, we get the relations for (the universal enveloping algebra of) $\mathfrak{sl}_2$

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H. \quad (5.16)$$

This formal manipulation can be given solid grounding by giving the h -adic topology to $U_q \mathfrak{sl}_2$, which makes H an element of the h -adic completion of $U_q \mathfrak{sl}_2$.

There is a natural Hopf algebra structure inherited from $U \mathfrak{sl}_2$, with the comultiplication Δ , counit ϵ , and antipode S given by

$$\begin{aligned} \Delta(K) &= K \otimes K, & \Delta(E) &= E \otimes K + 1 \otimes E, & \Delta(F) &= F \otimes 1 + K^{-1} \otimes F, \\ \epsilon(K) &= 1, & \epsilon(E) &= \epsilon(F) = 0, & \text{and} \\ S(K) &= K, & S(E) &= -EK^{-1}, & S(F) &= -KF, \end{aligned} \quad (5.17)$$

and extended by (anti)multiplicativity and $\mathbb{C}$ -linearity.

Remark 5.21. Let A be a Hopf algebra and let V and W be vector spaces carrying representations of A , i.e., A -modules. We can define the action of A on the tensor product $V \otimes W$ by using the comultiplication. For any element $\alpha \in A$, we have $\Delta(\alpha) = \sum_j a_j \otimes b_j \in A \otimes A$ for some $a_j, b_j \in A$. For any $v \otimes w \in V \otimes W$, we define the action of α by

$$\alpha \cdot (v \otimes w) := \sum_j a_j v \otimes b_j w \quad (5.18)$$

and extend to $V \otimes W$ by linearity.

Remark 5.22. Let A be a Hopf algebra and let V be a vector space carrying a representation of A , i.e., an A -module. We can define the action of A on the dual V^* by using the antipode. For any element $\alpha \in A$ and any $f \in V^*$, we define the action to be such that

$$(\alpha \cdot f, v) = (f, S(\alpha)v) \text{ for every } v \in V, \quad (5.19)$$

where $(\cdot, \cdot)$ is the pairing $V^* \otimes V \rightarrow \mathbb{F}$. Note that if S were a homomorphism and not an antihomomorphism, we would have $(\alpha\beta) \cdot f = \beta \cdot (\alpha \cdot f)$ instead of $(\alpha\beta) \cdot f = \alpha \cdot (\beta \cdot f)$.

There is a special invertible element in (the $\hbar$ -adic completion of) $U_q \mathfrak{sl}_2 \otimes U_q \mathfrak{sl}_2$ called the universal R -matrix given by

$$R = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} (q - q^{-1}) E^n \otimes F^n, \quad (5.20)$$

where we have introduced the *quantum integer*

$$[n]_q := \frac{\{n\}_q}{\{1\}_q} = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{with} \quad \{n\}_q := q^n - q^{-n} \quad (5.21)$$

for any $n \in \mathbb{Z}$, and a quantum version of the factorial $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$.

Remark 5.23. Quantum integers are a q -deformation of the integers because in the classical limit $q \rightarrow 1$, we have $[n]_q \rightarrow n$. We can also use the notation of quantum integers with non-integer arguments. For example, one of the relations in the definition of $U_q \mathfrak{sl}_2$ can be written as $EF - FE = [H]_q$.

Remark 5.24. Let (A, R) be a quasitriangular Hopf algebra, let V and W be vector spaces carrying representations of A , and let $P_{V,W} : V \otimes W \rightarrow W \otimes V$ be the flip operator which acts as $v \otimes w \mapsto w \otimes v$ on elementary tensors. We can define a braiding by the action of $P_{V,W} \circ R$ on $V \otimes W$. Explicitly, if $R = \sum_j \alpha_j \otimes \beta_j$ for some $\alpha_j, \beta_j \in A$, then for any $v \in V$ and $w \in W$, the braiding $c_{V,W}$ is given by

$$c_{V,W}(v \otimes w) = \sum_j \beta_j w \otimes \alpha_j v, \quad (5.22)$$

and extended to $V \otimes W$ by linearity.

Finally, there is a special invertible element in (the $\hbar$ -adic completion of) $U_q \mathfrak{sl}_2$ called the universal twist given by

$$\mu = q^{-H^2/2} \sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}}{[n]_q!} (q - q^{-1})^n F^n K^{-n-1} E^n, \quad (5.23)$$

which makes $U_q \mathfrak{sl}_2$ a ribbon Hopf algebra.

Remark 5.25. Let (A, R, μ) be a ribbon Hopf algebra and let V be a vector space carrying a representation of A . We can define the twist isomorphism θ_V on V as the action of μ^{-1} .

Remark 5.26. For any ribbon Hopf algebra A , the category of finite dimensional representations $\text{Rep } A$ is a ribbon Ab-category whose ground ring coincides with the base ring of A .

Remark 5.27. There is a distinguished element $\nu = \sum_j S\beta_j \alpha_j \in A$ in every quasitriangular Hopf algebra A , where α_j, β_j are as in Remark 5.24. If $f : V \rightarrow V$ is an A -linear endomorphism of the A -module V , then $\text{tr}_q f = \text{tr}_V(\nu \mu^{-1} f)$, where $\text{tr}_q f$ is the trace of

f as a morphism of the ribbon Ab-category $\text{Rep } \mathcal{A}$ and $\text{tr}_V(\nu\mu^{-1}f)$ is the trace of $\nu\mu^{-1}f$ as a linear map. As a corollary, the dimension of V as an object in $\text{Rep } \mathcal{A}$ is $\text{tr}_V(\nu\mu^{-1})$.

Remark 5.28. In the ribbon Hopf algebra $U_q\mathfrak{sl}_2$ described above, it can be shown that $\nu\mu^{-1} = K$ so that $\dim_q V = \text{tr}_V K$.

5.3.2 Representations of $U_q\mathfrak{sl}_2$ and the modular category

We recall the well-known fact that the Lie algebra $\mathfrak{sl}_2$ has a unique n -dimensional irreducible representation for every $n \in \mathbb{Z}_{>0}$. For every n , the $\mathfrak{sl}_2$ -module V_n is isomorphic to $\mathbb{C}^n$ and $\mathfrak{sl}_2$ acts on it by

$$E \mapsto \begin{bmatrix} 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad F \mapsto \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & n-1 & 0 \end{bmatrix}, \quad (5.24)$$

$$H \mapsto \begin{bmatrix} n-1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(n-3) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -(n-1) \end{bmatrix}, \quad (5.25)$$

in an eigenbasis of H . More compactly, there is an eigenbasis $\{v_0, v_1, \dots, v_{n-1}\}$ of H such that [49, § 4.4]

$$\begin{aligned} Ev_j &= (n-j)v_{j-1}, & Ev_0 &= 0, \\ Fv_j &= (j+1)v_{j+1}, & Fv_{n-1} &= 0, \quad \text{and} \\ Hv_j &= (n-1-2j)v_j. \end{aligned} \quad (5.26)$$

Remark 5.29. The largest eigenvalue of H is called the highest weight of the representation. Sometimes representations are labeled by their highest weights instead of by dimensions, especially when there are non-equivalent representations in the same dimension.

For the quantum version, we can q -deform the above representation to get

$$\begin{aligned} Ev_j &= [n-j]_q v_{j-1}, & Ev_0 &= 0, \\ Fv_j &= [j+1]_q v_{j+1}, & Fv_{n-1} &= 0, \quad \text{and} \\ Kv_j &= q^{n-1-2j} v_j. \end{aligned} \quad (5.27)$$

This is indeed an irreducible representation of the quantum group $U_q \mathfrak{sl}_2$, and the module V_n is a simple object in $\text{Rep } U_q \mathfrak{sl}_2$. By Remark 5.28 the quantum dimension of V_n can be computed as $\dim_q V_n = \text{tr}_{V_n} K = [n]_q$.

Remark 5.30. When the R -matrix and universal twist act on these modules, the infinite sums in their formulae terminate because E and F act nilpotently on V_n .

Remark 5.31. Since the modules V_n are self-dual, we have $V_n^* \cong V_n$ after a choice of isomorphism. In the category of tangles colored by these modules, we have $\downarrow_{V_n} = \uparrow_{V_n^*} = \uparrow_{V_n}$, and therefore the associated link invariant is independent of orientations.

As we noted above, the category $\text{Rep } U_q \mathfrak{sl}_2$ has an infinite number of simple objects. To get a modular category, we restrict the parameter q to be a primitive root of unity. Let r and s be positive integers so that $\zeta := e^{\pi i s/r}$ is a primitive $2r$ -th root of unity, and let $\xi = \zeta^2 = e^{2\pi i s/r}$. We will follow convention so that $\zeta^\alpha = e^{\pi i s \alpha/r}$ for any $\alpha \in \mathbb{C}$.

We define the *small quantum group* $\tilde{U}_\xi \mathfrak{sl}_2$ as the quotient of $U_\xi \mathfrak{sl}_2$ by the two-sided ideal generated by E^r , F^r , and $K^{2r} - 1$. The quotient is a Hopf algebra because this ideal is a Hopf ideal. Specializing to a root of unity and doing this quotient makes the Hopf algebra finite dimensional over $\mathbb{C}$ and kills objects of $\text{Rep } U_\xi \mathfrak{sl}_2$ that have non-integer highest weights. We refer to [56, 55], and references therein for details on the representation theory of quantum $\mathfrak{sl}_2$.

The full representation theory of $U_\xi \mathfrak{sl}_2$ is subtle, but the relevant fact is that, for a given r , the modules V_n of $U_q \mathfrak{sl}_2$ for $n = 1, 2, \dots, r-1$ descend to irreducible modules of the small quantum group. It was shown in [55, § 8] that the modules $\{V_n\}_{n=1, \dots, r-1}$, make $\tilde{U}_\xi \mathfrak{sl}_2$ a modular Hopf algebra.

Given a modular Hopf algebra $\mathcal{A}$, with a finite family of simple $\mathcal{A}$ -modules $\{V_i\}_{i \in I}$, one considers the subcategory of $\text{Rep } \mathcal{A}$ whose objects are (quasi)dominated by $\{V_i\}_{i \in I}$. The semisimplification of this subcategory has the structure of a modular category with simple objects $\{V_i\}_{i \in I}$ [61, Chapter XI § 6].

For the purposes of defining the manifold invariant, we consider the modular category that is the subcategory of $\text{Rep } \tilde{U}_\xi \mathfrak{sl}_2$ whose objects are dominated by the irreducible modules $\{V_n\}_{n=1, \dots, r-1}$. We also have to pick a rank of this modular category. The rank is a square root of

$$\mathfrak{D}^2 = \sum_{n=1}^{r-1} (\dim_\xi V_n)^2 = \sum_{n=1}^{r-1} [n]_\xi^2 = \frac{r}{2} \left(\frac{1}{\sin(\pi s/r)} \right)^2, \quad (5.28)$$

and we pick $\mathfrak{D}^{-1} = \sqrt{2/r} \sin(\pi s/r)$.

To summarize, if we let $J_{n_1, \dots, n_V}(K, \zeta)$ be the colored Jones polynomial of the framed link K , where the component $K_j \subset K$ is colored with V_{n_j} , then with the intermediate quantity

$$F(K, \zeta) := \sum_{n \in \{1, \dots, r-1\}^V} J_n(K, \zeta) \prod_{K_j \subset K} [n_j]_\xi, \quad (5.29)$$

the $\mathfrak{sl}_2$ WRT invariant for the manifold $X = S^3(K)$ is given by

$$\tau_X(\zeta) = \frac{1}{\mathfrak{D}^{b_0}} \frac{F(K, \zeta)}{F(U^{+1}, \zeta)^{b_+} F(U^{-1}, \zeta)^{b_-}}, \quad (5.30)$$

where $b_{\pm}$ is the number of $\pm$ ve eigenvalues of the linking matrix of K and b_0 is the number of zero eigenvalues of the linking matrix of K .

CHAPTER 6

Modular forms and theta functions

We have already encountered the modular group $\mathrm{SL}(2, \mathbb{Z})$ as the mapping class group of the torus in § 3.1.2, and in § 5.2.1 we saw how it acts (projectively) on the simple objects of a modular category. In this chapter, we introduce modular forms, theta functions, and their Eichler integrals, which are all functions whose transformation properties under the action of $\mathrm{SL}(2, \mathbb{Z})$ are varying degrees of nice.

6.1 Definitions and examples

The modular group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is the group of 2×2 integer matrices with unit determinant. We recall that it has the presentation $\langle s, t \mid s^4 = 1_\Gamma, (ts)^3 = s^2 \rangle$ and is generated by matrices

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (6.1)$$

As a discrete subgroup of $\mathrm{SL}(2, \mathbb{R})$, it acts on the upper half plane, $\mathfrak{H}$, by Möbius transformations, i.e.,

$$\text{for any } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \quad \text{we have} \quad \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \quad (6.2)$$

where $\tau \in \mathfrak{H}$. Roughly speaking, modular forms are functions on the upper half plane that transform “nicely” under the action of the modular group.

Let k be a half-integer and let $\rho : \Gamma \rightarrow \mathbb{C}$ be a map such that $\rho(-I_2) = e^{\pi i k}$. We define an *automorphic factor* of weight k as a map $\tilde{\rho}_k : \Gamma \times \mathfrak{H} \rightarrow \mathbb{C}$ given by $\tilde{\rho}_k(\gamma, \tau) = \rho(\gamma)(c\tau + d)^k$ such that

$$\tilde{\rho}_k(\gamma_1 \gamma_2, \tau) = \tilde{\rho}_k(\gamma_1, \gamma_2 \tau) \tilde{\rho}_k(\gamma_2, \tau) \text{ for every } \gamma_1, \gamma_2 \in \Gamma. \quad (6.3)$$

The map $\rho : \Gamma \rightarrow \mathbb{C}$ is a one-dimensional projective representation of Γ and is called a *multiplier system*. A *weakly holomorphic modular form* of weight k with multiplier system ρ is a holomorphic function on the upper half plane, $f : \mathfrak{H} \rightarrow \mathbb{C}$, that satisfies

$$f(\gamma \tau) = \tilde{\rho}_k(\gamma, \tau) f(\tau). \quad (6.4)$$

A *modular form* is a weakly holomorphic modular form that is bounded as $\tau \rightarrow i\infty$, and a *cusp form* is a weakly holomorphic modular form that vanishes as $\tau \rightarrow i\infty$.

Remark 6.1. We define the *Petersson slash operator* of weight k on a function $f : \mathfrak{H} \rightarrow \mathbb{C}$ as $(f|_k \gamma)(\tau) = \bar{\rho}_k(\gamma)^{-1} f(\gamma\tau) = \rho(\gamma)^{-1} (c\tau + d)^{-k} f(\gamma\tau)$. The requirement (6.4) can be stated as $(f - f|_k \gamma)(\tau) = 0$.

Remark 6.2. Sets of weakly holomorphic modular forms, modular forms, and cusp forms of a given weight and multiplier system are $\mathbb{C}$ -vector spaces and denoted by $M_k^!(\rho)$, $M_k(\rho)$, and $S_k(\rho)$ respectively. We also have the inclusions $S_k(\rho) \subset M_k(\rho) \subset M_k^!(\rho)$.

Example 6.1 (Dedekind eta function and the modular discriminant). For $\tau \in \mathfrak{H}$, let $q = e^{2\pi i \tau}$ and define the *Dedekind eta function* as the following infinite product

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \quad (6.5)$$

The Dedekind eta function is a weight-1/2 modular form with a multiplier system and undergoes the following transformations [16, Proposition 1.2.5]

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau), \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (6.6)$$

Its twenty-fourth power, $\Delta(\tau) := \eta(\tau)^{24}$, is called the *modular discriminant*, and is a weight-12 modular form.

We can define vector-valued modular forms if we let the map $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{C})$ be a d -dimensional projective representation of Γ . A *weakly holomorphic vector-valued modular form* of weight k with multiplier system ρ is a holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}^d$ that satisfies (6.4) with $\bar{\rho}_k(\gamma) = (c\tau + d)^k \rho(\gamma)$. As in the rank one case, we can drop *weakly holomorphic* if we require each component of f to be bounded as $\tau \rightarrow i\infty$, and we call f a cusp form if each of its components vanishes as $\tau \rightarrow i\infty$. The integer d is called the *rank* of the vector-valued modular form.

Example 6.2 (Jacobi theta functions). Jacobi theta functions defined as

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} \quad \text{and} \quad \theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad (6.7)$$

are components of a weight-1/2 vector-valued modular form of rank two. We have

$$\begin{bmatrix} \theta_2(\tau + 1) \\ \theta_3(\tau + 1) \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_2(\tau) \\ \theta_3(\tau) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \theta_2(-1/\tau) \\ \theta_3(-1/\tau) \end{bmatrix} = \sqrt{\frac{-i\tau}{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_2(\tau) \\ \theta_3(\tau) \end{bmatrix}. \quad (6.8)$$

While the T -transformation formula is quite straightforward, deriving the rule for S -transformations is a little less obvious and follows from a standard argument involving the Poisson summation formula.

In what follows, we will come across theta functions twisted by a Dirichlet character. We remind the reader that, for positive integer M , a *Dirichlet character* of modulus M is an M -periodic function $\chi : \mathbb{Z} \rightarrow \mathbb{C}$ such that (i) $\chi(ab) = \chi(a)\chi(b)$ for every $a, b \in \mathbb{Z}$ (ii) and $\chi(a) = 0$ if and only if $\gcd(a, M) > 1$. A theta function twisted by a Dirichlet character is defined as

$$\theta_\chi(\tau) = \sum_{n \in \mathbb{Z}} \chi(n) n^\nu q^{n^2}, \quad (6.9)$$

where $\nu = 0, 1$ such that $\chi(-1) = (-1)^\nu$. The theta function θ_χ is a component of a weight- $(\nu + 1/2)$ vector-valued modular form.

Example 6.3 (Theta functions $\Psi_p^{(a)}$). Let P be a positive integer, and for every integer $a = 1, \dots, P-1$ define $2P$ -periodic functions

$$\psi_{2P}^{(a)}(l) = \begin{cases} \pm 1 & \text{if } l = \pm a \pmod{2P}, \\ 0 & \text{otherwise.} \end{cases} \quad (6.10)$$

Each $\psi_{2P}^{(a)}$ has an associated theta function

$$\Psi_p^{(a)}(\tau) = \sum_{l \in \mathbb{Z}} l \psi_{2P}^{(a)}(l) q^{l^2/4P} \quad \text{with } q = e^{2\pi i \tau} \quad (6.11)$$

which are components of a weight- $3/2$ vector-valued modular form of rank $P-1$. Under T and S modular transformations, they behave as [28, § 2.2]

$$\Psi_p^{(a)}(\tau + 1) = e^{\pi i a^2/2P} \Psi_p^{(a)}(\tau) \quad \text{and} \quad (6.12)$$

$$\Psi_p^{(a)}(-1/\tau) = (-i\tau)^{3/2} \sum_{b=1}^{P-1} M_b^a(P) \Psi_p^{(b)}(-1/\tau) \quad (6.13)$$

respectively, where

$$M_b^a(P) = \sqrt{\frac{2}{P}} \sin \frac{ab}{P} \pi. \quad (6.14)$$

Partly because of their importance and partly because of their complicated definition, we put the next class of examples in its own section.

6.1.1 A nice basis of theta functions

Following [28, 27, 30], we describe theta functions that span a subspace of half-integer weight vector-valued modular forms. This basis of theta functions is nice because radial limits of their Eichler integrals are related to quantum invariants of Seifert homology spheres (see Proposition 7.2).

Given a triple, $\vec{p} = (p_1, p_2, p_3)$, let $P = p_1 p_2 p_3$, and for every triple, $\vec{a} = (a_1, a_2, a_3)$, with integers a_j such that $0 < a_j < p_j$, define the $2P$ -periodic functions

$$\phi_{\vec{p}}^{\vec{a}}(l) = \begin{cases} -\epsilon_1 \epsilon_2 \epsilon_3 & \text{if } l = P \left(1 + \sum_{j=1}^3 \frac{\epsilon_j a_j}{p_j} \right) \bmod 2P, \\ 0 & \text{otherwise,} \end{cases} \quad (6.15)$$

where $\epsilon_j = \pm 1$. However, not all $\phi_{\vec{p}}^{\vec{a}}$'s are independent. Indeed, for $j = 1, 2, 3$ define the involutions

$$\sigma_1(\vec{a}) = (p_1 - a_1, a_2, a_3), \quad \sigma_2(\vec{a}) = (a_1, p_2 - a_2, a_3), \quad \text{and } \sigma_3(\vec{a}) = (a_1, a_2, p_3 - a_3), \quad (6.16)$$

and note that for $j \neq k$ we have $\phi_{\vec{p}}^{\sigma_j \circ \sigma_k(\vec{a})}(l) = \phi_{\vec{p}}^{\vec{a}}(l)$, so that the number of such independent periodic functions is

$$D_{\vec{p}} = \frac{1}{4}(p_1 - 1)(p_2 - 1)(p_3 - 1). \quad (6.17)$$

For later use, we also note the following properties of the $\phi_{\vec{p}}^{\vec{a}}$'s,

$$\phi_{\vec{p}}^{\vec{a}}(-l) = -\phi_{\vec{p}}^{\vec{a}}(l) \quad \text{and} \quad \sum_{l=0}^{2P-1} \phi_{\vec{p}}^{\vec{a}}(l) = 0. \quad (6.18)$$

For each independent $\phi_{\vec{p}}^{\vec{a}}$, define theta functions on the upper half plane

$$\Phi_{\vec{p}}^{\vec{a}}(\tau) = \frac{1}{2} \sum_{l \in \mathbb{Z}} l \phi_{\vec{p}}^{\vec{a}}(l) q^{l^2/4P} \quad \text{with } q = e^{2\pi i \tau}, \quad (6.19)$$

which are components of a $D_{\vec{p}}$ -dimensional vector-valued modular form of weight $3/2$. Under T and S modular transformations, these theta functions behave as [27, Proposition 2]

$$\Phi_{\vec{p}}^{\vec{a}}(\tau + 1) = T^{\vec{a}}(\vec{p}) \Phi_{\vec{p}}^{\vec{a}}(\tau) \quad \text{and} \quad \Phi_{\vec{p}}^{\vec{a}}(\tau) = \left(\frac{i}{\tau} \right)^{3/2} \sum_{\vec{b}} S_{\vec{b}}^{\vec{a}}(\vec{p}) \Phi_{\vec{p}}^{\vec{b}}(-1/\tau) \quad (6.20)$$

respectively, where

$$T^{\vec{a}} = \exp \left(\frac{P}{2} \left(1 + \sum_{j=1}^3 \frac{a_j}{p_j} \right)^2 \pi i \right) \quad (6.21)$$

and

$$S_{\vec{b}}^{\vec{a}} = -\frac{8}{\sqrt{2P}} (-1)^{P \left(1 + \sum_j \frac{a_j + b_j}{p_j} \right) + P \sum_{j \neq k} \frac{a_j b_k}{p_j p_k}} \prod_j \sin \left(P \frac{a_j b_j}{p_j^2} \pi \right). \quad (6.22)$$

6.2 Eichler integrals and false theta functions

For modular forms of integer weight $k \geq 2$, there is a differential operator $D^{k-1} : M_{2-k}^! \rightarrow M_k^!$ that takes weakly holomorphic modular forms of weight $-(2-k)$ to forms of weight k given by (see [7])

$$D^{k-1} := \frac{1}{(2\pi i)^{k-1}} \frac{\partial^{k-1}}{\partial \tau^{k-1}}. \quad (6.23)$$

At the level of Fourier expansions, if f is a weakly holomorphic modular form of weight $2-k$, then the differential gives

$$D^{k-1}f(\tau) = \sum_n a_n n^{k-1} q^n \in M_k^! \quad \text{if} \quad f(\tau) = \sum_n a_n q^n. \quad (6.24)$$

As a formal inverse of this differential operator, Eichler considered the $(k-1)$ -th antiderivative defined for $f \in M_k^!$ as

$$\tilde{f}(\tau) = \frac{(-2\pi i \tau)^{k-1}}{\Gamma(k-1)} \int_{\tau}^{i\infty} dz f(z) (z - \tau)^{k-2}, \quad (6.25)$$

the so-called *Eichler integral*. If f is a cusp form of weight k , then the obstruction to modularity of its Eichler integral can be computed as

$$(\tilde{f} - \tilde{f}|_{2-k}\gamma)(\tau) = \frac{(-2\pi i \tau)^{k-1}}{\Gamma(k-1)} \int_{\gamma^{-1}(i\infty)}^{i\infty} dz f(z) (z - \tau)^{k-2}, \quad (6.26)$$

and using properties of the differential operator D^{k-1} , we can show that

$$D^{k-1}(\tilde{f} - \tilde{f}|_{2-k}\gamma) = 0. \quad (6.27)$$

We see that the Eichler integral of a cusp form of integer weight k is nearly modular of weight $2-k$, and its obstruction to modularity is a polynomial of degree at most $k-2$ called the *period polynomial*. As before, we see that at the level of q -series, the Eichler integral of a cusp form is given by

$$\tilde{f}(\tau) = \sum_{n=1}^{\infty} a_n n^{1-k} q^n \quad \text{if} \quad f(\tau) = \sum_{n=1}^{\infty} a_n q^n. \quad (6.28)$$

When the weight is not an integer, the interpretation of differential operators and antiderivative no longer works, but we can still define an Eichler integral for the q -series as in (6.28).

Example 6.4 (False theta functions $\tilde{\theta}_\chi$). The theta function twisted by a Dirichlet character, θ_χ , is a modular form of weight $k = \nu + 1/2$, where ν is either 0 or 1 such that

$\chi(-1) = (-1)^\nu$. Its Eichler integral is the false theta function

$$\tilde{\theta}_\chi(\tau) = \sum_{n=1}^{\infty} \chi(n) n^{1-\nu} q^{n^2}. \quad (6.29)$$

We will soon argue that the limit of these false theta functions as τ approaches rational numbers is nearly modular of weight $\nu + \frac{(-1)^\nu}{2}$, but the obstruction to modularity is no longer a polynomial.

It will be useful to define another variant of the Eichler integral as an antiderivative like in (6.25)

$$\theta_\chi^*(\tau) = \frac{(-2\pi i)^{k-1}}{\Gamma(k-1)} \int_{\tau^*}^{\infty} dz \theta_\chi(z) (z - \tau)^{k-2}, \quad (6.30)$$

defined when τ is in the lower half plane, $\Im \tau^-$. It is relatively straightforward to deduce the modular behaviour and period function of θ_χ^* from the known modularity of θ_χ . We can use this modular behaviour and the fact that the series expansion of θ_χ^* and $\tilde{\theta}_\chi$ agree to all orders near $\alpha \in \mathbb{Q}$,

$$\tilde{\theta}_\chi(\alpha + i\epsilon) = \sum_{n=0}^{\infty} b_n \epsilon^n \quad \text{and} \quad \theta_\chi^*(\alpha - i\epsilon) = \sum_{n=0}^{\infty} b_n (-\epsilon)^n \quad (6.31)$$

as $\epsilon \rightarrow 0^+$, to derive the modular behaviour of $\tilde{\theta}_\chi$ near $\alpha \in \mathbb{Q}$. We refer the reader to [68, 27, 28, 7] for details.

Example 6.5 (Eichler integral of the nice basis). Eichler integral of the theta function $\Phi_{\vec{p}}^{\vec{a}}$ is given by the q -series

$$\tilde{\Phi}_{\vec{p}}^{\vec{a}}(\tau) = \sum_{l=0}^{\infty} \phi_{\vec{p}}^{\vec{a}}(l) q^{l^2/4P}, \quad (6.32)$$

convergent in the upper half plane $\text{Im } \tau > 0$, whose limits at rational numbers $\tau \rightarrow s/r$ follow from a proposition of [68] (see also [27, Proposition 3])

$$\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r) = \lim_{t \rightarrow 0^+} \tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r + it) = \frac{1}{2} \sum_{l=0}^{2Pr} \phi_{\vec{p}}^{\vec{a}}(l) \xi^{l^2/4P} \left(1 - \frac{l}{Pr}\right). \quad (6.33)$$

From the formula above, it is straightforward to deduce the behavior of these functions under a modular T -transformation, $\tilde{\Phi}_{\vec{p}}^{\vec{a}}(\tau + 1) = T^{\vec{a}}(\vec{p}) \tilde{\Phi}_{\vec{p}}^{\vec{a}}(\tau)$, where $T^{\vec{a}}(\vec{p})$ is a fractional power of ξ . And while the false theta functions do not have good transformation behavior under the action of the full modular group, their limits at cusps, $\tau \rightarrow r/s$, satisfy the following quantum modular property under an S -transformation as $r \rightarrow \infty$ [27, Proposition 8]

$$\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r) + \sqrt{\frac{r}{is}} \sum_{\vec{b}} S_{\vec{b}}^{\vec{a}}(\vec{p}) \tilde{\Phi}_{\vec{p}}^{\vec{b}}(-r/s) \simeq \sum_{k=0}^{\infty} \frac{L(-2k, \phi_{\vec{p}}^{\vec{a}})}{k!} \left(\frac{\pi is}{2Pr}\right)^k, \quad (6.34)$$

where the sum over $\vec{b}$ includes only the $D_{\vec{p}}$ independent false theta functions, and the coefficients of the power series on the right-hand side are

$$L(-2k, \phi_{\vec{p}}^{\vec{a}}) = -\frac{(2P)^{2k}}{2k+1} \sum_{l=1}^{2P} \phi_{\vec{p}}^{\vec{a}}(l) B_{2k+1}(l/2P), \quad (6.35)$$

where B_{2k+1} denotes the $(2k+1)$ -th Bernoulli polynomial, are values of the analytically continued L -function for the Dirichlet character $\phi_{\vec{p}}^{\vec{a}}$.

Remark 6.3. In Appendix B, we prove an integrality result for the radial limits of these false theta functions (Proposition B.1).

Example 6.6 (False theta functions $\tilde{\Psi}_p^{(a)}$). We define false theta functions, which are the Eichler integrals of $\Psi_p^{(a)}$

$$\tilde{\Psi}_p^{(a)}(\tau) = \sum_{l=0}^{\infty} \psi_{2P}^{(a)}(l) q^{l^2/4P}, \quad (6.36)$$

whose limit at rational numbers

$$\tilde{\Psi}_p^{(a)}(s/r) = \frac{1}{2} \sum_{l=0}^{2Pr} \psi_{2P}^{(a)}(l) \xi^{l^2/4P} \left(1 - \frac{l}{Pr}\right), \quad (6.37)$$

satisfies the following property under a modular S -transformation [28, Proposition 2]

$$\tilde{\Psi}_p^{(a)}(s/r) + \sqrt{\frac{r}{is}} \sum_{b=1}^{P-1} M_b^a(P) \tilde{\Psi}_p^{(b)}(-r/s) \simeq \sum_{k=0}^{\infty} \frac{L(-2k, \psi_{2P}^{(a)})}{k!} \left(\frac{\pi is}{2Pr}\right)^k, \quad (6.38)$$

where the coefficients of the power series on the right-hand side are

$$L(-2k, \psi_{2P}^{(a)}) = -\frac{(2P)^{2k}}{2k+1} \sum_{l=1}^{2P} \psi_{2P}^{(a)}(l) B_{2k+1}(l/2P), \quad (6.39)$$

where B_{2k+1} denotes the $(2k+1)$ -th Bernoulli polynomial, are values of the analytically continued L -function for the Dirichlet character $\psi_{\vec{p}}^{\vec{a}}$.

CHAPTER 7

Quantum modularity conjecture

To state the quantum modularity conjecture, it will be convenient to normalize the WRT invariant in the following way; suppose that X is a rational homology sphere, let $H := |H_1(X, \mathbb{Z})|$ be the order of its first homology group, and define

$$W_X(\xi) := \sqrt{H} \left(\frac{H}{s} \right) (\xi - 1) \tau_X(\xi). \quad (7.1)$$

Since s is an odd integer, the Jacobi symbol (H/s) is well-defined.

When X is an integer homology sphere, $W_X(\xi) = (\xi - 1) \tau_X(\xi)$ and the statement takes the form of Conjecture 1.1. We propose the following generalization for rational homology spheres that have no $\mathbb{Z}_2$ -torsion in their homology.

Conjecture 7.1. *Let $X \cong M/\Gamma$ be a geometric rational homology sphere with no $\mathbb{Z}_2$ -torsion in homology and let $\xi^{1/4}$ be a primitive $4r$ -th root of unity. Specifically, take $\xi^{1/4} = e^{\pi i s/2r}$ for some $r \in 2\mathbb{Z}_{\geq 1} + 1$ and $s \in 4\mathbb{Z} + 1$. Moreover, assume that s is coprime with the order of the homology group $H = |H_1(X, \mathbb{Z})|$. Then*

1. *the WRT invariant admits the decomposition into a sum of terms labeled by abelian flat connections*

$$W_X(\xi) = \sum_{a \in \pi_0(M_{\text{flat}}^{ab}(X, \text{SL}(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[a]} W_X^a(\xi); \quad (7.2)$$

such that each term has the following asymptotic expansion as $r \rightarrow \infty$ along integers coprime with s and H :

$$e^{2\pi i \frac{r}{s} \text{CS}[a]} W_X^a(\xi) \simeq \sum_{A \in \pi_0(M_{\text{flat}}(X, \text{SL}(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[A]} P_A^a(\xi) I_A^a(s/r), \quad (7.3)$$

where $\tilde{\xi} = e^{-2\pi i r/s}$, $P_A^a(\tilde{\xi})$ are polynomials in $\tilde{\xi}$ (with s -dependent coefficients), and $I_A^a(s/r) \in (s/r)^{\delta_A/2} \mathbb{C}[[s/r]]$ for some $\delta_A \in \mathbb{Z}$. Moreover, $P_A^{triv}(\tilde{\xi}) \in \mathbb{Z}[\tilde{\xi}]$ and

$$P_A^a(\tilde{\xi}) = 0 \quad (7.4)$$

for $A \in \pi_0(M_{\text{flat}}^{ab}(X, \text{SL}(2, \mathbb{C}))) \setminus \{a\}$.

2. for the geometric flat connection A_*

$$P_{A_*}^{triv}(\xi) = \xi^\delta \sum_{a \in \pi_0(M_{\text{flat}}(X, U(1)))} W_X^a(\xi) - \chi_{S^3}(M) \quad (7.5)$$

where $\delta \in \mathbb{Q}$ and the function $\chi_{S^3}(M) = 1$ if $M = S^3$ and zero otherwise.

Remark 7.1. The decomposition (7.2) with respect to the abelian flat connections is motivated by the analogous decomposition of the WRT invariant at $\xi = e^{\frac{2\pi i}{r}}$ in its relation to the $\widehat{Z}$ -invariants [23, 21, 22, 46]. The sum on the right-hand side of 7.5 is the analogue of $\widehat{Z}_0$. In § A.1 we provide a conjectural formula that relates $\widehat{Z}$ -invariants to the WRT invariant at a general root of unity. Such a relation allows one to draw a direct parallel between the quantum modularity considered in this thesis and *holomorphic quantum modularity* of $\widehat{Z}$ -invariants.

7.1 Flat connections

Flat connections on a manifold are completely specified by their holonomies around non-contractible loops. Let X be a manifold and let $E \rightarrow X$ be a (trivial) principle G -bundle on X . The moduli space of flat G -connections on X is defined as

$$\mathcal{M}_{\text{flat}}(X, G) := \text{Hom}(\pi_1(X), G)/G, \quad (7.6)$$

where G acts on $\text{Hom}(\pi_1(X), G)$ by conjugation. In this section we note some features of this moduli space for $G = \text{SL}(2, \mathbb{C})$, describe how the false theta basis $\tilde{\Phi}_p^{\vec{a}}$ is in one-to-one correspondence with flat connections on Brieskorn homology spheres, and discuss the geometric flat connection.

In the moduli space of $\text{SL}(2, \mathbb{C})$ flat connections over a manifold X , there is a distinguished set of *abelian flat connections* defined as

$$\mathcal{M}_{\text{flat}}^{\text{ab}}(X, \text{SL}(2, \mathbb{C})) := \text{Hom}(\pi_1(X), \mathbb{C}^\times)/\{\pm 1\} \cong H^1(X, \mathbb{C}/\mathbb{Z})/\{\pm 1\}, \quad (7.7)$$

where $\{\pm 1\}$ should be identified with the Weyl group of $\text{SL}(2, \mathbb{C})$, and the last equivalence follows because $\mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$. Maps representing abelian flat connections $\pi_1(X) \rightarrow \text{SL}(2, \mathbb{C})$ can be conjugated into the maximal torus subgroup $\mathbb{C}^\times$ of $\text{SL}(2, \mathbb{C})$. We also

define the set of *non-abelian flat connections*

$$m_{\text{flat}}^{\text{non-ab}}(X, \text{SL}(2, \mathbb{C})) := m_{\text{flat}}(X, \text{SL}(2, \mathbb{C})) \setminus m_{\text{flat}}^{\text{ab}}(X, \text{SL}(2, \mathbb{C})), \quad (7.8)$$

as the complement of abelian flat connections.

We have the canonical identification

$$\pi_0(m_{\text{flat}}^{\text{ab}}(X, \text{SL}(2, \mathbb{C}))) = \text{Hom}(\text{Tor } H_1(X, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \text{Tor } H_1(X, \mathbb{Z})/\mathbb{Z}_2 \quad (7.9)$$

given by the linking pairing, $\text{lk} : \text{Tor } H_1(X, \mathbb{Z}) \otimes \text{Tor } H_1(X, \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$. The Chern–Simons action of an abelian flat connection $\alpha \in m_{\text{flat}}^{\text{ab}}(X, \text{SL}(2, \mathbb{C}))$, is given by the linking pairing, $\text{CS}[A_\alpha] = \text{lk}(\alpha, \alpha)$, using the above identification.

7.1.1 False theta functions and flat connections

Here, we review a characterization of the connected components of the moduli space of non-trivial $\text{SL}(2, \mathbb{C})$ flat connections on Seifert integer homology spheres, and in particular, Brieskorn homology spheres from [3, Proposition 5] (see also [27, § 4]). Let $\Sigma(p_1, p_2, p_3)$ be a Brieskorn homology sphere, and suppose (possibly after relabeling) that p_2 and p_3 are odd. Define $A_{\vec{p}} \subset \mathbb{Z}_{>0}^3$ as the set of triples (a_1, a_2, a_3) such that $1 \leq a_1 < p_1$ and $1 \leq a_j \leq (p_j - 1)/2$ for $j = 2$ and 3 , then there is an isomorphism

$$\pi_0(m_{\text{flat}}^{\text{non-ab}}(\Sigma(p_1, p_2, p_3), \text{SL}(2, \mathbb{C}))) \cong A_{\vec{p}}. \quad (7.10)$$

Each element $\vec{a} \in A_{\vec{p}}$ is called the *rotation number* of the flat connection it represents, and the Chern–Simons value of the corresponding flat connection is

$$\text{CS}[\vec{a}] = -\frac{P}{4} \left(1 + \sum_{j=1}^3 \frac{a_j}{p_j} \right)^2 \pmod{1}. \quad (7.11)$$

Note that the set $A_{\vec{p}}$ has cardinality

$$D_{\vec{p}} = \frac{1}{4} (p_1 - 1)(p_2 - 1)(p_3 - 1), \quad (7.12)$$

and the triples that are its elements, $\vec{a} \in A_{\vec{p}}$, pick out the $D_{\vec{p}}$ independent false theta functions $\tilde{\Phi}_{\vec{p}}^{\vec{a}}$. With this correspondence, we can write the T -matrix of the modular transformation as $T^{\vec{a}}(\vec{p}) = e^{-2\pi i \text{CS}[\vec{a}]}$. And the following result says that the limit of the false theta function $\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r)$ at a rational number is a polynomial in the root of unity $\xi = e^{\frac{2\pi i s}{r}}$ with an overall fractional power that equals $\text{CS}[\vec{a}]$.

7.1.2 Geometric flat connection

As stated before, the geometric flat connection for a geometric 3-manifold, $X = M/\Gamma$ is given by the conjugacy class of a certain homomorphism $\Gamma \cong \pi_1(X) \rightarrow \mathrm{SL}(2, \mathbb{C})$, which is a lift of the canonical inclusion $\Gamma \hookrightarrow \mathrm{Isom}_+ M$ with certain natural map $\mathrm{Isom}_+ M \rightarrow \mathrm{PSL}(2, \mathbb{C})$. We give candidates for the natural map from the isometry group to $\mathrm{PSL}(2, \mathbb{C})$ (summarized in Table 7.1).

Remark 7.2. Lifting the map $\pi_1(X) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ to $\pi_1(X) \rightarrow \mathrm{SL}(2, \mathbb{C})$ is analogous to lifting the frame bundle of the manifold to its double cover, the spin bundle. As such, the obstruction to lifting a $\mathrm{PSL}(2, \mathbb{C})$ connection to $\mathrm{SL}(2, \mathbb{C})$ connection is given by a version of the the Stiefel–Whitney class, and since every orientable 3-manifold is spin, such a lift always exists. Choice of the lift corresponds to a choice of the spin structure. All examples we consider in this thesis are $\mathbb{Z}_2$ -homology spheres, which have a unique spins structure, and therefore a unique lift.

M	Isometry group	$\longrightarrow \mathrm{PSL}(2, \mathbb{C})$
$\mathbb{H}^3$	$\mathrm{Isom}_+ M = \mathrm{PSL}(2, \mathbb{C})$	$\mathrm{id}_{\mathrm{PSL}(2, \mathbb{C})}$
S^3	$\mathrm{Isom}_+ M = \mathrm{SO}(4)$	$\twoheadrightarrow \mathrm{PSU}(2) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
$\mathbb{R}^3$	$\mathrm{Isom}_+ M = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$	$\twoheadrightarrow \mathrm{SO}(3) \cong \mathrm{PSU}(2) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
$S^2 \times \mathbb{R}$	$\mathrm{Isom}_0 M = \mathrm{SO}(3) \times \mathbb{R}$	$\twoheadrightarrow \mathrm{SO}(3) \cong \mathrm{PSU}(2) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
$\mathbb{H}^2 \times \mathbb{R}$	$\mathrm{Isom} M = \mathrm{Isom} \mathbb{H}^2 \times \mathrm{Isom} \mathbb{R}$	$\twoheadrightarrow \mathrm{PSL}(2, \mathbb{R}) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
$\overline{\mathrm{SL}(2, \mathbb{R})}$	$\mathbb{R} \rightarrow \mathrm{Isom}_+ M \rightarrow \mathrm{Isom} \mathbb{H}^2$	$\twoheadrightarrow \mathrm{PSL}(2, \mathbb{R}) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
Nil_3	$\mathbb{R} \rightarrow \mathrm{Isom}_0 M \rightarrow \mathbb{R}^2 \rtimes \mathrm{SO}(2)$	$\cong \mathbb{C} \rtimes \mathrm{U}(1) \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$
Sol_3	$\mathrm{Isom}_* \mathbb{R}^2 \rightarrow \mathrm{Isom}_+ M \rightarrow \mathrm{Isom} \mathbb{R}$	$\twoheadrightarrow \mathbb{R} \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$

Table 7.1: The eight Thurston geometries, the connected components of the identity of their isometry groups, and natural maps to $\mathrm{PSL}(2, \mathbb{C})$.

Hyperbolic Since the group of orientation preserving isometries for the hyperbolic space is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$, so we just take the map to $\mathrm{PSL}(2, \mathbb{C})$ to be the identity.

Euclidean The orientation preserving isometries of the Euclidean space is the semi-direct product $\mathbb{R}^3 \rtimes \mathrm{SO}(3)$. The natural map $\rightarrow \mathrm{PSL}(2, \mathbb{C})$ is given by a projection onto the $\mathrm{SO}(3) \cong \mathrm{PSU}(2)$ factor, followed by an inclusion into $\mathrm{PSL}(2, \mathbb{C})$.

Spherical For the 3-sphere, we have $\mathrm{Isom}_+ S^3 = \mathrm{SO}(4) \cong \mathrm{P}(\mathrm{SU}(2) \times \mathrm{SU}(2))$, and we require that $\Gamma \subset \mathrm{SO}(4)$ is contained in one of the $\mathrm{SU}(2)$ factors and the image of Γ under projection to the other factor is trivial. For such Γ , we can define the map $\rightarrow \mathrm{PSL}(2, \mathbb{C})$ to be the nontrivial projection onto $\mathrm{PSU}(2)$ followed by an inclusion into $\mathrm{PSL}(2, \mathbb{C})$.

We remark that the spherical geometry is exceptional because for the other seven geometries, the fixing Γ , uniquely determines the homeomorphism type of X . The above requirement on Γ restores this uniqueness and restricts it to be an ADE-type discrete subgroup of $\mathrm{SU}(2)$

$\mathbb{H}^2 \times \mathbb{R}$ For this product geometry, the group of isometries factors $\text{Isom}(\mathbb{H}^2 \times \mathbb{R}) \cong \text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{R})$, and we can define the map $\rightarrow \text{PSL}(2, \mathbb{C})$ by projecting to the $\text{Isom}(\mathbb{H}^2)$ factor followed by $\text{Isom}(\mathbb{H}^2) \twoheadrightarrow \text{PSL}(2, \mathbb{R}) \hookrightarrow \text{PSL}(2, \mathbb{C})$.

$\widetilde{\text{SL}(2, \mathbb{R})}$ This model geometry can be thought of as an $\mathbb{R}$ -fibration over the hyperbolic plane, $0 \rightarrow \mathbb{R} \hookrightarrow \widetilde{\text{SL}(2, \mathbb{R})} \twoheadrightarrow \mathbb{H}^2 \rightarrow 0$, which induces a short exact sequence of isometry groups, $0 \rightarrow \mathbb{R} \hookrightarrow \text{Isom}_+ \widetilde{\text{SL}(2, \mathbb{R})} \twoheadrightarrow \text{Isom } \mathbb{H}^2 \rightarrow 0$. As above, we define the map $\rightarrow \text{PSL}(2, \mathbb{C})$ by the composition $\text{Isom } \mathbb{H}^2 \twoheadrightarrow \text{PSL}(2, \mathbb{R}) \hookrightarrow \text{PSL}(2, \mathbb{C})$.

Sol₃ This model geometry can be thought of as an $\mathbb{R}^2$ -fibration over the line, which induces a short exact sequence of isometry groups, $0 \rightarrow \text{Isom}_* \mathbb{R}^2 \hookrightarrow \text{Isom}_+ \text{Sol}_3 \twoheadrightarrow \text{Isom } \mathbb{R} \rightarrow 0$. We define the map $\rightarrow \text{PSL}(2, \mathbb{C})$ by the composition $\text{Isom } \mathbb{R} \twoheadrightarrow \mathbb{R} \hookrightarrow \text{PSL}(2, \mathbb{C})$.

$\mathbb{S}^2 \times \mathbb{R}$ Like the other product geometry, the group of isometries factors $\text{Isom}_+(\mathbb{S}^2 \times \mathbb{R}) \cong \text{O}(3) \times \mathbb{R}$, but there is no canonical map from $\text{O}(3)$ to $\text{SO}^+(3, 1) \cong \text{PSL}(2, \mathbb{C})$. To get around this, we restrict to $\Gamma \subset \text{Isom}_0(\mathbb{S}^2 \times \mathbb{R}) = \text{SO}(3) \times \mathbb{R}$, and define the map $\rightarrow \text{PSL}(2, \mathbb{C})$ by projecting onto the $\text{SO}(3) \cong \text{PSU}(2)$ factor followed by the inclusion $\text{PSU}(2) \hookrightarrow \text{PSL}(2, \mathbb{C})$.

Nil₃ This model geometry can be thought of as an $\mathbb{R}$ -fibration over $\mathbb{R}^2$, which induces a short exact sequence of isometry group, $0 \rightarrow \mathbb{R} \hookrightarrow \text{Isom}_+ \text{Nil}_3 \twoheadrightarrow \text{Isom } \mathbb{R}^2 \rightarrow 0$. We have a map $\text{Isom}_+ \text{Nil}_3 \twoheadrightarrow \text{Isom } \mathbb{R}^2 \cong \mathbb{R}^2 \rtimes \text{O}(2)$, but there is no canonical map from $\text{O}(2)$ to $\text{SO}^+(3, 1) \cong \text{PSL}(2, \mathbb{C})$. To get around this, we restrict to $\Gamma \subset \text{Isom}_0 \text{Nil}_3 \cong \mathbb{R}^2 \rtimes \text{SO}(2)$, and define the map $\rightarrow \text{PSL}(2, \mathbb{C})$ as the inclusion $\mathbb{R}^2 \rtimes \text{SO}(2) \cong \mathbb{C} \rtimes \text{U}(1) \hookrightarrow \text{PSL}(2, \mathbb{C})$.

7.1.3 Values of the Chern–Simons functional

We present a short argument for the Chern–Simons functional at the geometric flat connection for spherical geometry. Let $X = S^3/\Gamma$, where we think of S^3 as the group $\text{SU}(2)$, and Γ as a discrete subgroup of $\text{SU}(2) \subset \text{SO}(4)$, acting on S^3 by (left/right) multiplication. For the geometric flat connection on X , we pick a connection one-form $A_* \in \Omega^1(X) \otimes \mathfrak{su}_2 \subset \Omega^1(X) \otimes \mathfrak{sl}_2$ such that its lift $\tilde{A}_* \in \Omega^1(S^3) \otimes \mathfrak{su}_2$ to S^3 is pure gauge, $\tilde{A}_* = g d(g^{-1})$ for some $g : S^3 \rightarrow \text{SU}(2)$. We can always do this because the only flat connection on S^3 is the trivial one. If we let $g : S^3 \cong \text{SU}(2) \rightarrow \text{SU}(2)$ be the identity map $\text{id}_{\text{SU}(2)}$, we get

$$\text{CS}[\tilde{A}_*] = \text{CS}[0] + \deg \text{id}_{\text{SU}(2)} = \pm 1, \quad (7.13)$$

whose sign is fixed by a choice of orientation. The Chern–Simons integral of $\tilde{A}_*$ over S^3 is equal to $|\Gamma|$ copies of the integral of A_* over the fundamental region $S^3/\Gamma \cong X$, so that $|\Gamma| \text{CS}[A_*] = \text{CS}[\tilde{A}_*] = \pm 1$ and therefore

$$\text{CS}[A_*] = \pm \frac{1}{|\Gamma|}. \quad (7.14)$$

For $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ geometry, it is known that the Chern–Simons functional at the geometric flat connection coincides with the so-called Seifert volume and the Godbillon–Vey invariant. Recall that every $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ manifold X is homeomorphic to a Seifert fibration. If we let $A_* \in \Omega^1(X) \otimes \mathfrak{sl}_2$ be a connection one-form for the geometric flat connection described above, the Chern–Simons value can be written in terms of the Seifert invariants $\chi = \chi(X/S^1)$ and $e = e(X)$,

$$\mathrm{CS}[A_*] = \pm \frac{\chi^2}{4e} \pmod{1}, \quad (7.15)$$

and, once again, the sign is fixed by a choice of orientation.

Remark 7.3. A direct calculation shows that $|\Gamma| = |4e/\chi|$ for spherical Seifert manifolds.

7.2 Proof of conjecture for Brieskorn homology spheres

We start by recalling a formula of the WRT invariant for Seifert integer homology spheres. Suppose that X is a Seifert integer homology sphere $\Sigma(p_1, \dots, p_m)$ with m exceptional fibers, where $p_j \geq 2$ are mutually coprime integers. As stated Section 3.2.1, Seifert invariants $p_1/q_1, \dots, p_m/q_m$, and b are determined from the requirement that the manifold is an integer homology sphere. Furthermore, we shall also pick a normalization in which $b = 0$ and the orientation such that $e > 0$. Let

$$P := \prod_{j=1}^m p_j, \quad H := P \sum_{j=1}^m \frac{q_j}{p_j} = |H_1(X, \mathbb{Z})|, \quad \phi := 3\sigma(H) + \sum_{j=1}^m \left(12s(q_j, p_j) - \frac{q_j}{p_j} \right), \quad (7.16)$$

where $s(\cdot, \cdot)$ is the Dedekind sum, then, with Seifert invariants normalized as above, the WRT invariant of X at $\xi^{1/4} = e^{\pi i s/2r}$ has the following form [40]

$$\xi^{\phi/4 - 1/2} (\xi - 1) \tau_X(\xi) = \left(\frac{Pr}{s} \right) \frac{e^{\pi i/4}}{2\sqrt{2Pr}} \sum_{\substack{n \in \mathbb{Z}/2Pr\mathbb{Z} \\ r \nmid n}} \xi^{-Hn^2/4P} \frac{\prod_{j=1}^m (\xi^{n/2p_j} - n^{-n/2p_j})}{(\xi^{n/2} - \xi^{-n/2})^{m-2}}. \quad (7.17)$$

We know from the integrality of the WRT invariant of integer homology spheres that $\tau_X(\xi) \in \mathbb{Z}[\xi]$. The following proposition says that the fractional part of the power of ξ on the left-hand side of (7.17) is equal to the Chern–Simons action at the geometric flat connection, $\mathrm{CS}[A_*] = -\chi^2/4e$.

Proposition 7.1. *Consider the Seifert homology sphere $\Sigma(p_1, p_2, \dots, p_m)$, then with ϕ as defined above, we have $\phi/4 - 1/2 = -\chi^2/4e \pmod{1}$.*

Proof. We begin by recalling how the quantity ϕ is related to the Casson invariant λ [40, §4]

$$-24\lambda = \phi + \frac{P}{H} \left(m - 2 - \sum_{j=1}^m \frac{1}{p_j^2} \right), \quad (7.18)$$

which implies that

$$\frac{\phi}{4} = -\frac{P}{4} \left(m - 2 - \sum_{j=1}^m \frac{1}{p_j^2} \right) \pmod{1}. \quad (7.19)$$

A short computation with the definition of the orbifold Euler characteristic, and the fact that $e = 1/P$ yields

$$\frac{\phi}{4} - \frac{1}{2} + \frac{\chi^2}{4e} = \frac{P}{2} \left(\frac{m(m+1)}{2} - 1 - m \sum_j \frac{1}{p_j} + \sum_{j < k} \frac{1}{p_j p_k} \right) - \frac{1}{2} \pmod{1}. \quad (7.20)$$

To show that the right-hand side is an integer (and therefore zero modulo 1), it suffices to show that the integer

$$P \left(\frac{m(m+1)}{2} - 1 - m \sum_j \frac{1}{p_j} + \sum_{j < k} \frac{1}{p_j p_k} \right) \quad (7.21)$$

is odd. Note that since p_j 's have to be mutually coprime because $\Sigma(p_1, p_2, \dots, p_m)$ is an integer homology sphere, at most one of them can be even. With this observation, we have

$$P \sum_j \frac{1}{p_j} = \begin{cases} 1 & \text{if } P \text{ is even} \\ m & \text{if } P \text{ is odd} \end{cases} \pmod{2}, \quad (7.22)$$

and

$$P \sum_{j < k} \frac{1}{p_j p_k} = \begin{cases} m-1 & \text{if } P \text{ is even} \\ \frac{m(m-1)}{2} & \text{if } P \text{ is odd} \end{cases} \pmod{2}, \quad (7.23)$$

which implies that (7.21) is odd. $\square$

Finally, we have all the tools to write the WRT invariant as limits of false theta functions and prove the quantum modularity conjecture for Brieskorn homology spheres.

Proposition 7.2. *For Brieskorn homology spheres, $\Sigma(p_1, p_2, p_3)$ with $1/p_1 + 1/p_2 + 1/p_3 < 1$, the WRT invariant at a generic primitive root of unity of odd order, $\xi = e^{\frac{2\pi i s}{r}}$, can be written as a limit of the false theta function,*

$$\xi^{\phi/4 - 1/2} (\xi - 1)^{\tau_{\Sigma(p_1, p_2, p_3)}}(\xi) = \frac{1}{2} \tilde{\Phi}_{\vec{p}}^{(1,1,1)}(s/r). \quad (7.24)$$

Proof. The proof here follows from results in [68], and is an adaptation of the one given in [27] to the case of generic roots of unity. We recall the formula (7.17) for the WRT invariant

$$\xi^{\phi/4-1/2}(\xi-1)\tau_{\Sigma(p_1,p_2,p_3)}(\xi) = \left(\frac{Pr}{s}\right) \frac{e^{\pi i/4}}{2\sqrt{2Pr}} \sum_{\substack{n \in \mathbb{Z}/2Pr\mathbb{Z} \\ r \nmid n}} \xi^{-n^2/4P} \frac{\prod_{j=1}^3 (\xi^{n/2p_j} - \xi^{-n/2p_j})}{\xi^{n/2} - \xi^{-n/2}}, \quad (7.25)$$

and use the fact that for $1/p_1 + 1/p_2 + 1/p_3 < 1$, there is a generating function for the rational function in the summand above in terms of the periodic function $\phi_{\vec{p}}^{(1,1,1)}$ from before,

$$\frac{\prod_{j=1}^3 (u^{P/p_j} - u^{-P/p_j})}{u^P - u^{-P}} = \sum_{l=0}^{\infty} \phi_{\vec{p}}^{(1,1,1)}(l) u^l, \quad (7.26)$$

to write it as a limit with $u = \xi^{n/2P} e^{-\pi t/2P}$,

$$\frac{\prod_{j=1}^3 (\xi^{n/2p_j} - \xi^{-n/2p_j})}{\xi^{n/2} - \xi^{-n/2}} = \lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} \phi_{\vec{p}}^{(1,1,1)}(l) u^l. \quad (7.27)$$

For the sum in (7.25), we have

$$\begin{aligned} \sum_{\substack{n \in \mathbb{Z}/2Pr\mathbb{Z} \\ r \nmid n}} \xi^{-n^2/4P} \frac{\prod_{j=1}^3 (\xi^{n/2p_j} - \xi^{-n/2p_j})}{\xi^{n/2} - \xi^{-n/2}} &= \lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} e^{-\pi t l/2P} \phi_{\vec{p}}^{(1,1,1)}(l) \sum_{n \in \mathbb{Z}/2Pr\mathbb{Z}} \xi^{-n^2/4P + nl/2P} \\ &= \lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} e^{-\pi t l/2P} \phi_{\vec{p}}^{(1,1,1)}(l) \sum_{n \in \mathbb{Z}/2Pr\mathbb{Z}} \xi^{-(n-l)^2/4P + l^2/4P} \\ &= \sum_{n \in \mathbb{Z}/2Pr\mathbb{Z}} \xi^{-n^2/4P} \lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} e^{-\pi t l/2P} \phi_{\vec{p}}^{(1,1,1)}(l) \xi^{l^2/4P}, \end{aligned}$$

where we have used a square completion to simplify the Gauss sum. To proceed, we can evaluate the quadratic Gauss sum in the last line above with a formula from Appendix A,

$$\sum_{n \in \mathbb{Z}/2Pr} \xi^{-n^2/4P} = \frac{1}{2} \sum_{n \in \mathbb{Z}/4Pr} \xi^{-n^2/4P} = \left(\frac{Pr}{s}\right) e^{-\pi i/4} \sqrt{2Pr}, \quad (7.28)$$

(remember that we have assumed $s \equiv 1 \pmod{4}$), and conclude by using a proposition from [68] for the first equality in the following

$$\lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} e^{-\pi t l/2P} \phi_{\vec{p}}^{(1,1,1)}(l) \xi^{l^2/4P} = \lim_{t \rightarrow 0^+} \sum_{l=0}^{\infty} e^{-\pi t l^2/2P} \phi_{\vec{p}}^{(1,1,1)}(l) \xi^{l^2/4P} \quad (7.29)$$

$$= \lim_{t \rightarrow 0^+} \tilde{\Phi}_{\vec{p}}^{(1,1,1)}(s/r + it) = \tilde{\Phi}_{\vec{p}}^{(1,1,1)}(s/r). \quad (7.30)$$

Putting all the pieces together, the result follows. $\square$

The only Brieskorn homology sphere for which $1/p_1 + 1/p_2 + 1/p_3 < 1$ does not hold is the Poincaré homology sphere, $\Sigma(2, 3, 5)$, and we will deal with it separately here. When $\vec{p} = (2, 3, 5)$, the only difference from the above calculation is due to

$$\frac{\prod_{j=1}^3 (u^{P/p_j} - u^{-P/p_j})}{u^P - u^{-P}} = u + u^{-1} + \sum_{l=0}^{\infty} \phi_{(2,3,5)}^{(1,1,1)}(l) u^l, \quad (7.31)$$

which results in an extra constant in the analogous formula for the WRT invariant,

$$\xi^{121/120} (\xi - 1) \tau_{\Sigma(2,3,5)}(\xi) = \xi^{1/120} + \frac{1}{2} \tilde{\Phi}_{(2,3,5)}^{(1,1,1)}(s/r). \quad (7.32)$$

Theorem 7.1. *The quantum modularity conjecture holds for Brieskorn homology spheres.*

Proof. With the exception of $\Sigma(2, 3, 5)$ which is spherical, all other Brieskorn homology spheres $X = \Sigma(p_1, p_2, p_3)$ have $1/p_1 + 1/p_2 + 1/p_3 < 1$ and are modeled on $\widetilde{\text{SL}(2, \mathbb{R})}$. We start by writing the WRT invariant of these manifolds in terms of the false theta function as in Proposition 7.2,

$$W_X(\xi) = (\xi - 1) \tau_X(\xi) = \frac{1}{2} \xi^{1/2 - \phi/4} \tilde{\Phi}_{\vec{p}}^{(1,1,1)}(s/r), \quad (7.33)$$

use the modular S -transformation of the false theta functions, and arrange the terms into an expansion around saddles in the following way

$$W_X(\xi) \simeq -\frac{1}{2} \sqrt{\frac{s}{ir}} \xi^{1/2 - \phi/4} \sum_{\vec{a}} S_{\vec{a}}^{(1,1,1)}(\vec{p}) \tilde{\Phi}_{\vec{p}}^{\vec{a}}(-r/s) + \frac{1}{2} \xi^{1/2 - \phi/4} \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{\pi i s}{2Pr} \right)^k \quad (7.34)$$

$$= \sum_{A \in \pi_0(M_{\text{flat}}(X, \text{SL}(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[A]} P_A(\tilde{\xi}) I_A(s/r), \quad (7.35)$$

where $P_{\text{triv}}(\tilde{\xi}) = 1$, and

$$P_A(\tilde{\xi}) = \frac{1}{2} \tilde{\xi}^{\text{CS}[A]} \tilde{\Phi}_{\vec{p}}^{\vec{a}}(-r/s) \quad (7.36)$$

if A is the flat connection represented by rotation numbers (a_1, a_2, a_3) ; and

$$I_{\text{triv}}(s/r) = \frac{1}{2} \xi^{\frac{1}{2} - \frac{\phi}{4}} \sum_{k=0}^{\infty} \frac{L(-2k, \phi_{\vec{p}}^{(1,1,1)})}{k!} \left(\frac{\pi i s}{2Pr} \right)^k, \quad (7.37)$$

and

$$I_A(s/r) = -\sqrt{\frac{r}{is}} S_{\vec{a}}^{(1,1,1)}(\vec{p}) \xi^{1/2 - \phi/4} \quad (7.38)$$

if A is the flat connection represented by rotation numbers (a_1, a_2, a_3) .

We have identified the asymptotic series in s/r that appears as the failure of modularity in the S -transformation of $\tilde{\Phi}_{\vec{p}}^{(1,1,1)}$ as the contribution from the trivial flat connection, and used the correspondence between non-trivial flat connections and the $D_{\vec{p}}$ independent theta functions to write the sum over the false theta functions as a sum over the connected components of $\mathcal{M}_{\text{flat}}(X, SL(2, \mathbb{C}))$ in (7.35).

Asymptotic series I_A are of the expected type, and the integrality of the polynomials $P_A(\tilde{\xi}) \in \mathbb{Z}[\tilde{\xi}]$ is implied directly by Proposition B.1.

Finally, it follows from (7.15) that, for the geometric flat connection on $\widetilde{SL(2, \mathbb{R})}$ manifolds, $\text{CS}[A_*] = -\chi^2/4e$. Comparing the Chern–Simons values tells us that A_* is represented by rotation numbers $(1, 1, 1)$, and therefore it follows from Propositions 7.1 and 7.2 that $P_{A_*}(\tilde{\xi}) = \xi^\delta W_X(\tilde{\xi})$ for some integer δ .

The case of $\Sigma(2, 3, 5)$ is completely analogous, and we have

$$W_{\Sigma(2,3,5)}(\xi) = \xi^{-1} + \frac{1}{2}\xi^{-121/120}\tilde{\Phi}_{(2,3,5)}^{(1,1,1)}(s/r) \quad (7.39)$$

$$\simeq \sum_{A \in \pi_0(\mathcal{M}_{\text{flat}}(X, SL(2, \mathbb{C})))} e^{\frac{2\pi i r}{s} \text{CS}[A]} P_A(\tilde{\xi}) I_A(s/r), \quad (7.40)$$

with $P_{\text{triv}}(\tilde{\xi}) = 1$, and

$$P_A(\tilde{\xi}) = \frac{1}{2}\tilde{\xi}^{-1/120}\tilde{\Phi}_{\vec{p}}^{\vec{a}}(-r/s), \quad (7.41)$$

if A is the flat connection represented by rotation numbers (a_1, a_2, a_3) ; and

$$I_{\text{triv}}(s/r) = \xi^{-1} + \frac{1}{2}\xi^{-121/120} \sum_{k=0}^{\infty} \frac{L(-2k, \phi_{(2,3,5)}^{(1,1,1)})}{k!} \left(\frac{\pi i s}{60r} \right)^k \quad (7.42)$$

and

$$I_A(s/r) = -\sqrt{\frac{r}{is}} S_{\vec{a}}^{(1,1,1)} \xi^{-121/120} \quad (7.43)$$

if A is the flat connection represented by rotation numbers (a_1, a_2, a_3) . We identify the geometric flat connection by its Chern–Simons value, $\text{CS}[A_*] = -1/120$, and notice that

$$P_{A_*}(\tilde{\xi}) = \frac{1}{2}\tilde{\xi}^{-1/120}\tilde{\Phi}_{(2,3,5)}^{(1,1,1)}(-r/s) = \tilde{\xi} W_{\Sigma(2,3,5)}(\tilde{\xi}) - 1, \quad (7.44)$$

as required. $\square$

7.3 Some rational homology sphere examples

7.3.1 Lens spaces $L(p, 1)$

For any positive integer p , the lens space $L(p, 1)$ has a realization as surgery on the p -framed unknot in S^3 . Computation of the WRT invariant of these spaces is essen-

tially an application of the reciprocity of quadratic Gauss sums. We start by noting that $J_n(U^p, q) = q^{p(n^2-1)/4} [n]_q$, and therefore for $L(p, 1) = S^3(U^p)$, we need the intermediate quantity

$$F(U^p, \xi) = \sum_{n=1}^{r-1} \xi^{p \frac{n^2-1}{4}} [n]_\xi^2 = \frac{1}{2} \sum_{n=-r}^{r-1} \xi^{p \frac{n^2-1}{4}} [n]_\xi^2 = \frac{1}{2} \sum_{n \in \mathbb{Z}/2r\mathbb{Z}} \xi^{p \frac{n^2-1}{4}} [n]_\xi^2 \quad (7.45)$$

$$= \frac{\xi^{-p/4}}{2(\xi^{1/2} - \xi^{-1/2})^2} \sum_{n \in \mathbb{Z}/2r\mathbb{Z}} \xi^{\frac{pn^2}{4}} (\xi^n + \xi^{-n} - 2), \quad (7.46)$$

which can be simplified by applying the reciprocity formula as

$$\sum_{n \in \mathbb{Z}/2r\mathbb{Z}} \xi^{\frac{pn^2}{4}} = \frac{e^{\pi i/4} \sqrt{2r}}{\sqrt{s} p} \sum_{y \in \mathbb{Z}/s p \mathbb{Z}} e^{-\frac{2\pi i r}{s p} y^2} = \frac{e^{\pi i/4} \sqrt{2r}}{\sqrt{s} p} \sum_{l \in \mathbb{Z}/s \mathbb{Z}} \xi^{p l^2} \sum_{a \in \mathbb{Z}/p \mathbb{Z}} e^{2\pi i \frac{r}{s} \left(-\frac{(sa)^2}{p}\right)}, \quad (7.47)$$

and

$$\sum_{n \in \mathbb{Z}/2r\mathbb{Z}} \xi^{\frac{pn^2}{4} \pm n} = \frac{e^{\pi i/4} \sqrt{2r}}{\sqrt{s} p} \sum_{y \in \mathbb{Z}/s p \mathbb{Z}} e^{-\frac{2\pi i r}{s p} (y \pm \frac{r}{r})^2} \quad (7.48)$$

$$= \frac{e^{\pi i/4} \sqrt{2r}}{\sqrt{s} p} \sum_{l \in \mathbb{Z}/s \mathbb{Z}} \xi^{p l^2} \sum_{a \in \mathbb{Z}/p \mathbb{Z}} e^{2\pi i \frac{r}{s} \left(-\frac{(sa)^2}{p}\right) \mp \frac{4\pi i s a}{p}} \xi^{-1/p}, \quad (7.49)$$

where we have used $(p, s) = 1$ to decompose the sum over $\mathbb{Z}/s p \mathbb{Z} \cong \mathbb{Z}/s \mathbb{Z} \oplus \mathbb{Z}/p \mathbb{Z}$. After this decomposition, the sum above over l is precisely a quadratic Gauss sum, for which there is a closed expression

$$\sum_{l \in \mathbb{Z}/s \mathbb{Z}} \xi^{p l^2} = \overline{G(rp, s)} = \left(\frac{rp}{s}\right) \sqrt{s}, \quad (7.50)$$

where we have used the fact that $s \equiv 1 \pmod{4}$. We can put these pieces together, and normalize with $F(U^{+1}, \xi)$ for the WRT invariant

$$\xi^{\frac{1-p}{4}} W_{L(p,1)}(\xi) = \sum_{a \in \mathbb{Z}/p \mathbb{Z}} e^{2\pi i \frac{r}{s} \left(-\frac{s^2 a^2}{p}\right)} \left(\xi^{-1/p} \cos \frac{4\pi s a}{p} - 1 \right). \quad (7.51)$$

Since, under our assumptions, s is coprime with $|H_1| = p$, there exists an integer s^* such that $s s^* \equiv 1 + n p$ for some $(s\text{-dependent}) n \in \mathbb{Z}$. After making the change $a \mapsto s^* a$ of the summation variable, we can read off the decomposition into abelian flat connections

as

$$W_{L(p,1)}(\xi) = \xi^{\frac{p-1}{4}} \sum_{a=0}^{p-1} e^{2\pi i \frac{r}{s} \left(-\frac{a^2}{p} + 2na^2 + n^2 pa\right)} \left(\xi^{-1/p} \cos \frac{4\pi a}{p} - 1 \right) \quad (7.52)$$

$$= \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(L(p,1), SL(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[a]} W_{L(p,1)}^a(\xi), \quad (7.53)$$

where, as before, $\text{CS}[a]$ means a lift of the CS value to $\mathbb{Q}$. Using $\pi_0(M_{\text{flat}}^{\text{ab}}(L(p,1), SL(2, \mathbb{C}))) \cong \{0, 1, \dots, (p-1)/2\}$, we have

$$W_{L(p,1)}^a(\xi) = \begin{cases} \xi^{\frac{p-1}{4}} \left(\xi^{-\frac{1}{p}} - 1 \right) & \text{if } a = 0, \\ 2\xi^{\frac{p-1}{4}} \left(\xi^{-\frac{1}{p}} \cos \frac{4\pi sa}{p} - 1 \right) & \text{if } a \neq 0. \end{cases} \quad (7.54)$$

Since the geometric flat connection for lens spaces is abelian and has CS value $-1/p$, we have $P_{A_*}^{\text{triv}}(\xi) = 0$ because of the way the decomposition is defined. On the other hand, since

$$\sum_{a=0}^{(p-1)/2} W_{L(p,1)}^a(\xi) = -p\xi^{\frac{p-1}{4}}, \quad (7.55)$$

we verify that

$$P_{A_*}^{\text{triv}}(\xi) = \xi^{\frac{1-p}{4}} \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(L(p,1), SL(2, \mathbb{C})))} W_{L(p,1)}^a(\xi) + p. \quad (7.56)$$

Thus the conjecture holds for these lens spaces.

We shall consider Seifert rational homology spheres fibered over S^2 with three exceptional fibers and Seifert invariants b and $p_1/q_1, p_2/q_2, p_3/q_3$, where $q_j = \pm 1$ for every $j = 1, 2, 3$. In what follows, we use p_j to denote p_j/q_j . Such manifolds are described as surgery on a link of unknots that has the linking matrix

$$B = \begin{bmatrix} b & 1 & 1 & 1 \\ 1 & p_1 & 0 & 0 \\ 1 & 0 & p_2 & 0 \\ 1 & 0 & 0 & p_3 \end{bmatrix}, \quad (7.57)$$

and the $n = (n_0, n_1, n_2, n_3)$ -colored Jones polynomial

$$J_n(L, q) = \frac{q^{b(n_0^2-1)/4}}{[n_0]_q^2} \prod_j q^{p_j(n_j^2-1)/4} [n_0 n_j]_q \quad (7.58)$$

$$= \frac{1}{q^{1/2} - q^{-1/2}} \frac{q^{b(n_0^2-1)/4}}{(q^{n_0/2} - q^{-n_0/2})^2} \prod_j q^{p_j(n_j^2-1)/4} (q^{n_0 n_j/2} - q^{n_0 n_j/2}). \quad (7.59)$$

Using the surgery formula for WRT invariants, we get

$$\xi^{\phi'/4-1/2}(\xi-1)\tau_X(\xi) = (-1)^{b_+} \frac{e^{-\pi i \sigma(B)/4}}{2(2r)^2} \sum_{\substack{n_0 \in \mathbb{Z}/2r\mathbb{Z} \\ n_0 \neq 0, r}} \frac{\xi^{bn_0^2/4}}{\xi^{n_0/2} - \xi^{-n_0/2}} \\ \times \prod_j \sum_{n_j \in \mathbb{Z}/2r\mathbb{Z}} \xi^{p_j n_j^2/4 + n_0 n_j/2} \left(\xi^{n_j/2} - \xi^{-n_j/2} \right),$$

where $\phi' = \text{tr } B - 3\sigma(B)$. To get the above formula in a form similar to (7.17), we apply the reciprocity formula for quadratic Gauss sums (Theorem A.1) to the sums over n_j , and use the assumption $(s, p_j) = 1$ on s to simplify, so that

$$\xi^{\phi'/4-1/2}(\xi-1)\tau_X(\xi) = -\frac{(-1)^{b_+}}{2\sqrt{2|P|r}} \left(\frac{Pr}{s} \right) \frac{e^{\frac{\pi i}{4} \sum_j \sigma(p_j)}}{e^{\frac{\pi i \sigma(B)}{4}}} \left(\dots \right), \quad (7.60)$$

where $\phi = \phi' + \sum_j \frac{1}{p_j}$ and

$$\left(\dots \right) = \sum_{\substack{n_0 \in \mathbb{Z}/2r\mathbb{Z} \\ n_0 \neq 0, r}} \frac{\xi^{bn_0^2/4}}{\xi^{n_0/2} - \xi^{-n_0/2}} \prod_j \sum_{n_j \in \mathbb{Z}/p_j\mathbb{Z}} \xi^{-\frac{(n_0+2rn_j)^2}{4p_j}} \left(\xi^{\frac{n_0+2rn_j}{2p_j}} - \xi^{-\frac{n_0+2rn_j}{2p_j}} \right). \quad (7.61)$$

Following [29, Proposition 2], we notice that the sum above is invariant under simultaneous change $n_0 \rightarrow n_0 + 2r$ and $n_j \rightarrow n_j - 1$ for every $j = 1, 2, 3$. If p_1 and p_2 are coprime, we can use this invariance to simplify the sums

$$\sum_{n_0=0}^{2r-1} \sum_{n_1 \in \mathbb{Z}/p_1\mathbb{Z}, n_2 \in \mathbb{Z}/p_2\mathbb{Z}, n_3 \in \mathbb{Z}/p_3\mathbb{Z}} = \sum_{n_0=0}^{2p_1 p_2 r-1} \sum_{n_1=0, n_2=0, n_3 \in \mathbb{Z}/p_3\mathbb{Z}}. \quad (7.62)$$

In the following examples, we shall use the above formula for the WRT invariant at generic roots of unity to write it in terms of false theta functions.

7.3.2 Tetrahedral space

For the Seifert manifold $S^2(1; 2, 3, 3)$, we can readily compute

$$e = \frac{1}{6}, \quad \chi = \frac{1}{6}, \quad \phi = \frac{25}{6}, \quad H_1(X, \mathbb{Z}) \cong \text{coker } B \cong \mathbb{Z}/3\mathbb{Z}. \quad (7.63)$$

Since $e \neq 0$ and $\chi > 0$, this manifold must be spherical. Indeed, it can also be thought of as the quotient of S^3 by the binary tetrahedral group. Apart from the Poincaré homology sphere and lens spaces $L(p, 1)$ for odd p , this is the only ADE type spherical manifold that is a mod-2 homology sphere.

Its WRT invariant is given by $\xi^{13/24}(\xi - 1)\tau_X(\xi) =$

$$\frac{1}{6\sqrt{r}} \left(\frac{18r}{s} \right) \sum_{\substack{n_0 \in \mathbb{Z}/12r\mathbb{Z} \\ r \nmid n_0}} \xi^{-n_0^2/24} \frac{(\xi^{n_0/4} - \xi^{-n_0/4})(\xi^{n_0/6} - \xi^{-n_0/6})}{\xi^{n_0/2} - \xi^{-n_0/2}} \\ \times \sum_{n_3 \in \mathbb{Z}/3\mathbb{Z}} \xi^{\frac{-r^2 n_3^2 + r n_3 n_0}{3}} \left(\xi^{\frac{n_0 + 2r n_3}{6}} - \xi^{-\frac{n_0 + 2r n_3}{6}} \right). \quad (7.64)$$

Analogously to the case of Brieskorn homology spheres, we can proceed by writing the rational function in the summand above as the limit of a holomorphic function on the upper half plane, and then complete squares in the exponent of ξ to simplify the Gauss sum over n_0 (see [29] for a similar calculation for the special root of unity $\xi = e^{\frac{2\pi i}{r}}$). After a bit of work, and, as in the case of lens spaces, the fact that s is coprime with 3, we can show that

$$\xi^{13/24} W_X(\xi) = -\frac{1}{2} \sum_{a \in \mathbb{Z}/3\mathbb{Z}} e^{2\pi i \frac{r}{s} \left(\frac{s^2 a^2}{3} \right)} \left(\tilde{\Psi}_6^{(1)+(5)}(s/r) + 2\xi^{ra/3} \tilde{\Psi}_6^{(3)}(s/r) - 2\xi^{1/24} \right) \quad (7.65)$$

$$= \sum_{a \in \{0,1\} \cong \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[a]} W_X^a(\xi) \quad (7.66)$$

where we have used the shorthand notation $\tilde{\Psi}^{n_a(a)+n_b(b)+\dots} = n_a \Psi^{(a)} + n_b \Psi^{(b)} + \dots$. Components of the decomposition are

$$\xi^{13/24} W_X^0(\xi) = -\frac{1}{2} \tilde{\Psi}_6^{(1)+2(3)+(5)}(s/r) + \xi^{1/24} \quad \text{and} \quad (7.67)$$

$$\xi^{13/24} W_X^1(\xi) = -\tilde{\Psi}_6^{(1)-(3)+(5)}(s/r) + 2\xi^{1/24}. \quad (7.68)$$

Asymptotic behavior of the above functions as $r \rightarrow \infty$ follows from the modular S -transformation of false theta functions, for example,

$$W_X^0(\xi) \simeq -\frac{1}{2} \tilde{\Psi}_6^{3(1)+3(5)}(-r/s) \left(-\sqrt{\frac{r}{3is}} \xi^{-13/24} \right) + \xi^{-1/2} - \xi^{-13/24} \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\pi is}{12r} \right)^k \quad (7.69)$$

$$= e^{2\pi i \frac{r}{s} \left(-\frac{1}{24} \right)} P_{-1/24}^0(\tilde{\xi}) I_{-1/24}^0(s/r) + P_0^0(\tilde{\xi}) I_0^0(s/r), \quad (7.70)$$

where we have denoted the flat connection by its CS value and arranged the terms in the following way

$$P_{-1/24}^0(\tilde{\xi}) = -\frac{1}{2} \tilde{\xi}^{-1/24} \tilde{\Psi}_6^{3(1)+3(5)}(-r/s) \quad \text{and} \quad I_{-1/24}^0(s/r) = -\sqrt{\frac{r}{3is}} \xi^{-13/24}, \quad (7.71)$$

ⁱWe can label the components of the moduli space of flat connections by the respective CS values, when they are uniquely determined by them, as in this case.

$$\approx 100 \ln 2$$

and for the trivial flat connection

$$P_0^0(\tilde{\xi}) = 1 \quad \text{and} \quad I_0^0(s/r) = \xi^{-1/2} - \xi^{-13/24} \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\pi i s}{12r} \right)^k, \quad (7.72)$$

with coefficients $c_k = L(-2k, \psi_{12}^{(1)+2(3)+(5)})$. Similarly, it can be verified that the asymptotic expansion for $W_X^1(\xi)$ is of the required form.

Finally, we recall that for the geometric flat connection A_* , we have $\text{CS}[A_*] = -1/24$, and a direct calculation shows that

$$P_{-1/24}^0(\tilde{\xi}) = \tilde{\xi}^{1/2} \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} W_X^a(\tilde{\xi}) - 3. \quad (7.73)$$

Thus, we have verified that the Conjecture 7.1 holds for this manifold.

7.3.3 -3 surgery on the right-handed trefoil

For the Seifert manifold $S^2(-1; -2, -3, -9)$, we calculate

$$e = \frac{1}{18}, \quad \chi = -\frac{1}{18}, \quad \phi = -\frac{71}{18}, \quad H_1(X, \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}. \quad (7.74)$$

Since $e \neq 0$ and $\chi < 0$, this manifold must be modeled on $\widetilde{\text{SL}(2, \mathbb{R})}$ geometry. To verify that the conjecture holds for this manifold, we proceed as in the previous example and write the WRT invariant in terms of false theta functions. We start with the decomposition

$$\xi^{-107/72} W_X(\xi) = \frac{1}{2} \sum_{a \in \mathbb{Z}/3\mathbb{Z}} e^{2\pi i \frac{a}{s} \left(\frac{s^2 a^2}{3} \right)} \left(\tilde{\Psi}_{18}^{(1)+(17)}(s/r) - \xi^{ra/3} \tilde{\Psi}_{18}^{(5)+(13)}(s/r) \right) \quad (7.75)$$

$$= \sum_{a \in \{0,1\} \cong \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} e^{2\pi i \frac{a}{s} \text{CS}[a]} \xi^{-107/72} W_X^a(\xi), \quad (7.76)$$

where

$$\xi^{-107/72} W_X^0(\xi) = \frac{1}{2} \tilde{\Psi}_{18}^{(1)-(5)-(13)+(17)}(s/r) \quad \text{and} \quad (7.77)$$

$$\xi^{-107/72} W_X^1(\xi) = \frac{1}{2} \tilde{\Psi}_{18}^{2(1)+(5)+(13)+2(17)}(s/r). \quad (7.78)$$

As before, we can compute the asymptotic expansion of the functions W_X^a by using the modular transformation property of false theta functions

$$W_X^0(\xi) \simeq \sum_{A \in \pi_0(M_{\text{flat}}(X, SL(2, \mathbb{C})))} e^{2\pi i \frac{A}{s} \text{CS}[A]} P_A^0(\tilde{\xi}) I_A^0(s/r), \quad (7.79)$$

where

$$\begin{aligned} P_{-1/72}^0(\xi) &= \frac{1}{2} \xi^{-1/72} \tilde{\Psi}_{18}^{3(1)+3(17)}(-r/s), & I_{-1/72}^0 &= -\frac{2}{9} \sqrt{\frac{r}{is}} \left(\sin \frac{\pi}{18} - \sin \frac{5\pi}{18} \right) \xi^{107/72}, \\ P_{-25/72}^0(\xi) &= \frac{1}{2} \xi^{-25/72} \tilde{\Psi}_{18}^{3(5)+3(13)}(-r/s), & I_{-25/72}^0 &= -\frac{2}{9} \sqrt{\frac{r}{is}} \left(\sin \frac{5\pi}{18} + \sin \frac{7\pi}{18} \right) \xi^{107/72}, \\ P_{-49/72}^0(\xi) &= \frac{1}{2} \xi^{-49/72} \tilde{\Psi}_{18}^{3(7)+3(11)}(-r/s), & I_{-49/72}^0 &= -\frac{2}{9} \sqrt{\frac{r}{is}} \left(\sin \frac{\pi}{18} + \sin \frac{7\pi}{18} \right) \xi^{107/72}, \end{aligned}$$

for the trivial flat connection, we have

$$P_{\text{triv}}^0(\xi) = 1, \quad I_{\text{triv}}^0(s/r) = \sum_{k=0}^{\infty} \frac{L(-2k, \psi_{18}^{(1)-(5)-(13)+(17)})}{k!} \left(\frac{\pi is}{36r} \right)^k, \quad (7.80)$$

as usual, and $P_{-19/24}^0(\xi) = 0$ because $W_X^0(\xi)$ does not receive any contribution from the non-abelian flat connection with $\text{CS} = -19/24$. It can be verified that the other piece $W_X^1(\xi)$ also has an asymptotic expansion of the expected type.

Finally, since for the geometric flat connection $\text{CS}[A_*] = -1/72 \bmod 1$, we can verify that

$$P_{-1/72}(\xi) = \xi^{-3/2} \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} W_X^a(\xi). \quad (7.81)$$

7.3.4 Infinite family

As a final example, we consider the family of Seifert manifolds $X = S^2(0; p, -2p - 1, -2p - 1)$ for positive integers $p \geq 2$. We can calculate

$$e = \frac{1}{p(2p+1)}, \quad \chi = -\frac{2p^2 - 3p - 1}{p(2p+1)}, \quad H_1(X, \mathbb{Z}) = \mathbb{Z}/(2p+1)\mathbb{Z}. \quad (7.82)$$

These manifolds are mod-2 homology spheres, and since $e \neq 0$ and $\chi < 0$ they must be modeled on $\text{SL}(2, \mathbb{R})$. The calculation of the WRT invariant in terms of false theta functions proceeds as in the earlier examples. Let $H = |H_1(X, \mathbb{Z})| = 2p + 1$, $P = p(2p + 1)$, and $\Delta_p = \phi/4 - 1/2$. Also define three integers $u = P - 4p - 1$, $v = P - 2p - 1$, and $w = P - 1$ that lie between 0 and P . We have the WRT invariant

$$\xi^{\Delta_p} W_X(\xi) = \frac{1}{2} \sum_{a \in \mathbb{Z}/H\mathbb{Z}} e^{2\pi i \frac{r}{s} \left(\frac{s^2 a^2}{H} (p+1) \right)} \left(\tilde{\Psi}_p^{(u)+(w)}(s/r) - 2\xi^{ra/H} \tilde{\Psi}_p^{(v)}(s/r) \right) \quad (7.83)$$

$$= \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} e^{2\pi i \frac{r}{s} \text{CS}[a]} \xi^{\Delta_p} W_X^a(\xi), \quad (7.84)$$

where $\pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C}))) \cong \{0, 1, \dots, p\}$, and we can read off the terms

$$\xi^{\Delta_p} W_X^a(\xi) = \begin{cases} \frac{1}{2} \tilde{\Psi}_P^{(u)-2(v)+(w)}(s/r) & \text{if } a = 0, \\ \tilde{\Psi}_P^{(u)+(w)}(s/r) - 2 \cos \frac{2\pi s a}{H} \tilde{\Psi}_P^{(w)}(s/r) & \text{otherwise.} \end{cases} \quad (7.85)$$

As the terms $W_X^a(\xi)$ are given by false theta functions (up to an overall fractional power of ξ), their asymptotics are determined by the behavior of false theta functions under a modular S -transformation. As before the resulting terms can be interpreted as expansions around flat connections. Of particular interest is the term

$$\begin{aligned} W_X^0(\xi) &\simeq -\sqrt{\frac{r}{is}} \xi^{-\Delta_p} \sum_{a=0}^{P-1} (M_a^u - 2M_a^v + M_a^w) \frac{1}{2} \tilde{\Psi}_P^{(a)}(-r/s) + \sum_{k=0}^{\infty} \frac{c_k}{k!} \left(\frac{\pi i s}{2Pr} \right)^k \\ &= e^{\frac{2\pi i s}{r} \left(-\frac{(P-1)^2}{4P} \right)} \tilde{\xi}^{-\frac{(P-1)^2}{4P}} \frac{1}{2} \tilde{\Psi}_P^{H(u)+H(w)}(-r/s) \left[\dots \right] + \text{other flat connections,} \end{aligned} \quad (7.86)$$

$$(7.87)$$

because for the geometric flat connection $\text{CS}[A_*] = -\chi^2/4e = -(P-1)^2/4P \bmod 1$. And we can read off

$$P_{A_*}^0(\tilde{\xi}) = \tilde{\xi}^{-\frac{(P-1)^2}{4P}} \frac{1}{2} \tilde{\Psi}_P^{H(u)+H(w)}(-r/s) \quad (7.88)$$

and

$$I_{A_*}^0(s/r) = -\frac{1}{H} \sqrt{\frac{r}{is}} \xi^{-\Delta_p} (M_u^u(P) - 2M_u^v(P) + M_u^w(P)), \quad (7.89)$$

and verify that

$$P_{A_*}^0(\tilde{\xi}) = \tilde{\xi}^{\Delta_p - \frac{(P-1)^2}{4P}} \sum_{a \in \pi_0(M_{\text{flat}}^{\text{ab}}(X, SL(2, \mathbb{C})))} W_X^a(\xi), \quad (7.90)$$

as in Conjecture 7.1.

We also refer to [48] for the analysis of the asymptotic expansion of WRT invariant at general roots of unity for some other manifolds.

CHAPTER 8

Unified quantum invariants

In this chapter, we shall briefly describe some preliminary results from ongoing work on the unification of radial limits of $\widehat{Z}$ -invariants of 3-manifolds.

8.1 Strange identities and the Habiro ring

The Reshetikhin–Turaev construction gives a numerical invariant of closed 3-manifolds for every primitive root of unity. *A priori*, it is not obvious how these WRT invariants at different roots of unity should be related to each other, but an important intermediate step in proving the quantum modularity conjecture for Brieskorn homology spheres, was to write the WRT invariant as the radial limit of a q -series defined inside the unit disk. For example, when X is the Poincaré homology sphere, we have

$$\xi(\xi - 1)\tau_X(\xi) = 1 + \frac{1}{2} \lim_{q \rightarrow \xi} q^{-1/120} \check{\Phi}_{(2,3,5)}^{(1,1,1)}(q), \quad (8.1)$$

where $\check{\Phi}_{(2,3,5)}^{(1,1,1)}(q)$ is a q -series

$$\check{\Phi}_{(2,3,5)}^{(1,1,1)}(q) = q^{1/120} (1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - q^{29} + q^{31} + \cdots). \quad (8.2)$$

One point that was maybe glossed over earlier, is that these false theta q -series *unify* all WRT invariants at different roots of unity in a single object. Given such a q -series, we can extract WRT invariant at any root of unity by taking a radial limit $q \rightarrow \xi$.

There is an algebraic analog of the above story given by Habiro [25]. Consider the following formula for the WRT invariant of the Poincaré homology sphere,

$$\tau_X(q) = \frac{1}{1-q} \sum_{m=1}^{\infty} q^{m-1} (q^m; q)_m, \quad (8.3)$$

where $(x; q)_n = (1-x)(1-xq) \cdots (1-xq^{n-1})$ is the q -Pochhammer symbol. The right-hand side of (8.3) is well-defined when q is a root of unity, but it does not make sense

as a function on any open subset of the complex plane; it does not even make sense as a formal power series in q . Contrast this with the q -series in (8.1), which converges inside the unit disk, but is ill defined when $|q| = 1$. We have the following *strange identity*

$$1 + \frac{1}{2}q^{-1/120}\tilde{\Phi}_{(2,3,5)}^{(1,1,1)}(q) \text{ “=” } - \sum_{m=1}^{\infty} q^m (q^m; q)_m. \quad (8.4)$$

Remark 8.1. The term strange identity was introduced in [66], to describe a relation like the one above between the so-called Kontsevich–Zagier function $F(q) = \sum_{n=0}^{\infty} (q; q)_n$ and the Eichler integral of the Dedekind η -function $\eta(q) = q^{1/24}(q; q)_{\infty}$, which is a weight-1/2 modular form. The Kontsevich–Zagier function also appeared in [67] as an example of a quantum modular form.

8.1.1 The Habiro ring

Expressions like the Kontsevich–Zagier series and the right-hand side of (8.3) do not make sense as elements of $\mathbb{Z}[[q]]$, but they converge in a different completion of the ring of polynomials $\mathbb{Z}[q]$ first defined by Habiro [25]. The *Habiro ring* is defined as the following inverse limit

$$\widehat{\mathbb{Z}[q]} := \varprojlim_n \frac{\mathbb{Z}[q]}{(q; q)_n} = \varprojlim_n \frac{\mathbb{Z}[q]}{(1-q)(1-q^2)\cdots(1-q^n)}, \quad (8.5)$$

along with the projections $p_m : \widehat{\mathbb{Z}[q]} \rightarrow \mathbb{Z}[q]/(q; q)_m$. Elements of the Habiro ring can be written as

$$f(q) = \sum_{n=0}^{\infty} a_n(q)(q; q)_n, \quad (8.6)$$

for some $a_n(q) \in \mathbb{Z}[q]$, and the expansion is unique if $\deg a_n \leq n$. The projection p_m acts on the expansion by

$$\sum_{n=0}^{\infty} a_n(q)(q; q)_n \xrightarrow{p_m} \sum_{n=0}^{m-1} a_n(q)(q; q)_n \mod ((q; q)_m) \quad (8.7)$$

Remark 8.2. A more common completion of $\mathbb{Z}[q]$ is $\varprojlim_n \mathbb{Z}[q]/(x^n)$, also known as the ring of power series $\mathbb{Z}[[q]]$.

Remark 8.3. The definition of the Habiro ring is reminiscent of p -adic integers. Recall that the *ring of p -adic integers* is defined as $\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, and a p -adic integer has a formal expansion

$$r = \sum_{j=0}^{\infty} a_j p^j, \quad (8.8)$$

which is unique when $a_j \in \{0, 1, \dots, p-1\}$ for every j .

Remark 8.4. A classical analog to the Habiro ring is the *ring of profinite integers*—defined as the inverse limit $\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n!\mathbb{Z}$ —because the q -Pochhammer symbol is a q -deformation of the factorial. Profinite integers have a formal expansion

$$r = \sum_{k=1}^{\infty} a_k k!, \quad (8.9)$$

which is unique when $a_k \in \{0, 1, \dots, k\}$ for every k .

Let ξ be an r -th root of unity, then there is an evaluation map $\text{ev}_{\xi} : \widehat{\mathbb{Z}[q]} \rightarrow \mathbb{Z}[\xi]$,

$$f(q) = \sum_{n=0}^{\infty} a_n(q)(q; q)_n \xrightarrow{\text{ev}_{\xi}} \sum_{n=0}^{r-1} a_n(\xi)(\xi; \xi)_n = f(\xi). \quad (8.10)$$

Remark 8.5. If ξ is a primitive r -th root of unity, then $\mathbb{Z}[\xi] \cong \mathbb{Z}[q]/(\Phi_r(q))$, where Φ_r is the r -th cyclotomic polynomial.

To get back to topology, we quote the following theorem about WRT invariants of an integer homology sphere.

Theorem 8.1 (Habiro [25]). *Let X be an integer homology sphere. There is a well-defined topological invariant $I_X(q)$, valued in $\widehat{\mathbb{Z}[q]}$ such that*

$$\tau_X(\xi) = \text{ev}_{\xi} I_X(q), \quad (8.11)$$

for any root of unity ξ .

Remark 8.6. Note that the above theorem implies $\tau_X(\xi) \in \mathbb{Z}[\xi]$ whenever X is an integer homology sphere.

Remark 8.7. We have numerical evidence which suggests that the polynomials $P_{\alpha}(\xi) \in \mathbb{Z}[\xi]$ —that appear attached to flat connections in the asymptotic expansion (1.10) of the WRT invariant of integer homology spheres—can be unified into an element of the Habiro ring. More precisely, for every flat connection α , there is an element of the Habiro ring $I_{\alpha} \in \widehat{\mathbb{Z}[q]}$ such that $\text{ev}_{\xi} I_{\alpha}(q) = P_{\alpha}(\xi)$. In particular this is true for the geometric flat connection α_* because $P_{\alpha_*}(\xi) = (1 - \xi)\tau_X(\xi)$.

8.2 $\widehat{\mathbb{Z}}$ -invariants of manifolds

An important idea in quantum topology—and in algebraic topology more broadly—is that of categorification. Perhaps the most famous example of categorification is singular homology theory which categorifies the Euler characteristic that was originally thought of as a numerical invariant. A more relevant example is the Khovanov homology which categorifies the Jones polynomial. In particular, Khovanov homology assigns doubly-graded

homology groups $\mathrm{Kh}^{\cdot,\cdot}(K)$ to links in $\mathbb{R}^3$ such that the graded Euler characteristic

$$\chi(\mathrm{Kh}(K)) = \sum_{i,j} (-1)^i q^j \dim \mathrm{Kh}^{i,j}(K) = V_K(q), \quad (8.12)$$

is equal to the Jones polynomial of K .

Various referred to as BPS q -series, homological blocks, Gukov–Putrov–Pei–Vafa (GPPV) invariants or $\widehat{Z}$ -invariants, these (conjectured) $\mathbb{Z}[[q]]$ -valued invariants of 3-manifolds provide an important step in the categorification of the WRT invariant.

Let X be a negative-definite plumbed manifold. There exist q -series invariants, $\widehat{Z}_b(q)$, of X , labeled by $b \in \mathrm{Spin}^{\mathbb{C}}(X)$, such that [46]

1. $\widehat{Z}_b(q) \in q^{\Delta_b} \mathbb{Z}[[q]]$ for some rational number Δ_b ,
2. under the action of $\mathbb{Z}_2$ on $\mathrm{Spin}^{\mathbb{C}}(X)$ by conjugation $\widehat{Z}_{-b}(q) = \widehat{Z}_b(q)$
3. $\widehat{Z}_b(q)$ converges as a power series in q inside the unit disk $|q| < 1$, and
4. the WRT invariant of X for $\xi = e^{2\pi i/r}$ admits a decomposition

$$\tau_X(\xi) = \frac{1}{2(\xi^{1/2} - \xi^{-1/2})} \sum_{a \in H_1(X, \mathbb{Z})} e^{-2\pi i r \mathrm{lk}(a, a)} \left(\lim_{q \rightarrow \xi} \sum_{b \in \mathrm{Spin}^{\mathbb{C}}(X)} S_{ab} \widehat{Z}_b(q) \right) \quad (8.13)$$

where the limit $q \rightarrow \xi$ is taken radially and

$$S_{ab} = \frac{e^{-2\pi i \mathrm{lk}(a, b)}}{\sqrt{|H_1(X, \mathbb{Z})|}}. \quad (8.14)$$

Remark 8.8. There is an isomorphism $\mathrm{Spin}^{\mathbb{C}}(X) \cong H_1(X, \mathbb{Z})$ but this isomorphism is not canonical.

Remark 8.9. The decomposition in (8.13) was conjectured in [22] and later proved in [46]. In § A.1, we derive a refinement of the radial limit conjecture for limits to generic roots of unity $q \rightarrow e^{2\pi i s/r}$.

These q -series invariants are thought of as a non-perturbative completion of complex Chern–Simons theory. Alternatively, they can be understood to be counting BPS states of a certain 3d $\mathcal{N} = 2$ theory—dual to complex Chern–Simons theory on X —with a “boundary condition” labeled by b

$$\widehat{Z}_b(q) = q^{\Delta_b} \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim \mathcal{H}_b^{i,j}(X), \quad (8.15)$$

where the doubly-graded vector spaces $\mathcal{H}_b^{\cdot,\cdot}(X)$ are Hilbert spaces of BPS states in the supersymmetric theory. The vector spaces $\mathcal{H}_b^{\cdot,\cdot}(X)$ are a physical construction of what would be “Khovanov homology for 3-manifolds”.

8.2.1 Strange identities and the $\widehat{Z}$ -invariant

On integer homology spheres, there is a unique $\text{Spin}^{\mathbb{C}}$ structure, a single $\widehat{Z}$ -invariant, and therefore the radial limit formula (8.13) becomes

$$\tau_X(\xi) = \frac{1}{2(\xi^{1/2} - \xi^{-1/2})} \lim_{q \rightarrow \xi} \widehat{Z}_0(q). \quad (8.16)$$

With Habiro's universal invariant, we have the strange identity

$$\frac{1}{2(q^{1/2} - q^{-1/2})} \widehat{Z}_0(q) \text{ " = " } I_X(q). \quad (8.17)$$

8.3 Unifying radial limits

At this point, the following question is hopefully well-motivated: For a given manifold X , are there elements $\widehat{W}_b(q) \in \widehat{\mathbb{Z}}[q]$ of the Habiro ring labeled by $b \in \text{Spin}^{\mathbb{C}}(X)$ such that $\text{ev}_{\xi} \widehat{W}_b(q) = \lim_{q \rightarrow \xi} \widehat{Z}_b(q)$? Our numerical experiments indicate that the answer (modulo some technical details) is yes.

Before we state the conjecture however, we will describe a linear relation between radial limits of $\widehat{Z}$ -invariants and the so-called non-semisimple quantum invariants. Non-semisimple invariants sit on solid mathematical ground and, unlike $\widehat{Z}$, they do not suffer the drawback of being undefined outside of certain classes of manifolds. If the reader is uncomfortable with the ill-definedness of $\widehat{Z}$ -invariants, the linear relation can be taken as the definition of radial limits.

8.3.1 Defining radial limits via non-semisimple invariants

In Chapter 5, we discussed the construction of manifold invariants from modular categories. There is a generalization of this construction to non-semisimple categories that gives a recipe for constructing homeomorphism invariants of 3-manifolds decorated with a cohomology class. For details on these non-semisimple invariants—which are also called CGP or BCGP invariants after Blanchet, Costantino, Geer, and Patureau-Mirand—we refer the reader to [13, 5, 14].

As before, let $\zeta = e^{\pi i/r}$ be a $2r$ -th root of unity and let $\xi = \zeta^2 = e^{2\pi i/r}$. The relevant facts about these invariants are that link components are colored with $\alpha \in \mathbb{C}$ instead of integers, and there is a *modified quantum dimension*

$$d(\alpha) = \frac{\zeta^{\alpha} - \zeta^{-\alpha}}{\zeta^{r\alpha} - \zeta^{-r\alpha}}, \quad (8.18)$$

for the module with highest weight α , that we use instead of the usual quantum dimension.

Let K be a link in S^3 with components $K_1, \dots, K_V$, and let $\alpha = (\alpha_1, \dots, \alpha_V) \in \mathbb{C}^V$. We shall use notation K_α for the link K whose component K_j is colored with the module $\mathcal{W}_{\alpha_j}$. The non-semisimple link invariant is denoted by $N_r(K_\alpha)$.

We also define analogs of $\mathfrak{D}_\pm$ for this category, with

$$\mathfrak{G}_- = \begin{cases} 0 & r = 0 \bmod 4 \\ i\zeta^{3/2}r^{1/2} & r = 1 \bmod 4 \\ (1-i)\zeta^{3/2}r^{1/2} & r = 2 \bmod 4 \\ -\zeta^{3/2}r^{1/2} & r = 3 \bmod 4 \end{cases} \quad (8.19)$$

and $\mathfrak{G}_+ = \bar{\mathfrak{G}}_-$.

Let X be a 3-manifold realized as surgery on a framed link K , let $\omega \in H^1(X, \mathbb{C}/2\mathbb{Z}) \setminus H^1(X, \mathbb{Z}/2\mathbb{Z})$ be a cohomology class, and let $\mu_j = \omega(m_j) \in \mathbb{C}/2\mathbb{Z}$, where $m_j \in H_1(X, \mathbb{Z})$ is the homology class of the meridian of K_j . The non-semisimple invariant of the pair (X, ω) is given by the following surgery formula

$$N_r(M, \omega) = \frac{1}{\mathfrak{G}_+^{b_+} \mathfrak{G}_-^{b_-}} \sum_{k \in H_r^V} \prod_{j=1}^V d(\mu_j + k_j) N_r(K_{\mu+k}), \quad (8.20)$$

where $b_\pm$ are the number of $\pm$ ve eigenvalues of the linking matrix of K .

Radial limits of $\widehat{Z}$ -invariants are related to CGP invariants as follows

$$N_r(X, \omega) = \lim_{q \rightarrow \xi} \sum_{\mathfrak{s} \in \text{Spin}^\mathbb{C}(X)} c_{\omega, b} \widehat{Z}_s(q), \quad (8.21)$$

where the $\text{Spin}^\mathbb{C}$ -structure is determined by $\mathfrak{s} = (b, s)$ for a $b \in H_1(X, \mathbb{Z})$ and $s \in \text{Spin}(X)$, and for $r = \pm 1 \bmod 4$

$$c_{\omega, (b, s)} = -\frac{\mathcal{T}(X, [\omega])}{|H_1(X, \mathbb{Z})|} e^{-\frac{\mp \pi i \mu(X, s)}{2}} \sum_{a, f \in H_1(X, \mathbb{Z})} e^{2\pi i \left(-\frac{r \mp 1}{4} \text{lk}(a, a) \pm \text{lk}(a, f \mp b) - \frac{1}{2} \omega(a) \pm \text{lk}(f, f) \right)}, \quad (8.22)$$

where $\mathcal{T}(X, [\omega])$ is an appropriate version of the torsion, and $\mu(X, s)$ is a mod 4 reduction of the Rokhlin invariant.

When X is a rational homology sphere, the number of $\widehat{Z}$ -invariants is $|\text{Spin}^\mathbb{C}(X)| = |H_1(X, \mathbb{Z})|$, and when there is no $\mathbb{Z}_2$ -torsion in homology, the number of CGP invariants is only one fewer

$$|H^1(X, \mathbb{C}/2\mathbb{Z}) \setminus H^1(X, \mathbb{Z}/2\mathbb{Z})| = |H_1(X, \mathbb{Z})| - 1. \quad (8.23)$$

There is a linear system of equations for the $|H_1(X, \mathbb{Z})|$ unknowns $\lim_{q \rightarrow \xi} \widehat{Z}(q)$ determined by (8.21), which gives $|H_1(X, \mathbb{Z})| - 1$ constraints, and (8.13) which gives the one

missing constraint. When this linear relation is invertible, we can take the solution of this system for $\lim_{q \rightarrow \xi} \widehat{Z}(q)$ to be a definition of the radial limits.

8.3.2 Unification conjecture and supporting evidence

We can now make a statement about the unification of radial limits of $\widehat{Z}$ -invariants.

Conjecture 8.1. *Let X be a rational homology sphere with no $\mathbb{Z}_2$ -torsion in homology, let $\xi = e^{2\pi i/r}$ with $r \in 2\mathbb{Z}_{\geq 0} + 1$ be an odd root of unity. Then for every $b \in \text{Spin}^{\mathbb{C}}(X)$, we have*

1. *there is an integer $Q_b > 0$ independent of r such that for every r , the scaled limit $\lim_{q \rightarrow \xi} Q_b \widehat{Z}_b(q) \in \mathbb{Z}[\xi]$ is a polynomial in ξ with integer coefficients*
2. *there is an element $\widehat{W}_b(q)$ in the ring completion $\widehat{\mathbb{Z}[q]} := \varprojlim_n \mathbb{Z}[q]/(q; q^2)_n$ such that $\text{ev}_{\xi} \widehat{W}_b(q) = \lim_{q \rightarrow \xi} Q_b \widehat{Z}_b(q)$.*

We have done numerical experiments to verify this conjecture in the examples $X = S_{-p}^3(K)$ for surgery coefficient $p = 3, 5, 7, 9, 11, 13, 15$ when K is the trefoil knot 3_1 , the figure-eight knot 4_1 , the three-twist knot 5_2 , and the Stevedore knot 6_1 .

Computations involved in these experiments involve the following steps. Fix some integer $N \simeq 20$, let $r = 2j + 1$ and do the following for every $j = 0, 1, \dots, N$.

1. Compute the CGP invariants, WRT invariant, and the matrix of linear relations $C_{\omega, b}$ to high-precision as fixed point numbers.
2. Solve for $z_b = \lim_{q \rightarrow \xi} \widehat{Z}_b(q)$ by inverting the linear relation.
3. Solve the integer relation problem $a_b z_b^r = a_0 + a_1 \xi + \dots + a_{\phi(r)-1} \xi^{\phi(r)-1}$ for some integers $a_b, a_0, a_1, \dots, a_{\phi(r)-1}$ not all zero, using the LLL algorithm. Note that fixed precision numbers are important for this step—the usual 64-bit floating point numbers do not have enough digits for this step to work reliably.

Verify that there is an integer q_b independent of r such that $q_b z_b^r(\xi) \in \mathbb{Z}[\xi]$ for every $r = 2j + 1$.

To check if these polynomials can be unified into a Habiro element. Consider an element of the modified Habiro ring

$$f(q) = \sum_{n=0}^{\infty} a_n(q)(q; q^2)_n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{2n} a_{n,m} q^m \right) (q; q^2)_n, \quad (8.24)$$

where $a_{n,m}$ are integers. Evaluation of this element at an odd root of unity is well-defined. For every $r = 2j + 1$ define the evaluation

$$f_r(q) = \sum_{n=0}^j \left(\sum_{m=0}^{2n} a_{n,m} q^m \right) (q; q^2)_n \pmod{(q^r - 1)}, \quad (8.25)$$

and pick a residue class modulo $(q^r - 1)$ such that $\deg f_r = r - 1$. We set up a linear system for integers $a_{n,m}$ by equating coefficients of polynomials

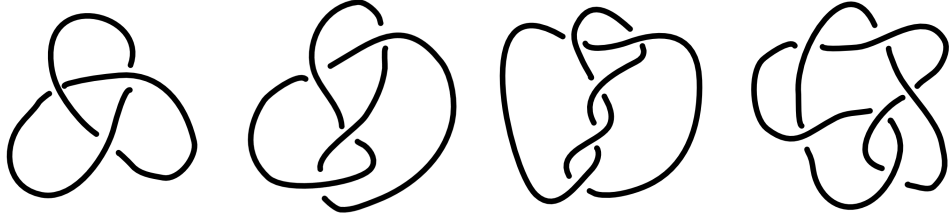
$$f_r(q) = q_b z_b^r(q) \quad (8.26)$$

for every $r = 2j + 1$. If $N = 0$, we have only one equation $f_0(q) = a_{0,0} = q_b z_b^1$. Every time we increment $N \rightarrow N + 1$, we introduce $2N + 3$ new unknowns,

$$a_{N+1,0}, a_{N+1,1}, \dots, a_{N+1,2(N+1)}, \quad (8.27)$$

and $2N + 3$ new constraints because $\deg f_{2(N+1)+1} = 2N + 2$.

If this linear system has integer solutions, it is an indication that the polynomials $z_b^r(q)$ can be unified into an element of the Habiro ring. In every example we considered, the matrix of the linear system had unit determinant.



APPENDIX A

Computing quadratic Gauss sums

For a positive integer r , and an integer s , the *quadratic Gauss sum* is defined as

$$G(s, r) := \sum_{n \in \mathbb{Z}/r\mathbb{Z}} e^{\frac{2\pi i s n^2}{r}}. \quad (\text{A.1})$$

When s and r are coprime, the Gauss sum $G(s, r)$ can be evaluated explicitly [47],

$$G(s, r) = \begin{cases} \left(\frac{r}{s}\right) (1 + i^s) \sqrt{r} & r \equiv 0 \pmod{4}, \\ \left(\frac{s}{r}\right) \sqrt{r} & r \equiv 1 \pmod{4}, \\ 0 & r \equiv 2 \pmod{4}, \\ \left(\frac{s}{r}\right) i \sqrt{r} & r \equiv 3 \pmod{4}, \end{cases} \quad (\text{A.2})$$

where $(\cdot)$ is the *Jacobi symbol*. If s and r are not coprime, let $g = \gcd(s, r)$ and define $s_1 = s/g$, $r_1 = r/g$ so that s_1 and r_1 are coprime, and a short calculation shows that $G(s, r) = g G(s_1, r_1)$.

We shall also come across quadratic Gauss sums with a linear piece in its exponent like

$$\sum_{n \in \mathbb{Z}/r\mathbb{Z}} \xi^{Pn^2 + 2An}, \quad (\text{A.3})$$

where $\xi = e^{\frac{2\pi i s}{r}}$ is a primitive r -th root of unity and P, A are integers. Such sums can be evaluated with a square completion argument as follows. Suppose that P is coprime with r (we shall relax this condition in the next paragraph) and that r is odd. Since we are working modulo r in the exponent to the root of unity ξ , we have

$$Pn^2 + 2An = P(n + 2^* P^* A)^2 - 4^* P^* A^2, \quad (\text{A.4})$$

where * denotes inversion in $\mathbb{Z}/r\mathbb{Z}$, so that

$$\sum_{n \in \mathbb{Z}/r\mathbb{Z}} \xi^{Pn^2 + 2An} = \xi^{-A^2/4P} \sum_{n \in \mathbb{Z}/r\mathbb{Z}} \xi^{P(n + 2^* P^* A)^2} = \xi^{-A^2/4P} G(sP, r), \quad (\text{A.5})$$

where $-A^2/4P$ in the exponent should be understood as an element of $\mathbb{Z}/r\mathbb{Z}$.

If P is not coprime with r (and therefore not invertible in $\mathbb{Z}/r\mathbb{Z}$), let $g = \gcd(r, P)$, define $r_1 = r/g$, $P_1 = P/g$. And instead of summing over $n \in \mathbb{Z}/r\mathbb{Z}$, let $n = r_1 j + k$ and sum over $j = 0, \dots, g-1$ and $k = 0, \dots, r_1-1$ for

$$\sum_{n=0}^{r-1} \xi^{Pn^2+An} = \sum_{j=0}^{g-1} \xi^{r_1 A j} \sum_{k=0}^{r_1-1} \xi_1^{P_1 k^2 + A_1 k} = g \delta_{g|A} \xi_1^{-A_1^2/4P_1} G(sP_1, r_1). \quad (\text{A.6})$$

where the sum over j is a sum of g -th roots of unity $\xi^{r_1 A}$, and is zero unless g divides A , and the sum over k is like the one we started out with, in which r , P , and A are replaced by r_1 , P_1 , and A_1 respectively. Note that the expression on the right-hand side vanishes unless A_1 is an integer, so we can use the square completion argument as before.

Using the above formulas, we have

$$F(U^{\pm 1}, q) = \mp \frac{q^{\mp 3/4}}{q^{1/2} - q^{-1/2}} \frac{G_{\pm}(q^{1/4})}{2} = \mp \left(\frac{r}{s}\right) \left(\frac{1 \pm i^s}{\sqrt{2}}\right) \sqrt{2r} \frac{q^{\mp 3/4}}{q^{1/2} - q^{-1/2}}. \quad (\text{A.7})$$

To see why the first equality is true, consider the Gauss sum involved in $F(U^{\pm 1}, q)$, for $\sigma = \pm 1$,

$$\sum_{n=1}^{r-1} q^{\sigma(n^2-1)/4} [n]^2 = \frac{q^{-\sigma/4}}{(q^{1/2} - q^{-1/2})^2} \sum_{n=1}^{r-1} q^{\sigma n^2/4} (q^n + q^{-n} - 2) \quad (\text{A.8})$$

$$= \frac{q^{-\sigma/4}}{(q^{1/2} - q^{-1/2})^2} \sum_{n \in \mathbb{Z}/2r\mathbb{Z}} q^{\sigma n^2/4} (q^n - 1) \quad (\text{A.9})$$

$$= \frac{q^{-\sigma/4}}{(q^{1/2} - q^{-1/2})^2} (q^{-\sigma} - 1) \sum_{n \in \mathbb{Z}/2r\mathbb{Z}} q^{\sigma n^2/4} \quad (\text{A.10})$$

$$= -\sigma \frac{q^{-3\sigma/4}}{q^{1/2} - q^{-1/2}} \sum_{n \in \mathbb{Z}/2r\mathbb{Z}} q^{\sigma n^2/4}. \quad (\text{A.11})$$

With this, the normalization in (5.30) can be written as

$$F(U^{\pm 1}, q)^{b_+} F(U^{\pm 1}, q)^{b_-} = (-1)^{b_+} \left(\frac{r}{s}\right)^N \left(\frac{1 + i^s}{\sqrt{2}}\right)^{\sigma(B)} \frac{(2r)^{N/2} q^{-3\sigma(B)/4}}{(q^{1/2} - q^{-1/2})^N}, \quad (\text{A.12})$$

where $\sigma(B) = b_+ - b_-$ is the signature of the linking matrix.

Theorem A.1 (Deloup and Turaev [15]). *Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{R}^N$, $B : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a self-adjoint automorphism, $\psi \in \mathbb{R}^N$, and r a positive integer. Assume that $B(\mathbb{Z}^N) \subseteq \mathbb{Z}^N$ and*

$$\frac{r}{2} \langle x, Bx \rangle, \langle x, Bx' \rangle, \langle x, rx' \rangle, r \langle x, \psi \rangle \in \mathbb{Z} \quad \text{for all } x, x' \in \mathbb{Z}^N. \quad (\text{A.13})$$

Then

$$\begin{aligned} \sum_{x \in \mathbb{Z}^N / r\mathbb{Z}^N} \exp \left(\frac{\pi i}{r} \langle x, Bx \rangle + 2\pi i \langle x, \psi \rangle \right) \\ = \frac{e^{\pi i \sigma / 4} r^{N/2}}{|\det B|^{1/2}} \sum_{y \in \mathbb{Z}^N / B\mathbb{Z}^N} \exp \left(-\pi i r \langle y + \psi, B^{-1}(y + \psi) \rangle \right), \end{aligned} \quad (\text{A.14})$$

for arbitrary lifts of x, y to $\mathbb{Z}^N$.

A.1 A refinement of the radial limit conjecture

In this section, we shall comment on the conjectural relationship between WRT invariants at generic roots of unity and radial limits of $\widehat{Z}$ -invariants. The following result about a higher rank version of quadratic Gauss sums will be important in simplifying the resulting relation.

Proposition A.1. *For any positive integer N , let $B : \mathbb{Z}^N \rightarrow \mathbb{Z}^N$ be a nondegenerate, symmetric, $\mathbb{Z}$ -linear map, and let r be an odd positive integer, coprime with $\det B$. Then*

$$\sum_{x \in \mathbb{Z}^N / r\mathbb{Z}^N} e^{\frac{2\pi i}{r} \langle x, Bx \rangle} = \begin{cases} \left(\frac{\det B}{r} \right) r^{N/2} & r \equiv 1 \pmod{4}, \\ \left(\frac{\det B}{r} \right) i^N r^{N/2} & r \equiv 3 \pmod{4}. \end{cases} \quad (\text{A.15})$$

Proof. Let $r = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the prime factorization of r where p_j are distinct odd primes and $n_j \in \mathbb{Z}_{>0}$ for all $j = 1, 2, \dots, k$. We will prove this result by induction on k . For $k = 1$, we use a diagonalization result for symmetric, invertible bilinear forms over local rings in which 2 is a unit [43, Corollary 3.4]. The matrix B is invertible over $\mathbb{Z}/p_1^{n_1}\mathbb{Z}$ because $\det B$ is coprime with r , and 2 is a unit in $\mathbb{Z}/p_1^{n_1}\mathbb{Z}$ because r is odd. We have $SBS^T = \text{diag}(b_1, \dots, b_N)$ for some orthonormal matrix S , and the result follows from a short calculation

$$\sum_{x \in \mathbb{Z}^N / p_1^{n_1}\mathbb{Z}^N} e^{\frac{2\pi i}{p_1^{n_1}} \langle x, Bx \rangle} = \prod_{j=1}^N \sum_{x_j \in \mathbb{Z}/p_1^{n_1}\mathbb{Z}} e^{\frac{2\pi i b_j}{p_1^{n_1}}} = \left(\frac{\det B}{p_1^{n_1}} \right) G(1, p_1^{n_1})^N. \quad (\text{A.16})$$

For the induction step, we factor the sum over $\mathbb{Z}^N / r\mathbb{Z}^N$ into $\mathbb{Z}^N / r'\mathbb{Z}^N \oplus \mathbb{Z}^N / p_k^{n_k}\mathbb{Z}^N$, where $r' = r/p_k^{n_k}$

$$\sum_{x \in \mathbb{Z}^N / r\mathbb{Z}^N} e^{\frac{2\pi i}{r} \langle x, Bx \rangle} = \sum_{x \in \mathbb{Z}^N / r'\mathbb{Z}^N} e^{\frac{2\pi i}{r'} \langle x, Bx \rangle} \sum_{y \in \mathbb{Z}^N / p_k^{n_k}\mathbb{Z}^N} e^{\frac{2\pi i}{p_k^{n_k}} \langle y, By \rangle} \quad (\text{A.17})$$

$$= \left(\frac{\det B}{r} \right) \left[\left(\frac{p_k^{n_k}}{r'} \right) \left(\frac{r'}{p_k^{n_k}} \right) \right]^N G(1, r')^N G(1, p_k^{n_k})^N. \quad (\text{A.18})$$

To see that the sign of the Gauss sum is correct, we need to use the reciprocity of Jacobi symbols

$$\left(\frac{r}{s}\right)\left(\frac{s}{r}\right) = (-1)^{\frac{r-1}{2}\frac{s-1}{2}}. \quad (\text{A.19})$$

□

Suppose that M is a 3-manifold obtained from a genus zero plumbing on a tree graph, and let B be its linking matrix, which we assume to be negative definite. Let $\xi = e^{\frac{2\pi i s}{r}}$ be a primitive r -th root of unity. With manipulations similar to [22, Appendix A], we get (see also [39, 12])

$$\begin{aligned} \tau_M(\xi) = \lim_{q \rightarrow \xi} \frac{1}{2(q^{1/2} - q^{-1/2}) |\det B|^{1/2} G(r, s)^N} \\ \times \sum_{a \in \text{coker } sB} e^{-2\pi i \frac{r}{s} \langle a, B^{-1}a \rangle} \sum_{b \in 2 \text{ coker } B + \delta} e^{-2\pi i \langle a, B^{-1}b \rangle} \widehat{Z}_b(q), \end{aligned} \quad (\text{A.20})$$

If we let s be coprime with $\det B$, it is possible to factor the sum over $\text{coker } sB$ —which is not invariant under Kirby moves—to sums over $\mathbb{Z}^N / s\mathbb{Z}^N$ and $\text{coker } B$ using the isomorphism $\mathbb{Z}^N / s\mathbb{Z}^N \oplus \text{coker } B \ni (x, a) \mapsto Bx + sa \in \text{coker } sB$. The two sums above become

$$\sum_{x \in \mathbb{Z}^N / s\mathbb{Z}^N} e^{-2\pi i \frac{r}{s} \langle x, Bx \rangle} \sum_{a \in \text{coker } B} e^{-2\pi i \frac{r}{s} s^2 \langle a, B^{-1}a \rangle} \sum_{b \in 2 \text{ coker } B + \delta} e^{-2\pi i s \langle a, B^{-1}b \rangle} \widehat{Z}_b(q). \quad (\text{A.21})$$

We can use the above formulas to evaluate the Gauss sums in the denominator, and if we assume that s is odd, we can use Proposition A.1 to evaluate the sum over x , so that

$$\begin{aligned} \left(\frac{\det(-B)}{s}\right) \tau_M(\xi) = \lim_{q \rightarrow \xi} \frac{1}{2(q^{1/2} - q^{-1/2}) |\det B|^{1/2}} \\ \times \sum_{a \in \text{coker } B} e^{-2\pi i \frac{r}{s} s^2 \langle a, B^{-1}a \rangle} \sum_{b \in 2 \text{ coker } B + \delta} e^{-2\pi i s \langle a, B^{-1}b \rangle} \widehat{Z}_b(q). \end{aligned} \quad (\text{A.22})$$

Using the assumption that s is coprime with $|\det B| = |H_1(M)|$ we can rewrite the formula in the form independent of the plumbing representation (cf. [14]):

$$\begin{aligned} \left(\frac{|H_1(M)|}{s}\right) \tau_M(\xi) = \lim_{q \rightarrow \xi} \frac{1}{2(q^{1/2} - q^{-1/2}) |H_1(M)|^{1/2}} \times \\ \sum_{a \in H_1(M)} e^{-2\pi i \frac{r}{s} \overline{\text{lk}(a, a)}} \sum_{\mathfrak{s} \in \text{Spin}^c(M)} e^{-2\pi i \text{lk}(a, \det(\mathfrak{s}))} \widehat{Z}_{\mathfrak{s}}(q). \end{aligned} \quad (\text{A.23})$$

ⁱWe expect that the argument below can be in principle generalized to the case of *weakly negative-definite* plumbings. However, this is redundant, since, as was shown in [26], any weakly negative-definite rational homology sphere plumbing has a negative-definite representative in its homeomorphism class.

where $\widetilde{\text{lk}}(a, a) = \text{CS}[a]$ is a lift of the value of the linking pairing to $\mathbb{Q}$. For $s = 1$ this relation reduces to the one conjectured in [22] and later proved in [46].

APPENDIX B

Integrality of false theta limits

This appendix is dedicated to proving the following integrality result for these false theta limits.

Proposition B.1. *Let $\xi = e^{\frac{2\pi i s}{r}}$ be a root of unity of odd order, then*

$$\frac{1}{2}\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r) \in \xi^{-\text{CS}[\vec{a}]} \mathbb{Z}[\xi]. \quad (\text{B.1})$$

Recall the definition of $\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r)$

$$\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r) = \frac{1}{2} \sum_{l=1}^{2Pr} \phi_{\vec{p}}^{\vec{a}}(l) \left(1 - \frac{l}{Pr}\right) \xi^{l^2/4P}, \quad (\text{B.2})$$

where the $2P$ -periodic function $\phi_{\vec{p}}^{\vec{a}}$ is defined by

$$\phi_{\vec{p}}^{\vec{a}}(l) = \begin{cases} -\epsilon_1 \epsilon_2 \epsilon_3 & \text{if } l = P \left(1 + \sum_{j=1}^3 \frac{\epsilon_j a_j}{p_j}\right) \bmod 2P, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{B.3})$$

for $\epsilon_j = \pm 1$. With the antisymmetry property $\phi_{\vec{p}}^{\vec{a}}(-l) = -\phi_{\vec{p}}^{\vec{a}}(l)$, we have

$$\sum_{l=1}^{2Pr} \phi_{\vec{p}}^{\vec{a}}(l) \xi^{l^2/4P} = 0, \quad (\text{B.4})$$

so that

$$\frac{1}{2}\tilde{\Phi}_{\vec{p}}^{\vec{a}}(s/r) = \frac{1}{4Pr} \sum_{l=1}^{2Pr} \phi_{\vec{p}}^{\vec{a}}(l) l \xi^{l^2/4P}. \quad (\text{B.5})$$

Note that all nonzero summands in the above sum have the same power of ξ modulo 1 because for $l = P \left(1 + \sum_j \frac{\epsilon_j a_j}{p_j}\right) \bmod 2P$ we have

$$\frac{l^2}{4P} = \frac{P}{4} \left(1 + \sum_j \frac{\epsilon_j a_j}{p_j}\right)^2 = \frac{P}{4} \left(1 + \sum_j \frac{a_j}{p_j}\right)^2 = -\text{CS}[\vec{a}] \bmod 1, \quad (\text{B.6})$$

where $\text{CS}[\vec{a}]$ is the Chern–Simons action of the flat connection on $\Sigma(p_1, p_2, p_3)$ that is denoted by rotation numbers (a_1, a_2, a_3) .

Next, we want to use the $2P$ -periodicity of $\phi_{\vec{p}}^{\vec{a}}$ to write the sum over l above as a sum over $L = 0, \dots, r-1$ and $\epsilon \in \{\pm 1\}^3$. To do so, we make the replacements $l \rightarrow 2PL + P + P \sum_j \frac{\epsilon_j a_j}{p_j}$ and $\phi_{\vec{p}}^{\vec{a}}(l) \rightarrow -\epsilon_1 \epsilon_2 \epsilon_3$, and observe that the sums

$$\frac{1}{4Pr} \sum_{l=0}^{2Pr-1} \phi_{\vec{p}}^{\vec{a}}(l) l \xi^{l^2/4P} \quad (\text{B.7})$$

and

$$-\frac{1}{4Pr} \sum_{L=0}^{r-1} \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \left(2PL + P + P \sum_{j=1}^3 \frac{\epsilon_j a_j}{p_j}\right) \xi^{\frac{P}{4} \left(1 + \sum_j \frac{\epsilon_j a_j}{p_j}\right)^2}, \quad (\text{B.8})$$

though not quite equal, differ by a polynomial in ξ (up to an overall fractional power) with integer coefficients. So, to prove the proposition, it will suffice to show that the second sum over $L = 1, \dots, r-1$ and $\epsilon \in \{\pm 1\}^3$ is an element of $\xi^{-\text{CS}[\vec{a}]} \mathbb{Z}[\xi]$.

It will be convenient to get rid of this fractional part of the exponent, and we will do so by multiplying $\xi^{-l_0^2/4P}$ with $l_0 = P \left(1 - \sum_j \frac{a_j}{p_j}\right)$. Define the quadratic function F_ϵ by

$$\frac{l^2 - l_0^2}{4P} = P \left(L + \sum_{j=1}^3 \frac{\epsilon_j + 1}{2} \frac{a_j}{p_j}\right) \left(L + 1 + \sum_{j=1}^3 \frac{\epsilon_j - 1}{2} \frac{a_j}{p_j}\right) =: F_\epsilon(L), \quad (\text{B.9})$$

and write the sum of interest, after removing the fraction exponent, as

$$\frac{1}{4Pr} \sum_{L=0}^{r-1} \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \left(2PL + P + P \sum_{j=1}^3 \frac{\epsilon_j a_j}{p_j}\right) \xi^{F_\epsilon(L)}. \quad (\text{B.10})$$

We will deal with each term in the parentheses above one by one. We start with the second term.

Lemma B.1. *With F_ϵ as defined above and ξ an r -th root of unity, we have*

$$\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} \xi^{F_\epsilon(L)} = 0. \quad (\text{B.11})$$

Proof. We use the fact that F_ϵ is a well-defined quadratic form on $\mathbb{Z}/r\mathbb{Z}$ to write the above sum as

$$\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)} = \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)} - \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_{-\epsilon}(L)}, \quad (\text{B.12})$$

where $E = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (-1, 1, 1)\}$, and conclude that the right-hand side is zero because $F_{-\epsilon}(L) = F_\epsilon(-L - 1) \pmod{r}$. $\square$

To deal with the other terms, we need to understand how the sums $\sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)}$ for different ϵ 's are related to each other. A short calculation gives the coefficients of the quadratic polynomial $F_\epsilon(L) = PL^2 + A_\epsilon L + B_\epsilon$,

$$A_\epsilon = P \left(1 + \sum_{j=1}^3 \frac{\epsilon_j a_j}{p_j} \right) \text{ and } B_\epsilon = P \sum_{j=1}^3 \frac{\epsilon_j + 1}{2} \frac{a_j}{p_j} + P \sum_{i,j} \frac{\epsilon_i + 1}{2} \frac{\epsilon_j - 1}{2} \frac{a_i a_j}{p_i p_j}. \quad (\text{B.13})$$

Like in Appendix A, let $g = \gcd(r, P)$ and define $r' = r/g$, $P' = P/g$, and let $g_j = \gcd(r, p_j)$ and define $p'_j = p_j/g_j$ for $j = 1, 2, 3$. Instead of summing over $L = 0, \dots, r-1$, write $L = r'k + l$ and sum over $k = 0, \dots, g-1$ and $l = 0, \dots, r'-1$ to write

$$\sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)} = \sum_{k=0}^{g-1} \xi^{r' A_\epsilon k} \sum_{l=0}^{r'-1} \xi^{F_\epsilon(l)} = g \delta_{g|A_\epsilon} \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_\epsilon(l)}, \quad (\text{B.14})$$

where $\xi' = \xi^g$ as before and $F'_\epsilon = F_\epsilon/g$. To see why the last sum over l can be written as a sum on $\mathbb{Z}/r'\mathbb{Z}$, note that the right-hand side is zero unless g divides A_ϵ , and from the expressions of A_ϵ and B_ϵ above and using the fact that p_j 's are mutually coprime, one can deduce that $g \mid A_\epsilon$ implies $g \mid B_\epsilon$.

Indeed, it is true that $g \mid A_\epsilon$ if and only if $g_j \mid a_j$ for every $j = 1, 2, 3$. If this is the case, common factors in F'_ϵ can be canceled and

$$F'_\epsilon(l) = P' \left(l + \sum_{j=1}^3 \frac{\epsilon_j + 1}{2} \frac{a'_j}{p'_j} \right) \left(l + 1 + \sum_{j=1}^3 \frac{\epsilon_j - 1}{2} \frac{a'_j}{p'_j} \right), \quad (\text{B.15})$$

where $a'_j = a_j/g_j$ for $j = 1, 2, 3$. We have the following relations (modulo r') between F'_ϵ for different ϵ 's,

$$F'_\epsilon(l) = F'_\epsilon(-l - 1), \quad (\text{B.16})$$

$$F'_{++}(l) = F'_{+++}(l - a'_1 p_1'^*), \quad (\text{B.17})$$

$$F'_{+-}(l) = F'_{+++}(l - a'_2 p_2'^*), \quad (\text{B.18})$$

$$F'_{+-}(l) = F'_{+++}(l - a'_3 p_3'^*), \quad (\text{B.19})$$

which let us relate all F'_ϵ 's to each other by an appropriate shift of the argument. In particular,

$$\sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_\epsilon(l)} = \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_\varepsilon(l)} \quad (\text{B.20})$$

for any $\epsilon, \varepsilon \in \{-1, 1\}^3$.

With these properties of the quadratic form F_ϵ understood, the following lemma—which implies that the third term is zero—can be proved.

Lemma B.2. *With F_ϵ as defined above and ξ an r -th root of unity, we have*

$$\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} \epsilon_j \xi^{F_\epsilon(L)} = 0, \quad \text{for every } j = 1, 2, 3. \quad (\text{B.21})$$

Proof. Here we will show what happens for $j = 1$ (other cases are completely analogous). As in the above discussion, we decompose the sum over L to write

$$\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} \epsilon_1 \xi^{F_\epsilon(L)} = g_{\delta_{g_j|a_j}, j=1,2,3} \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_2 \epsilon_3 \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_\epsilon(l)}. \quad (\text{B.22})$$

Next, using the relations between F'_ϵ 's and the fact that $\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_2 \epsilon_3 = 0$, we have

$$\sum_{\epsilon \in \{\pm 1\}^3} \epsilon_2 \epsilon_3 \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_\epsilon(l)} = \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_2 \epsilon_3 \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi'^{F'_{+++}(l)} = 0. \quad (\text{B.23})$$

□

Finally, for the first term, we have the following.

Lemma B.3. *With F_ϵ as defined above and ξ a root of unity of odd order r , the polynomial*

$$\frac{1}{2r} \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)}, \quad (\text{B.24})$$

in ξ has integer coefficients.

Proof. First, we get rid of the factor of half by noting that

$$\frac{1}{2} \sum_{\epsilon \in \{\pm 1\}^3} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)} = \frac{1}{2} \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)} - \frac{1}{2} \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_{-\epsilon}(L)} \quad (\text{B.25})$$

$$= \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)} - \frac{r-1}{2} \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)} \quad (\text{B.26})$$

$$= \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)}, \quad (\text{B.27})$$

where $E = \{(1, 1, 1), (1, 1, -1), (1, -1, -1), (1, -1, 1)\}$. We can show

$$\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{L \in \mathbb{Z}/r\mathbb{Z}} \xi^{F_\epsilon(L)} = g \delta_{g_j | a_j, j=1,2,3} \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{l \in \mathbb{Z}/r'\mathbb{Z}} \xi^{F'_{+++}(l)} = 0, \quad (\text{B.28})$$

by using relations among different F'_ϵ 's and $\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 = 0$. Note that the choice of the index set E above is not unique, and we can pick any $E \subset \{-1, 1\}^3$ such that $E \cap -E = \emptyset$ and $E \cup -E = \{-1, 1\}^3$.

Next, we decompose the sum over L as before

$$\frac{1}{r} \sum_{L=0}^{r-1} L \xi^{F_\epsilon(L)} = \frac{1}{r} \sum_{k=0}^{g-1} \sum_{l=0}^{r'-1} (r'k + l) \xi^{F_\epsilon(r'k+l)} \quad (\text{B.29})$$

$$= \frac{1}{g} \sum_{k=0}^{g-1} k \xi^{r' A_\epsilon k} \sum_{l=0}^{r'-1} \xi^{F_\epsilon(l)} + \delta_{g | A_\epsilon} \frac{1}{r'} \sum_{l=0}^{r'-1} l \xi^{F'_\epsilon(l)}. \quad (\text{B.30})$$

Since $g = g_1 g_2 g_3$ and g_j 's are mutually coprime, we can decompose the sum over k in the first term in the following symmetric way, $k \rightarrow g \sum_{j=1}^3 \frac{k_j}{g_j}$, where k_j 's all run from 0 to $g_j - 1$,

$$\frac{1}{g} \sum_{k=0}^{g-1} k \xi^{r' A_\epsilon k} = \sum_{k_1=0}^{g_1-1} \sum_{k_2=0}^{g_2-1} \sum_{k_3=0}^{g_3-1} \left(\sum_{j=1}^3 \frac{k_j}{g_j} \right) \xi^{r \frac{\epsilon_1 a_1 k_1}{g_1} + r \frac{\epsilon_2 a_2 k_2}{g_2} + r \frac{\epsilon_3 a_3 k_3}{g_3}}. \quad (\text{B.31})$$

We will work out the term $j = 1$ in detail; terms $j = 2, 3$ are completely analogous. We have

$$\frac{1}{g_1} \sum_{k_1=0}^{g_1-1} \sum_{k_2=0}^{g_2-1} \sum_{k_3=0}^{g_3-1} k_1 \xi^{r \frac{\epsilon_1 a_1 k_1}{g_1} + r \frac{\epsilon_2 a_2 k_2}{g_2} + r \frac{\epsilon_3 a_3 k_3}{g_3}} = -g_2 g_3 \delta_{g_2 | a_2} \delta_{g_3 | a_3} \frac{1}{1 - \xi^{r \frac{\epsilon_1 a_1}{g_1}}}, \quad (\text{B.32})$$

where we used the identity for the sum of roots of unity for sums over k_2 and k_3 , and the identity

$$\sum_{n=0}^{h-1} n\chi^n = -\frac{h}{1-\chi}, \quad (\text{B.33})$$

where χ is an h -th root of unity, for the sum over k_1 . With the sums over ϵ and l , this term becomes

$$-g_2 g_3 \delta_{g_j | a_j, j=2,3} \frac{1}{1 - \xi^{r \frac{a_1}{g_1}}} \sum_{\epsilon \in E} \epsilon_2 \epsilon_3 \sum_{l=0}^{r'-1} \xi^{F_\epsilon(l)}, \quad (\text{B.34})$$

where we have chosen $E \subset \{-1, 1\}^3$ such that $\epsilon_1 = 1$ for every $\epsilon \in E$. To complete the proof, it suffices to show that $\sum_{\epsilon \in E} \sum_{l=0}^{r'-1} \xi^{F_\epsilon(l)}$ is divisible by $1 - \xi^{r a_1 / g_1}$ in $\mathbb{Z}[\xi]$. Since the term above is nonzero even when g_1 does not divide a_1 , we cannot cancel factors in F_ϵ , and F'_ϵ might not be well-defined on $\mathbb{Z}/r'\mathbb{Z}$. But we still have

$$F_\epsilon(l + r'k) = F_\epsilon(l) + r' A_\epsilon k, \quad (\text{B.35})$$

and the relations,

$$F_{++-}(l) = F_{+++}(l - a'_3 p_3^*), \quad (\text{B.36})$$

$$F_{+--}(l) = F_{+-+}(l - a'_3 p_3^*) \quad (\text{B.37})$$

modulo r , where $0 < p_3^* < r$ is an integer such that $p_3' p_3^* = 1 \pmod{r}$.

To show that the sum in (B.34) is divisible by $1 - \xi^{r a_1 / g_1}$, we use (B.36) for

$$\sum_{l=0}^{r'-1} \xi^{F_{+++}(l)} - \sum_{l=0}^{r'-1} \xi^{F_{++-}(l)} = \sum_{l=0}^{r'-1} \xi^{F_{+++}(l)} - \sum_{l=-a'_3 p_3^*+r'-1}^{-a'_3 p_3^*+r'-1} \xi^{F_{+++}(l)}, \quad (\text{B.38})$$

and note that $1 - \xi^{r' A_{+++}}$ is a factor of the above because for every $l' \in \{-a'_3 p_3^*, -a'_3 p_3^* + 1, \dots, -a'_3 p_3^* + r' - 1\}$, there exists an $l \in \{0, 1, \dots, r' - 1\}$ such that $l' = l + r'k$ for some integer k , and due to (B.35) $F_{+++}(l') - F_{+++}(l) = r' A_{+++} k$. Similarly, we can use (B.37) and (B.35) to see that $1 - \xi^{r' A_{+++}}$ is also a factor of $\sum_{l=0}^{r'-1} \xi^{F_{+--}(l)} - \sum_{l=0}^{r'-1} \xi^{F_{+-+}(l)}$, so that $1 - \xi^{r' A_{+++}}$ is a factor of the sum in (B.34). But since

$$\frac{1 - \xi^{r' A_{+++}}}{1 - \xi^{r \frac{a_1}{g_1}}} = \frac{1 - \xi^{r \frac{a_1 p_2' p_3'}{g_1}}}{1 - \xi^{r \frac{a_1}{g_1}}} = 1 + \xi^{r \frac{a_1}{g_1}} + \dots + \xi^{r \frac{a_1 p_2' p_3'}{g_1}} \in \mathbb{Z}[\xi], \quad (\text{B.39})$$

the term (B.34) is an element of $\mathbb{Z}[\xi]$.

In the second term (B.30), we can assume $g_j \mid a_j$ for all $j = 1, 2, 3$ because it vanishes unless g divides A_ϵ , and therefore F'_ϵ is a well-defined quadratic form on $\mathbb{Z}/r'\mathbb{Z}$. If 2 is invertible in $\mathbb{Z}/r'\mathbb{Z}$, then we can write this quadratic polynomial in the following diagonal form $F'_\epsilon(l) = P'(l - D_\epsilon)^2 + C_\epsilon$ modulo r' , for some integers $0 \leq C_\epsilon, D_\epsilon < r'$. Using the

relations (B.16–B.19) among different F'_ϵ 's, we can deduce $C_\epsilon = C_\varepsilon$ for any $\epsilon, \varepsilon \in \{-1, 1\}^3$, and the following relations among D_ϵ 's,

$$D_{-\epsilon} = 1 - D_\epsilon, \quad (\text{B.40})$$

$$D_{-++} = D_{+++} + a'_1 p_1^*, \quad (\text{B.41})$$

$$D_{+-+} = D_{+++} + a'_2 p_2^*, \quad (\text{B.42})$$

$$D_{+--} = D_{+++} + a'_3 p_3^*. \quad (\text{B.43})$$

With the diagonal form of F'_ϵ as above, we write

$$\frac{1}{r'} \sum_{l=0}^{r'-1} l \xi'^{F'_\epsilon(l)} = \frac{1}{r'} \sum_{l=0}^{r'-1} l \xi'^{P'(l-D_\epsilon)^2+C_\epsilon} = \frac{1}{r'} \sum_{l=-D_\epsilon}^{-D_\epsilon+r'-1} l \xi'^{P'l^2+C_\epsilon} + \frac{D_\epsilon}{r'} \sum_{l=0}^{r'-1} \xi'^{F'_\epsilon(l)}. \quad (\text{B.44})$$

But we can show

$$\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 D_\epsilon \sum_{l=0}^{r'-1} \xi'^{F'_\epsilon(l)} = \sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 D_\epsilon \sum_{l=0}^{r'-1} \xi'^{F'_{+++}(l)} = 0, \quad (\text{B.45})$$

using (B.20), and $\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 D_\epsilon = 0$, which follows from the relations (B.40–B.43) between D_ϵ 's. To deal with the other term, we use

$$\sum_{l=-r+1}^{r-1} l \xi'^{P'l^2+C_\epsilon} = 0, \quad (\text{B.46})$$

to write

$$-\sum_{l=-D_\epsilon}^{-D_\epsilon+r'-1} l \xi'^{P'l^2+C_\epsilon} = \sum_{l=-r'+1}^{-D_\epsilon-1} l \xi'^{P'l^2+C_\epsilon} + \sum_{l=-D_\epsilon+r'}^{r'-1} l \xi'^{P'l^2+C_\epsilon} \quad (\text{B.47})$$

$$= \sum_{l=1}^{-D_\epsilon+r'-1} (l-r') \xi'^{P'(l-r')^2+C_\epsilon} + \sum_{l=-D_\epsilon+r'}^{r'-1} l \xi'^{P'l^2+C_\epsilon} \quad (\text{B.48})$$

$$= \sum_{l=1}^{r'-1} l \xi'^{P'l^2+C_\epsilon} - \sum_{l=1}^{-D_\epsilon+r'-1} r' \xi'^{P'l^2+C_\epsilon}, \quad (\text{B.49})$$

so that

$$\frac{1}{r'} \sum_{l=-D_\epsilon}^{-D_\epsilon+r'-1} l \xi'^{P'l^2+C_\epsilon} = \sum_{l=1}^{-D_\epsilon+r'-1} \xi'^{P'l^2+gC_{+++}} - \frac{1}{r'} \sum_{l=0}^{r'-1} l \xi'^{P'l^2+C_{+++}}, \quad (\text{B.50})$$

where we have used that C_ϵ 's are the same for every ϵ . We notice that the first term is an element of $\mathbb{Z}[\xi]$ and since the second term is independent of ϵ , we have

$$\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 \sum_{l=0}^{r'-1} l \xi^{P' l^2 + C_{+++}} = 0, \quad (\text{B.51})$$

because $\sum_{\epsilon \in E} \epsilon_1 \epsilon_2 \epsilon_3 = 0$. □

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