

An anharmonic alliance: exact WKB meets EPT

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ABSTRACT: Certain quantum mechanical systems with a discrete spectrum, whose observables are given by a transseries in \hbar , were shown to admit \hbar_0 -deformations with Borel resumable expansions which reproduce the original model at $\hbar_0 = \hbar$. Such expansions were dubbed Exact Perturbation Theory (EPT). We investigate how the above results can be obtained within the framework of the exact WKB method by studying the spectrum of polynomial quantum mechanical systems. Within exact WKB, energy eigenvalues are determined by exact quantization conditions defined in terms of Voros symbols a_{γ_i} , γ_i being their associated cycles, and generally give rise to transseries in \hbar . After reviewing how the Borel summability of energy eigenvalues in the quartic anharmonic potential emerges in exact WKB, we extend it to higher order anharmonic potentials with quantum corrections. We then show that any polynomial potential can be \hbar_0 -deformed to a model where the exact quantization condition reads simply $a_\gamma = -1$ and leads to the EPT Borel resumable series for all energy eigenvalues.

KEYWORDS: Large-Order Behaviour of Perturbation Theory, Renormalons, Nonperturbative Effects

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1 Introduction

Quantum mechanical models have been, and still are, an ideal playground to understand the nature of perturbation theory. Thanks to the Schrödinger equation we have easy access to perturbative terms at large orders. Starting from the seminal papers by Bender and Wu [1, 2], this allowed us to get accurate estimates of large order behaviours of perturbative asymptotic series. Famously, the Schrödinger equation can also be studied using a Wentzel-Kramers-Brillouin (WKB) approximation [3–5] which in fact played a key role also in [1, 2]. The proper use of the WKB approximation was found much later [6, 7] using Borel resummation and resurgence techniques [8]. In particular, building on previous works [9, 10], Voros upgraded the WKB approximation into an exact method, exact WKB (EWKB), which was further developed and laid on solid mathematical foundation in [11–15]. In EWKB, energy eigenvalues of a quantum mechanical system are determined

by exact quantization conditions (EQCs) defined in terms of so called Voros symbols a_{γ_i} , where γ_i are the associated periods in complexified space between the turning points of the classical potential. In certain quantum mechanical models EQCs were guessed based on truncated transseries in multi-instantons computations [16], and then rigorously established using EWKB [13]. In general, EQCs involve both classically allowed (perturbative) and disallowed (non-perturbative) periods and generally give rise to transseries in \hbar , $\exp(-1/\hbar)$ and possibly $\log \hbar$.

Several connections between EWKB and other subjects have been worked out. These include connections with $\mathcal{N} = 2$ supersymmetric gauge theories [17, 18], topological strings [19, 20], thermodynamic Bethe ansatz [21] via the ordinary differential equations/integrable models (ODE/IM) correspondence [22], generalizations in terms of the geometry of quantum periods [23] and so on. It is fair to say that EWKB represents one of the most interesting and established applications of resurgence in theoretical physics. How to unpack the EQCs to efficiently write transseries for energy eigenvalues or other observables, and their relation to transseries coming from multi-instantons in a path integral approach, has also been the subject of some activity in the last years, see e.g. [24–28].

Independently of EWKB it has been shown that several one-dimensional quantum mechanical models with a discrete spectrum admit (in general infinite) \hbar_0 -deformations (the original system is recovered by setting $\hbar_0 = \hbar$) such that the path integral is exactly determined by a single saddle-point [29, 30]. If we Borel resum the \hbar_0 -deformed perturbative series in \hbar at *fixed* \hbar_0 and *after* we set $\hbar_0 = \hbar$, the exact result is recovered. The “exact perturbation theory” (EPT) in the \hbar_0 -model is then able to capture the full result in models which are known to receive instanton corrections, such as the (supersymmetric) double well. This somewhat surprising and powerful result was obtained using path integrals and steepest-descent methods (Lefschetz thimbles).

The aim of this paper is to understand how EPT emerges from an EWKB analysis.

We start in section 2 with a brief review on EWKB, with an emphasis on how to determine EQCs. In general, EQCs are a constraint of the form

$$F[a_{\gamma_i}(E)] = 0, \tag{1.1}$$

where

$$a_{\gamma_i} = \exp\left(\frac{1}{\hbar} \oint_{\gamma_i} P_{\text{even}}(z) dz\right) \tag{1.2}$$

are the Voros symbols associated to the different cycles connecting classical turning points, and P_{even} is the Borel resummation of the \hbar series starting with the classical contribution $\sqrt{2(V - E)}$ followed by quantum corrections. At fixed parameters of the potential V , the Voros symbols depend only on the energy E entering the Schrödinger equation (2.1). Energy eigenvalues can be determined exactly as those values for which (1.1) is satisfied. However, if one is interested in determining their associated asymptotic series (or transseries) in \hbar , (1.1) should be “downgraded” to its formal power series form \tilde{F} . In \tilde{F} we replace E by a transseries \tilde{E} which is determined by demanding $\tilde{F} = 0$ order by order in \hbar , and possibly in the transseries parameters $\exp(-1/\hbar)$ and $\log \hbar$. Importantly, EQCs are

not uniquely determined and depend on E and $\arg \hbar$, because Voros symbols are subject to Stokes jumps.

In section 3 we study in some detail the EQCs in the quartic anharmonic oscillator as a function of $\arg \hbar$ and uncover a Stokes jump occurring for small $\arg \hbar$ when two turning points approach each other in the limit $E \rightarrow 0$. Because of this discontinuity, the limits $\arg \hbar \rightarrow 0$ and $E \rightarrow 0$ do not commute. By taking $E \rightarrow 0$ first, we get the simple EQC

$$a_\gamma = -1, \tag{1.3}$$

where γ is the perturbative cycle, which is shown to be Borel resummable and to lead to Borel resummable energy eigenvalues for sufficiently small \hbar , reproducing in this way a result of [13] but bypassing the complications related to double turning points. This analysis is then generalized to more general anharmonic potentials of the form (3.27) in section 3.4.

Using EPT, which we briefly review in 4.1, any bounded polynomial potential can be reduced to the form (3.27) plus a quantum potential which includes the remaining terms. We then prove in sections 4.2 and 4.3 that energy eigenvalues of an arbitrary quantum mechanical model with a bounded polynomial potential admit \hbar_0 -deformed EPT series which are Borel resummable with EQC given by (1.3). We conclude in section 5. We report in appendices A and B the derivation of so called connection matrices for simple and double turning points, which are important ingredients to determine EQCs. In appendix C we discuss the transseries associated to the energy eigenvalues in the pure quartic model (with no use of EPT) and prove that the radius of convergence of the partial series of exponential corrections is finite.

A note on notation. We carefully distinguish actual functions from their formal asymptotic power series expression by putting a tilde on the latter. A hat denotes the associated Borel function:

$$\tilde{f} = \sum_{n=0} c_n \hbar^n, \quad \hat{f} = \sum_{n=0} \frac{c_n}{n!} t^n. \tag{1.4}$$

If \tilde{f} is a Gevrey-1 series (i.e. $c_n \sim (n!)^1$ for large n), then \hat{f} is analytic in the Borel t -plane in a disc around the origin and is analytically continuable in the Borel plane. We define the Laplace transform in the direction θ as

$$f^\theta(\hbar) = s_\theta(\tilde{f}) = \frac{1}{\hbar} \int_0^{e^{i\theta}\infty} dt \hat{f}(t) e^{-\frac{t}{\hbar}}. \tag{1.5}$$

If $|\hat{f}(te^{i\theta})| \leq e^{at}$ for any $t \geq 0$, then $f^\theta(\hbar)$ is analytic for $\text{Re}(e^{i\theta}\hbar^{-1}) > a$. If the ray $\theta = 0$ is not a Stokes line, then f^0 is well-defined and might reconstruct the original function f . In the complex \hbar -plane we often have wedges, delimited by Stokes lines, labelled by roman numbers I, II, ... We then write

$$f^N = f^\theta, \quad \theta \in \text{wedge } N, \quad N = \text{I, II, } \dots \tag{1.6}$$

2 Exact WKB basics

In this section we briefly review basics of EWKB. As mentioned in the introduction, the spectrum of a given quantum mechanical system is encoded in the Voros symbols (1.2) and

energy eigenvalues are given by the solutions of the EQC of the form (1.1). Wave functions do not enter in EQCs, but the determination of the latter requires an understanding of the former. We then review the EWKB rules to derive EQCs. We discuss the ansatz for the wave function $\psi(z)$ in section 2.1, review how the z -plane splits into so called Stokes region and how Voros symbols and wave functions jump in section 2.2, and finally the rules of how to determine EQCs for an arbitrary polynomial potential in the case in which all its zeros are simple in section 2.3. We report in appendix A the computation of connection matrices for pure monomial potentials. Determining EQCs in presence of higher order zeros is more complicated and there are no general rules as for simple turning points. Nevertheless we report in appendix B the connection matrices for a deformed quadratic turning point, which will be useful in our considerations. There are no new results in this section, which contains standard material, though some emphasis might differ with respect to other presentations. More details on EWKB can be found e.g. in [31]. For a more elementary textbook presentation see [32].

2.1 Ansatz for the wave function

The starting point of a WKB analysis is the (complexified) Schrödinger equation

$$-\hbar^2 \frac{d^2 \psi(z)}{dz^2} + Q(z) \psi(z) = 0, \tag{2.1}$$

where $\hbar, z \in \mathbf{C}$ and

$$Q(z) = 2(V(z) - E). \tag{2.2}$$

For simplicity we assume in the following a potential V given by an entire function, where the only singularity is at $z = \infty$. In general the potential V could be a quantum potential, namely it can depend on \hbar :

$$Q(z, \hbar) = \sum_{n=0}^{\infty} Q_n(z) \hbar^n. \tag{2.3}$$

The fundamental WKB ansatz is

$$\psi(z) = c e^{\frac{1}{\hbar} \int_{z_0}^z P(w, \hbar) dw}, \tag{2.4}$$

where c is a constant, which could be reabsorbed in the definition of z_0 , but we prefer to keep it explicitly. Plugging (2.4) in (2.1) gives rise to a Riccati equation for P :

$$(\hbar P' + P^2 - Q) = 0, \tag{2.5}$$

where $f' \equiv df/dz$. We look for asymptotic solutions of P in the form

$$\tilde{P} = \sum_{n=0}^{\infty} P_n \hbar^n. \tag{2.6}$$

The coefficient functions $P_n(z)$ satisfy the following recursion relations:

$$2P_0 P_{n+1} = Q_{n+1} - P_n' - \sum_{k=1}^n P_k P_{n+1-k}, \quad n \geq 0. \tag{2.7}$$

Once P_0 is found, the whole series is fixed by (2.7). We then get two series, depending on which classical term P_0 is selected:

$$P_0^\eta = \pm\sqrt{Q_0}, \quad \eta = \pm 1. \quad (2.8)$$

The first two terms beyond P_0^η read

$$\begin{aligned} P_1^\eta &= -\frac{Q_0'}{4Q_0} + \eta \frac{Q_1}{2\sqrt{Q_0}}, \\ P_2^\eta &= \frac{\eta}{32Q_0^{\frac{5}{2}}} \left(-5Q_0'^2 + 4Q_0Q_0'' - 4Q_0Q_1^2 + 16Q_0^2Q_2 \right) + \frac{Q_1Q_0' - Q_0Q_1'}{4Q_0^2}. \end{aligned} \quad (2.9)$$

When $Q = Q_0$ (classical potential only) we have

$$P_{2n+1}^- = P_{2n+1}^+, \quad P_{2n}^- = -P_{2n}^+, \quad n \geq 0. \quad (2.10)$$

For now let us assume $Q = Q_0$ and write

$$\begin{aligned} \tilde{P}_{\text{even}} &= \frac{\tilde{P}^+ - \tilde{P}^-}{2} = \sum_{n=0}^{\infty} \hbar^{2n} P_{2n}, \quad (Q = Q_0) \\ \tilde{P}_{\text{odd}} &= \frac{\tilde{P}^+ + \tilde{P}^-}{2} = \sum_{n=0}^{\infty} \hbar^{2n+1} P_{2n+1}, \quad (Q = Q_0) \end{aligned} \quad (2.11)$$

We plug (2.11) in (2.5) and split even and odd terms in \hbar :

$$\hbar \tilde{P}'_{\text{even}} + 2\tilde{P}_{\text{even}}\tilde{P}_{\text{odd}} = 0, \quad (2.12)$$

$$\hbar \tilde{P}'_{\text{odd}} + \tilde{P}_{\text{even}}^2 + \tilde{P}_{\text{odd}}^2 - Q_0 = 0. \quad (2.13)$$

The first equation gives

$$\tilde{P}_{\text{odd}} = -\frac{\hbar}{2} \frac{d \log \tilde{P}_{\text{even}}}{dz}, \quad (2.14)$$

and hence

$$\tilde{\psi}_\pm = c \sqrt{\frac{\tilde{P}_{\text{even}}(z_0)}{\tilde{P}_{\text{even}}(z)}} e^{\pm \frac{1}{\hbar} \int_{z_0}^z \tilde{P}_{\text{even}} dw}. \quad (2.15)$$

For generic Q , (2.10) does not apply, but we can still define \tilde{P}_{even} and \tilde{P}_{odd} as

$$\begin{aligned} \tilde{P}_{\text{even}} &= \frac{\tilde{P}^+ - \tilde{P}^-}{2} = \sum_{n=0}^{\infty} \hbar^n P_{\text{even},n}, \\ \tilde{P}_{\text{odd}} &= \frac{\tilde{P}^+ + \tilde{P}^-}{2} = \sum_{n=1}^{\infty} \hbar^n P_{\text{odd},n}. \end{aligned} \quad (2.16)$$

They can no longer be expressed as even and odd powers of \hbar , respectively, but they still satisfy (2.14). Indeed, the Riccati equations $(\hbar \tilde{P}'^\pm + (\tilde{P}^\pm)^2 - Q) = 0$ turn into

$$\begin{aligned} \hbar \tilde{P}'_{\text{even}} + \hbar \tilde{P}'_{\text{odd}} + (\tilde{P}_{\text{even}} + \tilde{P}_{\text{odd}})^2 - Q &= 0, \\ -\hbar \tilde{P}'_{\text{even}} + \hbar \tilde{P}'_{\text{odd}} + (\tilde{P}_{\text{even}} - \tilde{P}_{\text{odd}})^2 - Q &= 0, \end{aligned} \quad (2.17)$$

and the difference between the two equations gives (2.14). We then learn that (2.15) applies also for *general* quantum potentials Q . We will generally choose z_0 to be a (simple) turning point and

$$c = \frac{1}{\sqrt{\tilde{P}_{\text{even}}(z_0)}}, \tag{2.18}$$

so that

$$\tilde{\psi}_{\pm}(z) = \frac{1}{\sqrt{\tilde{P}_{\text{even}}(z)}} e^{\pm \frac{1}{\hbar} \int_{z_0}^z \tilde{P}_{\text{even}}(w) dw}. \tag{2.19}$$

2.2 Stokes lines and periods

The solutions (2.19) of the Schrödinger equation are only formal asymptotic series which require resummation. In general we cannot have a unique resummation which applies globally. This can be seen by noting that while the actual solution has to be single valued in the z -plane for sufficiently regular potentials with a lower bound, in general (2.19) are not. Stokes phenomena occur and we have to determine the relations between the Borel resummations of (2.19) in different sectors of the complex plane. As we will see, such relations are encoded in 2×2 connection matrices. Stokes phenomena are determined by the zeroes and poles of Q_0 . We mostly focus on polynomial potentials where the only pole is at infinity, while we can have simple or higher-order zeros named turning points. Let us consider the generic situation of a point in moduli space (parameters of the potential and the energy E) where Q_0 has only simple turning points. The complex z -plane is divided in sectors delimited by Stokes lines defined as

$$\text{Im} \left\{ \frac{1}{\hbar} \int_A^z \sqrt{Q_0(w)} dw \right\} = 0, \tag{2.20}$$

where A denotes a simple turning point of Q_0 . Stokes lines are denoted regular if they start at point A and end at infinity, and singular if they start at A and end at another turning point B .¹ Configurations with singular Stokes lines lead to ambiguities and should be avoided by a proper deformation. We will generally avoid singular Stokes lines by assigning a phase to \hbar .

The key objects in EWKB are the so called periods, which are integrals between two turning points A and B . The various turning points of the potential makes generally the plane z into a Riemann surface \mathcal{M} of genus g (depending on the potential). The contours around turning points are non-trivial cycles in $H^1(\mathcal{M}, \mathbb{C})$. The classical period is defined as

$$\Pi_{0,AB} = 2 \int_A^B \sqrt{Q_0(z)} dz = \oint_{\gamma} \sqrt{Q_0(z)} dz, \tag{2.21}$$

where γ is a cycle encircling the points A and B .² The quantum periods are defined as

$$\Pi_{AB} = s(\tilde{\Pi}_{AB}), \quad \tilde{\Pi}_{AB} = 2 \int_A^B \tilde{P}_{\text{even}}(z) dz = \oint_{\gamma} \tilde{P}_{\text{even}}(z) dz, \tag{2.22}$$

¹When Q_0 has poles (in addition to the one at infinity) more Stokes trajectories are possible. See e.g. [31] for a clear exposition.

²As it is evident from (2.21), a square root branch-cut emanates from a simple turning point, so the contour integral does not vanish.

where s denotes Borel resummation in the appropriate wedge of the complex plane where γ sits. For each period we define the Voros symbol

$$a_{AB} = e^{\frac{1}{\hbar}\Pi_{AB}} = a_{BA}^{-1}. \quad (2.23)$$

Sometimes we use the notation Π_γ and a_γ instead of Π_{AB} and a_{AB} . Note that the definition of periods is done unambiguously in the Riemann surface, but the labelling of periods through the turning points presupposes a choice of principal sheet to specify the orientation.

If the Stokes lines are all regular, all the quantum periods Π_{AB} are Borel resummable and well-defined.³ As we vary the phase of \hbar , or move in moduli space, singular Stokes lines can emerge and lead to a change of configurations of Stokes lines in the z -plane. Singular Stokes lines can be seen as periods themselves as they start and end at simple turning points. Let γ be the cycle associated to a singular Stokes line and α be the phase of the associated classical period:

$$\alpha \equiv \arg \Pi_{\gamma,0} = \arg \oint_\gamma \sqrt{Q_0(z)} dz. \quad (2.24)$$

According to (2.20), the period (2.24) can correspond to a (singular) Stokes line only for $\hbar = |\hbar|e^{i\alpha}$. We can deform from the singular configuration by changing the phase of \hbar as $\alpha \rightarrow \alpha \pm \epsilon$, with $0 < \epsilon \ll 1$. The Voros symbols a_λ of a cycle λ computed in the configurations α^- and α^+ are not the same, but related through the celebrated Dillinger-Delabaere-Pham (DDP) formula [12, 14]

$$s_{\alpha^-}(\tilde{a}_\lambda) = \prod_{\arg \Pi_{\gamma_i,0} = \alpha} \left(1 + s_{\alpha^+}(\tilde{a}_{\gamma_i}^{-1})\right)^{-\langle \gamma_i, \lambda \rangle} s_{\alpha^+}(\tilde{a}_\lambda). \quad (2.25)$$

In (2.25) the product runs over all possible singular cycles with associated classical period of phase α and $\langle \gamma_i, \lambda \rangle$ denotes the intersection number of the two cycles.⁴ If γ is a singular Stokes line at α , the opposite cycle is a singular Stokes line at $\pi + \alpha$:

$$\arg \Pi_{\gamma^{-1},0} = \arg \Pi_{\gamma,0} + \pi. \quad (2.27)$$

The jump of a_λ is different, because in the r.h.s. of (2.25) $\tilde{a}_{\gamma_i}^{-1} \rightarrow \tilde{a}_{\gamma_i}$ and $\langle \gamma_i, \lambda \rangle \rightarrow -\langle \gamma_i, \lambda \rangle$. Without loss of generality we can then restrict the range of α in (2.25) to $-\pi/2 < \alpha \leq \pi/2$. Note that the \tilde{a}_{γ_i} appear as purely non-perturbative corrections since $\tilde{a}_\gamma^{-1} \sim e^{-\frac{1}{\hbar}\Pi_{\gamma,0}}(1 + \dots)$ and $\frac{1}{\hbar}\Pi_{\gamma,0}$ is positive and real. Moreover, unless a singular Stokes cycle γ is non-trivially linked with another singular cycle, (2.25) implies that $s_{\alpha^+}(\tilde{a}_\gamma) = s_{\alpha^-}(\tilde{a}_\gamma)$, since $\langle \gamma, \gamma \rangle = 0$. We will make extensive use of (2.25) in this paper.

³If a given period happens to cross a regular Stokes line, we can decompose it in terms of products of periods defined in wedges without Stokes lines intersection.

⁴The intersection number is topological and thus rotational invariant, anti-symmetric between the two cycles and swaps sign if the orientation of a cycle is reversed. We take

$$\langle \rightarrow, \uparrow \rangle = +1, \quad \langle \leftarrow, \uparrow \rangle = -1, \quad \langle \rightarrow, \downarrow \rangle = -1, \quad \langle \leftarrow, \downarrow \rangle = +1. \quad (2.26)$$

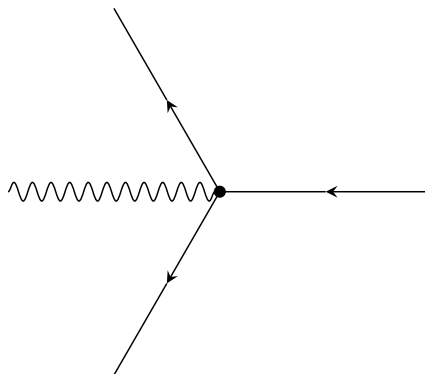


Figure 1. Stokes lines and their orientation for the Airy function. The wavy black line represents the branch-cut.

Whenever we cross a Stokes line, either ψ_+ or ψ_- undergoes a jump. The connection matrices for the wave-functions were found by Voros for a pure quartic potential [6]. His analysis has been further generalized and formalized in [11–14]. For simple turning points the full connection matrices can be split into “local” connection matrices which encode the Stokes automorphisms around a given simple turning point and the Voros symbols (1.2), which connect wave functions in “distant” regions, associated to different turning points. The local connection matrices around a simple turning point coincide with those given by the Airy differential equation, (A.1) with $q = 1$. See appendix A for the explicit computation. In the Airy case, the complex z -plane splits in three equal wedges, as depicted in figure 1. An arrow entering (exiting) the turning point corresponds to Stokes lines where ψ_+ (ψ_-) jumps. It is also convenient not to talk about the jump of the solutions ψ_{\pm} but of the jump of their coefficients. Namely we write

$$\psi^I = c_+^I \psi_+ + c_-^I \psi_-, \quad \psi^{\text{II}} = c_+^{\text{II}} \psi_+ + c_-^{\text{II}} \psi_-, \quad (2.28)$$

where ψ_{\pm} are the wave-functions in a chosen reference wedge.⁵ The local connection matrices as we pass a Stokes line or a branch-cut can be expressed as 2×2 matrices acting on the coefficients (c_+^N, c_-^N) as follows

$$c^{\text{II}} = A c^I, \quad c^N \equiv \begin{pmatrix} c_+^N \\ c_-^N \end{pmatrix}, \quad (2.29)$$

where A is any of the 2×2 matrices below (see figure 2):

$$S_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad S_- = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (2.30)$$

The relations (2.30) apply locally around an arbitrary simple turning point.

Voros symbols connect wave functions associated to different turning points. If A and B are two simple turning points, from (2.23) we have

$$\psi_{\pm}^{(B)}(z) = \psi_{\pm}^{(A)}(z) a_{AB}^{\mp \frac{1}{2}}, \quad (2.31)$$

⁵Unless specified otherwise, we take this reference wedge to be the one containing the asymptotic positive real axis, or its upper part, if the latter is on a Stokes line.

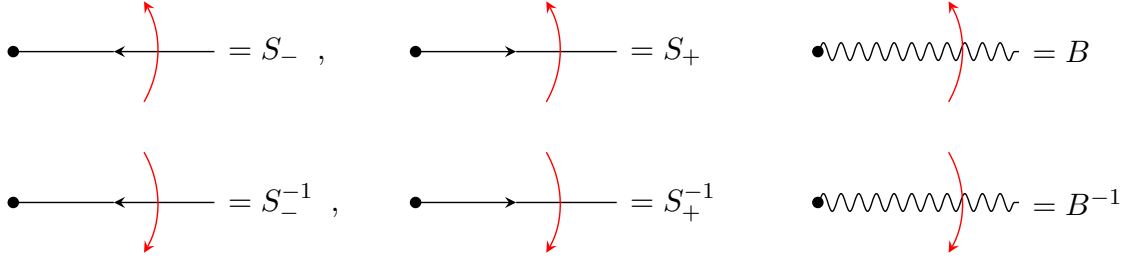


Figure 2. Local connection matrices as we pass an oriented Stokes line (straight lines with arrows) or a branch-cut (wavy lines) for a simple turning point.

where $\psi_{\pm}^{(A,B)}$ are the Borel resummations (in the appropriate wedge of the complex plane) of the formal wave functions (2.19) normalized at $z_0 = A, B$. It is useful to define the Voros connection matrix \mathcal{V}_{γ} as

$$\mathcal{V}_{\gamma} = \begin{pmatrix} a_{\gamma}^{\frac{1}{2}} & 0 \\ 0 & a_{\gamma}^{-\frac{1}{2}} \end{pmatrix}. \quad (2.32)$$

A similar relation is used to define the corresponding formal power series $\tilde{\mathcal{V}}_{\gamma}$.

2.3 Rules to determine EQCs

Exact quantization conditions are conditions that we impose on a combination of ψ_{\pm} in order to have a well-defined physical system. For bounded polynomial potentials with a stable discrete spectrum, the case considered in this paper, EQCs are determined by demanding that the wave functions are square integrable over the real axis.

Given a potential $Q(z)$ with only simple turning points, we start by determining all the Stokes lines using the definition (2.20). Possible singular Stokes lines are avoided by assigning a phase to \hbar . The determination of the exact wave functions $\psi_{\pm}(z)$, Borel resummation of the formal series (2.19), follow from the matrices (2.30) and (2.32). Let us denote by $\psi^{(P)} = c_+^{(P)}\psi_+ + c_-^{(P)}\psi_-$ and $\psi^{(N)} = c_+^{(N)}\psi_+ + c_-^{(N)}\psi_-$ the wave functions in the wedges containing the positive (P) and negative (N) real axis. If any of them is a Stokes line, we can infinitesimally shift the point above or below in the z -plane. The ending result does not depend on which shift we choose to make. We define $c^{(N)} = \mathcal{M}c^{(P)}$, where

$$\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad (2.33)$$

is the total monodromy matrix along a given path connecting a point in the asymptotic positive real axis and a point in the negative asymptotic real axis. For real potentials square integrability requires that $c_+^{(P)} = 0$, which determines the EQCs. Depending on the potential and in which Riemann sheet we end up when moving along the path, the requirement is

$$m_{12} = 0 \quad \text{or} \quad m_{22} = 0. \quad (2.34)$$

Since the coefficients m_{ij} are products of the matrices (2.30) and (2.32), we see that eventually EQCs depend on Voros symbols only. A good sanity check for the correctness of the

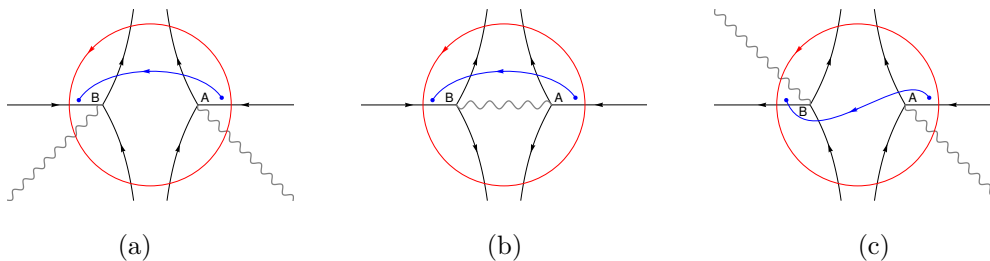


Figure 3. Three arbitrary choices of branch-cuts and connecting paths (blue lines) to determine EQCs in the harmonic oscillator. The total monodromy around the red circles are given in (2.36), the EQCs in (2.37).

procedure is to consider a closed path in the z -plane, in which case the connection matrix should reduce to the identity, given that wave functions are single-valued.

It is useful to consider a simple example, the harmonic oscillator. Setting $m = \omega = 1$, we have

$$Q_0 = z^2 - 2E, \tag{2.35}$$

where $E \geq 0$. For $E \neq 0$, we get two turning points at $A = \sqrt{2E}$ and $B = -\sqrt{2E}$. We report in figure 3 the decomposition of the z -plane in the different Stokes regions delimited by regular Stokes lines. Three possible choices of branch-cuts (wavy lines) and connection path (blue line) are shown.

As a first check, let us verify that the total monodromy is trivial. Starting from the region above the positive real axis in figure 3, we verify in the three cases (to be read from right to left as we move along the arrow)

$$\begin{aligned} (a) : & \quad S_- B S_- \mathcal{V}_\gamma^{-1} S_- B S_- S_+ \mathcal{V}_\gamma S_+ = I, \\ (b) : & \quad B^2 S_- S_+ \mathcal{V}_\gamma S_+ S_- S_+ \mathcal{V}_\gamma S_+ = I, \\ (c) : & \quad S_- B S_- \mathcal{V}_\gamma^{-1} S_- S_+ B S_+ \mathcal{V}_\gamma S_+ = I. \end{aligned} \tag{2.36}$$

We determine the EQC associated to the three blue paths in figure 3. In the region slightly above the positive real axis the normalizable wave function is ψ_- . So, we start with $c_+^P = 0$ and proceed along the blue curve towards the negative real axis, slightly above or below it. Demanding convergence of the wave function requires that $c_+^N = 0$ in cases (a) and (b), while in case (c) we require $c_-^N = 0$. We have

$$\begin{aligned} (a) : & \quad (S_+ \mathcal{V}_\gamma S_+)_{12} = 0, \\ (b) : & \quad (S_+ \mathcal{V}_\gamma S_+)_{12} = 0, \\ (c) : & \quad (S_-^{-1} \mathcal{V}_\gamma S_+)_{22} = 0. \end{aligned} \tag{2.37}$$

The EQCs above are all equivalent to

$$a_\gamma = -1. \tag{2.38}$$

At a first sight it seems that the computation of a_γ requires to consider the full expansion of P_{even} in \hbar , but, luckily enough, all terms but one vanish (after all, this is expected since

the harmonic oscillator is exactly solvable!). A simple way to show this is to blow up the contour γ to a parametrically large circle in the z -plane. Since $P_0 \propto z$, we see from the recursion relation (2.7) and the form of the first coefficients (2.9) that $P_{2n} \propto z^{1-4n}$ for large z . Hence all contributions, but P_0 , vanish. The integral over P_0 is elementary and gives the correct energy eigenvalues:

$$a_\gamma = e^{\frac{2i\pi E}{\hbar}} \implies E_n = \hbar \left(n + \frac{1}{2} \right), \quad n \in \mathbb{N}. \quad (2.39)$$

A few comments are in order.

1. The wave-function that jumps over the positive real axis is ψ_+ and this fixes the arrow in the Stokes line as in the Airy case. All other arrows follow. Two nearby Stokes lines emanating from a simple turning point have equal (opposite) orientation if a branch-cut (does not) divides them. Asymptotic Stokes lines in the same direction must have the same orientation.
2. Branch-cuts can be inserted between turning points or go at infinity. As far as periods are concerned both options are equivalent, since the branch-cuts are of square root type. For wave functions, however, this is not the case, because of the extra factor $\tilde{P}_{\text{even}}^{-1/2}$ in (2.19), which is not automatically taken into account when the branch-cuts are inserted between turning points. In the harmonic oscillator, $P_{\text{even}}^{-1/2} \sim z^{-1/2}$ for large z , and hence we have a total monodromy around infinity given by $B^2 = -I$. This explains the origin of the B^2 factor in case (b) of (2.36).
3. The (asymptotic) positive and negative real axis are Stokes lines. The orientation of the Stokes line in the negative real axis determines the EQC. We have $c_{12} = 0$ or $c_{22} = 0$ if ψ_+ or ψ_- respectively jump over the real negative axis. The same applies for more general potentials with real parameters for which the real axis is on Stokes lines.
4. Whenever the cycle between A and B does not have a branch-cut, the integral from A to B is opposite to the one from B to A , while it is equal if there is a branch-cut and the path circles through the opposite side of it. That is why in (2.36) we have \mathcal{V}_γ and \mathcal{V}_γ^{-1} in (a) and (c), but only \mathcal{V}_γ in (b).

We also have

$$S_\pm B = B S_\mp, \quad B \mathcal{V}_\gamma = \mathcal{V}_\gamma^{-1} B. \quad (2.40)$$

We will make extensive use of these rules to derive EQCs in the next sections.

As anticipated in the introduction, there are at least two ways to make use of EQCs in exact WKB to determine the energy spectrum E_n of the system:

1. We can directly determine E_n as those values of E for which (1.1) is satisfied. In this approach E is taken to be of $\mathcal{O}(1)$, and a_{γ_i} are determined from (2.22) and (1.2). Typically such E_n are found numerically.

2. We can consider the “downgraded” form

$$F[\tilde{a}_{\gamma_i}(E)] = 0, \tag{2.41}$$

where we undo the Borel resummation and Voros symbols in (1.1) are replaced by their formal power series. In (2.41) we then replace $E \rightarrow \tilde{E}_n$, where \tilde{E}_n are in general transseries which are obtained by demanding (2.41) order by order in \hbar and $\exp(-1/\hbar)$. The final energy spectrum is obtained by an appropriate Borel resummation of \tilde{E}_n : $E_n = s(\tilde{E}_n)$.

Since we are interested in the series expansion of energy eigenvalues, approach 2. is the one which will be mostly considered in this paper. However it is worth noting that there are systems, such as the anharmonic oscillator with no mass term, whose spectrum does not possess a transseries representation in \hbar , which makes approach 2. fail.⁶ Appendix C studies such a system in detail.

3 Anharmonic oscillators

In this section we analyze EQCs for anharmonic oscillators in some detail. We mostly focus on the quartic case, and then generalize to higher order anharmonic oscillators. We show that, at fixed moduli of the quartic anahamornic potential, there exist eight EQCs in different wedges in the \hbar complex plane.⁷ They include a “sweet spot” region where the EQC is particularly simple. We show how the Borel summability of the asymptotic series associated to the energy eigenvalues \tilde{E}_n follows from the existence of this “sweet spot” region in the limit where two simple turning points collapse to a double turning point. More specifically, we show how Borel summability of \tilde{E}_n can be established without making direct use of connection matrices for double turning points, as done in [13]. We then generalize our findings to higher order potentials.

3.1 The quartic anharmonic potential in the \hbar complex plane

We fix the moduli of the quartic anharmonic potential and take in (2.2)

$$V(z) = \frac{1}{2} (z^2 + z^4), \tag{3.1}$$

and E real and positive. For any $E > 0$ we have four simple turning points, two reals and two purely imaginary. We report in figure 4 the four phases of $-\pi/2 < \arg \hbar \leq \pi/2$ for which singular Stokes lines appear. Four mirror phases are present for $\pi/2 \leq |\arg \hbar| \leq \pi$, as given by (2.27). The turning points are labelled as in figure 4. We have the “perturbative

⁶By this we mean within ordinary perturbation theory. As we will see, approach 2 within EPT works also in these cases.

⁷Among other things, a classification of the different EQCs for the quartic oscillator already appeared in [33] (see in particular figure 5 there). The perspective in [33] was however a bit different as they were mostly interested on the ramifications of the energy eigenvalues as the (complex) moduli of the potential are varied. Here the moduli are held fixed and we classify the regions depending on the values of the energy and the phase of \hbar . The two analysis are however not totally independent.

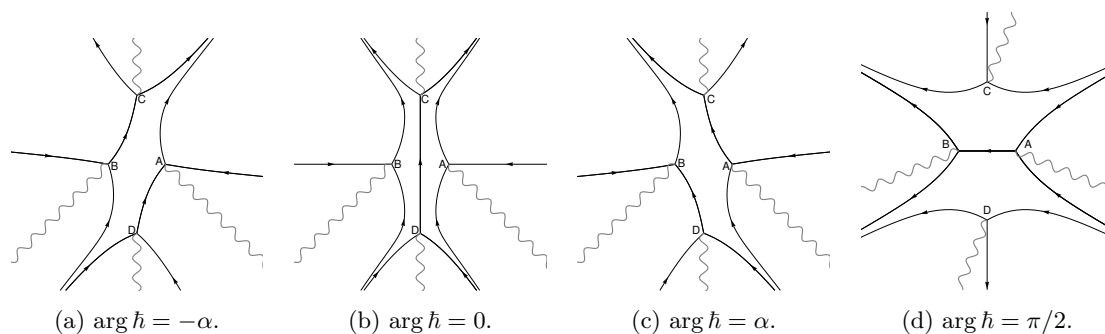


Figure 4. Phases of \hbar with $-\pi/2 < \arg \hbar \leq \pi/2$ for which critical Stokes lines appear in the anharmonic oscillator (3.1) with $E > 0$.

period” a_{AB} , the “non-perturbative period” a_{DC} , and four “diagonal” periods. Putting our branch cuts from the turning point to infinity as in figure 4, we have

$$\begin{aligned} \arg \Pi_{BA} &= \frac{\pi}{2}, & \arg \Pi_{CD} &= 0, \\ \arg \Pi_{CA} &= \arg \Pi_{BD} = -\arg \Pi_{AD} = -\arg \Pi_{CB} = \alpha(E). \end{aligned} \quad (3.2)$$

The angle $\alpha(E)$ will play an important role in our analysis. We do not report its full analytic expression, which is complicated. As far as our analysis is concerned, what really matters is its expansion for small $E > 0$, which reads

$$\alpha(E) = \arctan \left(\frac{\Pi_{0,BA}(E)}{i\Pi_{0,CD}(E)} \right) \approx \frac{3\pi}{8}E + O(E^2). \quad (3.3)$$

The key feature, which crucially relies on the mass term in the potential, is that $\alpha(E \rightarrow 0) = 0$.

The eight singular Stokes line configurations define corresponding eight wedges, or regions, in the complex \hbar plane, reported in figure 5. In each wedge, we can vary freely the argument of \hbar without any Stokes jump. In the upper half plane we label

- Region I: $0 < \arg \hbar < \alpha(E)$,
- Region II: $\alpha(E) < \arg \hbar < \frac{\pi}{2}$,
- Region III: $\frac{\pi}{2} < \arg \hbar < \pi - \alpha(E)$,
- Region IV: $\pi - \alpha(E) < \arg \hbar < \pi$.

The reflections of these regions in the lower half plane are labeled by bars, as in figure 5. It is useful to report the Stokes jumps of some relevant periods, derived using the DDP formula (2.25). The perturbative cycle is subject to the following jumps:

$$\begin{aligned} a_{AB}^{\text{II}} &= (1 + a_{AC}^{\text{I}})^{-1} (1 + a_{DB}^{\text{I}})^{-1} a_{AB}^{\text{I}}, \\ a_{AB}^{\text{I}} &= (1 + a_{DC}^{\text{I}})^{-2} a_{AB}^{\text{I}}, \\ a_{AB}^{\text{I}} &= (1 + a_{DA}^{\text{II}})^{-1} (1 + a_{BC}^{\text{II}})^{-1} a_{AB}^{\text{II}}. \end{aligned} \quad (3.4)$$

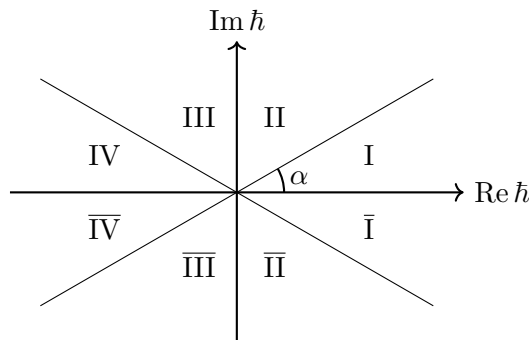


Figure 5. The eight regions delimited by singular Stokes lines in the complex \hbar plane for $E > 0$.

We also have that

$$\begin{aligned} a_{DB}^I &= (1 + a_{DC}^{\bar{I}})^{-1} a_{DB}^{\bar{I}}, \\ a_{AC}^I &= (1 + a_{DC}^{\bar{I}})^{-1} a_{AC}^{\bar{I}}, \end{aligned} \quad (3.5)$$

and

$$a_{DC}^{\text{III}} = (1 + a_{AB}^{\text{II}})^2 a_{DC}^{\text{II}}. \quad (3.6)$$

The Stokes jumps for other cycles can be obtained from the ones above by composition of periods and geometric properties of intersection numbers. Stokes jumps occurring in the regions with $\text{Re } \hbar < 0$ are determined using (2.27). For example, from the first relation in (3.4) we get

$$a_{AB}^{\bar{\text{III}}} = (1 + a_{CA}^{\bar{\text{IV}}})(1 + a_{BD}^{\bar{\text{IV}}}) a_{AB}^{\bar{\text{IV}}}, \quad (3.7)$$

and similarly for the other cases.

3.2 Exact quantization conditions

We can now use the techniques of section 2 to find the EQCs for the anharmonic oscillator. The associated connection paths in the z plane are reported as blue lines in figure 6 in the various regions. In regions II and $\bar{\text{II}}$ the EQCs are quite simple. We have

$$\begin{aligned} (S_+ \mathcal{V}_{AB} S_+ S_-)_{12} = 0 &\Rightarrow a_{AB}^{\text{II}} + 1 = 0, \\ (S_- S_+ \mathcal{V}_{AB} S_+)_{12} = 0 &\Rightarrow a_{AB}^{\bar{\text{II}}} + 1 = 0. \end{aligned} \quad (3.8)$$

This is similar to the purely perturbative quantization condition of the harmonic oscillator. In regions I and $\bar{\text{I}}$ we obtain more complicated expressions, namely

$$\begin{aligned} (S_+ \mathcal{V}_{DB} S_+ \mathcal{V}_{CD} S_-^{-1} \mathcal{V}_{AC} S_+)_{12} = 0 &\Rightarrow 1 + a_{AC}^I + a_{DB}^I + a_{AC}^I (1 + a_{CD}^I) a_{DB}^I = 0, \\ (S_+ \mathcal{V}_{CB} S_-^{-1} \mathcal{V}_{DC} S_+ \mathcal{V}_{AD} S_+)_{12} = 0 &\Rightarrow 1 + (1 + a_{AD}^{\bar{I}})(1 + a_{CB}^{\bar{I}}) a_{DC}^{\bar{I}} = 0. \end{aligned} \quad (3.9)$$

By geometrically composing the periods we can rewrite (3.9) as

$$\begin{aligned} 1 + (1 + a_{AC}^I)^{-1} (1 + a_{DB}^I)^{-1} a_{AB}^I &= 0, \\ 1 + (1 + a_{DA}^{\bar{I}}) (1 + a_{BC}^{\bar{I}}) a_{AB}^{\bar{I}} &= 0. \end{aligned} \quad (3.10)$$

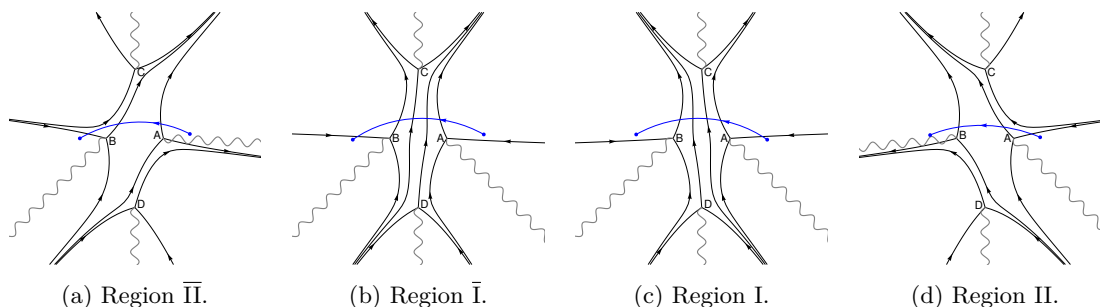


Figure 6. EQCs and connection paths (blue lines) for the anharmonic oscillator (3.1) with $E > 0$ for the four regions with $\text{Re } \hbar > 0$.

Alternatively, the EQCs can be written in terms of a choice of only two periods, though this results in a less compact form. In the form (3.10) it is manifest that the EQCs in region I/ \bar{I} are compatible with those in region II/ \bar{II} as long as one accounts for the Stokes jumps listed in (3.4). One can further check that they are also compatible with each other through (3.5).

We then note that the wedges II and \bar{II} are “sweet spot” regions where we get a simple EQC involving only the perturbative Voros symbol. This region is somehow hidden by the fact that, for any finite $E > 0$, the limit $\arg \hbar \rightarrow 0$ leads us to region I. The EQCs in this region include non-perturbative corrections, which might at first seem puzzling, due to the long known Borel summability of the \hbar expansion of the anharmonic potential [34]. If we determine the spectrum using EWKB as discussed in point 1. below (2.40), the exact spectrum is in fact determined by (3.10) where the non-perturbative cycles should be included.

On the other hand, if we want to relate EQCs with the asymptotic series of energy eigenvalues, we should consider the approach in point 2. below (2.40). We undo the Borel resummation implicit in the above EQCs, and turn them into formal power series equations, with

$$E \rightarrow \tilde{E} = \tilde{E}_P + \tilde{E}_{NP}, \quad \tilde{E}_P = e_0 + \sum_{k=1}^{\infty} e_k \hbar^k \quad (3.11)$$

and \tilde{E}_{NP} the non-perturbative transseries terms. Independently of the region we start in, the first result one obtains when asymptotically solving the EQCs is $e_0 = 0$, i.e. $\tilde{E}_P \sim O(\hbar)$. At fixed argument of \hbar , the limit $e_0 \rightarrow 0$ forces us to be in regions II/ \bar{II} /III/ \bar{III} , respectively, since $\alpha(0) = 0$. See figure 7 for an illustration of the complex \hbar plane in this limit. Thus, the EQCs with non-perturbative terms are never realized unless one takes some very unnatural limit. The absence of non-perturbative Voros symbols in the EQCs in the surviving regions implies that we can have solutions where $\tilde{E}_{NP} = 0$, i.e. perturbative asymptotic series with no associated transseries. In fact, for $e_0 \rightarrow 0$ the two turning points A and B collapse to a double turning point at $z_0 = 0$ and the EQC (3.8) applies in the entire \hbar complex plane with the exception of the real axis. However, as we will see, the EQCs derived before are *smooth* in the limit $e_0 \rightarrow 0$ and there is no need to derive EQCs where double turning points are present to start with, as done in [13].

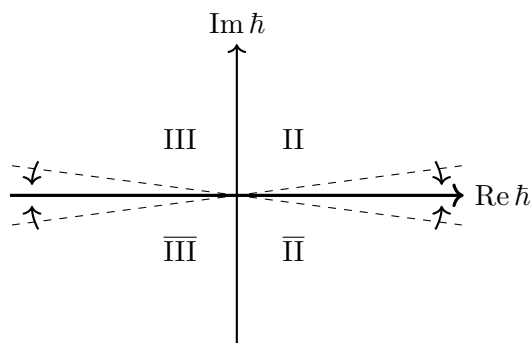


Figure 7. The regions of the complex \hbar plane in the limit $e_0 \rightarrow 0$.

We parametrise the Voros cycle a_{AB} as

$$a_{AB} \equiv e^{-2\pi i(t + \frac{1}{2})}. \quad (3.12)$$

The downgraded version of the EQC (3.8) for $e_0 \rightarrow 0$ reads then

$$\tilde{t}(\tilde{E}; \hbar) = n, \quad n \in \mathbb{N}. \quad (3.13)$$

Equation (3.13) allows us to determine the asymptotic series for the energy eigenvalues. Starting from finite E , we compute the period $\tilde{\Pi}_{AB}$ perturbatively, and then replace E as in (3.11). For the potential (3.1) we get⁸

$$\tilde{t} = \left(e_1 - \frac{1}{2} \right) + \hbar \left(e_2 - \frac{3}{16}(1 + 4e_1^2) \right) + \hbar^2 \left(e_3 - \frac{3}{2}e_2e_1 + \frac{5}{64}e_1(17 + 28e_1^2) \right) + \mathcal{O}(\hbar^3). \quad (3.14)$$

Demanding (3.13) order by order in \hbar fixes all energy coefficients e_k and allows us to determine all the asymptotic series \tilde{E}_n at once for any n :

$$\tilde{E}_n = \left(n + \frac{1}{2} \right) + \frac{3}{16} \left(1 + 4 \left(n + \frac{1}{2} \right)^2 \right) \hbar - \frac{1}{64} \left(n + \frac{1}{2} \right) \left(68 \left(n + \frac{1}{2} \right)^2 + 67 \right) \hbar^2 + \mathcal{O}(\hbar^3). \quad (3.15)$$

Since only the perturbative cycle \tilde{a}_{AB} is involved, no non-perturbative terms appear. Proving the Borel summability of the series \tilde{E}_n requires a further analysis of the $\hbar \in \mathbb{R}^+$ case, which is discussed next.

3.3 Borel summability

Borel summability of the energy eigenvalues \tilde{E}_n requires that

$$s_{0+}(\tilde{E}_n) = s_{0-}(\tilde{E}_n). \quad (3.16)$$

By means of (A.22) and (A.24), this is equivalent to show that

$$\dot{\Delta}_A \tilde{E}_n = 0, \quad \forall A \in \mathbb{R}^+, \quad (3.17)$$

⁸As shown in [13], by taking $e_0 = 0$ directly, the series (3.14) can also be determined by looking at the residue of the simple pole in $\tilde{P}(z)$ at $z = 0$.

where $\dot{\Delta}_A$ is the dotted alien derivative defined in (A.24). Applying $\dot{\Delta}_A$ to (3.13) and, using simple alien calculus rules,⁹ we have [13]

$$0 = \dot{\Delta}_A \tilde{t}(\tilde{E}_n(\hbar); \hbar) = \dot{\Delta}_A \tilde{t}(e_1; \hbar)|_{e_1=\tilde{E}_n} + (\dot{\Delta}_A \tilde{E}_n) \partial_{e_1} \tilde{t}(e_1; \hbar)|_{e_1=\tilde{E}_n}, \quad (3.18)$$

where $\tilde{t}(e_1; \hbar)$ is the asymptotic series (3.14) where all e_k but e_1 are set to zero. The condition $e_1 = \tilde{E}_n$ means replacing e_1 with the asymptotic series \tilde{E}_n , solution of (3.13), whose first order terms are given in (3.15). As explained before, for $e_0 \rightarrow 0$ regions II and $\bar{\text{II}}$ cover the whole cut complex plane. Since $\partial_{e_1} \tilde{t}(e_1; \hbar)$ can never vanish, establishing (3.17) is equivalent to establish $\dot{\Delta}_A \tilde{t} = 0$, namely

$$(s_{0+} - s_{0-}) \tilde{t}(e_1; \hbar)|_{e_1=\tilde{E}_n} = \lim_{e_0 \rightarrow 0} (t^{\text{II}}(e_0 + \hbar e_1; \hbar) - t^{\bar{\text{II}}}(e_0 + \hbar e_1; \hbar))|_{e_1=\tilde{E}_n} = 0. \quad (3.19)$$

The relation between t^{II} and $t^{\bar{\text{II}}}$ can be derived by combining the jumps at $\arg \hbar = -\alpha, 0, \alpha$ between regions II and $\bar{\text{II}}$ determined in the previous section. We get

$$a_{AB}^{\bar{\text{II}}} = \left(1 + a_{DA}^{\text{II}} \left(1 + a_{AB}^{\text{II}}\right)\right) \left(1 + a_{BC}^{\text{II}} \left(1 + a_{AB}^{\text{II}}\right)\right) a_{AB}^{\text{II}}. \quad (3.20)$$

Taking the logarithm, we write

$$\lim_{e_0 \rightarrow 0} (t^{\text{II}} - t^{\bar{\text{II}}}) = \lim_{e_0 \rightarrow 0} \frac{1}{2\pi i} \left\{ \log \left(1 + a_{DA}^{\text{II}} \left(1 + a_{AB}^{\text{II}}\right)\right) + \log \left(1 + a_{BC}^{\text{II}} \left(1 + a_{AB}^{\text{II}}\right)\right) \right\}, \quad (3.21)$$

Since the potential is even, $V(z) = V(-z)$, one can further simplify

$$\lim_{e_0 \rightarrow 0} (t^{\text{II}} - t^{\bar{\text{II}}}) = \lim_{e_0 \rightarrow 0} \frac{1}{\pi i} \log \left(1 + \sqrt{\frac{a_{DC}^{\text{II}}}{a_{AB}^{\text{II}}}} \left(1 + a_{AB}^{\text{II}}\right)\right). \quad (3.22)$$

When the EQC (3.8) is imposed, the right hand side of (3.22) vanishes unless a_{DC} is singular. The regularity of a_{DC} , for sufficiently small \hbar , has been formally proven in [14]. It is however useful to work out \tilde{a}_{DC} to one loop level. It is convenient to determine a_{DC} in terms of the perturbative cycle a_{AB} , which amounts to changing variables from E to t . To leading order, inverting the relation E as a function of t is merely

$$\tilde{E}(t; \hbar) = \mu \hbar + O(\hbar^2), \quad \mu \equiv t + \frac{1}{2}. \quad (3.23)$$

Assuming finite E , we calculate first the series for the non-perturbative period.¹⁰ Then we replace E by (3.23) to find

$$\begin{aligned} \tilde{\Pi}_{DC}(E(t; \hbar); \hbar) &= \left(-\frac{4}{3} + 2\mu \hbar \log\left(\frac{\hbar}{8}\right) + 2\mu \hbar (\log(\mu) - 1) + O(\hbar^2)\right) \\ &+ \left(-\frac{\hbar}{12\mu} + O(\hbar^2)\right) + \left(\frac{7\hbar}{1440\mu^3} + O(\hbar^2)\right) + \left(-\frac{31\hbar}{20160\mu^5} + O(\hbar^2)\right) + \dots \end{aligned} \quad (3.24)$$

⁹For a physics oriented introduction to alien calculus see e.g. the pedagogical reviews [35, 36], while for a more mathematics oriented presentation see e.g. [37].

¹⁰This series can be obtained to high order using, for example, the differential operator method introduced in [38], see e.g. appendix C of [39] for a concrete implementation in a similar case.

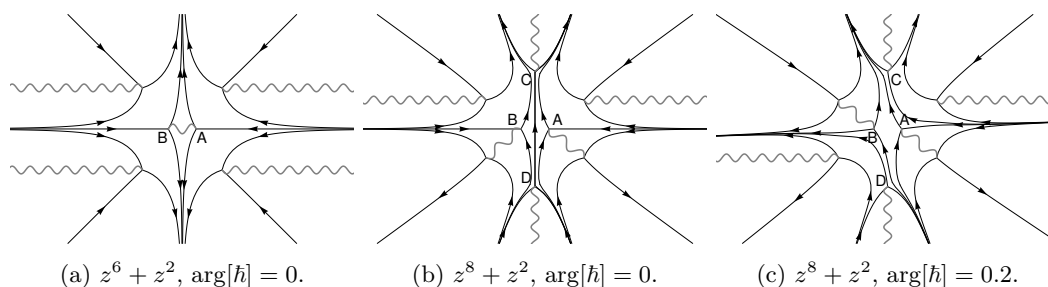


Figure 8. Stokes graphs for higher order anharmonic potentials when E is small.

We note that, in terms of μ , all orders in \hbar of $\tilde{\Pi}_{DC}(E; \hbar)$ contribute at 1-loop order. The resummation of these terms leads to¹¹

$$\tilde{a}_{DC}^{\text{II}}(t; \hbar) = e^{-\frac{4}{3\hbar}} \left[\frac{\Gamma(t+1)}{\sqrt{2\pi}} \right]^2 \left(\frac{8}{\hbar} \right)^{-2(t+\frac{1}{2})} (1 + O(\hbar)). \quad (3.25)$$

Using (3.25), we can expand the “symbol” controlling the jump at $\arg \hbar = 0$ in (3.22). We find

$$\sqrt{\frac{\tilde{a}_{DC}^{\text{II}}}{\tilde{a}_{AB}^{\text{II}}}} (1 + \tilde{a}_{AB}^{\text{II}}) = e^{-\frac{2}{3\hbar}} \frac{\sqrt{2\pi}}{\Gamma(-t)} \left(\frac{8}{\hbar} \right)^{-(t+\frac{1}{2})} (1 + O(\hbar)). \quad (3.26)$$

Theorem 4.1.1 and Lemma 4.1.3 of [14] imply that the resummation of the formal power series given by the $(1 + O(\hbar))$ terms in (3.26) gives rise to a bounded function in \hbar for sufficiently small \hbar , regular when t is a positive integer. The results of [14] apply close to a double turning point of general potentials, in particular generic polynomial potentials. In this way we have shown that the right hand side of (3.22) vanish when the EQCs are imposed and (3.17) is verified. Borel summability of the energy eigenvalues is then proven for sufficiently small \hbar . Our proof has the same structure of the one in [13] but uses only single turning point techniques. Equation (3.26) corresponds to what is denoted $a^{[L]}$ in [13].

3.4 Higher order potentials

The considerations made for the quartic anharmonic potentials are easily generalizable to higher order potentials. As long as the potential is bounded with a single global minimum z_0 there exist regions where the EQCs read as (3.8), with a_{AB} the perturbative cycle. If we normalize the potential such that $V(z_0) = 0$, the “sweet spot” is found in the region where $\text{Arg } \hbar \gtrsim ce_0$ for some model dependent constant c . For concreteness, we discuss the class of anharmonic potentials of the form

$$V(z) = \frac{1}{2} (z^2 + z^{2q}), \quad q > 2. \quad (3.27)$$

The analysis is different for $q = \text{even}$ and $q = \text{odd}$. For $q = \text{odd}$ the potentials (3.27) cannot have purely imaginary turning points which lead to singular Stokes line crossing the perturbative cycle a_{AB} at the origin $z_0 = 0$. For sufficiently small e_0 and real \hbar , in fact

¹¹This value can also be obtained by working directly with double turning points, as reviewed in appendix B.

no singular Stokes line crosses the cycle a_{AB} . Furthermore, there are no $\arg \hbar = 0$ periods connecting to a_{AB} and thus no Stokes jump. We are then automatically in region II where EQCs read as (3.8) and Borel summability of a_{AB} is guaranteed. See figure 8, panel (a), for an illustration of the Stokes line configurations for $q = 3$.

For $q = \text{even}$ the analysis is similar to the one of the quartic oscillator. For real \hbar , the non-perturbative cycle a_{DC} turns into a singular Stokes line crossing a_{AB} . See figure 8, panel (b), for the $q = 4$ case. The complex \hbar plane splits in several wedges with different EQCs. Crucially, we still have a region II like in the quartic case where the EQC reads (3.8). This region is adjacent to region I as in figure 5, while the additional regions appear between regions II and III in figure 5. This implies that the combined Stokes jumps between regions $\bar{\text{II}}$ and II is as in section 3.3, and (3.22) applies. The specific form of the series for the period $\tilde{\Pi}_{DC}$ changes, but we still have

$$\sqrt{\frac{\tilde{a}_{DC}^{\text{II}}}{\tilde{a}_{AB}^{\text{II}}}} (1 + \tilde{a}_{AB}^{\text{II}}) = e^{-\frac{c_1}{\hbar}} \frac{\sqrt{2\pi}}{\Gamma(-t)} \left(\frac{c_2}{\hbar}\right)^{-(t+\frac{1}{2})} (1 + O(\hbar)) , \quad (3.28)$$

where c_1 and c_2 are two model-dependent parameters. They are entirely determined from the leading and next to leading terms of the classical period. For the potentials (3.27), we have

$$c_1 = \frac{\sqrt{\pi}}{q+1} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{q-1}\right)} , \quad c_2 = 2^{\frac{q+1}{q-1}} . \quad (3.29)$$

Note that (3.28) applies for any potential with a unique global minimum, not only for those of the kind (3.27). The universality of the factor

$$\frac{\sqrt{2\pi}}{\Gamma(-t)} \hbar^{t+\frac{1}{2}} \quad (3.30)$$

is easily established in the strict double turning point limit $e_0 = 0$, where the factor (3.30) appears in connection matrices of certain Stokes jump in double turning points [13, 14]. See appendix B for its derivation. The right hand side of (3.22) then vanishes when the EQCs are imposed and (3.17) is verified. The analysis can be generalized for an arbitrary polynomial potential with a unique global minimum. Borel summability of the energy eigenvalues is then proven for sufficiently small \hbar .

4 How EPT Borel summability emerges from EWKB

In this section we show how to get Borel resumable series for energy eigenvalues of a general polynomial potential even in the cases where we would expect a transseries expansion. We briefly review in section 4.1 the results of [29, 30] and then show how the “sweet spot” regions for the anharmonic oscillators (3.27) allows us to reproduce the findings of [29, 30] in EWKB. We finally discuss the leading large order behaviour of the EPT asymptotic series.

4.1 Exact perturbation theory (EPT)

It has been shown in [29, 30] that for an arbitrary one-dimensional bounded polynomial potential V we can define a perturbative expansion such that all energy eigenvalues (and other observables as well) are reconstructable from a single Borel resumable perturbative series, dubbed EPT in [29, 30]. The idea is extremely simple and powerful at the same time. Let

$$V = V_0 + \Delta V \quad (4.1)$$

be the sum of two potentials V_0 and ΔV such that V_0 has a single non-degenerate minimum and

$$\lim_{|x| \rightarrow \infty} \frac{\Delta V}{V_0} = 0. \quad (4.2)$$

We take $\min(V) = \min(V_0) = 0$. Consider then the auxiliary potential

$$V_{\text{EPT}} = V_0 + \frac{\hbar}{\hbar_0} \Delta V \equiv V_0 + \hbar V_1, \quad (4.3)$$

where \hbar_0 is an arbitrary positive constant. Using path integral methods and Lefschetz thimbles it has been shown in [30] that the \hbar expansion in the deformed quantum mechanical model with potential V_{EPT} , at fixed \hbar_0 , is Borel resumable. Note that the condition (4.2) guarantees that the asymptotic behaviour of the wave function in the deformed model is the same as that in the original model. Let $\tilde{E}_n^{\text{EPT}}(\hbar, \hbar_0)$ be the EPT asymptotic series of the energy eigenvalues of the deformed model. Then the original energy eigenvalues E_n can be computed as

$$E_n(\hbar) = s_0 \left(\tilde{E}_n^{\text{EPT}}(\hbar, \hbar_0) \right)_{\hbar_0 = \hbar}. \quad (4.4)$$

The decomposition (4.1) is generally far from being unique. EPT is defined as an expansion around the minimum of V_0 , which does not need to be a minimum of the original potential V . The number of interaction terms present in EPT depends on the particular decomposition performed. Any choice of EPT is theoretically equivalent to any other, though in numerical computations with truncated series some choices might lead to more accurate results than others. EPT can also be used when the ordinary \hbar expansion is Borel resumable, in order to improve the efficiency of the resummation at strong coupling. Its more dramatic consequences however apply for quantum mechanical systems where, due to instanton configurations, energy eigenvalues are expected to be given by the Borel resummation of a transseries in \hbar , $\exp(-1/\hbar)$ and possibly $\log \hbar$.

The numerical efficacy of EPT has been extensively discussed in [29, 30] (see also [40], and [41] for an application in 2d QFT) and will not be further considered in this paper, where we focus more on analytical aspects. For this reason, we will consider a specific decomposition (4.3) for concreteness. Given an arbitrary polynomial potential V of the form $V = \sum_{i=2}^{2q} v_i x^i$, normalized so that $v_{2q} = 1/2$, we take V_0 as in (3.27):

$$V_0(z) = \frac{1}{2} \left(z^2 + z^{2q} \right), \quad q > 1. \quad (4.5)$$

In order to see how (4.4) follows from an EWKB analysis requires a slight generalization of the results of section 3 to quantum deformed potentials V_1 . Note that all possible moduli of the original potential V are encoded in V_1 . The analysis in this section extends to non-polynomial V_1 as long it is polynomially bounded at infinity and regular on a infinitesimal neighbourhood of the interval $[-i, i]$. One such example is $z/\cosh(z)$. This naturally extends the applicability of EPT by broadening the class of original potentials V .

4.2 Anharmonic oscillators in the presence of a quantum potential

We assume that the quantum potential is a generic polynomial of degree strictly less than $2q$, so that condition (4.2) is satisfied. Since the pattern of Stokes lines and jumps is determined from the classical potential, the analysis made in section 3 applies. In particular the EQC in region II is still given by (3.8). The discussion of Borel summability also carries over almost entirely from 3. The quantum periods are however affected by the quantum deformed potentials and hence we should verify that (3.21) vanishes when the EQC is imposed.

For even quantum potentials, at one-loop (3.21) simplifies to (3.22), so we should determine the leading order form of $\tilde{a}_{DC}^{\text{II}}$ only. We get

$$\tilde{a}_{DC}^{\text{II}}(t; \hbar) = e^{-\frac{2c_1}{\hbar}} e^{2c_3} \left[\frac{\Gamma(t+1)}{\sqrt{2\pi}} \right]^2 \left(\frac{c_2}{\hbar} \right)^{-2(t+\frac{1}{2})} (1 + O(\hbar)), \quad (4.6)$$

where c_1 and c_2 are given by (3.29) and are unaffected by V_1 , while the new real parameter c_3 in (4.6) reads

$$c_3 = \frac{1}{2} \lim_{\epsilon_0 \rightarrow 0} \oint_{\gamma_{DC}} P_{\text{even},1}(z) dz = i \int_{-1}^1 \frac{V_1(it)}{\sqrt{Q_0(it)}} \Big|_{E=0} dt, \quad (4.7)$$

and is governed by V_1 . For an arbitrary polynomial potential V_1 , the integral in (4.7) is finite and $|c_3|$ is bounded. Hence Borel summability is not affected. For non-even quantum potentials, one obtains instead (3.21). A calculation similar to the one leading to (3.25) applies for the diagonal periods, such as

$$\tilde{a}_{DA}^{\text{II}}(t; \hbar) \approx e^{i\pi(t+\frac{1}{2})} e^{-\frac{c_1}{\hbar}} e^{c_3 - i\pi\vartheta} \frac{\Gamma(t+1)}{\sqrt{2\pi}} \left(\frac{c_2}{\hbar} \right)^{-(t+\frac{1}{2})} (1 + O(\hbar)). \quad (4.8)$$

Here ϑ stands for an additional contribution from non-even quantum potentials,

$$\vartheta = \frac{1}{2\pi i} \lim_{E \rightarrow 0} \oint_{\gamma_{AD}} \{P_{\text{even},1}(z) - P_{\text{even},1}(-z)\} dz = \frac{1}{\pi} \int_0^1 \frac{V_1(it) - V_1(-it)}{\sqrt{Q_0(it)}} \Big|_{E=0} dt, \quad (4.9)$$

where $P_{\text{even},1}$ is the next to leading coefficient of \tilde{P}_{even} as defined in (2.16) More generally, we can use the symmetries of the problem to decompose the diagonal periods as

$$\begin{aligned} \tilde{a}_{CA} &= \tilde{a}_{DC}^{-\frac{1}{2}} \tilde{a}_{AB}^{-\frac{1}{2}} e^{-i\pi\tilde{\vartheta}(h)}, & \tilde{a}_{BD} &= \tilde{a}_{DC}^{-\frac{1}{2}} \tilde{a}_{AB}^{-\frac{1}{2}} e^{i\pi\tilde{\vartheta}(h)}, \\ \tilde{a}_{DA} &= \tilde{a}_{DC}^{\frac{1}{2}} \tilde{a}_{AB}^{-\frac{1}{2}} e^{-i\pi\tilde{\vartheta}(h)}, & \tilde{a}_{BC} &= \tilde{a}_{DC}^{\frac{1}{2}} \tilde{a}_{AB}^{-\frac{1}{2}} e^{i\pi\tilde{\vartheta}(h)}, \end{aligned} \quad (4.10)$$

where

$$\tilde{\vartheta}(\hbar) = \frac{1}{2\pi i \hbar} \lim_{E \rightarrow 0} \int_{AD} \left\{ \tilde{P}_{\text{even}}(z) - \tilde{P}_{\text{even}}(-z) \right\} dz \sim \vartheta + \mathcal{O}(\hbar). \quad (4.11)$$

It can be shown that $\tilde{\vartheta}(\hbar)$ is real to all orders. Putting everything together, (3.21) gives rise to

$$s_0^{-1} \lim_{e_0 \rightarrow 0} (t^{\text{II}} - t^{\overline{\text{II}}}) = \frac{1}{2\pi i} \left\{ \log \left(1 + e^{-\frac{c_1}{\hbar}} e^{c_3 - i\pi\vartheta} \frac{\sqrt{2\pi}}{\Gamma(-t)} \left(\frac{c_2}{\hbar} \right)^{-(t+\frac{1}{2})} (1 + \mathcal{O}(\hbar)) \right) + \text{h.c.} \right\}, \quad (4.12)$$

and vanishes when EQC is imposed, $t = n$. It then follows from (4.12) that

$$\dot{\Delta}_A \tilde{E}_n^{\text{EPT}} = 0, \quad \forall A \in \mathbb{R}^+, \quad (4.13)$$

and hence Borel summability of the EPT series of energy eigenvalues for sufficiently small \hbar is established.

4.3 Large order behaviour for quartic oscillators with quantum potentials

We discuss in this section the large order behavior of the coefficient terms of the EPT asymptotic series $\tilde{E}_n^{\text{EPT}} = \sum_k e_{k,n}^{\text{EPT}} \hbar^k$. For simplicity we focus on quartic oscillators. In absence of a quantum potential V_1 , the large order behaviour associated to the potential (3.1) has been famously found long ago by Bender-Wu [2] and more rigorously established using alien calculus in [13]. As well-known, all Borel singularities are along the negative real axis. We extend here the analysis of [13] to include a broad class of quantum potentials, so that the large order behaviour of \tilde{E}_n^{EPT} follows.

Like in the analysis of Borel summability, we need to study in more detail what happens to the non-perturbative periods after applying the EQCs. To obtain the large order behavior in the $e_0 \rightarrow 0$ limit we must compare regions III and $\overline{\text{III}}$, which in the vanishing limit correspond respectively to $s_{\pi-}$ and $s_{\pi+}$. Using (2.25), we derive

$$a_{AB}^{\overline{\text{III}}} = \left(1 + (1 + a_{AB}^{\text{III}}) a_{CA}^{\text{III}} \right) \left(1 + (1 + a_{AB}^{\text{III}}) a_{BD}^{\text{III}} \right) a_{AB}^{\text{III}}. \quad (4.14)$$

Unlike the jump for $\arg \hbar = 0$ studied in the previous sections where all terms were regular, here a_{CA}^{III} is in fact singular when we replace t by n (equivalently, E_1 by \tilde{E}_n). To see why, consider the jump at $\arg \hbar = \pi/2$,

$$a_{CA}^{\text{III}} = \left(1 + a_{AB}^{\text{II}} \right)^{-1} a_{CA}^{\text{II}}. \quad (4.15)$$

While it is a non-perturbative term when expressed in terms of $(E; \hbar)$, it becomes an overall coefficient for series parametrized by t . After changing variables in (4.15) from E to $t = t^{\text{II}}$ and then expanding at small \hbar (i.e. “downgrading” it), we find

$$\tilde{a}_{CA}^{\text{III}}(E(t; \hbar); \hbar) = \left(1 - e^{-2\pi i t} \right)^{-1} \tilde{a}_{CA}^{\text{II}}(E(t; \hbar); \hbar). \quad (4.16)$$

Notice that, due to the change variables from E to t , “perturbative” and “non-perturbative” are rearranged and the asymptotic series in the different regions differ. In the $e_0 \rightarrow 0$ limit

we obtain, to leading non-perturbative order,

$$\begin{aligned}
 s_\pi^{-1} \lim_{\epsilon_0 \rightarrow 0} (t^{\overline{\text{III}}} - t^{\text{III}}) &= \frac{1}{2\pi i} \log \left((1 + \tilde{a}_{CA}^{\text{II}}) (1 + \tilde{a}_{BD}^{\text{II}}) \right) \\
 &= \cos(\pi\vartheta) e^{\frac{2}{3\hbar}} \frac{\sqrt{2\pi}}{\pi i \Gamma(t+1)} \left(\frac{8e^{i\pi}}{\hbar} \right)^{(t+\frac{1}{2})} e^{-c_3} (1 + O(\hbar)) + \dots,
 \end{aligned} \tag{4.17}$$

from which we can extract the large order behavior:¹²

$$e_{k,n}^{\text{EPT}} \sim (-1)^{k+1} e^{-c_3} \cos(\pi\vartheta) \frac{2^{3n+1} \Gamma(k+n+\frac{1}{2})}{\pi^{3/2} \Gamma(n+1)} \left(\frac{3}{2} \right)^{k+n+\frac{1}{2}}. \tag{4.18}$$

In the absence of a quantum potential, formula (4.17) matches the equivalent formula in [13], where it corresponds to $a^{\mathcal{L}}$ in the notation therein, and (4.18) becomes the Bender-Wu formula.

The quantum potential affects the leading large order behaviour only in modifying the overall constant factor through the coefficients c_3 and θ . Non-perturbative effects in EPT are hidden in the negative real axis of the Borel plane and are essentially encoded in c_3 and θ , as well as higher order corrections, which in general depend on \hbar_0 . An illustrative example of the above formula is the EPT model:

$$V_0(z) = \frac{1}{2} (z^2 + z^4), \quad V_1(z) = az - \frac{z^2}{\hbar_0}. \tag{4.19}$$

This model has the anharmonic oscillator as its classical “base”, but can be used to reproduce some interesting test cases. For $a = 0$, at $\hbar_0 = \hbar$ it reduces to the symmetric double well, while at $\hbar_0 = 2\hbar$ it becomes the pure quartic potential. Meanwhile if $a = 1/2$ and $\hbar_0 = \hbar$ it becomes the *supersymmetric* double well. The perturbative series for the energy levels is given by

$$\begin{aligned}
 \tilde{E}_n^{\text{EPT}} &= \left(n + \frac{1}{2} \right) \hbar + \left(-\frac{a^2}{2} - \frac{(n+\frac{1}{2})}{\hbar_0} + \frac{3}{8}(2n(n+1)+1) \right) \hbar^2 \\
 &- \left((2n+1) (-48a^2 + 17n(n+1) + 21) + \frac{8}{\hbar_0} (4a^2 - 6n(n+1) - 3) + \frac{16n+8}{32\hbar_0^2} \right) \hbar^3 + \dots.
 \end{aligned} \tag{4.20}$$

In the EPT model (4.19) the coefficients c_3 and θ read

$$e^{-c_3} = e^{-\frac{2}{\hbar_0}}, \quad \vartheta = a. \tag{4.21}$$

We explicitly see that when we set $\hbar_0 = \hbar$, the leading singularity at $2/3$ becomes multiplied by the non-perturbative factor $e^{-2/\hbar}$ in the double-well potential and the quartic oscillator.

¹²This is obtained by expanding both the Stokes automorphism in the left hand side of (4.17) in terms of dotted alien derivatives using (A.24) and the log in the right hand side of (4.17), matching terms order by order in $\exp(2p/(3\hbar))$, for $p \in \mathbb{N}$. We determine in this way $\hat{\Delta}_A \tilde{E}_n^{\text{EPT}}$ for $A = -2p/3 \in \mathbb{R}^-$ and then the large order behaviour (4.18) (as well as higher order terms) follows from standard resurgence relations (see e.g. [35, 36]).

As for the supersymmetric potential, we see that even at the level of EPT the value of $a = 1/2$ is special, since it puts the leading singularity further away. The next contribution to the large behavior is

$$e_{k,n}^{\text{EPT}} \approx (-1)^{k+1} e^{-\frac{4}{\hbar_0}} \frac{4^{3n+1}}{\pi} \frac{\Gamma(k+2n+1)}{\Gamma(n+1)^2} \left(\frac{3}{4}\right)^{k+2n+1}, \quad (4.22)$$

which follows from (4.17). Consequently, the EPT perturbative series never vanishes, in contrast with the ordinary perturbative series in supersymmetric quantum mechanics. This is expected, since the EPT perturbative series must capture the non-perturbative effects that give rise to $E_{n=0} > 0$ in the exact case.¹³

5 Conclusions

We have shown in this paper how the results of [29, 30] about EPT, obtained using path integral and steepest-descent considerations, nicely fit with resurgence and EWKB methods, providing an alternative (and more rigorous) proof of the validity of EPT which applies to all energy eigenvalues at once. EPT in [29, 30] is based on the existence of deformations for which the Lefschetz thimble decomposition of the path integral trivializes and correspondingly observables can be expressed in terms of a single Borel resumable asymptotic series. In EWKB, at fixed moduli of the potential, EQCs depend in particular on E and $\arg \hbar$. In theories where instanton contributions occur, there is no way to get a EQC of the form (1.3). Non-perturbative cycles necessarily enter the EQCs and in particular take into account of the instanton contributions. The expansion of an observable in \hbar in this case is generally given by a transseries. However, the very same deformations above, provided E is carefully chosen, are able to lead to a trivialization of the EQC and to a single Borel resumable asymptotic series for energy eigenvalues.

We have focused in this paper to polynomial potentials and it would be interesting to extend our results to more general potentials. It would also be interesting to see if and to what extent one can reconstruct the transseries expression for an observable (in terms of the undeformed model) by unzipping its associated EPT series. As we mentioned in the introduction, connections between EWKB and several other theoretical subjects have been worked out. It would be very interesting to see if EPT in EWKB can be extended in this more general context, in particular in the context of 4d $\mathcal{N} = 2$ gauge theories.

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¹³ $E_{n=0}$ is the exact ground state energy and it should not be confused with e_0 defined in (3.11).

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A Connection matrices for monomial potentials

We compute in this appendix the connection matrices for wave functions as they cross a regular Stokes line. For simple turning points this connection matrix is given by the Airy function case with $Q(z) = z$. Since it is straightforward to generalize to pure monomial potentials, we report the connection matrices for potentials $Q(z) = z^q$, with $q \in \mathbb{N}$. It should be stressed that such general connection matrices with $q > 1$ are not directly useful to determine the connection matrices for higher order turning points in a *generic* potential. Determining such matrices is in fact a non-trivial task.

The differential equation reads

$$\hbar^2 \psi''(z) = z^q \psi(z), \quad q \in \mathbb{N}. \quad (\text{A.1})$$

The leading terms of the expansion are given by a WKB approximation:

$$\tilde{\psi}_\pm \approx z^{-\frac{q}{4}} e^{\pm \frac{z^\alpha}{\alpha \hbar}}, \quad (\text{A.2})$$

where

$$\alpha = \frac{q+2}{2}. \quad (\text{A.3})$$

We look for asymptotic solutions of the form

$$\tilde{\psi}_\pm = z^{-\frac{q}{4}} e^{\pm \frac{z^\alpha}{\alpha \hbar}} \tilde{\phi}_\pm, \quad \tilde{\phi}_\pm = \sum_{n=0}^{\infty} c_n^\pm(z) \hbar^n, \quad (\text{A.4})$$

where $c_0(z) = 1$. We have

$$c_n^\pm(z) = d_n^\pm z^{-n\alpha}, \quad (\text{A.5})$$

and after some algebra we get the recursion relation satisfied by d_n^\pm :

$$d_{n+1}^\pm = \frac{\pm d_n^\pm}{(n+1)(q+2)} \left[\frac{q}{4} \left(\frac{q}{4} + 1 \right) + n(n+1) \left(\frac{q}{2} + 1 \right)^2 \right], \quad (\text{A.6})$$

whose solution is

$$d_n^\pm = \sin \left(\frac{\pi q}{2q+4} \right) \left(\pm \frac{q+2}{4} \right)^n \frac{\Gamma \left(n + \frac{q}{2(q+2)} \right) \Gamma \left(n + \frac{q+4}{2(q+2)} \right)}{\pi \Gamma(n+1)}. \quad (\text{A.7})$$

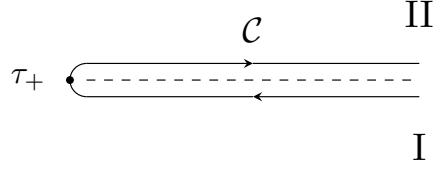


Figure 9. Hankel contour passing through the singular point τ_+ . The dashed line delimits the two adjacent regions II and I in the complex z -plane.

The Borel transforms of $\tilde{\phi}_\pm$ read

$$\hat{\phi}_\pm(\tau) = {}_2F_1\left(\frac{q}{4+2q}, \frac{4+q}{4+2q}, 1; \pm \frac{q+2}{4}\tau\right), \quad \tau = tz^{-\alpha}. \quad (\text{A.8})$$

The actual expansion parameter of the formal series in (A.4) is $\hbar z^{-\alpha}$ and correspondingly τ in (A.8) is the associated Borel variable. The singularities occur at $\tau_+ = 4/(q+2)$ for $\hat{\phi}_+(\tau)$ and $\tau_- = -4/(q+2)$ for $\hat{\phi}_-(\tau)$. If $z = |z|\exp(i\theta)$, with \hbar and t real and positive, Stokes lines occur for

$$\theta = \frac{2\pi}{q+2}n, \quad n = 0, 1, \dots, q+1, \quad (\text{A.9})$$

and the complex z -plane splits in $q+2$ wedges with opening angle $2\pi/(q+2)$. The Borel resummation in a generic wedge X is defined as

$$\phi_\pm^X = s_\theta(\hat{\phi}_\pm), \quad z \in X. \quad (\text{A.10})$$

The expansion of $\hat{\phi}_\pm$ around τ_\pm equals

$$\hat{\phi}_\pm|_{\tau=\tau_\pm} = -\frac{\eta_q}{2\pi} \log(\tau - \tau_\pm) \hat{\phi}_\mp|_{\tau=0} + \text{reg}, \quad \eta_q = 2 \cos\left(\frac{\pi}{q+2}\right), \quad (\text{A.11})$$

where reg stands for analytic terms in a neighborhood of τ_\pm . Let ϕ_\pm^I and ϕ_\pm^{II} be the Laplace transforms of $\hat{\phi}_\pm$ in two adjacent wedges separated by a Stokes line for $\hat{\phi}_+$. We have

$$\begin{aligned} \phi_+^{II}(z) - \phi_+^I(z) &= \left(-\frac{\eta_q}{2\pi\hbar}\right) \int_{\mathcal{C}} e^{-\frac{t}{\hbar}} \log(\tau - \tau_+) \hat{\phi}_-(\tau - \tau_+) \\ &= \frac{i\eta_q}{\hbar} \int_{\tau_+}^{\infty} dt e^{-\frac{t}{\hbar}} \hat{\phi}_-(\tau - \tau_+) = e^{-\frac{2z\alpha}{\alpha\hbar}} \frac{i\eta_q}{\hbar} \int_0^{\infty} dt e^{-\frac{t}{\hbar}} \hat{\phi}_-(\tau), \end{aligned} \quad (\text{A.12})$$

where \mathcal{C} is the Hankel contour shown in figure 9. Hence

$$\psi_+^{II} - \psi_+^I = i\eta_q z^{-\frac{q}{4}} e^{-\frac{z\alpha}{\hbar}} \frac{1}{\hbar} \int_0^{\infty} dt e^{-\frac{t}{\hbar}} \hat{\phi}_-(\tau) = i\eta_q \psi_-^I. \quad (\text{A.13})$$

Since ψ_-^I is regular when ψ_+ jumps, we have

$$\psi_-^{II} = \psi_-^I. \quad (\text{A.14})$$

A similar analysis applies in two adjacent wedges separated by a Stokes line for $\hat{\phi}_-$. In that case we get

$$\begin{aligned} \psi_-^{II} - \psi_-^I &= i\eta_q \psi_+^I, \\ \psi_+^{II} &= \psi_+^I. \end{aligned} \quad (\text{A.15})$$

We attach an arrow to Stokes lines to distinguish the two cases. An arrow entering (exiting) the turning point corresponds to Stokes lines where ψ_+ (ψ_-) jumps. We have also to take into account from (A.4) the presence of the branch-cut singularity at $z = 0$, which is different depending on whether q is even or odd. In terms of the coefficients defined in (2.29), we get

$$B^{(q)} = e^{-\frac{i\pi q}{2}} \frac{1}{2} \begin{pmatrix} 1 + (-1)^q & 1 - (-1)^q \\ 1 - (-1)^q & 1 + (-1)^q \end{pmatrix}, \quad (\text{A.16})$$

whereas the Stokes jumps (A.13)–(A.15) are encoded in the connection matrices

$$S_+^{(q)} = \begin{pmatrix} 1 & 2i \cos\left(\frac{\pi}{q+2}\right) \\ 0 & 1 \end{pmatrix}, \quad S_-^{(q)} = \begin{pmatrix} 1 & 0 \\ 2i \cos\left(\frac{\pi}{q+2}\right) & 1 \end{pmatrix}. \quad (\text{A.17})$$

For integer q , solutions of the differential equations are analytic over the complex plane and so under a full rotation over the complex z -plane

$$\psi_{\pm}^X(e^{2i\pi} z) = \psi_{\pm}^X(z), \quad (\text{A.18})$$

where X is the wedge of the complex plane which includes the starting point z . Under a 2π rotation, the total monodromy is given by the product of the $q + 2$ connection matrices (A.17) and the branch-cut matrix in (A.16). In total we indeed get

$$(S_+^{(q)} S_-^{(q)})^{\lfloor \frac{q+2}{2} \rfloor} (S_+^{(q)})^{\frac{1-(-1)^q}{2}} B^{(q)} = I, \quad \forall q \in \mathbb{N}, \quad (\text{A.19})$$

where $\lfloor x \rfloor$ is the integer part of x . The result does not depend on the location of the branch cut. For even q , $B^{(q)} = (-1)^{q/2}$, while

$$S_{\pm}^{(q)} B^{(q)} = B^{(q)} S_{\mp}^{(q)}, \quad \text{odd } q. \quad (\text{A.20})$$

The most relevant case is $q = 1$, so for simplicity we define

$$S_+ \equiv S_+^{(1)}, \quad S_- \equiv S_-^{(1)}, \quad B \equiv B^{(1)}, \quad (\text{A.21})$$

which are the matrices (2.30) in the main text.

Define the Stokes automorphism

$$\mathfrak{S}_{\theta} \equiv s_{\theta}^{-1} \cdot s_{\theta+}, \quad (\text{A.22})$$

where $\theta^{\pm} = \theta \pm \epsilon$, $\epsilon \ll 1$, and $s_{\theta\pm}$ represent the lateral Borel resummations defined in (1.5). In our case the two non-trivial points where a Stokes jump occurs are $\tau = \tau_+$ and $\tau = \tau_-$, corresponding respectively to $\theta_{\tau_+} = 2\pi(2n)/(q+2)$ and $\theta_{\tau_-} = 2\pi(2n+1)/(q+2)$ in the z -plane, with n an integer. In terms of $\tilde{\psi}_{\pm}$, we get

$$\begin{aligned} \mathfrak{S}_{\theta_{\tau_+}} \tilde{\psi}_+ &= \tilde{\psi}_+ + i\eta_q \tilde{\psi}_-, \\ \mathfrak{S}_{\theta_{\tau_-}} \tilde{\psi}_- &= \tilde{\psi}_- + i\eta_q \tilde{\psi}_+, \end{aligned} \quad (\text{A.23})$$

so that the connection formulas for the wave functions can be interpreted in terms of Stokes automorphisms of the associated formal power series. The Stokes automorphism can also be defined in terms of alien derivatives (dotted and undotted versions):

$$\mathfrak{S}_\theta \equiv \exp\left(\sum_{\{\omega_\theta\}} \dot{\Delta}_{\omega_\theta}\right), \quad \dot{\Delta}_{\omega_\theta} \equiv e^{-\frac{\omega_\theta}{\hbar}} \Delta_{\omega_\theta}, \quad (\text{A.24})$$

where $\{\omega_\theta\}$ are all the possible singularities (just one in our case) along the ray with angle θ . If ω is an analytic point of the Borel transform associated to an asymptotic series function $\tilde{f}(\hbar)$, then $\Delta_\omega \tilde{f}(\hbar) = 0$. It easily follows from (A.23) that

$$\begin{aligned} \Delta_{\tau_+} \tilde{\phi}_+ &= i\eta_q \tilde{\phi}_-, & \Delta_{\tau_+} \tilde{\phi}_- &= 0, \\ \Delta_{\tau_-} \tilde{\phi}_+ &= 0, & \Delta_{\tau_-} \tilde{\phi}_- &= i\eta_q \tilde{\phi}_+. \end{aligned} \quad (\text{A.25})$$

B Double turning points

In contrast to simple zeros, whose connection matrices can be reduced to that of the Airy function, double zeroes are more complicated. Intuitively it is simple to understand the origin of the complication. A double turning point can be seen as the annihilation of two simple turning points as we move parameters in the function $Q(z)$. No matter how close the turning points are, the Voros symbol $a_{\gamma_{\text{vc}}}$ associated to the vanishing cycle γ_{vc} will in general be non-vanishing. In contrast to simple turning points, we then expect that the connection matrices of double turning points cannot be universal but depend on a variable λ defined as

$$a_{\gamma_{\text{vc}}} \equiv e^{-2i\pi(\lambda + \frac{1}{2})}. \quad (\text{B.1})$$

It has been shown in [13, 14] (see also [11, 15]) that the full connection matrices for double turning points split into a universal and a model-dependent term. The universal term coincides with the connection matrices for the Weber model defined by the equation

$$\hbar^2 \psi''(z) = \left(\frac{z^2}{4} - \hbar\left(\lambda + \frac{1}{2}\right)\right) \psi(z). \quad (\text{B.2})$$

Aim of this appendix is to derive the connection matrices for the Weber model (B.2). Although this is a known result, we believe it can be useful to report here a comprehensive and detailed derivation aimed at physicists.¹⁴

The leading terms of the WKB expansion are found from (2.9). We have

$$P_0^\eta = \eta \frac{z}{2}, \quad P_1^\eta = \frac{1}{2z} - \eta \frac{1 + 2\lambda}{2z}, \quad (\text{B.3})$$

and hence

$$\tilde{\psi}_\pm = z^{-\frac{1}{2} \mp (\lambda + \frac{1}{2})} e^{\pm \frac{z^2}{4\hbar}} \tilde{\phi}_\pm, \quad \tilde{\phi}_\pm = \sum_{n=0}^{\infty} c_n^\pm(z) \hbar^n, \quad (\text{B.4})$$

¹⁴A similar, but less detailed, derivation appears in [27]. In particular the constraints given by imposing trivial total monodromy of the wave function are not considered there.

where $c_0(z) = 1$. It is easy to see that for any integer n

$$c_n^\pm(z) = d_n^\pm z^{-2n}. \quad (\text{B.5})$$

Plugging the ansatz (B.4) in (B.2) allows us to fix all the coefficients d_n^\pm . We get the following recursion relations for d_n^\pm :

$$d_{n+1}^+ = \frac{2d_n^+}{n+1} \left(n + \frac{\lambda+1}{2} \right) \left(n + \frac{\lambda+2}{2} \right), \quad d_{n+1}^- = -\frac{2d_n^-}{n+1} \left(n - \frac{\lambda}{2} \right) \left(n - \frac{\lambda-1}{2} \right), \quad (\text{B.6})$$

which are solved for

$$d_n^+ = \frac{2^{n+\lambda} \Gamma\left(n + \frac{\lambda+1}{2}\right) \Gamma\left(n + \frac{\lambda+2}{2}\right)}{\sqrt{\pi} \Gamma(n+1) \Gamma(\lambda+1)}, \quad d_n^- = \frac{2^{n-\lambda-1}}{\sqrt{\pi}} (-1)^n \frac{\Gamma\left(n - \frac{\lambda}{2}\right) \Gamma\left(n - \frac{\lambda-1}{2}\right)}{\Gamma(n+1) \Gamma(-\lambda)}. \quad (\text{B.7})$$

We compute the Borel transforms of $\tilde{\phi}_\pm$. Since the powers of z in front of $\tilde{\psi}_+$ and $\tilde{\psi}_-$ in the first relation in (B.4) are different, it is useful to define more general Borel-Leroy transforms:

$$\begin{aligned} \hat{\phi}_+^{a_+}(\tau) &= \sum_{n=0}^{\infty} \frac{c_n^+(z)}{\Gamma(n+1+a_+)} t^n = \frac{1}{\Gamma(1+a_+)} {}_2F_1\left(\frac{1+\lambda}{2}, \frac{2+\lambda}{2}, 1+a_+; \tau\right), \\ \hat{\phi}_-^{a_-}(\tau) &= \sum_{n=0}^{\infty} \frac{c_n^-(z)}{\Gamma(n+1+a_-)} t^n = \frac{1}{\Gamma(1+a_-)} {}_2F_1\left(\frac{1-\lambda}{2}, -\frac{\lambda}{2}, 1+a_-; -\tau\right), \end{aligned} \quad (\text{B.8})$$

where

$$\tau = \frac{2t}{z^2}, \quad (\text{B.9})$$

and a_\pm are parameters which we conveniently fix as explained below. The only singularities occur at $\tau = 1$ for $\hat{\phi}_+(\tau)$ and $\tau = -1$ for $\hat{\phi}_-(\tau)$. When $a \neq 0$, (1.5) generalizes to

$$f^\theta(\hbar) = \frac{1}{\hbar} \int_0^{e^{i\theta\infty}} dt \hat{f}^a(t) \left(\frac{t}{\hbar}\right)^a e^{-\frac{t}{\hbar}}. \quad (\text{B.10})$$

If $z = |z| \exp(i\theta)$, with \hbar and t real and positive, Stokes lines occur for

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}. \quad (\text{B.11})$$

The computation of the discontinuity $s_{\theta^+} - s_{\theta^-}$ is more involved than that for pure monomials. In particular, care should be paid in keeping track of phases. We first consider $(s_{0^+} - s_{0^-})\psi_+$, with z taken real and positive so that no extra phases appear. In other words, the discontinuity is expected to be purely imaginary (like in the Airy case) for any real value of λ . First of all we determine a convenient choice of a_\pm which simplifies the computation. This is achieved by the use of the hypergeometric identity

$$\begin{aligned} {}_2F_1(a, b, c, z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b+1-c, 1-z) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, 1+c-a-b, 1-z), \end{aligned} \quad (\text{B.12})$$

which immediately shows that the non-analytic term of the expansion around $z = 1$ is related to the expansion around $z = 0$ of another hypergeometric function. Demanding that the non-analytic behaviour of $\hat{\phi}_+^{a_+}$ around $\tau = 1$ is governed by $\hat{\phi}_-^{a_-}$ around $\tau = 0$ allows us to uniquely fix the coefficients a_{\pm} by means of (B.12). We get $a_+ = 0$, $a_- = -\lambda - 1/2$, resulting in

$$\hat{\phi}_+(\tau)\Big|_{\tau=1} = \kappa(\lambda)(1-\tau)^{-\lambda-\frac{1}{2}}\hat{\phi}_-(\tau-1) + \text{regular}, \quad (\text{B.13})$$

where $\hat{\phi}_+ \equiv \hat{\phi}_+^0$, $\hat{\phi}_- \equiv \hat{\phi}_-^{-\lambda-1/2}$, and

$$\kappa(\lambda) = \frac{\Gamma\left(-\lambda-\frac{1}{2}\right)\Gamma\left(-\lambda+\frac{1}{2}\right)}{\Gamma\left(\frac{1+\lambda}{2}\right)\Gamma\left(\frac{2+\lambda}{2}\right)}. \quad (\text{B.14})$$

Taking

$$1-\tau = e^{-i\pi}(\tau-1), \quad z = \rho \in \mathbb{R} \quad (\text{B.15})$$

we have

$$\begin{aligned} (s_{0+} - s_{0-})\tilde{\psi}_+ &= \rho^{-\lambda-1} e^{\frac{\rho^2}{4\hbar}} e^{i\pi(\lambda+\frac{1}{2})} \left(1 - e^{-2i\pi(\lambda+\frac{1}{2})}\right) \kappa(\lambda) \frac{1}{\hbar} \int_{\frac{\rho^2}{2}}^{\infty} dt e^{-\frac{t}{\hbar}} (\tau-1)^{-\lambda-\frac{1}{2}} \hat{\phi}_-(\tau-1) \\ &= \rho^\lambda e^{-\frac{\rho^2}{4\hbar}} \left(e^{i\pi(\lambda+\frac{1}{2})} - e^{-i\pi(\lambda+\frac{1}{2})}\right) \kappa(\lambda) \frac{2^{-\lambda-\frac{1}{2}}}{\hbar} \int_0^{\infty} dt e^{-\frac{t}{\hbar}} t^{-\lambda-\frac{1}{2}} \hat{\phi}_-(\tau) \\ &= \frac{i\sqrt{2\pi\hbar}^{-\lambda-\frac{1}{2}}}{\Gamma(1+\lambda)} \psi_-, \quad z \in \mathbb{R}_+, \end{aligned} \quad (\text{B.16})$$

where in the second identity the change of variable of integration $t \rightarrow t + \rho^2/2$ has been performed.

When z is real negative (B.13) still applies, but now $z = \rho e^{\pm i\pi}$. The choice of phase depends on how we split the Stokes line and the branch-cut for $z < 0$. Taking $z = \rho e^{i\pi}$ means that if we rotate counterclockwise from the positive real axis we first encounter the Stokes line and then the branch-cut. Viceversa for $z = \rho e^{-i\pi}$. For definiteness we choose the first assignment. With respect to the previous analysis, an extra phase is acquired in rewriting

$$(\rho e^{i\pi})^{-\lambda-1} \rho^{2\lambda+1} = (\rho e^{i\pi})^\lambda e^{-2i\pi(\lambda+\frac{1}{2})}, \quad (\text{B.17})$$

so that

$$(s_{0+} - s_{0-})\tilde{\psi}_+ = \frac{i\sqrt{2\pi} e^{-2i\pi(\lambda+\frac{1}{2})} \hbar^{-\lambda-\frac{1}{2}}}{\Gamma(1+\lambda)} \psi_-, \quad z \in \mathbb{R}_-. \quad (\text{B.18})$$

Consider now $(s_{\pi+} - s_{\pi-})\tilde{\psi}_-$. We determine again a_{\pm} by demanding that the non-analytic behaviour of $\hat{\phi}_-^{a_-}$ around $\tau = -1$ is given by $\hat{\phi}_+^{a_+}$ around $\tau = 0$. We find $a_- = 0$, $a_+ = \lambda + 1/2$, resulting in

$$\hat{\phi}_-(\tau)\Big|_{\tau=-1} = \tilde{\kappa}(\lambda)(1+\tau)^{\lambda+\frac{1}{2}}\hat{\phi}_+(\tau+1) + \text{regular}, \quad (\text{B.19})$$

where $\hat{\phi}_- \equiv \hat{\phi}_-^0$, $\hat{\phi}_+ \equiv \hat{\phi}_+^{\lambda+1/2}$, and

$$\tilde{\kappa}(\lambda) = \frac{\Gamma\left(-\lambda-\frac{1}{2}\right)\Gamma\left(\lambda+\frac{3}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2}\right)}. \quad (\text{B.20})$$

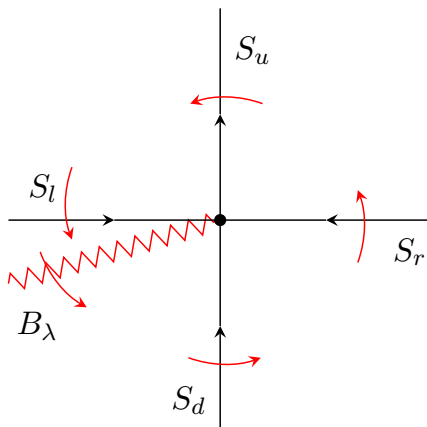


Figure 10. Connection matrices (including the branch-cut) for the Weber equation (B.2) (deformed quadratic turning point).

For z pure imaginary positive, $z = \rho \exp(i\frac{\pi}{2})$, we get

$$\begin{aligned}
 (s_{\pi+} - s_{\pi-})\tilde{\psi}_- &= z^\lambda e^{-\frac{z^2}{4\hbar}} \left(e^{-i\pi(\lambda+\frac{1}{2})} - e^{i\pi(\lambda+\frac{1}{2})} \right) \tilde{\kappa}(\lambda) \frac{1}{\hbar} \int_{-\frac{z^2}{2}}^{\infty} dt e^{-\frac{t}{\hbar}} (\tau+1)^{\lambda+\frac{1}{2}} \hat{\phi}_+(\tau+1) \\
 &= (\rho e^{i\frac{\pi}{2}})^\lambda \rho^{-\lambda-1} e^{+\frac{z^2}{4\hbar}} \left(e^{-i\pi(\lambda+\frac{1}{2})} - e^{i\pi(\lambda+\frac{1}{2})} \right) \tilde{\kappa}(\lambda) \frac{1}{\hbar} \int_0^{\infty} dt e^{-\frac{t}{\hbar}} t^{\lambda+\frac{1}{2}} \hat{\phi}_+(\tau) \\
 &= \frac{-e^{i\pi\lambda} \sqrt{2\pi\hbar}^{\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \psi_+, \quad \text{Im } z > 0.
 \end{aligned} \tag{B.21}$$

When z is pure imaginary negative, $z = \rho \exp(-i\frac{\pi}{2})$, we get

$$(s_{\pi+} - s_{\pi-})\tilde{\psi}_- = \frac{e^{-i\pi\lambda} \sqrt{2\pi\hbar}^{\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \psi_+, \quad \text{Im } z < 0. \tag{B.22}$$

In total we have 5 different connection matrices, as depicted in figure 10:

$$\begin{aligned}
 S_r &= \begin{pmatrix} 1 & 0 \\ \frac{i\sqrt{2\pi\hbar}^{-\lambda-\frac{1}{2}}}{\Gamma(1+\lambda)} & 1 \end{pmatrix}, & S_u &= \begin{pmatrix} 1 & \frac{-e^{i\pi\lambda} \sqrt{2\pi\hbar}^{\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \\ 0 & 1 \end{pmatrix}, \\
 S_l &= \begin{pmatrix} 1 & 0 \\ -\frac{i\sqrt{2\pi\hbar} e^{-2i\pi\lambda} \hbar^{-\lambda-\frac{1}{2}}}{\Gamma(1+\lambda)} & 1 \end{pmatrix}, & S_d &= \begin{pmatrix} 1 & \frac{e^{-i\pi\lambda} \sqrt{2\pi\hbar}^{\lambda+\frac{1}{2}}}{\Gamma(-\lambda)} \\ 0 & 1 \end{pmatrix}, & B_\lambda &= \begin{pmatrix} e^{-2i\pi\lambda} & 0 \\ 0 & e^{2i\pi\lambda} \end{pmatrix}.
 \end{aligned} \tag{B.23}$$

The subscripts r , u , l and d refer to the four Stokes lines in the positive real axis (r), positive imaginary axis (u), negative real axis (l) and negative imaginary axis (d) respectively. For simplicity we omit to write their dependence on λ . B_λ is the monodromy matrix due to the branch-cut at $z = 0$. Solutions of the differential equations are analytic over the complex plane and so under a full rotation over the complex z -plane $\psi_\pm \rightarrow \psi_\pm$. Indeed we have

$$S_r S_d B_\lambda S_l S_u = I. \tag{B.24}$$

As a sanity check we reproduce the pure case by setting $\lambda = -1/2$, in which case we have

$$S_r = S_l = S_-^{(2)}, \quad S_u = S_d = S_+^{(2)}, \quad B_\lambda = -I, \tag{B.25}$$

where $S_{\pm}^{(2)}$ are the connection matrices (A.17) for $q = 2$. Equation (B.24) and the $\lambda = -1/2$ match provide a quite non-trivial consistency check of the connection matrices (B.23). Note that (B.24) depends on the location of the branch-cut. For instance, by changing the order of the Stokes line and branch-cut for negative z , (B.17) changes in

$$(\rho e^{-i\pi})^{-\lambda-1} \rho^{2\lambda+1} = (\rho e^{-i\pi})^{\lambda} e^{2i\pi(\lambda+\frac{1}{2})}, \quad (\text{B.26})$$

and correspondingly

$$S_l \rightarrow S'_l = \begin{pmatrix} 1 & 0 \\ -\frac{i\sqrt{2\pi}e^{2i\pi\lambda}\hbar^{-\lambda-\frac{1}{2}}}{\Gamma(1+\lambda)} & 1 \end{pmatrix} = B_{\lambda} S_l B_{\lambda}^{-1}, \quad (\text{B.27})$$

so that

$$S_u B_{\lambda} S'_l S_d S_r = I. \quad (\text{B.28})$$

We recognize that the factors $\sqrt{2\pi}\hbar^{\lambda+1/2}/\Gamma(-\lambda)$ and $\sqrt{2\pi}\hbar^{-\lambda-1/2}/\Gamma(\lambda+1)$ appearing respectively in $S_{u,d}$ and $S_{r,l}$ precisely reproduce the similar factors in (3.28) and (4.17), upon identifying λ with t . In other words, when two simple turning points A and B collapse, the non-perturbative combination of Voros symbols $\sqrt{a_{DC}}$ and $\sqrt{a_{DC}/a_{AB}}(1+a_{AB})$ acquire a universal model-independent contribution which can be reinterpreted as a factor entering the connection matrix for the Weber equation (B.2). The remaining model-dependent terms, parametrized e.g. in the formal power series (3.28) by the constants $c_{1,2}$ and by the coefficients of the asymptotic series in \hbar hidden in the $\mathcal{O}(\hbar)$ terms, can be fixed by e.g. using the so called uniform WKB method. This method was introduced back in [42–44] and more recently reconsidered, see e.g. [45] for anharmonic potentials and [32] for a pedagogical introduction. In our context it essentially consists in finding a change of coordinates which brings the original Schrödinger equation (2.1) to the Weber form (B.2). See also [15, 46] for a closely related way to fix such terms. We will not discuss here how to determine the model-dependent factors using such techniques, since we can compute them by working with simple turning points in the limit $e_0 \rightarrow 0$, as explained in the main text.

C Transseries in the pure quartic anharmonic

In this appendix we discuss in some detail the quartic anharmonic potential with no mass term as an example of a model where perturbation theory in \hbar is ill-defined. Notably this is the first model where EWKB has been applied [6, 10]. A simple scaling argument shows that the energy levels have the form $E_n = \hbar^{4/3}\gamma_n$, where γ_n are real numbers independent of \hbar . Although the \hbar -expansion does not exist, we can consider a semiclassical expansion in $1/n$. We show below that energy eigenvalues E_n are given by a transseries in $1/n$ and $\exp(-\pi n)$, and we study its convergence properties. Since the quartic model can be easily obtained from EPT, as in (4.19), this also illustrates how an observable can be packaged in a dramatically different way by EPT.

Like in the massive quartic anharmonic model discussed in the main text, the pure quartic admit eight different wedges in the \hbar complex plane delimited by Stokes jumps. Unlike the massive quartic, however, the Stokes jump separating region I from region II is

independent of E and occurs at $\arctan\left(\frac{\Pi_{0,BA}(E)}{i\Pi_{0,CD}(E)}\right) = \frac{\pi}{4}$. No sweet spot can be defined and hence we focus our analysis just on the region I delimiting the real \hbar -axis. The EQC in this region can be read from the first line of (3.10) and can be written by simple manipulations of periods as

$$\sqrt{a_{BA}^I} = \pm i - \sqrt{a_{DC}^I}, \quad (\text{C.1})$$

for even/odd levels. The two periods can be written purely as functions of $x \equiv \hbar/E^{3/4}$ in terms of a single function R :

$$\sqrt{a_{BA}^I} = e^{iR(x)}, \quad \sqrt{a_{DC}^I} = e^{-iR(ix)}. \quad (\text{C.2})$$

The nontrivial relation between the perturbative and nonperturbative cycle, which allows us to express both in terms of the same function $R(x)$, can be seen at each order in n noting that the nonperturbative cycle is obtained from a complex rotation $z \rightarrow iz$ of the perturbative one, while the integrand has the form $f_k(z^4)$ for k even, $z^2 f_k(z^4)$ for k odd, thus picking up a minus sign for odd terms only.

Since the Voros symbol a_{BA} is not Borel resummable, so will be the formal asymptotic series

$$\tilde{R} = \sum_{k=0} a_k x^{2k-1} \quad (\text{C.3})$$

associated to R (see [47] for an explicit analysis). Neglecting the nonperturbative contribution given by a_{DC}^I , the EQC (C.1) implies

$$\tilde{R}(x) = \nu, \quad \nu \equiv \pi \left(n + \frac{1}{2} \right). \quad (\text{C.4})$$

Inverting the above formal series, we get

$$\tilde{x}_P(\nu) = \tilde{R}^{-1}(\nu) \sim \frac{a_0}{\nu} + \frac{a_0 a_1}{\nu^3} + \dots \quad (\text{C.5})$$

As we will see below, the presence of the nonperturbative cycle turns the asymptotic series \tilde{x}_P in (C.5) in a transseries $\tilde{x}(\nu)$ defined as

$$\tilde{x} = \sum_{j,k=0}^{\infty} c_{j,k} \frac{\epsilon^j}{\nu^{2k+1}}, \quad \epsilon \equiv e^{-(1+i)\nu}, \quad (\text{C.6})$$

where $c_{0,0} = a_0$, $c_{0,1} = a_0 a_1$, \dots , from which the resummed energy eigenvalues can be obtained through Borel resummation as

$$E_n = \hbar^{\frac{4}{3}} s (\tilde{x})^{-\frac{4}{3}} = \hbar^{\frac{4}{3}} \left(\sum_{j=0}^{\infty} (\mp i)^j e^{-j\pi(n+\frac{1}{2})} s_+ \left(\sum_{k=0}^{\infty} \frac{c_{j,k}}{\left(\pi(n+\frac{1}{2})\right)^{2k+1}} \right) \right)^{-\frac{4}{3}}, \quad (\text{C.7})$$

where s_+ indicates a lateral Borel resummation slightly above the positive \hbar real axis (region I) and \mp refers respectively to even and odd n . The appearance of imaginary terms in the physical energy eigenvalues should not surprise. They are there precisely to compensate imaginary terms arising from the lateral Borel resummation in the spirit of resurgence.

Naturally, we require that the sum over j results in a finite number. It has been observed in the literature on resurgence, see e.g. [8, 48], that in many cases the sum over j is convergent for sufficiently large n , or ν . A related observation is that if we formally reorganise \tilde{x} as

$$\tilde{x} = \sum_{k=0}^{\infty} \frac{\tilde{S}_k}{\nu^{2k+1}}, \quad \tilde{S}_k \equiv \sum_{j=0}^{\infty} c_{j,k} e^{-j(1+i)\nu}, \quad (\text{C.8})$$

the partial sums \tilde{S}_k can themselves be convergent. For example, in [49] it is shown that a broad class of non-linear ODEs admit formal solutions in terms of transseries where the analogous partial series \tilde{S}_k are all convergent within the radius of convergence of \tilde{S}_0 . We show below that this also happens with the partial series \tilde{S}_k in (C.8). We conjecture that this is evidence of the fact that the transseries (C.6) (i.e. the sum over j in (C.7)) is convergent in approximately the same region.

We now show that (C.6) applies and study the convergent properties of the factors \tilde{S}_k in (C.8). We massage (C.1) into

$$e^{iR(x)} = e^{i\nu} \left(1 - e^{-iR(ix)-i\nu}\right) \Rightarrow R(x) = \nu - i \ln \left(1 - e^{\nu-iR(ix)} e^{-(1+i)\nu}\right), \quad (\text{C.9})$$

where $e^{\nu-iR(ix)} \sim \mathcal{O}(1)$ for parametrically small x (or large ν). If we assume that \tilde{x} is known up to order ϵ^N and to sufficiently high order in ν^{-1} , then $R(ix)$ and its exponential are known to $\mathcal{O}(\epsilon^N)$. The additional factor ϵ inside the logarithm then allows us to compute to order ϵ^{N+1} . Iterating the process, (C.9) is satisfied to each order by the transseries ansatz (C.6).

Let us denote the radius of convergence of $\tilde{S}_k(\epsilon)$ as r_k . To obtain it, first notice that the coefficient of $1/\nu^{2k+1}$ is determined from the \tilde{R} series up to the term a_k . Thus to get the series \tilde{S}_0 it is sufficient to truncate $\tilde{R}(x)$ to $\frac{a_0}{x}$, neglecting $\mathcal{O}(x)$. We leave higher order terms for later. We replace $-i\tilde{R}(ix), -\tilde{R}(x) \rightarrow -\mathcal{R}$ and (C.9) gives

$$w = 1 - \epsilon e^{i \ln w}, \quad w \equiv e^{i(\mathcal{R}-\nu)}. \quad (\text{C.10})$$

Note that the x dependence is now completely implicit. We can see from (C.10) that all derivatives of $w(\epsilon)$ are bounded around the origin and hence analyticity is guaranteed at $\epsilon = 0$. Due to the logarithm, any $\bar{\epsilon}$ such that $w(\bar{\epsilon}) = 0$, must be at the edge of the radius of convergence of the series $w(\epsilon)$ defined by (C.10). Noting that $w(0) = 1$, we now consider all paths $\gamma \in \Gamma$, in the complex w plane, going from $w = 1$ to $w = 0$ and the corresponding value of $|\epsilon|$ along each path. The radius of convergence will be $\min_{\gamma \in \Gamma} \max_{w \in \gamma} |\epsilon(w)|$, because if it were greater then we could construct the curves $\epsilon(t), w(\epsilon(t))$ such that w approaches 0 and $\epsilon(t)$ is always in the radius of convergence. Conversely, there are no other singularities in (C.10) obstructing convergence.

Letting $w(t) = L(t)e^{i\theta(t)}$, (C.10) gives

$$|\epsilon|^2 = e^{2\theta} \left(1 + L^2 - 2L \cos \theta\right). \quad (\text{C.11})$$

Any curve that connects $w = 0$ to $w = 1$ must pass by the line $\text{Re}(w) = 1/2$, along which $|\epsilon|^2$ has a single global minimum, see figure 11. Since $\nabla|\epsilon|^2$ vanishes at $L = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$,

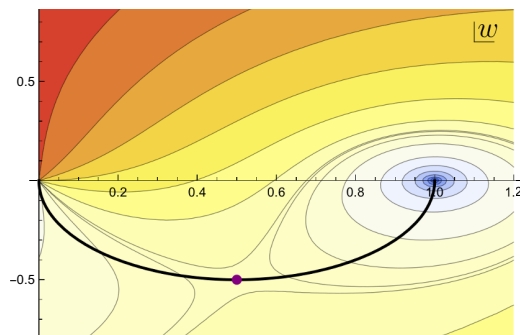


Figure 11. Level curves of the function $|\epsilon(w)|^2$. In black an optimal curve γ^* . The purple dot is where the maximum value of $|\epsilon|$ is achieved.

the minimum value of $|\epsilon|^2$ in this line is at $w = \frac{1-i}{2}$, and thus $\min_{\gamma \in \Gamma} \max_{w \in \gamma} |\epsilon(w)| \geq |\epsilon(\frac{1-i}{2})|$. To reduce the bound to an equality, consider the curve γ^* such that $L = \cos \theta$; $\max_{w \in \gamma^*} |\epsilon|$ is easily computed to be exactly at $w = \frac{1-i}{2}$. Thus, the radius of convergence r_0 of the series $\tilde{S}_0(\epsilon)$ is

$$r_0 \equiv \max_{w \in \gamma^*} |\epsilon| = \frac{e^{-\pi/4}}{\sqrt{2}}, \tag{C.12}$$

a result we numerically confirmed. Importantly, all quantized values of ν lie inside this radius.

We now want to extend the result to $\tilde{S}_{k>0}$. Recalling that to determine \tilde{S}_k we only need finitely many terms in ν^{-1} , let us truncate \tilde{x} as it is defined in equation (C.8),

$$\tilde{x}^{(k+1)} \equiv \sum_{k'=0}^{k+1} \frac{\tilde{S}_{k'}(\epsilon)}{\nu^{2k'+1}} = \tilde{x}^{(k)} + \frac{\tilde{S}_{k+1}}{\nu^{2k+3}}. \tag{C.13}$$

Plugging this ansatz for \tilde{x} into $\tilde{R}(x)$ (equation (C.3)) and then $\tilde{R}(x)$ into equation (C.9), we obtain \tilde{S}_{k+1} . It is expressed as a sum of products of the previously encountered $\{\tilde{S}_i(\epsilon)\}_{i \leq k}$ so, by Merten's theorem, the radius of convergence r_{k+1} of \tilde{S}_{k+1} is $r_{k+1} \geq \min_{i \leq k} \{r_i\}$.

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References

- [1] C.M. Bender and T.T. Wu, *Anharmonic oscillator*, *Phys. Rev.* **184** (1969) 1231 [[INSPIRE](#)].
- [2] C.M. Bender and T.T. Wu, *Anharmonic oscillator. 2: a study of perturbation theory in large order*, *Phys. Rev. D* **7** (1973) 1620 [[INSPIRE](#)].
- [3] G. Wentzel, *Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik* (in German), *Z. Phys.* **38** (1926) 518 [[INSPIRE](#)].
- [4] H.A. Kramers, *Wellenmechanik und halbzahlige Quantisierung* (in German), *Z. Phys.* **39** (1926) 828 [[INSPIRE](#)].

- [5] L. Brillouin, *La mécanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations successives* (in French), *Compt. Rend. Hebd. Seances Acad. Sci.* **183** (1926) 24 [[INSPIRE](#)].
- [6] A. Voros, *The return of the quartic oscillator. The complex WKB method*, *Annales de l'I.H.P. Physique Théorique* **39** (1983) 211.
- [7] H.J. Silverstone, *JWKB connection-formula problem revisited via Borel summation*, *Phys. Rev. Lett.* **55** (1985) 2523.
- [8] J. Écalle, *Les fonctions résurgentes* (in French), *Publ. Math. d'Orsay/Univ. de Paris, Dep. de math.* (1981).
- [9] R. Balian and C. Bloch, *Solution of the Schrödinger equation in terms of classical paths*, *Annals Phys.* **85** (1974) 514 [[INSPIRE](#)].
- [10] R. Balian, G. Parisi and A. Voros, *Discrepancies from asymptotic series and their relation to complex classical trajectories*, *Phys. Rev. Lett.* **41** (1978) 1141.
- [11] T. Aoki, T. Kawai and Y. Takei, *The Bender-Wu analysis and the Voros theory*, in the proceedings of the *ICM-90 satellite conference proceedings*, Tokyo, Japan (1991), p. 1.
- [12] H. Dillinger, E. Delabaere and F. Pham, *Résurgence de Voros et périodes des courbes hyperelliptiques* (in French), in the proceedings of the *Annales de l'institut Fourier*, (1993), p. 163.
- [13] E. Delabaere, H. Dillinger and F. Pham, *Exact semiclassical expansions for one-dimensional quantum oscillators*, *J. Math. Phys.* **38** (1997) 6126.
- [14] E. Delabaere and F. Pham, *Resurgent methods in semi-classical asymptotics*, *Annales I.H.P.* **71** (1999) 1.
- [15] T. Aoki, T. Kawai and Y. Takei, *The Bender-Wu analysis and the Voros theory. II*, *Adv. Stud. Pure Math.* **54** (2009) 19.
- [16] J. Zinn-Justin, *Multi-instanton contributions in quantum mechanics. 2*, *Nucl. Phys. B* **218** (1983) 333 [[INSPIRE](#)].
- [17] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, *Adv. Math.* **234** (2013) 239 [[arXiv:0907.3987](#)] [[INSPIRE](#)].
- [18] A. Mironov and A. Morozov, *Nekrasov functions and exact Bohr-Zommerfeld integrals*, *JHEP* **04** (2010) 040 [[arXiv:0910.5670](#)] [[INSPIRE](#)].
- [19] S. Codesido and M. Marino, *Holomorphic anomaly and quantum mechanics*, *J. Phys. A* **51** (2018) 055402 [[arXiv:1612.07687](#)] [[INSPIRE](#)].
- [20] S. Codesido, M. Marino and R. Schiappa, *Non-perturbative quantum mechanics from non-perturbative strings*, *Annales H. Poincaré* **20** (2019) 543 [[arXiv:1712.02603](#)] [[INSPIRE](#)].
- [21] K. Ito, M. Mariño and H. Shu, *TBA equations and resurgent quantum mechanics*, *JHEP* **01** (2019) 228 [[arXiv:1811.04812](#)] [[INSPIRE](#)].
- [22] P. Dorey and R. Tateo, *Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations*, *J. Phys. A* **32** (1999) L419 [[hep-th/9812211](#)] [[INSPIRE](#)].
- [23] J. Gu and M. Marino, *On the resurgent structure of quantum periods*, *SciPost Phys.* **15** (2023) 035 [[arXiv:2211.03871](#)] [[INSPIRE](#)].

- [24] G.V. Dunne and M. Ünsal, *Uniform WKB, multi-instantons, and resurgent trans-series*, *Phys. Rev. D* **89** (2014) 105009 [[arXiv:1401.5202](#)] [[INSPIRE](#)].
- [25] G. Basar, G.V. Dunne and M. Ünsal, *Quantum geometry of resurgent perturbative/nonperturbative relations*, *JHEP* **05** (2017) 087 [[arXiv:1701.06572](#)] [[INSPIRE](#)].
- [26] N. Sueishi, S. Kamata, T. Misumi and M. Ünsal, *On exact-WKB analysis, resurgent structure, and quantization conditions*, *JHEP* **12** (2020) 114 [[arXiv:2008.00379](#)] [[INSPIRE](#)].
- [27] N. Sueishi, S. Kamata, T. Misumi and M. Ünsal, *Exact-WKB, complete resurgent structure, and mixed anomaly in quantum mechanics on S^1* , *JHEP* **07** (2021) 096 [[arXiv:2103.06586](#)] [[INSPIRE](#)].
- [28] S. Kamata, T. Misumi, N. Sueishi and M. Ünsal, *Exact WKB analysis for SUSY and quantum deformed potentials: quantum mechanics with Grassmann fields and Wess-Zumino terms*, *Phys. Rev. D* **107** (2023) 045019 [[arXiv:2111.05922](#)] [[INSPIRE](#)].
- [29] M. Serone, G. Spada and G. Villadoro, *Instantons from perturbation theory*, *Phys. Rev. D* **96** (2017) 021701 [[arXiv:1612.04376](#)] [[INSPIRE](#)].
- [30] M. Serone, G. Spada and G. Villadoro, *The power of perturbation theory*, *JHEP* **05** (2017) 056 [[arXiv:1702.04148](#)] [[INSPIRE](#)].
- [31] K. Iwaki and T. Nakanishi, *Exact WKB analysis and cluster algebras*, *J. Phys. A* **47** (2014) 474009 [[arXiv:1401.7094](#)].
- [32] M. Mariño, *Advanced topics in quantum mechanics*, Cambridge University Press, Cambridge, U.K. (2021).
- [33] E. Delabaere and F. Pham, *Unfolding the quartic oscillator*, *Annals Phys.* **261** (1997) 180.
- [34] S. Graffi, V. Grecchi and B. Simon, *Borel summability: application to the anharmonic oscillator*, *Phys. Lett. B* **32** (1970) 631 [[INSPIRE](#)].
- [35] D. Dorigoni, *An introduction to resurgence, trans-series and alien calculus*, *Annals Phys.* **409** (2019) 167914 [[arXiv:1411.3585](#)] [[INSPIRE](#)].
- [36] I. Aniceto, G. Basar and R. Schiappa, *A primer on resurgent transseries and their asymptotics*, *Phys. Rept.* **809** (2019) 1 [[arXiv:1802.10441](#)] [[INSPIRE](#)].
- [37] D. Sauzin, *Resurgent functions and splitting problems*, *Res. Inst. Math. Sci.* **1493** (2006) 48 [[arXiv:0706.0137](#)].
- [38] M.-X. Huang, *On gauge theory and topological string in Nekrasov-Shatashvili limit*, *JHEP* **06** (2012) 152 [[arXiv:1205.3652](#)] [[INSPIRE](#)].
- [39] Y. Emery, *TBA equations and quantization conditions*, *JHEP* **07** (2021) 171 [[arXiv:2008.13680](#)] [[INSPIRE](#)].
- [40] J. Jaeckel and S. Schenk, *Exploring high multiplicity amplitudes: the quantum mechanics analogue of the spontaneously broken case*, *Phys. Rev. D* **99** (2019) 056010 [[arXiv:1811.12116](#)] [[INSPIRE](#)].
- [41] M. Serone, G. Spada and G. Villadoro, $\lambda\phi_2^4$ theory. Part II. The broken phase beyond $NNNN(NNNN)LO$, *JHEP* **05** (2019) 047 [[arXiv:1901.05023](#)] [[INSPIRE](#)].
- [42] R.E. Langer, *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point*, *Trans. Amer. Math. Soc.* **67** (1949) 461.

- [43] T.M. Cherry, *Uniform asymptotic formulae for functions with transition points*, *Trans. Amer. Math. Soc.* **68** (1950) 224.
- [44] S.C. Miller and R.H. Good, *A WKB-type approximation to the Schrödinger equation*, *Phys. Rev.* **91** (1953) 174.
- [45] G. Álvarez and C. Casares, *Uniform asymptotic and JWKB expansions for anharmonic oscillators*, *J. Phys. A* **33** (2000) 2499.
- [46] Y. Takei, *On the connection formula for the first Painlevé equation: from the viewpoint of the exact WKB analysis*, in *Painlevé transcendents and asymptotic analysis*, <http://hdl.handle.net/2433/59957> (1995), p. 70.
- [47] A. Grassi, M. Marino and S. Zakany, *Resumming the string perturbation series*, *JHEP* **05** (2015) 038 [[arXiv:1405.4214](https://arxiv.org/abs/1405.4214)] [[INSPIRE](https://inspirehep.net/literature/1405421)].
- [48] O. Costin, *Exponential asymptotics, transseries, and generalized Borel summation for analytic, nonlinear, rank-one systems of ordinary differential equations*, *Int. Math. Res. Not.* **1995** (1995) 377.
- [49] O. Costin and R.D. Costin, *On the formation of singularities of solutions of nonlinear differential systems in antistokes directions*, *Invent. Math.* **145** (2001) 425.