

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Topological Quantum Field Theories: gravity and finite group gauge theories

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Abstract

This thesis develops two applications of three-dimensional topological quantum field theories. The first concerns a toy model of topological quantum gravity defined through a sum over 3-manifold topologies with fixed boundary. By expressing the partition function of an abelian TQFT in terms of a finitely generated abelian group A , a non-degenerate bilinear form on $\text{Tor} A$ and a tuple of elements of A , we reformulate the topological sum as a sum over algebraic data and derive sufficient conditions on the weights in the sum for convergence. Under additional assumptions, we characterize an ensemble of two-dimensional TQFTs whose averaged boundary partition functions reproduce the gravitational amplitudes, in agreement with the ensemble version of the holographic principle. For suitable weights, the analysis factorizes into independent contributions associated with each prime.

The second part investigates a conjectural relation between twisted $\text{SL}(2, \mathbb{F}_q)$ Dijkgraaf–Witten theory at large q and $\text{SU}(2)$ Chern–Simons theory at large level. We construct the relevant twist using second Chern classes and test the proposed relation on lens spaces, torus bundles, Seifert fibered spaces, higher-genus plumbings and other geometric 3-manifolds. The examples support a correspondence between the leading asymptotic data for non-hyperbolic geometries, while a hyperbolic example demonstrates a limitation of the strongest formulation. For spherical manifolds, we prove a weaker statement which supports the conjecture.

These results illuminate interactions among topology, arithmetic and quantum theories and motivate further studies on topological quantum field theories in three dimensions.

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Chapter 1

Introduction and Background

1.1 Motivation

Topological Quantum Field Theories, from now on TQFTs, are a very rich class of Quantum Field Theories, sparking the interest of both low and high-energy physicists and pure and applied mathematicians. Despite lacking local propagating degrees of freedom, they are the perfect testing grounds for several interesting physical phenomena like symmetries and defects [1, 2]. Furthermore, they arise as low-energy theories of fractional quantum Hall effect [3], and describe topological quantum orders [4]. On the other hand, their apparent simplicity allowed mathematicians to formulate a rigorous definition in terms of a functor from the bordism category to the category of vector spaces, allowing for the axiomatization of TQFTs [5]. They turned out to play a crucial role in knot theory, and led to a deep understanding of the Jones polynomial [6, 7] and other powerful new invariants.

The word topological means that the quantum theory does not depend on any structure other than the topology of the spacetime. In other words, all the observables of the theory are topological invariants. However, this is far from being the only effect of diffeomorphism invariance. For example, it follows that the Hilbert space of quantum states of a TQFT is finite-dimensional and that the theory behaves well under standard operations in differential topology, like cutting and gluing. This explains why it is much simpler to work with a topological theory and why they are a useful tool to address open problems of quantum theories. Despite their apparent simplicity, such theories exhibit non-trivial features and can lead to exciting new mathematics.

By axiomatizing some of these properties, one can get a rigorous definition of a TQFT which can then help to answer precisely questions like how to classify all possible theories. Under suitable assumptions, the datum of a 3-dimensional TQFT is equivalent to a Modular Tensor Category [8], which is a purely algebraic object and hence is more suited to classification problems. A complete classification still evades us, and a crucial ingredient is being able to distinguish inequivalent TQFTs and MTCs. Being able to rephrase harder

problems algebraically is a crucial technique, and will be applied to our work as well.

This thesis develops two applications of the framework of 3-dimensional TQFTs. The first concerns a model of topological quantum gravity based on a sum over topologies and explores the implications of an ensemble version of the holographic principle. The second establishes an asymptotic correspondence between finite and continuous group gauge theories for an explicit choice of the groups: $\mathrm{SL}(2, \mathbb{F}_q)$ and $\mathrm{SU}(2)$ respectively.

The results shown in this work have been obtained by the author of this thesis, in collaboration with his supervisor, Pavel Putrov, and with Johann Quenta-Raygada, PhD student at SISSA. The results discussed in this thesis have already appeared on the arXiv, see [9] and [10] respectively.

Plan of the Thesis

The rest of this Chapter is dedicated to contextualizing and stating the main results obtained in this work. We introduce the two projects separately, formulating the research question and making the results clear. Then, we review some aspects of abelian TQFTs and Dijkgraaf–Witten theories that are needed in the remainder of the thesis, and we fix our conventions.

Chapter 2 develops a model of three-dimensional topological quantum gravity. We reformulate the sum over manifold topologies as a sum over algebraic data and discuss its convergence, then we derive the holographic dual ensemble of two-dimensional TQFTs under specified assumptions.

Chapter 3 investigates a conjectural relation between $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf–Witten theory and $\mathrm{SU}(2)$ Chern–Simons theory. We test the relation on several classes of 3-manifolds and present a weaker result for spherical manifolds.

Chapter 4 summarizes the results we have obtained and their limitations. We discuss some possible directions for further research and a current work in progress.

In the Appendices we discuss some technicalities that do not fit naturally into the main body of the thesis.

1.1.1 Topological Quantum Gravity

One of the biggest unresolved questions in modern physics concerns the so-called theory of everything, a theory able to tie together the microscopic and macroscopic behaviours of our universe. A crucial step in this quest is the formulation of a quantum theory of gravitation, which we call Quantum Gravity. The most naive attempt consists in defining a hypothetical path integral that mimics the path integral of a quantum field theory. The field in question would consist of the metric tensor, but one should also take into account the topology of the spacetime as something that needs to be “integrated over”. We mean that, for a fixed spacetime manifold, the path integral would consist of integration over all metrics admissible for that topology. Then one can consider summation over all possible

diffeomorphism classes of the spacetime manifolds, often referred to as the “sum over topologies”, which has a physical interpretation in terms of the creation and annihilation of “baby universes” [11–14].

The sum over topologies, in a sense, is analogous to the sum over isomorphism classes of principal bundles in gauge theories. Both the integration and the summation part are in general ill-defined, especially in dimensions $d > 2$. For the integration part, apart from the problem — already present in QFTs — that the integration measure is mathematically not well defined, it is already not known what exactly the space of metrics that are integrated over should be. In particular, it is not clear whether one should allow singular metrics of a certain kind. For the summation part, even though the set of diffeomorphism classes of compact manifolds is countable, their explicit classification is a hard mathematical problem. Moreover, the sum is, in general, divergent and therefore requires some regularization.

In the Chapter 2, we consider a toy model of a 3-dimensional topological quantum gravity. This greatly simplifies the theory, since there is no metric dependence and hence no path integral needs to be performed. In this model, a contribution of a given 3-manifold is given by the partition function of an abelian TQFT, with a topological boundary condition at the boundary. Using the fact that the TQFT partition function depends only on the first homology group of the 3-manifold with some additional structure, the sum over all 3-manifolds with fixed boundary can be rewritten as a sum over finitely generated abelian groups (and the extra structure). Furthermore, we present bounds on the universal weights in the sum, that is, the measure on the set of isomorphism classes of finitely generated abelian groups (with the extra structure) sufficient for the sum to be convergent.

Another supposed feature of quantum gravity is the holographic principle. Namely, the path integral is performed over the bulk geometries compatible with some fixed boundary conditions, thus yielding a new theory on the boundary. However, allowing for disconnected bulk geometries spoils the factorization property of the boundary theory. This is a well-known issue and there is evidence in the literature that quantum gravity exhibits a version of the holographic principle where an ensemble of quantum theories emerges at the boundary. In this context, ensemble holography has been discussed in [15, 16].

In our framework, we explicitly explore the ensemble of holographic dual theories to topological quantum gravity. In particular, with further assumptions on the universal weights, we determine the distribution of 2d TQFTs such that the sum over topologies yields the ensemble average of their partition functions evaluated on the boundary. Notice that the topological nature of the holographic dual theories is a direct consequence of the topological boundary conditions considered on the bulk.

Furthermore, we argue that for a certain kind of measure, the sum over the abelian groups can be factorized into sums over abelian p -groups, and the analysis can be performed independently for each prime p .

1.1.2 On the correspondence between finite and continuous gauge group theories

Topological gauge theories, both with continuous and finite gauge groups, play an important role in condensed matter and high-energy physics. In three dimensions, Chern-Simons theory is the archetypical case of a gauge theory with a continuous group [6]. In particular, the partition function on a 3-manifold is formally defined via a path integral over the space of G -connections on the 3-manifold, for G a Lie group (ordinarily assumed to be semisimple and compact). The mathematical definition of the partition function was provided by Reshetikhin and Turaev [7] through the Dehn surgery representation of the 3-manifold and using the representation theory of the quantum group $\bar{U}_\xi(\mathfrak{g})$ for ξ being a root of unity and $\mathfrak{g}$ the Lie algebra of G . The result gives a powerful topological invariant of 3-manifolds, commonly known as Witten-Reshetikhin-Turaev (WRT) invariant. The invariants of 3-manifolds and their link counterparts constructed from quantum groups are also often referred to as *quantum invariants*.

On the other hand, Dijkgraaf and Witten [17] proposed a toy model for Chern-Simons theory in which the gauge group G is a finite group. In this setting, the path integral reduces to a finite sum over flat G -connections and the corresponding invariant can be computed from a triangulation of the manifold. These invariants further depend on a choice of 3-cocycle $\omega \in H^3(G, \mathbb{U}(1))$, also known as a twist, which plays the role of the topological action of the theory. In contrast, Chern-Simons theory for a compact simply connected simple group G (such as $\mathrm{SU}(2)$) depends on an integer parameter k , known as the level. The corresponding quantum parameter in the WRT construction is¹ $\xi = e^{\frac{2\pi i}{k}}$. More precisely, this integer is to be understood as an element of $H^4(G, \mathbb{Z}) \simeq \mathbb{Z}$, whose generator is the second Chern class used to define the action of the theory. Since $H^4(G, \mathbb{Z}) \simeq H^3(G, \mathbb{U}(1))$ for a finite group G , a choice of level is then analogous to a choice of twist in the discrete scenario.

In Chapter 3, we will be interested in comparing the topological invariants (physically realized by partition functions) obtained from the continuous and discrete cases, where an appropriate choice of gauge groups (and topological actions) makes such a comparison meaningful. More specifically, we will consider the algebraic group $\mathrm{SU}(2)$ on the continuous side, and the same algebraic group, but over a finite field, on the discrete side. We recall that finite fields $\mathbb{F}_q$ are completely classified by their number of elements, q , which must be a power of a prime: $q = p^r$. The field has a Frobenius automorphism $x \mapsto x^p$ which has order r . Note that the definition of the ordinary $\mathrm{SU}(2) \equiv \mathrm{SU}(2, \mathbb{C})$ as a subgroup of 2×2 invertible matrices $\mathrm{GL}(2, \mathbb{C})$, apart from the usual algebraic operations, involves complex conjugation – an order 2 automorphism of $\mathbb{C}$. Because of this, its finite field analogue $\mathrm{SU}(2, \mathbb{F}_{q^2})$ can be defined only over the field with q^2 elements, where the analogue of conjugation is played by the order 2 automorphism $x \mapsto x^q$ (in a sense, the field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ is analogous to the extension $\mathbb{C}/\mathbb{R}$). However it is known that $\mathrm{SU}(2, \mathbb{F}_{q^2}) \cong \mathrm{SL}(2, \mathbb{F}_q)$. Similarly, one can

¹We use k for the renormalized level, which differs by 2 from the classical one.

consider $\mathrm{SL}(2, \mathbb{C})$ as a complexification of $\mathrm{SU}(2)$, with $\mathrm{SU}(2)$ being the maximal compact subgroup. The gauge group $\mathrm{SL}(2, \mathbb{C})$ plays a crucial role in the *analytic continuation* of the $\mathrm{SU}(2)$ Chern-Simons theory [18, 19], and will do so also in our work.

We find that the asymptotics of the DW theory when the number of elements of the finite field is large recovers the leading asymptotics of the CS theory at large level when the 3-manifold contains no hyperbolic components. A priori, it is not obvious why a relation between these two kinds of invariants should exist. One motivation comes from the celebrated Weil conjectures, which in particular imply that there is a close relationship between the algebraic varieties over complex numbers and finite fields. Namely, some topological information about the complex variety can be recovered from the count of points of the same variety over finite fields. Algebraic varieties appear naturally in the context of topological gauge theories as representation/character varieties of fundamental groups, when the group is algebraic. In particular, for the group over a finite field, the count of points of the representation variety, up to a normalization factor, equals the partition function of the corresponding untwisted DW theory. On the other hand, for the group over complex numbers, the variety is the space of critical points – flat connections – of the Chern-Simons functional, which plays a crucial role in the semi-classical expansion of the path-integral. We note that the $\mathrm{SL}(2, \mathbb{F}_q)$ character varieties of 3-manifolds have been studied in numerous contexts [20–25], but to the best of our knowledge, our work is the first instance in which a direct parallel with the $\mathrm{SU}(2)$ Chern-Simons theory is drawn. The count of points of $\mathrm{SL}(2, \mathbb{F}_q)$ character varieties of 2-manifolds (Riemann surfaces) was considered in particular in [26, 27].

As a byproduct, we develop efficient techniques to count the number of points over finite fields $\mathbb{F}_q$ of $\mathrm{SL}(2)$ representation varieties of fundamental groups of 3-manifolds, possibly with weights pulled back from a chosen class in $H^3(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1))$.

1.2 Preliminary material

Being relevant in so many areas of physics and mathematics, much could be said about TQFTs and their properties. For the sake of brevity, we refer to the rich literature on the topic, while trying to maintain a self-contained approach. This section aims to provide some background on abelian 3-dimensional TQFTs and Dijkgraaf-Witten TQFTs, which will be the main objects of study in the following chapters, mainly as a way to fix conventions. Useful references for what follows are [8, 28–32].

1.2.1 Abelian TQFTs

The first kind of 3-dimensional TQFTs that appear in this work are abelian TQFTs. These theories constitute a rich subsector of more tractable TQFTs, in the sense that they are easier to work with and depend nicely on the topology of the spacetime, making them the

perfect framework to study the properties of a naive realization of topological quantum gravity.

The world abelian refers to the anyons having abelian fusion rules, meaning that the fusion is group-like. More explicitly, they can be characterized in various ways, for example by requiring that the quantum dimension of each line operator is one. Here, we follow the approach of Deloup [33], describing abelian TQFTs by the defining data of the pair (D, q) of a finite abelian group and a non-degenerate even quadratic form on it. This approach is very explicit and useful when it comes to evaluate the partition function of any 3-manifold geometry, which is a crucial step in reaching our ultimate goal of evaluating the sum over topologies. Moreover, it will help us building some intuition on abelian theories before moving on to more general TQFTs.

The partition function of an arbitrary 3d abelian TQFT

An arbitrary 3d abelian bosonic TQFT can be specified by a finite abelian group D with a non-degenerate even quadratic form $q: D \rightarrow \mathbb{Q}/\mathbb{Z}$, see e.g. [34–36]. The physical theory also depends on the chiral central charge, an integer, with its value modulo 8 fixed by D and q . A quadratic form is said to be non-degenerate and even if the associate bilinear form b_q is so. The group D is called the discriminant group and classifies the bulk line operators of the theory, i.e. the anyons, while q gives the braiding, i.e. the additional phase acquired by two anyons when exchanged. The elements of the group D also give a basis in the Hilbert space of the TQFT assigned to a 2-torus. Namely, a state corresponding to an element $c \in D$ is realized by a solid torus with the corresponding line operator inserted at its core (a circle in the interior to which the solid torus deformation retracts).

Classically, such a TQFT can always be expressed as an even level K abelian Chern-Simons with gauge group the torus $U(1)^n$. By even level K we mean an $n \times n$ integral even symmetric bilinear form, that can be understood as the matrix of coefficients of the $U(1)$ -gauge fields in the Chern-Simons action. Indeed, one can always find the bilinear form K such that $b_q = K^{-1} \bmod 1$, $D = \text{coker } K$, and the chiral central charge is $\sigma(K) - \text{the signature of } K - \text{modulo } 8$.

We introduce the partition function following the approach of Deloup² as in [37] and [33]. We first define the τ -invariant, that is a topological invariant of a pair $(Y, \mathcal{L})$ of a closed, oriented, connected 3-manifold Y and an oriented framed link $\mathcal{L}$ with components colored by the elements of D . The partition function of an abelian TQFT is equal to it up to a normalization, which we will fix later.

Given a closed, oriented, connected 3-manifold Y and an oriented framed g -component link $\mathcal{L} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_g$ in Y , we call link presentation of the pair $(Y, \mathcal{L})$ a pair of links $(\mathcal{J}, \mathcal{J}')$ in S^3 where $\mathcal{J} = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_m$ is the surgery link of Y , while $\mathcal{J}'$ is a link yielding $\mathcal{L}$ after surgery. We will use J and J' to denote the linking matrices of $\mathcal{J}$ and $\mathcal{J}'$ respectively,

²Our convention is slightly different since we replace the τ -invariant with its complex conjugate to make contact with the physics literature.

and L to denote the linking matrix of the total link $\mathcal{J} \cup \mathcal{J}'$. Then, the τ -invariant for the TQFT (D, q) of the manifold $(Y, \mathcal{L})$ with the components of the link colored by $c \in D^g$ is defined in the following way:

$$\tau(Y, \mathcal{L}; D, q; c) = \frac{\exp\left[\frac{\pi i}{4}\sigma(K)\sigma(L)\right]}{|D|^{m/2}} \sum_{x \in \mathbb{Z}^m \otimes D} \exp[-2\pi i(L \otimes q)(x \oplus c)], \quad (1.2.1)$$

where $L \otimes q$ is the unique quadratic form on $\mathbb{Z}^{m+g} \otimes D$ satisfying:

$$(L \otimes q)(x \otimes y) = x^T L x \cdot q(y). \quad (1.2.2)$$

We stress that by definition τ does not distinguish between unknots and non-trivial knots, thus we may assume that $\mathcal{J} \cup \mathcal{J}'$ is a link of unknots.

In the case of a closed 3-manifold without a link, i.e. with $\mathcal{L} = \emptyset$, the formula reads:

$$\tau(Y; D, q) \equiv \tau(Y, \emptyset; D, q; 0) = \frac{\exp\left[\frac{\pi i}{4}\sigma(K)\sigma(L)\right]}{|D|^{m/2}} \sum_{x \in \mathbb{Z}^m \otimes D} \exp[-2\pi i(L \otimes q)(x)], \quad (1.2.3)$$

while the partition function in the physical normalization is:

$$Z(Y; D, q) = \frac{1}{|D|^{(m+1)/2}} \sum_{x \in \mathbb{Z}^m \otimes D} \exp[-2\pi i(L \otimes q)(x)]. \quad (1.2.4)$$

The difference in the two normalizations is such that on a 3-sphere τ evaluates to 1 in any theory, while $Z(S^3) = |D|^{-1/2}$. Moreover, $Z(Y; D, q)$ generally suffers from framing anomaly, which means its phase is ambiguous, unless one fixes the 2-framing of the tangent bundle Y , or the signature of a 4-manifold X bounded by $Y = \partial X$ [6, 38]. The latter can be identified with $\sigma(L)$ ³. Later on, we will restrict ourselves to the case of zero chiral central charge $\sigma(K) = 0$, so that this ambiguity will disappear.

Now, consider the case when $\mathcal{L} \neq \emptyset$. The τ -invariant, up to normalization, then gives the partition function on the compact manifold $M = Y \setminus \bar{U}$, where U is a tubular neighborhood of $\mathcal{L}$. The boundary of M is $\partial M \cong (T^2)^{\sqcup g}$, disjoint union of g -copies of a 2-torus T^2 . This construction allows us to consider the partition function on any compact oriented connected 3-manifold, the boundary of which consists of a disjoint union of any number of tori. We shall enter into the details of this correspondence, as well as the relation to the manifolds with more general boundary components, later in this section. In this interpretation, the elements $c_i \in D$, $i = 1, \dots, g$ serve the role of the basis elements in the Hilbert spaces of the TQFT assigned to each torus.

³The surgery representation of Y provides a 4-manifold X bounded by Y , obtained by attaching 2-handles to the 4-ball D^4 along the tubular neighborhoods of the components of $\mathcal{L} \subset S^3 = \partial D^4$. The signature of this 4-manifold coincides with $\sigma(L)$.

We can then retrieve the correct physical normalization in the presence of boundaries from the knowledge of the physical partition function on a closed manifold. It suffices to enforce the following prescription:

$$\frac{1}{|D|^{g/2}} \sum_{c \in D^g} Z(Y, \mathcal{L}; D, q; c) = Z(Y'; D, q), \quad (1.2.5)$$

where Y' is the closed 3-manifold obtained by surgery on the link $\mathcal{J} \cup \mathcal{J}'$. This fixes the normalization to be:

$$Z(Y, \mathcal{L}; D, q; c) = \frac{1}{|D|^{(m+1)/2}} \sum_{x \in \mathbb{Z}^m \otimes D} \exp[-2\pi i (L \otimes q)(x \oplus c)]. \quad (1.2.6)$$

Once the TQFT is fixed, a crucial result of Deloup [33] says that $\tau(Y, \mathcal{L}; D, q; c)$ is determined by the following data:

- the first homology group $H_1(Y)$;
- the linking pairing $\ell k: \text{Tor}H_1(Y) \otimes \text{Tor}H_1(Y) \rightarrow \mathbb{Q}/\mathbb{Z}$ on the torsion subgroup;
- the framed 1-cycle $\widehat{\theta} = \sum_k c_k \otimes \mathcal{L}_k$ in Y with coefficients in D .

Note the the dependence on $H_1(Y)$ can be split into the dependence on the pair $(b_1(Y), \text{Tor}H_1(Y))$, where $b_1(Y)$ is the first Betti number $b_1(Y)$, which equals to the rank of $H_1(Y)$.

Denote by θ its corresponding class in the homology group $H_1(Y; D) \cong D \otimes H_1(Y)$. As was shown in [33], the invariant τ vanishes unless $\theta \in D \otimes \text{Tor}H_1(Y)$ (which is always satisfied for Y a rational homology sphere). Moreover, if this condition holds, then either τ vanishes or:

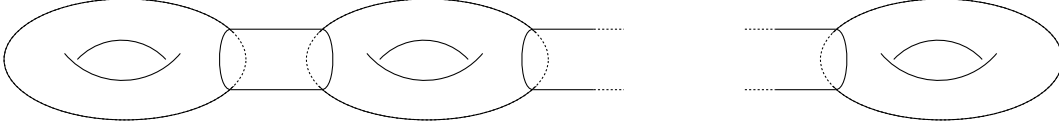
$$\tau(Y, \mathcal{L}; D, q; c) = |H^1(Y; D)|^{1/2} \exp \left[2\pi i \alpha(\text{Tor}H_1(Y), \ell k, \widehat{\theta}) \right], \quad (1.2.7)$$

with α taking value in $\mathbb{Q}/\mathbb{Z}$. We refer to [33] for an explicit expression of α and a characterization of the vanishing of the τ -invariant.

Similar considerations also apply to the partition function Z and can be deduced from the τ -invariant.

Manifolds with general boundary

In order to consider manifolds with boundary components of any genus, we will proceed as follows. First, let us fix a particular solid torus V_1 (i.e. a specific representative in the homeomorphism class). Then its boundary gives a particular 2-torus $T^2 := -\partial V_1$, where, as before, the minus refers to the opposite orientation. Next, we construct a genus g handle-body V_g as a gluing of g copies of V_1 and $g - 1$ copies of the solid cylinder

Figure 1.2.1: The genus g handle-body V_g .

$D^2 \times [0, 1]$, as depicted in Figure 1.2.1. Namely, we choose an embedding of D^2 into the first and the g -th copies of V_1 and a pair of non-overlapping embeddings of D^2 into each of the other copies. Using such embeddings, we attach the first cylinder to the first and the second copies of V_1 , the second cylinder to the second and the third copies of V_1 , and so on. The boundary of the resulting handle-body V_g gives a particular genus g surface $\Sigma_g := -\partial V_g$. This provides a realization of the TQFT Hilbert space $\mathcal{H}(\Sigma_g)$ in terms of the line operators. Namely, for an abelian TQFT, the basis elements can be given by the insertion of a line colored by $c_i \in D$ at the core of the i -th copy of V_1 in V_g . That is we have fixed the isomorphisms:

$$\mathcal{H}(\Sigma_g) \cong \mathcal{H}(T^2)^{\otimes g} \cong \left(\bigoplus_{c \in D} \mathbb{C}|c\rangle \right)^{\otimes g}. \quad (1.2.8)$$

Now let us consider a compact manifold M with a choice of isomorphism $\phi : \partial M \xrightarrow{\sim} \Sigma = \Sigma_{g_1} \sqcup \Sigma_{g_2} \sqcup \dots$. As was already considered in the introduction, any such pair can be realized as a complement $Y \setminus \iota(V)$, in a closed manifold Y , with an embedding $\iota : V \rightarrow Y$, where $V = V_{g_1} \sqcup V_{g_2} \sqcup \dots$, so that $\phi = \iota|_{\partial V}$. The embedding ι can be restricted to the solid tori V_1 in the handle-bodies V_{g_i} . Such a restriction would produce a different manifold⁴ $M^{f.t.} := Y \setminus \iota(V_1^{\sqcup(g_1+g_2+\dots)})$ with $\phi^{f.t.} := \iota|_{\partial V_1^{\sqcup(g_1+g_2+\dots)}} : (T^2)^{\sqcup(g_1+g_2+\dots)} \xrightarrow{\sim} \partial M^{f.t.}$. Equivalently, one can obtain $M^{f.t.}$ by attaching $(g_1-1) + (g_2-1) + \dots$ copies of the solid cylinder $D^2 \times [0, 1]$ to M along the cylinders $S^1 \times [0, 1]$ at the boundary, according to the way Σ_{g_i} were defined above. The manifolds M and $M^{f.t.}$ are schematically illustrated in Figure 1.2.2.

Let $|M, \phi\rangle \in \bigotimes_i \mathcal{H}(\Sigma_{g_i})$ be the state provided by the TQFT on the manifold M and $\langle c^{(i)}| \in \mathcal{H}(-\Sigma_{g_i}) \cong \mathcal{H}(\Sigma_{g_i})^*$, $c^{(i)} \in D^{g_i}$, the state provided by V_{g_i} with the lines $c_j^{(i)}$, $j = 1, \dots, g_i$ inserted at the cores of the g_i solid tori V_1 inside V_{g_i} . We then have:

$$\left(\langle c^{(1)}| \otimes \langle c^{(2)}| \otimes \dots \right) |M, \phi\rangle = \left(\bigotimes_{i,j} \langle c_j^{(i)}| \right) |M^{f.t.}, \phi^{f.t.}\rangle = Z \left(Y, \mathcal{L}; D, q; \bigoplus_i c^{(i)} \right), \quad (1.2.9)$$

where $\mathcal{L}$ is the framed $g = g_1 + g_2 + \dots$ -component link in Y obtained by the restriction of the embedding $\iota : V \hookrightarrow Y$ to the cores of solid tori V_1 inside $V = \sqcup_i V_{g_i}$ and their normal bundles.

⁴*f.t.* stands for “filled tubes”.



Figure 1.2.2: The complement $Y \setminus \iota(V) = M$ on the left and the filled-tubes manifold $M^{f,t}$ on the right.

Reciprocity formulae for Gauss sums

Expressions like the one in the partition function (1.2.1) are called quadratic Gauss sums, or generalizations thereof. They are quite challenging to evaluate directly. Luckily, we have powerful tools at our disposal, called reciprocity formulae for Gauss sums. Not only are they crucial in computing our partition functions but also they give rise to highly non-trivial dualities among abelian TQFTs.

The first reciprocity formula we state is due to Deloup [37]. Let d be a non-zero integer and A an even $m \times m$ matrix of integers of rank r . We can always find a non-singular matrix $\tilde{A}$ such that there exists a unimodular matrix P satisfying $P^T A P = \tilde{A} \oplus 0_{m-r}$. Then, the reciprocity formula is:

$$d^{-m/2} \sum_{x \in (\mathbb{Z}/d\mathbb{Z})^m} \exp \left[\frac{\pi i}{d} \hat{x}^T A x \right] = \frac{d^{(m-r)/2} e^{\pi i/4 \sigma(A)}}{|\det \tilde{A}|^{1/2}} \sum_{y \in \text{coker } \tilde{A}} \exp \left[-\pi i d \hat{y}^T \tilde{A}^{-1} \hat{y} \right], \quad (1.2.10)$$

where $\hat{x}$ and $\hat{y}$ are lifts in $\mathbb{Z}^m$ and $\mathbb{Z}^r$ respectively. We stress that this formula can be helpful in determining the partition function for closed manifolds, as we shall show in what follows.

A non-empty boundary introduces a linear term in (1.2.1) in addition to the quadratic one. Thus, we need a more general formula that has been proven in [39]. Let A and d be as above, with $P^T A P = \tilde{A} \oplus 0_{m-r}$. Given $\psi = \psi_1 \oplus \psi_2 \in \mathbb{R}^r \oplus \mathbb{R}^{m-r}$ such that $d\psi \in \mathbb{Z}^m$, the reciprocity formula is:

$$\begin{aligned} d^{-m/2} \sum_{x \in (\mathbb{Z}/d\mathbb{Z})^m} \exp \left[\frac{\pi i}{d} (\hat{x}^T A x + 2d \hat{x}^T \psi) \right] &= \\ &= \delta(\psi_2 \in \mathbb{Z}^{m-r}) \frac{d^{(m-r)/2} e^{\pi i/4 \sigma(A)}}{|\det \tilde{A}|^{1/2}} \sum_{y \in \text{coker } \tilde{A}} \exp \left[-\pi i d (\hat{y} + \psi_1)^T \tilde{A}^{-1} (\hat{y} + \psi_1) \right]. \end{aligned} \quad (1.2.11)$$

The latter reciprocity formula can be used, in particular, to verify that the surgery formula (1.2.4) for the partition function of an abelian TQFT agrees with other explicit formulas for abelian Chern-Simons theories that have appeared in the literature. Indeed, recall that the partition function of a level K $U(1)^n$ Chern-Simons theory on a closed connected manifold Y can be written in the appropriate normalization⁵ as:

$$Z_K(Y) = \frac{|\det K|^{(b_1(Y)-1)/2}}{|\mathrm{Tor} H_1(Y)|^{n/2}} e^{-\frac{\pi i}{4} \sigma(K)} \sum_{x \in \mathrm{Tor} H_1(Y) \otimes \mathbb{Z}^n} \exp[\pi i (\ell k \otimes K)(x)]. \quad (1.2.12)$$

See for example [40]. This expression looks different from the one for $Z(Y; D, q)$ given above. However, it follows from applying the first reciprocity formula that the two expressions coincide: $Z_K(Y) = Z(Y; D, q)$ given the appropriate D and q .

Moreover, even though it might not be evident, the reciprocity formulae establish a duality between abelian TQFTs. Their effect is to exchange the role of the quadratic form q with the linking matrix L associated with a closed manifold, up to an overall phase proportional to the anomaly. To be more specific, this duality acts on abelian spin-TQFTs, since it can map bosonic theories to fermionic ones. See, for example, [36] for a more detailed discussion of this duality. We remark that because of this correspondence, the sum over closed 3-manifolds could be interpreted as a sum over abelian TQFTs.

Abelian Dijkgraaf-Witten TQFT

So far, we have not specified any particular abelian 3-dimensional TQFT. However, to make our discussion concrete, we will consider abelian Dijkgraaf-Witten TQFTs, which are anomaly-free and admit topological boundary conditions, a feature that will be needed later. Later, we will discuss such theories more thoroughly, as they play a crucial role in the rest of our work.

First introduced by Dijkgraaf and Witten in [17], DW TQFTs are topological gauge theories with a finite gauge group G . On a closed manifold Y , the partition function is:

$$Z(Y) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(Y), G)} \langle \gamma^* \omega, [Y] \rangle, \quad (1.2.13)$$

where $\omega \in H^3(G, U(1))$ and $\gamma^*: H^3(G, U(1)) \rightarrow H^3(Y, U(1))$. We have implicitly used that a map $\gamma \in \mathrm{Hom}(\pi_1(Y), G)$ determines a map $\gamma: Y \rightarrow BG$ (up to homotopy), where BG is the classifying space. We say that the DW theory is twisted by ω unless ω is the trivial cohomology class; in that case, we call the theory untwisted.

The definition of the partition function on a manifold M with boundary is a bit more involved. Let $\phi: \Sigma \xrightarrow{\sim} \partial M \subset M$ and recall that the states in $\mathcal{H}(\Sigma)$ are generated by the elements $c \in \mathrm{Hom}(\pi_1(\Sigma), G)$. Then, the partition function evaluated on such c should only

⁵Such that $Z(S^1 \times S^2) = 1$.

get contributions from maps γ that restrict to c on the boundary, i.e. $\gamma \circ \phi_* = c$, where ϕ_* is the map between the fundamental groups induced by ϕ . However, one also has to deal with the problem that the action of the theory is no longer defined by the formula:

$$e^{2\pi i S[\gamma]} = \langle \gamma^* \omega, [Y] \rangle \quad (1.2.14)$$

due to the absence of the fundamental class $[Y] \in H_3(Y, \mathbb{Z})$ when $\partial Y \neq \emptyset$. This can be resolved in various ways, see e.g. [17, 41, 42]. However, we will not dive into the details of this procedure yet. For a thorough discussion on DW TQFTs we refer to Section 1.2.2.

Let us focus on the case of an untwisted Dijkgraaf-Witten theory with an abelian gauge group G . Since the action is now trivial, the partition function on a closed manifold Y becomes:

$$Z(Y) = \frac{|\text{Hom}(H_1(Y), G)|}{|G|}. \quad (1.2.15)$$

Notice that we are allowed to replace the fundamental group $\pi_1(Y)$ with its abelianization $H_1(Y)$, since G is abelian.

Furthermore, the partition function on a manifold with boundary M , with $\phi : \Sigma \xrightarrow{\sim} \partial M \subset M$, evaluated on a boundary state $c \in \text{Hom}(H_1(\Sigma), G)$ is given simply by:

$$\langle c|M, \phi \rangle = \frac{1}{|G|} |\{\gamma \in \text{Hom}(H_1(M), G) : \gamma \circ \phi_* = c\}|. \quad (1.2.16)$$

The theory always admits the topological boundary condition corresponding to the *Neumann boundary condition* on the gauge fields, given by the boundary state $\langle \mathfrak{B} | = \sum_{c \in \text{Hom}(H_1(\Sigma), G)} \langle c |$. The TQFT partition function with such a boundary condition is:

$$Z(M; \mathfrak{B}) \equiv \langle \mathfrak{B} | M, \phi \rangle = \frac{|\text{Hom}(H_1(M), G)|}{|G|}. \quad (1.2.17)$$

The result is independent of ϕ , as expected. Notice that, without loss of generality, one can consider $\Sigma = (T^2)^{\sqcup g}$.

To see the agreement of (1.2.17) with the general analysis on the Deloup's τ -invariant, we need to rewrite the formula in terms of the triple $(H_1(Y), \ell k, f)$, where, as before, Y is the closed manifold obtained from M by gluing back g solid tori V_1 and $f = \iota_*$ is the map induced in the first homology by the inclusion $\iota : V_1^{\sqcup g} \hookrightarrow Y$ of these solid tori in Y .

First, we observe that $\text{Hom}(H_1(M), G) \cong H_2(M, \partial M; G)$ by the universal coefficient theorem and Poincaré duality. Then, consider the long exact sequence for the relative homology of the pair (Y, U) with G -coefficients, $U = \iota(V_1^{\sqcup g})$ being the image of the solid tori in Y :

$$\cdots \rightarrow H_2(U; G) \rightarrow H_2(Y; G) \rightarrow H_2(Y, U; G) \rightarrow H_1(U; G) \rightarrow H_1(Y; G) \rightarrow \cdots \quad (1.2.18)$$

Applying the universal coefficient theorem, we get:

$$\cdots \rightarrow 0 \longrightarrow H_2(Y; G) \longrightarrow H_2(Y, U; G) \longrightarrow G^g \xrightarrow{f \otimes G} H_1(Y) \otimes G \rightarrow \cdots \quad (1.2.19)$$

From the exactness of the above sequence, we have:

$$|H_2(Y, U; G)| = |H_2(Y; G)| \cdot |\ker(f \otimes G)|. \quad (1.2.20)$$

Applying Poincaré duality and the excision theorem, it follows that $H_2(Y; G) \cong H^1(Y; G) \cong \text{Hom}(H_1(Y); G)$ and $H_2(Y, U; G) \cong H_2(M, \partial M; G) \cong H^1(M; G) \cong \text{Hom}(H_1(M), G)$. Therefore:

$$\begin{aligned} Z(M; \mathfrak{B}) &= |G|^{-1} |\text{Hom}(H_1(Y); G)| \cdot |\ker(f \otimes G)| = \\ &= |G|^{b_1(Y)-1} |\text{Tor} H_1(Y) \otimes G| \cdot |\ker(f \otimes G)|. \end{aligned} \quad (1.2.21)$$

In terms of the general classification of abelian TQFTs, such a theory has the discriminant group $D = G \oplus G^*$, where $*$ stands for the Pontryagin dual, i.e. $G^* = \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$, and the quadratic form $q : D \rightarrow \mathbb{Q}/\mathbb{Z}$ is given by $q(a, \omega) = \omega(a)$. Finally we remark that the considered Neumann boundary condition corresponds to the Lagrangian subgroup $\mathfrak{L} = G \subset D$. In the next section, we will show how the result above can be reproduced via the surgery formula (1.2.6) for abelian TQFTs.

Untwisted Dijkgraaf-Witten TQFT with a cyclic gauge group

For simplicity, we consider the case of an untwisted DW theory with cyclic gauge group $G = \mathbb{Z}/N\mathbb{Z}$. The general case of G being the direct sum of cyclic groups follows immediately by properties of the τ -invariant. Indeed, one easily checks that the partition function factorizes with respect to the direct sum of the discriminant group.

The discriminant group is given by $D = G \oplus G^* \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$, with the quadratic function $q(u, v) = uv/N \pmod{1}$, $u, v \in \mathbb{Z}/N\mathbb{Z}$. Such a theory can be realized as a $U(1)^2$ Chern-Simons theory with the level matrix:

$$K_{\text{DW}} = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}. \quad (1.2.22)$$

Indeed, we have:

$$D = \text{coker } K_{\text{DW}} = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}, \quad (1.2.23)$$

and:

$$q(u, v) = 1/2 (u, v)^T K_{\text{DW}}^{-1} (u, v) \pmod{1} = uv/N \pmod{1}. \quad (1.2.24)$$

We can then compute the partition function on a closed manifold Y using the formula (1.2.12) for a general abelian CS theory. First of all, notice that the theory is anomaly-free since the signature of K_{DW} is vanishing. We recover the usual formula (1.2.15) for untwisted

Dijkgraaf-Witten TQFT as follows:

$$\begin{aligned}
Z(Y) &= \frac{N^{b_1(Y)-1}}{|\mathrm{Tor} H_1(Y)|} \sum_{x,y \in \mathrm{Tor} H_1(Y)} e^{2\pi i N \ell k(x,y)} \\
&= N^{b_1(Y)-1} |\{x \in \mathrm{Tor} H_1(Y) : Nx = 0\}| \\
&= N^{b_1(Y)-1} |\mathrm{Tor} H_1(Y) \otimes \mathbb{Z}/N\mathbb{Z}| = |G|^{-1} |\mathrm{Hom}(\pi_1(Y), G)|. \tag{1.2.25}
\end{aligned}$$

More generally, given a bordered manifold (M, ϕ) with boundary $(T^2)^{\sqcup g}$, we adopt the description in terms of a pair of framed links $(\mathcal{J}, \mathcal{J}')$ in S^3 . Then, the general surgery formula (1.2.6) for the partition function gives us:

$$\langle c|M, \phi \rangle = Z(\mathcal{J}, \mathcal{J}'; D, q; c) = \frac{1}{N^{m+1}} \sum_{x \in \mathbb{Z}^m \otimes (\mathbb{Z}/N\mathbb{Z})^2} \exp[-2\pi i (L \otimes q)(x, c)]. \tag{1.2.26}$$

Recall that the linking matrix L has the following form:

$$L = \begin{pmatrix} J & F \\ F^T & J' \end{pmatrix}. \tag{1.2.27}$$

Thus, taking $x = (u, v) \in (\mathbb{Z}/N\mathbb{Z})^m \oplus (\mathbb{Z}/N\mathbb{Z})^m$ and $c = (a, b) \in (\mathbb{Z}/N\mathbb{Z})^g \oplus (\mathbb{Z}/N\mathbb{Z})^g$, we observe that:

$$(L \otimes q)((u, v), (a, b)) = \frac{1}{N} (u^T J v + a^T J' b + v^T F a + u^T F b) \pmod{1}. \tag{1.2.28}$$

Next, we impose a topological boundary condition. As we have already stressed, Dijkgraaf-Witten TQFTs always admit topological boundary conditions. In particular, in the untwisted case, one can always choose a Lagrangian subgroup $\mathfrak{L} \subseteq D = G \oplus G^*$ of the form $\mathfrak{L} = G$, which is equivalent to the Neumann boundary condition considered in the previous section. In the case of the cyclic gauge group, the corresponding boundary state then reads:

$$\langle \mathfrak{L} | = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^g} \langle (a, 0) |, \tag{1.2.29}$$

and the quadratic form (1.2.28) can be reduced to:

$$(L \otimes q)((u, v), (a, 0)) = \frac{1}{N} (u^T J v + v^T F a) \pmod{1}. \tag{1.2.30}$$

The next step is applying a more general reciprocity formula for Gauss sums proven by Deloup and Turaev in [39]. Doing so, we find:

$$\begin{aligned}
\langle (a, 0) | M, \phi \rangle &= \frac{1}{N |\det \tilde{J}|} \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^{b_1(Y)}} \exp \left[\frac{2\pi i}{N} x^T F_{\mathrm{free}} a \right] \times \\
&\quad \sum_{r, s \in \mathrm{coker} \tilde{J}} \exp \left[2\pi i (N r^T \tilde{J}^{-1} s + r^T \tilde{J}^{-1} F_{\mathrm{tor}} a) \right], \tag{1.2.31}
\end{aligned}$$

where $\tilde{J}$ is the non-singular matrix such that $P^T J P = \tilde{J} \oplus \mathbf{0}_{b_1(Y)}$ for some unimodular matrix P , and $F_{\text{tor}} \oplus F_{\text{free}} = P^T F$. Moreover, the result does not depend on $F_{\text{tor}} a$ but rather on its image θ_{tor} in $\text{coker } \tilde{J} \otimes \mathbb{Z}/N\mathbb{Z}$. Observe that $\text{coker } \tilde{J}$ and $(\tilde{J}^{-1} \bmod 1)$ are respectively the torsion part of the first homology group and the linking form of the closed 3-manifold Y obtained by gluing back g solid to M according to the map ϕ .

The first sum forces $\theta_{\text{free}} := (F_{\text{free}} a \bmod N) = 0 \in \mathbb{Z}^{b_1(Y)} \otimes \mathbb{Z}/N\mathbb{Z}$, otherwise the partition function vanishes. Similarly, the second sum forces $\theta_{\text{tor}} = 0 \in \text{Tor} H_1(Y) \otimes \mathbb{Z}/N\mathbb{Z}$. We obtain:

$$\langle (a, 0) | M, \phi \rangle = N^{b_1(Y)-1} |\text{Tor} H_1(Y) \otimes \mathbb{Z}/N\mathbb{Z}| \delta(\theta \equiv \theta_{\text{tor}} \oplus \theta_{\text{free}} = 0). \quad (1.2.32)$$

Recall that F , composed with the projection to $\text{coker } J$, gives the map $f: H_1(V_1^{\sqcup g}) \cong \mathbb{Z}^g \rightarrow H_1(Y)$. Therefore, the vanishing condition on θ is equivalent to the request that a belongs to the kernel of $f \otimes \mathbb{Z}/N\mathbb{Z}$.

Finally, evaluating the partition function on the boundary state (1.2.29) yields:

$$Z(M; \mathfrak{L}) \equiv \langle \mathfrak{L} | M, \phi \rangle = N^{b_1(Y)-1} |\text{Tor} H_1(Y) \otimes \mathbb{Z}/N\mathbb{Z}| \cdot |\ker(f \otimes \mathbb{Z}/N\mathbb{Z})|, \quad (1.2.33)$$

which is in agreement with (1.2.21).

1.2.2 Dijkgraaf-Witten theories

We have already encountered Dijkgraaf-Witten theories in the context of abelian TQFTs, where we noticed how their simplicity made them the perfect choice for defining a toy model of topological quantum gravity. However, DW TQFTs have in general non-abelian fusion rules and deserve to be studied in more detail.

As we have already anticipated, Dijkgraaf-Witten theories are gauge theories with a finite gauge group G . Their partition function (1.2.13) on an oriented closed 3-manifold M is a topological invariant, depending on the choice of a group cohomology class $\omega \in H^3(G, U(1))$. It is a fact that Dijkgraaf-Witten theory satisfies the axioms of a 3-dimensional TQFT [41].

When the cohomology class is trivial⁶, for an algebraic G , the partition function simply counts the number of points in the G *representation variety* $\text{Hom}(\pi_1(M), G)$ of the manifold, up to the overall normalization factor⁷. We note that this count is closely related, but not exactly the same, as the count of points in the *character variety* $\text{Hom}(\pi_1(M), G) // G$.

We now proceed to study in the detail the Dijkgraaf-Witten theories and how to explicitly compute their partition functions in the twisted case. The first step in computing (1.2.13) consists in determining the space of flat connections, $\text{Hom}(\pi_1(M), G)$. Suppose

⁶The modular tensor category that describes the untwisted version was studied in [43]. In our context, however, we will be interested in computing partition functions on closed manifolds in the twisted version.

⁷More naturally, the ratio $|\text{Hom}(\pi_1(M), G)|/|G|$ can be understood as the “number of points” in the $\text{Hom}(\pi_1(M), G)/G$ stack.

that we are given a finite presentation of the fundamental group of M , consisting of m generators x_i and n relations $R_j(x_1, \dots, x_m)$. The set of flat connections is in one-to-one correspondence with the set of solutions to the system

$$R_j(X_1, \dots, X_m) = \mathbb{1}, \quad \forall j \in \{1, \dots, n\}, \quad (1.2.34)$$

where the X_i are elements in G . This system of equations is generally hard to solve analytically. However, we will showcase several examples in which this is possible in Section 3.4, when we will specify to $G = \text{SL}(2, \mathbb{F}_q)$.

The next step is to compute the topological action $\langle \gamma^* \omega, [M] \rangle$. To this end, we resort to an alternative definition of Dijkgraaf-Witten theory as a *lattice gauge theory*. Fix a triangulation of the 3-manifold M as well as an ordering of its N vertices $\{v_1, \dots, v_N\}$. A G -coloring of M is an assignment of elements $g_{ij} \in G$ to each link ℓ_{ij} , oriented from vertex v_j to vertex v_i . For consistency, we define $\ell_{ji} = \ell_{ij}^{-1}$. Such an assignment defines a flat connection over M if and only if the holonomies around 2-simplices are all trivial. That is, for any 2-simplex with vertices v_i, v_j and v_k , the equation

$$g_{ij}g_{jk}g_{ki} = 1 \quad (1.2.35)$$

holds. Given a G -coloring φ , we can then compute the phase $\langle \gamma^* \omega, [M] \rangle$ of the corresponding flat connection γ as follows. The ordering on the vertices of the triangulation of M induces an ordering on the vertices of each tetrahedron. If τ is a tetrahedron with ordered vertices $\{v_{i_0}, v_{i_1}, v_{i_2}, v_{i_3}\}$, we assign to it the phase factor $W(\tau; \varphi) = \omega(g_{i_0 i_1}, g_{i_1 i_2}, g_{i_2 i_3})$, where we understand ω as a group cochain from $G^3 \rightarrow \text{U}(1)$. The topological action is then a product of the single contributions from each tetrahedron in the triangulation,

$$\langle \gamma^* \omega, [M] \rangle = \prod_{\tau} W(\tau; \varphi)^{\epsilon_{\tau}}, \quad (1.2.36)$$

where $\epsilon_{\tau} = +1$ if τ has the same orientation as M , and -1 otherwise. From the lattice gauge theory perspective, the Dijkgraaf-Witten partition function reads

$$Z_{\text{DW}}(M) = \frac{1}{|G|^N} \sum_{G\text{-colorings } \varphi} \prod_{\tau} W(\tau; \varphi)^{\epsilon_{\tau}}. \quad (1.2.37)$$

Naturally, this object does not depend on the particular choice of triangulation of M nor does it depend on the choice of representative for ω [42]. Notice also that different colorings may correspond to the same flat connection, which explains the normalization factor in the partition function.

In practice, triangulations for various kinds of 3-manifolds can be obtained with the help of software packages such as Regina [44] or SnapPy [45]. However, we would like to stress that not all triangulations obtained this way are suitable for computing Dijkgraaf-Witten invariants. Indeed, ordering the vertices (and consequently the links) of a triangulation

introduces an additional structure known as a *branching structure*. A branching consists of a choice of orientation for the links such that no 2-simplex has a cyclic edge orientation. A tetrahedron in a branched triangulation can then be of essentially two types, shown in Figure 1.2.3. Branched triangulations generally require a larger number of tetrahedra

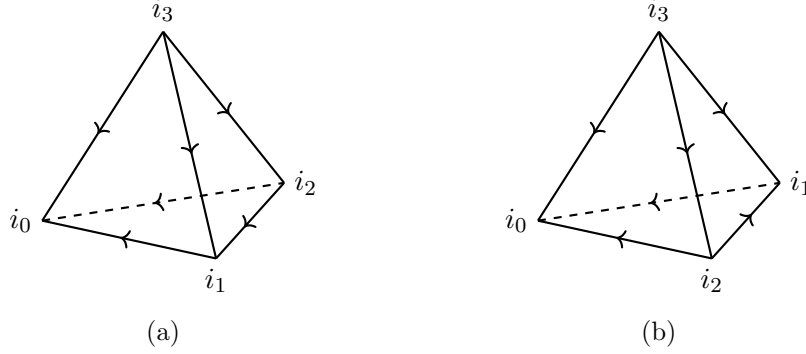


Figure 1.2.3: A tetrahedron in a branched triangulation can be of two types: (a) positively oriented, or (b) negatively oriented.

compared to unbranched ones, making the calculation of $Z_{\text{DW}}(M)$ more demanding computationally. A way to circumvent the necessity for a branched triangulation is to consider a choice of representative for ω satisfying

$$\omega(g, g^{-1}, h) = \omega(g, h, h^{-1}) = 1, \quad (1.2.38)$$

for all $g, h \in G$ [17]. Within this prescription, the phases $W(\tau; \phi)$ are invariant under reorderings of the vertices. Unfortunately the existence of such a representative is not always guaranteed, and depends on certain divisibility conditions which we review in Appendix B. Luckily, we are able to bypass the need for a triangulation by considering a large class of examples, i.e. plumbed manifolds, whose partition functions can be easily computed by other means. We will shortly review plumbed manifolds later in this section.

States and Operators

The first insight on the states of the theory comes from the dimension of the Hilbert space $\mathcal{H}(\Sigma_g)$, which coincides with the partition function $Z_{\text{DW}}(S^1 \times \Sigma_g)$. One immediately finds that $\mathcal{H}(S^2)$ is one-dimensional, while the result for the torus is non-trivial and needs some preliminaries. Given a normalized cocycle⁸ $\omega \in H^3(G, \text{U}(1))$, we can always define a family

$$\beta_a(g, h) = \frac{\omega(a, g, h)\omega(g, h, (gh)^{-1}agh)}{\omega(g, g^{-1}ag, h)} \quad (1.2.39)$$

⁸In practice we often work with explicit representatives of cohomology classes, but it should be clear that every result is independent of this choice.

of twisted 2-cocycles of G for each $a \in G$, meaning they satisfy a twisted version of the cocycle condition:

$$\beta_a(g, h)\beta_a(gh, k) = \beta_a(g, hk)\beta_{g^{-1}ag}(h, k). \quad (1.2.40)$$

However, notice that if we restrict g to be in the centralizer $C_G(a)$ of a , then β_a gives rise to a 2-cocycle of $C_G(a)$.

It is important to observe that a 2-cocycle $\beta \in H^2(G, \mathbb{U}(1))$ can be used to define projective β -representations, namely maps $\tilde{\rho} : G \rightarrow \text{GL}(V)$ such that $\tilde{\rho}(x)\tilde{\rho}(y) = \beta(x, y)\tilde{\rho}(xy)$. Clearly, the untwisted version recovers ordinary linear representations. Therefore, projective β -characters are analogously defined as the traces of projective β -representations.

The result for the torus partition function can be written as

$$Z_{\text{DW}}(S^1 \times S^1 \times S^1) = \sum_{a \in C} r(C_G(a), \beta_a), \quad (1.2.41)$$

where $r(G; \beta)$ is the number of linearly inequivalent irreducible β -representations of G and the sum is taken over the conjugacy classes of G .

From this piece of data we can extract a basis for the Hilbert space $\mathcal{H}(\mathbb{T}^2)$. It is given by pairs $(a, \tilde{\chi})$, where a is a representative of a conjugacy class K_a in G and $\tilde{\chi}$ is an irreducible projective β_a -character of $C_G(a)$. We stress that this makes sense since $\beta_a \in H^2(C_G(a), \mathbb{U}(1))$. A special role is played by the element $0 := (\mathbb{1}, \chi_{\text{triv}})$ which corresponds to the vacuum state in $\mathcal{H}(\mathbb{T}^2)$. Notice that G acts by conjugation on states

$$g \cdot (a, \tilde{\chi}) = (gag^{-1}, \tilde{\chi}^g), \quad (1.2.42)$$

where $\tilde{\chi}^g(h) = \tilde{\chi}(g^{-1}hg)$.

Then, the operators of the theory are 1-dimensional topological defects labeled by group-theoretic data. Moreover, we can compute expectation values as follows. Suppose we are given a framed link $\mathcal{L}$ in S^3 , with each component colored by a state $(a, \tilde{\chi})$, and call μ_i and λ_i the meridians and longitudes of its components. They can be canonically chosen by requiring that the longitude is drawn parallel to the knot in blackboard framing and that the meridian has linking number one with the longitude. Therefore, $[\mu_i, \lambda_i] = e$ in $\pi_1(S^3 \setminus \mathcal{L})$ and we have

$$\langle \mathcal{L}; (a_i, \tilde{\chi}_i) \rangle := \sum_{\substack{\phi \in \text{Hom}(\pi_1(S^3 \setminus \mathcal{L}), G) \\ \phi(\mu_i) = h_i a_i h_i^{-1} \in K_{a_i}}} \prod_i \tilde{\chi}_i(h_i^{-1} \phi(\lambda_i) h_i). \quad (1.2.43)$$

Notice that since $[\phi(\lambda_i), \phi(\mu_i)] = e$, we have that $h_i^{-1} \phi(\lambda_i) h_i$ belongs to $C_G(a_i)$.

Modular data

In a TQFT, much information is encoded in the action of the mapping class group of the torus on its Hilbert space of quantum states $\mathcal{H}(\mathbb{T}^2)$. Recall that the mapping class group of the torus is $\mathrm{SL}(2, \mathbb{Z})$, whose generators are usually denoted by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (1.2.44)$$

satisfying $S^2 = -\mathbb{1}$ and $(ST)^3 = \mathbb{1}$. The S matrix corresponds to swapping the longitude and the meridian of the torus, while the T matrix encodes the action of a Dehn twist along the meridian. The $\mathrm{SL}(2, \mathbb{Z})$ -action on the Hilbert space $\mathcal{H}(\mathbb{T}^2)$ is then represented by matrices, which we also denote by S and T , related to the expectation values of the Hopf link and framed unknot respectively:

$$\frac{S_{ab}}{S_{00}} = \langle \textstyle\bigcirc_a \bigcirc_b \rangle, \quad \frac{T_{aa} S_{0a}}{S_{00}} = \langle \textstyle\bigcirc_a^{+1} \rangle. \quad (1.2.45)$$

The overall data of the finite-dimensional Hilbert space and the pair of S and T matrices acting on it is known as the *modular data* of the TQFT.

Before delving into the details, we remark that S and T matrices will allow us to compute the DW partition function for a large class of 3-manifolds, bypassing the problem of finding a triangulation. This is the case for plumbed manifolds and torus bundles.

We review the modular data for any finite group G , following [46]. The S and T matrices are expressed in terms of the data associated with the states, but the general formula is quite involved. However, we shall describe an important simplification that occurs when β_a is a coboundary on $C_G(a)$ for any $a \in G$.

Assuming that β_a is cohomologically trivial, the projective characters are the same as the ordinary characters – thus, in the basis for $\mathcal{H}(\mathbb{T}^2)$, we can replace the pair $(a, \tilde{\chi})$ with the pair (a, χ) , with χ an irreducible character of $C_G(a)$. Furthermore, by definition there exists a 1-cochain $\varepsilon_a : C_G(a) \rightarrow \mathrm{U}(1)$ for which

$$\beta_a(g, h) = \frac{\varepsilon_a(g)\varepsilon_a(h)}{\varepsilon_a(gh)}, \quad \varepsilon_{x^{-1}ax}(x^{-1}gx) = \frac{\beta_a(x, x^{-1}gx)}{\beta_a(g, x)}\varepsilon_a(g), \quad (1.2.46)$$

for all $g, h \in C_G(a)$ and $x \in G$. Then the modular matrices read

$$S_{(a, \chi), (b, \psi)} = \frac{1}{|C_G(a)||C_G(b)|} \sum_{g \in G(a, b)} \chi(gbg^{-1})^* \psi(g^{-1}ag)^* \sigma(a | bgb^{-1})^*, \quad (1.2.47)$$

$$T_{(a, \chi), (b, \psi)} = \delta_{a, b} \delta_{\chi, \psi} \frac{\chi(a)}{\chi(\mathbb{1})} \varepsilon_a(a), \quad (1.2.48)$$

where we introduced $\sigma(h | g) = \varepsilon_h(g)\varepsilon_g(h)$ and $G(a, b) = \{g \in G \mid agbg^{-1} = bgb^{-1}a\}$.

The partition function of plumbed manifolds

The modular data by itself allows us to compute the partition function of a large class of 3-manifolds, called plumbed manifolds, which are described in terms of some combinatorial data, i.e. the plumbing graph.

For the purposes of this work, we restrict our attention to plumbing graphs that are trees, i.e. connected and acyclic graphs, which are moreover closed and oriented in the sense of [47]. An example of such graphs is provided in Figure 1.2.4a. To each vertex i of the graph Γ we assign a non-negative integer g_i , the genus, and an arbitrary integer f_i , the weight. We may now construct a *plumbed* 3-manifold from this graph as follows. To each vertex i we associate a circle bundle over a genus g_i Riemann surface with Euler class f_i . Next, to an edge between vertices i and j we associate the following operation. Cut out a small disk from the base space of each circle bundle — this creates a torus boundary on the total space of each fibration. We then glue the two torus boundaries by the map $S \in \text{SL}(2, \mathbb{Z})$. We denote the resulting 3-manifold by M_Γ .

Some useful properties of the plumbed manifold M_Γ are easily deduced from its plumbing graph Γ . For instance, the *linking matrix* B_Γ of M_Γ is obtained as follows. The off-diagonal elements of B correspond to the number of edges connecting i and j , which are either 0 or 1 in our case. Diagonal elements are instead given by $B_{ii} = f_i$. Furthermore, one can easily deduce the first homology group of M_Γ to be

$$H_1(M_\Gamma) = \mathbb{Z}^2 \sum_i g_i \oplus \text{Coker}(B). \quad (1.2.49)$$

When all genera are zero, the plumbed 3-manifold also admits an alternative representation in terms of Dehn surgery on a framed link as follows. To each vertex we assign an unknot with framing f_i , and two unknots form a Hopf link if the corresponding vertices are connected by an edge. Otherwise, the two unknots are unlinked (see Figure 1.2.4b).

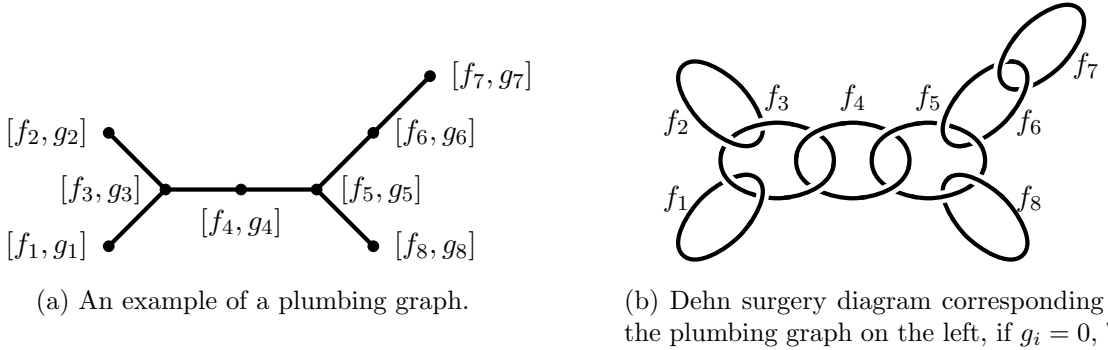


Figure 1.2.4: A plumbing graph and the corresponding framed link.

Remarkably, the partition function of a plumbed manifold M_Γ can be easily expressed

in terms of the modular data, see [48]. Call V and E the collections of vertices and edges of Γ , then we can write

$$Z_{\text{DW}}(M_\Gamma) = \sum_{\{s(v)\}_{v \in V}} \prod_{v \in V} T_{s(v), s(v)}^{f(v)} S_{0, s(v)}^{2-2g(v)-\deg(v)} \prod_{e \in E} S_{s(e_1), s(e_2)}. \quad (1.2.50)$$

Let us explain this expression. To each vertex $v \in V$ we associate a state $s(v) = (a, \tilde{\chi})$ in the basis of $\mathcal{H}(\mathbb{T}^2)$ and we sum over all possible choices. We recall that 0 stands for the vacuum, while $\deg(v)$ is the degree of a vertex $v \in V$, i.e. the number of edges that end in v . Finally, e_1 and e_2 are the two ends of an edge $e \in E$ and $[f(v), g(v)]$ are the weight and the genus of the vertex v , respectively.

Chapter 2

Topological Quantum Gravity

As we have already anticipated in the previous chapter, we are interested in studying the properties of a putative theory of Quantum Gravity by inspecting the consequences of summing over topologies. Indeed, the path integral of this naive realization of Quantum Gravity consists of an integration over all metrics at a fixed spacetime, together with the sum over all possible diffeomorphism classes of the spacetime manifolds, i.e. the “sum over topologies”.

We consider the summation over different topologies separately from the integration over metrics. In particular, we choose a topological action for our Quantum Gravity, which trivializes the integration over the metrics. Hence, we call this hypothetical theory Topological Quantum Gravity, which is perfect for focusing just on the problem of the sum over topologies.

Notice that the sum is in general ill-defined, especially in dimensions $d > 2$. Even though the set of diffeomorphism classes of compact manifolds is countable, their explicit classification is a hard mathematical problem. Moreover, the sum is in general divergent and therefore requires some regularization.

2.1 Introduction

Let us now enter into more details. In d dimensions, for a fixed topology M , one can consider the state $|M, \phi\rangle_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}(\Sigma)$ given by a d -dimensional TQFT $\mathcal{T}$ in its Hilbert space associated to a $(d-1)$ -manifold Σ . Here we explicitly indicate the choice of the diffeomorphism $\phi : \Sigma \xrightarrow{\sim} \partial M \subset M$ as a part of the data on which the state depends. That is, the pair (M, ϕ) defines a bordism from $\emptyset$ to Σ , which is sometimes also referred to as a bordered manifold with boundary Σ . If one understands $\mathcal{T}$ as a functor $\mathbf{Bord}_{d,d-1} \rightarrow \mathbf{Vect}_{\mathbb{C}}$ from the bordism category to the category of complex vector spaces, we have $|M, \phi\rangle_{\mathcal{T}} \equiv \mathcal{T}(M, \phi) \in \text{Hom}(\mathbb{C}, \mathcal{T}(\Sigma)) \cong \mathcal{T}(\Sigma) \equiv \mathcal{H}_{\mathcal{T}}(\Sigma)$. One can then define the d -dimensional topological quantum gravity state corresponding to the transition

from nothing to Σ as a sum:

$$|\Sigma; \mu\rangle_{\mathcal{T}} := \sum_{[M, \phi: \Sigma \xrightarrow{\sim} \partial M]} \mu([M, \phi]) |M, \phi\rangle_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}(\Sigma), \quad (2.1.1)$$

with certain weights $\mu([M, \phi]) \in \mathbb{C}$. Assuming $\mu([M, \phi]) \geq 0$, the collection of all weights can be understood as a measure μ on the set of equivalence classes $[M, \phi]$. The sum is performed over the diffeomorphism classes of bordisms, namely $(M, \phi) \sim (M', \phi')$ if and only if there is a diffeomorphism $\Phi: M \rightarrow M'$ such that $\Phi \circ \phi = \phi'$. In writing such a sum we use the fact that, by definition, the value of the TQFT is the same for equivalent bordisms. We are assuming that the weights are universal, in the sense that they do not depend on the choice of the TQFT $\mathcal{T}$. At the moment we would like to be very generic and we make no additional assumptions; in particular, so far there is no assumption about the independence of the weights on the choice of the diffeomorphism $\phi: \Sigma \xrightarrow{\sim} \partial M$ for a fixed diffeomorphism class $[M]$ of M . The implementation of such an assumption will be considered later in the present Chapter. Note that just naively requiring $\mu([M, \phi])$ to be independent of ϕ may render the sum divergent, as in general there can be infinitely many pairs $[M, \phi]$ for a given $[M]$, so some regularization is required.

The summation over topologies is in tension with the holographic principle. In the most basic form it tells that when a local boundary condition $\mathfrak{B}$ in the theory $\mathcal{T}$ is imposed on Σ , the partition function of a d -dimensional quantum gravity should be equal to a partition function of a $(d-1)$ -dimensional theory $\mathcal{B}$ supported on the boundary: $\langle \mathfrak{B} | \Sigma \rangle_{\mathcal{T}} = Z_{\mathcal{B}}(\Sigma)$, where $\langle \mathfrak{B} |$ is the corresponding boundary state in $\mathcal{H}_{\mathcal{T}}(-\Sigma) \cong \mathcal{H}_{\mathcal{T}}(\Sigma)^*$. Here and throughout our work we use the minus sign to denote the orientation change. When the boundary consists of disconnected components, say $\Sigma = \Sigma_1 \sqcup \Sigma_2$, one expects the factorization $Z_{\mathcal{B}}(\Sigma) = Z_{\mathcal{B}}(\Sigma_1) Z_{\mathcal{B}}(\Sigma_2)$, while a priori $|\Sigma; \mu\rangle_{\mathcal{T}} \neq |\Sigma_1; \mu\rangle_{\mathcal{T}} \otimes |\Sigma_2; \mu\rangle_{\mathcal{T}}$ due to the presence of “wormhole” topologies, that is connected d -manifolds M with boundary $\partial M \cong \Sigma = \Sigma_1 \sqcup \Sigma_2$. One proposed solution to this issue is that one has to consider an *ensemble* of $(d-1)$ -dimensional boundary theories so that the gravity partition function equals to the average of their partition functions with respect to a certain probability distribution of the boundary theories $\mathcal{B}$:

$$\langle \mathfrak{B} | \Sigma; \mu \rangle_{\mathcal{T}} = \langle Z_{\mathcal{B}}(\Sigma) \rangle. \quad (2.1.2)$$

In two dimensions, such a relation between a sum of 2d TQFT partition functions and an ensemble average of 1d TQFTs was worked out explicitly in [15, 49–51] (see also [52] for a general treatment of the sum of partition functions of 2d TQFTs). In three dimensions, the relation between a sum of 3d TQFT partition functions over a certain class of 3-manifolds (with a conformal boundary condition) and an ensemble of 2d Conformal Field Theories (CFT) was considered in [53–58]. The evidence for such a relation for the sum over all 3-manifolds was recently provided in [59–61].

In this chapter, we consider a simplified version of such a holographic relation for 3d topological quantum gravity. For the 3d “bulk” theory, we will have a unitary bosonic abelian TQFT with a *topological* boundary condition on the 2d boundary. It was shown in [36] that this class of 3d TQFTs coincides with abelian Dijkgraaf-Witten TQFTs. We will also argue that, under certain assumptions, the complete state (2.1.1) can be recovered by considering all possible topological boundary conditions. In this setting, one can recast the sum over topologies into a sum over some simple algebraic data. In order to do that, the first step is to realize the manifolds M with boundary Σ as the complements of a collection of handlebodies in a closed manifold. Namely, let us fix a collection of handlebodies V , the boundaries of which are connected components of Σ with orientation reversed: $\partial V = -\Sigma$. Consider its embedding in a closed 3-manifold Y , $\iota : V \hookrightarrow Y$. Any pair (M, ϕ) in the sum (2.1.1) can be realized as $M = Y \setminus \iota(V)$ with $\phi = \iota|_{\partial V} : \Sigma \xrightarrow{\sim} \partial M \subset M$. The diffeomorphism¹ class of (M, ϕ) is the same for the pairs (Y, ι) and (Y', ι') such that there exists a diffeomorphism $\Phi : Y \rightarrow Y'$ satisfying $\iota' = \Phi \circ \iota$, so we will consider the corresponding equivalence relation on the pairs. Up to this equivalence relation the pair (Y, ι) is uniquely determined by the class $[[M, \phi]]$: $(Y, \iota) = (M \cup_{\phi} V, \iota_V)$ where ι_V is the inclusion of V into the union. A different representative (M', ϕ') results in the same class, as $\Phi : M \rightarrow M'$ with $\Phi \circ \phi = \phi'$ extends to $M \cup_{\phi} V \rightarrow M' \cup_{\phi'} V$ by the identity on V . This provides a bijection between the classes $[M, \phi]$ and $[Y, \iota]$.

The sum (2.1.1) then can be rewritten in the following form:

$$|\Sigma; \mu\rangle_{\mathcal{T}} = \sum_{[Y, \iota: V \hookrightarrow Y]} \mu([Y, \iota]) |Y \setminus \iota(V), \iota|_{\partial V}\rangle_{\mathcal{T}}. \quad (2.1.3)$$

where

$$\mu([Y, \iota]) := \mu([Y \setminus \iota(V), \iota|_{\partial V}]). \quad (2.1.4)$$

Next, one can use the fact that the partition function of an arbitrary abelian TQFT with a topological boundary condition $\mathfrak{B}$ depends on the pair (Y, ι) only through a certain algebraic data $\mathcal{A}$, namely the first homology group $H_1(Y)$, the $\mathbb{Q}/\mathbb{Z}$ -valued linking bilinear form on its torsion subgroup, and the induced homomorphism $\iota_* : H_1(V) \cong \mathbb{Z}^g \rightarrow H_1(Y)$ (where g is the sum of genera of the components of Σ). Therefore, at least formally, one can replace the sum (2.1.3) with the sum over the algebraic data, after having evaluated it on $\mathfrak{B}$. In this way, each term can be understood to contain the sum over the pairs (Y, ι) that have the same algebraic data (the corresponding equivalence classes are denoted with double brackets):

$$\langle \mathfrak{B} | \Sigma; \mu \rangle_{\mathcal{T}} = \sum_{\mathcal{A}} \mu_{\text{alg}}(\mathcal{A}) \langle \mathfrak{B} | Y \setminus \iota(V), \iota|_{\partial V} \rangle_{\mathcal{T}} |_{[[Y, \iota]] = \mathcal{A}}, \quad \mu_{\text{alg}}(\mathcal{A}) = \sum_{[Y, \iota]: [[Y, \iota]] = \mathcal{A}} \mu'([Y, \iota]). \quad (2.1.5)$$

¹In 3 dimensions, diffeomorphism classes can be identified with homeomorphism classes, as any topological 3-manifold admits a unique smooth structure.

Assuming non-negativity, μ_{alg} can be understood as a measure on $\{\mathcal{A}\}$, the set of possible algebraic data. Abstractly, the measure μ_{alg} is the pushforward of the measure μ on the set of diffeomorphism classes of closed manifolds with embedded handlebodies $[Y, \iota]$ (with respect to the projection map $[Y, \iota] \rightarrow [[Y, \iota]]$), or equivalently on the null-bordisms $[M, \phi]$. In the resulting (2.1.5) the physical spacetimes are effectively replaced with a certain algebraic data. This can be understood as applying the general philosophy of algebraic topology – replacing topological objects with algebraic ones – in the context of topological quantum gravity. Physically, going from μ to μ_{alg} can be interpreted as “integrating out” certain degrees of freedom. Note that for any given μ_{alg} there exists a measure μ the pushforward of which is μ_{alg} . For example, for each $\mathcal{A}$ choose one $[Y, \iota]$ with $[[Y, \iota]] = \mathcal{A}$ and set $\mu([Y, \iota]) = \mu_{alg}(\mathcal{A})$, and zero on all other classes $[Y, \iota]$.

We remark that in principle it is possible to rewrite directly the sum (2.1.1) over manifolds M with boundary in terms of certain other algebraic data, without passing to a sum over closed manifolds or imposing any particular boundary condition. Such data is the long exact sequence of homology groups associated to the pair (M, Σ) , together with bilinear pairings $H_1(M, \Sigma) \times H_1(M) \rightarrow \mathbb{Z}$ and $\text{Tor } H_1(M, \Sigma) \times \text{Tor } H_1(M) \rightarrow \mathbb{Q}/\mathbb{Z}$. The equivalence relation between manifolds having isomorphic data is known as homology-isomorphism, Y-equivalence, or Borromean surgery equivalence [62]. However, due to various constraints on the pieces of the data, it is harder to organize explicitly the sum over the equivalence classes. We are not taking this route in our discussion.

Because we choose a topological boundary condition (not a generic conformal one), the holographic dual will be an ensemble of 2d TQFTs (and not of conformal theories). The fact that the bulk TQFT is abelian provides a universal (i.e. independent of the actual choice of TQFT) relation between the gravity partition functions for disjoint unions of 2-manifolds and their connected sum. This gives a strong constraint on the possible form of the distribution of the boundary 2d TQFTs. In particular, we show that the distribution can be completely and explicitly determined in terms of the average of the partition function on a connected surface, assuming its dependence on the genus is of a certain form – a Dirichlet series with non-negative coefficients. We also provide sufficient conditions on the measure for the Dirichlet series assumption to hold for an arbitrary bulk abelian 3d TQFT, while we managed to prove the non-negativity of the coefficients only in the case of untwisted Dijkgraaf-Witten theories with Neumann boundary conditions.

2.2 The algebraic data

In Section 1.2.1, we have seen how to define and compute the partition function of an abelian TQFT. We will now specify to the case of theories with topological boundary conditions. In this case, the partition function will satisfy some additional properties and it will crucially depend on the spacetime manifold only through some algebraic data, amounting to a triple (A, ℓ, f) of a finitely generated abelian group A , a non-degenerate

bilinear form $\ell: \text{Tor} A \times \text{Tor} A \rightarrow \mathbb{Q}/\mathbb{Z}$, and a map $f: \mathbb{Z}^g \rightarrow A$. Such triples give the algebraic data anticipated in the introduction, that is $\mathcal{A} = (A, \ell, f)$ in the summation formula (2.1.5).

We are interested in computing the sum over manifolds with a fixed boundary (2.1.1), recasted into the sum over closed manifolds (2.1.3), of partition functions of an abelian TQFT with a topological boundary condition. Part of this goal will then amount to the classification and sum over all possible triples of algebraic data.

Topological boundary conditions

The topological boundary conditions are known to be in one-to-one correspondence with Lagrangian subgroups $\mathfrak{L} \subset D$ with respect to the quadratic form q [35]. Imposing the boundary condition corresponding to a Lagrangian subgroup $\mathfrak{L}_i$ on the i -th boundary component Σ_{g_i} of the bordism $(M = Y \setminus \iota(V), \phi = \iota|_{\partial V})$ results in pairing the state $|M, \phi\rangle$ with the boundary state:

$$\langle \mathfrak{L}_i | := \sum_{c \in \mathfrak{L}_i^{g_i} \subset D^{g_i}} \langle c | \in \mathcal{H}(-\Sigma_{g_i}). \quad (2.2.1)$$

Such a state is in particular invariant under the action of the mapping class group of Σ_{g_i} on $\mathcal{H}(-\Sigma_{g_i})$. The pairing results in a quantity independent of the restriction of $\phi: \partial M \xrightarrow{\sim} \Sigma \equiv \sqcup_i \Sigma_{g_i}$ on the corresponding boundary component.

When the same boundary condition is imposed for all the components, i.e. $\mathfrak{L}_i \equiv \mathfrak{L} \forall i$, writing the boundary state $\langle \mathfrak{B} | = \bigotimes_i \langle \mathfrak{L} |^{(i)} \in \bigotimes_i \mathcal{H}(-\Sigma_{g_i})$, we have for connected M :

$$Z(M; \mathfrak{B}) \equiv \langle \mathfrak{B} | M, \phi \rangle = \left(\bigotimes_i \langle \mathfrak{L} |^{(i)} \right) |M, \phi\rangle = \sum_{c \in \mathfrak{L}^g} Z(Y, \mathcal{L}; D, q; c), \quad (2.2.2)$$

where $\mathcal{L}$ is the g -component link in Y ($g = \sum_i g_i$) defined by the embedding $\iota: V \rightarrow Y$ as above.

We recall that a Lagrangian subgroup of a pair (D, q) is an isotropic and coisotropic subgroup of D with respect to the bilinear form b_q associated with q . In particular, this implies that $|\mathfrak{L}|^2 = |D|$ and $\sigma(K) = 0$, as was argued in [35]. Therefore, a TQFT admitting topological boundary conditions has no framing anomaly. It's worth mentioning that in the literature (see [36], for example) the vanishing of the central charge is sometimes required only mod 8. This is because they consider a TQFT up to invertible TQFTs, and one can always stack E_8 factors to obtain a genuine gapped boundary, i.e. vanishing central charge. In this work, we stick with the condition that the central charge is identically zero. Finally, it was shown in [36] that an abelian bosonic TQFT admits topological boundary conditions if and only if it is an abelian Dijkgraaf-Witten theory.

We now observe an additional characterization of an abelian TQFT admitting topological boundary conditions due to [36]. First of all, notice that any abelian TQFT

$\mathcal{T} = (D, q)$ can be factorized into prime factors $\mathcal{T}_{p_i} = (D_{p_i}, q_{p_i})$, where D_{p_i} are the Sylow p -subgroups of D , and since such splitting is orthogonal with respect to q , the quadratic form decomposes accordingly. Then, an abelian TQFT $\mathcal{T}$ admits topological boundary conditions if and only if each factor $\mathcal{T}_{p_i}$ does.

The hypothesis of the existence of topological boundary conditions simplifies the analysis at the beginning of this section. First of all, the relation between the TQFT partition function and the Deloup τ -invariant does not involve the ambiguous phase factor in this case, i.e. one simply has $Z = |D|^{-1/2} \tau$. Moreover, when all components of $c \in D^g$ are in the same Lagrangian subgroup, as it happens in the sum in the right-hand side of (2.2.2), the invariant depends on the homology class $\theta = \sum_{i=1}^g c_i \otimes [\mathcal{L}_i] \in H_1(Y, \mathfrak{L}) \subset H_1(Y, D)$, but not on the choice of a framed 1-cycle $\hat{\theta}$ representing it. This is a consequence of the invariance under the action of the mapping class group of the boundary of $M = Y \setminus \iota(V)$, but it can also be seen explicitly as follows. Recall that L is the linking matrix of the link $\mathcal{J} \cup \mathcal{J}'$ in S^3 and is related to the linking matrices of $\mathcal{J}$ and $\mathcal{J}'$ through:

$$L = \begin{pmatrix} J & F \\ F^T & J' \end{pmatrix}, \quad (2.2.3)$$

where $F: \mathbb{Z}^g \rightarrow \mathbb{Z}^m$ gives the linking numbers between the link components of $\mathcal{J}$ and $\mathcal{J}'$. The exponent in the surgery formula (1.2.1) is expressed in terms of those blocks of L as follows:

$$(L \otimes q)(x \oplus c) = (J \otimes q)(x) + (F \otimes b_q)(x, c) + (J' \otimes q)(c). \quad (2.2.4)$$

The homology group $H_1(M) \cong \text{coker } J$ and the linking form ℓk on its torsion part are determined by J . Notice that J has rank equal to $m - b_1$. Then, there exists a non-singular integer matrix $\tilde{J}$ and a unimodular matrix P such that $P^T J P = \tilde{J} \oplus 0_{b_1}$, so that $\text{Tor} H_1(Y) \cong \text{coker } \tilde{J}$ and $\ell k = \tilde{J}^{-1} \pmod{\mathbb{Z}}$. The dependence on the framed 1-cycle is through the blocks F and J' . Namely, the homology class θ is fixed by F while the choice of the framed representative $\hat{\theta}$ is determined by J' . When all c_i are in the same Lagrangian subgroup, the last term in (2.2.4) identically vanishes and the dependence on J' , and therefore on framing, disappears. The homology class of θ is determined by the collection $f_i := [\mathcal{L}_i] \in H_1(Y)$, $i = 1, \dots, g$, which can be combined into an element $f \in H_1(Y)^g$ or, equivalently, into the map $\mathbb{Z}^g \rightarrow H_1(Y)$. The latter is the pushforward map $f = \iota_* : H_1(V) \cong H_1(\mathcal{L}) \rightarrow H_1(Y)$, where the generators of $H_1(V) \cong \mathbb{Z}^g$ are given by the cores inside the copies of the solid torus V_1 in V . In terms of the surgery data, we have $f = p \circ F$, where p is the projection map to $\text{coker } J$.

Therefore, when the same topological boundary condition is imposed on all boundary components, for any choice of the abelian 3d TQFT, the partition function depends on $(Y, \iota : V \hookrightarrow Y)$ only through the triple $(H_1(Y), \ell k, f)$. That is, one can write:

$$Z(Y, \mathcal{L}; D, q; c) \equiv Z(H_1(Y), \ell k, f; D, q; c). \quad (2.2.5)$$

On the other hand, any triple (A, ℓ, f) , where A is a finitely generated abelian group, ℓ is a non-degenerate bilinear form $\text{Tor} A \times \text{Tor} A \rightarrow \mathbb{Q}/\mathbb{Z}$, and $f : \mathbb{Z}^g \rightarrow A$, can be realized

by some 3-manifold Y and an embedding of a collection of handle-bodies $\iota : V \hookrightarrow Y$ with total genus g . Indeed, any pair (A, ℓ) determines (not uniquely) a symmetric matrix J such that $A = \text{coker } J$ and $\ell = \tilde{J}^{-1} \pmod{\mathbb{Z}}$, where $\tilde{J}$ is defined as above. This is the content of Lemma 2.1 in [37], part (a). Thus, we can interpret J as the linking matrix of a link of unknots in S^3 and Y is obtained by surgery along this link. Finally, $f \in A^g \equiv H_1(Y)^g$ corresponds to a collection of singular 1-cycles in Y which can be smoothed out to produce a link in Y . The handle-body required is obtained as the tubular neighborhood of each link component, glued together along some tubes.

Such triples give the algebraic data anticipated above, that is $\mathcal{A} = (H_1(Y), \ell k, f)$.

Moreover, the TQFT partition function depends on $f \in \text{Hom}(\mathbb{Z}^g, A) \cong A^g$ only through $\theta \equiv \sum_{i=1}^g c_i \otimes f_i \in \mathcal{L} \otimes A \subset D \otimes A$, so one can write:

$$Z(A, \ell, f; D, q; c) \equiv Z(A, \ell; D, q; \sum_{i=1}^g c_i \otimes f_i). \quad (2.2.6)$$

Factorization properties

The TQFT partition function (equivalently, the τ -invariant) behaves nicely under the connected sum operation. Indeed, given a link $\mathcal{L}_1$ in Y_1 and a link $\mathcal{L}_2$ in Y_2 , it satisfies:

$$Z(Y_1 \# Y_2, \mathcal{L}_1 \cup \mathcal{L}_2; D, q; c_1 \oplus c_2) = \frac{Z(Y_1, \mathcal{L}_1; D, q; c_1) \cdot Z(Y_2, \mathcal{L}_2; D, q; c_2)}{Z(S^3, \emptyset; D, q; 0)}. \quad (2.2.7)$$

When all the components of c_1 and c_2 are in the same Lagrangian subgroup, one can write this relation in terms of the algebraic data used in (2.2.5) as follows:

$$Z(A_1 \oplus A_2, \ell_1 \oplus \ell_2, f_1 \oplus f_2; D, q; c_1 \oplus c_2) = \frac{Z(A_1, \ell_1, f_1; D, q; c_1) \cdot Z(A_2, \ell_2, f_2; D, q; c_2)}{Z(0, 0, 0; D, q; 0)}. \quad (2.2.8)$$

Moreover, the partition function is multiplicative with respect to the direct sum of TQFTs. By this we mean that, given $(D, q; c) = (D_1 \oplus D_2, q_1 \oplus q_2; c_1 \oplus c_2)$, we have:

$$Z(Y, \mathcal{L}; D, q; c) = Z(Y, \mathcal{L}; D_1, q_1; c_1) \cdot Z(Y, \mathcal{L}; D_2, q_2; c_2). \quad (2.2.9)$$

One can also consider a different operation that preserves the number of boundary components. Consider a pair $(Y, \mathcal{L})$ realized via surgery on a pair of framed links in S^3 of the form $(\mathcal{J}_1 \cup \mathcal{J}_2, \mathcal{J}')$, where J_1 and J_2 are unlinked with each other. Let the pairs $(Y_1, \mathcal{L}_1)$ and $(Y_2, \mathcal{L}_2)$ be realized by the pairs $(\mathcal{J}_1, \mathcal{J}')$ and $(\mathcal{J}_2, \mathcal{J}')$ respectively, and notice that for closed manifolds without links, this implies $Y = Y_1 \# Y_2$. In an abelian TQFT, without loss of generality one can assume that $\mathcal{L} = \mathcal{L}_1 \# \mathcal{L}_2$, i.e. that each component of $\mathcal{L}$ is a connected sum of the corresponding components of $\mathcal{L}_1$ and $\mathcal{L}_2$. From the surgery formula, we then have:

$$Z(Y_1 \# Y_2, \mathcal{L}_1 \# \mathcal{L}_2; D, q; c) = \frac{Z(Y_1, \mathcal{L}_1; D, q; c) \cdot Z(Y_2, \mathcal{L}_2; D, q; c)}{Z(S^3, \mathcal{J}'; D, q; c)}. \quad (2.2.10)$$

When all the components of c are in the same Lagrangian subgroup, the relation simplifies as the dependence on the linking matrix of $\mathcal{J}'$ disappears:

$$Z(Y_1 \# Y_2, \mathcal{L}_1 \# \mathcal{L}_2; D, q; c) = \frac{Z(Y_1, \mathcal{L}_1; D, q; c) \cdot Z(Y_2, \mathcal{L}_2; D, q; c)}{Z(S^3, \emptyset; D, q; 0)}. \quad (2.2.11)$$

We will refer to this relation as the surgery link factorization formula. We stress that this construction is important since it does not affect the boundary and will allow us to reduce the computation of the partition function on an arbitrary manifold to the evaluation of simpler building blocks.

In terms of the algebraic data, the surgery link factorization formula reads:

$$Z(A_1 \oplus A_2, \ell_1 \oplus \ell_2, f_1 \oplus f_2; D, q; c) = \frac{Z(A_1, \ell_1, f_1; D, q; c) \cdot Z(A_2, \ell_2, f_2; D, q; c)}{Z(0, 0, 0; D, q; 0)}. \quad (2.2.12)$$

Note that, unlike in (2.2.8), here the direct sum $f_1 \oplus f_2$ is performed only with respect to the codomain, but not the domain, which is now fixed to $\mathbb{Z}^g$. Because of this, the summation over the Lagrangian subgroup in (2.2.2) spoils the factorization property (2.2.11), but not the one in (2.2.7).

However, in terms of the dependence on $\theta \in A \otimes \mathfrak{L}$ as in (2.2.6), both (2.2.8) and (2.2.12) can be written simply as:

$$Z(A_1 \oplus A_2, \ell_1 \oplus \ell_2; D, q; \theta_1 + \theta_2) = \frac{Z(A_1, \ell_1; D, q; \theta_1) \cdot Z(A_2, \ell_2; D, q; \theta_2)}{Z(0, 0, 0; D, q; 0)}. \quad (2.2.13)$$

Classification of finite bilinear forms

We briefly review the classification of bilinear forms on finite abelian groups following [63]. For the original works, we refer to [64] and [65].

The classification is essential for our purposes since, thanks to the surgery link factorization formula, the partition function factorizes compatibly with the decomposition of finite abelian groups and bilinear forms on them.

We study the classification of pairs (A, ℓ) of a finite abelian group A and a symmetric bilinear form $\ell: A \otimes A \rightarrow \mathbb{Q}/\mathbb{Z}$ up to equivalence. Two pairs are equivalent, that is $(A, \ell) \sim (A', \ell')$, if there exists an isomorphism $\phi: A \rightarrow A'$ such that $\ell = \ell' \circ (\phi \otimes \phi)$. The first step consists of writing A as a product of its Sylow p -subgroups and noticing that the bilinear form splits accordingly:

$$(A, \ell) \simeq \bigoplus_{p \text{ prime}} (A_p, \ell_p). \quad (2.2.14)$$

Next, for each prime, one considers the decomposition into irreducible components of the following form:

$$(A_p, \ell_p) \simeq \bigoplus_{i \in I_p} ((\mathbb{Z}/p^{m_i}\mathbb{Z})^{d_i}, \ell_{p, m_i, d_i}^{(i)}) \quad (2.2.15)$$

where $\ell_{p,m_i,d_i}^{(i)}$ is a bilinear form on $(\mathbb{Z}/p^{m_i}\mathbb{Z})^{d_i}$, for some lists of positive integers $\{m_i, d_i\}_{i \in I_p}$.

Then, we are left to consider the classification of irreducible forms $\ell_{p,m,d}$ on $(\mathbb{Z}/p^m\mathbb{Z})^d$. The discussion is considerably different depending on whether p is an odd prime. If that is the case, there are only two possible choices of irreducible bilinear forms on $\mathbb{Z}/p^m\mathbb{Z}$:

$$X_{p^m} \text{ on } \mathbb{Z}/p^m\mathbb{Z}: \quad \ell_{p,m,1}(x, y) = \frac{xy}{p^m} \pmod{1}, \quad (2.2.16)$$

$$Y_{p^m} \text{ on } \mathbb{Z}/p^m\mathbb{Z}: \quad \ell_{p,m,1}(x, y) = r \frac{xy}{p^m} \pmod{1}, \quad (2.2.17)$$

for r any quadratic non-residue mod p . We remark that if $p = 3 \pmod{4}$, then $r = -1$ is a quadratic non-residue; however, there is no canonical choice for r when $p = 1 \pmod{4}$.

These two bilinears generate any other bilinear form on the Sylow p -subgroup. However, the decomposition (2.2.15) is not unique in general, and therefore we must also consider possible relations among the generators. The only such relation is:

$$2X_{p^m} = 2Y_{p^m} \quad (2.2.18)$$

on $\mathbb{Z}/p^m\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}$. Thus, dX_{p^m} and $(d-1)X_{p^m} + Y_{p^m}$ are the only two inequivalent bilinear forms on $(\mathbb{Z}/p^m\mathbb{Z})^d$.

On the other hand, if $p = 2$ the classification is much more involved. All the possible bilinear forms ℓ_2 on the Sylow 2-subgroup are generated by:

$$A_{2^m} \text{ on } \mathbb{Z}/2^m\mathbb{Z}: \quad \ell_{2,m,1}(x, y) = \frac{xy}{2^m} \pmod{1}, \quad (2.2.19)$$

$$B_{2^m} \text{ on } \mathbb{Z}/2^m\mathbb{Z}: \quad \ell_{2,m,1}(x, y) = -\frac{xy}{2^m} \pmod{1}, \quad (2.2.20)$$

$$C_{2^m} \text{ on } \mathbb{Z}/2^m\mathbb{Z}: \quad \ell_{2,m,1}(x, y) = 5\frac{xy}{2^m} \pmod{1}, \quad (2.2.21)$$

$$D_{2^m} \text{ on } \mathbb{Z}/2^m\mathbb{Z}: \quad \ell_{2,m,1}(x, y) = -5\frac{xy}{2^m} \pmod{1}, \quad (2.2.22)$$

$$E_{2^m} \text{ on } (\mathbb{Z}/2^m\mathbb{Z})^2: \quad \ell_{2,m,2}(x, y) = x^T \begin{pmatrix} 0 & 1/2^m \\ 1/2^m & 0 \end{pmatrix} y \pmod{1}, \quad (2.2.23)$$

$$F_{2^m} \text{ on } (\mathbb{Z}/2^m\mathbb{Z})^2: \quad \ell_{2,m,2}(x, y) = x^T \begin{pmatrix} 1/2^{m-1} & 1/2^m \\ 1/2^m & 1/2^{m-1} \end{pmatrix} y \pmod{1}. \quad (2.2.24)$$

Unfortunately, the relations among these generators are rather involved. We avoid listing here all the possibilities.

Thus, thanks to the surgery link factorization formulas (2.2.11)-(2.2.12), it is enough to compute the partition function on the generators of finite bilinear forms.

Counting finite abelian groups with bilinear forms

As a prototypical algebraic version of the sum over all 3-manifolds, one can consider the following generating Dirichlet series for the numbers of isomorphism classes of abelian

groups of given order (see e.g. [66]):

$$\sum_{n \geq 1} \# \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of finite abelian groups} \\ \text{of order } n \end{array} \right\} n^{-s} = \prod_{p \in \{\text{primes}\}} \sum_{\text{partitions } \Lambda} p^{-s|\Lambda|} = \prod_{p \in \{\text{primes}\}} \prod_{m \geq 1} \frac{1}{1 - p^{-sm}} = \prod_{m \geq 1} \zeta(ms), \quad (2.2.25)$$

which is convergent for $\text{Re}(s) > 1$.

For our purpose, however, it is more relevant to consider its analogue counting non-equivalent pairs (A, ℓ) , where A is a finite abelian group and ℓ is a bilinear form on it, as considered in the previous section. Let $a(n)$ denote the number of such pairs with $|A| = n$. Then, according to the classification reviewed above, we have:

$$\Xi(s) := \sum_{n \geq 1} a(n) n^{-s} = \prod_{p \in \{\text{primes}\}} \Xi_p(s), \quad (2.2.26)$$

where:

$$\Xi_p(s) := \sum_{n \geq 1} a(p^k) p^{-ks}. \quad (2.2.27)$$

For odd p , using the fact that for each $(\mathbb{Z}/p^m\mathbb{Z})^d$ factor with $d \geq 1$ there are two inequivalent forms, we have:

$$\Xi_p(s) = \prod_{m \geq 1} \frac{1 + p^{-ms}}{1 - p^{-ms}}, \quad p \neq 2. \quad (2.2.28)$$

For $p = 2$ we can use the results about the normal forms by Miranda [67] to write the generating series (each normal form contributes with $2^{-xs}/(1 - 2^{-ys})$ for some x and y):

$$\begin{aligned} \Xi_2(s) &= \left(\frac{1}{1 - 2^{-s}} + \frac{2^{-2s}}{1 - 2^{-2s}} \right) \cdot \left(\frac{1 + 2^{-2s} + 2^{-4s} + 2^{-6s}}{1 - 2^{-2s}} + \frac{2 \cdot 2^{-4s}}{1 - 2^{-4s}} \right) \times \\ &\prod_{m \geq 3} \left(1 + 2^{-ms} + 2^{-2ms} + 2^{-3ms} + \frac{2 \cdot 2^{-2ms}}{1 - 2^{-2ms}} + \frac{3 \cdot 2^{-ms} + 2 \cdot 2^{-2ms} + 2 \cdot 2^{-3ms} + 2^{-4ms}}{1 - 2^{-ms}} \right) = \\ &\frac{1 + 2^{-s} + 2^{-2s}}{1 - 2^{-2s}} \cdot \frac{1 + 2 \cdot 2^{-2s} + 4 \cdot 2^{-4s} + 2 \cdot 2^{-6s} + 2^{-8s}}{1 - 2^{-4s}} \times \\ &\prod_{m \geq 3} \frac{1 + 4 \cdot 2^{-ms} + 7 \cdot 2^{-2ms} + 4 \cdot 2^{-3ms} + 2^{-4ms}}{1 - 2^{-2ms}}. \quad (2.2.29) \end{aligned}$$

Note that the details of the expression for $\Xi(s)$ are not going to be relevant for further analysis. However, what will be important is that the Dirichlet series (2.2.26) converges for $\text{Re}(s) > 1$, similarly to (2.2.25), as standard arguments show.

2.3 Summing over topologies

One of the main goals of this work is to analyze the convergence properties, as well as the dependence on various input parameters, of the sum over topologies, rewritten as a sum over certain algebraic data (2.1.5). In the previous section, in the case when the same topological boundary condition is imposed on all connected components of the boundary and the 3-manifold itself is connected, we described what this algebraic data is and the dependence of the TQFT partition function on it. When the 3-manifold has non-empty boundary, it is a triple $\mathcal{A} \equiv [A, \ell, f : \mathbb{Z}^g \rightarrow A]$ considered up to an isomorphism, or, when the manifold is closed, a couple $\mathcal{A} \equiv [A, \ell]$. Note that we want to formally distinguish the cases when the boundary is a non-empty collection of spheres, that is $g = 0$ and f is the unique map from the trivial group, and when the boundary is empty, that is, when f is absent. The equivalence with respect to isomorphisms $(A, \ell, f) \sim (A', \ell', f')$ is the existence of an isomorphism $\psi : A \rightarrow A'$ such that $\ell' \circ (\psi \otimes \psi) = \ell$, $f' = \psi \circ f$. For triples related by such an isomorphism, the partition function $Z(A, \ell, f; D, q; c)$ is necessarily the same. Moreover, the class $[Y, \iota]$ determines (A, ℓ, f) only up to such an automorphism. Namely $(Y, \iota) \sim (Y', \iota')$ give triples that differ by an isomorphism $\psi = \Phi_*$, $\Phi : Y \rightarrow Y'$.

When the 3-manifold is disconnected, the algebraic data $\mathcal{A}$ is then a collection of such triples and couples for each connected component. Namely, in (2.1.5) let $Y = \sqcup_i Y^{(i)}$, where each $Y^{(i)}$ is connected. Respectively, we have $\iota = \sqcup_i \iota^{(i)}$ with $\text{Im } \iota^{(i)} \subset Y^{(i)}$. Then $\mathcal{A} = \{\mathcal{A}^{(i)}\}_i$ where $\mathcal{A}^{(i)}$ is a triple/couple corresponding to the pair $(Y^{(i)}, \iota^{(i)})$. That is $\mathcal{A}^{(i)} = [H_1(Y^{(i)}), \ell k(Y), \iota_*^{(i)}|_{H_1}]$. In order to make things as explicit and computable as possible, we will make various assumptions on the summation weights $\mu_{\text{alg}}(\mathcal{A})$.

The first important assumption, which is already implicitly made in writing (2.1.5), is that $\mu_{\text{alg}}(\mathcal{A})$ is, as a function of the algebraic data $\mathcal{A}$, the same for different choices of Σ compatible with that algebraic data. In particular, for a contribution of a connected 3-manifold, it depends only on the total genus of Σ : that is, it is the same for connected $\Sigma = \Sigma_g$ and disconnected $\Sigma = \Sigma_{g_1} \sqcup \Sigma_{g_2} \sqcup \dots$ with $g = g_1 + g_2 + \dots$. By default, we will also assume that $\mu_{\text{alg}}(\mathcal{A}) \geq 0$, so that μ_{alg} can be understood as a choice of measure.

Next, we make the natural assumption that the measure factorizes with respect to the disjoint union, up to the symmetry factor corresponding to the permutation of the closed connected components:

$$\mu_{\text{alg}}(\mathcal{A}) = \frac{1}{\prod_m N_m!} \prod_i \mu_{\text{alg}}(\mathcal{A}^{(i)}). \quad (2.3.1)$$

Here N_m are the numbers of closed connected components with the same algebraic data, that is, the numbers of repeating triples in $\mathcal{A}$. Such a factor is not introduced for components with boundary, because the components of Σ are considered ordered. This assumption is along the lines of what has been previously considered in the literature (see e.g. [52]).

Classification of *triples* up to isomorphisms instead of just *pairs* (A, ℓ) is much more

involved and is out of the scope of this paper. Moreover, for practical reasons, it is much more convenient to work with representative triples (A, ℓ, f) . More concretely, we fix a representative pair (A, ℓ) for each equivalence class $[A, \ell]$. After such a choice, for each $[A, \ell]$ we consider all possible $f : \mathbb{Z}^g \rightarrow A$. The union over all $[A, \ell]$ has a projection to the set of the algebraic data:

$$\bigsqcup_{[A, \ell]} \text{Hom}(\mathbb{Z}^g, A) \longrightarrow \{\mathcal{A} = [A, \ell, f : \mathbb{Z}^g \rightarrow A]\} \quad (2.3.2)$$

The fibers are the orbits of $\text{Aut}(A, \ell)$ in $\text{Hom}(\mathbb{Z}^g, A)$. We then want to lift the measure μ_{alg} to a measure, which we will simply denote as μ , on the fibration. That is, μ should be such that μ_{alg} is its pushforward with respect to the fibration above. Practically, we want to consider a (positive) function μ on the triples (A, ℓ, f) , with (A, ℓ) being a representative pair, which satisfies:

$$\sum_{f' \in \text{Aut}(A, \ell) \cdot f} \mu(A, \ell, f') = \mu_{\text{alg}}([A, \ell, f]). \quad (2.3.3)$$

Obviously, for any given positive μ_{alg} there exists a (non-unique) positive μ that satisfies this condition. Note however, that *unless* $\text{rank } A \leq 1$ or $\text{Im } f \subset \text{Tor } A$, the orbit is not finite, and thus a priori one *cannot* simply take

$$\mu(A, \ell, f) = \frac{\mu_{\text{alg}}([A, \ell, f])}{|\text{Aut}(A, \ell) \cdot f|}. \quad (2.3.4)$$

2.3.1 Definition and basic properties

With the assumptions and the conventions above, for the topological boundary condition corresponding to a Lagrangian subgroup $\mathfrak{L} \subset D$, the sum (2.1.5) for general Σ can be expressed through the sums:

$$\mathcal{Z}_g(D, q; \mathfrak{L}; \mu) := \sum_{[A, \ell]} \sum_{f \in \text{Hom}(\mathbb{Z}^g, A)} \mu(A, \ell, f) \sum_{c \in \mathfrak{L}^g} Z(A, \ell, f; D, q; c) \quad (2.3.5)$$

corresponding to contributions of *connected* 3-manifolds with the boundary of total genus g . Namely, the original full sum for the general boundary Σ can be expressed as a polynomial in $\mathcal{Z}_g$ for different g , see Section 2.4 for details. We will refer to $\mathcal{Z}_g$ as the *connected* gravitational partition function. As before, in the formula above, the pair (D, q) determines the bulk 3d abelian TQFT. The summand in the right-hand side is the partition function (2.2.5). Since it depends only on the equivalence class of the triple (A, ℓ, f) , the full sum depends on μ_{alg} , but not on the choice of the lift μ .

It is worth clarifying the summation range and order. The most external sum is over equivalence classes of pairs $[A, \ell]$ (for each of which an explicit representative (A, ℓ) is chosen). We order such pairs first by the rank (a non-negative integer) of A , and then by the

size of the torsion subgroup $\text{Tor} A \subset A$. With the rank and the size of the torsion subgroup fixed, there is a finite number of equivalence classes $[A, \ell]$, so any order can be chosen there. For a fixed pair (A, ℓ) , the summation is performed over $f \in \text{Hom}(\mathbb{Z}^g, A) \cong A^g$. We can order this set lexicographically, using an already chosen order on A .

We will now derive sufficient conditions on the measure μ for the series to be convergent, and also for the dependence on the genus g to have the form of a Dirichlet series:

$$\mathcal{Z}_g(D, q; \mathfrak{L}; \mu) = \sum_{n=1}^{\infty} b_n n^g, \quad (2.3.6)$$

where the coefficients b_n are g -independent. The latter property is crucial and will be used in Sections 2.4 and 2.4.8.

Let us focus on the sums over f and c for a fixed pair (A, ℓ) . As was argued in Section 2.2, the TQFT partition function depends on $c \in \mathfrak{L}^g \subset D^g$ and $f \in \text{Hom}(\mathbb{Z}^g, A) \cong A^g$ only through $\theta \equiv \sum_{i=1}^g c_i \otimes f_i \in \mathfrak{L} \otimes A$, so one can write:

$$Z(A, \ell, f; D, q, c) \equiv Z(A, \ell; D, q; \sum_{i=1}^g c_i \otimes f_i). \quad (2.3.7)$$

Therefore, the sum over f and c in (2.3.5) for a given pair (A, ℓ) can be rewritten as follows:

$$\begin{aligned} \mathcal{Z}_g^{(A, \ell)}(D, q; \mathfrak{L}; \mu) &= \sum_{f \in \text{Hom}(\mathbb{Z}^g, A)} \mu(A, \ell, f) \sum_{c \in \mathfrak{L}^g} Z(A, \ell, f; D, q; c) = \\ &= \sum_{\substack{f \in A^g \\ c \in \mathfrak{L}^g}} \sum_{\theta \in \mathfrak{L} \otimes A} \sum_{\alpha \in (\mathfrak{L} \otimes A)^*} \frac{\mu(A, \ell, f)}{|\mathfrak{L} \otimes A|} Z(A, \ell; D, q; \theta) e^{2\pi i \alpha(\theta - \sum_{i=1}^g c_i \otimes f_i)} = \\ &= \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A)^* \\ \cong \text{Hom}(A, \mathfrak{L}^*)}} \frac{|\mathfrak{L}|^g \mu(A, \ell, (\ker \alpha)^g)}{|\mathfrak{L} \otimes A|} \sum_{\theta \in \mathfrak{L} \otimes A} Z(A, \ell; D, q; \theta) e^{2\pi i \alpha(\theta)}, \end{aligned} \quad (2.3.8)$$

where:

$$\mu(A, \ell, S^g) := \sum_{f \in S^g \subset A^g} \mu(A, \ell, f) \equiv \mu(\{(A, \ell, f)\}_{f \in S^g}), \quad S \subseteq A, \quad (2.3.9)$$

is the measure of the subset $\{(A, \ell, f)\}_{f \in S^g}$. In the last equality, $\ker \alpha$ is understood as the kernel of the map $\alpha : A \rightarrow \mathfrak{L}^*$. From the above, one can see that $\mu(A, \ell, \cdot)$, understood as a measure on A^g , actually only needs to be defined on the σ -algebra generated by the subgroups of the form $S^g \subseteq A^g$.

First, let us note that the sum:

$$Z_{\mathfrak{L}}(A, \ell; D, q; \alpha) := \frac{|\mathfrak{L}|}{|\mathfrak{L} \otimes A|} \sum_{\theta \in \mathfrak{L} \otimes A} Z(A, \ell; D, q; \theta) e^{2\pi i \alpha(\theta)} \quad (2.3.10)$$

is the partition function of the 3d TQFT with gauged non-anomalous 1-form symmetry $\mathfrak{L} \subset D$, since $\theta \in \mathfrak{L} \otimes A \cong H_1(Y; \mathfrak{L}) \cong H^2(Y; \mathfrak{L})$. It has dual 0-form symmetry $\mathfrak{L}^*$, and $\alpha \in \text{Hom}(A, \mathfrak{L}^*) \cong H^1(Y; \mathfrak{L}^*)$ plays the role of its background gauge field (see e.g. [68] for the general description of gauging higher form symmetries in TQFTs in terms of cohomology groups). Since a Lagrangian subgroup is gauged, the resulting theory is trivial in the absence of the background 0-form field. This trivial theory can be understood as the theory on the other side of a boundary with topological boundary condition corresponding to $\mathfrak{L}$. Considered in the presence of a background 0-form symmetry $\mathfrak{L}^*$, the theory is an invertible TQFT (an SPT in condensed matter terminology), and we have:

$$|Z_{\mathfrak{L}}(A, \ell; D, q; \alpha)| = 1. \quad (2.3.11)$$

We remark that gauging a Lagrangian subgroup of 1-form symmetry has also been considered in [69] in the context of 3d quantum gravity, as an alternative to summing over topologies.

The property (2.3.11) can be used to obtain an absolute bound for the total sum in (2.3.8). Assuming the positivity of the measure, we have:

$$\begin{aligned} \left| \mathcal{Z}_g^{(A, \ell)}(D, q; \mathfrak{L}; \mu) \right| &\leq |\mathfrak{L} \otimes A| |\mathfrak{L}|^{g-1} \mu(A, \ell, A^g) = \\ &|\mathfrak{L} \otimes \text{Tor} A| |\mathfrak{L}|^{g-1+\text{rank } A} \mu(A, \ell, A^g) \\ &\leq |\text{Tor} A|^{I(D)} |D|^{\frac{g-1+\text{rank } A}{2}} \mu(A, \ell, A^g), \end{aligned} \quad (2.3.12)$$

where $I(D)$ is the minimal number of generators of the group D (i.e. the minimal number of cyclic group factors) and we used the facts that $I(\mathfrak{L}) \leq I(D)$ and $|D| = |\mathfrak{L}|^2$. Now, assume that the measure satisfies the following bound:

$$\mu(A, \ell, A^g) \equiv \mu(\{(A, \ell, f)\}_{f \in A^g}) \equiv \mu_{\text{alg}}(\{[A, \ell, f]\}_{[A, \ell] \text{ fixed}}) \leq C \lambda^{\text{rank } A} |\text{Tor} A|^{-s}, \quad (2.3.13)$$

for some $s \in \mathbb{R}$, $\lambda \in \mathbb{R}_{\geq 0}$ and $C > 0$. This immediately implies the following absolute bound on the series (2.3.5):

$$|\mathcal{Z}_g(D, q; \mathfrak{L}; \mu)| \leq \sum_{b_1 \geq 0} \sum_{n \geq 1} C |D|^{\frac{g+b_1-1}{2}} \lambda^{b_1} a(n) n^{I(D)-s} = \frac{C |D|^{\frac{g-1}{2}}}{1 - \lambda |D|^{\frac{1}{2}}} \Xi(s - I(D)), \quad (2.3.14)$$

where $a(n)$ are the coefficients in the generating Dirichlet series (2.2.26). The sums are therefore absolutely convergent for $s > I(D) + 1$ and $\lambda < |D|^{-1/2}$. These conditions depend on the choice of group D , that is, the choice of the 3d abelian TQFT. To ensure convergence for all D with the same measure μ , it is sufficient to require that for any λ and s , there exists C such that (2.3.13) is satisfied for any finite abelian group A . That is,

$\mu(A, \ell, A^g) \rightarrow 0$ faster than exponentially as $\text{rank } A \rightarrow \infty$ and faster than a power law as $|\text{Tor } A| \rightarrow \infty$.

Having obtained sufficient conditions for the absolute convergence of the sum (2.3.5), we will now present sufficient conditions for the result to be of the form of a Dirichlet series (2.3.6). First, assume the measure factorizes symmetrically with respect to the g boundary tori, that is:

$$\mu(A, \ell, f) = \mu(A, \ell) \prod_{i=1}^g \nu(A, \ell, f_i), \quad (2.3.15)$$

for some f -independent $\mu(A, \ell)$ and $\nu(A, \ell, f_i)$ depending on a single component $f_i \in A$. It then follows that:

$$\mu(A, \ell, S^g) = \mu(A, \ell) \nu(A, \ell, S)^g, \quad (2.3.16)$$

where:

$$\nu(A, \ell, S) := \sum_{\phi \in S \subset A} \nu(A, \ell, \phi). \quad (2.3.17)$$

The form (2.3.6) is then achieved if:

$$\nu(A, \ell, S) \in \mathbb{Z}_{\geq 0}, \quad \forall S \subseteq A. \quad (2.3.18)$$

The coefficients of the Dirichlet series are then given by:

$$b_n = \sum_{(A, \ell)} \frac{\mu(A, \ell)}{|\mathfrak{L} \otimes A|} \sum_{\substack{\theta \in \mathfrak{L} \otimes \text{Tor } A \\ \alpha \in \text{Hom}(A, \mathfrak{L}^*): \nu(A, \ell, \ker \alpha) |\mathfrak{L}| = n}} Z(A, \ell; D, q; \theta) e^{2\pi i \alpha(\theta)}, \quad (2.3.19)$$

where we used the fact that the partition function vanishes unless $\theta \in \mathfrak{L} \otimes \text{Tor } A$ [33]. Note that $b_n = 0$ unless n is a multiple of $|\mathfrak{L}|$.

2.3.2 An example of measure

In this section, we will present some additional conditions on the measure for which the sum over topologies can be simplified to the point that it can be evaluated for a simple enough choice of the 3d TQFT.

The integrality condition (2.3.18) can be satisfied, for example, if:

$$\nu(A, \ell, \phi) = \begin{cases} 1 & \text{if } \phi \in \text{Tor } A, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3.20)$$

so that:

$$\nu(A, \ell, S) = |\text{Tor } S| \in \mathbb{Z}_{\geq 0}. \quad (2.3.21)$$

Note that in this case the measure μ is supported only on triples with $\text{Im } f \subset \text{Tor } A$, and invariant under the action of $\text{Aut}(A, \ell)$, so one can make the choice (2.3.4)². Moreover, since $\text{Tor } \ker \alpha = \ker \alpha|_{\text{Tor } A}$, it is possible to reduce the sum over all finitely generated abelian groups $A \cong \mathbb{Z}^{b_1} \oplus \text{Tor } A$ to just finite abelian groups $\tilde{A} = \text{Tor } A$:

$$\begin{aligned} \mathcal{Z}_g(D, q; \mathfrak{L}; \mu) = \\ \sum_{\tilde{A}, \ell} \sum_{b_1 \geq 0} \mu(\mathbb{Z}^{b_1} \oplus \tilde{A}, \ell) |D|^{\frac{b_1}{2}} \sum_{\theta \in \mathfrak{L} \otimes \tilde{A}} \sum_{\substack{\alpha \in (\mathfrak{L} \otimes \tilde{A})^* \\ \cong \text{Hom}(\tilde{A}, \mathfrak{L}^*)}} \frac{|\mathfrak{L}|^g |\ker \alpha|^g}{|\mathfrak{L} \otimes \tilde{A}|} Z(\tilde{A}, \ell; D, q; \theta) e^{2\pi i \alpha(\theta)}, \end{aligned} \quad (2.3.22)$$

where we also used the factorization property of the partition function (2.2.12) with $Z(\mathbb{Z}, 0; D, q; 0) \equiv Z(S^2 \times S^1, \emptyset; D, q; 0) = 1$.

The sum can be further evaluated if we assume the following factorization property of the measure:

$$\mu(\mathbb{Z}^{b_1} \oplus \tilde{A}) = \mu(\mathbb{Z}, 0)^{b_1} \prod_{p \in \text{primes}} \mu(A_p, \ell_p), \quad (2.3.23)$$

where A_p is the p -torsion subgroup of:

$$A = \mathbb{Z}^{b_1} \oplus \tilde{A} = \mathbb{Z}^{b_1} \oplus \bigoplus_{p \in \text{primes}} A_p, \quad (2.3.24)$$

and ℓ_p is the corresponding component of the bilinear form $\ell = \bigoplus_p \ell_p$ on $\tilde{A} = \text{Tor } A$, as in Section 2.2.

The data (D, q) describing the 3d abelian TQFT, and the Lagrangian subgroup $\mathfrak{L} \subset D$ can also be split with respect to the prime numbers. Namely, the discriminant group D decomposes into its Sylow p' -subgroups $D = \bigoplus_{p'} D_{p'}$ and, in a similar fashion, $\mathfrak{L} = \bigoplus_{p'} \mathfrak{L}_{p'}$. Notice that all but a finite number of $D_{p'}$ and $\mathfrak{L}_{p'}$ are trivial groups. This splitting is orthogonal with respect to the quadratic form q , meaning that it decomposes as $q = \bigoplus_{p'} q_{p'}$. We then decompose $\theta \in \tilde{A} \otimes \mathfrak{L}$ respectively as $\theta = \sum_{p, p'} \theta_{p, p'}$, with $\theta_{p, p'} \in A_p \otimes \mathfrak{L}_{p'}$. Note that $A_p \otimes \mathfrak{L}_{p'} = 0$ unless $p = p'$.

Now, making use of the factorization properties of the TQFT partition function reviewed in Section 2.2, we have:

$$\begin{aligned} Z(\tilde{A}, \ell; D, q; \theta) = \prod_{p': D_{p'} \neq 0} |D_{p'}|^{-\frac{1}{2}} \prod_{p: A_p \neq 0} |D_{p'}|^{\frac{1}{2}} Z(A_p, \ell_p; D_{p'}, q_{p'}; \theta_{p, p'}) = \\ \prod_{p: D_p \neq 0} Z(A_p, \ell_p; D_p, q_p; \theta_{p, p}), \end{aligned} \quad (2.3.25)$$

²More generally, one can naturally satisfy the integrality condition in the way compatible with (2.3.4) by taking $\nu(A, \ell, S) = |B \cap S|$ where B is a finite $\text{Aut}(A, \ell)$ -stable multiset. The example considered in this section is a special case $B = \text{Tor } A$.

where we used that for $p \neq p'$, we have:

$$Z(A_p, \ell_p; D_{p'}, q_{p'}; \theta_{p,p'}) = Z(A_p, \ell_p; D_{p'}, q_{p'}; 0) = |D_{p'}|^{-\frac{1}{2}}, \quad (2.3.26)$$

where the second equality follows from the statement that the closed-manifold partition function of a TQFT admitting topological boundary conditions satisfies $Z(A, \ell; D, q; 0) = Z(0, 0; D, q) = |D|^{-\frac{1}{2}}$, whenever $|A|$ and $|D|$ are coprime (see e.g. [36]).

Using both the factorization of the measure and the factorization of the TQFT partition function, the sum (2.3.22) can also be decomposed with respect to the primes:

$$\begin{aligned} \mathcal{Z}_g(D, q; \mathfrak{L}; \mu) &= \frac{1}{1 - \mu(\mathbb{Z}, 0)|D|^{\frac{1}{2}}} \cdot \prod_{p \in \text{primes}} \sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} \mu(A_p, \ell_p) \times \\ &\quad \sum_{\theta_p \in \mathfrak{L}_p \otimes A_p} \sum_{\substack{\alpha_p \in (\mathfrak{L}_p \otimes A_p)^* \\ \cong \text{Hom}(A_p, \mathfrak{L}_p^*)}} \frac{|\mathfrak{L}_p|^g |\ker \alpha_p|^g}{|\mathfrak{L}_p \otimes A_p|} Z(A_p, \ell_p; D_p, q_p; \theta_p) e^{2\pi i \alpha_p(\theta_p)}, \end{aligned} \quad (2.3.27)$$

having introduced $\theta_p \equiv \theta_{p,p}$. We consider now a very specific choice of measure that satisfies the factorization property above, with ν as in (2.3.20) and:

$$\mu(A, \ell) = \lambda^{\text{rank } A} |\text{Tor } A|^{-s}. \quad (2.3.28)$$

This choice moreover saturates the bound (2.3.13) with $C = 1$ and s there replaced with $s - g$. We then have:

$$\begin{aligned} \mathcal{Z}_g(D, q; \mathfrak{L}; \mu) &= \frac{1}{1 - \lambda |D|^{\frac{1}{2}}} \cdot \prod_{\substack{p \in \text{primes} \\ p \nmid |D|}} \Xi_p(s - g) \times \\ &\quad \prod_{\substack{p \in \text{primes} \\ p \mid |D|}} \sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} |A_p|^{-s} \sum_{\theta_p \in \mathfrak{L}_p \otimes A_p} \sum_{\substack{\alpha_p \in (\mathfrak{L}_p \otimes A_p)^* \\ \cong \text{Hom}(A_p, \mathfrak{L}_p^*)}} \frac{|\mathfrak{L}_p|^g |\ker \alpha_p|^g}{|\mathfrak{L}_p \otimes A_p|} Z(A_p, \ell_p; D_p, q_p; \theta_p) e^{2\pi i \alpha_p(\theta_p)}, \end{aligned} \quad (2.3.29)$$

where Ξ_p is the generating function considered in Section 2.2. In particular, for the trivial TQFT with $D = 0$ we have:

$$\mathcal{Z}_g(0, 0; 0; \mu) = \frac{\Xi(s - g)}{1 - \lambda}. \quad (2.3.30)$$

In the case of untwisted Dijkgraaf-Witten TQFT with Neumann boundary conditions, the expression (2.3.29) can be simplified further. Let G the gauge group, so that $D = G \oplus G^*$

and $\mathfrak{L} = G$. The partition function of untwisted DW theory has the following simple form (see Section 1.2.1):

$$Z(A, \ell; D, q; \theta) = \frac{|G \otimes A|}{|G|} \delta(\theta). \quad (2.3.31)$$

We then have:

$$\begin{aligned} \mathcal{Z}_g(D, q; \mathfrak{L}; \mu) &= \frac{1}{1 - \lambda |D|^{\frac{1}{2}}} \cdot \prod_{\substack{p \in \text{primes} \\ p \nmid |G|}} \Xi_p(s - g) \times \\ &\quad \prod_{\substack{p \in \text{primes} \\ p \mid |G|}} |G_p|^{g-1} \sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} |A_p|^{-s} \sum_{\alpha_p \in \text{Hom}(A_p, G_p^*)} |\ker \alpha_p|^g. \end{aligned} \quad (2.3.32)$$

Let us now consider the simplest non-trivial choice of the gauge group: $G = \mathbb{Z}/p\mathbb{Z}$ for some prime $p \neq 2$. The factor in (2.3.32) corresponding to this particular prime reads:

$$\begin{aligned} p^{g-1} \sum_{(A_p, \ell_p)} |A_p|^{-s} \sum_{\alpha_p \in \text{Hom}(A_p, \mathbb{Z}/p\mathbb{Z})} |\ker \alpha_p|^g &= \\ p^{g-1} \sum_{(A_p, \ell_p)} |A_p|^{-s} \left((|A_p|p^{-1})^g (p^{\dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z})} - 1) + |A_p|^g \right) &= \\ p^{-1}(p^g - 1) \sum_{(A_p, \ell_p)} |A_p|^{g-s} + p^{-1} \sum_{(A_p, \ell_p)} |A_p|^{g-s} p^{\dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z})} &= \\ p^{-1}(p^g - 1) \Xi_p(s - g) + p^{-1} \prod_{m \geq 1} \frac{1 + p^{1-m(s-g)}}{1 - p^{1-m(s-g)}}, \end{aligned} \quad (2.3.33)$$

where in the first equality we used the fact that for all maps $\alpha_p : A_p \rightarrow \mathbb{Z}/p\mathbb{Z}$, except the zero one, we have $|\ker \alpha_p| = |A_p|p^{-1}$ and the total number of maps is equal to $p^{\dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z})}$, and in the last equality we used a refined version of (2.2.28):

$$\sum_{(A_p, \ell_p)} |A_p|^{-s} x^{\dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z})} = \prod_{m \geq 1} \frac{1 + xp^{-ms}}{1 - xp^{-ms}}. \quad (2.3.34)$$

Note that $\dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z}) \equiv b$ in the decomposition:

$$A_p = \bigoplus_{i=1}^b \mathbb{Z}/p^{m_i}\mathbb{Z}. \quad (2.3.35)$$

We arrive at the following explicit expression for the connected gravitational partition function:

$$\mathcal{Z}_g(D, q; \mathfrak{L}; \mu) = \frac{\Xi(s-g)}{1 - \lambda|D|^{\frac{1}{2}}} \cdot \left(p^{-1}(p^g - 1) + p^{-1} \prod_{m \geq 1} \frac{1 + p^{1-m(s-g)}}{1 - p^{1-m(s-g)}} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \right). \quad (2.3.36)$$

2.4 Distribution of 2d TQFTs for general abelian 3d TQFTs

Naively applying the holographic principle, we would expect that summing over all the possible bulk topologies with a fixed boundary yields a new theory on the boundary itself. However, we should be careful in identifying the theory living on the boundary, since considering possibly disconnected bulks spoils the factorization property of the theory with respect to the disjoint union. Indeed, recall that one expects the partition function Z of a theory to factorize in the following way:

$$Z(M \sqcup N) = Z(M) \cdot Z(N). \quad (2.4.1)$$

However, this property will fail to hold when applied to the sum over topologies, due to the presence of wormhole topologies, that is, connected bordisms between disconnected components of the possibly disconnected boundary. A proposed generalization of the holographic principle suggests promoting the dual theory to a statistical ensemble of theories. In the following, we will analyze the consequences of such a bulk-boundary correspondence.

Since we were interested in abelian 3d TQFTs on the bulk, we shall focus on ensembles of 2d theories. Moreover, we have already argued that the dual theories can be considered topological when appropriate boundary conditions are imposed. Thus, in this section, we aim to find some constraints on the boundary ensemble of theories, given an arbitrary abelian 3d TQFT on the bulk. We will see that the condition of being abelian will play a crucial role.

2.4.1 Bulk-boundary correspondence

Given a possibly disconnected closed and oriented 2-manifold Σ , we expect a correspondence:

$$\mathcal{ZG}(\Sigma; \mathcal{T}) \leftrightarrow \langle T_\xi(\Sigma) \rangle_\xi, \quad (2.4.2)$$

where $\mathcal{ZG}(\Sigma; \mathcal{T})$ is the “gravity” partition function with abelian 3d TQFT $\mathcal{T}$ in the bulk, while T_ξ denotes a family of 2d TQFTs on the boundary, parameterised by some data ξ . We use $\langle \cdot \rangle_\xi$ for the average over the statistical ensemble of theories T_ξ . For the gravity side, we will consider the sum over bulk topologies $\mathcal{ZG}(\Sigma; \mathcal{T}) = \langle \mathfrak{B}|\Sigma; \mu \rangle_{\mathcal{T}}$ as in (2.1.5),

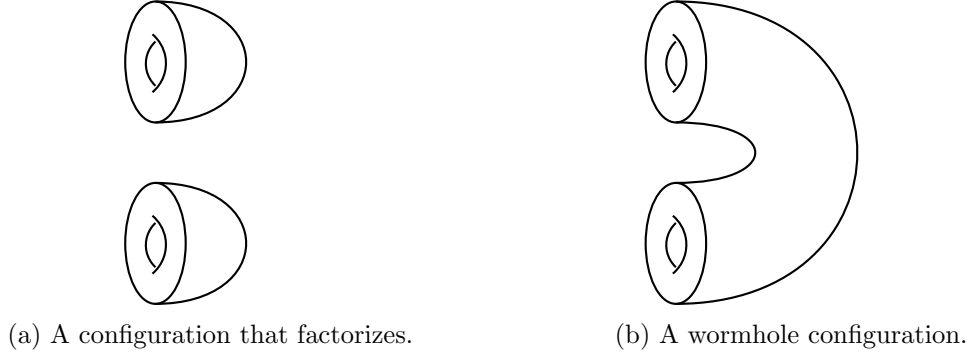


Figure 2.4.1: Two different kind of terms contribute to $\mathcal{ZG}(T^2 \sqcup T^2)$.

in the case where the same topological boundary condition $\mathfrak{L}$ is imposed on all connected boundary components. Note that $\langle \mathfrak{B} | \equiv \bigotimes_i \langle \mathfrak{L} |^{(i)}$ implicitly depends on the number of connected components of Σ . In Section 2.3 we analyzed its convergence and pointed out that it can be expressed through the connected gravitational partition functions (2.3.5).

We now address the problem of the lack of factorization of $\mathcal{ZG}$, which explains the need for a statistical ensemble of theories on the boundary. By lack of factorization, we mean that $\mathcal{ZG}$ is not multiplicative under taking a disjoint union of boundary 2-manifolds, hence it cannot be a partition function of a single 2d theory.

Specifically, consider the sum over topologies $\mathcal{ZG}(\Sigma)$ and suppose that Σ has two disconnected components $\Sigma = \Sigma_g \sqcup \Sigma_{g'}$, then $\mathcal{ZG}(\Sigma)$ will get contribution from two different kinds of terms, which are depicted in Figure 2.4.1. Any 3-manifold in the sum will be a disjoint union of a number n of closed components, while its boundary will be either bounding a single 3-manifold or it will be split into two different connected components. Then, the algebraic data $\mathcal{A}$ is actually a list $\{\mathcal{A}^{(i)}\}_i$ of n couples corresponding to closed components, together with either one or two triples corresponding to the manifolds with boundary. Therefore, recalling that we assumed the measure μ to factorize with respect to disjoint union, as in (2.3.1), we obtain:

$$\mathcal{ZG}(\Sigma) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{Z}_0^n \cdot (\mathcal{Z}_{g+g'} + \mathcal{Z}_g \mathcal{Z}_{g'}) = e^{\mathcal{Z}_0} (\mathcal{Z}_{g+g'} + \mathcal{Z}_g \mathcal{Z}_{g'}), \quad (2.4.3)$$

where $\mathcal{Z}_g$ is the connected gravitational partition function (2.3.5), depending on the TQFT data $\mathcal{T} = (D, q)$ and on the topological boundary condition $\mathfrak{L}$ that we kept implicit to lighten the notation. As pointed out in Section 2.2, the partition function of an abelian TQFT on a connected 3-manifold with a topological boundary condition depends only on the total genus of the boundary component (up to a possibly decoupled 2d TQFT supported on the boundary). Assuming the same holds for the summation measure (as in Section 2.3), the connected gravitational partition function only depends on the total genus of the 2-manifold. This is a key step in obtaining (2.4.3).

We remark that similar expressions can be obtained for Σ with any number of connected components. The computation ultimately reduces to enumerating all possible ways of connecting the boundary components via 3-dimensional bordisms. In particular, the gravitational partition function of the vacuum is given by $\mathcal{ZG}(\emptyset) = e^{\mathcal{Z}_0}$.

On the other hand, if $\mathcal{ZG}$ were to factorize, we would expect $\mathcal{ZG}(\Sigma_g \sqcup \Sigma_{g'})$ to be equal to:

$$\mathcal{ZG}(\Sigma_g) \mathcal{ZG}(\Sigma_{g'}) = e^{2\mathcal{Z}_0} \mathcal{Z}_g \mathcal{Z}_{g'}. \quad (2.4.4)$$

Thus, the lack of factorization can be understood as a consequence of the presence of the so-called wormhole configurations. This peculiar choice of words is dictated by the interpretation of each connected component of the boundary as a parallel universe. Hence, a wormhole is a bulk configuration connecting two different universes.

Nonetheless, one could be tempted to force the factorization of $\mathcal{ZG}$. However, we show that requiring factorization is equivalent to $\mathcal{Z}_g = 0$ for any closed surface of genus g . The fact that this condition is sufficient for factorization is straightforward since everything trivializes. On the other hand, assuming factorization for $\Sigma = (S^2)^{\sqcup k}$, one can easily check that it implies $\mathcal{Z}_0 = 0$. Then, we can extend this result to any other genus g by requiring the factorization for $S^2 \sqcup \Sigma_g$. We stress that we have extensively used the property that the connected gravitational partition function depends only on the total genus g of Σ .

We note that untwisted Dijkgraaf-Witten theories cannot satisfy this vanishing condition on $\mathcal{Z}_g$. Hence, the lack of factorization is intrinsic to the sum over topologies and invalidates the usual holography principle. The way around it is that the bulk theory is not dual to a boundary theory, but rather to an ensemble of theories living on the boundary, as this would explain the failure of $\mathcal{ZG}$ to factorize.

Let us formulate the bulk-boundary correspondence more precisely. Define $\zeta := \mathcal{Z}_0$. Thus, the bulk-boundary correspondence is given by the following identification:

$$e^{-\zeta} \mathcal{ZG}(\Sigma) = \langle T_\xi(\Sigma) \rangle_\xi. \quad (2.4.5)$$

The extra factor on the left-hand side is required to satisfy the normalization condition for the distribution of the boundary theories:

$$\langle T_\xi(\emptyset) \rangle_\xi \equiv \langle 1 \rangle_\xi = 1, \quad (2.4.6)$$

where we used the fact that the partition function of any 2d TQFT on an empty spacetime is 1.

2.4.2 Brief review of 2d TQFTs

We shall explain in which way T_ξ constitutes a statistical ensemble of theories. To do so, we first recall some basic facts about 2d TQFTs.

It is well-known that (non-extended) 2d TQFTs are in one-to-one correspondence with commutative Frobenius algebras. Since we are dealing with unitary 2d TQFTs,

the algebras get promoted to semi-simple Frobenius algebras. We briefly describe how this correspondence works. Suppose that T is a unitary 2d TQFT, then $\mathcal{V} := T(S^1)$ is a finite-dimensional vector space, which has the physical meaning of the Hilbert space of states on a circle. Moreover, $\mathcal{V}$ is a commutative Frobenius algebra, where the multiplication $m(\cdot, \cdot)$ is given by T applied to a pair of pants and the trace $\theta(\cdot)$ is given by T applied to a disk.

For a semi-simple algebra, we are free to choose a basis of idempotents e_i for $i = 1, \dots, N$. Next, denoting $\theta_i = \theta(e_i)$, one can show that:

$$T_{N, \{\theta_i\}}(\Sigma_g) = \sum_{i=1}^N \theta_i^{1-g}, \quad (2.4.7)$$

for any Riemann surface Σ_g of genus g .

Hence, we see that the data specifying our boundary 2d TQFT is $\xi = (N, \{\theta_i\})$. Notice that $(N, \{\theta_i\})$ characterises the whole TQFT, even though we just need the value of its partition function on closed 2-manifolds.

2.4.3 Statistical ensemble of theories

Now that the data defining the family of boundary 2d TQFTs are clear, we focus on the meaning of this statistical ensemble. In the most general situation, we will have that the dimension of the Hilbert space N is a random variable following a discrete probability distribution supported on the natural numbers and, similarly, we will have N random variables $\{\theta_i\}$ that follow the same, perhaps continuous, probability distribution supported on the reals. A priori, there is no reason to suppose that the θ_i 's are independent from each other and N . Moreover, we will argue later that assuming such an independence condition is inconsistent with our bulk theory. For the moment, let us work in the above assumptions and denote the joint probability density function $f_{N, \theta_1, \dots, \theta_N}(n, \alpha_1, \dots, \alpha_n)$. We write:

$$f_{N, \theta_1, \dots, \theta_N}(n, \alpha_1, \dots, \alpha_n) = f_{\theta_1, \dots, \theta_N | N}(\alpha_1, \dots, \alpha_n | n) f_N(n), \quad (2.4.8)$$

to rephrase the problem in terms of the conditional probability. Indeed, we are trying to model an ensemble of commutative Frobenius algebras, which poses some interesting challenges. An n -dimensional algebra will need exactly n real parameters, hence the distribution we are looking for will have a non-fixed number of random variables. This is best understood on the conditional probability, since at a fixed value $N = n$ of the dimension, the number of trace variables is fixed.

Without loss of generality (because the partition functions (2.4.7) are symmetric in θ_i), we will assume that distribution $f_{\theta_1, \dots, \theta_N | N}(\alpha_1, \dots, \alpha_n | n)$ is symmetric with respect to the permutations of θ_i .

The average of the partition functions of the boundary theories on a connected Riemann surface Σ_g reads:

$$\begin{aligned} \langle T_\xi(\Sigma_g) \rangle_\xi &= \sum_{n=0}^{\infty} \int d\alpha_1 \cdots d\alpha_n f_{N, \theta_1, \dots, \theta_N}(n, \alpha_1, \dots, \alpha_n) T_{n, \{\alpha_i\}}(\Sigma_g) \\ &= \mathbb{E}[N\theta^{1-g}], \end{aligned} \quad (2.4.9)$$

where $\theta \equiv \theta_i$ for some i .

For a disjoint union of two Riemann surfaces, $\Sigma = \Sigma_g \sqcup \Sigma_{g'}$, we have:

$$\begin{aligned} \langle T_\xi(\Sigma_g \sqcup \Sigma_{g'}) \rangle_\xi &= \sum_{n=0}^{\infty} f_N(n) \int \left(\prod_{i=1}^n d\alpha_i \right) f_{\theta_1, \dots, \theta_N|N}(\alpha_1, \dots, \alpha_n|n) \sum_{j=1}^n \alpha_j^{1-g} \sum_{k=1}^n \alpha_k^{1-g'} \\ &= \sum_{n=0}^{\infty} f_N(n) \left(\sum_{j \neq k} \int d\alpha_j d\alpha_k f_{\theta, \theta'|N}(\alpha_j, \alpha_k|n) \alpha_j^{1-g} \alpha_k^{1-g'} \right. \\ &\quad \left. + \sum_j \int d\alpha_j f_{\theta|N}(\alpha_j|n) \alpha_j^{2-g-g'} \right) \\ &= \mathbb{E}[(N^2 - N)\theta^{1-g}\theta'^{1-g'}] + \mathbb{E}[N\theta^{2-g-g'}], \end{aligned} \quad (2.4.10)$$

where $\theta \equiv \theta_i$ and $\theta' \equiv \theta_j$ for some $i \neq j$, and we use

$$f_{\theta_{i_1}, \dots, \theta_{i_k}|N}(\alpha_{i_1}, \dots, \alpha_{i_k}|n) := \int \prod_{i \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}} d\alpha_i f_{\theta_1, \dots, \theta_N}(\alpha_1, \dots, \alpha_n|n) \quad (2.4.11)$$

to denote the reduced probability densities.

The partition function of a disjoint union of k Riemann surfaces is hard to express in a closed form. Nonetheless, it will be useful to manipulate its expression though we postpone the discussion until the next section.

We conclude with another simple example. Consider the disjoint union of k tori:

$$\begin{aligned} \langle T_\xi((T^2)^{\sqcup k}) \rangle_\xi &= \sum_{n=0}^{\infty} f_N(n) \int \left(\prod_{i=1}^n d\alpha_i \right) f_{\theta_1, \dots, \theta_N|N}(\alpha_1, \dots, \alpha_n|n) \prod_{l=1}^k \sum_{j=1}^n 1 \\ &= \sum_{n=0}^{\infty} f_N(n) n^k \\ &= \mathbb{E}[N^k]. \end{aligned} \quad (2.4.12)$$

Notice that such configurations yield all the statistical moments of the random variable N .

2.4.4 The abelian condition

It turns out that the bulk theory being abelian is a strong condition on the ensemble of dual theories. We introduce the *abelian condition* as the set of constraints that the ensemble of 2d TQFTs has to satisfy to be consistent with an abelian theory on the bulk.

The crucial property of an abelian 3d TQFT is that it does not distinguish between disjoint unions and connected sums on the boundary (see Section 2.2). Hence (assuming the summation weights also respect this property, see Section 2.3), the connected gravitational partition function $\mathcal{Z}_g$ only depends on the total genus of the boundary surface.

For example, we deduce the following equations from the disjoint union of two Riemann surfaces and the disjoint union of k tori, respectively:

$$\langle T_\xi(\Sigma_g \sqcup \Sigma_{g'}) \rangle_\xi = \langle T_\xi(\Sigma_g) \rangle_\xi \langle T_\xi(\Sigma_{g'}) \rangle_\xi + \langle T_\xi(\Sigma_{g+g'}) \rangle_\xi, \quad (2.4.13)$$

$$\langle T_\xi((T^2)^{\sqcup k}) \rangle_\xi = B_k(\langle T_\xi(T^2) \rangle_\xi, \langle T_\xi(\Sigma_2) \rangle_\xi, \dots, \langle T_\xi(\Sigma_k) \rangle_\xi), \quad (2.4.14)$$

where $B_k(x_1, \dots, x_k)$ are the complete exponential Bell polynomials.

These two equations are already very constraining on the distributions for N and θ . Indeed, plugging into these the results of the previous section we get:

$$\mathbb{E} \left[(N^2 - N) \theta^{1-g} \theta^{1-g'} \right] + \mathbb{E} [N \theta^{2-g-g'}] = \mathbb{E} [N \theta^{1-g}] \mathbb{E} [N \theta^{1-g'}] + \mathbb{E} [N \theta^{1-g-g'}], \quad (2.4.15)$$

$$\mathbb{E} [N^k] = B_k \left(\mathbb{E} [N], \mathbb{E} [N \theta^{-1}], \dots, \mathbb{E} [N \theta^{1-k}] \right). \quad (2.4.16)$$

The general condition cannot be expressed in closed form, but we can still work out a list of constraints.

Proposition 2.4.1 (The abelian condition). *Consider a collection of k Riemann surfaces $\{\Sigma_{g_i}\}$ of genus g_i . The abelian condition then reads as follows:*

$$\sum_{p \in P(\{g_1, \dots, g_k\})} \prod_{q \in p} \mathbb{E} \left[N \theta^{1-\sum_{g \in q} g} \right] = \sum_{p \in P(\{g_1, \dots, g_k\})} \mathbb{E} \left[\frac{N!}{(N-|p|)!} \prod_{q \in p} \theta_q^{\sum_{g \in q} (1-g)} \right], \quad (2.4.17)$$

where $P(\{g_1, \dots, g_k\})$ denotes the set of partitions of the set $\{g_1, \dots, g_k\}$ and, for a fixed partition p , $\theta_q \equiv \theta_{i_q}$ for some i_q such that $i_q \neq i_{q'}$ for $q \neq q' \in p$.

Indeed, the left-hand side comes from taking into account all possible ways of joining the disconnected surfaces via connected sum. To be more precise, each contribution corresponds to a partition p of the set $\{\Sigma_{g_i}\}_i$ of Riemann surfaces, where each part p_j of p corresponds to the connected sum of all the surfaces in that part. Thus:

$$\langle T_\xi(\sqcup_{i=1}^k \Sigma_{g_i}) \rangle_\xi = \sum_{p \in P(\{g_1, \dots, g_k\})} \prod_{q \in p} \langle T_\xi(\Sigma_{\sum_{g \in q} g}) \rangle_\xi = \sum_{p \in P(\{g_1, \dots, g_k\})} \prod_{q \in p} \mathbb{E} \left[N \theta^{1-\sum_{g \in q} g} \right]. \quad (2.4.18)$$

Similarly, for the right-hand side we first directly compute:

$$\langle T_\xi(\sqcup_{i=1}^k \Sigma_{g_i}) \rangle_\xi = \mathbb{E} \left[\prod_{i=1}^k \sum_{j_i=1}^N \theta_{j_i}^{1-g_i} \right]. \quad (2.4.19)$$

To re-express this term, we notice that we have different contributions depending on whether or not certain j_i 's are equal. This is explicitly specified by the partition p of the sequence $\{j_1, \dots, j_k\}$, where the blocks consist of equal j_i 's. In particular, having equal indices corresponds to the contribution of the genera of the corresponding surfaces to add up. Finally, notice that any combination of indices has to be counted $N(N-1) \cdots (N-|p|+1)$ times. It follows that:

$$\mathbb{E} \left[\prod_{i=1}^k \sum_{j_i=1}^N \theta_{j_i}^{1-g_i} \right] = \sum_{p \in P(\{g_1, \dots, g_k\})} \mathbb{E} \left[\frac{N!}{(N-|p|)!} \prod_{q \in p} \theta_q^{\sum_{g \in q} (1-g)} \right], \quad (2.4.20)$$

which gives the equality between the right-hand and the left-hand sides of the abelian condition stated in the proposition.

2.4.5 Independent variables

As we anticipated above, we want to rule out the possibility that the random variables N and θ are independent. Indeed, we will deduce that this assumption and the abelian condition are too constraining together.

Assume that all random variables θ_i are independent of each other and also independent of the random variable N , i.e.:

$$f_{\theta_1, \dots, \theta_N | N}(\alpha_1, \dots, \alpha_N | n) = \prod_{i=1}^n f_\theta(\alpha_i). \quad (2.4.21)$$

Then, we may rewrite the abelian conditions (2.4.15) and (2.4.16):

$$(N_2 - N_1) \theta_{(1-g)} \theta_{(1-g')} + N_1 \theta_{(2-g-g')} = N_1^2 \theta_{(1-g)} \theta_{(1-g')} + N_1 \theta_{(1-g-g')}, \quad (2.4.22)$$

$$N_k = B_k(N_1, N_1 \theta_{(-1)}, \dots, N_1 \theta_{(1-k)}), \quad (2.4.23)$$

where we have introduced the moments of the two distributions, namely $N_k := \mathbb{E}[N^k]$ and $\theta_{(k)} := \mathbb{E}[\theta^k]$.

Assuming that $N_1 > 0$, these conditions can only be solved by a degenerate distribution supported on the single value $\theta \equiv 1/\theta_{(-1)}$. Moreover, the N -distribution has to satisfy:

$$N_k = B_k(N_1, N_1 \lambda, \dots, N_1 \lambda^{k-1}), \quad (2.4.24)$$

having denoted $\lambda := \theta_{(-1)}$. This implies that the probability density function for N is:

$$f_N(n) = \begin{cases} e^{-N_1/\lambda} \frac{1}{k!} \left(\frac{N_1}{\lambda}\right)^k & \text{if } n = k\lambda \text{ for } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.4.25)$$

that is a Poisson-like distribution supported on $\lambda \cdot \mathbb{N}$. This fixes λ to be an integer since we require N to take value in $\mathbb{N}$.

This distribution is consistent with the abelian condition (2.4.17). To sum up, we have seen that the abelian condition is quite powerful, at least assuming independence among the random variables. It constrains the distribution rules of N and θ , leaving only two parameters, N_1 and λ , that depend on the explicit theory in the bulk. We stress that λ has to be a nonzero natural number to make sense of this result. We conclude this section by remarking that this solution is too restrictive for the holographic principle to work at least in some generality, ruling out the possibility that N and θ are independent random variables.

2.4.6 Solving the abelian condition

We aim to solve the abelian condition in full generality, with the only assumption being the symmetry of the distribution with respect to the permutation of θ_i .

Looking at the abelian condition, there is a clear hierarchy. Indeed, the equation corresponding to $\Sigma = \sqcup_{i=1}^k \Sigma_{g_i}$ depends on the distributions $f_N, f_{\theta|N}, \dots, f_{\theta_1, \dots, \theta_l|N}$ for l equal to the number of Riemann surfaces of genus different from one in Σ . Thus, we can opt for an inductive approach.

The first step consists of considering k copies of tori, for which the abelian condition gives the equation (2.4.16). Hence, suppose we know $x_k := \mathbb{E}[N\theta^{1-k}]$ for any $k \geq 1$, then this allows us to compute the moment-generating function for the N -distribution:

$$M_N(t) := \mathbb{E}[e^{tN}] = \sum_{k=0}^{\infty} N_k \frac{t^k}{k!} = \sum_{k=0}^{\infty} B_k(x_1, \dots, x_k) \frac{t^k}{k!} = \exp \left[\sum_{k=1}^{\infty} x_k \frac{t^k}{k!} \right], \quad (2.4.26)$$

where we used some basic facts about Bell polynomials. This assumes that the N -distribution admits a moment-generating function, which is not necessarily true. For example, the higher moments might be divergent. This is indeed what can happen in our setting, as we will point out later. Furthermore, we stress that from this expression it is not apparent that $M_N(t)$ corresponds to a discrete distribution, as it will turn out later.

For the time being, we will not worry about these problems and assume that the moment-generating function exists. Hence, we are also supposing that the power series in the argument of the exponential is convergent in a sufficiently small disk centred at $t = 0$. This, in turn, would imply that all the moments are well-defined and the distribution is uniquely determined by them.

Next, we shall consider surfaces of the form $\Sigma = \Sigma_g \sqcup (T^2)^{\sqcup k}$. Directly from the definition, we can check that:

$$\langle T_\xi(\Sigma_g \sqcup (T^2)^{\sqcup k}) \rangle_\xi = \mathbb{E}[N^{k+1} \theta^{1-g}]. \quad (2.4.27)$$

On the other hand, performing some combinatorics:

$$\langle T_\xi(\Sigma_g \sqcup (T^2)^{\sqcup k}) \rangle_\xi = \sum_{n=0}^k \binom{k}{n} \langle T_\xi((T^2)^{\sqcup k-n}) \rangle_\xi \langle T_\xi(\Sigma_{g+n}) \rangle_\xi, \quad (2.4.28)$$

so that:

$$\mathbb{E}[N^{k+1} \theta^{1-g}] = \sum_{n=0}^k \binom{k}{n} \mathbb{E}[N^{k-n}] \mathbb{E}[N \theta^{1-g-n}]. \quad (2.4.29)$$

Now, define $\Phi(t, r) := \mathbb{E}[N \theta \exp(tN + r/\theta)]$. It is straightforward to obtain the moment-generating function for N and θ^{-1} in terms of Φ :

$$M_{N, \theta^{-1}}(t, r) \equiv \mathbb{E}[\exp(tN + r/\theta)] = \int_{-\infty}^t d\tau \frac{d}{dr} \Phi(\tau, r). \quad (2.4.30)$$

Hence, it is enough to compute $\Phi(t, r)$. We start by noticing that after some manipulations, we may write:

$$\Phi(t, r) = M_N(t) (\mathbb{E}[N\theta] + \log M_N(t + r)). \quad (2.4.31)$$

All the information is now encoded in the moment-generating function for N . Finally, plugging this into the right-hand side of (2.4.30) yields:

$$M_{N, \theta^{-1}}(t, r) = \int_{-\infty}^t d\tau \frac{d}{dr} M_N(\tau) (\mathbb{E}[N\theta] + \log M_N(\tau + r)) = \int_{-\infty}^t d\tau \frac{M_N(\tau)}{M_N(\tau + r)} \frac{d}{dr} M_N(\tau + r). \quad (2.4.32)$$

Discrete dimensions

Before continuing, let us deal with the problem of imposing a discrete distribution supported on integers for the dimension. Imposing such a condition at the level of the moment-generating function amounts to requiring it to be $2\pi i$ periodic. However, determining whether a power series is of a periodic function can be quite challenging. We avoid this problem by considering the following sufficient condition, which will be satisfied by a large class of examples. Namely, we require the x_k to be a Dirichlet series:

$$x_k = \sum_{n=1}^{\infty} b_n n^k, \quad (2.4.33)$$

with non-negative coefficients $b_n \geq 0$. We remark that negative coefficients would lead to negative probabilities.

We also require these series to be convergent for any $k \leq k_* \in \mathbb{N}$, possibly with $k_* = +\infty$. We remark that a finite value of k_* means that our distribution only admits a finite number of moments. This is not a problem a priori, and indeed happens if the bound (2.3.13) is not satisfied for arbitrary s (as in the simple example of the measure considered in Section 2.3.2). However, some of the infinite series below should be understood formally, unless $k_* = +\infty$. Thus, even though we assume $k_* = +\infty$ to derive the unique distribution of dual theories, one can check a posteriori that such a distribution gives a canonical – but not unique – solution even if k_* is finite. Indeed, the resulting distribution itself is well-defined at $k_* < +\infty$, and reproduces the correct x_k for $k \leq k_*$.

We now check that this condition implies that N is supported on the integers. Indeed, we can plug this expression into the moment-generating function and, assuming that the series is absolutely convergent in an open neighborhood of $t = 0$, we obtain:

$$M_N(t) = e^{-x_0} \exp \left[\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} b_n \frac{(tn)^k}{k!} \right] = e^{-x_0} \exp \left[\sum_{n=1}^{\infty} b_n e^{tn} \right]. \quad (2.4.34)$$

Thus, the moment-generating function is manifestly $2\pi i$ periodic as we want. Moreover, this also implies that the distribution are normalized since $M_N(0) = 1$, as long as the x_0 is well-defined, i.e. $k_* \geq 0$.

Under this additional hypothesis, we can explicitly determine the probability density functions:

$$f_N(n) = e^{-x_0} c_n, \quad (2.4.35)$$

$$f_{N,\theta^{-1}}(n, m) = \frac{e^{-x_0}}{n} m b_m c_{n-m}, \quad (2.4.36)$$

where we have defined the c_k 's to be the coefficients in the series expansion:

$$\exp \left[\sum_{k=1}^{\infty} b_k x^k \right] = \sum_{k=0}^{\infty} c_k x^k. \quad (2.4.37)$$

Notice that it is intended $c_k = b_k = 0$ for $k < 0$. Moreover, for $k \geq 0$, c_k can be expressed as a polynomial in b_i , $i = 1, \dots, k$. The obtained distribution for θ^{-1} turns out to be discrete and supported on positive integers, so $f_{N,\theta^{-1}}$ in (2.4.36) means the distribution function where θ^{-1} is considered as a *discrete* random variable, although a priori it was a continuous one³. Similarly, the distributions for multiple θ_i^{-1} will be considered as distributions of discrete variables from now on.

³Explicitly, in terms of the previously considered continuous distributions, we have:

$$f_{N,\theta^{-1}}(n, m) \delta(\alpha_1 - 1/m) \equiv f_N(n) f_{\theta_1|N}(\alpha_1|n) = f_N(n) \int d\alpha_2 \dots d\alpha_N f_{\theta_1, \dots, \theta_N|N}(\alpha_1, \alpha_2, \dots, \alpha_N|n). \quad (2.4.38)$$

The full solution

One could repeat the above procedure to solve the abelian condition with $\Sigma = \Sigma_g \sqcup \Sigma_{g'} \sqcup (T^2)^{\sqcup k}$ and so on. By doing so, we infer the general solution.

Proposition 2.4.2. *Suppose that the coefficients b_n are non-negative and consider the following function for $n \geq k$:*

$$f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}(n, m_1, \dots, m_k) = e^{-x_0} \frac{(n-k)!}{n!} \sum_{p \in P(\{1, \dots, k\})} c_{n - \sum_{i=1}^l m_{p_i}} \prod_{i=1}^l \delta(p_i) b_{m_{p_i}} \frac{m_{p_i}!}{(m_{p_i} - |p_i|)!}, \quad (2.4.39)$$

where $p = \{p_1, \dots, p_l\}$ is a partition of the set $\{1, \dots, k\}$ of length l and:

$$\delta(\{\mu_1, \dots, \mu_r\}) \equiv \prod_{\alpha=1}^{r-1} \delta_{m_{\mu_\alpha}, m_{\mu_{\alpha+1}}}, \quad m_{\{\mu_1, \dots, \mu_r\}} \equiv m_{\mu_1} = \dots = m_{\mu_r}. \quad (2.4.40)$$

Notice that it agrees with $f_{N, \theta^{-1}}$ defined as above, when $k = 1$.

Then, such a function satisfies:

$$\sum_{m_{l+1}} \dots \sum_{m_k} f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}(n, m_1, \dots, m_k) = f_{N, \theta_1^{-1}, \dots, \theta_l^{-1}}(n, m_1, \dots, m_l), \quad (2.4.41)$$

when both sides are well-defined, i.e. $n \geq k \geq l$. Moreover, when $l = 0$, the right-hand side of this formula gives f_N , as defined above.

The distribution corresponding to $f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}$ satisfies the abelian condition for disjoint unions of up to k Riemann surfaces (that are not tori) with an arbitrary amount of tori.

We postpone the proof of this proposition to the appendix A, as it is more technical than conceptual.

The functions $f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}(n, m_1, \dots, m_k)$ are well-defined only when $n \geq k$, however we shall remember that the corresponding distributions are just formal solutions to the abelian condition. Indeed, the ensemble of 2d TQFTs is described by the conditional density functions defined in (2.4.8), which depend only on the case $n = k$.

2.4.7 Some remarks on the distribution

We managed to find a solution to the abelian condition, which determines (non-uniquely) a distribution of 2d TQFTs through (2.4.8). We remark that this result depends on two key assumptions: the Dirichlet conditions (2.4.33) and the reality and non-negativity of the coefficients b_n . The first one guarantees that N is supported on the integers, while the second one makes sure that we avoid complex and negative probabilities. Moreover, the distribution exists even for finite values of k_* .

In the remainder of the section, we will point out some properties of the ensemble of 2d TQFTs that corresponds to such distributions. First of all, the random variables specifying the ensemble are not independent in general, as can be seen from the explicit formulas for the distribution.

As was already pointed out above, the distributions for θ are also discrete. Their values are supported on $1/k$ for $k = 1, \dots, n$, where n is the value assumed by the random variable N . More conditions on the allowed values for θ can be deduced from the expression for the conditional probability.

Expression for the conditional probability

Consider the conditional probability, given $N = n$ for some $n \in \mathbb{N}$:

$$f_{\theta_1^{-1}, \dots, \theta_n^{-1}|N}(m_1, \dots, m_n|n) = \frac{1}{n!} c_n \sum_{p \in P(\{1, \dots, n\})} c_{n - \sum_{q \in p} m_q} \prod_{q \in p}^l \delta(q) b_{m_q} \frac{m_q!}{(m_q - |q|)!}. \quad (2.4.42)$$

We will now argue that this expression simplifies significantly. First of all, we show that the only partitions with a non-vanishing contribution have to satisfy:

$$n = \sum_{q \in p} m_q. \quad (2.4.43)$$

In particular, we show that if $\sum_{q \in p} m_q \geq n + 1$ or $\sum_{q \in p} m_q \leq n - 1$, then the contribution is zero. The first case is straightforward since the argument of c_n has to be positive. In the second case, notice that $n = \sum_{q \in p} |q|$, hence $\sum_{q \in p} (m_q - |q|)$ is negative. This implies that there exists a $q \in p$ such that $m_q - |q|$ is negative and therefore:

$$\frac{m_q!}{(m_q - |q|)!} = 0. \quad (2.4.44)$$

Now, the presence of the factor:

$$\delta(q) \frac{m_q!}{(m_q - |q|)!} \quad (2.4.45)$$

implies that a specific partition contributes zero unless it selects $|q|$ equal entries $m_q \equiv m_{q_1} = \dots = m_{q_{|q|}}$. However, if $|q| \geq m_q + 1$, then the contribution is again zero due to the factorial in the denominator.

Moreover, we now show that $|q|$ needs to be exactly equal to m_q . Indeed, if this was not the case, then there would exist a $\bar{q} \in p$ such that $|\bar{q}| \leq m_{\bar{q}} - 1$. On the other hand, the others should satisfy $|q| \leq m_q$. Hence:

$$n = \sum_{q \in p} |q| \leq \left(\sum_{q \in p} m_q \right) - 1 = n - 1, \quad (2.4.46)$$

yielding a contradiction.

Thus, the only non-vanishing combinations of $\vec{m} = (m_1, \dots, m_n)$ are of the following form:

$$\vec{m} = (\underbrace{m_1, \dots, m_1}_{a_1 \cdot m_1 \text{ times}}, \underbrace{m_2, \dots, m_2}_{a_2 \cdot m_2 \text{ times}}, \dots, \underbrace{m_r, \dots, m_r}_{a_r \cdot m_r \text{ times}}), \quad (2.4.47)$$

and permutations thereof, with probability:

$$\begin{aligned} f_{\theta_1^{-1}, \dots, \theta_n^{-1} | N}(\vec{m} | n) &= \frac{1}{n! c_n} \sum_p \prod_{i=1}^r (m_i! b_{m_i})^{a_i} \\ &= \frac{1}{n! c_n} \prod_{i=1}^r (m_i! b_{m_i})^{a_i} \frac{(a_i m_i)!}{(m_i!)^{a_i} a_i!} \\ &= \frac{1}{n! c_n} \prod_{i=1}^r b_{m_i}^{a_i} \frac{(a_i m_i)!}{a_i!}. \end{aligned} \quad (2.4.48)$$

Notice that, up to permutations, the possible choices of $\vec{m}$ — i.e. those with non-zero probability — are in a one-to-one correspondence with possible choices of positive integers m_i and a_i such that $n = \sum_{i=1}^r a_i m_i$ for some r .

Explicit formulae for low dimensions

We derive explicit expressions for the conditional probabilities, given $n = 1, 2, 3$. When $n = 1$, the only non-zero probability is for $m = 1$ and it is consistent with the normalization condition $f_{\theta^{-1} | N}(1 | 1) = 1$.

For $n = 2$, only two configurations are non-vanishing:

$$f_{\theta_1^{-1}, \theta_2^{-1} | N}(1, 1 | 2) = \frac{b_1^2}{b_1^2 + 2b_2}, \quad f_{\theta_1^{-1}, \theta_2^{-1} | N}(2, 2 | 2) = \frac{2b_2}{b_1^2 + 2b_2}. \quad (2.4.49)$$

While, in the case $n = 3$, we have:

$$\begin{aligned} f_{\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1} | N}(1, 1, 1 | 3) &= \frac{b_1^3}{b_1^3 + 6b_1b_2 + 6b_3}, & f_{\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1} | N}(1, 2, 2 | 3) &= \frac{2b_1b_2}{b_1^3 + 6b_1b_2 + 6b_3}, \\ f_{\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1} | N}(3, 3, 3 | 3) &= \frac{6b_3}{b_1^3 + 6b_1b_2 + 6b_3}, \end{aligned} \quad (2.4.50)$$

and permutations thereof.

2.4.8 Recovering the distribution for a given 3d TQFT

Now, we will combine the results of this section with the analysis performed in Section 2.3. We have shown that under some mild hypotheses that we are going to verify, it is enough to

compute $\mathbb{E}[N\theta^{1-k}]$. If furthermore one can show that the coefficients b_n are non-negative, they determine a (non-unique) distribution of commutative Frobenius algebras.

As already argued at the beginning of this section, one can express the sum over topologies via terms that only include connected manifolds. The case of $\Sigma = (T^2)^{\sqcup k}$ is simple enough to obtain a closed relation:

$$\mathcal{ZG}((T^2)^{\sqcup k}; \mathcal{T}) = e^\zeta B_k(\mathcal{Z}_1, \dots, \mathcal{Z}_k). \quad (2.4.51)$$

On the other hand, by the abelian condition (2.4.16), we have:

$$\langle T_\xi((T^2)^{\sqcup k}) \rangle_\xi = B_k \left(\mathbb{E}[N], \mathbb{E}[N\theta^{-1}], \dots, \mathbb{E}[N\theta^{1-k}] \right). \quad (2.4.52)$$

Thus, by the holographic principle, we obtain:

$$x_g = \mathbb{E}[N\theta^{1-g}] = \mathcal{Z}_g \equiv \mathcal{Z}_g(D, q; \mathfrak{L}; \mu). \quad (2.4.53)$$

That is, the moments of the distribution are closely related to the connected gravitational partition function we have analyzed in Section 2.3. In particular, it was shown there that the hypothesis (2.4.33) is satisfied under certain assumptions on the measure. The coefficients b_n are explicitly given by the formula (2.3.19). The reality of the coefficients is automatic, assuming that the measure is real μ and respects the orientation reversal of the manifolds, that is $\mu(A, \bar{\ell}) = \mu(A, \ell)$ and $\nu(A, \ell, S) = \nu(A, \bar{\ell}, S)$, where $\bar{\ell}$ is the representative linking pairing equivalent to $-\ell$. The reality then follows from the unitarity of the partition function of the 3d TQFT (cf. [33]):

$$Z(A, \bar{\ell}; D, q; -\theta) = \overline{Z(A, \ell; D, q; \theta)}. \quad (2.4.54)$$

We were not able to prove in generality the non-negativity of the coefficients b_n . However, in the case of untwisted DW theory with Neumann boundary conditions, it is quite straightforward to do so, assuming the measure is positive. From the expression (2.3.31) for the partition function we have:

$$b_n = \sum_{(A, \ell)} \frac{\mu(A, \ell)}{|\mathfrak{L} \otimes A|} |\{\alpha \in \text{Hom}(A, \mathfrak{L}^*) : \nu(A, \ell, \ker \alpha) |\mathfrak{L}| = n\}| > 0. \quad (2.4.55)$$

2.5 Generalizations

In this section, we discuss the dependence of the sum over 3-manifolds on the choice of the topological boundary conditions. We also evaluate the sum with the special measure considered in Section 2.3.2 for a case of untwisted DW with a boundary condition which is neither Neumann nor Dirichlet, and also for a case of twisted DW theory.

2.5.1 Dependence on the boundary condition

In the previous sections, we have considered the sum over topologies of the partition function of an abelian TQFT $\mathcal{T}$ with a fixed topological boundary condition $\mathfrak{L}$, that is, the sum:

$$\langle \mathfrak{L} | \Sigma; \mu \rangle_{\mathcal{T}} := \sum_{[M, \phi]: \Sigma \xrightarrow[\phi]{\sim} \partial M} \mu([M, \phi]) \langle \mathfrak{L} | M, \phi \rangle_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}(\Sigma), \quad (2.5.1)$$

which was given precise meaning by rewriting it as a sum over certain algebraic data.

We will now discuss the dependence on the choice of the topological boundary condition. We will argue that, under some additional assumptions, one can recover the actual state given by the sum of the values of the TQFT on the bordisms:

$$|\Sigma; \mu\rangle_{\mathcal{T}} := \sum_{[M, \phi]: \Sigma \xrightarrow[\phi]{\sim} \partial M} \mu([M, \phi]) |M, \phi\rangle_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}(\Sigma), \quad (2.5.2)$$

from the knowledge of $\langle \mathfrak{L} | \Sigma, \mu \rangle_{\mathcal{T}}$ for all possible topological boundary conditions $\mathfrak{L}$. The knowledge of such a state in particular provides us the value of the sum over topologies $\langle \mathfrak{B} | \Sigma; \mu \rangle_{\mathcal{T}}$ with an arbitrary, possibly non-topological, boundary condition $\mathfrak{B}$.

The additional assumption that we impose for this to be achievable can be formulated as follows. First, as it was already done in the algebraic setting in Section 2.3, we assume that the measure $\mu([M, \phi])$ in (2.5.1) factorizes with respect to disjoint union of 3-manifolds (up to the combinatorial factor – one over the number of permutations of homeomorphic closed components). The total sum (2.5.1) then can be expressed through the sums over connected 3-manifolds:

$$|\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} := \sum_{\substack{[M, \phi]: \Sigma \xrightarrow[\phi]{\sim} \partial M \\ \text{connected } M}} \mu([M, \phi]) |M, \phi\rangle_{\mathcal{T}} \in \mathcal{H}_{\mathcal{T}}(\Sigma), \quad (2.5.3)$$

so that we can focus on such sums only. Let Σ , the boundary surface, decompose into its connected component as $\Sigma = \sqcup_i \Sigma_i$, and $\phi_i := \phi|_{\Sigma_i}$ be the inclusion of the i -th component into the boundary of the connected 3-manifold M . The sum over the bordisms equivalence classes $[M, \phi]: \emptyset \rightarrow \Sigma$ in particular includes sums over ϕ_i 's for a fixed diffeomorphism class M . Let ψ_i be some self-diffeomorphisms $\psi_i: \Sigma_i \rightarrow \Sigma_i$ of the connected components of Σ . One can then consider a change of the summation of variables $\phi_i = \phi'_i \circ \psi_i$ in the sum

above:

$$\begin{aligned}
|\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} &= \sum_{\substack{[M, \phi'] : \Sigma \xrightarrow[\phi']{\sim} \partial M \\ \text{connected } M}} \mu([M, \{\phi'_i \circ \psi_i\}_i]) |M, \{\phi'_i \circ \psi_i\}_i\rangle_{\mathcal{T}} = \\
&= \bigotimes_i \mathcal{T}(\psi_i) \sum_{\substack{[M, \phi'] : \Sigma \xrightarrow[\phi']{\cong} \partial M \\ \text{connected } M}} \mu([M, \{\phi'_i\}_i]) |M, \{\phi'_i\}_i\rangle_{\mathcal{T}} \in \bigotimes_i \mathcal{H}_{\mathcal{T}}(\Sigma_i), \quad (2.5.4)
\end{aligned}$$

where $\mathcal{T}(\psi_i) : \mathcal{H}_{\mathcal{T}}(\Sigma_i) \rightarrow \mathcal{H}_{\mathcal{T}}(\Sigma_i)$ is the representation of the action of the mapping class group of Σ_i on the corresponding Hilbert space. If we further assume that the measure is invariant under the action of the mapping group on the boundary, that is:

$$\mu([M, \{\phi'_i \circ \psi_i\}_i]) = \mu([M, \{\phi'_i\}_i]), \quad (2.5.5)$$

we obtain that the total state must be invariant under the action of the mapping class group on the Hilbert space:

$$|\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} = \left(\bigotimes_i \mathcal{T}(\psi_i) \right) |\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} \in \bigotimes_i \mathcal{H}_{\mathcal{T}}(\Sigma_i). \quad (2.5.6)$$

The TQFT states $|\mathfrak{L}_i\rangle$ corresponding to the Lagrangian subgroups are invariant under the action of the mapping class group. Here we assume that the converse holds⁴: any invariant state is a linear combination of $|\mathfrak{L}_i\rangle$. This implies that the state must be of the following form:

$$|\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} = \sum_{\{\text{Lagrangian } \mathfrak{L}_i\}_i} \left(\bigotimes_i |\mathfrak{L}_i\rangle \right) \mathcal{P}_{\{\mathfrak{L}_i\}_i}^{\text{conn}}(\Sigma) \in \bigotimes_i \mathcal{H}_{\mathcal{T}}(\Sigma_i). \quad (2.5.7)$$

where the sum happens over all Lagrangian subgroups $\mathfrak{L}_i$ of the 3d TQFT discriminant group for each connected component Σ_i . Similarly, the total sum, which includes possible disconnected 3-manifolds, can be written as:

$$|\Sigma; \mu\rangle_{\mathcal{T}} = \sum_{\{\text{Lagrangian } \mathfrak{L}_i\}_i} \left(\bigotimes_i |\mathfrak{L}_i\rangle \right) \mathcal{P}_{\{\mathfrak{L}_i\}_i}(\Sigma) e^{\zeta} \in \bigotimes_i \mathcal{H}_{\mathcal{T}}(\Sigma_i). \quad (2.5.8)$$

where, as before, $\zeta \equiv \mathcal{Z}_0$ is the sum over connected closed 3-manifolds. The coefficients $\mathcal{P}$ can be expressed as a polynomial in $\mathcal{P}^{\text{conn}}$. For example, when $\Sigma = \Sigma_1 \sqcup \Sigma_2$ has two connected components, we have:

$$\mathcal{P}_{\mathfrak{L}_1, \mathfrak{L}_2}(\Sigma_1 \sqcup \Sigma_2) = \mathcal{P}_{\mathfrak{L}_1, \mathfrak{L}_2}^{\text{conn}}(\Sigma_1 \sqcup \Sigma_2) + \mathcal{P}_{\mathfrak{L}_1}^{\text{conn}}(\Sigma_1) \mathcal{P}_{\mathfrak{L}_2}^{\text{conn}}(\Sigma_2). \quad (2.5.9)$$

⁴We are not aware of any counterexample in the setting of Abelian 3d TQFTs.

The next assumption that we make is less trivial, since we require that the final state in the sum over connected topologies only receives a nonzero contribution if all connected components share the same Lagrangian subgroup, that is:

$$\mathcal{P}_{\{\mathfrak{L}_i\}_i}^{\text{conn}}(\Sigma) = \mathcal{P}_{\mathfrak{L}_1}^{\text{conn}}(\Sigma) \delta(\mathfrak{L}_1 = \mathfrak{L}_2 = \dots), \quad (2.5.10)$$

from which we deduce:

$$|\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} = \sum_{\text{Lagrangian } \mathfrak{L}} \left(\bigotimes_i |\mathfrak{L}\rangle^{(i)} \right) \mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma) \in \bigotimes_i \mathcal{H}_{\mathcal{T}}(\Sigma_i), \quad (2.5.11)$$

where the sum is now over a single Lagrangian subgroup $\mathfrak{L}$, with $|\mathfrak{L}\rangle^{(i)}$ being the corresponding state in $\mathcal{H}_{\mathcal{T}}(\Sigma_i)$.

The assumed property (2.5.10), or, equivalently, the special form (2.5.11) of the state can be in principle deduced from the following assumption of the measure μ on the set of equivalence classes $[M, \phi]$, i.e. null-bordisms of Σ . A partial application of the “filling tubes” operation defined in Section 1.2.1 provides a map from the set of connected null-bordisms of $\Sigma_1 \# \Sigma_2 \# \dots$ to the set of connected null bordisms of $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \dots$. We then require that the measure on the latter is the pushforward of the measure on the former. We then have

$$\begin{aligned} |\Sigma; \mu\rangle_{\mathcal{T}}^{\text{conn}} &= \sum_{\substack{[M, \phi]: \Sigma \xrightarrow[\phi]{\sim} \partial M \\ \text{connected } M}} \mu([M, \phi]) |M, \phi\rangle_{\mathcal{T}} = \\ &= \sum_{\substack{[M, \phi]: \Sigma \xrightarrow[\phi]{\sim} \partial M \\ \text{connected } M}} \sum_{\substack{[\widetilde{M}, \widetilde{\phi}]: \Sigma_1 \# \Sigma_2 \# \dots \xrightarrow[\phi]{\sim} \partial \widetilde{M} \\ (\widetilde{M}^{f \cdot t}, \widetilde{\phi}^{f \cdot t}) = (M, \phi)}} \mu([\widetilde{M}, \widetilde{\phi}]) \mathcal{T}(FT) |\widetilde{M}, \widetilde{\phi}\rangle_{\mathcal{T}} = \\ &= \mathcal{T}(FT) \sum_{\substack{[\widetilde{M}, \widetilde{\phi}]: \Sigma_1 \# \Sigma_2 \# \dots \xrightarrow[\phi]{\sim} \partial \widetilde{M} \\ \text{connected } \widetilde{M}}} \mu([\widetilde{M}, \widetilde{\phi}]) |\widetilde{M}, \widetilde{\phi}\rangle_{\mathcal{T}} = \\ &= \mathcal{T}(FT) \sum_{\mathfrak{L}} \mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma_1 \# \Sigma_2 \# \dots) |\mathfrak{L}\rangle = \sum_{\mathfrak{L}} \mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma_1 \# \Sigma_2 \# \dots) \bigotimes_i |\mathfrak{L}\rangle^{(i)} \quad (2.5.12) \end{aligned}$$

where FT is the cobordism realizing “filling tubes”, which can be explicitly defined as follows. Start with a handlebody that has boundary $\Sigma_1 \# \Sigma_2 \# \dots$ and remove from it smaller versions of the handlebodies that have Σ_i as their boundaries. The resulting manifold gives the bordism $FT: \Sigma_1 \# \Sigma_2 \# \dots \rightarrow \Sigma_1 \sqcup \Sigma_2 \sqcup \dots = \Sigma$. The last two equalities in (2.5.12) follow from (2.5.7) in the case of the connected boundary surface and the fact that

$$\mathcal{T}(FT) |\mathfrak{L}\rangle = \bigotimes_i |\mathfrak{L}\rangle^{(i)}, \quad (2.5.13)$$

which can be defined, for example, using the explicit realization of $|\mathfrak{L}\rangle$ as the state given by a handlebody with an insertion of the algebra object $\mathfrak{L}$ at its core.

Under the assumptions above, and an additional a posteriori assumption below, the full state can be recovered from the sums $\mathcal{Z}_{g;\mathfrak{L}}$ that were considered in Section 2.3 (in the subscript we now include explicit dependence on the choice of the topological boundary condition $\mathfrak{L}$). Namely, we have:

$$\begin{aligned} \mathcal{Z}_{g;\mathfrak{L}'} \equiv \langle \mathfrak{L}' | \Sigma; \mu \rangle_{\mathcal{T}}^{\text{conn}} &= \sum_{\text{Lagrangian } \mathfrak{L}} \langle \mathfrak{L}' | \left(\bigotimes_i |\mathfrak{L}\rangle^{(i)} \right) \mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma) = \\ &= \sum_{\text{Lagrangian } \mathfrak{L}} |\mathfrak{L}' \cap \mathfrak{L}|^g \mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma), \end{aligned} \quad (2.5.14)$$

where $g = g_1 + g_2 + \dots$ is the total genus of $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \dots$. We then have:

$$\mathcal{P}_{\mathfrak{L}}^{\text{conn}}(\Sigma) = \sum_{\text{Lagrangian } \mathfrak{L}'} \Pi_{g;\mathfrak{L},\mathfrak{L}'} \mathcal{Z}_{g;\mathfrak{L}'}, \quad (2.5.15)$$

where $\Pi_{g;\mathfrak{L},\mathfrak{L}'}$ are elements of the matrix Π_g which is inverse to the matrix with components $|\mathfrak{L}' \cap \mathfrak{L}|^g$, assuming the latter is invertible (which, in particular, always fails for $g = 0$ if there are multiple Lagrangian subgroups). Note that when Σ itself is connected, the operator:

$$\sum_{\mathfrak{L},\mathfrak{L}'} |\mathfrak{L}\rangle \Pi_{g;\mathfrak{L},\mathfrak{L}'} \langle \mathfrak{L}' | \quad (2.5.16)$$

plays the role of the projector on the subspace of states invariant under the mapping class group action. In the case of a connected Σ , the formula (2.5.15) then follows just from invariance of the measure under self-diffeomorphisms of Σ .

We will now consider the implications of the above assumptions to the holographic correspondence. In Sections 2.4 and 2.4.8, we considered a holographically dual distribution of boundary 2d topological theories $T_{\xi;\mathfrak{L}}$, where ξ denotes the distribution variables and $\mathfrak{L}$ now explicitly indicates the choice of the topological boundary conditions. The holographic correspondence worked in the following way:

$$e^{-\zeta} \langle \mathfrak{L} | \Sigma; \mu \rangle_{\mathcal{T}} = \langle T_{\xi;\mathfrak{L}}(\Sigma) \rangle_{\xi}, \quad (2.5.17)$$

where $T_{\xi;\mathfrak{L}}(\Sigma)$ is the partition function of the 2d TQFT on Σ , $\langle \dots \rangle_{\xi}$ denotes the ensemble average, and $\zeta \equiv \mathcal{Z}_0$ is the sum over connected closed 3-manifolds.

Suppose the 2d TQFTs in the ensemble are of the following form⁵:

$$T_{\xi;\mathfrak{L}} = \bigoplus_{\mathfrak{L}'} T_{\mathfrak{L},\mathfrak{L}'} \otimes \hat{T}_{\xi;\mathfrak{L}'}, \quad (2.5.18)$$

⁵Recall the definition of the direct sum of d -dimensional TQFTs: it is the TQFT for which the Hilbert space associated to a *connected* $(d-1)$ -dimensional manifold is the direct sum of the Hilbert spaces of the summands, and the value on a *connected* d -manifold is the sum of the values of the summands.

where $T_{\mathfrak{L}, \mathfrak{L}'}$ is the TQFT obtained by the reduction of the 3d TQFT $\mathcal{T}$ on the interval with topological boundary conditions $\mathfrak{L}$ and $\mathfrak{L}'$ at the ends. More explicitly, its partition function on a surface Σ of total genus g is given by $T_{\mathfrak{L}, \mathfrak{L}'}(\Sigma) = \langle \mathfrak{L} | \mathfrak{L}' \rangle = |\mathfrak{L} \cap \mathfrak{L}'|^g$.

Plugging (2.5.18) and (2.5.8) into the left and right-hand side of (2.5.17) respectively, we obtain:

$$\mathcal{P}_{\{\mathfrak{L}_i\}_i}(\sqcup_i \Sigma_i) = \left\langle \prod_i \widehat{T}_{\mathfrak{L}_i}(\Sigma_i) \right\rangle_{\xi}. \quad (2.5.19)$$

The condition (2.5.10) that the contribution to $\mathcal{P}_{\{\mathfrak{L}_i\}}(\Sigma)$ from connected topologies vanishes unless all $\mathfrak{L}_i$ are the same then can be interpreted as the condition that the distributions of the 2d TQFTs $\widehat{T}_{\mathfrak{L}; \mathfrak{L}}$ are independent for different $\mathfrak{L}$.

Moreover, the correlators of the partition functions of $\widehat{T}_{\mathfrak{L}; \mathfrak{L}}$ must satisfy the same abelian condition property as the one considered in Section 2.4.4 for the correlators of the partition functions of $T_{\mathfrak{L}; \mathfrak{L}}$. Thus, the distribution of $\widehat{T}_{\mathfrak{L}; \mathfrak{L}}$ can be recovered from the knowledge of the expectation values $\langle \widehat{T}_{\mathfrak{L}; \mathfrak{L}}(\Sigma_g) \rangle_{\xi}$ for connected Σ_g in the same way the distribution of $T_{\mathfrak{L}; \mathfrak{L}}$ was recovered from $\langle T_{\mathfrak{L}; \mathfrak{L}}(\Sigma_g) \rangle_{\xi} = \mathcal{Z}_g = x_g$ in Section 2.4.6. However, unlike the previously obtained distribution of $T_{\mathfrak{L}; \mathfrak{L}}$, the distribution of $\widehat{T}_{\mathfrak{L}; \mathfrak{L}}$ will be generically only formal: it will be non-normalizable and will lack positivity. This is because:

$$\langle \widehat{T}_{\mathfrak{L}; \mathfrak{L}}(\Sigma_g) \rangle_{\xi} = \sum_{\text{Lagrangian } \mathfrak{L}'} \Pi_{g; \mathfrak{L}, \mathfrak{L}'} \mathcal{Z}_{g; \mathfrak{L}'}, \quad (2.5.20)$$

and the matrix $\Pi_{g; \mathfrak{L}, \mathfrak{L}'}$, which is the inverse to the matrix with the components $|\mathfrak{L} \cap \mathfrak{L}'|^g$, has negative components and is singular when $g = 0$.

2.5.2 An explicit example of general topological boundary conditions

In the previous sections, we have focused on the most natural choice for the topological boundary conditions in untwisted Dijkgraaf-Witten theories with gauge group G . Namely, we have adopted the Neumann boundary condition, which is equivalent to $\mathfrak{L}_{\text{Neu}} = G \subset G \oplus G^*$. Another standard choice is the Dirichlet boundary condition, corresponding to $\mathfrak{L}_{\text{Dir}} = G^*$. Notice that untwisted DW theories always admit both boundary conditions, though these do not in general exhaust all possible choices. Indeed, depending on G , there may be more than just two Lagrangian subgroups. In particular, in the case of $G = \mathbb{Z}/p^r\mathbb{Z}$, the 3d theory admits a family of Lagrangian subgroups:

$$\mathfrak{L}_k = \{(cp^{r-k}, dp^k) \mid c \in \mathbb{Z}/p^k\mathbb{Z}, d \in \mathbb{Z}/p^{r-k}\mathbb{Z}\} \simeq \mathbb{Z}/p^k\mathbb{Z} \oplus (\mathbb{Z}/p^{r-k}\mathbb{Z})^* \subset G \oplus G^*. \quad (2.5.21)$$

Notice that the Neumann and Dirichlet boundary conditions are included in this family as extremal cases: $\mathfrak{L}_{\text{Neu}} \equiv \mathfrak{L}_r$ and $\mathfrak{L}_{\text{Dir}} \equiv \mathfrak{L}_0$.

We plan to demonstrate how the procedure described in Section 2.5.1 can be carried over in a simple example. In particular, we compute the connected gravitational partition function, given any topological boundary condition, for a specific choice of the gauge group G .

Let p be an odd prime and consider an untwisted Dijkgraaf-Witten theory with gauge group $G = \mathbb{Z}/p^2\mathbb{Z}$ and topological boundary condition given by $\mathfrak{L} \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \subset G \oplus G^*$. It is straightforward to repeat the computations of Section 1.2.1 with boundary state $c = (c_1p, c_2p) \in \mathfrak{L}^g$, for any $c_1, c_2 \in (\mathbb{Z}/p\mathbb{Z})^g$.

As discussed in Section 2.2, it is enough to compute the partition function for the bilinear forms $X_{q^a}, Y_{q^a}, A_{2^a}, B_{2^a}, C_{2^a}, D_{2^a}, E_{2^a}$ and F_{2^a} .

Since p is an odd prime, the partition function on a 3-manifold with the first homology group being 2-torsion is trivial:

$$Z(\mathbb{Z}/2^a\mathbb{Z}, A_{2^a}, f; D, q; c) = 1/|G|, \quad (2.5.22)$$

and the same holds by replacing A_{2^a} with any other bilinear form on $\mathbb{Z}/2^a\mathbb{Z}$ or $(\mathbb{Z}/2^a\mathbb{Z})^2$. A similar reasoning shows:

$$Z(\mathbb{Z}/q^a\mathbb{Z}, X_{q^a}, f; D, q; c) = Z(\mathbb{Z}/q^a\mathbb{Z}, Y_{q^a}, f; D, q; c) = 1/|G|, \quad (2.5.23)$$

whenever $q \neq p$.

We are now left with the case $q = p$. For the quadratic forms on $\mathbb{Z}/p^a\mathbb{Z}$ with $a \geq 2$, we get:

$$Z(\mathbb{Z}/p^a\mathbb{Z}, X_{p^a}, f; D, q; c) = Z(\mathbb{Z}/p^a\mathbb{Z}, Y_{p^a}, f; D, q; c) = \frac{|G \otimes A_p|}{|G|} \delta(\theta_1) \delta(\theta_2), \quad (2.5.24)$$

where $\theta = \sum_{i=1}^g c_i \otimes f_i \in \mathfrak{L} \otimes \mathbb{Z}/p^a\mathbb{Z}$ splits as $\theta = \theta_1 \oplus \theta_2$. On the other hand, when $a = 1$ we obtain:

$$Z(\mathbb{Z}/p\mathbb{Z}, X_p, f; D, q; c) = \frac{|G \otimes \mathbb{Z}/p\mathbb{Z}|}{|G|} \exp[2\pi i X_p(\theta_1, \theta_2)], \quad (2.5.25)$$

$$Z(\mathbb{Z}/p\mathbb{Z}, Y_p, f; D, q; c) = \frac{|G \otimes \mathbb{Z}/p\mathbb{Z}|}{|G|} \exp[2\pi i Y_p(\theta_1, \theta_2)], \quad (2.5.26)$$

where it is understood that $\theta_i \in \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z}$.

For later convenience, we recombine these results in the partition function of the generic pair $(A_p, \ell_p) = ((\mathbb{Z}/p\mathbb{Z})^l \oplus \tilde{A}_p, xX_p \oplus yY_p \oplus \tilde{\ell}_p)$, where $\tilde{A}_p$ consists of copies of $\mathbb{Z}/p^a\mathbb{Z}$ for $a > 1$. Accordingly decomposing $\theta \oplus \tilde{\theta} \in \mathfrak{L} \otimes ((\mathbb{Z}/p\mathbb{Z})^l \oplus \tilde{A}_p)$, we have:

$$Z(A_p, \ell_p; D, q; \theta \oplus \tilde{\theta}) = \frac{|G \otimes A_p|}{|G|} \delta(\tilde{\theta}_1) \delta(\tilde{\theta}_2) \exp[2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)]. \quad (2.5.27)$$

Obtaining the gravitational partition function for this choice of topological boundary conditions is now straightforward, assuming the measure (2.3.28). Let us consider the factor corresponding to the particular prime p in the expression (2.3.29) and plug into the

TQFT partition function (2.5.27), which results in:

$$\sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} |A_p|^{-s} \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} |G \otimes A_p| \times \\ \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A_p)^* \\ \cong \text{Hom}(A_p, \mathfrak{L}^*)}} |\ker \alpha|^g \exp [2\pi i \alpha (\theta \oplus 0) + 2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)]. \quad (2.5.28)$$

Note that we can understand α as a $2 \times b$ matrix over the finite field $\mathbb{Z}/p\mathbb{Z}$, where $b \equiv \dim_{\mathbb{Z}/p\mathbb{Z}}(A_p \otimes \mathbb{Z}/p\mathbb{Z})$ and the rows are given by $\alpha_i \in (\mathbb{Z}/p\mathbb{Z})^b$ in $\alpha = \alpha_1 \oplus \alpha_2$. Then, $|\ker \alpha| = |A_p|$, $|A_p|p^{-1}$, or $|A_p|p^{-2}$ when α is the zero matrix, the rows of α are proportional one to each other, or the rows are linearly independent over $\mathbb{Z}/p\mathbb{Z}$ respectively. Then, we can rewrite the sum as:

$$(1 - p^{-2g}) \sum_{(A_p, \ell_p)} |A_p|^{g-s} \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} |G \otimes A_p| \times \\ \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)] + \\ (p^{-g} - p^{-2g}) \sum_{(A_p, \ell_p)} |A_p|^{g-s} \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} |G \otimes A_p| \times \\ \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)] \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A_p)^* \neq 0 \\ \alpha_1 \alpha \alpha_2}} \exp [2\pi i \alpha (\theta \oplus 0)] + \\ p^{-2g} \sum_{(A_p, \ell_p)} |A_p|^{g-s} \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} |G \otimes A_p| \times \\ \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)] \sum_{\alpha \in (\mathfrak{L} \otimes A_p)^*} \exp [2\pi i \alpha (\theta \oplus 0)]. \quad (2.5.29)$$

Now, we proceed to compute separately each summand. First, we easily evaluate the Gauss sum:

$$\sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i (xX_p \oplus yY_p)(\theta_1, \theta_2)] = p^l. \quad (2.5.30)$$

Then, we are left with the term corresponding to the sum over proportional rows in α :

$$\begin{aligned}
& \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp[2\pi i(xX_p \oplus yY_p)(\theta_1, \theta_2)] \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A_p)^* \neq 0 \\ \alpha_1 \propto \alpha_2}} \exp[2\pi i\alpha(\theta \oplus 0)] = \\
& 2 \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp[2\pi i(xX_p \oplus yY_p)(\theta_1, \theta_2)] \sum_{\alpha \in (\mathbb{Z}/p\mathbb{Z} \otimes A_p)^* \neq 0} \exp[2\pi i\alpha(\theta_1 \oplus 0)] + \\
& \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp[2\pi i(xX_p \oplus yY_p)(\theta_1, \theta_2)] \sum_{\alpha \in (\mathbb{Z}/p\mathbb{Z} \otimes A_p)^* \neq 0} \exp[2\pi i\alpha(\theta_1 + \lambda\theta_2 \oplus 0)] = \\
& 2 \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} |\mathbb{Z}/p\mathbb{Z} \otimes A_p| \delta(\theta_1) - (p+1) \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp[2\pi i(xX_p \oplus yY_p)(\theta_1, \theta_2)] + \\
& \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} |\mathbb{Z}/p\mathbb{Z} \otimes A_p| \delta(\theta_1 + \lambda\theta_2) \exp[2\pi i(xX_p \oplus yY_p)(\theta_1, \theta_2)] = \\
& 2p^l \sqrt{|\mathfrak{L} \otimes A_p| - (p+1)p^l} + \sqrt{|\mathfrak{L} \otimes A_p|} \sum_{\lambda \in (\mathbb{Z}/p\mathbb{Z})^\times} (-1)^y \varepsilon_p^l p^{l/2} \left(\frac{(-\lambda)^l}{p} \right), \quad (2.5.31)
\end{aligned}$$

where in the last line we have evaluated the Gauss sum:

$$\sum_{\theta \in (\mathbb{Z}/p\mathbb{Z})^l} \exp[-2\pi i(xX_p \oplus yY_p)(\lambda\theta, \theta)] = (-1)^y \varepsilon_p^l p^{l/2} \left(\frac{(-\lambda)^l}{p} \right). \quad (2.5.32)$$

Putting all together, we obtain:

$$\begin{aligned}
& \sum_{(A_p, \ell_p)} |A_p|^{g-s} \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} |G \otimes A_p| \left((1 - p^{-2g})p^l + \right. \\
& \left. (p^{-g} - p^{-2g}) \left(2p^l \sqrt{|\mathfrak{L} \otimes A_p| - (p+1)p^l} + (p-1) \delta(A_p = 0) \right) + p^{-2g} |\mathfrak{L} \otimes A_p| \right), \quad (2.5.33)
\end{aligned}$$

after having noticed that the factor $(-1)^y$ localizes the sum over all pairs (A_p, ℓ_p) on the trivial group $A_p = 0$. Indeed, recall that on each $(\mathbb{Z}/p^m\mathbb{Z})^d$ there are only two quadratic forms with $y = 0$ and $y = 1$, respectively. Finally, introducing the quantities b and l as in the decomposition:

$$A_p = \bigoplus_{i=1}^b \mathbb{Z}/p^{m_i}\mathbb{Z} = (\mathbb{Z}/p\mathbb{Z})^l \oplus \tilde{A}_p, \quad (2.5.34)$$

we can rewrite the above as:

$$\begin{aligned}
p^{-2} \sum_{(A_p, \ell_p)} |A_p|^{g-s} & \left((p^g - 1)(p^g - p) + 2(p^g - 1)p^b + (p - 1)(p^g - 1) \delta(A_p = 0) + p^{2b-l} \right) = \\
p^{-2} & \left((p^g - 1)(p^g - p) \Xi_p(s - g) + 2(p^g - 1) \prod_{m \geq 1} \frac{1 + p^{1-m(s-g)}}{1 - p^{1-m(s-g)}} \right. \\
& \left. + (p - 1)(p^g - 1) + \frac{1 + p^{1-(s-g)}}{1 - p^{1-(s-g)}} \prod_{m \geq 2} \frac{1 + p^{2-m(s-g)}}{1 - p^{2-m(s-g)}} \right). \tag{2.5.35}
\end{aligned}$$

In the last equality, we have used a refined version of (2.2.28):

$$\sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} |A_p|^{-s} x^b y^l = \frac{1 + xy p^{-s}}{1 - xy p^{-s}} \prod_{m \geq 2} \frac{1 + xp^{-ms}}{1 - xp^{-ms}}. \tag{2.5.36}$$

Combining the factors for different primes, we arrive at the final result for the sum over connected topologies:

$$\begin{aligned}
\mathcal{Z}_g(D, q; \mathfrak{L}; \mu) = & \frac{\Xi(s - g)}{1 - \lambda |D|^{\frac{1}{2}}} \cdot p^{-2} \left((p^g - 1)(p^g - p) + 2(p^g - 1) \prod_{m \geq 1} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \frac{1 + p^{1-m(s-g)}}{1 - p^{1-m(s-g)}} \right. \\
& \left. + (p - 1)(p^g - 1) \prod_{m \geq 1} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} + \frac{1 + p^{1-(s-g)}}{1 - p^{1-(s-g)}} \frac{1 - p^{-(s-g)}}{1 + p^{-(s-g)}} \prod_{m \geq 2} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \frac{1 + p^{2-m(s-g)}}{1 - p^{2-m(s-g)}} \right). \tag{2.5.37}
\end{aligned}$$

For completeness, we include the results corresponding to Neumann and Dirichlet boundary conditions. We avoid diving into the details of the computation since it is a slight modification of the above derivation. We find:

$$\begin{aligned}
\mathcal{Z}_g(D, q; \mathfrak{L}_{\text{Neu}}; \mu) = & \frac{\Xi(s - g)}{1 - \lambda |D|^{\frac{1}{2}}} \cdot p^{-2} \left((p^{2g} - p^g) + (p^g - 1) \prod_{m \geq 1} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \frac{1 + p^{1-m(s-g)}}{1 - p^{1-m(s-g)}} \right. \\
& \left. + \frac{1 + p^{1-(s-g)}}{1 - p^{1-(s-g)}} \frac{1 - p^{-(s-g)}}{1 + p^{-(s-g)}} \prod_{m \geq 2} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \frac{1 + p^{2-m(s-g)}}{1 - p^{2-m(s-g)}} \right). \tag{2.5.38}
\end{aligned}$$

Notice that imposing Dirichlet boundary conditions would yield the same connected gravitational partition function since $\mathfrak{L}_{\text{Dir}} = (\mathfrak{L}_{\text{Neu}})^*$ are dual to each other.

2.5.3 A simple twisted Dijkgraaf-Witten theory

One should be interested also in twisted Dijkgraaf-Witten theories because, together with the untwisted theories, they represent all the possible abelian 3d TQFTs admitting topological boundary conditions. Indeed, it was proven in [36] that a theory admitting topological boundary conditions is necessarily a Dijkgraaf-Witten theory. The problem, however, is that the expression for partition function of such theories is more involved, so that the explicit evaluation of the corresponding gravitational partition function becomes more complicated. Thus we only include a simple example of a twisted DW to show that in principle it is still possible to explicitly compute the gravitational partition function for a simple choice of measure.

Let p be an odd prime. Consider an abelian Chern-Simons theory with level:

$$K = \begin{pmatrix} 0 & p \\ p & 2 \end{pmatrix}. \quad (2.5.39)$$

This theory realizes a twisted Dijkgraaf-Witten with gauge group $\mathbb{Z}/p\mathbb{Z}$. See e.g. [40] for a review of the correspondence between these two description of the 3d theory. For our purposes, we just remark that the discriminant group is given by $D = \mathbb{Z}/p^2\mathbb{Z}$, while the quadratic form is:

$$q(x) = \begin{cases} x^2/p^2 & \text{if } p \equiv 1 \pmod{4}, \\ -x^2/p^2 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (2.5.40)$$

Being a DW TQFT, this theory admits topological boundary conditions. In particular, we fix the Lagrangian subgroup $\mathfrak{L} = \mathbb{Z}/p\mathbb{Z} \subseteq D$, given by the inclusion $1 \mapsto p$.

As in the previous subsection, it is sufficient to consider the partition function on 3-manifolds with the first homology group being p -torision, for the chosen prime p . Let c be an element of $\mathfrak{L}^g$. Then, for $\mathbb{Z}/p^a\mathbb{Z}$ with $a \geq 2$, we obtain:

$$Z(\mathbb{Z}/p^a\mathbb{Z}, X_{p^a}, f; D, q; c) = Z(\mathbb{Z}/p^a\mathbb{Z}, Y_{p^a}, f; D, q; c) = \frac{1}{|\mathfrak{L}|} \sqrt{|\mathbb{Z}/p^a\mathbb{Z} \otimes D|} \delta(\theta), \quad (2.5.41)$$

where we recall $\theta = \sum_{i=1}^g c_i \otimes f_i \in \mathfrak{L} \otimes \mathbb{Z}/p^a\mathbb{Z}$. Finally, we are left with the case $a = 1$:

$$Z(\mathbb{Z}/p\mathbb{Z}, X_p, f; \mathbb{Z}_{p^2}, q; c) = \frac{1}{|\mathfrak{L}|} \sqrt{|\mathbb{Z}/p\mathbb{Z} \otimes D|} \begin{cases} \exp[2\pi i(b \otimes X_p)(\theta, \theta)] & \text{if } p \equiv 1 \pmod{4}, \\ i \exp[-2\pi i(b \otimes X_p)(\theta, \theta)] & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (2.5.42)$$

$$Z(\mathbb{Z}/p\mathbb{Z}, Y_p, f; \mathbb{Z}_{p^2}, q; c) = -\frac{1}{|\mathfrak{L}|} \sqrt{|\mathbb{Z}/p\mathbb{Z} \otimes D|} \begin{cases} \exp[2\pi i(b \otimes Y_p)(\theta, \theta)] & \text{if } p \equiv 1 \pmod{4}, \\ i \exp[-2\pi i(b \otimes Y_p)(\theta, \theta)] & \text{if } p \equiv 3 \pmod{4}, \end{cases} \quad (2.5.43)$$

where we have introduced the function $b : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathbb{Z}$, defined by $b(x, y) = \widehat{x\hat{y}}$, given a lift $\widehat{(\cdot)}$ from $\mathfrak{L}$ to $\mathbb{Z}$.

For later convenience, as in the previous subsection, we recombine these results in the partition function of the generic pair $(A_p, \ell_p) = ((\mathbb{Z}/p\mathbb{Z})^l \oplus \tilde{A}_p, xX_p \oplus yY_p \oplus \tilde{\ell}_p)$. Accordingly decomposing $\theta \oplus \tilde{\theta} \in \mathfrak{L} \otimes ((\mathbb{Z}/p\mathbb{Z})^l \oplus \tilde{A}_p)$, we have:

$$Z(A_p, \ell_p; D, q; \theta \oplus \tilde{\theta}) = \frac{\sqrt{|A_p \otimes D|}}{|\mathfrak{L}|} (-1)^y \varepsilon_p^l \delta(\tilde{\theta}) \exp [2\pi i \varepsilon_p^2 (b \otimes (xX_p \oplus yY_p))(\theta, \theta)], \quad (2.5.44)$$

where ε_p is either 1 or i , depending whether $p \equiv 1 \pmod{4}$ or $p \equiv 3 \pmod{4}$.

We now assume the measure (2.3.28) and compute the gravitational partition functions for this specific twisted DW TQFT. It is enough to consider the factor corresponding to the particular prime p in the expression (2.3.29). Plugging (2.5.44) into the partition function, yields:

$$\begin{aligned} & \sum_{\substack{(A_p, \ell_p): \\ \text{finite abelian} \\ p\text{-groups} \\ \text{with pairings}}} |A_p|^{-s} (-1)^y \varepsilon_p^l \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} \sqrt{|D \otimes A_p|} \times \\ & \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A_p)^* \\ \cong \text{Hom}(A_p, \mathfrak{L}^*)}} |\ker \alpha|^g \exp [2\pi i \alpha(\theta \oplus 0) + 2\pi i \varepsilon_p^2 (b \otimes (xX_p \oplus yY_p))(\theta, \theta)]. \end{aligned} \quad (2.5.45)$$

Now, the trick is again to notice that all maps $\alpha_p : A_p \rightarrow \mathbb{Z}/p\mathbb{Z}$ have kernel $|\ker \alpha| = |A_p|p^{-1}$, except the zero map. Then, we reorganize the sum as:

$$\begin{aligned} & (1 - p^{-g}) \sum_{(A_p, \ell_p)} |A_p|^{g-s} (-1)^y \varepsilon_p^l \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} \sqrt{|D \otimes A_p|} \times \\ & \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i \varepsilon_p^2 (b \otimes (xX_p \oplus yY_p))(\theta, \theta)] + \\ & p^{-g} \sum_{(A_p, \ell_p)} |A_p|^{g-s} (-1)^y \varepsilon_p^l \frac{|\mathfrak{L}|^{g-1}}{|\mathfrak{L} \otimes A_p|} \sqrt{|D \otimes A_p|} \times \\ & \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i \varepsilon_p^2 (b \otimes (xX_p \oplus yY_p))(\theta, \theta)] \sum_{\substack{\alpha \in (\mathfrak{L} \otimes A_p)^* \\ \cong \text{Hom}(A_p, \mathfrak{L}^*)}} \exp [2\pi i \alpha(\theta \oplus 0)]. \end{aligned} \quad (2.5.46)$$

Finally, we make use of the following equality for quadratic Gauss sums:

$$(-1)^y \varepsilon_p^l \sum_{\theta \in \mathfrak{L} \otimes (\mathbb{Z}/p\mathbb{Z})^l} \exp [2\pi i \varepsilon_p^2 (b \otimes (xX_p \oplus yY_p))(\theta, \theta)] = p^{l/2}, \quad (2.5.47)$$

and obtain:

$$|\mathfrak{L}|^{g-1} \sum_{(A_p, \ell_p)} |A_p|^{g-s} \left((1 - p^{-g}) + p^{-g} (-1)^y \varepsilon_p^l \sqrt{|D \otimes A_p|} \right) = |\mathfrak{L}|^{g-1} \left((1 - p^{-g}) \Xi_p(s - g) + p^{-g} \right). \quad (2.5.48)$$

In the last equality, we have used a refined version of (2.2.28):

$$\sum_{(A_p, \ell_p)} |A_p|^{g-s} (-1)^y \varepsilon_p^l \sqrt{|D \otimes A_p|} = 1. \quad (2.5.49)$$

We again notice the factor $(-1)^y$, which kills every contribution but the one corresponding to the trivial group $A_p = 0$.

Combining the factors for different primes, we arrive at the final result for the sum over topologies:

$$\mathcal{Z}_g(D, q; \mathfrak{L}; \mu) = \frac{\Xi(s - g)}{1 - \lambda |D|^{\frac{1}{2}}} \cdot \left(p^{-1} (p^g - 1) + p^{-1} \prod_{m \geq 1} \frac{1 - p^{-m(s-g)}}{1 + p^{-m(s-g)}} \right). \quad (2.5.50)$$

Chapter 3

Finite and Continuous Gauge Group Theories

In this chapter, we raise the question of which kind of relation there is between finite and continuous gauge group theories, if any. The perfect setting for studying this problem is that of TQFTs, since it allows for many non-trivial instances of finite and continuous gauge group theories, which are simple enough to be analyzed in detail.

We have already noticed such a relation between abelian Chern-Simons and Dijkgraaf-Witten theories, which will be reviewed in a moment. Furthermore, there has been some interest in the physical literature concerning the realization of non-abelian finite gauge group theories, e.g. dihedral DW theories, from continuous CS or BF theories [70, 71]. The main motivation behind these works has been the realization of non-abelian topological orders for quantum computing applications.

Our focus will be a putative correspondence between $SU(2)$ Chern-Simons theory and twisted $SL(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory. The choice of the twist will play a fundamental role and is based on a novel construction by Nosaka [72].

3.1 Introduction

As anticipated above, we review some known examples of correspondences between finite group and continuous gauge theories. In the simplest case where $G = U(1) \equiv U(1, \mathbb{C}) \subset GL(1, \mathbb{C})$, the Chern-Simons partition function localizes to a sum over flat connections and can be computed explicitly to be (see e.g. [33, 40])

$$Z_{\text{CS}}^{U(1),k}(M) = k^{\frac{b_1-1}{2}} \sum_{a \in \text{Tor}H_1(M, \mathbb{Z})} e^{2\pi i k \ell k(a,a)}. \quad (3.1.1)$$

Here b_1 is the first Betti number of M and ℓk denotes the linking pairing between (torsion) 1-cycles of M . On the discrete side, consider $U(1, \mathbb{F}_{q^2})$, which is a cyclic group

$\mathbb{Z}/N\mathbb{Z}$ with $N = q + 1$. Alternatively, one can consider $\mathrm{GL}(1, \mathbb{F}_q) \cong \mathbb{F}_q^\times \cong \mathbb{Z}/N\mathbb{Z}$ for $N = q - 1$. In general, for $G = \mathbb{Z}/N\mathbb{Z}$ we have $H^3(G, \mathrm{U}(1)) \simeq \mathbb{Z}/N\mathbb{Z}$, so let k denote an integer corresponding to a particular choice of twist. The partition function of the Dijkgraaf-Witten theory reads (see e.g. [40])

$$Z_{\mathrm{DW}}^{\mathbb{Z}/N\mathbb{Z}, k}(M) = N^{b_1-1} \sum_{\substack{a \in \mathrm{Tor} H_1(M, \mathbb{Z}) \\ o(a) | N}} e^{2\pi i k \ell k(a, a)}, \quad (3.1.2)$$

where by $o(a)$ we denote the order of the torsion cycle a . One explicitly observes that the discrete gauge theory encodes the same topological information as the continuous theory does, namely the first Betti number and the linking pairing evaluated on equal cycles. Moreover, when, for a given 3-manifold, N is “large enough” in the sense that the direct limit of $\mathbb{Z}/N\mathbb{Z}$ groups gives¹ $\mathbb{Q}/\mathbb{Z}$ (cf. [73]), the condition $o(a) | N$ is automatically satisfied. Then one can formally identify the expressions in the right-hand sides of (3.1.1) and (3.1.2) by separately matching the order of the overall coefficients up to a factor of $1/2$ and the phases of each contribution.

The above results prompt us to conjecture the existence of a similar relation in the non-abelian case. In this context, the path integral does not always localize to a sum over flat connections, and so we will only consider the large k limit in which such an approximation is valid. The *asymptotic expansion conjecture* [6, 74–76] states that the $\mathrm{SU}(2)_k$ Chern-Simons partition function behaves as

$$Z_{\mathrm{CS}}^{\mathrm{SU}(2)_k}(M) \sim \sum_{\gamma \in \pi_0(\mathcal{M}(M, \mathrm{SU}(2)))} e^{2\pi i k \mathrm{CS}[\gamma]} \sum_{n \geq 0} \frac{a_n^\gamma}{k^{n+\delta_\gamma}}, \quad (3.1.3)$$

as we take the limit $k \rightarrow \infty$ through integer numbers. Here $\mathcal{M}(M, \mathrm{SU}(2))$ denotes the moduli space of flat $\mathrm{SU}(2)$ connections over M . We can also analytically continue CS theory, i.e. consider k to be a complex-valued level, and take the limit $k \rightarrow \infty$ in the complex plane. It has been conjectured that the corresponding asymptotic expansion must take into account not only flat $\mathrm{SU}(2)$ connections, but more generally flat $\mathrm{SL}(2, \mathbb{C})$ connections [18, 19, 77–79] — we will review this proposal in more detail later on (see Section 3.3). Alternatively, one can consider the asymptotic expansion of the WRT invariant for a general root of unity ξ , which formally corresponds to taking the CS level $k \in \mathbb{Q}$ [79–81]. Within the context of our work, these general versions of the asymptotic expansion conjecture turn out to be more relevant for a comparison with the discrete case. Note that in the abelian case, the sequence of cyclic groups $\mathbb{Z}/N\mathbb{Z}$ with $N \rightarrow \infty$ can be considered as densely filling $\mathrm{U}(1)$. In the case of $\mathrm{SU}(2)$, however, no such sequence of finite subgroups exists, which follows from the known ADE classification.

Our conjecture is then formulated as follows. We claim that for an appropriate choice of twist in $H^3(G, \mathrm{U}(1))$, both the leading powers of k in the asymptotic expansion and

¹E.g. by considering the sequence with $N = n!$, $n \rightarrow \infty$.

the twelvefold values of the Chern-Simons invariants can be recovered from the $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten partition function. We do not have a general proof for these statements, except for the particular case of spherical 3-manifolds, where we propose some ideas on which a full proof could be based. Instead, we provide considerable evidence for our conjecture in a variety of examples, most of which are *geometric* 3-manifolds. Interestingly enough, it is only for hyperbolic 3-manifolds — perhaps the most interesting type of geometric 3-manifolds — that our conjecture seems not to hold, at least not in the strongest form.

3.2 The finite groups $\mathrm{SL}(2, \mathbb{F}_q)$

We start by reviewing some basic facts about finite fields. Let p be a prime number and r a positive integer. There exists a unique (up to isomorphism) finite field of order $q = p^r$, denoted by $\mathbb{F}_q$. Conversely, any finite field is isomorphic to $\mathbb{F}_q$ for appropriate p and r . For $r = 1$ we identify $\mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}$ with the standard addition and multiplication modulo p , while for $r > 1$ an explicit construction is given by the quotient $\mathbb{F}_p[x]/(f)$, with f an irreducible polynomial of degree r .

The multiplicative subgroup $\mathbb{F}_q^\times$ is isomorphic to $\mathbb{Z}/(q-1)\mathbb{Z}$; we will usually denote a generator of this group by θ . If r divides s , the field $\mathbb{F}_{p^r}$ embeds into $\mathbb{F}_{p^s}$. It is often convenient to choose compatible generators for their multiplicative subgroups $\langle \theta \rangle$ and $\langle \vartheta \rangle$, meaning that $\theta = \vartheta^{(p^s-1)/(p^r-1)}$ after the embedding. This can be achieved by choosing the polynomials f defining $\mathbb{F}_{p^r}$ to be e.g. the Conway polynomials (see for example [82] for a detailed account). Clearly, finite fields are never algebraically closed: the algebraic closure of $\mathbb{F}_{p^r}$ can be obtained as the direct limit $\overline{\mathbb{F}}_p = \bigcup_{k=1}^{\infty} \mathbb{F}_{p^k}$.

In what follows, we restrict ourselves to the case where p is an odd prime². Consider now the special linear group of 2×2 matrices over $\mathbb{F}_q$,

$$\mathrm{SL}(2, \mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{F}_q, ad - bc = 1 \right\}. \quad (3.2.1)$$

It has order $|\mathrm{SL}(2, \mathbb{F}_q)| = q(q^2 - 1)$ and center $Z = \{1, -1\}$. It will also be useful to define the subgroups B, T , and U consisting of upper triangular, diagonal, and unipotent upper triangular matrices respectively. Their elements have the general form

$$u(\lambda, x) = \begin{pmatrix} \lambda & x \\ 0 & \lambda^{-1} \end{pmatrix}, \quad d(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad u(1, x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad (3.2.2)$$

where $\lambda \in \mathbb{F}_q^\times$ and $x \in \mathbb{F}_q$. Note that since the field $\mathbb{F}_q$ is not algebraically closed, there exist matrices that can be diagonalized over the quadratic extension $\mathbb{F}_{q^2}$ but not over $\mathbb{F}_q$.

²When $p = 2$ the general structure of the group differs slightly, but the details are not important for our purposes.

Their form can be written explicitly as follows. Choose an element $z \in \mathbb{F}_{q^2}^\times$ satisfying $z + z^q = 0$, and denote by μ_{q+1} the set of all $\xi \in \mathbb{F}_{q^2}^\times$ satisfying $\xi^{q+1} = 1$. For such ξ , define

$$d'(\xi) = \begin{pmatrix} \frac{\xi + \xi^q}{2} & z \frac{\xi - \xi^q}{2} \\ \frac{\xi - \xi^q}{2z} & \frac{\xi + \xi^q}{2} \end{pmatrix} \quad (3.2.3)$$

We will informally refer to these matrices as *codiagonal matrices*. They belong to $\mathrm{SL}(2, \mathbb{F}_q)$, but have eigenvalues in the quadratic extension unless $\xi = \pm 1$. We denote by T' the subgroup of all matrices $d'(\xi)$ with $\xi \in \mu_{q+1}$, which is isomorphic to $\mathbb{Z}/(q+1)\mathbb{Z}$.

Conjugacy classes and irreducible representations

From the above considerations one concludes that every matrix in $\mathrm{SL}(2, \mathbb{F}_q)$ is either ± 1 , or is conjugate to a diagonal, codiagonal, or unipotent upper triangular matrix. A more detailed study shows that there exist $q+4$ conjugacy classes, which can be characterized as follows [83]. Define equivalence relations identifying $\lambda \sim \lambda^{\pm 1}$ in $\mathbb{F}_q^\times$ and $\xi \sim \xi^{\pm 1}$ in $\mathbb{F}_{q^2}^\times$, and let z_0 be a fixed non-square element in $\mathbb{F}_q$. A complete set of representatives is then given by

$$\begin{cases} \pm 1, \\ d(\lambda), & \lambda \in (\mathbb{F}_q^\times / \sim) \setminus \{\pm 1\}, \\ d'(\xi), & \xi \in (\mu_{q+1} / \sim) \setminus \{\pm 1\}, \\ \epsilon u_\tau, & \epsilon = \pm 1 \text{ and } \tau = \pm 1, \end{cases} \quad (3.2.4)$$

where we defined $u_+ = u(1, 1)$ and $u_- = u(1, z_0)$. We summarize the structure of the corresponding conjugacy classes in Table 3.2.1.

Representative	± 1	$d(\lambda)$	$d'(\xi)$	ϵu_τ
Number of classes	2	$\frac{q-3}{2}$	$\frac{q-1}{2}$	4
Cardinality	1	$q(q+1)$	$q(q-1)$	$\frac{q^2-1}{2}$
Centralizer	$\mathrm{SL}(2, \mathbb{F}_q)$	T	T'	ZU

Table 3.2.1: Conjugacy classes of $\mathrm{SL}(2, \mathbb{F}_q)$.

The (irreducible) characters of $\mathrm{SL}(2, \mathbb{F}_q)$ are also well-known in the literature and are constructed as follows [84]. Let α be a character of $\mathbb{F}_q^\times \simeq \mathbb{Z}/(q-1)\mathbb{Z}$. We can lift it

to a character of B , acting as $\alpha(u(\lambda, x)) = \alpha(\lambda)$. The *principal series representations* of $\mathrm{SL}(2, \mathbb{F}_q)$ are then defined as the induced representations $R(\alpha) = \mathrm{Ind}_B^G \alpha$. They are irreducible unless α is of order 2, in which case they decompose into irreps as

$$R(1) = 1 + \mathrm{St}, \quad R(\alpha_0) = R_+(\alpha_0) + R_-(\alpha_0), \quad (3.2.5)$$

where α_0 denotes the non-trivial character of order 2. One further checks $R(\alpha) \simeq R(\alpha^{-1})$.

Now let β be a character of $\mu_{q+1} \simeq \mathbb{Z}/(q+1)\mathbb{Z}$. The *cuspidal representations* $R'(\beta)$ are given as follows. For any character $\alpha \in \widehat{\mathbb{F}_q^\times}$ agreeing with β on ± 1 , we define the virtual character $R'(\beta) = \mathrm{Ind}_{T'}^G \alpha - R(\alpha) - \mathrm{Ind}_T^G \beta$, where we understand α and β as being lifted to characters of T and T' in the obvious way. One can show they correspond to actual representations of $\mathrm{SL}(2, \mathbb{F}_q)$, which are moreover irreducible, whenever β is not of order 2. Otherwise, one has (virtual) decompositions into irreps

$$R'(1) = 1 - \mathrm{St}, \quad R'(\beta_0) = R'_+(\beta_0) + R'_-(\beta_0), \quad (3.2.6)$$

where β_0 denotes the non-trivial character of order 2. Just as for principal series representations, one checks $R'(\beta) \simeq R'(\beta^{-1})$.

These two constructions exhaust all irreducible representations of $\mathrm{SL}(2, \mathbb{F}_q)$. Thus, we conclude that a complete set of characters is given as follows:

$$\left\{ \begin{array}{ll} 1, \\ \mathrm{St}, \\ R(\alpha), & \alpha \in (\widehat{\mathbb{F}_q^\times} / \sim) \setminus \{\pm 1\}, \\ R'(\beta), & \beta \in (\widehat{\mu_{q+1}} / \sim) \setminus \{\pm 1\}, \\ R_\sigma(\alpha_0), & \sigma = \pm 1, \\ R'_\sigma(\beta_0), & \sigma = \pm 1, \end{array} \right. \quad (3.2.7)$$

where we defined an equivalence relation identifying characters $\alpha \sim \alpha^{\pm 1}$ in $\widehat{\mathbb{F}_q^\times}$ and $\beta \sim \beta^{\pm 1}$ in $\widehat{\mu_{q+1}}$. Their values on conjugacy class representatives are given in Table 3.2.2.

3.2.1 Group cohomology and second Chern classes

In this section, we proceed with a discussion of the group cohomology of $\mathrm{SL}(2, \mathbb{F}_q)$, as well as explicit constructions of representatives for group cocycles in $H^3(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1))$. The latter will become essential in our study of $\mathrm{SL}(2, \mathbb{F}_q)$ gauge theories in Section 3.3.

We start by pointing out the fact that the abelianization of $\mathrm{SL}(2, \mathbb{F}_q)$ is trivial for $q > 3$. This implies that

$$H^1(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1)) = 0 \quad (3.2.8)$$

in those cases. It turns out that also

$$H^2(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1)) = 0 \quad (3.2.9)$$

	$\epsilon \mathbb{1}$	$d(\lambda)$	$d'(\xi)$	ϵu_τ
1	1	1	1	1
St	q	1	-1	0
$R(\alpha)$	$(q+1)\alpha(\epsilon)$	$\alpha(\lambda) + \alpha(\lambda)^{-1}$	0	$\alpha(\epsilon)$
$R'(\beta)$	$(q-1)\beta(\epsilon)$	0	$-\beta(\xi) - \beta(\xi)^{-1}$	$-\beta(\epsilon)$
$R_\sigma(\alpha_0)$	$\frac{(q+1)\alpha_0(\epsilon)}{2}$	$\alpha_0(\lambda)$	0	$\alpha_0(\epsilon) \frac{1+\sigma\tau\sqrt{q_0}}{2}$
$R'_\sigma(\beta_0)$	$\frac{(q-1)\beta_0(\epsilon)}{2}$	0	$-\beta_0(\xi)$	$\beta_0(\epsilon) \frac{-1+\sigma\tau\sqrt{q_0}}{2}$

Table 3.2.2: Character table of $\mathrm{SL}(2, \mathbb{F}_q)$. Here $q_0 = \alpha_0(-1)q$.

for $q \notin \{4, 9\}$ [85]. The third group cohomology is however non-trivial and can be determined as follows. The Bockstein map β associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 0 \quad (3.2.10)$$

induces an isomorphism between $H^3(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1))$ and $H^4(\mathrm{SL}(2, \mathbb{F}_q), \mathbb{Z})$, which is in turn isomorphic to $H_3(\mathrm{SL}(2, \mathbb{F}_q), \mathbb{Z})$ by the universal coefficient theorem. The third group homology was computed in [86] and equals $\mathbb{Z}/(q^2 - 1)\mathbb{Z}$ whenever $q \notin \{2, 3, 4, 5, 8, 9, 27\}$. Hence we conclude that also

$$H^3(\mathrm{SL}(2, \mathbb{F}_q), \mathrm{U}(1)) = \mathbb{Z}/(q^2 - 1)\mathbb{Z} \quad (3.2.11)$$

for such values of q . Throughout this chapter, we will make the assumption that q is not any of the special values for which the group cohomology behaves differently.

Let us now introduce a procedure to find explicit group 3-cocycles constructed from second Chern classes of representations of $\mathrm{SL}(2, \mathbb{F}_q)$, following previous work by Nosaka [72]. Let G be a finite group. Given a representation $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$, one defines the Chern classes $c_i(\rho) \in H^{2i}(G, \mathbb{Z})$ of ρ as the Chern classes of its associated vector bundle over BG . Brauer's induction theorem tells us that for any representation ρ , there exist finitely many subgroups $H_i \subset G$ and corresponding 1-dimensional representations $\phi_i : H_i \rightarrow \mathbb{C}^\times$ such that ρ decomposes, as a virtual representation, into

$$\rho = \sum_{i=1}^N n_i(\rho) \mathrm{Ind}_{H_i}^G(\phi_i), \quad (3.2.12)$$

with $n_i(\rho) \in \mathbb{Z}$. Proposition 2.2 in [72] states that the twelffold second Chern class $12\beta^{-1}c_2(\rho) \in H^3(G, \mathrm{U}(1))$, where β is the Bockstein map associated to (3.2.10), can be

written as

$$12\beta^{-1}c_2(\rho) = 6 \sum_{k=1}^N n_k(\rho) \left(\left(\sum_{j=1}^N n_j(\rho) \mathrm{Tr}_{H_j}^G(\phi_j) \cup \beta \circ \mathrm{Tr}_{H_k}^G(\phi_k) \right) - \mathrm{Tr}_{H_k}^G(\phi_k \cup \beta\phi_k) \right). \quad (3.2.13)$$

Some clarifications are in order. First, note that we are expressing $12\beta^{-1}c_2(\rho)$ as a group cocycle with $\mathbb{Q}/\mathbb{Z}$ coefficients. This is possible since the group cohomology of finite groups is pure torsion, and hence the cohomologies with $\mathbb{Q}/\mathbb{Z}$, $\mathrm{U}(1)$, or even $\mathbb{C}^\times$ coefficients are all isomorphic³. A 1-dimensional representation $\phi : H \rightarrow \mathbb{C}^\times$ can thus be interpreted as a 1-cocycle in $H^1(H, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$, and applying the Bockstein map yields a 2-cocycle with coefficients in $\mathbb{Z}$. The cup product $\cup$ is the one associated to the product $\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ sending $(m, [a]) \mapsto [ma]$. Finally, Tr_H^G denotes the transfer map $H^n(H, A) \rightarrow H^n(G, A)$. A formal definition can be found in [87], but we will present concrete expressions in what follows.

Let us briefly recall the algebraic description of group cocycles. Cochains in $C^n(G, A)$ are represented by maps $\omega : G^n \rightarrow A$. The coboundary operator δ is defined as

$$\begin{aligned} \delta\omega(g_1, \dots, g_{n+1}) &= (-1)^{n+1}\omega(g_1, \dots, g_n) + \omega(g_2, \dots, g_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i \omega(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \end{aligned} \quad (3.2.14)$$

in additive notation. Elements of $H^n(G, A)$ are then represented by cocycles ω satisfying $\delta\omega = 0$, modulo coboundaries. Moreover, it is always possible to find *normalized* representatives satisfying $\omega(g_1, \dots, g_n) = 0$ whenever any one of the g_i equals 1 [88].

With these definitions, it is easy to describe the cohomology operations appearing in (3.2.13) directly on cochains. Consider a 1-cocycle $\phi : H \rightarrow \mathbb{Q}/\mathbb{Z}$. The 2-cocycle $\beta\phi$ is represented by a function $H \times H \rightarrow \mathbb{Z}$, which reads

$$\beta\phi(h_1, h_2) = \overline{\phi(h_1)} + \overline{\phi(h_2)} - \overline{\phi(h_1) + \phi(h_2)} = \begin{cases} 1, & \text{if } \overline{\phi(h_1)} + \overline{\phi(h_2)} \geq 1, \\ 0, & \text{if } \overline{\phi(h_1)} + \overline{\phi(h_2)} < 1, \end{cases} \quad (3.2.15)$$

where $\bar{x} \in [0, 1)$ denotes a lift of x from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}$. Now let R be a complete set of right coset representatives for $H \backslash G$, such that the identity represents the trivial coset H . For any $g \in G$, we denote by $\bar{g}$ the representative of the coset Hg . Notice then that $\bar{r}g\bar{r}g^{-1} \in H$ for any $r \in R$. For $\omega \in H^n(H, A)$, applying the transfer map yields an n -cocycle $\mathrm{Tr}_H^G(\omega) \in H^n(G, A)$ represented by a function $G^n \rightarrow A$ as [89]

$$\mathrm{Tr}_H^G(\omega)(g_1, \dots, g_n) = \sum_{r \in R} \omega(\bar{r}g_1\bar{r}g_1^{-1}, \bar{r}g_1g_2\bar{r}g_1g_2^{-1}, \dots, \bar{r}g_1 \cdots g_{n-1}g_n\bar{r}g_1 \cdots g_n^{-1}), \quad (3.2.16)$$

³We will employ additive or multiplicative notation for the coefficient group interchangeably, understanding that group cocycles are expressed with coefficients in either $\mathbb{Q}/\mathbb{Z}$ or $\mathrm{U}(1)$ respectively.

again in additive notation. Combining these results allows us to find explicit representatives for the twelvefold Chern classes (3.2.13) as functions $G^3 \rightarrow \mathbb{Q}/\mathbb{Z}$.

We now specialize to $G = \mathrm{SL}(2, \mathbb{F}_q)$. For simplicity, let us assume that we start with a representation ρ given as a linear combination of induced representations. Since the first and second group cohomologies of $\mathrm{SL}(2, \mathbb{F}_q)$ vanish, we can always discard the first term in (3.2.13), considerably simplifying the expression for the twelvefold second Chern classes. Let us now focus on representations induced from those of T and T' , which we shall use extensively throughout the rest of this work. Let θ and ϑ be compatible generators of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$ respectively. Consider the induced representations

$$\begin{cases} \mathrm{Ind}_T^G \phi, & \phi : d(\theta^j) \longrightarrow \frac{j}{q-1} \bmod 1, \end{cases} \quad (3.2.17)$$

$$\begin{cases} \mathrm{Ind}_{T'}^G \phi', & \phi' : d'(\vartheta^{(q-1)j}) \longrightarrow \frac{j}{q+1} \bmod 1. \end{cases} \quad (3.2.18)$$

In particular, we will be interested in the linear combination

$$\rho = \mathrm{Ind}_T^G \phi + \mathrm{Ind}_{T'}^G \phi', \quad (3.2.19)$$

whose twelvefold second Chern class reads

$$12\beta^{-1}c_2(\rho) = -6\mathrm{Tr}_T^G(\phi \cup \beta\phi) - 6\mathrm{Tr}_{T'}^G(\phi' \cup \beta\phi'). \quad (3.2.20)$$

The motivation for said choice will become evident a posteriori in Section 3.3.3, where we state our main conjecture.

3.2.2 The transfer map on 3-cocycles

Let us study the properties of the transfer map for some important subgroups of $\mathrm{SL}(2, \mathbb{F}_q)$, namely the centralizers $C_G(g)$ for $g \in \mathrm{SL}(2, \mathbb{F}_q)$, which are isomorphic to T , T' , ZU or $\mathrm{SL}(2, \mathbb{F}_q)$ itself, depending on g . For our purposes, it will suffice to evaluate the transfer map Tr_H^G restricted to the centralizers.

Fix $\psi \in H^3(H, \mathbb{Q}/\mathbb{Z})$. The restriction of the transfer map satisfies the following crucial identity [87]:

$$\mathrm{Tr}_H^G(\psi)|_{H'} = \sum_{g \in E} \mathrm{Tr}_{H' \cap gHg^{-1}}^{H'}(g \star \psi|_{H' \cap gHg^{-1}}), \quad (3.2.21)$$

where E is a set of representatives for the double cosets in $H' \backslash G/H$ and $\star$ is the action induced by conjugation on cocycles, i.e.

$$(g \star \psi)(h_1, h_2, h_3) = \psi(g^{-1}h_1g, g^{-1}h_2g, g^{-1}h_3g). \quad (3.2.22)$$

In the following, H and H' will be either T , T' or ZU . Then, notice that $H' \cap gHg^{-1}$ is equal to the center Z whenever $H \neq H'$, while if instead $H = H'$, it might also happen

that $H \cap gHg^{-1} = H$. Therefore, when $H \neq H'$ we have:

$$\mathrm{Tr}_H^G(\psi)|_{H'} = \sum_{g \in E} \mathrm{Tr}_Z^{H'}(g \star \psi|_Z). \quad (3.2.23)$$

The other three possibilities have to be treated separately. Suppose $H = H' = T$, then $T \cap gTg^{-1} = T$ if and only if $g \in T$ or if g is of the form

$$g = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}, \quad (3.2.24)$$

for some $\mu \in \mathbb{F}_q^\times$. Notice that the latter are exactly the matrices satisfying $gd(\lambda)g^{-1} = d(\lambda^{-1})$. We shall now determine whether or not these matrices are included in E and hence contribute to (3.2.21). It turns out that it is enough to include one from each family — for example $\mathbb{1}$ and s are a natural choice of representatives, with

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.2.25)$$

Therefore, we obtain

$$\mathrm{Tr}_T^G(\psi)|_T = \psi + \psi \circ \iota + \sum_{g \in E \setminus \{\mathbb{1}, s\}} \mathrm{Tr}_Z^T(g \star \psi|_Z), \quad (3.2.26)$$

where ι is the inversion map on triples of elements:

$$\iota(h_1, h_2, h_3) = (h_1^{-1}, h_2^{-1}, h_3^{-1}). \quad (3.2.27)$$

Similarly, we deduce the following expressions for the last two cases:

$$\mathrm{Tr}_{T'}^G(\psi)|_{T'} = \psi + \psi \circ \iota + \sum_{g \in E \setminus \{\mathbb{1}, s'\}} \mathrm{Tr}_Z^{T'}(g \star \psi|_Z), \quad (3.2.28)$$

$$\mathrm{Tr}_{ZU}^G(\psi)|_{ZU} = \sum_{\lambda \in \mathbb{F}_q^\times / \sim} d(\lambda) \star \psi + \sum_{g \in E \setminus \{d(\lambda) : \lambda \in \mathbb{F}_q^\times / \sim\}} \mathrm{Tr}_Z^{ZU}(g \star \psi|_Z), \quad (3.2.29)$$

where s' is any matrix realizing the inversion map on T' through conjugation and $\lambda \sim \lambda'$ if and only if $\lambda' \in \{\lambda, -\lambda\}$. Finally, notice that multiplication by 2 kills the term $\mathrm{Tr}_Z^H(g \star \psi|_Z)$ for any H , since $H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This remark is crucial since we will later specialize to $\psi = -6\phi \cup \beta\phi$, which vanishes when restricted to the center.

3.3 $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

We refer to the Section 1.2.2 for a thorough review of Dijkgraaf-Witten TQFT. Here, we focus on the case of $G = \mathrm{SL}(2, \mathbb{F}_q)$, with the twist given by $\omega = 12\beta^{-1}c_2(\rho)$ as in (3.2.20).

3.3.1 The Modular Data

Recall that the analysis of the modular data associated with twisted DW theory greatly simplifies when the family of twisted 2-cocycles β_a , defined in (1.2.39), is cohomologically trivial for all $a \in G$. This is indeed our case.

A sufficient condition to have β_a trivial is that $H^2(C_G(a), \mathbb{U}(1))$ vanishes for all $a \in G$. If we suppose that q is odd and different from 9, all the relevant second cohomology groups vanish, except for $H^2(ZU, \mathbb{U}(1))$. Nonetheless, we shall see that all the β_a are cohomologically trivial if we choose $\omega = 12\beta^{-1}c_2(\rho)$ as above.

Let us consider β_a for a a representative of some conjugacy class in $\mathrm{SL}(2, \mathbb{F}_q)$. First, suppose $a \neq \pm 1$. Then, we should consider equation (1.2.39) with $g, h \in C_G(a)$. In particular, notice that β_a only depends on the restriction of the twist ω on $C_G(a)$. Then, making use of the identities (3.2.23), (3.2.26), (3.2.28) and (3.2.29), we compute:

$$\beta_{d(\lambda)} = -6\phi(\lambda)(\beta\phi - \beta\phi \circ \iota), \quad (3.3.1)$$

$$\beta_{d'(\xi)} = -6\phi'(\xi)(\beta\phi' - \beta\phi' \circ \iota), \quad (3.3.2)$$

$$\beta_{\epsilon u_\tau} = 0, \quad (3.3.3)$$

where $\iota(h_1, h_2) = (h_1^{-1}, h_2^{-1})$ is the inversion map on pairs of elements. From these, one can easily deduce the expressions for ε_a . Indeed, if we call θ and ϑ the (compatible) generators of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$ respectively, we get:

$$\varepsilon_{d(\theta^a)}(d(\theta^b)) = -6 \frac{\bar{a}\bar{b} + \overline{-a-b}}{(q-1)^2} \mod 1, \quad (3.3.4)$$

$$\varepsilon_{d'(\vartheta^{a(q-1)})}(d'(\vartheta^{b(q-1)})) = -6 \frac{\bar{a}\bar{b} + \overline{-a-b}}{(q+1)^2} \mod 1, \quad (3.3.5)$$

$$\varepsilon_{\epsilon u_\tau}(\epsilon' u_{\tau'}) = 0 \mod 1, \quad (3.3.6)$$

where $\bar{a}$ is a lift of $a \in \mathbb{Z}/n\mathbb{Z}$ to $\{0, 1, \dots, n-1\} \subset \mathbb{Z}$. Notice that we have solved the problem of $H^2(ZU, \mathbb{U}(1)) \neq 0$ by choosing a twist that vanishes when restricted to ZU . We are left with the case $a = \pm 1$. It is easy to check that

$$\beta_{\pm 1}(g, h) = -6\phi(\pm 1)\mathrm{Tr}_T^G \beta\phi(g, h) - 6\phi'(\pm 1)\mathrm{Tr}_{T'}^G \beta\phi'(g, h) = 0, \quad (3.3.7)$$

since $2\phi(\pm 1) = 2\phi'(\pm 1) = 0$. Therefore

$$\varepsilon_{\pm 1}(g) = 0 \mod 1. \quad (3.3.8)$$

We are now ready to write down the explicit expression for the S matrix. First of all, notice that a and b must be commuting, otherwise the set $G(a, b)$ is empty and the corresponding S matrix entries are vanishing. One possibility is that at least one of the two entries belongs to the center Z of G . In this case we have

$$S_{(\pm 1, \chi), (a, \psi)} = \frac{1}{|C_G(a)|} \chi(a)^* \psi(\pm 1)^*, \quad (3.3.9)$$

where it is implicit that χ is a character of G , while ψ is a character of $C_G(a)$. In the following, we will not specify which group each character is a character of, unless it is not clear from the context. Another non-vanishing entry of the S matrix corresponds to having both $a, b \in T$ or T' :

$$S_{(a,\chi),(b,\psi)} = \frac{1}{|C_G(a)|} (\chi(b)\psi(a)\sigma(a | b) + \chi(b^{-1})\psi(a^{-1})\sigma(a | b^{-1}))^*. \quad (3.3.10)$$

Finally, the last possibility is that both $a, b \in ZU$:

$$S_{(a,\chi),(b,\psi)} = \frac{1}{|C_G(a)|} \sum_{\lambda \in \mathbb{F}_q^\times / \sim} \chi(d(\lambda)bd(\lambda)^{-1})^* \psi(d(\lambda)^{-1}ad(\lambda))^*, \quad (3.3.11)$$

where $\lambda \sim \lambda'$ if and only if $\lambda' \in \{\lambda, -\lambda\}$.

3.3.2 Naturality of the twist

We will now assume that the twist ω is of the form mentioned in Section 3.2.1, i.e. $\omega = 12\beta^{-1}c_2(\rho)$, with $\rho = \mathrm{Ind}_T^G(\phi) + \mathrm{Ind}_{T'}^G(\phi')$ for ϕ and ϕ' the two 1-dimensional representations of T and T' defined in (3.2.17) and (3.2.18). One crucial property of the above twist is its naturality. Indeed, consider the embedding $i_r : \mathrm{SL}(2, \mathbb{F}_q) \rightarrow \mathrm{SL}(2, \mathbb{F}_{q^r})$, and realize $\mathbb{F}_q$ and $\mathbb{F}_{q^r}$ via Conway polynomials so that the respective multiplicative generators are compatible. If ϕ_r and ϕ'_r are the one-dimensional representations in (3.2.17) and (3.2.18) corresponding to q^r then $i_r^*(\phi_r) = \phi$ and $i_r^*(\phi'_r) = \phi'$. Then, we claim that ω is natural in the following sense:

$$i_r^*(\omega_r) = 12 i_r^* \beta^{-1} c_2(\mathrm{Ind}_{T_r}^{G_r}(\phi_r) + \mathrm{Ind}_{T'_r}^{G_r}(\phi'_r)) = 12 \beta^{-1} c_2(\mathrm{Ind}_T^G(\phi) + \mathrm{Ind}_{T'}^G(\phi')) = \omega, \quad (3.3.12)$$

where $G = \mathrm{SL}(2, \mathbb{F}_q)$, $G_r = \mathrm{SL}(2, \mathbb{F}_{q^r})$ and T_r, T'_r are respectively the subgroups of diagonal and codiagonal matrices in G_r . It is important to note that naturality under pullbacks does not hold in general for induced representations and transfer maps, meaning that i_r^* does not simply commute with Ind . However, it follows from the definition that Ind is left adjoint to the restriction functor. Consequently, the transfer map possesses certain naturality properties under restrictions in the sense of Equation (3.2.21). This is the main ingredient in the proof of the naturality of ω .

We start by considering $i_r^* \mathrm{Tr}_{T_r}^{G_r}(\phi_r \cup \beta \phi_r)$. To apply (3.2.21), we need to study the double cosets $G \backslash G_r / T_r$ and the intersection $G \cap g T_r g^{-1}$ for g a representative of $G \backslash G_r / T_r$. As one might have expected, there is a crucial difference depending on the parity of r . Indeed, if r is even, codiagonal matrices in G are similar to elements of T_r in G_r . Let r be odd. An element in $g T_r g^{-1}$ has order dividing $q^r - 1$, therefore its intersection with $\mathrm{SL}(2, \mathbb{F}_q)$ can only consist of elements of the form $h T h^{-1}$ where now $h \in \mathrm{SL}(2, \mathbb{F}_q)$ – otherwise it equals the center. However, $G \cap g T_r g^{-1} = h T h^{-1}$ implies that either $h^{-1}g$ is in T_r or it

is any matrix realizing the inversion map on T_r through conjugation. Either way, we can write $g = \tilde{h} \cdot t$, where $\tilde{h} \in \mathrm{SL}(2, \mathbb{F}_q)$ and $t \in T_r$, meaning that g is trivial as a representative of double cosets $G \backslash G_r / T_r$. On the other hand, it is clear that for any other representative g we have $G \cap gT_r g^{-1} = Z$. Then, using the fact that multiplication by 2 kills cocycles restricted to the center, we have

$$i_r^* \mathrm{Tr}_{T_r}^{G_r}(6\phi_r \cup \beta\phi_r) = \mathrm{Tr}_T^G(i_r^*(6\phi_r \cup \beta\phi_r)) = \mathrm{Tr}_T^G(6\phi \cup \beta\phi). \quad (3.3.13)$$

The case of r even is similar, except that elements of order dividing $q^r - 1$ can also be of the form $hT'h^{-1}$. Call m any element of $\mathrm{SL}(2, \mathbb{F}_{q^r})$ which diagonalizes codiagonal matrices in the quadratic extension, then $G \cap gT_r g^{-1} = hT'h^{-1}$ if and only if $g = \tilde{h} \cdot m \cdot t$ for some $t \in T_r$ and $\tilde{h} \in \mathrm{SL}(2, \mathbb{F}_q)$. As above, it suffices to take $g = m$. Thus, in this case we obtain

$$\begin{aligned} i_r^* \mathrm{Tr}_{T_r}^{G_r}(6\phi_r \cup \beta\phi_r) &= \mathrm{Tr}_T^G(6\phi \cup \beta\phi) + \mathrm{Tr}_{T'}^G(i_r^*(m \star (6\phi_r \cup \beta\phi_r))) \\ &= \mathrm{Tr}_T^G(6\phi \cup \beta\phi) + \mathrm{Tr}_{T'}^G(6\phi' \cup \beta\phi') \end{aligned} \quad (3.3.14)$$

Notice that there is a non-trivial mixing of the contributions of the two representations $\mathrm{Ind}_T^G(\phi)$ and $\mathrm{Ind}_{T'}^G(\phi')$. Finally, we have to consider the term $i_r^* \mathrm{Tr}_{T_r}^{G_r}(\phi'_r \cup \beta\phi'_r)$. It is straightforward to check that when r is even, elements of $\mathrm{SL}(2, \mathbb{F}_q)$ cannot have order dividing $q^r + 1$ unless they belong to the center, hence the restriction of the transfer to $\mathrm{SL}(2, \mathbb{F}_q)$ is trivial. On the other hand, when r is odd $G \cap gT'_r g^{-1}$ might coincide with $hT'h^{-1}$. As above, it is enough to consider $g = \mathbb{1}$, so that

$$i_r^* \mathrm{Tr}_{T'_r}^{G_r}(6\phi'_r \cup \beta\phi'_r) = \mathrm{Tr}_{T'}^G(i_r^*(6\phi'_r \cup \beta\phi'_r)) = \mathrm{Tr}_{T'}^G(6\phi' \cup \beta\phi'). \quad (3.3.15)$$

Putting together the contributions from the two pieces concludes the proof of $\omega = i_r^*(\omega_r)$.

One may wonder what happens when we consider a different representation ρ instead of (3.2.19). First, it is useful to rephrase this question in terms of the irreducible representations (3.2.7). Notice that we may rewrite

$$\mathrm{Ind}_T^G \phi = \mathrm{St} \otimes R(\phi), \quad \mathrm{Ind}_{T'}^G \phi' = \mathrm{St} \otimes R'(\phi'), \quad (3.3.16)$$

see e.g. Exercise 5.1 in [83]. Thus, the only contributions that we are possibly missing are those from $R_\sigma(\alpha_0)$ and $R'_\sigma(\beta_0)$. However, we expect these to contribute only for non-generic primes p , since they have non-trivial values only on matrices of order p or $2p$, see Table 3.2.2. Lastly, we remark that the choice of the particular form of ϕ and ϕ' is not canonical, since it depends on the choice of compatible generators of $\mathbb{F}_q^\times$ and $\mathbb{F}_{q^2}^\times$. This can be understood as a consequence of the freedom of acting on $\mathbb{Q}/\mathbb{Z}$ via its automorphism group.

3.3.3 A conjectural relation with $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The main contribution of this chapter is a conjectural correspondence between $\mathrm{SU}(2)$ Chern-Simons and $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theories. By definition, the DW partition function on a closed 3-manifold M counts homomorphisms from its fundamental group to G , weighted by a non-trivial phase arising from the twist. In the case of $G = \mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory twisted by the 3-cocycle $\omega \in H^3(G, \mathrm{U}(1))$ chosen above, we claim that the phase factor and the counting are closely related to the asymptotic expansion of $\mathrm{SU}(2)$ Chern-Simons, at least for manifolds with no hyperbolic components in their Thurston geometric decomposition. We will provide strong evidence supporting this statement through several examples that cover a wide variety of closed oriented 3-manifolds. In Section 3.5, we will prove a weaker statement for spherical manifolds, which could shed some light on the relation between continuum and discrete gauge theories.

Let us be more precise. Recall that the partition function of $\mathrm{SU}(2)_k$ Chern-Simons can be expressed as a perturbative series as the level k is sent to infinity. Indeed, it was argued in [6] that in such a limit the path integral localizes on the flat connections. Thus, the partition function gets a contribution from each connected component of the moduli space of flat connections $\mathcal{M}(M, \mathrm{SU}(2)) = \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$. A mathematically rigorous realization of the Chern-Simons partition function is given by the WRT invariant [7]. Then, the asymptotic expansion conjecture [6, 74–76] states that the WRT invariant evaluated at a root of unity $\xi = e^{\frac{2\pi i}{k}}$ can be expressed as

$$\mathrm{WRT}(M; \xi) \sim \sum_{Q \in \mathcal{S}_{\mathrm{SU}(2)}} e^{2\pi i k Q} I_Q \left(\frac{1}{k} \right), \quad (3.3.17)$$

in the limit $k \rightarrow \infty$. The sum is taken over the finite set $\mathcal{S}_{\mathrm{SU}(2)}$ of the different values $\mathrm{CS}[\gamma]$ of the Chern-Simons invariant for γ in the moduli space of $\mathrm{SU}(2)$ flat connections $\mathcal{M}(M, \mathrm{SU}(2))$. We have also introduced $I_Q \in k^{-\Delta_Q} \mathbb{C}[[1/k]]$, where it is understood that the term of order $-\Delta_Q$ has non-zero coefficient. As the WRT invariant is well defined at any primitive root of unity $\xi = e^{\frac{2\pi i \varsigma}{\tau}}$, assuming that ς is odd, the asymptotic expansion conjecture can be further refined to

$$\mathrm{WRT}(M; \xi) \sim \sum_{Q \in \mathcal{S}} e^{\frac{2\pi i \varsigma}{\tau} Q} I_Q^{(\varsigma, \tau \bmod \varsigma)} \left(\frac{\varsigma}{\tau} \right), \quad (3.3.18)$$

in the limit $\tau \rightarrow \infty$ while keeping ς and $\tau \bmod \varsigma$ fixed and $\mathrm{gcd}(\tau, \varsigma) = 1$. Here, the sum is taken over some finite subset $\mathcal{S}$ of $\{z \in \mathbb{C} : 0 \leq \Re(z) < 1\}$, while $I_Q^{(\varsigma, \tau \bmod \varsigma)} \in (\tau/\varsigma)^{-\Delta_Q} \mathbb{C}[[\varsigma/\tau]]$ with coefficients dependent on ς and τ modulo ς . Various versions of this conjecture already appeared in the literature, see for example [79–81]. Here, we have stated the weakest version that is needed to formulate our conjectural correspondence between Chern-Simons and Dijkgraaf-Witten theories.

We will now make some additional remarks about (3.3.18), and formulate a more refined version of the asymptotic expansion statement, to give some context. Similarly to how we can interpret Equation (3.3.17) as the asymptotics of the $SU(2)_k$ Chern-Simons partition function, one would expect (3.3.18) to match with the asymptotics of a Chern-Simons theory at rational level $k = \tau/s \in \mathbb{Q}$. Obviously, such a Chern-Simons theory would be ill-defined as the usual action would no longer be gauge-invariant. Still, it is conjectured that one can make the partition function at rational level non-ambiguous and then interpret (3.3.18) as its asymptotic expansion at large τ . The set $\mathcal{S}$ would then be understood as the finite set of the different values $CS[\gamma]$ of the Chern-Simons invariant for γ in the moduli space of $SL(2, \mathbb{C})$ flat connections $\mathcal{M}(M, SL(2, \mathbb{C})) = \text{Hom}(\pi_1(M), SL(2, \mathbb{C}))/SL(2, \mathbb{C})$.

Let us mention how one can make sense of $SU(2)$ Chern-Simons theory at rational level. The idea is that Chern-Simons theory can be analytically continued, and the level $k \in \mathbb{Z}$ gets promoted to $k \in \mathbb{C}$ [18, 19, 77]. The partition function can be formulated as the path integral

$$Z_{\text{CS}}^{(\Gamma)}(M) = \int_{\Gamma} \mathcal{D}A e^{2\pi i k \text{CS}[A]}, \quad (3.3.19)$$

where Γ is a middle-dimensional contour in the universal cover of the space of $SL(2, \mathbb{C})$ flat connections modulo gauge equivalence. When $k = \tau/s \in \mathbb{Q}$ it is also sufficient to consider the s -fold cover instead of the universal one. Notice that the gauge ambiguity of the analytically continued theory is now reflected in the ambiguity of the contour Γ . However, it is possible to choose Γ such that it recovers the partition function of the usual $SU(2)_k$ Chern-Simons theory, when k is taken to be an integer in (3.3.19). Then, if one assumes that the contour decomposes as

$$\Gamma = \sum_{\alpha} n_{\alpha} \Gamma_{\alpha} \quad (3.3.20)$$

into the Lefschetz thimbles Γ_{α} , each one corresponding to a connected component of the moduli space⁴, the path integral (3.3.19) has a well-defined perturbative expansion. Indeed, denote by Z_{α} the path integral on the Lefschetz thimble Γ_{α} :

$$Z_{\alpha} = \int_{\Gamma_{\alpha}} \mathcal{D}A e^{2\pi i k \text{CS}[A]}. \quad (3.3.21)$$

Each path integral has an asymptotic expansion for large k as follows:

$$Z_{\alpha} = e^{2\pi i k \text{CS}_{\alpha}} I_{\alpha} \left(\frac{1}{k} \right), \quad (3.3.22)$$

where $I_{\alpha} \in k^{-\delta_{\alpha}} \mathbb{C}[[1/k]]$, and CS_{α} is the value of the CS invariant of the corresponding component of the moduli space. Notice that δ_{α} is the leading order of k in the expansion

⁴More generally, Lefschetz thimbles correspond to compact middle-dimensional cycles in the connected components.

corresponding to the thimble α . Let γ_α be any connection in the connected component of the moduli space associated with α . It was argued in [74, 75, 90, 91] that δ_α can be expressed, under certain assumptions, by the formula

$$\delta_\alpha = \frac{1}{2}(\dim H^0(M, d + \gamma_\alpha) - \dim H^1(M, d + \gamma_\alpha)). \quad (3.3.23)$$

Therefore, it is expected to be a half-integer. Moreover, if one assumes that the CS functional, considered on the space of $SL(2, \mathbb{C})$ connections modulo *based* gauge transformations, behaves as a Morse-Bott function in the vicinity of the corresponding connected component $\widetilde{\mathcal{M}}_\alpha \subset \text{Hom}(\pi_1(M), SL(2, \mathbb{C}))$ in the $SL(2, \mathbb{C})$ representation variety, one can argue that [77]

$$\delta_\alpha = \frac{1}{2} \left(\dim_{\mathbb{C}} SL(2, \mathbb{C}) - \dim_{\mathbb{C}} \widetilde{\mathcal{M}}_\alpha \right) \quad (3.3.24)$$

Finally, this yields the following asymptotic expansion for the partition function:

$$Z_{\text{CS}}^{(\Gamma)}(M) = \sum_{\alpha} n_{\alpha} Z_{\alpha} \sim \sum_{\alpha} n_{\alpha} e^{2\pi i k \text{CS}_{\alpha}} I_{\alpha} \left(\frac{1}{k} \right). \quad (3.3.25)$$

It was argued in [78] that the integral contour Γ might get additional contributions from Lefschetz thimbles that do not correspond to usual connected components of the moduli space of $SL(2, \mathbb{C})$ flat connections. The authors observed that these contributions arise from the non-compactness of $SL(2, \mathbb{C})$, as there might exist sequences of connections that become flat at infinity, while attaining a finite limit of the Chern-Simons functional, the so-called *flat connections at infinity*.

We can now compare the asymptotic expansion (3.3.18) of the WRT invariant at a root of unity $\xi = e^{\frac{2\pi i \omega}{\tau}}$ with the asymptotics (3.3.25) of the partition function of analytically continued $SU(2)$ Chern-Simons at level $k = \tau/\omega$. As already mentioned, one expects $\mathcal{S}$ to be the set of values of the CS functional on $SL(2, \mathbb{C})$ flat connections, and we cannot exclude that it may contain contributions from flat connections at infinity as well. Moreover, this association allows us to identify the leading orders of I_{α} and I_Q :

$$\Delta_Q = \min_{\alpha: \text{CS}_{\alpha}=Q} \delta_{\alpha}. \quad (3.3.26)$$

It is well-known that the Chern-Simons functional of a manifold M is valued in $\mathbb{Q}$ whenever M does not contain any hyperbolic component in its geometric decomposition. Thus, $\mathcal{S}$ is expected to be a subset of $\mathbb{Q}/\mathbb{Z}$ in this case.

Given $\mathcal{S}$ as above, define $\mathcal{S}_{\text{DW}} \subset \mathbb{Q}/\mathbb{Z}$ as the image of $\mathcal{S}$ under the map induced by multiplication by 12 – note that it is not injective in general. Suppose that q is a power of a *generic* prime, for which each element of $\mathcal{S}$ is contained either in the image of ϕ or in the image of ϕ' . This condition can be equivalently stated as follows: for each $Q \in \mathcal{S}$, the order of Q , $o(Q)$, divides $q^2 - 1$. We say that q is a power of a *special* prime p , if p divides $o(Q)$ for some $Q \in \mathcal{S}$. With this setting, we state the following conjecture:

Conjecture. *Given M a closed oriented 3-manifold with no hyperbolic components in its geometric decomposition, the partition function of $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory, twisted by the 3-cocycle $k\omega$, where ω is as in the beginning of this section, can be written as*

$$Z_{\mathrm{DW}}(M) = \sum_{Q \in \mathcal{S}_{\mathrm{DW}}} e^{2\pi i k Q} T_Q(q), \quad (3.3.27)$$

where T_Q is a function of q valued in $|\mathrm{SL}(2, \mathbb{F}_q)|^{-1}\mathbb{Z}$ such that for large q we have

$$T_Q \sim q^{-2\Delta'_Q}, \quad \Delta'_Q = \min_{Q' \in \mathcal{S}: 12Q' = Q} \Delta_{Q'}. \quad (3.3.28)$$

Notice this is consistent with the conjectural (3.3.24) and the fact that the number of points of a variety of dimension n over $\mathbb{F}_q$ grows as $\sim q^n$. We remark that in all the examples we have considered $|\mathrm{SL}(2, \mathbb{F}_q)| T_Q(q)$ is a polynomial in q . Furthermore, it is implicit in (3.3.27) that $\mathcal{S}_{\mathrm{DW}}$ does not contain the eventual contributions from flat connections at infinity. We stress that the condition that $o(Q)$ divides $q^2 - 1$ for any $Q \in \mathcal{S}$ can be relaxed — however, it is crucial that we avoid special primes. In that case, only some elements of $\mathcal{S}_{\mathrm{DW}}$ are detected by the Dijkgraaf-Witten partition function, and the DW functional computes $Q \in \mathcal{S}_{\mathrm{DW}}$ only up to a multiple, depending on $\gcd(q^2 - 1, o(Q))$. We note, that as in the abelian case reviewed in Section 3.1, one can formally identify $q \sim k^{\frac{1}{2}}$ to match the leading terms in the expansions of CS and DW partition functions.

We remark that hyperbolic manifolds, which are excluded by the assumptions in the conjecture, constitute the richest and most interesting class of geometric 3-manifolds. However, there is no hope that our conjecture holds for such manifolds in its current form: the Chern-Simons functional at flat connections is in general complex-valued, and Dijkgraaf-Witten theory cannot reproduce non-rational phases. In the following section, we will consider a hyperbolic manifold as a concrete counterexample and show that there are no evident relations between the partition functions of the two theories. However, we will notice an interesting correspondence between the representation varieties over $\mathbb{C}$ and $\mathbb{F}_q$ for some specific values of q , and we will match the leading order Δ_Q with Δ'_Q .

Finally we remark on another potential direction of study of $\mathrm{SL}(2, \mathbb{F}_q)$ DW partition functions. In the untwisted case we have the following interpretation in terms of Behrend's generalization of the Grothendieck–Lefschetz trace formula:

$$Z_{\mathrm{DW}}(M) = \#X(\mathbb{F}_q) = \sum_i (-1)^i \mathrm{Tr}_{H_c^i(X(\overline{\mathbb{F}_q}); \mathbb{Q}_\ell)} \mathrm{Fr}^r, \quad (3.3.29)$$

where X is the algebraic stack $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(2))/\mathrm{SL}(2)$ and Fr is the induced action of the Frobenius automorphism on the cohomology. When the theory is twisted by $r\omega$ it is then suggestive to ask about existence of a rank 1 sheaf $\mathcal{F}$ with $\mathrm{Fr}|_{\mathcal{F}_\gamma} = \langle \gamma^* \omega, [M] \rangle$ such that for the twisted version we have

$$Z_{\mathrm{DW}}(M) \stackrel{?}{=} \sum_i (-1)^i \mathrm{Tr}_{H_c^i(X(\overline{\mathbb{F}_q}); \mathcal{F})} \mathrm{Fr}^r. \quad (3.3.30)$$

In this way $H_c^*(X(\overline{\mathbb{F}}_q); \mathcal{F})$ would provide a categorification of the DW invariant of 3-manifolds. It would be also interesting to study the analytic properties of the corresponding zeta function, which would be a generating function for the DW partition functions for different $q = p^r$.

3.4 Examples

In order to support our conjecture, we shall present several computations of the $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten partition function on different kinds of 3-manifolds. In what follows, for simplicity, we consider the twist to be given by ω defined in Section 3.3.2 itself, rather than a multiple of it. To cover a wide variety of manifolds, we will consider at least one example for each one of the eight Thurston geometries. See Table 3.4.1 for a comprehensive list of the examples with global geometry to be discussed throughout this section.

Global geometry	Examples	Section(s)
S^3	Lens spaces, Poincaré homology sphere	3.4.1 , 3.4.3
E^3	—	See below
$\mathbb{H}^3$	$-\frac{1}{2}$ -surgery on the figure-eight knot	3.4.6
$S^2 \times \mathbb{R}$	—	See below
$\mathbb{H}^2 \times \mathbb{R}$	$M(-1; (5, -2), (5, -2), (5, -1))$	3.4.5
$\widetilde{\mathrm{SL}(2, \mathbb{R})}$	Brieskorn homology spheres ($\neq \Sigma(2, 3, 5)$), degree- n circle bundles over $\Sigma_{g>1}$	3.4.3 , 3.4.4
Nil	T^n torus bundles	3.4.2
Sol	$T^n S$ torus bundles	3.4.2

Table 3.4.1: Examples treated for each of the eight Thurston geometries.

Notice that we did not include any example of Euclidean or $S^2 \times \mathbb{R}$ geometry. In both cases, there are only a few compact manifolds with that specific global geometry. For Euclidean geometry, five out of the six possible manifolds are torus bundles of a finite-order element of $\mathrm{SL}(2, \mathbb{Z})$. They are left out of the analysis since two large families of torus bundles are already considered in Section 3.4.2. For $S^2 \times \mathbb{R}$ geometry, the only possible manifolds are $S^2 \times S^1$, $S^2 \tilde{\times} S^1$ and $\mathbb{RP}^3 \# \mathbb{RP}^3$. The first two are, respectively, the torus bundles of the identity map and the antipodal map of S^2 , while in the last case it is

enough to consider $\mathbb{RP}^3$, which is a lens space. These cases are uninteresting, and nothing remarkable happens.

Recall that one way to state Thurston's geometrization conjecture is that any closed prime 3-manifold has a decomposition along tori into pieces admitting one of the eight possible geometric structures. Thus, it is not enough to check our conjecture on manifolds with global geometries. In general, a closed oriented 3-manifold whose geometric decomposition has no hyperbolic pieces is called a graph manifold. In [47] Neumann showed that graph manifolds can be obtained by plumbing. However, Neumann's definition of a plumbed manifold also allows for plumbing graphs with cycles. For completeness, in Section 3.4.3, we have included two general examples of plumbed manifolds with a non-trivial decomposition into Seifert fibered spaces.

3.4.1 Lens spaces

Let M be the lens space $L(n, 1)$, which is defined as follows. Embed the 3-sphere S^3 into $\mathbb{C}^2$, and consider the free $\mathbb{Z}/n\mathbb{Z}$ action sending

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{n}} z_1, e^{\frac{2\pi i}{n}} z_2).$$

The lens space $L(n, 1)$ is then given as the quotient $S^3/(\mathbb{Z}/n\mathbb{Z})$, and its fundamental group is

$$\mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n = 1 \rangle. \quad (3.4.1)$$

For simplicity, we will consider n to be an odd prime, but our results can be easily generalized to arbitrary n .

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The moduli space of $\mathrm{SL}(2, \mathbb{C})$ flat connections in this case is easily identified with the conjugacy classes of matrices of order n . Note that their images necessarily fall into $\mathrm{SU}(2)$, since upper triangular matrices have infinite order. The moduli space then consists of $\frac{n+1}{2}$ isolated points; one of them corresponds to the trivial flat connection γ_0 , while the other $\frac{n-1}{2}$ correspond to the representations

$$\gamma_\ell : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{SU}(2), \quad x \mapsto d(\zeta_n^\ell), \quad \ell = 1, \dots, \frac{n-1}{2}, \quad (3.4.2)$$

where we defined $\zeta_n = e^{\frac{2\pi i}{n}}$. Under conjugation, their orbits are all isomorphic to $\mathbb{C}^\times$ (except for the trivial connection, whose orbit is a single point). From these considerations, one concludes that the powers of k appearing in the asymptotic expansion are $\delta_{\gamma_0} = 3/2$ and $\delta_{\gamma_\ell} = 1/2$. Furthermore, the values of the CS functional on each flat connection read [74, 92]

$$\mathrm{CS}[\gamma_\ell] = \frac{\ell^2}{n}. \quad (3.4.3)$$

SL(2, $\mathbb{F}_q$) Dijkgraaf-Witten theory

The space of $\text{SL}(2, \mathbb{F}_q)$ flat connections is now identified with the set of matrices in $\text{SL}(2, \mathbb{F}_q)$ of order n . First, it is easy to see that n should divide either one of $q-1$, $q+1$ or q in order to find non-trivial flat connections. That is because matrices of type $d(\lambda)$ (respectively $d'(\xi)$, ϵu_τ) have order dividing $q-1$ (respectively $q+1$, $p \cdot \text{ord}(\epsilon)$). Notice that these three conditions are mutually exclusive, since $n > 2$. It is also clear that $p = n$ is a special prime, so let us focus on the other two cases.

If $n \mid q-1$, the equation $d(\lambda)^n = \mathbb{1}$, $\lambda^2 \neq 1$ has $\frac{n-1}{2}$ distinct solutions up to conjugation. In terms of a generator θ of $\mathbb{F}_q^\times$, they are

$$d(\theta^{\ell \frac{q-1}{n}}), \quad \ell = 1, \dots, \frac{n-1}{2}. \quad (3.4.4)$$

Similarly, when $n \mid q+1$, the equation $d'(\xi)^n = \mathbb{1}$, $\xi^2 \neq 1$ has $\frac{n-1}{2}$ distinct solutions up to conjugation. In terms of a generator ϑ of $\mathbb{F}_{q^2}^\times$, they read

$$d'(\vartheta^{\ell \frac{q^2-1}{n}}), \quad \ell = 1, \dots, \frac{n-1}{2}. \quad (3.4.5)$$

However, the orbit size differs between each case, being $q(q+1)$ for the connections of d type and $q(q-1)$ for the ones of d' type (see Table 3.2.1).

The partition function (1.2.13) is easy to find in terms of the S and T matrices defined in Section 1.2.2. The lens space $L(n, 1)$ has a simple surgery description as the $-n$ -surgery on the unknot. Hence,

$$Z_{\text{DW}}(L(n, 1)) = \sum_s S_{0,s}^2 T_{s,s}^{-n} = \frac{1}{q(q^2-1)} \left(1 + q(q \mp 1) \sum_{\ell=1}^{\frac{n-1}{2}} e^{2\pi i \frac{12\ell^2}{n}} \right), \quad n \mid q \pm 1. \quad (3.4.6)$$

Comparing with the complex case, we observe that the phase factors in the finite field partition function exactly reproduce the twelvefold Chern-Simons invariants $12\text{CS}[\gamma_\ell]$. Moreover, the polynomials in q multiplying these phases behave at leading order as $T_{\gamma_0}(q) \sim q^{-3}$ and $T_{\gamma_\ell}(q) \sim q^{-1}$, in agreement with the conjectured relation $T_\gamma(q) \sim q^{-2\Delta_\gamma}$.

3.4.2 Torus bundles

Consider a surface Σ and an orientation-preserving diffeomorphism $\beta : \Sigma \rightarrow \Sigma$. The *torus bundle* Σ_β is constructed by gluing the two ends of the manifold $\Sigma \times [0, 1]$ via β , as

$$\Sigma_\beta = \frac{\Sigma \times [0, 1]}{(x, 0) \sim (\beta(x), 1)}. \quad (3.4.7)$$

We will consider Σ to be the torus $\mathbb{T}^2$, in which case orientation-preserving diffeomorphisms are determined up to isotopy by a matrix $U \in \text{SL}(2, \mathbb{Z})$. Note also that two torus bundles

Σ_U and $\Sigma_{U'}$ are diffeomorphic if and only if U is conjugate to $U'^{\pm 1}$ in $\mathrm{GL}(2, \mathbb{Z})$. If we let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the fundamental group of Σ_U can be easily written as

$$\pi_1(\Sigma_U) = \langle x, y, z \mid [x, y] = 1, xzx^{-1} = x^a y^b, yzy^{-1} = x^c y^d \rangle. \quad (3.4.8)$$

The $\mathrm{SU}(2)$ Chern-Simons theory on torus bundles has been studied extensively in the literature [90, 92]. Non-trivial values of the CS functional are well-known (at least for flat connections of the type we will study in this section), as is the asymptotic behavior of the CS partition function. The latter can actually be computed exactly by using the S and T matrices of the representation category of $\bar{U}_\xi(\mathfrak{sl}_2)$ for a primitive ε -th root of unity $\xi = e^{\frac{2\pi i \varepsilon}{\varepsilon}}$:

$$\begin{aligned} S_{jl} &= \frac{1}{2i} \sqrt{\frac{2}{\varepsilon}} \left(\xi^{\frac{j^2}{2}} - \xi^{-\frac{j^2}{2}} \right), \\ T_{jl} &= \delta_{jl} e^{-\frac{\pi i}{4}} \xi^{\frac{j^2}{4}}. \end{aligned} \quad (3.4.9)$$

For $\varepsilon = 1$ this realizes the $\mathrm{SU}(2)$ Chern-Simons theory at (renormalized) integral level $k = \varepsilon$. The indices j, l above run from 1 to $\varepsilon - 1$, the dimension of the torus Hilbert space. Then, to compute the partition function of a torus bundle with gluing map U , we simply compute its trace.

For the purposes of our conjecture, we would like to compare with the asymptotics of the WRT invariant at rational levels, which as argued in Section 3.3 includes $\mathrm{SL}(2, \mathbb{C})$ connections. However, in contrast to the integral level case, in which only $\mathrm{SU}(2)$ flat connections appear, we generally do not have complete information on the single contributions to the asymptotic expansion coming from each $\mathrm{SL}(2, \mathbb{C})$ flat connection. This is of course not necessary in the context of our conjecture, but it could be instructive to analyze how a stronger statement may apply for these particular 3-manifolds.

We will focus on essentially two types of torus bundles, constructed via diffeomorphisms of the form $U = T^n$ and $U = T^n S$. The reason for these choices is that of studying 3-manifolds with different Thurston geometries, as mentioned at the start of this section. Torus bundles Σ_U have Nil geometry if U is a Dehn twist (such as T^n), while they have Sol geometry if U is an Anosov map (i.e. if U has two real distinct eigenvalues, such as $T^n S$ with $|n| > 2$). Note that the Σ_{T^n} torus bundles can also be understood as higher-genus plumbings (see Section 3.4.4), more concretely as plumbed manifolds constructed from graphs with one single vertex of genus 1 and weight n .

Nil torus bundles

We first focus on the nilmanifolds Σ_{T^n} . Let us consider n to be an odd prime, although the case of general n is straightforward to treat. The fundamental group simplifies to

$$\pi_1(\Sigma_{T^n}) = \langle x, y, z \mid [x, y] = 1, [z, y] = 1, xzx^{-1} = xy^n \rangle. \quad (3.4.10)$$

In what follows, we denote by X, Y and Z the images of the generators in $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SL}(2, \mathbb{F}_q)$, depending on the context.

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections can be determined analytically quite easily. The possible solutions (up to conjugacy) to the fundamental group relations are shown in Table 3.4.2, ordered by first fixing the conjugacy class of X . Note that there exist

Conjugacy class of X	Possible solutions
$X = \pm \mathbb{1}$	$Y = \mathbb{1}, \quad Z \in \mathrm{SL}(2, \mathbb{C})$ $Y = d(\zeta_n^\ell), \quad Z = d(*)$
$X = d(\lambda)$	$Y = \mathbb{1}, \quad Z = d(*)$ If $\lambda = i$: $Y = -\mathbb{1}, \quad Z = \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$ $Y = d(\zeta_n^\ell), \quad Z = d(*)$
$X = u(\epsilon, 1)$	$Y = \mathbb{1}, \quad Z = u(\pm 1, *)$

Table 3.4.2: Moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections on Σ_{T^n} .

families of flat connections with upper triangular images, meaning they are not $\mathrm{SU}(2)$ connections. Their CS invariants are however trivial. Indeed, this is true for all the flat connections with $Y = \mathbb{1}$: either they are abelian, or they can be continuously connected to abelian connections. Remember that for such connections the CS invariants are given by the linking form of Σ_{T^n} , which only depends on the torsion part of the homology. From the presentation of the fundamental group we find $\mathrm{Tor}H_1 = \mathbb{Z}/n\mathbb{Z} = \langle y \mid y^n = 1 \rangle$, implying that when $Y = \mathbb{1}$ the CS invariant vanishes.

Instead, the non-trivial values of the CS functional read

$$\mathrm{CS}[\gamma] \in \left\{ -\frac{\ell^2}{n} \mid \ell = 1, \dots, \frac{n-1}{2} \right\} \cup \left\{ \frac{n}{4} \right\}. \quad (3.4.11)$$

The first $\frac{n-1}{2}$ values correspond to flat connections where $Y = d(\zeta_n^\ell)$, while the last one corresponds to the unique (up to conjugacy) flat connection with $Y = -\mathbb{1}$. Note already that the latter will be invisible to the discrete gauge theory, since $12\mathrm{CS}[\gamma]$ vanishes in that case.

On the other hand, we do not have a way to obtain δ_γ for each individual flat connection. Instead, we will compute the partition function exactly at rational levels $\tau/\mathfrak{s}$ using the S and T matrices in Eq. (3.4.9), with $\xi = e^{\frac{2\pi i \mathfrak{s}}{\tau}}$, and study its asymptotic expansion as $\tau \rightarrow \infty$. This then determines the exponents Δ_Q associated to each CS value (see Eq. (3.3.18)). Assuming that $\mathfrak{s}$ and τ are coprime and odd, and that τ is fixed mod $\mathfrak{s}$, the asymptotic behavior of the WRT invariant reads

$$\text{WRT}(\Sigma_{T^n}; \xi) \sim e^{-\frac{\pi i n}{4}} \sqrt{\frac{i\tau}{2n\mathfrak{s}}} \sum_{\ell=0}^{n-1} e^{-2\pi i \frac{\xi}{\mathfrak{s}} \frac{\ell^2}{n}} \left(\sum_{\lambda=0}^{\mathfrak{s}-1} e^{-2\pi i \frac{\xi}{\mathfrak{s}} (\lambda^2 n + 2\lambda \ell)} \right) - \frac{1}{2} e^{2\pi i \frac{\xi}{\mathfrak{s}} \frac{n}{4}} \left(e^{2\pi i \frac{\xi}{\mathfrak{s}} \frac{\mathfrak{s}^2-1}{4}} \right). \quad (3.4.12)$$

This result is in agreement⁵ with the asymptotic expansion conjecture of Eq. (3.3.18). For the CS value of $-\frac{\ell^2}{n}$ we find $\Delta_Q = -1/2$, as well as for the trivial CS value. For $\text{CS} = \frac{n}{4}$, we find $\Delta_Q = 0$ instead.

$\text{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

The possible solutions to the fundamental group relations in the finite case are shown in Table 3.4.3, again ordered by first fixing the conjugacy class of X . Note we have excluded the special prime $p = n$ from our analysis. By direct comparison, all flat connections have their analog in the complex case if we assume $n \mid q^2 - 1$. Observe also that 4 always divides either one of $q - 1$ or $q + 1$, so there is always a single flat connection with $Y = -\mathbb{1}$ for any value of q . Moreover, we highlight the existence of connections with image in ZU , corresponding to the upper triangular $\text{SL}(2, \mathbb{C})$ connections.

The partition function is straightforward to compute, being equal to the trace of the gluing matrix T^n :

$$Z_{\text{DW}}(\Sigma_{T^n}) = q + 5 + (q \pm 1) \sum_{m=1}^{\frac{n-1}{2}} e^{-2\pi i \frac{12m^2}{n}}, \quad n \mid q \pm 1. \quad (3.4.13)$$

We find that the finite field partition function reproduces well the (twelfold) CS values in (3.4.11), in agreement with our conjecture. The leading powers of the polynomials $T_Q(q)$ above are also in agreement with the values of $-2\Delta_Q$, as expected.

Sol torus bundles

We now focus on the solvmanifolds $\Sigma_{T^n S}$. In this case, we will only require n to be an odd number for simplicity. The fundamental group reads

$$\pi_1(\Sigma_{T^n S}) = \langle x, y, z \mid [x, y] = 1, z x z^{-1} = x^n y^{-1}, z y z^{-1} = x \rangle. \quad (3.4.14)$$

Again, we denote by X, Y and Z the images of the generators in $\text{SL}(2, \mathbb{C})$ or $\text{SL}(2, \mathbb{F}_q)$ accordingly.

⁵Note that the function accompanying the phase factor $e^{2\pi i \frac{\xi}{\mathfrak{s}} \frac{n}{4}}$ is well defined for $\tau \bmod \mathfrak{s}$, since $\mathfrak{s}$ is odd.

Conjugacy class of X	Possible solutions
$X = \pm \mathbb{1}$	$Y = \mathbb{1}, \quad Z \in \mathrm{SL}(2, \mathbb{F}_q)$ $Y = d(\theta^{\ell \frac{q-1}{n}}), \quad Z = d(*)$ $Y = d'(\vartheta^{\ell \frac{q^2-1}{n}}), \quad Z = d'(*)$
$X = d(\lambda)$	$Y = \mathbb{1}, \quad Z = d(*)$ If $4 \mid q-1, \lambda = \theta^{\frac{q-1}{4}}: Y = -\mathbb{1}, \quad Z = \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$ If $n \mid q-1: Y = d(\theta^{\ell \frac{q-1}{n}}), \quad Z = d(*)$
$X = d'(\xi)$	$Y = \mathbb{1}, \text{ and } Z = d'(*)$ If $4 \mid q+1, \xi = \vartheta^{\frac{q^2-1}{4}}: Y = -\mathbb{1}, \quad Z = A \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix} A^{-1}$ If $n \mid q+1: Y = d'(\vartheta^{\ell \frac{q^2-1}{n}}), \quad Z = d'(*)$
$X = \epsilon u_\tau$	$Y = \mathbb{1}, \text{ and } Z = u(\pm 1, *)$

Table 3.4.3: Moduli space of $\mathrm{SL}(2, \mathbb{F}_q)$ connections on Σ_{T^n} . Here A is a matrix in $\mathrm{SL}(2, \mathbb{F}_{q^2})$ diagonalizing $d'(\xi)$, and $\eta \in \mathbb{F}_{q^2}^\times$ such that $Z \in \mathrm{SL}(2, \mathbb{F}_q)$.

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

Again in this case the moduli space can be determined analytically. The possible solutions (up to conjugacy) to the fundamental group relations are shown in Table 3.4.4, ordered by first fixing the conjugacy class of X . Just as in the Nil case, we can again argue that the flat connections with upper triangular images, as well as those with $X = \mathbb{1}$, have trivial CS invariant. Following the same line of thought, these connections are either abelian or continuously connected to abelian connections, hence their CS invariants are given by the linking form of $\Sigma_{T^n S}$. From the presentation one finds $\mathrm{Tor} H_1 = \mathbb{Z}/(n-2)\mathbb{Z} = \langle x \mid x^{n-2} = 1 \rangle$, which allows us to conclude that their CS values vanish. The non-trivial values of the CS functional on $\mathrm{SU}(2)$ connections are instead given as follows:

$$\mathrm{CS}[\gamma] \in \left\{ -\frac{\ell^2}{n-2} \mid \ell = 1, \dots, \frac{n-3}{2} \right\} \cup \left\{ -\frac{\ell^2}{n+2} \mid \ell = 1, \dots, \frac{n+1}{2} \right\}, \quad (3.4.15)$$

Conjugacy class of X	Possible solutions
$X = \pm \mathbb{1}$	If $X = \mathbb{1}$: $Y = \mathbb{1}$, $Z \in \mathrm{SL}(2, \mathbb{C})$
$X = d(\lambda)$	If $\lambda = \zeta_{n-2}^\ell$: $Y = X$, $Z = d(*)$ If $\lambda = \zeta_{n+2}^\ell$: $Y = X^{-1}$, $Z = \begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$
$X = u(\epsilon, 1)$	If $\epsilon = 1$: $Y = u(1, \alpha^{-2})$, $Z = u(\alpha, *)$, with $\alpha^2 + \alpha^{-2} = n$

Table 3.4.4: Moduli space of flat $\mathrm{SL}(2, \mathbb{C})$ connections on $\Sigma_{T^n S}$.

the first group corresponding to flat connections where $X = d(\zeta_{n-2}^\ell)$, and the second group to those where $X = d(\zeta_{n+2}^\ell)$.

As for the leading order exponents, it is known that for all $\mathrm{SU}(2)$ connections one has $\delta_\gamma = 0$ (see Prop. 5.6(a) in [90]), but unfortunately we cannot determine them for $\mathrm{SL}(2, \mathbb{C})$ connections. We then proceed as in the Nil case by computing the large τ behavior of the WRT invariant for rational levels. With the same assumptions — namely that τ and $\mathfrak{s}$ are coprime and odd, and that τ is fixed mod $\mathfrak{s}$ — we find

$$\mathrm{WRT}(\Sigma_{T^n S}; \xi) \sim e^{-\frac{\pi i n}{4}} \sum_{\pm} \pm \frac{1}{2\sqrt{i(n \pm 2)}} \sum_{\ell=0}^{(n \pm 2)-1} e^{-2\pi i \frac{\xi}{\mathfrak{s}} \frac{\ell^2}{n \pm 2}} \left(\frac{1}{\sqrt{\mathfrak{s}}} \sum_{\lambda=0}^{\mathfrak{s}-1} e^{-2\pi i \frac{\xi}{\mathfrak{s}} (\lambda^2(n+2) + 2\lambda\ell)} \right), \quad (3.4.16)$$

which again is in agreement with the asymptotic expansion conjecture of Eq. (3.3.18). From here we read that $\Delta_Q = 0$ for all CS values.

$\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

In the finite setting, we expect the existence of flat connections to depend on various divisibility conditions between $n \pm 2$ and $q \pm 1$. Indeed, let us define the quantities

$$\begin{aligned} \kappa &= \gcd(q-1, n-2), & \kappa' &= \gcd(q-1, n+2), \\ \tilde{\kappa} &= \gcd(q+1, n-2), & \tilde{\kappa}' &= \gcd(q+1, n+2). \end{aligned} \quad (3.4.17)$$

We show the possible solutions to the fundamental group relations in Table 3.4.5. All abelian flat connections have their analog in the complex case whenever both $n-2$ and $n+2$ divide q^2-1 . As for the upper triangular ones, their total number (either 0, 2 or 4) will depend on the specific values of n and q , which govern how many solutions exist for the equation $\alpha^2 + \alpha^{-2} = n$ in $\mathbb{F}_q^\times$. In contrast, in the complex case there always exist 4 distinct (and real) solutions.

Conjugacy class of X	Possible solutions
$X = \pm \mathbb{1}$	If $X = \mathbb{1}$: $Y = \mathbb{1}$, $Z \in \mathrm{SL}(2, \mathbb{F}_q)$
$X = d(\lambda)$	If $\kappa \neq 1$, $\lambda = \theta^{\ell \frac{q-1}{\kappa}}$: $Y = X$, $Z = d(*)$ If $\kappa' \neq 1$, $\lambda = \theta^{\ell \frac{q-1}{\kappa'}}$: $Y = X^{-1}$, $Z = \begin{pmatrix} 0 & * \\ -\kappa^{-1} & 0 \end{pmatrix}$
$X = d'(\xi)$	If $\tilde{\kappa} \neq 1$, $\xi = \vartheta^{\ell \frac{q^2-1}{\tilde{\kappa}}}$: $Y = X$, $Z = d'(*)$ If $\tilde{\kappa}' \neq 1$, $\xi = \vartheta^{\ell \frac{q^2-1}{\tilde{\kappa}'}}$: $Y = X^{-1}$, $Z = A \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix} A^{-1}$
$X = \epsilon u(1, a)$	If $\epsilon = 1$: $Y = u(1, a\alpha^{-2})$, $Z = u(\alpha, *)$, with $\alpha^2 + \alpha^{-2} = n$

Table 3.4.5: Moduli space of $\mathrm{SL}(2, \mathbb{F}_q)$ connections on $\Sigma_{T^n S}$. Here A is a matrix in $\mathrm{SL}(2, \mathbb{F}_{q^2})$ diagonalizing $d'(\xi)$, and $\eta \in \mathbb{F}_{q^2}^\times$ such that $Z \in \mathrm{SL}(2, \mathbb{F}_q)$.

We now compute the finite field partition function by taking the trace of the gluing matrix $T^n S$:

$$\begin{aligned}
Z_{\mathrm{DW}}(\Sigma_{T^n S}) = & 1 + \sum_{\ell=1}^{\frac{\kappa-1}{2}} e^{-2\pi i \frac{n-2}{\kappa} \frac{12\ell^2}{\kappa}} + \sum_{\ell'=1}^{\frac{\kappa'-1}{2}} e^{-2\pi i \frac{n+2}{\kappa'} \frac{12\ell'^2}{\kappa'}} \\
& + \sum_{\tilde{\ell}=1}^{\frac{\tilde{\kappa}-1}{2}} e^{-2\pi i \frac{n-2}{\tilde{\kappa}} \frac{12\tilde{\ell}^2}{\tilde{\kappa}}} + \sum_{\tilde{\ell}'=1}^{\frac{\tilde{\kappa}'-1}{2}} e^{-2\pi i \frac{n+2}{\tilde{\kappa}'} \frac{12\tilde{\ell}'^2}{\tilde{\kappa}'}} + |\{\alpha \in \mathbb{F}_q^\times \mid \alpha^2 + \alpha^{-2} = n\}|. \quad (3.4.18)
\end{aligned}$$

This expression exactly agrees with our conjecture in the case where both $n \pm 2$ divide $q^2 - 1$, as all factors defined in Eq. (3.4.17) equal $n \pm 2$ accordingly. Otherwise, the roots of unity corresponding to each flat connection γ have order a divisor of $o(\gamma)$ instead. Of course, this is to be seen as a slight generalization of our conjecture to the case where $o(\gamma)$ does not exactly divide $q^2 - 1$. In any case, such divisibility conditions do not change the leading factors of q corresponding to each Chern-Simons value — all of them go as $T_Q(q) \sim 1$, as expected from the continuous case.

3.4.3 Seifert fibered homology spheres

Seifert fibered homology spheres⁶ over S^2 constitute a rich family of homology spheres parameterized by a list of pairwise coprime integers. The Poincaré homology sphere and the Brieskorn homology spheres are particular cases of these manifolds.

A Seifert fibered space over S^2 is denoted by $M = M(b; (a_1, b_1), \dots, (a_n, b_n))$. The data $(b; (a_1, b_1), \dots, (a_n, b_n))$ are called the Seifert invariants of M , while n is the number of exceptional fibers. The 3-manifolds M can be obtained by performing rational Dehn surgery on the link in Figure 3.4.1.

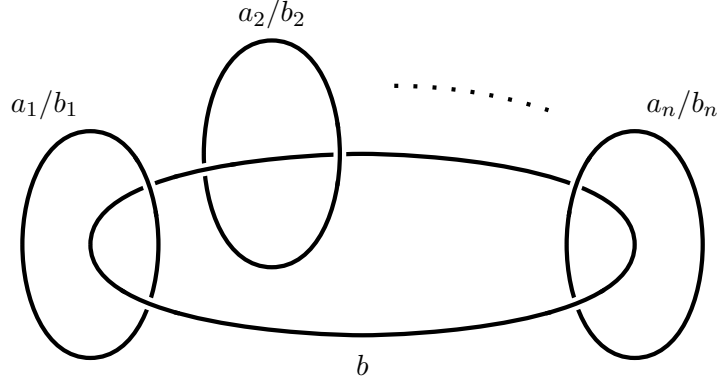


Figure 3.4.1: A surgery presentation of the manifold $M(b; (a_1, b_1), \dots, (a_n, b_n))$.

Such manifolds are not in general homology spheres, but one can check that $H_1(M, \mathbb{Z})$ vanishes if and only if

$$\sum_{i=1}^n \frac{b_i}{a_i} = \frac{1}{a_1 \cdots a_n} + b. \quad (3.4.19)$$

This implies that the a_i are relatively prime pairwise and the b_i are determined modulo a_i . Moreover, b can be set to zero without loss of generality. Thus, the previous condition together with the values of a_i , uniquely determines the homology 3-sphere $M = \Sigma(a_1, \dots, a_n)$. Notice that we can always restrict $a_2, \dots, a_n$ to be odd, so that $b_2, \dots, b_n$ can be chosen to be even. Furthermore, the fundamental group has the following presentation:

$$\pi_1(\Sigma(a_1, \dots, a_n)) = \langle x_1, \dots, x_n, h \mid x_1 \cdots x_n = 1, x_i^{a_i} h^{-b_i} = 1, [x_i, h] = 1, i = 1, \dots, n \rangle, \quad (3.4.20)$$

which can be deduced from the surgery presentation.

Much is known about the $SU(2)$ Chern-Simons theory on Seifert homology spheres. It was shown in [94] that the moduli space of $SU(2)$ flat connections consists of

⁶The interested reader can find more details in [93].

even-dimensional components of dimension at most $2n - 6$. The $\mathrm{SL}(2, \mathbb{C})$ theory has also been studied; in [95] it was shown that the set of connected components of the moduli space of non-trivial $\mathrm{SL}(2, \mathbb{C})$ -flat connections $\mathcal{M}^*(M, \mathrm{SL}(2, \mathbb{C}))$ is one-to-one with the set $L(a_1, \dots, a_n)$ defined as follows. Its elements are n -tuples of at least three non-vanishing integers $\vec{l} = (l_1, \dots, l_n)$ such that $0 \leq l_1 \leq a_1$ and $0 \leq l_i \leq (a_i - 1)/2$ for all $i = 2, \dots, n$. Let us give a quick idea of how this correspondence works, as the same line of reasoning will be repeated when looking at the case over finite fields. For each $\vec{l} \in L(a_1, \dots, a_n)$ one can show that there exists a non-trivial representation $\gamma_{\vec{l}}: \pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{C})$ such that

$$\gamma_{\vec{l}}(h) = (-1)^{l_1}, \quad \gamma_{\vec{l}}(x_1) \in \left[d(e^{\pi i l_1 / a_1}) \right], \quad \gamma_{\vec{l}}(x_i) \in \left[d(e^{2\pi i l_i / a_i}) \right] \text{ for } i \neq 1, \quad (3.4.21)$$

where square brackets denote the conjugacy class of a matrix in $\mathrm{SL}(2, \mathbb{C})$. The opposite implication is straightforward once we notice that M being an integer homology sphere forces the image of h to lie in the center of $\mathrm{SL}(2, \mathbb{C})$.

The set $L(a_1, \dots, a_n)$ also characterizes the values of the Chern-Simons action. Indeed, a further result of [95] states that

$$\mathrm{CS}[\gamma_{\vec{l}}] = -\frac{a_1 \cdots a_n}{4} \left(\frac{l_1}{a_1} + 2 \sum_{i=2}^n \frac{l_i}{a_i} \right)^2 \pmod{1} \quad (3.4.22)$$

where $\gamma_{\vec{l}}$ is the flat connection corresponding to $\vec{l}$. We remark that the computation of the CS functional is based on the method introduced by Kirk and Klassen in [92], generalized to $\mathrm{SL}(2, \mathbb{C})$ -flat connections. Furthermore, the leading order of k in the asymptotic expansion can be obtained by studying the connected components of flat connections in $\mathcal{M}^*(M, \mathrm{SL}(2, \mathbb{C}))$. In particular, their dimension is $2n - 6 - 2m$ where m is the number of zero components of the vector $\vec{l}$ corresponding to a flat connection in that connected component.

Seifert spaces with three exceptional fibers

For simplicity, we restrict to the case of three exceptional fibers. The manifold $\Sigma(a_1, a_2, a_3)$ is also known as a Brieskorn homology sphere, which was originally defined as the intersection between the unit sphere in $\mathbb{C}^3$ and the complex algebraic surface $z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0$.

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The family of Brieskorn homology spheres is very special since the moduli space of non-trivial flat connections $\mathcal{M}^*(\Sigma(a_1, a_2, a_3), \mathrm{SL}(2, \mathbb{C}))$ is finite, with cardinality given by the $\mathrm{SL}(2, \mathbb{C})$ Casson invariant introduced by Curtis [96, 97]:

$$\lambda_{\mathrm{SL}(2, \mathbb{C})}(M) = \frac{(a_1 - 1)(a_2 - 1)(a_3 - 1)}{4}. \quad (3.4.23)$$

For reference, in [94] Fintushel and Stern describe a combinatorial formula for the cardinality of the $\mathrm{SU}(2)$ moduli space. Consider triples of integers $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that $0 < n_i < a_i$ for $i = 1, 2, 3$ and introduce the quantities

$$\sigma_+ = \left| \left\{ (n_1, n_2, n_3) : 0 < \frac{n_1}{a_1} + \frac{n_2}{a_2} + \frac{n_3}{a_3} < 1 \pmod{2} \right\} \right|, \quad (3.4.24)$$

$$\sigma_- = \left| \left\{ (n_1, n_2, n_3) : -1 < \frac{n_1}{a_1} + \frac{n_2}{a_2} + \frac{n_3}{a_3} < 0 \pmod{2} \right\} \right|. \quad (3.4.25)$$

Then, the cardinality of $\mathcal{M}^*(\Sigma(a_1, a_2, a_3), \mathrm{SU}(2))$ is given by $(\sigma_- - \sigma_+)/4$.

We can apply this formula in two simple examples. For the Poincaré homology sphere, i.e. $M = \Sigma(2, 3, 5)$, we find that both $\mathcal{M}^*(M, \mathrm{SL}(2, \mathbb{C}))$ and $\mathcal{M}^*(M, \mathrm{SU}(2))$ contain two points. However, the first non-trivial example is $M = \Sigma(2, 3, 7)$, which still has only two non-trivial inequivalent $\mathrm{SU}(2)$ flat connections, while it has an additional point in $\mathcal{M}^*(M, \mathrm{SL}(2, \mathbb{C}))$, corresponding to an orbit of $\mathrm{SL}(2, \mathbb{R})$ flat connections as argued in [95]. Thus, this example provides a great benchmark to test whether the $\mathrm{SL}(2, \mathbb{F}_q)$ DW theory detects this additional contribution.

Thus, we have all the ingredients to compare with the asymptotic expansion of the $\mathrm{SL}(2, \mathbb{C})$ CS partition function on a Brieskorn homology sphere $\Sigma(a_1, a_2, a_3)$. Indeed, the value of the action on critical points is easily evaluated from (3.4.22). Moreover, the leading order of k is $k^{-3/2}$ for the trivial connection and k^0 for the other flat connections, see [74].

$\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

Consider now the DW theory on Brieskorn homology spheres $\Sigma(a_1, a_2, a_3)$. Suppose for simplicity that each a_i is prime. While we will not use the lattice gauge theory description of Dijkgraaf-Witten theory, we still believe that it is interesting to study the moduli space of flat connections $\mathrm{Hom}(\pi_1(\Sigma(a_1, a_2, a_3)), \mathrm{SL}(2, \mathbb{F}_q))$. Remarkably, the same ideas that appeared in [95] and were presented above for the $\mathrm{SL}(2, \mathbb{C})$ case can be generalized to $\mathrm{SL}(2, \mathbb{F}_q)$ with a bit of caution. Then, it is not too hard to prove that

$$|\mathrm{Hom}(\pi_1(\Sigma(a_1, a_2, a_3)), \mathrm{SL}(2, \mathbb{F}_q))| = 1 + \frac{(\gcd(a_1, q^2 - 1) - 1)(\gcd(a_2, q^2 - 1) - 1)(\gcd(a_3, q^2 - 1) - 1)}{4} |\mathrm{SL}(2, \mathbb{F}_q)| \quad (3.4.26)$$

away from special primes, i.e. q must not be a power of any prime number in the decomposition of $a_1 a_2 a_3$. Notice that if $a_1 a_2 a_3 \mid q^2 - 1$ then the factor multiplying $|\mathrm{SL}(2, \mathbb{F}_q)|$ is equal to the $\mathrm{SL}(2, \mathbb{C})$ Casson invariant $\lambda_{\mathrm{SL}(2, \mathbb{C})}$.

Brieskorn homology spheres, and actually all Seifert fibrations over S^2 , are plumbed manifolds with star-shaped plumbing graphs. For example, the plumbing graph for the manifold $M(b; (a_1, b_1), \dots, (a_n, b_n))$ can be easily deduced from Figure 3.4.1 and it is shown in Figure 3.4.2. Here, we admit vertices with rational weights, which have to be

understood as a sequence of l vertices with integer weights $m_1, \dots, m_l$ where $[m_1, \dots, m_l]$ is the continued fraction expression of the rational weight, as shown in Figure 3.4.3. Recall that given a rational number a/b , we say that $[m_1, \dots, m_l]$ is a continued fraction for it if

$$\frac{a}{b} = m_l - \frac{1}{m_{l-1} - \frac{1}{\dots - \frac{1}{m_1}}}. \quad (3.4.27)$$

The direct consequence of this fact is that we can compute the partition function of any Brieskorn homology sphere by simply applying the formula in Equation (1.2.50).

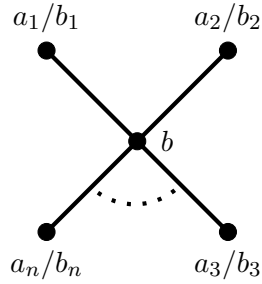


Figure 3.4.2: The plumbing graph of $M(b; (a_1, b_1), \dots, (a_n, b_n))$.

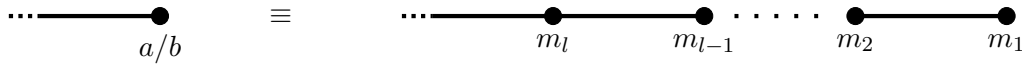


Figure 3.4.3: When the weight of a vertex is rational, the vertex gets replaced with a sequence of vertices corresponding to its continued fraction expression.

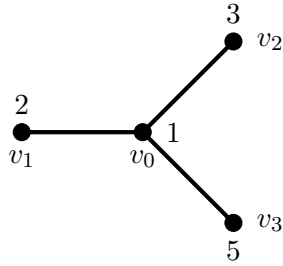


Figure 3.4.4: The plumbing graph of the Poincaré homology sphere. We denote by s_i the state corresponding to the vertex v_i .

Let us consider the Poincaré homology sphere $\Sigma(2, 3, 5)$ as an example of how to carry on the computation of the partition function. Its plumbing graph is shown in Figure 3.4.4

and it allows us to find

$$Z_{\text{DW}}(\Sigma(2, 3, 5)) = \sum_{s_0, s_1, s_2, s_3} S_{s_0, s_1} S_{s_0, s_2} S_{s_0, s_3} S_{0, s_0}^{-1} S_{0, s_1} S_{0, s_2} S_{0, s_3} T_{s_0, s_0} T_{s_1, s_1}^2 T_{s_2, s_2}^3 T_{s_3, s_3}^5. \quad (3.4.28)$$

Being a homology sphere, the state corresponding to the trivalent vertex is forced to be of the form $s_0 = (\epsilon \mathbb{1}, \chi)$, ϵ being a sign. This is in agreement with the fact that the image of the generator $h \in \pi_1(\Sigma(2, 3, 5))$ must belong to the center of $\text{SL}(2, \mathbb{F}_q)$. Then, if we remove the trivial contribution corresponding to having all the s_i of the form $(\pm \mathbb{1}, \chi)$, we find that s_1, s_2 and s_3 have to be the form (a, ψ) , where a is a diagonal or codiagonal conjugacy class representative, and ψ is a character of T or T' , respectively (notice that this is what fails when $p = 2, 3, 5$). Hence, the partition function can be written as:

$$Z_{\text{DW}}(\Sigma(2, 3, 5)) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + \frac{|\text{SL}(2, \mathbb{F}_q)|}{q_{\pm}^6} \sum_{\epsilon = \pm} \sum_{a_1, a_2, a_3} \varepsilon_{a_1}(a_1)^2 \varepsilon_{a_2}(a_2)^3 \varepsilon_{a_3}(a_3)^5 \times \\ \sum_{\chi} \frac{\chi(a_1) \chi(a_2) \chi(a_3) \chi(\epsilon \mathbb{1})}{\chi(\mathbb{1})^2} \sum_{\psi_1, \psi_2, \psi_3} \psi_1(\epsilon a_1^2) \psi_2(\epsilon a_2^3) \psi_3(\epsilon a_3^5), \quad (3.4.29)$$

where $q_{\pm}$ is either $q + 1$ or $q - 1$. The sum over ψ_1 forces a_1 to satisfy $a_1^2 = \epsilon \mathbb{1}$, however the case $\epsilon = 1$ is ruled out since $a_1 \neq \mathbb{1}, -\mathbb{1}$. We get similar conditions for the other a_i . Then, we obtain:

$$Z_{\text{DW}}(\Sigma(2, 3, 5)) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + \frac{|\text{SL}(2, \mathbb{F}_q)|}{q_{\pm}^3} \sum_{a_1, a_2, a_3} \varepsilon_{a_1}(a_1)^2 \varepsilon_{a_2}(a_2)^3 \varepsilon_{a_3}(a_3)^5 \times \\ \delta(a_1^2 = -\mathbb{1}) \delta(a_2^3 = -\mathbb{1}) \delta(a_3^5 = -\mathbb{1}) \sum_{\chi} \frac{\chi(a_1) \chi(a_2) \chi(a_3) \chi(-\mathbb{1})}{\chi(\mathbb{1})^2}. \quad (3.4.30)$$

The evaluation of the sum over the $\text{SL}(2, \mathbb{F}_q)$ characters is quite tricky and we discuss it in Appendix C. There we prove that

$$\frac{|\text{SL}(2, \mathbb{F}_q)|}{q_{\pm}^3} \sum_{\chi} \frac{\chi(a_1) \chi(a_2) \chi(a_3) \chi(-\mathbb{1})}{\chi(\mathbb{1})^2} = 1. \quad (3.4.31)$$

Plugging this result into the expression above, the partition function for the Poincaré homology sphere is

$$Z_{\text{DW}}(\Sigma(2, 3, 5)) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + (e^{2\pi i/10} + e^{-2\pi i/10}) \delta(5|q^2 - 1). \quad (3.4.32)$$

We now consider the general case of the Brieskorn homology sphere $\Sigma(a_1, a_2, a_3)$, assuming only that the a_i are prime. This implicitly fixes b_1, b_2, b_3 to satisfy $b_1/a_1 +$

$b_2/a_2 + b_3/a_3 = 1/(a_1 a_2 a_3)$, which makes the manifold a homology sphere. For simplicity, we will only consider the case of $a_1 a_2 a_3 \mid q^2 - 1$. A similar computation as in the case of the Poincaré homology sphere yields

$$Z_{\text{DW}}(\Sigma(a_1, a_2, a_3)) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + \sum_{k_1, k_2, k_3} e^{-6\pi i \left(\frac{b_1}{a_1} k_1^2 + \frac{b_2}{a_2} k_2^2 + \frac{b_3}{a_3} k_3^2 \right)} + \sum_{m_1, m_2, m_3} e^{-6\pi i \left(\frac{b_1}{a_1} m_1^2 + \frac{b_2}{a_2} m_2^2 + \frac{b_3}{a_3} m_3^2 \right)}, \quad (3.4.33)$$

where the first sum is over odd k_i with $1 \leq k_1 \leq a_1 - 1$ and $1 \leq k_i \leq a_i - 2$ for $i = 2, 3$, while the second sum is over even m_i with $2 \leq m_i \leq a_i - 1$ for $i = 1, 2, 3$. Notice that when $a_1 = 2$, the second sum does not contribute.

The phases appearing in the DW partition function have to be compared with the values of the CS functional in (3.4.22), which can be written as:

$$12\text{CS}(l_1, l_2, l_3) = -3a_1 a_2 a_3 \left(\frac{l_1}{a_1} + \frac{2l_2}{a_2} + \frac{2l_3}{a_3} \right)^2 \mod 1, \quad (3.4.34)$$

with $\vec{l} \in L(a_1, a_2, a_3)$. The correspondence between these two sets of values is not evident, but it is easy to check. Hence, this result is in agreement with the statement of our conjecture, since the leading power of q also matches with $-2\Delta_Q$.

A simple example with four exceptional fibers

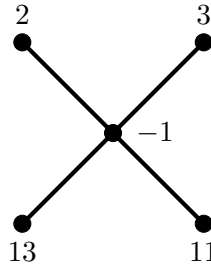


Figure 3.4.5: The plumbing graph of $\Sigma(2, 3, 11, 13)$.

For completeness, we consider a simple case of a Seifert homology sphere with four fibers, namely $\Sigma(2, 3, 11, 13)$ with plumbing graph as in Figure 3.4.5. In the case of four exceptional fibers, the moduli space of flat connections contains connected components of dimension two, in addition to isolated points. Then, in the DW partition function we expect contributions of order q^2 from these flat connections.

The computation proceeds as in the case of three exceptional fibers with the exception that up to one of the four generators $x_i \in \pi_1(\Sigma(2, 3, 11, 13))$ can be chosen to be sent to

± 1 . Then, we obtain the following partition function:

$$Z_{\text{DW}}(\Sigma(2, 3, 11, 13)) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + 2 \sum_{l_3, l_4} e^{-6\pi i \left(\frac{l_3^2}{11} + \frac{l_4^2}{13} \right)} + \sum_{l_3, l_4} e^{-6\pi i \left(\frac{1}{2} + \frac{l_3^2}{11} + \frac{l_4^2}{13} \right)} + \sum_{l_4} e^{-6\pi i \left(\frac{1}{2} + \frac{l_4^2}{13} \right)} + \sum_{l_3} e^{-6\pi i \left(\frac{1}{2} + \frac{l_3^2}{11} \right)} + (q^2 + 1) \sum_{l_3, l_4} e^{-6\pi i \left(\frac{1}{2} + \frac{l_3^2}{11} + \frac{l_4^2}{13} \right)}, \quad (3.4.35)$$

where the sums are over odd l_3 and l_4 with $1 \leq l_3 \leq 9$ and $1 \leq l_4 \leq 11$. The phases are again in agreement with 12CS as one can check via (3.4.22). Moreover, the leading order of q in the prefactors of each contribution also matches with $-2\Delta_Q$ as expected.

Homology spheres that are not Seifert spaces

More general homology spheres may be constructed from plumbing graphs satisfying certain conditions — from (1.2.49) it is evident that M_Γ is an integer homology sphere if and only if all the $g_i = 0$ and $\det B_\Gamma = \pm 1$.

Let us now discuss a concrete example. Consider the plumbing graph Γ_1 shown in Figure 3.4.6. It is immediate to check that it is an integer homology sphere. Furthermore, it cannot be realized as a Seifert fibered space as its plumbing graph is in normal form and is not star-shaped, see [47]. Thus, it is an example of a manifold that does not admit a global geometry in the sense of Thurston's geometrization conjecture.

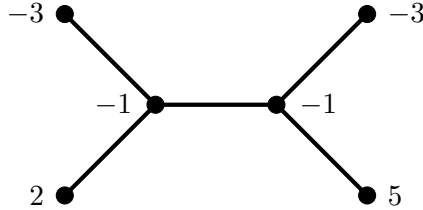


Figure 3.4.6: An example of a plumbed integer homology sphere that is not a Seifert manifold. Note its plumbing graph Γ_1 is not star-shaped.

The possible values of the Chern-Simons functional on flat connections were determined in [78] through a *finite-dimensional model* of analytically continued $\text{SU}(2)$ Chern-Simons theory. Here we list their results, organized into four different kinds of contributions:

- One contribution comes from the trivial flat connection, and has the usual leading order $\Delta_Q = 3/2$ as the manifold is a homology sphere.
- The second kind consists of flat connections such that $12\text{CS} \in \{1/26, 3/26, 9/26, 17/26, 23/26, 25/26\}$, with leading order in k given by $\Delta_Q = 0$.

- The third kind consists of flat connections such that $12\text{CS} \in \{6/35, 19/35, 24/35, 26/35, 31/35, 34/35\}$, with leading order in k given by $\Delta_Q = 0$.
- Finally, the fourth kind is such that $12\text{CS} \in \{1/10, 9/10\}$, and has leading order in k given by $\Delta_Q = -1/2$.

Remarkably, the last kind of contribution is part of a phenomenon that was first noticed in [78], namely the existence of the so-called *flat connections at infinity*. A flat connection at infinity is a discrete (or continuous) family of $\text{SL}(2, \mathbb{C})$ connections $\{\gamma_\varepsilon\}$ which is singular in the limit $\varepsilon \rightarrow \infty$, but the curvature tends to 0 and $\text{CS}[\gamma_\varepsilon]$ has a finite limit $\text{CS}[\gamma_\infty]$. In this work, the authors study some necessary conditions for their existence on manifolds obtained as ± 1 -surgery on knots. It can be proved that the fourth kind of contribution actually comes from flat connections at infinity, by explicitly constructing the family of $\text{SL}(2, \mathbb{C})$ flat connections $\{\gamma_\varepsilon\}$.

On the other hand, we can consider the $\text{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten partition function. If we suppose that $5 \cdot 7 \cdot 13 \mid q^2 - 1$, Equation (1.2.50) yields

$$Z_{\text{DW}}(M_{\Gamma_1}) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + \sum_{\substack{l=1 \\ \text{odd}}}^{11} e^{2\pi i(\frac{1}{2} - \frac{6}{13}l^2)} + \sum_{\substack{l=1 \\ \text{odd}}}^5 \sum_{\substack{l'=1 \\ \text{odd}}}^3 e^{2\pi i(\frac{4}{7}l^2 + \frac{2}{5}l'^2)}. \quad (3.4.36)$$

Notice that this is in agreement with our conjecture for the standard $\text{SL}(2, \mathbb{C})$ flat connections, but it does not detect flat connections at infinity.

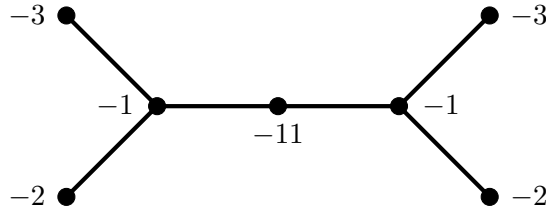


Figure 3.4.7: The plumbing graph Γ_2 of the 1-surgery on the right-handed granny knot $3_1 \# 3_1$.

We conclude this section with one last example. Consider the manifold obtained as the 1-surgery on the right-handed granny knot $3_1 \# 3_1$, whose plumbing graph Γ_2 is shown in Figure 3.4.7. This example was already considered in [78] where it was stated that it admits flat connections at infinity.

The following results are obtained by applying the finite-dimensional model of [78]. The asymptotic expansion gets contributions from:

- The trivial flat connection.

- A family of flat connections with $12\text{CS} \in \{0, 1/11, 3/11, 4/11, 5/11, 9/11\}$ and $\Delta_Q = -1/2$.
- Two additional families with $12\text{CS} \in \{1/2, 1/10, 9/10\}$ and $\Delta_Q = 0$.

After a detailed analysis, we found out that this manifold actually admits three flat connections at infinity. However, this is not in disagreement with [78] since, as the authors claim, only one of these actually contributes to the asymptotic expansion of the Chern-Simons theory.

The moduli spaces of $\text{SL}(2, \mathbb{C})$ flat connections has been also studied in the context of $\text{SL}(2, \mathbb{C})$ Floer cohomology of [98]. In particular, explicit description of the moduli space for surgeries on the granny knot can be found in [99]. In particular, for the $+1$ -surgery we expect 4 points and 5 copies of $\mathbb{C}^\times$, whereas above we listed a total of 6 points and 6 copies of $\mathbb{C}^\times$. It was already noted in [78] that this method does not detect flat connections at infinity. However, we can compare the two results to count the number of such connections.

Finally, we compute the DW partition function using (1.2.50). Assuming that $5 \cdot 11 \mid q^2 - 1$, we get

$$Z_{\text{DW}}(M_{\Gamma_2}) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} + (q \pm 1) \sum_{l=1}^5 e^{2\pi i \frac{l^2}{11}} + 2 \sum_{l=1,3} e^{2\pi i (\frac{1}{2} + \frac{3}{5}l^2)}, \quad \text{if } 11 \mid q \pm 1. \quad (3.4.37)$$

Notice that the factor 2 in the last sum corresponds to two families of connections with the same phase factor. This result is again in complete agreement with our conjecture.

3.4.4 Higher-genus plumbings

The simplest example of a higher-genus plumbing is given by a graph Γ consisting of one single vertex of genus $g > 1$ and weight n . The corresponding 3-manifold M is then a degree n -circle bundle over the Riemann surface Σ_g of genus g . Its fundamental group reads

$$\pi_1(M) = \langle a_1, \dots, a_g, b_1, \dots, b_g, h \mid \prod_{i=1}^g [a_i, b_i] = h^n, [a_i, h] = 1, [b_i, h] = 1 \rangle. \quad (3.4.38)$$

It is worth noting that these manifolds are all geometric of type $\widetilde{\text{SL}(2, \mathbb{R})}$. In what follows, we consider n to be an odd prime for simplicity.

The asymptotic behavior of the $\text{SU}(2)_k$ Chern-Simons partition function can be derived, for example, from the results for general Seifert fibered manifolds in [100]. Indeed, M can be represented as the Seifert manifold with base Σ_g and one single exceptional fiber of type $(1, n)$. Alternatively, one can use the finite-dimensional model of [78], which also includes $\text{SL}(2, \mathbb{C})$ flat connections.

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The set of all possible CS values on flat connections reads

$$\mathrm{CS} \in \left\{0, \frac{n}{4}\right\} \cup \left\{-\frac{a^2}{n} \mid a = 1, \dots, \frac{n-1}{2}\right\}. \quad (3.4.39)$$

The corresponding leading powers of k are given as follows. For the trivial CS value, $\Delta_Q = 3 - 3g$. The asymptotic analysis also reveals a subleading contribution of order $k^{2g - \frac{3}{2}}$, which can be singled out through the methods of [100], or via the finite-dimensional model in [78] (in which case it manifests itself as an additional Lefschetz thimble contributing to the term with trivial CS value in the asymptotic expansion). For the value $\mathrm{CS} = \frac{1}{4}$ we have $\Delta_Q = 3 - 3g$ as well, while for the last set of values $\Delta_Q = \frac{1}{2} - g$ instead.

 $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

We now assume that $n \mid q^2 - 1$ in order to obtain non-trivial results (hence $n \neq p$). The finite field partition function can be computed via Eq. (1.2.50) — in our case, we have

$$\begin{aligned} Z_{\mathrm{DW}}(M) &= \sum_{s \in \mathcal{H}(\mathbb{T}^2)} T_{s,s}^n S_{0,s}^{2-2g} \\ &= |G|^{2g-2} \left(1 + q^{2-2g} + \left(\frac{q-3}{2} + 2^{2g-1} \right) (q+1)^{2-2g} + \left(\frac{q-1}{2} + 2^{2g-1} \right) (q-1)^{2-2g} \right) \\ &\quad + |G|^{2g-2} \times \begin{cases} 1 + q^{2-2g} + (2^{2g-1} - 1)(q+1)^{2-2g} - 2^{2g-1}(q-1)^{2-2g}, & q \equiv 1 \pmod{4}, \\ 1 + q^{2-2g} + (2^{2g-1} - 1)(q-1)^{2-2g} - 2^{2g-1}(q+1)^{2-2g}, & q \equiv 3 \pmod{4}, \end{cases} \\ &\quad + (q \pm 1)^{2g-1} \sum_{a=1}^{\frac{n-1}{2}} e^{-2\pi i \frac{12a^2}{n}}, \quad n \mid q \pm 1. \end{aligned} \quad (3.4.40)$$

We have written the result in this particular way to highlight the contributions from each of the states $s = (a, \chi)$ coming from different conjugacy classes: the first, second, and third lines correspond to contributions from states with $a = \mathbb{1}$, $a = -\mathbb{1}$, and $a \in T$ or $a \in T'$ respectively (the latter depending on $o(a) \mid q - 1$ or $o(a) \mid q + 1$).

First of all, we find contributions of leading order q^{6g-6} in the first and second lines, agreeing with the asymptotics of the two components with CS value 0 and $\frac{n}{4}$ (which is trivial multiplied by 12) as per our conjecture. In the first line we can further identify particular contributions of order q^{4g-3} , reminiscent of the subleading contributions of order $k^{2g - \frac{3}{2}}$ to the term with trivial CS value in the asymptotic expansion. Finally, the last line corresponds to the remaining values of the CS invariant, with a coefficient of leading order q^{2g-1} as expected.

3.4.5 An example with $\mathbb{H}^2 \times \mathbb{R}$ geometry

In the previous sections, we have considered six out of the eight types of geometric structures. Except for the hyperbolic case, only the geometry $\mathbb{H}^2 \times \mathbb{R}$ is left. A large class of manifolds with this geometric structure is given by Seifert fibered spaces over S^2 with negative orbifold Euler characteristic $\chi = 2 - \sum_i (1 - 1/a_i)$ and vanishing Seifert Euler number $E = b + \sum_i b_i/a_i$.

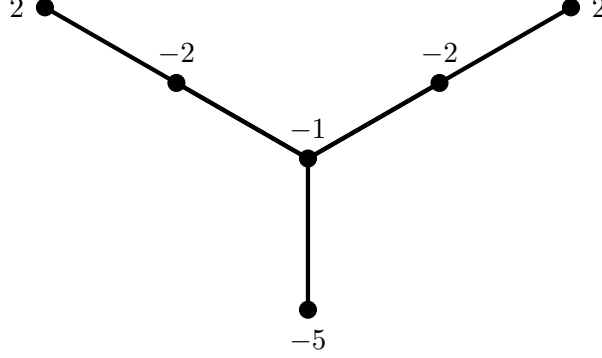


Figure 3.4.8: The plumbing graph of $M(-1; (5, -2), (5, -2), (5, -1))$.

We focus on the manifold $M = M(-1; (5, -2), (5, -2), (5, -1))$, schematically given by the plumbing graph in Figure 3.4.8. Notice that the fundamental group, which can be deduced from the plumbing graph, is given by

$$\pi_1(M) = \langle x, y, z, h \mid x^5 h^2 = y^5 h^2 = z^5 h = xyzh = [x, h] = [y, h] = [z, h] = 1 \rangle, \quad (3.4.41)$$

and its abelianization gives the first homology group:

$$H_1(M, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}. \quad (3.4.42)$$

A crucial property of manifolds with $E = 0$ is that their linking matrix is singular, as $b_1 > 0$. Thus, we will not be able to easily compute the asymptotics of the $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons partition function from the finite-dimensional model of [78].

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

The $\mathrm{SU}(2)$ Chern-Simons partition functions of Seifert fibered spaces are pretty well-known in the literature, see for example [101]. Unfortunately, $\mathrm{SL}(2, \mathbb{C})$ did not get a similar interest. So far, we managed to bypass this issue by employing the finite-dimensional method of [78]. However, this fails when the linking matrix is singular.

In [100], general formulas for the asymptotic expansion of the $\mathrm{SU}(2)$ partition function of Seifert manifolds appear. From these, we can deduce that $\mathrm{SU}(2)$ flat connections contribute

with a phase factor given by $\text{CS} \in \{0, 1/5, 4/5\}$ and leading order in k that is $\Delta_Q = 0$ for each Chern-Simons value. We speculate that the same argument given for $G = \text{SU}(2)$ also applies to $G = \text{SL}(2, \mathbb{C})$. Then, the genuine $\text{SL}(2, \mathbb{C})$ flat connections, which do not arise from $\text{SU}(2)$ flat connections, are expected to contribute with $\text{CS} \in \{2/5, 3/5\}$ and $\Delta_Q = 0$. We remark that the proof in [100] is based on [102] and on the method introduced by Kirk and Klassen in [92]. Generalizations of the latter to $\text{SL}(2, \mathbb{C})$ already appeared in [95].

$\text{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

In this example, we witness a new peculiar feature of the theory. Already at the untwisted level, there is a qualitative difference between the cases $5 \mid q+1$ and $5 \nmid q+1$, which is a novelty. Indeed, up to a normalization constant, the untwisted partition function for a generic prime power q is:

$$|\text{Hom}(\pi_1(M), \text{SL}(2, \mathbb{F}_q))| = \begin{cases} |G|, & 5 \nmid q^2 - 1, \\ 7|G|, & 5 \mid q+1, \\ 31|G|, & 5 \mid q-1. \end{cases} \quad (3.4.43)$$

This feature has consequences even in the twisted theory. Indeed, Equation (1.2.50) yields

$$\begin{aligned} Z_{\text{DW}}(M) = & \frac{2}{|G|} + \frac{2(q^2 - 1)}{|G|} + \frac{12}{q \pm 1}(\zeta_5^2 + \zeta_5^{-2}) \\ & + \begin{cases} \frac{q-3}{2(q-1)}, & 5 \nmid q-1, \\ \frac{q-11}{q-1} \cdot \frac{5}{4}(\zeta_5^2 + \zeta_5^{-2}), & 5 \mid q-1 \end{cases} + \begin{cases} \frac{q-1}{2(q+1)}, & 5 \nmid q+1, \\ \frac{q-9}{q+1} \cdot \frac{5}{4}(\zeta_5^2 + \zeta_5^{-2}), & 5 \mid q+1 \end{cases} \\ & + \left(\sum_{l_i=1,2} e^{2\pi i \left(-\frac{l_1^2}{5} - \frac{l_2^2}{5} + 2\frac{l_3^2}{5} \right)} + \sum_{l_i=1,3} e^{2\pi i \left(\frac{l_1^2}{5} + \frac{l_2^2}{5} - 2\frac{l_3^2}{5} \right)} \right) \begin{cases} \frac{2q-1}{q-1}, & 5 \mid q-1, \\ \frac{1}{q+1}, & 5 \mid q+1, \end{cases} \end{aligned} \quad (3.4.44)$$

which results in

$$Z_{\text{DW}}(M) = \begin{cases} \frac{17q-7}{2(q-1)} + 2(\zeta_5 + \zeta_5^{-1}) + \frac{37q-47}{4(q-1)}(\zeta_5^2 + \zeta_5^{-2}), & 5 \mid q-1, \\ \frac{q+11}{2(q+1)} + 2(\zeta_5 + \zeta_5^{-1}) + \frac{5q-1}{4q+1}(\zeta_5^2 + \zeta_5^{-2}), & 5 \mid q+1. \end{cases} \quad (3.4.45)$$

Thus, also this case is in agreement with our conjecture, as $\Delta_Q = 0$ for each value.

3.4.6 Hyperbolic manifold counterexample

At last, we consider an example — or even better, a counterexample — of a hyperbolic 3-manifold. In particular, we study the $-1/2$ surgery on the figure-eight knot 4_1 , depicted in Figure 3.4.9. Such a manifold is a closed, hyperbolic, integer homology sphere $M = S_{-1/2}(4_1)$.

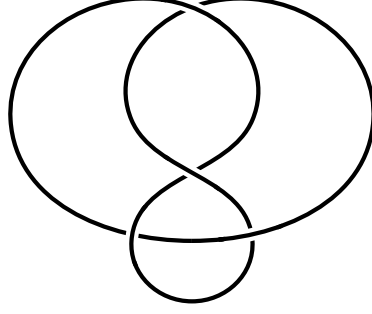


Figure 3.4.9: The figure eight knot.

We remark that plumbed manifolds are not hyperbolic, meaning that their geometric decomposition does not contain a hyperbolic component. Hence, the partition function of hyperbolic manifolds is not computed via S and T matrices as in (1.2.50). For how to deal with this problem, we refer back to the discussion at the end of Section 1.2.2. In short, we fix a triangulation of $S_{-1/2}(4_1)$ and bypass the necessity for a branched triangulation by choosing a cocycle representing $\omega = 24\beta^{-1}c_2(\rho)$. Indeed, it turns out that a representative satisfying condition (1.2.38) can always be found for $\omega = 24\beta^{-1}c_2(\rho)$, but not for our usual choice of (3.2.13). Further details can be found in Appendix B.

Asymptotic expansion of $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory

Let us study the moduli space of $\mathrm{SL}(2, \mathbb{C})$ -flat connections. In order to do so, consider the presentation of the fundamental group of the manifold:

$$\pi_1(S_{-1/2}(4_1)) = \langle x, y \mid x = \omega^{-1}\nu\omega^{-1}\nu, y = \nu\omega^{-1}\nu\omega^{-1} \rangle, \quad (3.4.46)$$

where $\omega = [x^{-1}, y]$ and $\nu = [x, y^{-1}]$ are such that $\omega^{-1}\nu$ is a canonical choice for the longitude of the knot complement $S^3 \setminus 4_1$. Non-trivial elements of $\mathrm{Hom}(\pi_1(S_{-1/2}(4_1)), \mathrm{SL}(2, \mathbb{C}))$ are in correspondence with solutions of the equation

$$\lambda^8 + \lambda^{-8} - 2 = \lambda^4 + \lambda^{-4} + \lambda + \lambda^{-1}, \quad \lambda \neq -1 \quad (3.4.47)$$

over $\mathbb{C}$. This is just the request that the A -polynomial [103] of the figure-eight knot,

$$A(\lambda, \mu) = -2 + \mu^4 + \mu^{-4} - \mu^2 - \mu^{-2} - \lambda - \lambda^{-1}, \quad (3.4.48)$$

vanishes when we enforce the gluing condition $\lambda^2 = \mu$. Note that if λ solves the equation, then also λ^{-1} does. Moreover, we have to discard the solution $\lambda = -1$, which has multiplicity 2, leaving us with a degree 14 polynomial equation. Thus, we find 14 solutions in 7 pairs (λ, λ^{-1}) , since $\mathbb{C}$ is algebraically closed, which correspond to the 7 isolated points

of the moduli space of non-trivial $\mathrm{SL}(2, \mathbb{C})$ flat connections $\mathcal{M}^*(S_{-1/2}(4_1), \mathrm{SL}(2, \mathbb{C}))$. These are well-known and already appeared in the literature, see for example [79]. The values of the Chern-Simons action $\mathrm{CS}[\gamma]$ are listed in Table 3.4.6. Notice in particular that $\mathrm{CS}[\gamma_6]$ and $\mathrm{CS}[\gamma_7]$ are complex.

Flat connection	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7
Approx. $\mathrm{CS}[\gamma]$	0.514	0.828	0.997	0.235	0.054	$0.123 - 0.035i$	$0.123 + 0.035i$

Table 3.4.6: Approximate values of the Chern-Simons action on the $\mathrm{SL}(2, \mathbb{C})$ flat connections for $M = S_{-1/2}(4_1)$.

$\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory

In looking for all the elements of $\mathrm{Hom}(\pi_1(S_{-1/2}(4_1)), \mathrm{SL}(2, \mathbb{F}_q))$, we remark that it is not enough to consider Equation (3.4.47) over $\mathbb{F}_q$. Indeed, this would find only solutions for which the longitude and meridian are represented by diagonal matrices in $\mathrm{SL}(2, \mathbb{F}_q)$, while we should also take care of the other possible conjugacy classes. The case of codiagonal matrices is easily dealt with by solving Equation (3.4.47) over the quadratic extension $\mathbb{F} = \mathbb{F}_{q^2}$. Finally, one can check that there exist solutions of upper triangular form if and only if q is a power of $p = 7$. We discard such a case as it would violate the assumption that q is generic. Thus, for a generic q , we find at most 7 non-trivial orbits of $\mathrm{SL}(2, \mathbb{F}_q)$ flat connections, corresponding to pairs of solutions (λ, λ^{-1}) of Equation (3.4.47) over $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$. Each orbit has size equal to $|\mathrm{SL}(2, \mathbb{F}_q)|$.

For completeness, we notice that $p = 1143067$ is the smallest prime for which the polynomial splits over $\mathbb{F}_p$ and all seven flat connections are in diagonal form. We computed the value of the Dijkgraaf-Witten action on these flat connections. Results are listed in Table 3.4.7. We already noted that our conjecture could not be generalized to hyperbolic 3-manifolds, as Dijkgraaf-Witten theories cannot reproduce complex phases. Furthermore, no evident relation between the $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons and $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten values emerged. Even though the values of the CS and DW invariants do not match, we observe a remarkable correspondence between the leading orders. Indeed, for trivial flat connections γ_0 and β_0 we have $\delta_{\gamma_0} = 3/2$ and $\Delta'_{\beta_0} = 3/2$, while $\delta_{\gamma_i} = 0$ and $\Delta'_{\beta_i} = 0$ for all the non-trivial connections. However, this correspondence makes sense only when q is such that Equation (3.4.47) splits over $\mathbb{F}_{q^2}$. Unlike for manifolds with no hyperbolic components, such a condition cannot be realized as a divisibility condition between q and the order of the values of the CS invariant (which might be infinite).

Flat connection	β_1	β_2	β_3	β_4	β_5	β_6	β_7
DW $[\beta]$	$\frac{173068}{190511}$	$\frac{141395}{190511}$	$\frac{109168}{190511}$	$\frac{105685}{190511}$	$\frac{93571}{190511}$	$\frac{20011}{190511}$	$\frac{119146}{190511}$

Table 3.4.7: Exact values of the Dijkgraaf-Witten action on the $\mathrm{SL}(2, \mathbb{F}_q)$ -flat connections for $M = S_{-1/2}(4_1)$ and $q = 1143067$.

3.5 A proof of a weak version for spherical manifolds

In the previous section, we have seen several examples of spherical manifolds: lens spaces were considered in Section 3.4.1 and the Poincaré homology sphere appeared as a special case of a Seifert homology sphere in Section 3.4.3. For all of them we managed to check our conjecture as stated in Section 3.3.3. However, the proof that the conjecture holds for any 3-manifold is beyond our reach.

In this section, our goal is to prove for all spherical 3-manifolds a statement that is weaker than (3.3.27). It can be stated as follows.

Theorem 3.5.1. *Let M be a spherical manifold and $\mathcal{S}$ be defined as in Section 3.3.3. Suppose that q is a power of a generic prime, such that the order of Q , $o(Q)$, divides $q^2 - 1$ for each $Q \in \mathcal{S}$. The partition function of $\mathrm{SL}(2, \mathbb{F}_q)$ Dijkgraaf-Witten theory, twisted by the 3-cocycle ω as in Section 3.3.2, can be expressed as*

$$Z_{\mathrm{DW}}(M) = \sum_{Q \in \mathcal{S}_{\mathrm{DW}}} e^{2\pi i Q} T_Q(q), \quad (3.5.1)$$

where $\mathcal{S}_{\mathrm{DW}}$ is as in (3.3.27), and the $T_Q(q)$ are some rational numbers in $|\mathrm{SL}(2, \mathbb{F}_q)|^{-1}\mathbb{Z}$.

That is, we are only able to prove the result for the phase factor, but not for the leading order in the asymptotic expansion. Notice that for spherical manifolds there are no additional contributions from $\mathrm{SL}(2, \mathbb{C})$ flat connections, as $\pi_1(M)$ is finite and the finite order elements of $\mathrm{SL}(2, \mathbb{C})$ sit in its $\mathrm{SU}(2)$ subgroup. Then, in the notation of Section 3.3.3, we may assume $\mathcal{S} = \mathcal{S}_{\mathrm{SU}(2)}$.

The main idea of the proof is to apply a result by Nosaka [72]:

Proposition 3.5.1. *Let $\rho : \mathrm{SL}(2, \mathbb{F}_q) \rightarrow \mathrm{SL}(N, \mathbb{C})$ be a representation. Call $\mathrm{CCS} \in H^3(\mathrm{SL}(N, \mathbb{C}), \mathbb{C}/\mathbb{Z})$ the 3-cocycle that computes the CS invariant via pairing with the fundamental class of M , i.e. $\mathrm{CS}_{\mathrm{SL}(N, \mathbb{C})}[\gamma] = \langle \gamma^* \mathrm{CCS}_{\mathrm{SL}(N, \mathbb{C})}, [M] \rangle$. Then*

$$\rho^*(\mathrm{CCS}_{\mathrm{SL}(N, \mathbb{C})}) = \beta^{-1} c_2(\rho) \in H^3(\mathrm{SL}(2, \mathbb{F}_q), \mathbb{Q}/\mathbb{Z}), \quad (3.5.2)$$

and the Dijkgraaf-Witten invariant with twist $\beta^{-1} c_2(\rho)$ is given by

$$\frac{1}{|\mathrm{SL}(2, \mathbb{F}_q)|} \sum_{f \in \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2, \mathbb{F}_q))} \exp(2\pi i \mathrm{CS}_{\mathrm{SL}(N, \mathbb{C})}[\rho \circ f]). \quad (3.5.3)$$

If the representation is $\rho = \text{Ind}_T^G \phi + \text{Ind}_{T'}^G \phi'$, we argue that, when M is spherical, we can replace $\rho \circ f$ with $\text{Sym}^{2q^2-1} \circ \widehat{f}$ for some $\widehat{f} : \pi_1(M) \rightarrow \text{SL}(2, \mathbb{C})$. Here, Sym^N is the N^{th} symmetric power of the $\text{SL}(2, \mathbb{C})$ -representation given by matrix multiplication. Then, using that

$$(\text{Sym}^N)^* \text{CCS}_{\text{SL}(N+1, \mathbb{C})} = \frac{N(N+1)(N+2)}{6} \text{CCS}_{\text{SL}(2, \mathbb{C})}, \quad (3.5.4)$$

we obtain

$$\text{DW}[f] = \text{CS}_{\text{SL}(2q^2, \mathbb{C})}[\rho \circ f] = \frac{q^2(4q^4 - 1)}{3} \text{CS}_{\text{SL}(2, \mathbb{C})}[\widehat{f}], \quad (3.5.5)$$

where we can interpret the integer prefactor $q^2(4q^4 - 1)/3$ as the level of the $\text{SL}(2, \mathbb{C})$ Chern-Simons. In this argument, it looks like we don't need the twelvefold second Chern class of ρ , $\omega = 12\beta^{-1}c_2(\rho)$. Indeed, everything works just fine if we use $\beta^{-1}c_2(\rho)$. However, we don't want the value Q in (3.5.1) to depend on q , i.e. we would like the prefactor in (3.5.5) to satisfy

$$\frac{q^2(4q^4 - 1)}{3} Q = Q \pmod{1} \quad (3.5.6)$$

for any $Q \in \mathcal{S}$.

For spherical manifolds, it is known⁷ that the order $o(Q)$ for all the values $Q \in \mathcal{S}$ of the Chern-Simons invariant divides $|\pi_1(M)|$. Thus we may assume $q^2 = 1 \pmod{o(Q)}$. Notice that this very condition already appeared in the hypothesis of our main conjecture (3.3.27). However, this is not enough as one can check that the condition (3.5.6) is satisfied only if $3 \nmid o(Q)$. Here, using the twelvefold second Chern class of ρ is again crucial. Indeed, everything gets multiplied by 12 and the condition (3.5.6) becomes

$$4q^2(4q^4 - 1)Q = 12Q \pmod{1}, \quad (3.5.7)$$

which is true for any $Q \in \mathcal{S}$. Thus, this would imply

$$Z_{\text{DW}}(M) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} \sum_{f \in \text{Hom}(\pi_1(M), \text{SL}(2, \mathbb{F}_q))} \exp\left(2\pi i 12 \text{CS}_{\text{SL}(2, \mathbb{C})}[\widehat{f}]\right), \quad (3.5.8)$$

and $12 \text{CS}_{\text{SL}(2, \mathbb{C})}[\widehat{f}] \in \mathcal{S}_{\text{DW}}$ by construction. This completes the proof of the Theorem 3.5.1, and as a byproduct we obtain a formula for the coefficients

$$T_Q(q) = \frac{1}{|\text{SL}(2, \mathbb{F}_q)|} \left| \left\{ f : \Gamma \rightarrow \text{SL}(2, \mathbb{F}_q) \mid 12 \text{CS}_{\text{SL}(2, \mathbb{C})}[\widehat{f}] = Q \right\} \right| \quad (3.5.9)$$

⁷It suffices to notice that any flat connection A can be lifted to the unique flat connection A_0 of S^3 via its universal covering $p : S^3 \rightarrow M$. Then, the Chern-Simons invariant of A_0 must be equal to the Chern-Simons invariant of p^*A . This, on the other hand, is equal to $|\pi_1(M)|\text{CS}[A]$, so that we have obtained $|\pi_1(M)|\text{CS}[A] = 0 \pmod{1}$.

which clearly belong to $|\mathrm{SL}(2, \mathbb{F}_q)|^{-1}\mathbb{Z}$. Notice that it might happen that $T_Q(q)$ is zero for some $Q \in \mathcal{S}_{\mathrm{DW}}$. This means that the corresponding critical points of the CS functional do not come from representations over the finite field.⁸

We now proceed with the proof of the claim $\rho \circ f \simeq \mathrm{Sym}^{2q^2-1} \circ \hat{f}$. For this argument, it is essential that M is a spherical manifold. Recall that a spherical manifold is given as a quotient of S^3 by a finite subgroup $\Gamma \subset \mathrm{SO}(4)$ and hence, spherical manifolds have fundamental group $\pi_1(M) = \Gamma$. The possible choices for Γ are completely classified, see for example [104].

Consider the following diagram:

$$\begin{array}{ccc}
 & \pi_1(M) \equiv \Gamma & \\
 f \swarrow & & \searrow \hat{f} \\
 \mathrm{SL}(2, \mathbb{F}_q) & & \mathrm{SL}(2, \mathbb{C}) \\
 \rho \downarrow & & \downarrow \mathrm{Sym}^{2q^2-1} \\
 \mathrm{SL}(2q^2, \mathbb{C}) & \xrightarrow{\sim} & \mathrm{SL}(2q^2, \mathbb{C})
 \end{array}$$

Our goal is to find the putative automorphism of $\mathrm{SL}(2q^2, \mathbb{C})$ that makes the diagram commute. Notice that $\rho \circ f$ and $\mathrm{Sym}^{2q^2-1} \circ \hat{f}$ are two complex representations of the finite group Γ . Hence, two such representations are equivalent if and only if their characters coincide [105].

Let us compute the character of $\rho \circ f$. Clearly, $f(g) \in \mathrm{SL}(2, \mathbb{F}_q)$ has finite order and lies in some conjugacy class of $\mathrm{SL}(2, \mathbb{F}_q)$. Hence

$$\chi_{\rho \circ f}(g) = \begin{cases} \phi(\lambda) + \phi(\lambda^{-1}) & \text{if } f(g) \sim d(\lambda), \\ \phi'(\xi) + \phi'(\xi^{-1}) & \text{if } f(g) \sim d'(\xi), \\ \pm 2q^2 & \text{if } f(g) = \pm \mathbb{1}. \end{cases} \quad (3.5.10)$$

Notice that $f(g)$ cannot be similar to $\epsilon u_{\pm}$ since we are away from special primes, i.e. $p \nmid |\Gamma|$. This expression is reminiscent of the Brauer character of the representation f [105, 106]. Indeed, given a representation $f : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{F}_q)$ and a homomorphism $\hat{\phi} : \overline{\mathbb{F}_q}^{\times} \rightarrow \mathbb{C}^{\times}$, we can use the latter to lift the eigenvalues of $f(g)$ to $\mathbb{C}^{\times}$. Then, the Brauer character of f is

$$\chi_B(g) = \hat{\phi}(\lambda_g) + \hat{\phi}(\lambda_g^{-1}), \quad (3.5.11)$$

⁸In every example we have checked, the assumption that $o(Q)$ divides $q^2 - 1$ seems to be what guarantees that Q can be obtained by lifting a representation over $\mathbb{F}_q$ to a complex one. Unfortunately, we are not aware of a proof of this statement.

where λ_g and λ_g^{-1} are the two eigenvalues of $f(g)$. In this particular case, it is enough to fix $\widehat{\phi} : \mathbb{F}_{q^2}^\times \rightarrow \mathbb{C}^\times$, since $\lambda_g \in \mathbb{F}_{q^2}^\times$, and it is clear that such a map can be obtained by combining ϕ and ϕ' as above. This observation is crucial since the Brauer character is the fundamental ingredient in the correspondence between representations over $\mathbb{F}_q$ and $\mathbb{C}$. We remark that the representation theories in characteristic p and 0 are very different unless $p \nmid |\Gamma|$, which is in agreement with our definition of p being not special.

A direct implication of the Brauer correspondence is that χ_B is the character of a representation $\widehat{f} : \Gamma \rightarrow \mathrm{SL}(2, \mathbb{C})$, i.e. $\widehat{f}(g)$ has eigenvalues $\widehat{\phi}(\lambda_g^{\pm 1})$. Here, the fact that $p \nmid |\Gamma|$ is crucial since it guarantees that χ_B is an ordinary – non-virtual – character. Finally, this is the map $\widehat{f}$ we were looking for. Indeed, $\chi_{\mathrm{Sym}^{2q^2-1} \circ \widehat{f}}(g) = \mathrm{tr}(\mathrm{Sym}^{2q^2-1}(\widehat{f}(g))) = U_{2q^2-1}(\mathrm{tr} d(\widehat{\phi}(\lambda_g)/2))$, where U_n are Chebyshev polynomials of the second kind. Then

$$\chi_{\mathrm{Sym}^{2q^2-1} \circ \widehat{f}}(g) = \begin{cases} \phi(\lambda) + \phi(\lambda^{-1}) & \text{if } f(g) \sim d(\lambda), \\ \phi'(\xi) + \phi'(\xi^{-1}) & \text{if } f(g) \sim d'(\xi), \\ \pm 2q^2 & \text{if } f(g) = \pm \mathbb{1}. \end{cases} \quad (3.5.12)$$

We now have two complex representations of the finite group Γ with the same character. It is a standard result of representation theory that the two representations are equivalent, as claimed. This ends the proof.

We conclude this section with the obvious question of what fails in this proof when we consider non-spherical geometries. The crucial observation is that a complex representation might have infinite order, since $\Gamma = \pi_1(M)$ needs not be finite. This happens for example in the case of the Brieskorn homology sphere $M = \Sigma(2, 3, 7)$. Then, the Brauer correspondence breaks down and we are not able to lift representations over $\mathbb{F}_q$ to complex ones.

Chapter 4

Conclusions and outlook

This thesis has tried to answer two precise research questions in the framework of three-dimensional topological quantum field theories. We summarize the results we have obtained and their implications, with particular attention to possible new directions and ideas to overcome their current limitations.

4.1 Topological quantum gravity

We developed a theory of topological quantum gravity in three dimensions, mimicking the path-integral formulation of quantum field theories. In the topological setting, the path-integral reduces to a “sum over topologies”. The main question was how to properly define this sum and discuss its convergence. Indeed, similar ideas had already been considered in the literature in the two-dimensional case, see [15], and the extension to higher dimensions was deemed problematic in [52].

The key observation that allowed us to obtain some progress was to consider an abelian TQFT in the bulk. Then, the sum over topologies is understood as a sum over some algebraic data, and we no longer have to deal with the classification of three-dimensional manifolds. This is only possible by introducing the correct measure on the space of three-dimensional manifolds, which has many purposes. The first is to be able to recast the sum over topologies as a sum over the algebraic data, which otherwise would lead to an undercounting, as inequivalent 3-manifolds might correspond to the same algebraic data, and to regularize the sum.

On the other hand, this measure is also crucial for the holographic interpretation, as the putative distribution of boundary theories depends on this choice, and it takes care of divergence issues. Remarkably, we found sufficient conditions that guarantee convergence of the sum and that make the holographic interpretation possible. These conditions have a simple interpretation as a bound on the measure and good factorization properties with respect to boundary tori.

To make our discussion more concrete, we specialized to a precise choice of measure. This allowed us to obtain explicit expressions. Remarkably, we control the convergence of the sum by tuning a regularizing parameter in the measure which has a clear interpretation in the holographic dual theory. It turns out that it corresponds to the maximal order up to which the statistical moments of the distribution of boundary theories are well-defined and finite.

Once we made sense of the sum over topologies, we decided to test the possibility of it admitting a holographic dual theory. Here, the choice of an abelian 3d TQFT in the bulk is fundamental. Indeed, an abelian 3d TQFT does not distinguish between disjoint unions and connected sums on the boundary, hence the connected gravitational partition function $\mathcal{Z}_g$ only depends on the total genus of the boundary surface. This condition turned out to be very restrictive and allowed us to fix the distribution of boundary theories. Remarkably, this property holds as long as the summation weights in the measure factorize with respect to boundary tori, which is what we had already assumed above. Furthermore, the same property is sufficient to show that the distribution of boundary theories is physically meaningful, i.e. the distribution of the dimensions of the Hilbert spaces is supported on non-negative integers.

Our result depends explicitly on the choice of a topological boundary condition. Finally, we argued that the full state in the quantum gravity theory can be reconstructed from knowledge of the gravitational partition function for each possible choice of a topological boundary condition, at least formally. This result depends on an additional assumption that fails when the boundary is a collection of spheres. Via a series of examples, we showed that this computation can be performed in practice.

4.2 Finite and continuous gauge theory correspondence

Relations and correspondences between different physical theories can provide powerful insights and can lead to exciting new findings. For this reason, we explored the relation between finite and continuous gauge theories, focusing on $SU(2)$ Chern-Simons and $SL(2, \mathbb{F}_q)$ Dijkgraaf-Witten TQFTs. We chose to work with $SU(2)$ Chern-Simons because it has received a lot of interest in the literature and it gives rise to a powerful invariant, the WRT invariant. It is a very rich topological invariant of a 3-manifold and can be complicated to compute explicitly, even though a lot of progress has been made over the years to extract its asymptotics at large level for several classes of manifolds. These techniques have been our starting point in comparing the CS theory with DW TQFT.

In our work, we developed an efficient method to compute the partition function of twisted $SL(2, \mathbb{F}_q)$ DW TQFT on plumbed manifolds, which allowed us to compare the DW partition function with the asymptotics of the WRT invariant. The results we gathered led us to formulate a conjectural relation between the two theories. It is stated in Section 3.3.3, but the takeaway is that, under suitable hypotheses, the DW invariant contains information

about the leading order of the asymptotics of the WRT invariant.

The suitable hypotheses are not a secondary aspect of the conjecture. Indeed, the conjecture is only valid for geometric 3-manifolds with no hyperbolic components in their geometric decomposition. The reason for this restriction is clear. The Chern-Simons functional evaluated at a flat connection $\gamma \in \text{Hom}(\pi_1(M), \text{SL}(2, \mathbb{C}))$ is in general a complex number when M is hyperbolic. On the other hand, the Dijkgraaf-Witten functional can only produce rational numbers. This means that the conjecture cannot hold for hyperbolic manifolds in its current form, which is unfortunate as hyperbolic manifolds account for the majority of all 3-manifolds.

The evidence supporting the conjecture is convincing. Several classes of manifolds have been taken into account and they are all in agreement with our statement. Furthermore, we managed to prove part of the conjecture for spherical manifolds. Spherical geometry accounts for only a small subset of 3-manifolds, however it contains many familiar examples of manifolds. All lens spaces are spherical, as is the famous Poincaré homology sphere. They have the property that their fundamental group is finite, which was a crucial ingredient in the proof of the weaker version of the conjecture.

The lack of a general proof is unsatisfying. Some aspects of the argument for spherical manifolds could easily be generalized but this is not sufficient to establish the conjecture in full. A general proof might shed light on the underlying mechanism and could lead to a deeper understanding of the relation between finite and continuous gauge theories.

4.3 Outlook

Our work on topological quantum gravity has some clear limitations. First of all, we have only considered abelian TQFTs in the bulk. It would be very interesting to study more general TQFTs, however the properties of abelian TQFTs were a key ingredient in many steps of our analysis and it seems very difficult to progress without any additional assumption. A similar argument applies also to the question of more general boundary conditions. It is known that CS theories always admit conformal boundary conditions, hence we would expect them to yield a *conformal* quantum gravity theory. However, much of our work breaks down when boundaries are no longer topological.

Another possible direction of research consists of extending the holographic interpretation to operators. This would require a *sum over topologies with defects*, whose ending points are fixed points on the boundary. Holography is expected to relate such operators to an ensemble of operators inserted on punctures on the boundary theories. Just the definition of the operators poses new challenges that we decided to avoid.

Regarding the question on the correspondence between finite and continuous gauge theories, much work still has to be done. Possible directions of research include the proof of the conjecture even in the non-spherical case and the generalization to hyperbolic manifolds. We have briefly discussed these possibilities in the previous section. Here, we anticipate

the findings of a current work in progress with Pavel Putrov and Johann Quenta-Raygada, where we explore the generalization of the correspondence between finite and continuous gauge theories to other gauge groups.

As already mentioned, the correspondence is well established for abelian gauge groups, while the work in this thesis dealt with the generalization to simple groups like $SU(2)$. However, another meaningful generalization to non-abelian groups can be obtained using nilpotent groups. From a mathematical viewpoint, it makes sense as abelian groups are trivially nilpotent. However, the definition of a gauge theory with a nilpotent, and hence non-compact, gauge group poses new problems.

In a future work, we will give a recipe to produce a nilpotent quadratic Lie algebra from a nilpotent finite p -group G , and define the Chern-Simons theory from the Lie algebra. Remarkably, we observe the emergence of a topological subsector at level $k = p$. We then conjecture that this subsector is in correspondence with the fusion categories of line operators in the untwisted G Dijkgraaf-Witten theory.

It is a generalization of the work presented in this thesis in two different ways. First of all, it expands the relation between finite and continuous gauge theories to a new class of gauge groups, and at the same time it goes beyond the mere comparison of partition functions as it analyzes the equivalence between the fusion categories of line operators.

Appendices

Appendix A

Proof of the Proposition 2.4.2

We give a proof of the Proposition 2.4.2, starting with the first statement. Let us introduce:

$$\Gamma(p) := c_{n-\sum_i m_{p_i}} \prod_i \delta(p_i) \frac{m_{p_i}!}{(m_{p_i} - |p_i|)!}, \quad (\text{A.1})$$

so that:

$$f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}(n, m_1, \dots, m_k) = e^{-x_0} \frac{(n-k)!}{n!} \sum_{p \in P(\{1, \dots, k\})} \Gamma(p). \quad (\text{A.2})$$

Given a partition q of the set $\{1, \dots, k+1\}$, we can always obtain a partition $p = \{p_1, \dots, p_l\}$ of $\{1, \dots, k\}$ by removing the element $k+1$. In particular, q can only have two different forms:

$$\begin{aligned} q^{(j)} &= \{p_1, \dots, p_j \cup \{k+1\}, \dots, p_l\}, & j &= 1, \dots, l, \\ \bar{q} &= \{p_1, \dots, p_l, \{k+1\}\}. \end{aligned} \quad (\text{A.3})$$

Thus, instead of summing over $q \in P(\{1, \dots, k+1\})$, we may take the sum over $p \in P(\{1, \dots, k\})$ and then, at a fixed p , sum over the $l+1$ ways to include $\{k+1\}$ in p .

Having to evaluate:

$$\sum_{m_{k+1}} f_{N, \theta_1^{-1}, \dots, \theta_{k+1}^{-1}}(n, m_1, \dots, m_{k+1}), \quad (\text{A.4})$$

we start with computing:

$$\sum_{m_{k+1}} \Gamma(q^{(j)}) = (m_{p_j} - |p_j|) \Gamma(p), \quad (\text{A.5})$$

$$\sum_{m_{k+1}} \Gamma(\bar{q}) = (n - \sum_i m_{p_i}) \Gamma(p). \quad (\text{A.6})$$

It is then easy to deduce:

$$\begin{aligned} \sum_{m_{k+1}} f_{N, \theta_1^{-1}, \dots, \theta_{k+1}^{-1}}(n, m_1, \dots, m_{k+1}) &= e^{-x_0} \frac{(n-k-1)!}{n!} (n-k) \sum_{p \in P(\{1, \dots, k\})} \Gamma(p) \\ &= f_{N, \theta_1^{-1}, \dots, \theta_k^{-1}}(n, m_1, \dots, m_k), \end{aligned} \quad (\text{A.7})$$

as claimed.

Next, we show that the proposed distribution satisfies the abelian condition:

$$\sum_{p \in P(\{g_1, \dots, g_k\})} \prod_{q \in p} \mathbb{E} \left[N \theta^{1 - \sum_{g \in q} g} \right] = \sum_{p \in P(\{g_1, \dots, g_k\})} \mathbb{E} \left[\frac{N!}{(N - |p|)!} \prod_{q \in p} \theta_q^{\sum_{g \in q} (1-g)} \right]. \quad (\text{A.8})$$

The first step is to consider the addend on the left-hand side.

Lemma A.1. *Fix a partition $p = \{p_1, \dots, p_l\} \in P(\{g_1, \dots, g_k\})$, then:*

$$\prod_{i=1}^l \mathbb{E} \left[N \theta^{1 - \sum_{g \in p_i} g} \right] = e^{-x_0} \sum_{n \geq l} \sum_{m_1 + \dots + m_l = n} c_{n-m_1-\dots-m_l} \prod_{i=1}^l b_{m_i} m_i^{\sum_{g \in p_i} g}. \quad (\text{A.9})$$

Moreover, it is convergent as long as x_k – defined as in (2.4.33) – is absolutely convergent, where $k = \max_i(\sum_{g \in p_i} g)$.

Proof. We start by looking at the case $l = 1$:

$$\begin{aligned} \mathbb{E} \left[N \theta^{1 - \sum_{i=1}^r g_i} \right] &= \\ &= \sum_{n \geq 1} \sum_{m=1}^n n m^{\sum_{i=1}^r g_i - 1} f_{N, \theta^{-1}}(n, m) = e^{-x_0} \sum_{n \geq 1} \sum_{m=1}^n m^{\sum_{i=1}^r g_i} b_m c_{n-m}. \end{aligned} \quad (\text{A.10})$$

We want to prove the statement by induction on l . However, first notice that we can also write:

$$\mathbb{E} \left[N \theta^{1-G} \right] = e^{-x_0} \sum_{n \geq 1} \sum_{m=1}^n m^G b_m c_{n-m} = e^{-x_0} \sum_{m \geq 1} m^G b_m \sum_{n \geq m} c_{n-m} = \sum_{m \geq 1} m^G b_m = x_G, \quad (\text{A.11})$$

since, by normalization, we have:

$$e^{-x_0} \sum_{n \geq 0} c_n = \mathbb{E}[1] = 1. \quad (\text{A.12})$$

Suppose that the claim holds for l , we show that it also holds for $l + 1$:

$$\begin{aligned} \prod_{i=1}^{l+1} \mathbb{E} \left[N \theta^{1 - \sum_{g \in p_i} g} \right] &= \mathbb{E} \left[N \theta^{1 - \sum_{g \in p_{l+1}} g} \right] \prod_{i=1}^l \mathbb{E} \left[N \theta^{1 - \sum_{g \in p_i} g} \right] = \\ &= e^{-x_0} \sum_{n \geq l} \sum_{m_1 + \dots + m_l = l}^n c_{n-m_1-\dots-m_l} \prod_{i=1}^l b_{m_i} m_i^{\sum_{g \in p_i} g} \sum_{m_{l+1} \geq 1} m_{l+1}^{\sum_{g \in p_{l+1}} g} b_{m_{l+1}}, \quad (\text{A.13}) \end{aligned}$$

by having used the inductive hypothesis and the remark about the case $l = 1$. Finally, we use the Cauchy product of series to get:

$$\prod_{i=1}^{l+1} \mathbb{E} \left[N \theta^{1 - \sum_{g \in p_i} g} \right] = e^{-x_0} \sum_{n \geq l+1} \sum_{m_{l+1}=1}^{n-l} \sum_{m_1 + \dots + m_l = l}^{n-m_{l+1}} c_{n-m_1-\dots-m_l-m_{l+1}} \prod_{i=1}^{l+1} b_{m_i} m_i^{\sum_{g \in p_i} g}. \quad (\text{A.14})$$

□

Let us introduce the following notation:

$$a(n, \{p_1, \dots, p_l\}) = \sum_{m_1 + \dots + m_l = l}^n c_{n-m_1-\dots-m_l} \prod_{i=1}^l b_{m_i} m_i^{\sum_{g \in p_i} g}, \quad (\text{A.15})$$

for each addend in the series in equation (A.9).

Now, we focus on the study of the addends of the right-hand side of (A.8). We simply substitute for the expression of the probability density function. Let us do it for the following addend:

$$\begin{aligned} b(n, \{p_1, \dots, p_l\}) &= e^{x_0} \sum_{m_1, \dots, m_l} f_{N, \theta_1^{-1}, \dots, \theta_l^{-1}}(n, m_1, \dots, m_l) \frac{n!}{(n-l)!} \prod_{i=1}^l m_i^{\sum_{g \in p_i} (g-1)} \quad (\text{A.16}) \\ &= \sum_{m_1, \dots, m_l} \sum_{\rho \in P(\{1, \dots, l\})} c_{n - \sum_{q \in \rho} m_q} \prod_{q \in \rho} \delta(q) b_{m_q} \frac{m_q!}{(m_q - |q|)!} \prod_{i=1}^l m_i^{\sum_{g \in p_i} (g-1)}. \end{aligned}$$

Notice that, taking the sum over all the m_i 's, the effect of $\delta(\rho_\alpha)$ is to identify some of them. In particular, $\delta(\rho_\alpha = \{\mu_1, \dots, \mu_r\})$ identifies $m_{\mu_1} = \dots = m_{\mu_r} =: m_\alpha$, so that the sum over the m_i 's is restricted to a sum over the m_α 's, $\alpha = 1, \dots, \lambda$. This is equivalent to *fusing* the partitions $p = \{p_1, \dots, p_l\}$ and ρ into a new partition $F_p(\rho) \in P(\{g_1, \dots, g_k\})$, which we call the fusion of p along ρ .

Definition A.1. Define the fusion function as follows:

$$\begin{aligned} F: P(\{g_1, \dots, g_k\}) \times P(\{1, \dots, k\}) &\rightarrow P(\{g_1, \dots, g_k\}), \\ (\{p_1, \dots, p_l\}, \{\rho_1, \dots, \rho_\lambda\}) &\mapsto \{\tilde{p}_1, \dots, \tilde{p}_\lambda\} \end{aligned} \quad (\text{A.17})$$

where $\tilde{p}_\alpha = \bigcup_{i \in \rho_\alpha} p_i$, with $\tilde{p}_\alpha$ possibly empty if $i \geq |p| = l$ for all $i \in \rho_\alpha$.

In our case ρ will always be an element of $P(\{1, \dots, l\})$, which can be seen as a subset of $P(\{1, \dots, k\})$ by appending $\{l+1, \dots, k\}$ to ρ .

Then, noticing that:

$$m_{\alpha}^{\sum_{g \in \tilde{p}_{\alpha}} (g-1)} = m_{\alpha}^{\sum_{i=1}^{|\rho_{\alpha}|} \sum_{g \in p_i} (g-1)} = \prod_{i=1}^{|\rho_{\alpha}|} m_{\alpha}^{\sum_{g \in p_i} (g-1)}, \quad (\text{A.18})$$

we can use the fusion to rewrite:

$$\begin{aligned} b(n, \{p_1, \dots, p_l\}) &= \sum_{\rho \in P(\{1, \dots, l\})} \sum_{m_1, \dots, m_{\lambda}} c_{n - \sum_{\alpha=1}^{\lambda} m_{\alpha}} \prod_{\alpha=1}^{\lambda} b_{m_{\alpha}} \frac{m_{\alpha}!}{(m_{\alpha} - |\rho_{\alpha}|)!} m_{\alpha}^{\sum_{g \in \tilde{p}_{\alpha}} (g-1)} \\ &= \sum_{\rho} \sum_{m_1, \dots, m_{|F(p, \rho)|}} c_{n - \sum_{q \in F(p, \rho)} m_q} \prod_{q \in F(p, \rho)} b_{m_q} \frac{m_q!}{(m_q - |\rho_q|)!} m_q^{\sum_{g \in q} (g-1)}. \end{aligned} \quad (\text{A.19})$$

In other words, b does not depend directly on p but on $F(p, \rho)$. Hence, we would like to perform a smart change of summation indices. We have the following:

Lemma A.2. *The function*

$$\begin{aligned} P(\{g_1, \dots, g_k\}) \times P(\{1, \dots, k\}) &\rightarrow P(\{g_1, \dots, g_k\}) \times P(\{g_1, \dots, g_k\}), \\ (p, \rho) &\mapsto (F(p, \rho), p) \end{aligned} \quad (\text{A.20})$$

establishes a one-to-one correspondence between pairs (p, ρ) with $\rho \in P(\{1, \dots, |p|\}) \subseteq P(\{1, \dots, k\})$ and pairs (p', q) with $p' \in P(\{g_1, \dots, g_k\})$ and $q \in \mathcal{F}(p') \subseteq P(\{g_1, \dots, g_k\})$. We have introduced $\mathcal{F}(p')$ as the set of possible partitions that can yield p via fusion, i.e.:

$$\mathcal{F}(p') = \{q \in P(\{g_1, \dots, g_k\}) \mid \exists \rho \in P(\{1, \dots, |p'|\}): F(q, \rho) = p'\}. \quad (\text{A.21})$$

Proof. Given the pair (p, ρ) , we can construct the pair (p', q) as $(F(p, \rho), p)$. Clearly, $p \in \mathcal{F}(F(p, \rho))$. Moreover, this pair is unique since $F(p, \rho) = F(p, \rho')$ implies $\rho = \rho' \in P(\{1, \dots, |p|\})$. Notice that this last statement would not hold in $P(\{1, \dots, k\})$.

We easily invert this construction. Fix a pair (p', q) with $q \in \mathcal{F}(p')$. Then, there exists some ρ such that the pair (q, ρ) yields $(p' = F(q, \rho), q)$. $\square$

Therefore, instead of summing over (p, ρ) , we can sum over pairs (p', q) with $p' \in P(\{g_1, \dots, g_k\})$ and $\rho \in \mathcal{F}(p')$. In particular, notice that $q \in \mathcal{F}(p')$ means that we can write $q = \{\bar{q}_1, \dots, \bar{q}_{|p'|}\}$, where each $\bar{q}_{\alpha}$ is itself a partition of the set p'_{α} , with $p' = \{p'_1, \dots, p'_{|p'|}\}$. Performing this change of variables yields:

$$\sum_{p \in P(\{g\})} b(n, p) = \sum_{p \in P(\{g\})} \sum_{m_1, \dots, m_{|p|}} c_{n - \sum_{\alpha \in p} m_{\alpha}} \prod_{\alpha \in p} b_{m_{\alpha}} m_{\alpha}^{\sum_{g \in \alpha} (g-1)} \sum_{q \in \mathcal{F}(p)} \prod_{\alpha=1}^{|p|} \frac{m_{\alpha}!}{(m_{\alpha} - |q_{\alpha}|)!}. \quad (\text{A.22})$$

To prove proposition 2.4.2, we will show the equality between the coefficients of the two series:

$$\sum_{p \in P(\{g\})} a(n, p) = \sum_{p \in P(\{g\})} b(n, p). \quad (\text{A.23})$$

It is enough to manipulate the expression for b . Indeed, notice that for a fixed partition p we have:

$$\sum_{q \in \mathcal{F}(p)} \prod_{\alpha=1}^{|p|} \frac{m_\alpha!}{(m_\alpha - |q_\alpha|)!} = \prod_{\alpha=1}^{|p|} \sum_{q \in P(p_\alpha)} \frac{m_\alpha!}{(m_\alpha - |q|)!}. \quad (\text{A.24})$$

Another useful result is the following:

Lemma A.3. *The following identity holds true:*

$$\sum_{q \in P(p)} \frac{m!}{(m - |q|)!} = m^{|p|}. \quad (\text{A.25})$$

Proof. First of all, decompose $P(p)$ into sets of partitions of fixed length k , denoted by $P_k(p)$. Then:

$$\sum_{q \in P(p)} \frac{m!}{(m - |q|)!} = \sum_{k=0}^{|p|} \frac{m!}{(m - k)!} \sum_{q \in P_k(p)} 1. \quad (\text{A.26})$$

Notice that the number of partitions of length k of a set of order $|p|$ is equal to the Stirling numbers of second kind $S(|p|, k)$. Thus, the result follows by a fundamental property of the Stirling numbers:

$$\sum_{q \in P(p)} \frac{m!}{(m - |q|)!} = \sum_{k=0}^{|p|} \frac{m!}{(m - k)!} S(|p|, k) = m^{|p|}. \quad (\text{A.27})$$

□

Using the above remarks, we get:

$$\begin{aligned} \sum_{p \in P(\{g\})} b(n, p) &= \sum_{p \in P(\{g\})} \sum_{m_1, \dots, m_{|p|}} c_{n - \sum_{\alpha \in p} m_\alpha} \prod_{\alpha \in p} b_{m_\alpha} m_\alpha^{\sum_{g \in \alpha} g} \prod_{\alpha=1}^{|p|} m_\alpha^{-|p_\alpha|} m_\alpha^{|p_\alpha|} \\ &= \sum_{p \in P(\{g\})} \sum_{m_1, \dots, m_{|p|}} c_{n - \sum_{\alpha \in p} m_\alpha} \prod_{\alpha \in p} b_{m_\alpha} m_\alpha^{\sum_{g \in \alpha} g}. \end{aligned} \quad (\text{A.28})$$

This concludes the proof of the Proposition 2.4.2. Notice that this guarantees in particular the convergence of the right-hand side of (A.8) as long as the left-hand side converges.

Appendix B

Antisymmetric cocycles

Before delving into the definition of antisymmetric cocycles, it will be useful to represent group cochains as *homogeneous* cochains. An homogeneous n -cochain in $C^n(G, A)$ is a map $F : G^{n+1} \rightarrow A$ which transforms under a G -action as follows:

$$(g \cdot F)(g_0, \dots, g_n) = F(gg_0, \dots, gg_n). \quad (\text{B.1})$$

The differential $\delta : C^n(G, A) \rightarrow C^{n+1}(G, A)$ is then defined as

$$(\delta F)(g_0, \dots, g_{n+1}) = \sum_{i=0}^n (-1)^i F(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}), \quad (\text{B.2})$$

where the hat indicates we are removing the corresponding entry from F . We can go back and forth between homogeneous and inhomogeneous cochains (the latter defined as in Section 3.3) through the relation

$$f(g_1, g_2, \dots, g_n) = F(1, g_1, g_1g_2, \dots, g_1g_2 \dots g_n). \quad (\text{B.3})$$

We will use capital letters to denote homogeneous cochains and the corresponding small letters for their inhomogeneous versions.

We now define *antisymmetric* group cochains as follows. An n -cochain F is antisymmetric if, for any permutation $\sigma \in S_{n+1}$, we have

$$F(g_{\sigma(0)}, \dots, g_{\sigma(n)}) = \varepsilon(\sigma) F(g_0, \dots, g_n), \quad (\text{B.4})$$

where $\varepsilon(\sigma)$ denotes the sign of σ . It is straightforward to translate these conditions into their inhomogeneous form. For example, one finds Eq. (1.2.38) in the case $n = 3$.

For an arbitrary (normalized) n -cochain F , it is not always possible to find an antisymmetric cochain F' cohomologous to it, unless certain divisibility conditions are satisfied. For example, it is well known that any 2-cochain can be made antisymmetric

whenever every element in $\text{Im}(F)$ has a half inside A (not necessarily unique), i.e. if $\text{Im}(F)$ is 2-divisible inside A . If this holds, an explicit expression is given by

$$f'(g, h) = f(g, h) - \delta\alpha(g, h), \quad \alpha(g) = \frac{1}{2}f(g, g^{-1}). \quad (\text{B.5})$$

For $n = 3$, we find that the analogous condition on $\text{Im}(F)$ is that it is both 2, 3 and 4-divisible inside A . If that is the case, an explicit expression reads

$$f'(g, h, k) = f(g, h, k) - \delta\alpha(g, h, k), \quad (\text{B.6})$$

where

$$\begin{aligned} \alpha(g, h) = & \frac{1}{2}f(g, g^{-1}, gh) - \frac{1}{3}f(gh, h^{-1}, h) + \frac{1}{3}f(h, (gh)^{-1}, gh) - \frac{1}{6}f((gh)^{-1}, gh, h^{-1}) \\ & + \frac{1}{6}f(h^{-1}, h, (gh)^{-1}) - \frac{1}{4}f(g, g^{-1}, g) + \frac{1}{4}f(h, h^{-1}, h) - \frac{1}{4}f(gh, (gh)^{-1}, gh). \end{aligned} \quad (\text{B.7})$$

To the best of our knowledge, such an expression for the $n = 3$ case has not been computed in the existing literature.

Let us now focus on our particular setting, where the cocycle $\omega = 12\beta^{-1}c_2(\rho)$ has image $6\mathbb{Z}_{q^2-1} \subset \text{U}(1)$, with q a prime power. While the operations $1/2, 1/3$ are always well-defined for $\text{Im}(\omega)$, in general the $1/4$ operation is not. For this reason, we instead switch to the cocycle $\omega = 24\beta^{-1}c_2(\rho)$, for which Eq. (B.7) can be readily applied.

Appendix C

Class multiplication coefficients

Given a group G with conjugacy classes $C_1, \dots, C_k$, one can introduce the conjugacy class sums as

$$\mathcal{C}_i := \sum_{g \in C_i} g. \quad (\text{C.1})$$

Then, the class multiplication coefficients c_{ijk} are defined by the formula

$$\mathcal{C}_i \mathcal{C}_j = \sum_k c_{ijk} \mathcal{C}_k, \quad (\text{C.2})$$

see [107] for more details. An equivalent characterization is given by

$$c_{ijk} = |\{(g_i, g_j, g_k) \in C_i \times C_j \times C_k : g_i g_j g_k^{-1} = e\}|. \quad (\text{C.3})$$

It is not hard to prove that the class multiplication coefficient can also be expressed in terms of the irreducible characters of the group G . This is known as the Frobenius character sum formula:

$$c_{ijk} = \frac{|C_1||C_2||C_3|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(C_1)\chi(C_2)\chi(C_3)^*}{\chi(e)}. \quad (\text{C.4})$$

The equivalent characterization (C.3) prompts us to generalize the definition of the class multiplication coefficients to more than three conjugacy classes. Then, the Frobenius formula holds in a similar fashion. Indeed, let

$$\mathcal{N}(C_1, \dots, C_k) := |\{(g_1, \dots, g_k) \in C_1 \times \dots \times C_k : g_1 \dots g_k = e\}| \quad (\text{C.5})$$

be the generalization of the class multiplication coefficients, where $\mathcal{N}(C_i, C_j, C_k)$ recovers¹ c_{ijk} . One can check that the Frobenius formula becomes

$$\mathcal{N}(C_1, \dots, C_k) = \frac{|C_1| \dots |C_k|}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(C_1) \dots \chi(C_k)}{\chi(e)^{k-2}}. \quad (\text{C.6})$$

¹Note that one should be careful in comparing c_{ijk} with $\mathcal{N}(C_i, C_j, C_k)$, as we follow a different notation where g_3^{-1} is replaced with g_3 for a neater result.

Notice that expressions of this form appear in the DW partition function of plumbed manifolds at any vertex of degree at least three. In particular, this is the case for the partition function of Brieskorn homology spheres considered in Section 3.4.3. Consider a_i be as in (3.4.30), we have:

$$\frac{|\mathrm{SL}(2, \mathbb{F}_q)|}{q_{\pm}^3} \sum_{\chi} \frac{\chi(a_1)\chi(a_2)\chi(a_3)\chi(-1)}{\chi(1)^2} = \frac{1}{|\mathrm{SL}(2, \mathbb{F}_q)|} \mathcal{N}(a_1, a_2, a_3, -1). \quad (\text{C.7})$$

The computation of $\mathcal{N}(a_1, a_2, a_3, -1)$ is highly non-trivial. One way to do it is based on the same approach that appeared in [95] to find elements of $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(2, \mathbb{C}))$. Then, applying the same idea to $\mathrm{SL}(2, \mathbb{C})$, we can prove that

$$\mathcal{N}(a_1, a_2, a_3, -1) = |\mathrm{SL}(2, \mathbb{F}_q)|. \quad (\text{C.8})$$

A different approach is possible and would consist in evaluating the sum (C.6) directly using the character table of $\mathrm{SL}(2, q)$ (see Table 3.2.2). This procedure is more straightforward but is harder to generalize to an arbitrary number of conjugacy classes k .

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