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(Article begins on next page)

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Symplectic capacities of domains close to the ball and Banach–Mazur geodesics in the space of contact forms

Alberto Abbondandolo*, Gabriele Benedetti†, and Oliver Edtmair‡

Abstract

We prove that all normalized symplectic capacities coincide on smooth domains in $\mathbb{C}^n$ which are C^2-close to the Euclidean ball, whereas this fails for some smooth domains which are just C^1-close to the ball. We also prove that all symplectic capacities whose value on ellipsoids agrees with that of the n-th Ekeland–Hofer capacity coincide in a C^2-neighborhood of the Euclidean ball of $\mathbb{C}^n$. These results are deduced from a general theorem about contact forms which are C^2-close to Zoll ones, saying that these contact forms can be pulled back to suitable "quasi-invariant" contact forms. We relate all this to the question of the existence of minimizing geodesics in the space of contact forms equipped with a Banach–Mazur pseudo-metric. Using some new spectral invariants for contact forms, we prove the existence of minimizing geodesics from a Zoll contact form to any contact form which is C^2-close to it. This paper also contains an appendix in which we review the construction of exotic ellipsoids by the Anosov–Katok conjugation method, as these are related to the above mentioned pseudo-metric.

Introduction

Normalized symplectic capacities. Let

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

be the standard symplectic form on $\mathbb{C}^n$, endowed with coordinates $z_j = x_j + iy_j$, for $j = 1, \dots, n$. In this paper, by a symplectic capacity on $(\mathbb{C}^n, \omega_0)$ we mean a function

$$c : \{\text{open subsets of } \mathbb{C}^n\} \to [0, +\infty]$$

satisfying the following conditions:

*Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstraße 150, 44801 Bochum, Germany, alberto.abbondandolo@rub.de

†Department of Mathematics, Vrije Universiteit Amsterdam, De Boelelaan 1111, 1081 HV Amsterdam, g.benedetti@vu.nl

‡Department of Mathematics, University of California at Berkeley, Berkeley, CA, 94720, USA, oliver_edtmair@berkeley.edu

- (monotonicity) if there exists a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\phi(A) \subset A'$, then $c(A) \leq c(A')$;
- (conformality) $c(rA) = r^2 c(A)$ for every $r > 0$;
- (non-triviality) $c(B) > 0$ and $c(Z) < \infty$.

Here, B denotes the unit Euclidean ball in $\mathbb{C}^n$ and Z the cylinder $\{z \in \mathbb{C}^n \mid |z_1| < 1\}$. The symplectic capacity c is said to be normalized if the non-triviality condition is upgraded by the following requirement:

- (normalization) $c(B) = c(Z) = \pi$.

Symplectic capacities were introduced by Ekeland and Hofer in [EH89] building on Gromov's work [Gro85]. What we here call a symplectic capacity on $(\mathbb{C}^n, \omega_0)$ is called by some authors a *relative* or *extrinsic* symplectic capacity, whereas a symplectic capacity should be *intrinsic*, meaning that the monotonicity axiom should hold more generally for all symplectic embeddings of A into A', see [MS17, Chapter 12]. With this terminology, any symplectic capacity is a relative symplectic capacity, but there are interesting relative symplectic capacities, such as Hofer's displacement energy from [Hof90], which are not symplectic capacities. In this paper, we wish to consider also these more general functions and hence work with the above weaker definition, but, to avoid to burden the terminology, we omit the word "relative".

Gromov's non-squeezing theorem implies that the functions

$$\underline{c}(A) := \sup\{\pi r^2 \mid \exists \text{ symplectomorphism } \phi : \mathbb{C}^n \to \mathbb{C}^n \text{ such that } \phi(rB) \subset A\},$$
$$\overline{c}(A) := \inf\{\pi r^2 \mid \exists \text{ symplectomorphism } \phi : \mathbb{C}^n \to \mathbb{C}^n \text{ such that } \phi(A) \subset rZ\},$$

are normalized symplectic capacities. They are known as *ball capacity* (or Gromov width) and *cylindrical capacity*, respectively. From the above axioms, one easily sees that they are the smallest and largest normalized capacities: any normalized symplectic capacity c satisfies the inequalities

$$\underline{c} \leq c \leq \overline{c}. \tag{1}$$

The ball and cylindrical capacities are difficult to compute, as this would involve understanding symplectic embeddings, which is precisely what one would use a symplectic capacity for. Over the last decades, several more computable symplectic capacities based on periodic Hamiltonian orbits, pseudo-holomorphic curves, Lagrangian submanifolds, or a combination of these ingredients have been constructed. See the survey [CHLS07] and references therein. Many of these symplectic capacities are *spectral*, meaning that the symplectic capacity of a bounded domain A with smooth boundary of restricted contact type coincides with the action of some closed characteristic on ∂A (these hypersurfaces always admit closed characteristics, as first proven in [Vit87]).

It is important to observe that the normalization axiom does not determine c uniquely. Indeed, different normalized symplectic capacities may have different values on domains

belonging to the set

$$\mathcal{A} \coloneqq \{\text{bounded open neighborhoods of the origin in } \mathbb{C}^n \text{ whose boundary is smooth and transverse to the radial direction}\}.$$

The first examples are due to Hermann, who in [Her98b] constructed Reinhardt domains in $\mathcal{A}$ with arbitrarily small volume, and hence arbitrarily small ball capacity, but large displacement energy, and hence large cylindrical capacity. Examples of domains in $\mathcal{A}$ whose cylindrical capacity is strictly larger than the displacement energy are constructed in [Ham02]. Examples of domains whose ball capacity does not coincide with any spectral capacity are given by the domains in $\mathcal{A}$ with fixed volume and arbitrarily large systole which are constructed in [ABHS18] and [Sağ21]. Here, the systole of $A \in \mathcal{A}$ is the positive number

$$\operatorname{sys}(A) \coloneqq \min\{\text{action of closed characteristics on } \partial A\}.$$

A long standing open question is whether all normalized symplectic capacities coincide on bounded convex domains, see e.g. [Her98b, Conjecture 1.9] and [MS17, Section 14.9, Problem 53]. A positive answer to this question would imply that for any normalized symplectic capacity and any bounded convex domain C the inequality

$$c(C)^n \leq n! \operatorname{vol}(C) \tag{2}$$

holds, as the above inequality trivially holds for the ball capacity. Inequality (2) was conjectured by Viterbo in [Vit00] and, for this reason, the coincidence of all normalized symplectic capacities on bounded convex domains is in the recent literature referred to as the strong Viterbo conjecture. Note that the equality holds in (2) when C is symplectomorphic to a ball, and Viterbo actually conjectured that these are the only bounded convex domains for which the equality holds (this part of Viterbo's conjecture would not follow from the coincidence of all normalized capacities on bounded convex domains). See [Ost14] and [GHR22] for more information about these conjectures and their consequences in convex geometry.

Evidence for a positive answer to the strong Viterbo conjecture is given by the fact that for many normalized symplectic capacities c it has been shown that

$$c(C) = \operatorname{sys}(C) \tag{3}$$

whenever C is a bounded convex domain with smooth boundary. Identity (3) has been proven in [Vit89] for the Ekeland–Hofer capacity from [EH89], for the Hofer–Zehnder capacity in [HZ90], and in [AK22, Iri22] for the capacity from symplectic homology introduced in [FHW94], which by Hermann's work [Her04] coincides also with Viterbo's generating functions capacity from [Vit92].

Thanks to (1), proving the equivalence of all normalized capacities on bounded convex domains amounts to prove that $\underline{c} = \overline{c}$ for these domains, for instance by showing that both these capacities satisfy (3). For an arbitrary bounded convex domain C, it is not known whether (3) holds for $\underline{c}$ or for $\overline{c}$.

In this paper, we restrict our analysis to smooth convex domains that are C^k-close to the ball B. More precisely, we observe that the domains in $\mathcal{A}$ are precisely the sets of the form

$$A_f \coloneqq \{rz \mid z \in S^{2n-1},\ 0 \le r < f(z)\},$$

where f is an arbitrary smooth positive function on the unit sphere S^{2n-1} of $\mathbb{C}^n$, and define the C^k-distance between domains in $\mathcal{A}$ as the C^k-distance between the corresponding smooth functions on S^{2n-1}. With the notation above, $B = A_1$.

Our first main result is the following theorem.

THEOREM 1. *There exists a C^2-neighborhood $\mathcal{B}$ of B in $\mathcal{A}$ such that for every A in $\mathcal{B}$ the following facts hold:*

(i) There is a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ mapping A into the cylinder whose systole coincides with the systole of A. In particular, $\overline{c}(A) = \mathrm{sys}(A)$.

(ii) There is a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ mapping the ball whose systole coincides with the systole of A into A. In particular, $\underline{c}(A) = \mathrm{sys}(A)$.

Consequently, all normalized symplectic capacities coincide on $\mathcal{B}$.

Statement (i) is an improvement of Proposition 4 from [AB23], where the analogous result is proven for domains that are C^3-close to the ball. The proof of (i) (as that of [AB23, Proposition 4]) is elementary. It uses generating functions together with ideas from [Edt22a] for constructing symplectic embeddings starting with suitable Hamiltonian diffeomorpisms. In [Edt22a], one of us showed that in dimension four the equality $\overline{c} = \mathrm{sys}$ holds also under non-perturbative assumptions: indeed, this equality holds whenever the boundary of the smooth bounded convex domain $C \subset \mathbb{C}^2$ admits a closed characteristic of minimal action which is unknotted as a closed curve on $\partial C \cong S^3$ and has self-linking number -1. The latter condition holds for domains that are C^2-close to B, but it may actually hold for any A in $\mathcal{A}$ which is convex, see the open question concluding the introduction in [HWZ98].

Statement (ii) is more subtle. In dimension four, [Edt22a] proves the equality $\underline{c} = \mathrm{sys}$ on a C^3-neighborhood of the ball using global surfaces of section for the characteristic foliation of ∂A. The novelty here is that we shall prove this in all dimensions, while at the same time improving the closeness assumption from C^3 to C^2. Note that domains that are sufficiently C^2-close to the ball are convex, whereas any C^1-neighborhood of B contains non-convex domains. This is consistent with the following statement which implies that the C^2-closeness assumption in Theorem 1 (ii) is optimal.

PROPOSITION 1. *There exists a smooth family of domains $\{A_\lambda\}_{\lambda \in (0,\lambda_0]}$ in $\mathcal{A}$ which C^1-converges to the ball B for $\lambda \to 0$, and such that*

$$\underline{c}(A_\lambda) < \mathrm{sys}(A_\lambda) \le \overline{c}(A_\lambda) \qquad \forall \lambda \in (0, \lambda_0]. \tag{4}$$

Here, the second inequality holds for any domain in $\mathcal{A}$, just because there exist normalized capacities which are spectral. So the new statement is the first strict inequality.

In the proof of Theorem 1 (ii), we exploit the S^1-symmetry of the ball. Recall that if the domain $A \in \mathcal{A}$ is convex and invariant under the S^1-action on $\mathbb{C}^n$ given by multiplication by unitary complex numbers, then the equivalence of all normalized symplectic capacities for A follows from a short argument of Ostrover: by the S^1-invariance, the largest ball centered at the origin and contained in A touches ∂A along a closed characteristic, proving that $\underline{c}(A) \geq \mathrm{sys}(A)$. By convexity, A is contained in the cylinder over the disk spanned by this closed characteristic, proving that $\overline{c}(A) \leq \mathrm{sys}(A)$, and the equivalence of all normalized symplectic capacities for A follows from (1). Under a stronger symmetry assumption, namely the invariance under the n-torus action given by multiplication of each coordinate in $\mathbb{C}^n$ by unitary complex numbers, the equivalence of all normalized symplectic capacities holds also by replacing convexity by a weaker monotonicity property (see [GHR22] for the 4-dimensional case and [CGH23] for the general case).

We will prove Theorem 1 (ii) by showing that any $A \in \mathcal{A}$ which is C^2-close to B is symplectomorphic to a domain A' which, albeit not being S^1-invariant, is such that the largest ball centered at the origin and contained in A' touches $\partial A'$ along a closed characteristic. This fact will be deduced from a general theorem on "quasi-invariant" contact forms, which is described further down in this introduction. The same theorem implies that the smallest ball centered at the origin and containing A' touches $\partial A'$ along a closed characteristic. The latter fact has a consequence for higher symplectic capacities, which we now describe.

Higher symplectic capacities. In [EH90], Ekeland and Hofer used the S^1-invariance of the Hamiltonian action functional in order to define a sequence of symplectic capacities, only the first of which is normalized (it is the already mentioned Ekeland–Hofer capacity from [EH89]). In [GH18], Gutt and Hutchings introduced a sequence of symplectic capacities using S^1-equivariant symplectic homology, the first of which is the already mentioned normalized symplectic capacity from [FHW94]. Conjecturally, these two sequences of symplectic capacities coincide.

Let $\widehat{c}_k$ be either the k-th Ekeland–Hofer capacity or Gutt–Hutchings capacity. The considerations below hold for both of them, as these symplectic capacities do coincide on ellipsoids. Indeed, any ellipsoid in $\mathbb{C}^n$ is linearly symplectic equivalent to an ellipsoid of the form

$$E(a_1, \ldots, a_n) \coloneqq \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \textstyle\sum_{j=1}^n \frac{\pi}{a_j} |z_j|^2 < 1 \right\},$$

where the a_j are positive numbers, and

$$\begin{aligned} \widehat{c}_k(E(a_1, \ldots, a_n)) = {} & k\text{-th element in the sequence obtained by ordering the list} \\ & \big(ha_j\big)_{h \in \mathbb{N}, j=1,\ldots,n} \text{ in increasing order, allowing repetitions,} \end{aligned}$$

as proven in [EH90, Proposition 4] and [GH18, Example 1.8]. Building on this, we shall say that a symplectic capacity c is *k-normalized* if it satisfies the following condition:

- (k-th normalization) $c(E) = \widehat{c}_k(E)$ for every ellipsoid E.

Normalized symplectic capacities, as defined at the beginning of this introduction, are 1-normalized. The k-th Ekeland–Hofer and Gutt–Hutchings symplectic capacities are k-normalized. The functions

$$\underline{c}_k(A) \coloneqq \sup\{\widehat{c}_k(E) \mid E \text{ ellipsoid}, \exists \phi : \mathbb{C}^n \to \mathbb{C}^n \text{ symplectomorphisms with } \phi(E) \subset A\},$$
$$\overline{c}_k(A) \coloneqq \inf\{\widehat{c}_k(E) \mid E \text{ ellipsoid}, \exists \phi : \mathbb{C}^n \to \mathbb{C}^n \text{ symplectomorphisms with } \phi(A) \subset E\},$$

are k-normalized capacities, and any k-normalized capacity c satisfies

$$\underline{c}_k \leq c \leq \overline{c}_k.$$

The coincidence of all k-normalized capacities for the bounded open set $A \subset \mathbb{C}^n$ is equivalent to the equality $\underline{c}_k(A) = \overline{c}_k(A)$ and hence to the fact that A contains the symplectic image of an ellipsoid E and can be symplectically embedded into an ellipsoid E' such that the difference $\widehat{c}_k(E') - \widehat{c}_k(E)$ is arbitrarily small. Our next result concerns the case $k = n$, where $2n$ is the dimension of the symplectic vector space we are considering.

THEOREM 2. *There exists a C^2-neighborhood $\mathcal{B}$ of B in $\mathcal{A}$ such that for any A in $\mathcal{B}$ and any n-normalized symplectic capacity c the following facts hold:*

(i) There exists an ellipsoid E with $c(E) = c(A)$ and a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\phi(E) \subset A$.

(ii) There exists a symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ mapping A into the ball B' such that $c(B') = c(A)$.

In particular, all n-normalized symplectic capacities coincide on $\mathcal{B}$.

It is natural to ask whether all k-normalized capacities coincide on bounded convex domains of $\mathbb{C}^n$. As we shall see, the answer is negative for most pairs (n, k), but besides the pairs $(n, 1)$ (corresponding to the strong Viterbo conjecture) there are other pairs for which we do not know the answer. The next result shows that all 2-normalized capacities of polydiscs in $\mathbb{C}^2$ coincide and that this is not true anymore for $k \geq 3$.

PROPOSITION 2. *Let $P(a, b)$ be the following four-dimensional polydisk:*

$$P(a, b) \coloneqq \{(z_1, z_2) \in \mathbb{C}^2 \mid \tfrac{\pi}{a}|z_1|^2 < 1,\ \tfrac{\pi}{b}|z_2|^2 < 1\},$$

where $a, b > 0$. Then:

(i) All 2-normalized symplectic capacities coincide on $P(a, b)$: if c is such a capacity, then

$$c(P(a, b)) = 2\min\{a, b\}.$$

(ii) For every $k \in \mathbb{N}$ we have

$$\underline{c}_k(P(1,1)) = \widehat{c}_k(E(1,2)) \leq k = \overline{c}_k(P(1,1)),$$

and the inequality is strict if and only if $k \geq 3$.

More generally, Gutt and Ramos have very recently proved that all 2-normalized capacities coincide on convex toric domains in $\mathbb{C}^2$, see [GR23]. They also showed that when $k \geq \max\{n, 3\}$ the following strict inequality holds

$$\underline{c}_k(P(1,\dots,1)) < \widehat{c}_k(P(1,\dots,1)) = k,$$

where

$$P(1,\dots,1) \coloneqq \{z \in \mathbb{C}^n \mid \pi|z_j|^2 < 1 \ \forall j = 1,\dots,n\}$$

is the equilateral polydisc in $\mathbb{C}^n$. Therefore, for these pairs (n,k) the k-normalized capacities do not coincide on bounded convex domains. In particular, the open C^2-neighborhood $\mathcal{B}$ of Theorem 2 does not contain all (smooth bounded) convex domains.

These results suggest that the case $k = n = 2$ may be somehow special. As far as we know, it is indeed possible that all 2-normalized capacities coincide on bounded convex domains of $\mathbb{C}^2$. Notice that this would imply the following inequality

$$\widehat{c}_2(A)^2 \leq 4\,\mathrm{vol}(A) \tag{5}$$

since the ellipsoid $E(1,2)$ maximizes $\widehat{c}_2$ among all ellipsoids with the same volume. Some evidence for the validity of (5) comes from a recent paper of Baracco, Bernardi, Lange and Mazzucchelli: in [BBLM23], the authors studied the Ekeland–Hofer capacities on smooth starshaped domains in $\mathbb{C}^2$ and for each k identified the local maximizers of the ratio

$$\frac{\widehat{c}_k(A)^2}{\mathrm{vol}(A)}$$

with respect to the C^3-topology on $\mathcal{A}$. In particular, their analysis shows that the inequality (5) holds for every A in a C^3-neighborhood of $E(1,2)$, with equality if and only if A is symplectomorphic to a rescaled copy of the ellipsoid $E(1,2)$.

The proofs of Theorems 1 and 2 are based on a general result about contact forms which are close to Zoll ones, which might have independent interest and is described in the next part of this introduction.

Quasi invariant contact forms. Let M be a $(2n-1)$-dimensional closed manifold, $n \geq 1$, and α a contact form on M, that is, a smooth 1-form such that $\alpha \wedge d\alpha^{n-1}$ is nowhere vanishing. The corresponding Reeb vector field R_α is defined by the identities

$$\imath_{R_\alpha} d\alpha = 0, \qquad \imath_{R_\alpha}\alpha = 1.$$

The flow of R_α is called the Reeb flow of α. The set of periods of the closed orbits of the Reeb flow of α is closed in $\mathbb{R}$ and we denote by $\mathrm{sys}(\alpha) > 0$ its minimum (setting $\mathrm{sys}(\alpha) \coloneqq +\infty$ if R_α has no closed orbits, a possibility which never occurs if the Weinstein conjecture holds true). If A belongs to the set $\mathcal{A}$ defined above, then the standard primitive

$$\lambda_0 \coloneqq \frac{1}{2}\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j) \tag{6}$$

of ω_0 restricts to a contact form on ∂A, the Reeb vector field of $\lambda_0|_{\partial A}$ spans the characteristic distribution of ∂A, and we have the identity

$$\mathrm{sys}(A) = \mathrm{sys}(\lambda_0|_{\partial A}),$$

which justifies the choice of a common notation.

The contact form α is said to be Zoll if all the orbits of its Reeb flow are closed and have the same minimal period, that is, if the Reeb flow of α defines a free S^1-action on M. We shall deduce Theorems 1 and 2 from the following theorem.

THEOREM 3. *Let α_0 be a Zoll contact form on the closed manifold M. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that, for any contact form α on M with $\|\alpha - \alpha_0\|_{C^2} < \delta$, there exists a diffeomorphism $\varphi : M \to M$ with the property that*

$$\varphi^*\alpha = Te^h\alpha_0, \tag{7}$$

where:

(a) T is a smooth positive function on M invariant under the free S^1-action defined by the Reeb flow of α_0;

(b) h is a smooth real function on M such that h and dh vanish on the critical set of T;

(c) $\min_M Te^h = \min_M T$ and $\max_M Te^h = \max_M T$;

(d) any closed orbit of R_α is either long*, meaning that its minimal period is larger than $\frac{1}{\epsilon}$, or* short*, meaning that its minimal period belongs to the interval $(\mathrm{sys}(\alpha_0) - \epsilon, \mathrm{sys}(\alpha_0) + \epsilon)$;*

(e) the short closed orbits of R_α are precisely the images via φ of those orbits γ of the free S^1-action defined by the Reeb flow of α_0 consisting of critical points of T, and the minimal period of such an orbit is $\mathrm{sys}(\alpha_0)T(\gamma)$.

Moreover, the map $\alpha \mapsto (\varphi, T, h)$ is smooth and maps α_0 to $(\mathrm{id}, 1, 0)$.

The smoothness mentioned at the end of the statement is meant in the diffeological sense: if $\{\alpha_t\}_{t\in\mathbb{R}^k}$ is a smooth family of contact forms having C^2-distance less than δ from α_0, then the diffeomorphism φ_t and the functions T_t and h_t which are given by the above theorem depend smoothly on $t \in \mathbb{R}^k$. This immediately implies that if $\ker\alpha = \ker\alpha_0$ then

the diffeomorphism φ belongs to the identity component in the contactomorphism group of $(M, \ker \alpha_0)$.

By statement (e), the S^1-invariant function T in the above theorem gives us a finite dimensional variational principle for detecting the short closed Reeb orbits of α and, together with (c), implies the identity

$$\mathrm{sys}(\alpha) = \mathrm{sys}(\alpha_0) \min_M T = \mathrm{sys}(\alpha_0) \min_M T e^h.$$

An immediate consequence of Theorem 3 is then the inequality

$$\frac{\mathrm{sys}(\alpha)^n}{\mathrm{vol}(M, \alpha \wedge d\alpha^{n-1})} = \mathrm{sys}(\alpha_0)^n \frac{\min_M T^n e^{nh}}{\int_M T^n e^{nh}\, \alpha_0 \wedge d\alpha_0^{n-1}} \leq \frac{\mathrm{sys}(\alpha_0)^n}{\mathrm{vol}(M, \alpha_0 \wedge d\alpha_0^{n-1})}, \tag{8}$$

which gives us the following corollary.

Corollary 1. *Zoll contact forms on the closed manifold M are C^2-local maximizers of the systolic ratio*

$$\rho_{\mathrm{sys}}(\alpha) := \frac{\mathrm{sys}(\alpha)^n}{\mathrm{vol}(M, \alpha \wedge d\alpha^{n-1})},$$

and if α is C^2-close to the Zoll contact form α_0 and satisfies $\rho_{\mathrm{sys}}(\alpha) = \rho_{\mathrm{sys}}(\alpha_0)$, then there is a diffeomorphism $\varphi : M \to M$ such that $\varphi^* \alpha = c\, \alpha_0$ for some constant $c > 0$.

The latter assertion follows from the fact that if the equality holds in (8) then the function T is constant and hence by Theorem 3 (b) and (7) the diffeomorphism φ has the required property. Under a stronger closeness assumption, this systolic inequality is proven in [AB23] using arguments which will also be used in the proof of Theorem 3.

A modification of the proof of Proposition 1 shows that the C^2-closeness assumption is sharp in Corollary 1.

Proposition 3. *For every Zoll contact form α_0 there exists a smooth family of contact forms $\{\alpha_\lambda\}_{\lambda \in (0, \lambda_0]}$ which C^1-converges to α_0 for $\lambda \to 0$, and satisfies*

$$\rho_{\mathrm{sys}}(\alpha_\lambda) > \rho_{\mathrm{sys}}(\alpha_0) \qquad \forall \lambda \in (0, \lambda_0].$$

In this paper, our interest in Theorem 3 is that it allows us to construct symplectic embedding of balls and into balls, thus proving Theorems 1 and 2.

Contact forms of the form $Te^h \alpha_0$ with α_0 Zoll and T, h satisfying the conditions (a)-(e) of Theorem 3 might be called "quasi-invariant", the "quasi" referring to the presence of the non-S^1-invariant function e^h. Note the similarity with the notion of quasi-autonomous Hamiltonian: a compactly supported time-dependent Hamiltonian $H = H(t, x)$ on a symplectic manifold (M, ω) is said to be quasi-autonomous if the sets of maximizers and minimizers of the functions $H(t, \cdot)$ do not depend on t. As shown in [BP94] for $(\mathbb{C}^n, \omega_0)$ and in [LM95] for closed symplectic manifolds, any compactly supported Hamiltonian diffeomorphism which is C^1-close to the identity is generated by a quasi-autonomous Hamiltonian.

The proof of this fact is based on the Weinstein tubular neighborhood theorem and on a version of the Hamilton–Jacobi equation.

Theorem 3 seems much more subtle. Our proof is based on a normal form for contact forms which are closed to Zoll ones (proved by two of us in [AB23]), and uses averaging techniques and Moser's homotopy argument.

Paths in the Hamiltonian group which are generated by quasi-autonomous Hamiltonians are minimizing geodesics in the Hofer metric, see again [BP94] and [LM95]. Results of the same flavour hold also for a Banach–Mazur pseudo-metric on the space of contact forms on a given contact manifold. In the next subsection, we discuss these results.

A contact Banach–Mazur pseudo-metric. Let ξ be a co-oriented contact structure on the closed manifold M. We denote by $\mathcal{F}(M,\xi)$ the set of contact forms α defining ξ, meaning that $\ker\alpha = \xi$ and α is positive on tangent vectors which are positively transverse to ξ. We denote by $\mathrm{Cont}_0(M,\xi)$ the identity component of the contactomorphism group of (M,ξ), i.e. the set of all diffeomorphisms of M which are smoothly isotopic to the identity by a path of diffeomorphisms whose differential maps ξ to itself. The following function

$$d(\alpha,\beta) \coloneqq \inf\{\max f - \min f \mid \exists\varphi \in \mathrm{Cont}_0(M,\xi) \text{ such that } \varphi^*\beta = e^f\alpha\} \tag{9}$$

is readily seen to be a pseudo-metric on $\mathcal{F}(M,\xi)$. The group $G \coloneqq \mathbb{R}\times\mathrm{Cont}_0(M,\xi)$ acts on $\mathcal{F}(M,\xi)$ by

$$(s,\varphi)\cdot\alpha \mapsto e^s\varphi^*\alpha,$$

and the pseudo-metric d is clearly invariant under this action, meaning that

$$d(e^s\alpha,e^t\beta) = d(\alpha,\beta), \quad \forall s,t\in\mathbb{R}, \qquad d(\varphi^*\alpha,\psi^*\beta) = d(\alpha,\beta), \quad \forall\varphi,\psi\in\mathrm{Cont}_0(M,\xi),$$

for every $\alpha,\beta\in\mathcal{F}(M,\xi)$. In particular, the pseudo-distance of any two elements of the same G-orbit is zero. More precisely, d measures the minimal distance between the G-orbits of two contact forms in $\mathcal{F}(M,\xi)$ in the C^0-topology and can be therefore considered as a contact analogue of the Banach–Mazur pseudo-metric on the set of convex bodies in $\mathbb{R}^n$ (there, the action is given by the group of affine transformations).

Remark 1. Banach–Mazur pseudo-metrics on contact forms were introduced by Rosen and Zhang in [RZ21], where they considered the slightly different pseudo-metric

$$d'(\alpha,\beta) \coloneqq \inf\{\max|f| \mid \exists\varphi\in\mathrm{Cont}_0(M,\xi) \text{ such that } \varphi^*\beta = e^f\alpha\}.$$

This pseudo-metric is invariant only under the action of $\mathrm{Cont}_0(M,\xi)$ and satisfies

$$d'(e^s\alpha,e^t\alpha) = |t-s|, \qquad d'(e^s\alpha,e^t\beta)\le|t-s|+d'(\alpha,\beta), \qquad \forall s,t\in\mathbb{R}.$$

Here, we prefer to work with the pseudo-metric d which is invariant also under rescalings, but all the results below hold for d' as well. A different pseudo-metric on $\mathcal{F}(M,\xi)$ is studied in Melistas' PhD thesis [Mel21]. Banach–Mazur pseudo-metrics on contact forms are strictly related to symplectically invariant Banach–Mazur pseudo-metrics on domains, whose definition was suggested by Ostrover and Polterovich and which are studied in [PRSZ20, SZ21, Ush22].

By invariance, d descends to the quotient of $\mathcal{F}(M,\xi)$ by G. This quotient pseudo-metric need not be a genuine metric either. Indeed, by construction $d(\alpha,\beta)$ vanishes if and only if β belongs to the C^0-closure of the G-orbit of α. Therefore, the element $G\cdot\alpha$ in the orbit space $\mathcal{F}(M,\xi)/G$ has positive pseudo-distance from all the other elements if and only if $G\cdot\alpha$ is C^0-closed in $\mathcal{F}(M,\xi)$. The next examples show that some G-orbits are indeed C^0-closed, but other ones are not even C^∞-closed.

In both examples, we consider the standard tight contact structure ξ_{st} on S^3 which is defined by the restriction of the one-form λ_0 to $S^3\subset\mathbb{C}^2$. For every pair of positive numbers a,b we denote by $\varepsilon_{a,b}\in\mathcal{F}(S^3,\xi_{\mathrm{st}})$ the contact form which is given by the pull-back of the restriction of λ_0 to the boundary of the ellipsoid $E(a,b)$ by the radial projection $S^3\to\partial E(a,b)$.

Example 1. If $\frac{a}{b}$ is rational then the G-orbit of $\varepsilon_{a,b}$ is C^0-closed in $\mathcal{F}(S^3,\xi_{\mathrm{st}})$. Here is a proof based on Hutchings' embedded contact homology (ECH), for which we refer to [Hut14]. Let (s_j,φ_j) be a sequence in G such that $e^{s_j}\varphi_j^*\varepsilon_{a,b}$ C^0-converges to some contact form α in $\mathcal{F}(S^3,\xi_{\mathrm{st}})$. Then the total volume of this sequence of contact forms converges to the total volume of α. This implies that the sequence s_j converges to a real number s, and hence $\varphi_j^*\varepsilon_{a,b}$ converges to the contact form $\beta\coloneqq e^{-s}\alpha$. Denote by $\sigma_k:\mathcal{F}(S^3,\xi_{\mathrm{st}})\to\mathbb{R}$ the ECH-spectral invariant which is associated with the generator of degree $2k$ of the embedded contact homology of (S^3,ξ_{st}). Since the ECH-spectral invariants are invariant under contactomorphisms and C^0-continuous, we have $\sigma_k(\beta)=\sigma_k(\varepsilon_{a,b})$ for every k. As shown by Cristofaro-Gardiner and Mazzucchelli in [CGM20, Lemma 3.1], any contact form β in $\mathcal{F}(S^3,\xi_{\mathrm{st}})$ having the same ECH-spectral invariants of $\varepsilon_{a,b}$ with $\frac{a}{b}$ rational is equivalent to $\varepsilon_{a,b}$: there exists $\psi\in\mathrm{Cont}_0(S^3,\xi_{\mathrm{st}})$ such that $\beta=\psi^*\varepsilon_{a,b}$ (see also [MR23]). Therefore, $\alpha=e^s\psi^*\varepsilon_{a,b}$ belongs to the G-orbit of $\varepsilon_{a,b}$, which is then C^0-closed.

Example 2. If $\frac{a}{b}$ is irrational and not Diophantine then the G-orbit of $\varepsilon_{a,b}$ is not C^∞-closed in $\mathcal{F}(S^3,\xi_{\mathrm{st}})$. Indeed, by the Anosov–Katok conjugation method from [AK70b] and [Kat73] one can construct a sequence (φ_j) in $\mathrm{Cont}_0(S^3,\xi_{\mathrm{st}})$ such that $(\varphi_j^*\varepsilon_{a,b})$ C^∞-converges to a contact form $\alpha\in\mathcal{F}(S^3,\xi_{\mathrm{st}})$ whose Reeb flow has a dense orbit. See Theorem A.1 and Remark A.2 in the appendix below. The contact form α cannot belong to the G-orbit of $\varepsilon_{a,b}$ because the Reeb flow of the latter contact form does not have dense orbits.

Note however that the domain in $\mathbb{C}^2$ which corresponds to α in the correspondence between domains in $\mathcal{A}$ and elements of $\mathcal{F}(S^3,\xi_{\mathrm{st}})$ is symplectomorphic to the open ellipsoid $E(a,b)$, see Theorem A.3 in the appendix below. This is a case of "invisible symplectic boundaries", the first examples of which were found by Eliashberg and Hofer in [EH96].

If $\frac{a}{b}$ is Diophantine we do not know whether the G-orbit of $\varepsilon_{a,b}$ is closed in some C^k-topology. Nor we know whether there are domains in $\mathcal{A}$ which are symplectomorphic to the open ellipsoid $E(a,b)$ but such that the restriction of λ_0 to ∂A is not conjugate to $\varepsilon_{a,b}$. These questions are related to an open question of Hermann about area-preserving diffeomorphisms of the annulus (see [Her98a, Question 3.2]), whose reformulation for flows is the following: Is a Reeb flow on (S^3,ξ_{st}) with only two closed orbits having periods with Diophantine ratio smoothly conjugated to the Reeb flow of an irrational ellipsoid?

It is interesting to investigate the situation in which the infimum defining $d(\alpha,\beta)$ is achieved, as this fact implies the existence of common invariant probability measures and of minimizing geodesics connecting α and β in $\mathcal{F}(M,\xi)$. In this context, a continuous curve $\gamma:[0,1]\to\mathcal{F}(M,\xi)$ is said to be a minimizing geodesic if its length

$$\text{Length}(\gamma) \coloneqq \sup\Big\{\sum_{j=1}^{k} d(\gamma(t_{j-1}),\gamma(t_j)) \;\Big|\; k\in\mathbb{N},\ 0=t_0<t_1<\cdots<t_k=1\Big\}$$

coincides with $d(\gamma(0),\gamma(1))$. The precise result is as follows.

THEOREM 4. *Let $\alpha,\beta\in\mathcal{F}(M,\xi)$ and assume that*

$$d(\alpha,\beta)=\max_M f-\min_M f$$

for some $f\in C^\infty(M)$ and $\varphi\in\text{Cont}_0(M,\xi)$ such that $\varphi^*\beta=e^f\alpha$. Then the following facts hold:

(i) There are probability measures which are supported in the level sets $f^{-1}(\min f)$ and $f^{-1}(\max f)$ and are invariant with respect to the Reeb flows of both α and $\varphi^*\beta$.

(ii) For every path of contactomorphisms $\{\psi_t\}_{t\in[0,1]}\subset\text{Cont}_0(M,\xi)$ such that $\psi_0=\text{id}$ and $\psi_1=\varphi^{-1}$, the path

$$\gamma(t)\coloneqq\psi_t^*(e^{tf}\alpha),\quad t\in[0,1],$$

is a minimizing geodesic with $\gamma(0)=\alpha$ and $\gamma(1)=\beta$.

In particular, if the contact form α in the above proposition is Zoll, then (i) implies that the Reeb flow of β has a closed orbit which is the image by φ of a closed orbit of the Reeb flow of α and whose β-action is $e^{\min f}$ times its α-action. As an immediate consequence, we obtain the following statement: If $\alpha\in\mathcal{F}(M,\xi)$ is a Zoll contact form then any β in $\mathcal{F}(M,\xi)$ such that the infimum in the definition of $d(\alpha,\beta)$ is achieved satisfies the contact systolic inequality $\rho_{\text{sys}}(\beta)\le\rho_{\text{sys}}(\alpha)$.

The fact that any co-oriented contact structure ξ admits defining contact forms with arbitrarily large systolic ratio (see [Sağ21]) shows that the infimum in the definition of $d(\alpha,\beta)$ may not be achieved. Actually, Example 3 shows that $d(\alpha,\beta)$ may not be achieved for contact forms β which are C^1-close to a Zoll contact form α. It is however achieved when the contact form β is C^2-close enough to the Zoll contact form α. Indeed, our last result states the following,

THEOREM 5. *Let α_0 be a Zoll contact form defining a co-oriented contact structure ξ on the closed manifold M. If the contact form $\alpha\in\mathcal{F}(M,\xi)$ is C^2-close enough to α_0 then there exist $f\in C^\infty(M)$ and $\varphi\in\text{Cont}_0(M,\xi)$ such that*

$$\varphi^*\alpha=e^f\alpha_0\quad\textit{and}\quad d(\alpha_0,\alpha)=\max_M f-\min_M f=\log\frac{T_{\max}(\alpha)}{T_{\min}(\alpha)},$$

where $T_{\max}(\alpha)$ and $T_{\min}(\alpha)=\text{sys}(\alpha)$ are the maximum and minimum periods of the short closed orbits of α, in the sense of Theorem 3 (d). In particular, there are minimizing geodesics connecting α_0 to α.

This result will be deduced from Theorem 3 using a sequence of "elementary" spectral invariants for elements of $\mathcal{F}(M,\xi)$, whose definition mimics analogous spectral invariants for domains and Hamiltonian diffeomorphisms which were introduced by McDuff and Siegel in [MS23], Hutchings in [Hut22a, Hut22b] and one of us in [Edt22b].

Organization of the paper. In Section 1, we show how suitable compactly supported Hamiltonian diffeomorphisms of the unit ball in $\mathbb{C}^{n-1}$ can be lifted as characteristic flows on the boundary of certain domains in $\mathbb{C}^n$. We use this construction in Section 2 to prove Propositions 1 and 3, and in Section 3 to prove part (i) of Theorem 1. In Section 4, we recall and complement the main result from [AB23], which is then used in Section 5 to prove Theorem 3. Building on this theorem, we prove statement (ii) of Theorem 1 in Section 6. The proof of Theorem 2 is similar to the proof of Theorem 1 and is sketched in Section 7. Proposition 2 is proved in Section 8 and Theorem 4 is proved in Section 9. In Section 10, we introduce the spectral invariants which are mentioned above. In Section 11, we prove Theorem 5. Appendix A contains a discussion of the results which are mentioned in Example 2.

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1 From Hamiltonian diffeomorphisms to domains

In this section, which is inspired by analogous 4-dimensional arguments from [Edt22a], we show how suitable compactly supported Hamiltonian diffeomorphisms of the unit ball in $\mathbb{C}^{n-1}$ can be lifted as characteristic flows on the boundary of certain domains in $\mathbb{C}^n$. These arguments will be used here to prove Propositions 1 and 3 in Section 2 and to prove part (i) of Theorems 1 and 2 in Sections 3 and 7, respectively.

Setting $\mathbb{T} \coloneqq \mathbb{R}/\mathbb{Z}$, we consider the domain

$$\Omega \coloneqq \{(s,t,w) \in \mathbb{R} \times \mathbb{T} \times \mathbb{C}^{n-1} \mid s > \pi(|w|^2 - 1)\}$$

and the smooth map

$$\Phi : \Omega \to \mathbb{C}^n, \qquad \Phi(s,t,w) \coloneqq e^{2\pi i t}\left(\sqrt{1 + \tfrac{s}{\pi} - |w|^2}, w\right), \tag{1.1}$$

which is a diffeomorphism onto $\mathbb{C}^* \times \mathbb{C}^{n-1}$. Recall that λ_0 denotes the standard primitive (6) of the standard symplectic form ω_0 of $\mathbb{C}^n$. Denoting by $\widehat{\lambda}_0$ and $\widehat{\omega}_0$ the analogous forms on $\mathbb{C}^{n-1}$, we have

$$\Phi^* \lambda_0 = (\pi + s)\, dt + \widehat{\lambda}_0, \tag{1.2}$$

where (s,t) denote the coordinates in $\mathbb{R} \times \mathbb{T}$. By differentiating (1.2), we obtain that Φ is a symplectomorphism from $(\Omega, ds \wedge dt + \widehat{\omega}_0)$ to $(\mathbb{C}^* \times \mathbb{C}^{n-1}, \omega_0)$. Since

$$|\Phi(s,t,w)| = \sqrt{1 + \tfrac{s}{\pi}}, \tag{1.3}$$

this symplectomorphism maps each domain $\{(s,t,w) \in \Omega \mid s < s_0\}$ to a ball centered at 0 minus the linear subspace $\{z_1 = 0\}$. In particular, the hypersurface $\{(s,t,w) \in \Omega \mid s = 0\} = \{0\} \times \mathbb{T} \times \widehat{B}$, where $\widehat{B}$ denotes the open unit ball in $\mathbb{C}^{n-1}$, is mapped onto the unit sphere ∂B minus the linear subspace $\{z_1 = 0\}$.

Given a smooth compactly supported time-periodic Hamiltonian $H : \mathbb{T} \times \widehat{B} \to \mathbb{R}$ such that

$$H(t,w) > -\pi(1 - |w|^2) \qquad \forall (t,w) \in \mathbb{T} \times \widehat{B}, \tag{1.4}$$

we consider the subset of $\mathbb{C}^n$ given by

$$D(H) \coloneqq \Phi\big(\{(s,t,w) \in \Omega \mid w \in \widehat{B},\ s < H(t,w)\}\big) \cup \{z \in \mathbb{C}^n \mid z_1 = 0,\ |z| < 1\},$$

which is readily seen to be an open neighborhood of 0 diffeomorphic to a ball. The boundary of $D(H)$ is smooth and is given by the closure of the image of the graph of H by Φ:

$$\partial D(H) = \overline{\Phi(\Gamma(H))} = \Phi(\Gamma(H)) \cup \{z \in \mathbb{C}^n \mid z_1 = 0,\ |z| = 1\},$$

where

$$\Gamma(H) \coloneqq \{(s,t,w) \in \mathbb{R} \times \mathbb{T} \times \widehat{B} \mid s = H(t,w)\}.$$

The domain $D(H)$ coincides with B near the subspace $\{z_1 = 0\}$ and its boundary is transverse to the vector field $\nabla |z_1|$ away from $\{z_1 = 0\}$. Conversely, any smooth domain with these properties has the form $D(H)$ for a suitable compactly supported smooth function H on $\mathbb{T} \times \widehat{B}$ satisfying (1.4). Moreover, $D(H)$ is C^k-close to B when H is C^k-small and supported uniformly away from the boundary of $\mathbb{T} \times \widehat{B}$. Denote by

$$\phi_H^t : \widehat{B} \to \widehat{B}, \qquad \phi_H^0 = \mathrm{id},$$

the smooth path of compactly supported Hamiltonian diffeomorphisms of $(\widehat{B}, \widehat{\omega}_0)$ which is obtained by integrating the time-periodic Hamiltonian vector field X_H given by

$$\imath_{X_{H_t}} \widehat{\omega}_0 = dH_t, \qquad H_t(w) \coloneqq H(t,w).$$

We recall that the Calabi invariant of $\varphi \coloneqq \phi_H^1$ is the real number

$$\mathrm{Cal}(\varphi) \coloneqq \int_{\mathbb{T} \times \widehat{B}} H\, dt \wedge \widehat{\omega}_0^{n-1}.$$

The notation is justified by the fact that the above integral does not depend on the choice of the compactly supported Hamiltonian H defining φ. The Calabi invariant defines a real-valued homomorphism on the group of compactly supported Hamiltonian diffeomorphisms of $(\widehat{B}, \widehat{\omega}_0)$. The action of a fixed point w of $\varphi = \phi_H^1$ is given by

$$\mathcal{A}_\varphi(w) \coloneqq \int_{\substack{t \mapsto \phi_H^t(w) \\ t \in [0,1]}} \widehat{\lambda}_0 + \int_0^1 H(t, \phi_H^t(w))\, dt.$$

This quantity does not depend on the choice of the compactly supported Hamiltonian H defining φ either. See [MS17, Chapter 9 and 10] for the proofs of the above mentioned facts about the Calabi invariant and the action. The next result relates the properties of the Hamiltonian diffeomorphism determined by H to those of the domain $D(H)$.

PROPOSITION 1.1. *Let $H : \mathbb{T} \times \widehat{B} \to \mathbb{R}$ be a smooth compactly supported Hamiltonian satisfying* (1.4) *and set $\varphi \coloneqq \phi_H^1$. Then:*

(i) $\operatorname{vol}(D(H)) = \dfrac{\pi^n}{n!} + \dfrac{1}{(n-1)!}\operatorname{Cal}(\varphi) = \operatorname{vol}(B) + \dfrac{1}{(n-1)!}\operatorname{Cal}(\varphi).$

(ii) There is a one-to-one correspondence between the periodic points w of φ and the closed characteristics γ on the boundary of $D(H)$ other than those which foliate the submanifold $\partial D(H) \cap \{z_1 = 0\} = \partial B \cap \{z_1 = 0\}$. The corresponding actions are related by

$$\int_\gamma \lambda_0 = k\pi + \mathcal{A}_{\varphi^k}(w),$$

where $k \in \mathbb{N}$ denotes the minimal period of w.

(iii) Let $\{H^\lambda\}_{\lambda \in [0,1]}$ be a smooth family of compactly supported Hamiltonians on $\mathbb{T} \times \widehat{B}$ satisfying (1.4) *and such that $\varphi \coloneqq \phi_{H^\lambda}^1$ does not depend on λ. Then there exists a compactly supported symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\psi(D(H^0)) = D(H^1)$.*

Proof. (i) The map

$$\psi : \mathbb{T} \times \widehat{B} \to \partial D(H), \quad \psi(t, w) \coloneqq \Phi(H(t, w), t, w),$$

is a diffeomorphism onto $\partial D(H) \setminus \{z_1 = 0\}$ and hence

$$n! \operatorname{vol}(D(H)) = \int_{D(H)} \omega_0^n = \int_{\partial D(H)} \lambda_0 \wedge \omega_0^{n-1} = \int_{\mathbb{T} \times \widehat{B}} \psi^*(\lambda_0 \wedge \omega_0^{n-1}).$$

From (1.2) we deduce

$$\psi^* \lambda_0 = (\pi + H)\, dt + \widehat{\lambda}_0, \qquad \psi^* \omega_0 = d\psi^* \lambda_0 = dH \wedge dt + \widehat{\omega}_0,$$

and hence

$$\psi^*(\lambda_0 \wedge \omega_0^{n-1}) = (\pi + H) dt \wedge \widehat{\omega}_0^{n-1} + (n-1) dH \wedge dt \wedge \widehat{\lambda}_0 \wedge \widehat{\omega}_0^{n-2}.$$

Since $dH \wedge dt \wedge \widehat{\lambda}_0 \wedge \widehat{\omega}_0^{n-2}$ differs from $H\,dt \wedge \widehat{\omega}_0^{n-1}$ by an exact form, we obtain

$$n!\,\mathrm{vol}(D(H)) = \pi \int_{\mathbb{T}\times\widehat{B}} dt \wedge \widehat{\omega}_0^{n-1} + n \int_{\mathbb{T}\times\widehat{B}} H\,dt \wedge \widehat{\omega}_0^{n-1} = \pi^n + n\,\mathrm{Cal}(\varphi),$$

which implies (i).

(ii) The hypersurface $\Gamma(H)$ is the zero level set of the autonomous Hamiltonian

$$\widetilde{H}(s,t,w) \coloneqq H(t,w) - s \tag{1.5}$$

on the symplectic manifold $(\mathbb{R}\times\mathbb{T}\times\widehat{B}, ds\wedge dt + \widehat{\omega})$, whose Hamiltonian vector field is easily seen to be

$$X_{\widetilde{H}}(s,t,w) = \partial_t H(t,w)\,\partial_s + \partial_t + X_{H_t}(w). \tag{1.6}$$

The closed characteristics of $\Gamma(H)$ are the closed orbits of the flow of $X_{\widetilde{H}}$ on the energy level $\{\widetilde{H}=0\}$. Every orbit of this flow intersects the hypersurface $\{t=0\}$ transversally infinitely many times, and the orbit of $(s,0,w)$ is easily seen to be given by

$$\phi_{\widetilde{H}}^\tau(s,0,w) = \big(s + H(\tau,\phi_H^\tau(w)) - H(0,w), \tau, \phi_H^\tau(w)\big), \qquad \tau\in\mathbb{R}. \tag{1.7}$$

In particular, the orbit of a point $(s,0,w)$ in $\Gamma(H)$ is the curve

$$\widehat{\gamma}(\tau) = \big(H(\tau,\phi_{X_H}^\tau(w)), \tau, \phi_H^\tau(w)\big), \qquad \tau\in\mathbb{R}.$$

This orbit is closed if and only if w is a k-periodic point of ϕ_X^1, for some $k\in\mathbb{N}$, and in this case

$$\int_{[0,k]} \widehat{\gamma}^*(\Phi^*\lambda_0) = \int_{[0,k]} \widehat{\gamma}^*((\pi+s)\,dt + \widehat{\lambda}_0) = \pi k + \mathcal{A}_{\phi_H^k}(w).$$

Statement (ii) follows by taking $\gamma \coloneqq \Phi(\widehat{\gamma})$.

(iii) For each $(\lambda,t)\in[0,1]\times\mathbb{T}$, consider the Hamiltonian diffeomorphism

$$\psi_\lambda^t \colon \widehat{B}\to\widehat{B}, \qquad \psi_\lambda^t \coloneqq \phi_{H^\lambda}^t \circ (\phi_{H^0}^t)^{-1}.$$

By assumption, we have

$$\psi_\lambda^0 = \psi_\lambda^1 = \mathrm{id}_{\widehat{B}}, \qquad \forall\lambda\in[0,1]. \tag{1.8}$$

For each $\lambda\in[0,1]$, the loop of Hamiltonian diffeomorphisms $t\mapsto\psi_\lambda^t$, $t\in\mathbb{T}$, based at $\mathrm{id}_{\widehat{B}}$ is generated by the Hamiltonian

$$G^\lambda\colon \mathbb{T}\times\widehat{B}\to\mathbb{R}, \qquad G^\lambda(t,w) = H^\lambda\#\overline{H^0}(t,w) = H^\lambda(t,w) - H^0(t,(\psi_\lambda^t)^{-1}(w)). \tag{1.9}$$

Notice that G^λ is indeed 1-periodic in t since the same is true for ψ_λ^t. Moreover, $G^0=0$. We define the diffeomorphism

$$\tilde{\psi}_\lambda\colon \mathbb{R}\times\mathbb{T}\times\widehat{B}\to\mathbb{R}\times\mathbb{T}\times\widehat{B}, \quad \tilde{\psi}_\lambda(s,t,w) = (s + G^\lambda(t,\psi_\lambda^t(w)), t, \psi_\lambda^t(w)). \tag{1.10}$$

We have $\tilde{\psi}_0 = \mathrm{id}$ and, using the formula for G^λ given in (1.9), that $\widetilde{H}^\lambda \circ \tilde{\psi}_\lambda = \widetilde{H}^0$, where $\widetilde{H}^\lambda$ is induced by H^λ as in (1.5). In particular, $\tilde{\psi}_\lambda$ maps $\Gamma(H^0)$ to $\Gamma(H^\lambda)$.

We now make the claim that the path $\lambda \mapsto \tilde{\psi}_\lambda$, $\lambda \in [0,1]$, is generated by a smooth path of Hamiltonians $F^\lambda \colon \mathbb{R} \times \mathbb{T} \times \widehat{B} \to \mathbb{R}$ which are supported in a set of the form $\mathbb{R} \times \mathbb{T} \times K$, where K is a compact subset of $\widehat{B}$. If the claim is true, we can consider the multiplication $E^\lambda \coloneqq \chi^\lambda F^\lambda$ of F^λ by a cut-off function χ^λ supported in an arbitrarily small neighborhood of $\Gamma(H^\lambda)$. Since F^λ has support in $\mathbb{R} \times \mathbb{T} \times K$, the function E^λ has compact support. Since Φ is a symplectomorphism, the function $E^\lambda \circ \Phi^{-1}$ yields a compactly supported function of the whole $\mathbb{C}^n$. The time-one map $\psi \colon \mathbb{C}^n \to \mathbb{C}^n$ of the Hamiltonian flow of $\{E^\lambda \circ \Phi^{-1}\}_{\lambda \in [0,1]}$ yields the desired symplectomorphism.

We are left with proving the claim. As a first step, consider the compactly supported Hamiltonian function $F_t^\lambda \colon \widehat{B} \to \mathbb{R}$, which, for every $t \in \mathbb{R}$, generates the path of compactly supported Hamiltonian diffeomorphisms $\lambda \mapsto \psi_\lambda^t$, $\lambda \in [0,1]$. By [Ban97, Prop. 3.1.5], we have

$$\partial_t F_t^\lambda - \partial_\lambda G_t^\lambda = \{F_t^\lambda, G_t^\lambda\}, \tag{1.11}$$

where $\{F_t^\lambda, G_t^\lambda\} = dG_t^\lambda[X_{F_t^\lambda}] = -dF_t^\lambda[X_{G_t^\lambda}] = -\{G_t^\lambda, F_t^\lambda\}$ is the Poisson bracket.

We now make the subclaim that

$$F_1^\lambda = F_0^\lambda, \qquad \forall \lambda \in [0,1]. \tag{1.12}$$

If the subclaim is true, then the function

$$F^\lambda \colon \mathbb{R} \times \mathbb{T} \times \widehat{B} \to \mathbb{R}, \qquad F^\lambda(s,t,w) = F_t^\lambda(w)$$

is well-defined and smooth, by the smoothness of $t \mapsto \psi_\lambda^t$ on $\mathbb{T}$. This function has the desired property. Indeed, by (1.10) and (1.11), we get

$$(\partial_\lambda \tilde{\psi}_\lambda) \circ (\tilde{\psi}_\lambda)^{-1} = \big(\partial_\lambda G_t^\lambda + \{F_t^\lambda, G_t^\lambda\}\big)\partial_s + X_{F_t^\lambda} = \partial_t F_t^\lambda \partial_s + X_{F_t^\lambda}$$

and

$$\imath_{\partial_t F_t^\lambda \partial_s + X_{F_t^\lambda}}(ds \wedge dt + \widehat{\omega}) = \partial_t F_t^\lambda dt + dF_t^\lambda = dF^\lambda.$$

We are now left with proving the subclaim (1.12). Since G^λ has compact support and $\phi^1_{G^\lambda}(w) = w$ for all $w \in \widehat{B}$, we have $\mathcal{A}_{\phi^1_{G^\lambda}}(w) = 0$. Differentiating this equality with respect to λ and setting

$$\gamma_\lambda \colon \mathbb{T} \to \widehat{B}, \qquad \gamma_\lambda(t) \coloneqq \phi^t_{G^\lambda}(w), \ \forall t \in \mathbb{T},$$

using the definition of the action we get

$$\begin{aligned} 0 = \partial_\lambda \mathcal{A}_{\phi^1_{G^\lambda}}(w) &= \int_0^1 \Big(\widehat{\omega}(\partial_\lambda \gamma_\lambda(t), \dot{\gamma}_\lambda(t)) + dG_t^\lambda(\gamma_\lambda(t))[\partial_\lambda \gamma_\lambda(t)]\Big)dt + \int_0^1 (\partial_\lambda G_t^\lambda)(\gamma_\lambda(t))dt \\ &= \int_0^1 (\partial_\lambda G_t^\lambda)(\gamma_\lambda(t))dt, \end{aligned} \tag{1.13}$$

where we used that γ_λ is a Hamiltonian trajectory for G^λ. On the other hand, by (1.11), we get

$$\begin{aligned}\int_0^1 (\partial_\lambda G_t^\lambda)(\gamma_\lambda(t))dt = \int_0^1 \big(\partial_t F_t^\lambda + dF_t^\lambda[\dot\gamma_\lambda]\big)(\gamma_\lambda(t))dt &= \int_0^1 \frac{d}{dt}\Big(F_t^\lambda(\gamma_\lambda(t)\Big)dt \\ &= F_1^\lambda(\gamma_\lambda(1)) - F_0^\lambda(\gamma_\lambda(0)) \\ &= F_1^\lambda(w) - F_0^\lambda(w),\end{aligned}$$

which, in combination with (1.13), yields $0 = F_1^\lambda(w) - F_0^\lambda(w)$ for all $\lambda \in [0,1]$ and $w \in \widehat{B}$, as stated in the subclaim (1.12). □

Remark 1.2. Let $\epsilon > 0$ and $r \in (0,1)$ be such that $(-\epsilon,\epsilon) \times \mathbb{T} \times \widehat{B}_r$ is contained in Ω and set

$$V \coloneqq \Phi((-\epsilon,\epsilon) \times \mathbb{T} \times \widehat{B}_r).$$

The above proof shows that if $H^\lambda - H^0$ is supported in $\mathbb{T} \times \widehat{B}_r$ and $|H^\lambda| < \epsilon$ for every $\lambda \in [0,1]$, then the symplectomorphism ψ in (iii) can be chosen to be supported in V using suitable cut-off functions χ^λ.

2 Proof of Proposition 1 and Proposition 3

In this section we show how the first two statements of Proposition 1.1 can be used to prove Proposition 1 from the Introduction. The proof of Proposition 3 goes along similar lines and is sketched in Remark 2.1 at the end of this section.

We consider a compactly supported smooth Hamiltonian $H : \mathbb{T} \times \widehat{B} \to \mathbb{R}$ such that all the fixed points of ϕ_H^1 have non-negative action while the Calabi invariant of ϕ_H^1 is negative. Following [ABHS18], a Hamiltonian with these properties can be constructed in the following way. Consider an autonomous radial Hamiltonian on $\widehat{B}$ of the form

$$F(w) \coloneqq f(|w|^2),$$

where $f : [0,1) \to \mathbb{R}$ is a smooth compactly supported monotonically decreasing function such that $f'(r^2) = -\frac{\pi}{2}$ for every $r \in [0, \frac{1}{2}]$. The identity

$$\phi_F^1(w) = e^{-2if'(|w|^2)}w$$

shows that $\phi_F^1(w) = -w$ if $|w| \le \frac{1}{2}$. Next, consider an open ball $U \subset \widehat{B}$ contained in the ball of radius $\frac{1}{2}$ centered at the origin and such that $\phi_F^1(U) = -U$ does not intersect U. Let G be a smooth function on $\widehat{B}$ with support in U and such that

$$\mathrm{Cal}(\phi_G^1) = \int_{\widehat{B}} G\,\widehat{\omega}^{n-1} < -\mathrm{Cal}(\phi_F^1). \tag{2.1}$$

The Hamiltonian diffeomorphism $\phi_F^1 \circ \phi_G^1$ is generated by the compactly supported Hamiltonian

$$H(t,w) = (F\#G)(t,w) = F(w) + G(t,(\phi_F^t)^{-1}(w)).$$

The fact that ϕ_F^1 displaces the support of G from itself implies that the only fixed points of ϕ_H^1 are fixed points of ϕ_F^1, and if w is such a fixed point we have

$$\mathcal{A}_{\phi_H^1}(w) = \mathcal{A}_{\phi_F^1}(w) = f(|w|^2) - |w|^2 f'(|w|^2) \geq 0.$$

Finally, (2.1) implies that the Calabi invariant of ϕ_H^1 is negative. This shows that H has the required properties. The k-periodic points of ϕ_H^1 may have negative action, but by the definition of action we have a bound of the form

$$\left|\mathcal{A}_{\phi_H^k}(w)\right| \leq ck \qquad \forall k \in \mathbb{N}, \tag{2.2}$$

for every k-periodic point w of ϕ_H^1, for a suitable number c.

We now see H as a smooth Hamiltonian on $\mathbb{T} \times \mathbb{C}^{n-1}$ with support in $\mathbb{T} \times \widehat{B}$. The family of rescaled Hamiltonians

$$H^\lambda(t,w) \coloneqq \lambda^2 H\big(t, \tfrac{w}{\lambda}\big), \quad \lambda > 0,$$

C^1-converges to the zero function for $\lambda \to 0$. Therefore, there exists $\lambda_0 > 0$ such that H^λ satisfies (1.4) and $A_\lambda \coloneqq D(H^\lambda)$ belongs to $\mathcal{A}$ if $\lambda \in (0, \lambda_0]$, and A_λ C^1-converges to B for $\lambda \to 0$. The identity

$$X_{H^\lambda}(t,w) = \lambda X_H\big(t, \tfrac{w}{\lambda}\big)$$

implies that the conformally symplectic diffeomorphism $w \mapsto \lambda \cdot w$ conjugates the Hamiltonian dynamics of H and H^λ, meaning that

$$\phi_{H^\lambda}^t(w) = \lambda \cdot \phi_H^t\big(\tfrac{w}{\lambda}\big).$$

Therefore, $w \in \widehat{B}$ is a fixed point of $\phi_{H^\lambda}^1$ if and only if $\frac{w}{\lambda}$ is a fixed point of ϕ_H^1, and in this case we have

$$\mathcal{A}_{\phi_{H^\lambda}^1}(w) = \lambda^2 \mathcal{A}_{\phi_H^1}\big(\tfrac{w}{\lambda}\big) \geq 0.$$

If γ is the closed characteristic on ∂A_λ corresponding to a fixed point of $\phi_{H^\lambda}^1$, statement (ii) in Proposition 1.1 gives us the bound

$$\int_\gamma \lambda_0 = \pi + \mathcal{A}_{\phi_{H^\lambda}^1}(w) \geq \pi.$$

If γ is the closed characteristic on ∂A_λ corresponding to a periodic point of $\phi_{H^\lambda}^1$ of minimal period $k \geq 2$, then using also (2.2) we obtain

$$\int_\gamma \lambda_0 = k\pi + \mathcal{A}_{\phi_{H^\lambda}^k}(w) = k\pi + \lambda^2 \mathcal{A}_{\phi_H^k}\big(\tfrac{w}{\lambda}\big) \geq k\pi - \lambda^2 ck \geq \pi,$$

provided that $\lambda^2 \leq \frac{\pi}{2c}$. Together with the fact that the characteristics of ∂A_λ in $\{z_1 = 0\}$ are closed with action π, up to reducing the size of λ_0 we deduce that $\mathrm{sys}(A_\lambda) = \pi$ for every $\lambda \in (0, \lambda_0]$. On the other hand,

$$\mathrm{Cal}(\phi^1_{H^\lambda}) = \lambda^{2n} \mathrm{Cal}(\phi^1_H) < 0,$$

so by statement (i) in Proposition 1.1 we obtain

$$\mathrm{vol}(A_\lambda) < \frac{\pi^n}{n!} \qquad \forall \lambda \in (0, \lambda_0],$$

and hence

$$\mathrm{sys}(A_\lambda)^n = \pi^n > n! \,\mathrm{vol}(A_\lambda) \geq \underline{c}(A_\lambda)^n \qquad \forall \lambda \in (0, \lambda_0],$$

proving the strict inequality

$$\underline{c}(A_\lambda) < \mathrm{sys}(A_\lambda) \qquad \forall \lambda \in (0, \lambda_0],$$

which is stated in Proposition 1.

Remark 2.1. The proof of Proposition 3 is similar and we just sketch it. Assume without loss of generality that the orbits of the Zoll contact form α_0 have minimal period 1. Then we can find an embedding $\varphi : \mathbb{T} \times \widehat{B}_r \to M$ such that

$$\varphi^* \alpha_0 = dt + \widehat{\lambda}_0.$$

Here, $\widehat{B}_r$ denotes the open ball of radius r in $\mathbb{C}^{n-1}$. We modify α_0 inside the image of φ and obtain a smooth family of contact forms $\{\alpha_\lambda\}_{\lambda \in (0, \lambda_0]}$ such that

$$\varphi^* \alpha_\lambda = (1 + H^\lambda)\, dt + \widehat{\lambda}_0,$$

where $\{H^\lambda\}$ is the family of rescaled Hamiltonians constructed above and $\lambda_0 > 0$ is so small that H^λ is supported in $\mathbb{T} \times \widehat{B}_r$ for all $\lambda \in (0, \lambda_0]$. The contact forms α_λ converge to α_0 in the C^1-topology for $\lambda \to 0$. By the properties of H^λ, we have $\mathrm{sys}(\alpha_\lambda) = 1 = \mathrm{sys}(\alpha_0)$ but the volume of $(M, \alpha_\lambda \wedge d\alpha_\lambda^{n-1})$ is strictly smaller than that of $(M, \alpha_0 \wedge d\alpha_0^{n-1})$.

3 Proof of Theorem 1 (i)

The proof of statement (i) of Theorem 1 uses the construction from Section 1 together with the following result.

Proposition 3.1. *Let $m \in \mathbb{N}$ and denote by B_r the open ball of radius r centered at 0 in $\mathbb{C}^m$ and by B_r^{c} its complement. For every $r > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that if H is a compactly supported Hamiltonian on $\mathbb{T} \times \mathbb{C}^m$ satisfying $\|H\|_{C^2} < \delta$ and such that 0 is a fixed point of ϕ^1_H with $\mathcal{A}_{\phi^1_H}(0) = 0$, then there exists a smooth family of Hamiltonians $\{H^\lambda\}_{\lambda \in [0,1]} \subset C^\infty(\mathbb{T} \times \mathbb{C}^m)$ such that:*

(i) $H^0 = H$;

(ii) $H^\lambda = H$ *on* $\mathbb{T} \times B^c_r$ *for every* $\lambda \in [0,1]$;

(iii) $\phi^1_{H^\lambda} = \phi^1_H$ *for every* $\lambda \in [0,1]$;

(iv) $\|H^\lambda\|_{C^0} < \epsilon$ *for every* $\lambda \in [0,1]$;

(v) $|H^1(t,z)| \leq \epsilon |z|^2$ *for every* $(t,z) \in \mathbb{T} \times \mathbb{C}^m$.

The proof of this proposition is given at the end of this section. Here, we show how Propositions 1.1 and 3.1 imply statement (i) of Theorem 1.

Our aim is to prove that if $A \in \mathcal{A}$ is C^2-close enough to the unit ball B, then there exists a symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ mapping A into the cylinder whose systole coincides with the systole of A. Up to rescaling, we may assume that $\mathrm{sys}(A) = \pi$, so that ψ is required to map A into the standard cylinder $Z = \{z \in \mathbb{C}^n \mid |z_1| < 1\}$. A closed characteristic γ on ∂A achieving $\mathrm{sys}(A)$ is C^0-close to some closed characteristic on ∂B. Thus, up to applying a unitary transformation of $\mathbb{C}^n$ and a small translation we may assume that γ passes through the point $(1,0,\dots,0) \in \mathbb{C}^n$ and is C^0-close to the curve

$$\gamma_0 : \mathbb{T} \to \mathbb{C}^n, \qquad t \mapsto (e^{2\pi i t}, 0, \dots, 0).$$

Let $\Phi : \Omega \to \mathbb{C}^* \times \mathbb{C}^{n-1}$ be the symplectomorphism introduced in (1.1). Noting that $\Phi(0,t,0) = \gamma_0(t)$ and that the set

$$U \coloneqq \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \times \mathbb{T} \times \widehat{B}_{\frac{2}{3}} \tag{3.1}$$

is contained in Ω, we consider the open neighborhood $V \coloneqq \Phi(U)$ of γ_0. Here, $\widehat{B}_r$ denotes the open ball of radius r centered at the origin of $\mathbb{C}^{n-1}$. Being close to γ_0, the curve γ is contained in V. Moreover, the form of Φ implies the identities

$$B \cap V = \Phi(U \cap \{s < 0\}), \qquad Z \cap V = \Phi(U \cap \{s < \pi |w|^2\}). \tag{3.2}$$

Since A is C^2-close to B, the hypersurface $\partial A \cap V$ is the image via Φ of the graph of a C^2-small smooth function on $\mathbb{T} \times \widehat{B}_{\frac{2}{3}}$. We can extend this function to a C^2-small smooth function H on $\mathbb{T} \times \mathbb{C}^{n-1}$ having support in $\mathbb{T} \times \widehat{B}$ and satisfying (1.4), thus obtaining

$$A \cap V = D(H) \cap V, \tag{3.3}$$

where $D(H)$ denotes the domain which we introduced in the previous section. As it passes through $(1,0,\dots,0) = \Phi(0,0,0)$, the closed characteristic γ_0 on $\partial A \cap V = \partial D(H) \cap V$ corresponds via Φ to the fixed point 0 of ϕ^1_H and

$$\mathcal{A}_{\phi^1_H}(0) = \int_\gamma \lambda_0 - \pi = 0,$$

as shown in Proposition 1.1 (ii). Assuming that A is C^2-close enough to B - and hence H is C^2-small enough - we apply Proposition 3.1 with $m = n-1$ to get a smooth family of Hamiltonian functions $H^\lambda : \mathbb{T} \times \mathbb{C}^{n-1} \to \mathbb{R}$, $\lambda \in [0,1]$, such that:

(i) $H^0 = H$;

(ii) $H^\lambda = H$ on $\mathbb{T} \times \widehat{B}^{\mathrm{c}}_{\frac{1}{3}}$ for every $\lambda \in [0,1]$;

(iii) $\phi^1_{H^\lambda} = \phi^1_H$ for every $\lambda \in [0,1]$;

(iv) $\|H^\lambda\|_{C^0} < \frac{\pi}{2}$ for every $\lambda \in [0,1]$;

(v) $H^1(t,w) \leq \pi |w|^2$ for every $(t,w) \in \mathbb{T} \times \mathbb{C}^{n-1}$.

By properties (i) and (iii), Proposition 1.1 (iii) implies that there exists a symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\psi(D(H)) = D(H^1)$. Thanks to properties (ii), (iv) and Remark 1.2, we can assume that ψ is supported in V and hence (3.3) implies

$$\psi(A) = (A \setminus V) \cup (D(H^1) \cap V).$$

Since the closure of $B \setminus V$ is contained in Z, C^0-closeness of A to B implies that $A \setminus V$ is contained in Z. By (v) and the second identity in (3.2), $D(H^1) \cap V$ is also contained in Z. We conclude that $\psi(A)$ is contained in Z, hence proving statement (i) in Theorem 1.

There remains to prove Proposition 3.1. The proof of this proposition makes use of some properties of generating functions which we now recall, referring to [BP94] for more details. Every compactly supported symplectomorphism $\varphi : \mathbb{C}^m \to \mathbb{C}^m$ which is sufficiently C^1-close to the identity is represented by a unique C^2-small, compactly supported, smooth generating function $S : \mathbb{C}^m \to \mathbb{R}$ which is characterized by the equation

$$i(z - \varphi(z)) = \nabla S\big(\tfrac{z+\varphi(z)}{2}\big) \qquad \forall z \in \mathbb{C}^m. \tag{3.4}$$

Conversely, every C^2-small, compactly supported, smooth function S defines by the above identity a compactly supported symplectomorphism $\varphi : \mathbb{C}^m \to \mathbb{C}^m$ which is C^1-close to the identity. If $\{\varphi_t\}_{t\in[0,1]}$ is a smooth isotopy starting at the identity and consisting of compactly supported symplectic diffeomorphisms of $\mathbb{C}^m$ which are C^1-close to the identity, then $\varphi_t = \phi^t_H$, where the smooth family of compactly supported Hamiltonians $\{H_t\}_{t\in[0,1]}$ is related to the smooth family $\{S_t\}_{t\in[0,1]}$ of compactly supported generating functions for $\{\varphi_t\}_{t\in[0,1]}$ by the Hamilton–Jacobi equation

$$\partial_t S_t(z) = H\big(t, z + \tfrac{i}{2}\nabla S_t(z)\big) \qquad \forall (t,z) \in [0,1] \times \mathbb{C}^m. \tag{3.5}$$

Let $\varphi : \mathbb{C}^m \to \mathbb{C}^m$ be a compactly supported symplectomorphism which is C^1-close to the identity and denote by $S : \mathbb{C}^m \to \mathbb{R}$ the corresponding generating function. Let z be a fixed point of φ. By (3.4), z is a critical point of S. The path of generating functions $\{tS\}_{t\in[0,1]}$ produces a compactly supported symplectic isotopy from the identity to φ such that z is a fixed point of all maps in this isotopy. If H denotes the compactly supported Hamiltonian on $[0,1] \times \mathbb{C}^m$ associated with this isotopy, (3.5) implies that $H(t,z) = S(z)$ for every $t \in [0,1]$ and hence

$$\mathcal{A}_\varphi(z) = S(z). \tag{3.6}$$

After these preliminaries, we can prove Proposition 3.1.

Proof of Proposition 3.1. In this proof, whenever we say that some function is C^k-small, we understand "provided that H is C^2-small".

We first prove this proposition under the additional assumption that H_t is identically zero for every t in a neighborhood of 0 in $\mathbb{T}$.

Since H is C^2-small, each symplectomorphism ϕ_H^t is C^1-close to the identity and hence generated by a unique C^2-small generating function S_t^0. It follows from the Hamilton–Jacobi equation (3.5) that S_t^0 is also C^2-small when viewed as a function on $[0,1]\times\mathbb{C}^m$. By our additional assumption on H, $\phi_H^t = \mathrm{id}$ for t close to 0 and $\phi_H^t = \phi_H^1$ for t close enough to 1. Therefore, $S_t^0 = 0$ for t close to 0 and $S_t^0 = S_1^0$ for t close to 1.

Fix a smooth monotonically increasing function $\eta : [0,1] \to [0,1]$ such that $\eta(t) = 0$ for t close to 0 and $\eta(t) = 1$ for t close to 1. By modifying S_t^0, we can find a smooth family of functions S_t^1 such that:

(a) $S_t^1 = \eta(t) S_1^0$ inside the ball $B_{\frac{r}{3}}$, for every $t \in [0,1]$;

(b) $S_t^1 = S_t^0$ outside the ball $B_{\frac{2}{3}r}$, for every $t \in [0,1]$;

(c) $S_t^1 = 0$ for t close to 0 and $S_t^1 = S_1^0$ for t close to 1;

(d) S_t^1 is C^2 small when viewed as a function on $[0,1]\times\mathbb{C}^m$.

For every $\lambda \in [0,1]$, we define

$$S_t^\lambda \coloneqq (1-\lambda) S_t^0 + \lambda S_t^1.$$

By (d), this function is C^2-small when viewed as a function on $[0,1]^2 \times \mathbb{C}^m$. Moreover, $S_t^\lambda = 0$ for t sufficiently close to 0 and $S_t^\lambda = S_1^0$ for t sufficiently close to 1. For every $\lambda \in [0,1]$, let H^λ be the unique compactly supported Hamiltonian generating the Hamiltonian isotopy associated with the family of generating functions $\{S_t^\lambda\}_{t\in[0,1]}$. It follows from the Hamilton–Jacobi equation that H^λ is C^1-small, and in particular we can ensure that (iv) holds. Clearly, $H^0 = H$ and by (b) H_t^λ agrees with H_t outside B_r, for every $t \in [0,1]$, so (i) and (ii) hold. By the second assertion in (c), we have $\phi_{H^\lambda}^1 = \phi_H^1$ for all $\lambda \in [0,1]$, which proves (iii). Since S_t^λ is constant in t for t close to 0 and for t close to 1, the Hamiltonian H_t^λ vanishes for t near 0 and 1. Therefore, it descends to a 1-periodic Hamiltonian. It remains to check (v). By (a), the Hamilton–Jacobi equation (3.5) implies that

$$H^1\big(t, z + \tfrac{i}{2}\nabla S_t^1(z)\big) = \partial_t S_t^1(z) = \eta'(t) S_1^0(z) \tag{3.7}$$

for every $z \in B_{\frac{r}{3}}$. Since 0 is a fixed point of ϕ_H^1 of zero action, the function S_1^0 and its derivative both vanish at 0, see (3.6). Therefore, we get

$$|S_1^0(z)| \le \tfrac{1}{2}\|S_1^0\|_{C^2}|z|^2, \qquad \forall z \in \mathbb{C}^m.$$

The fact that S_1^0 is C^2-small implies that the diffeomorphism

$$\theta_t \colon \mathbb{C}^m \to \mathbb{C}^m, \quad \theta_t(z) \coloneqq z + \tfrac{i}{2}\nabla S_t^1(z), \ \forall z \in \mathbb{C}^m,$$

is C^1-close to the identity and fixes the origin. Thus, the same is true for the inverse diffeomorphism, and we get

$$|\theta_t^{-1}(w)| \leq 2|w|, \qquad \forall w \in \mathbb{C}^m.$$

Using (3.7), we obtain for all $(t, w) \in \mathbb{T} \times \mathbb{C}^m$

$$|H^1(t,w)| = |\eta'(t) S_1^0(\theta_t^{-1}(w))| \leq \|\eta\|_{C^1} \tfrac{1}{2} \|S_1^0\|_{C^2} |\theta_t^{-1}(w)|^2 \leq \|\eta\|_{C^1} \tfrac{1}{2} \|S_1^0\|_{C^2} 4|w|^2,$$

which implies the bound (v) since S_1^0 is C^2-small.

We now show how the general case can be deduced from the special one considered above. Using again the function η whose properties are described above, we introduce the Hamiltonian

$$\widehat{H}(t,z) \coloneqq \eta'(t) H(\eta(t), z), \qquad \forall (t,z) \in [0,1] \times \mathbb{C}^m,$$

which vanishes for t close to 0 and for t close to 1, and in particular can be seen as 1-periodic in time, is C^2-small and satisfies $\phi^1_{\widehat{H}} = \phi^1_H$. We claim that there exists a family of smooth Hamiltonians $K^\lambda : \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$, $\lambda \in [0,1]$, such that:

(a') $K^0 = H$;

(b') $K^1 = \widehat{H}$ on $\mathbb{T} \times B_{\frac{r}{2}}$;

(c') $K^\lambda = H$ on $\mathbb{T} \times B_r^{\mathrm{c}}$ for every $\lambda \in [0,1]$;

(d') $\phi^1_{K^\lambda} = \phi^1_H$ for every $\lambda \in [0,1]$;

(e') $\|K^\lambda\|_{C^0} < \epsilon$ for every $\lambda \in [0,1]$.

Once this is proven, the conclusion of Proposition 3.1 in the general case follows. Indeed, by applying the above special case to the Hamiltonian $\widehat{H}$, we obtain a smooth family of Hamiltonians $\{\widehat{H}^\lambda\}_{\lambda \in [0,1]}$ on $\mathbb{T} \times \mathbb{C}^m$ which satisfies:

(i') $\widehat{H}^0 = \widehat{H}$;

(ii') $\widehat{H}^\lambda = \widehat{H}$ on $\mathbb{T} \times B^{\mathrm{c}}_{\frac{r}{4}}$ for every $\lambda \in [0,1]$;

(iii') $\phi^1_{\widehat{H}^\lambda} = \phi^1_{\widehat{H}}$ for every $\lambda \in [0,1]$;

(iv') $\|\widehat{H}^\lambda\|_{C^0} < \epsilon$ for every $\lambda \in [0,1]$;

(v') $|\widehat{H}^1(t,z)| \leq \epsilon |z|^2$ for every $(t,z) \in \mathbb{T} \times \mathbb{C}^m$.

For $\lambda \in [0,1]$, we define

$$L^\lambda := \begin{cases} \widehat{H}^\lambda & \text{on } \mathbb{T} \times B_{\frac{r}{2}}, \\ K^1 & \text{on } \mathbb{T} \times B^{\mathrm{c}}_{\frac{r}{4}}. \end{cases}$$

This yields a smooth family of Hamiltonians $\{L^\lambda\}_{\lambda \in [0,1]}$ on $\mathbb{T} \times \mathbb{C}^m$. One easily verifies that $K^1 = L^0$ and that the smooth concatenation

$$H^\lambda := \begin{cases} K^{\eta(2\lambda)} & \text{for } \lambda \in [0, \frac{1}{2}], \\ L^{\eta(2\lambda-1)} & \text{for } \lambda \in [\frac{1}{2}, 1], \end{cases}$$

satisfies the desired properties (i)-(v).

There remains to construct a family of smooth Hamiltonians $K^\lambda : \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$ satisfying (a')-(e'). For $\lambda \in [0,1]$, we define $\eta^\lambda(t) := (1-\lambda)t + \lambda\eta(t)$. Note that η^λ descends to a smooth map from $\mathbb{T}$ to itself. Therefore, we may define a smooth family of Hamiltonians $\widetilde{H}^\lambda : \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$, $\lambda \in [0,1]$, by the formula

$$\widetilde{H}^\lambda(t,z) := (\eta^\lambda)'(t) H(\eta^\lambda(t), z).$$

Note that $\widetilde{H}^0 = H$ and $\widetilde{H}^1 = \widehat{H}$ and that $\phi^1_{\widetilde{H}^\lambda} = \phi^1_H$ for all λ. Choose a family of smooth functions $F^\lambda : \mathbb{T} \times \mathbb{C}^m \to \mathbb{R}$, $\lambda \in [0,1]$, such that

$$F^\lambda = \widetilde{H}^\lambda \quad \text{on } \mathbb{T} \times B_{\frac{7}{10}r}, \qquad F^\lambda = H \quad \text{on } \mathbb{T} \times B^{\mathrm{c}}_{\frac{8}{10}r}. \tag{3.8}$$

We can choose F^0 to be equal to H everywhere because $\widetilde{H}^0 = H$. Since H and $\widetilde{H}^\lambda$ are C^2-small, F^λ can be made C^2-small as well. Therefore, the compactly supported symplectic diffeomorphism

$$\psi^\lambda := (\phi^1_{F^\lambda})^{-1} \circ \phi^1_H$$

is C^1-close to the identity and hence has a C^2-small compactly supported generating function $S^\lambda : \mathbb{C}^m \to \mathbb{R}$. By choosing H to be C^2-small enough, we can ensure that

$$\|X_H\|_{C^0} < \tfrac{r}{10}, \qquad \|X_{F^\lambda}\|_{C^0} < \tfrac{r}{10}, \tag{3.9}$$

and, together with (3.8) and the fact that the time-1-map of the Hamiltonian $\widetilde{H}^\lambda$ is equal to ϕ^1_H, we deduce that ψ^λ is the identity on $B_{\frac{6}{10}r}$ and on $B^{\mathrm{c}}_{\frac{9}{10}r}$. The identity (3.4) guarantees that ∇S^λ vanishes on these two sets and hence

$$S^\lambda = c^\lambda \quad \text{on } B_{\frac{6}{10}r}, \qquad S^\lambda = 0 \quad \text{on } B^{\mathrm{c}}_{\frac{9}{10}r}, \tag{3.10}$$

for a suitable constant $c^\lambda \in \mathbb{R}$ which we will later show to be zero. Now let ψ^λ_t be the symplectomorphism which is generated by $\eta(t)S^\lambda$. Then ψ^λ_t depends smoothly on t, $\psi^\lambda_t = \mathrm{id}$ for t close to 0 and $\psi^\lambda_t = \psi^\lambda$ for t close to 1.

Let $G^\lambda : [0,1] \times \mathbb{C}^m \to \mathbb{R}$ be the compactly supported Hamiltonian associated with the symplectic isotopy $\{\psi_t^\lambda\}_{t\in[0,1]}$. Then $G_t^\lambda = 0$ for t close to either 0 or 1, and hence we can see G^λ as a smooth function on $\mathbb{T} \times \mathbb{C}^m$. The Hamilton–Jacobi equation (3.5) reads

$$\eta'(t) S^\lambda(z) = G_t^\lambda\big(z + \tfrac{i}{2}\eta(t)\nabla S^\lambda(z)\big).$$

It implies that G^λ is C^1-small. Moreover, thanks to (3.10) it implies:

$$G^\lambda(t,z) = c^\lambda\,\eta'(t) \qquad \forall (t,z) \in \mathbb{T} \times B_{\frac{6}{10}r}, \tag{3.11}$$

$$G^\lambda(t,z) = 0 \qquad \forall (t,z) \in \mathbb{T} \times B^{\mathrm{c}}_{\frac{9}{10}r}. \tag{3.12}$$

We set

$$K^\lambda(t,z) \coloneqq (F^\lambda \# G^\lambda)(t,z) = F^\lambda(t,z) + G^\lambda\big(t, (\phi^t_{F^\lambda})^{-1}(z)\big).$$

This function is smooth on $\mathbb{T} \times \mathbb{C}^m$, compactly supported and satisfies

$$\phi^1_{K^\lambda} = \phi^1_{F^\lambda} \circ \phi^1_{G^\lambda} = \phi^1_{F^\lambda} \circ \psi^\lambda = \phi^1_H,$$

proving (d'). By (3.8), (3.9) and (3.12), K^λ satisfies (c'). By (3.8), (3.9) and (3.11), we obtain

$$K^\lambda(t,z) = \widetilde{H}^\lambda(t,z) + c^\lambda\,\eta'(t) \qquad \forall (t,z) \in \mathbb{T} \times B_{\frac{r}{2}}.$$

The Hamiltonians K^λ and $\widetilde{H}^\lambda$ induce the same time-one map, so the independence of the action of the fixed point 0 from the choice of the Hamiltonian implies that the number c^λ in the above identity is zero, as claimed above. This shows that K^1 satisfies (b'). Property (e') follows from the fact that F^λ and G^λ are both C^0-small. Since $F^0 = H$, we have $G^0 = 0$ and thus $K^0 = H$, showing (a'). This concludes the proof of Proposition 3.1. □

4 Averaging and a preliminary normal form

We denote by $\Omega^k(M)$, $k \geq 0$, the space of smooth k-forms on the manifold M. We assume that M is closed, odd dimensional and endowed with a Zoll contact form $\alpha_0 \in \Omega^1(M)$. We denote by R_{α_0} the Reeb vector field of α_0 and by θ_t its flow. By the Zoll property, the flow θ_t defines a free, smooth S^1-action on M. Here, we identify S^1 with $\mathbb{R}/T_0\mathbb{Z}$, where $T_0 = \mathrm{sys}(\alpha_0)$ is the minimal period of the Reeb orbits of α_0. Whenever we talk about S^1-invariant or S^1-equivariant objects on M, we refer to this S^1-action.

On the space $\Omega^k(M)$ we have the averaging operator mapping each $\beta \in \Omega^k(M)$ to the S^1-invariant k-form

$$\overline{\beta} \coloneqq \frac{1}{T_0}\int_0^{T_0} \theta_t^*\beta\, dt.$$

Averaging commutes with differentiation:

$$\overline{d\beta} = d\overline{\beta} \qquad \forall \beta \in \Omega^k(M).$$

Moreover, if Y is an S^1-invariant vector field on M (for instance $Y = R_{\alpha_0}$), we have

$$\overline{\imath_Y \beta} = \imath_Y \overline{\beta} \qquad \forall \beta \in \Omega^k(M).$$

We shall also need to average linear endomorphisms $F : T^*M \to T^*M$ lifting the identity. The averaged endomorphism $\overline{F} : T^*M \to T^*M$ is defined by

$$\overline{F} \coloneqq \frac{1}{T_0} \int_0^{T_0} \theta_t^* F \, dt,$$

where the pull-back of F by a diffeomorphism $\theta : M \to M$ is given by

$$(\theta^* F)(x)[p] \coloneqq d\theta(x)^* \Big(F(\theta(x))[p \circ d\theta(x)^{-1}] \Big), \qquad \forall x \in M, \ \forall p \in T_x^* M.$$

If F is as above and $\beta \in \Omega^1(M)$ is S^1-invariant, then the 1-form $F[\beta]$ satisfies

$$\overline{F[\beta]} = \overline{F}[\beta].$$

We fix an arbitrary S^1-invariant Riemannian metric on M. The C^k-norms of tensors on M and the covariant derivatives we shall occasionally use in our proofs are induced by this metric.

Our proof of Theorem 3 relies on the following normal form which is proven in [AB23, Theorem 2].

THEOREM 4.1. *Let α_0 be a Zoll contact form on a closed manifold M with orbits having minimal period T_0. There is $\delta_0 > 0$ such that if α is a contact form on M satisfying $\|\alpha - \alpha_0\|_{C^2} < \delta_0$, then there exists a diffeomorphism $u : M \to M$ such that*

$$u^* \alpha = S \alpha_0 + \eta + df, \tag{4.1}$$

where:

(i) S is a smooth positive function on M that is invariant under the Reeb flow of α_0;

(ii) f is a smooth function on M with average zero along each orbit of R_{α_0};

(iii) η is a smooth one-form on M satisfying $\iota_{R_{\alpha_0}} \eta = 0$;

(iv) $\iota_{R_{\alpha_0}} d\eta = F[dS]$ for a smooth endomorphism $F : T^*M \to T^*M$ lifting the identity;

(v) $\iota_{R_{\alpha_0}} df = \iota_Z dS$ for a smooth vector field Z on M taking values in the contact distribution $\ker \alpha_0$ and having average zero along each orbit of R_{α_0};

(vi) $dS = -B[V]$, where V is a smooth S^1-invariant section of $\ker \alpha_0$ and $B : \ker \alpha_0 \to (\ker \alpha_0)^*$ is a smooth isomorphism lifting the identity.

Moreover, for every integer $k \geq 0$, there is a modulus of continuity ω_k such that

$$\begin{aligned}\max\big\{\mathrm{dist}_{C^{k+1}}(u,\mathrm{id}), \|S-1\|_{C^{k+1}}, \|f\|_{C^{k+1}}, \|\eta\|_{C^k}, \|d\eta\|_{C^k}, \|F\|_{C^k}, \|Z\|_{C^k},\\ \|V\|_{C^{k+1}}, \|B-B_0\|_{C^k}\big\} \leq \omega_k(\|\alpha-\alpha_0\|_{C^{k+2}}),\end{aligned} \tag{4.2}$$

where $B_0 : \ker\alpha_0 \to (\ker\alpha_0)^*$ is the isomorphism given by the non-degenerate bilinear form $d\alpha_0|_{\ker\alpha_0\times\ker\alpha_0}$. Finally, the map $\alpha \mapsto (u, S, \eta, f)$ is smooth and maps α_0 to $(\mathrm{id}, 1, 0, 0)$.

Here, by modulus of continuity we mean a monotonically increasing continuous function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$.

Remark 4.2. Actually, statement (vi) and the corresponding bounds for V and B in (4.2) are not present in the formulation of [AB23, Theorem 2], but they are easily recoverable from its proof (see in particular equation (2.25) together with (2.4) and (2.24) therein).

The smoothness which is mentioned at the end of the theorem is not explicitly stated in [AB23, Theorem 2] either, but can be deduced from its proof. This amounts to checking that in [AB23, Theorem B.1] the vector fields U, V and the function h depend smoothly on the vector field X, and take the values $U = 0$, $V = 0$, $h = 1$ when $X = X_0$. The triplet (U, V, h) is determined by X by means of a functional equation of the form $\Phi_X(U, V, h) = 0$, where the dependence of Φ on X is affine and $\Phi_0(0, 0, 1) = 0$. A standard argument involving the parametric inverse mapping theorem and [AB23, Lemma B.4] gives us the required smooth dependence on X of the solution (U, V, h) of this equation.

We will also need the following additional fact.

Proposition 4.3. *In the setting of Theorem 4.1, the following additional properties hold: for every $\epsilon > 0$ there is a positive number $\delta \leq \delta_0$ such that if $\|\alpha - \alpha_0\|_{C^2} < \delta$ then:*

(vii) any closed orbit of R_α is either long, meaning that its minimal period is larger than $\frac{1}{\epsilon}$, or short, meaning that its minimal period is contained in the interval $(T_0 - \epsilon, T_0 + \epsilon)$;

(viii) the short closed orbits of R_α are precisely the images via u of those orbits γ of the free S^1-action θ_t consisting of critical points of S, and the minimal period of such an orbit is $T_0 S(\gamma)$.

The above proposition sharpens [AB23, Proposition 1 and Remark 3.1], in which the fact that all short closed orbits of R_α are given by the critical points of S is proven assuming that α is C^3-close to α_0. The proof of Proposition 4.3 uses the following lemma.

Lemma 4.4. *There exists $\nu > 0$ such that the following is true. Let V be an autonomous smooth vector field on $\mathbb{R}^k$ with $\|V\|_{C^1} < \nu$ and $A : [0, 1] \times \mathbb{R}^k \to \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^k)$ a smooth map into the space of linear endomorphisms of $\mathbb{R}^k$ with $\|A\|_{C^0} < \nu$. Let us define the time-dependent vector field $W_t := (\mathrm{id} + A_t)V$ and let $(\chi_t)_{t\in[0,1]}$ be its flow. Then the fixed points of χ_1 are precisely the zeros of the vector field V.*

Proof. Every zero of V is also a zero of W_t for all t and hence a fixed point of χ_1. Let $p \in \mathbb{R}^k$ be a point such that $V(p) \neq 0$. We need to show that χ_1 does not fix p if ν is small enough. We claim that there is $R > 0$ such that for all $(t, x) \in [0, 1] \times B_R(p)$, we have

$$\text{(a)} \quad |W_t(x)| < R, \qquad \text{(b)} \quad |W_t(x) - V(p)| < |V(p)|,$$

where $B_R(p)$ denotes the open ball of radius R centered at p. Let us show how to use the claim to prove the lemma. Call $\gamma\colon [0, 1] \to \mathbb{R}^k$ the trajectory of W_t starting at p, that is, $\gamma(t) = \chi_t(p)$ for all $t \in [0, 1]$. We need to show that γ is not closed. Let $s \in [0, 1]$ be such that $\gamma([0, s])$ is contained in $B_R(p)$. By (a),

$$|\gamma(s) - p| = |\gamma(s) - \gamma(0)| \leq \int_0^s |\dot{\gamma}(t)|dt = \int_0^s |W_t(\gamma(t))|dt < Rs.$$

Thus, $\gamma(s) \in B_{sR}(p)$ and we deduce that $\gamma([0, 1])$ is contained in $B_R(p)$. Denoting by $\langle \cdot, \cdot \rangle$ the euclidean inner product in $\mathbb{R}^k$, for all $t \in [0, 1]$ we have by (b)

$$\begin{aligned} \frac{d}{dt}\langle \gamma(t), V(p)\rangle = \langle \dot{\gamma}(t), V(p)\rangle = \langle W_t(\gamma(t)), V(p)\rangle &\geq |V(p)|^2 - |W_t(\gamma(t) - V(p)||V(p)| \\ &> |V(p)|^2 - |V(p)||V(p)| \\ &= 0. \end{aligned}$$

Therefore, $t \mapsto \langle \gamma(t), V(p)\rangle$ is a strictly increasing function and hence γ is not closed.

We are left to prove the claim. Set $r \coloneqq |V(p)| > 0$. Let $\nu > 0$ and write $R = ar$ for some positive number $a > 0$ to be determined. We have

$$|V(x) - V(p)| \leq R\|V\|_{C^1} < a\nu r, \qquad \forall x \in B_R(p).$$

Thus,

$$|V(x)| \leq |V(x) - V(p)| + |V(p)| < (1 + a\nu)r, \qquad \forall x \in B_R(p).$$

For all $(t, x) \in [0, 1] \times B_R(p)$, it follows that

$$\begin{aligned} |W_t(x)| &< (1 + \nu)(1 + a\nu)r, \\ |W_t(x) - V(p)| &< |A_t(x)V(x)| + a\nu r < (1 + a + a\nu)\nu r. \end{aligned}$$

To achieve (a) and (b), we need to show that for every sufficiently small $\nu > 0$, there exists $a > 0$ such that

$$(1 + \nu)(1 + a\nu) < a \quad \text{and} \quad (1 + a + a\nu)\nu < 1.$$

If $\nu > 0$ is sufficiently small, these two inequalities are equivalent to

$$\frac{1 + \nu}{1 - \nu - \nu^2} < a < \frac{1 - \nu}{\nu(1 + \nu)}$$

and clearly possess a solution $a > 0$. □

Proof of Proposition 4.3. If the contact form α is C^2-close to α_0 then the Reeb vector field R_α is C^1-close to R_{α_0}. Statement (vii) is then a consequence of a general fact about C^1-perturbations of vector fields inducing a free S^1-action, see [Ban86, Corollary 1].

By differentiating (4.1) and contracting along R_{α_0}, we obtain the identity

$$\imath_{R_{\alpha_0}} u^*\alpha = \imath_{R_{\alpha_0}}(dS \wedge \alpha_0 + S\, d\alpha_0 + d\eta) = -dS + F[dS],$$

which shows that the Reeb vector field of $u^*\alpha$ is parallel to R_{α_0} on the critical set of S. Therefore, every orbit γ of θ_t consisting of critical points of the S^1-invariant function S is a closed orbit of $R_{u^*\alpha}$ of minimal period

$$\int_\gamma u^*\alpha = \int_\gamma (S\alpha_0 + \eta + df) = T_0 S(\gamma).$$

By (4.2), this number is close to T_0 when $\|\alpha - \alpha_0\|_{C^2}$ is small, so $u(\gamma)$ is a short closed orbit of R_α if $\|\alpha - \alpha_0\|_{C^2} < \delta$ with $\delta = \delta(\epsilon)$ small enough. There remains to show that, up to reducing the number δ if necessary, every short closed orbit of R_α is obtained in this way or, equivalently, that every short closed orbit of u^*R_α is an orbit of the S^1-action θ_t consisting of critical points of S.

The diffeomorphism u is given by a version of a theorem of Bottkol [Bot80] which is proven in [AB23, Theorem 2.1]. By this result, u satisfies

$$h\, u^*R_\alpha = R_{\alpha_0} - Q[V], \tag{4.3}$$

where h is a smooth S^1-invariant function on M, V is a smooth S^1-invariant vector field which is orthogonal to R_{α_0} with respect to the chosen S^1-invariant metric on M, and Q is an endomorphism of TM lifting the identity. Moreover

$$\max\{\|h-1\|_{C^1}, \|V\|_{C^1}, \|Q - \mathrm{id}\|_{C^0}\} \leq \omega(\|\alpha - \alpha_0\|_{C^2}), \tag{4.4}$$

for a suitable modulus of continuity ω, and the zeros of the S^1-invariant vector field V are precisely the critical points of the S^1-invariant function S, see [AB23, Equation (2.25)].

Therefore, we have to show that if $\|\alpha - \alpha_0\|_{C^2}$ is small enough, then the short closed orbits of the vector field u^*R_α given by (4.3), where h, Q and V satisfy the above conditions, are orbits of θ_t which are contained in the set of zeroes of V.

The free S^1-action θ_t defines the smooth S^1-bundle

$$S^1 = \mathbb{R}/T_0\mathbb{Z} \to M \xrightarrow{\pi} B.$$

Let $\{B_j\}_{j\in\{1,\dots,k\}}$ be a covering of B consisting of open embedded balls and consider a second covering $\{B'_j\}_{j\in\{1,\dots,k\}}$ with $\overline{B'_j} \subset B_j$ for every j. By (4.4), the vector field u^*R_α is C^0-close to R_{α_0} when $\|\alpha - \alpha_0\|_{C^2}$ is small, so we can assume that any short closed orbit of u^*R_α which meets $\pi^{-1}(B'_j)$ is fully contained in $\pi^{-1}(B_j)$. This allows us to fix j and work in $\pi^{-1}(B_j)$.

By a suitable bundle trivialization, we can identify $\pi^{-1}(B_j)$ with the product $S^1 \times B_j$ in such a way that, denoting by t the variable in S^1, we have

$$R_{\alpha_0} = \partial_t \qquad \text{and} \qquad V(t,x) = V(x) \in T_x B_j \quad \forall (t,x) \in S^1 \times B_j.$$

From (4.3), we obtain that on $\pi^{-1}(B_j) \cong S^1 \times B_j$ the vector field $u^* R_\alpha$ satisfies

$$g\, u^* R_\alpha = \partial_t + Q'[V],$$

where the smooth function $g : S^1 \times B_j \to \mathbb{R}$ is C^0-close to the constant function 1 and Q' is an endomorphism of TB_j which lifts the identity and is C^0-close to the identity. Therefore, up to a time reparametrization which is determined by the function g, the short closed orbits of $u^* R_\alpha$ which are contained in $\pi^{-1}(B_j)$ correspond to the T_0-periodic orbits of the non-autonomous T_0-periodic vector field $Q'(t,x)[V(x)]$ which are contained in B_j. The desired conclusion that these T_0-periodic orbits are just the zeros of V when $\|\alpha - \alpha_0\|_{C^2}$ is sufficiently small now follows from Lemma 4.4. This concludes the proof of statement (viii). □

5 Proof of Theorem 3

In this section, we prove Theorem 3. Let α be a contact form on M which is C^2-close to the Zoll contact form α_0. By Theorem 4.1, there exist a diffeomorphism $u \in \mathrm{Diff}(M)$, a positive S^1-invariant function $S \in C^\infty(M)$, a 1-form $\eta \in \Omega^1(M)$ and a function $f \in C^\infty(M)$ satisfying the properties listed in Theorem 4.1. In particular,

$$u^* \alpha = S\alpha_0 + \eta + df.$$

We claim that $S\alpha_0 + \eta$ is a contact form strictly contactomorphic to $S\alpha_0 + \eta + df$, i.e., that there exists a diffeomorphism $v : M \to M$ such that

$$v^*(S\alpha_0 + \eta + df) = S\alpha_0 + \eta. \tag{5.1}$$

To see this, let us define the family of 1-forms

$$\beta_t \coloneqq S\alpha_0 + \eta + t\, df, \qquad t \in [0,1].$$

Note that since α is C^2-close to α_0, it follows from (4.2) that $S - 1$ and f are C^1-small and that η and $d\eta$ are C^0-small. This fact implies that β_t is C^0-close to α_0 and that $d\beta_t$ is C^0-close to $d\alpha_0$. Therefore, β_t is a contact form for all $t \in [0,1]$. Hence there exists a unique time-dependent vector field B_t on M which is parallel to the Reeb vector field R_{β_t} and satisfies $\iota_{B_t}\beta_t + f = 0$. Let v_t denote the flow generated by B_t. We compute:

$$\partial_t (v_t^* \beta_t) = v_t^* (\mathcal{L}_{B_t}\beta_t + \partial_t \beta_t) = v_t^* (d\iota_{B_t}\beta_t + \iota_{B_t} d\beta_t + df) = v_t^* d(\iota_{B_t}\beta_t + f) = 0.$$

Therefore, we have $v_t^* \beta_t = \beta_0$ for all $t \in [0,1]$ and hence the diffeomorphism $v \coloneqq v_1$ satisfies identity (5.1).

Next, let us define the family of 1-forms

$$\gamma_t \coloneqq S\alpha_0 + (1-t)\overline{\eta} + t\eta, \qquad t \in [0,1],$$

where $\overline{\eta}$ is the average of η with respect to the S^1-action. Again, since $S-1$ is C^1-small and η and $d\eta$ are C^0-small, γ_t is a contact form for all $t \in [0,1]$. We apply Moser's homotopy argument: Let C_t be the unique time-dependent vector field on M satisfying

$$\begin{cases} C_t \in \ker \gamma_t, \\ (\iota_{C_t} d\gamma_t + \partial_t \gamma_t)|_{\ker \gamma_t} = 0. \end{cases}$$

Let w_t be the flow generated by C_t. Then

$$w_t^* \gamma_t = e^{g_t} \gamma_0$$

where g_t is the unique solution of

$$\begin{cases} \partial_t g_t = w_t^* (\iota_{R_{\gamma_t}} \partial_t \gamma_t) \\ g_0 = 0. \end{cases} \tag{5.2}$$

Set $g \coloneqq g_1$ and $w \coloneqq w_1$. Then we have

$$w^*(S\alpha_0 + \eta) = e^g (S\alpha_0 + \overline{\eta}).$$

We proceed with the following claim:

Claim 5.1. *The function g satisfies the pointwise bound*

$$|g| \leq \sigma(\|\alpha - \alpha_0\|_{C^2}) \cdot |dS|^2, \tag{5.3}$$

for some modulus of continuity σ. In particular, g and dg vanish on the critical set of S.

Proof. In order to prove (5.3), we begin by showing that

$$|\eta - \overline{\eta}| \leq \frac{1}{2} T_0 \|F\|_{C^0} \cdot |dS|. \tag{5.4}$$

It follows from the properties of η and F listed in Theorem 4.1 that

$$\mathcal{L}_{R_{\alpha_0}} \eta = d\iota_{R_{\alpha_0}} \eta + \iota_{R_{\alpha_0}} d\eta = F[dS].$$

Together with the S^1-invariance of S, we deduce the pointwise estimate

$$|\theta_t^* \eta - \eta| = \Big| \int_0^t \partial_\tau \theta_\tau^* \eta \, d\tau \Big| = \Big| \int_0^t \theta_\tau^* (F[dS]) \, d\tau \Big| \leq |t| \cdot \|F\|_{C^0} \cdot |dS|,$$

from which the desired estimate (5.4) follows:

$$|\eta - \overline{\eta}| = \Big| \frac{1}{T_0} \int_0^{T_0} (\eta - \theta_t^* \eta) \, dt \Big| \leq \frac{1}{T_0} \int_0^{T_0} |t| \cdot \|F\|_{C^0} \cdot |dS| \, dt \leq \frac{1}{2} T_0 \|F\|_{C^0} \cdot |dS|.$$

Recall that the vector field C_t is characterized by $C_t \in \ker \gamma_t$ and

$$(\iota_{C_t} d\gamma_t + \eta - \overline{\eta})|_{\ker \gamma_t} = 0.$$

Since γ_t is C^0-close to α_0 and $d\gamma_t$ is C^0-close to $d\alpha_0$, this implies together with (5.4) that

$$|C_t| \leq b \, \|F\|_{C^0} \cdot |dS| \tag{5.5}$$

for some constant $b > 0$ which can be chosen uniform among all α which are sufficiently C^2-close to α_0.

Let us split $R_{\gamma_t} = X_t + Y_t$ into a vector field X_t parallel to R_{α_0} and a vector field Y_t taking values in the contact distribution $\ker \alpha_0$. We can write $X_t = a_t \cdot R_{\alpha_0}$ where $a_t \in C^\infty(M)$ is C^0-close to the constant function 1. We compute

$$\begin{aligned} \iota_{X_t} d\gamma_t &= \iota_{X_t}(dS \wedge \alpha_0 + S d\alpha_0 + (1-t) d\overline{\eta} + t d\eta) \\ &= a_t(-dS + (1-t)\overline{F}[dS] + tF[dS]). \end{aligned}$$

Here we use that $\iota_{R_{\alpha_0}} dS = 0$ because S is S^1-invariant. Moreover, we use the identity $\iota_{R_{\alpha_0}} d\eta = F[dS]$ from Theorem 4.1. Combining the above computation with the identity $0 = \iota_{R_{\gamma_t}} d\gamma_t = \iota_{X_t + Y_t} d\gamma_t$, we obtain

$$\iota_{Y_t} d\gamma_t = -a_t(-dS + (1-t)\overline{F}[dS] + tF[dS]).$$

Since $\ker \gamma_t$ is C^0-close to $\ker \alpha_0$ and $d\gamma_t$ is C^0-close to $d\alpha_0$, this implies the pointwise estimate

$$|Y_t| \leq b|dS|$$

for some constant $b > 0$ independent of the contact form α. In particular, we obtain, in combination with (5.4), that

$$|\iota_{R_{\gamma_t}} \partial_t \gamma_t| = |\iota_{X_t + Y_t}(\eta - \overline{\eta})| = |\iota_{Y_t}(\eta - \overline{\eta})| \leq |Y_t||\eta - \overline{\eta}| \leq b\|F\|_{C^0} \cdot |dS|^2,$$

for some constant b independent of α. Using (5.2), this yields the estimate

$$|g| \leq \int_0^1 |\iota_{R_{\gamma_t}} \partial_t \gamma_t| \circ w_t \, dt \leq b\|F\|_{C^0} \int_0^1 |dS(w_t)|^2 \, dt \tag{5.6}$$

In order to bound the integrand in the above expression, we argue as follows. By assertion (vi) in Theorem 4.1, we have

$$dS = -B[V],$$

where, provided α is C^{k+2}-close to α_0, V is a C^{k+1} small, S^1-invariant section of $\ker \alpha_0$, and $B : \ker \alpha_0 \to (\ker \alpha_0)^*$ is an isomorphism which is C^k-close to the isomorphism B_0 induced by the non-degenerate bilinear form $d\alpha_0$ on $\ker \alpha_0$. Since we are assuming that α is C^2-close to α_0, V is C^1-small and B is C^0-close to B_0. In particular, both B and B^{-1} are uniformly bounded and we obtain the pointwise bounds

$$\frac{1}{c}|V| \leq |dS| \leq c|V|, \tag{5.7}$$

for a suitable number $c \geq 1$. Therefore, from (5.6) we deduce the bound

$$|g| \leq bc^2 \|F\|_{C^0} \int_0^1 |V(w_t)|^2 \, dt.$$

Next we notice that

$$\begin{aligned} \frac{d}{dt}|V(w_t)|^2 &= 2\langle V, \nabla_{C_t} V\rangle \circ w_t \leq 2\|V\|_{C^1}|V(w_t)| \cdot |C_t(w_t)| \\ &\leq 2b\|V\|_{C^1}\|F\|_{C^0}|V(w_t)| \cdot |dS(w_t)| \leq 2bc\|V\|_{C^1}\|F\|_{C^0}|V(w_t)|^2, \end{aligned}$$

where we have used estimate (5.5) and the second inequality in (5.7). It follows from Gronwall's inequality that, for all $t \in [0,1]$,

$$|V(w_t)|^2 \leq e^{2bc\|V\|_{C^1}\|F\|_{C^0} t} \cdot |V|^2.$$

Thus

$$|g| \leq bc^2\|F\|_{C^0} \cdot \int_0^1 e^{2bc\|V\|_{C^1}\|F\|_{C^0} t} \, dt \cdot |V|^2 \leq bc^2\|F\|_{C^0} \cdot e^{2bc\|V\|_{C^1}\|F\|_{C^0}} \cdot |V|^2. \tag{5.8}$$

From (4.2) it follows that

$$\max\big\{\|V\|_{C^1}, \|F\|_{C^0}\big\} \leq \omega_0(\|\alpha - \alpha_0\|_{C^2}).$$

Inequality (5.8) and the first estimate in (5.7) therefore imply a pointwise bound of the form

$$|g| \leq \sigma(\|\alpha - \alpha_0\|_{C^2}) \cdot |dS|^2,$$

for some modulus of continuity σ. This concludes the proof of Claim 5.1. □

Our next task is to prove the following claim.

Claim 5.2. *If $\|\alpha - \alpha_0\|_{C^2}$ is small enough, then*

$$\min_M Se^g = \min_M S, \qquad \max_M Se^g = \max_M S. \tag{5.9}$$

Proof. We claim that if $\|\alpha - \alpha_0\|_{C^2}$ is small enough, then S satisfies the pointwise bound

$$S \geq \min S + \frac{1}{4c^4}|dS|^2, \tag{5.10}$$

where c is the number appearing in (5.7).

As already recalled, the vector field V is C^1-small when $\|\alpha - \alpha_0\|_{C^2}$ is small. Therefore, when $\|\alpha - \alpha_0\|_{C^2}$ is sufficiently small the following fact holds: for any $p \in M$ with $r := |V(p)| > 0$, setting $R := 3rc$ we have

$$\frac{r}{2} \leq |V| \leq 2r \qquad \text{on } B_R(p), \tag{5.11}$$

where $B_R(p) \subset M$ denotes the ball of radius R centered at p.

The bound (5.10) trivially holds at critical points of S. Let $p \in M$ be a point such that $dS(p) \neq 0$. By (5.7) the number $r = |V(p)|$ is positive and (5.11) holds with $R = 3rc$. Now let $\gamma : \mathbb{R} \to M$ be the solution of the Cauchy problem

$$\gamma'(t) = -\nabla S(\gamma(t)), \qquad \gamma(0) = p.$$

If $\gamma([0, s])$ is contained in $B_R(p)$ for some $s \in [0, 1]$, then by (5.7) and (5.11) we have

$$\begin{aligned}\operatorname{dist}(\gamma(s), p) = \operatorname{dist}(\gamma(s), \gamma(0)) &\leq \int_0^s |\gamma'(t)|\, dt = \int_0^s |dS(\gamma(t))|\, dt \\ &\leq c \int_0^s |V(\gamma(t))|\, dt \leq 2rcs \leq 2rc < R.\end{aligned}$$

The above inequality implies that $\gamma([0, 1]) \subset B_R(p)$. Therefore, using again (5.7) and (5.11), we obtain the chain of inequalities

$$\begin{aligned}\min S \leq S(\gamma(1)) = S(p) - \int_0^1 |dS(\gamma(t))|^2\, dt &\leq S(p) - \frac{1}{c^2} \int_0^1 |V(\gamma(t))|^2\, dt \\ &\leq S(p) - \frac{1}{4c^2} r^2 = S(p) - \frac{1}{4c^2} |V(p)|^2 \leq S(p) - \frac{1}{4c^4} |dS(p)|^2,\end{aligned}$$

proving (5.10).

We can now verify the first identity in (5.9). Let $q \in M$ be a point at which S achieves its minimum. Then $dS(q) = 0$ and hence $g(q) = 0$ thanks to Claim 5.1. From this, we obtain the inequality

$$\min_M S e^g \leq S(q) e^{g(q)} = S(q) = \min_M S.$$

In order to prove the opposite inequality, we consider the pointwise bound

$$S e^g \geq S(1 + g) \geq S - 2|g| \geq \min_M S + \Big(\frac{1}{4c^4} - 2\sigma(\|\alpha - \alpha_0\|_{C^2}) \Big) |dS|^2. \tag{5.12}$$

Here, the first inequality follows from the convexity of the exponential function, the second one from the fact that we can assume that $S \leq 2$, as by (4.2) S is C^0-close to the constant function 1 when $\|\alpha - \alpha_0\|_{C^2}$ is small, and the third inequality follows from (5.3) and (5.10).

If $\|\alpha - \alpha_0\|_{C^2}$ is small enough, then the term in brackets at the end of (5.12) is non-negative and hence

$$\min_M S e^g \geq \min_M S.$$

This proves the first identity in (5.9). The proof of the second one is analogous. □

Let us define the family of one-forms

$$\delta_t \coloneqq S\alpha_0 + t\overline{\eta}, \qquad t \in [0, 1].$$

Since $S-1$ is C^1-small and η and $d\eta$ are C^0-small when α is C^2-close to α_0, see (4.2), this is a contact form for all $t \in [0,1]$. Let D_t denote the time-dependent vector field on M characterized by

$$\begin{cases} D_t \in \ker \delta_t, \\ (\iota_{D_t} d\delta_t + \partial_t \delta_t)|_{\ker \delta_t} = 0. \end{cases}$$

We observe that since both S and $\overline{\eta}$ are S^1-invariant, the vector field D_t is S^1-invariant as well. Let ψ_t be the flow generated by D_t. Since D_t is S^1-invariant, this flow is S^1-equivariant. Moreover, we have

$$\psi_t^* \delta_t = e^{d_t} \delta_0,$$

where $d_t \in C^\infty(M)$ is the unique solution of

$$\begin{cases} \partial_t d_t = \psi_t^* (\iota_{R_{\delta_t}} \partial_t \delta_t) \\ d_0 = 0. \end{cases}$$

We claim that $\psi_1^* \delta_1 = (S \circ \psi_1)\alpha_0$. Indeed, we compute

$$\iota_{R_{\alpha_0}} \psi_1^* \delta_1 = \psi_1^* (\iota_{(\psi_1)_* R_{\alpha_0}} \delta_1) = \psi_1^* (\iota_{R_{\alpha_0}} \delta_1) = S \circ \psi_1.$$

On the other hand, we have

$$\iota_{R_{\alpha_0}} \psi_1^* \delta_1 = \iota_{R_{\alpha_0}} e^{d_1} \delta_0 = S e^{d_1}.$$

Therefore, we have $S \circ \psi_1 = S e^{d_1}$. This implies that

$$\psi_1^* \delta_1 = e^{d_1} \delta_0 = S e^{d_1} \alpha_0 = (S \circ \psi_1)\alpha_0.$$

Let us abbreviate $\psi \coloneqq \psi_1$. Then the above computations show:

$$\psi^* (S\alpha_0 + \overline{\eta}) = (S \circ \psi)\alpha_0.$$

Let us define the diffeomorphism φ of M by $\varphi \coloneqq u \circ v \circ w \circ \psi$. Then

$$\begin{aligned} \varphi^* \alpha &= \psi^* w^* v^* u^* \alpha = \psi^* w^* v^* (S\alpha_0 + \eta + df) = \psi^* w^* (S\alpha_0 + \eta) = \psi^* (e^g (S\alpha_0 + \overline{\eta})) \\ &= \big((e^g S) \circ \psi\big)\alpha_0. \end{aligned}$$

Therefore, setting

$$T \coloneqq S \circ \psi, \qquad h \coloneqq g \circ \psi,$$

we obtain that the diffeomorphism φ satisfies the desired identity

$$\varphi^* \alpha = T e^h \alpha_0.$$

We now check that properties (a)-(e) from Theorem 3 hold. Since S is S^1-invariant and ψ is S^1-equivariant, the function T is S^1-invariant, proving (a). By Claim 5.1, both g and dg vanish on the critical set of S. Therefore, the same is true for the pull-backs

h and T under the diffeomorphism ψ, proving property (b). Property (c) follows from Claim 5.2. Property (d) follows from property (vii) in Proposition 4.3. By property (viii) from the same proposition, the short closed orbits of $R_{u^*\alpha}$ are precisely the orbits γ of the free S^1-action θ_t that consist of critical points of S, and the minimal period of such an orbit is $T_0 S(\gamma) = T_0 T(\psi^{-1}(\gamma))$. Since $u^*\alpha = S\alpha_0 + \eta + df$ and $v^*u^*\alpha = S\alpha_0 + \eta$ induce the same Reeb flow up to a time reparametrization which preserves the periods of closed orbits, the same is true for the short closed Reeb orbits of $v^*u^*\alpha$. By estimate (5.5), the diffeomorphism w fixes the critical set of S, so the same fact continues to hold for the short closed Reeb orbits of $w^*v^*u^*\alpha$. By applying the S^1-equivariant diffeomorphism ψ, we obtain that the short closed Reeb orbits of $\varphi^*\alpha$ are the orbits of θ_t forming the image by ψ^{-1} of the critical set of S. The latter set is precisely the critical set of T and property (e) follows.

The last claim in Theorem 3 is that the map $\alpha \mapsto (\varphi, T, h)$ we have just constructed is smooth and maps α_0 to $(\mathrm{id}, 1, 0)$. This follows from the corresponding statement about the map $\alpha \mapsto (u, S, \eta, f)$ in Theorem 4.1 and from the fact that the tuple (v, w, g, ψ) we constructed above depends smoothly on the defining data (S, η, f) and takes the value $(\mathrm{id}, \mathrm{id}, 0, \mathrm{id})$ for $(S, \eta, f) = (1, 0, 0)$. This concludes the proof of Theorem 3.

6 Proof of Theorem 1 (ii)

In this section, we deduce Theorem 1 (ii) from Theorem 3. Recall that $\mathcal{A}$ denotes the set of domains in $\mathbb{C}^n$ of the form

$$A_f = \{rz \mid z \in S^{2n-1},\ 0 \le r < f(z)\},$$

where f is a positive smooth function on S^{2n-1}, and that the C^k-distance on $\mathcal{A}$ is induced by the C^k-distance on the space of positive smooth functions on S^{2n-1}. For every A in $\mathcal{A}$, we denote by

$$\alpha_A \coloneqq \lambda_0|_{\partial A}$$

the restriction of the standard primitive of ω_0 to the boundary of A. In the case of the unit ball $A_1 = B$, we find the standard contact form

$$\alpha_0 \coloneqq \alpha_B = \lambda_0|_{S^{2n-1}}$$

on S^{2n-1}, which is Zoll with all orbits of period π. Given a domain $A = A_f$ in $\mathcal{A}$, the radial projection

$$\rho_A : S^{2n-1} \to \partial A, \qquad z \mapsto f(z)z,$$

satisfies

$$\rho_A^* \alpha_A = f^2 \alpha_0. \tag{6.1}$$

This identity shows that $A \in \mathcal{A}$ is C^k-close to B if and only if $\rho_A^*\alpha_A$ is C^k-close to α_0.

We now proceed with the proof of statement (ii) in Theorem 1. Given a smooth positive function f on S^{2n-1}, we denote by

$$\{f_t \coloneqq 1 + t(f-1)\}_{t\in[0,1]}$$

the smooth homotopy connecting the constant function 1 to f and we consider the corresponding smooth path of domains $\{A_{f_t}\}_{t\in[0,1]}$ in $\mathcal{A}$, connecting B to A_f. We assume that the domain $A \coloneqq A_f$ is sufficiently C^2-close to B, so that

$$\|\rho^*_{A_{f_t}}\alpha_{A_{f_t}} - \alpha_0\|_{C^2} = \|(f_t^2 - 1)\alpha_0\|_{C^2} < \delta \quad \forall t \in [0,1],$$

where δ is the positive number appearing in Theorem 3. Note that $\{\rho^*_{A_{f_t}}\alpha_{A_{f_t}}\}_{t\in[0,1]}$ is a smooth path of contact forms on S^{2n-1} connecting α_0 to $\rho_A^*\alpha_A$. By Theorem 3, there exists a smooth family of diffeomorphism $\{\varphi_t : S^{2n-1} \to S^{2n-1}\}_{t\in[0,1]}$ such that

$$\varphi_t^*(\rho^*_{A_{f_t}}\alpha_{A_{f_t}}) = T_t e^{h_t}\,\alpha_0, \tag{6.2}$$

where $\{T_t\}_{t\in[0,1]}$ and $\{h_t\}_{t\in[0,1]}$ are smooth families of functions on S^{2n-1} satisfying properties (a)-(e) of Theorem 3 together with $\varphi_0 = \mathrm{id}$, $T_0 = 1$ and $h_0 = 0$. By (e) we have

$$\mathrm{sys}(A) = \mathrm{sys}(\alpha_A) = \mathrm{sys}(\rho_A^*\alpha_A) = \mathrm{sys}(\rho^*_{A_1}\alpha_{A_1}) = \mathrm{sys}(\alpha_0)\min_{S^{2n-1}} T_1 = \pi \min_{S^{2n-1}} T_1.$$

Therefore, Theorem 1 (ii) will be proven if we can find a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ mapping the open ball B' of radius

$$R \coloneqq \sqrt{\min_{S^{2n-1}} T_1}$$

centered at the origin into A. By property (c) of Theorem 3, we have

$$R = \sqrt{\min_{S^{2n-1}} T_1 e^{h_1}} = \min_{S^{n-1}} \sqrt{T_1} e^{\frac{h_1}{2}}$$

and hence the domain

$$A' \coloneqq A_{\sqrt{T_1}e^{\frac{h_1}{2}}}$$

contains B'. Hence, it is enough to find a symplectomorphism $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\phi(A') = A$. Consider the smooth path of domains

$$\{A'_t \coloneqq A_{\sqrt{T_t}e^{\frac{h_t}{2}}}\}_{t\in[0,1]},$$

which satisfies $A'_0 = B$ and $A'_1 = A'$. By (6.2) and (6.1), the diffeomorphisms

$$\psi_t : \partial A'_t \to \partial A_{f_t}, \qquad \psi_t \coloneqq \rho_{A_{f_t}} \circ \varphi_t \circ \rho^{-1}_{A'_t},$$

satisfy

$$\psi_t^*(\lambda_0|_{\partial A_{f_t}}) = (\rho^{-1}_{A'_t})^*(\varphi_t^*(\rho^*_{A_{f_t}}\alpha_{A_{f_t}})) = (\rho^{-1}_{A'_t})^*(T_t e^{h_t}\alpha_0) = \alpha_{A'_t} = \lambda_0|_{\partial A'_t}.$$

The fact that the diffeomorphism ψ_t preserves the restriction of λ_0 implies that its positively 1-homogeneous extension

$$\tilde{\psi}_t : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}, \qquad \tilde{\psi}_t(rz) \coloneqq r\psi_t(z) \quad \forall z \in \partial A'_t,\ \forall r > 0,$$

preserves λ_0. By construction, $\tilde{\psi}_t$ depends smoothly on t and satisfies

$$\tilde{\psi}_0 = \mathrm{id}, \qquad \tilde{\psi}_1(A' \setminus \{0\}) = A \setminus \{0\}.$$

Let X_t be the vector field generating $\tilde{\psi}_t$. Since $\tilde{\psi}_t$ preserves λ_0, we have

$$0 = \mathcal{L}_{X_t}\lambda_0 = \imath_{X_t}\omega_0 + d\imath_{X_t}\lambda_0,$$

and hence X_t is the Hamiltonian vector field of the Hamiltonian $H_t \coloneqq -\lambda_0(X_t)$ on $\mathbb{C}^n \setminus \{0\}$. Multiplying H_t by a smooth function on $\mathbb{C}^n$ which is supported in $\mathbb{C}^n \setminus \{0\}$ and equals 1 outside of a sufficiently small neighborhood of 0, we obtain a new time-dependent Hamiltonian on $\mathbb{C}^n$ whose time-one map is a global symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ which coincides with $\tilde{\psi}_1$ on $\mathbb{C}^n \setminus A'$, and hence satisfies $\phi(A') = \tilde{\psi}_1(A') = A$. This concludes the proof of Theorem 1 (ii).

7 Proof of Theorem 2

Let $A \in \mathcal{A}$ be C^2-close to B. Using the notation of Section 6, we deduce that the contact form $\rho_A^*\alpha_A$ on S^{2n-1} is C^2-close to the Zoll contact form α_0 and hence there exists a diffeomorphism $\varphi : S^{2n-1} \to S^{2n-1}$ such that

$$\varphi^*(\rho_A^*\alpha_A) = Te^h\,\alpha_0, \tag{7.1}$$

where the smooth functions $T : S^{2n-1} \to \mathbb{R}$ and $h : S^{2n-1} \to \mathbb{R}$ satisfy all the requirements of Theorem 3. In particular, ∂A has a closed characteristic γ whose action equals the number

$$T_{\max}(A) \coloneqq \pi \max_{S^{2n-1}} T.$$

Up to rescaling, we can assume that

$$T_{\max}(A) = T_{\max}(B) = \pi, \tag{7.2}$$

and we shall prove the following facts:

(i') There exists a symplectomorphism $\phi : \mathbb{C}^n \to \mathbb{C}^n$ mapping the ellipsoid $E \coloneqq E(\pi, \frac{\pi}{2}, \ldots, \frac{\pi}{2})$ into A.

(ii') There exists a symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ mapping A into B.

From the monotonicity property of the capacity c we then have

$$c(E) \leq c(A) \leq c(B).$$

Since c is n-normalized, its values at E and B coincide with the corresponding values of the n-th Ekeland-Hofer capacity, which assigns to both E and B the value π. Therefore, $c(E) = c(B)$ and both inequalities in the above expression are equalities. This proves claims (i) and (ii) from Theorem 2. There remains to prove (i') and (ii').

The proof of (i') is similar to the proof of Theorem 1 (i) from Section 3, and here we use the notation which we introduced there. Up to the application of a unitary isomorphism, we may assume that the closed characteristic γ, which by (7.2) has action π, passes through the point $(1, 0, \ldots, 0) \in \mathbb{C}^n$ and is contained in the open set $V = \Phi(U)$, see (1.1) and (3.1). By the explicit form (1.1) of the symplectomorphism Φ, we have the identities

$$B \cap V = \Phi(U \cap \{s < 0\}), \qquad E \cap V = \Phi\big(U \cap \{s < -\pi|w|^2\}\big). \tag{7.3}$$

Since A is C^2-close to B, we can find a C^2-small smooth function $H : \mathbb{T} \times \mathbb{C}^{n-1} \to \mathbb{R}$ with support in $\mathbb{T} \times \widehat{B}$ satisfying (1.4) and such that

$$A \cap V = D(H) \cap V. \tag{7.4}$$

The closed characteristic γ of action π corresponds to the fixed point 0 of ϕ_H^1, which has action 0, see Proposition 1.1 (ii). By Proposition 3.1, there exists a smooth family of Hamiltonians $\{H^\lambda\}_{\lambda \in [0,1]}$ on $\mathbb{T} \times \mathbb{C}^{n-1}$ which satisfies:

(i) $H^0 = H$;

(ii) $H^\lambda = H$ on $\mathbb{T} \times \widehat{B}^{\mathrm{c}}_{\frac{1}{3}}$ for every $\lambda \in [0, 1]$;

(iii) $\phi^1_{H^\lambda} = \phi^1_H$ for every $\lambda \in [0, 1]$;

(iv) $|H^\lambda| < \frac{\pi}{2}$ for every $\lambda \in [0, 1]$;

(v) $H^1(t, w) \geq -\pi|w|^2$ for every $(t, w) \in \mathbb{T} \times \mathbb{C}^{n-1}$.

By (i) and (iii), Proposition 1.1 (iii) gives us a symplectomorphism $\widetilde{\phi} : \mathbb{C}^n \to \mathbb{C}^n$ such that $\tilde{\phi}(D(H)) = D(H^1)$, which by (ii), (iv) and Remark 1.2 is supported in V and by (7.4) satisfies

$$\tilde{\phi}(A) = (A \setminus V) \cup (D(H^1) \cap V).$$

Since the closure of $E \setminus V$ is contained in B, C^0-closeness of A to B implies that $E \setminus V$ is contained in $A \setminus V$. By (v) and the second identity in (7.3), $E \cap V$ is contained in $D(H^1) \cap V$, so the above identity implies that E is contained in $\tilde{\phi}(A)$. We conclude that the symplectomorphism $\phi \coloneqq \tilde{\phi}^{-1}$ satisfies $\phi(E) \subset A$, proving (i').

The proof of (ii') is similar to the proof of Theorem 1 from Section 6. Arguing as in that proof, we can apply Theorem 3 to a 1-parameter family of contact forms joining α_0 and $\rho_A^*\alpha_A$ and obtain a symplectomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ mapping A onto the domain

$$A' \coloneqq A_{\sqrt{T}e^{\frac{h}{2}}},$$

see (7.1). By (7.2) and the fact that the maximum of Te^h coincides with the maximum of T (see statement (c) in Theorem 3), we get that

$$\max_{S^{2n-1}}(\sqrt{T}e^{\frac{h}{2}}) = 1.$$

Hence, $A' \subset A_1 = B$, proving (ii').

8 Proof of Proposition 2

We now prove Proposition 2. Up to exchanging the coordinates and rescaling, we may assume that $a = 1 \leq b$, so that

$$\widehat{c}_k(P(1,b)) = k \min\{1, b\} = k, \qquad \forall k \in \mathbb{N},$$

see [EH90, Proposition 5] and [GH18, Example 1.7].

Next we recall that the ellipsoid $E(1,2)$ symplectically embeds into $P(1,1)$ (see [FM15, Theorem 1.3] and [Sch18, Lemma 8.2]). The existence of this symplectic embedding is quite remarkable, because $E(1,2)$ and $P(1,1)$ have the same volume and are not symplectomorphic, as $\widehat{c}_3(E(1,2)) = 2$ and $\widehat{c}_3(P(1,1)) = 3$.

By the "extension after retraction principle" from [Sch02, Proposition 1.3], for every $\theta < 1$ there is a global symplectomorphism of $\mathbb{C}^2$ mapping $E(\theta, 2\theta)$ into $P(1,1)$, and hence also into $P(1,b)$. Therefore,

$$\underline{c}_k(P(1,b)) \geq \widehat{c}_k(E(\theta, 2\theta)) \qquad \forall k \in \mathbb{N},$$

and by taking the supremum over all $\theta < 1$ we obtain

$$\underline{c}_k(P(1,b)) \geq \widehat{c}_k(E(1,2)) \qquad \forall k \in \mathbb{N}. \tag{8.1}$$

For every $\alpha > 1$ we choose β large enough, so that $P(1,b) \subset E(\alpha, \beta)$. Then for every $k \in \mathbb{N}$ we have

$$\overline{c}_k(P(1,b)) \leq \widehat{c}_k(E(\alpha,\beta)) \leq k\alpha,$$

and by taking the infimum over all $\alpha > 1$ we obtain

$$\overline{c}_k(P(1,b)) \leq k \qquad \forall k \in \mathbb{N}. \tag{8.2}$$

On the other hand, $k = \widehat{c}_k(P(1,b)) \leq \overline{c}_k(P(1,b))$ and we conclude that

$$\overline{c}_k(P(1,b)) = k, \qquad \forall k \in \mathbb{N}.$$

Since $\widehat{c}_2(E(1,2)) = 2$ and every 2-normalized capacity c satisfies $\underline{c}_2 \leq c \leq \overline{c}_2$, the inequalities (8.1) and (8.2) for $k = 2$ imply that

$$c(P(1,b)) = 2 = 2\min\{1,b\},$$

proving statement (i) of Proposition 2.

In order to prove statement (ii), we use Hutchings' ECH-capacities c_k^{ECH} for four-dimensional domains from [Hut11]. By [Hut11, Proposition 1.2], the k-th ECH-capacity of the ellipsoid $E(a,b)$ is the $(k+1)$-th smallest element in the list $(ha + jb)_{h,j\geq 0}$, again allowing repetitions. In particular, c_k^{ECH} is not a k-normalized capacity in the sense of this paper.

Let $E(\alpha,\beta)$ be an ellipsoid which symplectically embeds into $P(1,1)$. As observed in [Hut11, Remark 1.8], the ECH-capacities $P(1,1)$ and $E(1,2)$ coincide, and hence

$$c_k^{\mathrm{ECH}}(E(\alpha,\beta)) \leq c_k^{\mathrm{ECH}}(P(1,1)) = c_k^{\mathrm{ECH}}(E(1,2)) \qquad \forall k \in \mathbb{N}.$$

As proven by McDuff in [McD11], the ECH-capacities form a complete list of obstructions for the problem of symplectically embedding an ellipsoid into another, and hence the above inequalities imply that $E(\alpha,\beta)$ symplectically embeds into $E(1,2)$. Therefore,

$$\widehat{c}_k(E(\alpha,\beta)) \leq \widehat{c}_k(E(1,2)) \qquad \forall k \in \mathbb{N},$$

and by taking the supremum over the space of all ellipsoids $E(\alpha,\beta)$ which embed into $P(1,1)$ by a globally defined symplectomorphism of $\mathbb{C}^n$ we obtain the inequalities

$$\underline{c}_k(P(1,1)) \leq \widehat{c}_k(E(1,2)) \qquad \forall k \in \mathbb{N}.$$

Together with (8.1), we deduce the identities

$$\underline{c}_k(P(1,1)) = \widehat{c}_k(E(1,2)) \qquad \forall k \in \mathbb{N}.$$

Moreover, $\widehat{c}_k(E(1,2)) \leq k$ with equality if and only if k equals 1 or 2. This proves statement (ii) of Proposition 2.

9 Proof of Theorem 4

The proof of statement (i) in Theorem 4 makes use of the following well known alternative (see [Sul76, Theorem II.26] for a proof based on the Hahn–Banach theorem and the appendix of [LS94] for a more elementary proof).

THEOREM 9.1. *Let X be a smooth vector field on the closed manifold M and let K be a compact subset of M. Then there exists either a probability measure which is supported in K and is invariant under the flow of X or a smooth real function h on M such that $dh[X] > 0$ on K.*

Let ξ be a co-oriented contact structure on the closed manifold M. A smooth vector field on M is said to be a contact vector field if its flow consists of contactomorphisms of (M,ξ). Contact vector fields are in one-to-one correspondence with smooth real functions on M: A smooth real function H on M (the contact Hamiltonian) and a defining contact form α for ξ define the contact vector field X by the identities

$$\imath_X d\alpha = \big(\imath_{R_\alpha} dH\big)\alpha - dH, \qquad \imath_X \alpha = H, \tag{9.1}$$

see [Gei08, Section 2.3]. We shall make use of the following formula for the conformal factor of the flow of contactomorphisms in terms of the generating contact Hamiltonian.

LEMMA 9.2. *Let X be the contact vector field induced by the contact Hamiltonian function $H \in C^\infty(M)$ together with the contact form $\alpha \in \mathcal{F}(M,\xi)$, and denote by ϕ_X^t its flow. Then*

$$(\phi_X^t)^*\alpha = \exp\Big(\int_0^t (\imath_{R_\alpha} dH) \circ \phi_X^s \, ds\Big)\alpha,$$

for every $t \in \mathbb{R}$.

Proof. We define h_t by the identity $(\phi_X^t)^*\alpha = e^{h_t}\alpha$. Then $h_0 = 0$ and differentiation in t gives us

$$\begin{aligned}\partial_t h_t \, e^{h_t}\alpha = \frac{d}{dt}(\phi_X^t)^*\alpha = (\phi_X^t)^*\mathcal{L}_X\alpha = (\phi_X^t)^*\big(\imath_X d\alpha + d\imath_X\alpha\big)\\ = (\phi_X^t)^*\big((\imath_{R_\alpha} dH)\alpha\big) = (\imath_{R_\alpha} dH)\circ\phi_X^t \, e^{h_t}\alpha,\end{aligned}$$

where we have used (9.1). Then

$$h_t = \int_0^t \partial_s h_s \, ds = \int_0^t (\imath_{R_\alpha} dH)\circ\phi_X^s \, ds,$$

as claimed. □

We can now prove Theorem 4.

Proof of Theorem 4. We are assuming that the contact forms $\alpha, \beta \in \mathcal{F}(M,\xi)$ satisfy

$$d(\alpha,\beta) = \max_M f - \min_M f \quad \text{with} \quad \varphi^*\beta = e^f\alpha,$$

for some $\varphi \in \mathrm{Cont}_0(M,\xi)$.

(i) If f is constant, then $R_{\varphi^*\beta}$ is a constant multiple of R_α and hence the probability measure on M which is determined by the volume form $\alpha \wedge d\alpha^{n-1}$ is invariant for the flows of both these Reeb vector fields. Therefore, we can assume that the compact sets

$$K_{\min} \coloneqq f^{-1}(\min_M f) \qquad \text{and} \qquad K_{\max} \coloneqq f^{-1}(\max_M f)$$

are disjoint. Since df vanishes on $K_{\min}$ and $K_{\max}$, the identity

$$d(\varphi^*\beta) = d(e^f\alpha) = e^f(df \wedge \alpha + d\alpha)$$

implies that

$$\begin{aligned} R_{\varphi^*\beta} &= e^{-\min f} R_\alpha \quad \text{on} \quad K_{\min}, \\ R_{\varphi^*\beta} &= e^{-\max f} R_\alpha \quad \text{on} \quad K_{\max}. \end{aligned}$$

Therefore, any invariant measure for the Reeb flow of $\varphi^*\beta$ which is supported in either $K_{\min}$ or $K_{\max}$ is invariant also for the Reeb flow of α. We shall show the existence of a $R_{\varphi^*\beta}$-invariant probability measure supported in $K_{\min}$, the case of $K_{\max}$ being analogous.

We argue by contradiction and assume that the flow of $R_{\varphi^*\beta}$ admits no invariant probability measure which is supported in $K_{\min}$. Then Theorem 9.1 implies the existence of a smooth real function H on M such that

$$dH\big[R_{\varphi^*\beta}\big] > 0 \quad \text{on} \quad K_{\min}.$$

By multiplication by a smooth function having the value 1 near $K_{\min}$ and 0 near $K_{\max}$, we can assume that $H = 0$ on a neighborhood V of $K_{\max}$. Let X be the contact vector field which is induced by the contact Hamiltonian H and the contact form $\varphi^*\beta = e^f\alpha$. By Lemma 9.2, we have

$$(\phi_X^t)^*(\varphi^*\beta) = e^{g_t}\alpha,$$

where

$$g_t \coloneqq f + \int_0^t \big(\imath_{R_{\varphi^*\beta}} dH\big) \circ \phi_X^s \, ds = f + t\, dH\big[R_{\varphi^*\beta}\big] + O(t^2)$$

for $t \to 0$. We can find an open neighborhood U of $K_{\min}$ and a positive number δ such that

$$\begin{aligned} dH\big[R_{\varphi^*\beta}\big] &\geq \delta && \text{on} \quad U, \\ f &\geq \min f + \delta && \text{on} \quad M \setminus U, \\ f &\leq \max f - \delta && \text{on} \quad M \setminus V. \end{aligned}$$

On U, we then have

$$g_t \geq \min f + \delta t + O(t^2),$$

while on $M \setminus U$ we have

$$g_t \geq \min f + \delta - ct + O(t^2),$$

where c is the supremum norm of the function $dH\big[R_{\varphi^*\beta}\big]$. We conclude that

$$g_t \geq \min f + \tfrac{\delta}{2} t \quad \text{on} \quad M, \tag{9.2}$$

for $t \geq 0$ small enough. Since H vanishes on V, so does X and hence $g_t = f$ on V for every $t \in \mathbb{R}$. On $M \setminus V$ we have

$$g_t \leq \max f - \delta + ct + O(t^2).$$

Therefore,

$$g_t \leq \max f \quad \text{on} \quad M,$$

for $t \geq 0$ small enough. Together with (9.2), this implies that for $t > 0$ sufficiently small the contactomorphism

$$\psi := \varphi \circ \phi_X^t \in \mathrm{Cont}_0(M, \xi)$$

satisfies

$$\psi^* \beta = e^{g_t} \alpha$$

with

$$\max g_t - \min g_t < \max f - \min f,$$

contradicting the assumption that $d(\alpha, \beta) = \max f - \min f$.

(ii) From the definition of the pseudo-metric d we obtain

$$d(e^{g_1}\alpha, e^{g_2}\alpha) = d(e^{g_1}\alpha, e^{g_2 - g_1} e^{g_1}\alpha) \leq \max_M (g_2 - g_1) - \min_M (g_2 - g_1), \tag{9.3}$$

for every g_1 and g_2 in $C^\infty(M)$. Given the path $\{\psi_t\}_{t\in[0,1]}$ in $\mathrm{Cont}_0(M, \xi)$ such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \varphi^{-1}$, we consider the curve

$$\gamma(t) := \psi_t^*(e^{tf}\alpha), \qquad t \in [0, 1],$$

which satisfies $\gamma(0) = \alpha$ and $\gamma(1) = \beta$. For any subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ we have, by the invariance of the psedo-metric d under the action of $\mathrm{Cont}_0(M, \xi)$ and by (9.3),

$$\begin{aligned} d(\alpha, \beta) &\leq \sum_{j=1}^k d(\gamma(t_{j-1}), \gamma(t_j)) = \sum_{j=1}^k d(\psi_{t_{j-1}}^*(e^{t_{j-1}f}\alpha), \psi_{t_j}^*(e^{t_j f}\alpha)) = \sum_{j=1}^k d(e^{t_{j-1}f}\alpha, e^{t_j f}\alpha) \\ &\leq \sum_{j=1}^k \max_M (t_j f - t_{j-1} f) - \min_M (t_j f - t_{j-1} f) = \max_M f - \min_M f = d(\alpha, \beta). \end{aligned}$$

The above inequalities are then equalities and hence γ has length $d(\alpha, \beta)$ and is therefore a minimizing geodesic. □

10 Elementary spectral invariants

The proof of Theorem 5 in Section 11 is based on a sequence of spectral invariants for contact forms on a closed contact manifold, whose definition and properties are discussed in the present section. These spectral invariants are inspired by the symplectic capacities defined by McDuff–Siegel in [MS23], by those introduced by Hutchings in [Hut22a, Hut22b] and by the spectral invariants for Hamiltonian diffeomorphisms defined by one of us in [Edt22b]. They are more elementary than, say, the spectral invariants which are induced by contact homology, in the sense that their definition and the proof of their properties require

only compactness results for pseudoholomorhpic curves. Their computation involves some knowledge of Gromov–Witten invariants, see Proposition 10.3 below.

Before discussing them, we recall some basic facts and fix notation concerning the symplectization of a contact manifold. Let ξ be a co-oriented contact structure on the $(2n-1)$-dimensional manifold M. A cotangent vector $x \in T^*_pM$ is said to define $\xi(p)$ if $\ker x = \xi(p)$ and x is positive on vectors in T_pM which are positively transverse to $\xi(p)$. We denote by

$$\widetilde{M} \coloneqq \{x \in T^*M \mid x \text{ defines } \xi(\pi(x))\}$$

the symplectization of M, where $\pi : T^*M \to M$ denotes the footpoint projection. The symplectization $\widetilde{M}$ is a $2n$-dimensional submanifold of T^*M. It carries a Liouville 1-form $\widetilde{\lambda}$ and a symplectic form $\widetilde{\omega} = d\widetilde{\lambda}$ given by the restriction of the standard Liouville 1-form and symplectic form of T^*M. Moreover, $\widetilde{M}$ carries a canonical $\mathbb{R}$-action given by

$$\mathbb{R} \times \widetilde{M} \to \widetilde{M}, \qquad (a, \eta) \mapsto e^a \eta.$$

We denote by ∂_a the vector field on $\widetilde{M}$ generating this action. The choice of a contact form $\alpha \in \mathcal{F}(M, \xi)$ induces the diffeomorphism

$$\varphi_\alpha : \mathbb{R} \times M \to \widetilde{M}, \qquad (a, p) \mapsto e^a \alpha(p),$$

such that $\varphi_\alpha^* \widetilde{\omega} = d(e^a \alpha)$ where a is the variable in the first factor of $\mathbb{R} \times M$. Under the identification of $\widetilde{M}$ with $\mathbb{R} \times M$ defined by φ_α, the canonical action of $\mathbb{R}$ is given by translations in the first factor (this justifies the choice of denoting the generating vector field by ∂_a) and the contact-type hypersurface

$$M_\alpha \coloneqq \{\alpha(p) \mid p \in M\} \subset \widetilde{M}$$

corresponds to the set $\{a = 0\}$. The contact form α and the Reeb vector field R_α can be seen as a translation invariant 1-form and vector field on $\mathbb{R} \times M$, and by the diffeomorphism φ_α induce an $\mathbb{R}$-invariant 1-form and an $\mathbb{R}$-invariant vector field on $\widetilde{M}$:

$$\widetilde{\alpha} \coloneqq \varphi_{\alpha *} \alpha, \qquad \widetilde{R}_\alpha \coloneqq \varphi_{\alpha *} R_\alpha.$$

The vector field $\widetilde{R}_\alpha$ is tangent to the hypersurface M_α and all its images by the canonical $\mathbb{R}$-action, and spans the characteristic distribution of these hypersurfaces.

Given contact forms $\alpha, \beta \in \mathcal{F}(M, \xi)$ with $\alpha < \beta$, we denote by $\widetilde{M}^\beta_\alpha \subset \widetilde{M}$ the symplectic cobordism which is given by the region in $\widetilde{M}$ bounded by M_α and M_β, that is,

$$\widetilde{M}^\beta_\alpha \coloneqq \{x \in \widetilde{M} \mid \alpha(\pi(x)) < x < \beta(\pi(x)) \text{ on vectors which are positively transverse to } \xi(\pi(x))\}.$$

The diffeomorphism φ_β exhibits the region $\widetilde{M}^+_\beta \subset \widetilde{M}$ above the hypersurface M_β as a positive cylindrical end attached to the symplectic cobordism $\widetilde{M}^\beta_\alpha$. Similarly, we can regard the region $\widetilde{M}^-_\alpha \subset \widetilde{M}$ below the hypersurface M_α as a negative cylindrical end of $\widetilde{M}^\beta_\alpha$ via the diffeomorphism φ_α.

Let us recall the following terminology from [BEH$^+$03]. An almost complex structure J on $\widetilde{M}$ is called *adjusted* to a contact form $\alpha \in \mathcal{F}(M, \xi)$ if it satisfies the following properties:

1. J is invariant under the canonical $\mathbb{R}$-action on $\widetilde{M}$.
2. $J\partial_a = \widetilde{R}_\alpha$.
3. The codimension-two distribution $\xi_\alpha := \varphi_{\alpha *}((0) \times \xi)$ is invariant under J. Moreover, the restriction of J to ξ_α is compatible with $d\widetilde{\alpha}$.

Suppose that J is an α-adjusted almost complex structure on $\widetilde{M}$. Let $u : (\Sigma, j) \to (\widetilde{M}, J)$ be a J-holomorphic map whose domain is a punctured Riemann surface. We define the *action* $\mathcal{A}_\alpha(u)$ to be

$$\mathcal{A}_\alpha(u) := \int_\Sigma u^* d\widetilde{\alpha} \in [0, +\infty].$$

Note that since u is J-holomorphic and J is adjusted to α, the 2-form $u^* d\widetilde{\alpha}$ is everywhere non-negative on Σ. Thus $\mathcal{A}_\alpha(u)$ is indeed well-defined an non-negative. Moreover, we define

$$E_\alpha(u) := \sup_\rho \int_\Sigma u^* \big(\rho \circ a_\alpha \, da_\alpha \wedge \widetilde{\alpha}\big) \in [0, +\infty], \qquad \text{with } a_\alpha := a \circ \varphi_\alpha^{-1},$$

where the supremum is taken over all compactly supported functions $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ whose integral is equal to 1. As before, the 2-form $u^*(\rho \circ a_\alpha) da_\alpha \wedge \widetilde{\alpha}$ is everywhere non-negative because J is α-adjusted and u is J-holomorphic. We say that u is a *finite-energy* curve if both $\mathcal{A}_\alpha(u)$ and $E_\alpha(u)$ are finite. If u is a finite-energy curve and α is non-degenerate, then u is positively or negatively asymptotic to trivial cylinders over periodic Reeb orbits of α at every puncture of Σ which is not a removable singularity of u. We define the *positive action* $\mathcal{A}_\alpha^+(u)$ to be the sum of the actions (with respect to α) of the positive asymptotic Reeb orbits of u. Similarly, we define the *negative action* $\mathcal{A}_\alpha^-(u)$ to be the sum of the actions of the negative asymptotic Reeb orbits of u. By Stokes' theorem, we have $\mathcal{A}_\alpha(u) = \mathcal{A}_\alpha^+(u) - \mathcal{A}_\alpha^-(u)$.

Let $\alpha < \beta$ be contact forms and assume that J is a compatible almost complex structure on $\widetilde{M}$ whose restriction to $\widetilde{M}_\beta^+$ is β-adjusted and whose restriction to $\widetilde{M}_\alpha^-$ is α-adjusted. We define the *action*

$$\mathcal{A}_\alpha^\beta(u) := \int_{u^{-1}(\widetilde{M}_\beta^+)} u^* d\widetilde{\beta} + \int_{u^{-1}(\widetilde{M}_\alpha^\beta)} u^* \widetilde{\omega} + \int_{u^{-1}(\widetilde{M}_\alpha^-)} u^* d\widetilde{\alpha} \in [0, +\infty].$$

All three summands in this expression are well-defined and non-negative. For the terms involving the cylindrical ends $\widetilde{M}_\beta^+$ and $\widetilde{M}_\alpha^-$ this follows from the fact that J is adjusted and for the cobordism $\widetilde{M}_\alpha^\beta$ this uses that J is compatible. In addition, we define

$$E_\alpha^\beta(u) := \sup_{\rho^+} \int_{u^{-1}(\widetilde{M}_\beta^+)} u^* \big(\rho^+ \circ a_\beta \, da_\beta \wedge \widetilde{\beta}\big) + \sup_{\rho^-} \int_{u^{-1}(\widetilde{M}_\alpha^-)} u^* \big(\rho^- \circ a_\alpha \, da_\alpha \wedge \widetilde{\alpha}\big) \in [0, +\infty].$$

Here the suprema are taken over all compactly supported, non-negative functions $\rho^\pm$ on $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$, respectively, with integral equal to 1. Again, a curve u is said to have

finite energy if both $\mathcal{A}_\alpha^\beta(u)$ and $E_\alpha^\beta(u)$ are finite. If u has finite energy and α and β are non-degenerate, then u is positively asymptotic to periodic Reeb orbits of β or negatively asymptotic to periodic Reeb orbits of α at punctures which are not removable singularites. The positive action $\mathcal{A}_\beta^+(u)$ and the negative action $\mathcal{A}_\alpha^-(u)$ are defined as before and we have $\mathcal{A}_\alpha^\beta(u) = \mathcal{A}_\beta^+(u) - \mathcal{A}_\alpha^-(u)$ by Stokes' theorem.

Given a contact form $\alpha \in \mathcal{F}(M,\xi)$, define $\mathcal{J}(\alpha)$ to be the set all compatible almost complex structures J on the symplectization $\widetilde{M}$ with the property that there exist real numbers $s^- < s^+$ such that J is adjusted to α on both $\widetilde{M}^\pm_{e^{s^\pm}\alpha}$. Here the numbers $s^\pm$ are allowed to depend on J. For a given $J \in \mathcal{J}(\alpha)$, there are many choices of $s^\pm$ making it adjusted to α on $\widetilde{M}^\pm_{e^{s^\pm}\alpha}$. While the action $\mathcal{A}^{e^{s^+}\alpha}_{e^{s^-}\alpha}$ and the energy $E^{e^{s^+}\alpha}_{e^{s^-}\alpha}$ themselves depend on the choice of $s^\pm$, the property of being a finite-energy J-holomorphic curve does not.

Suppose now that $\alpha \in \mathcal{F}(M,\xi)$ is non-degenerate, that is, all the periodic orbits of the Reeb vector field R_α are non-degenerate. Given an almost complex structure $J \in \mathcal{J}(\alpha)$ and k points $x_1,\dots,x_k \in \widetilde{M}$, we define $\mathcal{M}^J(x_1,\dots,x_k)$ to be the set of all pseudoholomorphic curves $u : (\Sigma, j) \to (\widetilde{M}, J)$ with the following properties:

1. The domain (Σ, j) is a finite union of Riemann spheres with finitely many punctures.
2. The map u is non-constant on every connected component of Σ.
3. The points $x_1,\dots,x_k$ are contained in the image of u.
4. u has finite energy.
5. u has at least one negative asymptotic orbit.

By Stokes' theorem and exactness of $\widetilde{M}$, any curve $u \in \mathcal{M}^J(x_1,\dots,x_k)$ has at least one positive asymptotic orbit. We recall that the positive action $\mathcal{A}_\alpha^+(u)$ is the sum of the α-actions of these orbits. Given a non-degenerate $\alpha \in \mathcal{F}(M,\xi)$, we define

$$c_k(\alpha) \coloneqq \sup_{\substack{J\in\mathcal{J}(\alpha)\\ x_1,\dots,x_k\in\widetilde{M}}} \inf_{u\in\mathcal{M}^J(x_1,\dots,x_k)} \mathcal{A}_\alpha^+(u) \in (0,+\infty]. \tag{10.1}$$

The proof of the basic properties of the functions c_k which are listed in the next theorem is similar to the analogous proofs in [MS23, Hut22a, Hut22b, Edt22b].

THEOREM 10.1. *The functions given by* (10.1) *have unique extensions* $c_k\colon \mathcal{F}(M,\xi) \to [0,+\infty]$, $k \geq 0$, *satisfying the following properties:*

1. *(Scaling)* $c_k(r\cdot\alpha) = r\cdot c_k(\alpha)$ *for all positive real numbers* r.
2. *(Increasing)* $c_k(\alpha) \leq c_\ell(\alpha)$ *for* $k \leq \ell$.
3. *(Sublinearity)* $c_{k+\ell}(\alpha) \leq c_k(\alpha) + c_\ell(\alpha)$.
4. *(Invariance)* $c_k(\varphi^*\alpha) = c_k(\alpha)$ *for all* $\varphi \in \mathrm{Cont}(M,\xi)$.

5. *(Monotonicity): $c_k(\alpha) \leq c_k(\beta)$ whenever $\alpha \leq \beta$.*

6. *(Lipschitz): If c_k is finite, then c_k is Lipschitz for the C^0-norm on $\mathcal{F}(M,\xi)$.*

7. *(Spectrality): If $c_k(\alpha)$ is finite, then it is a finite sum of actions of periodic Reeb orbits of α.*

8. *(Packing): Suppose that $\alpha < \beta$ and assume that the ball $B^{2n}(A)$ of symplectic width A admits a symplectic embedding into $\widetilde{M}_\alpha^\beta$. Then*

$$c_{k+1}(\beta) \geq c_k(\alpha) + A.$$

REMARK 10.2. It follows from the monotonicity and scaling properties that, for fixed (M,ξ) and k, the spectral invariant c_k is either finite on all of $\mathcal{F}(M,\xi)$ or identically equal to $+\infty$. If c_k is finite, then the Weinstein conjecture holds for (M,ξ).

Proof. It is enough to check the above properties for non-degenerate contact forms. Then, the Lipschitz property allows us to extend c_k continuously to all, possibly degenerate, contact forms and, by continuity, all properties of c_k continue to hold in the degenerate case.

Let f_r be the diffeomorphism of $\mathbb{R} \times M$ given by $f_r(a,p) = (ra,p)$. The pull-back by f_r yields a bijection

$$f_r^* : \mathcal{J}(r\alpha) \to \mathcal{J}(\alpha)$$

and, for every $J \in \mathcal{J}(r\alpha)$ and every $x_1,\dots,x_k \in \widetilde{M}$, a bijection

$$f_r^* : \mathcal{M}^J(x_1,\dots,x_k) \to \mathcal{M}^{f_r^*J}(f_r^{-1}(x_1),\dots,f_r^{-1}(x_k)).$$

For every $u \in \mathcal{M}^J(x_1,\dots,x_k)$, the positive action of u with respect to $r\alpha$ is related to the positive action of $f_r^*(u)$ with respect to α by $\mathcal{A}^+_{r\alpha}(u) = r\mathcal{A}^+_\alpha(f_r^*u)$. This implies the scaling property $c_k(r\alpha) = rc_k(\alpha)$.

It is clear from the definition that c_k is increasing in k since adding point constraints can only make the set of relevant curves smaller.

Sublinearity is also clear because the union of two curves with k and ℓ point constraints, respectively, yields a curve with $k+\ell$ point constraints.

Invariance follows because a contactomorphism $\varphi \in \mathrm{Cont}(M,\xi)$ lifts to a symplectomorphism of the symplectization which induces bijections between the relevant sets of auxiliary data and curves.

The Lipschitz property follows from scaling and monotonicity.

Spectrality follows from the fact that the action spectrum of α is a closed subset of $\mathbb{R}$.

The proofs of the monotonicity and packing property are more substantial and rely on a neck-stretching argument and the SFT compactness theorem. Compared to analogous arguments given in [Hut22a, Edt22b], the proofs are slightly simplified because we have an apriori genus bound and can apply the SFT compactness theorem directly. As these two proofs are very similar, we just sketch the proof of the packing property.

We fix $\alpha < \beta$ and a symplectically embedded ball $B = B^{2n}(A)$ in $\widetilde{M}_\alpha^\beta$. Moreover, we fix an almost complex structure $J \in \mathcal{J}(\alpha)$ and k points $x_1, \dots, x_k \in \widetilde{M}$. Let $\epsilon > 0$ be arbitrarily small. Our goal is to find a J-holomorphic curve $u \in \mathcal{M}^J(x_1, \dots, x_k)$ satisfying

$$\mathcal{A}_\alpha^+(u) + A \leq c_{k+1}(\beta) + \epsilon.$$

Since the constant ϵ, the almost complex structure J, and the point constraints are arbitrary, this implies the desired packing inequality.

Note that if we translate the almost complex structure J and the points x_i using the canonical $\mathbb{R}$-action on $\widetilde{M}$, the new moduli space $\mathcal{M}^J(x_1, \dots, x_k)$ is obtained from the old one by applying the same translation. The positive action $\mathcal{A}_\alpha^+(u)$ is not affected by translation. After translating J and the points x_i sufficiently far downwards, we can therefore assume w.l.o.g. that J is adjusted to α on an open neighbourhood of $\widetilde{M}_\alpha^+$ and that all points x_i lie below the hypersurface M_α.

Fix a real number S such that J is adjusted to α on an open neighbourhood of $\widetilde{M}_{e^S\alpha}^-$ and such that all x_i lie above the hypersurface $M_{e^S\alpha}$. Let $T \in \mathbb{R}$ be such that the hypersurface $M_{e^T\beta}$ lies below the hypersurface $M_{e^S\alpha}$. Let J_α^+ denote the α-adjusted almost complex structure whose restriction to $\widetilde{M}_\alpha^+$ agrees with J and let J_α^- denote the α-adjusted almost complex structure whose restriction to $\widetilde{M}_{e^s\alpha}^-$ agrees with J.

Let J_α^β be a compatible almost complex structure which agrees with J_α^+ on a neighbourhood of $\widetilde{M}_\alpha^-$ and which is equal to a β-adjusted almost complex structure J_β^+ on $\widetilde{M}_\beta^+$. Moreover, we assume that the restriction of J_α^β to the ball B agrees with the standard integrable almost complex structure on Euclidean space.

Let J_β^α be a compatible almost complex structure agreeing with J_α^- on a neighbourhood of $\widetilde{M}_{e^S\alpha}^+$ and whose restriction to $\widetilde{M}_{e^T\beta}^-$ agrees with a β-adjusted almost complex structure J_β^-. Let us fix small tubular neighbourhoods $\mathrm{nb}(M_\alpha)$ and $\mathrm{nb}(M_{e^S\alpha})$ and sequences of diffeomorphisms

$$\varphi_\nu^+ : \widetilde{M}_{e^{-\nu}\alpha}^{e^\nu\alpha} \to \mathrm{nb}(M_\alpha), \qquad \varphi_\nu^- : \widetilde{M}_{e^{-\nu}\alpha}^{e^\nu\alpha} \to \mathrm{nb}(M_{e^S\alpha})$$

with the following properties:

1. $\varphi_\nu^\pm$ maps translates of the hypersurface M_α to translates of M_α.
2. $\pi \circ \varphi_\nu^\pm = \pi$ where $\pi : \widetilde{M} \to M$ is the canonical projection.
3. $d\varphi_\nu^\pm \partial_a = \partial_a$ near the boundary of $\widetilde{M}_{e^{-\nu}\alpha}^{e^\nu\alpha}$.

Let us now define a sequence of almost complex structures $J^\nu \in \mathcal{J}(\beta)$ as follows:

1. J^ν agrees with J_α^β above $\mathrm{nb}(M_\alpha)$.
2. J^ν agrees with $(\varphi_\nu^+)_* J_\alpha^+$ on $\mathrm{nb}(M_\alpha)$.
3. J^ν agrees with J between $\mathrm{nb}(M_\alpha)$ and $\mathrm{nb}(M_{e^S\alpha})$.

4. J^ν agrees with $(\varphi_\nu^-)_* J_\alpha^-$ on $\mathrm{nb}(M_{e^s\alpha})$.

5. J^ν agrees with J_β^α below $\mathrm{nb}(M_{e^s\alpha})$.

Let x_{k+1} denote the center of B. By the definition of $c_{k+1}(\beta)$ given in (10.1), there exists a sequence of J^ν-holomorphic curves $u^\nu \in \mathcal{M}^{J^\nu}(x_1, \ldots, x_{k+1})$ such that

$$\mathcal{A}_\beta^+(u^\nu) \leq c_{k+1}(\beta) + \epsilon.$$

The SFT compactness theorem implies that, after possibly passing to a subsequence, the sequence of curves u^ν converges to a pseudo-holomorphic building $\underline{u} = (u_1, \ldots, u_N)$. Each u_i consists of finitely many pseudoholomorphic nodal punctured finite-energy spheres. In the following we will ignore nodal points and ghost bubbles and regard each u_i as a finite union of non-constant pseudoholomorphic punctured finite-energy spheres. There exist indices $1 \leq r < s < t \leq N$ such that

1. u_i is J_β^+-holomorphic and positively and negatively asymptotic to closed orbits of R_β for all $1 \leq i < r$.

2. u_r is J_α^β-holomorphic and passes through x_{k+1}. It is positively asymptotic to closed orbits of R_β and negatively asymptotic to closed orbits of R_α.

3. u_i is J_α^+-holomorphic and positively and negatively asymptotic to closed orbits of R_α for all $r < i < s$.

4. u_s is J-holomorphic and passes through $x_1, \ldots, x_k$. It is positively and negatively asymptotic to closed orbits of R_α.

5. u_i is J_α^--holomorphic and positively and negatively asymptotic to closed orbits of R_α for all $s < i < t$.

6. u_t is J_β^α-holomorphic. It is positively asymptotic to closed orbits of R_α and negatively asymptotic to closed orbits of R_β.

7. u_i is J_β^--holomorphic and positively and negatively asymptotic to closed orbits of R_β for all $t < i \leq N$.

Note that since every u^ν has at least one negative end, each u_i must have at least one negative end. Therefore, $u_s \in \mathcal{M}^J(x_1, \ldots, x_k)$. We need to estimate the positive action $\mathcal{A}_\alpha^+(u_s)$. For every i, the negative asymptotic orbits of u_i match the positive asymptotic orbits of u_{i+1}. This implies that

$$\sum_{i=1}^{r-1} \mathcal{A}_\beta(u_i) + \mathcal{A}_\alpha^\beta(u_r) + \sum_{i=r+1}^{s-1} \mathcal{A}_\alpha(u_i) + \mathcal{A}_\alpha^+(u_s) = \mathcal{A}_\beta^+(u_1).$$

Since $\mathcal{A}_\beta^+(u^\nu) \leq c_{k+1}(\beta) + \varepsilon$ for all ν and the positive ends of u^ν are converging to the positive ends of u_1, we have

$$\mathcal{A}_\beta^+(u_1) \leq c_{k+1}(\beta) + \varepsilon.$$

The actions $\mathcal{A}_\beta(u_i)$ are non-negative for $1 \leq i < r$. Similarly, the actions $\mathcal{A}_\alpha(u_i)$ are non-negative for $r < i < s$. Since u_r passes through the center of the ball B equipped with the standard complex structure, $\mathcal{A}^\beta_\alpha(u_r) \geq A$ by the monotonicity lemma. Combining the above estimates yields

$$\mathcal{A}^+_\alpha(u_s) + A \leq c_{k+1}(\beta) + \varepsilon$$

as desired. □

In the next result, we determine the value of the first two spectral invariants for Zoll contact forms.

PROPOSITION 10.3. *Let $\alpha_0 \in \mathcal{F}(M, \xi)$ be a Zoll contact form with minimal period* $\mathrm{sys}(\alpha_0)$. *Then we have*

$$c_0(\alpha_0) = c_1(\alpha_0) = \mathrm{sys}(\alpha_0).$$

Proof. Let α be a C^∞-small non-degenerate perturbation of α_0. Fix an almost complex structure $J \in \mathcal{J}(\alpha)$ and a point $x \in \widetilde{M}$. Our goal is to show that there exists a J-holomorphic curve $u \in \mathcal{M}^J(x)$ with positive action bounded from above by

$$\mathcal{A}^+_\alpha(u) \leq \tfrac{3}{2}\,\mathrm{sys}(\alpha_0).$$

Since J and x are arbitrary, this implies that $c_1(\alpha) \leq \frac{3}{2}\,\mathrm{sys}(\alpha_0)$. It then follows from continuity of c_1 that $c_1(\alpha_0) \leq \frac{3}{2}\,\mathrm{sys}(\alpha_0)$. The increasing and the spectrality properties imply that we must have $c_0(\alpha_0) = c_1(\alpha_0) = \mathrm{sys}(\alpha_0)$.

We construct the pseudoholomorphic curve u by neck-stretching pseudoholomorphic spheres in a ruled symplectic manifold (X, ω) which contains M_α as a contact type hypersurface. The symplectic manifold (X, ω) is constructed by performing a so-called symplectic cut, see [Ler95], as follows. Identify the symplectization $\widetilde{M}$ with $\mathbb{R} \times M$ by means of the diffeomorphism φ_{α_0}, so that $\widetilde{\omega}$ is given by $d(e^s\alpha_0)$. Fix a constant $S > 0$. The boundary components of the compact region $[-S, S] \times M$ in the symplectization are foliated by closed characteristic leaves. The symplectic manifold (X, ω) is obtained from $[-S, S] \times M$ by collapsing each of these leaves to a point. Let B denote the quotient of M by the free S^1-action induced by the Zoll Reeb flow. The space X naturally fibers over B. The fibers are symplectic 2-spheres of the form $([-S, S] \times \gamma)/\sim$ where γ is a closed Reeb orbit of α_0 and $\sim$ collapses the circles $\{\pm S\} \times \gamma$ to points. Under the quotient, the two boundary components of $[-S, S] \times M$ yield smooth symplectic divisors $B_\pm$ of X both diffeomorphic to B. Clearly, X contains M_α as a contact type hypersurface. The function $(s, p) \mapsto -\mathrm{sys}(\alpha_0)e^s$ on $\mathbb{R} \times M$ induces a smooth Hamiltonian H on X which generates a Hamiltonian S^1-action. The two divisors $B_\pm$ are fixed point sets of this action and the restriction of the action to the complement $X \setminus (B_+ \cup B_-)$ is free. In particular, the assumptions in [McD09, Proposition 4.3] are satisfied. Let $F \in H_2(X; \mathbb{Z})$ denote the homology class of a fiber. Then [McD09, Proposition 4.3] implies that the Gromov–Witten invariant $\langle \mathrm{pt} \rangle^X_F$ does not vanish. We conclude that, for every ω-compatible almost complex structure $\widetilde{J}$ on X and for every point $p \in X$, there exists a nodal $\widetilde{J}$-holomorphic sphere passing through p and representing the homology class F.

Let us now choose a constant S' large enough such that the almost complex structure $J \in \mathcal{J}(\alpha)$ fixed at the beginning of the proof is adjusted to α on neighbourhoods of the sets $\widetilde{M}^{\pm}_{e^{\pm S'}\alpha}$. Moreover, we assume that the point x strictly lies between the two hypersurfaces $M_{e^{\pm S'}\alpha}$. We set the constant S appearing in the construction of X to be $S := S' + \ln(3/2)$. Since α is C^∞-close to α_0, the hypersurfaces $M_{e^{\pm S'}\alpha}$ lie between the hypersurfaces $M_{e^{\pm S}\alpha_0}$ and do not intersect them.

We define a sequence of compatible almost complex structures J^ν on X as follows. Let $J^{\pm}_\alpha$ denote the α-adjusted almost complex structure whose restriction to $\widetilde{M}^{\pm}_{e^{\pm S'}}$ agrees with J. Let $X_\pm \subset X$ denote the connected component of $X \setminus M_{e^{\pm S'}\alpha}$ containing $B_\pm$. Let $\widehat{X}_\pm$ denote the symplectic completion of $X_\pm$ obtained by attaching the cylindrical end $\widetilde{M}^{\mp}_{e^{\pm S'}\alpha}$. Let $J_\pm$ be a compatible almost complex structure on $\widehat{X}_\pm$ agreeing with $J^{\pm}_\alpha$ in a neighbourhood of the cylindrical end $\widetilde{M}^{\mp}_{e^{\pm S'}\alpha}$.

Let us fix small tubular neighbourhoods $\mathrm{nb}(M_{e^{\pm S'}\alpha})$ and sequences of diffeomorphisms

$$\varphi^{\pm}_\nu : \widetilde{M}^{e^\nu \alpha}_{e^{-\nu}\alpha} \to \mathrm{nb}(M_{e^{\pm S'}\alpha})$$

with the following properties:

1. $\varphi^{\pm}_\nu$ maps translates of the hypersurface M_α to translates of M_α.
2. $\pi \circ \varphi^{\pm}_\nu = \pi$ where $\pi : \widetilde{M} \to M$ is the canonical projection.
3. $d\varphi^{\pm}_\nu \partial_a = \partial_a$ near the boundary of $\widetilde{M}^{e^\nu \alpha}_{e^{-\nu}\alpha}$.

Let us now define the compatible almost complex structure J^ν on X as follows:

1. J^ν agrees with J_+ above $\mathrm{nb}(M_{e^{S'}\alpha})$.
2. J^ν agrees with $(\varphi^+_\nu)_* J^+_\alpha$ on $\mathrm{nb}(M_{e^{S'}\alpha})$.
3. J^ν agrees with J between $\mathrm{nb}(M_{e^{S'}\alpha})$ and $\mathrm{nb}(M_{e^{-S'}\alpha})$.
4. J^ν agrees with $(\varphi^-_\nu)_* J^-_\alpha$ on $\mathrm{nb}(M_{e^{-S'}\alpha})$.
5. J^ν agrees with J_- below $\mathrm{nb}(M_{e^{-S'}\alpha})$.

By the above discussion, we may choose, for every ν, a J^ν-holomorphic sphere u^ν representing the homology class of the fiber F and passing through the point $x \in X$. By the SFT compactness theorem, this sequence converges, after possibly passing to a subsequence, to a pseudoholomorphic building $\underline{u} = (u_1, \ldots, u_N)$. Every u_i consists of finitely many nodal punctured finite-energy pseudoholomorphic spheres. As before, we ignore nodes and ghost bubbles and regard each u_i as a finite union of punctured finite energy spheres.

The homological intersection number $F \cdot B_\pm$ is equal to 1. This implies that every u^ν intersects both $B_\pm$. Therefore, u_1 is a J_+-holomorphic curve in $\widehat{X}_+$ and its intersection number with B_+ is equal to 1. Similarly, u_N is a J_--holomorphic curve in $\widehat{X}_-$ intersecting B_- with intersection number 1. Since the negative asymptotic limit orbits of u_i match the

positive asymptotic limit orbits of u_{i+1}, we conclude that every u_i for $1 < i < N$ must have both positive and negative asymptotic orbits. Let $1 < r < N$ be the index such that u_r is a J-holomorphic curve in $\widetilde{M}$ passing through $x \in \widetilde{M}$. Since u_r has a negative asymptotic orbit, we have $u_r \in \mathcal{M}^J(x)$. It remains to estimate $\mathcal{A}^+_\alpha(u)$.

The complement of B_+ in $\widehat{X}_+$ is exact. Indeed, restricting the natural Liouville 1-form $\widetilde{\lambda}$ on $\widetilde{M}$ to the interior of $\widetilde{M}^-_{e^S\alpha_0}$ yields a primitive λ of the symplectic form ω_+ on $\widehat{X}_+ \setminus B_+$. Clearly, the primitive λ does not extend over B_+. In fact, if Σ is a compact oriented surface, possibly with boundary, and $v : \Sigma \to \widehat{X}_+$ is a smooth map sending $\partial\Sigma$ into the complement of B_+, then

$$\int_\Sigma v^*\omega_+ = \int_{\partial\Sigma} v^*\lambda + e^S \operatorname{sys}(\alpha_0) v \cdot B_+.$$

In order to make sense of the homological intersection numer $v \cdot B_+$, we regard v as a 2-chain with fixed boundary in the complement of B_+. Applying this identity to u_1 yields

$$\int_{u_1^{-1}(\widetilde{M}^-_{e^{S'}\alpha})} (\varphi_\alpha^{-1} \circ u_1)^* d\alpha + \int_{u_1^{-1}(X_+)} u_1^*\omega_+ = -\mathcal{A}^-_{e^{S'}\alpha}(u_1) + e^S \operatorname{sys}(\alpha_0).$$

Since u_1 is pseudoholomorphic, the left-hand side of this identity is non-negative. We deduce that

$$e^{S'}\mathcal{A}^-_\alpha(u_1) = \mathcal{A}^-_{e^{S'}\alpha}(u_1) \le e^S \operatorname{sys}(\alpha_0). \tag{10.2}$$

It follows from our choice $S = S' + \ln(3/2)$ that $\mathcal{A}^-_\alpha(u_1) \le \frac{3}{2}\operatorname{sys}(\alpha_0)$.

The curve u_i is a J^+_α-holomorphic curve in $\widetilde{M}$ for $1 < i < r$. We have

$$\sum_{i=2}^{r-1} \mathcal{A}_\alpha(u_i) + \mathcal{A}^+_\alpha(u_r) = \mathcal{A}^-_\alpha(u_1).$$

Since the actions $\mathcal{A}_\alpha(u_i)$ are non-negative for $1 < i < r$, we deduce that

$$\mathcal{A}^+_\alpha(u_r) \le \mathcal{A}^-_\alpha(u_1) \le \frac{3}{2}\operatorname{sys}(\alpha_0).$$

This concludes the proof of the proposition. □

Next, we compute the first two spectral invariants for contact forms which are C^2-small, S^1-invariant perturbations of a Zoll one.

PROPOSITION 10.4. *Let $\alpha_0 \in \mathcal{F}(M,\xi)$ be a Zoll contact form with minimal period* $\operatorname{sys}(\alpha_0)$. *If the smooth function $S : M \to \mathbb{R}$ is C^2-close to the constant function 1 and S^1-invariant, then*

$$c_0(S\alpha_0) = \operatorname{sys}(\alpha_0) \min_M S, \qquad c_1(S\alpha_0) = \operatorname{sys}(\alpha_0) \max_M S.$$

Proof. We abbreviate $\max S \coloneqq \max_M S$ and $\min S \coloneqq \min_M S$. Our proof is based on the following claims.

Claim 1. If S is C^2-close to 1 and the positive number $\overline{S} > \max S$ is close to $\max S$, then there is a symplectic embedding of the $2n$-dimensional ball of symplectic width $\mathrm{sys}(\alpha_0)(\max S - \min S)$ into the cobordism $\widetilde{M}^{\overline{S}\alpha_0}_{S\alpha_0}$.

Claim 2. If S is C^2-close to 1 and the positive number $\underline{S} < \min S$ is close to $\min S$, then there is a symplectic embedding of the $2n$-dimensional ball of symplectic width $\mathrm{sys}(\alpha_0)(\max S - \min S)$ into the cobordism $\widetilde{M}^{S\alpha_0}_{\underline{S}\alpha_0}$.

Before proving these claims, we show how they allow us to determine $c_0(S\alpha_0)$ and $c_1(S\alpha_0)$. By Claim 1, the packing property applied to $\widetilde{M}^{\overline{S}\alpha_0}_{S\alpha_0}$, the scaling property and Proposition 10.3, we have

$$c_0(S\alpha_0) + \mathrm{sys}(\alpha_0)(\max S - \min S) \leq c_1(\overline{S}\alpha_0) = \overline{S}\, c_1(\alpha_0) = \overline{S}\,\mathrm{sys}(\alpha_0),$$

and hence, letting $\overline{S}$ tend to $\max S$, we get

$$c_0(S\alpha_0) \leq \mathrm{sys}(\alpha_0) \min S.$$

Since the right-hand side is the minimum of the action spectrum of $S\alpha_0$, the spectrality property implies that the above inequality is an equality. Similarly, the packing property applied to $\widetilde{M}^{S\alpha_0}_{\underline{S}\alpha_0}$, the scaling property and Proposition 10.3 imply

$$c_1(S\alpha_0) \geq c_0(\underline{S}\alpha_0) + \mathrm{sys}(\alpha_0)(\max S - \min S) = \underline{S}\, c_0(\alpha_0) + \mathrm{sys}(\alpha_0)(\max S - \min S).$$

Letting $\underline{S}$ tend to $\min S$, we get $c_1(S\alpha_0) \geq \mathrm{sys}(\alpha_0) \max S$. By monotonicity, we have

$$c_1(S\alpha_0) \leq c_1((\max S)\,\alpha_0) = \mathrm{sys}(\alpha_0) \max S,$$

and the desired formula for $c_1(S\alpha_0)$ follows.

There remains to prove the above claims. We prove Claim 1, the proof of Claim 2 being analogous. Up to rescaling, we may assume that $\mathrm{sys}(\alpha_0) = 1$. We identify the symplectization $\widetilde{M}$ with $\mathbb{R} \times M$ via φ_{α_0}, so that $\widetilde{\omega} = d(e^a \alpha_0)$ and

$$M^{\overline{S}\alpha_0}_{S\alpha_0} = \{(a, q) \in \mathbb{R} \times M \mid S(q) < e^a < \overline{S}\}.$$

Denote by $p : M \to Y$ the $\mathbb{T}$-bundle, $\mathbb{T} \coloneqq \mathbb{R}/\mathbb{Z}$, whose base is the quotient of M by the free action of the Reeb flow of α_0. Then Y is a $(2n-2)$-dimensional manifold with a symplectic form ω satisfying $p^*\omega = d\alpha_0$. By means of a Darboux trivializing chart, we can identify a neighborhood (U_y, ω) of any y in Y with a ball $\widehat{B}_\rho$ of radius $\rho > 0$ in $(\mathbb{C}^{n-1}, \widetilde{\omega}_0)$, where ρ is independent of y. Moreover, we can identify $p^{-1}(U_y)$ with $\mathbb{T} \times \widehat{B}_\rho$ in such a way that α_0 corresponds to $\widehat{\lambda}_0 + dt$. Here, t denotes the variable in $\mathbb{T}$ and $\widehat{\lambda}_0$ is the standard primitive of the standard symplectic form $\widehat{\omega}_0$ on $\mathbb{C}^{n-1}$ (we are using the notation from Section 1).

Correspondingly, we obtain an identification of the open subset $\widetilde{U}_y \coloneqq \mathbb{R} \times p^{-1}(U_y)$ of $\widetilde{M}$ with $\mathbb{R} \times \mathbb{T} \times \widehat{B}_\rho$, with respect to which the 1-form $e^a\alpha_0$ reads $e^a(dt + \widehat{\lambda}_0)$. The diffeomorphism

$$\Psi_y : \{(s,t,w) \in (0,+\infty) \times \mathbb{T} \times \mathbb{C}^{n-1} \mid |w| < s^{\frac{1}{2}}\rho\} \to \mathbb{R} \times \mathbb{T} \times \widehat{B}_\rho = \widetilde{U}_y \subset \widetilde{M}$$

given by $\Psi_y(s,t,w) \coloneqq (\log s, t, s^{-\frac{1}{2}}w)$ satisfies

$$\Psi_y^*(e^a\alpha_0) = \Psi_y^*(e^a(dt + \widehat{\lambda}_0)) = s\,dt + \widehat{\lambda}_0,$$

and hence

$$\Psi_y^*\widetilde{\omega} = \Psi_y^* d(e^a\alpha_0) = ds \wedge dt + \widetilde{\omega}_0.$$

Now we choose $y \in Y$ so that the $\mathbb{T}$-invariant function S achieves its minimum at the circle $p^{-1}(y)$. In the above identifications, we then have $S(t,0) = \min S$ for every $t \in \mathbb{T}$,

$$\widetilde{M}^{\overline{S}\alpha_0}_{S\alpha_0} \cap \widetilde{U}_y = \{(a,t,u) \in \mathbb{R} \times \mathbb{T} \times \widehat{B}_\rho \mid S(t,u) < e^a < \overline{S}\},$$

and hence it is enough to symplectically embed a $2n$-ball of symplectic width $\max S - \min S$ into the open subset

$$V \coloneqq \Psi_y^{-1}(\widetilde{M}^{\overline{S}\alpha_0}_{S\alpha_0} \cap \widetilde{U}_y) = \{(s,t,w) \in (0,+\infty) \times \mathbb{T} \times \mathbb{C}^{n-1} \mid S(t, s^{-\frac{1}{2}}w) < s < \overline{S},\ |w| < s^{\frac{1}{2}}\rho\}$$

of $((0,+\infty) \times \mathbb{T} \times \mathbb{C}^{n-1}, ds \wedge dt + \widehat{\omega}_0)$. Since $S(t,0) = \min S$, we have

$$S(t,u) \le \min S + c\|\nabla^2 S\|_{C^0}|u|^2 \qquad \forall (t,u) \in \mathbb{T} \times \widehat{B}_\rho,$$

for a suitable constant $c > 0$. By this bound, the inequality $s > S(t, s^{-\frac{1}{2}}w)$ appearing in the representation of V is implied by

$$s > \min S + c\|\nabla^2 S\|_{C^0}\frac{|w|^2}{s},$$

which is equivalent to

$$s > \frac{1}{2}\min S\left(1 + \sqrt{1 + \frac{4c\|\nabla^2 S\|_{C^0}}{(\min S)^2}|w|^2}\right).$$

Since $\sqrt{1+r} \le 1 + \frac{r}{2}$ for every $r \ge 0$, the above inequality is implied by

$$s > \min S + \frac{c}{\min S}\|\nabla^2 S\|_{C^0}|w|^2,$$

and we conclude that V contains the set

$$V' \coloneqq \{(s,t,w) \in (0,+\infty) \times \mathbb{T} \times \mathbb{C}^{n-1} \mid f(|w|^2) < s < \overline{S}\},$$

where

$$f(r) \coloneqq \max\left\{\min S + \frac{c\|\nabla^2 S\|_{C^0}}{\min S} r, \frac{r}{\rho^2}\right\}.$$

If S is C^2-close to 1 and $\overline{S}$ is close to $\max S$, then both $\overline{S} - \min S$ and $\|\nabla^2 S\|_{C^0}$ are small and we can guarantee that

$$f(r) \le \min S + \pi r \qquad \forall r \in \big[0, \tfrac{1}{\pi}(\overline{S} - \min S)\big].$$

This implies that V' contains the set

$$V'' \coloneqq \big\{(s,t,w) \in (0,+\infty) \times \mathbb{T} \times \mathbb{C}^{n-1} \mid \min S + \pi|w|^2 < s < \overline{S}\big\}.$$

Let $\Phi : \Omega \to \mathbb{C}^* \times \mathbb{C}^{n-1}$ be the symplectomorphism defined in (1.1). By (1.3), the symplectomorphism

$$(s,t,w) \mapsto \Phi\big(s - \min S - \pi, t, w\big)$$

maps V'' onto $B(\rho) \setminus \{z_1 = 0\}$, where $B(\rho)$ is the ball of $\mathbb{C}^n$ of symplectic width $\rho \coloneqq \overline{S} - \min S$ and $\{z_1 = 0\}$ is a complex hyperplane. By Traynor's embedding construction (see Construction 3.2 in [Tra95]), for every $\rho' \in (0,\rho)$, there is a symplectic embedding $\Psi\colon B(\rho') \to B(\rho) \setminus \{z_1 = 0\}$. In the simple case we are considering here, the embedding is constructed as follows. First, let $\psi\colon D(\rho') \to D(\rho) \setminus \{0\}$ be an area preserving embedding such that

$$\pi|\psi(\zeta)|^2 \le \pi|\zeta|^2 + (\rho - \rho'), \qquad \forall\, \zeta \in D(\rho').$$

Then,

$$\Psi(\zeta_1, \dots, \zeta_n) \coloneqq (\psi(\zeta_1), \zeta_2, \dots, \zeta_n), \qquad (\zeta_1, \dots, \zeta_n) \in B(\rho')$$

is the required symplectic embedding. Choosing $\rho' = \max S - \min S < \overline{S} - \min S = \rho$, we deduce that $V \supset V''$ contains a symplectic ball of width $\max S - \min S$. □

11 Proof of Theorem 5

Let $\alpha \in \mathcal{F}(M,\xi)$ be C^2-close to the Zoll contact form $\alpha_0 \in \mathcal{F}(M,\xi)$. By Theorem 3, there is $\varphi \in \mathrm{Cont}_0(M,\xi)$ such that

$$\varphi^*\alpha = Te^h\alpha_0,$$

where the functions T and h satisfy the conditions (a)-(e) listed in that theorem. We denote by $T_{\min}(\alpha) = \mathrm{sys}(\alpha)$ and $T_{\max}(\alpha)$ the minimum and the maximum of the periods of the short closed orbits of R_α, which by (e) satisfy

$$T_{\min}(\alpha) = \mathrm{sys}(\alpha_0) \min T, \qquad T_{\max}(\alpha) = \mathrm{sys}(\alpha_0) \max T. \tag{11.1}$$

Then $\varphi^*\alpha = e^f\alpha_0$ with

$$f \coloneqq h + \log T,$$

and by property (c) of Theorem 3 we have

$$\max_M f - \min_M f = \log \frac{T_{\max}(\alpha)}{T_{\min}(\alpha)}.$$

This shows that

$$d(\alpha_0, \alpha) \leq \log \frac{T_{\max}(\alpha)}{T_{\min}(\alpha)}.$$

In order to conclude our proof, we need to prove the reverse inequality

$$d(\alpha_0, \alpha) \geq \log \frac{T_{\max}(\alpha)}{T_{\min}(\alpha)}. \tag{11.2}$$

We shall deduce this from the following claim.

CLAIM. $c_0(\alpha) = T_{\min}(\alpha)$ and $c_1(\alpha) = T_{\max}(\alpha)$.

Postponing its proof, we show how this claim implies (11.2). Let $\psi \in \mathrm{Cont}_0(M, \xi)$ and let g be the smooth positive function such that

$$\psi^* \alpha = e^g \alpha_0.$$

Denoting by

$$a := e^{\min_M g} = \min_M e^g, \qquad b := e^{\max_M g} = \max_M e^g,$$

we have

$$a\alpha_0 \leq \psi^* \alpha \leq b\alpha_0.$$

Using Proposition 10.3, the scaling, monotonicity and invariance properties of the spectral invariants, and the above claim, we compute

$$a \operatorname{sys}(\alpha_0) = c_0(a\,\alpha_0) \leq c_0(\psi^* \alpha) = c_0(\alpha) = T_{\min}(\alpha),$$

and

$$b \operatorname{sys}(\alpha_0) = c_1(b\,\alpha_0) \geq c_1(\psi^* \alpha) = c_1(\alpha) = T_{\max}(\alpha).$$

Therefore

$$\max_M g - \min_M g = \log \frac{b}{a} \geq \log \frac{T_{\max}(\alpha)}{T_{\min}(\alpha)}.$$

Since the contactomorphism $\psi \in \mathrm{Cont}_0(M, \xi)$ was arbitrary, the above inequality implies (11.2).

There remains to prove the claim. In the notation from the proof of Theorem 3, we have

$$v^* u^* \alpha = S\alpha_0 + \eta.$$

By invariance, this implies that

$$c_k(\alpha) = c_k(S\alpha_0 + \eta),$$

where c_k is viewed as a function on $\mathcal{F}(M, \ker(S\alpha_0+\eta))$ on the right hand side. For $t \in [0,1]$, we define

$$\gamma_t \coloneqq S\alpha_0 + t\eta.$$

Since S is C^1-close to 1 and η and $d\eta$ are both C^0-small, γ_t is a contact form for all t. Moreover, γ_t and $d\gamma_t$ are C^0-close to α_0 and $d\alpha_0$, respectively.

Claim. The short closed Reeb orbits of γ_t and their periods are independent of $t \in [0,1]$.

Proof. Let us split the Reeb vector field R_{γ_t} of γ_t into a component X_t parallel to R_{α_0} and a component Y_t tangent to $\xi = \ker \alpha_0$. Since γ_t and $d\gamma_t$ are C^0-close to α_0 and $d\alpha_0$, the component X_t is C^0-close to R_{α_0} and the component Y_t is C^0-small. Let us write $X_t = a_t R_{\alpha_0}$ for some function a_t which is C^0-close to 1. Then a direct computation shows that

$$\iota_{X_t} d\gamma_t = a_t(-dS + tF[dS])$$

and therefore

$$\iota_{Y_t} d\gamma_t = a_t(dS - tF[dS]).$$

Recall that $dS = -B[V]$ where $B : \ker \alpha_0 \to (\ker \alpha_0)^*$ is C^0-close to the isomorphism induced by $d\alpha_0$ and V is a section of $\ker \alpha_0$ which is C^1-small. Let $B_t : \ker \alpha_0 \to (\ker \alpha_0)^*$ denote the isomorphism induced by $d\gamma_t$, which is also C^0-close to the isomorphism induced by $d\alpha_0$. We have

$$Y_t = a_t B_t^{-1}(B[V] - tF[B[V]]) = a_t Q_t[V]$$

where Q_t is a suitable isomorphism of $\ker \alpha_0$ which is C^0-close to the identity. We may therefore write $R_{\gamma_t} = a_t(R_{\alpha_0} + Q_t[V])$. The argument in the proof of Proposition 4.3 (ii) shows that the short closed Reeb orbits of γ_t are precisely the zero set of V or equivalently the critical set of S. The action of such an orbit γ is simply given by $S(\gamma)\,\mathrm{sys}(\alpha_0)$ and in particular independent of t. □

The above claim shows that the short action spectrum of γ_t is independent of t. The function $t \mapsto c_k(\gamma_t)$ for $k \in \{0,1\}$ is continuous and takes values in this short action spectrum. Since the action spectrum has measure zero, this implies that $t \mapsto c_k(\gamma_t)$ is constant for $k \in \{0,1\}$. In particular, we obtain

$$c_0(\alpha) = c_0(S\alpha_0) \quad \text{and} \quad c_1(\alpha) = c_1(S\alpha_0).$$

Next, we define $\beta_t \coloneqq (1-t)S\alpha_0 + t\alpha_0$. Since S is C^1-close to 1, this is a contact form for all $t \in [0,1]$.

Claim. The short closed Reeb orbits of β_t are independent of $t \in (0,1]$. If γ is such a short closed Reeb orbit, then its action with respect to β_t is given by $((1-t)S(\gamma) + t)\,\mathrm{sys}(\alpha_0)$.

Proof. The proof of this claim is similar to the proof of the claim above. □

It follows from continuity of the spectral invariants, the above claim and the fact that action spectra have measure zero that

$$c_k(\beta_t) = (1-t)c_k(S\alpha_0) + t\,\mathrm{sys}(\alpha_0) \quad \text{for } k \in \{0,1\}.$$

If $t > 0$ is sufficiently small, then $(1-t)S + t$ is C^2-small. Therefore, we may apply Proposition 10.4 to the contact form β_t for $t > 0$ small. We conclude that

$$c_0(\beta_t) = \min_M((1-t)S + t)\,\mathrm{sys}(\alpha_0) \quad \text{and} \quad c_1(\beta_t) = \max_M((1-t)S + t)\,\mathrm{sys}(\alpha_0).$$

In combination with the above expression for $c_k(\beta_t)$, this implies

$$c_0(S\alpha_0) = \min_M S\,\mathrm{sys}(\alpha_0) = T_{\min}(\alpha) \quad \text{and} \quad c_1(S\alpha_0) = \max_M S\,\mathrm{sys}(\alpha_0) = T_{\max}(\alpha).$$

This concludes the proof of the claim.

A Appendix: Construction of exotic ellipsoids

The statements. Denote by $\alpha_0 = \lambda_0|_{S^3}$ the standard contact form on the 3-sphere, see (6), by $\xi_{\mathrm{st}} = \ker\alpha_0$ the induced co-oriented contact structure, and by $\mathcal{F}(S^3, \xi_{\mathrm{st}})$ the space of contact forms on S^3 which define ξ_{st}. This space can be identified with the space of positive smooth functions on S^3 and is endowed with the C^∞-topology. If (a,b) is a pair of positive real numbers, we denote by $E(a,b)$ the open ellipsoid

$$E(a,b) := \Big\{(z_1,z_2) \in \mathbb{C}^2 \mid \frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2 < 1\Big\},$$

and by $\varepsilon_{a,b} \in \mathcal{F}(S^3, \xi_{\mathrm{st}})$ the contact form which is obtained by pulling the restriction of λ_0 to $\partial E(a,b)$ back to S^3 by the radial projection, i.e.

$$\epsilon_{a,b}(z) = \Big(\frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2\Big)^{-1}\alpha_0(z), \qquad \forall z = (z_1,z_2) \in S^3.$$

The Reeb flow of $\varepsilon_{a,b}$ has the form

$$\phi^t_{\varepsilon_{a,b}}(z_1,z_2) = \big(e^{\frac{2\pi}{a}ti}z_1, e^{\frac{2\pi}{b}ti}z_2\big).$$

The circles $\Gamma_1 := \{z_2 = 0\} \subset S^3$ and $\Gamma_2 := \{z_1 = 0\} \subset S^3$ are closed Reeb orbits of $\varepsilon_{a,b}$ of period a and b, respectively. The Reeb flow of $\varepsilon_{a,b}$ preserves the 2-tori

$$\{|z_1| = \cos\beta,\ |z_2| = \sin\beta\} \subset S^3, \qquad \forall \beta \in \big(0, \tfrac{\pi}{2}\big),$$

on which the flow is periodic if and only if the number $\frac{a}{b}$ is rational.

The first aim of this Appendix is to present a self-contained proof of the following result.

THEOREM A.1. *Let (a_0, b_0) be a pair of positive numbers. For every neighborhood U of (a_0, b_0) in $(0, +\infty)^2$ and every neighborhood $\mathcal{U}$ of ε_{a_0,b_0} in $\mathcal{F}(S^3, \xi_{\text{st}})$ there exist $(a, b) \in U$, a contact form α in $\mathcal{U}$ and a sequence (φ_j) in $\text{Cont}_0(S^3, \xi_{\text{st}})$, each having support away from $\Gamma_1 \cup \Gamma_2$, such that:*

(i) $(\varphi_j^* \varepsilon_{a,b})$ converges to α in the C^∞-topology;

(ii) the only closed Reeb orbits of α are Γ_1 and Γ_2, with periods a and b, respectively;

(iii) the Reeb flow of α has a dense orbit.

By (iii), the Reeb flow of α is not conjugate to the one of $\varepsilon_{a,b}$. Together with (i), this implies that the orbit of $\varepsilon_{a,b}$ by the action of $\text{Cont}_0(S^3, \xi_{\text{st}})$ is not C^∞-closed in $\mathcal{F}(S^3, \xi_{\text{st}})$, see Example 2 in the Introduction. The above result can be deduced from Theorem A in [Kat73], where Katok used the conjugacy method which he had introduced together with Anosov in [AK70a] to construct ergodic Hamiltonian flows arbitrarily close to suitably integrable flows. Therefore, (iii) can be replaced by the stronger condition:

(iii') the Reeb flow of α is ergodic with respect to the invariant measure induced by the volume form $\alpha \wedge d\alpha$.

Specializing the argument to the perturbation of ε_{a_0,b_0} and showing just the existence of a dense orbit, instead of ergodicity, allows us to give here a short self-contained proof, still based on the Anosov-Katok conjugacy method. See also [CS16] for a reformulation of Katok's result in the Reeb setting.

REMARK A.2. In the proof of Theorem A.1, the pair (a, b) is obtained as a limit of pairs of positive numbers (a_j, b_j) with rational ratio which is required to converge extremely fast and we do not control which kind of limiting pairs (a, b) we can produce. It is actually possible to strengthen the above result by prescribing the pair (a, b): For any pair of positive numbers (a, b) whose ratio is irrational and not Diophantine, there exist a contact form $\alpha \in \mathcal{F}(S^3, \xi_{\text{st}})$ and a sequence (φ_j) in $\text{Cont}_0(S^3, \xi_{\text{st}})$ which satisfy (i), (ii) and (iii) (or even (iii')) with respect to (a, b). This follows from a corresponding result for area-preserving diffeomorphisms of the 2-disk which is proven in [FS05] and by a construction from [AGZ22], which allows to lift such diffeomorphisms to Reeb flows on (S^3, ξ_{st}).

The condition on (a, b) is optimal. Indeed, if the ratio $\frac{a}{b}$ is rational, then any contact form α which satisfies (i) (even just with C^0-convergence) is smoothly conjugate to $\varepsilon_{a,b}$, as discussed in Example 1 in the Introduction, so cannot satisfy (iii). If the ratio $\frac{a}{b}$ is irrational and Diophantine and the contact form α satisfies (i), then the first-return map to a local Poincaré section transverse to Γ_1 of the Reeb flow of α is an area-preserving diffeomorphism with an elliptic and Diophantine fixed point. In his "last geometric theorem", Herman proved that arbitrarily near to such a fixed point there are smooth invariant circles, see [FK09, Theorem 4], and hence the Reeb flow of α has invariant tori. Since these tori disconnect S^3, α cannot satisfy (iii).

Writing the contact form α of Theorem A.1 as $\alpha = f^2\alpha_0$ for a suitable smooth positive function on S^3, we consider the starshaped domain $A \subset \mathbb{C}^2$ which is given by

$$A \coloneqq \{rz \mid z \in S^3,\ 0 \le r < f(z)\}.$$

The pullback by the radial projection $S^3 \to \partial A$ of $\lambda_0|_{\partial A}$ to S^3 is then α. We can complement Theorem A.1 with the following result.

THEOREM A.3. *The domains A and $E(a,b)$ are symplectomorphic but the contact forms which are given by the restrictions of λ_0 to their boundaries are not conjugate.*

The first examples of pairs of starshaped domains in $\mathbb{C}^n$, $n > 1$, having the above properties were constructed by Eliashberg and Hofer in [EH96]. The construction we present here is somehow simpler, but builds on some results which are known to hold only in dimension four.

Proof of theorem A.1. By slightly perturbing (a_0, b_0), we may assume that the ratio $\frac{a_0}{b_0}$ is rational. Starting from the pair (a_0, b_0) and from $\varphi_0 = \mathrm{id}$, we shall inductively construct sequences (a_j, b_j) in U with $\frac{a_j}{b_j} \in \mathbb{Q}$ and (φ_j) in $\mathrm{Cont}_0(S^3, \xi_{\mathrm{st}})$ such that (a_j, b_j) converges to some (a, b) in U and the sequence of contact forms

$$\alpha_j \coloneqq \varphi_j^* \varepsilon_{a_j, b_j}$$

converges to some contact form α in $\mathcal{U}$. Assuming that (a_j, b_j) and φ_j have been chosen, we choose φ_{j+1} of the form

$$\varphi_{j+1} \coloneqq \psi_j \circ \varphi_j,$$

for some ψ_j in $\mathrm{Cont}_0(S^3, \xi_{\mathrm{st}})$ satisfying $\psi_j^* \varepsilon_{a_j, b_j} = \varepsilon_{a_j, b_j}$ whose properties are discussed below. With these choices, we have for every $(a', b') \in (0, +\infty)^2$

$$\begin{aligned} &\varphi_{j+1}^* \varepsilon_{a', b'} - \varphi_j^* \varepsilon_{a_j, b_j} = \varphi_{j+1}^* \varepsilon_{a', b'} - (\psi_j^{-1} \circ \varphi_{j+1})^* \varepsilon_{a_j, b_j} \\ &= \varphi_{j+1}^* \varepsilon_{a', b'} - \varphi_{j+1}^* ((\psi_j^{-1})^* \varepsilon_{a_j, b_j}) = \varphi_{j+1}^* (\varepsilon_{a', b'} - \varepsilon_{a_j, b_j}). \end{aligned} \tag{A.1}$$

Once ψ_j, and hence φ_{j+1}, has be chosen, the above identity shows that by choosing (a_{j+1}, b_{j+1}) close to (a_j, b_j) we can make the C^k-norm of

$$\alpha_{j+1} - \alpha_j = \varphi_{j+1}^* (\varepsilon_{a_{j+1}, b_{j+1}} - \varepsilon_{a_j, b_j})$$

arbitrarily small. Therefore, we can ensure the convergence of (a_j, b_j) to some $(a, b) \in U$ and of (α_j) to some $\alpha \in \mathcal{U}$ in the C^∞-topology. Moreover, (A.1) shows that by making the convergence of (a_j, b_j) fast enough and by a diagonal argument we can also ensure that the sequence $(\varphi_j^* \varepsilon_{a,b})$ converges to α, as claimed in (i).

Next, we remark that any open condition which is satisfied by α_j can be transmitted to α by making the convergence of (a_j, b_j) sufficiently fast. More precisely: if $\mathcal{V}_j$ is some open neighborhood of α_j in $\mathcal{F}(S^3, \xi_{\mathrm{st}})$, we can fix a closed neighborhood $\mathcal{W}_j \subset \mathcal{V}_j$ of α_j and

make all the subsequent choices of (a_i, b_i) in such a way that for every $i > j$ the contact form α_i belongs to $\mathcal{W}_j$. By doing this, we can ensure that also the limiting contact form α belongs to $\mathcal{V}_j$.

We now discuss the choice of ψ_j. By choosing each ψ_j to be supported away from the link $\Gamma := \Gamma_1 \cup \Gamma_2$, we can ensure that α agrees up to every order with $\varepsilon_{a,b}$ on Γ, and hence Γ_1 and Γ_2 are closed Reeb orbits of α of period a and b, respectively.

Write $\frac{a_j}{b_j} = \frac{p_j}{q_j}$, where p_j and q_j are coprime natural numbers. By choosing (a_{j+1}, b_{j+1}) close to (a_j, b_j), but not equal to it, we have that the sequences (p_j) and $q_j)$ diverge.

We claim that if the convergence of (a_j, b_j) is sufficiently fast then Γ_1 and Γ_2 are the only closed Reeb orbits of α. Let V_j be a sequence of open neighborhoods of Γ whose intersection is Γ. Since the Reeb orbits of α_j other than Γ_1 and Γ_2 are all closed with period $p_j b_j = q_j a_j$, the Reeb flow of α_j has no closed orbits of period $\tau \leq \tau_j := p_j b_j - 1$ which pass through $S^3 \setminus V_j$. The set of contact forms with the latter property is C^1-open, so by the remark above we can guarantee that the Reeb flow of α has no closed orbits of period $\tau \leq \tau_j$ which pass through $S^3 \setminus V_j$, for every $j \in \mathbb{N}$. Since the sequence (V_j) shrinks to Γ and (τ_j) diverges, Γ_1 and Γ_2 are the only two closed Reeb orbits of α, as claimed in (ii).

Fix some infinitesimal sequence (ϵ_j) of positive numbers. We claim that by a suitable choice of the defining data we can ensure that for every $j \geq 1$ the Reeb flow of α_j has an orbit which is ϵ_j-dense. Here, a subset of S^3 is said to be ϵ-dense if it meets any open ball of radius ϵ, with respect to some fixed metric d on S^3.

Indeed, once this is achieved we can argue as follows. Having an ϵ_j-dense orbit is a C^0-open condition in the space of smooth vector fields, and hence a C^1-open condition in $\mathcal{F}(S^3, \xi_{\text{st}})$. Therefore, the above property of each α_j can be transmitted to α, whose Reeb flow then has an ϵ_j-dense orbit for every j. When non-empty, the set O_j of ϵ_j-dense orbits of a flow is open and ϵ_j-dense. Using that $\epsilon_j \to 0$, a standard argument, which is analogous to the proof of Baire's theorem, shows that $\cap_{k \in \mathbb{N}} O_{j_k}$ is non-empty for a suitable subsequence (j_k). Every element of this intersection has a dense orbit and, therefore, α satisfies (iii).

There remains to prove the following claim, in which ϵ is an arbitrary positive number: Once φ_j and (a_j, b_j) have been chosen, by a suitable choice of the diffeomorphism ψ_j, which we recall is required to preserve the contact form ε_{a_j, b_j} and to be supported away from Γ, and by choosing (a_{j+1}, b_{j+1}) sufficiently close to (a_j, b_j), we can make sure that the Reeb flow of

$$\alpha_{j+1} = \varphi_j^* (\psi_j^* \varepsilon_{a_{j+1}, b_{j+1}})$$

has an orbit which is ϵ-dense. The proof of this fact uses the following lemma.

LEMMA A.4. *Let (a, b) be a pair of positive numbers with rational ratio. Let $(\gamma_1, \dots, \gamma_k)$ and $(\gamma'_1, \dots, \gamma'_k)$ be k-tuples of distinct Reeb orbits of $\varepsilon_{a,b}$ in $S^3 \setminus \Gamma$. Then there exists $\psi \in \text{Cont}_0(S^3, \xi_{\text{st}})$ which preserves the contact form $\varepsilon_{a,b}$, is supported away from Γ and satisfies $\psi(\gamma_j) = \gamma'_j$ for every $j = 1, \dots, k$.*

Postponing the proof of this lemma, we show how it implies the above claim. Note that

the Reeb flow of α_{j+1} has the form

$$\phi^t_{\alpha_{j+1}} = \varphi_j^{-1} \circ \psi_j^{-1} \circ \phi^t_{\varepsilon_{a_{j+1},b_{j+1}}} \circ \psi_j \circ \varphi_j, \tag{A.2}$$

where $\phi^t_{\varepsilon_{a_{j+1},b_{j+1}}}$ denotes the Reeb flow of $\varepsilon_{a_{j+1},b_{j+1}}$. Let $\delta > 0$ be such that

$$d(z,z') < \delta \qquad \Rightarrow \qquad d\big(\varphi_j^{-1}(z), \varphi_j^{-1}(z')\big) < \epsilon, \tag{A.3}$$

and let $A \subset S^3 \setminus \Gamma$ be a finite set which is $\frac{\delta}{2}$-dense in S^3. Fix some invariant torus T in $S^3 \setminus \Gamma$, such as for instance

$$T \coloneqq \big\{(z_1,z_2) \in S^3 \mid |z_1|^2 = |z_2|^2 = \tfrac{1}{2}\big\}.$$

Since this torus consists of infinitely many Reeb orbits of ε_{a_j,b_j}, by the above lemma there exists $\psi_j \in \mathrm{Cont}_0(S^3,\xi_{\mathrm{st}})$ which preserves ε_{a_j,b_j} and is supported away from Γ such that $\psi_j^{-1}(T)$ contains the finite set A, and is therefore $\frac{\delta}{2}$-dense in S^3.

Now let $\sigma > 0$ be such that

$$d(z,z') < \sigma \qquad \Rightarrow \qquad d\big(\psi_j^{-1}(z), \psi_j^{-1}(z')\big) < \tfrac{\delta}{2}. \tag{A.4}$$

Let (a',b') be a pair of positive numbers with rational ratio $\frac{a'}{b'} = \frac{p}{q}$, where p and q are coprime natural numbers. Each Reeb orbit of $\varepsilon_{a',b'}$ in T is σ-dense, provided that p and q are large enough. Therefore, by choosing (a_{j+1},b_{j+1}) with the already required properties and sufficiently close to (a_j,b_j), we obtain that each Reeb orbit of $\varepsilon_{a_{j+1},b_{j+1}}$ is σ-dense in T.

Let z be a point in $\varphi_j^{-1} \circ \psi_j^{-1}(T)$ and set $w \coloneqq \psi_j \circ \varphi_j(z) \in T$. Then the orbit of w by the Reeb flow of $\varepsilon_{a_{j+1},b_{j+1}}$ is σ-dense in T. By (A.4), the image of this orbit by ψ_j^{-1} is $\frac{\delta}{2}$-dense in $\psi_j^{-1}(T)$ and hence δ-dense in S^3. Together with (A.2) and (A.3), we conclude that the orbit of z by the Reeb flow of α_{j+1} is ϵ-dense. This concludes the proof of Theorem A.1.

Proof of Lemma A.4. Since $\frac{a}{b}$ is rational, the Reeb flow of $\varepsilon_{a,b}$ defines a free S^1-action on $S^3 \setminus \Gamma$. The orbit space of this action is a smooth surface Σ which is endowed with a symplectic form ω satisfying $\pi^*\omega = d\varepsilon_{a,b}$, where

$$\pi : S^3 \setminus \Gamma \to \Sigma$$

denotes the quotient projection. Let $(p_1,\dots,p_k)$ and $(p'_1,\dots,p'_k)$ be the k-tuples of distinct points in Σ such that

$$\gamma_j = \pi^{-1}(p_j), \qquad \gamma'_j = \pi^{-1}(p'_j) \qquad \forall j = 1,\dots,k.$$

It is easy to show that the group of compactly supported Hamiltonian diffeomorphisms of (Σ,ω) acts k-transitively: There exists a smooth function $H : [0,1] \times \Sigma \to \mathbb{R}$ with compact support such that the time-1 map of the induced Hamiltonian flow maps p_j to p'_j, for every $j = 1,\dots,k$ (see e.g. [Ban97, p. 109]). Lift H to a smooth function $K : [0,1] \times S^3 \to \mathbb{R}$

supported away from Γ and let X be the contact vector field on S^3 which is induced by the Hamiltonian K and the contact form $\varepsilon_{a,b}$, i.e., the unique vector field satisfying

$$\imath_X d\varepsilon_{a,b} = \big(\imath_{R_{\varepsilon_{a,b}}} dK\big)\varepsilon_{a,b} - dK, \qquad \imath_X \varepsilon_{a,b} = K,$$

where $R_{\varepsilon_{a,b}}$ denote the Reeb vector field of $\varepsilon_{a,b}$. The flow of X consists of contactomorphisms of (S^3, ξ_{st}) preserving $\varepsilon_{a,b}$ which are supported away from Γ and whose time-1 map ψ maps γ_j to γ'_j, for every $j = 1, \dots, k$. □

Proof of theorem A.3. By composition with the radial projections, a diffeomorphism from $\partial E(a,b)$ to ∂A which intertwines the restrictions of λ_0 would induce a smooth conjugacy between $\varepsilon_{a,b}$ and α on S^3, which cannot exist because the Reeb flow of α has a dense orbit while the one of $\varepsilon_{a,b}$ does not. So we only have to prove that $E(a,b)$ and A are symplectomorphic.

Set for simplicity $E \coloneqq E(a,b)$. Write $\varphi_j^* \varepsilon_{a,b} = f_j^2\, \alpha_0$ and set

$$A_j \coloneqq \{rz \mid z \in S^3,\ 0 \le r < f_j(z)\}.$$

By composing the contactomorphism $\varphi_j^{-1} : S^3 \to S^3$, which isisotopic to the identity, with the radial projections $\partial E \to S^3$ and $S^3 \to \partial A_j$, we obtain a diffeomorphism from ∂E to ∂A_j which intertwines the restrictions of λ_0. The positively 1-homogeneous extension of this map is a symplectomorphism of $\mathbb{C}^2 \setminus \{0\}$ isotopic to the identity via such maps, and mapping $rE \setminus \{0\}$ to $rA_j \setminus \{0\}$, for every $r > 0$. Repeating the argument at the end of the proof of Theorem 1 (ii) in Section 6, this map can be smoothed near the origin producing a global symplectomorphism $\phi_j : \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$\phi_j(rE) = rA_j \qquad \forall r \ge \tfrac{1}{2}.$$

The C^0-convergence of (f_j) to f implies that, up to passing to a subsequence, we can find a strictly increasing sequence (r_j) in the interval $(\frac{1}{2}, 1)$ which converges to 1 and satisfies

$$r_j \overline{A}_j \subset r_{j+1} A_{j+1}, \qquad \bigcup_{j \ge 0} r_j A_j = A. \tag{A.5}$$

Starting with $\psi_0 \coloneqq \phi_0$ and arguing inductively, we wish to construct symplectomorphisms $\psi_j : \mathbb{C}^2 \to \mathbb{C}^2$ such that for every $j \ge 0$:

$$\psi_{j+1} = \psi_j \qquad \text{on } r_j E, \tag{A.6}$$

$$\psi_{j+1} = \phi_{j+1} \qquad \text{on } \mathbb{C}^2 \setminus r_{j+1} E. \tag{A.7}$$

Once $\psi_0, \dots, \psi_j$ have been constructed, the construction of ψ_{j+1} goes as follows. The symplectomorphism $\theta \coloneqq \phi_{j+1}^{-1} \circ \psi_j$ maps $r_j E$ to the domain $\phi_{j+1}^{-1}(r_j A_j)$, whose closure satisfies

$$\overline{\phi_{j+1}^{-1}(r_j A_j)} = \phi_{j+1}^{-1}(r_j \overline{A_j}) \subset \phi_{j+1}^{-1}(r_{j+1} A_{j+1}) = r_{j+1} E.$$

Since the space of symplectic embeddings between ellipsoids is connected, see [McD09], we can find a symplectic isotopy $\theta_t : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\theta_0 = \mathrm{id}$, $\theta_1 = \theta$ and

$$\theta_t(r_j\overline{E}) \subset r_{j+1}E \qquad \forall t \in [0,1].$$

Let $H_t : \mathbb{C}^2 \to \mathbb{R}$ be a smooth path of Hamiltonians generating θ_t. Let $\chi : \mathbb{C}^2 \to \mathbb{R}$ be a smooth function with support in $r_{j+1}E$ and taking the value 1 on $\bigcup_{t\in[0,1]} \theta_t(r_j\overline{E})$. Denote by $\eta : \mathbb{C}^2 \to \mathbb{C}^2$ the time-1 map of the Hamiltonian isotopy induced by the non-autonomous Hamiltonian χH. Then

$$\eta = \theta_1 = \phi_{j+1}^{-1} \circ \psi_j \quad \text{on } r_jE, \qquad \eta = \mathrm{id} \quad \text{on } \mathbb{C}^2 \setminus r_{j+1}E,$$

and hence $\psi_{j+1} \coloneqq \phi_{j+1} \circ \eta$ satisfies (A.6) and (A.7).

Note that by (A.7) we have

$$\psi_j(rE) = rA_j \quad \forall r \geq r_j, \qquad \forall j \geq 0. \tag{A.8}$$

By (A.6), the sequence $\psi_j(z)$ stabilizes for every $z \in E$, and hence the map

$$\psi : E \to \mathbb{C}^2, \qquad \psi(z) \coloneqq \lim_{j\to\infty} \psi_j(z)$$

is a well defined symplectic embedding, which by (A.5) and (A.8) has image A. The maps ψ is then the required symplectomorphism from $E(a,b)$ to A.

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