Physics Area - PhD course in Theoretical Particle Physics

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Symmetries and their Holographic Aspects in Quantum Field Theory

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The only good defects are topological defects

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Abstract

In this thesis we study generalized global symmetries in Quantum Field Theory. In the modern definition, symmetries are viewed as extended topological operators. On the one hand this new paradigm efficiently encodes all the information a symmetry usually comes with (e.g. 't Hooft anomalies) and on the other it allows for many interesting generalizations. After the Introduction, the first chapter is a brief review of basic concepts used throughout the thesis. The rest of this work is conceptually divided into two parts. In the first one, we consider conventional symmetries in exotic theories characterized by randomly distributed interaction couplings as well as non-invertible symmetries in well-known 2 dimensional theories corresponding to Calabi-Yau non-linear sigma models. In the second part instead we showcase two applications of the holographic approach to symmetries, the SymTFT. In particular we show how this tool can be used efficiently to discuss anomalies (defined as obstructions to gauging) of non-invertible symmetries in higher dimensions, and also how it can be used to establish holographic correspondences relating a Topological Quantum Field Theories and the universal effective theories describing the spontaneous breaking phases of continuous symmetries.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

The discussion is based on the following published works:

- A. Antinucci, G. Galati, G. Rizi , M. Serone, *Symmetries and topological operators, on average*, SciPost Phys. 15 (2023) 3, 125 [2305.08911] [1]

- A. Antinucci, F. Benini, C. Copetti, G. Galati, G. Rizi *Anomalies of non-invertible self-duality symmetries: fractionalization and gauging*, [2308.11707] [2]

- C. Córdova, G. Rizi, *Non-Invertible Symmetry in Calabi-Yau Conformal Field Theories*, [2312.17308] [3]

- A. Antinucci, F. Benini, G. Rizi, *Holographic duals of symmetry broken phases*, [2408.01418] [4]

I also coauthored these works, which are not part of this thesis:

- F. Benini, G. Rizi, *Superconformal index of low-rank gauge theories via the Bethe Ansatz*, JHEP 05 (2021) 061, [2102.03638] [5]

- A. Antinucci, G. Galati, G. Rizi *On continuous 2-category symmetries and Yang-Mills theory*, JHEP **12** (2022) 061, [2206.05646] [6]

- A. Antinucci, F.Benini, C.Copetti, G. Galati, G. Rizi, *"The holography of non-invertible self-duality symmetries*, [2210.09146] [7].

- A. Antinucci, C.Copetti, G. Galati, G. Rizi, *"Zoology" of non-invertible duality defects: the view from class S*, JHEP 04 (2024) 036, [2212.09549] [8] .

Contents

Chapter 1

Introduction

This thesis is devoted to the study of symmetries in Quantum Field Theory (QFT). Symmetries and conservation laws play a crucial role in physics, particularly in QFT, where their significance is even more pronounced, as they can offer insights into strongly coupled regimes that are otherwise inaccessible. The first distinction we have to make when discussing symmetries is between spacetime and internal symmetries. The first class consists of all transformations that act on the spacetime manifold, such as parity and time reversal, or those generated by the conformal or supersymmetry algebras. Internal symmetries act on states and operators of the theory while commuting with all the space-time symmetries generators. Another very important distinction is between global and gauge symmetries. Gauge symmetries are a redundancy of the theory. Namely, their action relates indistinguishable physical configurations, and to obtain a physical theory, we have to break them to single out one element for each gauge orbit. Global symmetries instead relate different physical configurations, which happen to share the same observables. This distinction is important in understanding how symmetries put constraints on a theory. For instance, gauge symmetries are not matched across dualities or along renormalization group (RG) flows, while global symmetries are. A caveat to this distinction between global and gauge symmetries arises when we consider theories on manifolds with boundaries. Indeed, in the presence of boundaries, the theory needs further data to be completely specified, that is, we need to specify boundary conditions. In the case of a gauge symmetry in the bulk, it is important to define the behavior of gauge transformations at the boundary, with the standard approach being to allow only those transformations that become trivial at the boundary. In these situations, the symmetry becomes global on the boundary and does have consequences on the dynamics of the theory.

In this thesis the focus is on internal global symmetries. Whenever a theory enjoys such a structure there are many consequences on can draw from it. First of all, these structures provide an organization principle not only for the spectrum of the theory but also for its infrared (IR) behavior. This is the (generalized) Landau Paradigm program, whose aim is to understand all possible phases of a theory in terms of its global symmetries and how those are realized on the vacuum of the theory. Symmetries themselves are RG invariants and dictate which operators can be generated in the low energy effective actions. As we will describe more extensively in the next chapter, a symmetry is equivalent to the presence of topological extended operators and, in the case of continuous symmetries, of conserved currents. The behavior of these operators inside correlation functions is completely fixed by the symmetry structure and leads to selection rules, which, in the most general form, can be understood as relations between correlators containing different operators related by the symmetry action. A crucial method for employing symmetries as a diagnostic tool for the properties of a theory involves the use of background gauge fields. Indeed, a set of observables that is most easily detected using background fields are anomalies; these can be defined as the failure of the partition function of the theory to be invariant, up to counterterms, under background gauge transformations. Importantly, anomalies are preserved under RG flow, and hence their matching between UV and IR imposes constraints on the possible IR phases. In absence of anomalies, a global symmetry can be gauged, namely, we can consistently make the background gauge fields dynamical. For continuous symmetries this generically changes the dynamics of the theory in a dramatic way, while gauging discrete symmetries changes only the global aspects of the theory. In particular the gauging of discrete symmetries is a reversible operation, namely, the gauged theory always has a dual (or "quantum") symmetry which can be gauged to retrieve the original theory.

In recent years many efforts have been made to generalize the notion of symmetry. This comes from the realization that all conventional symmetries can be comfortably described in terms of topological operators [9]. This presentation of the symmetry has the advantage of being an efficient packaging of all the information that a symmetry contains. As we will detail in the next chapter, all the previously mentioned features, including background gauge fields and anomalies, can be represented in terms of these operators. Another benefit of this presentation is that it lends itself particularly well to generalizations. Conventional group-like symmetries are represented by codimension 1 operators whose fusion rules follow the group law; then either considering operators supported on lower dimensional manifolds or allowing more general fusion rules leads to interesting generalizations, which have been intensively investigated in the last few years. In particular, topological operators supported on a codimension $(p+1)$ manifold generate a p-form symmetry, which naturally acts on p-dimensional extended operators by linking. On the other hand, topological operators whose fusion rules are not group-like and do not admit inverses, are called non-invertible. As it turns out, the mathematical framework that best encapsulates the structures that one obtains from these generalizations is that of (higher) category theory [10–14]. In 2 and 3 dimensions the relevant categorical structures, at least for finite symmetries, have been known for quite some time in the physics communities (see e.g. [11, 12, 15–38]), more recently efforts have been made to understand the relevant higher-categories in higher dimensions as well [6–8, 14, 39–54]. These efforts, fueled by the known results in $2d$, resulted in the discovery and analysis of many generalized symmetry structures in higher dimensions [55–89] as well as their applications [12, 30, 90–102] to study the dynamics of the theory (see also [103–107] and [108] for reviews and more complete lists of references). These generalizations are also important to bring to completion the generalized Landau paradigm program. Indeed, some phases are outside of this paradigm if one sticks to conventional symmetries, a prime example being confinining or deconfining phases of $4d$ non-Abelian gauge theories. In this case the order parameter that distinguishes the two phases is not a local operator. Instead, it is a Wilson line, whose vacuum expectation value signals the breaking of a 1-form symmetry as well as a perimeter law behavior, hence a deconfined phase [9]. By now many other examples of vacuum and phases structures dictated by generalized symmetries have been discussed in various dimensions [12, 80, 109–114]. As conventional symmetries (aka invertible 0-form symmetries), higher-form and/or non-invertible symmetries impose selection rules and can have anomalies, which are again best described using the topological defects themselves. Therefore, the modern take on symmetries is to define them as topological operators. The topological nature implies that, in any quantization scheme, if they are placed on a space-like slice, they become operators on the Hilbert space commuting with the Hamiltonian, recovering the standard notion of symmetry in quantum systems [115][1].

A very fruitful perspective that has been adopted when studying symmetries is that of the Symmetry Topological Field Theory (SymTFT) [9, 116–119]. This is a Topological Quantum Field Theory (TQFT) in one dimension higher than the physical QFT that we want to study, which contains all of the information on the symmetry structures of the physical theory. For $2d$ QFTs with a (unitary) fusion category symmetry the SymTFT picture can be made rigorous [120–123] (see section 2.3.3 for more details), and it shares some similarities with the Chern-Simons/WZW correspondence [124, 125] although it applies to $3d$ TQFTs with gapped boundaries and general $2d$ QFTs, not necessarily conformal invariant. Rigorous generalizations in higher dimensions have also been considered [126–130]. The holographic flavor of the SymTFT picture is reminiscent of the AdS/CFT correspondence. The two setups are indeed related and the SymTFTs for holographic theories can be derived from a full-fledged string theory [7, 67, 69, 117, 131–134]. Other, somewhat related, holographic approaches to symmetries involve brane and/or geometric engineering [14, 67–72, 82, 117, 132, 134–174] or the realization of symmetry defects directly as branes in the gravity theory (that become topological in some limit) [69–72, 175, 176].

This thesis is organized as follows.

Chapter 2. We review some background material to explain the basic concepts used throughout the thesis. We start by introducing symmetries as topological operators in general dimension, and then we focus on the $2d$ case to discuss more in detail the mathematical description of non-invertible symmetries. We proceed with a review on the holographic approach to symmetries, starting with a discussion of TQFTs to conclude with a description of the SymTFT and how this captures the symmetries of a physical QFT.

Chpater 3. This is a transcription of the original works [1] and [3], in which we either investigated symmetries in exotic QFTs [1] or exotic symmetries in well-know theories [3]. More precisely in [3] we studied the non-invertible symmetries along the conformal manifold of some Calabi-Yau sigma models. Starting from the Gepner point, where the theory is rational and has a large amount of non-invertible line defects, we can use tools discussed in Chaper 2 to find the symmetries preserved by the exactly marginal deformations. This analysis shows that

[1] Non-invertible symmetry defects apparently evade Wigner theorem being non-invertible, hence non unitary, when acting only on the untwisted Hilbert space. The expectation is that a notion of unitarity is recovered considering the action on the full Hilbert space that includes twisted sectors, see e.g. [36].

along many submanifolds of the conformal manifold there are preserved non-invertible lines which can be useful to impose constraints on correlation functions and also that non-invertible symmetries are not special to rational CFTs.

In [1] instead we studied the fate of invertible 0-form symmetries explicitly broken by some randomly distributed interaction. There are two scenarios that we considered: the disorder case in which the random couplings are taken to be space-dependent or the average case in which the random couplings are homogeneous. The results are pretty different in the two cases. Symmetries in average theories turn out to behave very similar to standard symmetries in non-disordered QFTs, while the non-locality introduced by the average over homogeneous couplings turns out to have a rather profound impact on the structure of the symmetries in an average of QFTs. In particular in the second scenario we find that topological defects implementing the symmetries are not genuine and need a d-dimensional bulk attached to be well defined. The selection rules imposed by those operators have a compelling interpretation in terms of a higher-dimensional gravity dual and its wormholes that prevent factorization of the observables.

Chapter 4. This Chapter in a transcription of the original works [2] and [4], both of which can be regarded as applications of the SymTFT setup. In [2] we discuss anomalies for a special type of non-invertible symmetries, the duality defects, in 4 dimensions. We define 't Hooft anomalies as obstruction to consistently gauge the symmetry via a Lagrangian algebra insertion in the path integral. This is a stronger definition with respect to the one appeared in the literature according to which an anomaly is an incompatibility with a trivially gapped phase. More precisely our definition regards as non-anomalous situations in which there is no compatible trivially gapped phase but there is a compatible TQFT, while an anomaly in our definition also implies incompatibility with a trivially gapped phase. We study the problem using the SymTFT approach reviewed in Chapter 2. This is useful as it allows us to rephrase the gauging as an operation in steps, which in turn gives two successive layers of obstruction to gauging. The first necessary condition is the presence of a duality invariant gapped boundary condition for the SymTFT on which the duality defect becomes invertible. Thus the first step to gauge the non-invertible symmetry is to condense a subgroup of the 1-form symmetry in order to reach this particular global variant of the theory. The second and final step is to gauge the now invertible duality defect, thus the second necessary condition is the absence of anomalies for this symmetry. In this context the notion of symmetry fractionalization comes into play as it allows us to mix the couplings to backgrounds in order to find an anomaly free symmetry. We make these two conditions very explicit in terms of the 1-form symmetry group and the other data appearing in the definition of a duality defect.

In [4] we investigate non-topological boundary conditions for the SymTFTs describing continuous symmetries. These boundary conditions induce dynamical edge modes giving rise to a, possibly interacting, boundary theory. We show that gauging a suitable Lagrangian algebra in the SymTFT we are able to trivialize the bulk theory on closed manifolds and establish a full-fledged holographic duality. We argue that these types of holographic correspondences involving a TQFT in the bulk can be dual only to specific boundary theories. More precisely we conjecture, confirming our expectation with many examples, that when the TQFT involved is the SymTFT for a certain symmetry the boundary theory is the effective field theory (EFT) for the spontaneous breaking of that symmetry. This conjecture allows us to derive the EFT for the spontaneous breaking of a non-Abelian 2-group, which has implications for the low energy dynamics of $U(N)$ QCD.

We collect in two appendices some of the more techincal materials.

Chapter 2

Background Material

In this first chapter we review standard material on symmetries in QFT. Many more complete reviews on the subject have already appeared [103–107].

2.1 Symmetries and Topological Defects

We begin to discuss the relation between symmetries and topological defects showing how topological operators can be constructed out of conserved currents. We will then describe what properties of the operators can be generalized to accommodate both higher-form and non-invertible symmetries. Consider a standard $U(1)$ symmetry in a d-dimensional euclidean QFT. This is implemented by a one-form current $J^{(1)}$ that satisfies the conservation equation $d * J^{(1)} = 0$. The operator

$$Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} *J^{(1)} \tag{2.1.1}$$

is topological, indeed, upon a small deformation of its support, we have

$$Q(\Sigma'_{d-1}) - Q(\Sigma_{d-1}) = \int_{B_d} d * J^{(1)} = 0\,, \tag{2.1.2}$$

where B_d is a manifold bounded by Σ_{d-1} and Σ'_{d-1}. We can then construct the topological operator

$$U_\alpha(\Sigma_{d-1}) = e^{i\alpha Q(\Sigma_{d-1})}\,. \tag{2.1.3}$$

For a $U(1)$ symmetry all charges are integer multiples of a fundamental unit, therefore, assuming a properly normalized current, the parameter α is identified modulo 2π. For a general continuous group G we are always able to construct, from the integrals of the currents, a set of topological operators $U_g(\Sigma_{d-1})$ labelled by group elements $g \in G$.

The arguments we just presented give an intuitive picture as to why symmetries are related to topological operators; however, before discussing possible generalizations, let us see how we can make more precise statements in QFT by considering its correlation functions. The first two most important features of these topological operators are the action on local operators and fusion rules, both of which can be derived from the Ward identities associated with the symmetry. These identities dictate how the $U_g(\Sigma_{d-1})$ behave inside correlation functions, the basic relation is[1]

$$\langle d * J^{(1)}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \sum_{i=1}^{n}\delta(x-x_i)\langle\mathcal{O}_1(x_1)\dots\delta\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle \qquad (2.1.4)$$

where $\delta\mathcal{O}_i$ is the infinitesimal G-symmetry action on the operator $\mathcal{O}_i$. Integrating this relation in x over a d-dimensional ball B_d bounded by Σ_{d-1} we obtain the action of the charge operator, on all those local operators inserted at points $x_i \in B_d$, namely all local operators linking with Σ_{d-1}. In particular

$$\langle Q(\Sigma_{d-1})\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \sum_{i|x_i\in B_d}\langle\mathcal{O}_1(x_1)\dots\delta\mathcal{O}_i(x_i)\dots\mathcal{O}_n(x_n)\rangle\,, \qquad (2.1.5)$$

$Q_{\Sigma_{d-1}}$ implements the action of the Lie algebra of G, the full group action is obtained by exponentiation and is implemented by the operators $U_g(\Sigma_{d-1})$

$$\langle U_g(\Sigma_{d-1})\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \left(\prod_{i|x_i\in B_d} R_i(g)\right)\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle\,, \qquad (2.1.6)$$

where R_i is the representation of G in which $\mathcal{O}_i$ transforms. Since this is a relation valid in any correlation function it can be promoted to an operator equation

$$U_g(\Sigma_{d-1})\mathcal{O}_i(x) = R_i(g)\cdot(\mathcal{O}_i)(x)U_g(\Sigma'_{d-1}) \qquad (2.1.7)$$

where Σ_{d-1} links with x while Σ'_{d-1} does not.Thus as we sweep the operator $U_g(\Sigma_{d-1})$ past $\mathcal{O}_i(x)$ we act on it with the element $g \in G$. These relations prove that the operators $U_g(\Sigma_{d-1})$ are topological, namely that correlation functions do not change under small deformations of their support. Large deformations can cross the insertion points of local operators, and correlation functions do change but in a very mild and controlled way. Starting from the Ward Identity with two current insertions it is easy to show that the fusion of two topological defects follows the group law, namely, in any correlation function

$$U_g(\Sigma_{d-1})U_{g'}(\Sigma_{d-1}) = U_{gg'}(\Sigma_{d-1})\,. \qquad (2.1.8)$$

The support of the operators $U_g(\Sigma_{d-1})$ is an oriented manifold Σ_{d-1}, in particular reversing its orientation is an involution on the set of topological operators which we take to correspond to inversion in the group

$$U_g(\overline{\Sigma}_{d-1}) = U_{g^{-1}}(\Sigma_{d-1}). \qquad (2.1.9)$$

For more general symmetries we will not have a notion of inverse element, but this involution induced by orientation reversal will provide the closest analog possible.

One can consider more intricate geometrical configurations of topological defects, one of the simplest ones is a junction at which three operators meet in codimension 2. This configuration is consistent, for the type of defects we have described so far, only if the labels of the three operators satisfy (with all three orientations going towards the junction) $g_1g_2g_3 = 1 \in G$. In other words, this configuration is another way to describe the fusion of defects. We can go higher

[1]Here and in the rest of this chapter we are assuming that the symmetry is not spontaneously broken.

in codimension considering more defects meeting on a submanifold and generically there are many possible different-looking configurations one can consider at some fixed codimension. The expectation is that, since the defects are topological, all those configurations are equivalent, but this is not always the case. This is important especially when we reach the maximum codimension and consider operators meeting at points. In fact, as we will see later, violations of this equivalence can signal an anomaly of the symmetry.

An immediate generalization from the case of continuous group symmetries is that to the discrete case. In all key equations (2.1.9), (2.1.8), and (2.1.9) it is not necessary for the group to be continuous and can be applied equally to G discrete. Thus, from now on, we take these to be the defining properites of invertible, group-like symmetries for any group G.

Another interesting generalization one can consider goes in a somewhat orthogonal direction than the ones we will discuss below: instead of considering more exotic defects we can consider more exotic theories. Interesting classes of unusual theories, which however have many applications, is that of disordered/averaged theories. The fate of symmetries in these setups is investigated in 3.

Higher-form symmetries

A first important generalization is to allow supports of different codimensions for the topological defects. Consider a topological defect $U_g(\Sigma_{d-p-1})$ supported on an oriented manifold of codimension $p+1$ and labeled by $g \in G$ for some group G. There are two immediate consequences. First, it is clear that $U_g(\Sigma_{d-p-1})$ cannot act on operators of dimensions lower than p, indeed if an operator is supported on a manifold of dimension less than p, there is always room for $U_g(\Sigma_{d-p-1})$ to move around it without ever interacting. Therefore, the lowest dimension an operator needs to have to be charged under $U_g(\Sigma_{d-p-1})$ is p, from here the name p-form symmetry for the structure carried by the operators $U_g(\Sigma_{d-p-1})$. In general an operator of codimension $p+1$ can act on operators supported in dimension p and higher, for simplicity here we limit ourselves to the simplest action by linking. The second important consequence is that higher-form symmetries are necessarily Abelian; indeed there is no notion of ordering for $(d-p-1)$-dimensional submanifolds in d-dimensions, namely we can always exchange the order of fusion for higher-form symmetry defects,

$$U_g(\Sigma_{d-p-1})U_{g'}(\Sigma_{d-p-1}) = U_{g'}(\Sigma_{d-p-1})U_g(\Sigma_{d-p-1}) = U_{gg'}(\Sigma_{d-p-1}) \tag{2.1.10}$$

then $gg' = g'g$ and G is necessarily Abelian. Apart from these two extra constraints higher-form symmetries behave analogously to 0-form symmetries. We have Ward identities that can be uplifted to operator equations

$$U_g(\Sigma_{d-p-1})V_i(\gamma_p) = g^{q_i}V_i(\gamma_p)U_g(\Sigma'_{d-p-1})\,, \tag{2.1.11}$$

where we denoted by q_i the charge of V_i. The fusion rules follow the group law and we have again an involution induced by orientation reversal. Continuous p-form symmetries imply the presence of a $p+1$-form current $J^{(p+1)}$ that is conserved $d * J^{(p+1)} = 0$.

Non-invertible symmetries

The second interesting generalization is to consider topological operators with more general fusion rules. These generically take the form

$$U_a(\Sigma_{d-p-1})U_b(\Sigma_{d-p-1}) = \sum_c N^c_{a,b}(\Sigma_{d-p-1})U_c(\Sigma_{d-p-1})\,, \tag{2.1.12}$$

here a, b, c are labels for the topological defects, while $N^c_{a,b}(\Sigma_{d-p-1})$ are the fusion coefficients, which generically depend on the topology of the support Σ_{d-p-1}. Generically these fusion coefficients can be interpreted as partition functions of $(d-p-1)$-dimensional Topological Quantum Field theory. Importantly, the defects U_a generically do not have an inverse. We still have a dual defect obtained via orientation reversal, denoted $U_{\overline{a}}$, and the fusion of dual defects always contains the identity with coefficient 1

$$U_a(\Sigma_{d-p-1})U_{\overline{a}}(\Sigma_{d-p-1}) = \sum_c N^c_{a,\overline{a}}(\Sigma_{d-p-1})U_c(\Sigma_{d-p-1}) \supset \mathbb{1}\,. \tag{2.1.13}$$

A consequence of these fusion rules is that shrinking the defects down to a point we get numbers, called quantum dimensions, that satisfy the same fusion ring. Invertible defects always have quantum dimension 1, while non-invertible operators have dimensions greater than 1 (in unitary theories). Being topological the defects $U_a(\Sigma_{d-p-1})$ still imply selection rules on correlators, these however can be somewhat complicated due to the non-invertible action. For instance, as we mentioned in the introduction, an hallmark of non-invertibility is that defects map genuine operators into non-genuine ones, thus selection rules can relate correlators of operators of the untwisted sector to those of the various twisted sectors. The mathematical structure that properly describes these symmetries is that of higher category theory. Intuitively, we label n-dimensional topological defects with objects in a higher category, morphisms between objects are represented by $n-1$-dimensional defects functioning as interfaces on the worldvolumes of the defects. Higher morphisms are lower dimensional defects separating morphisms on one degree lower and so on. Therefore the expectation is that general symmetries are described by appropriate higher-categories [6, 10–14, 57, 65, 118]. Relatively simple examples of these structures arise when we only consider invertible defects, for instance forming 2-groups [177–179]. These can be tought of as a group extension involving a 0-form and an higher-form symmetry.

We will not delve into the structure of higher categories in this thesis. In the next section, we will consider the case of 1-categories that describe line defects.

2.2 The $2d$ case: Topological Defect Lines

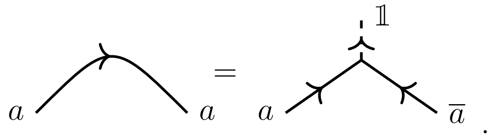
The setting in which generalized symmetries are best understood is that of 2-dimensional QFTs. The reason for this is that in 2 dimensions the most general situation we can consider is that of 0-form symmetries with non-invertible fusion rules. In principle we can also have 1-form symmetries, but those are generated by local operators and always lead to decomposition and universes [76, 180] (as any $d-1$-form symmetry in d dimensions). All universes are decoupled, and in each one we can at most have a 0-form symmetry. Moreover, 0-form symmetries are generated by line operators, which are clearly the simplest type of extended operator to consider. This is due to their internal structure being entirely determined by local operators living on the worldline. In the next two subsections we first briefly describe the mathematical structure formed by these defect lines and then discuss a class of examples.

2.2.1 Fusion Categories

The mathematical structure that describes 0-form symmetries is that of fusion categories [11,12, 17,29,31,181–184]. Let us motivate the various ingredients in the construction from a physical perspective. As we tried to motivate in the previous section, a 0-form symmetry is equivalent to the presence of a collection of topological line operators, which we denote by L_a for a in some labeling set. We will assume to be dealing with a finite number of defects in this section, techincally, this means that the category is finite and semisimple. On line operators we can define an operation of direct sum $\oplus$, in terms of correlation functions of the QFT we set

$$\langle L_a \oplus L_b \ldots \rangle = \langle L_a \ldots \rangle + \langle L_b \ldots \rangle \tag{2.2.1}$$

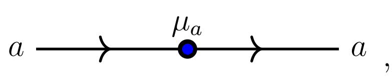
where the dots denote any other possible insertion. We will consider oriented line defects, reversing the orientation induces involution which we denote as $a \mapsto \overline{a}$. We can think of this orientation reversal as a way of bending the lines (which is of course allowed since these defects are topological)



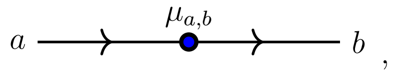
The topological property of these line operators allows us to define their fusion, denoted $\otimes$. Physically, we can bring a line L_a on top of another line L_b, and no divergence arises from this procedure. This necessarily produces another topological line operator, which in the most general case can be written as a direct sum. This leads to the operator equation

$$L_a \otimes L_b = \bigoplus_c N_{a,b}^c L_c \tag{2.2.2}$$

valid in correlation functions as long as we can deform the two lines to be on top on each other in a continuous way, namely without crossing any local operator. The fusion coefficients $N_{a,b}^c$ are positive integers, they depend on the basis we choose to describe the line defects. On any given line operator there exist topological local operators. One way of realising this is the case is to consider the line operators as a (topological) quantum mechanics coupled to the bulk theory, then this quantum mechanics has its own operators and we can consider their insertion in the path integral. Intuitively there are two types of these operators to consider. There are operators μ_a that live on a specified line L_a



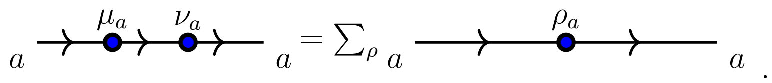
we can represent those as maps $\mu_a : L_a \to L_a$, but there are also operators $\mu_{a,b}$ that separate two different lines L_a, L_b



those can be thought of as maps $\mu_{a,b} : L_a \to L_b$. In general, we can define vector spaces $\mathrm{Hom}(L_a, L_b)$ containing the morphisms between lines. Local operators on a fixed line defect L_a have additional structure as we have a multiplication $\otimes_a$ given by the operator product expansion

$$\mu_a \times_a \nu_a = \sum_\rho \rho_a \,, \tag{2.2.3}$$

or, pictorially,



This is important as it leads us to define a very useful basis for line defects. In particular for any algebra of local operators we can find a basis of projectors (idempotents) π_i such that

$$\pi_i \otimes \pi_j = \delta_{ij}\pi_i \,. \tag{2.2.4}$$

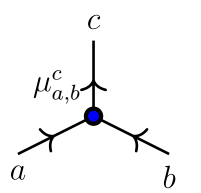
This means that if a line defect has a non-trivial algebra of topological local operators on it we can split it into other line defects. To see this consider a line L_a with an algebra of local operators generated by projectors $\pi_{i,a}$. Inserting π_i on the line L_a produces another defect $L_{i,a}$ which, thanks to the fusion rules of the projectors, has no topological local operators on it besides π_i (which behaves as the identity). In particular writing the identity on L_a as

$$\mathbb{1}_a = \sum_i \pi_{i,a} \tag{2.2.5}$$

we can rewrite the line L_a itself as a direct sum

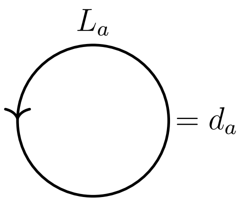
$$L_a = \bigoplus_i L_{i,a} \,. \tag{2.2.6}$$

The lines $L_{i,a}$ cannot be split any further, this leads us to considering a basis of lines generated by those defects that cannot be written as a sum of other lines. These are called simple line defects. In the basis of simple lines, which we still label by latin letters, we can unambiguously talk of fusion coefficients $N^c_{a,b}$. Notice that the fusion of two simple lines is generically non simple, in particular on the line $L_a \otimes L_b$ we have local topological operators that can be organized into projectors $\pi_{a,b;c}$ onto the simple lines L_c appearing in the fusion. Therefore the coefficient $N^c_{a,b}$ counts how many different operators living on $L_a \otimes L_b$ project on the line L_c. We can give another interpretation to $N^c_{a,b}$ considering higher junctions of line defects. A natural configuration to consider is the trivalent junction at which the simple lines a, b, c meet



At the junction there is a topological local operator insertion $\mu^c_{a,b}$, in particular bringing the lines a and b together we see that $\mu^c_{a,b}$ is a map $L_a \otimes L_b \to L_c$ and belongs to $\mathrm{Hom}(L_a \otimes L_b, L_c)$. Since $L_a \otimes L_b$ is non-simple we can also think of $\mu^c_{a,b}$ as an endormophism of $L_a \otimes L_b$, hence we see that it corresponds to one of the operators $\pi_{a,b;c}$ projecting onto L_c. Since $N^c_{a,b}$ counts the number of different projectors available, it also counts the different junction operators $\mu^c_{a,b}$ that we have, in other words $N^c_{a,b} = \dim\left(\mathrm{Hom}(L_a \otimes L_b, L_c)\right)$. Notice that the fusion of a line a with its orientation reversal $\overline{a}$ always contains the identiy, i.e. $N^{\mathbb{1}}_{a,\overline{a}} = 1$.

Another important quantity that we can associate to these line defects is their quantum dimension. We can consider correlation functions with an insertion of a loop of some line L_a that encircles no local operator. Shrinking the loop generically produces a number



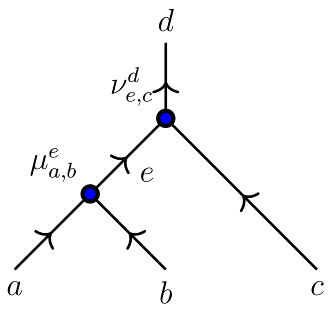
,

called quantum dimension of L_a. These numbers have several important properties. First one can show that they have to satisfy the same fusion rules as the simple lines, namely

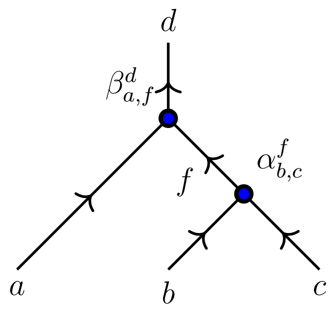
$$d_a d_b = \sum_c N^c_{a,b} d_c \,. \tag{2.2.7}$$

In conformal theories one can also show that $d_a \geq 0$ for every line L_a and, if the CFT is also compact, one has the stronger constraint $d_a \geq 1$.

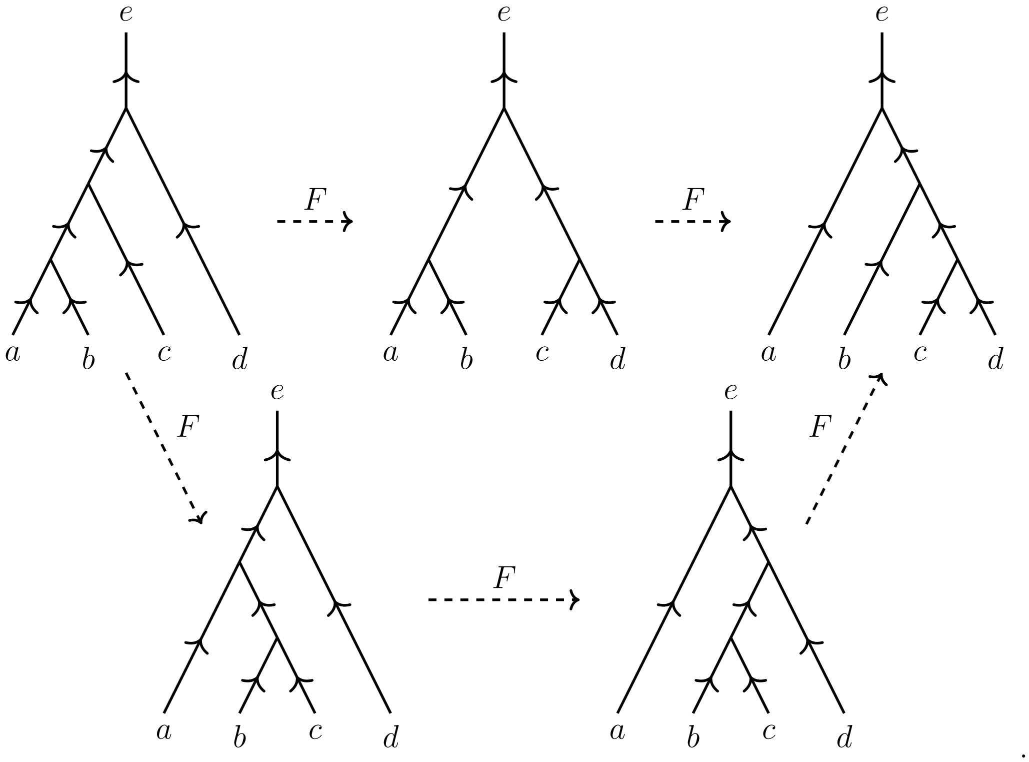
We could move on and consider even higher junctions, but this is not really needed, in the sense that an arbitrary network of lines can always be resolved in terms of trivalent junctions only. Therefore to define correlation functions with a network of lines inserted we only need to understand how to relate different meshes with only trivalent junctions. It turns out that we can do so by using repeatedly the so-called F-move. This operation defines the F-symbols of the symmetry



$$= \sum_{f,\alpha,\beta} \left[F^d_{a,b,c}\right]_{(e,\mu,\nu);(f,\alpha,\beta)} \tag{2.2.8}$$



This collection of numbers is subject to the Pentagon equations, which guarantee that, starting from the same configuration and using F-moves, we always obtain the same result. In pictures,



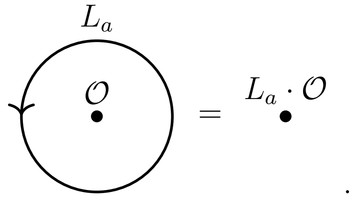
(2.2.9)

The corresponding equations are

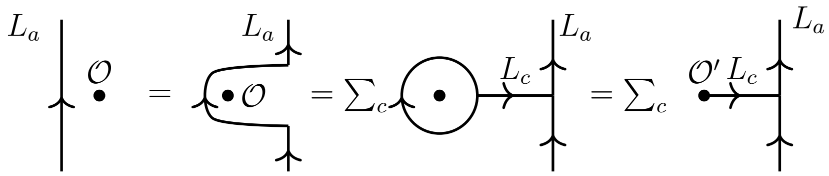
$$\sum_{\lambda} \left[F^{e}_{abf}\right]_{(g,\beta,\gamma);(\ell,\sigma,\lambda)} \left[F^{e}_{\ell cd}\right]_{(f,\alpha,\lambda);(k,\psi,\rho)} = \\ \sum_{h} \sum_{\delta\mu\nu} \left[F^{k}_{abc}\right]_{(h,\delta,\nu);(\ell,\sigma,\psi)} \left[F^{e}_{ahd}\right]_{(g,\mu,\gamma);(k,\nu,\rho)} \left[F^{g}_{bcd}\right]_{(f,\alpha,\beta)(h,\delta,\mu)} . \tag{2.2.10}$$

The data we have discussed so far define a fusion category $\mathcal{C}$: the set $\mathcal{S}$ corresponds to the objects of the category $\mathrm{Obj}(\mathcal{C})$, with the local operators living on the lines being morphisms between objects, the fusion product $\otimes$ is a tensor product on $\mathcal{C}$ whose is given by the F-symbols. We also should point out that there is a gauge freedom in the F-symbols given by the possibility of changing basis in the Hom spaces of trivalent junctions; only equivalence classes under these changes of basis give inequivalent F-symbols.

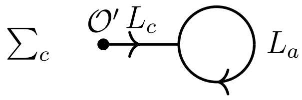
Action on local operators and twisted sectors. Lines link with local operators. This allows to define an action of any line L_a on a local operator $\mathcal{O}$, the result is, generically, a new local operator $L_a \cdot \mathcal{O}$:



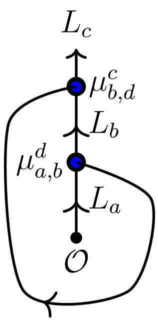
This action by linking should not be confused with the action we obtain sweeping a line L_a past a local operator



here the sum over c runs over all lines in the fusion channel $L_a \otimes L_{\overline{a}}$ and $\mathcal{O}'$ is an operator in the twisted sector of L_c. We can retrieve the linking action by closing the line L_a on itself, this produces the tadpole diagrams

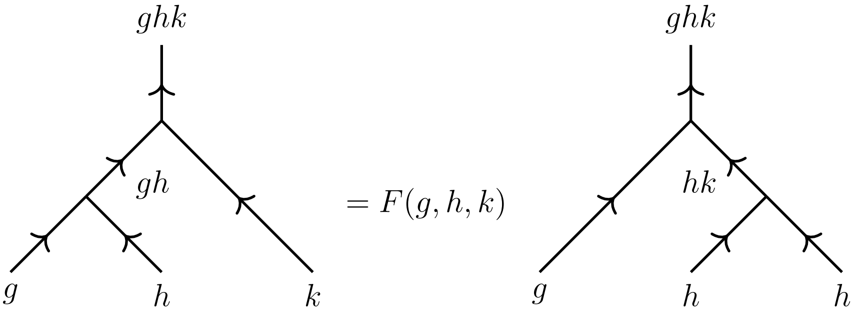


Assuming faithfulness of the action of line operators the empty loop of L_a vanishes unless $L_c = \mathbb{1}$, and we recover the standard linking action. For some local operators it may happen that $L_a \cdot \mathcal{O} = 0$, this is the situation we mentioned in the introduction, where a non-invertible line maps genuine operators in twisted sectors. From the action we obtained sweeping a line past a local operator we see that the most general action is encoded in the lasso diagrams [12]



these define the elements of the Tube algebra associated to the fusion category [185, 186], see also [36] for a more physical perspective and [51–53, 81] for generalisations to higher dimensions. Operators, both genuine and twisted ones, transform in representations of this algebra from which selection rules follow [36].

Invertible symmetries. When we are dealing with only invertible lines all the structure above collapses to a simpler one. Indeed as we already know the fusion product reduces to the group operation, and all line operators associated to a fixed group element $g \in G$ are simple and have quantum dimension $d_g = 1$. The only extra datum are the F-symbols which take the particularly simple form



(2.2.11)

for $g, h, k \in G$. One can check that the pentagon equation reduce to the condition

$$\frac{F(g,h,k)F(h,k,l)F(g,hk,l)}{F(gh,k,l)F(g,h,kl)} = 1 \tag{2.2.12}$$

while gauge transformations allow the transformation

$$F(g,h,k) \mapsto F(g,h,k)\frac{f(h,k)f(g,hk)}{f(gh,k)f(g,h)} \,. \tag{2.2.13}$$

The most convenient framework to understand these conditions is that of group cohomology (see e.g. [187]), and can be summarized saying that $F(g, h, k)$ defines a class in the third cohomology of G valued in $U(1)$, namely $F \in H^3(G, U(1))$. This group classifies the possible anomalies of a 0-form symmetry G in 2 dimensions [188, 189]. Thus the symmetry category for a group G, conventionally denoted $\mathrm{Vec}(G)_\alpha$ also encodes the anomaly α. This also shows that gauge inequivalent solutions to the Pentagon equations are isolated, this is a general fact valid in Fusion Categories known as Oceanu's rigidity [181].

Example: Tambara-Yamagami Categories. To keep the topic of fusion categories from being too abstract let us discuss a concrete set of examples provided by Tambara-Yamagami categories. These are constructed starting from an Abelian group $\mathbb{A}$ and adding a defect $\mathcal{D}$ with fusion rules

$$g \otimes h = gh \,, \quad g \otimes \mathcal{D} = \mathcal{D} \otimes g = \mathcal{D} \,, \quad \mathcal{D} \otimes \mathcal{D} = \bigoplus_{g \in \mathbb{A}} g \,. \tag{2.2.14}$$

The remaining data specifying the category are provided by the F-symbols. Specifically the F-symbols to consider are

$$\left[F^{\mathcal{D}}_{g,\mathcal{D},h}\right]_{\mathcal{D};\mathcal{D}} \equiv \gamma(g.h) \,, \quad \left[F^{ghk}_{g,h,k}\right]_{gh;hk} \equiv \alpha(g.h,k) \,, \quad \left[F^{\mathcal{D}}_{\mathcal{D},\mathcal{D},\mathcal{D}}\right]_{g,h} \equiv \chi(g,h) \,, \tag{2.2.15}$$

the Pentagon equations constraint these data as follows. The function $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$ is constrained to be a non-degenerate symmetric bicharacter, namely it satisfies

$$\gamma(g,h)\gamma(h,g)^{-1} = 1 \,, \tag{2.2.16}$$

the cocycle $\alpha(g, h, k)$ instead needs to be trivial, so that the symmetry $\mathbb{A}$ is non-anomalous. Finally one has

$$\chi(g,h) = \frac{\epsilon}{\sqrt{|\mathbb{A}|}}\gamma(g,h)^{-1} \,, \tag{2.2.17}$$

where $\epsilon = \pm 1$ is the Frobenius-Schur indicator of the defect $\mathcal{D}$. The FP indicator is a quantity that can be defined for every self-dual defect as $\kappa_a = d_a \left[F^a_{a,a,a}\right]_{0,0}$, we then see that the quantum dimension of the non invertible line is

$$d_{\mathcal{D}} = \sqrt{|\mathbb{A}|}\,. \tag{2.2.18}$$

The usual notation to denote TY categories is then $\text{TY}(\mathbb{A})_{\gamma,\epsilon}$. This symmetry structure arises in theories that are self-dual under gauging the symmetry $\mathbb{A}$ [29, 57, 63, 65, 66], a prominent example being the critical Ising model.

2.2.2 RCFTs and Verlinde Lines

A concrete class of examples where we can understand in detail the non-invertible symmetry structure is that of Rational Conformal Field Theories in $2d$. In this subsection we briefly review these particular cases, with an emphasis on the "bootstrap" approach one can take to find the set of topologial defect lines of the theory [12, 16, 31, 190]. RCFTs are CFTs with an extended symmetry algebra and a finite number of primaries of the extended symmetry algebra $\mathcal{A}$. This means that the Hilbert space of the theory is a finite sum of irreducible representations of the chiral algebra $\mathbb{H} = \bigoplus_{i,\bar{i}} M_{i,\bar{i}} \mathbb{H}_i \otimes \mathbb{H}_{\bar{i}}$. The torus partition function is

$$\mathcal{Z}(\tau,\overline{\tau}) = \sum_{i,\bar{i}} M_{i,\bar{i}} \chi_i(\tau)\chi^*_{\bar{i}}(\overline{\tau}) = \text{Tr}_{\mathbb{H}}\left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}\right) \tag{2.2.19}$$

where τ is the complex modulus of the torus and

$$\chi_i(\tau) = \text{Tr}_{\mathbb{H}_i}\left(q^{L_0 - \frac{c}{24}}\right) \tag{2.2.20}$$

is the character of the i-th representation of $\mathcal{A}$. The coefficients $M_{i,\bar{i}}$ are positive integers chosen to have a modular invariant torus partition function. To an RCFT we can associate an S matrix and a T matrix studying the modular properties of the characters

$$S \cdot \chi_i(\tau) = \chi_i\left(-\frac{1}{\tau}\right) = \sum_j S_{i,j}\chi_j(\tau)\,, \qquad T \cdot \chi_i(\tau) = \chi_i\left(\tau + 1\right) = \sum_j T_{i,j}\chi_j(\tau) \tag{2.2.21}$$

and the partition function is required to be invariant under both transformations. The bootstrap idea is to find constraints on the action of the topological line defects of the theory, in RCFTs this program shows its full potential as the constraints one obtains are actually enough to fully determine a large set of topological lines and their action on local operators, sometimes called Verlinde lines [15, 16, 191, 192]. The only caveat is that we can only bootstrap lines that commute with the chiral algebra, so that they do not mix different representations. One could still only require that the lines commute with the Virasoro algebra contained in $\mathcal{A}$, but the equations obtained are very difficult to solve. Let us stick to the standard situation and only consider lines that commute with $\mathcal{A}$. The simplest bootstrap equation is derived considering a putative line L inserted along the spatial cycle of the torus. The line acts on the Hilbert space of the theory and the path integral produces the twined partition function[2]

$$\mathcal{Z}(L) = \text{Tr}_{\mathbb{H}}\left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} L\right)\,, \tag{2.2.22}$$

[2]We omit the dependence on τ where redundant.

via an S transformation we can turn this into the L-twisted partition function

$$S \cdot \mathcal{Z}(L) = \mathcal{Z}_L \tag{2.2.23}$$

and the constraint is that the twisted Hilbert space should have a well defined interpretation, namely it should split in irreducible representations of $\mathcal{A}$ with positive integer multiplicities. We can parametrize the action of the line L with some unknown coefficients $X_{i,\bar{i}}$ so that

$$L \cdot \Phi_{i,\bar{i}} = X_{i,\bar{i}} \Phi_{i,\bar{i}} \tag{2.2.24}$$

where $\Phi_{i,\bar{i}}$ is a physical primary of the theory. Notice that this is the practical step where we are using that L commutes with the chiral algebra. Then we have

$$\mathcal{Z}(L) = \sum_{i,\bar{i}} M_{i,\bar{i}} X_{i,\bar{i}} \chi_i \chi^*_{\bar{i}} \ \stackrel{S}{\to} \ \mathcal{Z}_L = \sum_{i,\bar{i},j,\bar{j}} M_{i,\bar{i}} X_{i,\bar{i}} S_{i,j} S^*_{\bar{i},\bar{j}} \chi_j \chi^*_{\bar{j}} \equiv \sum_{j,\bar{j}} N_{j,\bar{j}} \chi_j \chi^*_{\bar{j}} \tag{2.2.25}$$

where $N_{j,\bar{j}} \in \mathbb{N}$ are the multiplicities in the twisted sector. The constraint is

$$\sum_{i,\bar{i}} M_{i,\bar{i}} X_{i,\bar{i}} S_{i,j} S^*_{\bar{i},\bar{j}} = N_{j,\bar{j}} \,, \tag{2.2.26}$$

which in general can be difficult to solve. For the diagonal modular invariant $M_{i,\bar{i}} = \delta_{i,\bar{i}}$ however the Verlinde formula provides a solution

$$X_i = \frac{S_{k,i}}{S_{0,i}} \,, \tag{2.2.27}$$

so that we have a line L_k for every representation k of $\mathcal{A}$ acting as

$$L_k \cdot \Phi_{i,i} = \frac{S_{k,i}}{S_{0,i}} \Phi_{i,i} \,. \tag{2.2.28}$$

The twisted sector multiplicities for the line L_k are given by

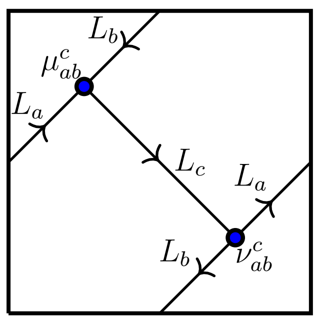
$$N^k_{j,\bar{j}} = \sum_i \frac{S_{k,i} S_{i,j} S^*_{i,\bar{j}}}{S_{0i}} \,, \tag{2.2.29}$$

and are positive integers. The fusion of these lines is also governed by the same coefficients

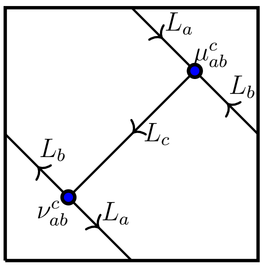
$$L_k \otimes L_j = \sum_i N^i_{k,j} L_i \,. \tag{2.2.30}$$

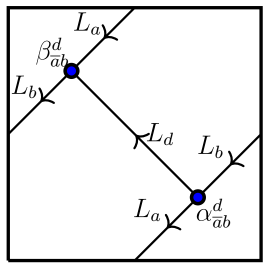
This analysis provides us with the set of lines as well as their action on local operators, however this is not the full story, as we also have an action on twisted sectors. It is not hard to see that using T on the twined partition function we obtain the action of a line L on its own twisted sector, but to see how L acts on other twisted Hilbert spaces requires more work. We can repeat the bootstrap analysis considering a more general network of lines on the torus. Representing the torus as a rectangle with opposite sides identified we define as

$$[\mathcal{Z}_{L_b,L_c}(L_a)]_{\mu,\nu} =$$



the L_b-twisted partition function acted on by L_a (we take time running from bottom to top). The line L_c is introduced to resolve the network into trivalent junctions, thus we need $N_{a,b}^c > 0$. The morphisms $\mu_{a,b}^c$ and $\nu_{a,b}^c$ are also arbitrary and each choice corresponds to a different partition function. Via an S transformation, which amounts to a rotation of 90 degrees, we obtain

$S \cdot [\mathcal{Z}_{L_b,L_c}(L_a)]_{\mu,\nu} =$ 

$= \sum_{d,\alpha,\beta} \left[F^{\overline{a}}_{\overline{b},\overline{a},b}\right]_{(\overline{c},\mu,\nu);(d,\alpha,\beta)}$  ,

therefore, in formulas,

$$S \cdot [\mathcal{Z}_{L_b,L_c}(L_a)]_{\mu,\nu} = \sum_{d,\alpha,\beta} \left[F^{\overline{a}}_{\overline{b},\overline{a},b}\right]_{(\overline{c},\mu,\nu);(d,\alpha,\beta)} \left[\mathcal{Z}_{L_a,L_{\overline{d}}}(L_{\overline{b}})\right]_{\alpha,\beta} . \tag{2.2.31}$$

These equations contain the basic one we have discussed above and generalize it to bootstrap the action of topological lines on twisted sectors. Even in RCFTs these equations can be hard to solve in complete generality, thus in irrational theories, unless one focuses on some particular type of defects making an ansatz for their action, there is very little hope of being able to solve them. Nevertheless there are ways of hunting for non-invertible line defects in irrational theories, at least for those having a conformal manifold, more on this in 3.

The importance of defects. An important point that is highlighted by this approach is that to properly define a symmetry defect, it is not enough to define a topological operator, but there are more constraints this operator has to fulfill (see also [107]). Indeed, in a CFT setting, we could define a topological operator simply as any operator that acts on conformal families and commutes with the Virasoro generators, with no constraints on its action. However, as we have seen above, there are bootstrap equations the defect need to satisfy. Concretely, we require the twisted sectors to have a well-defined Hilbert space interpretation. If a defect $\mathcal{L}$ satisfies this constraint, another defect that differs only by an overall normalization, $\mathcal{L}' = x\mathcal{L}$ with x a generic real number, does not. Hence $\mathcal{L}'$ does not define a good defect. As an example of why this is important consider the case of the Ising CFT, this has three topological defect line operators $\mathbb{1}, \eta, \mathcal{D}$ with fusion rules

$$\eta \otimes \eta = \mathbb{1}\,, \quad \eta \otimes \mathcal{D} = \mathcal{D} \otimes \eta = \mathcal{D}\,, \quad \mathcal{D} \otimes \mathcal{D} = \mathbb{1} + \eta\,, \tag{2.2.32}$$

corresponding to a Tambara-Yamagami category with symmetry $\mathbb{Z}_2$ generated by η [12]. The linking action on the three primaries $1, \sigma, \epsilon$ is

	1	σ	ϵ
η	1	-1	1
$\mathcal{D}$	$\sqrt{2}$	0	$-\sqrt{2}$

and the two nontrivial defects have a well-defined twisted Hilbert space. Consider now the linear combination

$$L = \frac{1}{2}\eta + \frac{1}{\sqrt{2}}\mathcal{D} - \frac{1}{2}\mathbb{1}\,, \tag{2.2.33}$$

it is easy to check that this line is invertible:

$$L \otimes L = \mathbb{1}\,. \tag{2.2.34}$$

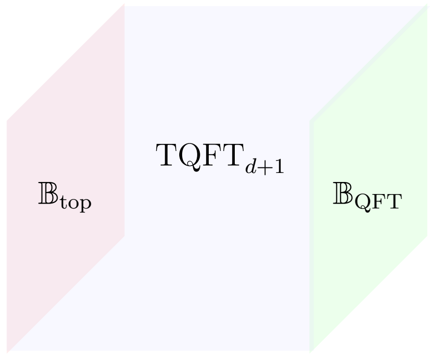
What is preventing us from taking this to be a defect of the theory instead of the Tambara-Yamagami line $\mathcal{D}$? As we have probably stressed enough in this section the reason why this line does not define a consistent defect in the theory is that it does not solve the bootstrap equations, hence does not have a good twisted sector. In general only linear combinations with positive integer coefficients guarantee that the defects we obtain are well defined. As operators on the Hilbert space any linear combination of defects is well defined, but the bootstrap equations impose further constraints.

2.3 The Holographic perspective on symmetries

In this section we discuss a very useful tool to describe the symmetry structure of general QFTs in general dimensions: the Symmetry Topological Field Theory [9, 116–119][3]. This construction provides a very powerful framework to study the realization of symmetry and anomalies in QFT as well as possible phases of the theory [2,7,8,67,69,98,99,109–112,133,195–198]. The general idea is somewhat a generalization of anomaly inflow [199–201], according to which the anomalies of a d-dimensional theory are captured by an invertible field theory (i.e. with a 1-dimensional Hilbert space on every manifold) in $(d+1)$-dimension. Indeed, instead of considering an invertible theory, we can couple the QFT to a full-fledged TQFT in one dimension higher. Intuitively one can think that the coupling occurs via the symmetries of the boundary QFT. Namely we can imagine turning on a background on the boundary and extend it dynamically to one dimension higher, where by dynamically we mean that we are essentially gauging the symmetry of the boundary theory in $(d+1)$ dimensions. Imposing Dirichlet boundary conditions $\mathbb{B}_{\text{QFT}}$ on the TQFT we can interpret the boundary value of the bulk fields as a background for the QFT. This constructs a $(d+1)$ dimensional theory, which depends on the extension in the bulk, and is a reminiscient set-up of that of relative theories [202]. To recover the original d-dimensional theory we need to trivialize the dependence on the extra dimension. This can be done imposing a topological boundary condition $\mathbb{B}_{\text{top}}$ on the other side

[3]see also [193, 194] for a condensed matter theory perspective.

of the set-up, resulting in the so-called the sandwich picture



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Since the extra boundary condition is topological we can shrink the interval and recover the d-dimensional theory. The TQFT$_{d+1}$ is called the Symmetry theory (or SymTFT in short) for the original quantum field theory and it captures all the information contained in the boundary symmetry. We should, however, be a bit more precise. In general a QFT may have various symmetries and the construction outlined above can be applied to any of those, not necessarily the full symmetry. Therefore it is more correct to refer to the TQFT$_{d+1}$ as the SymTFT for a specific symmetry of the boundary theory. Symmetries in QFT are described by (higher) categories, therefore this construction provides a physical argument as to why there should exist, for a given symmetry category $\mathcal{C}$ in d-dimensions, a TQFT $\mathcal{Z}(\mathcal{C})_{d+1}$ in one dimension more interpreted as its SymTFT.

Generally a TQFT can have more than one topological boundary condition, thus we often have many possibilities for $\mathbb{B}_{\text{top}}$, each of which corresponds to a global variant of the theory. Here we define the global variants of the theory as the allowed boundary conditions of the SymTFT.

We still have not explained the meaning of the SymTFT or how it captures the information of the boundary symmetry. Moreover, we have only given a physical construction of this theory without a very precise definition. In order to fill those gaps we need to give a quick review of what TQFTs are, what are their observables, and also what we mean by topological boundary conditions.

2.3.1 Topological Quantum Field Theories

TQFTs are among those few QFTs that can be given a rigorous axiomatic definition [203] (see e.g. [204] for a nice review) which is also computationally useful, especially in low dimensions. Intuitively TQFTs are QFTs which do not depend on the metric on spacetime, hence have a vanishing stress energy tensor. This implies that all their observables are independent of distances on spacetime and only the topological classes of the objects involved are meaningful. We start by giving a lightning review of the axiomatic definition, then proceed and discuss examples that highlight aspects important for our discussion.

Axiomatic TQFTs. Formally TQFTs in d dimensions are defined as symmetric monoidal functors from a category of oriented d-dimensional bordisms to that of complex vector spaces.

Let us quickly introduce those ingredients. Objects in the category of bordisms are $(d-1)$-dimensional manifolds. In general we can equip those manifold with extra structures depending on the TQFT. For instance we can add a G-principal bundle, for G a discrete group, if we want to describe backgrounds for a G 0-form symmetry or we can input a spin structure to describe spin TQFTs. For simplicity let us focus on oriented manifolds only. A morphism between two manifolds M_{d-1} and M'_{d-1} in the bordisms category is d-dimensional manifolds M_d with boundaries such that $\partial M_d = M_{d-1} \sqcup \overline{M'}_{d-1}$, where the bar denotes orientation reversal. We can view the d-dimensional manifold as a map $M_d : M_{d-1} \to \overline{M'}_{d-1}$ with a particular choice of an in and an out boundary. The usual notation for the category of d-dimensional oriented bordisms is $\text{Bord}_d^{\text{SO}}$, with SO standing for the structure group of the tangetn bundle. In presence of extra structures on M_{d-1} those are extended in d-dimensions and one enriches the notation to denote the category of the associated bordisms. The category of (finite dimensional) complex vector spaces instead has as objects complex vector spaces, which are all built as direct products of the unique simple object $\mathbb{C}$. Morphisms instead are linear maps between vector spaces and the category is denoted by $\text{Vec}_{\mathbb{C}}$. Then a d-dimensional TQFT is a functor

$$Z : \text{Bord}_d^{\text{SO}} \to \text{Vec}_{\mathbb{C}} \,. \tag{2.3.1}$$

Physically this means the following. To every object in $\text{Bord}_d^{\text{SO}}$ the TQFT assigns an object of $\text{Vec}_{\mathbb{C}}$, this is simply the fact that to every $(d-1)$-dimensional manifold the theory assigns an Hilbert space

$$Z(M_{d-1}) = \mathbb{H}(M_{d-1}) \,. \tag{2.3.2}$$

To a morphism $M_d = M_{d-1} \sqcup \overline{M'}_{d-1}$ of $\text{Bord}_d^{\text{SO}}$ the functor assigns a linear map between the vector spaces associated to M_{d-1} and M'_{d-1},

$$Z(M_d) : \mathbb{H}(M_{d-1}) \to \mathbb{H}(M'_{d-1}) \,, \tag{2.3.3}$$

physically this is the time evolution of a state in $\mathbb{H}(M_{d-1})$ to a state in $\mathbb{H}(M'_{d-1})$. The adjectives symmetric and monoidal attached to the functor Z are needed to guarantee that the path integral on disjoint manifolds factorizes on the various components, that it is compatible with the tensor product of Hilbert spaces and that gluing manifolds along a common boundary amounts to the compoisition of linear maps.

We can make contact with the path integral of the theory as follows. Consider a d-dimensional manifold N_d with $\partial N_d = \overline{M}_{d-1}$, the path integral on M_d produces the wave-functional of a state in $\mathbb{H}(M_{d-1})$. This is the same result as the action of the functor Z which produces the map

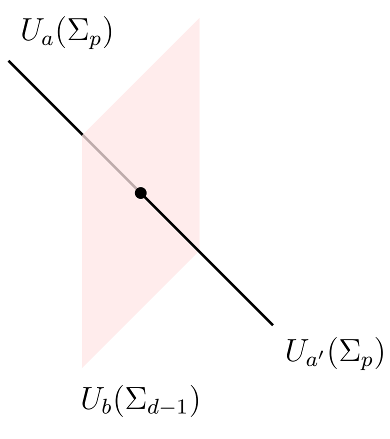
$$Z(N_d) : \mathbb{C} \to \mathbb{H}(M_{d-1}) \,. \tag{2.3.4}$$

A special choice is $N_d = D_d$ the d-dimensional disk, the path integral on this topology produces the Hartle-Hawking state. Changing the internal topology of N_d with a fixed boundary, we can probe the Hilbert space on ∂N_d. In general dimensions the difficulties in concretely using this approach lies in the complexity of the classification of the topological classes of d-dimensional bordisms (even more so if we add more structures). In low dimensions however the axiomatic approach can be fruitful. For instance in $2d$ one has a finite amount of data that uniquely specify the TQFT (this stems from the possibility of constructing every $2d$ manifolds gluing pairs of pants), and one can show important results such as the equivalence of TQFTs and symmetric Frobenius algebras [205].

TQFTs, symmetries and operators. The axiomatic definition is a very convenient framework as it packages in an efficient way the properties of the path integral. However, for our purposes, it is more useful to understand the operator content and correlation of the theory. As we already mentioned, TQFTs have a vanishing stress-energy tensor thus all the operators are necessarily topological, then, following the modern philosophy, TQFTs are essentially determined by their symmetries. In other words all operators in a TQFT are generators of a symmetry and, at the same time, are charged under another. Let us be more precise. In general a TQFT can have operators of all possible codimensions, each corresponding to a higher-form symmetry (not necessarily invertible). Consider for instance operators $U_a(\Sigma_{d-p-1})$ and $U_{\hat{a}}(\Sigma_p)$, in codimensions $p+1$ and $d-p$ respectively, both these classes of operators are topological hence generate a p-form and a $(d-p-1)$-form symmetries. However the two classes also have a natural linking configuration which imply that one class of operators is charged under the other and viceversa. Explicitly we can write, taking Σ_{d-p-1} and Σ'_p linking once,

$$U_a(\Sigma_{d-p-1})U_{\hat{a}}(\Sigma'_p) = \chi(a,\hat{a})U_{\hat{a}}(\Sigma_p)U_a(\Sigma_{d-p-1}) \tag{2.3.5}$$

where $\chi(a,\hat{a})$ is a, generically complex, number encoding the action of one operator on the other. Notice that the labels $a, \hat{a}$ can be interpreted as the charges of the operators. Indeed, in order for an operator to be detectable in correlation functions of a TQFT (hence distinguishable from the identity), it has to be charged under a symmetry. Thus far in this thesis we have only considered linking configurations as possible interactions of topological defects. In $2d$ this is not a problem if we also discuss junctions of line defects. However in higher dimensions this is no longer true: there are other configurations of topological defects of various dimensions which we need to consider. In TQFTs we can always interpret those configurations as a topological defect acting on another one, for instace we can have a 0-form symmetry acting on lower dimensional extended operators. This is realized by a defect $U_a(\Sigma_p)$ piercing a 0-form symmetry generator $U_b(\Sigma_{d-1})$ and getting transformed to a new defect $U_{a'}(\Sigma_p)$



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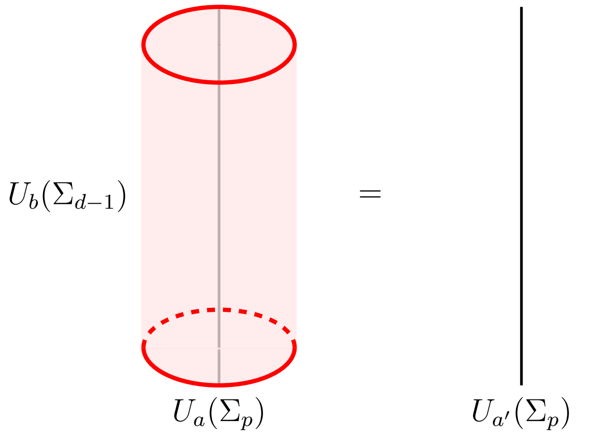
Alternatively we can consider a cylinder of $U_b(\Sigma_{d-1})$ around $U_a(\Sigma_p)$ (with other directions compactified), shrinking the cylinder results in a new defect $U_{a'}(\Sigma_p)$



0-form symmetries act by linking on local operators, however, in TQFTs we can often get rid of local operators simply focusin on a specific vacuum of the theory. One might be then tempted to say that in each vacuum the 0-form symmetries act trivially but, due to the actions we have just mentioned, this is not necessarily true. Moreover, in the absence of local operators, it is not necessary for the 0-form symmetry defects to be of codimension 1: they can be constructed as a mesh of lower-dimensional defects. This is the idea of condensation defects [13]: we can gauge a discrete higher form symmetry in higher codimension to construct a topological defect. Condensation defects are important for many reasons. For once, especially in higher dimensions, certain non-invertible defects are non invertible precisely for the appereance of condensation defects in fusion rules (see e.g. [7, 8, 57, 63, 65]). Another reason is that in $3d$ TQFTs with only line operators it is a theorem that all 0-form (unitary) symmetries are implemented by condensation defects [13], this has also implications for the classifications of modular invariants in RCFT [17].

Besides 0-form symmetries acting on lower dimensional defects we can have more intricate configurations. As an example consider surface defects in $4d$, these link with lines and hence generate a 1-form symmetry. However we can also consider the configuration of three surfaces $\Sigma_2^{(i)}$, $i = 1, 2, 3$ such that $\Sigma_2^{(3)}$ links with the line $\Sigma_1 = \Sigma_2^{(1)} \cap \Sigma_2^{(2)}$ on which the two other surfaces intersect. This triple linking configuration carries information that can distinguish TQFTs with the same operator content [206](we will expand on this in the following subsections). We will not attempt to classify all possible non-trivial configurations, which seems a rather complicated task. Rather, to provide a more concrete perspective on the topics we just discussed, we now discuss examples of TQFTs that are relevant for the rest of this thesis.

Discrete Abelian gauge theories. A gauge theory for a discrete Abelian 0-form symmetry $\mathbb{A}$ has two types of extended operators. The first one can be thought of as the Wilson lines for the gauge field of $\mathbb{A}$, these are labelled by representations of $\mathbb{A}$. In the discrete Abelian case these representations are elements of the Pontryagin dual group $\widehat{\mathbb{A}} = \text{Hom}(\mathbb{A}, U(1)) \simeq \mathbb{A}$, where the isomorphism is non-canonical, we will denote these Wilson lines as $W_{\hat{a}}(\Sigma_1)$. The second type of operators can be defined as disorder operators for the Wilson lines. This means that we pick a submanifold Σ_{d-2} in spacetime and the insertion of an operator $U_a(\Sigma_{d-2})$ is defined as the integral path of the configurations of the gauge field $\mathbb{A}$ such that, for every cycle Σ_1 linking with Σ_{d-2} we have

$$W_{\hat{a}}(\Sigma_1) = \chi_{\hat{a}}(a)\,, \tag{2.3.6}$$

where $\chi_{\hat{a}}(a)$ is the character of the element $a \in \mathbb{A}$ of the representation $\hat{a} \in \widehat{\mathbb{A}}$. Thus the discrete $\mathbb{A}$ gauge theory has a symmetry $\mathbb{A}^{[1]} \times \widehat{\mathbb{A}}^{[d-2]}$ with the apices denoting the degree of the higher-form symmetries. By definition there is a non-trivial linking, or braiding,

$$\mathcal{B}_{(a,\hat{a}),(b,\hat{b})} = \chi_{\hat{a}}(b)\chi_{\hat{b}}(a)\,, \qquad (a,\hat{a}),(b,\hat{b}) \in \mathbb{A} \times \widehat{\mathbb{A}}\,. \tag{2.3.7}$$

which simply indicates that the generators of one symmetry are charged under the other and viceversa. Depending on $\mathbb{A}$ the theory can also have 0-form symmetries. One that is always present is charge conjugation, namely

$$C \cdot (a,\hat{a}) = (a^{-1}, \hat{a}^{-1}) \tag{2.3.8}$$

which respects the braiding in the sense that

$$\mathcal{B}_{C\cdot(a,\hat{a}),C\cdot(b,\hat{b})} = \mathcal{B}_{(a,\hat{a}),(b,\hat{b})}\,, \tag{2.3.9}$$

and is implemented by a condensation defect.

Now, every Abelian group is isomorphic to a product of $\mathbb{Z}_n$'s with different n's. In this sense the $\mathbb{Z}_n$ gauge theory is a building block for discrete Abelian gauge theories. To this extent it is often useful to use the lagrangian formulation of the $\mathbb{Z}_n$ gauge theory given in [207]. One simply writes (in Euclidean signature)

$$S = \frac{in}{2\pi}\int_{M_d} A_1 \wedge dA_{d-2}\,, \tag{2.3.10}$$

where A_1 and A_{d-2} are a standard $U(1)$ gauge field and an higher-form one. The action is gauge invariant if and only if $n \in \mathbb{Z}$. In this formulation the operator content is clear: there are no gauge invariant local operators, we only have Wilson lines and Wilson surfaces, both of which are labelled by elements of $\mathbb{Z}_n$

$$W_a(\Sigma_1) = e^{ia\int_{\Sigma_1} A_1}\,, \qquad U_b(\Sigma_{d-2}) = e^{ib\int_{\Sigma_{d-2}} A_{d-2}}\,, \quad a,b \in \mathbb{Z}_n\,. \tag{2.3.11}$$

One can see that the operators W_n and U_n are trivial in correlation functions computing the linking of a Wilson surface and a Wilson line,

$$\langle W_a(\Sigma_2)U_b(\Sigma_{d-2})\rangle = e^{2i\pi \mathrm{Lk}(\Sigma_2,\Sigma_{d-2})\frac{ab}{n}}\,, \tag{2.3.12}$$

where $\mathrm{Lk}(\Sigma_2,\Sigma_{d-2})$ is the linking number of the surfaces involved. This justifies labeling operators by elements of $\mathbb{Z}_n$. Charge conjugation now acts reversing the signs of both A_1 and A_{d-2}, and is an evident 0-form symmetry of the theory acting on lines by $(a,b) \mapsto (-a,-b)$. This discussion can be repeated for the gauge theory of an higher-form symmetry. For instance we can write the action

$$S = \frac{in}{2\pi}\int_{M_d} A_p \wedge dA_{d-p-1} \tag{2.3.13}$$

for the $\mathbb{Z}_n$ p-form gauge theory, from which the operator content and braiding follow as in the 0-form symmetry case.

For certain choices of d and p we might be able to write more general topological actions. For instance take $d=3$ and $p=1$, then we may write

$$S = \frac{in}{2\pi}\int_{M_3} A_1 \wedge dB_1 + \frac{ik}{4\pi}\int_{M_3} A_1 \wedge dA_1 \tag{2.3.14}$$

and we have modified the initial BF action with the addition of a Chern-Simons term. Gauge transformations are standard an impose $n.k \in \mathbb{Z}$. Gauge invariant operators are Wilson lines of A_1 and B_1, which satisfy the constraints

$$e^{in\int_{\Sigma_1} A_1} = e^{ik\int_{\Sigma_1} A_1} = e^{in\int_{\Sigma_1} B_1} = 1\,. \tag{2.3.15}$$

We have a $\mathbb{Z}_n$ 1-form symmetry generated by the lines of B_1, while the Wilson lines of A_1 generate a $\mathbb{Z}_{\mathrm{gcd}(k,n)}$ 1-form symmetry. The Chern-Simons term also modifies the linking of the lines, inducing a self-linking for the Wilson lines of A_1.

Another example is for $d=4$ and $p=2$ in which case we can write

$$S = \frac{in}{2\pi}\int_{M_4} B_2 \wedge dC_1 + \frac{inp}{4\pi}\int_{M_4} B_2 \wedge B_2\,, \tag{2.3.16}$$

this extra term modifies gauge transformations of the 1-form C_1 mixing them with those of B_2:

$$B_2 \mapsto B_2 + d\lambda_1\,, \qquad C_1 \mapsto C_1 - p\lambda_1 + d\lambda_0\,. \tag{2.3.17}$$

The quantization condition for p is $np \in 2\mathbb{Z}$ on general manifolds and we also have the identification $p \sim p + 2n$ [207]. The gauge invariant operators of the theory are the surfaces

$$U_a(\Sigma_2) = e^{ia\int_{\Sigma_2} B_2}\,, \tag{2.3.18}$$

and the non-genuine lines

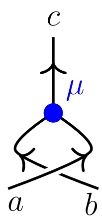
$$\widetilde{W}_b(\Sigma_1, \Sigma_2) = e^{ib\int_{\Sigma_1} C_1 + ibp\int_{\Sigma_2} B_2} \tag{2.3.19}$$

with $\partial\Sigma_2 = \Sigma_1$. Since the surface operator $U_n(\Sigma_2)$ is trivial the genuine lines are those for which bp is a multiple of n, namely

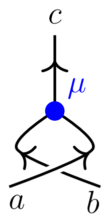
$$W_a(\Sigma_1) = \left(\widetilde{W}_{n/\mathrm{gcd}(n,p)}\right)^a \tag{2.3.20}$$

with $a \in \mathbb{Z}_{\mathrm{gcd}(p,n)}$. Notice that when a is a multiple of p the surface operators $U_a(\Sigma_2)$ can be cut open on a Wilson line, so that there are $\mathrm{gcd}(p,n)$ non-trivial closed surface operators.

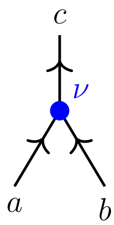
$3d$ **TQFTs and Modular Tensor Categories.** In low dimensions $d = 2, 3$ TQFTs are very well understood. We already mentioned that in $d = 2$ we even have a classification of TQFTs in terms of Frobenius algebras, we now quickly describe the $3d$ case. As we have already mentioned we can assume that the TQFT has no local operators without loss of generality, moreover we know that surface operators in $3d$ implement 0-form symmetries and, in a theory with only line defects, the expectation is that every 0-form can be constructed by condensing an appropriate set of lines [13]. Then let us focus only on the line content of the theory, the punchline is that those defects are described by a Modular Tensor Category (MTC), see e.g. [17,28,192,208,209] for an extensive treatment. In $2d$ we have seen that line defects form a Fusion category, in $3d$ this is still true, namely we again have a fusion product and a set of F-symbols, but we have more structure to take into account. In particular, the presence of an extra dimension allows us to consider configuration of braided lines which are all built from the basic diagram



The possibility of distentangling lines leads to the definition of a new datum in the category called the R-matrix:

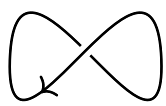


$$= \sum_\nu \left[R^c_{ab}\right]_{\mu,\nu}$$



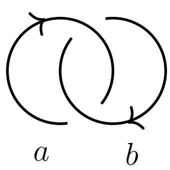
This additional datum has to satisfy a compatibility condition with the F-symbols called the Hexagon equation, see e.g. [28]. Similarly to the F-symbols the R-matrix depends on the basis we choose for the morphisms in trivalent junctions. An gauge equivalence class of compatible F-symbols and R-symbols gives a braided category, in order to obtain a full fledged MTC there are more constraints to be satisfied. To explain those we have to define two crucial observables, which play a central role in $3d$ TQFTs, and are invariant under chenges of basis in trivalent vertices. One observable is given by the spins, a set of numbers θ_a we can attach to each simple line a. These are defined by the diagram

$$d_a \theta_a =$$



where d_a is the quantum dimension of the simple line a. Two lines linking define the S-matrix S_{ab}

$$\mathcal{D} S_{ab} =$$



Here $\mathcal{D}$ is the total quantum dimension of the category defined as

$$\mathcal{D} = \sqrt{\sum_a d_a^2} \tag{2.3.21}$$

where the sum runs over the simple lines of the category. In an MTC the spins are phases $|\theta_a| = 1$ and the S-matrix is unitary. The adjective modular comes from the fact that the S matrix, togheter with the T matrix defined as

$$T_{ab} = e^{-2i\pi c_-/24}\theta_a \delta_{ab} \tag{2.3.22}$$

form a representation of $SL(2,\mathbb{Z})$, namely

$$(ST)^3 = S^2 = C\,, \qquad C^2 = \mathbb{1}\,. \tag{2.3.23}$$

The number c_- is the chiral central charge and is defined as

$$\frac{1}{\mathcal{D}}\sum_a \theta_a d_a^2 = e^{2i\pi\frac{c_-}{8}}\,, \tag{2.3.24}$$

while C is the charge conjugation matrix. If the data F and R, which we can use to express both the spins and the S-matrix, satisfy these extra conditions we obtain an MTC that describes line operators in generic $3d$ TQFTs (at least when we are considering a finite number of lines). Quantities such as the S-matrix and the spins, being gauge invariant, can be computed as correlators in the TQFT.

The framework of MTCs can be enriched including 0-form symmetries acting as automorphisms of the set of simple lines, this is described by G-crossed categories [28].

Non-Compact TQFTs. As a final class of examples of TQFTs we want to describe a simple case of a TQFT with a continuous/non-compact spectrum of topological operators. These are relevant for the SymTFT description of continuous symmetries [206, 210, 211] (see also [212]), have nice holographic properties (see the second half of chapter 4 in this thesis) and also come into play in the structure of generalized symmetries in 4 dimensions [213]. To our knowledge, there is no axiomatic definition for this type of theories, which in fact seem to have divergent partition functions on general manifolds (see B.8). However normalized correlators are always valid observables, and, from a SymTFT perspective, these are the objects of central importance. The idea to describe these TQFTs is to consider actions that look very similar to those of finite group gauge theories, but relax the compactness of one (or both) the fields involved. As an example consider

$$S = \frac{i}{2\pi}\int_{M_3} a_1 \wedge dA_1\,, \tag{2.3.25}$$

where A_1 is a standard $U(1)$ gauge field while a_1 is an $\mathbb{R}$ gauge field. The difference between $\mathbb{R}$ and a $U(1)$ gauge fields lies in their gauge transformations: $U(1)$-valued gauge transformations include what are called large gauge transformations. In the case of a 1-form gauge field A_1 these are gauge parameters λ_0 such that $\int_{\Sigma_1} d\lambda \neq 0$ for a 1-cycle Σ_1. Of course their integral cannot be arbitrary but is quantized in certain units, generically one takes $\int_{\Sigma_1} d\lambda \in 2\pi\mathbb{Z}$.

For higher form gauge fields one can think of large gauge transformations as not globally defined $U(1)$ gauge fields of lower degree. Instead, $\mathbb{R}$ gauge fields do not have such large gauge transformation. This has an important impact on the spectrum of operators of the theory, indeed the quantization of the charges of Wilson lines is a direct consequence of the presence of large gauge transformations. If we disallow those we have a valid line operator for every real number. This means that the gauge invariant operators of the theory we are considering are

$$W_n(\Sigma_1) = e^{in\int_{\Sigma_1} A_1}\,, \qquad U_\alpha(\Sigma_1) = e^{i\alpha\int_{\Sigma_1} a_1} \tag{2.3.26}$$

with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. However it turns out that not all values of α give independent operators, indeed computing the linking correlator

$$\langle W_n(\Sigma_1) U_\alpha(\Sigma_1')\rangle = e^{2i\pi \mathrm{Lk}(\Sigma_1,\Sigma_1')n\alpha} \tag{2.3.27}$$

we see that α and $\alpha+1$ are identified. This leaves us with a $U(1)^{[1]} \times \mathbb{Z}^{[1]}$ symmetry in the theory. Many other interesting examples are discussed, from a different perspective, in the second half of Chapter 4.

2.3.2 Boundary conditions

A central ingredient in any holographic construction is the boundary conditions. In the setup outlined at the beginning of this section, we mentioned both a topological boundary condition and a non topological one (often called physical). The study of boundary conditions in general QFTs is very complicated, many results are known in $2d$ CFTs, where it is possible, in certain cases (especially in RCFTs), to classify all conformal boundary conditions that preserve some symmetry [191, 214–218]. In higher dimensions the situation is much more complicated, and there are no equivalent results. However, in the SymTFT setup, there are a couple of caveats. On the one hand, the non-topological boundary is not really a boundary condition for the TQFT, but we have to allow for more general possibilities. Indeed, as we have already mentioned, it is best to think of the physical boundary as a coupling of a QFT to the TQFT in one dimension higher, which is equivalent to saying that we are not only putting some Dirichlet boundary condition for the TQFT fields but also adding some edge modes. This is because we want the SymTFT to be universal, namely to be shared by all those QFTs with same symmetry. It is of course possible, and interesting, to study what happens when we do not add those edge modes and only consider the dynamics induced by a non-topological boundary for the SymTFT. This is explored in the second half of 4. We will not discuss here the details of the general coupling between the SymTFT and the boundary degrees of freedom. Rather we will assume that such coupling allows all the bulk defects to consistently end on lower-dimensional, and possibly non-topological, operators in the physical boundary and also that we can push a bulk defect onto the boundary retaining its topological nature. In this part of the thesis we will be mostly interested in the topological boundary condition, which can be much better characterized. The intuitive reason for this is that a topological boundary preserves as much as possible of the bulk symmetries, including topological invariance. Basically a topological boundary condition is such that correlators of the whole theory do not depend on the location, or the geometry, of the boundary, but only on its topology. Let us now consider the properties we expect from such boundary conditions. In a $(d+1)$-dimensional TQFT we can have defects $U_a(\Sigma_{d-p})$ for every codimension $p = 1, \dots d-2$ (we can again not discuss local operators). In presence of a topological boundary we can push all bulk defects onto the boundary without changing correlation functions. A natural way in which we can characterize a topological boundary is in terms of the category of defects it hosts, or, equivalently, in terms of which defects become trivial on it. In this sense we can think of the operation of pushing a bulk defect onto the boundary as a map f from the category of bulk defects to those of the boundary. Intuitively, we expect this map to be surjective, namely all boundary defects are induced by those of the bulk, but not injective, i.e. some of the bulk defects are mapped to the identity on the boundary. The name of the game then is to understand which consistency conditions we have to impose on a set of bulk defects in order for them to be trivializable on the boundary. This is quite a difficult task in general, so let us proceed by examples.

Topological boundaries in $3d$ $\mathbb{Z}_n$ gauge theories. Consider the action

$$S = \frac{in}{2\pi} \int_{M_3} A_1 \wedge dB_1 \tag{2.3.28}$$

The bulk defects are Wilson lines labelled by a pair $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_n$, with braiding

$$\mathcal{B}_{(a,b),(a',b')} = e^{2i\pi \frac{ab'+a'b}{n}} , \tag{2.3.29}$$

based on our general consideration we now should look for defects that we can consistently set to the identity on the boundary. The key observation here is that if two defects link non-trivially we cannot have both of them become trivial on the boundary. The reason is that, given a link in the bulk, we can resolve it in two ways, which have to give the same result. We can first unlink them in the bulk, possibly getting a phase, and then close the slab mving the topological boundary, or we can close the slab first and unlink them after. Another condition that we have to require is that this set has to be maximal, namely all those lines not included in it have to link non-trivially with at least one line of the set. This follows from requiring that the symmetry hosted on the boundary acts fatihfully, we will see this later. These sets of defects are in one to one correspondence with subgroups $\mathbb{B} \subseteq \mathbb{Z}_n$ and, thinking of the Wilson lines of B_1 as labelled by elements of the Pontryagin dual $\widehat{\mathbb{Z}}_n$, we have

$$\mathcal{L}_{\mathbb{B}} = \{(a, \chi) \in \mathbb{Z}_n \times \widehat{\mathbb{Z}}_n : \ a \in \mathbb{B} \, , \ \chi \in N(\mathbb{B})\} \tag{2.3.30}$$

where $N(\mathbb{B})$, the normalizer of $\mathbb{B}$, is defined as

$$N(\mathbb{B}) = \{\chi \in \widehat{\mathbb{Z}}_n \ : \ \chi(b) = 1 \ \forall \ b \in \mathbb{B}\} \, . \tag{2.3.31}$$

It is not difficult to generalize this to a generic discrete gauge theory, see e.g. Appendix B.2. For a general n there can be many subgroups of $\mathbb{Z}_n$, but two choices that always exist are $\mathbb{B} = \varnothing$ and $\mathbb{B} = \mathbb{Z}_n$, which correspond to boundary conditions which trivialize respectively all Wilson lines of B_1 or A_1. In this simple case we also have a direct approach to study boundary conditions as literal boundary conditions for the fields A_1 and B_1, for instance we can set $A_1 = 0$

which corresponds to the boundary hosting all Wilson lines of B_1, but using topological defects directly allows to understand also more complicated situations in which no simple lagrangian presentation is known.

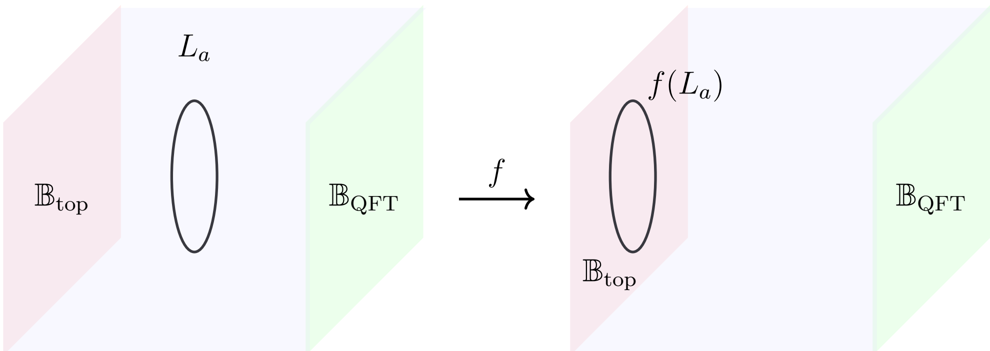
Algebras, boundary conditions and gauging The example of $3d$ discrete gauge theories can be easily generalized to higher dimensions with arbitrary Abelian groups, and the result is always that topological boundary conditions can be labeled by maximal sets of trivially braiding defects. These sets identify what are called Lagrangian Algebra objects in the category of defects [17, 219]. From a physical perspective the requirement of having trivial braiding translates into the condition for the symmetry of being anomaly free, while the maximality implies that there are no defects invariant under the action of the algebra. Let us explain this in some more detail. To gauge a symmetry we need to couple the theory to a dynamical background, for discrete symmetries this is equivalent to the insertion of topological defects along the various cycles of spacetime. In order for the gauged partition function to be well defined it should not depend on the details of the mesh of defects we are inserting. Requiring this to be the case constraints the type of defects we can insert, the precise conditions depend on both the dimensionality of defects and that of spacetime and it is difficult to make general statements for general non-invertible symmetries. For general line defect operators in 2 and 3 dimensions these conditions are very well-known. Gaugable symmetries in $2d$ are given by algebra objects in the fusion category [11], which generalize the choice of non-anomalous subgroups and of discrete torsion. In $3d$ gaugable 1-form symmetries correspond to connected commutative special Frobenius algebras [17, 219], which are not necessarily maximal. When also maximality is obeyed they are called Lagrangian Algebras. Some more techincal details on these procedures in 2 and 3 dimensions are collected in B.1.Note that, by definition, after gauging a Lagrangian algebra the theory becomes trivial as there are no defects invariant under it. Therefore no object of the original category survives the gauging. One might object that, as it always happens for discrete symmetries, the gauged theory necessarily has a dual symmetry, however in the cases we are considering this dual symmetry has no non-trivial charged objects on which it can act.

When the symmetries involved are invertible, one can proceed "by hand" and figure out the set of topological boundary conditions and Lagrangian algebras also in higher dimensions, however the non-invertible cases are more difficult to treat, though there exist definitions for algebras in certain higher categories [43, 44]. The general picture we get is that there is a correspondence between topological boundary conditions and Lagrangian algebra objects, namely maximal gaugable symmetries. Physically we can think of this correspondence as follows. Divide space-time in two halves, separated by the trivial interface. On one half we can gauge a Lagrangian algebra, namely insert a fine enough mesh of defects, this has the effect of trivializing the theory on that half of spacetime. On the interface we allow the defects of the Lagrangian algebra to terminate topologically. What was originally the trivial interface now separates the trivial theory from the starting TQFT, thus providing a topological boundary condition [26, 220–222]. One might be tempted now to state that there is a one-to-one correspondence between topological boundary conditions and Lagrangian algebras. This, however, is not true as just stated, due to the freedom of stacking on the topological boundary a decoupled TQFT to obtain a different boundary. Moreover, since stacking of TQFTs is not an equivalence relation, we cannot eliminate this freedom by taking equivalence classes. Therefore, to keep our discussion simple, in what follows we will focus directly on lagrangian algebras rather than boundary conditions.

2.3.3 The SymTFT

We are finallyready to present the SymTFT approach. The setup is the same discussed at the beginning of this section, a slab with two boundaries. The topological boundary $\mathbb{B}_{\text{top}}$ corresponds to a bulk Lagrangian algebra $\mathcal{L}$ whose defects are allowed to terminate on the boundary. As the SymTFT is determined by the symmetry $\mathcal{C}$ of the QFT, which coincides with the category of defects hosted by $\mathbb{B}_{\text{top}}$, we will denote by $\mathcal{Z}(\mathcal{C})$ the category of bulk defects.

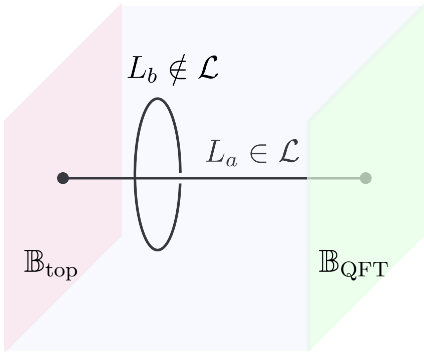
Symmetry generators and Charged objects. The defects of the bulk TQFT which are not allowed to terminate on the topological boundary (i.e. do not participate in the Lagrangian algebra $\mathcal{L}$), hence remain as non-trivial defects on the boundary, generate the symmetry $\mathcal{C}$ that acts on the QFT.



Roughly speaking we can think of those defects as elements of the "quotient" of the set of bulk defects with respect to the Lagrangian algebra corresponding to the boundary, namely $\mathcal{C} = \mathcal{Z}(\mathcal{C})/\mathcal{L}$. With the intuition that, if two bulk defects are related by fusion with a defect in the algebra, they must be identified on the boundary. For invertible symmetries this is precise, since we can define the quotient of the group of bulk symmetries with respect to one of its Lagrangian subgroups, for non-invertible symmetries the situation is technically more complicated. For $3d$ TQFTs of lines the defects confined on a topological boundary $\mathbb{B}_{\text{top}}$ form the module category of the Lagrangian algebra $\text{Mod}_{\mathcal{Z}(\mathcal{C})}(\mathcal{L})$ [219], so that $\text{Mod}_{\mathcal{Z}(\mathcal{C})}(\mathcal{L}) = \mathcal{C}$ as fusion categories. The expectation is that a similar statement should also hold in higher dimensions and for higher categories [39, 50, 223, 224]. In $3d$ the map $f : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ induced by pushing the bulk defects onto the boundary is a (forgetful) functor between the bulk MTC and the boundary fusion category (see e.g. [36]).

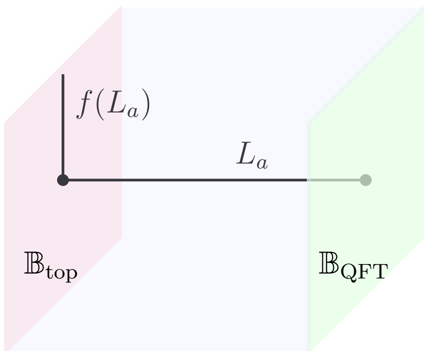
Charged objects are realized as bulk defects of the Lagrangian algebra stretching between the two boundaries. Upon closing the slab these correspond to non-topological operators of the

QFT. The action of $\mathcal{C}$ on those operators is obtained by letting the defects link in the bulk



resolving this configuration in terms of the categorical data in $\mathcal{Z}(\mathcal{C})$ yields the action of $\mathcal{C}$. In general a bulk defect could have various consistent endpoints on the two boundaries, so that the picture drawn can describe families of operators on the boundary. For example, the lines of a $3d$ SymTFT for a $2d$ CFT can end, at the physical boundary, on a whole conformal family and not just on the primary operators. Similarly, there could be several possible topological endpoints on $\mathbb{B}_{\text{top}}$, leading to physical operators with different properties.

In this picture we can also describe twisted sectors operators and the symmetry action on them. To construct a twisted sector operator we can take a bulk defect L_a not participating in the algebra and let it end on the physical non-topological boundary. Since this bulk defect cannot end topologically on $\mathbb{B}_{\text{top}}$ it does not become trivial on the boundary, rather, it reduces to a defect $f(L_a)$ of $\mathcal{C}$



Closing the slab we obtain an operator in the twisted sector of the $f(L_a)$. The action of $\mathcal{C}$ on those operators can again be described by appropriately linking the bulk defects with the configuration above. Also in this context there could be several choices for the endpoints of the defects on the two boundaries that lead to different physical operators. Another description of non-local boundary operators arises compactifying the SymTFT to lower dimensions, in this setup twisted sector operators are described by topological boundary conditions for the reduced TQFT [224].

Global Variants. A SymTFT can have several Lagrangian algebras[4], thus to any given SymTFT we can associate various QFTs in one dimension less. These define the possible global variants of the boundary theory, and the common lore is that any two such variants are related by a generalized discrete gauging operation (or topological manipulation). In other words, starting from any given global variant with a certain symmetry category, we can reach all the other ones by successive gaugings, possibly with the inclusion of discrete torsion. This can be explicitly checked in SymTFTs with invertible defects, see e.g. [7, 66, 116, 133] and Chapter 4 of this thesis for some examples. Again the situation is more involved for non-invertible symmetries, and most of the results are in low dimensions as we will discuss momentarily. Still the common lore is that there is a one-to-one correspondence between topological manipulations in the boundary theory and lagrangian algebras of the SymTFT. This lore applies at least to categories in which there is no charged object which is also topological. When some of the charged objects are topological one must allow for a more general notion of Lagrangian algebra, that includes also non-genuine defects [88].

The $2d - 3d$ case. As we have already mentioned generalised symmetries of $2d$ QFTs are described by fusion categories, at least as far as we are interested in discrete and finite symmetries. In these cases there is a mathematically rigorous construction of the SymTFT. Indeed, for any unitary fusion category $\mathcal{C}$, it is possible to construct a $3d$ TQFT, the Turaev-Viro theory [120–122], that admits at least one topological boundary hosting topological line defects described by the seed category $\mathcal{C}$[5]. The line defects of the Turaev-Viro theory are described by a (finite and semisimple) modular tensor category called Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ [122, 123]. As we have discussed in the section 2.2, the $2d$ perspective is that the action of a fusion category $\mathcal{C}$ on the operators of a $2d$ QFT is via the lasso actions, which generate the Tube algebra of $\mathcal{C}$. The SymTFT provides an alternative description via linking in $3d$, the connection between the two is that representations of the Tube algebra, which label the operators of the QFT, are in one to one correspondence with the anyons of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ [225–227]. Physically the correspondence is realized noticing that the lasso actions, from the SymTFT perspective, happen on the topological boundary $\mathbb{B}_{\text{top}}$ and hence cannot change the bulk anyon attached to the local operators, which is then an invariant of the Tube algebra representation. See [36] for a complete account of this correspondence from a physical point of view.

In this context, it is also possible to render precise the correspondence between bulk Lagrangian algebras and topological manipulations in the boundary fusion category. Possible gaugings in the boundary fusion category $\mathcal{C}$ are given by algebra objects. More precisely inequivalent gaugings are labelled by module categories $\mathcal{M}$ over $\mathcal{C}$, and two algebras corresponding to the same module category are said to be Morita equivalent [11]. After gauging an algebra

[4]By definition a TQFT to be interpreted as a SymTFT must have at least one topological boundary condition. There are situations in which we have a $(d+1)$-dimensional TQFT that does not adimt any topological boundary consition attached to a d-dimensional theory, these are generally called relative theories [202]. Famous examples of this are chiral WZW models in $2d$ [124, 125, 208] and $\mathcal{N} = (2, 0)$ theories in $6d$ [131], which need, respectively, a $3d$ or $7d$ Chern-Simons theory to be well defined.

[5]Generalizations of the state sum procedure used to construct the Turaev-Viro theory in $3d$ have also been considered in higher dimensions, especially in both math and condensed matter theory literature [126–130].

in a certain Morita equivalence class we get a dual symmetry $\mathcal{C}'$, and it can be shown that two unitary fusion categories $\mathcal{C}$ and $\mathcal{C}'$ are dual in this sense if and only if they have the same Drinfeld center $\mathcal{Z}(\mathcal{C})$ [11, 182]. This result can also be stated in a slightly different form. The notion of Morita equivalence can be generalised to fusion categories, and physically amount to the statement that two fusion categories $\mathcal{C}$ and $\mathcal{C}'$ are Morita equivalent if and only if they are connected by a discrete gauging (see e.g. [228, 229]). Then the correspondence between bulk lagrangian algebras and boundary topological manipulations amounts to the statement that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is the unique invariant of the fusion category $\mathcal{C}$ under categorical Morita equivalence[6].

[6] The study of higher-categories from the point of view of Morita equivalence classes and how those are related to Drinfeld centers has appeared in the math literature [42, 230–233].

Chapter 3

Generalized Symmetries in Quantum Field Theory

In this chapter we study how symmetries are realized in QFT. In the first half we study non-invertible symmetries preserved along some branch of the conformal manifold of a Calabi-Yau sigma model. In the second half instead we study how conventional 0-form are realized in disordered or averaged theories.

3.1 Non-Invertible Symmetries in Calabi-Yau conformal field theories

Despite their seemingly exotic nature, non-invertible symmetries exist in many familiar quantum field theories. In particular, as we have explained in the first chapter of this thesis, one of the arenas in which generalized symmetries are best understood is that of 2-dimensional CFTs. The Verlinde lines we have described in 2.2.2 are a useful example to study non-invertible symmetry in part because the associated rational CFTs are essentially exactly solvable. Looking beyond these examples, our goal here is instead to find *irrational* CFTs that still possess non-invertible global symmetry, where the existence of new symmetries may yield new insight into the dynamics. Below we will carry this out by constructing non-invertible global symmetries for supersymmetric CFTs described by non-linear sigma models with Calabi-Yau target spaces, specifically, K3 surfaces, and quintic threefolds [234–236]. These models are interesting in that they provide some of the first examples of irrational theories with non-invertible symmetries (excluding interesting symmetries of the compact boson and orbifolds at generic radii [31, 237, 238].) Moreover, in string theory such CFTs can describe the string worldsheet and it is expected, though not proven, that any worldsheet global symmetry should give rise to a spacetime gauge symmetry. If that is the case we expect string theory on the spacetimes we describe below to exhibit novel gauge structures, perhaps implying the existence of novel branes as in [171, 239–241], which is an interesting target for future investigation.

Summary of Results

Our starting point is the observation that Calabi-Yau CFTs come in moduli spaces known as conformal manifolds. Along these manifolds the scaling dimensions and operator product coefficients of generic operators vary. The typical point in the conformal manifold is an irrational CFT with a current algebra given by the appropriate (extended) superconformal algebra. The moduli space is parameterized by exactly marginal local operators, which moreover preserve the supercharges. It is locally factorized into the Kähler moduli, corresponding to size deformations of the target space, and complex structure moduli corresponding to deformations in the shape of the target space (or more pragmatically deformations in the coefficients of the equations defining the target space.)

At special loci in these conformal manifolds, some Calabi-Yau CFTs admit Gepner points [242–244] where the models are described formally by Fermat Calabi-Yau's with a suitable discrete gauging (orbifolding). At these Gepner points the CFT specializes and becomes rational. In such theories, there are many non-invertible symmetries described by the Verlinde lines reviewed above. Our first task, carried out in Sections 3.1.1 and 3.1.2 below is thus to construct explicitly the symmetries of these Gepner models by reviewing their operator content and S-matrices following [234, 235, 245–249]. We achieve this by using a Cardy condition and demanding that the twisted sector Hilbert spaces, describing operators at the end of the lines, has a well-defined positive integral dimension at each energy level and spin [16, 31, 190, 214]. We also provide an equivalent perspective using an associated three-dimensional topological field theory which encodes the symmetries of these models [9, 69, 116–119, 133, 195].

In Section 3.1.3 we derive our main results. Specifically, we move away from the Gepner point by deforming the action by exactly marginal operators. We then track which non-invertible symmetries are preserved by these deformations. We content ourselves to an investigation of the fusion rules, leaving a detailed study of the F-symbols to future work. Remarkably, we find that at many special loci in the moduli space preserve some of the non-invertible symmetry. These theories are generically irrational, and correspondingly little is known about their non-BPS spectrum of scaling dimensions and OPE coefficients. While the exact non-invertible symmetry depends in detail on the special locus investigated, we highlight several special cases to get a feeling for our results:

- We begin with a warm up of the torus SCFT. The Gepner point of interest corresponds to a square torus of unit volume. As we deform away from is point, we find that an interesting non-invertible symmetry is preserved at any modulus τ as long as the B field is taken to always vanish and the volume (defined by the metric G) obeys:

$$\tau = x + iy, \quad \det(G) = \frac{1 + x^2 + y^2}{2y}. \tag{3.1.1}$$

  Specifically, along this locus in moduli space there are topological defect lines (TDLs) with fusion algebra of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami category [31, 250, 251]. This fusion category has invertible lines η_i generating the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ as well as a single non-invertible line $\mathcal{D}$ obeying:

$$\mathcal{D} \times \mathcal{D}^\dagger = \mathbb{1} + \eta_1 + \eta_2 + \eta_1 \times \eta_2\,. \tag{3.1.2}$$

- In the case of the K3 CFT we find a variety of subloci in moduli space preserving non-invertible symmetries. For instance along a four-dimensional subspace preserving the full $\mathcal{N} = 4$ superconformal algebra we find a fusion algebra containing of a Tambara-Yamagami $\mathbb{Z}_2^4$ category

$$\mathcal{D} \times \mathcal{D}^\dagger = \mathbb{1} + \sum_{i=1}^{4} \eta_i + \sum_{i<j} \eta_i \times \eta_j \sum_{i<j<k} \eta_i \times \eta_j \times \eta_k + \eta_1 \times \eta_2 \times_3 \times \eta_4\,. \tag{3.1.3}$$

  As discussed in [12,31,37,57,62,63] Tambara-Yamagami fusion category symmetries arise when a model is self-dual under gauging invertible symmetries. Thus, our identification of Tambara-Yamagami fusion category symmetry in the K3 sigma model implies that at these special loci there are new self-dualities of these CFTs.

- In the case of the quintic threefold, we again find many loci preserving non-invertible symmetry. Notably in this case, these are all loci in the complex structure moduli space; the unique Kähler modulus is frozen to its Gepner value. As a particular highlight, we mention that along various ten-dimensional loci in complex structure moduli space we find (at least) a fusion category symmetry characterized by Fibonacci line W obeying the fusion rule:

$$W \times W = \mathbb{1} + W. \tag{3.1.4}$$

  The fact that non-invertible symmetry appears in the irrational CFTs discussed above implies new constraints on these models which may be useful, for example, in constraining correlation functions or in a bootstrap type analysis of their spectral data [190, 252–255]. To this end in Section 3.1.3 we review results of [36] which characterize these selection rules in representation theoretic terms. Finally we then apply these considerations in Section 3.1.3 to conformal perturbation theory of the K3 sigma model near its Gepner point as recently studied in [256]. In particular, we show that the non-invertible symmetry at the Gepner point implies that only certain powers of the coupling can appear with non-zero coefficients in calculations of the perturbed scaling dimensions. This is consistent with observations of similar phenomena in [256].

3.1.1 LG/CY Correspondence and Supersymmetric Minimal Models

In this section we briefly review the Landau-Ginzburg/Calabi-Yau correspondence to provide the necessary context, then we prepare the stage to study the topological defect lines in Gepner models analyzing the case of a single minimal model.

Landau-Ginzburg/Calabi-Yau Correspondence

Gepner [244, 257] was the first one to provide evidence relating $\mathcal{N} = 2$ minimal models with Calabi-Yau sigma models. We won't attempt to give an historically accurate account of the subsequent developments, the interested reader can consult [234,235]. Here we instead recall the arguments of [246], providing evidence for this relation. We consider $\mathcal{N} = (2,2)$ supersymmetric theories in two dimensions. The particular class of models we are going to look at are $U(1)$ gauge theories with $r+1$ chiral multiplets P and $X_1, \ldots, X_r$ with a specific superpotential

$$W = P\left(X_1^{k_1+2} + \cdots + X_r^{k_r+2}\right), \qquad k_i \in \mathbb{N}_0, \ \forall\, i = 1, \ldots, r \tag{3.1.5}$$

and a linear twisted superpotential

$$\widetilde{W} = t\Sigma \tag{3.1.6}$$

where $\Sigma = D_+ D_- V$, V being the vector multiplet, and $t = r - i\theta$ encodes the FI parameter r and the theta angle. The $U(1)$ gauge group transformations are

$$P \to \mathrm{e}^{-iH\lambda} P, \quad X_i \to \mathrm{e}^{iw_i\lambda} X_i \tag{3.1.7}$$

where

$$\begin{aligned} H &:= \operatorname{lcm}\{k_i + 2\}, \\ w_i &:= \frac{H}{k_i + 2}. \end{aligned} \tag{3.1.8}$$

Before analysing the phases of this system let's look at the symmetries of the action, in particular to the R-symmetries. In superspace notation the F-terms involving the two superpotentials are of the form

$$\int d\theta_+\, d\theta_- W(P, X_i) + \text{h.c} \qquad\qquad \int d\theta_+\, d\bar{\theta}_- \widetilde{W}(\Sigma) + \text{h.c}, \tag{3.1.9}$$

then, in order for the theory to enjoy both left and right moving R-symmetries we need to find charge assigments for the chiral multiplets such that W has charges $(-1,-1)$ under $U(1)_R^+ \times U(1)_R^-$. For the twisted superpotential instead one can check that a consistent charge assigments for the vector multiplet are possible only if $\widetilde{W}$ is linear [246]. This ensures that the R-symmetries are classical symmetries of the action, but they may still be affected by ABJ-anomalies as they couple to fermions of a given chirality only. It is simple to check that the anomaly cancellation condition is

$$-1 + \sum_{i=1}^{r} \frac{1}{k_i + 2} = 0, \tag{3.1.10}$$

i.e. that the gauge charges of the chirals sum to zero. If this condition is verified the twisted superpotential is not renormalized and the FI parameter r is a true parameter of the theory [235,246]. We now look at the space of classical vacua to try to understand the IR phases of this system. The scalar potential is (in the following the lowercase letters are the expectation values of the scalar components of the superfield denoted by the corresponding uppercase symbol)

$$\begin{aligned} U(x_i, \sigma, p) = &\left|\sum_i x_i^{k_i+2}\right|^2 + |p|^2 \sum_i |(k_i+2) x_i^{k_i+1}|^2 + \frac{e^2}{2}\left(\sum_i \frac{H|x_i|^2}{k_i+2} - H|p|^2 - r\right) \\ &+ 2|\sigma|^2 \left(\sum_i \frac{H^2 |x_i|^2}{(k_i+2)^2} + H^2 |p|^2\right). \end{aligned} \tag{3.1.11}$$

Notice that the second term vanishes if $|p| = 0$ and/or $|x_i| = 0$ for all $i = 1, \ldots r$[1]. We can identify two regions with qualitatively different behaviour. For $r > 0$ we see that the susy vacua necessarily have $|p| = |\sigma| = 0$, the leftover equations are

$$\begin{aligned} &\sum_i \frac{H}{k_i+2}|x_i|^2 = r \\ &\sum_i x_i^{k_i+2} = 0\,. \end{aligned} \tag{3.1.12}$$

The first one identifies a compact submanifold of $\mathbb{C}^r$, for instance in the case $k_i = k$ this is simply a sphere. Modding out by the $U(1)$ gauge group we find that the first equation identifies a complex weighted projective space $\mathbb{CP}[w_1, \ldots, w_n]$ with weights given by the gauge charges (notice that if $k_i = k$ for all i then $w_i = 1$). The second equation then identifies a complex hypersurface M inside the weighted projective space, in particular M is Kähler and the vanishing of its first Chern class turns out to be equivalent to the anomaly cancellation condition for the R-symmetries (3.1.10) (see e.g. Chapter 14 of [258]). The system at low energies reduces to a non-linear sigma model on M (one checks that all fields except the components of the x_i tangent to M get a mass at tree level), thus an anomaly free R-symmetry in the IR sigma model is equivalent to the CY condition. At least classically we found a region of the parameter space in which the low energy theory is a Calabi-Yau sigma model, although classical these computations are qualitatively correct also taking into account quantum corrections [246]. The other regime of interest is instead $r < 0$, in this case the susy vacua are

$$|\sigma| = |x_1| = \cdots = x_r = 0\,, \qquad |p| = \sqrt{\frac{-r}{H}} \tag{3.1.13}$$

around these vacua the only the x_i are massless (unless some $k_i = 0$) with interactions determined by a superpotential obtained integrating out P. This can be done replacing P with its expectation value, so that, modulo reabsorbing an overall constant in the x_i's, the new superpotential is simply

$$X_1^{k_1+2} + \cdots + X_r^{k_r+2}\,. \tag{3.1.14}$$

Notice that, since P is charged under the gauge group, we have an Higgsing from $U(1)$ to $\mathbb{Z}_H$ with the unbroken gauge group acting as

$$X_i \mapsto e^{iw_i\lambda} X_i \tag{3.1.15}$$

where $e^{iH\lambda} = 1$. Thus in this regime the theory below the breaking scale is a theory of only chirals with superpotential (3.1.14), *a.k.a* a Landau-Ginzburg model, orbifolded by the action (3.1.15). The theory of a single chiral superfield with superpotential $W = X^{k+2}$ flows in the IR to an $\mathcal{N} = 2$ minimal model M_k of central charge

$$c = \frac{3k}{k+2}\,, \tag{3.1.16}$$

[1]For a generic superpotential of the form $W = PG(X_1, \ldots, X_r)$, where G is a quasi homogeneous function, one requires that the equations $\partial_{X_i} G = 0$ for $i = 1, \ldots r$ have a unique common solution at $X_1 = X_2 = \ldots = X_r = 0$. In our example this transversality condition is automatically built in.

in particular to the so called A-series minimal model corresponding to choosing the diagonal modular invariant partition function [245]. Then our initial model, for negative values of the FI parameter, flows in the IR to an orbifolded product of minimal models

$$\prod_{i=1}^{r} M_{k_i} \Big/ \mathbb{Z}_H \tag{3.1.17}$$

this is the Gepner model.

From the perspective of the NLSM we discussed before this Gepner model should correspond to a particular value of the Kähler modulus, that is it should correspond to a particular point on the conformal manifold, the Gepner point. The path connecting a generic point and this special point is smooth and there are no singularities (at least as long as one keeps a generic value of the theta angle [246]). There is much more to this story, with many possible generalizations (for instance taking a gauge group $U(1)^n$) and further subtleties to be addressed, however this general picture is enough to provide context.

Minimal Models: Representations and Fusion Rules

Minimal models for the $\mathcal{N}=2$ superconformal algebra (see Appendix A.1 for our conventions) form a discrete series with central charge

$$c = \frac{3k}{k+2}\,, \tag{3.1.18}$$

where $k \in \mathbb{N}$ is the level. At these values of c unitarity allows only a finite number of superconformal primaries, these are labelled by their weight and $U(1)_R$ charges

$$\begin{aligned} h^{(\lambda)}_{l,m} &= \frac{l(l+2)}{4(k+2)} + \frac{\lambda^2}{2} - \frac{(m-2\lambda)^2}{4(k+2)} \\ q^{(\lambda)}_{l,m} &= \frac{m+k\lambda}{k+2} \end{aligned} \tag{3.1.19}$$

with $\lambda = 0$ in the NS sector and $\lambda = -1/2$ in the R sector. The variables l, m are valued in

$$(l,m) \in P_k = \{l = 0,\ldots,k\,,\ |m| \leq l\,,\ l+m = 0\ (\text{mod } 2)\}\ . \tag{3.1.20}$$

Chiral primaries in the NS sector have $m = l$ while antichirals have $m = -l$, Ramond sector ground states are obtained by spectral flow.

To each superconformal primary representation $\mathbb{H}^{(\lambda)}_{l,m}$ we associate two characters

$$\begin{aligned} \text{ch}^{(\lambda)}_{l,m}(\tau,z) &= \text{Tr}_{\mathbb{H}^{(\lambda)}_{l,m}} e^{2i\pi\tau L_0} e^{2i\pi z j_0} \\ \widetilde{\text{ch}}^{(\lambda)}_{l,m}(\tau,z) &= \text{Tr}_{\mathbb{H}^{(\lambda)}_{l,m}} (-1)^F e^{2i\pi\tau L_0} e^{2i\pi z j_0} = e^{-i\pi q^{(\lambda)}_{l,m}} \text{ch}^{(\lambda)}_{l,m}\left(\tau, z+\frac{1}{2}\right) \end{aligned} \tag{3.1.21}$$

where, since the fermionic modes have $U(1)_R$ charge ± 1, we represented fermion parity as $(-1)^{j_0 - q^{(\lambda)}_{l,m}}$. To discuss $SL(2,\mathbb{Z})$ transformation is convenient to work with characters of the bosonic subalgebra, this is equivalent to realize the minimal model as the coset $\widehat{\mathfrak{su}}(2)_k \times$

$\widehat{\mathfrak{u}}(1)_2/\widehat{\mathfrak{u}}(1)_{k+2}$ which we will discuss momentarily. The bosonic sub-representations include only states with a fixed fermion number mod 2, their characters are

$$\chi^{(\lambda)}_{l,m}(\tau,z)=\frac{1}{2}\left(\mathrm{ch}^{(\lambda)}_{l,m}(\tau,z)+\widetilde{\mathrm{ch}}^{(\lambda)}_{l,m}(\tau,z)\right) \tag{3.1.22}$$

which only contains states with even fermion numbers, and

$$\widetilde{\chi}^{(\lambda)}_{l,m}(\tau,z)=\frac{1}{2}\left(\mathrm{ch}^{(\lambda)}_{l,m}(\tau,z)-\widetilde{\mathrm{ch}}^{(\lambda)}_{l,m}(\tau,z)\right)\,. \tag{3.1.23}$$

Following [249] we relabel the primaries using new variables $a=l, c=m-2\lambda$ and $b=[a+c]=-2\lambda$, where here and in the following we define

$$[x]\equiv x \bmod 2. \tag{3.1.24}$$

We have

$$\begin{aligned} h_{a,c}&=\frac{a(a+2)-c^2}{4(k+2)}+\frac{[a+c]}{8}\\ q_{a,c}&=\frac{c}{k+2}-\frac{[a+c]}{2} \end{aligned} \tag{3.1.25}$$

and the new variables take values in

$$(a,c)\in P'_k=\{(a,c)\,|\,a=0,\dots,k\,,\ |c-[a+c]|\le a\}\,. \tag{3.1.26}$$

Now we set[2]

$$\chi^{(\lambda)}_{l,m}(\tau,z)=\chi_{a,c}(\tau,z)\qquad \widetilde{\chi}^{(\lambda)}_{l,m}(\tau,z)=\chi_{k-a,c+k+2}(\tau,z)\,. \tag{3.1.29}$$

Notice that for $(a,c)\in P'_k$ the pair $(k-a,c+k+2)$ does not belong to P'_k, thus we have to enlarge the indexing set for characters to

$$Q_k=P'_k\cup\{(k-a,c+k+2),(a,c)\in P'_k\}=\{(a,c)\,|\,0\le a\le k\,,\ 0\le c\le 2k+3\}. \tag{3.1.30}$$

It turns out that $\chi_{a,c}$ with $(a,c)\in Q_k$ do have nice modular properties and yield a unitary representation of $SL(2,\mathbb{Z})$ with S and T matrices

$$\begin{aligned} S_{ac;a'c'}&=\frac{1}{k+2}\sin\left(\frac{\pi(a+1)(a'+1)}{k+2}\right)e^{i\pi\frac{cc'}{k+2}}e^{-i\pi\frac{[a+c][a'+c']}{2}}\\ T_{ac;a'c'}&=e^{2i\pi\left(h_{a,c}-\frac{c}{24}\right)}\delta_{a,a'}\delta_{c,c'}\,. \end{aligned} \tag{3.1.31}$$

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[2] The notation

$$\widetilde{\chi}^{(\lambda)}_{l,m}(\tau,z)=\chi_{k-a,c+k+2}(\tau,z)\,. \tag{3.1.27}$$

is justified noticing that the states of lowest weight surviving the projection are obtained acting with $G^{\pm}_{-1/2}$ in the NS sector and $G^{\pm}_0$ in the R sector. It is then easy to check that

$$\begin{aligned} q_{a,c}\pm 1&=q_{k-a,c+k+2} \bmod 2\\ h_{k-a,c+k+2}-h_{a,c}&=\frac{a+c+1}{2} \bmod 1 \end{aligned} \tag{3.1.28}$$

then $h_{k-a,c+k+2}$ mod 1 is the eigenvalue of $T:\tau\mapsto\tau+1$ on $\chi_{k-a,c+k+2}(\tau,z)$.

As usual $S^2 = C$, $C^2 = \mathbb{1}$ with

$$C_{a'c';ac} = \delta_{a',a^+}\delta_{c',c^+}$$
$$(a^+, c^+) = \begin{cases} (a, -c \bmod 2(k+2)) & \text{if} \quad [a+c] = 0 \\ (k-a, k+2-c \bmod 2(k+2)) & \text{if} \quad [a+c] = 1\,. \end{cases} \tag{3.1.32}$$

Notice that in this basis of half-characters the T matrix is diagonal, while this would not be the case if we were working with the full characters $\mathrm{ch}^{(\lambda)}_{l,m}(\tau, z)$.

By means of the Verlinde formula we can obtain the fusion coefficients for the bosonic subrepresentations

$$\begin{aligned} N^{\alpha\gamma}_{ac;a'c'} &= \sum_{(d,f)\in Q_k} \frac{S_{ac;df}S_{a'c';df}S^*_{\alpha\gamma;df}}{S_{00;df}} \\ &= \left\{ \begin{array}{ll} \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}}(1)_{k+2}}\right)^{\gamma}_{c,c'}\,, & \text{if } [a+c]\,[a'+c'] = 0 \\ \left(N^{\widehat{\mathfrak{su}(2)}_k}\right)^{k-\alpha}_{a,a'} \left(N^{\widehat{\mathfrak{u}}(1)_{k+2}}\right)^{\gamma+k+2}_{c,c'}, & \text{if } [a+c]\,[a'+c'] = 1 \end{array} \right\} \\ &\text{for } (ac), (a'c')\,, (\alpha\gamma) \in Q_k\,, \end{aligned} \tag{3.1.33}$$

where

$$\begin{aligned} \left(N^{\widehat{\mathfrak{su}}(2)_k}\right)^{l}_{a,a'} &= \delta\left(|a-a'| \leq l \leq \min\left(a+a', 2k-a-a'\right)\right)\delta\left(a+a' \equiv l \bmod 2\right) \\ \left(N^{\widehat{\mathfrak{u}}(1)_{k+2}}\right)^{n}_{c,c'} &= \delta\left(c+c' \equiv n \bmod 2(k+2)\right) \end{aligned} \tag{3.1.34}$$

are the fusion coefficients for the $\widehat{\mathfrak{su}}(2)_k$ and $\widehat{\mathfrak{u}}(1)_{k+2}$ Kac-Moody algebras. One can check that this result is consistent with the conservation of the $U(1)_R$ charge in the OPE.

Symmetries and Orbifolds

In the rest of the paper we will only consider the minimal model with diagonal modular invariant, the torus partition function is

$$Z(\tau, z) = \sum_{(a,c)\in Q_k} |\chi_{a,c}(\tau,z)|^2 = \sum_{\substack{(l,m)\in P_k \\ \lambda=0,-1/2}} \mathrm{Tr}_{\mathbb{H}^{(\lambda)}_{l,m}} \left((1+(-1)^{F_L+F_R}) q^{L_0-\frac{c}{12}} \bar{q}^{\bar{L}_0-\frac{c}{12}} y^{j_0} \bar{y}^{\bar{j}_0} \right)\,, \tag{3.1.35}$$

with $q = e^{2i\pi\tau}, y = e^{2i\pi z}$. Requiring modular invariance automatically includes the GSO projection, and in the diagonal case the physical primaries have to be fermionic or bosonic on both sides. Notice in particular that among those the holomorphic and antiholomorphic supercurrents do not survive the projection, rather they can appear only when paired with another fermionic state on the other side.

We now construct the Verlinde lines of the theory, the simplest way to do so is bootstrap them from the partition function. In Appendix A.3 we also give an alternative derivation using the folding trick. The idea is to make an ansatz for the action of the line on the physical primaries of the theory, and then constrain it imposing concistency of the twisted Hilbert spaces. This is a well known construction, we repeat it here as a warm up for the more involved case of Gepner models. In the diagonal theory the circle Hilbert space is

$$\mathbb{H} = \bigoplus_{(a,c)\in Q_k} \mathbb{H}_{a,c} \otimes \overline{\mathbb{H}}_{a,c} \tag{3.1.36}$$

and a physical primary Φ_{ac} corresponds to the state $|a,c\rangle \otimes \overline{|a,c\rangle}$. A topological line $L_{r,s}$ commutes with the Virasoro generators, here we also require it to be supersymmetric, namely to commute with all the generators of the bosonic subalgebra. We parametrize the action on primaries as

$$L_{r,s}\Phi_{a,c} = X^{a,c}_{r,s}\Phi_{a,c} \tag{3.1.37}$$

and constrain it imposing that the twisted partition function $Z_{r,s}(\tau,z)$ admits a decomposition in characters of the bosonic subalgebra with integer multiplicities. We have

$$Z_{r,s}(\tau,z) = \sum_{\substack{(a,c),(a',c')\\(a'',c'')}} X^{a,c}_{r,s} S_{ac;a'c'} S^*_{ac;a''c''} \chi_{a',c'}(q,y)\chi_{a'',c''}(\bar{q},\bar{y}) \tag{3.1.38}$$

then we require

$$\sum_{(a,c)\in Q_k} X^{a,c}_{r,s} S_{ac;a'c'} S^*_{ac;a''c''} \in \mathbb{N}\,. \tag{3.1.39}$$

A natural solution for the multiplicities is given by the fusion coefficients, namely

$$X^{a,c}_{r,s} = \frac{S_{rs;ac}}{S_{00;ac}}\,, \qquad \sum_{(a,c)\in Q_k} X^{a,c}_{r,s} S_{ac;a'c'} S^*_{ac;a''c''} = N^{a''c''}_{rs;a'c'} \tag{3.1.40}$$

corresponding to

$$Z_{r,s}(\tau,z) = \sum_{(a',c')(a'',c'')} N^{a''c''}_{rs;a'c'} \chi_{a',c'}(q,y)\chi_{a'',c''}(\bar{q},\bar{y})\,. \tag{3.1.41}$$

The action by linking on physical primaries is the usual one

$$L_{r,s}\Phi_{a,c} = \frac{S_{rs;ac}}{S_{00;ac}}\Phi_{a,c} \tag{3.1.42}$$

and fusion immediately follows from the Verlinde formula

$$L_{rs} \times L_{r's'} = \sum_{(r'',s'')\in Q_k} N^{r''s''}_{rs;r's'} L_{r''s''}\,, \tag{3.1.43}$$

we therefore have $|Q_k| = 2(k+1)(k+2)$ topological defect lines.

The partition functions $Z_{r,s}(\tau,z)$ are traces over the twisted Hilbert spaces. The states in those spaces are mapped, by the state operator correspondence, to non-genuine local operators, namely local operators attached to the topological line $L_{r,s}$. When inserted in correlation functions these twist defects generically introduce branch cut singularities, corresponding to the action of the attached TDL. We also notice that, since $L_{r,s}$ acts non-trivially on the physical primaries of the theory the corresponding twisted Hilbert spaces cannot contain the identity operator, namely the ground state necessarily has positive Virasoro weights.

We first want to find the set of invertible lines. To this extent it is useful to recall that the fusion of a line and its orientation reversal always contains the identity. Reversing the orientation however is equivalent to act with charge conjugation on the labels of the line. From the explicit expression of the fusion coefficients it is simple to compute

$$N^{rs}_{ac;a^+c^+} = \begin{cases} \delta(0 \leq r \leq \min(2a, 2k-2a))\, \delta(r = 0 \bmod 2)\, \delta_{s,0} \quad \text{if} \quad [a+c] = 0 \\ \delta(0 \leq r \leq k - |2a-k|)\, \delta(r = 0 \bmod 2)\, \delta_{s,0} \quad \text{if} \quad [a+c] = 1 \end{cases} \tag{3.1.44}$$

which consistently obey $N^{00}_{ac;a^+c^+} = 1$. A line is invertible if *only* the identity appears in the fusion channel with its charge conjugate, we then see that the invertible lines are $L_{0,s}$ and $L_{k,s}$; where $s = 0, \ldots, 2k+3$. Thus there are $4(k+2)$ invertible lines. The fusion among those is controlled by the coefficients

$$N^{ac}_{0s;0s'} = \begin{cases} \delta_{a,0}\delta_{c,s+s'} \quad [s][s'] = 0 \\ \delta_{a,k}\delta_{c,s+s'+k+2} \quad [s][s'] = 1 \end{cases} \tag{3.1.45}$$

$$N^{ac}_{ks;ks'} = \begin{cases} \delta_{a,0}\delta_{c,s+s'} \quad [k+s][k+s'] = 0 \\ \delta_{a,k}\delta_{c,s+s'+k+2} \quad [k+s][k+s'] = 1 \end{cases} \tag{3.1.46}$$

$$N^{ac}_{ks;0s'} = \begin{cases} \delta_{a,k}\delta_{c,s+s'} \quad [k+s][s'] = 0 \\ \delta_{a,0}\delta_{c,s+s'+k+2} \quad [k+s][s'] = 1 \end{cases} = N^{ac}_{0s';ks}\,. \tag{3.1.47}$$

The group structure depends on k:

- k even. It is easy to see that

$$L^{2n}_{k,1} = L_{0,2n+n(k+2)} \qquad L^{2n+1}_{k,1} = L_{k,2n+1+n(k+2)} \tag{3.1.48}$$

  and in particular $L^{2(k+2)}_{k,1} = \mathbb{1}$, therefore $L_{k,1}$ generates a $\mathbb{Z}_{2(k+2)}$ group. We also have

$$L^2_{k,0} = \mathbb{1} \tag{3.1.49}$$

  so $L_{k,0}$ generates a $\mathbb{Z}_2$. By computing

$$L_{k,0} \cdot L_{k,1} \cdot L_{k,0} = L_{k,1} \tag{3.1.50}$$

  we see that the symmetry structure is a direct product. One also checks that all invertible lines can be obtained fusing $L_{k,0}$ and $L_{k,1}$. Thus the invertible lines for k even form a group $\mathbb{Z}_2 \times \mathbb{Z}_{2(k+2)}$ with the generators acting as

$$L_{k,0}\Phi_{ac} = (-1)^a \Phi_{ac}\,, \qquad L_{k,1}\Phi_{ac} = (-1)^a e^{\frac{2i\pi}{2(k+2)}\left(c - \frac{k+2}{2}[a+c]\right)} \Phi_{ac}\,. \tag{3.1.51}$$

- k odd. In this case fusing $L_{0,1}$ with itself we can generate all lines, in particular

$$L^n_{0,1} = \begin{cases} L_{0,n} \qquad n = 0,1 \bmod 4 \\ L_{k,n+(k+2)} \qquad n = 2,3 \bmod 4 \end{cases} \tag{3.1.52}$$

  so $L^{4(k+2)}_{0,1} = \mathbb{1}$. Thus the invertible lines form a $\mathbb{Z}_{4(k+2)}$ group, with the generator acting as

$$L_{0,1}\Phi_{ac} = e^{\frac{2i\pi}{4(k+2)}(2c-(k+2)[a+c])} \Phi_{ac}\,. \tag{3.1.53}$$

These match known symmetries of the minimal models see e.g. [234, 248].

For any value of k we have a $\mathbb{Z}_{k+2}$ subgroup generated by $L_{0,2}$

$$L_{0,2}\Phi_{ac} = e^{2i\pi\frac{c}{k+2}}\Phi_{ac} \tag{3.1.54}$$

notice that under this subgroup a full superconformal family transforms with the same charge since

$$L_{0,2}\Phi_{k-a;c+k+2} = e^{2i\pi\frac{c}{k+2}}\Phi_{k-a;c+k+2}\,. \tag{3.1.55}$$

Another symmetry present for all values of k is a $\mathbb{Z}_2$ generated by $L_{k,k+2}$. This acts by

$$L_{k,k+2}\Phi_{ac} = (-1)^{a+c}\Phi_{ac} \tag{3.1.56}$$

i.e. it leaves invariant the NS sector primaries while giving a sign on the R ones, one way of interpreting this is as the symmetry dual to $(-1)^F$ which has been trivialized by the GSO projection. Both these $\mathbb{Z}_{k+2}$ and $\mathbb{Z}_2$ will play an important role in the construction of the Gepner models.

As a warm-up for the next section we compute the partition functions of the diagonal minimal model orbifolded by the $\mathbb{Z}_{k+2}$ described above. To do so we need to twist and twine the partition function by the generator of $\mathbb{Z}_{k+2}$ and then sum over the group elements. The twined partition function is simple to write down, acting with (a s-th power of) the symmetry operator on the Hilbert space we get

$$Z(\tau,z,s) = \sum_{(a,c)\in Q_k} e^{2i\pi\frac{sc}{k+2}}\chi_{ac}(q,y)\chi_{ac}(\bar{q},\bar{y})\,. \tag{3.1.57}$$

The twisted partition function is obtained with a S modular transformation

$$\begin{aligned} Z_s(\tau,z) = Z\left(-\frac{1}{\tau},\frac{z}{\tau}\right) &= \sum_{(a,c),(a',c'),(a'',c'')\in Q_k} e^{2i\pi\frac{sc}{k+2}} S_{ac;a'c'}S^*_{ac;a''c''}\chi_{a'c'}(q,y)\chi_{a''c''}(\bar{q},\bar{y}) \\ &= \sum_{(a',c'),(a'',c'')\in Q_k} N^{a''c''}_{a'c';02s}\chi_{a'c'}(q,y)\chi_{a''c''}(\bar{q},\bar{y}) = \sum_{(a,c)\in Q_k}\chi_{ac}(q,y)\chi_{a,c+2s}(\bar{q},\bar{y}) \end{aligned} \tag{3.1.58}$$

Where we used the Verlinde formula and, in the last step, the explicit form of the fusion coefficients. To write down the orbifold partition function we need both twisting and twining, the simplest way to twine a twisted partition function is to use the T transformation. We have

$$T\cdot Z_s(\tau,z) \sum_{(a,c)\in Q_k} e^{2i\pi s\frac{2c+2s}{2(k+2)}}\chi_{ac}(q,y)\chi_{a,c+2s}(\bar{q},\bar{y})^*\,. \tag{3.1.59}$$

therefore

$$Z_s(\tau,z,r) = \sum_{(a,c)\in Q_k} e^{2i\pi r\frac{2c+2s}{2(k+2)}}\chi_{ac}(q,y)\chi_{a,c+2s}(\bar{q},\bar{y})\,. \tag{3.1.60}$$

The partition function of the gauged theory is then

$$Z_{\mathbb{Z}_{k+2}}(\tau,z) = \frac{1}{k+2}\sum_{s,r=0}^{k+1} Z_s(\tau,z,r) = \frac{1}{k+2}\sum_{s,r=0}^{k+1}\sum_{(a,c)\in Q_k} e^{2i\pi r\frac{2c+2s}{2(k+2)}}\chi_{ac}(q,y)\chi_{a,c+2s}(\bar{q},\bar{y}) \tag{3.1.61}$$

the sum over r sets

$$2c + 2s = 0 \bmod 2(k+2) \to c + 2s = -c \bmod 2(k+2) \tag{3.1.62}$$

thus

$$Z^{\mathbb{Z}_{k+2}}(\tau, z) = \sum_{(a,c)\in Q_k} \chi_{ac}(q,y)\chi_{a,-c}(\bar{q},\bar{y})\,. \tag{3.1.63}$$

This partition function is modular invariant[3] and defines a sensible SCFT. We now want to determine the symmetries of the orbifolded theory. The first thing we notice is that

$$N_{0,2s;ac}^{a'c'} = \delta_{a,a'}\delta_{c',c+2s} = N_{ac;02s}^{a'c'} \tag{3.1.64}$$

i.e. the fusion of the generic Verlinde line with the $\mathbb{Z}_{k+2}$ symmetry is abelian

$$L_{0,2s}L_{ac}L_{0,-2s} = L_{ac} \tag{3.1.65}$$

therefore we may hope that L_{ac} survives the gauging operation. Another independent way we have to study the symmetries of the orbifold is to use modular covariance. The Hilbert space of the gauged theory is

$$\mathbb{H}^{(\mathbb{Z}_{k+2})} = \bigoplus_{(a,c)\in Q_k} \mathbb{H}_{(a,c)} \otimes \overline{\mathbb{H}}_{(a,-c)} \tag{3.1.66}$$

then we make an ansatz for the action of some new TDL $\mathcal{L}_{(a,c)}$ on it. Denoting again a physical primary as Φ_{ac} we set

$$\mathcal{L}_{(r,s)}\Phi_{ac} = X_{rs}^{ac}\Phi_{ac}\,, \tag{3.1.67}$$

clearly this action preserves the full chiral algebra. We then constraint this ansatz by requiring that the TDL $\mathcal{L}_{(r,s)}$ gives a consistent twisted Hilbert space. Specifically, via a modular transformation of the twined partition function, we impose

$$\sum_{(a,c)\in Q_k} X_{rs}^{ac} S_{ac;a'c'} S^*_{a,-c;a''c''} \in \mathbb{N}\,. \tag{3.1.68}$$

Using the symmetry $S_{a-c;a'c'} = S_{ac;a'-c'}$ we see that there is an obvious solution

$$X_{rs}^{ac} = \frac{S_{ac;rs}}{S_{00;ac}} \tag{3.1.69}$$

corresponding to

$$\sum_{(a,c)} X_{rs}^{ac} S_{ac;a'c'} S^*_{a,-c;a''c''} = N_{rs;a'c'}^{a'',-c''}\,. \tag{3.1.70}$$

This shows that the lines $\mathcal{L}_{r,s}$ acting as

$$\mathcal{L}_{(r,s)}\Phi_{ac} = \frac{S_{ac;rs}}{S_{00;ac}}\Phi_{ac}\,, \tag{3.1.71}$$

yield a consistent twisted Hilbert space. Whenever we gauge a discrete symmetry we expect a dual one to show up in the gauged theory. Indeed one can easily show that this dual symmetry is generated by $\mathcal{L}_{0,2}$, which acts by

$$\mathcal{L}_{0,2s}\Phi_{ac} = e^{2i\pi s\frac{c}{k+2}}\Phi_{ac}. \tag{3.1.72}$$

[3] One can quickly check this using that $S_{a-c;a'c'} = S_{ac;a'-c'}$

giving charge to the new physical primaries coming from the old twisted sectors. It is also easy to see that gauging this new $\mathbb{Z}_{k+2}$ symmetry we get back to the original theory with the diagonal modular invariant. One can repeat a similar analysis for the $\mathbb{Z}_2$ generated by $L_{k,k+2}$ and also for the product $\mathbb{Z}_2 \times \mathbb{Z}_{k+2}$. In particular one can check that gauging the product we obtain the charge-conjugation invariant partition function

$$Z_{\mathbb{Z}_2\times\mathbb{Z}_{k+2}}(\tau, z) = \sum_{(a,c)\in Q_k} \chi_{ac}(q, y)\chi_{a^+c^+}(\bar{q}, \bar{y})^* \,. \tag{3.1.73}$$

This property is well known in the literature [234,242,259] and is important in the construction of mirror manifolds.

The $3d$ TQFT

A very useful realization of the $\mathcal{N} = 2$ minimal models, that we implicitly used throughout this section, is as the coset (see e.g. [260] for a complete list of references)

$$M_k = \frac{\widehat{\mathfrak{su}}(2)_k \times \widehat{\mathfrak{u}}(1)_2}{\widehat{\mathfrak{u}}(1)_{k+2}} \,. \tag{3.1.74}$$

In particular when the minimal model is presented in this form it is immediate to write down the $3d$ TQFT corresponding to it. This is simply the Chern-Simons theory with gauge group

$$G_k = \frac{SU(2)_k \times U(1)_2 \times U(1)_{-(k+2)}}{\mathbb{Z}_2^{(1)}} \tag{3.1.75}$$

where $\mathbb{Z}_2^{(1)}$ is the one-form symmetry deriving from common center of gauge group factors. The Wilson lines for $SU(2)_k \times U(1)_2 \times U(1)_{k+2}$ can be labelled by three integers (a, s, c) with $a = 0, \ldots, k$ for $SU(2)_k$, $s = 0, 1, 2, 3$ and $c = 0, \ldots, 2(k + 2) - 1$ for the two $U(1)$s. The fusion rules are

$$\mathcal{L}_{(a,s,c)} \times \mathcal{L}_{(a',s',c')} = \sum_{\substack{a''=|a-a'| \\ a''=a-a' \bmod 2}}^{\min(a+a',2k-a-a')} \mathcal{L}_{(a'',s+s',c+c')} \tag{3.1.76}$$

while the topological S matrix and spins are

$$\begin{aligned} S_{asc;a's'c'} &= \frac{1}{k+2} \sin\left(\frac{\pi(a+1)(a'+1)}{k+2}\right) e^{2i\pi\frac{ss'}{4}} e^{-2i\pi\frac{cc'}{2(k+2)}} \,. \\ \theta_{asc} &= e^{i\pi\frac{a(a+2)}{2(k+2)}} e^{i\pi\frac{s^2}{2}} e^{i\pi\frac{m^2}{k+2}} \end{aligned} \tag{3.1.77}$$

Also the F-symbols are factorized, see [17] for explicit expressions. The $\mathbb{Z}_2^{(1)}$ is generated by the line $(k, 2, k + 2)$, which has $\theta_{k,2,k+2} = 1$ for any k, hence can always be gauged. The eigenvalue of the action by linking on other lines is

$$\frac{S_{k,2,k+2;asc}}{S_{000;asc}} = (-1)^{a+c+s} \tag{3.1.78}$$

while fusion is

$$\mathcal{L}_{(k,2,k+2)} \times \mathcal{L}_{(a,s,c)} = \mathcal{L}_{(k-a,s+2,c+k+2)} \,. \tag{3.1.79}$$

Then in the theory with gauge group (3.1.75) we identify

$$(a, s, c) \sim (k - a, s + 2, c + k + 2) \tag{3.1.80}$$

and keep only gauge invariant lines with $a + c + s = 0 \bmod 2$. Notice that the action by fusion has no fixed points, therefore there is no doubled line. These of course match the field identifications and restrictions in the coset (3.1.74). In our analysis above we have chosen, following [249], a particular gauge fixing in which $s = 0, 1 = [a+c]$, then anyons can be labelled by a pair of integers $(a, c) \in Q_k$. The S-matrix, fusion rules and F-symbols are well defined on the anyons of the gauged theory, one only needs to be careful with the gauge fixing chosen when writing down explicit expressions.

When discussing the full CFT and not only a chiral half the coupled $3d$-$2d$ system consists of a $3d$ bulk given by a Chern-Simons theory with gauge group $G_k \times G_{-k}$ with two boundary conditions, a conformal one and a topological one. Anyons of the theory are labelled by four integers $(a_L, c_L, a_R, c_R) \in Q_k \times Q_k$ giving the labels of the Wilson lines for G_k and G_{-k} respectively. Lines can terminate on the conformal boundary giving rise to local, but not necessarily genuine, operators labeled by the same four integers (a_L, c_L, a_R, c_R). As usual the physical spectrum is determined by the topological boundary condition [17].

3.1.2 Symmetries of Gepner Models

By the Landau-Ginzburg/Calabi-Yau correspondence the Gepner model can be constructed as an orbifold of a tensor product of $\mathcal{N} = 2$ minimal models, in particular

$$\left(\bigotimes_{i=1}^{r} M_{k_i} \right) \Big/ \mathbb{Z}_H \tag{3.1.81}$$

where M_{k_i} is a minimal model at level k_i and $H = \mathrm{lcm}\{k_i + 2\}$, with the group $\mathbb{Z}_H$ being generated by the line operator

$$\bigotimes_{i=1}^{r} L_{0,2} \; . \tag{3.1.82}$$

The levels are chosen to satisfy the Calabi-Yau condition (3.1.10) so that the total central charge is a multiple of 3

$$c = \sum_{i=1}^{r} \frac{3k_i}{k_i + 2} = 3(r - 2) \, . \tag{3.1.83}$$

Before carrying out the $\mathbb{Z}_H$ orbifold we have to perform the correct GSO projection. This has to be imposed simultaneously on all the minimal models, namely we are going to allow only states whose components along the single minimal models are all either in the NS or in R sector. We detail the construction of the Gepner model via subsequent gaugings starting from the product of GSO-projected minimal models in the first subsection. In the rest of this section we study the spectrum of the model as well as its symmetries.

Construction of the Model

To construct the model we start from the product of GSO-projected minimal models, the partition function is simply

$$Z(\tau,z)=\prod_{i=1}^{r}\sum_{(a_i,c_i)\in Q_{k_i}}|\chi_{a_i,c_i}(q,y)|^2\ , \tag{3.1.84}$$

and the physical primaries are

$$\Phi_{\{a_i,c_i\}}=\bigotimes_{i=1}^{r}\Phi^{(i)}_{a_i,c_i}\ . \tag{3.1.85}$$

Clearly the TDLs of the tensor product theory are just tensor products of the lines of each minimal model, therefore in total we have

$$\prod_{i=1}^{r}2(k_1+1)(k_i+2) \tag{3.1.86}$$

topological line defects. To achieve the correct GSO projection consider the lines

$$L_{k_1,k_1+2}\otimes L_{k_i,k_i+2}\qquad i=2,\dots r\ , \tag{3.1.87}$$

these generate a $\mathbb{Z}_2^{r-1}$ group and act as

$$L_{k_1,k_1+2}\otimes L_{k_i,k_i+2}\Phi_{\{a_i,c_i\}}=(-1)^{a_1+c_1+a_i+c_i}\Phi_{\{a_i,c_i\}}\ . \tag{3.1.88}$$

Therefore gauging this $\mathbb{Z}_2^{r-1}$ enforces the correct projection, and the addition of the corresponding twisted sectors ensures modular invariance. Instead of going through the gauging procedure via the insertion of defects a quicker way to obtain the correct expression is to consider the diagonal modular invariant partition function one would write using the full superconformal characters

$$Z_{\text{GSO}}(\tau,z)=\sum_{\substack{\{l_i,m_i\}\\ \lambda=0,-1/2}}\prod_{i=1}^{r}|\text{ch}^{(\lambda)}_{l_i,m_i}(q,y)|^2+\prod_{i=1}^{r}|\widetilde{\text{ch}}^{(\lambda)}_{l_i,m_i}(q,y)|^2\ , \tag{3.1.89}$$

in which the proper NS/R alignement is imposed by hand and is manifestly S and T invariant. Now, rewriting it in terms of the half-characters χ_{ac}, we have

$$\begin{aligned}
Z_{\text{GSO}}(\tau,z)&=\sum_{A\in\mathcal{S}_r}Z_A(\tau,z)\\
Z_A(\tau,z)&=\sum_{\{a_i,c_i\}}P^{\text{GSO}}_{\{a_i,c_i\}}\prod_{i\in A^\perp}\chi_{a_i,c_i}(q,y)\chi_{a_i,c_i}(\bar q,\bar y)\prod_{i\in A}\chi_{a_i,c_i}(q,y)\chi_{k_i-a_i,c_i+k_i+2}(\bar q,\bar y)
\end{aligned} \tag{3.1.90}$$

where A is an ordered subset of $\{1,\dots,r\}$ of even order, $A^\perp$ is its complement and

$$P^{\text{GSO}}_{\{a_i,c_i\}}=\prod_{j>1}\frac{1+(-1)^{a_1+c_1+a_j+c_j}}{2} \tag{3.1.91}$$

enforces the proper projection. We have also denoted by $\mathcal{S}_r$ the set of all ordered subsets of $\{1,\dots,r\}$ of even order. The sum over A is the sum over twisted sectors of the $\mathbb{Z}_2^{r-1}$, indeed

$|\mathcal{S}_r| = 2^{r-1}$ and there are exactly $2^{r-1}-1$ non-empty ordered subsets of $\{1,\ldots,r\}$ of even order, one for each non-trivial element of $\mathbb{Z}_2^{r-1}$[4].

We now repeat the bootstrapping analysis for the TDLs of this theory. The physical primaries are labelled as $\Phi_{\{a_i,c_i\},A}$ and are subject to the NS/R alignement constraint $[a_1+c_1] = [a_i+c_i]\ \forall i = 2,\ldots,r$. A TDL $L_{\{r_i,s_i\},B}$ acts by

$$L_{\{r_i,s_i\},B}\Phi_{\{a_i,c_i\},A} = \left(\zeta_{AB}\prod_{i=1}^{r} X^{a_i,c_i}_{r_i,s_i}\right)\Phi_{\{a_i,c_i\},A}\,, \tag{3.1.93}$$

here we have added an extra sign ζ_{AB} which parametrizes the quantum $\mathbb{Z}_2^{r-1}$ symmetry acting on the twisted sectors, namely $\zeta_{AB} = (-1)^{\delta_{AB}}$ and $\zeta_{A\varnothing} = 1$. The constraint on multiplicities is

$$\sum_{A\in\mathcal{S}_r}\zeta_{AB}\sum_{\{a_i,c_i\}} P^{\rm GSO}_{\{a_i,c_i\}}\prod_{i\in A^\perp} X^{r_i,s_i}_{a_i,c_i} S_{a_i,c_i;a_i',c_i'} S^*_{a_i,c_i;a_i'',c_i''}\prod_{i\in A} X^{r_i,s_i}_{a_i,c_i} S_{a_i,c_i;a_i',c_i'} S^*_{k_i-a_i,c_i+k_i+2;a_i'',c_i''} \in \mathbb{N}\,. \tag{3.1.94}$$

We first notice that

$$\begin{aligned} P^{\rm GSO}_{\{a_i,c_i\}} &= \frac{1}{2^{r-1}}\sum_{A'\in\mathcal{S}_r}(-1)^{\sum_{i\in A'}a_i+c_i} = \frac{1}{2^{r-1}}\sum_{A'\in\mathcal{S}_r}\prod_{i\in A'}\frac{S_{k_i,k_i+2;a_i,c_i}}{S_{0,0;a_i,c_i}}\,,\\ S_{k_i-a_i,c_i+k_i+2;a_i',c_i'} &= (-1)^{a_i'+c_i'} S_{a_i,c_i;a_i',c_i'}\,. \end{aligned} \tag{3.1.95}$$

Then

$$(3.1.94) = \frac{1}{2^{r-1}}\sum_{A,A'\in\mathcal{S}_r}(-1)^{\sum_{i\in A}a_i''+c_i''}\zeta_{AB}\prod_{i\in A'}\left(\sum_{a_i,c_i} X^{r_i,s_i}_{a_i,c_i}\frac{S_{k_i,k_i+2;a_i,c_i}}{S_{0,0;a_i,c_i}} S_{a_i,c_i;a_i',c_i'} S^*_{a_i,c_i;a_i'',c_i''}\right)\times \prod_{i\in A'^\perp}\left(\sum_{a_i,c_i} X^{r_i,s_i}_{a_i,c_i} S_{a_i,c_i;a_i',c_i'} S^*_{a_i,c_i;a_i'',c_i''}\right)\,. \tag{3.1.96}$$

Now, setting

$$X^{r_i,s_i}_{a_i,c_i} = \frac{S_{r_i,s_i;a_i,c_i}}{S_{00;a_i,c_i}} \tag{3.1.97}$$

and using that $X^{r_i,s_i}_{a_i,c_i}$ is a one-dimensional representation of the fusion ring[5] we find the multiplicities

$$\frac{1}{2^{r-1}}\left(\sum_{A\in\mathcal{S}_r}(-1)^{\sum_{i\in A}a_i''+c_i''}\zeta_{AB}\right)\sum_{A'\in\mathcal{S}_r}\prod_{i\in A'^\perp} N^{a_i''c_i''}_{r_i,s_i;a_i',c_i'}\prod_{i\in A'} N^{a_i''c_i''}_{k_i-r_i,s_i+k_i+2;a_i',c_i'}\,. \tag{3.1.100}$$

---

[4] A generic element of $\mathbb{Z}_2^{r-1}$ corresponds to the line

$$\bigotimes_{j\in A} L_{k_j,k_j+2} \qquad \text{with} \qquad A\in\mathcal{S}_r\,. \tag{3.1.92}$$

[5] Concretely

$$\frac{S_{r,s;a,c}}{S_{00;a,c}}\frac{S_{r',s';a,c}}{S_{00;a,c}} = \sum_{(r'',s'')\in Q_k} N^{r''s''}_{rs;r's'}\frac{S_{r'',s'';a,c}}{S_{00;a,c}}\,. \tag{3.1.98}$$

In the case at hand

$$\frac{S_{r,s;a,c}}{S_{00;a,c}}\frac{S_{k,k+2;a,c}}{S_{00;a,c}} = \frac{S_{k-r,s+k+2;a,c}}{S_{00;a,c}} \tag{3.1.99}$$

which is equivalent to the statement that the line $L_{k,k+2}$ is invertible in the fusion ring of a single minimal model.

The factor

$$\frac{1}{2^{r-1}}\left(\sum_{A}(-1)^{\sum_{i\in A\in\mathcal{S}_r} a_i''+c_i''}\zeta_{AB}\right) \tag{3.1.101}$$

is again a projector, and the whole expression is always a positive integer. Clearly for $\zeta_{AB}=1$ and $r_i=s_i=0$ we get back the original partition function. We then can conclude that the GSO-projected theory has TDLs $L_{\{r_i,s_i\},B}$ acting as

$$L_{\{r_i,s_i\},B}\Phi_{\{a_i,c_i\},A}=\left(\zeta_{AB}\prod_{i=1}^{r}\frac{S_{r_i,s_i;a_i,c_i}}{S_{00;a_i,c_i}}\right)\Phi_{\{a_i,c_i\},A}\,, \tag{3.1.102}$$

however because of the NS/R alignement condition on physical primaries the parametrization above is redundant. Namely a line with labels $\{r_i,s_i\}$ and one in which we replace $(r_j,s_j)\mapsto(k_j-r_j,s_j+k_j+2)$ for any even number of values of $j=1,\dots,r$ act in the same way on physical primaries and can be identified. Nevertheless, because of the presence of the quantum symmetry, the total number of faithfully acting lines is preserved by the orbifold, and still equals (3.1.86). We shall comment further on these redundancies later.

Among the TDLs we found there is also the $\mathbb{Z}_H$ group that we have to gauge to obtain the Gepner model. Indeed setting $(r_i,s_i)=(0,2)$ for every $i=1,\dots r$ as well as $B=\varnothing$ we obtain a line $L_{\{0,2\},\varnothing}$ acting as

$$L_{\{0,2\},\varnothing}\Phi_{\{a_i,c_i\},A}=e^{2i\pi\sum_{i=1}^{r}\frac{c_i}{k_i+2}}\Phi_{\{a_i,c_i\},A}\,. \tag{3.1.103}$$

The final step is then to gauge this symmetry. We start by letting a line $L_{\{0,2s\},\varnothing}$ act on the circle Hilbert space, namely we insert it along the space cycle of the torus. This gives us the twined partition function

$$\begin{aligned}Z_{\mathrm{GSO}}(\tau,z,s)=&\sum_{A;\{a_i,c_i\}}P^{\mathrm{GSO}}_{\{a_i,c_i\}}e^{2i\pi s\sum_{i=1}^{r}\frac{c_i}{k_i+2}}\times\\&\times\prod_{i\in A^{\perp}}\chi_{a_i,c_i}(q,y)\chi_{a_i,c_i}(\bar q,\bar y)\prod_{i\in A}\chi_{a_i,c_i}(q,y)\chi_{k_i-a_i,c_i+k_i+2}(\bar q,\bar y)\end{aligned} \tag{3.1.104}$$

where $s\in\mathbb{Z}_H$ is the extra fugacity. By means of an S transformation on the expression above, or directly employing the multiplicities (3.1.100), we obtain the twisted partition functions

$$\begin{aligned}Z_{\mathrm{GSO},x}(\tau,z)=&\sum_{A;\{a_i,c_i\}}P^{\mathrm{GSO}}_{\{a_i,c_i\}}\times\\&\times\prod_{i\in A^{\perp}}\chi_{a_i,c_i}(q,y)\chi_{a_i,c_i+2x}(\bar q,\bar y)\prod_{i\in A}\chi_{a_i,c_i}(q,y)\chi_{k_i-a_i,c_i+k_i+2+2x}(\bar q,\bar y)\,,\end{aligned} \tag{3.1.105}$$

with $x\in\mathbb{Z}_H$. To combine both the twisting and twining operations we need to know how the symmetry acts on twisted sectors. For invertible abelian symmetries however this action is already completely encoded in the modular transformations. Indeed the T transformation mixes the two cycles of the torus, and acting with it on $Z_{\mathrm{GSO},x}$ twines the twisted partition function. Since T is diagonal on the half-characters the result of its action is the phase

$$\exp\left[2i\pi\left(\sum_{i\in A^{\perp}}h_{a_i,c_i}-h_{a_i,c_i+2x}+\sum_{i\in A}h_{a_i,c_i}-h_{k_i-a_i,c_i+2x+k_i+2}\right)\right] \tag{3.1.106}$$

multiplying the characters. From the explicit expression of the weights we have

$$
\begin{aligned}
h_{a_i,c_i} - h_{a_i,c_i+2x} &= x\frac{c_i+x}{k_i+2} \mod 1\,,\\
h_{a_i,c_i} - h_{k_i-a_i,c_i+2x+k_i+2} &= \frac{a_i+c_i+1}{2} + x\frac{c_i+x}{k_i+2} \mod 1\,.
\end{aligned}
\tag{3.1.107}
$$

Using the NS/R alignement constraint, the Calabi-Yau condition (3.1.10) and the fact that the order of A is always even the phase simplifies to

$$
e^{2i\pi x \sum_{i=1}^{r}\frac{c_i}{k_i+2}}\,. \tag{3.1.108}
$$

Acting multiple times with T we obtain power of this phase, therefore the twisted and twined partition function is

$$
\begin{aligned}
Z_{\mathrm{GSO},x}(\tau,z,s) = &\sum_{A\in\mathcal{S}_r;\{a_i,c_i\}} P^{\mathrm{GSO}}_{\{a_i,c_i\}} e^{2i\pi s\sum_{i=1}^{r}\frac{c_i}{k_i+2}}\times\\
&\times\prod_{i\in A^{\perp}}\chi_{a_i,c_i}(q,y)\chi_{a_i,c_i+2x}(\bar q,\bar y)\prod_{i\in A}\chi_{a_i,c_i}(q,y)\chi_{k_i-a_i,c_i+k_i+2+2x}(\bar q,\bar y)\,.
\end{aligned}
\tag{3.1.109}
$$

The Gepner model partition function is then obtained summing over $x, s \in \mathbb{Z}_H$

$$
Z_{\mathrm{Gep}}(\tau,z) = \frac{1}{H}\sum_{s,x\in\mathbb{Z}_H} Z_{\mathrm{GSO},x}(\tau,z,s) \tag{3.1.110}
$$

The sum over s produces a projector

$$
P^{\mathbb{Z}_H}_{\{a_i,c_i\}} = \frac{1}{H}\sum_{s\in\mathbb{Z}_H} e^{2i\pi s\sum_{i=1}^{r}\frac{c_i}{k_i+2}} = \delta\left(\sum_{i=1}^{r}\frac{c_i}{k_i+2} = 0 \bmod 1\right) \tag{3.1.111}
$$

we might then express the total partition function as a sum over the twisted sector contributions

$$
Z_{\mathrm{Gep}}(\tau,z) = \sum_{x\in\mathbb{Z}_H;A\in\mathcal{S}_r} Z_{x,A}(\tau,z) \tag{3.1.112}
$$

with

$$
\begin{aligned}
Z_{x,A}(\tau,z) = &\sum_{\{a_i,c_i\}} P^{\mathrm{GSO}}_{\{a_i,c_i\}} P^{\mathbb{Z}_H}_{\{a_i,c_i\}}\times\\
&\times\prod_{i\in A^{\perp}}\chi_{a_i,c_i}(q,y)\chi_{a_i,c_i+2x}(\bar q,\bar y)\prod_{i\in A}\chi_{a_i,c_i}(q,y)\chi_{k_i-a_i,c_i+k_i+2+2x}(\bar q,\bar y)\,.
\end{aligned}
\tag{3.1.113}
$$

This is the partition function of the Gepner model.

Another method to construct Gepner models is via simple current extensions [247, 258]. In particular starting from the tensor product of the GSO-projected minimal models (3.1.84) the simple current of interest are

$$
\begin{aligned}
J_i &= \Phi_{k_1,k_1+2}\otimes\mathbb{1}\ldots\otimes\Phi_{k_i,k_i+2}\otimes\ldots\mathbb{1} \qquad i=2,\ldots r\\
J_{\mathrm{orb}} &= \Phi_{k_1,k_1+4}\Phi^{3(r-2)}_{k_1,k_1+2}\bigotimes_{i=2}^{r}\Phi_{k_i,k_i+4}\,.
\end{aligned}
\tag{3.1.114}
$$

Extending the diagonal theory by these currents means to gauge the corresponding Verlinde lines. The J_i correspond to the $\mathbb{Z}_2^{r-1}$ group imposing the GSO projection, while J_{orb} corresponds to a line acting as

$$\Phi_{\{a_i,c_i\}} \mapsto (-1)^{3r(a_1+c_1)}(-1)^{\sum_{i=1}^r (a_i+c_i)} e^{2i\pi \sum_i \frac{c_i}{k_i+2}} \Phi_{\{a_i,c_i\}} \,. \tag{3.1.115}$$

In the theory before NS/R alignement this is not a $\mathbb{Z}_H$ action, rather its order depends on the various levels k_i [247]. However proceeding in steps as we did and performing first the $\mathbb{Z}_2^{r-1}$ gauging it is immediate to check that, on physical primaries of the NS/R aligned theory, the action above turns exactly in the $\mathbb{Z}_H$ we have considered. Another way to state this is that the identification of the $\mathbb{Z}_H$ in the theory (3.1.84) is ambiguous because of the redundancies we discussed around (3.1.102). The fact that the action of J_{orb} and $\mathbb{Z}_H$ coincide after GSO projection precisely means that the two are identified in the aligned theory.

Spectrum and Exactly Marginal Deformations

By inspection of the partition function we read off the circle Hilbert space

$$\begin{aligned} \mathbb{H} &= \bigoplus_{x\in\mathbb{Z}_H;A} \mathbb{H}^{(x,A)} \\ \mathbb{H}^{(x,A)} &= \bigoplus_{\{a_i,c_i\}} \bigotimes_{i\in A^\perp} \mathbb{H}_{a_i,c_i} \otimes \overline{\mathbb{H}}_{a_i,c_i+2x} \bigotimes_{i\in A} \mathbb{H}_{a_i,c_i} \otimes \overline{\mathbb{H}}_{k_i-a_i,c_i+k_i+2+2x} \end{aligned} \tag{3.1.116}$$

subject to the conditions

$$\sum_i \frac{c_i}{k_i+2} = 0 \mod 1\,, \qquad [a_1+c_1] = [a_2+c_2] = \ldots = [a_r+c_r]\,. \tag{3.1.117}$$

The first condition ensures that, for all states at the Gepner point, both the left and right $U(1)_R$ charges are integer for states coming from the NS sector (and half-integer for R sector states). This integrality condition is crucial in string compactification as it ensures spacetime supersymmetry. We start by noticing that this spectrum contains a single spectral flow operator. The vacuum state is unique and the spectral flow operator is obtained acting on it with 1/2 units of spectral flow. Clearly doing this only on a subsets of the minimal models violates NS/R alignement, only acting on all minimal models simultaneously we get a state that's still in the spectrum. The spectral flow operator corresponds to the state

$$\bigotimes_{i=1}^r |0,1\rangle \otimes \overline{|0,1\rangle} \in \mathbb{H}^{0,\varnothing} \tag{3.1.118}$$

which satisfies the constraints and has

$$h_L = h_R = \frac{c}{24} = \frac{r-2}{8} \qquad q_L = q_R = -\frac{c}{6} = -\frac{r-2}{2}\,. \tag{3.1.119}$$

We can also spectral flow by $-1/2$ unit obtaining a state with $h_L = h_R = (r-2)/8$ and $q_L = q_R = (r-2)/2$. Similarly with a ± 1 unit of spectral flow we obtain states with $h_L = h_R = (r-2)/2$ and $q_L = q_R = \pm(r-2)$. The CY sigma model has an (extended) $\mathcal{N} = (2,2)$ algebra, which at the Gepner point, is realized as the diagonal subalgebra of the tensor product of algebras of the single minimal models. Note that $h_L - h_R \in \mathbb{Z}$ consistently with T invariance of the partition function and that $q_L, q_R \in \frac{1}{2}\mathbb{Z}$, with the half-integer charge state being those in the R sector.

We are interested in the states corresponding to exactly marginal deformations, as those allow to probe the moduli space of the Calabi-Yau manifold. In order to preserve $\mathcal{N} = 2$ supersymmetry the deformation has to be BPS. The condition of marginality selects a subset of the BPS operators, the requirement being that the deformation preserves both the holomorphic and antiholomorphic R-symmetries. It turns out that the superconformal Ward identities imply that any $\mathcal{N} = 2$-supersymmetric marginal deformation is exactly marginal [234,261]. Therefore, modulo complex conjugation, there are two classes of operators to consider:

- Given a chiral-chiral primary $\phi(z, \bar{z})$ with $q_L = q_R = 1$ its descendant

$$\left(\overline{G}^{-}_{-1/2} G^{-}_{-1/2} \phi\right)(w, \bar{w}) \tag{3.1.120}$$

  has $h_L = h_R = 1$ and $q_L = q_R = 0$.

- Given an antichiral-chiral primary $\phi(z, \bar{z})$ with $q_L = -q_R = 1$ its descendant

$$\left(G^{+}_{-1/2} \overline{G}^{-}_{-1/2} \phi\right)(w, \bar{w}) \tag{3.1.121}$$

  has $h_L = h_R = 1$ and $q_L = q_R = 0$.

In both those cases, in order to have a real deformation one has to add the complex conjugate field given by (a descendant of) the antichiral-antichiral or chiral-antichiral primary. In terms of representations of the bosonic subalgebra notice that the exactly marginal deformation has the same total fermion parity of its primary state, thus to detect its presence in the spectrum it is enough to find the corresponding primary.

The BPS spectrum of the Gepner model, in the NS sector, consists of the four chiral rings. A primary of the Gepner model is chiral or antichiral if it is so under every $\mathcal{N} = 2$ subalgebra of to the single minimal models[6]. Let's then take the generic state in a sector with both $A, A^{\perp} \neq \varnothing$, requiring the holomorphic side to be chiral or antichiral we need to set $c_i = \pm a_i$ for all i. Requiring the antiholomorphic side to be of the same type as the holomorphic part

[6]To see this we consider the generator of the diagonal subalgebra

$$G^{\pm}_{-s} = \sum_{i=1}^{r} \mathbb{1} \otimes \ldots \otimes G^{(i);\pm}_{-s} \otimes \ldots \mathbb{1} \tag{3.1.122}$$

and let it act on a tensor product state

$$G^{\pm}_{-s} \bigotimes_{i=1}^{r} |\phi_i\rangle \tag{3.1.123}$$

assuming that the ϕ_i are orthonormal we see that the norm of such descendant state is

$$\left|\left| G^{\pm}_{-s} \bigotimes_{i=1}^{r} |\phi_i\rangle \right|\right|^2 = \sum_{i=1}^{r} \left|\left| G^{(i);\pm}_{-s} |\phi_i\rangle \right|\right|^2 . \tag{3.1.124}$$

Then the total state is annihilated if and only if all its components are.

we would need $c_i + 2x = \pm a_i$ for $i \in A^\perp$ and $c_i + k_i + 2 + 2x = \pm(k_i - a_i)$ for $i \in A$. Clearly this sets $x = 0$ and the remaining equation is

$$\pm a_i + k_i + 2 = \pm(k_i - a_i) \bmod 2(k_i + 2) \quad \Rightarrow \quad 2a_i + 2 = 0 \bmod 2(k_i + 2) \tag{3.1.125}$$

which has no solution in the range $a_i = 0, \dots k_i$. The first extremal case to consider, that is present for any r, is $A = \varnothing$. Here it is easy to find the BPS states

$$\bigotimes_{i=1}^{r} |a_i, \pm a_i\rangle \otimes \overline{|a_i, \pm a_i\rangle} \in \mathbb{H}^{(0,\varnothing)}\,, \tag{3.1.126}$$

of course only those with integral $U(1)_R$ charges survive the projection. The other extremal case to consider is when r is even and $A = \{1, \dots r\}$. Here again $c_i = \pm a_i$ for all i, but we are no longer forced to set $x = 0$, rather we need to solve the equations

$$2(a_i + 1 \pm x) = 0 \bmod 2(k_i + 2) \quad \forall i = 1, \dots r \tag{3.1.127}$$

which fix

$$\begin{aligned} a_i &= k_i + 1 - x\,, \\ a_i &= x - 1\,. \end{aligned} \tag{3.1.128}$$

for + and − signs respectively. In the range $a_i = 0, \dots, k_i$ we have $1 \leq s \leq \min(k_i) + 1$. Therefore for r even we find two extra conjugate BPS states for each non-zero $x \in \mathbb{Z}_H$

$$\begin{aligned} &\bigotimes_{i=1}^{r} |k_i + 1 - x, k_i + 1 - x\rangle \otimes \overline{|x - 1, x - 1\rangle} \in \mathbb{H}^{(s,\{1,\dots r\})} \\ &\bigotimes_{i=1}^{r} |x - 1, -x + 1\rangle \otimes \overline{|k_i + 1 - x, -k_i - 1 + x\rangle} \in \mathbb{H}^{(x,\{1,\dots r\})} \end{aligned} \tag{3.1.129}$$

which also satisfy the charge constraint. We conclude that the chiral-chiral or antichiral-antichiral states can only belong to the untwisted sector and, for r even to the twisted sectors with $A = \{1, \dots, r\}$. Marginal deformations correspond to those states with $|q_L| = |q_R| = 1$, in the untwisted sector this constraint is

$$\sum_{i=1}^{r} \frac{a_i}{k_i + 2} = 1 \tag{3.1.130}$$

due to the CY condition the state with $a_i = 1$ for all i are always a solution. The chiral-chiral and antichiral-antichiral states in $\mathbb{H}^{(s,\{1,\dots,r\})}$ have charges

$$\begin{aligned} &q_L = \sum_{i=1}^{r} \frac{k_i + 1 - x}{k_i + 2} = \frac{c}{3} + 1 - x = r - 1 - x\,; \quad q_R = x - 1 \\ &q_L = 1 - x\,; \quad q_R = -r + 1 + x \end{aligned} \tag{3.1.131}$$

respectively. Those giving rise to a marginal deformation have

$$r - 1 - x = x - 1 = 1 \iff x = 2\,, r = 4\,. \tag{3.1.132}$$

We can repeat the analysis for the chiral-antichiral states and their conjugates. For $A, A^{\perp} \neq \varnothing$ the equations are

$$
\begin{aligned}
& c_i = \pm a_i \ , \ \forall i \\
& c_i + 2x = \mp a_i \ , \ i \in A^{\perp} \\
& c_i + k_i + 2 + 2x = \mp (k_i - a_i) \ , \ i \in A
\end{aligned}
\tag{3.1.133}
$$

and one easily sees that there is no admissible solution. We again consider first the extremal case $A = \varnothing$, here we do not have to impose the third equation above, therefore we have

$$
c_i = \pm a_i \qquad x = \mp a_i \tag{3.1.134}
$$

hence $a_i = a$ for all i. The corresponding states are

$$
\bigotimes_{i=1}^{r} |a, \pm a\rangle \otimes \overline{|a, \mp a\rangle} \in \mathbb{H}^{(\mp a, \varnothing)} \, . \tag{3.1.135}
$$

The other extremal case is again $A = \{1, \ldots, r\}$ for r even. Here we have $c_i = \pm a_i$ and $x = \pm 1$, the states are

$$
\bigotimes_{i=1}^{r} |a_i, \pm a_i\rangle \otimes \overline{|k_i - a_i, \mp (k_i - a_i)\rangle} \in \mathbb{H}^{(\pm 1, \{1, \ldots, r\})} \, . \tag{3.1.136}
$$

Thus we have an antichiral-chiral BPS state for every $\mathbb{H}^{(x,\varnothing)}$ with appropriate x and, if r is even, we have more coming from the twisted sector $\mathbb{H}^{(\pm 1, \{1, \ldots r\})}$. Among those in the $\mathbb{H}^{(x,\varnothing)}$ sectors only for $x = \pm 1$ we have a marginal deformation. Instead the conditions on the R charge of the states showing up for r even are

$$
\sum_{i=1}^{r} \frac{a_i}{k_i + 2} = 1 \, , \qquad \sum_{i=1}^{r} \frac{k_i - a_i}{k_i + 2} = 1 \tag{3.1.137}
$$

if the first condition is met the second requires

$$
1 = \sum_{i=1}^{r} \frac{k_i - a_i}{k_i + 2} = r - 3 \tag{3.1.138}
$$

i.e $r = 4$. Summarizing, the marginal deformations are:

- chiral-chiral and antichiral-antichiral states. For any r

$$
\bigotimes_{i=1}^{r} |a_i, \pm a_i\rangle \otimes \overline{|a_i, \pm a_i\rangle} \in \mathbb{H}^{(0,\varnothing)} \qquad \sum_{i=1}^{r} \frac{a_i}{k_i + 2} = 1 \, . \tag{3.1.139}
$$

For $r = 4$ and $H > 2$ we also have

$$
\begin{aligned}
& \bigotimes_{i=1}^{4} |k_i - 1, k_i - 1\rangle \otimes \overline{|1, 1\rangle} \in \mathbb{H}^{(2, \{1, \ldots, 4\})} \\
& \bigotimes_{i=1}^{4} |1, -1\rangle \otimes \overline{|k_i - 1, 1 - k_i\rangle} \in \mathbb{H}^{(2, \{1, \ldots, 4\})} \, .
\end{aligned}
\tag{3.1.140}
$$

- antichiral-chiral and chiral-antichiral states. For any r there's only one with the correct R-charges

$$\bigotimes_{i=1}^{r} |1, \pm 1\rangle \otimes \overline{|1, \mp 1\rangle} \in \mathbb{H}^{(\mp 1, \varnothing)}. \tag{3.1.141}$$

For $r = 4$ and $H > 2$ we also have

$$\bigotimes_{i=1}^{4} |a_i, \pm a_i\rangle \otimes \overline{|k_i - a_i, \mp(k_i - a_i)\rangle} \in \mathbb{H}^{(\pm 1, \{1,\dots,4\})}, \qquad \sum_{i=1}^{r} \frac{a_i}{k_i + 2} = 1\,. \tag{3.1.142}$$

The case $r = 4$ is evidently special as it admits particular marginal deformations. The theory has $c = 3(r - 2) = 6$, corresponding to a sigma model on a $K3$ surface. We notice that for $r = 4$ we have the same number of chiral-chiral and antichiral-chiral marginal deformations. It is also known that there is an enhancement of supersymmetry and the theory enjoys an $\mathcal{N} = (4, 4)$ superconformal algebra, therefore the marginal deformations moving along the moduli space should preserve the full supersymmetry. Indeed the degeneracy between chiral-chiral and antichiral-chiral states meets the expectation that $\mathcal{N} = 2$ BPS states pair up to form a BPS multiplet for the larger algebra. For $r = 4$ we also have the states

$$\bigotimes_{i=1}^{r} |k_i, k_i\rangle \otimes \overline{|0, 0\rangle} \in \mathbb{H}^{(1,\{1,\dots 4\})} \qquad \bigotimes_{i=1}^{r} |k_i, -k_i\rangle \otimes \overline{|0, 0\rangle} \in \mathbb{H}^{(-1,\{1,\dots 4\})} \tag{3.1.143}$$

which correspond to holomorphic operators with $h_L = 1$ and $q_L = \pm 2$. Similar states exist also for the antiholomorphic side. Together with the R-symmetry generator we have three currents that transform in the adjoint of $SU(2)_R$, the R-symmetry of the $\mathcal{N} = 4$ algebra. It is known, see e.g. [262], that for any CY sigma model the chiral algebra is extended, for K3 the resulting symmetry is $\mathcal{N} = 4$ supersymmetry, while in general the algebra is the $\mathcal{N} = 2$ one extended by the square of the spectral flow operator [263]. One could then expect that also when r is odd new holomorphic and antiholomorphic states corresponding to this operator show up in the spectrum, this however is not the case. Indeed if the complex dimension of the CY is odd this operator is a fermion with half-integer weight, therefore, due to the GSO projection, it cannot appear as a purely left-moving state, rather it can only appear tensored with a right-moving fermion. When r is even instead these operators are bosonic, and appear tensored with the identity in the antiholomorphic sector precisely in the twisted sectors with $A = \{1, \dots, r\}$.

Topological Defect Lines and $3d$ TQFT

We now investigate the symmetries of the model. Again our strategy is to bootstrap the action of line operators on physical primaries imposing consistency of the twisted Hilbert spaces. The physical primaries appearing in (3.1.112) can be labelled as $\Phi_{\{a_i,c_i\},A,x}$ with the set of labels $\{a_i, c_i\}$ subject to

$$\sum_i \frac{c_i}{k_i + 2} = 0 \mod 1\,, \qquad [a_1 + c_1] = [a_2 + c_2] = \dots = [a_r + c_r]\,. \tag{3.1.144}$$

Since we obtained the Gepner model as a $\mathbb{Z}_H$ orbifold of the GSO-projected theory we can factor out the dual $\mathbb{Z}_H$ symmetry in our ansatz, we set

$$\mathcal{L}_{\{r_i,s_i\},B,\eta}\Phi_{\{a_i,c_i\},A,x} = \zeta_{AB} e^{2i\pi\frac{\eta x}{H}} \left(\prod_{i=1}^{r} X^{a_i,c_i}_{r_i,s_i}\right) \Phi_{\{a_i,c_i\},A,x}\,. \tag{3.1.145}$$

Inserting $\mathcal{L}_{\{r_i,s_i\},B,\eta}$ as an operator acting on the Hilbert space and acting with an S transformation we obtain the multiplicities in the twisted sector

$$\begin{aligned}&\sum_{x\in\mathbb{Z}_H;A\in\mathcal{S}_r}\zeta_{AB}e^{2i\pi\frac{\eta x}{H}}\sum_{\{a_i,c_i\}}P^{\text{GSO}}_{\{a_i,c_i\}}P^{\mathbb{Z}_H}_{\{a_i,c_i\}}\times\\&\times\prod_{i\in A^\perp}X^{a_i,c_i}_{r_i,s_i}S_{a_ic_i;a'_ic'_i}S^*_{a_i,c_i+2x;a''_i,c''_i}\prod_{i\in A}X^{a_i,c_i}_{r_i,s_i}S_{a_ic_i;a'_ic'_i}S^*_{k_i-a_i,c_i+k_i+2+2x;a''_i,c''_i}\,.\end{aligned}\tag{3.1.146}$$

To rewrite this in a more manageable form we use

$$\begin{aligned}S_{k-a,c+k+2+2x;a',c'}&=(-1)^{a'+c'}e^{2i\pi\frac{xc'}{k+2}}S_{ac;a'c'}\\P^{\mathbb{Z}_H}_{\{a_i,c_i\}}&=\frac{1}{H}\sum_{s\in\mathbb{Z}_H}e^{2i\pi s\sum_{i=1}^r\frac{c_i}{k_i+2}}=\frac{1}{H}\sum_{s\in\mathbb{Z}_H}\prod_{i=1}^r\frac{S_{0,2s;a_ic_i}}{S_{00;a_ic_i}}\end{aligned}\tag{3.1.147}$$

and also (3.1.95) for the GSO projector. We get

$$\begin{aligned}&(3.1.146)=\frac{1}{2^{r-1}H}\sum_{x\in\mathbb{Z}_H;A}\zeta_{AB}(-1)^{\sum_{i\in A}a''_i+c''_i}e^{2i\pi x\left(\frac{\eta}{H}+\sum_{i=1}^r\frac{c''_i}{k_i+2}\right)}\\&\sum_{\substack{\{a_i,c_i\};\\A',s\in\mathbb{Z}_H}}\prod_{i\in A'^\perp}X^{a_i,c_i}_{r_i,s_i}\frac{S_{0,2s;a_ic_i}}{S_{00;a_ic_i}}S_{a_ic_i;a'_ic'_i}S^*_{a_i,c_i;a''_i,c''_i}\prod_{i\in A'}X^{a_i,c_i}_{r_i,s_i}\frac{S_{0,2s;a_ic_i}}{S_{00;a_ic_i}}\frac{S_{k_i,k_i+2;a_i,c_i}}{S_{0,0;a_i,c_i}}S_{a_ic_i;a'_ic'_i}S^*_{a_i,c_i;a''_i,c''_i}\,.\end{aligned}\tag{3.1.148}$$

Again we see that there is a natural solution

$$X^{a_i,c_i}_{r_i,s_i}=\frac{S_{r_i,s_i;a_i,c_i}}{S_{00;a_i,c_i}}\tag{3.1.149}$$

that gives the integer multiplicities

$$\begin{aligned}&N^{\text{Gep}\,\{a''_i,c''_i\}}_{\{r_i,s_i\},\{a'_i,c'_i\}}=\frac{1}{2^{r-1}H}\sum_{x\in\mathbb{Z}_H;A\in\mathcal{S}_r}\zeta_{AB}(-1)^{\sum_{i\in A}a''_i+c''_i}e^{2i\pi x\left(\frac{\eta}{H}+\sum_{i=1}^r\frac{c''_i}{k_i+2}\right)}\\&\sum_{A'\in\mathcal{S}_r,s\in\mathbb{Z}_H}\prod_{i\in A'^\perp}N^{a''_i,c''_i}_{r_i,s_i+2s;a'_i,c'_i}\prod_{i\in A'}N^{a''_i,c''_i}_{k_i-r_i,s_i+k_i+2+2s;a'_i,c'_i}\,.\end{aligned}\tag{3.1.150}$$

We conclude that the Gepner model enjoys line defects $\mathcal{L}_{\{r_i,s_i\};B,\eta}$ acting as

$$\begin{aligned}\mathcal{L}_{\{r_i,s_i\};B,\eta}\Phi_{\{a_i,c_i\};A,s}&=\zeta_{AB}e^{2i\pi\frac{\eta x}{H}}\left(\prod_{i=1}^r\frac{S_{r_i,s_i;a_i,c_i}}{S_{00;a_i,c_i}}\right)\Phi_{\{a_i,c_i\},A,x}\\&=\zeta_{AB}e^{i\pi\left(\frac{2\eta x}{H}+\sum_i\frac{s_ic_i}{k_i+2}-\frac{[a_1+c_1][r_i+s_i]}{2}\right)}\prod_{i=1}^r\frac{\sin\left(\frac{\pi(r_i+1)(a_i+1)}{k_i+2}\right)}{\sin\left(\frac{\pi(a_i+1)}{k_i+2}\right)}\Phi_{\{a_i,c_i\},A,x}\,.\end{aligned}\tag{3.1.151}$$

By our analysis of the lines of a single minimal model we see that any line for which at least one $r_i\neq 0,k_i$ is non-invertible. The fusion ring is simple to describe, we have

$$\mathcal{L}_{\{r_i,s_i\};B,\eta}\times\mathcal{L}_{\{r'_i,s'_i\};B',\eta'}=\sum_{\{r''_i,s''_i\}}\prod_{i=1}^r N^{r''_i;s''_i}_{r_i,s_i;r'_i,s'_i}\mathcal{L}_{\{r''_i,s''_i\};BB',\eta+\eta'}\tag{3.1.152}$$

with the dual symmetry labels following a group law. The remarks concerning the redundancy of our parametrization apply also in this case. Namely the set of labels of faithfully acting lines is the quotient of $\bigoplus_i Q_{k_i}$, where the $2r$-tuple $\{r_i, s_i\}$ takes values, with respect to the equivalence relations

$$\begin{aligned} &(r_j, s_j) \sim (k_j - r_j, s_j + k_j + 2) \qquad \forall j \in A\,,\; A \in \mathcal{S}_r \\ &(r_i, s_i) \sim (r_i, s_i + 2) \qquad \forall i = 1, \ldots, r\,. \end{aligned} \tag{3.1.153}$$

Of course, accounting for the dual symmetry labels, the total number of lines still equals (3.1.86). If we identify these lines with those of the theory prior to GSO projection and orbifold the equivalences above derive by fusion with the gauged lines, which are invisible in the Gepner model. Both the multiplicities $N^{\text{Gep}\,\{a_i'',c_i''\}}_{\{r_i,s_i\},\{a_i',c_i'\}}$ and the fusion coefficients are well defined on the quotient. In the multiplicities of the twisted sectors this is guaranteed by the sum over $s \in \mathbb{Z}_H$ and $A' \in \mathcal{S}_r$, as changing the representative of the line only reshuffles the terms in the sums. For the fusion coefficients, since the identifications follow from fusing with invertible lines, changing representatives of the lines on the left hand side does not affect the fusion coefficients or the equivalence class of the result of the fusion.

Also in this case we can give a 3-dimensional description of this symmetry. Since we only performed orbifolds in $2d$ the $3d$ TQFT corresponding to the Gepner model is the same one corresponding to the product of GSO-projected minimal model. This is just the Chern-Simons theory with gauge group

$$G_{\text{Gep}} = G_{k_1} \times \ldots \times G_{k_r} \times G_{-k_1} \times \ldots \times G_{-k_r} \tag{3.1.154}$$

with G_{k_i} as in (3.1.75). The MTC data of this TQFT can be computed from those of a single factor. In particular anyons are labelled, in our choice of gauge, by a $4r$-tuple $\{(r_i, s_i); (\bar{r}_i, \bar{s}_i)\}$ with $(r_i, s_i), (\bar{r}_i, \bar{s}_i) \in Q_{k_i}$ and their fusion is

$$\mathcal{L}_{\{(r_i,s_i);(\bar{r}_i,\bar{s}_i)\}} \times \mathcal{L}_{\{(r_i',s_i');(\bar{r}_i{}',\bar{s}_i{}')\}} = \sum_{\{(r_i'',s_i'');(\bar{r}_i{}'',\bar{s}_i{}'')\}} \prod_{i=1}^{r} N^{r_i'',s_i''}_{r_i,s_i;r_i',s_i'} N^{\bar{r}_i{}'',\bar{s}_i{}''}_{\bar{r}_i,\bar{s}_i;\bar{r}_i{}',\bar{s}_i{}'} \mathcal{L}_{\{(r_i'',s_i''),(\bar{r}_i{}'',\bar{s}_i{}'')\}}\,. \tag{3.1.155}$$

From the TQFT perspective gauging a 0-form symmetry in the boundary amounts to change topological boundary condition. The topological boundary condition for the tensor product of GSO-projected minimal models is the diagonal one, corresponding to the diagonal lagrangian algebra $\bigoplus_{\{(r_i,s_i)\}} \mathcal{L}_{\{(r_i,s_i);(r_i,s_i)\}}$. In general we can read off the lagrangian algebra corresponding to a gapped boundary directly from the torus partition function. Consider the $3d$ TQFT on a solid torus with an inner and an outer torus boundaries, on the outer boundary we impose the conformal boundary condition, while on the inner one we set the topological boundary condition. Evaluating the path integral of the TQFT in this configuration produces the torus partition function of the CFT. Now, shrinking the inner torus leaves behind the lagrangian algebra corresponding to the chosen boundary condition, which we can write out as a sum of anyons, possibly with multiplicities. Since the path integral on a solid torus with the insertion of an anyon produces a character of the chiral algebra we can extract the anyons of the lagrangian algebra by comparing with the explicit expression of the partition function. For instance the

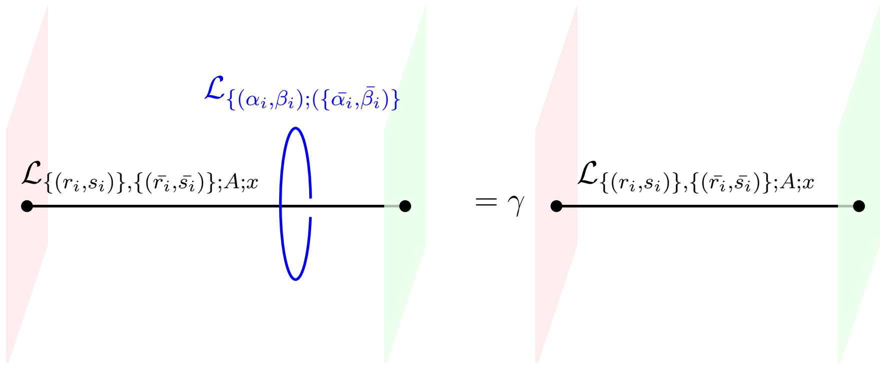


Figure 3.1: An anyon of $\mathfrak{L}_{\text{Gep}}$ can end on both the (red) topological and the (green) conformal boundaries, upon shrinking of the bulk we obtain a local operator in the CFT. The symmetry of the boundary CFT is captured by linking in the bulk.

lagrangian algebra imposing the GSO-projection is

$$\mathfrak{L}_{\text{GSO}} = \bigoplus_{\{(r_i,s_i)\},A\in\mathcal{S}_r} P^{\text{GSO}}_{\{r_i,s_i\}} \mathcal{L}_{\{(r_i,s_i);(r_i,s_i)\}} \times \mathcal{L}_A \tag{3.1.156}$$

where $\mathcal{L}_A$ is the bulk line with $r_i = s_i = 0\ \forall i$ and $\bar{r}_j = k_j, \bar{s}_j = k_j + 2\ \forall j \in A$. And similarly one can write down an expression for the lagrangian algebra corresponding to the $\mathbb{Z}_H$-orbifolded theory

$$\mathfrak{L}_{\text{Gep}} = \bigoplus_{\{(r_i,s_i)\},A\in\mathcal{S}_r,x\in\mathbb{Z}_H} P^{\text{GSO}}_{\{r_i,s_i\}} P^{\mathbb{Z}_H}_{\{r_i,s_i\}} \mathcal{L}_{\{(r_i,s_i);(r_i,s_i)\};A,x} \tag{3.1.157}$$

with

$$\mathcal{L}_{\{(r_i,s_i);(r_i,s_i)\};A,x} = \mathcal{L}_{\{(r_i,s_i);(r_i,s_i)\}} \times \mathcal{L}_A \times \mathcal{L}_{\{(0,0);(0,2x)\}}\,. \tag{3.1.158}$$

Anyons participating in a lagrangian algebra can end on both the conformal and topological boundaries, therefore producing local operators in the CFT.

Bulk lines that are not condensed on the topological boundary generate the symmetry of the CFT. In the bulk the symmetry action is detected linking an anyon of the lagrangian algebra with a generic bulk line. In our case, for the gapped boundary corresponding to the Gepner model, we consider the configuration in Fig.3.1, then

$$\begin{aligned}\gamma &= \prod_{i=1}^{r} \frac{S_{\alpha_i,\beta_i;r_i,s_i}}{S_{00;r_i,s_i}} \prod_{i\in A^\perp} \frac{S^*_{\bar{\alpha}_i,\bar{\beta}_i;r_i,s_i+2x}}{S_{00;r_i,s_i+2x}} \prod_{i\in A} \frac{S^*_{\bar{\alpha}_i,\bar{\beta}_i;k_i-r_i,s_i+k_i+2+2x}}{S_{00;k_i-r_i,s_i+k_i+2+2x}} \\ &= e^{-2i\pi x\left(\sum_{i=1}^{r}\frac{\bar{\beta}_i}{k_i+2}\right)}(-1)^{\sum_{i\in A}[\bar{\alpha}_i+\bar{\beta}_i]} \prod_{i=1}^{r} \frac{S_{\alpha_i,\beta_i;r_i,s_i} S^*_{\bar{\alpha}_i,\bar{\beta}_i;r_i,s_i}}{S^2_{00;r_i,s_i}}\,.\end{aligned} \tag{3.1.159}$$

When $\mathcal{L}_{\{\alpha_i,\beta_i\},\{\bar{\alpha}_i,\bar{\beta}_i\}}$ is also participating in $\mathfrak{L}_{\text{Gep}}$ the quantity above reduces to its quantum dimension, as it should given the commutativity of the lagrangian algebra. In (3.1.159) the two phases in front of the product give the action of the dual symmetries, with $\eta = H\sum_i \bar{\beta}_i/(k_i+2)$ and $\zeta_{AB} = (-1)^{\sum_{i\in A}[\bar{\alpha}_i+\bar{\beta}_i]}$. Due to the boundary condition many bulk lines will be mapped to the same boundary line, hence act on lines in $\mathfrak{L}_{\text{Gep}}$ with the same eigenvalue. As representatives of the faithfully acting lines we can take all those with $\alpha_i = \beta_i = 0$ and impose the identifications implied by the projectors P^{GSO} and $P^{\mathbb{Z}_H}$, then (3.1.159) matches what we found directly in the CFT.

3.1.3 Symmetric Marginal Deformations and Selection Rules

We found a wealth of non-invertible lines at the Gepner point of the Calabi-Yau sigma model. We now want to investigate the existence of continuous families of SCFTs which preserve some subcategory of lines. To this extent we look at marginal deformations invariant under the action of the lines described in the previous section. On general grounds, for an operator Φ of a $2d$ QFT to be invariant under the action of a TDL $\mathcal{L}$, the two have to commute [12]. Shrinking $\mathcal{L}$ produces a number, then the condition is

$$\mathcal{L}\Phi = \langle\mathcal{L}\rangle\Phi \tag{3.1.160}$$

where $\langle\mathcal{L}\rangle$ is the quantum dimension of $\mathcal{L}$, which can also be seen as the eigenvalue of $\mathcal{L}$ on the identity operator (we are assuming a CFT with a unique vaccum). Therefore a physical primary $\Phi_{\{a_i,c_i\},A,x}$ of the Gepner model is invariant under a line $\mathcal{L}_{\{r_i,s_i\},B,\eta}$ if

$$\zeta_{AB}e^{2i\pi\frac{s\eta}{H}}\prod_{i=1}^{r}\frac{S_{r_i,s_i;a_i,c_i}}{S_{00;a_i,c_i}} = \prod_{i=1}^{r}\frac{S_{r_i,s_i;00}}{S_{00;00}}\,. \tag{3.1.161}$$

Since we are going to consider only exactly marginal deformations we can take $\Phi_{\{a_i,c_i\};A,x}$ to be in the NS sector, then the invariance condition is slightly simpler

$$\zeta_{AB}e^{i\pi\left(\frac{2s\eta}{H}+\sum_i\frac{s_ic_i}{k_i+2}\right)}\prod_{i=1}^{r}\frac{\sin\left(\frac{\pi(r_i+1)(a_i+1)}{k_i+2}\right)}{\sin\left(\frac{\pi(a_i+1)}{k_i+2}\right)} = \prod_{i=1}^{r}\frac{\sin\left(\frac{\pi(r_i+1)}{k_i+2}\right)}{\sin\left(\frac{\pi}{k_i+2}\right)}\,. \tag{3.1.162}$$

The operator we add to the action is a descendant of the BPS primaries we discussed in the previous section. To obtain it we act on those primaries with a product of the supercharges $G^{\pm},\bar{G}^{\pm}$ of the diagonal $\mathcal{N}=(2,2)$ algebra. In terms of the supercharges of each tensor factor the operators we are interested in are

$$G^x\bar{G}^y = \sum_{i,j=1}^{r}\mathbb{1}\otimes\ldots\otimes G^{(i);x}\otimes\ldots\otimes\bar{\mathbb{1}}\otimes\ldots\otimes\bar{G}^{(j);y}\otimes\ldots\otimes\bar{\mathbb{1}}\,, \tag{3.1.163}$$

with $x,y=\pm$. As states in the Gepner model Hilbert space these operators correspond to

$$|G^x\bar{G}^y\rangle = \sum_{i,j=1}^{r}|0,0\rangle\otimes\ldots\otimes|k_i,k_i+2\rangle\otimes\ldots\overline{|0,0\rangle}\otimes\ldots\overline{|k_j,k_j+2\rangle}\otimes\ldots\otimes\overline{|0,0\rangle}\,. \tag{3.1.164}$$

The summands belong to the twisted Hilbert spaces $\mathbb{H}^{(0,A=\{i,j\})}$ if $i\neq j$, while are in the untwisted Hilbert space if $i=j$. A line $\mathcal{L}_{\{r_i,s_i\};B;\eta}$ acting on a component gives

$$\begin{cases}(-1)^{r_i+s_i}\zeta_{\{ij\},B} & i\neq j\\ (-1)^{r_i+s_i} & i=j\,.\end{cases} \tag{3.1.165}$$

A single term in the sum (3.1.163), upon acting on the BPS primary, gives raise to an operator participating in the deformation. For each such component the commutation condition is

$$\zeta_{\{i,j\},B}(-1)^{r_i+s_i}\zeta_{AB}e^{i\pi\left(\frac{2s\eta}{H}+\sum_l\frac{s_lc_l}{k_l+2}\right)}\prod_{l=1}^{r}\frac{\sin\left(\frac{\pi(r_l+1)(a_l+1)}{k_l+2}\right)}{\sin\left(\frac{\pi(a_l+1)}{k_l+2}\right)} = \prod_{l=1}^{r}\frac{\sin\left(\frac{\pi(r_l+1)}{k_l+2}\right)}{\sin\left(\frac{\pi}{k_l+2}\right)}\,,\quad\forall i,j\,. \tag{3.1.166}$$

We can take the line $\mathcal{L}_{\{r_i,s_i\},;B,\eta}$ to commute or anticommute with both the product of supercharges and the BPS primary. Requiring (anti-)commutativity with the supercharges we see that we have to set $(-1)^{r_i+s_i} = \pm 1$ for all $i = 1, \ldots, r$ and $\zeta_{\{ij\},B} = 1$.

Now, solving for the commutativity with the BPS primary in full generality is complicated. In order to continue our analysis we consider specific examples. In the following we shall only focus on chiral-chiral and antichiral-chiral deformations, as we can use charge conjugation to relate those to the antichiral-antichiral and chiral-antichiral ones. In the literature the usual way to denote Gepner models is as $\prod_y (k_y)^{m_y}$ where m_y is the number of times k_y appears in the list $(k_1, \ldots, k_r)$, we'll make use of this notation in the remainder of this section.

Torus

The simplest target space we can consider for the superconformal sigma model is the torus. The theory is free, and consists of a complex scalar X parametrizing the torus, a complex left moving fermion ψ and a complex right moving one λ. The conformal manifold is the Narain moduli space

$$\mathcal{M} = O(2,2,\mathbb{Z})\backslash O(2,2,\mathbb{R})/O(2) \times O(2)\,. \tag{3.1.167}$$

A point in $\mathcal{M}$ consist of a choice of metric $G_{ij} = G_{ji}$ and B-field $B_{ij} = -B_{ji}$ on the target torus, with $i, j = 1, 2$. The partition function of the theory on a torus worldsheet with modular parameter τ at a point $m \in \mathcal{M}$ is the product of the bosonic one and the GSO-projected fermion partition function. The R-symmetry charges are

	q	$\bar{q}$
ψ	1	0
λ	0	1
X	0	0

The tangent space of $\mathcal{M}$ can be probed deforming the action by the operator

$$\mathcal{O} = \left(\delta G_{ij}\delta^{\alpha\beta} + i\delta B_{ij}\epsilon^{\alpha\beta}\right)\partial_\alpha X^i \partial_\beta X^j \tag{3.1.168}$$

where we wrote $X = X^1 + iX^2$ and introduced coordinates σ^α on a flat worldsheet. In terms of X and $z = \sigma^1 + i\sigma^2$ we have

$$\mathcal{O} = g\partial X \bar{\partial} X + f \partial X \bar{\partial}\bar{X} + c.c. \tag{3.1.169}$$

where we introduced the complex parameters

$$g = \delta G_{11} - \delta G_{22} - 2i\delta G_{12}\,, \qquad f = \delta G_{11} + \delta G_{22} + 2i\delta B_{12}\,. \tag{3.1.170}$$

To compare $\mathcal{O}$ with the exactly marginal deformations obtained from the $\mathcal{N} = (2,2)$ algebra we write down the chiral rings using the sigma model fields. We have[7]

$$\begin{aligned}&(\text{chiral-chiral}) = \{\mathbb{1}, \psi, \lambda, \psi\lambda\}\\ &(\text{antichiral-chiral}) = \{\mathbb{1}, \bar{\psi}, \lambda, \bar{\psi}\lambda\}\end{aligned} \tag{3.1.171}$$

[7]Note that because of the GSO projection the fields ψ, λ and their conjugates cannot appear in the partition function on their own, to survive the projection they need to be tensored with another fermionic state on the other holomorphic half.

Using the explicit realization of the $\mathcal{N}=(2,2)$ algebra generators in terms of the sigma model fields one can check that the two marginal operators $\partial X\bar{\partial}X$ and $\partial\bar{X}\bar{\partial}X$ are descendants of the bosonic BPS primaries $\psi\lambda$ and $\bar{\psi}\lambda$ respectively. In other words g couples to the chiral-chiral deformation while f to the antichiral-chiral one.

On $\mathcal{M}$ there are three Gepner points: $(1)^3$, $(2)^2$ or $(1)(4)$, each of which has total central charge $c=3$. The last two cases however, as presented, do not fulfill the CY condition. In particular, for both $(2)^2$ and $(1)(4)$,

$$\sum_{i=1}^{2}\frac{1}{k_i+2}=\frac{1}{2}\,. \tag{3.1.172}$$

We can solve this issue in both cases adding a minimal model with $k=0$. On its own this is the trivial (spin) theory, with a single ground state in both the NS and R sectors with vanishing charges and weights. However its presence is detected by the $\mathbb{Z}_H$ orbifold: without it we would not project on NS sector states with integer $U(1)_R$ charges. The $(1)^3$ model is the least interesting as the Verlinde lines of the $k=1$ minimal model are all invertible. We shall then mainly focus on the $(2)^2(0)$ case. In this simple case one can check by hand that there is only one chiral-chiral marginal deformation corresponding to the state

$$(|2,2\rangle)^{\otimes 2}\otimes|0,0\rangle\otimes(\overline{|2,2\rangle})^{\otimes 2}\otimes\overline{|0,0\rangle}\in\mathbb{H}^{(0,\varnothing)}\,. \tag{3.1.173}$$

Also there's a unique antichiral-chiral marginal operator corresponding to

$$(|1,-1\rangle)^{\otimes 2}\otimes|0,0\rangle\otimes(\overline{|1,1\rangle})^{\otimes 2}\otimes\overline{|0,0\rangle}\in\mathbb{H}^{(3,\varnothing)}\,. \tag{3.1.174}$$

Now let's look at what lines are preserved by these deformations.

- A line $\mathcal{L}_{\{r_i,s_i\};B,\eta}$ (anti)commutes with the chiral-chiral operator if

$$e^{i\pi\frac{2s_1+2s_2}{4}}\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)}=\pm 1\,. \tag{3.1.175}$$

  To eliminate the imaginary part we set $2s_1+2s_2=0$ mod 4, then

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)}=\pm 1 \tag{3.1.176}$$

  which are solved by

$$\begin{aligned}(r_1,r_2)_+&=(0,0),(2,0),(0,2),(1,1),(2,2);\\(r_1,r_2)_-&=(0,1),(1,0),(2,1),(1,2)\,.\end{aligned} \tag{3.1.177}$$

  Any solution in which r_1 or r_2 is 1 corresponds to a non-invertible line. It is easy to see that the antichiral-antichiral state conjugate to (3.1.173) is also invariant under those lines.

- For $\mathcal{L}_{\{r_i,s_i\};B,\eta}$ to (anti-)commute with the antichiral-chiral operator we need instead

$$e^{i\pi\left(\frac{s_1+s_2}{4}+\eta\right)}\prod_{i=1}^{2}\frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)}=\pm 2\,. \tag{3.1.178}$$

We can eliminate the imaginary part with $s_1+s_2=\eta \bmod 4$, then

$$\prod_{i=1}^{2}\frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)}=\pm 2\,, \tag{3.1.179}$$

which are solved by

$$\begin{aligned}(r_1,r_2)_+ &= (0,0),(2,2);\\ (r_1,r_2)_- &= (2,0),(0,2)\,.\end{aligned} \tag{3.1.180}$$

Thus only invertible lines commute with this deformation.

By construction the symmetries of the chiral-chiral deformation form a fusion subcategory $\mathcal{S}_{CC}$ of the full symmetry of the Gepner model. For instance, in $\mathcal{S}_{CC}$ we have the line $\mathcal{D}=\mathcal{L}_{1,1,1,3;\varnothing,0}$, this fuses as

$$\mathcal{D}\times\mathcal{D}^\dagger=\mathbb{1}+\eta_1+\eta_2+\eta_1\times\eta_2 \tag{3.1.181}$$

where $\eta_1=\mathcal{L}_{0,2,0,0;\varnothing,0}$ and $\eta_2=\mathcal{L}_{0,0,2,0;\varnothing,0}$ are $\mathbb{Z}_2$ generators. These are the fusion of a Tambara-Yamagami category over a group $\mathbb{Z}_2\times\mathbb{Z}_2$. The Gepner model $(2)^2(0)$ is known to sit at the point in $\mathcal{M}$ corresponding to the square torus $G_{ij}=\delta_{ij}$, $B_{ij}=0$. Sitting at this point and deforming with a linear combination of the exactly marginal chiral-chiral and antichiral-antichiral operators, which we denote Φ_{CC} and $\Phi_{CC}^\dagger$ respectively, we preserve $\mathcal{S}_{CC}$. We first have to address the relation between Φ_{CC} and $\partial X\bar\partial X$. The latter is a superconformal descendant of the fermion product $\psi\lambda$, since the fermion action and the chiral rings are independent on the point in the conformal manifold it is natural to identify $\psi\lambda$ with the state (3.1.173). The relation between Φ_{CC} and $\partial X\bar\partial X$ could still depend on the moduli through the supercharges. However the Gepner point corresponds to the square torus with no B-field, then the supercurrents at this point are simply those of the free field realization of the $\mathcal{N}=(2,2)$ superconformal algebra, namely

$$\begin{aligned}G^-(z)&=\frac{1}{2}\bar\psi\partial X \quad G^+(z)=\frac{1}{2}\psi\partial\bar X\\ \bar G^-(\bar z)&=\frac{1}{2}\bar\lambda\bar\partial X \quad \bar G^+(\bar z)=\frac{1}{2}\lambda\bar\partial\bar X\,.\end{aligned} \tag{3.1.182}$$

From these expressions and using the free field OPEs one easily gets that the marginal descendant of $\psi\lambda$ is $\partial X\bar\partial X$, thus leading us to the conclusion that $\Phi_{CC}=\partial X\bar\partial X$. Then consider the two real operators

$$\mathcal{O}_1=g_1\left(\Phi_{CC}+\Phi_{CC}^\dagger\right)\,,\qquad \mathcal{O}_2=-ig_2\left(\Phi_{CC}-\Phi_{CC}^\dagger\right) \tag{3.1.183}$$

where $2g_1=\delta G_{11}-\delta G_{22}$ and $g_2=\delta G_{12}$. The values of the metric components as a function of g_1 and g_2 are

$$G_{11}=1+g_1\,,\qquad G_{22}=1-g_1\,,\qquad G_{12}=g_2\,. \tag{3.1.184}$$

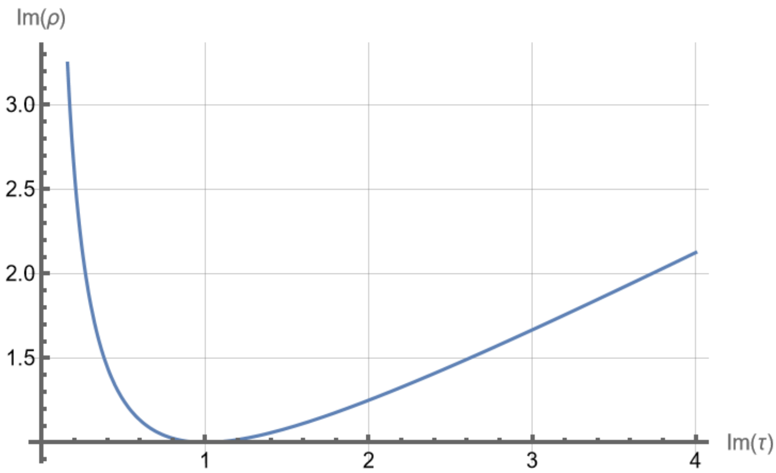


Figure 3.2: Setting $g_2 = 0$, so that $\mathrm{Re}(\tau) = \mathrm{Re}(\rho) = 0$ we can consider $-1 < g_1 < 1$. The submanifold on which $\mathcal{S}_{\mathcal{CC}}$ is preserved is given by the curve $\mathrm{Im}(\rho) = \frac{1}{2}\left(\mathrm{Im}(\tau) + \frac{1}{\mathrm{Im}(\tau)}\right)$ plotted here.

For G_{ij} to be positive definite we have to restrict g_i inside the disk $g_1^2 + g_2^2 < 1$. As we change g_1 and g_2 we trace out a 2-dimensional submanifold of $\mathcal{M}$ on which the symmetry $\mathcal{S}_{\mathcal{CC}}$ is preserved. A convenient presentation of $\mathcal{M}$ is obtained introducing two complex parameters [264]

$$\tau = \frac{1}{G_{11}}\left(G_{12} + i\sqrt{\det G}\right) , \qquad \rho = B_{12} + i\sqrt{\det G}\,; \tag{3.1.185}$$

corresponding respectively to the complex structure and complexified area of the target torus. Integer-valued change of coordinates on G, namely $G \mapsto M^T G M$ with $M \in SL(2,\mathbb{Z})$, induce $PSL(2,\mathbb{Z})$ transformations on τ leaving ρ invariant. The $PSL(2,\mathbb{Z})$ acting on ρ is generated by T-duality on both compact directions as well as shifts of the B-field by integers. T duality on only one direction exchanges ρ and τ, parity in both the worldsheet and the target torus induce further $\mathbb{Z}_2$ identifications. Then $\mathcal{M}$ is a product of two copies of the fundamental domain of $SL(2,\mathbb{Z})$ in the upper-half plane with some extra discrete $\mathbb{Z}_2$ identifications. In terms of g_1 and g_2 we have

$$\tau = \frac{1}{1+g_1}\left(g_2 + i\sqrt{1-g_1^2-g_2^2}\right) , \qquad \rho = i\sqrt{1-g_1^2-g_2^2}\,. \tag{3.1.186}$$

or, using an S-transformation on both to have $\mathrm{Im}\tau, \mathrm{Im}\rho > 1$

$$\tau = \frac{1}{1-g_1}\left(-g_2 + i\sqrt{1-g_1^2-g_2^2}\right) , \qquad \rho = \frac{i}{\sqrt{1-g_1^2-g_2^2}}\,. \tag{3.1.187}$$

Now for g_i inside the disk we span the whole τ fundamental domain while moving along the complex direction in the ρ plane from i to $i\infty$.

$K3$ surface

The first interacting case we consider is a sigma model on a $K3$ surface with central charge $c = 6$. The Gepner point can be realized as the product of four minimal models at level

$k = 2$, conventionally denoted $(2)^4$. We start looking in more detail to the BPS spectrum. The chiral-chiral states in the untwisted sector are

$$\bigotimes_{i=1}^{4} |a_i, a_i\rangle \bigotimes_{i=1}^{4} \overline{|a_i, a_i\rangle} \tag{3.1.188}$$

and 19 of those satisfy the charge requirement

$$\sum_i \frac{a_i}{4} = 1\,. \tag{3.1.189}$$

The values of a_i for these BPS states are (non-trivial) permutations of the following

$$(a_1, a_2, a_3, a_4) = (2, 2, 0, 0); (2, 1, 1, 0); (1, 1, 1, 1)\,. \tag{3.1.190}$$

For $r = 4$ we know that there is another chiral-chiral state with the appropriate $U(1)_R$ charges in the twisted component $\mathbb{H}^{(2,\{1,\ldots,4\})}$, this is given by

$$(|1,1\rangle)^{\otimes^4} \otimes \left(\overline{|1,1\rangle}\right)^{\otimes^4} \in \mathbb{H}^{(2,\{1,\ldots,4\})}\,. \tag{3.1.191}$$

In total we have 20 chiral-chiral states. Acting with charge conjugation on each of those states we obtain the 20 antichiral-antichiral states corresponding to marginal deformations. Notice that the state (3.1.191) has the same weight and R-charges of the chiral-chiral state

$$(|1,1\rangle)^{\otimes^4} \otimes \left(\overline{|1,1\rangle}\right)^{\otimes^4} \in \mathbb{H}^{(0,\varnothing)}\,. \tag{3.1.192}$$

appearing in the untwisted sector. These two however are not indistinguishable as the quantum numbers of the corresponding primaries under the dual symmetries are different.

The analysis of antichiral-chiral states corresponding to marginal deformations is essentially the same. In the twisted Hilbert spaces $\mathbb{H}^{(x,\varnothing)}$, we have antichiral-chiral states of the form

$$(|a,-a\rangle)^{\otimes^4} \otimes (|a,a\rangle)^{\otimes^4} \in \mathbb{H}^{(a,\varnothing)} \tag{3.1.193}$$

of those only the one with $a = 1$ has left and right R-charges equal to -1 and 1. The remaining antichiral-chiral states sit in $\mathbb{H}^{3,\{1,\ldots 4\}}$ and are of the form

$$\bigotimes_{i=1}^{4} |a_i, -a_i\rangle \otimes \overline{|2-a_i, 2-a_i\rangle} \in \mathbb{H}^{3,\{1,\ldots 4\}}\,. \tag{3.1.194}$$

The charge constraint

$$\sum_i \frac{a_i}{4} = 1 \tag{3.1.195}$$

selects 19 of them, with the values of a_i again given by permutations of the 4-tuples in (3.1.190). We then have 20 chiral-chiral and 20 antichiral-chiral states (plus conjugates), which pair up into BPS multiplets for the $\mathcal{N} = 4$ algebra, consistently with $h^{1,1} = 20$ for $K3$. Now let's look at the symmetries.

- For chiral-chiral states we look at each vector (3.1.190) separately.

– $(a_1, a_2, a_3, a_4) = (2, 2, 0, 0)$. The invariance condition is

$$e^{i\pi\frac{2s_1+2s_2}{4}}\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)} = \pm 1\,. \tag{3.1.196}$$

which is symmetric in r_1 and r_2. Note that this condition only constraints the action of the line on two minimal models, namely the parameters r_3, s_3, r_4, s_4, η are not fixed by this condition. This already shows that the deformation preserves non-invertible lines. To eliminate the imaginary part we set $2s_1 + 2s_2 = 0 \mod 4$, then

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)} = \pm 1 \tag{3.1.197}$$

which are solved by

$$\begin{aligned}(r_1, r_2)_+ &= (0, 0), (0, 2), (1, 1), (2, 2)\\ (r_1, r_2)_- &= (0, 1), (1, 2)\,.\end{aligned} \tag{3.1.198}$$

Here we reported only one element of the orbit of the swap symmetry $r_1 \leftrightarrow r_2$ on the solutions. Any solution in which r_1 or r_2 is 1 corresponds to a non-invertible line.

– $(a_1, a_2, a_3, a_4) = (2, 1, 1, 0)$. We impose

$$e^{i\pi\frac{2s_1+s_2+s_3}{4}}\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{2\pi(r_2+1)}{4}\right)\sin\left(\frac{2\pi(r_3+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)\sin\left(\frac{\pi(r_3+1)}{4}\right)} = \pm 2 \tag{3.1.199}$$

hence for $2s_1 + s_2 + s_3 = 0 \mod 4$ we have

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{2\pi(r_2+1)}{4}\right)\sin\left(\frac{2\pi(r_3+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)\sin\left(\frac{\pi(r_3+1)}{4}\right)} = \pm 2 \tag{3.1.200}$$

with solutions

$$\begin{aligned}(r_1, r_2, r_3)_+ &= (0, 0, 0); (0, 2, 2); (1, 0, 2); (2, 0, 0); (2, 2, 2)\\ (r_1, r_2, r_3)_- &= (0, 0, 2), (1, 0, 0); (1, 2, 2); (2, 0, 2)\end{aligned} \tag{3.1.201}$$

where we again reported only one representative for the swap symmetry of r_2 and r_3. Again we see that there are solutions in which at least one among r_1, r_2 or r_3 is 1, together with the freedom of choosing the parameter r_4, we see that there are preserved non-invertible TDLS along the deformations.

– $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$. This is the last chiral-chiral state we have to consider. Since it sits in a twisted component of the Hilbert space the invariance condition now also involve the dual symmetries and becomes

$$\zeta_{A;\{1,\ldots,4\}}e^{i\pi\left(\frac{s_1+s_2+s_3+s_4}{4}+\eta\right)}\prod_{i=1}^{4}\frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)} = \pm 4\,. \tag{3.1.202}$$

We can leave the dual symmetry labels A, η free and only require $s_1 + s_2 + s_3 + s_4 = 0 \bmod 4$. Then

$$\prod_{i=1}^{4} \frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)} = \pm 4 \tag{3.1.203}$$

with solutions

$$\begin{aligned} (r_1, r_2, r_3, r_4)_+ &= (0,0,0,0); (0,0,2,2); (2,2,2,2) \\ (r_1, r_2, r_3, r_4)_- &= (0,0,0,2), (0,2,2,2); \end{aligned} \tag{3.1.204}$$

where we again reported only one representative for the orbit of the permutation symmetry. This deformation is invariant only under invertible lines.

Let's take stock. The majority of the chiral-chiral deformations preserve at least one non-invertible line. It is interesting to notice also that we can deform the Gepner model with multiple chiral-chiral operators and still preserve non-invertible lines. For instance we can turn on simultaneously some of the deformations given by permutations of $(a_1, a_2, a_3, a_4) = (2, 2, 0, 0)$, still preserving some non-invertible lines.

- For antichiral-chiral states we again consider the vectors (3.1.190) separately, each giving rise to a state of the form (3.1.194).

  - $(a_1, a_2, a_3, a_4) = (2, 2, 0, 0)$. The invariance condition is

$$\zeta_{B,\{1,\ldots,4\}} e^{i\pi\left(\frac{2s_1+2s_2}{4}+\frac{3\eta}{2}\right)} \frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)} = \pm 1\,. \tag{3.1.205}$$

    We can leave B free and set $2s_1 + 2s_2 + 2\eta = 0 \bmod 4$. The remaining equation is

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{3\pi(r_2+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)} = \pm 1 \tag{3.1.206}$$

    Which is the same one we solved for the chiral-chiral deformation with $(a_1, a_2, a_3, a_4) = (2, 2, 0, 0)$. Notice that the lines preserving this deformation are not the same ones preserving the chiral-chiral one, but the two sets have a non-empty intersection. For instance lines with $\eta = 0, 4$ that preserve the chiral-chiral deformation will preserve also this one.

  - $(a_1, a_2, a_3, a_4) = (2, 1, 1, 0)$. We impose

$$\zeta_{B,\{1,\ldots,4\}} e^{i\pi\left(\frac{2s_1+s_2+s_3}{4}+\frac{3\eta}{2}\right)} \frac{\sin\left(\frac{3\pi(r_1+1)}{4}\right)\sin\left(\frac{2\pi(r_2+1)}{4}\right)\sin\left(\frac{2\pi(r_3+1)}{4}\right)}{\sin\left(\frac{\pi(r_1+1)}{4}\right)\sin\left(\frac{\pi(r_2+1)}{4}\right)\sin\left(\frac{\pi(r_3+1)}{4}\right)} = \pm 2 \tag{3.1.207}$$

    hence for $2s_1 + s_2 + s_3 + 2\eta = 0 \bmod 4$ we have again an equation we solved in the chiral-chiral case. Also here to lines with $\eta = 0, 4$ will preserve both chiral-chiral and antichiral-chiral deformations.

– $(a_1, a_2, a_3, a_4) = (1,1,1,1)$. The invariance condition is now

$$e^{i\pi\left(\frac{s_1+s_2+s_3+s_4}{4}+\eta\right)} \prod_{i=1}^{4} \frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)} = \pm 4\,. \tag{3.1.208}$$

We can leave the dual symmetry label η free and require $s_1+s_2+s_3+s_4 = 0 \bmod 4$. Then

$$\prod_{i=1}^{4} \frac{\sin\left(\frac{2\pi(r_i+1)}{4}\right)}{\sin\left(\frac{\pi(r_i+1)}{4}\right)} = \pm 4 \tag{3.1.209}$$

which we have solved above.

We see that the structure of the lines preserved by these antichiral-chiral deformations closely follows the one we found for chiral-chiral deformations.

Note that lines that preserve both the chiral-chiral and antichiral-chiral deformations given by the same vector in (3.1.190), will also preserve the exactly marginal $\mathcal{N}=4$ deformation we obtain taking their sum. Using what we found above we can also enlarge the submanifold of the moduli space that enjoys non-invertible symmetries. Consider both the chiral-chiral and antichiral-chiral deformations given by $(a_1, a_2, a_3, a_4) = (2,2,0,0), (0,0,2,2)$. These preserve the line with

$$\begin{aligned} &r_i = 1\,, \qquad i = 1,\dots,4 \\ &s_1 = s_3 = 1\,, s_2 = s_4 = 3 \end{aligned} \tag{3.1.210}$$

as well as $\eta = 0, 4$. Therefore we have a 4-dimensional subspace of the moduli space with a non-invertible symmetry. More precisely denoting $\mathcal{D} = \mathcal{L}_{1,1,1,3,1,1,1,3;\varnothing,0}$ we have

$$\mathcal{D}\times\mathcal{D}^\dagger = \mathbb{1} + \sum_{i=1}^{4}\eta_i + \sum_{i<j}\eta_i\times\eta_j \sum_{i<j<k}\eta_i\times\eta_j\times\eta_k + \eta_1\times\eta_2\times_3\times\eta_4 \tag{3.1.211}$$

where

$$\begin{aligned} \eta_1 &= \mathcal{L}_{2,0,0,0,0,0,0,0;\varnothing,0}\,, \qquad & \eta_2 &= \mathcal{L}_{0,0,2,0,0,0,0,0;\varnothing,0} \\ \eta_3 &= \mathcal{L}_{0,0,0,0,2,0,0,0;\varnothing,0}\,, \qquad & \eta_4 &= \mathcal{L}_{0,0,0,0,0,0,2,0;\varnothing,0} \end{aligned} \tag{3.1.212}$$

generate a $\mathbb{Z}_2^4$ group. Including $\mathcal{D}$ we have a $\mathbb{Z}_2^4$ Tambara-Yamagami symmetry.

Quintic Threefold

For the Quintic threefold the Gepner point is $(3)^5$, namely we take five minimal models with $k=3$. The central charge is $c=9$. Again we start with chiral-chiral primaries, the states of the form

$$\bigotimes_{i=1}^{5} |a_i, a_i\rangle \otimes \bigotimes_{i=1}^{5} \overline{|a_i, a_i\rangle} \tag{3.1.213}$$

for $a_i = 0,1,2,3$ are $4^5 = 1024$, of those the charge constraint

$$\sum_i \frac{a_i}{5} = 1 \tag{3.1.214}$$

selects 101. This matches $h^{2,1} = 101$ counting $(2,1)$ forms on the quintic. These correspond to complex structure deformations of the underlying Calabi-Yau. For each of those states the vector $(a_1, a_2, a_3, a_4, a_5)$ is a permutation of one of the following five

$$(a_1, a_2, a_3, a_4, a_5) = (3,2,0,0,0), (3,1,1,0,0), (2,2,1,0,0), (2,1,1,1,0), (1,1,1,1,1)\,. \quad (3.1.215)$$

The antichiral-chiral states are of the form

$$(|a,-a\rangle)^{\otimes^5} \otimes \left(\overline{|a,a\rangle}\right)^{\otimes^5} \in \mathbb{H}^{(a,\varnothing)} \quad (3.1.216)$$

but of those only the one with $a = 1$ has left R charge -1. Thus there is only one complex Khaler modulus, which agrees with $h^{1,1} = 1$ for the quintic. Now we look at the symmetries.

- For chiral-chiral states it is enough to find the solution of the invariance constraint (which is invariant under permutations) for each of the five vectors (3.1.215). We have
  - $(a_1, a_2, a_3, a_4, a_5) = (3,2,0,0,0)$. A line $\mathcal{L}_{\{r_i,s_i\},B,\eta}$ leaving invariant the deformation has to satisfy

$$e^{i\pi\frac{3s_1+2s_2}{5}} \frac{\sin\left(\frac{4\pi(r_1+1)}{5}\right)\sin\left(\frac{3\pi(r_2+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)} = \pm\frac{1}{2}\left(1+\sqrt{5}\right) \quad (3.1.217)$$

    with all labels other than r_1, s_1 and r_2, s_2 free. By choosing $3s_1 + 2s_2 = 0 \bmod 5$ we can look for solutions of

$$\frac{\sin\left(\frac{4\pi(r_1+1)}{5}\right)\sin\left(\frac{3\pi(r_2+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)} = \pm\frac{1}{2}\left(1+\sqrt{5}\right)\,. \quad (3.1.218)$$

    We find

$$\begin{aligned} (r_1, r_2)_+ &= (0,0); (0,3); (2,0); (2,3) \qquad 3s_1 + 2s_2 = 0 \bmod 10 \\ (r_1, r_2)_- &= (1,0); (1,3); (3,0); (3,3) \qquad 3s_1 + 2s_2 = 5 \bmod 10 \end{aligned} \quad (3.1.219)$$

    for the two signs. Therefore for each pair of $(r_1, r_2)_\pm$ we choose (s_1, s_2) accordingly while all other labels are free. Here, as long as r_i is either 1 or 2 the corresponding TDL is non-invertible.
  - $(a_1, a_2, a_3, a_4, a_5) = (3,1,1,0,0)$. Invariance requires

$$e^{i\frac{\pi}{5}(3s_1+s_2+s_3)} \frac{\sin\left(\frac{4\pi(r_1+1)}{5}\right)\sin\left(\frac{2\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)} = \pm\frac{1}{2}\left(3+\sqrt{5}\right) \quad (3.1.220)$$

    thus for $3s_1 + s_2 + s_3 = 0 \bmod 5$ we need to solve

$$\frac{\sin\left(\frac{4\pi(r_1+1)}{5}\right)\sin\left(\frac{2\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)} = \pm\frac{1}{2}\left(3+\sqrt{5}\right)\,. \quad (3.1.221)$$

    Note that this is symmetric in r_2 and r_3, in the solutions below we write only one solution per orbit of this swap symmetry. We find

$$\begin{aligned} (r_1, r_2, r_3)_+ &= (0,0,0); (0,3,3); (1,0,3); (2,0,0); (2,3,3); (3,0,3); (3,3,0) \\ (r_1, r_2, r_3)_- &= (0,0,3); (1,0,0); (1,3,3); (2,0,3); (3,0,0); (3,3,3)\,. \end{aligned} \quad (3.1.222)$$

– $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 1, 0, 0)$. Invariance requires

$$e^{i\frac{\pi}{5}(2s_1+2s_2+s_3)}\frac{\sin\left(\frac{3\pi(r_1+1)}{5}\right)\sin\left(\frac{3\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)} = \pm\left(2+\sqrt{5}\right) \quad (3.1.223)$$

thus for $2s_1 + 2s_2 + s_3 = 0 \bmod 5$ we need to solve

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{5}\right)\sin\left(\frac{3\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)} = \pm\left(2+\sqrt{5}\right). \quad (3.1.224)$$

Note that this is symmetric in r_1 and r_2, in the solutions below we write only one solution per orbit of this swap symmetry. We find

$$\begin{aligned}(r_1, r_2, r_3)_+ &= (0,0,0); (0,3,0); (3,3,0)\\ (r_1, r_2, r_3)_- &= (0,0,3); (0,3,3); (3,3,3).\end{aligned} \quad (3.1.225)$$

We see that a line preserving this deformation necessarily acts invertibly on it, altough it may act non-invertibly on other operators of the theory.

– $(a_1, a_2, a_3, a_4, a_5) = (2, 1, 1, 1, 0)$. Invariance requires

$$\begin{aligned}&e^{i\frac{\pi}{5}(2s_1+s_2+s_3+s_4)}\frac{\sin\left(\frac{3\pi(r_1+1)}{5}\right)\sin\left(\frac{2\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)\sin\left(\frac{2\pi(r_4+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)\sin\left(\frac{\pi(r_4+1)}{5}\right)}\\ &= \pm\frac{1}{2}\left(7+3\sqrt{5}\right)\end{aligned} \quad (3.1.226)$$

thus for $2s_1 + s_2 + s_3 + s_4 = 0 \bmod 5$ we need to solve

$$\frac{\sin\left(\frac{3\pi(r_1+1)}{5}\right)\sin\left(\frac{2\pi(r_2+1)}{5}\right)\sin\left(\frac{2\pi(r_3+1)}{5}\right)\sin\left(\frac{2\pi(r_4+1)}{5}\right)}{\sin\left(\frac{\pi(r_1+1)}{5}\right)\sin\left(\frac{\pi(r_2+1)}{5}\right)\sin\left(\frac{\pi(r_3+1)}{5}\right)\sin\left(\frac{\pi(r_4+1)}{5}\right)} = \pm\frac{1}{2}\left(7+3\sqrt{5}\right). \quad (3.1.227)$$

Note that this is symmetric in r_2, r_3 and r_4, in the solutions below we write only one solution per orbit of this permutation symmetry. We find

$$\begin{aligned}(r_1, r_2, r_3, r_4)_+ &= (0,0,0,0); (0,0,3,3); (3,0,0,0); (3,3,3,0)\\ (r_1, r_2, r_3, r_4)_- &= (0,0,0,3); (0,3,3,3); (3,3,0,0); (3,3,3,3).\end{aligned} \quad (3.1.228)$$

Again a line preserving this deformation necessarily acts invertibly on it.

– $(a_1, a_2, a_3, a_4, a_5) = (1, 1, 1, 1, 1)$. Invariance requires

$$e^{i\frac{\pi}{5}(s_1+s_2+s_3+s_4+s_5)}\prod_{i=1}^{5}\frac{\sin\left(\frac{2\pi(r_i+1)}{5}\right)}{\sin\left(\frac{\pi(r_i+1)}{5}\right)} = \pm\frac{1}{2}(11+5\sqrt{5}) \quad (3.1.229)$$

thus for $s_1 + s_2 + s_3 + s_4 + s_5 = 0 \bmod 5$ we need to solve

$$\prod_{i=1}^{5}\frac{\sin\left(\frac{2\pi(r_i+1)}{5}\right)}{\sin\left(\frac{\pi(r_i+1)}{5}\right)} = \pm\frac{1}{2}(11+5\sqrt{5}) \quad (3.1.230)$$

Note that this is symmetric in all r_i, in the solutions below we write only one solution per orbit of this permutation symmetry. We find

$$\begin{aligned}(r_1, r_2, r_3, r_4, r_5)_+ &= (0,0,0,0,0); (0,0,0,3,3); (0,3,3,3,3);\\ (r_1, r_2, r_3, r_4, r_5)_- &= (0,0,0,0,3); (0,0,3,3,3); (3,3,3,3,3)\,.\end{aligned} \tag{3.1.231}$$

We see that a line preserving this deformation is necessarily invertible.

- The symmetries preserved by the Khaler structure deformation obey

$$e^{i\frac{\pi}{5}\left(2\eta-\sum_i s_i\right)}\prod_{i=1}^{5}\frac{\sin\left(\frac{2\pi(r_i+1)}{5}\right)\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi(r_i+1)}{5}\right)\sin\left(\frac{2\pi}{5}\right)} = \pm 1 \tag{3.1.232}$$

  which, after having picked $2\eta = \sum_i s_i$ mod 5, is the same equation as the last case among the chiral-chiral deformations. Again only invertible symmetries preserve this operator.

Also in this example we can look for higher dimensional submanifolds preserving a non-invertible symmetry. As a simple illustration consider the chiral-chiral deformations with

$$(a_1, a_2, a_3, a_4, a_5) = (3,2,0,0,0); (3,0,2,0,0); (3,0,0,2,0); (3,0,0,0,2)\,, \tag{3.1.233}$$

from our analysis we see that each one of those commutes with the line $W = \mathcal{L}_{2,0} \otimes (\mathbb{1})^{\otimes 4}$, whose fusion rule is

$$W \times W = \mathbb{1} + W\,. \tag{3.1.234}$$

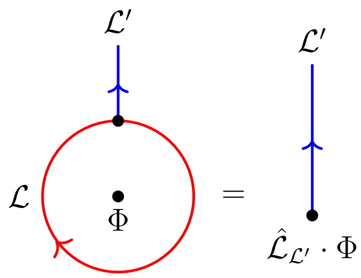
Thus on this 4-dimensional submanifold we have at least a Fibonacci category symmetry. We can also turn on the deformations

$$\begin{aligned}(a_1, a_2, a_3, a_4, a_5) =&(3,1,1,0,0); (3,1,0,1,0); (3,1,0,0,1);\\ &(3,0,1,1,0); (3,0,1,0,1); (3,0,0,1,1)\,,\end{aligned} \tag{3.1.235}$$

and W is still preserved, enlarging the Fibonacci-symmetric submanifold of the moduli space to 10 dimensions. We can also consider the submanifolds obtained turning on the deformations above with $a_1 \leftrightarrow a_i$ for $i = 2, 3, 4, 5$. On each of those 10-dimensional subspaces we have a different Fibonacci category symmetry.

Selection Rules

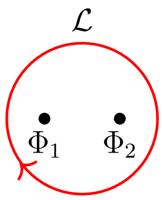
The presence of these topological defects at the Gepner point and along certain submanifolds of the moduli space imposes constraints on the dynamics of the theory. The presence of a fusion category symmetry in a $2d$ QFT implies degeneracies between twisted and untwisted sectors, as non-invertible lines transform local operators in twist defects. This is properly addressed using the tube algebra built out of the fusion category, see e.g. [36, 81]. The elements of such algebra correspond to the lasso actions [12]



. (3.1.236)

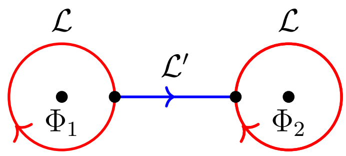
In general one also needs to specify a junction vector in $\mathrm{Hom}(\mathcal{L}\times\bar{\mathcal{L}},\mathcal{L}')$, however in our case all junction spaces are at most one-dimensional and we can omit this extra label. More general lasso actions can be obtained acting on operators in twisted sectors. The total Hilbert space, which includes the twisted Hilbert spaces for all the topological defect lines of the theory, splits in representations of the Tube algebra. Therefore operators, both local and twisted, will also be organized in such representations. Representations of the tube algebra are in canonical one-to-one correspondence with anyons of the Drinfeld center of the fusion category, with the fusion rules of the $3d$ TQFT anyons coinciding with tensor products of Tube algebra representations. Moreover the representation in which a local operator of the CFT transforms is determined by the $3d$ bulk anyon ending on it (see Fig.3.1). The most immediate consequence of the symmetry are selection rules. In case of a fusion category symmetry these state that a correlation function can be non-zero only if the tensor product of the Tube algebra representations of all operators contains the identity [36].

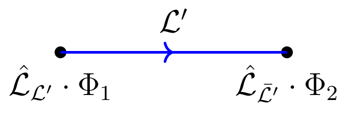
A subsets of interesting selection rules however can be accessed without employing the full power of the Tube algebra. Two important observables on the conformal manifold of a CFT are the two and three point functions of the exactly marginal deformations. The former gives the Zamolodchikov metric of the conformal manifold, while the latter encode information about the curvature [262]. We shall consider the CFT on a genus zero surface, this allows us to nucleate a non-invertible line defect linking with all operators in the correlator at the price of dividing by its quantum dimension. Let Φ_1 and Φ_2 be exactly marginal operators, and consider its their two point function. Opening an $\mathcal{L}$ loop and dividing by $\langle\mathcal{L}\rangle$ the correlator is unchanged, namely

$$\langle\Phi_1\Phi_2\rangle = \frac{1}{\langle\mathcal{L}\rangle}\Big\langle$$


$$\Big\rangle\,. \tag{3.1.237}$$

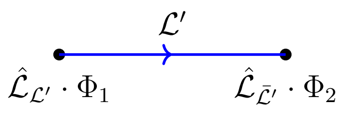
Now, pinching the line in between the locations of the two local operators and fusing we get[8]

$$\langle\Phi_1\Phi_2\rangle = \frac{1}{\langle\mathcal{L}\rangle}\sum_{\mathcal{L}'}\frac{\sqrt{\langle\mathcal{L}'\rangle}}{\langle\mathcal{L}\rangle}\Big\langle$$


$$\Big\rangle$$
$$= \frac{1}{\langle\mathcal{L}\rangle}\sum_{\mathcal{L}'}\frac{\sqrt{\langle\mathcal{L}'\rangle}}{\langle\mathcal{L}\rangle}\Big\langle$$


$$\Big\rangle \tag{3.1.238}$$

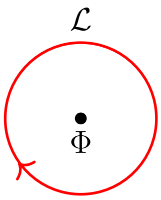
where the sum over $\mathcal{L}'$ runs over the lines appearing in the fusion channel $\mathcal{L}\times\bar{\mathcal{L}}$. Recall that this channel always contains the identity, so that

$$\langle\Phi_1\Phi_2\rangle = \frac{1}{\langle\mathcal{L}\rangle^2}\langle\hat{\mathcal{L}}\cdot\Phi_1\hat{\mathcal{L}}\cdot\Phi_2\rangle + \frac{1}{\langle\mathcal{L}\rangle}\sum_{\mathcal{L}'\neq\mathbb{1}}\frac{\sqrt{\langle\mathcal{L}'\rangle}}{\langle\mathcal{L}\rangle}\Big\langle$$


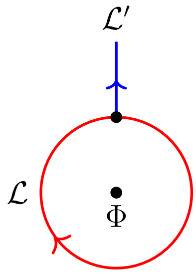
$$\Big\rangle\,. \tag{3.1.239}$$

[8]The coefficient $\frac{\sqrt{\langle\mathcal{L}'\rangle}}{\langle\mathcal{L}\rangle}$ ensures the proper normalization of the completeness relation, see e.g. [28]

This is the general selection rule implied by the presence of a non-invertible line on two point functions, we see that it relates correlators of local operators to those of the twisted sectors. From this expression one can show that if, say, Φ_1 is invariant under $\mathcal{L}$, i.e. $\hat{\mathcal{L}} \cdot \Phi_1 = \langle \mathcal{L} \rangle \Phi_1$, then all non-trivial lasso actions annihilate Φ_1 [261]. The proof is simple enough, take $\Phi_2 = \Phi_1$, then the first term on the right hand side of (3.1.239) already saturates the left hand side. The remaining terms then have to give zero, but since each can be interpreted as the norm squared of a vector in a twisted Hilbert space (we are assuming a unitary theory), they each vanish separately, implying that the image vector of Φ_1 under the lasso $\hat{\mathcal{L}}_{\mathcal{L}'}$ is null. In other words



$= \langle \mathcal{L} \rangle \Phi \quad \Rightarrow$



$= 0\,.$ (3.1.240)

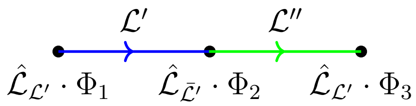
Now take the selection rule (3.1.239) with Φ_1 and Φ_2 different and suppose Φ_1 is invariant. By the argument above all contributions from twisted sectors vanish and we have

$$\langle \Phi_1 \Phi_2 \rangle = \frac{1}{\langle \mathcal{L} \rangle^2} \langle \hat{\mathcal{L}} \cdot \Phi_1 \hat{\mathcal{L}} \cdot \Phi_2 \rangle = \frac{1}{\langle \mathcal{L} \rangle} \langle \Phi_1 \hat{\mathcal{L}} \cdot \Phi_2 \rangle \,, \tag{3.1.241}$$

thus for the correlator to be non-zero also Φ_2 has to be invariant. This implies that, as we move away from the Gepner point preserving some non-invertible line, the mixed components of the Zamolodchikov metric involving the perturbation and any other marginal operator not invariant under the preserved lines vanish. A similar selection rule can be derived for three-point function on the sphere

$$\langle \Phi_1 \Phi_2 \Phi_3 \rangle = \frac{1}{\langle \mathcal{L} \rangle^2} \langle \hat{\mathcal{L}} \cdot \Phi_1 \hat{\mathcal{L}} \cdot \Phi_2 \hat{\mathcal{L}} \cdot \Phi_3 \rangle +$$

$$+ \frac{1}{\langle \mathcal{L} \rangle^2} \sum_{\substack{\mathcal{L}' \neq \mathbb{1} \\ \mathcal{L}'' \neq \mathbb{1}}} \sqrt{\langle \mathcal{L}' \rangle \langle \mathcal{L}'' \rangle} \langle$$



$\rangle\,.$ (3.1.242)

Now, if two out of three operators are invariant the correlator is non vanishing only if the also the third operator commutes with the line $\mathcal{L}$:

$$\langle \Phi_1 \Phi_2 \Phi_3 \rangle = \frac{1}{\langle \mathcal{L} \rangle} \langle \Phi_1 \Phi_2 \hat{\mathcal{L}} \cdot \Phi_3 \rangle \neq 0 \to \hat{\mathcal{L}} \cdot \Phi_3 = \langle \mathcal{L} \rangle \Phi_3 \,. \tag{3.1.243}$$

When the Φ_i are BPS operators this selection rule can be translated as a constraint on the moduli dependence of the chiral ring coefficients. It implies that, as we move along submanifolds of the moduli space, certain chiral ring coefficients are forced to vanish.

Constraints on Conformal Perturbation Theory

Besides selection rules in the deformed theory we can use the full symmetry at the Gepner point to simplify the use of conformal perturbation theory to compute corrections to conformal weights and 3-point function coefficients. For concreteness let us focus on 2-point correlators, but the method applies to higher point functions as well. Suppose we turn on the marginal deformation $\mathcal{O}$, in our case this will be expressed as $\mathcal{O} = G\bar{G}\Phi$ where Φ is a BPS primary and $G, \bar{G}$ are the appropriate supercharges. Two-point functions of the deformed theory can be written

$$\langle \Phi_1 \Phi_2 \rangle_\lambda = \langle \Phi_1 \Phi_2 e^{\lambda \int d^2 w \mathcal{O}(w)} \rangle \tag{3.1.244}$$

and the corrections to the weights of Φ_1 and Φ_2 form a power series in λ

$$h(\lambda) = \sum_{n=0}^{\infty} h^{(n)} \lambda^n \,, \tag{3.1.245}$$

with the n-th term determined by the integrated correlation function

$$\int d^2 w_1 \dots d^2 w_n \langle \Phi_1 \Phi_2 \mathcal{O}(w_1) \dots \mathcal{O}(w_n) \rangle \tag{3.1.246}$$

computed at the Gepner point. By using the selection rules implied by the Tube algebra we can find patterns of zeros in the series of correction. Consider for example the case of the $K3$ sigma model and take the deformation to be the one deriving from the chiral-chiral state

$$(|2,2\rangle)^{\otimes 2} \otimes (|0,0\rangle)^{\otimes 2} \otimes (\overline{|2,2\rangle})^{\otimes 2} \otimes (\overline{|0,0\rangle})^{\otimes 2} \,. \tag{3.1.247}$$

To use the selection rules implied by the Tube algebra we have to compute the tensor products of all the representations of the Tube algebra associated to the operator insertions. In the following we will always indicate a representation Γ of the Tube algebra by the corresponding $3d$ anyon, in particular we will write

$$\Gamma = \left(\vec{\mu}, \vec{\bar{\mu}}\right) \tag{3.1.248}$$

where $\vec{\mu}, \vec{\bar{\mu}}$ are $2r$-components vectors containing the Wilson lines labels

$$\vec{\mu} = ((a_1, c_1), \dots, (a_r, c_r)); \qquad (a_i, c_i) \in Q_{k_i} \tag{3.1.249}$$

Let's start from discussing the representation of $\mathcal{O}$. The primary Φ is in a representation

$$\Gamma_0 = (\vec{\mu}_0, \vec{\mu}_0) \qquad \vec{\mu}_0 = ((2,2), (2,2), (0,0), (0,0)) \tag{3.1.250}$$

which corresponds to an invertible line in $3d$. The supercharges of the diagonal superalgebra instead are in a reducible representation of the Tube algebra. More precisely, for a general Gepner model,

$$\Gamma_G = \bigoplus_{i=1}^{r} \left(\vec{\mu}_i, \vec{0}\right) \qquad \Gamma_{\bar{G}} = \bigoplus_{i=1}^{r} \left(\vec{0}, \vec{\mu}_i\right) \tag{3.1.251}$$

where

$$\vec{\mu}_i = ((0,0), \dots (k_i, k_i+2), \dots, (0,0)) \,. \tag{3.1.252}$$

Now, in a correlator with n insertion of $\mathcal{O}$ we have to compute the n-th tensor powers of the three representations Γ_0, Γ_G and $\Gamma_{\bar{G}}$. This is greatly simplified by the fact that the irreducible factors in Γ_G and $\Gamma_{\bar{G}}$ corresponds to lines that are not only invertible but also of order 2. We find

$$\begin{aligned}\Gamma_G^{\otimes n} &= \bigoplus_{k_1+\ldots+k_4=n} \binom{n}{k_1\ldots k_4} \left(\vec{\mu}_{k_1,\ldots,k_4}, \vec{0}\right) , \\ \vec{\mu}_{k_1,\ldots,k_4} &= \Big((2[k_1], 4[k_1]), (2[k_2], 4[k_2]), (2[k_3], 4[k_3]), (2[k_4], 4[k_4]), \vec{0}\Big)\end{aligned} \tag{3.1.253}$$

and similarly for $\Gamma_{\bar{G}}$. It is also easy to compute tensor powers of Γ_0, we have

$$\Gamma_0^{\otimes n} = \left(\vec{\mu}_0^{\otimes n}, \vec{\mu}_0^{\otimes n}\right) \qquad \vec{\mu}_0^{\otimes n} = ((2[n], 2n), (2[n], 2n), (0,0), (0,0)) , \tag{3.1.254}$$

notice that only n mod 4 matters as $\Gamma_0^{\otimes 4} = \mathbb{1}$. Then the representations entering in the correlators are

$$\Gamma_{\mathcal{O}}^{\otimes n} = \Gamma_G^{\otimes n} \otimes \Gamma_{\bar{G}}^{\otimes n} \otimes \Gamma_0^{\otimes n} = \bigoplus_{k_i,\bar{k}_i} \binom{n}{k_1\ldots k_4}\binom{n}{\bar{k}_1\ldots \bar{k}_4} \left(\vec{\mu}_{\mathcal{O}^n}, \vec{\bar{\mu}}_{\mathcal{O}^n}\right) \tag{3.1.255}$$

with

$$\vec{\mu}_{\mathcal{O}^n} = ((2[k_1+n], 4[k_1]+2n), (2[k_2+n], 4[k_2]+2n), (2[k_3], 4[k_3]), (2[k_4], 4[k_4])) \tag{3.1.256}$$

and similarly for $\vec{\bar{\mu}}_{\mathcal{O}^n}$. To give a concrete example we consider the 12 lightest non-BPS primaries corresponding to the states $|\phi_{ij}\rangle$, these are all of the form

$$|\phi_{12}\rangle = |1,1\rangle \otimes |1,-1\rangle \otimes (|0,0\rangle)^{\otimes 2} \otimes \overline{|1,1\rangle} \otimes \overline{|1,-1\rangle} \otimes (\overline{|0,0\rangle})^{\otimes 2} \tag{3.1.257}$$

with i and j denoting the tensor factor with $|1,1\rangle$ and $|1,-1\rangle$ respectively. Note that $\phi_{ij}^\dagger = \phi_{ji}$. These are all degenerate operators with $h = \bar{h} = 1/4$ and vanishing R-charges. We are interested in the two point functions $\langle \phi_{ij}^\dagger \phi_{lk}\rangle_\lambda$. The associated tube algebra representations $\Gamma_{\phi_{ij}}$ are of the form

$$\Gamma_{\phi_{12}} = (\vec{\mu}_{\phi_{12}}, \vec{\mu}_{\phi_{12}}) \qquad \vec{\mu}_{\phi_{12}} = ((1,1), (1,-1), (0,0), (0,0)) , \tag{3.1.258}$$

and correspond to non-invertible lines. The tensor products $\vec{\mu}_{\phi_{ji}} \otimes \vec{\mu}_{\phi_{kl}}$ contain the identity if and only if $k = i, l = j$, thus the only non-zero correlators at the Gepner point are $\langle \phi_{ji}\phi_{ij}\rangle$. Turning on λ we can have mixing among the operators, which is constrained by the selection rules. Since $\Gamma_{\mathcal{O}^n}$ only contains invertible anyons we see that a necessary condition for the identity to appear in $\vec{\mu}_{\phi_{ji}} \otimes \vec{\mu}_{\phi_{kl}} \otimes \vec{\mu}_{\mathcal{O}^n}$ is that $\vec{\mu}_{\phi_{ji}} \otimes \vec{\mu}_{\phi_{kl}}$ contains at least one invertible line. This immediately shows that the only non-vanishing 2-point functions are those of the form $\langle \phi_{ji}\phi_{ij}\rangle_\lambda$ or $\langle \phi_{ij}\phi_{ij}\rangle_\lambda$. For those correlators the relevant representations are of the form

$$\begin{aligned}&\Gamma_{12} \otimes \Gamma_{12} = \left(\vec{\mu}_{\phi_{12}}^{\otimes 2}, \vec{\mu}_{\phi_{12}}^{\otimes 2}\right) , \qquad \vec{\mu}_{\phi_{12}}^{\otimes 2} = ((0,2) \oplus (2,2), (0,-2) \oplus (2,-2), (0,0), (0,0)) \\ &\Gamma_{12} \otimes \Gamma_{21} = \left(\vec{\mu}_{\phi_{12}} \otimes \vec{\mu}_{\phi_{21}}, \vec{\mu}_{\phi_{12}} \otimes \vec{\mu}_{\phi_{21}}\right) , \\ &\qquad\qquad\qquad\qquad \vec{\mu}_{\phi_{12}} \otimes \vec{\mu}_{\phi_{21}} = ((0,0) \oplus (2,0), (0,0) \oplus (2,0), (0,0), (0,0)) .\end{aligned} \tag{3.1.259}$$

This conclusion holds for any deformation such that $\vec{\mu}_{\mathcal{O}^n}$ contains only invertible lines, in our specific example however we can do better. Indeed also all correlators of the form $\langle \phi_{ij}\phi_{ij}\rangle_\lambda$

vanish whenever i or j is different than 1 or 2. This is because when tensoring, say $\vec{\mu}_{\phi_{13}}^{\otimes 2}$ with $\vec{\mu}_{\mathcal{O}^n}$ we get terms that in the third tensor factor have the pair $(2[k_3], 4[k_3]+2)$, which never trivializes allowing the singlet representation. It follows that the correlators to consider are $\langle\phi_{12}\phi_{12}\rangle_\lambda$ and $\langle\phi_{ji}\phi_{ij}\rangle_\lambda$, namely the only mixing allowed by the deformation is between ϕ_{12} and itself. We can also study the power series in λ in more detail, starting from $\langle\phi_{12}\phi_{12}\rangle_\lambda$. The selection rule requires the product $\Gamma_{\phi_{12}}^{\otimes 2} \otimes \Gamma_{\mathcal{O}}^{\otimes n}$ to contain the identity, this forces us to choose $k_3, \bar{k_3}$ and $k_4, \bar{k_4}$ even, then any representation appearing in the decomposition in the tensor product is of the form $(\vec{\mu}_\phi^{\otimes 2} \otimes \vec{\mu}_{\mathcal{O}^n}, \vec{\mu}_\phi^{\otimes 2} \otimes \vec{\mu}_{\mathcal{O}^n})$ with

$$\begin{aligned}\vec{\mu}_\phi^{\otimes 2} \otimes \vec{\mu}_{\mathcal{O}^n} =&\Big((2[k_1+n], 4[k_1]+2n+2) \oplus (2[k_1+n+1], 4[k_1]+2n+2),\\ &(2[k_1+n], 4[k_2]+2n-2) \oplus (2[k_2+n+1], 4[k_2]+2n-2), (0,0), (0,0)\Big)\,.\end{aligned} \tag{3.1.260}$$

Now we notice that when $n = 0, 2 \bmod 4$ there is no value of k_1 or k_2 such that

$$4[k_1]+2n+2 = 4[k_1] \pm 2 = 0 \bmod 8 \tag{3.1.261}$$

and the singlet representation appears only when $n = 1, 3 \bmod 4$. Therefore, the power series in only contains the odd powers λ^{2m+1}. Another two-point function we can consider is $\langle\phi_{21}\phi_{12}\rangle_\lambda$. In this case, for the n-th order correction we find

$$\begin{aligned}\vec{\mu}_\phi \otimes \vec{\mu}_{\phi^\dagger} \otimes \vec{\mu}_{\mathcal{O}^n} =&\Big((2[k_1+n], 4[k_1]+2n) \oplus (2[k_1+n+1], 4[k_1]+2n),\\ &(2[k_1+n], 4[k_2]+2n) \oplus (2[k_2+n+1], 4[k_2]+2n), (0,0), (0,0)\Big)\end{aligned} \tag{3.1.262}$$

which shows that for $n = 1, 3 \bmod 4$ the identity does not appear in the tensor product, and the series is in even powers λ^{2m}. These result are compatible with those of [256]. Similar computations can be repeated for other operators, more complicated correlation functions or more general Gepner models.

3.2 Toplogical operators, on average

Global symmetries constitute an indispensable tool for studying physical systems, especially when the dynamics cannot be analyzed using exact techniques. The idea of symmetry is sometimes vaguely stated and often confused with slightly different concepts, such as *selection rules.* While these two ideas are often connected, they are logically distinct. Topological operators provide a clear definition of symmetry, which encodes all of the dynamical consequences. The aim of the rest of this chapteris to extend the formalization of symmetries of [9] to QFTs where the interactions are randomly distributed, for the case of 0-form global symmetries. We believe that a more systematic treatment of symmetries in QFTs of this kind can be useful, given the notorious difficulties in treating such systems. There are two relevant possibilities considered here.

1. The random couplings $h(x)$ vary in space and are distributed according to a probability functional $P[h]$.

2. The random couplings h are constant and drawn from a probability density $P(h)$.

Scenario 1. is relevant for statistical mechanical systems with impurities or disorder (for a review see [265]). There are two main variants of disorder QFT: *quenched* if the impurities are treated as external random sources and *annealed* if the impurities are taken dynamical. Physically the two situations depend on the time scale we are looking at. At extremely long time-scales, where the entire system reaches equilibrium, we should take the impurities dynamical. Since impurities have very long thermalizations time scales, quenching is useful for time-scales where the system essentially thermalizes, with the impurities taken fixed. In the quenched case, the properties of the QFT will of course depend on the impurities. If we assume that impurities are random, possible observables are taken by averaging over the impurities with the chosen distribution. In a lattice formulation an impurity is modelled by an interaction which is different at any site, and its presence is unpredictable. In the continuum limit it is often the case that we can describe such systems as the average over an ensemble of field theories where the coupling constants are space dependent. Particularly interesting is the case of the Ising model perturbed with a random magnetic field (dubbed as random field Ising model) [266] or with a random interaction between nearby spins (dubbed as random bond Ising model) [267]. See e.g. [268–271] for recent works on these models.

Scenario 2. is relevant for quantum gravity and has received significant attention lately. The connection between averaging and euclidean gravity path integrals dates back to [272, 273] in association to Euclidean wormholes. In the context of the AdS/CFT correspondence [274–276], the connection has been invoked in [277] as a possible way to interpret from a boundary point of view the origin of interactions between disconnected components of a boundary theory induced by bulk Euclidean wormholes (factorization puzzle). Further elaborations with concrete examples appear in [278]. Ensemble averaging features also in the Sachdev-Ye-Kitaev (SYK) model [279–281]. A concrete connection has recently been made in [282], where it has been shown that the sum over geometries in Jackiw-Teitelboim gravity [283, 284] with n disconnected boundaries is dual to the ensemble average of an n-point correlation function in a matrix model. Other notable examples of ensemble averaging after [282] include averages over free compact bosons in 2d [285, 286] (see also e.g. [287–291] for related studies and generalizations), averages over OPE coefficients in effective 2d CFTs [292, 293], averages over the gauge coupling in 4d $\mathcal{N} = 4$ super Yang-Mills theory [294].

In both scenarios 1. and 2. we focus on correlation functions of local operators with *quenched disorder* averaging. These include averages of products of correlators, which are effectively independent observables. In disconnected spaces, when h is constant, also averaged single correlators can lead to averages of products of correlators, which is the mechanism leading to the factorization puzzle in the context of AdS/CFT. In order to distinguish scenario 1. from scenario 2. we dub the first as "quenched disorder" and the latter as "ensemble average", but it should be kept in mind that quenching is involved also in scenario 2.

We start from a *pure theory*, that is an ordinary QFT with no disorder, and deform it with a certain local interaction. In the quenched disorder case the strength of this interaction varies from point to point, while it is constant in ensemble average. In both cases the interaction can break part of all of the global symmetries of the pure system, so that each specific realization generically has less symmetries and less predictive power than the pure theory. On the other hand it has been noticed in several examples that symmetries of the pure systems can be recovered after the average on the coupling is taken into account. These statements are mostly based on the observation that the averaged system satisfies selection rules which are not enjoyed by the generic specific realization. Intuitively speaking, even if the random coupling breaks the symmetry, this re-emerges provided we average over all the ensemble in a sufficiently symmetric way (in a sense to be clarified). For simplicity, in what follows we refer to such symmetries as *disordered symmetries* and *averaged symmetries* respectively in the context of quenched disorder and ensemble average. Note that this is distinct from the notion of emergent symmetries used in pure QFTs when a symmetry is approximately conserved in the IR. In the disorder case the symmetry is *exact* at all energy scales, but only on average.[9]

We will review these kind of arguments from a spurionic point of view at the beginning of section 3.2.1 for quenched disorder, and in section 3.2.5 for ensemble average, deriving under which condition the selection rules of the pure theory are satisfied after the average.

This is still an imprecise information since, as we emphasized, having a global symmetry is stronger than just observing the validity of some selection rule. This is crucial in order to get stronger dynamical constraints implied by 't Hooft anomalies, and eventually gauging the symmetry. Our goal is to clarify the sense in which these systems recover the symmetry, aiming to construct the analog of topological operators for both quenched disorder and ensemble average QFTs.

Sections 3.2.1, 3.2.2 and 3.2.3 focus on quenched disordered systems. We consider theories defined in the continuum and admitting a description in terms of an action (Hamiltonian) obtained from that of the pure theory S_0 by adding a local operator $\mathcal{O}_0(x)$ with a space-time dependent coupling $h(x)$:

$$S[h] = S_0 + \int d^d x\, h(x)\mathcal{O}_0(x) \ . \tag{3.2.1}$$

This is what we will call a specific realization. Correlation functions of local operators $\mathcal{O}_i$ for a given value of $h(x)$ are computed by a path integral:

$$\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle = \frac{\int \mathcal{D}\mu\, e^{-S[h]}\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)}{\int \mathcal{D}\mu\, e^{-S[h]}} \ . \tag{3.2.2}$$

Given a probability functional $P[h]$, a set of observables of the disordered system are the averaged correlation functions

$$\overline{\big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle} = \int \mathcal{D}h\, P[h]\, \big\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\big\rangle \,, \tag{3.2.3}$$

or more generally the averages of products of correlators

$$\overline{\prod_{j=1}^{N} \Big\langle \mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{n_j}^{(j)}(x_{k_j}^{(j)})\Big\rangle} \ . \tag{3.2.4}$$

The starting point for systematizing global symmetries of the disordered system which are not enjoyed by the specific realizations is to derive Ward identities for the averaged correlators.

[9]We can also have emergent symmetries in both senses, namely emerging after average and in the IR. We will discuss this case in section 3.2.3.

We do this in section 3.2.1, starting from the simplest case of continuous 0-form invertible symmetries. The Noether current J^{μ} associated to the symmetry of the pure theory is no longer conserved in the specific realizations if $\mathcal{O}_0(x)$ is charged under the symmetry. However we find that the *shifted current* in (3.2.37) leads to standard Ward identities (3.2.41) for averaged single correlators and the less standard identities (3.2.45) for averages of products of correlators.

In order to generalize our results to discrete symmetries, where a Noether current is unavailable, in section 3.2.1 we construct the symmetry operators, topological on average, which implements the finite group action. This is not as easy as in pure theories because of the disorder. The topological operator $\widetilde{U}_g$ is a complicated power series of integrated currents which however can be resummed to give the simple expression

$$\widetilde{U}_g = U_g \langle U_g \rangle^{-1} \,, \tag{3.2.5}$$

where U_g is the topological operator of the pure theory. Its action on averages of simple correlators is given in (3.2.58). In products of correlators (3.2.4) the operator $\widetilde{U}_g$ is topological on average only if inserted in all the correlators involved, as in (3.2.61). This characterizes intrinsically the disordered symmetries and implies somewhat exotic selection rules which are weaker with respect to symmetries not broken by the random interactions.

The Ward identities satisfied by $\widetilde{U}_g$, when the latter is supported on a compact surface $\Sigma^{(d-1)}$ are valid regardless of how the symmetry is realized on the vacuum. When the symmetry operator is well defined also on infinite surfaces the disordered symmetry is not spontaneously broken and implies selection rules. The same is not true for spontaneously broken symmetries, we will briefly discuss this situation in the final section 3.2.6.

Beyond selection rules, our analysis allows us to show that disordered symmetries (both continuous and discrete) can be coupled to external backgrounds, can be gauged, and can have 't Hooft anomalies, precisely like ordinary symmetries. We also argue that a symmetry of a pure system with a 't Hooft anomaly, when it reappears as disordered symmetry, enjoys the same 't Hooft anomaly thus implying the same constraints on the dynamics, and that a possible higher-group structure of the underlying 0-form symmetry with higher-form symmetries of the pure theory is recovered after average due to the topological nature of the higher-group structure. Symmetry Protected Topological (SPT) phases [189], protected by what we denoted disordered symmetries, appeared already in condensed matter, see e.g. [295–302]. Our findings can possibly provide a different theoretical QFT-based framework for such phases of matter.

In section 3.2.2 the above results, derived directly from the disordered theory, are reproduced using the replica trick, the standard way to deal with theories of this kind. Disordered symmetries manifest themselves as standard symmetries in the replica theory, thus offering a conceptual different viewpoint on these kind of symmetries. Aside from providing a sanity check of the results, the replica theory allows us to also study another scenario: disordered symmetries emerging at long distances, discussed in section 3.2.3. The effect of the disorder can now lead to the more exotic selection rules (3.2.104) and (3.2.105). The phenomenon manifests in the replica theory as two irreducible representations of the replica symmetry transforming in different representations of the emergent disordered symmetry. As an interesting application of this result we consider the prime example of an emergent symmetry, conformal invariance, and we show that as a consequence of these exotic selection rules, a quenched disordered system can flow in the IR to a fixed point described by a Logarithmic conformal field theory (LogCFT) [303–307].

We analyze ensemble average in section 3.2.5. While the intuitive idea that the average restores the symmetry is still true, and selection rules apply (section 3.2.5), the status of the averaged symmetry is drastically different. A hint already comes from the replica trick: when applied with constant couplings, the replica theory is non-local, and even if the symmetry is manifest its Noether current is not a local operator. This is problematic for constructing a topological operator. Indeed, independently of the replica trick, we imitate the analysis done for disordered theories, and we get the exotic topological charge operator (3.2.134). This is not really a co-dimension one operator, since it depends both on a closed surface $\Sigma^{(d-1)}$ and on a filling region $D^{(d)}$ such that $\partial D^{(d)} = \Sigma^{(d-1)}$. In particular the operator cannot be supported on homologically non-trivial cycles. Crucially, the operator $\widehat{Q}$ implies selection rules, because the second term in (3.2.134), when inserted on average of correlation functions of local operators, vanishes when integrated over the full space. If the space manifold is connected, there are only two possible filling regions of a homologically trivial $\Sigma^{(d-1)}$, and $\widehat{Q}$ is independent of the choice. On the other hand on a *disconnected* space there are several choices of filling region $D^{(d)}$, and the charge operator does depend on these choices. Nevertheless, we do have selection rules for averages of correlators, if one takes into account all the connected components of space, and we can construct operators (A.5.20) implementing the finite group action. In each connected component the selection rules can be violated, allowing the charges to escape from one component to the other. We have then the somewhat exotic situation of a 0-form symmetry in the sense of selection rules on correlation functions of local operators, but without having genuine topological operators (even after average). In contrast to ordinary symmetries and disordered symmetries in the quenched disordered case above, averaged symmetries cannot be coupled to background gauge fields in ordinary ways and hence cannot be gauged.

In section 3.2.5 we comment about the gravity interpretation of these results. Whenever the average theory admits a gravitational bulk dual, the local charge violation in presence of disconnected space has the natural interpretation in the bulk as charge violation induced by Euclidean wormholes configurations, as pointed out in [308–310]. The difficulty (impossibility) of gauging averaged boundary symmetries that we have found clarify why such symmetries cannot be identified with bulk gauge symmetries.

We conclude in section 3.2.6. In appendix A.4 we work out some specific examples for concreteness, and in appendix A.5 we explicitly construct the operator which implements the action of the group for averaged symmetries.

3.2.1 Symmetries in quenched disorder

In this section we study global 0-form symmetries in quenched disorder theories which arise only after the average. We start in section 3.2.1 by reviewing how Ward identities for ordinary 0-form symmetries are recasted in terms of topological operators in pure QFTs. We generalize the analysis to theories with quenched disorder in section 3.2.1 and construct the topological operator implementing the symmetry group action in section 3.2.1. We then discuss 't Hooft anomalies and gaugings for both continuous and discrete disordered symmetries in sections 3.2.1 and 3.2.1.

Pure theories and explicit symmetry breaking

Consider a standard d-dimensional Euclidean QFT described by the action S_0. If this theory is invariant under some continuous symmetry group G, correlation functions of local operators must satisfy the usual constraints imposed by the Ward-Takahashi identities:

$$i\big\langle \partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle = \sum_{l=1}^{k} \delta^{(d)}(x-x_l)\big\langle \mathcal{O}_1(x_1)\dots\delta\mathcal{O}_l(x_l)\dots\mathcal{O}_k(x_k)\big\rangle \,. \tag{3.2.6}$$

Here $J_\mu(x)$ is the Noether current[10] and $\delta\mathcal{O}_l(x_l)$ is the transformation of the local operator $\mathcal{O}_l$ under the action of the Lie algebra of G. For instance if $G = U(1)$ and $\mathcal{O}_l$ has charge q_l, then $\delta\mathcal{O}_l = iq_l\mathcal{O}_l$. Integrating over the full space $X^{(d)}$, the left hand side of (3.2.6) vanishes if $X^{(d)}$ has no boundary and the symmetry is not spontaneously broken, and we get selection rules on the correlators.

The modern approach [9] to interpret the same constraints consists in associating global symmetries to co-dimension one *topological operators* $U_g[\Sigma^{(d-1)}]$, $g \in G$, namely extended operators supported on some $(d-1)$-dimensional closed surface $\Sigma^{(d-1)}$, which are invariant under continuous deformations of their support. In the case of continuous symmetries such topological operators are simply[11]

$$U_g[\Sigma^{(d-1)}] = e^{i\alpha Q[\Sigma^{(d-1)}]} \,, \tag{3.2.7}$$

where $Q[\Sigma^{(d-1)}] = \int_{\Sigma^{(d-1)}} J_\mu n^\mu$ is the Noether operator which measures the charge enclosed within the region $D^{(d)}$ delimited by $\Sigma^{(d-1)}$ with $\partial D^{(d)} = \Sigma^{(d-1)}$. We can then write integrated Ward identities. For instance, if $G = U(1)$ we have

$$\Big\langle Q[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = \chi(\Sigma^{(d-1)})\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle \,, \tag{3.2.8}$$

with

$$\chi(\Sigma^{(d-1)}) = \sum_{l, x_l \in D^{(d)}} q_l \,. \tag{3.2.9}$$

The integrated Ward identity can be iterated using the fact that $J^\mu(x)$ is uncharged with respect to $Q[\Sigma^{(d-1)}]$,[12] resulting in

$$\Big\langle Q^n[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = \chi^n(\Sigma^{(d-1)})\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle \,. \tag{3.2.10}$$

This implies that the exponentiated operators (3.2.7) satisfy

$$\Big\langle U_g[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\Big\rangle = e^{i\alpha\chi(\Sigma^{(d-1)})}\big\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle \,, \quad g = e^{i\alpha} \,. \tag{3.2.11}$$

[10] For convenience we define the Noether current as $\delta S = i\int \epsilon(x)\partial^\mu J_\mu$. Notice that this has an extra factor of i with respect to the one obtained by Wick rotating the standard Minkowski current.

[11] In the following we suppress the group and algebra indices. In (3.2.7) the element $g \in G$ is the exponential of α valued in the dual of the Lie algebra of G.

[12] This is not true for non-abelian G. However with simple manipulations one can reach the same conclusion. Here we focus on the abelian case just for notational simplicity.

More generally the integrated Ward identities associated to a finite transformation $g \in G$ can be written as

$$\left\langle U_g[\Sigma^{(d-1)}]\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\right\rangle = \left\langle \mathcal{O}'{}_1(x_1)U_g[\Sigma'_{d-1}]\mathcal{O}_2(x_2)\dots\mathcal{O}_k(x_k)\right\rangle , \qquad (3.2.12)$$

where $\mathcal{O}'_1(x_1) = (R_1(g)\cdot\mathcal{O}_1)(x_1)$ is the transformed operator according to its representation R_1 under G, $\Sigma^{(d-1)}$ is a surface linking with the point x_1 and $\Sigma'^{(d-1)}$ is its deformation across the point. The selection rules on correlation functions now follow from the fact that a topological operator $U_g[\Sigma^{(d-1)}]$ supported on a very big surface at infinity is trivial, but shrinking it to a point, U_g passes and transforms all the local operators. We then get

$$\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = R_1(g)\dots R_n(g)\cdot\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle , \qquad (3.2.13)$$

which is the desired selection rule. A correlation function of local operators can be non-vanishing only if the direct product of representations contains the singlet representation. While $Q[\Sigma^{(d-1)}]$ and $U_g[\Sigma^{(d-1)}]$ enforce equivalent constraints on the theory, the advantage of using the exponentiated operator $U_g[\Sigma^{(d-1)}]$ is that in (3.2.12) we do not need to define the infinitesimal transformation $\delta\mathcal{O}$ so that the generalization to discrete symmetries is straightforward.

If we add a deformation of the pure theory which explicitly breaks G, the Ward identities (3.2.6) acquire a new term and, as expected, the operator $Q[\Sigma^{(d-1)}]$ (or equivalently $U_g[\Sigma^{(d-1)}]$) is no longer topological. For example, for $G = U(1)$ and a deformation described by the action (the term $h\mathcal{O}_0(x)$ is always paired with its hermitian conjugate, which we leave implicit)

$$S = S_0 + h\int d^dx\,\mathcal{O}_0(x) , \qquad (3.2.14)$$

where $\mathcal{O}_0(x)$ is a local operator with charge q_0 under $U(1)$ and h is a coupling, we get

$$\begin{aligned} i\big\langle\partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\big\rangle =& \sum_{l=1}^k \delta^{(d)}(x-x_l)\big\langle\mathcal{O}_1(x_1)\dots\delta\mathcal{O}_l(x_l)\dots\mathcal{O}_k(x_k)\big\rangle \\ & - ihq_0\big\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\mathcal{O}_0(x)\big\rangle . \end{aligned} \qquad (3.2.15)$$

Integrating over an open region $D^{(d)}$ with boundary $\Sigma^{(d-1)}$ we have

$$\left\langle Q[\Sigma^{(d-1)}]\mathcal{O}_1\dots\mathcal{O}_k\right\rangle = \chi(\Sigma^{(d-1)})\langle\mathcal{O}_1\dots\mathcal{O}_k\rangle - hq_0\int_{D^{(d)}} d^dx\,\langle\mathcal{O}_1\dots\mathcal{O}_k\mathcal{O}_0(x)\rangle . \qquad (3.2.16)$$

If the coupling h is irrelevant, at large distances and for sufficiently large surfaces $\Sigma^{(d-1)}$, the second term in the r.h.s. of (3.2.16) is suppressed with respect to the first one, and the operators $Q(\Sigma^{(d-1)})$ become approximately topological.[13] In this case we say that the symmetry G is *emergent* in the IR.

Quenched disorder and Ward identities

Theories with quenched disorder in the continuum limit can often be described starting from a pure theory S_0 and adding a perturbation like in (3.2.14) (see e.g. [311, 312]), where h is

[13]For a related discussion on approximate symmetries in the language of topological operators see [93].

taken to be space-dependent (again we always implicitly pair up $h(x)\mathcal{O}_0(x)$ with its hermitian conjugate):

$$S[h] = S_0 + \int d^d x\, h(x)\mathcal{O}_0(x)\,. \tag{3.2.17}$$

The random coupling is sampled from a distribution $P[h]$ and we should think of an ensemble of systems, each member being described by the action (3.2.17). Note that the considerations above on the explicit breaking are valid, with minor modifications, for each member of the ensemble.

A relevant example (which we will extensively consider in the sections 3.2.2 and 3.2.3) is the case of white noise, where $P[h]$ is Gaussian

$$P[h] \propto \exp\Big(-\frac{1}{2v}\int d^d x\, h^2(x)\Big)\,, \tag{3.2.18}$$

parametrized by a coupling v which governs the width of the Gaussian distribution. Dimensional analysis fixes the dimension of v to be

$$[v] = d - 2\Delta_{\mathcal{O}_0}\,, \tag{3.2.19}$$

where $\Delta_{\mathcal{O}_0}$ is the classical scaling dimension of the operator $\mathcal{O}_0$. The disorder is classically irrelevant in the RG sense when

$$\Delta_{\mathcal{O}_0} > \frac{d}{2}\,. \tag{3.2.20}$$

The equation (3.2.20) is called Harris criterion [313]. If the disorder is classically relevant or marginal, it has an important effect on the IR dynamics. For instance, other fixed points could emerge, so called random fixed points, which can also have logarithmic behavior (see section 3.2.3), or we could have no fixed points at all. When (3.2.20) is satisfied, the IR behaviour of the system is unaffected by the impurities.

Like in the pure theory case, if the coupling $h(x)$ breaks a symmetry G and is irrelevant, then the symmetry G will appear as an emergent symmetry in the IR theory. On the other hand, in disordered theories symmetries might also appear on average, but *exactly*, namely at all energy scales, independently on the scaling dimension of $h(x)$. It is important to keep into account this distinction in the considerations that will follow. The latter case is the one that we will call *disordered symmetries.*

The observables we are interested in are averaged correlation functions of local operators defined as (we adopt here the notation of [312])

$$\overline{\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_k(x_k)\rangle} = \int \mathcal{D}h\, P[h]\frac{\int\mathcal{D}\mu\, e^{-S[h]}\mathcal{O}_1(x_1)\ldots\mathcal{O}_k(x_k)}{\int\mathcal{D}\mu\, e^{-S[h]}}\,, \tag{3.2.21}$$

where μ is the path integral measure and $P[h]$ is an arbitrary distribution, not necessarily of the form (3.2.18). Correlation functions can be obtained as usual by coupling each local operator $\mathcal{O}_i$ to an external source K_i and by taking functional derivatives with respect to the K_i's of the averaged generating functional $Z_D[K_i]$ defined as

$$Z_D[K_i] := \int \mathcal{D}h\, P[h]\frac{\int\mathcal{D}\mu\, e^{-S[h]+\int K_i\mathcal{O}_i}}{\int\mathcal{D}\mu\, e^{-S[h]}} = \int\mathcal{D}h\, P[h]\frac{Z[K_i,h]}{Z[0,h]}\,. \tag{3.2.22}$$

We can also define the disordered free energy $W_D[K_i]$ as

$$W_D[K_i] := \int \mathcal{D}h \, P[h] \log Z[K_i, h] = \int \mathcal{D}h \, P[h] W[K_i, h] = \overline{W[K_i, h]} \,, \tag{3.2.23}$$

that generates averages of *connected* correlation functions

$$\overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \rangle_c} = \left. \frac{\delta^k W_D[K_i]}{\delta K_1(x_1) \dots \delta K_k(x_k)} \right|_{K_i=0} . \tag{3.2.24}$$

We stress that, unlike standard QFTs, in quenched disorder theories not all correlators can be determined from the connected ones and in particular

$$\overline{\langle \mathcal{O}_i(x) \rangle \langle \mathcal{O}_j(y) \rangle} \neq \overline{\langle \mathcal{O}_i(x) \rangle} \; \overline{\langle \mathcal{O}_j(y) \rangle} \,. \tag{3.2.25}$$

This is one of the crucial properties of disordered systems which will play an important role in the following. This motivates to introduce a more general generating functional

$$Z_D^{(N)}[K_i^{(1)}, ..., K_i^{(N)}] := \int \mathcal{D}h \, P[h] \prod_{j=1}^N \frac{Z[K_i^{(j)}, h]}{Z[0, h]} \tag{3.2.26}$$

whose functional derivatives produce the average of products of correlators. The generalization of (3.2.25) is

$$Z_D^{(N)}[K_i^{(1)}, ..., K_i^{(N)}] \neq \prod_{j=1}^N Z_D[K_i^{(j)}] \,. \tag{3.2.27}$$

Now suppose that the pure theory S_0 has some global 0-form invertible symmetry G. If the random deformation is $G-$invariant every realization of the system enjoys the symmetry, therefore G is a symmetry of the full disordered theory and it will show up in the averaged correlators. Indeed from the Ward identities of the theory in presence of a random source $h(x)$, by simply taking the average we immediately get the expected identities. This applies also to higher-form symmetries which cannot be broken by adding local operators to the action [9, 207].

If the random deformation breaks some or all of the symmetries of the pure theory, the story is more interesting. In this case we want to understand if and under which conditions the disordered theory still enjoys these symmetries. We start by considering an internal invertible continuous 0-form symmetry G, but our conclusions apply also in more general setups. In order to gain some intuition it is useful to use a spurionic argument. The path integral of the theory coupled to a random source $h(x)$ is

$$Z[h] = \int \mathcal{D}\mu \exp\left(-S_0 - \int h(x) \mathcal{O}_0(x) \right) \,. \tag{3.2.28}$$

Because of the explicit breaking the partition function obeys

$$Z[h] = Z[R_0^\vee(g) \cdot h], \qquad g \in G \tag{3.2.29}$$

where $\mathcal{O}_0$ transforms in representation R_0 of G, and $R_0^\vee$ is its transpose. Turning on sources K_i for operators of the pure theory we see that the generating functional satisfies

$$\begin{aligned} Z[K_i, h] &= \int \mathcal{D}\mu \exp\left(-S_0 - \int h(x) \mathcal{O}_0(x) + \int K_i(x) \mathcal{O}_i(x) \right) \\ &= Z[R_i^\vee(g) \cdot K_i, R_0^\vee(g) \cdot h]. \end{aligned} \tag{3.2.30}$$

Thus the correlators before averaging are not $G-$invariant but

$$\frac{\delta}{\delta K_1 \dots \delta K_n} Z[K_i,h]\bigg|_{K_i=0} = R_1(g)\dots R_n(g)\cdot \frac{\delta}{\delta K_1 \dots \delta K_n} Z[K_i, R_0^\vee(g)\cdot h]\bigg|_{K_i=0} \quad . \tag{3.2.31}$$

This implies that

$$\begin{aligned}\overline{\langle \mathcal{O}_1(x_1)\dots \mathcal{O}_n(x_n)\rangle} &= \int \mathcal{D}h\, P[h] \frac{1}{Z[h]} \frac{\delta Z[K_i,h]}{\delta K_1\dots\delta K_n}\bigg|_{K_i=0} \\ &= R_1(g)\dots R_n(g)\cdot \int \mathcal{D}h\, P[h]\frac{1}{Z[h]}\frac{\delta Z[K_i, R_0(g)^\vee\cdot h]}{\delta K_1\dots \delta K_n}\bigg|_{K_i=0} .\end{aligned} \tag{3.2.32}$$

We can now change variable in the h-path integral, $R_0(g^{-1})^\vee \cdot h(x) \to h(x)$. Crucially, *if the probability measure $\mathcal{D}h\, P[h]$ is invariant*, the averaged correlator obeys the G selection rules

$$\overline{\langle \mathcal{O}_1(x_1)\dots \mathcal{O}_n(x_n)\rangle} = R_1(g)\dots R_n(g)\cdot \overline{\langle \mathcal{O}_1(x_1)\dots \mathcal{O}_n(x_n)\rangle}\,, \tag{3.2.33}$$

but only on average. For example, a space-dependent coupling breaks translations, but if $P[h]$ is translation-invariant (like e.g. in (3.2.18)), then momentum conservation is recovered on average.

Although the above spurion analysis is enough to determine selection rules, it does not provide the explicit form of the conserved currents and which Ward identities are satisfied (and how). The existence of topological operators is not even guaranteed and the common lore which identifies symmetries with topological defects needs a more detailed analysis in order to be verified. Let us then derive the form of Ward identities for disordered symmetries. For notational simplicity we focus on $G = U(1)$, but the analysis can be extended to any Lie group. Consider the generating functional $Z_D[K_i]$ defined in (3.2.22). The usual Ward identities are derived by changing variables in the path integral at the numerator, transforming all the fields with a space-time dependent $U(1)$ element $e^{i\epsilon(x)}$, so that

$$S_0 \to S_0 + i\int \epsilon(x)\partial_\mu J^\mu(x)\ , \tag{3.2.34}$$

J^μ being the Noether current. Here the symmetry is broken by $h(x)$ in any specific realization, nevertheless we can modify the standard procedure by changing variable also in the path integral at the denominator. Since $h(x)$ is space dependent, Poincaré invariance is explicitly broken in each specific realization and generally $\langle J^\mu\rangle \neq 0$. This suggests that even if the symmetry is recovered on average the current must be modified somehow. The above-mentioned change of variable in both numerator an denominator, expanding at first order in $\epsilon(x)$ leads to

$$\int \mathcal{D}h\, P[h]\left(\big\langle -\partial_\mu J^\mu - q_0 h\mathcal{O}_0 + q_i K_i \mathcal{O}_i\big\rangle_K + \frac{Z[K_i,h]}{Z[0,h]}\big\langle \partial_\mu J^\mu + q_0 h \mathcal{O}_0\big\rangle\right) = 0\,. \tag{3.2.35}$$

By taking functional derivatives with respect to the sources K_i and then setting them to zero we get

$$\overline{\big\langle \partial_\mu \widetilde{J}^\mu(x)\mathcal{O}_1(x_1)\cdots\big\rangle} + q_0\overline{\big\langle h(x)\widetilde{\mathcal{O}}_0(x)\mathcal{O}_1(x_1)\cdots\big\rangle} = \sum_i q_i \delta^{(d)}(x-x_i)\overline{\big\langle \mathcal{O}_1(x_1)\cdots\big\rangle}\ , \tag{3.2.36}$$

where we introduced the shifted operators

$$\widetilde{J}^\mu(x) := J^\mu(x) - \langle J^\mu(x)\rangle \ , \qquad \widetilde{\mathcal{O}}_0(x) := \mathcal{O}_0(x) - \langle \mathcal{O}_0(x)\rangle \ . \tag{3.2.37}$$

The vacuum expectation values should be thought of as certain (generally non-local) functionals of $h(x)$, whose presence is important in the average.

Since $h(x)$ is integrated over all space-dependent configurations, the second term in (3.2.36) vanishes identically provided that the probability measure satisfies certain invariance conditions. Indeed we are allowed to perform the change of variable $h(x) \to e^{-iq_0\epsilon(x)}h(x)$ in the h path integral of (3.2.22), and if the probability measure is invariant under this formal transformation we obtain

$$q_0 \int \mathcal{D}hP[h] \left(\langle h\mathcal{O}_0 \rangle_{K_i} - \frac{Z[K_i,h]}{Z[0,h]} \langle h\mathcal{O}_0\rangle \right) = 0 \,. \tag{3.2.38}$$

Taking arbitrary functional derivatives with respect to the external sources and setting them to zero we find

$$q_0 \overline{\langle h(x)\widetilde{\mathcal{O}}_0(x)\mathcal{O}_1(x_1)\dots\rangle} = 0 \,, \tag{3.2.39}$$

which implies the vanishing of the second term in the left hand side of (3.2.36). By changing variables in the path integral, we also get the relation

$$\Big\langle \partial^\mu J_\mu(x) + q_0 h(x)\mathcal{O}_0(x) \Big\rangle = 0 \,, \tag{3.2.40}$$

valid before averaging. We are now ready to discuss Ward identities. If $q_0 = 0$, namely the $U(1)$ symmetry is unbroken in any realization of the ensemble, plugging (3.2.40) in (3.2.36) leads to the averaged version of the ordinary Ward identities (3.2.6). This is of course expected, given that (3.2.6) holds even before average in this case. More interestingly, for $q_0 \neq 0$, thanks to (3.2.39) we find the disordered Ward identities

$$i\overline{\langle \partial_\mu \widetilde{J}^\mu(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\rangle} = \sum_{i=1}^k i q_i \delta^{(d)}(x-x_i)\overline{\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\rangle} \ . \tag{3.2.41}$$

Several comments are in order.

- The relation we obtained has the same form of a standard Ward identity, but for a modified current $\widetilde{J}^\mu = J^\mu - \langle J^\mu\rangle$. The modification is proportional to the identity operator in any of the specific realization of the ensemble, and can be thought of as an $h-$dependent counterterm which restores the conservation in the disordered theory. Note that the Ward identities written as in (3.2.41) apply for arbitrary correlation functions of local operators which *do not* contain explicit powers of $h(x)$.

- Before averaging the current J^μ (as well as its shifted version $\widetilde{J}^\mu$) is sensitive to the UV renormalization of the theory, i.e. it acquires a non-vanishing anomalous dimension (in contrast to ordinary conserved currents in pure theories). A proper definition of J^μ would require a regularization of the theory and a choice of renormalization scheme. Luckily enough, if we are *only* interested in averaged correlators, we do not need to worry about these issues, since (3.2.41) guarantees that $\widetilde{J}^\mu$ is effectively conserved inside averaged correlators.

- The Ward identities (3.2.41) are valid independently of the behavior of the current at infinity. When the integral of $\partial_\mu \widetilde{J}^\mu$ over the full space diverges (this requires the space to be non compact) the disordered symmetry is spontaneously broken. We do not discuss spontaneous disordered symmetry breaking in detail in this paper. We briefly comment on it in the conclusions. If the symmetry is not spontaneously broken the integral of $\partial_\mu \widetilde{J}^\mu$ over the full space vanishes. Thus (3.2.41) implies the selection rules we already derived from the spurionic argument. However (3.2.41) is a more refined constraint being a local conservation equation: local currents can be used to discuss 't Hooft anomalies and eventually gauging the symmetry, as we will see shortly. Moreover we will show in the next subsection that, with some modification with respect to the usual story, the conservation of $\widetilde{J}^\mu$ leads to topological operators as in the pure case.

- Since the random coupling $h(x)$ is space dependent, in every member of the ensemble translational symmetry is explicitly broken. The analysis above can be repeated for the stress-energy tensor $T^{\mu\nu}$, showing that also traslational invariance is recovered in a theory with quenched disorder, provided $P[h]$ is translational invariant.

With simple modifications we have a similar identity for any Lie group G:

$$i\overline{\langle \partial_\mu \widetilde{J}^\mu_a(x) \mathcal{O}_1(x_1)\cdots \rangle} = \sum_i \delta^{(d)}(x-x_i)\overline{\langle \mathcal{O}_1(x_1)\cdots r_i(T_a)\cdot \mathcal{O}_i(x_i)\cdots \rangle} \ . \qquad (3.2.42)$$

Here T_a is a Lie algebra generator and r_i is the representation of the Lie algebra, induced by R_i, under which $\mathcal{O}_i$ transforms. A more general situation could take place, in which the disorder deformation does not break the full group, but leaves a subgroup $H \subset G$ unbroken. In this case any specific realization is $H-$symmetric, and thus the currents J^μ_α, with T_α generator of $\mathfrak{h} = \mathrm{Lie}(H)$, satisfy the standard Ward identity without the necessity of averaging. In particular $\langle \partial_\mu J^\mu_\alpha \rangle = 0$, even if the expectation value of the current itself is not necessarily vanishing due to the lack of Poincaré invariance. Even if G/H is generically not a group, the associated currents, which are not conserved in any specific realization, after the appropriate shift by their expectation values turn out to satisfy the Ward identity (3.2.42) in the disordered theory, and reconstruct the full group G.

Sometimes a 0-form symmetry G can form an higher-group structure with higher-form symmetries of the theory [177–179]. In this case G is not really a subgroup of the full symmetry structure, since the product of several $G-$elements can also produce an element of the higher-form symmetry. This kind of extension is classified by group-cohomology classes, the Postnikov classes: for instance in a 2-group, mixing G with a 1-form symmetry Γ, the relevant datum is a class $\beta \in H^3(BG, \Gamma)$, with BG the classifying space of G. The important thing is that this is a discrete datum and cannot change under continuous deformation. Suppose we add a disorder breaking G, and this re-emerges as a disordered symmetry. A natural question is whether the higher-group structure is also recovered. The answer is affirmative as a consequence of the discrete nature of this structure. Indeed the probability distributions $P[h]$ have some tunable continuous parameters, like v in the Gaussian case (3.2.18), such that the pure theory is recovered in some limit ($v \to 0$ in the Gaussian case). The cohomology class characterising the higher-group is discrete and cannot change with this continuous parameter. Since all these disordered theories are continuously connected to the pure one, the higher-group structure is unchanged.

Up to this point disordered symmetries seem to behave like ordinary global symmetries in pure theories. The difference arises by considering averages of products of correlators

$$\overline{\prod_{j=1}^{N}\left\langle \mathcal{O}_1^{(j)}(x_1^{(j)})\cdots\mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)})\right\rangle} \,. \tag{3.2.43}$$

Because of (3.2.27) these are independent correlators, and we do not expect them to satisfy Ward identities immediately implied by (3.2.41), or to be constrained by the usual selection rules. Let us consider the more general generating functional $Z_D^{(N)}[\{K_i^{(j)}\}]$ introduced in (3.2.26). With the same manipulations which led to (3.2.38), we get

$$q_0\sum_{j=1}^{N}\int\mathcal{D}hP[h]\left(\left(\langle h\mathcal{O}_0\rangle_{K_i^{(j)}}-\frac{Z[K_i^{(j)},h]}{Z[0,h]}\langle h\mathcal{O}_0\rangle\right)\prod_{l\neq j}\frac{Z[K_i^{(l)},h]}{Z[0,h]}\right)=0\,, \tag{3.2.44}$$

while the individual terms of the sum are generically non-vanishing. This implies that the only Ward identity we can prove from $Z_D^{(N)}[\{K_i^{(j)}\}]$ are obtained by changing variable in *all* the path integrals involved: if we try to change variables only in a subset of these path integrals, the extra term arising would be not be of the form (3.2.44), but the sum would be over that subset of indices. Repeating the steps above we obtain the Ward identities for averages of products of correlators:

$$\sum_{j=1}^{N}\overline{\langle\partial_\mu\widetilde{J}^\mu\mathcal{O}_1^{(j)}\cdots\mathcal{O}_{k_j}^{(j)}\rangle\left(\prod_{l\neq j}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle\right)}=\sum_{j=1}^{N}\sum_{i_j=1}^{k_j}q_{i_j}^{(j)}\delta^{(d)}(x-x_{i_j}^{(j)})\overline{\prod_{l=1}^{N}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle}\,. \tag{3.2.45}$$

These Ward identities imply weaker selection rules. For instance, the correlator

$$\overline{\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_{k_1}(x_{k_1})\rangle\langle\mathcal{O}_{k_1+1}(x_{k_1+1})\cdots\mathcal{O}_{k_1+k_2}(x_{k_1+k_2})\rangle} \tag{3.2.46}$$

can be non zero when $\sum_{i=1}^{k_1}q_i\neq 0$ and $\sum_{i=k_1+1}^{k_1+k_2}q_i\neq 0$, provided that $\sum_{i=1}^{k_1+k_2}q_i=0$.

In a theory with quenched disorder ordinary and disordered symmetries can be present at the same time, and we see that their different action shows up in looking at averages of products of correlators.

See appendix A.4.1 for an explicit derivation of (3.2.41) for a two-point ($k=2$) function in a simple solvable model.

Topological operators for disordered symmetries

We now address the question of whether there exist topological symmetry operators implementing disordered symmetries, placing them in the general framework of [9]. This is important to e.g. generalize to discrete symmetries, coupling them to backgrounds and discuss non-perturbative anomalies. For notational simplicity we again focus on the $G=U(1)$ case, but all the considerations can be extended to any Lie group. We introduce the modified charge operator

$$\widetilde{Q}[\Sigma^{(d-1)}] = \int_{\Sigma^{(d-1)}} \widetilde{J}_\mu n^\mu = Q[\Sigma^{(d-1)}] - \left\langle Q[\Sigma^{(d-1)}] \right\rangle \tag{3.2.47}$$

which satisfies the integrated Ward identity

$$\overline{\left\langle \widetilde{Q}[\Sigma^{(d-1)}] \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \right\rangle} = \chi(\Sigma^{(d-1)}) \overline{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \rangle} , \tag{3.2.48}$$

with $\chi(\Sigma^{(d-1)})$ as in (3.2.9), as well as the generalization to arbitrary products

$$\overline{\sum_{j=1}^{N} \left\langle \widetilde{Q}[\Sigma^{(d-1)}] \mathcal{O}_1^{(j)} \cdots \mathcal{O}_{k_j}^{(j)} \right\rangle \left(\prod_{l \neq j} \langle \mathcal{O}_1^{(l)} \cdots \mathcal{O}_{k_l}^{(l)} \rangle \right)} = \chi(\Sigma^{(d-1)}) \overline{\prod_{l=1}^{N} \langle \mathcal{O}_1^{(l)} \cdots \mathcal{O}_{k_l}^{(l)} \rangle} . \tag{3.2.49}$$

The reason why the naive procedure of constructing the symmetry operator by exponentiating $\widetilde{Q}[\Sigma^{(d-1)}]$ does not work can be already understood at the second order: $\widetilde{Q}^2[\Sigma^{(d-1)}]$ does not measure the square of the total charge. Let Φ be a generic product of local operators.[14] We have

$$\overline{\langle \widetilde{Q}^2 \Phi \rangle} = \overline{\langle \widetilde{Q}\, Q \Phi \rangle} - \overline{\langle Q \rangle \langle \widetilde{Q} \Phi \rangle} = \chi \overline{\langle\, Q \Phi \rangle} - \chi \overline{\langle Q \rangle \langle \Phi \rangle} + \overline{\langle \widetilde{Q} Q \rangle \langle \Phi \rangle} = \chi^2 \overline{\langle\, \Phi \rangle} + \overline{\langle \widetilde{Q} Q \rangle \langle \Phi \rangle} . \tag{3.2.50}$$

In the second step we used both the Ward identity (3.2.48) and (3.2.49) with $N = 2$. We deduce that what measures the total charge square is not $\widetilde{Q}^2$ but

$$\widetilde{Q}_2 := \widetilde{Q}^2 - \langle \widetilde{Q} Q \rangle = Q^2 - 2\langle Q \rangle Q + 2\langle Q \rangle^2 - \langle Q^2 \rangle . \tag{3.2.51}$$

In order to construct the topological symmetry operator we need, for any $n \in \mathbb{N}$, an operator $\widetilde{Q}_n$ such that

$$\overline{\langle \widetilde{Q}_n \mathcal{O}_1 \cdots \mathcal{O}_k \rangle} = \chi^n \overline{\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle} \tag{3.2.52}$$

and then define the symmetry operators as

$$\widetilde{U}_g = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \widetilde{Q}_n , \qquad g = e^{i\alpha} . \tag{3.2.53}$$

To prove that such operators exist, and show how to compute them, we start from $\overline{\langle Q^n \Phi \rangle}$ (again Φ denotes a generic product of local operators), and we rewrite one Q as $\widetilde{Q} + \langle Q \rangle$, so that we can use a linear Ward identity for $\widetilde{Q}$, and we iterate until we eliminate all the Qs:

$$\begin{aligned} \overline{\langle Q^n \Phi \rangle} &= \overline{\langle \widetilde{Q} Q^{n-1} \Phi \rangle} + \overline{\langle Q \rangle \langle Q^{n-1} \Phi \rangle} = \chi \overline{\langle Q^{n-1} \Phi \rangle} + \overline{\langle Q \rangle \langle Q^{n-1} \Phi \rangle} \\ &= \chi^2 \overline{\langle Q^{n-2} \Phi \rangle} + \chi \overline{\langle Q \rangle \langle Q^{n-2} \Phi \rangle} + \overline{\langle Q \rangle \langle Q^{n-1} \Phi \rangle} \\ &\vdots \\ &= \chi^n \overline{\langle \Phi \rangle} + \sum_{k=0}^{n-1} \chi^k \overline{\langle Q \rangle \langle Q^{n-k-1} \Phi \rangle} . \end{aligned} \tag{3.2.54}$$

[14] In order to avoid cluttering in the formulas, from now on we will adopt a lighter notation omitting often the support of local operators or indices.

The terms $\chi^k \overline{\langle Q\rangle\langle Q^{n-k-1}\Phi\rangle}$ can be managed as follows. We eliminate one χ by using the linear Ward identity for the averaged product of two correlators for $\widetilde{Q}$, which we then re-expand as $Q - \langle Q\rangle$:

$$\begin{aligned}\chi^k \overline{\langle Q\rangle\langle Q^{n-k-1}\Phi\rangle} &= \chi^{k-1}\Big(\overline{\langle \widetilde{Q}Q\rangle\langle Q^{n-k-1}\Phi\rangle} + \overline{\langle Q\rangle\langle \widetilde{Q}Q^{n-k-1}\Phi\rangle} \Big) \\ &= \chi^{k-1}\Big(\overline{\langle Q^2\rangle\langle Q^{n-k-1}\Phi\rangle} - 2\overline{\langle Q\rangle^2\langle Q^{n-k-1}\Phi\rangle} + \overline{\langle Q\rangle\langle Q^{n-k}\Phi\rangle} \Big)\,.\end{aligned} \tag{3.2.55}$$

Then we eliminate an other χ from each term, again using the linear Ward identity, in some terms with the product of two correlators, in others with the product of three correlators. We continue in this way until we eliminate all the χs, and remain with a sum of averages of products of expectation values of $\langle Q^a\rangle$ for various a, and $\langle Q^b\Phi\rangle$ for a certain b, generally different for each term. This defines the operator $\widetilde{Q}_n$. For instance

$$\widetilde{Q}_3 = Q^3 - 3\langle Q\rangle Q^2 - 3\langle Q^2\rangle Q + 6\langle Q\rangle^2 Q - \langle Q^3\rangle + 6\langle Q\rangle\langle Q^2\rangle - 6\langle Q\rangle^3\,. \tag{3.2.56}$$

While this seems very complicated, one can check until arbitrarily high order that the expansion can be beautifully resummed as

$$\widetilde{U}_g = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\widetilde{Q}_n = e^{i\alpha Q}\Big\langle e^{i\alpha Q}\Big\rangle^{-1}\,, \tag{3.2.57}$$

where $\widetilde{Q}_0 := 1$. Note that this is the only result consistent with $\big\langle \widetilde{U}_g\big\rangle = 1$, which must be true by construction since $\langle\widetilde{Q}_n\rangle = 0$ as a direct consequence of the Ward identities (3.2.52) satisfied by $\widetilde{Q}_n$ in absence of local operators.

The operator $\widetilde{U}_g$ in averaged correlators behaves as

$$\overline{\langle \widetilde{U}_g[\Sigma^{(d-1)}]\mathcal{O}_1\cdots\mathcal{O}_k\rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})}\overline{\langle \mathcal{O}_1\cdots\mathcal{O}_k\rangle} \tag{3.2.58}$$

and is hence a *topological symmetry operator, on average.* It satisfes the group law

$$\overline{\langle \widetilde{U}_g\widetilde{U}_h\Phi\rangle} = \overline{\langle \widetilde{U}_{gh}\Phi\rangle}\,, \tag{3.2.59}$$

Φ being an arbitrary product of local operators. As a consequence, the naive expectation that $e^{i\alpha Q}e^{i\beta Q} = e^{i(\alpha+\beta)Q}$ is wrong because of the disorder. Note that before averaging the operator $\widetilde{U}$ is subject to renormalization and its proper definition requires a choice of renormalization scheme. We do not need to keep track of these subtleties, however, because they are washed away after the average is taken.

We now consider how $\widetilde{U}_g$ behaves inside averages of products of correlators (3.2.43), extending (3.2.49) to finite symmetry actions. This is important because, as we mentioned, products of correlators is what really characterizes disordered symmetries with respect to ordinary ones, and we need the symmetry operator version of the criterion we discussed at the end of section 3.2.1. In principle one could explicitly construct the correct combination of charges $\widetilde{Q}_n$ entering the Ward identities using the results above. For example, in the average of products of two correlators, at quadratic order in the charges we have

$$\overline{\langle \widetilde{Q}_2\Phi_1\rangle\langle\Phi_2\rangle} + \overline{\langle\Phi_1\rangle\langle\widetilde{Q}_2\Phi_2\rangle} + 2\overline{\langle\widetilde{Q}_1\Phi_1\rangle\langle\widetilde{Q}_1\Phi_2\rangle} = \chi^2\overline{\langle\Phi_1\rangle\langle\Phi_2\rangle}\,, \tag{3.2.60}$$

$\Phi_{1,2}$ being two distinct generic products of local operators. Similarly for multiple products.

We claim that the correct Ward identities consist in inserting $\widetilde{U}_g$ in all the (un)factorized correlators under average:

$$\overline{\prod_{j=1}^{N} \left\langle \widetilde{U}_g[\Sigma^{(d-1)}] \mathcal{O}_1^{(j)}(x_1^{(j)}) \cdots \mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)}) \right\rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})} \overline{\prod_{j=1}^{N} \left\langle \mathcal{O}_1^{(j)}(x_1^{(j)}) \cdots \mathcal{O}_{k_j}^{(j)}(x_{k_j}^{(j)}) \right\rangle} \,. \quad (3.2.61)$$

This can be checked by expanding both members in powers of α, which gives a series of Ward identities for the $\widetilde{Q}_n$'s. For example, for two correlators ($N = 2$) we have

$$\sum_{l=0}^{k} \binom{k}{l} \overline{\left\langle \widetilde{Q}_l \Phi_1 \right\rangle \left\langle \widetilde{Q}_{k-l} \Phi_2 \right\rangle} = \chi^k \overline{\left\langle \Phi_1 \right\rangle \left\langle \Phi_2 \right\rangle} \,, \quad (3.2.62)$$

where $\chi = \chi_1 + \chi_2$ are the sum of charges of the local operators in the product $\Phi_{1,2}$ which are inside the support of the charge operators. Checking (3.2.62) directly is cumbersome, but we can proceed as follows. We rewrite the last term appearing in (3.2.54) using (3.2.62) (assuming its validity) with $\Phi_1 = Q$ and $\Phi_2 = Q^{n-k-1}\Phi$.[15] In this way we get

$$\widetilde{Q}_n = Q^n - \sum_{k=0}^{n-1} \sum_{l=0}^{k} \binom{k}{l} \langle \widetilde{Q}_l Q \rangle \widetilde{Q}_{k-l} Q^{n-k-1} = Q \widetilde{Q}_{n-1} - \sum_{l=0}^{n-1} \binom{n-1}{l} \langle \widetilde{Q}_l Q \rangle \widetilde{Q}_{n-l-1} \,. \quad (3.2.63)$$

This is a recursion formula which determines $\widetilde{Q}_n$ in terms of all the $\widetilde{Q}_m$ for $m < n$, and it is equivalent to (3.2.62). It can be checked that computing the topological charges with this formula gives the same result as computing them directly from the linear Ward identities, proving in this way the validity of (3.2.61) and (3.2.62).

For averages of multiple correlators the group law (3.2.59) generalizes to

$$\overline{\prod_{j=1}^{N} \left\langle \widetilde{U}_g \widetilde{U}_h \Phi_j \right\rangle} = \overline{\prod_{j=1}^{N} \left\langle \widetilde{U}_{gh} \Phi_j \right\rangle} \,. \quad (3.2.64)$$

We are finally able to characterize disordered symmetries in full generality. These are symmetries of theories with quenched disorder implemented by symmetry operators $\widetilde{U}_g$, $g \in G$, which become topological after quenched average. They satisfy the identity (3.2.58) and the group law (3.2.59) as operator equations valid in any averaged correlator. Differently from ordinary global symmetries, in averages of products of correlators like (3.2.43) they are topological only if inserted in each factor of the product, and satisfy the generalized group law (3.2.64) inside averaged correlators. Disordered symmetries are symmetries of the pure system broken by the disorder but with a symmetric probability measure. It is then not surprising that $\widetilde{U}_g$ can be written in terms of the corresponding topological operator U_g of the pure system as

$$\widetilde{U}_g[\Sigma^{(d-1)}] = U_g[\Sigma^{(d-1)}] \left\langle U_g[\Sigma^{(d-1)}] \right\rangle^{-1} \,. \quad (3.2.65)$$

However the characterization above is intrinsic and does not require to know the pure system. The resummation of the series (3.2.53) into the compact expression (3.2.65) allows us to immediately generalize the analysis to more general groups G, including discrete ones where there is no current or charge operator available.

[15] Note that Φ can include integrated current operators J_μ, hence powers of charges Q, but *not* powers of $\widetilde{Q}$. The latter is still the integral of a local operator, but with an explicit dependence on $h(x)$, in which case the analysis does not apply.

't Hooft anomalies for continuous disordered symmetries

We examine in this and the next subsections some general properties of disordered symmetries. We will argue that the concept of 't Hooft anomalies, for both continuous and discrete symmetries, extends to this context. In particular we show that disordered symmetries inherit the anomaly of their pure counterpart. This is important because we can use anomalies to constraint the IR dynamics of quenched disordered theories, whose flow is generally extremely complicated. We start discussing 't Hooft anomalies for continuous disordered symmetries, postponing to section 3.2.1 the case of discrete symmetries.

A theory with a global symmetry can be coupled to a background gauge field A which acts as an external source for the conserved current J, and results in a partition function $Z[A]$. A 't Hooft anomaly arises whenever $Z[A]$ is not invariant under gauge transformations of the background (see e.g. [314] for a modern review). Denoting by A^λ the gauge transformed background, we have

$$Z[A^\lambda] = e^{i\int_{X^{(d)}}\alpha(\lambda,A)} Z[A]\,, \tag{3.2.66}$$

where the phase in the exponent is the t'Hooft anomaly, a functional depending on λ and A, which cannot be cancelled by local counterterms. Coupling to backgrounds for disordered symmetries is more subtle, because the symmetry is explicitly broken in any specific realization of the ensemble. If the symmetry is restored on average, however, a coupling to an external background becomes possible via the shifted current $\widetilde{J}$ defined in (3.2.37), namely we define

$$\overline{Z[A]} = \int \mathcal{D}h\, P[h] \int \mathcal{D}\mu e^{-S_0 - \int h(x)\mathcal{O}_0(x) + \int A_\mu \widetilde{J}^\mu}\,. \tag{3.2.67}$$

A 't Hooft anomaly for a disordered symmetry G can be defined in close analogy with the ordinary case (3.2.66):

$$\overline{Z[A^\lambda]} = e^{i\int_{X^{(d)}}\alpha(\lambda,A)} \overline{Z[A]}\,. \tag{3.2.68}$$

Anomalies (both continuous and discrete) are known to be invariant under RG flow thanks to their topological nature (typically associated to a Chern-Simons level taking value in a cohomology group, see e.g. [315, 316]). In particular, the value of the anomaly cannot depend on possible continuous parameters entering in the disorder distribution $P[h]$, such as v in the Gaussian example (3.2.18). By adiabatically changing such parameters, we can make the distribution arbitrarily peaked around $h = 0$, in which case we effectively recover the pure theory.[16] We then expect that a 't Hooft anomaly (3.2.68) associated to a disordered symmetry G can only appear if the associated pure theory (before adding the disorder perturbation) had a 't Hooft anomaly for the same symmetry G. Moreover, the two anomalies must coincide. This can be easily verified for all anomalies which, from a path integral point of view, can be seen to derive from the non-invariance of the path integral measure [317]. Starting from the left hand side of (3.2.68) when λ is infinitesimal, we perform a change of variable in the path integral in $Z[A^\lambda]$, which corresponds to an x-dependent transformation under G such that $A^\lambda \to A$. As in pure theories, the non-invariance of the measure leads to the anomaly term. The derivative of the current coming from the action variation is cancelled by the explicit symmetry breaking

[16] For the gaussian case this is achieved by taking $v \to 0$.

term and we are left with the anomalous term only. Crucially, the latter does not depend on the disorder h and hence we immediately get the infinitesimal version of the right hand side of (3.2.68), where α is exactly the same as in the underlying pure theory. If the anomaly vanishes, the disordered symmetry can be gauged by making the gauge field A_μ in (3.2.67) dynamical.

We report in appendix A.4.2 an example of matching of 't Hooft anomalies between the pure and the disorder theories using the replica trick, which will be introduced in section 3.2.2, for the case of the $U(1)$ chiral anomaly in four dimensions.

Discrete disordered symmetries: 't Hooft anomalies and gauging

The topological operators $U_g[\Sigma^{(d-1)}]$ are crucial to handle discrete symmetries for which there is no current. In pure theories the coupling to background gauge fields associated to a discrete symmetry group G can be achieved by modifying the path integral with the topological symmetry operators [9]. There are several equivalent ways to introduce a background gauge field for a discrete symmetry group G. One of these (see e.g. [178] for further details) consists in taking an atlas $\{U_i\}$ of the d-dimensional space $X^{(d)}$ and assigning group-valued connections $A_{ij} \in G$ on $U_i \cap U_j$ such that $A_{ij} = A_{ji}^{-1}$ and $A_{ij}A_{jk}A_{ki} = 1$ on triple intersections $U_i \cap U_j \cap U_k$. A codimension one symmetry operator $U_{g_p}[\Sigma_p^{(d-1)}]$ assigns $A_{ij} = g_p$ (or g_p^{-1} depending on its orientation) if $\Sigma_p^{(d-1)}$ has a non trivial intersection number with the line dual to $U_i \cap U_j$ and $A_{ij} = 1$ otherwise.[17] The resulting sets of connections A_{ij} defines a background gauge field for G and can be represented by a cohomology class $A \in H^1(X^{(d)}, G)$. The operators $U_{g_p}[\Sigma_p^{(d-1)}]$ can intersect in three-valent junctions of codimension two provided that

$$g_i g_j g_k = 1\,, \tag{3.2.69}$$

or also in higher multi-valued junctions. The configuration described above requires few choices, and one must check independence on those. Since the operators are topological local changes in their support are immaterial. We could also change the mesh locally near the junctions, which corresponds to resolve a multi-valent junction in three-valent ones in different ways. This corresponds to background gauge transformations and a non-invariance under them signals a 't Hooft anomaly for discrete symmetries. In d dimensions a 't Hooft anomaly is classified by a class $\alpha \in H^{d+1}(BG, U(1))$.[18]

Consider now a theory T with quenched disorder, obtained by deforming a pure theory T_0, and denote by T_h the member of the ensemble with coupling $h(x)$. Suppose T has a discrete disordered symmetry G. As we have seen this is implemented by the operators

$$\widetilde{U}_g[\Sigma^{(d-1)}] = U_g[\Sigma^{(d-1)}]\Big\langle U_g[\Sigma^{(d-1)}]\Big\rangle^{-1}\,. \tag{3.2.70}$$

We introduce a fine-enough mesh of topological operators $\widetilde{U}_{g_i}[\Sigma_i^{(d-1)}]$ satisfying (on average) the cocycle condition (3.2.69) in the three-valent junctions. Since $\widetilde{U}_{g_i}[\Sigma_i^{(d-1)}]$ is not topological in T_h, the junctions (as well as the operators $\widetilde{U}$ themselves) are not really well-defined because of UV

[17]In the dual triangulation the charts U_i are points, the intersections $U_i \cap U_j$ are lines, and so on.

[18]Strictly speaking, this is the case for bosonic theories in $d < 3$ dimensions. More in general, anomalies are classified by a cobordism group [318].

divergences. However we can employ an arbitrary regularization scheme for these divergences, without the need of specifying a renormalization scheme to try to define the junctions and the operators $\widetilde{U}$ (recall the second comment after (3.2.41)). This is because we know that the operators become topological after the average and hence such divergences are expected to be washed away from the integration over $h(x)$. We define

$$Z_{T_h}\left[\,\{g_i\}\,,h\right] = \int \mathcal{D}\mu\, e^{-S[\phi]-\int h(x)\mathcal{O}_0(x)} \prod_i \widetilde{U}_{g_i}[\Sigma_i^{(d-1)}] = \Big\langle \prod_i \widetilde{U}_{g_i}[\Sigma_i^{(d-1)}] \Big\rangle \tag{3.2.71}$$

which, contrary to the pure case, *does* depend on the specific location of the planes $\Sigma_i^{(d-1)}$. At this point there is no notion of background gauge fields. However, as a consequence of the Ward identity discussed in section 3.2.1,

$$Z_T\left[\,\{g_i\}\,\right] = \int \mathcal{D}h\, P[h]\, Z_{T_h}\left[\,\{g_i\}\,,h\right] \tag{3.2.72}$$

is independent of the choice of location for Σ_i and hence the set of operators $\widetilde{U}_{g_i}$ inserted in (3.2.72) corresponds to a well-defined discrete gauge field $A \in H^1(X^{(d)}, G)$. It is important to emphasize here that the gauge field A arises only after the average over $h(x)$ is performed. Differently said, if a pure system has a symmetry G, perturbing it with quenched disorder and coupling it to a background are non-commutative operations. In what follows we denote the above partition function by $Z_T[A]$.

Local modifications of the three-valent junctions change the gauge field by an exact 1-cocycle $A \to A^\lambda = A + \delta\lambda$. This can change the partition function $Z_T[A]$ by a phase, which represents a class $\alpha \in H^{d+1}(BG, U(1))$: this is the diagnostic for an 't Hooft anomaly for a discrete disordered symmetry. Since the topological operator $\widetilde{U}_g[\Sigma^{(d+1)}]$ is different from the one in the pure theory by the stacking of an $h(x)-$dependent functional, it is not a priori obvious that the contact terms arising in the local moves are the same as those in the pure theory, precisely as it occurred in the continuous case discussed in section 3.2.1. However, the fact that anomalies are classified by classes in $H^{d+1}(BG, U(1))$, which are discrete, immediately proves that they cannot depend on the strength of the disorder and must be equal to those of the pure theory. As a result, a system with a disordered symmetry with a 't Hooft anomaly cannot be trivially gapped. This is in agreement with previous works in condensed matter where – mostly in the context of topological insulators [295–299, 301] but not only (see e.g. [300]) – SPT phases of matter where the symmetry is disordered were found. We see that in general disordered symmetries can lead to protected non-trivial topological phases (see [302] for a recent analysis).[19]

Now suppose that the 't Hooft anomaly vanishes. Then $Z_T[A]$ is well defined and is possible to gauge the symmetry by summing over all consistent insertions of symmetry operators, or equivalently over cohomology classes $A \in H^1(X^{(d)}, G)$. We denote the resulting theory by T/G,

[19] In [302] it is considered a Lorentizan theory with a disorder coupling depending on space but not in time. In this set-up it is found that purely disordered symmetries, i.e. in absence of pure symmetries, necessarily have a trivial t' Hooft anomaly. This is not in contradiction with our findings, based on Euclidean theories.

whose partition function is[20]

$$Z_{T/G} = \sum_{A \in H^1(X^{(d)},G)} Z_T[A] \ . \tag{3.2.73}$$

At this point everything is essentially the same as in the pure case (see e.g. [9, 11]). The operators of T with a counterpart in T/G are the gauge-invariant ones, while we also add the $(d-2)$ dimensional operators in the twisted sector of G. Indeed the topological operators $\widetilde{U}_g[\Sigma^{(d-1)}]$ become trivial in T/G, and their boundary operators turn into genuine operators (on average). Finally, since $A \in H^1(X^{(d)}, G)$ is dynamical, T/G has a dual symmetry generated by the Wilson lines of the G gauge field. This is a $(d-2)$-form symmetry whose charged objects are the operators coming from the twisted sectors of G. For G abelian the symmetry is the Pontryagin dual $G^\vee$, while it is a non-invertible symmetry in the non-abelian case [11].

3.2.2 Disordered symmetries and the replica trick

Disordered systems are often treated by means of the *replica trick*, which expresses the averaged correlation functions as certain limits of correlation functions of a standard QFT, the replica theory. In this section we interpret the disordered symmetries from the point of view of the replica theory. In addition to provide a sanity check of the results found in section 3.2.1, the method of replicas allows us to consider emergent symmetries in the disordered theory for which the results in the previous section do not apply. We will discuss emergent symmetries in section 3.2.3. For the rest of this section and the next section we assume a Gaussian probability distribution like (3.2.18) (and its generalization for complex h) with variance v.[21]

The replica trick

To fix our notation we briefly review the *replica trick*. This is a useful tool that allows to compute connected and full (i.e. both its connected and disconnected parts) correlators of the disordered theory as limits of correlators of a pure theory. The starting point of the replica trick is the identity

$$W = \log Z = \lim_{n\to 0} \left(\frac{\partial Z^n}{\partial n} \right) \ . \tag{3.2.74}$$

The idea is to replicate the pure system n times, indexing each copy with a label a

$$Z^n[h, K_i] = \int \prod_{a=1}^{n} \mathcal{D}\mu_a \, \exp \left(-\sum_a S_{0,a} - \sum_a \int h(x)\mathcal{O}_{0,a}(x) + \sum_{i,a} \int K_i(x)\mathcal{O}_{i,a}(x) \right) \ , \tag{3.2.75}$$

[20]In the pure case it is possible to modify this sum weighting the terms with phases. Consistency conditions related with associativity constraint these phases to be of the form $\int_{X^{(d)}} A^*\nu$, where $\nu \in H^d(BG, U(1))$ is a *discrete torsion class* and we think A as a homotopy class of maps $X^{(d)} \to BG$, so that $A^*\nu \in H^d(X^{(d)}, U(1))$. Since the same kind of constraints are valid also in the disordered theories, we expect the very same modification of the gauging procedure to be possible also in this context.

[21]Normalization factors of $P[h]$, which ensure that probabilities add to one, will not play a role in our considerations and are then left implicit.

with the same random field coupling h and external sources K_i for all replicas. When $P[h]$ is Gaussian the average over $h(x)$ can be performed explicitly and we get

$$W_n[K_i] := \int \mathcal{D}h\, P[h] Z^n[h, K^i] = \int \prod_{a=1}^{n} \mathcal{D}\mu_a\, e^{-S_{\rm rep}+\sum_{i,a}\int K_i \mathcal{O}_{i,a}}\,, \tag{3.2.76}$$

where

$$S_{\rm rep} = \sum_a S_{0,a} - \frac{v}{2}\sum_{a,b}\int d^d x\, \mathcal{O}_{0,a}(x)\mathcal{O}_{0,b}(x) \tag{3.2.77}$$

is the replica action. We see how a coupling between the replica theories has been generated after the average. Renormalization will possibly induce other couplings in the replica theory, all compatible with the symmetries of the system. Among these, importantly the replica theory enjoys an S_n replica symmetry that permutes the various copies of the pure theory. We now assume that W_n can be analytically continued for arbitrary values of n including the origin in the complex n-plane.[22] Using (3.2.74) we find

$$W_D = \lim_{n\to 0}\left(\frac{\partial W_n}{\partial n}\right), \tag{3.2.78}$$

where W_D is defined in (3.2.23), and thus

$$\overline{\big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\big\rangle}_c = \lim_{n\to 0}\partial_n \left(\Big\langle \sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\ldots\Big\rangle^{\rm rep}\right), \tag{3.2.79}$$

where we used the fact that

$$\lim_{n\to 0} W_n[K_i] = 1\,. \tag{3.2.80}$$

Note that the in the left hand side of (3.2.79) we have the connected part of the correlator (indicated with the subscript c) which is computed in the replica theory by a suitable limit of a full correlator. Moreover, we see from (3.2.79) that a local operator $\mathcal{O}$ inside connected correlators of the disordered theory is mapped in the replica theory to its S_n-singlet component $\sum_a \mathcal{O}_a$.

The replica trick is also useful to compute general correlation functions in the disordered theory. Denoting by

$$S_a[h] = S_{0,a} + \int h(x)\mathcal{O}_{0,a}(x)\,, \tag{3.2.81}$$

we have

$$\begin{aligned}\overline{\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_k(x_k)\rangle} &= \int \mathcal{D}h\, P[h]\frac{\int \mathcal{D}\mu\, e^{-S[h]}\mathcal{O}_1(x_1)\ldots\mathcal{O}_k(x_k)}{Z[h]} \\ &= \int \mathcal{D}h\, P[h]\frac{\int\prod_a \mathcal{D}\mu_a\, e^{-\sum_a S_a[h]}\mathcal{O}_{1,1}(x_1)\ldots\mathcal{O}_{k,1}(x_k)}{Z[h]^n}\,,\end{aligned} \tag{3.2.82}$$

which is an identity for any positive integer n. Assuming again that it can be analytically continued for $n\to 0$ we get[23]

$$\overline{\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_k(x_k)\rangle} = \lim_{n\to 0}\langle \mathcal{O}_{1,1}(x_1)\ldots\mathcal{O}_{k,1}(x_k)\rangle^{\rm rep}\,. \tag{3.2.83}$$

[22] This is a notoriously subtle limit. In particular we can have the phenomenon of spontaneous replica symmetry breaking (see [319] and references therein). We assume in what follows that the replica symmetry is not spontaneously broken.

[23] Note that we have actually taken the limit $n\to 0$ in the denominator of (3.2.82) ($Z^n[h]\to 1$) before integrating over h, while in the numerator it is kept after the integration over h.

In general correlators, in contrast to connected correlators, local operators are mapped to a specific copy (the same for all operators in the correlation function) in the replica theory. Equation (3.2.83) can easily be generalized to averages of products of general correlation functions. For example, omitting for simplicity the x-dependence of the local operators, we have

$$\begin{aligned}\overline{\Big\langle \prod_{i=1}^{k} \mathcal{O}_i^{(1)} \Big\rangle \Big\langle \prod_{j=1}^{l} \mathcal{O}_j^{(2)} \Big\rangle} &= \lim_{n\to 0} \int \mathcal{D}h \, P[h] \frac{\int \prod_a \mathcal{D}\mu_a \, e^{-\sum_a S_a[h]} \prod_{i=1}^k \mathcal{O}_{i,1}^{(1)} \prod_{j=1}^l \mathcal{O}_{j,2}^{(2)}}{Z^n[h]} \\ &= \lim_{n\to 0} \Big\langle \prod_{i=1}^{k} \mathcal{O}_{i,1}^{(1)} \prod_{j=1}^{l} \mathcal{O}_{j,2}^{(2)} \Big\rangle^{\text{rep}} ,\end{aligned} \tag{3.2.84}$$

and similarly for more than two products. The last observables which we need to evaluate are averages of products of N connected correlators. Before averaging, these correlators are obtained by taking functional derivatives of the product $W[K_i^{(1)}] \cdots W[K_i^{(N)}]$. For each of them we can use the replica trick to express this product as a unique path integral. We then have

$$\overline{\prod_{l=1}^{N} \Big\langle \prod_{j_l=1}^{k_l} \mathcal{O}_{j_l}^{(l)} \Big\rangle_c} = \Big(\prod_{k=1}^{N} \lim_{n_k \to 0} \frac{\partial}{\partial n_k} \Big) \Big\langle \prod_{l=1}^{N} \prod_{j_l=1}^{k_l} \sum_{a_{j_l}^{(l)}=1}^{n_l} \mathcal{O}_{j_l, a_{j_l}^{(l)}}^{(l)} \Big\rangle^{\text{rep}} , \tag{3.2.85}$$

where S_{rep} is the replica theory for $n = \sum_{i=1}^N n_i$ replicas. Note that averages of products of general or connected correlators in the disordered theory are always expressed in the replica theory as suitable limits of a *single general* correlator. Since any correlator can be expanded in its connected components, (3.2.85) is actually sufficient to compute generic correlation functions of the disordered theory. Any operator of the disordered theory gives rise to a multiplet transforming in the n-dimensional (natural) representation of S_n. Averages of connected correlators of operators of the disordered theory are given by the S_n singlet operators inside the natural representation in the replica theory. More general correlation functions of the disoredered theory are instead given by considering operators singlets under subgroups $S_{n_i} \subset S_n$ induced by the natural representation in the replica theory.

Disordered symmetries from replica theory

Our first task is to understand how disordered symmetries manifest themselves in the replica theory. For concreteness we consider again the case of a $G = U(1)$ symmetry, the replica action reads

$$S_{\text{rep}} = \sum_a S_{0,a} - \frac{v}{2} \sum_{a,b} \int d^d x \, \mathcal{O}_{0,a}(x) \overline{\mathcal{O}}_{0,b}(x) \, . \tag{3.2.86}$$

The $U(1)^n$ symmetry of the replicated pure part is broken by the disorder coupling to its diagonal $U(1)$ subgroup, which is then a symmetry of the replica theory. In particular there is a conserved current

$$J_D^\mu = \sum_a J_a^\mu \tag{3.2.87}$$

constructed as the S_n singlet out of the multiplet induced by the current J^μ of the disordered theory.

We can recover the Ward identities of the disordered symmetry from those produced by J^μ_D in the replica theory by using (3.2.85) for averages of products of connected correlators. The general key idea is to write a sum of averages of products of connected correlators with current insertions that, once mapped to correlators of the replica theory, reconstruct the complete diagonal current J^μ_D. Then we can use the Ward identity in the replica theory and finally we rewrite the results back in terms of the disordered theory.

Determining the Ward identities for averages of single connected correlators is simple, because the diagonal current J_D appears directly in the replica theory and we can immediately use the ordinary Ward identities there. We have

$$\begin{aligned}\overline{\big\langle \partial_\mu J^\mu(x)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\big\rangle_c} &= \lim_{n\to 0}\partial_n\Big(\Big\langle J^\mu_D(x)\sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\dots\Big\rangle^{\text{rep}}\Big)\\ &= \sum_i q_i\delta^{(d)}(x-x_i)\lim_{n\to 0}\partial_n\Big\langle\sum_a \mathcal{O}_{1,a}(x_1)\sum_b \mathcal{O}_{2,b}(x_2)\dots\Big\rangle^{\text{rep}}\\ &= \sum_i q_i\delta^{(d)}(x-x_i)\overline{\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\rangle_c}\,,\end{aligned}\tag{3.2.88}$$

which reproduces the connected version of (3.2.41). Averages of products of connected correlators are also easy to treat, because it is enough to consider a sum of correlators where the current is inserted in each term to reconstruct J_D in the replica theory and then use the Ward identities there. Skipping obvious steps, we get

$$\sum_{j=1}^N \overline{\langle\partial_\mu J^\mu(x)\mathcal{O}^{(j)}_1\dots\mathcal{O}^{(j)}_{k_j}\rangle_c\Big(\prod_{l\neq j}\langle\mathcal{O}^{(l)}_1\dots\mathcal{O}^{(l)}_{k_l}\Big\rangle_c\Big)} = \sum_{j=1}^N\sum_{i_j=1}^{k_j}\delta_{i^{(j)}_j}q^{(j)}_{i_j}\overline{\prod_{l=1}^N\langle\mathcal{O}^{(l)}_1\dots\mathcal{O}^{(l)}_{k_l}\rangle_c}\,,\tag{3.2.89}$$

which is similar to (3.2.45), but expressed in terms of connected correlators and the unshifted current.

Due to the different way the replica trick handles connected and general correlators, determining the Ward identities for the latter will produce the improved current $\widetilde{J}_\mu$. We use (3.2.83) to write

$$\begin{aligned}&\overline{\langle\partial^\mu J_\mu\mathcal{O}_1\cdots\mathcal{O}_n\rangle} = \lim_{n\to 0}\langle\partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\\ &= \lim_{n\to 0}\langle\partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}} - \lim_{n\to 0}\frac{1}{n-1}\langle\sum_{a=2}^n\partial^\mu J_{\mu,a}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\\ &+ \lim_{n\to 0}\langle\partial^\mu J_{\mu,2}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\,.\end{aligned}\tag{3.2.90}$$

In the last step, the last two terms add to zero due to the S_n symmetry enjoyed by the replica theory. In the limit $n\to 0$ we have

$$\begin{aligned}&\lim_{n\to 0}\langle\partial^\mu J_{\mu,1}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}} - \lim_{n\to 0}\frac{1}{n-1}\langle\sum_{a=2}^n\partial^\mu J_{\mu,a}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\\ &= \lim_{n\to 0}\langle\partial^\mu J_{\mu,D}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}}\end{aligned}\tag{3.2.91}$$

and

$$\lim_{n\to 0}\langle\partial^\mu J_{\mu,2}\mathcal{O}_{1,1}\dots\mathcal{O}_{k,1}\rangle^{\text{rep}} = \overline{\langle\partial^\mu J_\mu\rangle\langle\mathcal{O}_1\cdots\mathcal{O}_k\rangle}\,.\tag{3.2.92}$$

Therefore, by using the standard Ward identities of the replica theory, from (3.2.90) we get (3.2.41), as expected. The Ward identities (3.2.45) for products of generic correlators can be derived using a similar treatment:

$$\begin{aligned}
&\sum_{j=1}^{N}\overline{\langle\partial_\mu J^\mu \mathcal{O}_1^{(j)}\cdots\mathcal{O}_{k_j}^{(j)}\rangle\left(\prod_{l\neq j}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle\right)} = \lim_{n\to 0}\sum_{j=1}^{N}\langle\partial^\mu J_{\mu,j}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\text{rep}}\\
&=\lim_{n\to 0}\sum_{j=1}^{N}\langle\partial^\mu J_{\mu,j}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\text{rep}} - \lim_{n\to 0}\frac{N}{n-N}\langle\sum_{a=N+1}^{n}\partial^\mu J_{\mu,a}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\text{rep}}\\
&+\lim_{n\to 0} N\langle\partial^\mu J_{\mu,N+1}\prod_{j=1}^{N}(\mathcal{O}_{1,j}^{(j)}\cdots\mathcal{O}_{k_j,j}^{(j)})\rangle^{\text{rep}} \qquad (3.2.93)\\
&=\sum_{j=1}^{N}\sum_{i_j=1}^{k_j} q_{i_j}^{(j)}\delta^{(d)}(x-x_{i_j}^{(j)})\overline{\prod_{l=1}^{N}\langle\mathcal{O}_1^{(l)}\cdots\mathcal{O}_{k_l}^{(l)}\rangle} + N\overline{\langle\partial^\mu J_\mu\rangle\prod_i\langle\mathcal{O}_1^{(i)}\cdots O_{k_i}^{(i)}\rangle}\,.
\end{aligned}$$

The last term in the right-hand-side in the third row of (3.2.93) precisely combines with the left-hand-side to reproduce the shifted current $\widetilde{J}_\mu$ and hence the Ward identities (3.2.45).

The above analysis shows that the replica counterpart of the disordered symmetry is an ordinary symmetry generated by the diagonal current J_D^μ and all the Ward identities of the disordered theory reduce to Ward identities involving J_D^μ in the replica theory. The exotic selection rules (see discussion around (3.2.46)) of the disordered symmetry are a consequence of the non-trivial map between the observables of the replica theory and those in the theory with quenched disorder.

3.2.3 Disordered emergent symmetries and LogCFTs

Our analysis of Ward identities in section 3.2.1 applies for disordered symmetries, namely symmetries which are present in the underlying UV theory, are broken by the disorder, and get restored after disorder average. On the other hand, as in pure theories, we can have genuinely emergent symmetries in the IR, namely symmetries which are not present in the UV theory even before adding the disorder coupling. If the symmetry emerges for each theory in the ensemble, then we expect that it gives rise to approximate selection rules of the same kind as in pure theories with emergent symmetries in the IR. However, we could also have symmetries that emerge in the IR only *after* disorder average. By definition, this implies the existence of additional selection rules which are valid on average in the IR of the theory. For non-emergent, actual disordered symmetries such selection rules arise from a conserved current which is a shifted version of the current operator J^μ of the UV theory $\widetilde{J}^\mu = J^\mu - \overline{\langle J^\mu\rangle}$. For emergent symmetries we cannot determine its explicit form, as the description in terms of the UV action is useless, and the analysis in section 3.2.1 does not hold. However, as we will see, we can deduce which are the selection rules that the emergent disordered symmetry imposes on averaged correlation functions using the replica theory.

From a symmetry point of view, the key qualitative feature of the replica theory (for any finite n) is the presence of a S_n global permutation symmetry not present in the original theory with disorder. In the analysis in section 3.2.2 the internal symmetry G generated by the current J_D^μ commutes with S_n, namely the infinitesimal transformations $\delta\mathcal{O}_{j,a}$ of the fields do not mix different replicas. This is guaranteed by the fact that G in the replica theory is the diagonal subgroup of the G^n global symmetry of the replica theories when $v = 0$. On the other hand, in the case of an emergent symmetry this is not necessarily the case: each irreducible representation of S_n can sit in a different G-representation, or even more generally, the local operators could sit in representations of the semi-direct product $G \rtimes S_n$. We expect that emergent symmetries in the replica theory of this kind correspond to *disordered emergent symmetries* in the theory with disorder. As we will see below, even in the deep IR the resulting selection rules will be modified with respect to those coming from (3.2.41) and its generalizations. As an application we will show how these modified Ward Identities allow for logarithmic conformal field theories (LogCFTs) as IR fixed points of disordered systems.

3.2.4 Emergent disordered symmetries

Let us analyze in some detail the Ward Identities for emergent symmetries in the replica theory. We study theories in which the total symmetry is a direct product $G \times S_n$, since this particular case already exhibits interesting features. For further simplification, we consider $G = U(1)$ and correlators where only the singlet and the standard representations of S_n are involved. Generalizations to other representations of S_n or more general groups G should be straightforward.

Consider the average of a single correlation function of k local operators in the disordered theory. We consider both the general and the connected part of the correlator. Using (3.2.83) and (3.2.79), they are mapped in the replica to the $n \to 0$ limit of respectively $\langle \mathcal{O}_{1,1} \dots \mathcal{O}_{k,1} \rangle^{\text{rep}}$ and $\partial_n \langle \sum_{a_1} \mathcal{O}_{1,a_1} \dots \sum_{a_k} \mathcal{O}_{k,a_k} \rangle^{\text{rep}}$, omitting the space dependence of the operators in the correlators for simplicity. The replica theory is an ordinary pure theory and the emergent symmetry should manifest with the existence of a vector local operator J_D^μ, which becomes conserved in the IR. The operator J_D^μ is necessarily a singlet of S_n, since $U(1)$ commutes with S_n by definition. Note that we do not need to assume the knowledge of the full multiplet J_a^μ for which $J_D^\mu = \sum_{a=1}^n J_a^\mu$. Indeed, while in the UV, for weak disorder, the existence of vector operators in the natural representation of S_n is guaranteed, we do not need to keep track of the IR fate of the non-singlet components. Assuming that J_D^μ is conserved in the IR also at *finite* n, the following standard selection rules on k-point correlators apply:

$$\sum_{j=1}^{k} \langle \sum_{a_j=1}^{n} \delta\mathcal{O}_{j,a_j} \prod_{j \neq i=1}^{k} \sum_{a_i=1}^{n} \mathcal{O}_{i,a_i} \rangle^{\text{rep}} = 0 \,, \tag{3.2.94}$$

$$\sum_{j=1}^{k} \langle \delta\mathcal{O}_{j,1} \prod_{j \neq i=1}^{k} \mathcal{O}_{i,1} \rangle^{\text{rep}} = 0 \,. \tag{3.2.95}$$

The key point is now to look more closely to the variations $\delta\mathcal{O}_{j,a_j}$. Indeed, the natural representation of S_n is reducible and the $\mathcal{O}_i$'s split in

$$\mathcal{O}_i^{(S)} = \sum_{a=1}^{n} \mathcal{O}_{i,a} \,, \qquad \mathcal{O}_{i,a}^{(F)} = \mathcal{O}_{i,a} - \frac{1}{n} \mathcal{O}_i^{(S)} \,, \tag{3.2.96}$$

which transform in the singlet and in the standard, or fundamental, representation respectively.[24] The $U(1)$ symmetry acts as

$$\delta\mathcal{O}_i^{(S)} = q_{S,i}\mathcal{O}_i^{(S)}, \qquad \delta\mathcal{O}_{i,a}^{(F)} = q_{F,i}\mathcal{O}_{i,a}^{(F)}\,, \tag{3.2.97}$$

where the charges are generically different, $q_{S,i} \neq q_{F,i}$, and can possibly depend on n. The variations entering the Ward identities of the replica theory are then

$$\delta\mathcal{O}_{i,a} = \delta\mathcal{O}_{i,a}^{(F)} + \frac{1}{n}\delta\mathcal{O}_i^{(S)} = q_{F,i}\mathcal{O}_{i,a} + \frac{\Delta q_i}{n}\sum_{a=1}^{n}\mathcal{O}_{i,a}\,, \tag{3.2.98}$$

where

$$\Delta q_i := q_{i,S} - q_{i,F}\,. \tag{3.2.99}$$

Since in connected correlators we only have singlet components, plugging (3.2.98) in (3.2.94) gives simply

$$\sum_{j=1}^{k} q_{S,j}\langle\prod_{i=1}^{k}\sum_{a_i=1}^{n}\mathcal{O}_{i,a_i}\rangle^{\rm rep} = 0\,. \tag{3.2.100}$$

On the other hand, plugging (3.2.98) in (3.2.95) equals

$$\begin{aligned}
0 &= \sum_{j=1}^{k} q_{F,j}\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} + \sum_{j=1}^{k}\frac{\Delta q_j}{n}\langle\sum_{b=1}^{n}\mathcal{O}_{j,b}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} \\
&= \sum_{j=1}^{k}\Big(q_{F,j} + \frac{\Delta q_j}{n}\Big)\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} + \sum_{j=1}^{k}\frac{\Delta q_j}{n}\langle\sum_{b=2}^{n}\mathcal{O}_{j,b}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} \\
&= \sum_{j=1}^{k} q_{F,j}\langle\prod_{i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} + \sum_{j=1}^{k}\Delta q_j\langle\mathcal{O}_{j,2}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} \\
&+ \frac{1}{n}\sum_{j=1}^{k}\Delta q_j\Big(\langle\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep} - \langle\mathcal{O}_{j,2}\prod_{j\neq i=1}^{k}\mathcal{O}_{i,1}\rangle^{\rm rep}\Big)\,.
\end{aligned} \tag{3.2.101}$$

The existence of the limit $n \to 0$ requires that

$$\Delta q_j(n) = nK_j + O(n^2)\,, \qquad \text{as} \quad n \to 0\,, \tag{3.2.102}$$

where

$$K_j = \frac{\partial\Delta q_j}{\partial n}\bigg|_{n=0}\,. \tag{3.2.103}$$

We can use (3.2.102) to go back to the averaged correlators of the disordered theory and obtain the desired selection rules

$$\sum_{j=1}^{k} q_j\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle} + \sum_{j=1}^{k}K_j\Big(\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle} - \overline{\langle\mathcal{O}_j\rangle\langle\prod_{j\neq i=1}^{k}\mathcal{O}_i\rangle}\Big) = 0\,, \tag{3.2.104}$$

$$\sum_{j=1}^{k} q_j\overline{\langle\prod_{i=1}^{k}\mathcal{O}_i\rangle_c} = 0\,, \tag{3.2.105}$$

[24]More general representations arise for composite operators of the disordered theory which, once replicated, correspond to multiplets of S_n transforming in a (reducible) tensor product of two or more natural representations.

where

$$q_j = q_{F,j}|_{n=0} = q_{S,j}|_{n=0}\,, \quad j = 1,\ldots,k\,. \tag{3.2.106}$$

A similar analysis can be repeated for averages of products of correlation functions of the kind (3.2.43). We report here only the final result:

$$\begin{aligned}\sum_{m=1}^{N}\sum_{j=1}^{k_m}\Bigg[&\left(q_j^{(m)}+K_j^{(m)}\right)\overline{\prod_{l=1}^{N}\langle\Upsilon^{(l)}\rangle}\\ &+K_j^{(m)}\left(\sum_{a\neq m}\overline{\langle\Upsilon_j^{(m)}\rangle\langle\mathcal{O}_j^{(m)}\Upsilon^{(a)}\rangle\prod_{l\neq m,a}\langle\Upsilon^{(l)}\rangle}-N\overline{\langle\mathcal{O}_j^{(m)}\rangle\langle\Upsilon_j^{(m)}\rangle\prod_{l\neq m}\langle\Upsilon^{(l)}\rangle}\right)\Bigg]=0\end{aligned} \tag{3.2.107}$$

where we introduced the notations

$$\Upsilon^{(l)} = \prod_{i=1}^{k_l}\mathcal{O}_i^{(l)}\,, \quad \Upsilon_j^{(l)} = \prod_{i=1,i\neq j}^{k_l}\mathcal{O}_i^{(l)}\,. \tag{3.2.108}$$

When $K_j = 0$, the selection rules (3.2.104) are the standard ones associated to a $U(1)$ conserved symmetry, while for $K_j \neq 0$ we get additional terms which affect the disconnected component of the correlator only, given that the connected part satisfies the ordinary selection rule (3.2.105). The fact that (3.2.105) holds implies that in the disordered theory we have a notion of operators $\mathcal{O}_i$ carrying a definite $U(1)$ charge q_i, yet in disconnected correlators some effect is responsible for the appearance of the extra terms proportional to K_j. It would be interesting to understand the origin of these extra factors directly from the disordered theory.

For $k = 2$, (3.2.104) and (3.2.105) simplify and can be rewritten as

$$\begin{aligned}(q_1+q_2)\overline{\langle\mathcal{O}_1\rangle_c\langle\mathcal{O}_2\rangle_c}+(K_1+K_2)\overline{\langle\mathcal{O}_1\mathcal{O}_2\rangle_c}&=0\,,\\ (q_1+q_2)\overline{\langle\mathcal{O}_1\mathcal{O}_2\rangle_c}&=0\,.\end{aligned} \tag{3.2.109}$$

If $K_1 + K_2 \neq 0$, independently of the value of $q_1 + q_2$, the connected part of the 2-point function has to vanish and only a disconnected component is allowed. We are not aware of disordered theories with $K_j \neq 0$ for an internal global symmetry. On the other hand, we will show in the next section that the exotic selection rules derived above, applied to the case of emergent conformal symmetry, are at the origin of the possible appearance of logarithmic CFTs in the IR of disordered theories.

LogCFTs

Infrared fixed points of theories with quenched disorder can be described by non-unitary LogCFTs, first discussed in 2d [304, 305]. See e.g. [320] for a review of 2d LogCFTs or [306] for an introduction to LogCFTs in d dimensions from an axiomatic point of view. It was recognized in [305] that LogCFTs are intrinsically associated in having primary operators that are highest weight of indecomposable but not irreducible representations of the conformal group. A derivation of how LogCFTs can arise as random fixed points was given in [307] and more recently in [312] by means of (suitable generalizations of) Callan-Symanzik equations, in both cases using replica methods. We provide here an alternative derivation, working out the generalization of (3.2.109) when the emergent group is assumed to be the conformal one.

In the IR fixed point of the replica theory we have a dilatation current J_d^μ which yields the topological dilatation operator

$$D\left[\Sigma^{(d-1)}\right] = \int_{\Sigma^{(d-1)}} J_d^\mu n_\mu \,. \tag{3.2.110}$$

The conformal Ward identities applied to a primary operator $\mathcal{O}$ imply

$$D\left[\Sigma_x^{(d-1)}\right]\mathcal{O}(x) = \delta_D\mathcal{O}(x) + \mathcal{O}(x)D\left[\Sigma_{\text{no}\,x}^{(d-1)}\right] , \tag{3.2.111}$$

where

$$\delta_D\mathcal{O} = (\Delta + x^\mu\partial_\mu)\,\mathcal{O}(x)\,, \tag{3.2.112}$$

$(\Sigma_{\text{no}\,x}^{(d-1)})$ $\Sigma_x^{(d-1)}$ is a closed codimension 1 surface (not) encircling x. The dilatation operator acts diagonally only on the irreducible representations (3.2.96):

$$\delta_D\mathcal{O}_i^{(S)}(x) = (\Delta_{S,i}(n) + x^\mu\partial_\mu)\,\mathcal{O}_i^{(S)}(x)\ , \quad \delta_D\mathcal{O}_{i,a}^{(F)}(x) = (\Delta_{F,i}(n) + x^\mu\partial_\mu)\,\mathcal{O}_{i,a}^{(F)}(x)\ . \tag{3.2.113}$$

Thus on $\mathcal{O}_{i,a}(x)$ we have

$$\delta_D\mathcal{O}_{i,a}(x) = (\Delta_{F,i} + x^\mu\partial_\mu)\,\mathcal{O}_{i,a}(x) + \frac{\Delta_{S,i} - \Delta_{F,i}}{n}\sum_{\alpha=1}^{n}\mathcal{O}_{i,\alpha}(x)\,, \tag{3.2.114}$$

where in general $\Delta_{S,i}(n) \neq \Delta_{F,i}(n)$ for finite n. We plug the above transformations in (3.2.95) with $k = 2$ and equal operators. In this way we find the analogues of (3.2.109) for scaling transformations:

$$\begin{aligned} (x^\mu\partial_\mu + 2\Delta)\,\overline{\langle\mathcal{O}(x)\rangle_c\langle\mathcal{O}(0)\rangle_c} + 2K\overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_c} = 0\,, \\ (x^\mu\partial_\mu + 2\Delta)\,\overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_c} = 0\,, \end{aligned} \tag{3.2.115}$$

where

$$\Delta := \Delta_F|_{n=0} = \Delta_S|_{n=0}\,, \qquad\qquad K = \partial_n(\Delta_S - \Delta_F)|_{n=0}\,. \tag{3.2.116}$$

The general solution of (3.2.115) reads

$$\begin{aligned} \overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_c} &= \frac{c_1}{|x|^{2\Delta}} \\ \overline{\langle\mathcal{O}(x)\mathcal{O}(0)\rangle} &= \frac{c_2}{|x|^{2\Delta}} - \frac{c_1\log(\mu|x|)}{|x|^{2\Delta}}\,, \end{aligned} \tag{3.2.117}$$

where $c_{1,2}$ are two integration constants with mass dimension -2Δ and μ is an arbitrary mass scale. Note that in a LogCFT, due to the peculiar way dilatations act on operators, the presence of a mass scale is actually compatible with conformal symmetry (see e.g. [306] for a more detailed explanation). We see that the log term arises when $K \neq 0$, which acts as a source term in the second equation in (3.2.115).

Whenever the LogCFT has some internal global symmetry G which is not emergent in the IR but is an exact symmetry present along the whole RG flow (i.e. present for each member of the ensemble and not broken by the disorder), the derivation above shows that logarithms can only appear in two-point functions of operators singlets under G. Indeed, in the replica theory the symmetry G gets replicated in n (unbroken) copies G_a, while the conformal symmetry generally is not, being only emergent at the fixed point. A representation ρ of G acting on a primary operator $\mathcal{O}$ is then replicated into n copies ρ_a, each acting only on $\mathcal{O}_a$. Let $g \in G_a$, by simple manipulations we get

$$\begin{aligned}\rho_a(g) \cdot \mathcal{O}^{(S)} &= \rho_a(g) \cdot \mathcal{O}_a - \mathcal{O}_a + \mathcal{O}^{(S)} \\ &= (\rho_a(g) - \mathbb{1}) \cdot \mathcal{O}_a^{(F)} + \frac{1}{n} \left(\rho_a(g) + (n-1)\mathbb{1}\right) \cdot \mathcal{O}^{(S)} .\end{aligned} \tag{3.2.118}$$

Since G_a are internal symmetries, which necessarily commute with the dilatation operator D, we have

$$0 = [D, \rho_a(g)] \cdot \mathcal{O}^{(S)} = (\Delta_F - \Delta_S)\left(\rho_a(g) - \mathbb{1}\right) \cdot \mathcal{O}_a^{(F)} . \tag{3.2.119}$$

Unless ρ is in the trivial representation, the only solution of (3.2.119) is

$$\Delta_S(n) = \Delta_F(n) , \tag{3.2.120}$$

which implies that the factor K defined in (3.2.116) vanishes, and thus logharithms cannot appear in the two-point function of $\mathcal{O}$ at the IR fixed point.

3.2.5 Symmetries in ensemble average

We discuss in this section the case in which the random coupling is taken to be constant:

$$h(x) \to h . \tag{3.2.121}$$

Such set-up, which does not physically describe impurities as in quenched disorder, is particularly interesting in the light of the recent understanding of the role of average QFTs in the AdS/CFT correspondence [282]. As in the case of quenched disorder, we are interested in the situation where a symmetry is explicitly broken in any element of the ensemble and we want to see when and under which conditions it can emerge after the average. To distinguish them from the case of disordered systems, we will call these symmetries *averaged symmetries*. A notable example of this kind is the $O(N)$ symmetry in the SYK model [279–281] which rotates the N Majorana fermions, broken by the random fermion coupling, and restored after average (provided the average is taken with an $O(N)$-invariant distribution, as is often the case).

We will see that the simple replacement (3.2.121) leads to crucial differences with respect to the quenched disorder case. We discuss the importance of connectedness of the full space in section 3.2.5, we derive the Ward identities and the topological operators emerging after ensemble average in section 3.2.5, and finally in section 3.2.5 we comment on the implications of our results in the context of the AdS/CFT correspondence where the ensemble average is supposed to be the dual theory of a bulk theory of gravity in $d + 1$ dimensions.

Selection rules in disconnected spaces

The presence of a constant random coupling h over the entire space $X^{(d)}$ leads to a new effect, not present in the quenched disorder, which is the lack of factorization of correlation functions in *disconnected* spaces. For definiteness, consider a theory deformed by a random coupling h in a space $X^{(d)}$ which is the union of two spaces $X^{(d)} = X_1^{(d)} \sqcup X_2^{(d)}$, with $X_1^{(d)} \cap X_2^{(d)} = \emptyset$. At this stage we are not specifying whether the coupling is a constant or not, we only assume that it breaks a global 0-form symmetry G of the pure theory. For each element of the ensemble we can define a generating functional introducing sources K_i for the local operators $\mathcal{O}_i$. Since the space manifold is disconnected, for each local operator $\mathcal{O}$ we effectively need two sources, K_1 and K_2, defined in $X_1^{(d)}$ and $X_2^{(d)}$. For any h, constant or not, the total functional factorizes[25]

$$Z[X^{(d)}, K, h] = Z[X_1^{(d)}, K_1, h] \; Z[X_2^{(d)}, K_2, h] \,, \tag{3.2.122}$$

and so will do arbitrary correlation functions of local operators Φ:

$$\langle \Phi \rangle_X = \langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle_{X_2} \,, \tag{3.2.123}$$

with obvious notation. When h is space dependent (quenched disorder), its support and its probability measure splits into X_1 and X_2. Hence quenched averaged correlators factorize in the two distinct components:[26]

$$\overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle_{X_2}} = \overline{\langle \Phi_1 \rangle}_{X_1} \, \overline{\langle \Phi_2 \rangle}_{X_2} \,. \tag{3.2.124}$$

Thanks to this factorization, the selection rules of the disordered theory are realized independently on each connected component:

$$\overline{\langle \Phi_i \rangle}_{X_i} = R_i \, \overline{\langle \Phi_i \rangle}_{X_i} \,, \qquad i = 1, 2, \qquad (\text{quenched disorder}) \,, \tag{3.2.125}$$

where R_i are the direct products of the representations of the local operators in $X_i^{(d)}$, which should each contain a singlet to get a non-vanishing correlator.

Crucially, in the ensemble average case (3.2.124) cannot hold, because a constant h does not split on the connected components and the average *correlates* the operators across $X_1^{(d)}$ and $X_2^{(d)}$. In particular, we now get the selection rules

$$\overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle}_{X_2} = R_1 \cdot R_2 \, \overline{\langle \Phi_1 \rangle_{X_1} \langle \Phi_2 \rangle}_{X_2} \,, \qquad (\text{ensemble average}) \,. \tag{3.2.126}$$

In contrast to the quenched disorder case, *averages of single correlators in the ensemble average effectively turn into averages of products of correlators when the space is disconnected.* The constraint (3.2.126) is weaker than (3.2.125), obtained in the quenched average theory. In (3.2.126) we need the singlet to appear only in the product $R_1 \cdot R_2$, in (3.2.125) separately for R_1 and R_2. For symmetries that emerge after ensemble average, which we dub *average* symmetries, the charge is then *not* conserved on a single connected component of the manifold, but can "escape" to the other connected components (see the end of appendix A.4.1 for an explicit computation in a free scalar model). We will see how this relates to the violation of global symmetries by Euclidean wormholes in section 3.2.5. The above analysis is trivally generalized to a space with an arbitrary number of disconnected components and to arbitrary products of correlation functions of local operators.

[25] This follows from the observation that any map whose domain is disconnected can be written uniquely as a sum of maps each supported in a connected component.

[26] It should not be confused this factorization of correlators in disconnected space with the non-factorization of products of averaged correlators due to quenched disorder considered in section 3.2.1 and present in any space $X^{(d)}$, connected or not.

Ensemble average and Ward identities

The analysis presented in section 3.2.1 can be repeated in the case of constant h. For concreteness we consider again the case in which the pure theory has a $U(1)$ global symmetry under which $\mathcal{O}_0$ has charge q_0. We have one complex parameter h and the average generating functional is

$$\overline{Z[K_i]} = \int dh d\bar{h}\, P[\bar{h}h] \frac{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0 + c.c.)+\int K_i \mathcal{O}_i}}{\int \mathcal{D}\mu e^{-S_0-(h\int \mathcal{O}_0+c.c.)}} \,. \tag{3.2.127}$$

We derive identities between correlators by changing variables inside the various integrals in (3.2.127). By changing variable in the numerator with an infinitesimal space-dependent symmetry transformation of parameter $\epsilon(x)$, we get

$$\langle \partial_\mu J^\mu(x)\Phi\rangle = \sum_i \delta^{(d)}(x-x_i) q_i \langle\Phi\rangle + q_0 \langle \mathcal{D}(x)\Phi\rangle \,, \tag{3.2.128}$$

where the sum runs over all the local operators defining Φ and we have defined

$$\mathcal{D}(x) := -h\mathcal{O}_0(x) + \bar{h}\,\overline{\mathcal{O}}_0(x) \,. \tag{3.2.129}$$

Note that (3.2.128) holds *before* taking the average. Indeed, this is nothing else than the Ward identities one obtains in a pure theory for an explicitly broken symmetry. We are now not allowed to do a change of variable in the h integral to possibly prove the vanishing on average of the last term in (3.2.128). However, we can perform a *global* transformation $h \to e^{-iq_0\epsilon}h$, with ϵ constant, inside (3.2.127). In this way, we get

$$\int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\Phi\rangle} = \int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\rangle \langle\Phi\rangle} \,, \tag{3.2.130}$$

where $X^{(d)}$ is the *full* space manifold. Finally we can perform a space dependent $U(1)$ transformation only in the path integral in the denominator of (3.2.127), getting

$$\langle \partial_\mu J^\mu \rangle = q_0 \langle \mathcal{D}\rangle \,, \tag{3.2.131}$$

valid before ensemble average. From now on we will assume that $\mathcal{O}_0$ is a scalar under spatial rotations,[27] so that every element of the ensemble is $\mathfrak{so}(d)$ invariant. We then have $\langle J_\mu\rangle = 0$ and thanks to (3.2.131) the relation (3.2.130) simplifies to

$$\int_{X^{(d)}} \overline{\langle \mathcal{D}(x)\Phi\rangle} = 0 \,. \tag{3.2.132}$$

See appendix A.4.1 for an explicit derivation of (3.2.132) for a two-point function in a simple solvable model. The combination $\partial^\mu J_\mu - q_0\mathcal{D}(x)$ satisfies the condition

$$\int_{X^{(d)}} d^d x\, \overline{\langle \left(\partial^\mu J_\mu(x) - q_0 \mathcal{D}(x)\right)\Phi\rangle} = 0 \,, \tag{3.2.133}$$

[27] This assumption is not crucial. For non-scalar deformations, rotational invariance is broken before the average and we need to keep track of all the vacuum expectation values induced by the random variable, as done in the quenched disorder case. This can be repeated in the ensemble average case, but makes the analysis more involved.

which ensures that the Ward identities (3.2.128), when integrated over the full space and after ensemble average, imply charge conservation. As expected from a spurionic argument, the symmetry is restored after average.[28]

Let us now see if we can define more general operators $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$, topological after ensemble average. The natural choice from (3.2.133) is

$$\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}] = Q[\Sigma^{(d-1)}] - q_0 \int_{D^{(d)}} d^d x\, \mathcal{D}(x)\,, \qquad Q[\Sigma^{(d-1)}] := \int_{\Sigma^{(d-1)}} n_\mu J^\mu(x)\,, \tag{3.2.134}$$

where $D^{(d)}$ is an arbitrary region such that $\partial D^{(d)} = \Sigma^{(d-1)}$. Note that this requires $\Sigma^{(d-1)}$ to be homologically trivial otherwise, by definition, the surface $D^{(d)}$ does not exist. In the terminology of [321], the operator (3.2.134) is a *non-genuine* co-dimension one operator, since it requires a topological surface attached to it.[29]

We can discuss the dependence of $\widehat{Q}$ in (3.2.134) on the choice of the filling region $D^{(d)}$. Given another such manifold $D'^{(d)}$ we can glue it along $\Sigma^{(d-1)}$ with the orientation reversal of $D^{(d)}$ to form a closed manifold $Y^{(d)} = D'^{(d)} \sqcup \overline{D^{(d)}}$, and $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$ is independent on $D^{(d)}$ if and only if

$$\int_{Y^{(d)}} \overline{\langle \mathcal{D}(x)\Phi \rangle} = 0\ . \tag{3.2.135}$$

We see that (3.2.135) is not satisfied unless the space-time $X^{(d)}$ is connected, and we will generically refer to $\widehat{Q}$ as a non-genuine operator. On the other hand, if $X^{(d)}$ is connected any homologically trivial co-dimension one submanifold $\Sigma^{(d-1)}$ of $X^{(d)}$ divides $X^{(d)} - \Sigma^{(d-1)}$ in two disjoint connected components glued along $\Sigma^{(d-1)}$, hence necessarily $Y^{(d)} = X^{(d)}$ and (3.2.135) reduces to (3.2.132), showing the independence of $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$ on the filling region. $\widehat{Q}$ is still expressed with an integral over $D^{(d)}$, but the dependence of the non-genuine symmetry operator on the filling region is only apparent, and for all practical purposes this can be regarded as independent on the filling region. We refer to this situation as a quasi-genuine co-dimension one operator.

If $X^{(d)}$ has several connected components, $Y^{(d)}$ can be a proper sub-region, since adding or removing from it an entire connected component which does not intersect $\Sigma^{(d-1)}$ preserves the property that $Y^{(d)}$ is the union of regions glued along $\Sigma^{(d-1)}$. For instance if $X^{(d)}$ has two connected components $X_1^{(d)}$ and $X_2^{(d)}$, and suppose $\Sigma^{(d-1)}$ is entirely contained in $X_1^{(d)}$, the latter is divided by $\Sigma^{(d-1)}$ into two regions $D^{(d)}$ and $D'^{(d)}$, and choosing one or the other leads to different operators $\widehat{Q}[\Sigma^{(d-1)}]$, since (3.2.132) holds only in the entire space and not to each connected component:

$$\overline{\Big\langle \int_{D^{(d)}} \mathcal{D}(x)\Phi \Big\rangle} = \overline{\Big\langle \Big(\int_{D'^{(d)}} + \int_{X_2^{(d)}} \Big) \mathcal{D}(x)\Phi \Big\rangle} \neq \overline{\Big\langle \int_{D'^{(d)}} \mathcal{D}(x)\Phi \Big\rangle}\,. \tag{3.2.136}$$

In this case we cannot define a quasi-genuine co-dimension one topological operator and therefore, even if the total charge is conserved thanks to (3.2.133), we cannot measure it locally in a subregion of the entire (disconnected) space.

[28]In a pure theory the identities (3.2.128) apply but $\mathcal{D}(x)$ does not integrate to zero when inserted in arbitrary correlators. As a consequence no selection rules are implied, as expected for an explicitly broken symmetry!

[29]The requirement is however of different nature. In [321] (and subsequent works) the surface is required to

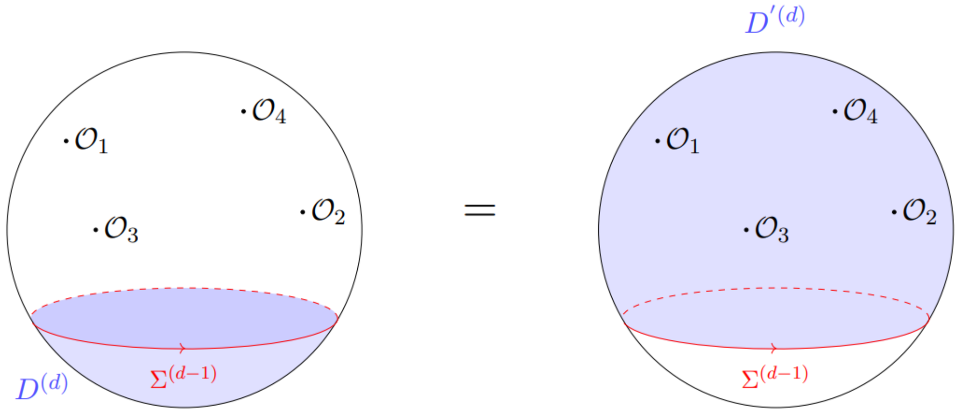


Figure 3.3: Selection rules (3.2.138) for correlators when $X^{(d)}$ is connected. The integral over the region $D^{(d)}$ in the left panel equals the integral over the region $D'^{(d)}$ in the right panel thanks to (3.2.132). When $D^{(d)}$ is shrunk to a point the region $D'^{(d)}$ extends to the whole $X^{(d)}$.

In order to measure the charge of operators in the whole space, we can consider $\widehat{Q}$ on a codimension 1 closed surface $\Sigma^{(d-1)} = \Sigma_1^{(d-1)} \sqcup \Sigma_2^{(d-1)}$, with $\Sigma_i^{(d-1)} \subset X_i^{(d)}$ $(i = 1, 2)$, and two regions $D_i^{(d)}$ such that $\partial D_i^{(d)} = \Sigma_i^{(d-1)}$. In each given connected component, the charge cannot be conserved, as we have seen, but if we simultaneously consider the two regions, then the Ward identities still apply. In the schematic notation of section 3.2.5 we have

$$\begin{aligned}\overline{\langle \widehat{Q}[\Sigma, D]\Phi\rangle_X} &= \overline{\langle \widehat{Q}[\Sigma_1, D_1]\Phi_1\rangle_{X_1}\langle\widehat{\Phi}_2\rangle_{X_2}} + \overline{\langle \Phi_1\rangle_{X_1}\langle \widehat{Q}[\Sigma_2, D_2]\Phi_2\rangle_{X_2}} \\ &= \Big(\chi_1(\Sigma_1) + \chi_2(\Sigma_2)\Big)\overline{\langle\widehat{\Phi}_1\rangle_{X_1}\langle\widehat{\Phi}_2\rangle_{X_2}} = \Big(\chi_1(\Sigma_1) + \chi_2(\Sigma_2)\Big)\overline{\langle\widehat{\Phi}\rangle_X}\,,\end{aligned} \tag{3.2.137}$$

where $\chi_{1,2}(\Sigma_{1,2})$ denotes the sum of the charges of the local operators $\Phi_{1,2}$ which are inside the surface $\Sigma_{1,2}^{(d-1)}$. Since $\Sigma^{(d-1)}$ depends now on $D^{(d)}$, it is crucial to consider the complement space in both connected spaces at the same time. The generalization to spaces $X^{(d)}$ with more than two connected components is obvious.

We refer the reader to appendix A.5 for a proof of the existence of the operator $\widehat{U}_g$ which implements the action of the group rather than the action of the corresponding Lie algebra. By definition, the operator $\widehat{U}_g$, given in (A.5.20), satisfies

$$\overline{\langle \widehat{U}_g[\Sigma^{(d-1)}, D^{(d)}]\mathcal{O}_1\cdots\mathcal{O}_n\rangle} = e^{i\alpha\chi(\Sigma^{(d-1)})}\overline{\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle}\,. \tag{3.2.138}$$

Since $\widehat{U}_g[\emptyset, X^{(d)}] = 1$, (3.2.138) implies the selection rules we derived from the spurion argument (see figure 3.3). The equivalent of (3.2.137) for a finite group action precisely reproduces the selection rule (3.2.126). With $\Sigma^{(d-1)}$ as in figure 3.4, we have

$$\overline{\langle\Phi\rangle}_X = \overline{\langle\widehat{U}_g[\Sigma, D]\Phi\rangle}_X = \overline{\langle\widehat{U}_g[\Sigma_1, D_1]\Phi_1\rangle_{X_1}\langle\widehat{U}_g[\Sigma_2, D_2]\Phi_2\rangle}_{X_2} = e^{i\alpha(\chi_1(\Sigma_1)+\chi_2(\Sigma_2))}\overline{\langle\Phi\rangle}_X \tag{3.2.139}$$

while, say,

$$\overline{\langle\Phi\rangle}_X = \overline{\langle\widehat{U}_g[\Sigma_1, D_1]\Phi_1\rangle_{X_1}\langle\Phi_2\rangle}_{X_2} \neq e^{i\alpha\chi_1(\Sigma_1)}\overline{\langle\Phi\rangle}_X\,. \tag{3.2.140}$$

have a well-defined gauge-invariant operator, here the surface is required to make the operator topological (on average).

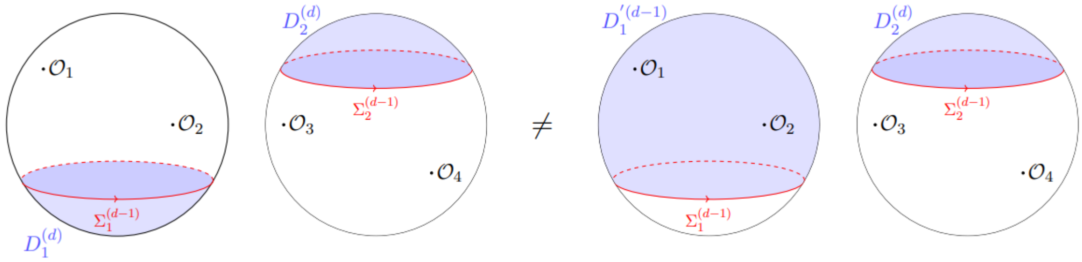


Figure 3.4: Violation of the selection rules (3.2.140) when $X^{(d)}$ is disconnected. The integral over the region $D_1^{(d)}$ in $X_1^{(d)}$ (left) is not equal to the integral over the region $D_1^{'(d)}$ in $X_1^{(d)}$ (right) because of the presence of the component $X_2^{(d)}$. An equality sign would require to reverse the region of integration also in $X_2^{(d)}$ (right) from $D_2^{(d)}$ to its complement.

We have then found an instance of a theory with a global zero-form symmetry in the sense of giving rise to selection rules for correlation functions of local operators, but with *no* genuine co-dimension one topological operator. Aside of being topological only on average, the operator $\widehat{U}_g[\Sigma, D]$ is not genuine and it can be defined only on homologically trivial cycles.

The local charge violation (3.2.140) in a single connected component of space when $X^{(d)}$ is an union of several connected components indicate the presence of non-local interactions in the theory. Their presence is manifest by using the replica trick. Consider a Gaussian random distribution $P[\bar{h}h] \propto \exp(-\bar{h}h/v)$ (e.g. as in the SYK model). Repeating the steps described in section 3.2.2 we find *non-local* interactions among replicas

$$S_{\text{rep}} = \sum_{a=1}^{n} S_{0,a} - v \int d^d x \int d^d y \sum_{a,b=1}^{n} \overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y)\,. \tag{3.2.141}$$

The replica theory enjoys a diagonal $U(1)_D$ global symmetry, but the naive diagonal current $J_D^\mu = \sum_a J_a^\mu$ does not satisfy standard Ward identities. By performing an infinitesimal $U(1)$ transformation with a local parameter $\alpha(x)$ we get

$$\begin{aligned}\delta S_{\text{rep}} &= \int dx\, \alpha(x)\partial_\mu J_D^\mu(x) - q_0 v \sum_{a,b} \int dx\, dy \Big(\alpha(y) - \alpha(x)\Big)\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) \\ &= \int_{X^{(d)}} dx\, \alpha(x)\Bigg(\partial_\mu J_D^\mu(x) + q_0 v \sum_{a,b} \int_{X^{(d)}} dy \left(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) - \mathcal{O}_{0,a}(x)\overline{\mathcal{O}}_{0,b}(y)\right)\Bigg)\,.\end{aligned} \tag{3.2.142}$$

Thus the Ward identities for the diagonal symmetry are modified by a non-local term and read

$$\begin{aligned}&\Big\langle \Big(\partial_\mu J_D^\mu(x) + q_0 v \sum_{a,b} \int_{X^{(d)}} d^d y \left(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) - \overline{\mathcal{O}}_{0,a}(y)\mathcal{O}_{0,b}(x)\right) \Big)\Phi \Big\rangle^{\text{rep}} \\ &\quad = \sum_i \delta^{(d)}(x - x_i) q_i \langle \Phi \rangle^{\text{rep}}\,.\end{aligned} \tag{3.2.143}$$

In the replica theory the operator

$$\partial_\mu J_D^\mu(x) + q_0 v \sum_{a,b} \int_{X^{(d)}} d^d y \,\left(\overline{\mathcal{O}}_{0,a}(x)\mathcal{O}_{0,b}(y) - \overline{\mathcal{O}}_{0,a}(y)\mathcal{O}_{0,b}(x)\right) \tag{3.2.144}$$

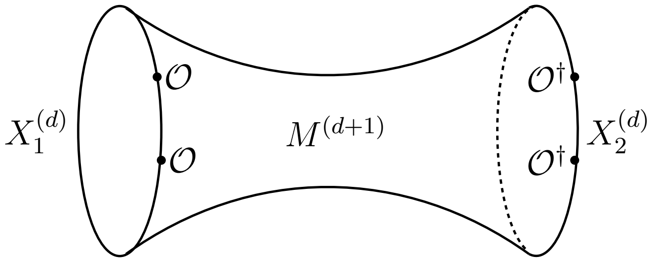


Figure 3.5: Example of a wormhole bulk geometry contributing to the average correlator $\overline{\langle \mathcal{O}\mathcal{O}\rangle\langle \mathcal{O}^\dagger\mathcal{O}^\dagger\rangle}$, with $\mathcal{O}$ a charged boundary operator.

satisfies the Ward identities and its integral over the full space evidently vanishes (inside arbitrary correlators), implying the $U(1)_D$ selection rules. This is how the properties of the averaged symmetry show up in the replica theory, where the non-local nature of the symmetry is manifest for Gaussian distributions. The property (3.2.132) of the operator $\mathcal{D}(x)$ defined in (3.2.129) is mapped to the property of the extra term in (3.2.144) of integrating to zero exactly as an operator equation. This is consistent with the dictionary between correlators of the averaged theory and the replica one.

A gravity discussion

We have found that averaged global symmetries are intrinsically different from ordinary global symmetries. They imply selection rules as dictated by the global symmetry but, in contrast to ordinary global symmetries, they do not admit genuine co-dimension one operators, topological after average. Even in a connected space such operators cannot be defined in homologically non-trivial cycles. As a result, these symmetries cannot consistently be coupled to an external background field, at least not in a natural way.[30] Note that this is different from the concept of 't Hooft anomalies. In the latter the obstruction is in making the gauge fields dynamical but there is a well defined notion of coupling the theory to backgrounds gauge fields. The difficulty of coupling the symmetry to an external background is clear in the replica theory from the presence of the second term in (3.2.144), which is non-local and not manifestly the divergence of a current.

The results have interesting consequences when applied to averaged theories which are assumed to have an holographic dual bulk gravitational theory in asymptotically AdS space-times.

In the ordinary AdS/CFT correspondence a given theory of gravity in asymptotically AdS space-time is dual to a given CFT. Ordinary global symmetries of the CFT become gauge symmetries in the bulk. This correspondence fits nicely with the widely accepted common lore that in quantum gravity unbroken global symmetries cannot exist [322–325]. A natural question then arises: when the dual theory is given by an ensemble average, what is the bulk interpretation of the symmetries emerging after average? In [308] (see also [309, 310]) it has

[30]For discrete symmetries, for example, coupling to an external background field corresponds to insert a mesh of symmetry defects on homologically non-trivial cycles of space.

been conjectured that boundary emergent symmetries correspond in the bulk to global, and not gauge, symmetries which are broken non-perturbatively by Euclidean wormhole configurations, which allow the global symmetry charge to flow from one connected component to another one, see figure 3.5. From the boundary point of view, this charge violation induced by bulk wormholes correspond to the lack of selection rules in the average theory that we have discussed before, when the space is not connected, in agreement with the findings in [308–310]. Since averaged symmetries simply cannot be gauged, our results clarify why they cannot be interpreted as gauge symmetries in the bulk, at least in the case where the average is of the form (3.2.14).[31]

Note that boundary emergent symmetries are compatible with recent works where, motivated by the connection with the lore of spectrum completeness in gravitational theories [326], "absence of global symmetries in gravitational theories" is replaced by "absence of topological operators", including those related to non-invertible symmetries [327, 328].

3.2.6 Conclusions

We have studied disordered QFTs where an ordinary symmetry of a pure QFT is explicitly broken by a random coupling, but the symmetry re-emerges after quenched average. We focused our attention to understand if and under what conditions we can have operators, topological on average, in analogy to ordinary QFTs [9]. We considered *quenched disorder theories*, where the pure theory is deformed with a space dependent coupling, and *ensemble average theories*, where the latter is kept constant.

In the quenched disordered case, we can write Ward identities for averages of products of correlators and construct the symmetry operator implementing the finite group action, topological after average. Such disordered symmetries can be coupled to external background, can be gauged, and can have 't Hooft anomalies (i.e. can exclude a trivially gapped phase at long distances), precisely like ordinary symmetries. Using the replica trick, we also discussed genuinely emergent symmetries in the IR after average, namely symmetries which are not present in the UV theory even before adding the disorder coupling. We pointed out that whenever a symmetry G is emergent in the IR, exotic selection rules can explain the origin of LogCFTs.

In ensemble average theories the analogy to pure QFTs is more loose. We still have selection rules for averages of correlators and we can construct operators implementing the finite group action, but the charge operator is not purely codimension-1 and cannot be defined if $\Sigma^{(d-1)}$ is homologically non-trivial. When the space is disconnected, the selection rules apply only globally and in each connected component charge violation can occur. Such averaged symmetries cannot be coupled to background gauge fields in ordinary ways. The difficulty (impossibility) of gauging emergent boundary symmetries clarify why such symmetries cannot be identified with bulk gauge symmetries when the average theory admits a gravitational bulk dual.

It would be interesting to analyze spontaneous breaking of disordered symmetries in more detail. There are essentially two ways in which the disordered symmetry could spontaneously

[31] In particular, our results do not straightforwardly apply when the average is over OPE coefficients, as e.g. discussed in [292, 293].

break: i) the symmetry is spontaneously broken in the pure theory before adding the random interaction, ii) the symmetry is unbroken in the pure theory and the random interaction induces a spontaneous breaking of the disordered symmetry. Let us consider the case of continuous symmetries. From the replica theory point of view, i) and ii) are distinguished by which components of the replica currents J_a^μ are subject to spontaneous breaking, all components in case i) and only the singlet $\sum_a J_a^\mu$ in case ii). Assuming the existence of the analytic continuation in n and of a smooth $n \to 0$ limit, we expect for $d > 2$ gapless excitations (Goldstone boosons) in the replica theory, giving rise to power-like correlators. From the disordered theory point of view, in case i) there is a Goldstone mode in the pure theory which acquires a mass in each specific realization of the ensemble, turning into a pseudo Goldstone boson. In contrast, no Goldstone boson is present in the pure theory in the more exotic case ii). In both cases it would be nice to identify which correlators (if any) exhibit power like-behavior on average as a result of the spontaneous breaking of the disordered theory.

It would be also interesting to generalize our findings to quantum disorder, namely to Lorentzian theories where the random coupling depends only on space. The natural extension of our analysis beyond 0-form symmetries does not seem straightforward. Higher-form symmetries can be broken only by non-local deformations, which should be also taken random. It is possibly easier to consider a set-up in $d = 2$ where non-invertible symmetries can be obtained by 0-form symmetries only, and see if and in what sense we can have a non-invertible symmetry re-emerging after average.

An important remark about the ensemble average case is that, in comparing our findings with the existing literature on the factorization problem in AdS/CFT, one should keep in mind that we only considered averaging over couplings. There are other setups, like averaging over OPE coefficients [292, 293] or over different modular invariants [34], where global symmetries could behave differently from our findings. In particular [34] discusses the gauging of a 1-form global symmetry in certain gravitational toy models, but this is not in contrast with our result about the impossibility of gauging average 0-form symmetries. It is a very interesting problem for the future to discuss the status of global symmetries in these other contexts, possibly finding a unified picture.

Chapter 4

Some applications of the holographic approach to symmetries

In this chapter we present two applications of the SymTFT. In the first half of the chapter we study 't Hooft anomalies of duality defects, defined as obstructions to gauge the symmetries. In the second half instead we employ the SymTFT as a bulk theory for a holographic duality, arguing that the boundary theories are effective theories describing the spontaneous breaking of a symmetry.

4.1 Anomalies of self-duality symmetries: fractionalization and gauging

An important step towards the applications of generalized symmetries is the development of a concrete characterization of 't Hooft anomalies and of their dynamical consequences for RG flows. While for invertible (higher) symmetries a complete classification of 't Hooft anomalies is given by the appropriate cobordism group [318, 329–331], for non-invertible symmetries the correct general framework remains unclear. A standard approach is to define anomalies as obstructions to the gauging of a symmetry $\mathcal{C}$. Gauging (or condensation) in higher fusion categories is however a subtle procedure, as it requires the specification of a certain type of consistent algebra objects $\mathcal{A} \in \mathcal{C}$. While the mathematical theory governing such objects has been developed for 1-categories [17, 219] and recently for 2-categories [43], a complete characterization of the required consistency conditions is to this day still missing. A more modern perspective would be to characterize 't Hooft anomalies as obstructions to the existence of a trivially-gapped $\mathcal{C}$-symmetric phase. As pointed out in [95], for non-invertible symmetries this latter definition of 't Hooft anomalies implies also that there is an obstruction to gauging, while the converse is generically not true. This observation has been recently reformulated as the existence of certain weakly (respectively, strongly) symmetric boundary conditions in [95].[1]

We focus on self-duality symmetries, which appear when a $d = 2n$ dimensional QFT $\mathcal{T}$ is

[1] The two notions coincide for invertible symmetries and the obstruction to define them is equivalent to an 't Hooft anomaly [332, 333]. In the non-invertible case the two notions bifurcate.

mapped back to itself after gauging a discrete $(n-1)$-form symmetry $\mathbb{A}$ [57, 63, 65, 66]:

$$\mathcal{T}/\mathbb{A} \cong \mathcal{T} \,, \tag{4.1.1}$$

possibly with a choice of discrete torsion which we leave implicit. Above $\cong$ means equivalence up to a change of duality frame. The corresponding symmetry category $\mathcal{C}$ is best described as a graded category, graded by the group G of self-dualities:

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g \,, \qquad \mathcal{C}_0 = n\text{Vec}_{\mathbb{A}} \,, \qquad \mathcal{C}_g = \{\mathcal{N}_g\} \,. \tag{4.1.2}$$

Here $n\text{Vec}_{\mathbb{A}}$ is the category describing an anomaly-free $(n-1)$-form symmetry $\mathbb{A}$, and in the last equality we meant that the connected component[2] $\pi_0(\mathcal{C}_g)$ has a single simple object $\mathcal{N}_g$ for $g \neq 0$. The fusion rules of the $\mathcal{N}_g$'s respect the G-grading up to condensates $C_{\mathbb{A}}$ of the symmetry $\mathbb{A}$ [13]. In particular

$$\mathcal{N}_g \times \overline{\mathcal{N}}_g = C_{\mathbb{A}} \,. \tag{4.1.3}$$

We will consider the cases of $G = \mathbb{Z}_2$ in $d = 2$ and $G = \mathbb{Z}_4, \mathbb{Z}_3$ in $d = 4$. Our analysis however can in principle be extended to more general cases.[3] Examples of theories with duality symmetries are the Ising CFT and the $c = 1$ boson in 2d [12, 31], and $\mathcal{N} = 4$ SYM and pure Maxwell theory (for specific values of the complexified gauge coupling) in 4d [57].[4]

We will define an 't Hooft anomaly for a non-invertible duality defect as the obstruction to constructing a condensable algebra $\mathcal{A}$ containing all the $\mathcal{N}_g$'s. It is believed (although not proven) that compatibility with a trivially-gapped phase is equivalent to the existence of such an algebra which furthermore contains the full category $\mathcal{C}$. In two dimensions for Tambara-Yamagami (TY) categories [250], this viewpoint has been examined in [29] by exploiting the concept of *fiber functor*. In this case, condensable algebras are of the form $\mathcal{A} = \mathbb{B} \oplus n_\nu \mathcal{N}$ with $\mathbb{B} \subset \mathbb{A}$ a subgroup and n_ν an integer. The symmetry admits a trivially-gapped realization only if $\mathbb{B} = \mathbb{A}$. If instead $\mathbb{B} \subsetneq \mathbb{A}$, the symmetry only admits a duality-invariant TQFT. We will regard $\mathcal{N}$ to be anomaly-free in both cases.

Our aim is to give a unified treatment of anomalies for duality symmetries which can be generalized to higher dimensions. A fundamental tool to this purpose is the Symmetry TFT $\mathcal{Z}(\mathcal{C})$ [118].[5] Given a fusion $(d-1)$-category $\mathcal{C}$, $\mathcal{Z}(\mathcal{C})$ is a $(d+1)$-dimensional TQFT which encodes the full categorical data, and in particular the anomalies, of the symmetry $\mathcal{C}$. Topological manipulations (generalized gaugings) in the QFT are belived to be in one-to-one

[2]Given a (higher) category $\mathcal{C}$, $\pi_0(\mathcal{C})$ denotes the set of simple objects of $\mathcal{C}$ modded out by the equivalence relation $x \sim y$ if $\text{Hom}(x, y)$ is nontrivial [334]. Physically, the modding procedure corresponds to the condensation of symmetries localized on the defects.

[3]In $d = 4$, theories of class $\mathcal{S}$ [335] can have self-duality defects with non-Abelian G [8, 67]. Moreover, it has been recently pointed out that there exist duality defects in 6d SCFTs [336].

[4]See also [80] for an extension to theories with lower amounts of SUSY, and [45] for the mathematical treatment of self-duality categories in 3d.

[5]See *e.g.* [7, 8, 53, 83, 133, 195] and references therein for recent applications. Notice that in the mathematics literature the notation $\mathcal{Z}(\mathcal{C})$ is used to denote the Drinfeld center [183] of a Fusion category $\mathcal{C}$, while here it denotes the Symmetry TFT for the symmetry $\mathcal{C}$. While the two concepts are equivalent in 2d, one must be careful when extending the correspondence to higher categories. See for example [334].

correspondence with topological boundary conditions of $\mathcal{Z}(\mathcal{C})$. Hence an 't Hooft anomaly corresponds to the absence of certain "magnetic" topological boundary conditions in $\mathcal{Z}(\mathcal{C})$ which would trivialize $\mathcal{N}$ on the boundary. A similar perspective has been considered recently in [99, 195].

The Symmetry TFT for duality defects has been identified as a $(2n + 1)$-dimensional Dijkgraaf-Witten (DW) theory, further gauged by a 0-form symmetry [7, 133]. In Section 4.1.1 we use this fact to lay out a general approach to identify obstructions to the existence of those magnetic boundary conditions. The resulting obstruction theory consist of two conditions:

1. The first one is the existence of a G-invariant Lagrangian algebra $\mathcal{L}_D$ in the ungauged DW theory. In the language of [133], the duality symmetry is "non-intrinsic". Hence intrinsically non-invertible symmetries are necessarily anomalous.

2. When that condition is satisfied, the second obstruction corresponds to the cancellation of a pure 't Hooft anomaly for an invertible symmetry. Given a G-invariant Lagrangian algebra $\mathcal{L}_D$, the group G generally acts on it through a nontrivial automorphism. This action is not unique and corresponds to a choice of *equivariantization* $\tilde{\eta}$ of $\mathcal{L}_D$ [337]. We propose and check in several examples that such a choice encodes symmetry fractionalization data for the boundary symmetry, which can sometimes be used to cancel the cubic 't Hooft anomaly (see [338] for other examples of this phenomenon). This allows to overcome the difficulty of generalizing the equivariantization procedure to higher categories.

In Section 4.1.2 we examine, in two dimensions, duality defects that are described by Tambara-Yamagami categories $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$. The full classification of their anomalies is known in both the mathematical [251, 339, 340] and physical literature [29]. We show how our prescription precisely reproduces the known results. We then generalize this strategy to analyze the anomalies of non-invertible duality defects in 4d [57,63,66] in Section 4.1.3. Notice that even though a complete definition of "gauging" for higher categories is still absent, the obstructions are nevertheless accessible from the Symmetry TFT. These defects are present, for instance, in $\mathcal{N} = 4$ SYM and the analysis of their anomalies is a crucial first step in understanding the dynamics of duality-preserving RG flows [74]. Gauging these symmetries in $\mathcal{N} = 4$ SYM also leads to certain $\mathcal{N} = 3$ SCFTs [341]. Our findings shed light on their consistency. In Section 4.1.4 we consider the compactification of the 4d/5d setup on a torus, obtaining a 2d/3d system with a Tambara-Yamagami symmetry associated with a finite group $\widetilde{\mathbb{A}} = \mathbb{A} \times \mathbb{A}$, where $\mathbb{A}$ is the 4d 1-form symmetry group. We show how in this case the 4d obstruction theory correctly descends to the 2d one. We conclude in Section 4.1.5 with several applications of our results and further directions. Technical material and general proofs are gathered in various appendices.

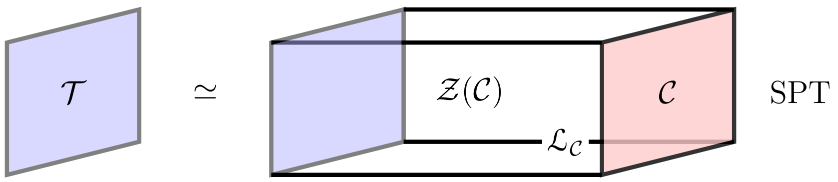
Notation We use additive notation for the Abelian groups $\mathbb{A}$, $\mathbb{B}$, *etc.* We indicate the Pontryagin dual to $\mathbb{A}$ as $\mathbb{A}^\vee = \mathrm{Hom}\big(\mathbb{A}, U(1)\big)$. For simplicity, we indicate group-cohomology groups as $H^p\big(\mathbb{A}, U(1)\big)$ as opposed to the lengthier (though equivalent) notation $H^p\big(B\mathbb{A}, U(1)\big)$ for the cohomology groups of the classifying space $B\mathbb{A}$ of $\mathbb{A}$. For group-cohomology classes, and more generally for cochains, we use multiplicative notation with values in $U(1)$, with the exception of integrals, examples or where otherwise stated where we use additive notation with values in

$\mathbb{R}/\mathbb{Z}$. In order to limit confusion, we sometimes use the notation $\underline{0}$ and $\underline{1}$ for the trivial element and the generator of $\mathbb{Z}_n$, respectively.

4.1.1 A proposal from the Symmetry TFT

A promising approach to analyze the structure and the anomalies of categorical symmetries is the Symmetry TFT [117, 118]. Given a symmetry category $\mathcal{C}$ in d dimensions, the associated Symmetry TFT is a $(d+1)$-dimensional TQFT $\mathcal{Z}(\mathcal{C})$ admitting a gapped boundary condition $\mathcal{L}_{\mathcal{C}}$, which we call electric, that gives rise to the symmetry $\mathcal{C}$ on the boundary. Formally this means that the category $\text{Mod}_{\mathcal{L}_{\mathcal{C}}}\big(\mathcal{Z}(\mathcal{C})\big)$ of $\mathcal{L}_{\mathcal{C}}$-modules, which describes topological operators confined to the gapped boundary, coincides with $\mathcal{C}$: $\text{Mod}_{\mathcal{L}_{\mathcal{C}}}\big(\mathcal{Z}(\mathcal{C})\big) = \mathcal{C}$. General gapped boundary conditions are in one-to-one correspondence with Lagrangian algebra objects $\mathcal{L}$ of the bulk category (see *e.g.* [342]). This correspondence is realized by noticing that $\mathcal{Z}(\mathcal{C})$ is trivialized (*i.e.*, it becomes an invertible TQFT) after condensing $\mathcal{L}$, and its unique state is the gapped boundary condition. Besides, topological operators charged under $\mathcal{L}$ are confined to the boundary where they form the category $\text{Mod}_{\mathcal{L}}\big(\mathcal{Z}(\mathcal{C})\big)$. The correspondence allows us to talk about maximal gaugings of the bulk theory and of gapped boundaries interchangeably, and to use the same symbol $\mathcal{L}$ for them.

This setup can be coupled to a dynamical theory $\mathcal{T}$ via a "sandwiching" procedure:



(4.1.4)

where the $(d+1)$-dimensional manifold is a slab with two boundaries, one supporting a dynamical theory (with free boundary conditions) and one on which we impose the gapped boundary condition $\mathcal{L}_{\mathcal{C}}$. Different choices of $\mathcal{L}$ give rise to symmetry categories belonging to the same Morita equivalence class $\mathcal{M}(\mathcal{C})$ of the symmetry $\mathcal{C}$. Physically, elements in the same class are related by (generalized) discrete gauging operations. The reason why the Symmetry TFT is a useful tool for detecting anomalies is that, at least in a large class of examples, the correspondence between gapped boundary conditions and elements of the Morita equivalence class $\mathcal{M}(\mathcal{C})$ is one-to-one:[6] all the allowed topological manipulations can be realized by condensing the appropriate Lagrangian algebra in the bulk. In QFT language we define an 't Hooft anomaly as an obstruction to gauging the symmetry $\mathcal{C}$. In the Symmetry TFT language this translates to the lack of an associated boundary condition $\mathcal{N}_{\mathcal{C}}$, which would implement the topological manipulation of "gauging $\mathcal{C}$". Notice that, in the absence of anomalies, one can expect multiple boundary conditions $\mathcal{N}_{\mathcal{C}}$ as there might be different gaugings that involve $\mathcal{C}$.

In this work we focus on self-duality defects [29,57,63,65,66], for which $\mathcal{Z}(\mathcal{C})$ is known [7,133]. The Symmetry TFT description of $\mathcal{C}$ is closely related to the one of $n\text{Vec}(\mathbb{A})$ (the category

[6]This is certainly true for maximally-degenerate categories, such as those considered in this work. Physically this means that there is no charged object which is also topological. In these cases the Symmetry TFT is constructed by a state-sum model and is a generalization of the Turaev-Viro theory [120] (see also [126–128,130] for concrete generalizations in both condensed matter physics and in mathematics).

describing an anomaly-free $(n-1)$-form symmetry $\mathbb{A}$). The Symmetry TFT for $n\mathrm{Vec}(\mathbb{A})$ is a generalized untwisted Dijkgraaf-Witten theory,

$$\mathcal{Z}\big(n\mathrm{Vec}(\mathbb{A})\big) = \mathrm{DW}(\mathbb{A}) \ , \tag{4.1.5}$$

and $n\mathrm{Vec}(\mathbb{A})$ is associated to the canonical (or electric) Dirichlet boundary condition in $\mathrm{DW}(\mathbb{A})$. Concretely, for $d = 2n$ dimensional boundaries, $\mathrm{DW}(\mathbb{A})$ is a $(d+1)$-dimensional pure n-form gauge theory for $\mathbb{A}$ with action

$$S = 2\pi i \int_{X_{d+1}} A \cup dB \ , \qquad A \in C^n(X_{d+1}, \mathbb{A}) \ , \qquad B \in C^n(X_{d+1}, \mathbb{A}^\vee) \ . \tag{4.1.6}$$

This theory has an n-form symmetry $\mathbb{A} \times \mathbb{A}^\vee$ generated by the Wilson surface operators of B and A, respectively. The canonical Dirichlet boundary condition simply sets to zero the pull-back of A to the boundary. The duality symmetry G is a subgroup of the full 0-form symmetry of the Dijkgraaf-Witten theory,[7] and it acts by exchanging electric and magnetic operators according to an isomorphism

$$\phi : \mathbb{A} \to \mathbb{A}^\vee \ . \tag{4.1.7}$$

In this work we will focus on the cases that G is isomorphic to $\mathbb{Z}_2$ for $n = 1$, and to either $\mathbb{Z}_3$ or $\mathbb{Z}_4$ for $n = 2$, corresponding to duality or triality symmetries. Therefore in the following we assume that G is Abelian (although a similar discussion could be made for non-Abelian G). Generically G also acts on boundary conditions through its action on the associated Lagrangian algebras $\mathcal{L}$. Gauging the symmetry G , possibly with discrete torsion ϵ, gives the sought-after Symmetry TFT $\mathcal{Z}(\mathcal{C})$:

$$\mathrm{DW}(\mathbb{A}) \ \underset{\mathrm{Rep}(G)}{\overset{G^\epsilon}{\rightleftarrows}} \ \mathcal{Z}(\mathcal{C}) \tag{4.1.8}$$

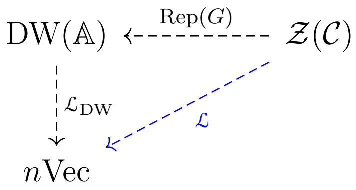
In these diagrams dashed lines represent gaugings in the bulk. The upper arrow indicates gauging with discrete torsion, while the lower one the "inverse" operation of gauging[8] the dual symmetry $\mathrm{Rep}(G)$ (for G Abelian, $\mathrm{Rep}(G) \cong G^\vee$). We will argue that the choice of ϵ acts as a kind of pure G anomaly for the duality defects. From the bulk perspective, the duality defects $\mathcal{N}_g$ are related to the liberated twist defects Σ_g of the 0-form symmetry G in $\mathrm{DW}(\mathbb{A})$ [7, 133] which are the objects carrying charge under the quantum symmetry $\mathrm{Rep}(G)$.

The gapped boundary conditions $\mathcal{L}_{\mathrm{DW}}$ for $\mathrm{DW}(\mathbb{A})$ are classified by the maximal (Lagrangian) sublattices $\mathcal{L}_{\mathrm{DW}}$ of mutually local charges. Condensing such objects leads in the bulk to a trivial theory (*i.e.*, an invertible TQFT, or SPT phase) whose unique state is the

[7] Depending on $\mathbb{A}$, in general there are other 0-form symmetries in the theory that map $\mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A} \times \mathbb{A}^\vee$ (*e.g.*, charge conjugation that acts separately on $\mathbb{A}$ and $\mathbb{A}^\vee$). Any subgroup G of the full symmetry could be considered. In this work, for simplicity, we focus on symmetries G that descend to "dualities" (and are thus controlled by the map (4.1.7)) and are generically present for any $\mathbb{A}$. However, the discussion that follows is quite general.

[8] Precisely, the gauging of $\mathrm{Rep}(G)$ should be accompanied by stacking with the inverse SPT ϵ^{-1}, that we leave implicit.

gapped boundary condition [222, 342]. From (4.1.8), we can always induce a gapped boundary condition $\mathcal{L}$ for $\mathcal{Z}(\mathcal{C})$ from a gapped boundary $\mathcal{L}_{\text{DW}}$ of $\text{DW}(\mathbb{A})$ by first condensing $\text{Rep}(G)$ and then $\mathcal{L}_{\text{DW}}$:

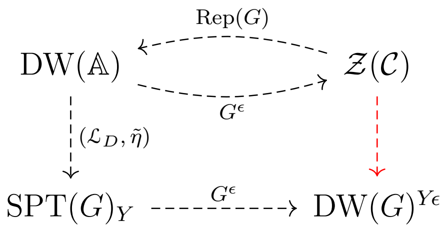


(4.1.9)

Here with $n\text{Vec}$ we denote the trivial $(d+1)$-dimensional theory obtained after condensing $\mathcal{L}_{\text{DW}}$ in $\text{DW}(\mathbb{A})$. When $\mathcal{L}_{\text{DW}}$ is the canonical Dirichelet boundary condition for $\text{DW}(\mathbb{A})$, this two-step gauging defines a canonical Dirichlet boundary condition $\mathcal{L}_{\mathcal{C}}$ for the Symmetry TFT $\mathcal{Z}(\mathcal{C})$. Since the liberated twist defects Σ_g in $\mathcal{Z}(\mathcal{C})$ are charged under the $\text{Rep}(G)$ symmetry, they are confined to the boundary $\mathcal{L}_{\mathcal{C}}$, which thus describes a system with a non-invertible self-duality symmetry. This construction was implicitly used in [7].

Gauging the non-invertible symmetry $\mathcal{N}_g$ on the boundary, on the contrary, must correspond to a gapped boundary on which the twist defects Σ_g are trivialized. Thus, in order to detect the absence of a self-duality anomaly, we must construct a different set of boundary conditions $\mathfrak{N}_{\mathcal{C}}$ for $\mathcal{Z}(\mathcal{C})$ whose symmetry $\text{Mod}_{\mathfrak{N}_{\mathcal{C}}}\big(\mathcal{Z}(\mathcal{C})\big)$ is trivially charged under $\text{Rep}(G)$. We will refer to these as Neumann boundary conditions, since the G gauge field remains dynamical on the boundary.

The crucial insight comes from considering — when it exists — a G-invariant Lagrangian algebra $\mathcal{L}_D$ in $\text{DW}(\mathbb{A})$. This ensures that gauging $\mathcal{L}_D$ leads, in the bulk, to an SPT phase for G, rather than to a completely trivial theory as in (4.1.9). The SPT phase is completely specified by an element Y living in the appropriate cobordism group.[9] It turns out that the datum Y cannot be fixed by the choice of $\mathcal{L}_D$ alone, but it requires a further piece of data, which we dub $\tilde{\eta}$, describing how the symmetry G acts on the algebra morphisms of $\mathcal{L}_D$. This is called an *equivariantization* of $\mathcal{L}_D$ [183, 337]. We denote the equivariantized algebra by a pair $(\mathcal{L}_D, \tilde{\eta})$. The SPT phase Y also contains a nonempty G-twisted sector with a unique simple object M_g for each $g \in G$. In the 3d setting, these can be formally described as local modules twisted by a G-action, see Appendix B.1. Since $\mathcal{L}_D$ is G-invariant, the operation of gauging G commutes with the condensation of $(\mathcal{L}_D, \tilde{\eta})$, and composing the two operations we end up with a bulk Dijkgraaf-Witten theory for G with twist $Y\epsilon$:



(4.1.10)

We stress that while $\text{DW}(\mathbb{A})$ is an n-form gauge theory, $\text{DW}(G)^{Y\epsilon}$ is always a standard (1-form) gauge theory. Its magnetic operators are the former twist defects M_g. In 3d their spin is

[9] For $d=2$ this is just a bosonic SPT $\in H^3\big(G, U(1)\big)$. For $d=4$ instead we will work on spin manifolds and the correct group to consider is either $\text{Tor}\big(\Omega_5^{\text{spin}_G}(\text{pt})\big)$ or $\text{Tor}\big(\Omega_5^{\text{spin}}(BG)\big)$ depending on whether $(-1)^F$ sits inside the duality group or not, respectively. The same observations apply to the discrete torsion ϵ.

determined by the total twist $Y\epsilon$.[10]

The theory $\mathrm{DW}(G)^{Y\epsilon}$ always admits a canonical (electric) Dirichlet boundary condition, corresponding to gauging $\mathrm{Rep}(G)$, that gives rise to an invertible symmetry G on the boundary (such occurrences have been dubbed *non-intrinsic* in [133]). This also coincides with one of the algebras $\mathcal{L}$ we have previously introduced in (4.1.9), in the special case that $\mathcal{L}_{\mathrm{DW}} = \mathcal{L}_D$ is G-invariant.

However, if $Y\epsilon = 1$, then also the magnetic defects M_g are condensable. This allows us to define another boundary condition, the Neumann boundary condition $\mathcal{N}_\mathcal{C}$ we were looking for:

$$\begin{array}{ccc} \mathrm{DW}(\mathbb{A}) & \xleftarrow{\ \mathrm{Rep}(G)\ } & \mathcal{Z}(\mathcal{C}) \\ \big\downarrow{\scriptstyle (\mathcal{L}_D,\tilde{\eta})} & & \\ \mathrm{SPT}(G)_Y & \xrightarrow{\ G^\epsilon\ } & \mathrm{DW}(G)^{Y\epsilon=1} \\ & & \big\downarrow{\scriptstyle \mathcal{N}_{\mathrm{DW}}} \\ & & n\mathrm{Vec} \end{array} \qquad \mathcal{Z}(\mathcal{C}) \xrightarrow{\ \mathcal{N}_\mathcal{C}\ } n\mathrm{Vec} \tag{4.1.11}$$

Thus, the existence of a duality-invariant Lagrangian algebra $(\mathcal{L}_D, \tilde{\eta})$ in $\mathrm{DW}(\mathbb{A})$ and the triviality of $Y\epsilon$ are sufficient conditions for the self-duality symmetry $\mathcal{C}$ to be anomaly-free.

Let us argue that they are also necessary.[11] Suppose that there exists an algebra $\mathcal{N}_\mathcal{C}$ of $\mathcal{Z}(\mathcal{C})$ containing the liberated twist defects Σ_g, *i.e.*, such that $\mathrm{Hom}(\mathcal{N}_\mathcal{C}, \Sigma_g) \neq \emptyset$. It follows that $\mathcal{N}_\mathcal{C}$ has a natural grading in terms of the charge of its elements under $\mathrm{Rep}(G)$:

$$\mathcal{N}_\mathcal{C} = \bigoplus\nolimits_{g\in G} \mathcal{N}^g\ . \tag{4.1.12}$$

Algebra objects come equipped with a product (or morphism) $\times_{\mathcal{N}_\mathcal{C}}$ (see Appendix B.1, where it is called m) mapping $\mathcal{N}^g \times_{\mathcal{N}_\mathcal{C}} \mathcal{N}^h \to \mathcal{N}^{gh}$ and respecting the grading. The consistency conditions for $\mathcal{N}_\mathcal{C}$ to be a (gaugeable) algebra object are also graded over G, and in particular imply that $\mathcal{N}^0$ must itself be an algebra, although not a maximal (*i.e.*, Lagrangian) one. They also imply that $\mathcal{N}^{g\neq 0}$ are (local) modules over $\mathcal{N}^0$, *i.e.*, they survive the condensation of $\mathcal{N}^0$. Physically this corresponds to the fact that one can gauge $\mathcal{N}_\mathcal{C}$ sequentially. One first condenses just $\mathcal{N}^0$. This preserves the $\mathrm{Rep}(G)$ symmetry, since $\mathcal{N}^0$ has trivial grading, and besides the $\mathcal{N}^g$'s become invertible G defects. Since the $\mathcal{N}^g$'s were part of the algebra $\mathcal{N}_\mathcal{C}$, the symmetry $\mathrm{Rep}(G)$ after the first condensation must be anomaly free.

Assuming that $\mathcal{N}^g$ with $g \in G$ and $\mathrm{Rep}(G)$ are the only defects surviving the condensation of $\mathcal{N}^0$, it follows that the resulting theory is the Dijkgraaf-Witten theory for G with trivial twist, $\mathrm{DW}(G)$. In other words, we can identify the vertical red arrow in (4.1.10) on the right with the condensation of $\mathcal{N}^0$, as in (4.1.13). The assumption can be rigorously proven in 3d,[12] whilst we do not know how to do that in 5d, which is why our argument remains heuristic. Now, further gauging the $\mathrm{Rep}(G)$ symmetry (and stacking with a discrete torsion ϵ^{-1}) leads

[10] In the 5d case, instead, the twist determines a triple linking between magnetic defects [195].

[11] See [86] for another argument, in the case $\mathbb{A} = \mathbb{Z}_n$, in favour of the necessity of a duality-invariant algebra.

[12] The assumption and its consequence can be proven for 3d MTCs. The fact that $\mathcal{N}^g$ are invertible as bimodules and the fact that $\mathcal{N}_\mathcal{C}$ is Lagrangian imply that $\dim(\mathcal{N}^g) = \dim(\mathcal{N}^0) = |\mathbb{A}|$. After condensing $\mathcal{N}^0$, the resulting MTC has dimension $\mathcal{D} = |G|$, which is saturated by the $|G|$ invertible lines $\mathcal{N}^g$ times the $|G|$ elements of $\mathrm{Rep}(G)$. The fact that the $\mathcal{N}^g$'s is charged under $\mathrm{Rep}(G)$ gives the canonical braiding of the DW theory.

us to an SPT phase $Y = \epsilon^{-1}$ for G. Chasing the diagram below shows that we can define a gauging of $\mathrm{DW}(\mathbb{A})$ by sequentially gauging G^ϵ - $\mathcal{N}^0$ - $\mathrm{Rep}(G)$:

$$
\begin{array}{ccc}
\mathrm{DW}(\mathbb{A}) & \xrightarrow{\ G^\epsilon\ } & \mathcal{Z}(\mathcal{C}) \\
\Big\downarrow{\scriptstyle (\mathcal{L}_D,\tilde{\eta})} & & \Big\downarrow{\scriptstyle \mathcal{N}^0} \\
\mathrm{SPT}(G)_{\epsilon^{-1}} & \xleftarrow{\ \mathrm{Rep}(G)\ } & \mathrm{DW}(G)
\end{array}
\tag{4.1.13}
$$

Since it produces an invertible TQFT with an action of G, the so-defined gauging must correspond to (*i.e.*, it must induce) a duality-invariant Lagrangian algebra $(\mathcal{L}_D, \tilde{\eta})$ in $\mathrm{DW}(\mathbb{A})$. Intuitively, one can think of $\mathcal{N}^0$ as the image of $\mathcal{L}_D$ under the gauging of G. We have thus argued that there necessarily is a duality-invariant Lagrangian algebra in $\mathrm{DW}(\mathbb{A})$.

As we stressed, our reasoning is mathematically rigorous only in the case of 3d TFTs, where the concepts above can be explicitly implemented. We however expect the same ideas to apply also to higher categorical settings, once a complete definition of the higher Symmetry TFT is given. We arrive at the following proposal for the (lack of) anomalies of duality defects:

First obstruction. There must exist, in $\mathrm{DW}(\mathbb{A})$, a G-invariant boundary condition $(\mathcal{L}_D, \tilde{\eta})$. This allows to make the symmetry G invertible. In the language of [133], the self-duality symmetry is non-intrinsic. We further explain in Appendix B.6 that this condition is equivalent to the existence of a duality-invariant TQFT [31, 78].[13]

Second obstruction. The invertible self-duality symmetry is anomaly-free. This is equivalent to the vanishing of the total Dijkgraaf-Witten twist, which depends on the equivariantization data $\tilde{\eta}$. In practice, the invertible self-duality symmetry suffers from a mixed anomaly with the symmetry $\mathcal{S}$ on the non-intrinsic boundary which can be computed explicitly. We conjecture (and show in examples) that the equivariantization data encodes how the 0-form symmetry fractionalizes with the $(n-1)$-form symmetry $\mathcal{S}$. Following [338], this can be used to shift the value of the anomaly ϵ, *i.e.*, to change the SPT phase Y in the bulk.

4.1.2 Anomalies of duality symmetries in 1+1 dimensions

This section is devoted to the discussion of anomalies in two-dimensional QFTs whose symmetries are described by Tambara-Yamagami (TY) categories [250]. We indicate such categories as $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ and we review their definition in Section 4.1.2. The results are already known both in the physics and mathematics literature [29, 340] but here we present their derivation from the point of view of the Symmetry TFT which can be generalized to higher dimensions. Physically $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ is the symmetry of a 2d theory which is self-dual under the gauging of an Abelian symmetry $\mathbb{A}$. Examples, especially in the realm of 2d CFTs, are ubiquitous, the most famous one being the realization of Kramers-Wannier duality in the Ising CFT [12, 22]

[13] A similar analysis connecting the two concepts when $\mathbb{A}$ is cyclic has recently appeared in [86]. Our results coincide with theirs when they overlap.

for $\mathbb{A} = \mathbb{Z}_2$. Many other examples come from either considering free theories[14] as described for instance in [31], or other minimal models which can flow to the Ising CFT, such as the $c = 7/10$ tricritical Ising CFT.

As a concrete example of anomalous versus non-anomalous theories, the $\mathrm{TY}(\mathbb{Z}_2)_{\gamma,1}$ symmetry of the Ising CFT is anomalous on non-spin manifolds,[15] while the diagonal $\mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\gamma,1}$ symmetry of $(\text{Ising})^2$ is anomaly-free. The question of which TY categories are anomalous is not a purely academic one, as it can imply strong constraints on duality-preserving RG flows [12]. For instance, the aforementioned tricritical Ising model has an *anomalous* non-invertible symmetry as well as a duality-preserving relevant deformation. As a direct consequence of the anomaly, the resulting RG flows cannot end in a trivially gapped theory. Depending on the sign of the deformation, the theory either flows to the gapless Ising model or to a gapped theory with three degenerate vacua.

Our Symmetry TFT analysis gives a simple characterization of the known obstruction theory in two steps, as already pointed out in Section 4.1.1. The first obstruction is equivalent to the absence of a duality-invariant Lagrangian algebra $\mathcal{L}_D$, which otherwise gives rise to a global variant with invertible symmetry $\mathcal{S} \rtimes_\rho \mathbb{Z}_2$. The second obstruction is instead the standard 't Hooft anomaly for $\mathbb{Z}_2$ subgroups of $\mathcal{S} \rtimes_\rho \mathbb{Z}_2$ in that global variant. When there exists such an anomaly-free $\mathbb{Z}_2$ subgroup, it can be gauged. The combined sequential gauging of $\mathcal{L}_D$ and of $\mathbb{Z}_2$ corresponds, in the original theory, to a gauging that involves the non-invertible symmetry defect.

Algebras in TY categories and anomalies

We start by reviewing the basic properties of Tambara-Yamagami categories $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ [250]. We assume that the reader has some familiarity with the theory of Fusion Categories, for which excellent reviews are [11,12,17,28,183,209]. Given an Abelian group $\mathbb{A}$, the Tambara-Yamagami symmetry is a $\mathbb{Z}_2$-graded fusion category

$$\mathcal{C} = \mathcal{C}_{\underline{0}} \oplus \mathcal{C}_{\underline{1}} \ , \tag{4.1.14}$$

where $\mathcal{C}_{\underline{0}} = \mathrm{Vec}_{\mathbb{A}}$ (the category of $\mathbb{A}$-graded vector spaces) describes an Abelian 0-form symmetry $\mathbb{A}$ with trivial anomaly,[16] while $\mathcal{C}_{\underline{1}}$ has a single simple object $\mathcal{N}$. The fusion rules consistent with the grading are uniquely fixed and read:

$$a \times b = (a+b) \ , \qquad a \times \mathcal{N} = \mathcal{N} \times a = \mathcal{N} \ , \qquad \mathcal{N} \times \mathcal{N} = \bigoplus_{a \in \mathbb{A}} a \ . \tag{4.1.15}$$

Here $a, b \in \mathbb{A}$ are the simple objects in $\mathcal{C}_{\underline{0}}$, and $+$ in the first equation is the binary group operation in $\mathbb{A}$. The category is uniquely determined by a triplet $(\mathbb{A}, \gamma, \epsilon)$ where $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$ is a non-degenerate symmetric bicharacter, whilst $[\epsilon] \in H^3\big(\mathbb{Z}_2, U(1)\big) \cong \mathbb{Z}_2$ is the Frobenius-Schur indicator of the self-dual defect $\mathcal{N}$. This data enters in the associator, or

[14]Examples include the $c = 1$ boson at squared radius $R^2 = 2k$, its $\mathbb{Z}_2$ orbifold, or multiple bosons at special points on the Narain moduli space.

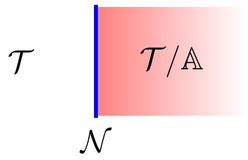
[15]On spin manifolds it can be fermionized to the $(-1)^{F_L}$ symmetry of the Majorana CFT.

[16]Such an anomaly would be represented by a trivial class $[0] \in H^3\big(\mathbb{A}, U(1)\big)$.

F-symbol, of the TY category:

$$\left[F_{\mathcal{N}}^{a,\mathcal{N},b}\right]_{\mathcal{N},\mathcal{N}} = \left[F_{b}^{\mathcal{N},a,\mathcal{N}}\right]_{\mathcal{N},\mathcal{N}} = \gamma(a,b) \ , \qquad \left[F_{\mathcal{N}}^{\mathcal{N},\mathcal{N},\mathcal{N}}\right]_{a,b} = \frac{\epsilon}{\sqrt{|\mathbb{A}|}}\,\gamma(a,b)^{-1} \ , \tag{4.1.16}$$

where $\epsilon = \pm 1$, while all other non-vanishing associators are equal to 1. The bicharacter γ has a nice physical interpretation (see, *e.g.*, [29]). Since a theory $\mathcal{T}$ with TY symmetry is mapped to itself under the gauging of $\mathbb{A}$, symbolically $\mathcal{T}/\mathbb{A} \cong \mathcal{T}$, we can consider constructing the defect $\mathcal{N}$ as a topological domain wall between $\mathcal{T}$ and $\mathcal{T}/\mathbb{A}$:

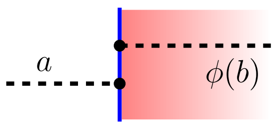


(4.1.17)

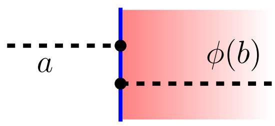
On the two sides of the wall the 0-form symmetries are $\mathbb{A}$ and $\mathbb{A}^\vee$, respectively. To identify them we must specify a group isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$ such that the associated bicharacter γ defined as

$$\gamma(a_1, a_2) = \phi(a_1)\, a_2 \quad \in U(1) \tag{4.1.18}$$

is symmetric.[17] We can then consider lines $a \in \mathbb{A}$ and $\beta \in \mathbb{A}^\vee$ ending on the defect $\mathcal{N}$ from the two sides. The endpoints of these objects are mutually nonlocal, and pick up a canonical phase $\beta(a)$ as they pass through each other. Indeed, from the point of view of the left side, a is a topological symmetry defect for the symmetry $\mathbb{A}$ and the endpoint of β is a charged operator (while vice versa from the point of view of the right side). Using the isomorphism ϕ this is converted into the symmetric bicharacter γ:

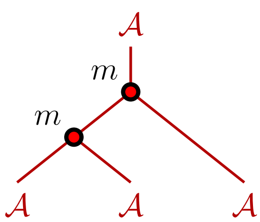


$= \gamma(a,b)$



(4.1.19)

We now review the classification of *bosonic* gaugeable symmetries $\mathcal{A}$ in $\text{TY}(\mathbb{A})_{\gamma,\epsilon}$, which are described by symmetric Frobenius algebras in $\mathcal{C}$ [17]. These are defined by an object $\mathcal{A} \in \mathcal{C}$ together with a choice of three-valent junction $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is strictly associative:



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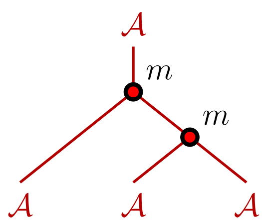

(4.1.20)

This diagram encodes the cancellation of 't Hooft anomalies for the symmetry $\mathcal{A}$. We give a brief review of the relevant concepts in Appendix B.1. The classification problem has been solved in the mathematics literature in [340] and given a physical interpretation from the viewpoint of TQFTs in [29].

[17] The requirement that γ be symmetric is equivalent to the canonical pairing between $\mathbb{A}^\vee$ and $\mathbb{A}$ being invariant under ϕ: $\alpha(a) = \phi(a)\big(\phi^{-1}(\alpha)\big)$ where a and α are elements of $\mathbb{A}$ and $\mathbb{A}^\vee$, respectively. Physically this translates into the fact that the electric and magnetic frames are equivalent due to self-duality.

Such algebras for the TY category can be divided into two types depending on whether $\mathcal{A}$ also contains the element $\mathcal{N}$ or not. If not, the gaugeable algebras correspond to gauging a subgroup $\mathbb{B}$ of $\mathbb{A}$ with discrete torsion $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$. From the latter one constructs its commutator[18]

$$\chi_\nu(b_1, b_2) = \frac{\nu(b_1, b_2)}{\nu(b_2, b_1)} \ . \tag{4.1.21}$$

This defines a map $[\nu] \to \chi_\nu$ from $H^2\big(\mathbb{B}, U(1)\big)$ to the group of alternating bicharacters which turns out to be an isomorphism [251]. The Frobenius algebra corresponding to the pair $\big(\mathbb{B}, [\nu]\big)$ is:

$$\mathcal{A} \equiv \mathcal{A}_{\mathbb{B},\,\nu} = \bigoplus\nolimits_{b\in\mathbb{B}} b \ , \qquad\qquad m_{b,\,b'}^{b+b'} = \nu(b, b') \ . \tag{4.1.22}$$

On the other hand, when including also $\mathcal{N}$ the most general algebra reads:[19]

$$\mathcal{A} \equiv \mathcal{A}_{\underline{0}} \oplus \mathcal{A}_{\underline{1}} = \mathcal{A}_{\mathbb{B},\,\nu} \oplus n_\nu \mathcal{N} \ , \qquad\qquad n_\nu \geq 1 \ . \tag{4.1.23}$$

The choices of $\mathbb{B}$, ν and n_ν for which such an object can be consistently defined on orientable 2-manifolds are severely restricted by the following two obstructions [29, 251, 340].

First obstruction. We introduce the subgroup $N(\mathbb{B}) \subset \mathbb{A}^\vee$ of characters annihilating $\mathbb{B}$:

$$N(\mathbb{B}) = \Big\{ \beta \in \mathbb{A}^\vee \ \Big| \ \beta(b) = 1 \ , \ \forall b \in \mathbb{B} \Big\} \ . \tag{4.1.24}$$

This group is canonically isomorphic to $(\mathbb{A}/\mathbb{B})^\vee$, while the quotient $\mathbb{A}^\vee/N(\mathbb{B})$ is canonically isomorphic to $\mathbb{B}^\vee$. We also define the *radical* $\mathrm{Rad}(\nu) \subset \mathbb{B}$ of the class $[\nu]$:

$$\mathrm{Rad}(\nu) = \Big\{ b \in \mathbb{B} \ \Big| \ \chi_\nu(b, b') = 1 \ , \ \forall b' \in \mathbb{B} \Big\} \ . \tag{4.1.25}$$

The alternating bicharacter χ_ν is non-degenerate on $\mathbb{B}/\mathrm{Rad}(\nu)$.

For the first obstruction to vanish these subgroups must be related as

$$\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B}) \ . \tag{4.1.26}$$

Besides, there must exist an involutive automorphism

$$\sigma : \mathbb{B}/\mathrm{Rad}(\nu) \ \to \ \mathbb{B}/\mathrm{Rad}(\nu) \qquad \text{with} \qquad \sigma^2 = 1 \tag{4.1.27}$$

such that the symmetric and alternating bicharacters, when restricted to $\mathbb{B}$ and then projected to $\mathbb{B}/\mathrm{Rad}(\nu)$, satisfy

$$\gamma\big(\sigma(a), b\big) = \chi_\nu(a, b) \qquad\qquad \text{for} \quad a, b \in \mathbb{B}/\mathrm{Rad}(\nu) \ . \tag{4.1.28}$$

[18] Notice that χ_ν is well defined (it is independent from the choice of representative ν), alternating (meaning that $\chi_\nu(a, a) = 1$) and antisymmetric (meaning that $\chi_\nu(a, b) = \chi_\nu(b, a)^{-1}$). One can prove that $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$ is bilinear (in the multiplicative sense), see for instance [343]. Then alternating implies antisymmetric, while the opposite is false and in fact dropping the alternating property one can describe fermionic Lagrangian algebras, see also after (4.1.58).

[19] Notice that, if we restrict to spin manifolds, there are more candidate algebras because it is possible to gauge discrete symmetries with a nontrivial Arf twist (*i.e.*, a discrete torsion constructed out of the spin connection, see *e.g.* [116, 344]). This difference will become apparent in the Symmetry TFT.

Note that the projections from $\mathbb{B}$ to $\mathbb{B}/\mathrm{Rad}(\nu)$ are well defined. One can prove, using the equation above, that $\nu(a,b)$ and $\nu\big(\sigma(b),\sigma(a)\big)$, when projected to $\mathbb{B}/\mathrm{Rad}(\nu)$, define equivalent cohomology classes in $H^2\big(\mathbb{B}/\mathrm{Rad}(\nu),U(1)\big)$.[20] Thus there exists a 1-cochain $\tilde{\eta} \in H^1\big(\mathbb{B}/\operatorname{Rad}(\nu),U(1)\big)$ such that[21]

$$\frac{\nu(a,b)}{\nu\big(\sigma(b),\sigma(a)\big)} = d\tilde{\eta}(a,b)\ , \qquad \tilde{\eta}(a)\,\tilde{\eta}\big(\sigma(a)\big) = 1\ . \tag{4.1.30}$$

From (4.1.24)–(4.1.26) it easily follows that

$$|\mathbb{A}|\, n_\nu^2 = |\mathbb{B}|^2\ , \tag{4.1.31}$$

where[22] $n_\nu^2 = \big|\mathbb{B}/\operatorname{Rad}(\nu)\big|$. The positive integer n_ν appearing here turns out to be the same as the one in (4.1.23). Notice in particular that a necessary condition to satisfy the first obstruction is that $|\mathbb{A}|$ is a perfect square. Since, as it follows from (4.1.15), the quantum dimension of $\mathcal{N}$ is $|\mathbb{A}|^{\frac{1}{2}}$, this reproduces the known fact that gauging is not possible in presence of non-integer quantum dimensions [12].

The rough idea that leads to these formulas is the following. Decomposing the defining equation of a Frobenius algebra into its graded components, one finds that $\mathcal{A}_{\underline{1}}$ must be an invertible $\mathcal{A}_{\underline{0}}$-bimodule: $\mathcal{A}_{\underline{1}} \times_{\mathcal{A}_{\underline{0}}} \mathcal{A}_{\underline{1}} = \mathbb{1}_{\mathcal{A}_{\underline{0}}}$, where $\times_{\mathcal{A}_{\underline{0}}}$ is the tensor product in the category of $\mathcal{A}_{\underline{0}}$-bimodules [17, 20, 183]. Physically this means that we can gauge $\mathcal{A}_{\underline{0}}$, and then $\mathcal{A}_{\underline{1}}$ will become an invertible $\mathbb{Z}_2$ global symmetry of the gauged theory. Eqns. (4.1.26)–(4.1.28) are necessary in order to endow $\mathcal{A}_{\underline{1}}$ with a bimodule structure, and in particular (4.1.30) ensures that the bimodule is invertible. This furthermore implies that $\dim(\mathcal{A}_{\underline{1}}) = \dim(\mathcal{A}_{\underline{0}})$ which reproduces (4.1.31) in terms of the integer n_ν in (4.1.23).

A beautiful alternative perspective on this condition was given in [29]. Suppose we want to construct a TQFT in which the duality symmetry generated by $\mathcal{N}$ is preserved. As the TQFT must have symmetry $\mathbb{A}$, it can be labelled by a doublet $(\mathbb{B},\nu)$ where $\mathbb{B}$ denotes the preserved (as opposed to spontaneously broken) subgroup while ν is an SPT phase for $\mathbb{B}$. The partition

[20] To prove it, one checks that the 2-cochains $\nu(a,b)$ and $\nu\big(\sigma(b),\sigma(a)\big)$ produce the same bicharacter χ_ν in (4.1.21) and so, by isomorphism, must be different representatives of the same cohomology class.

[21] It is always possible to choose $\tilde{\eta}$ such that it satisfies both relations. Consider the case $\mathbb{B} = \mathbb{A}$ and define the two subgroups $\mathbb{A}^\sigma = \{a \in \mathbb{A} \,|\, a = \sigma(a)\}$ as well as $\mathbb{A}_\sigma = \{a + \sigma(a) \,|\, a \in \mathbb{A}\}$, clearly $\mathbb{A}_\sigma \subset \mathbb{A}^\sigma \subset \mathbb{A}$. One checks (see [251]) that γ can be consistently reduced to a bicharacter $\bar{\gamma}$ on the quotient $\mathbb{A}^\sigma/\mathbb{A}_\sigma$: $\gamma\big(a + b + \sigma(b), a' + b' + \sigma(b')\big) = \gamma(a,a')$ for any $a, a' \in \mathbb{A}^\sigma$. Let μ^{-1} be a quadratic refinement of $\bar{\gamma}$ (see Sec. 4.1.2). Now, take the first equation and restrict it to $\mathbb{A}^\sigma$:

$$a,b \in \mathbb{A}^\sigma: \quad d\tilde{\eta}(a,b) = \frac{\nu(a,b)}{\nu(b,a)} = \chi_\nu(a,b) = \gamma(a,b) = \bar{\gamma}\big(\pi(a),\pi(b)\big) = d\mu\big(\pi(a),\pi(b)\big) \tag{4.1.29}$$

using (4.1.28), and π is the projection $\mathbb{A}^\sigma \xrightarrow{\pi} \mathbb{A}^\sigma/\mathbb{A}_\sigma$. It follows that $\tilde{\eta}\big|_{\mathbb{A}^\sigma} = \xi \cdot \pi^*\mu$ for some $\xi \in H^1\big(\mathbb{A}^\sigma, U(1)\big) \cong (\mathbb{A}^\sigma)^\vee$. Since the map $\mathbb{A}^\vee \to (\mathbb{A}^\sigma)^\vee$ given by restriction is surjective, it is always possible to find another solution $\tilde{\eta}' = \zeta \cdot \tilde{\eta}$ where $\zeta \in \mathbb{A}^\vee$ is a character such that $\zeta\big|_{\mathbb{A}^\sigma} = \xi^{-1}$ and thus $\tilde{\eta}'\big|_{\mathbb{A}^\sigma} = \pi^*\mu$. This implies $\tilde{\eta}'\big|_{\mathbb{A}_\sigma} = \tilde{\eta}'\big(a + \sigma(a)\big) = 1$ for all $a \in \mathbb{A}$. Using $d\tilde{\eta}\big(a,\sigma(a)\big) = 1$ from the first equation in (4.1.30), the second equation follows. The general case for $\mathbb{B} \subset \mathbb{A}$ is a straightforward generalization.

[22] Since χ_ν is a non-degenerate alternating bicharacter on $\mathbb{B}/\operatorname{Rad}(\nu)$ with values in $U(1)$, there exists an isotropic subgroup $\mathbb{G}$ such that $\mathbb{B}/\operatorname{Rad}(\nu) = \mathbb{G} \times \mathbb{G}^\vee$ and in particular $\big|\mathbb{B}/\operatorname{Rad}(\nu)\big| = |\mathbb{G}|^2 = n_\nu^2$ is a perfect square, where $n_\nu = |\mathbb{G}|$. See, *e.g.*, Lemma 5.2 in [345].

function is

$$Z[B] = \begin{cases} \exp\big(2\pi i \int B^*\nu\big) & \text{if } \pi(B) = 0 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4.1.32}$$

Here B is a background field coupled to $\mathbb{A}$, whilst π is the projection map of the short exact sequence $1 \to \mathbb{B} \xrightarrow{i} \mathbb{A} \xrightarrow{\pi} \mathbb{A}/\mathbb{B} \to 1$. The class $B^*\nu \in H^2\big(X_2, U(1)\big)$ (using now additive notation) is integrated over the spacetime 2-manifold X_2.[23] Imposing that the duality symmetry be unbroken means that

$$\mathcal{N} \cdot Z[B] \equiv \frac{1}{\sqrt{|H^1(X,\mathbb{A})|}} \sum_{a \in H^1(X,\mathbb{A})} \exp\left(2\pi i \int_X a \cup \phi(B)\right) Z[a] \stackrel{!}{=} Z[B] \ , \tag{4.1.33}$$

where we used the symmetric bicharacter to identify $\mathbb{A}^\vee$ with $\mathbb{A}$. It can be checked that (4.1.33) reproduces exactly the first obstruction condition. We report the detailed manipulation and its extension to the four-dimensional case in Appendix B.6.

Second obstruction. Having discussed the structure of $\mathcal{A}_{\underline{1}}$, we can simply gauge $\mathcal{A}$ sequentially by first gauging $\mathcal{A}_{\underline{0}}$ and then $\mathcal{A}_{\underline{1}}$. After the first step, $\mathcal{A}_{\underline{0}}$ becomes the identity defect while $\mathcal{A}_{\underline{1}}$ becomes an invertible $\mathbb{Z}_2$ symmetry: $\mathcal{A}_{\underline{1}}^2 = 1$. In order to be able to gauge the full algebra, it must happen that $\mathcal{A}_{\underline{1}}$ has a trivial self-anomaly ϵ_{tot}. This comes in two parts: a "bare" contribution from the original Frobenius-Schur indicator ϵ of the duality defect, and a further contribution Y coming from the bimodule morphism $\mathcal{A}_{\underline{1}} \times \mathcal{A}_{\underline{1}} \to \mathbb{1}$. The latter turns out to be given by the Arf invariant of $\tilde{\eta}$ restricted to the elements of $\mathbb{B}/\text{Rad}(\nu)$ invariant under the involution σ:

$$Y = \text{sign}\left(\sum_{\substack{b \in \mathbb{B}/\text{Rad}(\nu) \\ \sigma(b) = b}} \tilde{\eta}(b) \right) = \text{Arf}(\tilde{\eta}) \ . \tag{4.1.34}$$

We stress that here we are using multiplicative notation for $\tilde{\eta}$, so that Y is the sign of a sum of phases (alternatively, in (4.1.113) we indicate the correct normalization). The second obstruction then vanishes if and only if

$$\epsilon_{\text{tot}} = \epsilon\, Y = 1 \ . \tag{4.1.35}$$

Later on, around eqn. (4.1.140), we will find an alternative formula for the spectrum of values that Y can take as we explore the possible consistent choices of $\tilde{\eta}$ — the so-called fractionalization classes.

A note on quadratic refinements

At various points in this work we use the existence and properties of quadratic refinements.

[23]The class $B^*\nu$ can be thought of in two ways. Abstractly, B is a homotopy class of maps from X_2 to the classifying space $B\mathbb{A}$, while ν is a 2-form in $H^2\big(B\mathbb{A}, U(1)\big)$, so that the pull back $B^*\nu$ is a 2-form on X_2 that can be integrated. More concretely, $B \in H^1(X_2, \mathbb{A})$ is a 1-cochain in simplicial cohomology that to each edge (ij) of a triangulation of X_2 associates an element of $\mathbb{A}$, while $\nu : \mathbb{A} \times \mathbb{A} \to U(1)$ is an element of group cohomology $H^2\big(\mathbb{A}, U(1)\big)$, so that the 2-cochain $(B^*\nu)_{ijk} = \nu(b_{ij}, b_{jk}) \in H^2\big(X_2, U(1)\big)$ associates to each face (ijk) a value in $U(1)$ and can be integrated.

A function $q : \mathbb{A} \to U(1)$ (with $\mathbb{A}$ a finite Abelian group) is called a quadratic function if $q(a) = q(-a)$ and (using multiplicative notation)

$$\zeta(a,b) \equiv \frac{q(a+b)}{q(a)\,q(b)} \tag{4.1.36}$$

is a symmetric bicharacter. One easily derives that $q(0) = 1$, $q(ta) = q(a)^{t^2}$ for any $t \in \mathbb{Z}$, and

$$\zeta(a,a) = q(a)^2 \,. \tag{4.1.37}$$

Any quadratic function q, by definition, comes equipped with an associated symmetric bicharacter ζ as in (4.1.36). However also the converse is true: any symmetric bicharacter ζ arises from a (not necessarily unique) quadratic function q, which is called a quadratic refinement of ζ. The set of quadratic refinements forms a torsor over $\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2)$, indeed one easily proves that the ratio of two quadratic refinements is a $\mathbb{Z}_2$-valued character on $\mathbb{A}$.[24]

A closely related statement is that any symmetric 2-cocycle $\nu \in Z^2\big(\mathbb{A}, U(1)\big)$ is exact. This follows from the isomorphism between altenating bicharacters χ_ν (4.1.21) and classes $[\nu] \in H^2\big(\mathbb{A}, U(1)\big)$. In the special case that the symmetric 2-cocycle $\nu(a,b)$ is bilinear, exactness is equivalent to the existence of a quadratic refinement.

Symmetry TFT description

It is possible to reformulate the properties of Tambara-Yamagami categories $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ in terms of their 3d Symmetry TFT, *i.e.*, using the language of modular tensor categories (MTCs).[25] Let us review this fact, that will be useful in order to discuss anomalies in the following sections. In particular let us describe how the data $(\mathbb{A}, \gamma, \epsilon)$ appears from the bulk viewpoint.

One starts from a pure 3d gauge theory for $\mathbb{A}$ (*i.e.*, a Dijkgraaf-Witten theory for $\mathbb{A}$ with no torsion), which is the Symmetry TFT describing the invertible symmetry $\mathrm{Vec}_{\mathbb{A}}$.[26] The spectrum of lines of the $\mathbb{A}$ gauge theory is $\mathbb{A} \times \mathbb{A}^\vee$, the lines being labelled by pairs $(a, \alpha) \in \mathbb{A} \times \mathbb{A}^\vee$. All the F-symbols are trivial while the braiding is canonically determined by the pairing between $\mathbb{A}$ and $\mathbb{A}^\vee$:

$$\mathcal{B}_{(a_1,\alpha_1),(a_2,\alpha_2)} = \alpha_1(a_2)\,\alpha_2(a_1) \,. \tag{4.1.38}$$

It follows that the topological spins are

$$\theta_{(a,\alpha)} = \alpha(a) \,. \tag{4.1.39}$$

Crucially, the theory enjoys *electric-magnetic* (EM) duality due to $\mathbb{A}$ and $\mathbb{A}^\vee$ being isomorphic. More precisely, the choice of an isomorphism ϕ naturally induces an automorphism of the

[24] The set of quadratic functions $q : \mathbb{A} \to U(1)$ is an extension of the group of symmetric bicharacters $\zeta : \mathbb{A} \times \mathbb{A} \to U(1)$ by $\mathrm{Hom}(\mathbb{A}, \mathbb{Z}_2)$. For each bicharacter, a quadratic function is easily constructed. For $\mathbb{A} = \mathbb{Z}_n$ the bicharacters are $\zeta_r(a,b) = \exp\big(\frac{2\pi i r}{n} ab\big)$ with $r \in \mathbb{Z}_n$. Given one of them, a quadratic refinement is $q_r(a) = \exp\big(\frac{\pi i r(n+1)}{n} a^2\big)$. If n is odd then $r \in \mathbb{Z}_n$ and the quadratic function is unique. If n is even then $r \in \mathbb{Z}_{2n}$ and the quadratic functions for r and $r+n$ produce the same bicharacter $\mathbb{Z}_r$. The case that $\mathbb{A}$ is a product of cyclic factors is similar.

[25] Given a fusion category $\mathcal{C}$ as the symmetry of some 2d theory, the corresponding 3d Symmetry TFT is given via the Turaev-Viro construction [120] by the TQFT whose MTC is the *Drinfeld center* of $\mathcal{C}$ denoted $\mathcal{Z}(\mathcal{C})$.

[26] Mathematically this corresponds to the fact that the Drinfeld center of $\mathrm{Vec}_{\mathbb{A}}$ is $\mathbb{A} \times \mathbb{A}^\vee$.

Drinfeld center

$$\begin{array}{rccc} \Phi: & \mathbb{A}\times\mathbb{A}^\vee & \to & \mathbb{A}\times\mathbb{A}^\vee \\ & (a,\alpha) & \mapsto & \big(\phi^{-1}(\alpha),\phi(a)\big) \ . \end{array} \tag{4.1.40}$$

However not all choices of isomorphism are consistent EM dualities since Φ needs to preserve the braiding. This condition is equivalent to the bicharacter $\gamma(a,b)=\phi(a)\,b$ associated with ϕ being symmetric. Note that Φ squares to 1, so that the duality group is $G\cong\mathbb{Z}_2$.

If the boundary theory is self-dual under gauging, we can construct the full Symmetry TFT that includes the duality defect by gauging the duality symmetry G [133]. The gauging operation comes with a choice of discrete torsion $\epsilon\in H^3\big(G,U(1)\big)\cong\mathbb{Z}_2$ which translates to the Frobenius-Schur indicator of the duality defect $\mathcal{N}$ on the boundary. To summarize, the data $(\mathbb{A},\gamma,\epsilon)$ of the boundary Tambara-Yamagami category appears from the bulk viewpoint as the choice of a duality symmetry $G\cong\mathbb{Z}_2$ of the $\mathbb{A}$ Dijkgraaf-Witten theory and of a discrete torsion for the gauging.

To properly discuss the gauged theory, we first describe the data of the 3d Dijkgraaf-Witten theory enriched by the 0-form symmetry (a G-crossed category in the language of [28]). This includes data describing the topological twist defects for the $G\cong\mathbb{Z}_2$ symmetry. The full tensor category is graded:

$$\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\mathbb{Z}_2}=\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})\oplus\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\Phi}\ , \tag{4.1.41}$$

where $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\Phi}$ describes the twisted sector for the $\mathbb{Z}_2$ symmetry. The number of simple components of $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\Phi}$ is the same as the number of Φ-invariant anyons [28]. The latter are all of the form $\big(a,\phi(a)\big)$ with $a\in\mathbb{A}$. Thus there are $|\mathbb{A}|$ simple objects in $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\Phi}$ which we denote as σ_a, $a\in\mathbb{A}$ (not to be confused with the involution σ). The fusion and braiding data for the $\mathbb{Z}_2$ extension has been computed in [339], although we use here a slightly different notation similar to the one employed in [133]. We find

$$\begin{gathered} (a,\alpha)\times(b,\beta)=(a+b,\,\alpha+\beta)\ , \qquad\qquad (a,\alpha)\times\sigma_b=\sigma_{b+a+\phi^{-1}(\alpha)} \\ \sigma_a\times\sigma_b=\bigoplus_{c\in\mathbb{A}}\big(a+b+c,\,\phi(-c)\big)\ . \end{gathered} \tag{4.1.42}$$

These fusion rules are derived by realizing the G symmetry defects as 2d condensates [13] of the anti-diagonal lines $\big(a,\phi(-a)\big)$ (see *e.g.* [133] for the case $\mathbb{A}=\mathbb{Z}_n$).[27] Since the quantum dimension of (a,α) is 1, we also have

$$\dim(\sigma_a)=\sqrt{|\mathbb{A}|}\ . \tag{4.1.43}$$

The non-vanishing R-matrices, in a gauge, are given by [339]:

$$\begin{gathered} R^{(a_1+a_2,\,\alpha_1+\alpha_2)}_{(a_1,\alpha_1),\,(a_2,\alpha_2)}=\alpha_2(a_1)\ , \qquad R^{\sigma_{a_1+a_2+\phi^{-1}(\alpha_1)}}_{(a_1,\alpha_1),\,\sigma_{a_2}}=f_{a_2}(a_1)\ , \\ R^{\sigma_{a_1+a_2+\phi^{-1}(\alpha_2)}}_{\sigma_{a_1},\,(a_2,\alpha_2)}=1\ , \qquad R^{(a_3,\alpha_3)}_{\sigma_{a_1},\,\sigma_{a_2}}=f_{a_1}(-a_3)^{-1}\ . \end{gathered} \tag{4.1.44}$$

In the last entry, (a_3,α_3) must be a fusion channel of $\sigma_{a_1}\times\sigma_{a_2}$. Besides, $f_a:\mathbb{A}\to U(1)$ is a collection (for $a\in\mathbb{A}$) of functions given by $f_a=\phi(a)\cdot f_0$, or more explicitly $f_a(b)=\gamma(a,b)\,f_0(b)$,

[27] Indeed, the anti-diagonal lines are absorbed by the σ_b's, and $\sigma_b\times\sigma_{-b}$ is a 1d condensate of anti-diagonal lines.

Object	Definition	Dim	# of Objects	Spin θ
$L_{(a,x)}$	$\eta^x \times \big(a,\, \phi(a)\big)$	1	$2\,\lvert\mathbb{A}\rvert$	$\gamma(a,a)$
$X_{(a,b)}$	$\big(a,\, \phi(b)\big) \oplus \big(b,\, \phi(a)\big)$	2	$\lvert\mathbb{A}\rvert\,\big(\lvert\mathbb{A}\rvert-1\big)/2$	$\gamma(a,b)$
$\Sigma_{(a,x)}$	$\eta^x \times \sigma_a$	$\sqrt{\lvert\mathbb{A}\rvert}$	$2\lvert\mathbb{A}\rvert$	$(-1)^x \sqrt{\frac{\epsilon}{\lvert\mathbb{A}\rvert^{1/2}} \sum_{b\in\mathbb{A}} f_a(b)^{-1}}$

Table 4.1: Objects (lines) of the 3d Symmetry TFT $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$.

required to satisfy

$$f_a(b)\; f_a(b') = \gamma(b,b')\; f_a(b+b')\;. \tag{4.1.45}$$

Notice that the equations for different values of a are all equivalent. In these equations, distinct choices for f_0 differ by an $\mathbb{A}$-character and only reshuffle the f_a's, therefore the set of f_a's forms a torsor over $\mathbb{A}^\vee$. However f_0 should be chosen such that $f_0(b) = f_0(-b)$, in other words f_0 is a quadratic refinement of γ^{-1}, which alwasys exists (see Section 4.1.2). Possible different choices are related by $\mathrm{Hom}(\mathbb{A},\mathbb{Z}_2)$ and correspond to different gauge choices. The (gauge dependent) spins of the twisted sector lines are [339]:

$$\theta(\sigma_a) = \sqrt{\frac{1}{\lvert\mathbb{A}\rvert^{1/2}} \sum_{b\in\mathbb{A}}\; f_a(b)^{-1}}\;, \tag{4.1.46}$$

where the choice of sign for the square root is gauge.

We can now discuss the gauging of the symmetry $G \cong \mathbb{Z}_2$ with a twist $\epsilon \in H^3\big(G, U(1)\big) \cong \mathbb{Z}_2$. The gauged theory $\mathcal{Z}(\mathrm{Vec}_{\mathbb{A}})_{\mathbb{Z}_2}/\mathbb{Z}_2$ is isomorphic to $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$ and is graded by the quantum $\mathbb{Z}_2$ 1-form symmetry whose charged objects are the liberated twisted sectors σ_a. There are three types of objects, whose properties are summarized in Table 4.1. In the first line, $L_{(a,x)}$ arise from the Φ-invariant elements $\big(a, \phi(a)\big)$ in the ungauged theory. The label $x \in \{\underline{0}, \underline{1}\}$ specifies the dressing by the $\mathbb{Z}_2$ line $\eta \equiv L_{(0,\underline{1})}$ generating the dual 1-form symmetry $\mathrm{Rep}(\mathbb{Z}_2)$. The lines $X_{(a,b)}$ with $a \neq b$ arise from long orbits of generic invertible objects and absorb the $\mathbb{Z}_2$ line η. Finally, $\Sigma_{(a,x)}$ are the liberated twisted sectors, which are the charged objects under the dual $\mathrm{Rep}(\mathbb{Z}_2)$ symmetry and thus span the non-trivially graded component. The total dimension of the category is

$$\dim\Big(\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)\Big) = \left(\sum_{\ell\ \mathrm{simple}} \dim(\ell)^2\right)^{1/2} = 2\,\lvert\mathbb{A}\rvert\;. \tag{4.1.47}$$

The topological manipulations of the theory with TY symmetry correspond to Lagrangian algebras of this Symmetry TFT. By definition of Drinfeld center, there should exist a Lagrangian algebra corresponding to the global variant with full TY symmetry. As an object, this is given by

$$\mathcal{L}_{\mathrm{TY}} = \mathbb{1} \oplus \eta \oplus \bigoplus_{b\neq 0} X_{(0,b)}\;, \tag{4.1.48}$$

and indeed:

$$\dim\big(\mathcal{L}_{\mathrm{TY}}\big) = 2\,|\mathbb{A}| = \dim\Big(\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)\Big)\ . \tag{4.1.49}$$

This is the algebra induced by the electric Lagrangian subgroup $\mathcal{L}_{\mathrm{e}} = \bigoplus_{\alpha\,\in\,\mathbb{A}^\vee}(0,\alpha)$ in the pure $\mathbb{A}$ gauge theory, following our discussion in Section 4.1.1. While $\mathcal{L}_{\mathrm{e}}$ is clearly not duality invariant, it can be uplifted to an algebra in $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$ by adding to it its images under Φ.[28] The resulting object is well defined in $\mathcal{Z}\big(\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}\big)$, it has vanishing spin (see Table 4.1) and has dimension $2\,|\mathbb{A}|$ so it is Lagrangian. This provides an explicit realization of the sequential gauging procedure outlined in (4.1.9). The symmetry on the corresponding boundary can be computed using the sequential gauging prescription. In the trivially-graded sector $\mathcal{C}_0$ the simple objects are the elements of the quotient $(\mathbb{A}\times\mathbb{A}^\vee)/\mathbb{A}^\vee \simeq \mathbb{A}$. They generate the 0-form symmetry and we label them simply by a. On the other hand, in the $\mathcal{C}_\Phi$ sector all of the twist defects fall into a single orbit, without fixed points under fusion with $\mathcal{L}_{\mathrm{e}}$ as can be checked from (4.1.42). Let us denote this object by $\mathcal{N}$. The bulk fusion rules imply (4.1.15), giving back the $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ symmetry.

First obstruction and Lagrangian algebras

Our first goal is to describe how the first obstruction appears from the Symmetry TFT perspective. We have already mentioned in Section 4.1.1 that the first obstruction precludes the existence of a discrete gauging $(\mathbb{B},\nu)$ which renders the duality symmetry $\mathcal{N}$ invertible. Since, from the Symmetry TFT perspective, discrete gauging operations correspond to different choices of gapped boundary condition $\mathcal{L}$, it is natural to rephrase the first obstruction in the language of Lagrangian algebras of the DW theory. A similar logic has been followed recently in [99], where the obstructions to gauge the entire symmetry category (*i.e.* the case $\mathbb{B}=\mathbb{A}$ in our notation) when $|\mathbb{A}|$ is odd have been found counting the number of bulk lines with trivial spin. However such method is hard to generalize to higher dimensions (which is the main aim in our work) since it relies on the notion of topological spin which has no known analog in higher categories. In this and the next two sections, instead, we provide a complete bulk classification of the obstruction theory for $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ and besides we develop methods that allow us to extend the results to higher-dimensional cases.

The crucial point which makes this problem accessible is that the Symmetry TFT for the TY category is a $G\cong\mathbb{Z}_2$ gauging of the Dijkgraaf-Witten theory $\mathrm{DW}(\mathbb{A})$ [133]. By gauging G back and forth, we can rephrase the problem in terms of gauging Lagrangian algebras of a bulk theory that only consists of invertible symmetries. As already argued in Section 4.1.1, a sufficient condition for $\mathcal{N}$ to be anomalous is the absence of G-invariant Lagrangian algebras in $\mathrm{DW}(\mathbb{A})$, namely, of Lagrangian algebras $\mathcal{L}_D$ satisfying

$$\Phi(\mathcal{L}_D) = \mathcal{L}_D\ . \tag{4.1.50}$$

A duality-invariant Lagrangian algebra of $\mathrm{DW}(\mathbb{A})$ also gives rise to a duality-invariant boundary condition, where the duality symmetry becomes invertible. Hence we realize that, in the terminology of [133], *intrinsic non-invertible symmetries are anomalous.*

[28] If a line $\big(a,\,\phi(a)\big)$ (like the identity in this case) is duality invariant, we must add $L_{(a,\underline{0})}\oplus L_{(a,\underline{1})}$ to the algebra.

Notice that the obstruction we are discussing here is a priori distinct from the first obstruction we discussed in Section 4.1.2. However the main result of this section is to show that the two obstructions are equivalent. In order to do so, we make the first obstruction more explicit by classifying all Lagrangian algebras of DW($\mathbb{A}$) and providing explicit equivalent conditions for their duality invariance in terms of the data $(\mathbb{A}, \gamma)$.

The 3d theory DW($\mathbb{A}$) can be thought of as the Symmetry TFT of any theory with a non-anomalous 0-form symmetry $\mathbb{A}$, and as such the correspondence between topological manipulations and bulk Lagrangian algebras is particularly explicit, but yet non-trivial. The (bosonic) topological manipulations of the boundary are determined by two pieces of data [116]:

- The choice of a subgroup $\mathbb{B} \subset \mathbb{A}$ to be gauged.
- The choice of a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$ which plays the role of the discrete torsion.

The resulting symmetry after gauging is an extension of $\mathbb{A}/\mathbb{B}$ by the quantum symmetry $\mathbb{B}^\vee$ [346] (see Appendix B.2 for details).

On the other hand, global variants of the boundary theory correspond to different interfaces between DW($\mathbb{A}$) and the trivial 3d theory (*i.e.*, to gapped boundaries), and thus are specified by gauging a subgroup $\mathcal{L} \subset \mathbb{A} \times \mathbb{A}^\vee$, Lagrangian with respect to the braiding. Correspondingly, the lines of $\mathcal{L}$ can end on the boundary, and the topological lines of the boundary theory generating the 0-form symmetry are labelled by the quotient group

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L} \; . \tag{4.1.51}$$

Thus we expect a correspondence between pairs $\big(\mathbb{B}, [\nu]\big)$ and Lagrangian algebras $\mathcal{L}$ such that (4.1.51) coincides with the symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$. Notice that the braiding induces a canonical isomorphism[29]

$$\mathcal{L} \cong \mathcal{S}^\vee \; . \tag{4.1.52}$$

The simplest case is when $H^2\big(\mathbb{A}, U(1)\big) = 0$ (*e.g.*, if $\mathbb{A} = \mathbb{Z}_n$) so that the topological manipulations are simply labelled by the gauged subgroup $\mathbb{B} \subset \mathbb{A}$.[30] Then we consider

$$\mathcal{L}_\mathbb{B} \equiv \mathbb{B} \times N(\mathbb{B}) \subset \mathbb{A} \times \mathbb{A}^\vee \tag{4.1.53}$$

which has cardinality $|\mathbb{A}|$ and is made of lines of vanishing spin (in particular it trivializes the braiding, see (4.1.24)), hence it is Lagrangian. Moreover

$$\mathcal{S}_\mathbb{B} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_\mathbb{B} = (\mathbb{A}/\mathbb{B}) \times \mathbb{B}^\vee \tag{4.1.54}$$

is precisely the symmetry after gauging $\mathbb{B}$.

[29]This can be seen as follows. The braiding is a bilinear non-degenerate pairing on $\mathbb{A} \times \mathbb{A}^\vee$ and thus induces an isomorphism $\mathbb{A} \times \mathbb{A}^\vee \to (\mathbb{A} \times \mathbb{A}^\vee)^\vee$. Saying that $\mathcal{L}$ is Lagrangian is equivalent to saying that its image under this isomorphism is the subgroup of linear functions on $\mathbb{A} \times \mathbb{A}^\vee$ which vanish over $\mathcal{L}$. The latter is canonically isomorphic to the Pontryagin dual of $(\mathbb{A} \times \mathbb{A}^\vee)/\mathcal{L} = \mathcal{S}$.

[30]Indeed if $H^2\big(\mathbb{A}, U(1)\big) = 0$ then $H^2\big(\mathbb{B}, U(1)\big) = 0$ for every subgroup $\mathbb{B}$ of $\mathbb{A}$.

In the general case we define the linear map $\psi_\nu : \mathbb{B} \to \mathbb{B}^\vee$ associated to χ_ν:

$$\psi_\nu(b_1)\, b_2 = \chi_\nu(b_1, b_2) \; . \tag{4.1.55}$$

Given the pair $\big(\mathbb{B}, [\nu]\big)$ we construct the subgroup $\mathcal{L}_{\mathbb{B},[\nu]} \subset \mathbb{A} \times \mathbb{A}^\vee$ as follows. Since $\mathbb{B}^\vee = \mathbb{A}^\vee / N(\mathbb{B})$, any element of $\mathbb{A}^\vee$ can be presented as a pair $(\beta, \eta) \in N(\mathbb{B}) \times \mathbb{B}^\vee$ (even though the sum is different from the one in $\mathbb{A}^\vee$) and we denote this element simply as $\beta\eta \in \mathbb{A}^\vee$. The association is not canonical, however different choices agree on η (which is the projection from $\mathbb{A}^\vee$ to $\mathbb{B}^\vee$) while may differ on β. We denote by $\tilde{c} \in H^2\big(\mathbb{B}^\vee, N(\mathbb{B})\big)$ the cocycle which makes $\mathbb{A}^\vee$ an extension of $\mathbb{B}^\vee$ by $N(\mathbb{B})$. Then we construct

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b,\, \beta\psi_\nu(b)\big) \in \mathbb{A} \times \mathbb{A}^\vee \;\Big|\; b \in \mathbb{B}, \quad \beta \in N(\mathbb{B}) \Big\} \; . \tag{4.1.56}$$

This contains $N(\mathbb{B})$ as a subgroup (for $b = 0$), while its quotient by $N(\mathbb{B})$ is isomorphic to $\mathbb{B}$, hence $\mathcal{L}_{\mathbb{B},[\nu]}$ is a group extension

$$1 \;\to\; N(\mathbb{B}) \;\to\; \mathcal{L}_{\mathbb{B},[\nu]} \;\to\; \mathbb{B} \;\to\; 1 \tag{4.1.57}$$

whose corresponding cocycle is $\psi_\nu^*(\tilde{c}) \in H^2\big(\mathbb{B}, N(\mathbb{B})\big)$ (see Appendix B.2 for details). Moreover $\mathcal{L}_{\mathbb{B},[\nu]}$ has cardinality $|\mathbb{A}|$, and since χ_ν is alternating the spin of the lines is trivial:

$$\theta_{(b,\,\beta)} = \chi_\nu(b, b) = 1 \; . \tag{4.1.58}$$

Here (b, β) is a shorthand for $\big(b,\, \beta\psi_\nu(b)\big)$, and β does not contribute because it belongs to $N(\mathbb{B})$. One could weaken the alternating condition and just ask χ_ν to be antisymmetric. In that case the spins would be ± 1 and one would allow for fermionic Lagrangian algebras, which correspond to fermionizations of the boundary symmetry. We will not discuss such cases here, but note that they are a natural candidate to explain why certain duality symmetries — such as $\mathrm{TY}(\mathbb{Z}_2)_{\gamma,1}$ — can be gauged on spin manifolds.

We have thus shown that $\mathcal{L}_{\mathbb{B},[\nu]}$ is a Lagrangian algebra with respect to the braiding. In Appendix B.2 we prove that any Lagrangian algebra of $\mathbb{A} \times \mathbb{A}^\vee$ arises in this way. This classification of boundary conditions of the Dijkgraaf-Witten theory coincides with previously known results from category theory [347]. The boundary condition corresponding to $\mathcal{L}_{\mathbb{B},[\nu]}$ is obtained from the original one by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. Indeed the symmetry on that boundary is

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big) / \mathcal{L}_{\mathbb{B},[\nu]} \;\cong\; \big(\mathcal{L}_{\mathbb{B},[\nu]}\big)^\vee \; , \tag{4.1.59}$$

which is the group extension dual to (4.1.57), namely

$$1 \;\to\; \mathbb{B}^\vee \;\to\; \mathcal{S} \;\to\; \mathbb{A}/\mathbb{B} \;\to\; 1 \; . \tag{4.1.60}$$

The cocycle is $\psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$, where $c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B})$ determines $\mathbb{A}$ as an extension of $\mathbb{A}/\mathbb{B}$ by $\mathbb{B}$. One can show that this is indeed the symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$ (see Appendix B.2 for the proof).

We should now determine whether $\mathrm{DW}(\mathbb{A})$ admits duality-invariant Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$. In the simplest case that $[\nu] = 0$ and hence $\mathcal{L}_{\mathbb{B}} = \mathbb{B} \times N(\mathbb{B})$, duality invariance is simply equivalent to $\phi(\mathbb{B}) = N(\mathbb{B})$. Since $\big|N(\mathbb{B})\big| = |\mathbb{A}|/|\mathbb{B}|$ this requires $|\mathbb{B}|^2 = |\mathbb{A}|$ and in particular the cardinality of $\mathbb{A}$ must be a perfect square ($n_\nu = 1$ in (4.1.23) in this case). However this is in general not sufficient: $\phi(b) \in \mathbb{A}^\vee$ must vanish on $\mathbb{B}$, so that $\mathbb{B}$ must be a Lagrangian subgroup of $\mathbb{A}$ with respect to the symmetric bicharacter γ associated with ϕ.

In the cases with discrete torsion, we observe that

$$\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big) = \Big\{ \Big(\phi^{-1}\big(\beta\psi_\nu(b)\big), \, \phi(b)\Big) \in \mathbb{A}\times\mathbb{A}^\vee \,\Big|\, b \in \mathbb{B}, \quad \beta \in N(\mathbb{B}) \Big\} \tag{4.1.61}$$

is equal to $\mathcal{L}_{\mathbb{B},[\nu]}$ if and only if for all $b \in \mathbb{B}$ and $\beta \in N(\mathbb{B})$ there exist $b' \in \mathbb{B}$ and $\beta' \in N(\mathbb{B})$ such that

$$b' = \phi^{-1}\big(\beta\psi_\nu(b)\big) \,, \qquad\qquad b = \phi^{-1}\big(\beta'\psi_\nu(b')\big) \,. \tag{4.1.62}$$

Before stating the general condition under which these equations can be solved, consider the simpler case $\mathbb{B} = \mathbb{A}$ for which $N(\mathbb{B}) = 0$. Define the group homomorphism $\sigma = \phi^{-1}\circ\psi_\nu : \mathbb{A} \to \mathbb{A}$ in terms of which the two conditions become $b' = \sigma(b)$, $b = \sigma(b')$. They have a solution if and only if $\sigma^2 = 1$. In particular both σ and ψ_ν must be automorphisms.

When $\mathbb{B} \subsetneq \mathbb{A}$ is a proper subgroup, there are further conditions for duality invariance. The proof is technical and we report it in Appendix B.3.1. Let us remind that the *radical* of the class $[\nu]$ is

$$\mathrm{Rad}(\nu) = \mathrm{Ker}(\psi_\nu) \;\subset\; \mathbb{B} \,. \tag{4.1.63}$$

Besides, the projection of χ_ν to $\mathbb{B}/\,\mathrm{Rad}(\nu)$ being non-degenerate gives an isomorphism

$$\psi_\nu : \mathbb{B}/\,\mathrm{Rad}(\nu) \to \big(\mathbb{B}/\,\mathrm{Rad}(\nu)\big)^\vee \,. \tag{4.1.64}$$

Then duality invariance of $\mathcal{L}_{\mathbb{B},[\nu]}$ is equivalent to the following conditions:

1. $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$. In particular $N(\mathbb{B}) \subset \phi(\mathbb{B})$, and $|\mathbb{B}| = n_\nu\,|\mathbb{A}|^{1/2} \geq |\mathbb{A}|^{1/2}$. In other words, $\mathbb{B}$ cannot be smaller than Lagrangian and $|\mathbb{A}|$ must be a perfect square, hence reproducing the known obstruction induced by non-integer quantum dimensions [12].

2. Assuming condition 1., also ϕ projects to an isomorphism $\phi : \mathbb{B}/\,\mathrm{Rad}(\nu) \to \big(\mathbb{B}/\,\mathrm{Rad}(\nu)\big)^\vee$. Then we can define an automorphism

$$\sigma \;\equiv\; \phi^{-1}\circ\psi_\nu : \mathbb{B}/\mathrm{Rad}(\nu) \to \mathbb{B}/\mathrm{Rad}(\nu) \tag{4.1.65}$$

which, by construction, satisfies $\gamma\big(\sigma(a), b\big) = \chi_\nu(a,b)$. The second condition is that

$$\sigma^2 = 1 \,. \tag{4.1.66}$$

Notice that the conditions we obtained for $\mathcal{L}_{\mathbb{B},[\nu]}$ to be duality invariant are equivalent to the first obstruction we reviewed in Section 4.1.2. We thus arrive to the punchline of this section: the first obstruction is equivalent to the absence of duality-invariant Lagrangian algebras in $\mathrm{DW}(\mathbb{A})$, or in other words, to the non-invertible duality symmetry being intrinsic.

A straightforward consequence of the conditions above concerns the action of the duality symmetry G on the symmetry $\mathcal{S} = \big(\mathbb{A}\times\mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]}$ of the invariant boundary. To this purpose, it is convenient to present $\mathcal{S}$ as a group extension (4.1.60) and further view $\mathbb{B}^\vee$ as an extension of $\text{Rad}(\nu)^\vee$ by $\big(\mathbb{B}/\text{Rad}(\nu)\big)^\vee$, hence presenting the elements of $\mathcal{S}$ as triplets $(\beta, \eta, \tilde{a})$ with $\beta \in \big(\mathbb{B}/\text{Rad}(\nu)\big)^\vee$, $\eta \in \text{Rad}(\nu)^\vee$ and $\tilde{a} \in \mathbb{A}/\mathbb{B}$. Using that $\mathcal{S} = \mathcal{L}^\vee_{\mathbb{B},[\nu]}$ we find that the duality exchanges $\text{Rad}(\nu)^\vee$ with $\mathbb{A}/\mathbb{B}$, while it acts on $\big(\mathbb{B}/\text{Rad}(\nu)\big)^\vee$ as the automorphism $\sigma^\vee$:

$$\Phi : (\beta, \eta, \tilde{a}) \to \Big(\sigma^\vee(\beta),\ \phi(\tilde{a}),\ \phi^{-1}(\eta)\Big) . \tag{4.1.67}$$

When the data $\big(\mathbb{B}, [\nu]\big)$ defines a duality-invariant Lagrangian subgroup, using the definition of σ in (4.1.65) and $\sigma^2 = 1$ we can relate the symmetric and the antisymmetric bicharacters as

$$\chi_\nu(b_1, b_2) = \gamma\big(\sigma(b_1), b_2\big) , \qquad\qquad \gamma(b_1, b_2) = \chi_\nu\big(\sigma(b_1), b_2\big) . \tag{4.1.68}$$

This in turn implies a condition for the class $[\nu]$:

$$\nu(b_1, b_2)\, \nu\big(\sigma(b_1), \sigma(b_2)\big) = d\tilde{\zeta}(b_1, b_2) \qquad \text{or equivalently} \qquad \frac{\nu(b_1, b_2)}{\nu\big(\sigma(b_2), \sigma(b_1)\big)} = d\tilde{\eta}(b_1, b_2) . \tag{4.1.69}$$

This is because the l.h.s. of both equations is a symmetric 2-cocycle (see Section 4.1.2 or footnote 20). Those relations coincide with the known relation (4.1.30) (also appearing in the equivariantization of the algebras in TY categories, see Section 4.1.2).

We can neatly express the condition (4.1.69) by noticing that the action $\rho : G \to \text{Aut}(\mathbb{A})$ of any group G on a generic Abelian group $\mathbb{A}$ induces an action on $H^2\big(\mathbb{A}, U(1)\big)$ given by

$$(\rho_g\, \xi)(a_1, a_2) = \xi\big(\rho_g^{-1}(a_1),\ \rho_g^{-1}(a_2)\big) \tag{4.1.70}$$

for each $g \in G$ and $\xi \in H^2\big(\mathbb{A}, U(1)\big)$. Then (4.1.69) can be expressed as

$$\sigma\, [\nu] = \rho_{\underline{1}}\, [\nu] = [\nu^{-1}] , \tag{4.1.71}$$

where $\underline{1}$ is the generator of $G \cong \mathbb{Z}_2$. This reformulation will be convenient later on.

Examples

To make concrete the discussion above, we show a few examples. For convenience, here we use additive notation for the phases by thinking of them as elements of $\mathbb{R}/\mathbb{Z}$ instead of $U(1)$.

1. The simplest example is $\mathbb{A} = \mathbb{Z}_n$ for which there is no discrete torsion, and the subgroups are in correspondence with the divisors of n. Let $n = pq$, and $\mathbb{B} = \{qx \mid x = 0, \dots, p-1\} \cong \mathbb{Z}_p$ so that $N(\mathbb{B}) = \{py \mid y = 0, \dots, q-1\} \cong \mathbb{Z}_q$.

When we gauge $\mathbb{B}$ on the boundary, the global symmetry is the direct product of the dual symmetry $\mathbb{Z}_p$ and the quotient $\mathbb{Z}_q$. From the bulk perspective, the prescription is that this boundary condition is obtained by allowing the lines of the form (qx, py) to terminate on the boundary hence becoming transparent there. On the other hand, the 0-form symmetry is generated by the remaining lines stacked at the boundary, which indeed form the group $\mathbb{Z}_p \times \mathbb{Z}_q$.

For what concerns duality invariance, we need $\mathbb{B} \cong N(\mathbb{B})$ and hence $p = q$: this implies that $n = p^2$ must be a perfect square. Any symmetric bicharacter takes the form $\gamma(a, b) = rab/n \pmod 1$ for some $r \in \mathbb{Z}_n$ (r must be coprime with n for the bicharacter to be non-degenerate), and we notice indeed that $\mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ is Lagrangian in all cases:

$$\gamma(px, py) = 0 . \tag{4.1.72}$$

The integer coefficient introduced in (4.1.23) here is $n_\nu = |\mathbb{B}|/|\mathbb{A}|^{1/2} = 1$.

2. A less trivial example is $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with n a prime number. There are $n+3$ subgroups: the trivial one, the $n+1$ subgroups isomorphic to $\mathbb{Z}_n$ generated by $(1,0)$ and $(s,1)$ with $s = 0, \ldots, n-1$, and the full $\mathbb{A}$. Only the last one admits non-trivial discrete torsion $[\nu] \in H^2\big(\mathbb{Z}_n \times \mathbb{Z}_n, U(1)\big) \cong \mathbb{Z}_n$ which could be represented as

$$\nu\big((x_1,x_2),\,(y_1,y_2)\big) = \frac{r}{n}\, x_1 y_2 \qquad \text{or equivalently as} \qquad \nu\big((x_1,x_2),\,(y_1,y_2)\big) = -\frac{r}{n}\, x_2 y_1 \;. \tag{4.1.73}$$

The corresponding alternating bicharacter is given by the matrix

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} , \qquad\qquad \text{with} \qquad r \in \mathbb{Z}_n \;. \tag{4.1.74}$$

In total there are $2n+2$ boundary theories. One can explicitly see that these are in one-to-one correspondence with the Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ in $\mathbb{A} \times \mathbb{A}^\vee$.

Let us show that the induced global symmetry at the boundary is the one obtained by gauging $\mathbb{B}$ with discrete torsion ν. The cases $\mathbb{B} = \{0\}$ or $\mathbb{B} \cong \mathbb{Z}_n$ are similar to the one discussed above and the corresponding Lagrangian algebras are, respectively:

$$\begin{aligned}
\mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((0,0);(a_1,a_2)\big) \;\Big|\; a_1,a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} = \{0\} \;, \\
\mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((a_1,0);(0,a_2)\big) \;\Big|\; a_1,a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} \cong \mathbb{Z}_n \text{ generated by } (1,0) \;, \\
\mathcal{L}_{\mathbb{B},[0]} &= \Big\{ \big((sa_1,a_1);(a_2,-sa_2)\big) \;\Big|\; a_1,a_2 \in \mathbb{Z}_n \Big\} && \text{for } \mathbb{B} \cong \mathbb{Z}_n \text{ generated by } (s,1) \;.
\end{aligned} \tag{4.1.75}$$

When $\mathbb{B} = \mathbb{Z}_n \times \mathbb{Z}_n$ the resulting boundary theory has symmetry $\mathbb{B}^\vee \cong \mathbb{Z}_n \times \mathbb{Z}_n$ even for non-trivial discrete torsion. According to our prescription, and using that $N(\mathbb{B})$ is trivial and the map ψ_ν has the same matrix form of χ_ν defined in (4.1.74), the corresponding Lagrangian subgroup of $\mathbb{A} \times \mathbb{A}^\vee$ is

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big((a_1,a_2);(-ra_2,ra_1)\big) \;\Big|\; a_1,a_2 \in \mathbb{Z}_n \Big\} \;, \tag{4.1.76}$$

and indeed $(\mathbb{A} \times \mathbb{A}^\vee)/\mathcal{L} = \mathbb{Z}_n \times \mathbb{Z}_n$. To see the effect of the discrete torsion we use a Lagrangian description of the DW theory:

$$S = \frac{2\pi i}{n} \int_{X_3} \big(A_1 \cup dB_1 + A_2 \cup dB_2 \big) \;. \tag{4.1.77}$$

The generic line with charges $\big((a_1,a_2);(b_1,b_2)\big)$ is

$$\exp\left[\frac{2\pi i}{n} \int_\gamma \big(a_1 A_1 + a_2 A_2 + b_1 B_1 + b_2 B_2 \big) \right] . \tag{4.1.78}$$

Hence, as a boundary condition, $\mathcal{L}$ in (4.1.76) corresponds to $A_1 + rB_2 = A_2 - rB_1 = 0$. Changing variables according to $A_1 \to A_1 + rB_2$, $A_2 \to A_2 - rB_1$ we obtain the same bulk Lagrangian as in (4.1.77) but with an extra boundary term

$$\delta S_{\text{bdry}} = \frac{2\pi i r}{n} \int_{\partial X_3} B_1 \cup B_2 \;, \tag{4.1.79}$$

which is precisely the discrete torsion for the gauging on the boundary.

Let us discuss which of those algebras are duality invariant, and in particular which symmetric bicharacters admit duality-invariant algebras. There are two natural classes of non-degenerate symmetric bicharacters, diagonal and off-diagonal:

$$\gamma^{(\mathrm{D})}_{t_1,t_2} = \frac{1}{n}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \qquad \text{and} \qquad \gamma^{(\mathrm{O})}_{t} = \frac{1}{n}\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} . \tag{4.1.80}$$

Here non-degeneracy requires t_1, t_2, t to be invertible elements of $\mathbb{Z}_n$.[31] Note that $\mathbb{B} = \{0\}$ cannot lead to duality-invariant algebras because it is smaller than Lagrangian.

Consider the case of $\gamma^{(\mathrm{O})}_t$. First we look at Lagrangian algebras associated with subgroups $\mathbb{B} \cong \mathbb{Z}_n$ which, according to our general analysis, need to be Lagrangian with respect to $\gamma^{(\mathrm{O})}_t$ because $[\nu] = 0$. The two subgroups $\mathbb{B} = \big\langle (1,0) \big\rangle$, $\big\langle (0,1) \big\rangle$ are always Lagrangian, while $\mathbb{B} = \big\langle (s,1) \big\rangle$ is Lagrangian only if

$$2st = 0 \mod n \tag{4.1.81}$$

which can never be satisfied if n is odd. Then we look at the cases with $\mathbb{B} = \mathbb{A}$. In order to satisfy $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B}) = \{0\}$ in (4.1.26) we need a discrete torsion (4.1.74) with $r \neq 0$. From (4.1.65) we find

$$\sigma = \begin{pmatrix} t^{-1}r & 0 \\ 0 & -t^{-1}r \end{pmatrix} . \tag{4.1.82}$$

The duality-invariant condition $\sigma^2 = 1$ reads $(t^{-1}r)^2 = 1 \bmod n$, which can always be satisfied by the values $r = \pm t$.

Consider now the case of $\gamma^{(\mathrm{D})}_{t_1,t_2}$. The subgroups $\mathbb{B} \cong \mathbb{Z}_n$ are Lagrangian with respect to $\gamma^{(\mathrm{D})}_{t_1,t_2}$ only when $\mathbb{B} = \big\langle (s,1) \big\rangle$ with $t_1\, s^2 + t_2 = 0 \bmod n$. For $\mathbb{B} = \mathbb{A}$, instead, we need a non-vanishing discrete torsion (4.1.74), and since

$$\sigma = \begin{pmatrix} 0 & -t_1^{-1}r \\ t_2^{-1}r & 0 \end{pmatrix} , \tag{4.1.83}$$

the duality-invariance condition reads $r^2 = -t_1 t_2 \bmod n$. This equation and the previous one for s do not always have solutions. For instance, if $t_1 = t_2 = 1$, then r (or s) must be a square root of -1 which exists for $n = 2, 5, 13, \ldots$ but not for $n = 3, 7, 11, \ldots$

In summary, while $\mathrm{TY}(\mathbb{Z}_n \times \mathbb{Z}_n)_{\gamma,\epsilon}$ with off-diagonal bicharacter γ always trivializes the first obstruction, when the bicharacter is diagonal the category is necessarily anomalous for certain values of n for which the first obstruction forbids the gauging. We also notice that in all of these examples, when there is a duality-invariant Lagrangian algebra associated with $\mathbb{B} \cong \mathbb{Z}_n$ we have $n_\nu = 1$, while for $\mathbb{B} \cong \mathbb{A}$ we have $n_\nu = n$.

3. We conclude with a more complicated example which is representative of the general case $\mathbb{B} \subsetneq \mathbb{A}$ but $[\nu] \neq 0$, hence $\mathbb{B}$ is non-Lagrangian. Take $\mathbb{A} = \mathbb{Z}_4 \times \mathbb{Z}_4$ which, besides the subgroups we already considered, also has the subgroup

$$\mathbb{B} = \big\{ (x, 2y) \;\big|\; x \in \mathbb{Z}_4 ,\; y \in \mathbb{Z}_2 \big\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \tag{4.1.84}$$

[31] For n prime, they are just non-vanishing. However these two bicharacters will be used also for n non prime, hence t_1, t_2, t must be coprime with n.

(as well as the similar one with the two factors swapped) hence realizing

$$N(\mathbb{B}) = \big\{(0, 2\tilde{y}) \;\big|\; \tilde{y} \in \mathbb{Z}_2\big\} \cong \mathbb{Z}_2 \subset \mathbb{A}^\vee \,. \tag{4.1.85}$$

The most general alternating bicharacter on $\mathbb{B}$ is

$$\chi_\nu = \frac{1}{4}\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \qquad \text{with} \qquad 2(a+b) = 0 \mod 4 \,, \tag{4.1.86}$$

hence $a, b \in \mathbb{Z}_4$ must be either both even or both odd. If a, b are both even then $\mathrm{Rad}(\nu) = \mathbb{B}$ and duality invariance cannot be satisfied. If a, b are odd, instead,

$$\mathrm{Rad}(\nu) = \big\{(2z, 0) \;\big|\; z \in \mathbb{Z}_2\big\} \cong \mathbb{Z}_2 \subset \mathbb{Z}_4 \times \mathbb{Z}_2 \,. \tag{4.1.87}$$

The condition $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$ cannot be satisfied with the diagonal bicharacter $\gamma^{(\mathrm{D})}_{t_1, t_2}$, while with the off-diagonal one $\gamma^{(\mathrm{O})}_t$ the condition is met (for both the invertible elements $t = 1, 3$). The second condition for duality invariance involves

$$\sigma = \phi^{-1} \circ \psi_\nu = \begin{pmatrix} tb & 0 \\ 0 & ta \end{pmatrix} . \tag{4.1.88}$$

The condition $\sigma^2 = 1$ is equivalent to $(tb)^2 = (ta)^2 = 1$ which is automatically satisfied. In this case we get $n_\nu = 2$.

Second obstruction and equivariantization

In the previous section we rephrased the first obstruction to the gauging of a 2d duality symmetry in terms of the existence of a duality-invariant Lagrangian algebra $\mathcal{L}_D$ in the 3d TQFT DW($\mathbb{A}$). Gauging $\mathcal{L}_D$ leads to a bulk SPT phase $Y \in H^3\big(G, U(1)\big)$ for the duality symmetry $G \cong \mathbb{Z}_2$, which determines the DW twist of the corresponding G gauge theory as explained in (4.1.10). The total twist $\epsilon_{\mathrm{tot}} = \epsilon\, Y$ in turn determines whether a Neumann boundary condition is allowed (and $\mathcal{N}$ is anomaly-free).

In order to understand the origin of Y we must describe in detail how to make the gauging of $\mathcal{L}_D$ consistent with the presence of a 0-form symmetry. Naively this should amount to the requirement that $\mathcal{L}_D$ be G-invariant as an object: $\Phi(\mathcal{L}_D) = \mathcal{L}_D$ as stressed in (4.1.50). This is however not sufficient, as the algebra $\mathcal{L}_D$ comes with a specific choice of morphism $m : \mathcal{L}_D \times \mathcal{L}_D \to \mathcal{L}_D$ that is associative and commutative (see Appendix B.1 for the definitions) and a set of projections $\pi_x \in \mathrm{Hom}(\mathcal{L}_D, x)$. An *equivariantization* of $\mathcal{L}_D$ is the definition of a consistent action of the 0-form symmetry on the projections that leaves m invariant (for more details we refer the reader to Appendix B.1 and [337] for a thorough treatment). To define this structure the proper context is that of G-crossed MTCs [28]. In this framework a symmetry defect U_g acts on the junction spaces $V^z_{x,y}$, where $x, y, z \in \mathbb{A} \times \mathbb{A}^\vee$ label simple objects[32] , by a unitary automorphism $[\mathcal{U}_g]^z_{x,y} : V^z_{x,y} \to V^{g(z)}_{g(x),\, g(y)}$ as

$$U_g\big(v^z_{x,y}\big) = [\mathcal{U}_g]^z_{x,y} \cdot v^{g(z)}_{g(x),\, g(y)} \tag{4.1.89}$$

[32]Throughout this section we leave implicit that all simple objects are invertible and hence all junction spaces are one-dimensional.

where v is a chosen basis vector of $V^z_{x,y}$ (see Figure 4.1). The phases $[\mathcal{U}_g]^z_{x,y}$ have to satisfy several compatibility conditions with the data of the underlying category, in particular consistency with the braiding requires

$$[\mathcal{U}_g]^z_{x,\,y}\; R^z_{x,\,y} = R^{g(z)}_{g(x),\,g(y)}\; [\mathcal{U}_g]^z_{y,\,x}\ . \tag{4.1.90}$$

Using the R-matrices (4.1.44) and the $G \cong \mathbb{Z}_2$ action on elements of $\mathrm{DW}(\mathbb{A})$ one easily sees that this equation admits a simple solution

$$[\mathcal{U}_g]^{(a+b,\alpha+\beta)}_{(a,\alpha),\,(b,\beta)} = \alpha(b)\,, \tag{4.1.91}$$

for $g = \underline{1}$ the generator of $\mathbb{Z}_2$.

Now let us come to the equivariantization. For the algebras discussed in Section 4.1.2, a consistent[33] choice of m is

$$m^{x+x'}_{x,\,x'} = \nu(b',\,b) \qquad \text{where} \qquad x = \big(b,\; \beta\psi_\nu(b)\big) \in \mathcal{L}_D\ . \tag{4.1.92}$$

In the following we will use $x, y, z, \ldots$ to denote elements of $\mathcal{L}_D$ in order to lighten the notation. Working in components we expand

$$m = \bigoplus_{x,\,y} m^z_{x,\,y} \qquad \text{and} \qquad m^z_{x,\,y} \in V^z_{x,\,y} \tag{4.1.93}$$

where $z = x + y$. The defects U_g act on the projectors $\pi_x : \mathcal{L}_D \to x$ by an automorphism $\tilde{\eta}_g(x) : \pi_x \to \pi_{g(x)}$ as follows (see Figure 4.1)

$$U_g(\pi_x) = \tilde{\eta}_g(x) \cdot \pi_{g(x)} \tag{4.1.94}$$

Using these transformations, m is invariant if [34]

$$m^{g(z)}_{g(x),\,g(y)} = m^z_{x,\,y}\; [\mathcal{U}_g]^z_{x,\,y}\; \frac{\tilde{\eta}_g(z)}{\tilde{\eta}_g(x)\,\tilde{\eta}_g(y)}\ . \tag{4.1.95}$$

The equivariantization datum $\tilde{\eta}$ can be neatly interpreted in cohomology. First acting with gauge transformations $\pi_x \to \mu(x)\,\pi_x$ on the vector spaces associated to π_x and $\pi_{g(x)}$ we can identify

$$\tilde{\eta}_g(x) \;\sim\; \tilde{\eta}_g(x)\, \frac{\mu\big(g(x)\big)}{\mu(x)}\ . \tag{4.1.96}$$

Second, consistency with the group composition law demands that

$$\tilde{\eta}_g(x)\; \tilde{\eta}_h\big(g(x)\big) = \tilde{\eta}_{gh}(x)\ . \tag{4.1.97}$$

Interpreting $\tilde{\eta}_g$ as a cochain in $C^1\big(\mathcal{L}_D, U(1)\big)$, so that $\tilde{\eta} \in C^1\big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\big)$, we can rewrite (4.1.97) and (4.1.96) in terms of a differential. Using now additive notation, for the sake of clarity and for later convenience, they look, respectively, as

$$d_\rho \tilde{\eta} = 0\ , \qquad\qquad \tilde{\eta} \;\sim\; \tilde{\eta} + d_\rho \mu\ , \tag{4.1.98}$$

[33] Commutativity of the algebra requires $m^z_{x,\,x'} = m^z_{x',\,x} R^z_{x,\,x'}$, which, in our case, becomes $m^z_{x,\,x'}/m^z_{x',\,x} = \chi_\nu(b', b)$.

[34] Here we use that all objects in the algebra $\mathcal{L}_D$ are invertible and appear with multiplicity one in the $\mathrm{DW}(\mathbb{A})$ theory.

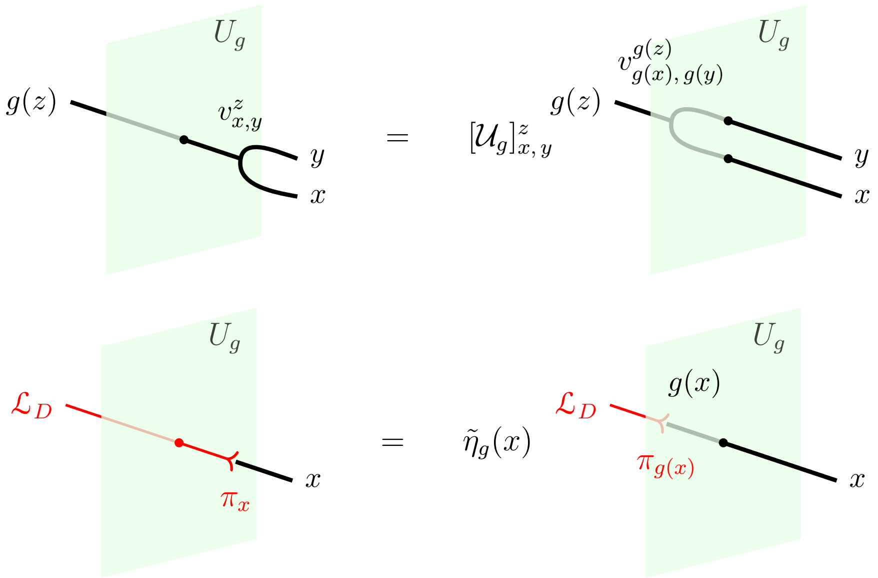

Figure 4.1: Graphical representation of the action of a symmetry defect U_g on the junction spaces $V^z_{x,y}$ (above) and on the projectors π_x (below).

for any $\mu \in C^0\big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\big) \cong C^1\big(\mathcal{L}_D, U(1)\big)$. Here d_ρ is a twisted differential, while ρ is the G-action on anyons. We obtain that $\tilde{\eta}$ is naturally an object in twisted group cohomology (see *e.g.* [178] and Appendix B.4 for a review):

$$\tilde{\eta} \in H^1_\rho\Big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\Big) \,. \tag{4.1.99}$$

Restricting the solution ((4.1.91)) to elements of $\mathcal{L}_D$ we find

$$[\mathcal{U}_g]^{x+x'}_{x,\,x'} = \chi_\nu\big(b, b'\big) = \chi_\nu\big(\sigma(b), \sigma(b')\big)^{-1} \tag{4.1.100}$$

where in the second step we used the relations between the symmetric and antisymmetric bicharacters (4.1.68). Since $m^{g(x+x')}_{g(x),\,g(x')} = \nu\big(\sigma(b'), \sigma(b)\big)$ from (4.1.92), then (4.1.95) becomes

$$\frac{\nu(b, b')}{\nu\big(\sigma(b'), \sigma(b)\big)} = d\tilde{\eta}_g \tag{4.1.101}$$

which we recognize as the first equation in (4.1.30) with a caveat. The set of solutions for $\tilde{\eta}_g$, with $g = \underline{1}$, form a torsor over $\mathcal{L}_D^\vee$ while the solutions of (4.1.30) are related by elements of $\big(\mathbb{B}/\mathrm{Rad}(\nu)\big)^\vee$, therefore, strictly speaking, the solutions sets of the two equations differ. However we will see below that the two sets of equations give rise to the same second obstruction.[35]

For later convenience we also notice that the set of solutions to (4.1.101) for $\tilde{\eta}$ forms a torsor over

$$H^1_\rho(G,\, \mathcal{L}_D^\vee) = H^1_\rho(G, \mathcal{S}) \,, \tag{4.1.102}$$

whose elements we denote by η. This will be useful for the upcoming reinterpretation of the second obstruction in terms of symmetry fractionalization in Section 4.1.2. All in all we found that an equivariantization of a duality-invariant Lagrangian algebra $\mathcal{L}_D$ is specified by the

[35] Physically the extra solutions correpond to symmetry fractionalization patterns between $\mathbb{Z}_2$ and $\mathcal{S}$ for which there is no mixed 't Hooft anomaly.

choice of an element $\tilde{\eta} \in H^1_\rho\big(G,\, C^1\big(\mathcal{L}_D, U(1)\big)\big)$ satisfying (4.1.101), and that any two choices differ by an element η of $H^1_\rho\big(G,\, \mathcal{L}_D^\vee\big)$.

Given an equivariantization $\tilde{\eta}$ of $\mathcal{L}_D$, we ask what is the SPT phase $Y \in H^3\big(G, U(1)\big)$ for G that we obtain after gauging $(\mathcal{L}_D, \tilde{\eta})$. Indeed, the theory after gauging has a single genuine line $\mathbb{1}$ (and thus is an invertible TQFT) but also a single non-genuine topological twist line M_g for each $g \in G$. The spins θ_{M_g} of such objects are gauge dependent by a G-character [28]. In the presence of a discrete torsion Y, the θ_{M_g}'s do not form a G-character: their deviation from being a character is physical and is induced by the SPT phase Y. In the present case that $G \cong \mathbb{Z}_2$,[36] θ_M does not square to 1 but instead

$$\theta_M = \sqrt{Y(\underline{1}, \underline{1}, \underline{1})}\ , \tag{4.1.103}$$

the sign of the square root being pure gauge. We can thus detect the $\mathbb{Z}_2$ SPT phase through the gauge-invariant quantity $\theta_M^2 = Y$. We now show how to reproduce (4.1.34). A key fact is that, given a choice of equivariantization $\tilde{\eta}$ for $\mathcal{L}_D$, there is a unique non-genuine twist line M after gauging $(\mathcal{L}_D, \tilde{\eta})$. It is then possible to show, using the defining equation (B.1.25) for a twisted local module M that

$$f_a(b)^{-1} = \tilde{\eta}(b) \qquad \text{for} \qquad b \in \mathbb{B}/\mathrm{Rad}(\nu) \qquad \text{and} \qquad \sigma(b) = b\ , \tag{4.1.104}$$

where f_a is the function introduced in (4.1.45). The equation holds for all the values of a for which $\mathrm{Hom}(\sigma_a,\, M) \neq 0$.[37] We will use the notation $M^{(a)}$ to account for the different choices one has for the equivariantization $\tilde{\eta}$: upon gauging, each choice leads to a theory with a unique non-genuine operator, however different choices lead to different SPT phases Y and the label a (whose possible values depend on $\mathcal{L}_D$ in a complicated way) keeps track of the equivariantization chosen.

Since θ_M^2 must be well defined, the spins squared of the components of M must coincide. Since, as an object, $M^{(a)}$ can be described as the orbit of the twist defect σ_a under fusion with the lines of $\mathcal{L}_D$, using the fusions in (4.1.42) we get:[38]

$$M^{(a)} = \bigoplus_u \sigma_{a+u} \qquad \text{where} \qquad u = b + \phi^{-1}\big(\beta\psi_\nu(b)\big) \qquad \text{for all} \qquad \big(b,\, \beta\psi_\nu(b)\big) \in \mathcal{L}_D\ . \tag{4.1.105}$$

Consistency with the existence of a unique local module requires that $\theta^2_{\sigma_a} = \theta^2_{\sigma_{a+u}}$, *i.e.*

$$\theta^2_{M^{(a)}} = \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_a(c)^{-1} \stackrel{!}{=} \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_{a+u}(c)^{-1} = \frac{1}{\sqrt{|\mathbb{A}|}} \sum_{c\,\in\,\mathbb{A}} f_a(c)^{-1}\, \gamma(u,c)^{-1}\ , \tag{4.1.106}$$

[36]In order not to clutter we will suppress the label g in what follows, as there is only one nontrivial G defect anyway.

[37]To show that the result holds, consider (B.1.25) and set $g(x_i) = x_i$. The matrix r_L can then be eliminated on the two sides. Decomposing the module M_g in its components and using the formulas (4.1.44) for the R matrix gives the desired result.

[38]Besides identifying twist defects related by fusion with the lines of $\mathcal{L}_D$, one also has to impose locality conditions, that depend on $\tilde{\eta}$ (see Appendix B.1). Together these constraints single out a unique twist defect for each choice of equivariantization.

from which we can extract some consequences. For our purposes it will be enough to consider $u = b + \phi^{-1}(\psi_\nu(b))$ with $b \in \mathbb{B}$, we then impose

$$
\begin{aligned}
\theta^2_{M^{(a)}} = \frac{1}{|\mathbb{B}|}\sum_{b\in\mathbb{B}} \theta^2_{M^{(a)}} &= \frac{1}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b\in\mathbb{B}\\ c\in\mathbb{A}}} f_a(c)^{-1}\,\gamma(b+\phi^{-1}(\psi_\nu(b)),c)^{-1} \\
&= \frac{1}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b\in\mathbb{B}\\ c\in\mathbb{A}}} f_a(c)^{-1}\,\gamma(b,c)^{-1}\,\gamma(\phi^{-1}(\psi_\nu(b)),c)^{-1}\,.
\end{aligned} \tag{4.1.107}
$$

Any $b \in \mathbb{B}$ can be split as

$$
b = \iota(\phi^{-1}(\beta)) + s(x) \tag{4.1.108}
$$

with $\beta \in N(\mathbb{B})$ and $x \in \mathbb{B}/\mathrm{Rad}(\nu)$. Here ι is the inclusion of $\mathrm{Rad}(\nu)$ in $\mathbb{B}$ and $s : \mathbb{B}/\mathrm{Rad}(\nu) \to \mathbb{B}$ is a section. Using linearity of γ and that $\psi_\nu(\phi^{-1}(\beta)) = 0$ we see that the only β-dependent factor in the summand is $\beta(c)$, so that the sum over β constraints $c \in \mathbb{B}$. We then have

$$
\theta^2_{M^{(a)}} = \frac{|\mathrm{Rad}(\nu)|}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{b'\in\mathbb{B}\\ x\in\mathbb{B}/\mathrm{Rad}(\nu)}} f_a(b')^{-1}\,\gamma(s(x),b')^{-1}\,\gamma(\sigma(s(x)),b')^{-1}\,. \tag{4.1.109}
$$

We now split also b' as (4.1.108) obtaining

$$
\theta^2_{M^{(a)}} = \frac{|\mathrm{Rad}(\nu)|}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{\beta'\in N(\mathbb{B})\\ x,x'\in\mathbb{B}/\mathrm{Rad}(\nu)}} f_a(\phi^{-1}(\beta'))^{-1} f_a(s(x'))^{-1}\,\gamma(s(x),s(x'))^{-1}\,\gamma(\sigma(s(x)),s(x'))^{-1} \tag{4.1.110}
$$

where we noticed that $f_a(\phi^{-1}(\beta)+b) = f_a(\phi^{-1}(\beta))f_a(b)$ for any $\beta \in N(\mathbb{B})$ and $b \in \mathbb{B}$. Because of this f_a restricted on $\mathrm{Rad}(\nu)$ is a character, hence the sum over β' yields $\theta^2_{M^{(a)}} = 0$ unless $f_a(\phi^{-1}(\beta')) = 1$ for any $\beta' \in N(\mathbb{B})$, *i.e.* f_a must restrict to the trivial character on $\mathrm{Rad}(\nu)$ to avoid an inconsistent answer. Plugging this in we get

$$
\theta^2_{M^{(a)}} = \frac{|\mathrm{Rad}(\nu)|^2}{|\mathbb{B}|\sqrt{|\mathbb{A}|}} \sum_{\substack{\beta'\in N(\mathbb{B})\\ x,x'\in\mathbb{B}/\mathrm{Rad}(\nu)}} f_a(s(x'))^{-1}\,\gamma(s(x)+\sigma(s(x)),s(x'))^{-1} \tag{4.1.111}
$$

notice that, due to the property of f_a mentioned above, this expression is independent of the sections chosen hence we shall drop them in the following. Using (4.1.68) we rewrite

$$
\gamma(\sigma(x),x') = \gamma(x,\sigma(x'))^{-1} \tag{4.1.112}
$$

so that summing over x constraints x' to be fixed by σ. All in all the spin of the twist defect is

$$
\theta^2_{M^{(a)}} = \frac{|\mathrm{Rad}(\nu)|}{\sqrt{|\mathbb{A}|}} \sum_{\substack{b\in\mathbb{B}/\mathrm{Rad}(\nu)\\ \sigma(b)=b}} f_a(b)^{-1} \tag{4.1.113}
$$

hence, due to (4.1.104), confirming that

$$
\theta^2_{M^{(a)}} = \mathrm{Arf}(\tilde{\eta}) = Y\,. \tag{4.1.114}
$$

Notice that this computation automatically provides with the proper normalization to ensure that $\mathrm{Arf}(\tilde{\eta}) = \pm 1$.

Example. Consider $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with off-diagonal bicharacter $\gamma_1^{(\mathrm{O})}$. The invariant algebra is

$$\mathcal{L}_D = \Big\{ \big((a_1, a_2); (-a_2, a_1) \big) \;\Big|\; a_1, a_2 \in \mathbb{Z}_n \Big\} \,. \tag{4.1.115}$$

Our choice for the functions f_a in (4.1.45) is

$$f_{(a_1,\,a_2)}(b_1, b_2) = \exp\!\left(-\frac{2\pi i}{n}\, b_1 b_2 \right) \gamma(a, b) \,. \tag{4.1.116}$$

From this it is simple to show that

$$\theta^2_{\sigma_a} = \exp\!\left(-\frac{2\pi i}{n}\, a_1 a_2 \right) \,. \tag{4.1.117}$$

A module $M^{(a)}$ is given, as an object, by

$$M^{(a)} = \begin{cases} \bigoplus_{b \,\in\, \mathbb{Z}_n} \sigma_{b,\, a_2} & \text{for } n \text{ odd}\,, \\ \bigoplus_{b \,\in\, \mathbb{Z}_n} \sigma_{a_1 + 2b,\, a_2} & \text{for } n \text{ even}\,. \end{cases} \tag{4.1.118}$$

Imposing the spin $\theta^2_{\sigma_a}$ to be constant on the orbit $M^{(a)}$ strongly constrains the possible local module candidates. One finds that for n odd there is only one consistent choice of module M, namely $M^{(0,0)}$ while for n even there are four, corresponding to $(a_1, a_2) = \big(s_1,\, \tfrac{n}{2} s_2\big)$ and $s_{1,2} \in \{0, 1\}$. Their spins squared are:

$M^{(a)}$	$M^{(0,\,0)}$	$M^{(1,\,0)}$	$M^{(0,\,n/2)}$	$M^{(1,\,n/2)}$
θ^2_M	1	1	1	-1

(4.1.119)

It is possible to check that all four satisfy the locality condition (B.1.25) for the four inequivalent choices of $\tilde{\eta}$, parametrized by $H^1_\rho(\mathbb{Z}_2,\, \mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_2 \times \mathbb{Z}_2$. We will see in the next section how the same result can be obtained in terms of symmetry fractionalization.

Second obstruction and symmetry fractionalization

The discussion in the previous section gave us a description of the second obstruction from a purely bulk perspective. It however requires precise knowledge of the full categorical data of the 3d MTC, hence it is hard to generalize it to higher-dimensional cases. Moreover it leaves one conceptual problem to address: what is the physical interpretation of the different choices of equivariantization from the point of view of the boundary? We make here a proposal that solves both issues: different choices of equivariantization in the bulk lead to different ways to couple the symmetry to backgrounds fields. This goes by the name of *symmetry fractionalization*.[39]

Even though we do not know how to turn on background fields for the non-invertible symmetry directly, we can use the vanishing of the first obstruction to reduce the problem to the discussion of inequivalent couplings to standard $\mathbb{Z}_2$ background fields on the invertible boundary. There we also have the 0-form symmetry $\mathcal{S} = \mathcal{Z}(\mathbb{A})/\mathcal{L}_D$, which crucially has a mixed

[39]This is a slight abuse of terminology since the term "symmetry fractionalization" is commonly used to indicate the decoration of defect junctions by higher-codimension defects, while here we mix two 0-form symmetries. Yet, we use the term in order to better uniformize the 2d discussion here with the 4d one.

anomaly with G. It is known [338, 348] that in such cases the cubic G anomaly might not have an intrinsic value: it can be changed by choosing different symmetry fractionalization classes. Analyzing this phenomenon will lead to the required condition for the vanishing of the second obstruction.

Let us start by determining the mixed anomaly between G and $\mathcal{S}$. The duality action Φ, which leaves $\mathcal{L}_D$ invariant, descends to an action on the quotient $\mathcal{S} = (\mathbb{A} \times \mathbb{A}^\vee)/\mathcal{L}_D$, which we already described in detail in Section 4.1.2. For simplicity we consider here the case $\mathbb{B} = \mathbb{A}$, so that $\mathcal{S} = \mathbb{A}^\vee$. The general case is qualitatively analogous and we report it in Appendix B.3.2. We use the duality isomorphism ϕ to write the background for $\mathbb{A}^\vee$ as $\phi(B)$ with $B \in H^1(X, \mathbb{A})$. The partition function of the invertible boundary theory coupled to a background B can be easily expressed in terms of the reference electric boundary:

$$Z_{\text{inv}}\big[\phi(B)\big] = \sum_{b \,\in\, H^1(X,\mathbb{A})} \exp\left[2\pi i \int_X b^*\nu + 2\pi i \int_X b \cup \phi(B)\right] Z_{\text{e}}[b] \; . \tag{4.1.120}$$

Here $b^*\nu \in H^2\big(X, U(1)\big)$ is the pull-back of $\nu \in H^2\big(\mathbb{A}, U(1)\big)$, understood in additive notation (see footnote 23). The duality maps Z_{e} to the partition function of the magnetic theory Z_{m}, which in turn can be expressed as a discrete gauging of the electric theory:

$$\Phi \cdot Z_{\text{e}}[b] = Z_{\text{m}}\big[\phi(b)\big] = \sum_{a \,\in\, H^1(X,\mathbb{A})} \exp\left[2\pi i \int_X \phi(a) \cup b\right] Z_{\text{e}}[a] \; . \tag{4.1.121}$$

The action of Φ on the invertible boundary can be derived combining (4.1.120) with (4.1.121), using that Φ only acts on the partition functions Z, and it reads

$$\Phi \cdot Z_{\text{inv}}\big[\phi(B)\big] = \exp\left[2\pi i \int_X B^*\nu\right] Z_{\text{inv}}\big[\phi(\sigma B)\big] \; . \tag{4.1.122}$$

The overall phase stems from a mixed 't Hooft anomaly between G and $\mathcal{S}$. Crucially, from (4.1.122) we find that $G \cong \mathbb{Z}_2$ acts non trivially on $\mathcal{S}$ through an automorphism $\rho : G \to \text{Aut}(\mathcal{S})$ such that

$$\rho_{\underline{1}}(B) = \sigma B \; , \tag{4.1.123}$$

so that the total symmetry of the invertible boundary is a semidirect product $\mathcal{S} \rtimes_\rho G$. Thanks to

$$\exp\left[2\pi i \int_X B^*\big(\nu \circ \sigma\big)\right] = \exp\left[-2\pi i \int_X B^*\nu\right] , \tag{4.1.124}$$

which is the integrated additive version of (4.1.69), the aforementioned anomaly is consistent with the $\mathbb{Z}_2$ symmetry. Let us write the inflow action for the anomalous phase, introducing a background filed $A \in H^1(X, \mathbb{Z}_2)$ for G. The general construction is detailed in Appendix B.4.1. The bottom line of that discussion is that such anomalies are classified by $\mu \in H^1_\rho\big(\mathbb{Z}_2, H^2\big(\mathbb{A}, U(1)\big)\big)$ in terms of which the 3d inflow action is

$$S_\mu = 2\pi i \int_{X_3} \mu(A) \cup B \cup B \; . \tag{4.1.125}$$

In components this is defined as

$$\Big(\mu(A) \cup B \cup B\Big)_{ijkl} = \mu(A_{ij})\big(\rho_{A_{ij}} B_{jk} \,,\, \rho_{A_{ik}} B_{kl}\big) \; . \tag{4.1.126}$$

A gauge variation $A \to A + d\lambda$ produces a boundary term

$$S_\mu \to S_\mu + 2\pi i \int_{\partial X_3} \mu(\lambda) \cup B \cup B \, . \tag{4.1.127}$$

The class μ can be thought of as a function $\mu : \mathbb{Z}_2 \to H^2\big(\mathbb{A}, U(1)\big)$ satisfying the twisted cocycle condition (using addivite notation)

$$\rho_g \, \mu(h) + \mu(g) = \mu(g+h) \, , \tag{4.1.128}$$

and subject to the the identification

$$\mu(g) \,\cong\, \mu(g) + \rho_g \, \xi - \xi \qquad\qquad \text{for any} \qquad \xi \in H^2\big(\mathbb{A}, U(1)\big) \, . \tag{4.1.129}$$

Because of the relation (4.1.69), which in additive notation reads $\sigma \cdot \nu = -\nu$, we can consistently choose

$$\mu(\underline{0}) = 0 \, , \qquad\qquad \mu(\underline{1}) = \nu \, . \tag{4.1.130}$$

Notice that this makes sense because Φ^2 leaves Z_{inv} invariant. With this choice, taking a background such that the pull-back of A to the boundary ∂X_3 is $\underline{0}$ and performing a gauge transformation $A \to A + d\lambda$ with $\lambda\big|_{\partial X_3} = \underline{1}$, one reproduces the anomalous phase (4.1.122). This construction also provides a convenient way to determine whether the anomalous phase (4.1.122) corresponds to a true anomaly or can be cancelled by a local counterterm. Indeed the latter situation occurs if and only if μ is cohomologically trivial, namely

$$\nu = \sigma \cdot \xi - \xi \tag{4.1.131}$$

for some $\xi \in H^2\big(\mathbb{A}, U(1)\big)$. In this case the anomalous phase is eliminated by modifying the action coupled to $B \in H^1(X_2, \mathbb{A})$ by the addition of the local counterterm

$$S_{\text{c.t.}} = 2\pi i \int_{X_2} B^* \xi \, . \tag{4.1.132}$$

If there exists no ξ satisfying (4.1.131) then the anomalous phase cannot be cancelled and there is an anomaly. To show the power of this method, let us discuss the example of $\mathbb{A} = \mathbb{Z}_n \times \mathbb{Z}_n$ with diagonal symmetric bicharacter $\gamma^{(\text{D})}_{1,1}$ and

$$\nu\big((x_1, x_2), \, (y_1, y_2)\big) = \frac{r}{n} \, x_1 y_2 \qquad \text{with} \qquad r^2 = -1 \mod n \, . \tag{4.1.133}$$

Then σ acts on $\mathbb{A}$ as $\sigma(x_1, x_2) = (r x_2, -r x_1)$, and since the most general $\xi \in H^2\big(\mathbb{A}, U(1)\big)$ is represented as $\xi\big((x_1, x_2), (y_1, y_2)\big) = \frac{s}{n} x_1 y_2$ or equivalently as $\xi\big((x_1, x_2), (y_1, y_2)\big) = -\frac{s}{n} x_2 y_1$ then

$$\big(\sigma \cdot \xi - \xi\big)\big((x_1, x_2), (y_1, y_2)\big) = -\frac{2s}{n} \, x_1 y_2 \, . \tag{4.1.134}$$

For n odd, we can always choose $s = -2^{-1} r$ hence the anomalous phase can be cancelled by a local counterterm and it is not an anomaly. On the other hand, for n even, r is necessarily odd and thus no choice of s can cancel the anomalous phase: in this case this is a genuine anomaly.

As already argued, the question of what is the value of the pure $G \cong \mathbb{Z}_2$ anomaly on the invertible boundary is not well-posed until we specify how G couples to a background field

$A \in H^1(X_2, G)$. In the boundary global variant where the full symmetry category is invertible, the presence of another 0-form symmetry $\mathcal{S}$ allows one to make discrete choices for that coupling labelled by a class

$$\eta \in H^1_\rho(G, \mathcal{S}) \ , \tag{4.1.135}$$

which satisfies the twisted cocycle condition $\rho_g \, \eta(h) + \eta(g) = \eta(g+h)$ and is subject to the identification $\eta(g) \cong \eta(g) + \rho_g \, c - c$ for any $c \in \mathcal{S}$, similarly to (4.1.128)–(4.1.129). So η specifies a (twisted) homomorphism from G to $\mathcal{S}$ which allows one to modify the minimal coupling prescription for A, declaring that the latter effectively couples to the diagonal subgroup of G and the image $\eta(G) \subset \mathcal{S}$. The anomaly cannot be unambiguously determined until we specify η because different choices correspond to different $\mathbb{Z}_2$ subgroups of the full 0-form symmetry group and, due to the mixed anomaly (4.1.125), they can have different anomalies.

This phenomenon is sometimes called *symmetry fractionalization*, even though the term is more often used for the mixing of a 0-form symmetry with higher-form symmetries [338,348][40] (which will be relevant for the 4d/5d case), but we will use the same terminology to emphasize a unified description. The crucial point is that in general there is no canonical choice and we can only talk about differences of anomalies induced by a certain class η. This is easy to implement at the level of background fields. When $A \in H^1(X_2, \mathbb{Z}_2)$ is activated, the symmetry fractionalization class changes the background $B \in H^1(X_2, \mathcal{S})$ to

$$B' = B + A^* \eta = B + \eta(A) \ . \tag{4.1.136}$$

By plugging this expression into the mixed anomaly (4.1.125) we change the pure $\mathbb{Z}_2$ anomaly, classified by $H^3\big(\mathbb{Z}_2, U(1)\big)$, by an extra piece

$$S_{\text{pure}} = 2\pi i \int_{X_3} \mu(A) \cup \eta(A) \cup \eta(A) \ \equiv \ 2\pi i \int_{X_3} A^* y \tag{4.1.137}$$

that can be written in terms of a class $y \in H^3\big(\mathbb{Z}_2, U(1)\big)$. An explicit expression for $y(g_1, g_2, g_3)$ can be derived by recasting $\mu(A) \cup \eta(A) \cup \eta(A)$ as

$$\Big(\mu(A) \cup \eta(A) \cup \eta(A)\Big)_{ijkl} = -\mu(-A_{ij}) \Big(\eta(A_{jk}) \ , \ \rho_{A_{jk}} \eta(A_{kl})\Big) \ . \tag{4.1.138}$$

This is useful because A appears with only three different pairs of indices, and we conclude that

$$y(g_1, g_2, g_3) = -\mu(-g_1) \Big(\eta(g_2) \ , \ \rho_{g_2} \eta(g_3)\Big) \ . \tag{4.1.139}$$

The possible non-triviality of this 3-cocycle is determined by its value at $g_1 = g_2 = g_3 = \underline{1}$, and we will denote simply by μ and η their values at $g = \underline{1}$. Since $\mu = \nu$ and $\rho \, \eta = -\eta$ we obtain

$$y \ \equiv \ y(\underline{1}, \underline{1}, \underline{1}) = \nu(\eta, \eta) \ . \tag{4.1.140}$$

Going back to multiplicative notation, we obtain that

$$Y = \nu(\eta, \, \eta) \ , \tag{4.1.141}$$

we will see in examples that coincides with the SPT phase obtained by the equivariantization procedure in Section 4.1.2.

[40] See, *e.g.*, [28,349–352] and references therein for discussions of symmetry fractionalization in the condensed matter literature.

Examples

Let us apply the general discussion to the previously discussed examples.

1. The first example is $\mathbb{A} = \mathbb{Z}_n$ where a duality-invariant lattice is present only for $n = p^2$. The choice of discrete torsion ν is trivial, so there is no way to shift the "bare" Frobenius-Schur indicator ϵ and the second obstruction vanishes if and only if $\epsilon = 1$.

2. Next we consider $\mathrm{TY}(\mathbb{Z}_n \times \mathbb{Z}_n)$. Choosing the diagonal bicharacter $\gamma_{1,1}^{(\mathrm{D})}$ in (4.1.80), the duality-invariant boundaries are obtained by gauging the full $\mathbb{A}$ with discrete torsion ν such that (see (4.1.74))

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} \qquad \text{with} \qquad r^2 = -1 \mod n \,. \tag{4.1.142}$$

Thus the action $\rho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{A}^\vee)$ is $\rho_{\underline{1}}(a_1, a_2) = (ra_2, -ra_1)$. We look for all possible symmetry fractionalization classes $\eta \in H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n)$, which are determined by $\eta \equiv \eta(\underline{1}) = (x_1, x_2)$ constrained by $x_2 = rx_1$. Taking into account the identification

$$(x_1, rx_1) \sim (x_1, rx_1) + (rc_2 - c_1, -rc_1 - c_2) \tag{4.1.143}$$

and setting $c_2 = 0$, $c_1 = -1$ we realize that $x_1 \sim x_1 + 1$ and hence all cocycles are exact:

$$H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n) = 0 \,. \tag{4.1.144}$$

Thus the phenomenon of symmetry fractionalization is absent in this case and there is only a single equivariantization for $\mathcal{L}_D$. The second obstruction again vanishes if and only if $\epsilon = 1$.

Choosing instead the off-diagonal bicharacter $\gamma_1^{(\mathrm{O})}$ is more interesting. As already discussed, the duality-invariant boundaries are associated with the alternating bicharacters

$$\chi_\nu = \frac{1}{n}\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} \qquad \text{with} \qquad r^2 = 1 \mod n \,. \tag{4.1.145}$$

Then $\rho_{\underline{1}}(a_1, a_2) = (-ra_1, ra_2)$ and the most general cocycle $\eta \in H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n)$ has $\eta = (x_1, x_2)$ with

$$(r-1)\, x_1 = 0 \mod n \,, \qquad\qquad (r+1)\, x_2 = 0 \mod n \,, \tag{4.1.146}$$

and is subject to the identifications $x_1 \sim x_1 - (r+1)c_1$, $x_2 \sim x_2 - (r-1)c_2$. Without loss of generality we can take $r = 1$, so that $2x_2 = 0$ and $x_1 \sim x_1 + 2$. Hence for n odd there is no symmetry fractionalization while for n even:

$$H^1_\rho(\mathbb{Z}_2, \mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad\qquad (n \text{ even}) \tag{4.1.147}$$

generated by $\eta_{s_1, s_2} = \big(s_1, \frac{n}{2} s_2\big)$ with $s_{1,2} \in \{0, 1\}$. A representative for ν is

$$\nu(a, b) = \exp\!\left(\frac{2\pi i}{n}\, a_1 b_2\right) \tag{4.1.148}$$

and therefore

$$Y = \nu(\eta, \eta) = \exp\big(\pi i\, s_1 s_2\big) = \mathrm{Arf}(\tilde{\eta}) \ . \tag{4.1.149}$$

Thus the second obstruction vanishes if and only if

$$\epsilon = 1 \quad \text{and} \quad s_1 s_2 = 0 \ , \qquad \text{or} \qquad \epsilon = -1 \quad \text{and} \quad s_1 s_2 = 1 \ , \tag{4.1.150}$$

in agreement with the discussion in [29] for the case $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and with our computations using the equivariantization of $\mathcal{L}_D$ around (4.1.119).

4.1.3 Anomalies of duality symmetries in 3+1 dimensions

We now extend the classification of anomalies for non-invertible duality defects to the four-dimensional case. As in 2d, we find that there are two obstructions to gauging a non-invertible duality symmetry. The first obstruction again hinges upon the absence of a duality-invariant bulk Lagrangian algebra $\mathcal{L}_D$. This maps to the fact that the 4d theory $\mathcal{T}$ coupled to the Symmetry TFT must admit a duality-invariant global variant. The second obstruction is the presence of a cubic anomaly:

$$\epsilon_{\text{tot}} \in \Omega_5^{\text{spin}}(BG) \ , \tag{4.1.151}$$

which can be contaminated by a mixed anomaly involving the 0-form symmetry G and a 1-form symmetry $\mathcal{S}$ through a symmetry fractionalization mechanism similar to the 2d case, now encoded in a class $\eta \in H^2_\rho(G, \mathcal{S})$.

Well-known examples of 4d theories with self-duality symmetries are the free Maxwell theory, super-Yang-Mills theories with $\mathcal{N} = 4$ supersymmetry and whose gauge algebra is invariant under Langlands duality (*i.e.*, ADEFG as well as $B_2 \cong C_2$) [7, 57, 63, 66] and various theories of class $\mathcal{S}$ [8, 68]. Understanding the anomalies in these symmetries has immediate interesting consequences. For example, it has been recently observed [80] that the $\mathcal{N} = 1^*$ massive deformation of $\mathcal{N} = 4$ SYM preserves a self-duality symmetry. The well-known results about vacuum degeneracy in $\mathcal{N} = 1^*$ can then be reinterpreted as anomaly matching conditions. A second natural application is to constrain which $\mathcal{N} = 3$ theories can be described through a discrete gauging of $\mathcal{N} = 4$, which we comment upon in the conclusions.

Duality defects

Much of our analysis in Section 4.1.2 can be generalized to self-duality defects in four-dimensional theories that are self-dual under the gauging of a 1-form symmetry $\mathbb{A}$, possibly with discrete torsion [57, 63, 66]. Again, the self-duality must be supplied with a choice of isomorphism $\phi : \mathbb{A} \to \mathbb{A}^\vee$. While a complete description of the underlying fusion 3-category $\mathcal{C}$ is still out of reach, some of the relevant data can be spelled out explicitly.[41] As stated in the introduction, this is a graded category with the grading being implemented by the duality group G, that for

[41] The Symmetry TFT analysis offers a complementary viewpoint on the data constituting the duality category on the boundary, which might be easier to handle. We explain how the data we describe here is matched between bulk and boundary in Section 4.1.3.

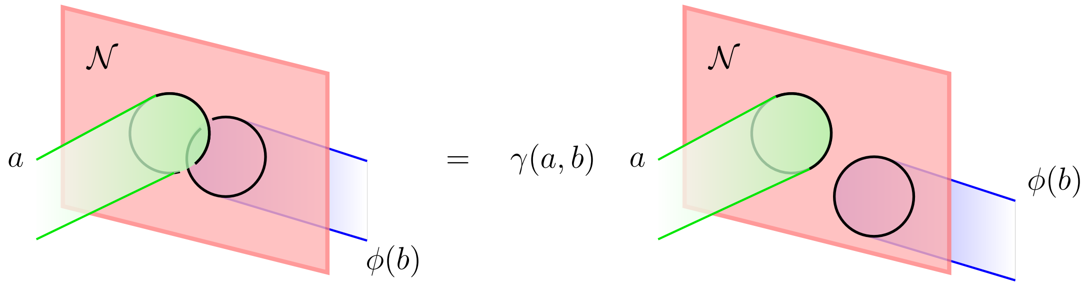

Figure 4.2: Braiding of lines W_a^L and $W_{\phi(b)}^R$ on the duality defect $\mathcal{N}$. Unlinking the line configuration gives rise to the symmetric bicharacter $\gamma(a,b)$.

now we take to be cyclic. The fusion rules take the form

$$a \times \mathcal{N}_g = \mathcal{N}_g \times a = \mathcal{N}_g \,, \qquad \mathcal{N}_g(\Sigma) \times \overline{\mathcal{N}}_g(\Sigma) = \sum_{a \,\in\, H_2(\Sigma, \mathbb{A})} a = C_{\mathbb{A}}(\Sigma) \,, \tag{4.1.152}$$

where $\mathcal{N}_{g\neq 0}$ are the duality interfaces, Σ the 3-manifold where they live, a a 1-form symmetry surface, and $C_{\mathbb{A}}(\Sigma)$ the condensate of $\mathbb{A}$ on Σ. The fusion of $\mathcal{N}_g \times \mathcal{N}_h$ is also known, and is group-like at the level of connected components, *i.e.*, forgetting the appearance of condensates (see footnote 2). It was analyzed in [7,8]. Assuming that G is cyclic, let $\mathcal{N}$ be a generator of it.

A first piece of categorical data can be obtained by noticing that the 1-form symmetry surfaces can end topologically on $\mathcal{N}$ thus defining topological line operators W_a^L and W_β^R, where L/R encode the side (Left or Right) on which the 1-form symmetry surfaces a, β end.[42] These line defects must compose according to the $\mathbb{A}$ group law, modulo undetectable decoupled objects:[43]

$$W_a^L \times W_b^L = W_{a+b}^L \tag{4.1.153}$$

and similarly for W_β^R. Following the same logic as in the Tambara-Yamagami case, we consider the braiding between endlines of 1-form symmetry surfaces a and $\beta = \phi(b)$ ending on the two sides of the duality defect $\mathcal{N}$. The endline of W_β^R is an 't Hooft line T_β^L from the point of view of the left side, and hence it braids canonically with W_a^L. We conclude that the braiding between W_a^L and $W_{\phi(b)}^R$ is given by a symmetric bicharacter γ:

$$\mathcal{B}_{W_a^L, W_{\phi(b)}^R} = \phi(b)\, a = \gamma(a,b) \,, \tag{4.1.154}$$

where the symmetry of γ follows from the fact that we should get the same result if we worked in the magnetic frame instead. The configuration is depicted in Figure 4.2.

The lines W_a^L and W_β^R form a 3d TQFT $\mathcal{A}$, but such a description is clearly non-minimal: lines of the form $\mathcal{K}_a = W_a^L \times W_{\phi(-a)}^R$ are decoupled from the bulk 1-form symmetry and constitute an undetectable sector $\mathcal{A}_0$. Quotienting this out gives the minimal description $\mathcal{A}_{\min}$ of the

[42] One could think of those as 2-morphisms $W_a^R : a \times \mathbb{1}_\mathcal{N} \to \mathbb{1}_\mathcal{N}$ and $W_\beta^L : \mathbb{1}_\mathcal{N} \times \beta \to \mathbb{1}_\mathcal{N}$, where $\mathbb{1}_\mathcal{N}$ is the identity endomorphism of $\mathcal{N}$.

[43] The importance of modding out such decoupled TQFTs has been recently emphasized in [54] in a related context.

category of lines living on the defect.[44] This produces, in general, a set of lines L_a forming a minimal TQFT $\mathcal{A}^{\mathbb{A},q}$ with 1-form symmetry $\mathbb{A}$ [353], where q a quadratic refinement of the symmetric bilinear form γ. This resonates with previous results obtained from the Symmetry TFT perspective [7].

Finally, as in the 2d case, we can associate to $\mathcal{N}$ a pure G anomaly ϵ. This is a higher analogue of the Frobenius-Schur indicator. Indeed, ϵ can be understood as a standard G 't Hooft anomaly on four-manifolds with trivial $H_2(X, \mathbb{A})$. On these manifolds, $\{\mathcal{N}_g\}$ behave as a standard invertible symmetry according to the fusion rules (4.1.152). At the level of the Symmetry TFT, the presence of a nontrivial ϵ gives a DW twist for the theory $\mathcal{Z}(\mathcal{C})$. All in all, we find that the known data defining a self-duality category in 4d, or at least a subset of it, is given by a pure anomaly ϵ for the self-duality group and a symmetric bicharacter $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$.

In the ensuing analysis we will make two simplifying assumptions. First, we will consider duality defects on spin manifolds, $w_2(TX) = 0$. The classification of discrete gauging operations (global variants of a gauge theory) is different on non-spin manifolds, as the set of discrete theta angles is larger.[45] Physically this amounts to the possibility of assigning a well-defined spin to lines as this cannot be screened by heavy neutral fermions [354]. This restriction has physical consequences on the obstruction theory outlined above: some duality defects can be anomaly free on spin manifolds, but anomalous in the presence of a nontrivial w_2.[46]. As a prototypical example, consider the $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM theory. This admits an S-invariant global variant $SO(3)_-$ on spin manifolds. On non-spin (but orientable) manifolds this variant splits into $SO(3)_-^{\text{b}}$ and $SO(3)_-^{\text{f}}$, where b/f (bosonic/fermionic) refer to the spin of the generator of the lattice of genuine lines. According to [354] (Appendix C) the two objects are interchanged by S. Thus, although the duality symmetry in $SU(2)$ $\mathcal{N} = 4$ SYM might be non-anomalous on spin manifolds, it is anomalous on generic orientable manifolds.[47]

Our second assumption is to consider duality defects for which G does not contain fermion parity. This for example excludes the vanilla S-duality of the $\mathcal{N} = 4$ SYM theory, for which $S^4 = (-1)^F$, but includes the situation where S is twisted by a discrete R-symmetry [80]. At the practical level, this implies that the relevant cobordism classification for cubic G anomalies is given by $\Omega_5^{\text{spin}}(BG)$ as opposed to $\Omega_5^{\text{spin}_G}(\text{pt})$. Both groups have been computed, *e.g.*, in [240, 355].

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[44]Formally one stacks $\mathcal{A}$ with the orientation reversal of $\mathcal{A}_0$ and gauges the diagonal symmetry $\mathbb{A} : \mathcal{A}_{\text{min}} = (\mathcal{A} \times \overline{\mathcal{A}}_0)/\mathbb{A}$.

[45]As an illuminating example, consider $\mathbb{A} = \mathbb{Z}_n$ with n even. On generic manifolds, discrete torsion terms are classified by $H^4\big(B^2\mathbb{Z}_n, U(1)\big) = \mathbb{Z}_{2n}$, while on spin manifolds the order-two element of this group vanishes due to the Wu formula $B \cup B = B \cup \big(w_2 + w_1^2\big)$ mod 2, where $w_j(TX)$ are the Stiefel-Whitney classes of X. This discussion generalizes to arbitrary $\mathbb{A}$ in a straightforward manner.

[46]Loosely speaking, this is some kind of mixed anomaly with gravity, due to the dependence on $w_2(TX)$.

[47]We will briefly comment on the interpretation of this fact from the point of view of gapped phases in Appendix B.6.

Symmetry TFT and Lagrangian algebras

The Symmetry TFT for 4d duality defects can be described in close analogy with the 2d case [7, 133]. We start from a 5d Dijkgraaf-Witten theory for a 1-form symmetry $\mathbb{A}$ with trivial twist. This has topological surface operators labelled by pairs $(a,\alpha) \in \mathbb{A} \times \mathbb{A}^\vee$ with antisymmetric canonical braiding

$$\mathcal{B}_{(a_1,\alpha_1),(a_2,\alpha_2)} = \alpha_1(a_2)\, \alpha_2(a_1)^{-1} \quad \in U(1) \; . \tag{4.1.155}$$

As in three dimensions, the 5d pure 2-form gauge theory for $\mathbb{A}$ enjoys *electric-magnetic* duality, corresponding to a choice of isomorphism ϕ.[48] There is an important difference, though, with respect to the 3d case. The most general ansatz for a duality is

$$\begin{array}{rccc} S: & \mathbb{A} \times \mathbb{A}^\vee & \to & \mathbb{A} \times \mathbb{A}^\vee \\ & (a,\alpha) & \mapsto & \big(I \circ \phi^{-1}(\alpha),\, \phi(a)\big) \end{array} \tag{4.1.156}$$

for some automorphism $I : \mathbb{A} \to \mathbb{A}$ to be determined. Let $\gamma : \mathbb{A} \times \mathbb{A} \to U(1)$ be the bicharacter associated with ϕ, namely $\gamma(a,b) = \phi(a)\, b$, then S preserves the braiding if and only if

$$\gamma\big(a,\, I \circ \phi^{-1}(\alpha)\big)\, \gamma\big(\phi^{-1}(\alpha),\, a\big) = 1 \; . \tag{4.1.157}$$

This equation may have multiple solutions, depending on the Abelian group $\mathbb{A}$, but here we limit ourselves to EM dualities that can be defined universally. If I is the identity then γ is an antisymmetric non-degenerate bicharacter, which however does not exist for all Abelian groups[49] and thus we will not study this case any further. On the other hand, if I is the inversion

$$I(a) = -a \tag{4.1.158}$$

then γ must be a symmetric non-degenerate bicharacter, which always exists. We will thus consider this case in the following.[50] With this choice of definition, S is an order-four automorphism:

$$S^2(a,\alpha) = (-a,-\alpha) \qquad \Rightarrow \qquad S^2 = C \; , \tag{4.1.159}$$

where we defined the charge-conjugation operator $C : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A} \times \mathbb{A}^\vee$. The 5d DW($\mathbb{A}$) theory enjoys a larger set of 0-form symmetries, for any group $\mathbb{A}$. Indeed we can define another generator

$$T : (a,\, \alpha) \mapsto \big(a + \phi^{-1}(\alpha),\, \alpha\big) \; , \tag{4.1.160}$$

and in this way construct an order-three automorphism of $\mathbb{A} \times \mathbb{A}^\vee$:

$$CST : (a,\, \alpha) \mapsto \big(\phi^{-1}(\alpha),\, -\alpha - \phi(a)\big) \qquad \text{such that} \qquad (CST)^3 = \mathbb{1} \; . \tag{4.1.161}$$

[48] This isomorphism appears in the construction of the element S of EM duality in the bulk (which might not be a symmetry element of the boundary theory), while the isomorphism we used in (4.1.154) was associated with the generator $\mathcal{N}$ of G on the boundary. As we discuss below, the two isomorphisms are essentially the same, possibly up to composition with charge conjugation.

[49] For instance $\mathbb{A} = \mathbb{Z}_n$ with $n \neq 2$ does not admit any.

[50] Other automorphisms I may exist and lead to EM dualities for certain Abelian groups $\mathbb{A}$.

The Symmetry TFT for the duality or triality defects is then defined by gauging the group G generated by S or CST, respectively. This gauging admits a choice of discrete torsion, which on spin manifolds is classified by

$$\epsilon \in \Omega_5^{\text{spin}}(BG) \,, \tag{4.1.162}$$

and can be thought of as the higher analogue of the Frobenius-Schur indicator we introduced before.

Notice that if we gauge the group $G = \mathbb{Z}_4$ generated by S, the generator maps $(b,0) \overset{S}{\mapsto} \big(0,\phi(b)\big)$ and thus the isomorphism ϕ appearing in (4.1.156) is precisely the one extracted from the boundary theory using (4.1.154). The same is true if we gauge $G = \mathbb{Z}_6$ generated by ST since $(b,0) \overset{ST}{\mapsto} \big(0,\phi(b)\big)$. On the other hand, if we gauge $G = \mathbb{Z}_3$ generated by CST, the isomorphisms in (4.1.156) and (4.1.154) differ by C.

The same argument for the first obstruction corresponding to the absence of G-invariant Lagrangian algebras in the DW($\mathbb{A}$) theory carries over to the 5d case. We are thus led to study the properties of gapped boundaries of the pure 2-form gauge theory for $\mathbb{A}$. These are labelled by two discrete choices, as in 2d:

- a subgroup $\mathbb{B} \subset \mathbb{A}$ to be gauged;
- a class $[\nu] \in H^4\big(B^2\mathbb{B}, U(1)\big)$ specifying the discrete torsion.

Recall that in 2d the discrete-torsion classes are classified by alternating bicharacters. The analog here is the identification of $H^4\big(B^2\mathbb{B}, U(1)\big)$ with the dual of the universal quadratic group $\Gamma(\mathbb{B})$ (see [178, 356] for details):

$$H^4\big(B^2\mathbb{B}, U(1)\big) \;\cong\; \Gamma(\mathbb{B})^\vee \,. \tag{4.1.163}$$

This means that any discrete torsion class $[\nu]$ is represented by a quadratic function $q_\nu : \mathbb{B} \to U(1)$. The group $\Gamma(\mathbb{B})$ is equipped with a quadratic function $\mathcal{Q} : \mathbb{B} \to \Gamma(\mathbb{B})$ and is such that for any Abelian group V, any quadratic function $q : \mathbb{B} \to V$ factorizes as $q = \tilde{q} \circ \mathcal{Q}$ with $\tilde{q} : \Gamma(\mathbb{B}) \to V$ a group homomorphism. Therefore, a quadratic function $q_\nu : \mathbb{B} \to U(1)$ is represented by a group homomorphism $\tilde{q}_\nu : \Gamma(\mathbb{B}) \to U(1)$. The topological term implementing the discrete torsion is

$$S_{\text{torsion}} = \int_{X_4} B^*\nu = \int_{X_4} \tilde{q}_\nu\big(\mathfrak{P}(B)\big) \,. \tag{4.1.164}$$

Here $\mathfrak{P} \in H^4\big(B^2\mathbb{B}, \Gamma(\mathbb{B})\big)$ is the special element whose representative homomorphism[51] is the identity map, $\tilde{q}_{\mathfrak{P}} : \Gamma(\mathbb{B}) \xrightarrow{\text{id}} \Gamma(\mathbb{B})$, called the universal Pontryagin square class. Then one constructs its pull back $\mathfrak{P}(B) \equiv B^*\mathfrak{P} \in H^4\big(X_4, \Gamma(\mathbb{B})\big)$ which is called the Pontryagin square of B, whilst $\tilde{q}_\nu \in \Gamma(\mathbb{B})^\vee$ is the homomorphism associated with the quadratic function q_ν.

As already explained in Section 4.1.2, each quadratic function q_ν has an associated symmetric bicharacter $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$. Crucially, if X_4 is a four-dimensional spin manifold, then two discrete torsions ν, ν' leading to two quadratic functions q_ν, $q_{\nu'}$ which are different quadratic refinements of the same bicharacter, lead to the same topological term [353, 354]:

[51] Indeed (4.1.163) generalizes to $H^4\big(B^2\mathbb{B}, \mathbb{C}\big) \cong \text{Hom}\big(\Gamma(\mathbb{B}), \mathbb{C}\big)$ for any Abelian group $\mathbb{C}$.

$\int_{X_4} B^*\nu = \int_{X_4} B^*\nu'$. Thus, by working on spin manifolds, we can safely label topological manipulations of the boundary theory in terms of a choice of subgroup $\mathbb{B} \subset \mathbb{A}$ and of a symmetric bicharacter χ_ν. Then most of the results will be closely analogous to the 2d/3d case, just replacing antisymmetric with symmetric bicharacters.

As explained, on spin manifolds we can label the Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ in terms of the data $(\mathbb{B}, \chi_\nu)$. The corresponding gapped boundary has a 1-form symmetry

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big)/\mathcal{L}_{\mathbb{B},[\nu]} \; . \tag{4.1.165}$$

One can easily adapt the 3d discussion in order to explicitly write the form of the general Lagrangian algebra. The symmetric bicharacter $\chi_\nu : \mathbb{B} \times \mathbb{B} \to U(1)$ induces a group homomorphism $\psi_\nu : \mathbb{B} \to \mathbb{B}^\vee$ as in the 3d case. Given a pair $(\mathbb{B}, \chi_\nu)$ we construct the Lagrangian algebra $\mathcal{L}_{\mathbb{B},[\nu]} \subset \mathbb{A} \times \mathbb{A}^\vee$ as

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b,\, \beta\psi_\nu(b)\big) \in \mathbb{A} \times \mathbb{A}^\vee \;\Big|\; b \in \mathbb{B}\,, \quad \beta \in N(\mathbb{B}) \Big\} \; . \tag{4.1.166}$$

This has cardinality $|\mathbb{A}|$ and is Lagrangian since $\mathcal{B}_{(b_1,\beta_1),(b_2,\beta_2)} = \chi_\nu(b_2, b_1)\, \chi_\nu(b_1, b_2)^{-1} = 1$, where (b, β) is a shorthand for $\big(b, \beta\psi_\nu(b)\big)$ and we used the symmetry of χ_ν. As in the 3d case (see Appendix B.2) one can show that all Lagrangian algebras of the 5d DW($\mathbb{A}$) theory are of this form.

First obstruction

After fixing a choice of electric-magnetic duality, we ask what are the conditions for a duality-invariant Lagrangian algebra $\mathcal{L}_D = \Phi(\mathcal{L}_D)$ to exist. We will study two cases: $\Phi = S$ (duality) and $\Phi = CST$ (triality). Other cyclic 0-form symmetry groups, when they exist, can be treated similarly. As we previously showed, all Lagrangian algebras are of the form (4.1.166). To verify whether a lattice is Φ-invariant, as in 3d, we impose that the pairing between $\mathcal{L}$ and $\Phi(\mathcal{L})$ be trivial. The analysis is analogous to the 3d case. For both choices of Φ, we find the necessary condition

$$\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B}) \; , \tag{4.1.167}$$

where $\mathrm{Rad}(\nu)$ is the kernel of ψ_ν. As in the 3d case, this implies that $|\mathbb{B}|^2 = k|\mathbb{A}|$ for some positive integer $k = \big|\mathbb{B}/\,\mathrm{Rad}(\nu)\big| \in \mathbb{N}$, and again $\mathbb{B}$ cannot be smaller than Lagrangian. Notice however that since χ_ν is now symmetric rather than antisymmetric, we cannot conclude that $|\mathbb{A}|$ (and in particular k) is a perfect square. Indeed we will see explicit counterexamples, hence showing that in higher categories the obstruction from non-integer quantum dimensions of [12] does not hold.

The remaining conditions depend on Φ and are listed below.

Duality. The automorphism $\sigma = \phi^{-1}\psi_\nu$ of $\mathbb{B}/\mathrm{Rad}(\nu)$ must satisfy

$$\sigma^2 = -1 \; . \tag{4.1.168}$$

In particular σ allows us to relate the two symmetric bicharacters as

$$\gamma\big(\sigma(a),\, b\big) = \chi_\nu(a, b) \; . \tag{4.1.169}$$

From the two equations above it follows that σ is an order-two automorphism of the group of symmetric bilinear forms on $\mathbb{B}/\text{Rad}(\nu)$:

$$\chi_\nu\big(\sigma(a),\sigma(b)\big)\ \chi_\nu(a,b) = 1\ . \tag{4.1.170}$$

Triality. The automorphism $\tau = \phi^{-1}\psi_\nu$ must satisfy

$$1 + \tau + \tau^2 = 0\ . \tag{4.1.171}$$

It is simple to show that the above implies that τ is an order-three operation: $\tau^3 = 1$. Also in this case, the restriction to $\mathbb{B}/\text{Rad}(\nu)$ of

$$\gamma\big(\tau(a),\, b\big) = \chi_\nu(a,b) \tag{4.1.172}$$

holds. Using the two above equations it follows that τ is an order-three automorphism of the group of symmetric bilinear forms on $\mathbb{B}/\text{Rad}(\nu)$:

$$\chi_\nu\big(\tau^2(a),\tau^2(b)\big)\ \chi_\nu\big(\tau(a),\tau(b)\big)\ \chi_\nu(a,b) = 1\ . \tag{4.1.173}$$

Examples

1. Let us study the case of $\mathbb{A} = \mathbb{Z}_n$ with the standard symmetric bicharacter $\gamma(a_1,a_2) = \exp\big(\frac{2\pi i}{n}\, a_1 a_2\big)$. Consider a factorization $n = pq$ and a subgroup

$$\mathbb{B} = \big\{ b\, q \,\big|\, b = 0,\ldots,p-1 \big\} \cong \mathbb{Z}_p \tag{4.1.174}$$

so that $N(\mathbb{B}) \cong \mathbb{Z}_q$. Since duality invariance requires $\mathbb{B}$ to contain $\phi^{-1}\big(N(\mathbb{B})\big)$ as a subgroup, q must divide p and we set $p = \ell q$. A choice of ψ_ν is associated with another symmetric bicharacter χ_ν defined on $\mathbb{B}$:

$$\chi_\nu(b_1,b_2) = \exp\left(\frac{2\pi i r}{p}\, b_1 b_2\right)\ , \tag{4.1.175}$$

where $r \in \{0,\ldots,p-1\}$. Notice that $\text{Rad}(\nu) \cong \mathbb{Z}_{\gcd(r,p)}$ hence imposing $\phi\big(\text{Rad}(\nu)\big) = N(\mathbb{B})$ forces $\gcd(r,p) = q$, namely $r = sq$ with $\gcd(s,\ell) = 1$. Furthermore, since the restriction of γ to $\mathbb{B}$ is $\gamma(qb_1,qb_2) = \exp\big(\frac{2\pi i}{\ell}\, b_1 b_2\big)$, over $\mathbb{B}/\text{Rad}(\nu) \cong \mathbb{Z}_p/\mathbb{Z}_q \cong \mathbb{Z}_\ell$ we have

$$\sigma(b) = \phi^{-1}\psi_\nu(b) = s\, b \mod \ell\ . \tag{4.1.176}$$

Thus we find that:

1. On spin manifolds, there is a duality-invariant $\mathcal{L}_D$ for $\mathbb{A} = \mathbb{Z}_n$ if and only if there exists an ℓ such that $n = \ell q^2$ and -1 is a quadratic residue mod ℓ, *i.e.*, there exists also an s such that

$$s^2 = -1 \mod \ell\ . \tag{4.1.177}$$

This equation has solutions for $\ell = 1, 2, 5, 10, 13, 17, 25, 26, \ldots$

2. On spin manifolds, there is a triality-invariant $\mathcal{L}_D$ for $\mathbb{A} = \mathbb{Z}_n$ if and only if there exist ℓ, s such that $n = \ell q^2$ and

$$s^2 + s + 1 = 0 \mod \ell \,. \tag{4.1.178}$$

This equation has solutions for $\ell = 1, 3, 7, 13, 19, 21, 31, 37, \ldots$

These results coincide with the recent classification [78] of 4d topological $\mathbb{Z}_n$ gauge theories that are duality or triality invariant on spin manifolds.[52] As in the 3d case, we provide a precise connection between the two approaches in Appendix B.6.

2. Another interesting case to consider is $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is the 1-form symmetry group of a Spin$(4k)$ gauge theory. On $\mathbb{Z}_2 \times \mathbb{Z}_2$ there are four symmetric non-degenerate quadratic forms:

$$\gamma^{(D)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{(O)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{-} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.1.179}$$

In this case -1 acts as the identity on $\mathbb{A}$ and duality is an involution. Thus given any choice of γ, the first obstruction is cancelled by choosing $\mathbb{B} = \mathbb{A}$, $\sigma = 1$ and $\chi_\nu = \gamma$. The case of triality is slightly more involved. Let us consider $\mathbb{B} = \mathbb{A}$. It is simple to show that the only two $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphisms τ satisfying $\tau^2 + \tau + 1 = 0$ are $\tau^{\pm} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which are inverses to each other. If $\gamma = \gamma^{(D)}$ we can solve the triality obstruction by taking $\chi_\nu = \gamma^{\pm}$ and $\tau = \tau^{\pm}$, similarly if $\gamma = \gamma^{\pm}$ we can take $\chi_\nu = \gamma^{(D)}$ and $\tau = \tau^{\mp}$. On the other hand, if $\gamma = \gamma^{(O)}$ then $\gamma\big(\tau^{\pm}(a), b\big)$ is not symmetric and the obstruction is present for $\mathbb{B} = \mathbb{A}$. Let us then consider $\gamma = \gamma^{(O)}$ and $\mathbb{B} = \mathbb{Z}_2$. Since $N(\mathbb{B})$ is also $\mathbb{Z}_2$, we must have that $\mathbb{B} = \phi\big(N(\mathbb{B})\big)$. It is simple to verify that taking $\mathbb{B}$ to be the diagonal $\mathbb{Z}_2$ this is indeed satisfied. We conclude that the first obstruction for $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ vanishes for both duality and triality.

This example, combined with the previous one, allows us to discuss the first obstruction for $\mathcal{N} = 4$ Spin$(2m)$ SYM (and its global variants). Recall that the 1-form symmetry group is

$$\mathbb{A} = \begin{cases} \mathbb{Z}_4 & \text{if } m = 2k+1 \,, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } m = 2k \,. \end{cases} \tag{4.1.180}$$

We thus find that the first obstruction vanishes in all cases.

Second obstruction and symmetry fractionalization

While in absence of duality- (or triality-) invariant Lagrangian algebras the non-invertible self-duality symmetry is anomalous, when such an invariant algebra does exist the anomalies are determined by those on the invariant boundary, where the symmetry is invertible. The philosophy is the same as in the 2d/3d case: due to a mixed anomaly between the 1-form

[52] Let us also notice a few facts. In the case of duality invariance, the possible values of ℓ are those that can be written as $\ell = x^2 + y^2$ for coprime x, y. The condition can never be satisfied by ℓ multiple of 4; indeed, if ℓ is even then s must be odd, but then $s^2 = 1 \bmod 4$. In the case of triality invariance, the possible values of ℓ are those that can be written as $\ell = x^2 + xy + y^2$ for coprime x, y. The condition can never be satisfied by ℓ multiple of 9.

symmetry $\mathcal{S} = \mathbb{A} \times \mathbb{A}^\vee/\mathcal{L}_D$ and the invertible duality symmetry G we can shift the value of the pure G anomaly by changing the symmetry fractionalization class $\eta \in H^2_\rho(G, \mathcal{S})$. We now determine the mixed anomaly in the simpler case $\mathbb{B} = \mathbb{A}$, the generalization to proper subgroups being straightforward but technically tedious.

Duality. In the case of $\Phi = S$ and so $G = \mathbb{Z}_4$ the invariant partition function is given by:[53]

$$Z_{\text{inv}}\big[\phi(B)\big] = \sum_{b \in H^2(X,\mathbb{A})} \exp\left(2\pi i \int_{X_4} b^*\nu + 2\pi i \int_{X_4} \phi(B) \cup b\right) Z_{\text{e}}[b] \,, \tag{4.1.181}$$

where Z_{e} is the partition function corresponding to the reference electric boundary condition, while ν is defined through a bicharacter χ_ν such that

$$\gamma\big(\sigma(a),\, b\big) = \chi_\nu(a,b) \qquad \text{and} \qquad \sigma^2 = -1 \,. \tag{4.1.182}$$

The action of S-duality on Z_{inv} is easily determined using the action of S-duality on the electric theory:

$$S \cdot Z_{\text{e}}[B] = \sum_{a \in H^2(X,\mathbb{A})} \exp\left(2\pi i \int_{X_4} \phi(B) \cup a\right) Z_{\text{e}}[a] \,. \tag{4.1.183}$$

We find

$$S \cdot Z_{\text{inv}}\big[\phi(B)\big] = G_\nu \exp\left(2\pi i \int_{X_4} B^*\nu\right) Z_{\text{inv}}\big[\phi(\sigma B)\big] \,, \tag{4.1.184}$$

where $G_\nu \equiv \sum_{b \in H^2(X,\mathbb{B})} \exp\big(2\pi i \int_X b^*\nu\big)$. Here, assuming that X_4 is spin, we used the simplifying relation

$$\exp\left(2\pi i \int_{X_4} B^*\big(\nu \circ \sigma\big)\right) = \exp\left(-2\pi i \int_{X_4} B^*\nu\right) . \tag{4.1.185}$$

Assuming that X_4 is simply connected (and thus $H^2(X_4, \mathbb{Z})$ has no torsion classes) and spin, one can show that the Gauss sum G_ν is equal to 1 [78]. In a similar way we can verify that

$$S^2 \cdot Z_{\text{inv}}\big[\phi(B)\big] = Z_{\text{inv}}\big[-\phi(B)\big] = C \cdot Z_{\text{inv}}\big[\phi(B)\big] \,. \tag{4.1.186}$$

Eqn. (4.1.184) implies that the $\mathbb{Z}_4$ symmetry generated by S acts on the 1-form symmetry of the theory through σ, *i.e.*, the symmetry is a split 2-group with nontrivial action $\rho : G \to \text{Aut}(\mathbb{A})$ [178, 356, 357] given by $\rho_{\underline{1}}(a) = \sigma\, a$. Furthermore, the overall phase $\exp\big(2\pi i \int_X B^*\nu\big)$ should be thought of as encoding a mixed anomaly

$$\mu \,\in\, H^1_\rho\Big(\mathbb{Z}_4,\, H^4\big(B^2\mathbb{A},\, U(1)\big)\Big) \qquad\qquad \text{where} \qquad\qquad \mu(\underline{1}) = \nu \tag{4.1.187}$$

and $\underline{1}$ is the generator of $G = \mathbb{Z}_4$, much as in the 2d case.

Triality. For $\Phi = CST$ and so $G = \mathbb{Z}_3$ we have the same expression (4.1.181) for $Z_{\text{inv}}\big[\phi(B)\big]$, but with the class ν now satisfying (4.1.173) in terms of a τ such that $\tau^2 + \tau + 1 = 0$. T-duality acts on the electric boundary as

$$T \cdot Z_e[B] \equiv \exp\left(-2\pi i \int_{X_4} B^*\gamma\right) Z_e[B] \,. \tag{4.1.188}$$

[53] For simplicity we omit the normalization factors due to gauging.

Then

$$(CST)\cdot Z_{\rm e}[B] = \exp\left(2\pi i \int_{X_4} B^*\gamma\right) \sum_{a\,\in\, H^2(X,\mathbb{A})} \exp\left(2\pi i \int_{X_4} \phi(B)\cup a\right) Z_{\rm e}[a] \tag{4.1.189}$$

with $B^*\gamma$ any class stemming from a quadratic refinement of γ, *i.e.* the Pontryagin square induced by γ, and we find

$$(CST)\cdot Z_{\rm inv}\big[\phi(B)\big] = G_{\gamma+\nu}\, \exp\left(2\pi i \int_{X_4} B^*\nu\right) Z_{\rm inv}\big[\phi(\tau B)\big] \tag{4.1.190}$$

Here we used that, on spin manifolds, $\exp\big[2\pi i \int B^*\big(\nu + \nu\circ\tau + \nu\circ\tau^2\big)\big] = 1$. It also holds that

$$(CST)^3 \cdot Z_{\rm inv}\big[\phi(B)\big] = Z_{\rm inv}\big[\phi(B)\big]\ . \tag{4.1.191}$$

As before, the result is interpreted by saying that the split 2-group is twisted by the $\mathbb{Z}_3$ symmetry and the overall phase comes from a mixed anomaly

$$\mu \,\in\, H^1_\rho\Big(\mathbb{Z}_3,\; H^4\big(B^2\mathbb{A},\, U(1)\big)\Big) \qquad\qquad \text{where} \qquad\qquad \mu(\underline{1}) = \nu\ . \tag{4.1.192}$$

We thus conclude that, similarly to the 3d case, the 5d mixed anomaly is determined by a class

$$\mu \in H^1_\rho\Big(G,\; H^4\big(B^2\mathbb{A}, U(1)\big)\Big) \cong H^1_\rho(G, \Gamma(\mathbb{A})^\vee) \tag{4.1.193}$$

namely a function from G to the group of quadratic functions over $\mathbb{A}$ satisfying

$$\rho_g\, \mu(h) + \mu(g) = \mu(g+h) \tag{4.1.194}$$

and subject to the the identification

$$\mu(g) \;\cong\; \mu(g) + \rho_g\,\xi - \xi \qquad\qquad \text{for any} \qquad\quad \xi \in H^4\big(B^2\mathbb{A}, U(1)\big)\ . \tag{4.1.195}$$

The full detailed derivation of the anomaly inflow is given in Appendix B.4.2 and we find

$$S_\mu = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(B)\ . \tag{4.1.196}$$

To reproduce the anomalous phase arising in the boundary theory we have to compare this phase with the boundary term arising in S_μ from $A + d\lambda$ when we set the pull-back of A to the boundary to zero, as well as the boundary value of λ equal to the element of the group G for which we compute the variation. This determines all the values of $\mu(g)$ for $g \in G$. We can check that the consistency (4.1.194) of these values is satisfied. In the case of duality $G = \mathbb{Z}_4$, since ν satisfies $\nu\big(\sigma(a), \sigma(b)\big) = -\nu(a,b)$, we deduce that

$$\mu(\underline{1}) = \mu(\underline{3}) = \nu\ , \qquad\qquad \mu(\underline{0}) = \mu(\underline{2}) = 0\ . \tag{4.1.197}$$

It is obvious that (4.1.194) is satisfied.

For triality $G = \mathbb{Z}_3$ the crucial relation is

$$\gamma(a,b) \;+\; \gamma\big(\tau(a), \tau(b)\big) \;+\; \gamma\big(\tau^2(a), \tau^2(b)\big) = 0\ . \tag{4.1.198}$$

By looking at the anomalous phases that we got this implies that

$$\mu(\underline{0}) = 0 \ , \qquad \mu(\underline{1})(a,b) = \gamma\big(\tau(a),\tau(b)\big) \ , \qquad \mu(\underline{2})(a,b) = \gamma\big(\tau(a),\tau(b)\big) + \gamma(a,b) \ . \tag{4.1.199}$$

Among the consistency relations (4.1.194), the only non-trivial (and independent) ones to check are: $\tau\mu(\underline{1}) + \mu(\underline{1}) = \mu(\underline{2})$, $\tau\mu(\underline{2}) + \mu(\underline{1}) = 0$ and $\tau^2\mu(\underline{2}) + \mu(\underline{2}) = \mu(\underline{1})$, which are indeed satisfied thanks to (4.1.198).

Given such a mixed anomaly, we are now able to discuss the pure G anomaly. The philosophy is the same as in the 2d/3d case: combining the choice of symmetry fractionalization with the mixed anomaly we can induce an extra contribution to the pure anomaly for the invertible duality symmetry. The details are however slightly different.

In 4d symmetry fractionalization is classified by $\eta \in H^2_\rho(G,\mathbb{A})$, which, as opposed to the 2d case where it corresponds to the choice of a G subgroup of the full symmetry, here it corresponds to the choice of a 1-form symmetry defect $\eta(g,h) \in \mathbb{A}$ inserted along the junction of the intersection of g,h and gh defects. This amounts to redefine the coupling of the 0-form symmetry to a background, prescribing that B is shifted to

$$B' = B + A^*\eta \in H^2_\rho(X,\mathbb{A}) \ . \tag{4.1.200}$$

By plugging this expression into the mixed anomaly (4.1.196) we shift the pure G anomaly by an extra piece

$$S_{\text{pure}} = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(A^*\eta) \ \equiv \ 2\pi i \int_{X_5} A^* y \tag{4.1.201}$$

that can be written in terms of a class $y \in H^5\big(G, U(1)\big)$. In order to work out an explicit expression for this class we rely on a working assumption. We note that the Pontryagin square operation, when the homology group $H_2(X_5,\mathbb{Z})$ is torsion-free, can be written as a cup product[54] [356]:

$$\mathfrak{P}_\rho(A^*\eta) = A^*\eta \cup A^*\eta \ . \tag{4.1.202}$$

On the other hand the pure G anomaly is non-trivial when the homology group $H_1(X_5,\mathbb{Z})$ contains torsion [240]. Therefore, in order to do the computation, we pick a bulk spin manifold X_5^* with torsion 1-cycles but with torsion-free 2-cycles so as to write (4.1.201) as

$$S_{\text{pure}} = 2\pi i \int_{X_5^*} \mu(A) \cup A^*\eta \cup A^*\eta \ . \tag{4.1.203}$$

Then it is easy to conclude that

$$y\big(g_1,g_2,g_3,g_4,g_5\big) = \langle -\mu(-g_1) \,, \eta(g_2,g_3)\, \rho_{g_2+g_3}\, \eta(g_4,g_5) \rangle \ . \tag{4.1.204}$$

where the product in the second entry should be interpreted as in footnote 54. When the second entry is the image of a quadratic function $\gamma : \mathbb{A} \to \Gamma(\mathbb{A})$ the above expression can

[54] The expression (4.1.202) should be interpreted as follows. One writes $\mathbb{A} = \oplus_i \mathbb{Z}_{n_i}$ and lift $A^*\eta$ to $\oplus_i \mathbb{Z}$, which is always possible for finite Abelian groups. In $\oplus_i \mathbb{Z}$ we can take the product among the various components of the lift, then (4.1.202) is obtained restricting the result to $\Gamma\left(\oplus_i \mathbb{Z}_{n_i}\right) = \bigoplus_i \Gamma\left(\mathbb{Z}_{n_i}\right) \oplus \bigoplus_{i<j} \mathbb{Z}_{n_i} \otimes \mathbb{Z}_{n_j}$. If X_5 has torsion 1-cycles the Pontryagin square is not a cup product and in order to write it in components we need Steenrod's cup products (see *e.g.* [178]).

be rewritten in a simpler form using the universal property defining $\Gamma(\mathbb{A})$ (see the discussion around (4.1.163)). In particular if we can find a representative for η that is invariant under the ρ action, setting $g_2 = g_4$ and $g_3 = g_5$, we have

$$y\big(g_1, g_2, g_3, g_2, g_3\big) = \langle -\mu(-g_1)\,, \eta(g_2, g_3)\, \eta(g_2, g_3)\rangle = -\mu(-g_1)(\eta(g_2, g_3))\;. \tag{4.1.205}$$

Examples

We now discuss how this general story applies to examples where $\mathbb{A} = \mathbb{Z}_n$ and G is either $\mathbb{Z}_4$ or $\mathbb{Z}_3$, namely duality and triality respectively. This has some consequence for the anomaly structure of $\mathcal{N} = 4$ SYM theories with gauge group $\mathrm{SU}(n)$ at $\tau = i, e^{\frac{2\pi i}{3}}$ respectively.

Several technical details on the computations of the twisted cohomology groups are based on the following known result (see *e.g.* [187]). If $G \cong \mathbb{Z}_k$, denoting $f = \rho_{\underline{1}} \in \mathrm{Aut}(\mathbb{A})$ (note that $f^k = 1$), then

$$H^n_\rho(G, \mathbb{A}) \cong \begin{cases} \dfrac{\mathrm{Ker}(1-f)}{\mathrm{Im}(1+f+f^2+\ldots+f^{k-1})} & \text{if } n \text{ is even} \\[2ex] \dfrac{\mathrm{Ker}(1+f+f^2+\ldots+f^{k-1})}{\mathrm{Im}(1-f)} & \text{if } n \text{ is odd} \end{cases} \tag{4.1.206}$$

The symmetry fractionalization classes are classified by $H^2_\rho(G, \mathbb{A})$, and we notice that in both the duality and triality examples we have

$$1 + f + f^2 + \ldots + f^{k-1} = 0 \tag{4.1.207}$$

by virtue of the relations $\sigma^2 = -1$, $\tau^2 + \tau + \tau = 1$. Hence for us

$$H^2_\rho(G, \mathbb{A}) = \mathrm{Ker}(1-f) = \{a \in \mathbb{A} \mid \rho_{\underline{1}}(a) = a\} = \mathrm{Fix}_{\rho_{\underline{1}}}(\mathbb{A})\;. \tag{4.1.208}$$

This also gives a hint for the form of the explicit representatives of the non-trivial twisted cocycles as

$$\eta_x(\underline{1}, \underline{1}) = x\;, \quad x \in \mathrm{Fix}_{\rho_{\underline{1}}}(\mathbb{A}) \tag{4.1.209}$$

Duality. For the case of duality $G \cong \mathbb{Z}_4$ we have

$$\rho_{\underline{1}}(a) = ta\;, \quad t^2 = -1 \bmod(n)\;. \tag{4.1.210}$$

Using (4.1.208) we get

$$H^2_\rho(\mathbb{Z}_4, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \tag{4.1.211}$$

and in the even case the cocycles can be represented

$$\eta_s(\underline{1}, \underline{1}) = \eta_s(\underline{3}, \underline{3}) = \eta_s(\underline{1}, \underline{3}) = \eta_s(\underline{3}, \underline{1}) = \frac{n}{2} s\;, \qquad s = 0, 1 \tag{4.1.212}$$

with all the other values vanishing. By setting $n = 2m$, the pure anomaly is determined by the value of the 5-cocycle $Y \in H^5(\mathbb{Z}_4, U(1))$ in $g_1 = \ldots = g_5 = \underline{1}$ and we get

$$Y = q_\nu(\eta_s(\underline{1}, \underline{1})) = e^{2\pi i t s^2 \frac{m}{4}}\;. \tag{4.1.213}$$

We conclude that for n odd the pure duality anomaly on the invertible boundary is the bare one, given by $\epsilon \in H^5(\mathbb{Z}_4, U(1)) \cong \mathbb{Z}_4$, while for n even the cancellation depends on the possible values of Y. Recall that the first obstruction never vanishes when m is even. Therefore the possible values of Y are

$$Y = \exp\left(\frac{\pi i}{2} t(2k+1)\right) \qquad \text{for } n = 2(2k+1)\,. \tag{4.1.214}$$

In the $\mathcal{N}=4$ theory with gauge group SU(n) at $\tau = i$ the non-invertible duality symmetry is anomalous whenever it is intrinsically non-invertible, on spin manifolds we have given the relevant condition for $\mathbb{A} = \mathbb{Z}_n$ around equation (4.1.177). If the defect is non-intrinsically non-invertible the anomaly automatically vanishes provided we combine the duality with an appropriate R-symmetry rotation in order to have a $\mathbb{Z}_4$ operation (see *e.g.* [341, 358]). Indeed following [359] and using that $\Omega_5^{spin}(B\mathbb{Z}_4) \cong \mathbb{Z}_4$ one gets [80]

$$\epsilon = 60(n-1) - 24(n^2-1) \bmod(4) = 0\,, \tag{4.1.215}$$

therefore one should choose the trivial fractionalization class to cancel the second obstruction. One could also consider other definitions of S-duality which do not involve the R-symmetry, in such cases the relevant bordism group $\Omega^{\text{spin}-\mathbb{Z}_8}(pt) = \mathbb{Z}_{32} \oplus \mathbb{Z}_2$ is larger and our techniques would need to be refined in order to appropriately account for the cubic anomaly.

A similar conclusion applies to Maxwell theory, for which $S^4 = 1$ and the anomaly $56 \bmod(4) = 0$ also identically vanishes.

Triality. In the triality case $G \cong \mathbb{Z}_3$,

$$\rho_{\underline{1}}(a) = ta\ , \qquad t^2 + t + 1 = 0 \tag{4.1.216}$$

for which we get

$$H^2_\rho(\mathbb{Z}_3, \mathbb{Z}_n) \cong \begin{cases} \mathbb{Z}_3 & \text{if } n = 0 \bmod(3) \\ 0 & \text{otherwise} \end{cases} \tag{4.1.217}$$

and the (non)trivial cocycles are

$$\eta_s(\underline{1},\underline{1}) = \eta_s(\underline{2},\underline{2}) = \eta_s(\underline{1},\underline{2}) = \eta_s(\underline{2},\underline{1}) = \frac{n}{3}s\ , \qquad s = 0,1,2 \tag{4.1.218}$$

with all the other values vanishing. Setting $n = 3m$, the class $Y \in H^5(\mathbb{Z}_3, U(1)) \cong \mathbb{Z}_3$ is determined by[55]

$$Y = \left[\mu(\underline{2})\Big(\eta_s(\underline{1},\underline{1}),\ \rho_{\underline{2}}\,\eta_s(\underline{1},\underline{1})\Big)\right]^{-1} = q_\gamma\left(\eta_s(\underline{1},\underline{1})\right) = \begin{cases} \exp\left(2\pi i \dfrac{k}{3}\right) & \text{if } m = 2k \\[2ex] \exp\left(2\pi i \dfrac{4k+2}{3}\right) & \text{if } m = 2k+1 \end{cases}\ . \tag{4.1.219}$$

[55] One can easily check that, when $n = 3m$ is also even, so $m = 0$ mod (2), the choice of quadratic refinement for q_γ is immaterial.

Again we can apply these results to the case of triality symmetry appearing in $\mathcal{N}=4$ SYM at $\tau = e^{2i\pi/3}$. The triality defect is non-intrinsic when there exist $t \in \mathbb{Z}_n$ such that $1+t+t^2 = 0 \text{ mod}(n)$. When this is the case we can ask about the second obstruction. To apply our methods we are not forced to combine the naive CST operation with an R-symmetry rotation to eliminate fermion parity, since $(CST)^3 = \mathbb{1}$. Then, by the same token as the duality case and knowning that $\Omega_5^{spin}(B\mathbb{Z}_3) \cong \mathbb{Z}_9$, we have

$$\epsilon = 60(n-1) \text{ mod}(9) = -3(n-1) \text{ mod}(9)\,. \tag{4.1.220}$$

Notice that Y is valued in the $\mathbb{Z}_3$ subgroup of the $\mathbb{Z}_9$ anomaly group, then to compare Y to the ϵ above we need to multiply by 3. When $n = 1 \text{ mod}(3)$ then $\epsilon = 0$ and there is no choice of fractionalization, therefore the second obstruction vanishes. For $n = 2 \text{ mod}(3)$ we find $\epsilon = 6$ and the triality defect is always anomalous. Finally when $n = 0 \text{ mod } 3$ we have $\epsilon = 3$ and a simple computation shows that the second obstruction can be cancelled only when $n = 3m$ with $m = 1 \text{ mod } (3)$.

In Maxwell theory instead the anomaly is $56 \text{ mod}(9) = 2$ and cannot be cancelled by any choice of symmetry fractionalization. We conclude that the triality symmetry in Maxwell theory is always anomalous due to the second obstruction.

4.1.4 A check from dimensional reduction

As a check of our results, we show that the obstruction theory of Section 4.1.3 is consistent with the one for Tambara-Yamagami categories upon dimensional reduction on an orientable 2-manifold W. We treat explicitly the case that W is a torus T^2, but the generalization to any Riemann surface $\Sigma_{\mathfrak{g}}$ is straightforward. Physically this should be expected, indeed the simplest example of a 4d theory enjoying self-duality is Maxwell theory, which upon compactification on T^2 reduces to the theory of two compact bosons.[56] In this example the complexified gauge coupling τ is mapped to the position of the 2d CFT on the Narain moduli space. Such a theory is well known to enjoy Tambara-Yamagami-type symmetries if the point on the conformal manifold is chosen appropriately [31].

Compactifying the 5d Dijkgraaf-Witten theory for $\mathbb{A}$ on the torus is a simple exercise. The resulting 3d TQFT has a 1-form symmetry $\widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}}^\vee$ where

$$\widetilde{\mathbb{A}} = \mathbb{A} \times \mathbb{A}\,, \tag{4.1.221}$$

together with a 0-form and a 2-form symmetry, both being $\mathbb{A} \times \mathbb{A}^\vee$, which we neglect in the following discussion. Given a choice ϕ for the isomorphism that enters into the 5d duality symmetry, the defect Φ also implements a $\mathbb{Z}_4$ symmetry in 3d:

$$\Phi(a_1,\, a_2;\, \alpha_1,\, \alpha_2) = \left(-\phi^{-1}(\alpha_2),\, \phi^{-1}(\alpha_1);\, -\phi(a_2),\, \phi(a_1)\right)\,, \tag{4.1.222}$$

[56] Plus a decoupled 2d Maxwell sector that we ignore. Such a sector has a 1-form and a (-1)-form symmetry (associated to a 2π shift of the theta angle), associated to the 0-form and 2-form symmetries of the Symmetry TFT.

where $(a_1, a_2) \in \widetilde{\mathbb{A}}$ and $(\alpha_1, \alpha_2) \in \widetilde{\mathbb{A}}^\vee$. To get a $\mathbb{Z}_2$ symmetry we compose this transformation with the internal S-duality of the torus, which also squares to charge conjugation and sends $(a_1,\, a_2;\, \alpha_1,\, \alpha_2) \to (a_2,\, -a_1;\, \alpha_2,\, -\alpha_1)$. The resulting $\mathbb{Z}_2$ symmetry, which we dub $\widetilde{\Phi}$ acts as:

$$\widetilde{\Phi}(a_1,\, a_2;\, \alpha_1,\, \alpha_2) = (\phi^{-1}(\alpha_1),\, \phi^{-1}(\alpha_2);\, \phi(a_1),\, \phi(a_2))\,, \tag{4.1.223}$$

or, using the $\widetilde{\mathbb{A}}$

$$\begin{array}{rcc} \widetilde{\Phi} : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}}^\vee & \longrightarrow & \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}}^\vee \\ (\widetilde{a}, \widetilde{\alpha}) & \longrightarrow & (\widetilde{\phi}^{-1}(\widetilde{\alpha}), \widetilde{\phi}(\widetilde{a})) \end{array} \tag{4.1.224}$$

with $\widetilde{\phi} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}^\vee \times \mathbb{A}^\vee$ given by $\widetilde{\phi}(a_1, a_2) = (\phi(a_1), \phi(a_2))$.

First and second obstruction upon dimensional reduction

We now discuss how the first obstruction in 5d is mapped to the first obstruction in 3d language after compactification. Clearly not all Lagrangian algebras $\mathcal{L}$ in the 3d description can descend from a 5d description, so we must first characterize them. Recall that, in 5d, algebras where described by a choice of subgroup $\mathbb{B}$ of $\mathbb{A}$ together with a discrete torsion $[\nu] \in H^4\big(B^2\mathbb{B},\, U(1)\big)$. Upon reduction on T^2 this should map to a specific class $[\tilde{\nu}] \in H^2\big(B\widetilde{\mathbb{B}},\, U(1)\big)$, where $\widetilde{\mathbb{B}} = \mathbb{B} \times \mathbb{B}$. Expanding the 5d background $B = B_1\theta_1 + B_2\theta_2$ with θ_i a basis of $H^1(T^2, \mathbb{Z})$ (we neglect the 0-form and 2-form symmetries), we find:

$$\int_{T^2} B^*\nu = B_1 \cup_\nu B_2 - B_2 \cup_\nu B_1\ , \tag{4.1.225}$$

where $\cup_\nu$ is the cup product induced by the symmetric bilinear form χ_ν. The bicharacter corresponding to $\tilde{\nu}$ is then, in matrix and additive notation,

$$\chi_{\tilde{\nu}} = \begin{pmatrix} 0 & \chi_\nu \\ -\chi_\nu & 0 \end{pmatrix}\,. \tag{4.1.226}$$

A 3d Lagrangian algebra $\widetilde{\mathcal{L}}$ induced from 5d then is of the form

$$\widetilde{\mathcal{L}} = \Big\{ (\tilde{b},\, \tilde{\beta}\psi_{\tilde{\nu}}(\tilde{b})) \,\Big|\, \tilde{b} \in \widetilde{\mathbb{B}}\,, \quad \tilde{\beta} \in N(\widetilde{\mathbb{B}}) \Big\}\,, \tag{4.1.227}$$

where $\psi_{\tilde{\nu}} : \widetilde{\mathbb{B}} \to \widetilde{\mathbb{B}}^\vee$ is the homomorphism associated with the antisymmetric bicharacter (4.1.226). Since $\mathrm{Rad}(\tilde{\nu}) = \mathrm{Rad}(\nu) \times \mathrm{Rad}(\nu)$ the 5d condition $\phi(N(\mathbb{B})) = \mathrm{Rad}(\nu)$ implies $\phi(N(\widetilde{\mathbb{B}})) = \mathrm{Rad}(\widetilde{\nu})$ in 3d. On the other hand, the map $\tilde{\sigma} = \tilde{\phi}^{-1}\psi_{\tilde{\nu}}$ is given by:

$$\tilde{\sigma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}\,, \tag{4.1.228}$$

which is an involution $\tilde{\sigma}^2 = 1$. We have thus shown that solutions to the first obstruction in 5d always descend to solutions to the first obstruction in 3d.

Let us now discuss the second obstruction. We notice that the 5d discrete torsion ϵ, when reduced on T^2, trivializes. This is because the torus (as well as any Riemann surface) does not have torsion 1-cycles. Thus it is not possible to detect the 5d second obstruction in 3d after compactification on a Riemann surface. Indeed, from the point of view of symmetry fractionalization, we have $G \cong \mathbb{Z}_n$ and for any Abelian group $\mathbb{A}$ we get

$$H^1_\rho(\mathbb{Z}_n,\,\mathbb{A}) = \mathrm{Ker}(1+f)/\mathrm{Im}(1-f)\;, \tag{4.1.229}$$

with $f = \rho_{\underline{1}}$. Applying this to the case $\mathbb{A} = \widetilde{\mathbb{B}}/N(\widetilde{\mathbb{B}})$ and $f = \tilde{\sigma}$ it is simple to prove that the twisted cohomology group is trivial for any choice of $\mathbb{B}$.[57] Thus there are no fractionalization classes and therefore the second obstruction always trivializes.

4.1.5 Conclusions and applications

Let us conclude by mentioning some immediate applications of our results, as well as some interesting open problems.

4d $\boldsymbol{\mathcal{N}} = 3$ theories. It has been appreciated in the past that a class of 4d $\mathcal{N} = 3$ theories may be obtained from a discrete gauging of the $\mathcal{N} = 4$ duality symmetry for special values of τ [341, 358]. More precisely, given a $\mathbb{Z}_k$ subgroup of $SL(2,\,\mathbb{Z})$ and a fixed coupling τ_k, where $k = 2, 3, 4, 6$,[58] we can combine this transformation with a $\mathbb{Z}_k$ R-symmetry rotation in the Cartan of $SU(4)$ so that the combined action preserves $\mathcal{N} = 3$ supersymmetry. As the gauge coupling $\tau = \tau_k$ must be fixed to its self-dual value, these theories have no exactly marginal deformation and are inherently strongly coupled. The case of $k = 2$ is special, as the symmetry is charge conjugation, hence it preserves the full $\mathcal{N} = 4$ supersymmetry, and is invertible. We will thus concentrate on the cases $k = 4$ (corresponding to the S transformation) and $k = 3$ (corresponding to the CST transformation) and gauge group $SU(n)$. As the duality symmetry is non-invertible, it must be gauged together with (a subgroup of) the $\mathbb{Z}_n$ 1-form symmetry and our results imply that this is only consistent if the first obstruction vanishes. Thus there is a severe constraint on the possible $\mathcal{N} = 3$ theories which can be obtained in this way. For example our results show that there is no such theory for $n = 3$ and $k = 4$. We must also check the vanshing of the second obstruction. The joint duality/R-symmetry anomaly is given by [80]:

$$60(n-1) - 24(n^2-1) \begin{cases} \bmod 4\,, & \text{if } k = 4 \\ \bmod 9\,, & \text{if } k = 3 \end{cases}. \tag{4.1.230}$$

For the duality case the cubic anomaly is identically trivial, thus the vanishing of the first obstruction is a sufficient condition for the gauging to be consistent. For triality instead it is given by 6 mod 9 when $n = 3m + 2$ and is zero otherwise. It has been checked in [80] that this anomaly identically trivializes when the first obstruction vanishes. Therefore also in the triality case the gauging is consistent if the first obstruction vanishes. This also implies that, when $n = 3m$, we must choose the trivial fractionalization class $\eta \in H^2_\rho(\mathbb{Z}_3,\,\mathbb{Z}_{3m})$.

---

[57] Using that $\sigma^2 = -1$ we find that $\mathrm{Ker}(1+\tilde{\sigma})$ is spanned by elements $(b_1, b_2) \in \widetilde{\mathbb{B}}/N(\widetilde{\mathbb{B}})$ such that $b_2 = \sigma(b_1)$. An element of $\mathrm{Im}(1-\tilde{\sigma})$ instead is of the form $(b_1, b_2) = \big(x - \sigma(y),\, \sigma(x) + y\big)$. A simple manipulation shows that this is equivalent to $b_2 = \sigma(b_1)$.

[58] To be precise, since the duality group is $\mathrm{Mp}(2,\,\mathbb{Z})$ the discrete groups are actually $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_{12}$ as charge conjugation squares to fermion number $C^2 = (-)^F$. The combined duality - R symmetry transformation however lies is $\mathbb{Z}_k$ with k as in the main text.

In some special cases the S-fold construction of [360] gives rise to discrete gaugings of $\mathcal{N} = 4$ SYM [361]. These are engineered by 2 D3-branes probing a $k = 3, 4, 6$ S-fold and lead to a discrete gauging of $SU(3)$, $SO(5)$ and G_2 $\mathcal{N} = 4$ SYM respectively. Our analysis can be applied to the first two cases which, following the discussed examples, indeed are free of anomalies for triality and duality respectively. It would certainly be interesting to understand whether our methods can give some insight also on $\mathcal{N} = 3$ theories which cannot be obtained by a discrete gauging procedure from $\mathcal{N} = 4$ and, in particular, if they enlarge the list of generalized symmetries of S-folds described recently in [362, 363].

A mixed anomaly. We have mentioned in Section 4.1.2 that the space of duality-invariant Lagrangian algebras is larger on spin manifolds. Similarly one can argue, for example following [78], that the first obstruction in the 4d case has less solutions if the spacetime X is not spin. This should be rephrased as the presence of a mixed 't Hooft anomaly between the non-invertible symmetry $\mathcal{N}$ and gravity, sourced by a nontrivial second Stiefel-Whitney class $w_2(X)$. A well known example is the symmetry $\mathrm{TY}(\mathbb{Z}_2)_{1,1}$ of the Ising CFT. As a bosonic symmetry this is anomalous as the first obstruction cannot be cancelled. However, if we consider it on spin manifolds X only, the obstruction is absent since the bulk algebra $\mathcal{L}_D = \{(0,0), (1,1)\}$ is manifestly duality invariant. Such an algebra can only be condensed on spin manifolds as $\theta_{(1,1)} = -1$. On the field theory side it is well known [12, 31, 364] that fermionizing the Ising CFT into a Majorana fermion the duality symmetry $\mathcal{N}$ becomes the invertible $(-1)^{F_L}$ which is anomaly free. A similar example in 4d, as already stated before, if the $\mathcal{N} = 4$ $SU(2)$ SYM theory, whose duality symmetry is anomaly-free on spin manifolds (after combining it with an R-symmetry rotation) it is anomalous by the first obstruction when X is non-spin. It would be nice to make this idea more precise.

Duality-invariant RG flows. In both 2d and 4d, duality-symmetric theories allow for a plethora of interesting RG flows which preserve the non-invertible symmetry. In the former case they have been studied in [31], while in the latter an initial study has appeared recently [80]. As in the 2d case, the anomalies for the duality symmetry can lead to strong constraints on the possible low energy phases. A simple example is the $\mathcal{N} = 1^*$ [365–369] deformation of $\mathcal{N} = 4$ SYM at $\tau = i$, which, in the presence of the first obstruction, necessarily leads in the IR either to spontaneous symmetry breaking of the non-invertible symmetry, or to a self-dual Coulomb phase [80]. A related problem deserving further study in the light of our results is the deformation of the $SU(2)$, $SU(3)$, $SU(4)$ $\mathcal{N} = 4$ theory by the Konishi operator. This must lead in the IR either to an $\mathcal{N} = 0$ CFT or to chiral symmetry breaking in order to match the cubic SU(4) anomaly. Consistency of these scenarios with the intricate pattern of non-invertible symmetries and their anomalies might allow to put stringent constraints on the possible IR phases. This problem is currently under investigation.

Intrinsic versus anomalous. In our work we have seen that, in the context of duality symmetries, the concept of "instrinsic" [133] implies that the duality symmetry is anomalous. Such concept is not unique to duality symmetries, and can be rephrased as the statement that the symmetry category $\mathcal{C}$ is not Morita equivalent to any category of the form $n\mathrm{Vec}_G$ for some (higher) group G. It would be interesting to understand how far the relationship between 't Hooft anomalies and intrinsic defects extends.

Duality-invariant boundary conditions. It is known [332, 333] that the presence of an 't Hooft anomaly for a symmetry $\mathcal{C}$ forbids the existence of a $\mathcal{C}$-invariant boundary condition.[59] Our results can in principle be used to constrain the existence of duality-invariant conformal boundary conditions, building on the results of [370–372][60] for $\mathcal{N} = 4$ SYM and free Maxwell theory, respectively.

4.2 Holographic duals of symmetry broken phases

A profound insight by E. Witten is that Topological Quantum Field Theories (TQFTs), due to their general covariance, can be seen as theories of quantum gravity [373]. Unlike in more conventional examples, general covariance is not achieved by integrating over metrics but rather by not introducing them at all. Consequently, these theories lack any semiclassical description involving weakly interacting gravitons. In traditional gravitational theories, one selects a background metric and expands around it, thereby breaking general covariance spontaneously. Therefore, TQFTs can be viewed as theories of quantum gravity with unbroken general covariance — where gravitons are, in a certain sense, confined.

This old story requires some important refinements. A full quantum-gravity theory should not depend on the background topology. TQFTs, on the other hand, are sensitive to spacetime topology through their global symmetries, broadly defined in terms of their topological operators [9], which are expected to form some higher category [6, 10–14, 57, 65, 88, 118]. One way to achieve such an independence is to sum over all topologies, which can be done in low dimensions [285, 286, 374–376]. Alternatively, one can use TQFTs that do not even depend on topology [34], hence that are free of global symmetries and then trivial (or invertible) [330, 377]. These can be obtained by gauging a maximal non-anomalous set of topological defects, called a *Lagrangian algebra*, in a nontrivial TQFT. Not all TQFTs have Lagrangian algebras (the typical example is 3d Chern–Simons theory), but those that have them admit topological (or gapped) boundary conditions. In fact, given a Lagrangian algebra $\mathcal{L}$, one can construct such a boundary condition as an interface between the TQFT and the gauged TQFT [26, 220–222]. Equivalently, the boundary condition is defined by allowing the defects inside $\mathcal{L}$ to end on the boundary.

TQFTs with topological boundary conditions have recently gained attention for their role as Symmetry Topological Field Theories (SymTFTs) in the context of generalized symmetries (see [104–107] for reviews). SymTFTs are $(d+1)$-dimensional TQFTs $\mathcal{Z}(\mathcal{C})$ associated with

[59] Strictly speaking the argument of [333] only applies to 2d. However, given a representation of the higher dimensional 't Hooft anomalies in terms of defect configurations, it should be possible to extend it to general symmetry categories.

[60] See also [84] for more details on the action of the non-invertible duality symmetry on boundary conditions in Maxwell theory.

symmetry structures $\mathcal{C}$ in d dimensions, capturing all properties of the symmetries regardless of the specific QFT$_d$ realizing them [9, 116–118]. The TQFT $\mathcal{Z}(\mathcal{C})$ is placed on a *slab* with two boundaries. The left one supports the physical QFT$_d$ of interest, coupled with the bulk. The right one is the topological boundary condition that one is free to choose, determined by a Lagrangian algebra $\mathcal{L}$. Defects inside $\mathcal{L}$ become trivial on the topological boundary, while all other ones (modulo those inside $\mathcal{L}$) give rise to topological operators of the symmetry $\mathcal{C}$, after the slab is squeezed. The endpoints of defects inside $\mathcal{L}$ inherit a braiding with the generators of $\mathcal{C}$ from the bulk braiding, hence they become the charges of the symmetry [52,53]. SymTFT has been shown to be a very powerful tool for studying global symmetries, also of non-invertible type [7,8,67,69,133] and their anomalies [2,98,99,195,196], as well as to characterize phases [109–112,197,198].

Although originally restricted to finite symmetries, the framework has been recently extended to continuous symmetries [206,210,211].[61] The prize to pay is to introduce a new type of TQFTs with gauge fields valued in both $U(1)$ and $\mathbb{R}$, and to have a continuous and/or non-compact spectrum of operators, thus going beyond the standard TQFTs well studied by mathematicians (we provide a more precise mathematical definition in Appendix B.8). This idea has been shown to be applicable to all possible non-finite and continuous symmetries, with or without anomalies, possibly with higher-group structures, and even including non-invertible and non-Abelian symmetries. By now the picture is that to *any* possible symmetry structure $\mathcal{C}$ in d dimensions one can canonically associate a $(d+1)$-dimensional TQFT $\mathcal{Z}(\mathcal{C})$.

Our aim here is to give a different interpretation to these TQFTs $\mathcal{Z}(\mathcal{C})$, not as SymTFTs but as theories of gravity. More precisely, we want to establish holographic dualities in which the bulk theory is a SymTFT. The main proposal of this paper is the following:

- Thought of as a theory of gravity, the SymTFT $\mathcal{Z}(\mathcal{C})$ for a symmetry $\mathcal{C}$ is the holographic dual to the universal effective field theory (EFT) that describes the spontaneous breaking of $\mathcal{C}$.

It is a general principle of quantum field theory that any theory with a certain continuous global symmetry that is spontaneously broken, in the far infrared (IR) flows to the same universal theory of Goldstone bosons [378,379]. This is roughly speaking always a sigma model, although the target space can be infinite dimensional (*e.g.*, it is the classifying space B^pG in the case of higher-form symmetries).[62] As for the SymTFT, this EFT is also canonically determined by the symmetry $\mathcal{C}$ without any further information. For this reason, it is natural to expect that, even though they appear to be completely different objects — a $(d+1)$-dimensional TQFT and a d-dimensional EFT — the two can be somehow related as they both have the same input datum. We will prove by means of many examples that this correspondence is holography.

A crucial part of the story is the proper choice of boundary conditions. These will be non-topological and of the Dirichlet type for some combination of the bulk fields. Since bulk fields are gauge fields A, these boundary conditions break some gauge invariance, making it a global

[61] See [212] for a different proposal involving non-topological theories.

[62] It is not clear to us how to make this precise for non-invertible symmetries, for instance for the $\mathbb{Q}/\mathbb{Z}$ chiral symmetry discovered in [61,64].

symmetry of the boundary theory. This agrees with the general principle in holography that boundary global symmetries correspond to bulk gauge fields. The non-triviality of the system really comes from the boundary conditions that, being non-topological, generate dynamics on the boundary. The boundary theory can be thought of as a theory of edge modes. Our setup has several similarities with, and may be understood as a generalization of, the Chern–Simons/WZW correspondence [124, 125] and its reinterpretation as a full-flagged holographic duality by means of bulk anyon condensation [34].

We find that for the simple Abelian TQFTs introduced in [206, 210] as the SymTFTs for $U(1)$, the dual boundary theory is a free theory of S^1 Goldstone bosons, or generalized Maxwell fields when the symmetry is of higher form. More precisely, these boundary theories have topological sectors (*e.g.*, winding for a compact scalar, or magnetic fluxes for a photon), and the nontrivial TQFT without gauging the Lagrangian algebra is only dual to a fixed topological sector. The latter is not a physical theory and is the non-chiral analog of the conformal blocks in the CS/WZW correspondence. The physical theory is obtained by summing over various topological sectors, and we will show that this sum is reproduced by the gauging of the Lagrangian algebra. These Abelian TQFTs have various interesting modifications describing chiral anomalies, higher groups, and non-invertible $\mathbb{Q}/\mathbb{Z}$ symmetries [206]. We include all of them in our analysis, showing that their holographic duals are the theories describing the spontaneous breaking of the corresponding symmetries. In particular the SymTFT for the non-invertible chiral symmetry is the gravity dual to axion-Maxwell theory.

For non-Abelian continuous symmetries G, the SymTFT was also conjectured in [206, 210] and further analyzed in [211]. In the simplest case, it is a TQFT introduced many years ago by Horowitz [380] and is written in terms of a G connection and a Lie-algebra-valued higher-form field in the adjoint of G. When employing this theory in our story, it proves to be the dual to a non-linear sigma model with target space G at the boundary. For $d = 4$ this in the pion Lagrangian describing the low-energy dynamics of massless QCD in the chiral symmetry breaking phase. We also show that including a term that describes an 't Hooft anomaly we obtain a WZW term in the sigma model [381].

A particularly interesting example is that of a non-Abelian 2-group in 4d, mixing a non-Abelian continuous symmetry G and a $U(1)$ 1-form symmetry [179]. The Goldstone theory for this symmetry structure was not determined before, and we use our holographic conjecture to derive it. It consists of a non-linear sigma model and a photon, coupled through a parity-violating interaction whose leading term is proportional to $k f_{abc} \epsilon^{\mu\nu\rho\sigma} A_\mu \, \partial_\nu \pi_a \, \partial_\rho \pi_b \, \partial_\sigma \pi_c$, where π_a are the pions, f_{abc} are the structure constants of G, while $k \in \mathbb{Z}$ is a quantized coefficient that governs the 2-group structure. This term encodes the coupling of the photon to the current for a topological 0-form symmetry of the sigma model. This result has a concrete application to the low-energy dynamics of 4d $U(N)$ QCD. For low enough number of flavors, the chiral symmetry is spontaneously broken and quarks form pion bound states as in $SU(N)$ QCD. However, here the theory also contains an Abelian gauge field A for the baryon number symmetry with quarks charged under it, hence in the IR this photon cannot be decoupled. The photon-pion term encodes the coupling of A to the baryon number current in the IR. We argue that the theory has a spontaneously-broken 2-group symmetry, implying that the leading photon-pion interaction coincides with the one we determined from our conjecture.

Since our work utilizes TQFTs with an infinite number of (simple) topological operators, as an aside in Appendix B.8 we explore some of their properties and show (in a simple example) that while their path integrals on closed Euclidean manifolds are divergent, the path integrals on open manifolds can be made finite.

The rest of the section is organized as follows. In Section 4.2.1 we explain the general setup and clarify some issues about holography with TQFTs in the bulk. In the rest of the sections we present several interesting examples. Section 4.2.2 concerns the vanilla example of Abelian symmetries without additional structures. In Section 4.2.3 we include chiral anomalies and higher group structures, showing that the Goldstone theory is the same as in the vanilla case but it couples differently to background fields, a fact that is interpreted in terms of *symmetry fractionalization*. The non-invertible example is discussed in Section 4.2.4 after we warm up with a similar but simpler example in 3d that produces Maxwell–Chern–Simons theory. The non-Abelian cases (including higher groups) are finally studied in Section 4.2.5.

4.2.1 Topological field theories as holographic duals

The bulk theories we use in this paper are TQFTs of the type introduced in [206, 210, 211] to describe SymTFTs for continuous symmetries. In the simplest cases, they have a Lagrangian formulation as[63]

$$S = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-p-1} \wedge dA_{p+1} \tag{4.2.1}$$

where A_{p+1} is a $U(1)$ $(p+1)$-form gauge field, while b_{d-p-1} is an $\mathbb{R}$ $(d-p-1)$-form gauge field. In the whole paper, we adopt this convention in which uppercase letters indicate $U(1)$ gauge fields, while lowercase letters indicate $\mathbb{R}$ gauge fields. Understood as a SymTFT, this describes a p-form $U(1)$ symmetry in d dimensions. The topological operators of the theory are [206]:

$$V_n(\gamma_{p+1}) = e^{in\int_{\gamma_{p+1}} A_{p+1}}\,, \quad U_\beta(\gamma_{d-p-1}) = e^{i\beta\int_{\gamma_{d-p-1}} b_{d-p-1}}\,, \quad n \in \mathbb{Z}\,, \quad \beta \in \mathbb{R}/\mathbb{Z} \cong U(1)\,. \tag{4.2.2}$$

The partition function of (4.2.1) on a generic closed manifold diverges, but infinities are avoided on certain classes of manifolds with boundaries (see Appendix B.8). These are the relevant ones for both the SymTFT and the holographic setup considered in this paper. Moreover, normalized correlators are always finite, and capture the braiding of topological defects:

$$\left\langle V_n(\gamma_{p+1})\, U_\beta(\gamma'_{d-p-1})\right\rangle = \exp\Big[2\pi i\, n\, \beta\, \mathrm{Link}\big(\gamma_{p+1}, \gamma'_{d-p-1}\big)\Big]\,. \tag{4.2.3}$$

In the following we will consider several modifications of the vanilla case (4.2.1) that take into account anomalies, higher groups, non-invertible symmetries, as well as extensions to non-Abelian groups. However let us focus here on this simplest case as an illustration of the basic ideas and setup.

In SymTFT, (4.2.1) is placed on a slab with two boundaries, one of which is topological and determines the symmetry after the slab is squeezed. This topological boundary is characterized

[63] We only consider Euclidean manifolds and normalize our actions so that the weight in the path integral is e^{-S}.

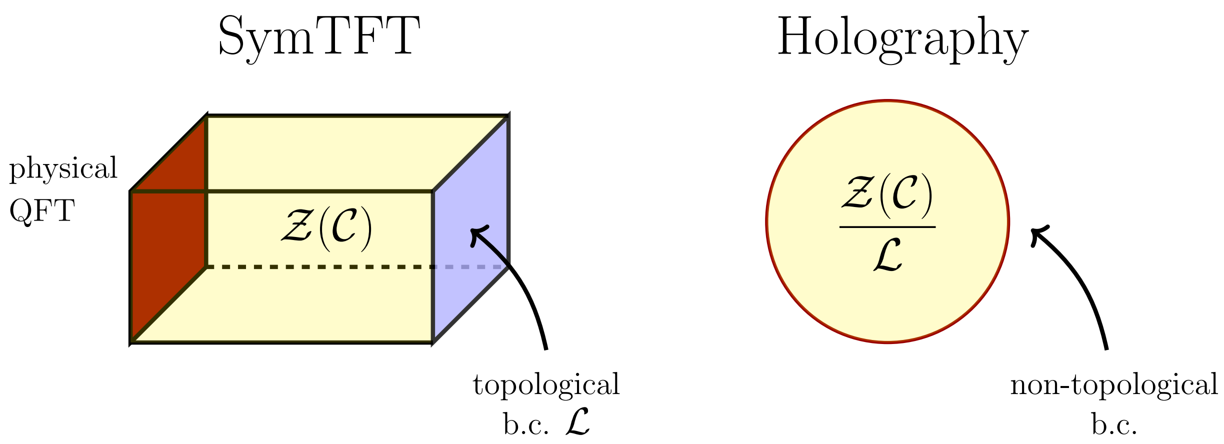


Figure 4.3: Left: the SymTFT setup. The TQFT is placed on a slab, whose right boundary is topological and determined by a Lagrangian algebra $\mathcal{L}$. Right: the holographic setup considered here. There is only one boundary with non-topological boundary conditions, while the Lagrangian algebra $\mathcal{L}$ is gauged to make the bulk invertible.

by a maximal set of mutually transparent objects, which we generically refer to as a Lagrangian algebra $\mathcal{L}$. In this example a natural Lagrangian algebra consists of all $V_n(\gamma_{p+1})$, while the $U_\beta(\gamma_{d-p-1})$ become the generators of the $U(1)$ p-form symmetry of the boundary theory.

In this paper, instead, we consider a different setting in which (4.2.1) is placed on a manifold X_{d+1} with a unique connected boundary $\mathcal{M}_d = \partial X_{d+1}$, which we endow with a Riemannian structure. On $\mathcal{M}_d$ we fix non-topological boundary conditions

$$A_{p+1} + iC \star b_{d-p-1} = \mathcal{A}_{p+1}\,. \tag{4.2.4}$$

Here $\star$ is the Hodge star operator of the boundary, $\mathcal{A}_{p+1}$ is a fixed $(p+1)$-form on the boundary, and C is a generically dimensionful constant with mass dimension $[C] = 2p+2-d$.[64] Moreover, the Lagrangian algebra $\mathcal{L}$ that was used to define the topological boundary in the SymTFT setup must now be gauged in the bulk X_{d+1}, and the final bulk theory $\mathcal{Z}(\mathcal{C})/\mathcal{L}$ is an invertible TQFT. See Fig. 4.3 for a comparison of the two setups.

In this second setup we want to establish a precise holographic duality with a certain local QFT$_d$ living on the boundary, which we need to determine. More precisely, the equality we need to show is the standard one [274–276]:

$$Z_{\mathrm{TQFT}_{d+1}}\big[\varphi\big|_\partial = \mathcal{A}\big] \;=\; Z_{\mathrm{QFT}_d}[\mathcal{M}_d, \mathcal{A}]\,. \tag{4.2.5}$$

Here TQFT$_{d+1}$ is the result of gauging $\mathcal{L}$ in $\mathcal{Z}(\mathcal{C})$, φ denotes generically some bulk fields (for instance $\varphi = A_{p+1} + iC \star b_{d-p-1}$ in the example (4.2.1)), while $\mathcal{A}$ is introduced as a boundary value from the bulk viewpoint and plays the role of a background field for the boundary QFT. Although SymTFT superficially resembles holography, the two are fundamentally different. SymTFT only captures symmetries and disregards dynamics, allowing any QFT with the

[64] The introduction of such a scale is necessary since the components of A_{p+1} have dimension $p+1$ while those of b_{d-p-1} have dimension $d-p-1$. In this way the forms A_{p+1} and b_{d-p-1} are dimensionless, the action in (4.2.1) is dimensionless, but $\star b_{d-p-1}$ has dimension $d-2p-2$.

specified symmetry. In contrast, in holography the dual QFT_d is uniquely determined by the bulk theory and its boundary conditions, encoding both symmetries and dynamics as in (4.2.5).

We will determine the dual QFT_d explicitly in the many examples considered below, providing strong evidence for the conjecture that the dual theory to $\mathcal{Z}(\mathcal{C})/\mathcal{L}$ is always the symmetry-breaking EFT for $\mathcal{C}$. Some of these checks are quite subtle and highly nontrivial. For instance, the Goldstone theory for a $U(1)$ symmetry with a cubic 't Hooft anomaly in 4d is still a compact boson with no additional terms as in the non-anomalous case,[65] but the background field for the symmetry is coupled non-minimally to the theory. We discuss this in Section 4.2.3 (in particular (4.2.62) is the additional coupling) to which we refer for more details. The SymTFT for a 4d anomalous $U(1)$ is [206]

$$S = \frac{i}{2\pi}\int_{X_5} b_3 \wedge dA_1 + \frac{ik}{24\pi^2}\int_{X_5} A_1 \wedge dA_1 \wedge dA_1 \,. \tag{4.2.6}$$

Forgetting about the boundary value $\mathcal{A}_1$ appearing in the boundary condition (4.2.4), the additional cubic term does not affect the dual boundary QFT_4. However we will show in Section 4.2.3 that keeping track of $\mathcal{A}_1$ we reproduce exactly the non-minimal coupling expected for an anomalous $U(1)$.

Before moving to the various examples, let us clarify a conceptual point. The assertion that certain dynamical QFTs have a TQFT as holographic dual might be perplexing at first. The origin of the confusion is that, even though TQFTs are good theories of gravity, the non-appearance of a metric tensor $g_{\mu\nu}$ is puzzling for holography: the metric should be dual to the stress-energy tensor $T_{\mu\nu}$ of the boundary QFT. While this observation is in general correct, in a few special cases it might have a loophole: the stress tensor might not be an independent operator. For instance, this is the case in the CS/WZW correspondence [124,125]. In 2d WZW models the stress tensor of the CFT, using the Sugawara construction, is made out of the currents which are dual to the gauge fields of the 3d Chern–Simons bulk theory. Something very similar happens in our examples. Indeed, the EFTs for symmetry breaking are very special QFTs in which everything, including the stress-energy tensor, is determined by the currents and their correlation functions. This is at the core of the *universality* of those EFTs. For instance, in the theory of a $U(1)$ Goldstone boson with action

$$S = \frac{R^2}{4\pi}\int_{\mathcal{M}_d} d\Phi \wedge \star\, d\Phi \,, \tag{4.2.7}$$

the $U(1)$ current is $J_\mu = \frac{iR^2}{2\pi}\partial_\mu \Phi$ and the stress tensor is a composite operator of J_μ:

$$T_{\mu\nu} = \frac{R^2}{4\pi}\left(\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2}\,\delta_{\mu\nu}\,(\partial\Phi)^2\right) = \frac{\pi}{R^2}\left(\frac{1}{2}\,\delta_{\mu\nu}\,J^2 - J_\mu J_\nu\right). \tag{4.2.8}$$

Through the boundary conditions, the bulk SymTFT provides background fields for the global symmetries of the boundary theory, which are sources for the boundary currents. Hence the TQFT can compute correlation functions of the currents, and by universality correlation functions of all operators, including those of the stress tensor, even without an explicit source $g_{\mu\nu}$.

[65] This is different from the non-Abelian case, in which an anomaly implies a WZW term in the sigma model.

This is a general statement: in the EFTs for spontaneous breaking the currents completely determine all operators and the holographic duals do not need a graviton field.

It is expected, however, that embedding our models into RG flows and taking into account non-universal features would require to reintroduce dynamical gravity into the game. Indeed, a related observation is that the boundary theories we obtain are either free or non-renormalizable. The reason why a TQFT, which is expected to be UV complete and finite, can be dual to a non-renormalizable theory is the choice of non-topological boundary conditions, which introduce an energy scale in the theory. This scale sets a limit below which both the bulk and boundary theories are well defined. Above this threshold, the boundary theory requires the inclusion of more and more operators to tame UV divergencies. This issue has to carry over to the bulk TQFT as well — albeit in a way unclear to us — making the TQFT description incomplete. The expectation is that, to make sense of the bulk theory above the scale of the boundary condition, one has to allow for dynamical gravity in the bulk in a way that is similar to the embedding of an EFT for spontaneous breaking into a UV complete theory. It would be interesting to understand this point better.

4.2.2 $U(1)$ Goldstone bosons

The simplest cases to test our conjecture are those of $U(1)$ symmetries of generic degree. We warm up with the textbook example of a spontaneously broken $U(1)$ 0-form symmetry in generic dimension and then move on to the case of higher-form symmetries, whose Goldstone bosons are (free) $U(1)$ higher-form gauge fields [9].

0-form symmetries

Consider the following TQFT in $d+1$ dimensions:

$$S = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-1} \wedge dA_1 \,, \tag{4.2.9}$$

where A_1 is a $U(1)$ gauge field while b_{d-1} is an $\mathbb{R}$ $(d-1)$-form gauge field. We endow the boundary $\mathcal{M}_d = \partial X_{d+1}$ with a Riemmanian metric and impose the boundary condition

$$\star A_1 = -\frac{i}{R^2}\, b_{d-1} + \star \mathcal{A}_1 \,. \tag{4.2.10}$$

Here R is a parameter of mass dimension $(d-2)/2$, while $\mathcal{A}_1$ is a fixed background 1-form on $\mathcal{M}_d$. Notice that only in $d = 2$ this boundary condition is conformally invariant. In order to get a consistent variational principle with this boundary condition we must add a boundary term S_∂ to (4.2.9). Indeed, the variation of the action produces a boundary piece

$$\delta S\big|_{\mathcal{M}_d} = (-1)^{d-1}\, \frac{i}{2\pi} \int_{\mathcal{M}_d} b_{d-1} \wedge \delta A_1 = \frac{1}{2\pi R^2} \int_{\mathcal{M}_d} b_{d-1} \wedge \star\, \delta b_{d-1} \,, \tag{4.2.11}$$

which requires a boundary term

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\mathcal{M}_d} b_{d-1} \wedge \star\, b_{d-1} \,. \tag{4.2.12}$$

Since the boundary condition (4.2.10) breaks gauge invariance on the boundary, we have to be careful in specifying the group of transformations we quotient by in the bulk: we choose to allow only gauge transformations that are trivial on the boundary. This implies that the bulk gauge symmetries become global on the boundary. For any global symmetry we should be able to turn on a background. In our setup this operation has a very natural realization: instead of freezing gauge transformations on the boundary, we allow them but transform the boundary data so as to render the boundary condition invariant. For instance, we can make (4.2.10) gauge invariant under gauge transformations of A_1 by demanding that $A_1 \mapsto A_1 + d\lambda_0$ is accompanied by a transformation of the fixed background $\mathcal{A}_1$:

$$\mathcal{A}_1 \mapsto \mathcal{A}_1 + d\lambda_0 \,. \tag{4.2.13}$$

With this choice, $\mathcal{A}_1$ is interpreted as a background gauge field for the global $U(1)$ symmetry on the boundary. Notice that with our choice of boundary term the whole system is gauge invariant.

We can also restore the gauge transformations $b_{d-1} \mapsto b_{d-1} + d\nu_{d-2}$ by transforming

$$\mathcal{A}_1 \;\mapsto\; \mathcal{A}_1 - (-1)^d \frac{i}{R^2} \star d\nu_{d-2} \,, \tag{4.2.14}$$

which however are not proper background gauge transformations. A clearer and equivalent possibility is to parametrize the boundary condition as

$$\star\, A_1 = -\frac{i}{R^2}\left(b_{d-1} - \mathcal{B}_{d-1}\right) , \tag{4.2.15}$$

where $\mathcal{B}_{d-1}$ is another fixed background on the boundary that transforms as $\mathcal{B}_{d-1} \mapsto \mathcal{B}_{d-1} + d\nu_{d-2}$. It can be understood as a background field for the global $(d-2)$-form symmetry on the boundary. Yet another possibility is to restore both gauge transformations, for instance through the parametrization

$$\star\left(A_1 - \mathcal{A}_1\right) = -\frac{i}{R^2}\left(b_{d-1} - \mathcal{B}_{d-1}\right) . \tag{4.2.16}$$

We can use it to discover information about the boundary theory. Indeed, with the choice of boundary term in (4.2.12), the system is not gauge invariant, rather under a gauge transformation we find

$$\delta(S+S_\partial) = (-1)^{d-1}\frac{i}{2\pi}\int_{\mathcal{M}_d} d\nu_{d-2}\wedge\mathcal{A}_1 - \frac{1}{4\pi R^2}\int_{\mathcal{M}_d}\left(2\, d\nu_{d-2}\wedge\star\,\mathcal{B}_{d-1} + d\nu_{d-2}\wedge\star\, d\nu_{d-2}\right) . \tag{4.2.17}$$

The second piece can be cancelled by modifying the boundary term with the addition of

$$\frac{1}{4\pi R^2}\int_{\mathcal{M}_d} \mathcal{B}_{d-1}\wedge\star\,\mathcal{B}_{d-1}\,, \tag{4.2.18}$$

that can be understood as a local counterterm. However the first piece in (4.2.17) cannot be removed while preserving background gauge invariance for the $U(1)$ 0-form symmetry. This is a sign that the two symmetries have a mixed 't Hooft anomaly. Indeed, as we are going to see, the theory we are describing is the holographic dual to a d-dimensional compact boson. In what follows we will turn on only the background for the $U(1)$ 0-form symmetry, *i.e.*, we will use the boundary condition (4.2.10).

In order to rewrite the path integral of this TQFT as that of the compact boson we proceed in analogy with [125, 382, 383] (see also [384]). We assume that X_{d+1} contains an S^1 factor parametrized by $t \sim t + \beta$, interpreted as Euclidean time, hence $X_{d+1} = X_d \times S^1$ and $\partial X_{d+1} \equiv \mathcal{M}_d = \mathcal{M}_{d-1} \times S^1$. For simplicity, we also choose the metric of ∂X_{d+1} to be diagonal in $\mathcal{M}_{d-1}$ and S^1 so that

$$\star\, dt = (-1)^{d-1}\,\mathrm{Vol}_{\mathcal{M}_{d-1}} \;\in\; \Omega^{d-1}(\mathcal{M}_{d-1}) \tag{4.2.19}$$

with $\mathrm{Vol}_{\mathcal{M}_{d-1}}$ the volume form of $\mathcal{M}_{d-1}$. We decompose the bulk fields as

$$A_1 = A_0^t\, dt + \widetilde{A}_1\,, \qquad\qquad b_{d-1} = b_{d-2}^t \wedge dt + \widetilde{b}_{d-1}\,, \tag{4.2.20}$$

where forms with a tilde live on the *spatial* manifold X_d. The time components A_0^t and b_{d-2}^t appear linearly and can be treated as Lagrange multipliers. Integrating them out enforces

$$\tilde{d}\widetilde{A}_1 = 0\,, \qquad\qquad \tilde{d}\,\widetilde{b}_{d-1} = 0\,. \tag{4.2.21}$$

We now make a choice for X_d and take it to be a d-dimensional ball so that $\mathcal{M}_d = S^{d-1} \times S^1$. Then (4.2.21) are solved by introducing a compact scalar Φ_0 and a $(d-2)$-form $\mathbb{R}$ gauge field ω_{d-2} as

$$\widetilde{A}_1 = \tilde{d}\Phi_0\,, \qquad\qquad \widetilde{b}_{d-1} = \tilde{d}\omega_{d-2}\,. \tag{4.2.22}$$

Rewriting both the bulk action and the boundary term using Φ_0 and ω_{d-2}, the system reduces to the boundary action

$$\begin{aligned} S = \frac{i}{2\pi}\int_{\mathcal{M}_d}\Bigg[&(-1)^d\,\tilde{d}\omega_{d-2}\wedge\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt \,+ \\ &-\frac{i}{2}\bigg(R^2\big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\big)\wedge\star\big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\big) + \frac{1}{R^2}\,\tilde{d}\omega_{d-2}\wedge\star\,\tilde{d}\omega_{d-2}\bigg)\Bigg]\,. \end{aligned} \tag{4.2.23}$$

This action is not covariant, and time derivatives appear linearly. For $d = 2$, the action contains two scalars and is a manifestly self-dual formulation of the compact boson known in the condensed matter literature as the Luttinger liquid Lagrangian (see, *e.g.*, [385] for a recent discussion). It has the advantage of making both $U(1)$ symmetries explicit, at the expense of hiding Lorentz invariance. The action (4.2.23) is a d-dimensional generalization of it and it makes both the 0-form and the $(d-2)$-form $U(1)$ symmetries manifest.

Path integrals with an action linear in time derivatives are interpreted as phase-space path integrals. One can typically obtain a configuration-space path integral by integrating out the momenta that appear quadratically. Indeed, here $\tilde{d}\omega_{d-2}$ is the conjugate momentum to Φ_0 and we can recast the theory in a Lorentz-invariant form by integrating out ω_{d-2}. An important observation is that the action has zero modes that need to be eliminated. One way to see this is via the equations of motion for ω_{d-2}. These are

$$\tilde{d}\left[\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt \,+\, (-1)^d\frac{i}{R^2}\star\tilde{d}\omega_{d-2}\right] = 0 \tag{4.2.24}$$

with solution

$$\tilde{d}\omega_{d-2} = iR^2 \left(\partial_t \Phi_0 - \mathcal{A}_0^t\right) \star dt - iR^2 \star \tilde{d}\gamma_0 \,. \tag{4.2.25}$$

Notice that, since $\left(\partial_t \Phi_0 - \mathcal{A}_0^t\right) \star dt$ is a $(d-1)$-form supported only on space, we have $\tilde{d}\star\tilde{d}\gamma_0 = 0$. The scalar γ_0 is integrated over but its path integral is naively divergent because γ_0 has vanishing action, *i.e.*, it is a zero-mode. Therefore in order to get a consistent theory we have to gauge fix $\gamma_0 = 0$. Plugging $\tilde{d}\omega_{d-2}$ in (4.2.23) we get the final action

$$S = \frac{R^2}{4\pi} \int_{\mathcal{M}_d} \left(d\Phi_0 - \mathcal{A}_1\right) \wedge \star \left(d\Phi_0 - \mathcal{A}_1\right) , \tag{4.2.26}$$

corresponding to a d-dimensional compact boson with radius R. Had we integrated out Φ_0 from (4.2.23), we would have found the dual formulation in terms of the $(d-2)$-form ω_{d-2}. The background field $\mathcal{A}_1$ corresponds to the $U(1)$ shift symmetry of the boson and the anomalous shift we discussed above corresponds to the mixed 't Hooft anomaly with the winding symmetry.

One might be puzzled by the fact that we have one bulk gauge symmetry $U(1)$, but we still obtain two global symmetries on the boundary, which might seem to clash with the usual holographic expectations. However, for the compact boson this is not really a contradiction: all correlation functions of one current can be obtained from those of the other. Indeed, the backgrounds of the two symmetries are obtained one from the other using the $\star$ operator (modulo counterterms, which correspond to contact terms in correlators); thus, functional derivatives of the partition function with respect to a single background already contain the information of all correlators of both currents (see [386] for a related discussion).

Before going on, let us mention an alternative, quicker way to arrive at the final result that does not pass through the Luttinger-liquid-like formulation (4.2.23). It requires X_{d+1} to be a ball, and hence $\mathcal{M}_d = S^d$. After determining the boundary conditions (4.2.10) and the boundary term (4.2.12), we just integrate the entire b_{d-1} out, imposing $dA_1 = 0$. Since the bulk is now topologically trivial, this is solved by $A_1 = d\Phi_0$. Using the boundary condition to express the boundary term (4.2.12) in terms of A_1, and plugging back $A_1 = d\Phi_0$, we immediately get (4.2.26).

Higher-form symmetries

The higher-form case is very similar and we only flash the 1-form symmetry example, just to highlight one small subtlety. The TQFT we start with has action

$$S = \frac{i}{2\pi} \int_{X_{d+1}} f_{d-2} \wedge dG_2 \,, \tag{4.2.27}$$

with f_{d-2} and G_2 being an $\mathbb{R}$ and $U(1)$ gauge field, respectively. On X_{d+1} with boundary $\mathcal{M}_d$, that we endow with a Riemannian metric (if $d = 4$ a conformal structure is enough) we set the boundary condition

$$\star G_2 = (-1)^{d+1} \frac{ie^2}{\pi} f_{d-2} + \star \mathcal{G}_2 \,, \tag{4.2.28}$$

where $[e^2] = 4 - d$. We must also add a boundary term

$$S_\partial = -\frac{e^2}{4\pi^2} \int_{\partial X_{d+1}} f_{d-2} \wedge \star f_{d-2} \,. \tag{4.2.29}$$

When solving the constraints imposed by the integral over time components as

$$\widetilde{f}_{d-2} = \tilde{d}\omega_{d-3}\,, \qquad\qquad \widetilde{G}_2 = \tilde{d}A_1\,, \tag{4.2.30}$$

we introduce (time-dependent) forms ω_{d-3} and A_1 only on the spatial manifold X_d, namely without time components. The boundary action one obtains is

$$\begin{aligned} S = \frac{i}{2\pi}\int_{\mathcal{M}_d}\Big[&(-1)^d\,\tilde{d}\omega_{d-3}\wedge\big(\partial_t A_1 + \mathcal{G}_1^t\big)\wedge dt + \\ &- \frac{i}{2}\bigg(\frac{e^2}{\pi}\,\tilde{d}\omega_{d-3}\wedge\star\,\tilde{d}\omega_{d-3} + \frac{\pi}{e^2}\big(\tilde{d}A_1 - \widetilde{\mathcal{G}}_2\big)\wedge\star\big(\tilde{d}A_1 - \widetilde{\mathcal{G}}_2\big)\bigg)\Big]\,. \end{aligned} \tag{4.2.31}$$

This is a higher-form generalization of (4.2.23) and integrating out ω_{d-3} we obtain

$$S = \frac{1}{4e^2}\int_{\mathcal{M}_d}\big(dA_1 - \mathcal{B}_2\big)\wedge\star\big(dA_1 - \mathcal{B}_2\big)\,, \tag{4.2.32}$$

where $\mathcal{B}_2 = -\mathcal{G}_1^t\wedge dt + \widetilde{\mathcal{G}}_2$ is a 2-form background field. This is a Maxwell action in d dimensions coupled to a background field $\mathcal{B}_2$ for its electric 1-form symmetry.

The subtlety we want to point out is that A_1 does not have the time component, hence this is a gauge-fixed Maxwell action.[66] There is a gauge choice that arises naturally in this reduction procedure, that is, the temporal gauge. The same story goes through for any higher-form gauge field: the boundary action is always a generalized Maxwell theory in the temporal gauge (see [384] for a discussion on this point). It is important to keep this small subtlety in mind when looking at more complicated TQFTs that produce further interactions involving the photon. For instance, in Section 4.2.4 we will obtain Chern–Simons terms on the boundary, and we will have to keep in mind that they always arise in the temporal gauge.

Lagrangian algebras and topological sectors

There is one very important caveat in the discussion of the previous two sections. Let us focus on the 0-form symmetry case for definiteness. We have shown that with the boundary condition we chose, the path integral of the TQFT can be rewritten as a path integral with the action of a compact boson (4.2.26). However, the domain is not the one of the physical theory. The reason is that when we solve (4.2.21) introducing Φ_0 and ω_{d-2} as in (4.2.22), these fields cannot wind around the time circle S^1. Hence what we established in Section 4.2.2 is that the TQFT partition function is equal to the zero-winding sector of a compact boson.[67]

However, it turns out that we can produce the path integral in *any* fixed winding sector, simply by inserting a Wilson line $e^{in\int_{S^1}A_1}$ along the time circle in the bulk. The line pierces the spatial manifold X_d at a point P, creating a nontrivial $(d-1)$-cycle $\Sigma_{d-1}\subset X_d$ and introducing a monodromy for $\widetilde{b}_{d-1}$ around it:

$$\int_{\Sigma_{d-1}}\widetilde{b}_{d-1} = 2\pi n\,. \tag{4.2.33}$$

[66] This subtlety does not arise in the quicker procedure described at the end of the last section.

[67] For $d=2$ the boundary spatial manifold is S^1, and since Φ_0 is compact the path integral includes a sum over all windings around that spatial circle, but not around the time circle.

To get the TQFT partition function with this insertion, consider a generator $\frac{\eta_{d-1}}{2\pi}$ of $H^{d-1}(X_d \setminus P;\mathbb{Z})$, namely $\int_{\Sigma_{d-1}} \eta_{d-1} = 2\pi$. The second equation in (4.2.21) is now solved by

$$\widetilde{b}_{d-1} = n\,\eta_{d-1} + \tilde{d}\omega_{d-2}\,. \tag{4.2.34}$$

With the same steps as before we obtain a path integral on boundary fields Φ_0 and ω_{d-2}, again over configurations of Φ_0 with zero winding around the time circle, but with a modified action with respect to (4.2.23):

$$\begin{aligned} S_n = \frac{i}{2\pi}\int_{\mathcal{M}_d}\Big[&(-1)^d\,\tilde{d}\omega_{d-2}\wedge\big(\partial_t\Phi_0 - \mathcal{A}_0^t\big)dt + \\ &-\frac{i}{2}\bigg(R^2\,\big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\big)\wedge\star\big(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\big) + \frac{1}{R^2}\,\tilde{d}\omega_{d-2}\wedge\star\,\tilde{d}\omega_{d-2}\bigg)\Big] + \\ -(-1)^d\,\frac{in}{2\pi}&\int_{\mathcal{M}_d}\mathcal{A}_0^t\,\widehat{\eta}_{d-1}\wedge dt + \frac{n^2}{4\pi R^2}\int_{\mathcal{M}_d}\widehat{\eta}_{d-1}\wedge\star\,\widehat{\eta}_{d-1}\,. \end{aligned} \tag{4.2.35}$$

Here $\widehat{\eta}_{d-1}$ is the pull-back of η_{d-1} on $\mathcal{M}_d$. It is a top form on $\partial X_d \equiv \mathcal{M}_{d-1}$ and one can make a choice for the representative η_{d-1} in (4.2.34) such that $\widehat{\eta}_{d-1} = \frac{2\pi}{v}\,\mathrm{Vol}_{\mathcal{M}_{d-1}}$ with $v = \int_{\mathcal{M}_{d-1}}\mathrm{Vol}_{\mathcal{M}_{d-1}}$ the volume of the boundary spatial slice. In particular $\star\,\widehat{\eta}_{d-1} = \frac{2\pi}{v}\,dt$. Plugging this back into (4.2.35) we obtain

$$S_n = S_0 - in\theta + \frac{\pi\beta n^2}{vR^2} \qquad \text{where} \qquad \theta = (-1)^d\int_{S^1}\mathcal{A}_0^t\,dt\,. \tag{4.2.36}$$

Here S_0 is the action (4.2.23) written in terms of the periodic scalar in the Luttinger liquid form, which could be rewritten in the Lorentz covariant form (4.2.26) that makes manifest its nature as a boson of radius R. Notice that $\theta \sim \theta + 2\pi$ has the interpretation of a chemical potential for the $U(1)$ 0-form symmetry. The partition function with the line inserted is then

$$Z_n = Z_{\text{pert}}\,\exp\bigg(in\theta - \frac{\pi\beta}{vR^2}\,n^2\bigg) \tag{4.2.37}$$

where Z_{pert} is the perturbative contribution due to a periodic boson.

We want to show our claim that, after we condense a Lagrangian algebra in the bulk, the partition function includes the sum over all topological sectors of the compact scalar, hence reproducing the physical partition function. The simplest Lagrangian algebra contains all the lines $W_n = e^{in\int A_1}$ and no surfaces $V_\alpha = e^{i\alpha\int b_{d-1}}$. Due to our choice of geometry, gauging this algebra is the same as summing over all lines inserted along the time circle, hence summing over all n in (4.2.37). The bulk interpretation of this sum is that we are computing the partition function of the SPT phase obtained by gauging the algebra, which we are taking as our theory of gravity. Hence using Poisson's summation formula we find[68]

$$Z_{\text{gravity}} = \sum_{n\in\mathbb{Z}} Z_n = Z_{\text{pert}}\sum_{w\in\mathbb{Z}}\exp\bigg[-\frac{\pi v R^2}{\beta}\Big(w + \frac{\theta}{2\pi}\Big)^2\bigg]\,. \tag{4.2.38}$$

The right hand side is precisely the partition function of a compact boson of radius R (with chemical potential θ).

[68]Here we are neglecting an extra factor $\sqrt{\beta/vR^2}$, since normalizations of the path integrals do not play a role in this paper. A similar factor is neglected in (4.2.39).

More generally, the bulk TQFT has other Lagrangian algebras consisting of the lines W_{km} and the surfaces $V_{m'/k}$ for an integer number $k \in \mathbb{Z}$. Condensing one of them produces a different SPT phase in the bulk, hence a different theory of gravity. In the SymTFT story this corresponds to gauging the $\mathbb{Z}_k$ subgroups of the $U(1)$ symmetry at the boundary [206]. Because of the chosen geometry, there are no $(d-1)$-cycles in the bulk and hence condensing this algebra simply means summing over all Wilson lines of charge multiple of k. The result is

$$Z'_{\text{gravity}} = \sum_{m \in \mathbb{Z}} Z_{km} = Z_{\text{pert}} \sum_{w \in \mathbb{Z}} \exp\left[-\frac{\pi v}{\beta} \left(\frac{R}{k} \right)^2 \left(w + \frac{k\theta}{2\pi} \right)^2 \right] \tag{4.2.39}$$

and the right-hand side can be interpreted as the partition function of a compact boson of radius $R' = R/k$. This is an orbifold of the previous boundary theory, which could be thought of as a different global form of the same theory.

We want to comment on a slightly different way to obtain a holographic dual to compact bosons, which also fits our proposal. We could have started with the TQFT of two $\mathbb{R}$ gauge fields described by the action

$$S = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-1} \wedge da_1 \,. \tag{4.2.40}$$

In this TQFT the charges of the Wilson lines $W_\alpha = e^{i\alpha \int a_1}$ are not quantized, and since there is no sum over fluxes,[69] there is no identification among the charges of $V_\beta = e^{i\beta \int b_{d-1}}$. The spectrum of bulk operators is then larger, labelled by $\mathbb{R} \times \mathbb{R}$, and the corresponding braiding is the phase $e^{2\pi i \alpha\beta}$. Lagrangian algebras are classified by the choice of a real number $Q \in \mathbb{R}_+$ and are given by [34]

$$\mathcal{L}_Q = \left\{ W_{Qn}, V_{Q^{-1}m} \;\middle|\; n, m \in \mathbb{Z} \right\} . \tag{4.2.41}$$

It was shown in [206] that this TQFT is the SymTFT for two $U(1)$ symmetries, namely a 0-form and a $(d-2)$-form, with a mixed anomaly. While this is a different symmetry structure from just a single $U(1)$, the second higher-form symmetry arises universally in the IR whenever the 0-form symmetry is spontaneously broken. Hence the two symmetry structures share the same EFT that describes the broken phase and, according to our proposal, they should both be the holographic dual to a compact boson. Indeed there is no much difference between the two theories: the non-topological boundary conditions can be chosen to be the same, and the computations of Section 4.2.2 give the same result.

The considerations explained in this section can be repeated for any higher-form symmetry. However, in order to detect the various global structures of a boundary p-form Maxwell theory, one needs to properly choose the geometry. Indeed the fluxes are supported on $(p+1)$-dimensional cycles, and thus a natural choice is to take $X_{d+1} = B_{d-p} \times T^{p+1}$ with B_{d-p} a ball. One of the S^1 factors of the torus plays the role of a time circle, and $X_d = B_{d-p} \times T^p$. The bulk TQFT has action

$$S = \frac{i}{2\pi} \int_{X_{d+1}} b_{d-p-1} \wedge dA_{p+1} \tag{4.2.42}$$

where b_{d-p-1} is an $\mathbb{R}$ gauge field whilst A_{p+1} is a $U(1)$ gauge field. One can obtain an SPT phase by gauging the Lagrangian algebra given by $W_n = e^{in\int A_{p+1}}$, and this is realized by inserting

[69] An $\mathbb{R}$ gauge field admits a gauge in which the connection is globally defined, therefore the field strength is an exact form and its integrals on compact submanifolds vanish.

these defects along the T^{p+1} factor in the bulk. This sum indeed reproduces the sum over fluxes of the p-form Maxwell theory on the boundary. The choice of other Lagrangian algebras modifies the value of the electric charge and corresponds to discrete gaugings of the 1-form symmetry.

4.2.3 Abelian anomalies and higher groups

We can enrich the analysis of $U(1)$ symmetries by including anomalies (Sections 4.2.3 and 4.2.3) or a 2-group structure (Section 4.2.3). We show here that, when doing it, the dual boundary theory gets coupled to background fields in a non-minimal way. In Sections 4.2.3 and 4.2.3 we provide a field-theoretic interpretation of our results in terms of symmetry fractionalization.

Chiral anomaly in 2d

The SymTFT for an anomalous $U(1)$ symmetry in 2d has action [206]:

$$S = \frac{i}{2\pi} \int_{X_3} b_1 \wedge dA_1 + \frac{ik}{4\pi} \int_{X_3} A_1 \wedge dA_1 \,. \tag{4.2.43}$$

The additional bulk Chern–Simons term significantly affects the consistent boundary conditions. To establish a proper variational principle with a non-topological boundary condition, it is essential to include the boundary term

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\partial X_3} \left(b_1 + \frac{k}{2}\, A_1 \right) \wedge \star \left(b_1 + \frac{k}{2}\, A_1 \right) \tag{4.2.44}$$

together with the following Dirichlet boundary condition:[70]

$$\star\, \delta A_1 = -\frac{i}{R^2}\, \delta \left(b_1 + \frac{k}{2}\, A_1 \right) . \tag{4.2.45}$$

In order to properly turn on a background for the boundary $U(1)$ symmetry we have to render the boundary condition invariant under gauge transformations of A_1. This is most naturally done by introducing a 1-form $\mathcal{A}_1$ as

$$\star\, (A_1 - \mathcal{A}_1) = -\frac{i}{R^2} \left(b_1 + \frac{k}{2}\, (A_1 - \mathcal{A}_1) \right) . \tag{4.2.46}$$

This boundary condition is invariant under $\delta \mathcal{A}_1 = \delta A_1 = d\lambda_0$, allowing us to interpret $\mathcal{A}_1$ as a background field for the $U(1)$ symmetry on the boundary. Notice that our choice does not modify (4.2.45) and is thus just a particularly convenient parametrization.

Before deriving the dual boundary theory, we can already establish that it has an 't Hooft anomaly. Indeed, under a gauge transformation $\delta A_1 = \delta \mathcal{A}_1 = d\lambda_0$ the total action $S + S_\partial$ transforms as

$$\delta(S + S_\partial) = -\frac{ik}{4\pi} \int_{\mathcal{M}_2} d\lambda_0 \wedge \mathcal{A}_1 - \frac{k^2}{16\pi R^2} \int_{\mathcal{M}_2} \Big(2\, d\lambda_0 \wedge \star\, \mathcal{A}_1 + d\lambda_0 \wedge \star\, d\lambda_0 \Big) \tag{4.2.47}$$

[70] One can check, by writing all possible boundary terms and imposing consistency of the variational principle, that these boundary data are the only possible choice.

where $\mathcal{M}_2 = \partial X_3$. The second term can be cancelled by adding the following counterterm to the boundary action:

$$S_{\text{c.t.}} = \frac{k^2}{16\pi R^2} \int_{\mathcal{M}_2} \mathcal{A}_1 \wedge \star \mathcal{A}_1 \,. \tag{4.2.48}$$

However the remaining total gauge variation

$$\delta\big(S + S_\partial + S_{\text{c.t.}}\big) = -\frac{ik}{4\pi} \int_{\mathcal{M}_2} d\lambda_0 \wedge \mathcal{A}_1 \tag{4.2.49}$$

cannot be cancelled by any local boundary counterterm: it is precisely the anomalous variation corresponding to a perturbative $U(1)$ anomaly.

To derive the boundary theory we follow the steps outlined in Section 4.2.2. The constraints imposed by the path integral over time components again allow us to write $\widetilde{A}_1 = \tilde{d}\Phi_0$ and $\widetilde{b}_1 = \tilde{d}\omega_0$. The boundary action expressed in terms of these variables, after introducing $\mathcal{F} = \mathcal{A}_1 - \frac{ik}{2R^2} \star \mathcal{A}_1$ for convenience, reads:

$$\begin{aligned} S = \frac{i}{2\pi} \int_{\mathcal{M}_2} \Big[&\Big(\tilde{d}\omega_0 + \tfrac{k}{2}\,\tilde{d}\Phi_0\Big)\big(\partial_t \Phi_0 - \mathcal{F}_0^t\big) \wedge dt + \\ &- \frac{i}{2}\bigg(R^2 \Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1\Big) \wedge \star \Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1\Big) + \frac{1}{R^2}\Big(\tilde{d}\omega_0 + \tfrac{k}{2}\,\tilde{d}\Phi_0\Big) \wedge \star \Big(\tilde{d}\omega_0 + \tfrac{k}{2}\,\tilde{d}\Phi_0\Big)\bigg)\Big] + S_{\text{c.t}} \,. \end{aligned} \tag{4.2.50}$$

This is the same action as in (4.2.23) for $d = 2$ but with $\omega_0 \mapsto \omega_0 + \frac{k}{2}\,\Phi_0$. Integrating ω_0 out we find

$$S = \frac{R^2}{4\pi} \int_{\mathcal{M}_2} \big(d\Phi_0 - \mathcal{A}_1\big) \wedge \star \big(d\Phi_0 - \mathcal{A}_1\big) + \frac{ik}{4\pi} \int_{\mathcal{M}_2} \Phi_0 \, d\mathcal{A}_1 \,. \tag{4.2.51}$$

This action describes a compact boson of radius R, but with an unusual coupling to a background for the momentum symmetry. Such a coupling reproduces the anomalous shift (4.2.49) that is indeed cancelled by the inflow action

$$S_{\text{inflow}} = -\frac{ik}{4\pi} \int_{\text{3d}} \mathcal{A}_1 \wedge d\mathcal{A}_1 \,. \tag{4.2.52}$$

Notice that the extra coupling $\Phi_0 \, d\mathcal{A}_1$ in (4.2.51) has a form similar to the coupling with the winding symmetry. In a sense, we are prescribing that a background $\mathcal{A}_1$ for the momentum symmetry also activates a background $\mathcal{B}_1 = k\mathcal{A}_1$ for the winding symmetry. In other words, $\mathcal{A}_1$ is not coupled with the momentum symmetry but rather with a diagonal combination of momentum and winding.[71] Since the two symmetries have a mixed anomaly, this diagonal $U(1)$ inherits a pure anomaly.

Chiral anomaly in 4d

The treatment of anomalies in higher dimensions presents a further conceptual difference. As a representative case, we consider $d = 4$ and the TQFT with action

$$S = \frac{i}{2\pi} \int_{X_5} b_3 \wedge dA_1 + \frac{ik}{24\pi^2} \int_{X_5} A_1 \wedge dA_1 \wedge dA_1 \,. \tag{4.2.53}$$

[71] More precisely, it is the diagonal combination between momentum and a $\mathbb{Z}_k$ extension of the winding symmetry.

To get a good variational principle we need to impose

$$\star\,\delta A_1 = -\frac{i}{R^2}\,\delta\Big(b_3 + \frac{k}{6\pi}A_1\wedge dA_1\Big) \tag{4.2.54}$$

and add a boundary term

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_5}\Big(b_3 + \frac{k}{6\pi}A_1\wedge dA_1\Big)\wedge\star\Big(b_3 + \frac{k}{6\pi}A_1\wedge dA_1\Big)\,. \tag{4.2.55}$$

These choices however do not allow us to turn on a background by simply changing the parametrization of the boundary condition, as we did in 2d. Indeed, if we try to restore the gauge transformations of A_1, the boundary condition shifts by terms that depend on the field A_1 itself and cannot be cancelled by adding counterterms in the background only. Turning on a background in $d > 2$ requires us to change the boundary data in a nontrivial way. In Appendix B.7 we explain an iterative procedure that, starting from the data above, produces a consistent variational principle together with a gauge-invariant boundary condition. The result for $d = 4$ is

$$\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i}{R^2}\Big(b_3 + \frac{k}{6\pi}\big(A_1-\mathcal{A}_1\big)\wedge dA_1 + \frac{k}{12\pi}\big(A_1-\mathcal{A}_1\big)\wedge d\mathcal{A}_1\Big) \tag{4.2.56}$$

with boundary term

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_5}\Big(b_3+\frac{k}{6\pi}\big(A_1-\mathcal{A}_1\big)\wedge dA_1+\frac{k}{12\pi}A_1\wedge d\mathcal{A}_1\Big)^2+\frac{ik}{24\pi^2}\int_{\partial X_5}\mathcal{A}_1\wedge A_1\wedge dA_1\,. \tag{4.2.57}$$

When setting $\mathcal{A}_1 = 0$ we recover the previous boundary data, but in general there are new terms that mix background and dynamical fields. As in 2d, one can show that the system has an anomaly performing a gauge transformation $\delta A_1 = \delta\mathcal{A}_1 = d\lambda_0$: up to a counterterm the gauge variation is

$$\delta\big(S + S_\partial + S_{\text{c.t.}}\big) = \frac{ik}{24\pi^2}\int_{\partial X_5}\lambda_0\, d\mathcal{A}_1\wedge d\mathcal{A}_1\,. \tag{4.2.58}$$

The procedure to determine the dual boundary theory is completely analogous to the examples we have already presented. Integrating the time components out, we introduce $\widetilde{A}_1 = \tilde{d}\Phi_0$ and $\widetilde{b}_3 = \tilde{d}\omega_2$. To simplify our expressions, we denote $\mathcal{F}_1 = \mathcal{A}_1 - \frac{ik}{12\pi R^2}\star(\mathcal{A}_1\wedge d\mathcal{A}_1)$. Then the boundary action, in its non-covariant presentation, is

$$\begin{aligned}S = \frac{i}{2\pi}\int_{\mathcal{M}_4}\Big[&\Big(\tilde{d}\omega_2 + \tfrac{k}{12\pi}\,\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\big(\partial_t\Phi_0 - \mathcal{F}_0^t\big)dt - \frac{i}{2}\Big(R^2\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1\Big)\wedge\star\Big(\tilde{d}\Phi_0 - \widetilde{\mathcal{F}}_1\Big) +\\ &+\frac{1}{R^2}\Big(\tilde{d}\omega_2 + \tfrac{k}{12\pi}\,\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\wedge\star\Big(\tilde{d}\omega_2 + \tfrac{k}{12\pi}\,\tilde{d}\Phi_0\wedge\tilde{d}\widetilde{\mathcal{A}}_1\Big)\Big)\Big] + S_{\text{c.t.}}\end{aligned} \tag{4.2.59}$$

where $\mathcal{M}_4 = \partial X_5$. As before we can integrate out ω_2 and the final action reads

$$S = \frac{R^2}{4\pi}\int_{\mathcal{M}_4}\big(d\Phi_0 - \mathcal{A}_1\big)\wedge\star\big(d\Phi_0 - \mathcal{A}_1\big) + \frac{ik}{24\pi^2}\int_{\mathcal{M}_4}\Phi_0\, d\mathcal{A}_1\wedge d\mathcal{A}_1\,. \tag{4.2.60}$$

This represents a compact scalar with a non-standard coupling to a background associated with the shift symmetry, akin to the situation in 2d. The additional interaction accounts for the anomalous variation described by (4.2.58). Nevertheless, unlike in the 2d scenario, we cannot view this altered interaction as a combination of the shift and winding symmetries since the two have different degree.

Anomaly matching in the broken phase

Let us provide a purely field-theoretic interpretation of the result in the previous section. For any Lie-group symmetry G, the Goldstone theory describing the symmetry breaking phase is a non-linear sigma model with target space G. In even spacetime dimensions d, the symmetry G can suffer from perturbative anomalies and the question is how these are matched in the sigma model.

For non-Abelian G it is well known that the anomaly is reproduced by a WZW term [381]. This is an additional interaction with important dynamical consequences. Perturbative anomalies are classified by $H^{d+2}(BG;\mathbb{Z})$, which determines a $(d+1)$-dimensional Chern–Simons action that cancels the anomaly by inflow. On the other hand, WZW terms in d dimensions are classified by $H^{d+1}(G;\mathbb{Z})$. Anomaly matching is mathematically represented by a map

$$\tau : H^{d+2}(BG;\mathbb{Z}) \to H^{d+1}(G;\mathbb{Z}) \tag{4.2.61}$$

called *transgression* [387]. For $d = 2$ this map also underlines the map of levels in the CS/WZW correspondence [188]. For the simple Lie group $G = SU(n)$, the transgression map τ is injective [188], meaning that any perturbative anomaly is matched by a WZW term.[72] However this is not the general case, and if τ has a nontrivial kernel, the corresponding anomalies require some new ingredient to be matched in the sigma model.

Here we focus on the extreme case $G = U(1)$ for which $H^{d+1}\big(U(1);\mathbb{Z}\big) = 0$, namely there is no WZW term at all, and any anomaly must be matched in a different way. From our holographic analysis we know the answer to this question: the dynamics of the sigma model is unchanged with respect to the non-anomalous case, but the symmetry is coupled non-minimally to the background $\mathcal{A}_1$ through the extra topological term

$$\frac{ik}{(2\pi)^{d/2}\left(\frac{d}{2}+1\right)!}\int_{\mathcal{M}_d} \Phi_0\left(d\mathcal{A}_1\right)^{d/2}. \tag{4.2.62}$$

This term reproduces the anomaly, but at this level it seems a bit ad hoc. We want to clarify why it arises from a UV viewpoint and how we understand it in the IR. This is important to understand why there is a difference in how anomaly matching works in the Abelian and non-Abelian cases.

We can show in a simple model that when the background field is turned on in the UV, the additional coupling (4.2.62) is generated along the RG flow by integrating out massive fields. Consider a 4d theory with a massless Dirac fermion ψ and a complex scalar ϕ, coupled via a Yukawa interaction:

$$\mathcal{L} \supset \phi\,\overline{\psi}\psi\,. \tag{4.2.63}$$

The theory has an axial symmetry $U(1)_A$ under which both Weyl components of ψ have charge 1, while ϕ has charge -2. $U(1)_A$ has a cubic anomaly with $k = 2$. Choosing a potential $V(\phi)$ that induces condensation of ϕ, the axial symmetry gets spontaneously broken to $\mathbb{Z}_2 = (-1)^F$. By decomposing $\phi = \rho\, e^{i\Theta}$ into its radial and angular parts, the VEV $\langle\rho\rangle = v$ gives mass to both ρ and ψ. The angular part Θ remains massless and is the only degree of freedom at low

[72] The transgression map is expected to be injective for all simple Lie groups.

energy: it is the Goldstone boson. The faithful symmetry in the IR is $U(1) = U(1)_A/\mathbb{Z}_2$ that shifts Θ. In order to reproduce the anomaly, the coupling to a background $\mathcal{A}$ must include the term

$$\frac{i}{24\pi^2}\,\Theta\,(d\mathcal{A})^2\,. \tag{4.2.64}$$

Indeed this term arises when integrating out the fermion. To see this notice that, for fixed ϕ and $\mathcal{A}$, if ϕ is real and positive then the fermion path integral can be regularized in a way such that the measure is positive [388–390]. Clearly this is not true on a generic configuration, but we can make it true by performing an axial rotation of parameter $e^{i\alpha}$, with $\alpha = -\frac{1}{2}\Theta$. A textbook computation [317, 391] shows that the path integral measure of the fermion changes by a phase

$$D[\psi] \;\mapsto\; D[\psi]\,\exp\!\left(\frac{ik}{24\pi^2}\int\alpha\,(d\mathcal{A})^2\right). \tag{4.2.65}$$

Setting $\alpha = -\frac{1}{2}\Theta$ this precisely reproduces the coupling (4.2.64). Now the Yukawa coupling becomes $\rho\,\overline{\psi}\psi$, that for fixed ρ is essentially a positive mass term for the fermion, hence integrating out the fermion becomes a safe operation that does not introduce extra phases.

Returning to the general case, we want to interpret the extra coupling (4.2.62) as specifying a (higher) symmetry fractionalization class for the $U(1)$ symmetry. This reinterpretation will be crucial to understand the analogous story for higher groups in the following sections. A 0-form symmetry G can fractionalize in the presence of a discrete 1-form symmetry Γ. This means that when two topological defects $g, h \in G$ fuse to produce $gh \in G$, their codimension-two junction gets covered by a topological defect $\omega(g,h) \in \Gamma$ of the 1-form symmetry [178, 338, 348], where $\omega \in H^2(BG;\Gamma)$. Equivalently, a background $\mathcal{A}_1$ for G turns on a background $\mathcal{B}_2 = \mathcal{A}_1^*\,\omega$ for the 1-form symmetry. In this formula, we think of $\mathcal{A}_1$ as a map $\mathcal{M}_d \to BG$ and of $\mathcal{B}_2$ as an element of $H^2(\mathcal{M}_d,\Gamma)$ so that we can use $\mathcal{A}_1$ to pull back ω. With this interpretation it becomes clear that, if G and Γ have a mixed anomaly, a non-trivial fractionalization class modifies the pure anomaly for G, possibly making it nontrivial even when it vanished originally [338, 348]. This has a natural generalization to the case that Γ is a discrete p-form symmetry: when $p+1$ topological defects $g_1, \ldots, g_{p+1} \in G$ fuse in generic position, they create a codimension-$(p+1)$ junction that can be dressed by a defect $\omega(g_1, \ldots, g_{p+1})$ of the p-form symmetry Γ, where ω is a class in $H^{p+1}(BG;\Gamma)$. Equivalently, a background $\mathcal{A}_1$ turns on a background $\mathcal{B}_{p+1} = \mathcal{A}_1^*\,\omega$ for Γ.

The compact boson theory that describes the breaking of a $U(1)$ 0-form symmetry also possesses a $U(1)$ $(d-2)$-form winding symmetry, and the two have a mixed anomaly. For this reason, a pure anomaly for the 0-form symmetry can be induced by fractionalizing it with the $(d-2)$-form symmetry. One minor modification with respect to what we described above is necessary because the p-form symmetry (here $p = d-2$) is continuous. Its most natural description is not in terms of a background potential $\mathcal{B}_{p+1}$, which is not a cohomology class in general, but in terms of its field strength $\frac{1}{2\pi}\,d\mathcal{B}_{p+1} \in H^{p+2}(\mathcal{M}_d;\mathbb{Z})$. As a consequence the fractionalization class, instead of being an element of $H^{p+1}\big(BU(1);U(1)\big)$, is more naturally an element of $H^{p+2}(BU(1);\mathbb{Z}) \cong \mathbb{Z}$. This is the datum that determines a $(p+1)$-dimensional Chern–Simons level, or equivalently the corresponding Chern class in $(p+2)$ dimensions. Hence, in analogy with the discrete case, we prescribe that a background $\mathcal{A}_1$ for the 0-form symmetry activates a background $\mathcal{B}_{d-1}$ for the $(d-2)$-form symmetry whose field strength is

$$\frac{1}{2\pi}\, d\mathcal{B}_{d-1} = \frac{k}{(2\pi)^{d/2}\left(\frac{d}{2}+1\right)!}\left(d\mathcal{A}_1\right)^{d/2}. \tag{4.2.66}$$

Recalling that the $(d-2)$-form symmetry is coupled to its background field through the action term $\frac{i}{2\pi}\int_{\mathcal{M}_d} \Phi_0\, d\mathcal{B}_{d-1}$, this reproduces the coupling (4.2.62) in agreement with our holographic result.

Abelian 2-groups

We consider a 2-group symmetry in four dimensions formed by a $U(1)$ 0-form symmetry and a $U(1)$ 1-form symmetry. This can be obtained by starting from a theory with two $U(1)$ 0-form symmetries with a cubic mixed anomaly and gauging the $U(1)$ that appears linearly in the anomaly polynomial [179, 346]. The 1-form symmetry participating in the 2-group structure is the magnetic symmetry of the photon. The SymTFT for such a 2-group symmetry has action [206]:

$$S = \frac{i}{2\pi}\int_{X_5}\left(b_3 \wedge dA_1 + h_2 \wedge dC_2 + \frac{k}{2\pi}\, h_2 \wedge A_1 \wedge dA_1\right). \tag{4.2.67}$$

Here A_1 and C_2 are $U(1)$ gauge fields, while b_3 and h_2 are $\mathbb{R}$ gauge fields. The topological operators that implement the symmetry are the Wilson surfaces of b_3 and h_2. On the other hand, the endpoints of $e^{i\int A_1}$ are local operators charged under the 0-form symmetry, and the endlines of $e^{i\int C_2}$ are 't Hooft lines charged under the magnetic 1-form symmetry. The gauge transformations are:[73]

$$\delta A_1 = d\lambda_0\,, \qquad \delta h_2 = d\xi_1\,, \qquad \delta b_3 = d\gamma_2 - \frac{k}{2\pi}\, d\xi_1 \wedge A_1\,, \qquad \delta C_2 = d\eta_1 + \frac{k}{2\pi}\, d\lambda_0 \wedge A_1\,. \tag{4.2.68}$$

We place this TQFT on a manifold with boundary, $X_5 = B_4 \times S^1$ for simplicity, and we interpret it as a theory of gravity, holographically dual to some 4d quantum field theory on the boundary. The last term in (4.2.67) contains a derivative, therefore it affects the boundary contribution to the variational principle, similarly to the case of chiral anomalies. To fix the boundary terms S_∂ and the boundary conditions on the fields, we use the same logic as in that case. We find the boundary conditions

$$\star\left(A_1 - \mathcal{A}_1\right) = -\frac{i}{R^2}\left[b_3 + \frac{k}{2\pi}\, h_2 \wedge \left(A_1 - \mathcal{A}_1\right)\right], \quad \star\, h_2 = \frac{ie^2}{\pi}\left(C_2 - \mathcal{C}_2 - \frac{k}{2\pi}\, \mathcal{A}_1 \wedge A_1\right) \tag{4.2.69}$$

and a corresponding boundary term

$$\begin{aligned} S_\partial = &-\frac{i}{2\pi}\int_{\partial X_5} h_2 \wedge \left(C_2 - \tfrac{k}{2\pi}\, \mathcal{A}_1 \wedge A_1\right) - \frac{e^2}{4\pi^2}\int_{\partial X_5}\left(C_2 - \tfrac{k}{2\pi}\, \mathcal{A}_1 \wedge A_1\right) \wedge \star\left(C_2 - \tfrac{k}{2\pi}\, \mathcal{A}_1 \wedge A_1\right) \\ &- \frac{1}{4\pi R^2}\int_{\partial X_5}\left[b_3 + \frac{k}{2\pi}\, h_2 \wedge \left(A_1 - \mathcal{A}_1\right)\right] \wedge \star\left[b_3 + \frac{k}{2\pi}\, h_2 \wedge \left(A_1 - \mathcal{A}_1\right)\right]. \end{aligned} \tag{4.2.70}$$

[73] There is some freedom in the choice of transformations that leave (4.2.67) invariant. In particular, the transformation $\delta A_1 = d\lambda_0$ could be accompanied by an action on both b_3 and C_2 as $\delta b_3 = -\epsilon \frac{k}{2\pi} d\lambda_0 \wedge h_2$ and $\delta C_2 = (1-\epsilon)\frac{k}{2\pi} d\lambda_0 \wedge A_1$ for any choice of ϵ. Here we chose $\epsilon = 0$ which matches the transformations in the boundary theory.

Here $\mathcal{A}_1$, $\mathcal{C}_2$ are fixed gauge fields on the boundary that transform as a proper 2-group background:

$$\delta\mathcal{A}_1 = d\lambda_0 \,, \qquad \delta\mathcal{C}_2 = d\eta_1 + \frac{k}{2\pi}\, d\lambda_0 \wedge \mathcal{A}_1 \,. \tag{4.2.71}$$

This makes the boundary conditions gauge invariant, provided we add a counterterm $\frac{e^2}{4\pi^2}\int_{\partial X_5} \mathcal{C}_2 \wedge \star\, \mathcal{C}_2$.

With the usual procedure, we obtain that the dual boundary theory has action:

$$\begin{aligned} S = &\frac{R^2}{4\pi}\int_{\partial X_5} \big(d\Phi_0 - \mathcal{A}_1\big) \wedge \star \big(d\Phi_0 - \mathcal{A}_1\big) + \frac{1}{4e^2}\int_{\partial X_5} da_1 \wedge \star\, da_1 \\ &+ \frac{i}{2\pi}\int_{\partial X_5} \mathcal{C}_2 \wedge da_1 + \frac{ik}{4\pi^2}\int_{\partial X_5} \Phi_0 \, da_1 \wedge d\mathcal{A}_1 \,. \end{aligned} \tag{4.2.72}$$

Naively one may think that a_1 is an $\mathbb{R}$ gauge field, because it comes from the trivialization of h_2. However, we have to take into account the condensation of the appropriate Lagrangian algebra in the bulk, necessary to trivialize the TQFT and making it independent of the topology. Specifically, here the relevant Lagrangian algebra is

$$\mathcal{L} = \left\{ e^{in\int A_1},\, e^{im\int C_2} \,\middle|\, n, m \in \mathbb{Z} \right\}. \tag{4.2.73}$$

Following the same logic as in Section 4.2.2, this introduces a sum over the fluxes of da_1 that effectively makes a_1 into a $U(1)$ gauge field.

Turning off the background $\mathcal{A}_1$ we obtain a free compact scalar and a free photon (coupled to a background field $\mathcal{C}_2$ for its magnetic symmetry), enjoying a $U(1)$ 0-form symmetry with conserved current $J_1 = \frac{iR^2}{2\pi}\, d\Phi_0$, and a $U(1)$ 1-form symmetry with conserved current $J_2 = \frac{1}{2\pi}\star da_1$, respectively. However, as soon as we turn on a background $\mathcal{A}_1$ for the 0-form symmetry, the 2-group structure manifests itself through the nonstandard coupling between the photon and the scalar, which modifies the currents and the background gauge transformations [179]. This is very similar to what happened in the case of the chiral anomaly, and we will provide a similar interpretation in terms of symmetry fractionalization in the next section.

Let us show that the theory in (4.2.72) reproduces the 2-group symmetry [179]. First, notice that the gauge transformation

$$\delta\Phi_0 = \lambda_0 \,, \qquad \delta\mathcal{A}_1 = d\lambda_0 \,, \qquad \delta\mathcal{C}_2 = d\eta_1 + \frac{k}{2\pi}\, d\lambda_0 \wedge \mathcal{A}_1 \tag{4.2.74}$$

leaves the action invariant. This is indeed the background gauge transformation for a 2-group. Second, in the presence of a background the currents get modified to:[74]

$$J_1 = \frac{iR^2}{2\pi}\big(d\Phi_0 - \mathcal{A}_1\big) + \frac{k}{4\pi} \star \big(d\Phi_0 \wedge da_1\big) \,, \qquad J_2 = \frac{1}{2\pi} \star da_1 \,, \tag{4.2.75}$$

and these satisfy modified conservation equations

$$d \star J_1 + \frac{k}{2\pi}\, d\mathcal{A}_1 \wedge \star J_2 = 0 \,, \qquad d \star J_2 = 0 \,, \tag{4.2.76}$$

that are the correct conservation equations for a 2-group symmetry.

[74] For a $U(1)$ p-form symmetry we use the convention that the current J_{p+1} is defined by $\star J_{p+1} = -i\frac{\delta S}{\delta \mathcal{A}_{p+1}}$ where $\mathcal{A}_{p+1}$ is the background field.

Abelian 2-groups in the broken phase

The unusual coupling to the background $\mathcal{A}_1$ in (4.2.72), responsible for the 2-group structure of the symmetry, is quite similar to the coupling (4.2.62) responsible for a chiral anomaly, that we interpreted in terms of symmetry fractionalization. Indeed we can give a similar interpretation here too. While it is intuitively clear why symmetry fractionalization can induce a pure anomaly, and this fact has been studied extensively [338, 348], the necessity of symmetry fractionalization to match higher-group structures has not been much appreciated. There is indeed one important difference, namely the nature of the symmetry used to fractionalize the $U(1)$ 0-form symmetry in question: it is a *composite symmetry* [392].

In general, if we have two $U(1)$ symmetries of degrees p and q with currents J_{p+1} and J_{q+1} respectively, if $p+q \geq d-1$ we can construct a third $U(1)$ symmetry simply because the current

$$J_{p+q-d+2} = \star\big((\star J_{p+1}) \wedge (\star J_{q+1})\big) \tag{4.2.77}$$

is automatically conserved. This symmetry is of degree $p+q-d+1$. In general, it is not a particularly interesting symmetry because its consequences are already implied by the constituent symmetries. However, it plays a role in our discussion. The IR theory of a 4d compact boson has an emergent 2-form symmetry: the winding symmetry of the scalar with current $J_3 = -\frac{1}{2\pi} \star d\Phi_0$. This is the symmetry we used to fractionalize the 0-form symmetry in the case of the chiral anomaly. In this case, since we also have the magnetic 1-form symmetry of the photon with current $J_2 = \frac{1}{2\pi} \star da_1$, we can construct

$$\widehat{J}_1 = \star\big((\star J_3) \wedge (\star J_2)\big) = \frac{1}{4\pi^2} \star \big(d\Phi_0 \wedge da_1\big) \tag{4.2.78}$$

that generates a 0-form symmetry. Using this symmetry to fractionalize the shift symmetry of the compact boson, as described in Section 4.2.3, we obtain precisely the non-canonical coupling in (4.2.72).

4.2.4 Boundary Chern–Simons-like terms

In this section we study bulk models obtained by adding terms without derivatives. These do not affect the boundary terms in the variational principle and hence do not modify the boundary conditions. Thus the dual theory couples minimally to the background fields, but it contains extra interactions, typically Chern–Simons-like terms. Our main motivation here is to verify our conjecture in a case with a non-invertible symmetry, the $\mathbb{Q}/\mathbb{Z}$ chiral symmetry in four dimensions [61, 64],[75] and to provide a framework to study aspects of its spontaneous breaking. We also consider in Section 4.2.4 a bulk 4d TQFT introduced in [206], which was argued to be related to 3d gauge theories with Chern–Simons interactions. We use our formalism to establish a precise holographic duality confirming the expectation of [206].

[75] See [213] for a recent proposal to recover the full $U(1)$ chiral symmetry.

Holographic dual to Maxwell–Chern–Simons theory

We consider the 4d TQFT with action

$$S=\frac{i}{2\pi}\int_{X_4}\left(A_1\wedge db_2+\frac{\phi}{4\pi}\,b_2\wedge b_2\right), \tag{4.2.79}$$

where b_2 is an $\mathbb{R}$ 2-from gauge field, A_1 is a standard $U(1)$ gauge field, and ϕ is a parameter. On closed manifolds the theory is invariant under the following gauge transformations:

$$\delta A_1=d\rho_0-\frac{\phi}{2\pi}\,\lambda_1\,,\qquad\qquad \delta b_2=d\lambda_1\,. \tag{4.2.80}$$

The gauge-invariant operators include surfaces $U_\alpha(\gamma_2)=e^{i\alpha\int_{\gamma_2}b_2}$ and the generically non-genuine lines $W_n(\gamma_1,D_2)=e^{in\int_{\gamma_1}A_1+\frac{in\phi}{2\pi}\int_{D_2}b_2}$ that need an attached two-disk D_2 bounded by γ_1. The label $\alpha\sim\alpha+1$ is circle valued, while $n\in\mathbb{Z}$. The coupling ϕ is 2π periodic. We will be mostly interested in the case

$$\phi=\frac{2\pi}{k}\qquad\text{with}\qquad k\in\mathbb{Z}\,. \tag{4.2.81}$$

In this case the lines W_{mk} become genuine, and an interesting Lagrangian algebra is obtained by taking all the genuine lines together with the surfaces $U_{l/k}$ with $l\in\mathbb{Z}_k$.

We place the theory on a manifold with boundary, where we impose the boundary condition

$$\star\left(A_1-\mathcal{A}_1\right)=-\frac{i\pi}{k^2e^2}\,b_2\,. \tag{4.2.82}$$

In order to have a good variational principle we must add the boundary term

$$S_\partial=-\frac{k^2e^2}{4\pi^2}\int_{\partial X_4}A_1\wedge\star A_1=\frac{1}{4k^2e^2}\int_{\partial X_4}\left(b_2+\frac{ik^2e^2}{\pi}\star\mathcal{A}_1\right)\wedge\star\left(b_2+\frac{ik^2e^2}{\pi}\star\mathcal{A}_1\right). \tag{4.2.83}$$

The gauge transformation $\delta A_1=d\rho_0$ is restored by $\delta\mathcal{A}_1=d\rho_0$ that makes (4.2.82) invariant. The full system is gauge invariant, provided that we also add a counterterm $S_{\text{c.t.}}=\frac{k^2e^2}{4\pi^2}\int_{\partial X_4}\mathcal{A}_1\wedge\star\mathcal{A}_1$.

We take the bulk to be the product of a three-dimensional ball B_3 and the time circle S^1, so that $\partial X_4\equiv\mathcal{M}_3=S^2\times S^1$. Integrating out the time components A_0^t, b_1^t we get delta functions imposing

$$\tilde{d}\,\widetilde{b}_2=0\,,\qquad\qquad \tilde{d}\widetilde{A}_1+\tfrac{1}{k}\,\widetilde{b}_2=0\,, \tag{4.2.84}$$

that are solved introducing Φ_0 and $\hat{a}_1$ through

$$\widetilde{b}_2=\tilde{d}\,\hat{a}_1\,,\qquad\qquad \widetilde{A}_1=\tilde{d}\Phi_0-\tfrac{1}{k}\,\hat{a}_1\,. \tag{4.2.85}$$

With this, the bulk path integral reduces to a boundary path integral with action

$$\begin{aligned}S+S_\partial+S_{\text{c.t.}}=\frac{i}{2\pi}\int_{\mathcal{M}_3}\Big[&\partial_t\hat{a}_1\wedge\left(\tilde{d}\Phi_0-\tfrac{1}{k}\,\hat{a}_1\right)\wedge dt+\frac{1}{2k}\,\hat{a}_1\wedge d\hat{a}_1+\tilde{d}\,\hat{a}_1\wedge\mathcal{A}_0^t\,dt+\\ &-\frac{i}{2}\left(\frac{\pi}{k^2e^2}\,\tilde{d}\,\hat{a}_1\wedge\tilde{d}\,\hat{a}_1+\frac{k^2e^2}{\pi}\left(\tilde{d}\Phi_0-\widetilde{\mathcal{A}}_1-\tfrac{1}{k}\,\hat{a}_1\right)\wedge\star\left(\tilde{d}\Phi_0-\widetilde{\mathcal{A}}_1-\tfrac{1}{k}\,\hat{a}_1\right)\right)\Big]\,.\end{aligned} \tag{4.2.86}$$

Attempting to integrate out $\hat{a}_1$ to derive a covariant action for the scalar field, as we did in Section 4.2.2, results in a non-local action.[76] However, there is no problem in integrating out Φ_0 from (4.2.86) and we obtain a local and covariant boundary theory with action

$$S = \frac{1}{4k^2e^2}\int_{\mathcal{M}_3} d\hat{a}_1 \wedge \star d\hat{a}_1 + \frac{i}{4\pi k}\int_{\mathcal{M}_3} \hat{a}_1 \wedge d\hat{a}_1 + \frac{i}{2\pi}\int_{\mathcal{M}_3} d\hat{a}_1 \wedge \mathcal{A}_1 \,. \tag{4.2.87}$$

This might seem like a $U(1)$ gauge theory with an improperly quantized Chern–Simons level. However we must be careful in identifying the correct $U(1)$ gauge field, by considering the condensation of the Lagrangian algebra that trivializes the bulk. This includes all genuine lines as well as k surfaces:

$$\mathcal{L} = \left\{ W_{km} = e^{ikm\int A_1} \,,\, U_{l/k} = e^{\frac{il}{k}\int b_2} \,\middle|\, m \in \mathbb{Z} \,,\;\; l \in \mathbb{Z}_k \right\}. \tag{4.2.88}$$

On the geometry that we are considering, condensing $\mathcal{L}$ amounts to inserting the lines W_{km} along the time circle and summing over m, while the surfaces have no effect. The insertion of W_{km} modifies the path integral so as to impose that $\int_{S^2} b_2 = 2\pi km$ for any two-sphere in B_3 that surrounds the Wilson line. This in particular includes the boundary spatial manifold. From the boundary theory viewpoint, this is a topological sector of the path integral with flux

$$\int_{S^2} \frac{d\hat{a}_1}{2\pi} = km \,. \tag{4.2.89}$$

Hence the canonically normalized $U(1)$ gauge field is $a_1 = \hat{a}_1/k$, in terms of which the boundary theory has action

$$S = \frac{1}{4e^2}\int_{\mathcal{M}_3} da_1 \wedge \star da_1 + \frac{ik}{4\pi}\int_{\mathcal{M}_3} a_1 \wedge da_1 + \frac{ik}{2\pi}\int_{\mathcal{M}_3} da_1 \wedge \mathcal{A}_1 \,. \tag{4.2.90}$$

This is Maxwell–Chern–Simons theory at level k, coupled to a background field for the topological $U(1)$ symmetry acting on monopoles. Precisely, the background field for this symmetry is $\mathcal{A}_1' = k\mathcal{A}_1$, while $\mathcal{A}_1$ is the background for a larger non-faithful $U(1)$ symmetry obtained by extending the topological symmetry with a trivially-acting $\mathbb{Z}_k$.[77]

Spontaneously broken non-invertible chiral symmetry

In 4d theories of massless Dirac fermions coupled with dynamical $U(1)$ gauge fields (QED-like theories) the classically conserved axial symmetry $U(1)_A$ suffers from an ABJ anomaly that spoils the conservation of the current: $d \star J_1^{(A)} = \frac{k}{8\pi^2} F_2 \wedge F_2$ [393, 394]. Traditionally, this was interpreted as the absence of $U(1)_A$ in the quantum theory. Recently [61, 64] showed that axial transformations labelled by rational numbers survive at the quantum level, but they obey

[76] A similar (even though less transparent) problem would have raised if we tried to obtain the boundary theory using the second method described at the end of Section 4.2.2, *i.e.*, by integrating out directly the whole b_2: it does not appear linearly in the bulk action.

[77] The reason why we got this coupling is that the TQFT we started with describes this larger symmetry, implemented by the operators $e^{i\alpha\int b_2}$, but the subgroup $\mathbb{Z}_k$ has been condensed in the bulk, and acts trivially in the boundary theory.

non-invertible fusion rules. The SymTFT for this non-invertible chiral symmetry was derived in [206]:

$$S = \frac{i}{2\pi}\int_{X_5}\left(b_3\wedge dA_1 + f_2\wedge dG_2 + \frac{k}{4\pi}\,A_1\wedge f_2\wedge f_2\right). \tag{4.2.91}$$

Here A_1, G_2 are $U(1)$ gauge fields, while b_3, f_2 are $\mathbb{R}$ gauge fields. The gauge transformations are

$$\begin{aligned} \delta A_1 &= d\rho_0\,, & \delta b_3 &= d\xi_2 - \frac{k}{4\pi}\,\lambda_1\wedge d\lambda_1 - \frac{k}{2\pi}\,\lambda_1\wedge f_2\,, \\ \delta f_2 &= d\lambda_1\,, & \delta G_2 &= d\eta_1 - \frac{k}{2\pi}\,\rho_0\left(f_2 + d\lambda_1\right) - \frac{k}{2\pi}\,\lambda_1\wedge A_1\,. \end{aligned} \tag{4.2.92}$$

As shown in [206], the gauge-invariant genuine topological defects are:

$$\begin{aligned} W_n(\gamma_1) &= e^{in\int_{\gamma_1}A_1}\,, & U_{\frac{p}{kq}}(\gamma_3) &= e^{i\frac{p}{kq}\int_{\gamma_3}b_3}\,\mathcal{A}^{q,p}(\gamma_3;f_2)\,, \\ V_\alpha(\gamma_2) &= e^{i\alpha\int_{\gamma_2}f_2}\,, & \mathcal{T}_m(\gamma_2) &= e^{im\int_{\gamma_2}G_2}\,\mathbb{Z}_{km}(\gamma_2;A_1,f_2)\,. \end{aligned} \tag{4.2.93}$$

Here $n,m\in\mathbb{Z}$ and $\alpha\in\mathbb{R}/\mathbb{Z}$, while $p/q\in\mathbb{Q}$ with $\gcd(p,q)=1$ and $p\sim p+kq$ so that the label $p/kq\in\mathbb{Q}/\mathbb{Z}$. Then $\mathbb{Z}_{km}(\gamma_2;A_1,f_2)$ denotes a pure 2d $\mathbb{Z}_{km}$ gauge theory on γ_2, whose 0-form and 1-form symmetries are coupled, respectively, to A_1 and f_2. Similarly, $\mathcal{A}^{q,p}(\gamma_3;f_2)$ is the minimal Abelian TQFT with $\mathbb{Z}_q$ 1-form symmetry and anomaly labeled by p introduced in [353], whose 1-form symmetry is coupled to f_2. Stacking these TQFTs is necessary in order to make the operators gauge invariant and topological. The theories $\mathcal{A}^{q,p}$ are nontrivial for any $q\neq 1$, so that only a $\mathbb{Z}_k$ subgroup of the operators $U_{\frac{p}{kq}}$ (those with $q=1$) are invertible, while the other ones obey non-invertible fusion rules. Similarly, $\mathcal{T}_m$ are non-invertible. In the SymTFT approach it is natural to choose topological boundary conditions associated with the Lagrangian algebra

$$\mathcal{L} = \left\{W_n\,,\,\mathcal{T}_m\;\middle|\;n,m\in\mathbb{Z}\right\}. \tag{4.2.94}$$

The remaining operators $U_{\frac{p}{kq}}(\gamma_3)$ and $V_\alpha(\gamma_2)$ implement the non-invertible symmetry and the magnetic 1-form symmetry, respectively.

Continuing with the approach we have followed so far, we want to consider a theory of gravity based on (4.2.91) with the condensation of $\mathcal{L}$ in the bulk. We place this theory on a manifold X_5 with a boundary and impose the non-topological boundary conditions

$$\star A_1 = -\frac{i}{R^2}\,b_3 + \star\mathcal{A}_1\,, \qquad \star G_2 = -\frac{i\pi}{e^2}\,f_2 + \star\mathcal{G}_2\,. \tag{4.2.95}$$

We need to add a boundary term:

$$S_\partial = -\frac{1}{4\pi R^2}\int_{\partial X_5} b_3\wedge\star b_3 - \frac{1}{4e^2}\int_{\partial X_5} f_2\wedge\star f_2\,. \tag{4.2.96}$$

As before we would like to assign gauge transformation rules to the boundary fields $\mathcal{A}_1$, $\mathcal{G}_2$ in order to restore some of the gauge transformations on the boundary, corresponding to the symmetries that become global there. However, while we can restore $\delta G_2 = d\eta_1$ by transforming $\delta\mathcal{G}_2 = d\eta_1$, the gauge transformation $\delta A_1 = d\rho_0$ cannot be restored. Indeed, while the first eqn. in (4.2.95) could be made gauge invariant by prescribing that $\delta\mathcal{A}_1 = d\rho_0$, the second one would not be invariant because G_2 transforms as $\delta G_2 = -\frac{k}{2\pi}\rho_0 f_2$. This term cannot be reabsorbed by modifying the gauge transformations of $\mathcal{G}_2$, since f_2 is a dynamical field. Thus the only way to make the boundary conditions gauge invariant is to freeze the boundary value of ρ_0, as those of λ_1 and ξ_2.

To get the boundary theory, as before, we integrate out the time components imposing

$$\tilde{d}\widetilde{A}_1 = 0\,, \qquad \tilde{d}\widetilde{f}_2 = 0\,, \qquad \tilde{d}\,\widetilde{b}_3 + \frac{k}{4\pi}\,\widetilde{f}_2 \wedge \widetilde{f}_2 = 0\,, \qquad \tilde{d}\,\widetilde{G}_2 + \frac{k}{2\pi}\,\widetilde{A}_1 \wedge \widetilde{f}_2 = 0\,, \tag{4.2.97}$$

that are solved by

$$\widetilde{A}_1 = \tilde{d}\Phi_0\,, \qquad \widetilde{f}_2 = \tilde{d}a_1\,, \qquad \widetilde{b}_3 = \tilde{d}\omega_2 - \frac{k}{4\pi}\,a_1 \wedge \tilde{d}a_1\,, \qquad \widetilde{G}_2 = \tilde{d}C_1 - \frac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1\,. \tag{4.2.98}$$

The total action reduces to a boundary theory with action:

$$\begin{aligned} S = \frac{i}{2\pi}\int_{\mathcal{M}_4}\Bigg[&\left(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1 \wedge \tilde{d}a_1\right) \wedge \left(\partial_t\Phi_0 - \mathcal{A}^t_0\right)dt - \left(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1\right) \wedge \partial_t a_1 \wedge dt \\ &- \frac{i}{2}\left(R^2\left(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\right) \wedge \star\left(\tilde{d}\Phi_0 - \widetilde{\mathcal{A}}_1\right) + \frac{1}{R^2}\left(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1 \wedge \tilde{d}a_1\right) \wedge \star\left(\tilde{d}\omega_2 - \tfrac{k}{4\pi}\,a_1 \wedge \tilde{d}a_1\right)\right) \\ &- \frac{i}{2}\left(\frac{\pi}{e^2}\,\tilde{d}a_1 \wedge \star\,\tilde{d}a_1 + \frac{e^2}{\pi}\left(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1 - \widetilde{\mathcal{G}}_2\right) \wedge \star\left(\tilde{d}C_1 - \tfrac{k}{2\pi}\,\Phi_0\,\tilde{d}a_1 - \widetilde{\mathcal{G}}_2\right)\right) \\ &+ \tilde{d}a_1 \wedge \mathcal{G}^t_1 \wedge dt + \frac{ik}{4\pi}\,\Phi_0\,da_1 \wedge da_1\Bigg] \end{aligned} \tag{4.2.99}$$

where $\mathcal{M}_4 = \partial X_5$. We can then integrate out both ω_2 and C_2 obtaining

$$S = \int_{\mathcal{M}_4}\left[\frac{R^2}{4\pi}\left(d\Phi_0 - \mathcal{A}_1\right) \wedge \star\left(d\Phi_0 - \mathcal{A}_1\right) + \frac{1}{4e^2}\,da_1 \wedge \star\,da_1 + \frac{ik}{8\pi^2}\,\Phi_0\,da_1 \wedge da_1 + \frac{i}{2\pi}\,da_1 \wedge \mathcal{G}_2\right]. \tag{4.2.100}$$

As in the cases of the Abelian 2-group and of Maxwell–Chern–Simons theory, gauging the Lagrangian algebra introduces fluxes for a_1 turning it into a standard $U(1)$ gauge field. The theory in (4.2.100) describes a compact boson Φ_0 and a photon a_1 interacting via an axion coupling. This is called axion-Maxwell theory, and the full structure of its symmetries (including some emergent ones) has been studied in great detail in [60]. From the discovery of the non-invertible chiral symmetry it was suspected that this theory universally describes its symmetry breaking. Our result confirms that. Notably, this is the first interacting boundary theory among the examples considered so far.

Some comments on the coupling to the background fields are in order. As we already noticed after (4.2.96), there is no sensible gauge transformation rules that we could assign to $\mathcal{A}_1$ and $\mathcal{G}_2$ to make the boundary condition invariant under $\delta A_1 = d\rho_0$, hence we needed to freeze it. In the action (4.2.100), $\mathcal{A}_1$ should not be thought of as the background field for the 0-form non-invertible symmetry, but rather just as an external source that couples with the operator $J_1^{(A)}$. This is enough for holography, but it might seem a bit unsatisfactory from a symmetry viewpoint. However, this is really the hallmark of the non-invertible nature of the symmetry: ordinary background gauge fields seem not to exist, and they are effectively replaced by boundary values of dynamical fields in one dimension higher [7]. The underlying reason is that non-invertible symmetries map untwisted sectors to twisted sectors, hence the gauge transformations of a background gauge field necessarily involve an interplay among backgrounds that do not exist simultaneously in the theory, but only in the SymTFT (or in holography) where all global variants are on the same footing. This is the reason why SymTFT is the main tool for discussing anomalies [2, 98, 99, 195].

4.2.5 Non-Abelian Goldstone bosons

A very interesting class of examples are those of spontaneously broken non-Abelian symmetries. In these cases the boundary EFTs that we derive are interacting and generically non-renormalizable. In the 2d/3d case we will be able to recover and somewhat generalize the CS/WZW correspondence outside of the conformal point, while in higher dimensions we will obtain the pion Lagrangian on the boundary. We start with the non-Abelian generalization of the theories considered in Section 4.2.2 and then add an anomaly term, which corresponds to WZW terms in various dimensions. Finally we show how our setup is able to produce an EFT for spontaneously broken non-Abelian 2-group symmetries.

Holographic dual to the pion Lagrangian

Let G be a connected and compact Lie group (with Lie algebra $\mathfrak{g}$). The SymTFT for a non-Abelian 0-form symmetry G in d dimensions is the TQFT with action [206, 210, 211]:[78]

$$S = \frac{i}{2\pi} \int_{X_{d+1}} \mathrm{Tr}\big(\mathrm{b}_{\mathrm{d}-1} \wedge \mathrm{F}_2\big) \,, \tag{4.2.101}$$

where $F_2 = dA_1 + iA_1 \wedge A_1$ is the field strength of a G connection A_1 while b_{d-1} is a $\mathfrak{g}$-valued $(d-1)$-form. The gauge transformations are

$$A_1 \;\mapsto\; \Lambda\, A_1\, \Lambda^{-1} + i\, d\Lambda\, \Lambda^{-1} \,, \qquad\qquad b_{d-1} \;\mapsto\; \Lambda\, b_{d-1}\, \Lambda^{-1} \tag{4.2.102}$$

as well as

$$b_{d-1} \;\mapsto\; b_{d-1} + D_A \lambda_{d-2} \,. \tag{4.2.103}$$

Here $D_A = d + i[A_1, \,\cdot\,]_\pm$ is the covariant derivative that acts on p-forms valued in the Lie algebra as

$$D_A \eta_p = d\eta_p + i\big(A_1 \wedge \eta_p - (-1)^p\, \eta_p \wedge A_1\big) \,. \tag{4.2.104}$$

The topological defects of this TQFT include the Wilson lines

$$W_{\mathfrak{R}}(\gamma_1) = \mathrm{Tr}_{\mathfrak{R}}\, \mathrm{Pexp}\left(\mathrm{i} \int_{\gamma_1} \mathrm{A}_1\right) \tag{4.2.105}$$

labelled by the irreducible representations $\mathfrak{R}$ of G, as well as $(d-1)$-dimensional Gukov-Witten operators $U_{[g]}(\gamma_{d-1})$ labelled by conjugacy classes $[g]$ of G and defined by prescribing that the holonomy of A_1 around $U_{[g]}$ be in $[g]$ [395]. The two classes of operators have a canonical linking given by the character $\chi_{\mathfrak{R}}([g])$. A natural Lagrangian algebra that we will condense consists of the Wilson lines in all representations of G.

[78] For $d = 3$ this theory was first considered by Horowitz [380]. Curiously, the motivation was to view it as an exactly solvable theory of gravity.

We use the following non-topological boundary condition and boundary term on $\mathcal{M}_d = \partial X_{d+1}$:

$$\star\left(A_1 - \mathcal{A}_1\right) = -\frac{i}{f_\pi^2}\, b_{d-1}\,, \qquad\qquad S_\partial = -\frac{1}{4\pi f_\pi^2}\int_{\mathcal{M}_d} \mathrm{Tr}\big(\mathrm{b_{d-1}} \wedge \star\, \mathrm{b_{d-1}}\big)\,. \tag{4.2.106}$$

We can recover the gauge transformations on the boundary by assigning the transformation rule $\mathcal{A}_1 \mapsto \Lambda\mathcal{A}_1\Lambda^{-1} + id\Lambda\Lambda^{-1}$ so that $\mathcal{A}_1$ is interpreted as a background field for a global symmetry G.[79] We can proceed with the usual steps to derive the dual boundary theory. Taking the spacetime to be $X_{d+1} = B_d \times S^1$, the path integral over time components imposes

$$\widetilde{F}_2 = 0\,, \qquad\qquad D_{\widetilde{A}_1}\widetilde{b}_{d-1} = 0\,. \tag{4.2.107}$$

The first equation can be solved in terms of a G-valued scalar field U as

$$\widetilde{A}_1 = i\,\tilde{d}U\,U^{-1}\,. \tag{4.2.108}$$

To solve the second one, since the covariant derivative with respect to a flat connection squares to zero (*i.e.*, it becomes a differential), we set

$$\widetilde{b}_{d-1} = \widetilde{D}\omega_{d-2} \tag{4.2.109}$$

where ω_{d-2} is a $\mathfrak{g}$-valued $(d-2)$-form, and $\widetilde{D}$ denotes the covariant derivative with respect to $i\,\tilde{d}U\,U^{-1}$. By plugging these back, the theory reduces to a boundary action:

$$\begin{aligned} S = (-1)^d\frac{i}{2\pi}\int_{\mathcal{M}_d} &\mathrm{Tr}\Big[\widetilde{\mathrm{D}}\omega_{\mathrm{d-2}} \wedge \Big(\mathrm{i}\,\partial_{\mathrm{t}}\mathrm{U}\,\mathrm{U}^{-1} - \mathcal{A}_0^{\mathrm{t}}\Big)\mathrm{dt}\Big] \\ &+ \frac{1}{4\pi}\int_{\mathcal{M}_d} \mathrm{Tr}\Big[\frac{1}{\mathrm{f}_\pi^2}\,\widetilde{\mathrm{D}}\omega_{\mathrm{d-2}} \wedge \star\,\widetilde{\mathrm{D}}\omega_{\mathrm{d-2}} + \mathrm{f}_\pi^2\Big(\mathrm{i}\,\tilde{\mathrm{d}}\mathrm{U}\,\mathrm{U}^{-1} - \widetilde{\mathcal{A}}_1\Big) \wedge \star\Big(\mathrm{i}\,\tilde{\mathrm{d}}\mathrm{U}\,\mathrm{U}^{-1} - \widetilde{\mathcal{A}}_1\Big)\Big]\,. \end{aligned} \tag{4.2.110}$$

One important difference with respect to the Abelian case is that U and ω_{d-2} do not appear symmetrically. While U appears in a complicated way, the action is still quadratic in ω_{d-2} that can thus be integrated out using its equation of motion

$$\widetilde{D}\Big(\partial_t U\,U^{-1} + i\mathcal{A}_0^t\Big) \wedge dt + \frac{(-1)^{d-1}}{f_\pi^2}\,\widetilde{D} \star \widetilde{D}\omega_{d-2} = 0\,. \tag{4.2.111}$$

Eliminating a zero-mode as in the Abelian case, we can use this equation to determine $\widetilde{D}\omega_{d-2}$, and we find the manifestly covariant form of the boundary theory:

$$S = \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_d} \mathrm{Tr}\Big[\Big(\mathrm{i}\,\mathrm{dU}\,\mathrm{U}^{-1} - \mathcal{A}_1\Big) \wedge \star\Big(\mathrm{i}\,\mathrm{dU}\,\mathrm{U}^{-1} - \mathcal{A}_1\Big)\Big]\,. \tag{4.2.112}$$

This describes a sigma model with target G, coupled to a background field $\mathcal{A}_1$ for the symmetry G that acts as $U \mapsto gU$ with $g \in G$. The sigma model is a non-renormalizable theory that provides the leading universal term in an expansion in number of derivatives (in 4d this is chiral perturbation theory), describing the EFT of any theory with spontaneously broken symmetry G [378, 379].

[79] Differently from the Abelian case, here we cannot turn on another background to rescue the other gauge symmetry as well. The reason is that the gauge transformation (4.2.103) of b_{d-1} cannot be reabsorbed in the boundary condition by replacing b_{d-1} with $b_{d-1} - \mathcal{B}_{d-1}$ and assigning a transformation rule to $\mathcal{B}_{d-1}$. Indeed, this transformation would necessarily involve the dynamical field A_1, instead of the background $\mathcal{A}_1$.

Non-Abelian chiral anomaly

For any even d we can add a Chern–Simons term to the bulk theory (4.2.101):[80]

$$S_{\text{CS}} = \frac{i\kappa_d}{2\pi}\int_{X_{d+1}} \text{Tr}\big(\text{CS}_{\text{d}+1}(\text{A}_1)\big)\,, \qquad \kappa_{\text{d}} = \frac{\text{k}}{(2\pi)^{\frac{\text{d}}{2}-1}\big(\frac{\text{d}}{2}+1\big)!}\,, \qquad \text{k}\in\mathbb{Z}\,, \tag{4.2.113}$$

that describes the presence of a perturbative anomaly for G. In this case, differently from the Abelian one, anomaly matching requires a WZW term in the spontaneously broken phase [381]. We want to show that this fact is implied by our conjecture. We also consider the case of $d = 2$ where, strictly speaking, our conjecture does not apply because there is no spontaneous breaking of a continuous symmetry in two dimensions.

Two dimensions

In the case of $d = 2$, we use the boundary condition

$$\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i}{f_\pi^2}\left(b_1 + \frac{k}{2}(A_1 - \mathcal{A}_1)\right) \tag{4.2.114}$$

that is gauge invariant under $A_1 \mapsto \Lambda A_1 \Lambda^{-1} + id\Lambda\,\Lambda^{-1}$, $\mathcal{A}_1 \mapsto \Lambda\mathcal{A}_1\Lambda^{-1} + id\Lambda\,\Lambda^{-1}$, and add the boundary term

$$S_\partial = -\frac{1}{4\pi f_\pi^2}\int_{\partial X_3} \text{Tr}\Big[\Big(\text{b}_1 + \tfrac{\text{k}}{2}\,\text{A}_1\Big)\wedge\star\Big(\text{b}_1 + \tfrac{\text{k}}{2}\,\text{A}_1\Big)\Big] \tag{4.2.115}$$

to make the variational principle well defined.

As a preliminary consistency check, we compute the gauge variation. The total gauge-transformed action differs by

$$\Delta\big(S + S_\partial + S_{\text{c.t.}}\big) = \frac{ik}{4\pi}\int_{\partial X_3}\text{Tr}\big(\mathcal{A}_1\wedge \text{i}\Lambda^{-1}\text{d}\Lambda\big) + \frac{\text{k}}{24\pi}\int_{\text{X}_3}\text{Tr}\big((\text{i}\Lambda^{-1}\text{d}\Lambda)^3\big) \tag{4.2.116}$$

from the original one.[81] Upon expanding $\Lambda = \mathbb{1} + \lambda_0$ and retaining only the linear order in λ_0, this reduces to the usual form of the consistent anomaly:

$$\delta\big(S + S_\partial + S_{\text{c.t.}}\big) = \frac{ik}{4\pi}\int_{\partial X_3}\text{Tr}\big(\mathcal{A}_1\wedge \text{id}\lambda_0\big)\,. \tag{4.2.117}$$

One can proceed in determining the dual boundary theory similarly to the non-anomalous case. Since the boundary condition is essentially the same (simply written in a different parametrization), the only difference is the bulk Chern–Simons term which gives rise to a WZW term in the boundary theory:

$$\begin{aligned} S = {}& \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_2}\text{Tr}\Big[\big(\text{i}\,\text{dU}\,\text{U}^{-1} - \mathcal{A}_1\big)\wedge\star\big(\text{i}\,\text{dU}\,\text{U}^{-1} - \mathcal{A}_1\big)\Big] + \frac{\text{k}}{12\pi}\int_{\text{X}_3}\text{Tr}\Big[\big(\text{i}\text{U}^{-1}\text{dU}\big)^3\Big] \\ & - \frac{ik}{4\pi}\int_{\mathcal{M}_2}\text{Tr}\Big[\mathcal{A}_1\wedge \text{i}\,\text{dU}\,\text{U}^{-1}\Big]\,. \end{aligned} \tag{4.2.118}$$

[80] Here we assume G to be simple and simply connected.

[81] Here $S_{\text{c.t.}} = \dfrac{k^2}{8\pi f_\pi^2}\displaystyle\int_{\partial X_3}\text{Tr}\big(\mathcal{A}_1\wedge\star\mathcal{A}_1\big)$ is a counterterm we add to simplify the final result.

We notice that there is also a non-standard coupling to the background field, that in our approach arises because of the boundary conditions, similarly to the Abelian case. Differently from that case, however, in a purely field theoretic analysis this is not interpreted as a coupling to a diagonal symmetry (since a winding symmetry is absent here), but rather it arises from the standard trial-and-error procedure to couple the G symmetry to a background in the presence of the WZW term, similarly to the 4d analysis in [381].

For generic values of f_π^2 the theory is not conformally invariant at the quantum level. However choosing $f_\pi^2 = \frac{k}{2}$ the theory has a conserved holomorphic current which generates a Kac–Moody symmetry algebra, and it displays conformal invariance [396]. In this case we recover a form of the CS/WZW correspondence, which is more general on one side, being valid even outside of the conformal point, but less general on the other side, since in the conformal case it automatically produces the full physical WZW model instead of its chiral halves.

Four dimensions

In the case of $d = 4$, the 5d Chern–Simons term is

$$\mathrm{Tr}\big(\mathrm{CS}_5(\mathrm{A}_1)\big) = \mathrm{Tr}\bigg(\mathrm{A}_1 \wedge (\mathrm{dA}_1)^2 + \frac{3\mathrm{i}}{2}\,\mathrm{A}_1^3 \wedge \mathrm{dA}_1 - \frac{3}{5}\mathrm{A}_1^5\bigg)\,. \tag{4.2.119}$$

As one might suspect already from the Abelian case, in order to obtain a gauge-invariant boundary condition with a consistent variational principle we need to introduce extra terms in the boundary condition that mix background and dynamical fields. We use the same iterative procedure discussed in Appendix B.7 for the Abelian anomaly, even though the computations are clearly more tedious here. We find the following solution. The boundary condition is

$$\star\left(A_1 - \mathcal{A}_1\right) - \frac{i\kappa_4}{f_\pi^2}\left(\frac{1}{2}\left(\mathcal{A}_1\,\mathcal{F}_2 + \mathcal{F}_2\,\mathcal{A}_1\right) - \frac{i}{2}\,\mathcal{A}_1^3\right) = -\frac{i}{f_\pi^2}\,\Omega_3 \tag{4.2.120}$$

where $\mathcal{F}_2$ is the field strength of $\mathcal{A}_1$ while

$$\Omega_3 = b_3 + \kappa_4\bigg(F_2\big(A_1 - \mathcal{A}_1\big) + \big(A_1 - \mathcal{A}_1\big)F_2 - \frac{i}{2}\Big(\big(A_1 - \mathcal{A}_1\big)^3 + \mathcal{A}_1\Big)^3 + \frac{1}{2}\big(A_1\mathcal{F}_2 + \mathcal{F}_2 A_1\big)\bigg) \tag{4.2.121}$$

and the boundary term is

$$\begin{aligned} S_\partial &= -\frac{1}{4\pi f_\pi^2}\int_{\partial X_5} \mathrm{Tr}\big(\Omega_3 \wedge \star\,\Omega_3\big) \;+\; \mathrm{S}_{\mathrm{top}} \;+\; \mathrm{S}_{\mathrm{c.t.}}\,, \\ S_{\mathrm{top}} &= \frac{i\kappa_4}{2\pi}\int_{\partial X_5} \mathrm{Tr}\bigg[\frac{1}{2}\,\mathrm{F}_2\,\mathcal{A}_1\,\mathrm{A}_1 + \frac{1}{2}\,\mathcal{A}_1\,\mathrm{F}_2\,\mathrm{A}_1 - \frac{\mathrm{i}}{4}\,\mathrm{A}_1\,\mathcal{A}_1\,\mathrm{A}_1\,\mathcal{A}_1 + \frac{\mathrm{i}}{2}\,\mathrm{A}_1^3\,\mathcal{A}_1\bigg]. \end{aligned} \tag{4.2.122}$$

The counterterm $S_{\text{c.t.}}$ is used to simplify the final expression, and it is convenient to choose it as

$$S_{\text{c.t.}} = \frac{\kappa_4^2}{4\pi f_\pi^2}\int_{\partial X_5} \mathrm{Tr}\Big[\phi(\mathcal{A}_1)\wedge\star\,\phi(\mathcal{A}_1)\Big]\,, \qquad \phi(\mathcal{A}_1) = \frac{1}{2}\Big(\mathcal{A}_1\wedge\mathrm{d}\mathcal{A}_1 + \mathrm{d}\mathcal{A}_1\wedge\mathcal{A}_1 + \mathrm{i}\mathcal{A}_1^3\Big)\,. \tag{4.2.123}$$

The boundary condition is gauge invariant under the transformation $A_1 \mapsto \Lambda A_1 \Lambda^{-1} + id\Lambda\,\Lambda^{-1}$, $\mathcal{A}_1 \mapsto \Lambda\mathcal{A}_1\Lambda^{-1} + id\Lambda\,\Lambda^{-1}$ and one can compute the total gauge variation

$$\begin{aligned} \Delta\big(S + S_\partial\big) = &-\frac{i\kappa_4}{2\pi}\int_{\partial X_5} \mathrm{Tr}\Big[\big(\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big)\wedge\phi(\mathcal{A}_1) + \frac{\mathrm{i}}{4}\big(\mathcal{A}_1\wedge\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big)^2 - \frac{\mathrm{i}}{2}\big(\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big)^3\wedge\mathcal{A}_1\Big] \\ &-\frac{i\kappa_4}{20\pi}\int_{X_5} \mathrm{Tr}\Big[\big(\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big)^5\Big]\,. \end{aligned} \tag{4.2.124}$$

Expanding $U = \mathbb{1} + \lambda_0$ to linear order, we recover the usual form of the consistent anomaly in four dimensions:

$$\delta\big(S + S_\partial\big) = -\frac{ik}{48\pi^2}\int_{\partial X_5} \mathrm{Tr}\Big[\mathrm{i}\mathrm{d}\lambda_0 \wedge \Big(\mathcal{A}_1 \wedge \mathrm{d}\mathcal{A}_1 + \mathrm{d}\mathcal{A}_1 \wedge \mathcal{A}_1 + \mathrm{i}\mathcal{A}_1^3\Big)\Big]\,. \tag{4.2.125}$$

We can then proceed, as before, with the reduction of the action on the boundary. We find

$$\begin{aligned} S = {} & \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_4} \mathrm{Tr}\Big[\Big(\mathrm{idU\,U^{-1}} - \mathcal{A}_1\Big) \wedge \star \Big(\mathrm{idU\,U^{-1}} - \mathcal{A}_1\Big)\Big] - \frac{\mathrm{ik}}{240\pi^2}\int_{\mathrm{X}_5} \mathrm{Tr}\Big[\big(\mathrm{iU^{-1}dU}\big)^5\Big] \\ & + \frac{ik}{48\pi^2}\int_{\mathcal{M}_4} \mathrm{Tr}\Big[\mathrm{idU\,U^{-1}} \wedge \Big(\mathcal{A}_1 \wedge \mathcal{F}_2 + \mathcal{F}_2 \wedge \mathcal{A}_1 - \mathcal{A}_1^3\Big)\Big] \\ & + \frac{k}{48\pi^2}\int_{\mathcal{M}_4} \mathrm{Tr}\Big[\frac{1}{2}\,\mathrm{idU\,U^{-1}} \wedge \mathcal{A}_1 \wedge \mathrm{idU\,U^{-1}} \wedge \mathcal{A}_1 - \big(\mathrm{idU\,U^{-1}}\big)^3 \wedge \mathcal{A}_1\Big]\,. \end{aligned} \tag{4.2.126}$$

Turning off the background gauge field $\mathcal{A}_1$ we recognize a non-linear sigma model with target space G with a properly normalized WZW term, that describes the dynamics of Goldstone bosons. The coupling to the background $\mathcal{A}_1$ is completely fixed by the requirement of a gauge-invariant boundary condition, and correctly captures the anomaly of the non-linearly realized G symmetry.

Non-Abelian 2-group symmetries

In 4d one can have 2-group symmetries whose 0-form part is a non-Abelian group G, while the 1-form part is $U(1)$. These symmetry structures arise, *e.g.*, if one starts from a theory with a 0-form symmetry group $U(1) \times G$ with an 't Hooft anomaly that is linear in $U(1)$ and quadratic in G:

$$S_{\text{inflow}} = \frac{ik}{8\pi^2}\int_{X_5} dV_1 \wedge \mathrm{Tr}\bigg(\mathrm{A}_1 \wedge \mathrm{dA}_1 + \frac{2\mathrm{i}}{3}\,\mathrm{A}_1^3\bigg)\,, \tag{4.2.127}$$

and then gauges the $U(1)$ symmetry [179]. The 1-form symmetry involved in the 2-group is the magnetic symmetry of the gauged $U(1)$. The SymTFT for this non-Abelian 2-group symmetry can be derived using the dynamical gauging procedure described in [206]. Indeed one starts from the SymTFT for the $U(1) \times G$ 0-form symmetry:

$$S' = \frac{i}{2\pi}\int_{X_5}\bigg[g_3 \wedge dV_1 + \mathrm{Tr}\big(\mathrm{b}_3 \wedge \mathrm{F}_2\big) + \frac{\mathrm{k}}{4\pi}\,\mathrm{dV}_1 \wedge \mathrm{Tr}\Big(\mathrm{A}_1 \wedge \mathrm{dA}_1 + \frac{2\mathrm{i}}{3}\mathrm{A}_1^3\Big)\bigg] \tag{4.2.128}$$

where g_3 and V_1 are an $\mathbb{R}$ and a $U(1)$ gauge field, respectively, b_3 is $\mathfrak{g}$-valued and A_1 is a G connection (F_2 is its field strength). Then one applies the map introduced in [206] that implements the dynamical gauging of $U(1)$ on the boundary from the viewpoint of the SymTFT. The net effect is the replacement $dV_1 \mapsto h_2$, $g_3 \mapsto dC_2$, thus the resulting SymTFT has action

$$S = \frac{i}{2\pi}\int_{X_5}\bigg[h_2 \wedge dC_2 + \mathrm{Tr}\big(\mathrm{b}_3 \wedge \mathrm{F}_2\big) + \frac{\mathrm{k}}{4\pi}\,\mathrm{h}_2 \wedge \mathrm{Tr}\Big(\mathrm{A}_1 \wedge \mathrm{dA}_1 + \frac{2\mathrm{i}}{3}\mathrm{A}_1^3\Big)\bigg]\,. \tag{4.2.129}$$

The gauge transformations are:[82]

$$
\begin{aligned}
&h_2 \mapsto h_2 + d\xi_1\,, && A_1 \mapsto \Lambda A_1 \Lambda^{-1} + id\Lambda\,\Lambda^{-1}\,,\\
&b_3 \mapsto b_3 - \frac{k}{4\pi}\,\xi_1 \wedge F_2\,, && C_2 \mapsto C_2 + d\eta_1 - \frac{k}{4\pi}\mathrm{Tr}\big(\mathrm{A}_1 \wedge \mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big) + \frac{\mathrm{ik}}{6\pi}\mathrm{Tr}\Omega_2\,,
\end{aligned}
\tag{4.2.131}
$$

where Ω_2 is a locally defined real 2-form with the property that $\mathrm{Tr}\big((\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda)^3\big) = \mathrm{d}\mathrm{Tr}\Omega_2$.

Again, we can use an iterative procedure to determine a set of gauge-invariant boundary conditions together with a boundary term that provide a good variation principle. The boundary conditions are

$$
\star\big(A_1 - \mathcal{A}_1\big) = -\frac{i}{R^2}\left(b_3 + \frac{k}{4\pi}(A_1 - \mathcal{A}_1)\right), \quad \star h_2 = \frac{ie^2}{\pi}\left(C_2 - \mathcal{C}_2 - \frac{k}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \mathrm{A}_1\big)\right) \tag{4.2.132}
$$

while the boundary term is

$$
\begin{aligned}
S_\partial = &-\frac{i}{2\pi}\int_{\partial X_5} h_2 \wedge \left(C_2 - \frac{k}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \mathrm{A}_1\big)\right)\\
&- \frac{e^2}{4\pi^2}\int_{\partial X_5}\left(C_2 - \frac{k}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \mathrm{A}_1\big)\right) \wedge \star\left(\mathrm{C}_2 - \frac{\mathrm{k}}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \mathrm{A}_1\big)\right)\\
&- \frac{1}{4\pi R^2}\int_{\partial X_5}\mathrm{Tr}\left[\left(\mathrm{b}_3 + \frac{\mathrm{k}}{4\pi}(\mathrm{A}_1 - \mathcal{A}_1)\right) \wedge \star\left(\mathrm{b}_3 + \frac{\mathrm{k}}{4\pi}(\mathrm{A}_1 - \mathcal{A}_1)\right)\right].
\end{aligned}
\tag{4.2.133}
$$

The boundary condition becomes gauge invariant by assigning the following transformations to the backgrounds $\mathcal{A}_1$ and $\mathcal{C}_2$:

$$
\mathcal{A}_1 \mapsto \Lambda\mathcal{A}_1\Lambda^{-1} + id\Lambda\,\Lambda^{-1}\,, \qquad \mathcal{C}_2 \mapsto \mathcal{C}_2 + d\eta_1 - \frac{ik}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \Lambda^{-1}\mathrm{d}\Lambda\big) + \frac{\mathrm{ik}}{12\pi}\mathrm{Tr}\Omega_2\,. \tag{4.2.134}
$$

These reproduce the background gauge transformation of [179] for a non-Abelian 2-group symmetry upon expanding $U = \mathbb{1} + \lambda_0$ at first order:

$$
\delta\mathcal{A}_1 = iD_{\mathcal{A}_1}\lambda_0\,, \qquad \delta\mathcal{C}_2 = d\eta_1 - \frac{ik}{4\pi}\mathrm{Tr}\big(\mathcal{A}_1 \wedge \mathrm{d}\lambda_0\big)\,. \tag{4.2.135}
$$

It is also easy to see that the whole bulk-boundary system is gauge invariant under transformations of A_1 and C_2 provided we add a counterterm $S_{\text{c.t.}} = \frac{e^2}{4\pi^2}\int_{\partial X_5}\mathcal{C}_2 \wedge \star\mathcal{C}_2$.

We can apply our usual machinery to get the dual boundary theory. We obtain a G-valued scalar field U from A_1, and a Maxwell field a_1 from h_2, with the following boundary action:

$$
\begin{aligned}
S = &\frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_4}\mathrm{Tr}\left[\left(\mathrm{idU\,U^{-1}} - \mathcal{A}_1\right) \wedge \star\left(\mathrm{idU\,U^{-1}} - \mathcal{A}_1\right)\right] + \frac{1}{4\mathrm{e}^2}\int_{\mathcal{M}_4}\mathrm{da}_1 \wedge \star\,\mathrm{da}_1\\
&+ \frac{k}{24\pi^2}\int_{\mathcal{M}_4} a_1 \wedge \mathrm{Tr}\left[\left(\mathrm{iU^{-1}dU}\right)^3\right]\\
&+ \frac{i}{2\pi}\int_{\mathcal{M}_4} da_1 \wedge \mathrm{Tr}\left[\mathcal{A}_1 \wedge \mathrm{iU^{-1}dU}\right] + \frac{\mathrm{i}}{2\pi}\int_{\mathcal{M}_4}\mathcal{C}_2 \wedge \mathrm{da}_1\,.
\end{aligned}
\tag{4.2.136}
$$

[82] Recall that the variation of the three-dimensional Chern–Simons term is:

$$
\mathrm{Tr}\big(\mathrm{CS}_3(\mathrm{A}_1)\big) \;\mapsto\; \mathrm{Tr}\big(\mathrm{CS}_3(\mathrm{A}_1)\big) + \mathrm{d}\mathrm{Tr}\big(\mathrm{A}_1 \wedge \mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big) - \frac{\mathrm{i}}{3}\mathrm{Tr}\Big(\big(\mathrm{i}\Lambda^{-1}\mathrm{d}\Lambda\big)^3\Big)\,. \tag{4.2.130}
$$

	Definition	x_i	$a_1\mathrm{Tr}\left[\left(\mathrm{iU}^{-1}\mathrm{dU}\right)^3\right]$	$\mathrm{Tr}\left[\left(\mathrm{iU}^{-1}\mathrm{dU}\right)^5\right]$
P_0	$x_i \mapsto -x_i$	-1	-1	-1
C_1	$a_1 \mapsto -a_1$	1	-1	1
C_2	$U \mapsto U^\mathsf{T}$	1	-1	1
$(-1)^{N_\pi}$	$U \mapsto U^{-1}$	1	-1	-1

Table 4.2: The four $\mathbb{Z}_2$ symmetries, and the corresponding phases acquired by the coordinates, the photon-pion coupling term, and the standard WZW term, respectively. Notice that while $\mathrm{Tr}\left[(\mathrm{iU}^{-1}\mathrm{dU})^5\right]$ is invariant under $U \mapsto U^\mathsf{T}$, the term $\mathrm{Tr}\left[(\mathrm{iU}^{-1}\mathrm{dU})^3\right]$ changes sign.

In the first line we recognize a non-linear sigma model with target space G and a Maxwell theory. The last line describes the coupling to the background field $\mathcal{C}_2$ for the magnetic $U(1)$ 1-form symmetry, as well as a nonstandard coupling to the background $\mathcal{A}_1$ for the symmetry G, similar to the one arising in the Abelian case in Section 4.2.3. The most interesting new thing here is the term in the second line that describes a coupling between the photon and the pions. This is a linear coupling of the photon to the current of a topological symmetry that exists in any sigma model with target G. According to our conjecture, this model is the universal EFT that describes the IR of any theory with a spontaneously broken non-Abelian 2-group symmetry. To the best of our knowledge, this universal EFT was not derived elsewhere.

Some comments on the extra Wess-Zumino-like coupling are in order. First, in any RG flow that breaks the 2-group spontaneously, this coupling must be generated as a consequence of the 2-group matching. In a sense, it is similar to the presence of the WZW term in the EFT of a spontaneously broken anomalous non-Abelian symmetry. Quite like that term, it breaks a symmetry of the EFT that would be there if $k = 0$. Indeed, for $k = 0$ the theory is separately invariant under four $\mathbb{Z}_2$ symmetries: parity $P_0 : x_i \mapsto -x_i$ for $i = 1, 2, 3$; photon charge conjugation $C_1 : a_1 \mapsto -a_1$; non-Abelian charge conjugation[83] $C_2 : U \mapsto U^\mathsf{T}$; pion number mod-2 $(-1)^{N_\pi} : U \mapsto U^{-1}$. All these four symmetries are violated by the photon-pion coupling, but the product of any two of them is preserved. Therefore the discrete symmetry for $k \neq 0$ is $(\mathbb{Z}_2)^3$ generated by

$$P = P_0\,(-1)^{N_\pi}\,, \qquad C = C_1\,C_2\,, \qquad \widetilde{C} = C_1\,(-1)^{N_\pi}\,. \tag{4.2.137}$$

The photon-pion coupling allows, for instance, a process involving three pions and one photon, which would have been forbidden otherwise. We summarize the various symmetry actions and charges in Table 4.2.

Second, the 2-group symmetry we started with could suffer from a perturbative cubic chiral anomaly for G as well. This would be described by the addition of a 5d Chern–Simons term

[83]The reason for this name will be clear in the upcoming discussion of $U(N)$ QCD.

(4.2.119) to the bulk action in (4.2.129), and would result in an extra WZW term $S_{\rm WZW} = -\frac{ik}{240\pi^2}\int_{X_5} \mathrm{Tr}\big[(\mathrm{iU}^{-1}\mathrm{dU})^5\big]$ in the 4d boundary action (4.2.136).[84] This term would further break the discrete symmetry of the EFT to $(\mathbb{Z}_2)^2$ generated by P and C, as it is clear from Table 4.2.

An application: $U(N)$ **QCD.** Let us present a concrete application of the effective action (4.2.136). Consider a 4d gauge theory with $U(N)$ gauge group and N_f flavors of massless Dirac fermions, so that there is a chiral symmetry $SU(N_f)_L \times SU(N_f)_R$. It can be obtained by gauging the baryon number symmetry $U(1)_B$ in ordinary $SU(N)$ QCD, hence it contains an Abelian gauge field A_μ on top of the non-Abelian gauge fields. Being weakly coupled at low energy, A_μ is not expected to drastically modify the strong coupling dynamics of the non-Abelian sector. Hence for N_f small enough, the quark bilinear takes VEV and spontaneously breaks the chiral symmetry:[85]

$$SU(N_f)_L \times SU(N_f)_R \;\to\; SU(N_f)_V \tag{4.2.138}$$

producing at low energy massless pions that interact as a non-linear sigma model with target space $SU(N_f)$. The pions are neutral under the non-Abelian gauge symmetry $SU(N)$, whose gluons are confined. However the Abelian gauge field A_μ remains even in the deep IR and there is no reason why it should be decoupled from the non-linear sigma model. Indeed, while the pion fields themselves are neutral under $U(1)$, being bound states of quarks it is a priori unclear whether there is a low-energy remnant of the quark-photon interaction.

We can answer this question using our result, and showing that the photon is not decoupled. Indeed there is a $U(1)$ magnetic 1-form symmetry from the Abelian gauge field (that is its Goldstone boson), which forms a non-trivial 2-group with $SU(N_f)_L$ (and also with $SU(N_f)_R$, but we can just focus on one of the two). To see this, we notice that there is a triangle anomaly $U(1)\text{-}SU(N_f)_L^2$ whose anomaly polynomial is

$$\mathcal{P}_{U(1)\text{-}SU(N_f)_L^2} = \frac{N}{8\pi^2}\, dA \wedge \mathrm{Tr}\big(\mathcal{F}\wedge\mathcal{F}\big)\,, \tag{4.2.139}$$

where $\mathcal{F} = d\mathcal{G} + i\,\mathcal{G}\wedge\mathcal{G}$ is the field strength of the background field $\mathcal{G}$ for $SU(N_f)_L$. The coefficient N comes because all left-moving fermions have charge 1 under $U(1)$ and are in the fundamental representation of the non-Abelian gauge symmetry $SU(N)$. By comparison with (4.2.127) we read off that the $U(1)$ 1-form symmetry and $SU(N_f)_L$ form a 2-group with $k = N$. Because of chiral symmetry breaking and spontaneous breaking of the 1-form symmetry, the 2-groups is fully broken and, from our result above, the low-energy EFT describing pions and photon is (4.2.136), plus the standard WZW term (also with coefficient N) for the pions due

[84] We did not work out the detailed form of the coupling to the background fields.

[85] Notice that the usual argument [397] based on 't Hooft anomaly matching in $SU(N)$ QCD is also valid here, hence we do not really need to make the assumption that the photon does not affect chiral symmetry breaking.

to the cubic $SU(N_f)_L$ anomaly:[86]

$$
\begin{aligned}
S_{\rm IR} = \frac{f_\pi^2}{4\pi}\int_{\mathcal{M}_4} {\rm Tr}\Big[\big({\rm idU\,U^{-1}}\big)\wedge\star\big({\rm idU\,U^{-1}}\big)\Big] + \frac{1}{4{\rm e}^2}\int_{\mathcal{M}_4} {\rm dA}\wedge\star{\rm dA} \\
+\frac{N}{24\pi^2}\int_{\mathcal{M}_4} A\wedge{\rm Tr}\Big[\big({\rm iU^{-1}dU}\big)^3\Big] - \frac{\rm iN}{240\pi^2}\int_{\rm X_5}{\rm Tr}\Big[\big({\rm iU^{-1}dU}\big)^5\Big]\,.
\end{aligned}
\tag{4.2.140}
$$

Thus, while the pions themselves are uncharged under the $U(1)$ gauge group, the photon A is coupled with an effective current

$$
J_B = -\frac{N}{24\pi^2}\star{\rm Tr}\Big[\big({\rm iU^{-1}dU}\big)^3\Big]. \tag{4.2.141}
$$

This current is conserved, and in the absence of the pion-photon interaction it generate a global $U(1)$ symmetry of the sigma model: the topological symmetry due to the non-trivial homotopy group $\pi_3\big(SU(N_f)\big)=\mathbb{Z}$. The integral of $\star\, J_B$ gives indeed the winding number:

$$
w(\mathcal{M}_3) = -\frac{i}{24\pi^2}\int_{\mathcal{M}_3}{\rm Tr}\Big[\big({\rm iU^{-1}dU}\big)^3\Big]\ \in\mathbb{Z}\,. \tag{4.2.142}
$$

In the $U(N)$ theory, configurations with nontrivial winding have a $U(1)$ gauge charge. These configurations are Skyrmions: solitonic objects which, in the $SU(N)$ theory, are identified with the baryons [381, 399]. This is confirmed by our finding: the $U(N)$ theory is obtained from ordinary $SU(N)$ QCD by gauging the baryon number symmetry, hence the baryons are no longer gauge invariant, but rather are coupled with A.

We can make this more precise as follows. In the absence of the photon-pion coupling, the operators charged under the topological $U(1)$ symmetry are local operators $\mathcal{B}_q(x)$ defined as disorder operators which impose that

$$
w(S^3) = q\in\mathbb{Z} \tag{4.2.143}
$$

on a 3-sphere S^3 that links with x. Similarly to the monopole operator in Chern–Simons theory, $\mathcal{B}_q(x)$ gets a gauge charge Nq due to the coupling with the photon.

Also, in the absence of the 2-group structure, the low-energy effective theory would have an emergent electric $U(1)$ 1-form symmetry shifting $A\to A+\lambda$ (with the periods of λ in the interval $[0,2\pi]$) and acting on the Wilson lines $W_n(\gamma)=e^{in\int_\gamma A}$. Because of the photon-pion coupling, however, only a $\mathbb{Z}_N\subset U(1)$ subgroup of this 1-form symmetry emerges. Indeed using the quantization (4.2.142), shifting $A\to A+\lambda$ leaves the exponentiated action invariant only if the periods of λ are multiples of $\frac{2\pi}{N}$. An equivalent way to see this is that the Wilson line $W_{n=N}$ can terminate on the Baryon operator $\mathcal{B}_1(x)$. Notice that the microscopic theory does not have this $\mathbb{Z}_N$ 1-form symmetry, because the quarks have unit charge under the gauged $U(1)_B$. The emergence of $\mathbb{Z}_N$ has a clear interpretation: the quarks are confined and the only dynamical particles charged under $U(1)_B$ at low energy are baryons, with charges multiple of N.

[86] 2-group structures in sigma models arising in the IR of QCD-like theories have been recently considered also in [398]. The IR there, however, is purely scalar, and the 2-group is not fully spontaneously broken (the 1-form symmetry is preserved). The interaction responsible for the 2-group is not a photon-pion coupling, but rather a coupling between pions parametrizing two different target spaces. Indeed the UV model studied in [398] can be obtained from $U(N)$ QCD by adding scalars charged under $U(1)_B$ that Higgs the Abelian gauge field.

As a final comment, notice that among the three $\mathbb{Z}_2$ symmetries $P, C, \widetilde{C}$ defined in (4.2.137) that are preserved by the photon-pion coupling, only P and C are preserved also by the standard WZW term, while $\widetilde{C}$ is explicitly broken (see Table 4.2). This has to do with the fact that in $U(N)$ QCD, $C_2 \colon U \mapsto U^{\mathsf{T}}$ is the low-energy remnant of the non-Abelian charge conjugation that, in the UV, also acts on the $SU(N)$ gauge bosons, confined in the IR. In the $U(N)$ theory this charge conjugation is not independent from the Abelian charge conjugation C_1 acting on the photon, since the fermions are in the fundamental representation of both. Hence, only the product $C = C_1 C_2$ is a symmetry of the theory.

Appendix A

Appendices for Chapter 3

A.1 Superconformal Symmetry

In this Appendix we collect several facts on the $\mathcal{N}=2$ superconformal algebra and summarize our conventions. The $\mathcal{N}=2$ multiplet containing the stress energy tensor $T_B(z)$ also includes two fermionic supercurrents $T_F^{\pm}(z)$ as well as a $U(1)_R$ current $J(z)$. The non zero OPEs are (equality below is up to regular terms)

$$\begin{aligned}
T_B(z)T_B(0) &= \frac{c}{2z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \\
T_B(z)T_F^{\pm}(0) &= \frac{3}{2z^2}T_F^{\pm}(0) + \frac{1}{z}\partial T_F^{\pm}(0) \\
T_B(z)J(0) &= \frac{1}{z^2}J(0) + \frac{1}{z}\partial J(0) \\
T_F^{+}(z)T_F^{-}(0) &= \frac{2c}{3z^2} + \frac{2}{z^2}J(0) + \frac{2}{z}\partial T_B(0) + \frac{1}{z}\partial J(0) \\
J(z)T_F^{\pm}(0) &= \pm\frac{1}{z}T_F^{\pm}(0) \\
J(z)J(0) &= \frac{c}{3z^2}
\end{aligned} \tag{A.1.1}$$

Besides the regular T_B OPE these tell us that $T_F^{\pm}$ are (Virasoro) primary fields with weight $3/2$ and $U(1)_R$ charge ± 1. As usual the conserved R-current $J(z)$ is a primary of weight 1. On the cylinder we can decompose these fields in Fourier modes, then mapping back to the punctured plane we have

$$T_B(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}\,, \qquad J(z) = \sum_{n\in\mathbb{Z}} \frac{j_n}{z^{n+1}}\,, \qquad T_F^{\pm}(z) = \sum_{r\in\mathbb{Z}\pm\nu} \frac{G_r^{\pm}}{z^{r+3/2}}\,. \tag{A.1.2}$$

where ν depends on the spin structure chosen: $\nu = 0$ corresponds to antiperiodic boundary conditions for the fermions (Ramond sector) while $\nu = 1/2$ gives periodic fermions (NS sector). The algebra of modes is

$$\begin{aligned}
&\left[L_m, G_r^{\pm}\right] = \left(\frac{m}{2} - r\right) G_{m+r}^{\pm}\,, \qquad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
&[L_m, j_n] = -n j_{m+n}\,, \qquad [j_m, j_n] = \frac{c}{3} m \delta_{m+n,0}\,, \qquad \left[j_m, G_r^{\pm}\right] = \pm G_{m+r}^{\pm}\,, \\
&\left\{G_r^{+}, G_s^{-}\right\} = 2L_{r+s} + (r-s)j_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,.
\end{aligned} \tag{A.1.3}$$

A convenient choice for the Cartan subalgebra is the pair L_0, j_0 so that states in representation spaces are labelled by both their conformal weight h and the $U(1)_R$ charge q

$$L_0|h,q\rangle = h|h,q\rangle \qquad j_0|h,q\rangle = q|h,q\rangle\,. \tag{A.1.4}$$

Irreducible representations of this algebra are lowest weight representations (LWR) built on top of a superconformal primary state $|h,q\rangle$ such that

$$L_n|h,q\rangle = j_n|h,q\rangle = G_r^{\pm}|h,q\rangle = 0 \qquad \forall\, n,r>0\,. \tag{A.1.5}$$

A unitary representation is one in which we have an hermitian conjugation operation with respect to which $(L_m)^\dagger = L_{-m}$, $(G_r^+)^\dagger = G_{-r}^-$ and $(j_n)^\dagger = j_{-n}$. Unitarity puts strong constraints on the spectrum of allowed weights and $U(1)_R$ charges for a given central charge. A simple unitarity bound in the NS sector is obtained imposing

$$0 \le \langle h,q|\left\{G^+_{\mp 1/2}, G^-_{\pm 1/2}\right\}|h,q\rangle = \langle h,q|\,(2L_0 \mp j_0)\,|h,q\rangle = 2h \mp q \tag{A.1.6}$$

that is states in a unitary representation in the NS sector obey $h \ge |q|/2$. For more details see e.g. [255, 400, 401]. Since the superconformal algebra includes Virasoro as a subaglebra we can split its representations in Virasoro irreps. This basically amounts to find states annihilated only by the positive Virasoro modes. Let's consider a superconformal primary $|h,q\rangle$ and its fermionic descendants $G^{\pm}_{-r}|h,q\rangle$ with $r>0$. We have

$$L_m G^{\pm}_{-r}|h,q\rangle = [L_m, G^{\pm}_{-r}]|h,q\rangle + G^{\pm}_{-r}L_m|h,q\rangle = \left(\frac{m}{2}+r\right)G^{\pm}_{m-r}|h,q\rangle \tag{A.1.7}$$

which vanishes only for $m>r$. Therefore these states are not Virasoro primaries. We can obtain further Virasoro primaries considering states obtained acting on $|h,q\rangle$ with products of fermionic generators $G^{\pm}_{-r}$ with different values of r. For instance in the NS sector one easily sees that $G^{\pm}_{-1/2}|h,q\rangle$ are Virasoro primaries while $G^{\pm}_{-3/2}|h,q\rangle$ are not. The next lowest weight Virasoro primary are instead $G^{\pm}_{-1/2}G^{\pm}_{-3/2}|h,q\rangle$, indeed

$$\begin{aligned} L_m G^{\pm}_{-1/2}G^{\pm}_{-3/2}|h,q\rangle &= [L_m, G^{\pm}_{-1/2}]G^{\pm}_{-3/2}|h,q\rangle + G^{\pm}_{-1/2}[L_m, G^{\pm}_{-3/2}]|h,q\rangle \\ &= \frac{m+1}{2}G^{+}_{m-1/2}G^{+}_{-3/2}|h,q\rangle + \frac{m+3}{2}G^{+}_{-1/2}G^{+}_{m-3/2}|h,q\rangle \end{aligned} \tag{A.1.8}$$

which vanishes for all $m>1$ due to $|h,q\rangle$ being a primary while for $m=1$ because the state $G^{+}_{-1/2}G^{+}_{-1/2}|h,q\rangle$ is actually null (as one would expect). Thus in general a superconformal family includes an infinite number of conformal ones, with all possible values of the $U(1)_R$ charge.

A.1.1 Chiral Ring and Spectral Flow

There are two useful features of the $\mathcal{N}=2$ superconformal symmetry, the first is the existence of shortened representations whose lowest weight state is called *chiral primary*, the second is the presence of an external automorphism of the algebra, the *spectral flow*. Chiral primaries and their ring are associated to the NS sector, here one defines chiral states as those such that

$$G^{+}_{-1/2}|h,q\rangle = 0\,. \tag{A.1.9}$$

In a $\mathcal{N}=(2,2)$ SCFT we have left and right chirals. If $|h,q\rangle$ is also a superconformal primary (A.1.6) shows that chiral primaries saturate the unitarity bound and have $h=q/2$. Interstingly one can also show the converse, thus for a state $|h,q\rangle$ being a chiral primary is equivalent to having $h=q/2$. Now consider the OPE of two chiral primaries ϕ_a and ϕ_b, this has the general form

$$\phi_a(z)\phi_b(w)=\sum_c\sum_{n\in\mathbb{N}_0}\frac{\partial^n\phi_c}{(z-w)^{h_a+h_b-h_c-n}}\,. \tag{A.1.10}$$

Since the R charge has to be conserved any operator appearing in the OPE must have $q_c=q_a+q_b$ and thus the unitarity bound implies

$$h_a+h_b-h_c=\frac{q_a+q_b}{2}-h_c=\frac{q_c}{2}-h_c\leq 0 \tag{A.1.11}$$

hence the OPE of two chiral primaries is free of singular terms. We can then define a product as the limit of coincident points of the OPE

$$\left(\phi_a\cdot\phi_b\right)(z)=\lim_{w\to z}\phi_a(z)\phi_b(w)=\sum_c C^c_{ab}\phi_c(z)\,. \tag{A.1.12}$$

The rhs of the product cannot contain terms with derivatives, indeed it can only involve operators with $q_c/2=h_c$, i.e. other chiral primaries. This product the closes on chiral primaries and endows them with a ring structure. In an $\mathcal{N}=(2,2)$ theory we have four of these rings depending on wheter we take a chiral or antichiral state on the left or on the right.

The other interesting feature of the $\mathcal{N}=2$ algebra is the spectral flow. This is the following one parameter deformation of the generators

$$\begin{aligned}
L'_n&=L_n+\eta j_n+\frac{\eta^2}{6}c\delta_{n,0}\\
j'_n&=j_n+\frac{c}{3}\eta\delta_{n,0}\\
G^{\pm\,\prime}_r&=G^{\pm}_{r\pm\eta}\,,
\end{aligned} \tag{A.1.13}$$

one easily checks that the primed generators satisfy the same algebra of the unprimed ones. Notice also that the flow changes the moding of the fermionic generators, so, for $\eta\in\mathbb{Z}/2$ it interpolates between NS and R sectors. Since this is an automorphism of the algebra it maps representations one into the other. Introducing a unitary operator U_η that implements the flow as

$$\begin{aligned}
L'_n&=U_\eta L_n U^\dagger_\eta\\
j'_n&=U_\eta j_n U^\dagger_\eta\\
G^{\pm\,\prime}_r&=U_\eta G^{\pm}_r U^\dagger_\eta\,,
\end{aligned} \tag{A.1.14}$$

we can spectrally flow a representation acting with U_η on the various states. In particular a state $|h,q\rangle$ is mapped to $U_\eta|h,q\rangle$, combining the relations above it is easy to show that

$$\begin{aligned}
L_0U_\eta|h,q\rangle&=\left(h-\eta q+\frac{\eta^2c}{6}\right)U_\eta|h,q\rangle\\
j_0U_\eta|h,q\rangle&=\left(q-\frac{\eta c}{3}\right)U_\eta|h,q\rangle
\end{aligned} \tag{A.1.15}$$

thus states in the spectrally flowed representations are still eigenstates of L_0 and j_0. Given a superconformal primary $|h,q\rangle$ we see that

$$L_m U_\eta |h,q\rangle = j_m U_\eta |h,q\rangle = 0 \qquad \forall\, m>0 \tag{A.1.16}$$

while

$$G_r^\pm U_\eta |h,q\rangle = U_\eta G^\pm_{r\mp\eta}|h,q\rangle \tag{A.1.17}$$

which vanishes only for $r\mp\eta>0$. Thus a LWR representation will be mapped to another LWR as long as we allows various moding of the fermionic generators. As an example let's consider the spectral flow with $\eta=1/2$ of a chiral primary representation. The chiral primary $|q/2,q\rangle$ flows to a state with weight $c/24$ and charge $q-c/6$ annihilated by all positive modes of T_B and J as well as the $G_r^\pm$ with $r\in\mathbb{N}$, i.e. a superconformal primary in the Ramond sector. As we choose different chiral primaries to flow we obtain degenerate states that differ for their R-charge. It is also easy to show that these are ground states in the R sector, we compute

$$0\le |G_0^+|h,q\rangle|^2+|G_0^-|h,q\rangle|^2 = \langle h,q|\{G_0^+,G_0^-\}|h,q\rangle = 2\left(h-\frac{c}{24}\right)||h,q\rangle|^2 \tag{A.1.18}$$

so unitarity implies $h\ge c/24$ and the ground states above saturate the bound. We can also define a spectral flow operator looking at the image under U_η of the NS vacuum $|0,0\rangle$, this then has weight $\eta^2 c/6$ and charge $-\eta c/3$.

A.2 Verlinde Formulas, and Modularity for Superconformal Primaries

In this Appendix we write down the modular transformations of the characters of the full superconformal representations and derive Verlinde like formulas for their fusion. From the modular S-matrix of the half-character is easy to derive the modular transformations of the full characters

$$\begin{aligned}
S\cdot \mathrm{ch}^{(\mathrm{NS})}_{l,m} &= \sum_{l',m'} S^{\mathrm{NSNS}}_{lm;l'm'}\mathrm{ch}^{(\mathrm{NS})}_{l',m'} & \qquad S\cdot \mathrm{ch}^{(\mathrm{R})}_{l,m} &= \sum_{l',m'} S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{lm;l'm'}\widetilde{\mathrm{ch}}^{(\mathrm{NS})}_{l',m'}\\
S\cdot \widetilde{\mathrm{ch}}^{(\mathrm{NS})}_{l,m} &= \sum_{l',m'} S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{lm;l'm'}\mathrm{ch}^{(\mathrm{R})}_{l',m'} & \qquad S\cdot \widetilde{\mathrm{ch}}^{(\mathrm{R})}_{l,m} &= \sum_{l',m'} S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{lm;l'm'}\widetilde{\mathrm{ch}}^{(\mathrm{R})}_{l',m'}
\end{aligned} \tag{A.2.1}$$

where

$$\begin{aligned}
S^{\mathrm{NSNS}}_{lm;l'm'} &= \frac{2}{k+2}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{i\pi\frac{mm'}{k+2}}\\
S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{lm;l'm'} &= \frac{2}{k+2}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{i\pi\frac{(m+1)m'}{k+2}}\\
S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{lm;l'm'} &= \frac{2}{k+2}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{i\pi\frac{m(m'+1)}{k+2}}\\
S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{lm;l'm'} &= -\frac{2i}{k+2}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{i\pi\frac{(m+1)(m'+1)}{k+2}}\,.
\end{aligned} \tag{A.2.2}$$

which correctly mimic the action of $SL(2,\mathbb{Z})$ on spin structures. One can verify that those S matrices are unitary and furnish a representation of $SL(2,\mathbb{Z})$, i.e. $S^4 = \mathbb{1}$. In verifying this last property one should be careful in taking into account the action on spin structures. In particular it makes no sense to square $S^{\mathrm{R}\widetilde{\mathrm{NS}}}$ or $S^{\widetilde{\mathrm{NS}}\mathrm{R}}$, rather the charge conjugation matrices in the R and $\widetilde{\mathrm{NS}}$ sectors are, respectively, $C^{\mathrm{R}} = S^{\mathrm{R}\widetilde{\mathrm{NS}}} S^{\widetilde{\mathrm{NS}}\mathrm{R}}$ and $C^{\widetilde{\mathrm{NS}}} = S^{\widetilde{\mathrm{NS}}\mathrm{R}} S^{\mathrm{R}\widetilde{\mathrm{NS}}}$, while $C^{\mathrm{NS}} = S^{\mathrm{NSNS}} S^{\mathrm{NSNS}}$ and $C^{\widetilde{\mathrm{R}}} = S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}} S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}$. Neither of these matrices is the identity, but they all square to it, thus at least one representations in each sector is not self-conjugate.

Knowing the fusion coefficients $N_{ac;a'c'}^{a''c''}$ of the half-families we can extract the fusion coefficients for the full superconformal families as simply

$$\widehat{N}_{ac;a'c'}^{\alpha,\gamma} = N_{ac;a'c'}^{\alpha\gamma} + N_{ac;a'c'}^{k-\alpha\gamma+k+2} \tag{A.2.3}$$

where now $(a,c), (a',c'), (\alpha,\gamma) \in P'_k$ label a superconformal primary rather than an half-family. We now want to separate out the NS and R sectors explicitly and write down Verlinde formulas in the various sectors. We first notice that

$$N_{ac;a'c'}^{\alpha\gamma} = \sum_{(d,f)\in P'_k} \frac{S_{ac;df} S_{a'c';df} S^*_{\alpha\gamma;df}}{S_{00;df}} (1 + (-1)^{a+c+a'+c'+\alpha+\gamma}) \tag{A.2.4}$$

and

$$N_{ac;a'c'}^{k-\alpha\,\gamma+k+2} = \sum_{(d,f)\in P'_k} \frac{S_{ac;df} S_{a'c';df} S^*_{\alpha\gamma;df}}{S_{00;df}} (1 + (-1)^{a+c+a'+c'+\alpha+\gamma})(-1)^{d+f} \tag{A.2.5}$$

so the fusion coefficients are non-zero only when $a+c+a'+c'+\alpha+\gamma = 0 \bmod 2$. Now switching to the (l,m,λ) parametrization we find

$$\begin{aligned} \widehat{N}_{lm\lambda;l'm'\lambda'}^{l'',m'',\lambda''} &= \left(1 + (-1)^{2(\lambda+\lambda'+\lambda'')}\right) \sum_{(r,s)\in P_k; x=0,-1/2} \frac{S_{lm\lambda;rsx} S_{l'm'\lambda';rsx} S^*_{l''m''\lambda'';rsx}}{S_{000;rsx}} (1 + (-1)^{2x}) \\ &= \left(1 + (-1)^{2(\lambda+\lambda'+\lambda'')}\right) \sum_{(r,s)\in P_k} \frac{S_{lm\lambda;rs0} S_{l'm'\lambda';rs0} S^*_{l''m''\lambda'';rs0}}{S_{000;rs0}} \,. \end{aligned} \tag{A.2.6}$$

Notice that the modular matrix $S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}$ can never appear in these expressions. Recalling that $\lambda = 0$ is NS and $\lambda = -1/2$ is R we see that there are four fusion channels

$$\begin{aligned} &\mathrm{NS} \times \mathrm{NS} = \mathrm{NS} \qquad \mathrm{R} \times \mathrm{R} = \mathrm{NS} \\ &\mathrm{R} \times \mathrm{NS} = \mathrm{R} \qquad \mathrm{NS} \times \mathrm{R} = \mathrm{R}\,, \end{aligned} \tag{A.2.7}$$

for each of those we have a Verlinde formula

- $\mathrm{NS} \times \mathrm{NS} = \mathrm{NS}$

$$\widehat{N}_{lm;l'm'}^{l''m'',\,\mathrm{NSNS}} = \sum_{(r,s)\in P_k} \frac{S^{\mathrm{NSNS}}_{lm;rs} S^{\mathrm{NSNS}}_{l'm';rs} \left(S^{\mathrm{NSNS}}_{l''m'';rs}\right)^*}{S^{\mathrm{NSNS}}_{00;rs}} \tag{A.2.8}$$

- $\mathrm{R} \times \mathrm{NS} = \mathrm{R}$

$$\widehat{N}_{lm;l'm'}^{l''m'',\,\mathrm{RNS}} = \sum_{(r,s)\in P_k} \frac{S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{lm;rs} S^{\mathrm{NSNS}}_{l'm';rs} \left(S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{l''m'';rs}\right)^*}{S^{\mathrm{NSNS}}_{00;rs}} \tag{A.2.9}$$

- NS × R = R

$$\widehat{N}^{l''m'',\,\mathrm{NSR}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\mathrm{NSNS}}_{lm;rs} S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{l'm';rs} \left(S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{l''m'';rs}\right)^*}{S^{\mathrm{NSNS}}_{00;rs}} \tag{A.2.10}$$

- R × R = NS

$$\widehat{N}^{l''m'',\,\mathrm{RR}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{lm;rs} S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{l'm';rs} \left(S^{\mathrm{NSNS}}_{l''m'';rs}\right)^*}{S^{\mathrm{NSNS}}_{00;rs}}\,. \tag{A.2.11}$$

from the explicit formulas of the S-matrices we see that $S^{\mathrm{R}\widetilde{\mathrm{NS}}}_{lm;rs} = e^{i\pi\frac{s}{k+2}} S^{\mathrm{NSNS}}_{lm,rs}$ then

$$\widehat{N}^{l''m'',\,\mathrm{NSNS}}_{lm;l'm'} = \widehat{N}^{l''m'',\,\mathrm{RNS}}_{lm;l'm'} = \widehat{N}^{l''m'',\,\mathrm{NSR}}_{lm;l'm'}\,. \tag{A.2.12}$$

The positive integers $\widehat{N}^{l''m'',\,\mathrm{RR}}_{lm;l'm'}$ can also be related to $\widehat{N}^{l''m'',\,\mathrm{NSNS}}_{lm;l'm'}$ albeit in a less trivial way. In examples we have checked that there exist a permutation of the labels of primaries $\sigma : P_k \to P_k$ such that

$$\widehat{N}^{l''m'',\,\mathrm{RR}}_{lm;l'm'} = \widehat{N}^{\sigma(l''m''),\,\mathrm{NSNS}}_{\sigma(lm);\sigma(l'm')}\,. \tag{A.2.13}$$

These integers have the interpretation of fusion coefficients for superconformal primaries. There are however other integers we can construct out of the S matrices by considering

$$\widehat{M}^{\alpha\gamma}_{ac;a'c'} = N^{\alpha\gamma}_{ac;a'c'} - N^{k-\alpha\gamma+k+2}_{ac;a'c'} \tag{A.2.14}$$

those are manifestly integers although not necessarily positive. However they obey Verlinde-like formulas:

- NS × NS = NS

$$\widehat{M}^{l''m'',\mathrm{NSNS}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{lm;rs} S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{l'm';rs} \left(S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{l''m'';rs}\right)^*}{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{00;rs}} \tag{A.2.15}$$

- R × NS = R

$$\widehat{M}^{l''m'',\,\mathrm{RNS}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{lm;rs} S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{l'm';rs} \left(S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{l''m'';rs}\right)^*}{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{00;rs}} \tag{A.2.16}$$

- NS × R = R

$$\widehat{M}^{l''m'',\,\mathrm{NSR}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{lm;rs} S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{l'm';rs} \left(S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{l''m'';rs}\right)^*}{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{00;rs}} \tag{A.2.17}$$

- R × R = NS

$$\widehat{M}^{l''m'',\,\mathrm{RR}}_{lm;l'm'} = \sum_{(r,s)\in P_k} \frac{S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{lm;rs} S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{l'm';rs} \left(S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{l''m'';rs}\right)^*}{S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{00;rs}} \tag{A.2.18}$$

.

Again, noticing that $S^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}_{lm;rs} = e^{i\pi\frac{s+1}{k+2}} S^{\widetilde{\mathrm{NS}}\mathrm{R}}_{lm;rs}$ one checks that

$$\widehat{M}^{l''m'',\,\mathrm{NSNS}}_{lm;l'm'} = \widehat{M}^{l''m'',\,\mathrm{RNS}}_{lm;l'm'} = \widehat{M}^{l''m'',\,\mathrm{NSR}}_{lm;l'm'}\,. \tag{A.2.19}$$

Also in this case there exist a permutation relating $\widehat{M}^{l''m'',\,\mathrm{RR}}_{lm;l'm'}$ to $\widehat{M}^{l''m'',\,\mathrm{NSNS}}_{lm;l'm'}$.

A.3 Supersymmetric Boundaries in Minimal Models and Folding Trick

In this Appendix we derive the supersymmetric Verlinde lines of a single minimal model using the folding trick [191]. The first step is to determine the supersymmetric boundary condition, see [214–218] for more details on boundary conditions in CFT. Supersymmetric boundaries in $\mathcal{N}=1$ minimal models have been worked out in [402]. The $\mathcal{N}=2$ superconformal algebra has an outer automorphism called mirror map

$$\Omega_M : \begin{cases} j_n \to -j_n \\ G_r^{\pm} \to G_r^{\mp} \end{cases} \tag{A.3.1}$$

thus there are two types of boundary states, the untwisted ones, or B-type

$$\begin{aligned} (L_n - \bar{L}_{-n})|\mathcal{B}_i\rangle\rangle_B &= (j_n + \bar{j}_{-n})|\mathcal{B}_i\rangle\rangle_B = 0 \\ (G_r^+ + i\eta\bar{G}_{-r}^+|\mathcal{B}_i\rangle\rangle_B &= (G_r^- + i\eta\bar{G}_{-r}^-)|\mathcal{B}_i\rangle\rangle_B = 0 \end{aligned} \tag{A.3.2}$$

and the twisted ones, or A-type

$$\begin{aligned} (L_n - \bar{L}_{-n})|\mathcal{B}_i\rangle\rangle_A &= (j_n - \bar{j}_{-n})|\mathcal{B}_i\rangle\rangle_A = 0 \\ (G_r^+ + i\eta\bar{G}_{-r}^-|\mathcal{B}_i\rangle\rangle_A &= (G_r^- + i\eta\bar{G}_{-r}^+)|\mathcal{B}_i\rangle\rangle_A = 0\,. \end{aligned} \tag{A.3.3}$$

In the S-dual channel the boundary conditions are

$$\begin{array}{lll} \text{A-type:} & J(z) = -\bar{J}(\bar{z}) & G^{\pm}(z) = \eta\bar{G}^{\mp}(\bar{z}) \\ \text{B-type:} & J(z) = +\bar{J}(\bar{z}) & G^{\pm}(z) = \eta\bar{G}^{\pm}(\bar{z}) \end{array} \quad . \tag{A.3.4}$$

Here η can be any phase in general, choosing $\eta = \pm 1$ one can see that both types of boundary conditions preserve an $\mathcal{N}=1$ subalgebra. As usual these boundary conditions preserve only one copy of the $\mathcal{N}=2$ algebra. For the B-type boundary conditions the preserved copy is the diagonal of the holomorphic and antiholomorphic algebras. The parameter η labels a continuous family of boundary conditions, let's be more precise about it. Consider an $\mathcal{N}=2$ SCFT on the upper half-plane and impose the boundary conditions

$$\begin{aligned} G^{\pm}(z) &= \bar{G}^{\pm}(\bar{z}) \qquad z = \bar{z} > 0 \\ G^{\pm}(z) &= \eta\bar{G}^{\pm}(\bar{z}) \qquad z = \bar{z} < 0\,. \end{aligned} \tag{A.3.5}$$

As in the doubling trick we can construct an holomorphic field on the whole complex plane by

$$\mathfrak{G}^{\pm}(z) = \begin{cases} G^{\pm}(z) & \text{Im}(z) > 0 \\ \bar{G}^{\pm}(\bar{z}) & \text{Im}(z) < 0 \end{cases} \tag{A.3.6}$$

which is not single valued in the complex plane as it obeys

$$\mathfrak{G}^{\pm}(e^{2i\pi}z) = \eta\mathfrak{G}^{\pm}(z)\,. \tag{A.3.7}$$

This can be interpreted as the insertion at the origin of a twist defect for the $U(1)_R$ symmetry, which comes with an attached topological defect line L_η implementing $\eta \in U(1)$. Since having different boundary conditions on the positive and negative real axes is interpreted as the insertion of a boundary changing operator at the origin, we see that, for the boundary conditions above, the boundary changing operator corresponds to a twist defect for the $U(1)_R$ symmetry. With more general boundary conditions

$$\begin{aligned} G^{\pm}(z) &= \eta' \bar{G}^{\pm}(\bar{z}) \qquad z = \bar{z} > 0 \\ G(z)^{\pm} &= \eta \bar{G}^{\pm}(\bar{z}) \qquad z = \bar{z} < 0\,. \end{aligned} \tag{A.3.8}$$

the extended field obeys

$$\widetilde{\mathfrak{G}}(e^{2i\pi} z) = \frac{\eta}{\eta'} \widetilde{\mathfrak{G}}(z)\,. \tag{A.3.9}$$

Thus the boundary changing operator is a twist defect attached to the line $L_{\eta/\eta'}$. Since the boundary conditions preserve the $U(1)_R$ symmetry, there exist well defined topological junctions between the $U(1)_R$ lines and the boundary. Therefore we can have, in the upper-half plane, TDLs homotopic to semi-circles stretching across the positive and negative real axes (eventually with trivalent junctions involving the boundary changing operator twist line). On the strip this configuration corresponds to a network of $U(1)_R$ lines, with a $U(1)_R$ line connecting the two boundaries and one running along the non-compact direction.

By the equations above we see that the boundary parameter η determines the mode expansion of the extended fermionic fields. Since those modes are used to construct the Hilbert space of the theory we see that having different values of η on the positive and negative axis leads to twisted interval Hilbert spaces. In particular when $\eta = -1$ the associated topological defect line implements $(-1)^F$, which is a $\mathbb{Z}_2$ subgroup of $U(1)_R$, and hence the theory with boundary conditions $\eta = 1$ and $\eta' = -1$ has a Ramond sector Hilbert space on the interval. The line L_γ stretching between the two boundaries instead acts on this Hilbert space.

The general case in the upper-half plane is to consider two boundary conditions B_η and $B_{\eta'}$, related by the twist defect of $L_{\eta/\eta'}$, as well as another TDL L_γ stretching between the boundaries. We map this configuration on the strip and compactify the extended direction, resulting in a finite cylinder. If we interpretet the compact direction as time (open sector) we have a trace over the Hilbert space with boundary conditions B_η and $B_{\eta'}$, i.e. a twisted Hilbert space, with an insertion of L_γ. In the S-dual channel (closed sector), with a periodic space direction, $L_{\eta/\eta'}$ acts on the boundary states while L_γ twists the Hilbert space, meaning that the boundary states will have components not in the vanilla circle Hilbert space $\mathbb{H}$ but in the twisted one $\mathbb{H}^{(\gamma)}$. In formulas

$$\langle B_\eta^{(\gamma)} | \tilde{q}^{L_0 - \frac{c}{24}} L_{\eta/\eta'} | B_{\eta'}^{(\gamma)} \rangle = \mathrm{Tr}_{\mathbb{H}(\eta,\eta')} L_\gamma q^{L_0 - \frac{c}{24}}\,. \tag{A.3.10}$$

Therefore choosing boundary conditions with $\eta \neq \eta'$ inevitably lead to a closed sector overlap involving a $L_{\eta/\eta'}$ insertion, or, equivalently, to an open sector tracing over a twisted Hilbert space. Similarly enriching the trace in the open sector with a fugacity for $U(1)_R$ can only correspond to an overlap of boundary states with components in a twisted Hilbert space. Another important fact is that boundary conditions preserving the superconformal algebra are invariant under $U(1)_R$, i.e. $L_{\eta'} |B_\eta^{(\gamma)}\rangle = |B_\eta^{(\gamma)}\rangle$, therefore we can forget about the insertion of $L_{\eta/\eta'}$ in the closed sector and simply write

$$\langle B_\eta^{(\gamma)} | \tilde{q}^{L_0 - \frac{c}{24}} | B_{\eta'}^{(\gamma)} \rangle = \mathrm{Tr}_{\mathbb{H}(\eta,\eta')} L_\gamma q^{L_0 - \frac{c}{24}} \tag{A.3.11}$$

Now recall that $(-1)^F$ is actually a subgroup of $U(1)_R$, so that the associated twist defect is the boundary changing operator between the boundary conditions with η and $\eta' = -\eta$. For simplicity let's stick to $\eta, \eta', \gamma = \pm 1$, then invariance under the S transformation requires

$$\begin{aligned}
\langle B_\pm^{(\mathrm{NS})} | \widetilde{q}^{L_0 - \frac{c}{24}} | B_\mp^{(\mathrm{NS})} \rangle &= \mathrm{Tr}_{\mathbb{H}^{(\mathrm{R},\pm)}} q^{L_0 - \frac{c}{24}} \\
\langle B_\pm^{(\mathrm{R})} | \widetilde{q}^{L_0 - \frac{c}{24}} | B_\pm^{(\mathrm{R})} \rangle &= \mathrm{Tr}_{\mathbb{H}^{(\mathrm{NS},\pm)}} (-1)^F q^{L_0 - \frac{c}{24}} \\
\langle B_\pm^{(\mathrm{NS})} | \widetilde{q}^{L_0 - \frac{c}{24}} | B_\pm^{(\mathrm{NS})} \rangle &= \mathrm{Tr}_{\mathbb{H}^{(\mathrm{NS},\pm)}} q^{L_0 - \frac{c}{24}} \\
\langle B_\pm^{(\mathrm{R})} | \widetilde{q}^{L_0 - \frac{c}{24}} | B_\mp^{(\mathrm{R})} \rangle &= \mathrm{Tr}_{\mathbb{H}^{(\mathrm{R},\pm)}} (-1)^F q^{L_0 - \frac{c}{24}}
\end{aligned} \tag{A.3.12}$$

which generalise Cardy's equations in an $\mathcal{N} = 2$ supersymmetric setting. To solve this conditions we need to introduce Ishibashi states. We can construct a unique Ishibashi state $|\mathcal{B}_{i,\pm}^{(X)}\rangle\rangle$ solving a given boundary constraint with $\eta = \pm 1$ for any irrep $\mathbb{H}_i^{(X)}$. The components of the states are elements of $\mathbb{H}_i^{(X)} \otimes \mathbb{H}_{\omega(i^+)}^{(X)}$. Here we are using a single label i to denote representations of the susy algebra and ω is the action of the automorphism defining the boundary conditions on the representations of the chiral algebra. Thus $\omega = \mathbb{1}$ for the B-type and $\omega = C$ for A-type. As usual we use Ishibashi states to construct physical boundary states that solve Cardy's condition. We set

$$|B_{a,\pm}^{(X)}\rangle = \sum_{i \in I_\Omega^{(X)}} B_{a,\pm}^{i,(X)} |\mathcal{B}_{i,\pm}^{(X)}\rangle\rangle \tag{A.3.13}$$

where X =NS, R and $I_\Omega^{(X)}$ labels the representations in the X sector that can be used to construct the Ishibashi states, namely it contains only those i for which $\mathbb{H}_i \otimes \mathbb{H}_{\omega(i^+)}$ appears in the circle Hilbert space. The overlaps of Ishibashi states are

$$\begin{aligned}
\langle\langle \mathcal{B}_{i,\pm}^{(X)} | \widetilde{q}^{L_0 - \frac{c}{24}} | \mathcal{B}_{j,\pm}^{(X)} \rangle\rangle &= \delta_{ij} \mathrm{ch}_i^{(X)}(\widetilde{q}) \\
\langle\langle \mathcal{B}_{i,\pm}^{(X)} | \widetilde{q}^{L_0 - \frac{c}{24}} | \mathcal{B}_{j,\mp}^{(X)} \rangle\rangle &= \delta_{ij} \widetilde{\mathrm{ch}}_i^{(X)}(\widetilde{q})
\end{aligned} \tag{A.3.14}$$

as one can check from their explicit definition. The interval Hilbert spaces with supersymmetric boundary conditions labeled by a, b are representations of the superconformal algebra, thus

$$\begin{aligned}
\mathrm{Tr}_{\mathbb{H}_{ab}^{(\mathrm{NS},\pm)}} q^{L_0 - \frac{c}{24}} &= \sum_{i \in I^{(\mathrm{NS})}} n_{ab;\pm}^i \mathrm{ch}_i^{(\mathrm{NS})}(q) \\
\mathrm{Tr}_{\mathbb{H}_{ab}^{(\mathrm{R},\pm)}} q^{L_0 - \frac{c}{24}} &= \sum_{i \in I^{(\mathrm{R})}} m_{ab;\pm}^i \mathrm{ch}_i^{(\mathrm{R})}(q) \\
\mathrm{Tr}_{\mathbb{H}_{ab}^{(\mathrm{NS},\pm)}} (-1)^F q^{L_0 - \frac{c}{24}} &= \sum_{i \in I^{(\mathrm{NS})}} \widetilde{n}_{ab;\pm}^i \widetilde{\mathrm{ch}}_i^{(\mathrm{NS})}(q) \\
\mathrm{Tr}_{\mathbb{H}_{ab}^{(\mathrm{R},\pm)}} (-1)^F q^{L_0 - \frac{c}{24}} &= \sum_{i \in I^{(\mathrm{R})}} \widetilde{m}_{ab;\pm}^i \widetilde{\mathrm{ch}}_i^{(\mathrm{R})}(q) \,.
\end{aligned} \tag{A.3.15}$$

Imposing (A.3.12) on the physical boundary states (A.3.13) we then obtain

$$\begin{aligned}
\sum_{i\in I_\Omega^{(\mathrm{NS})}} B_{a,\pm}^{i,\,(\mathrm{NS})} B_{b,\pm}^{i,\,(\mathrm{NS})} S_{ij}^{\mathrm{NSNS}} &= n_{ab;\pm}^{j} \\
\sum_{i\in I_\Omega^{(\mathrm{R})}} B_{a,\pm}^{i,\,(\mathrm{R})} B_{b,\pm}^{i,\,(\mathrm{R})} S_{ij}^{\mathrm{R}\widetilde{\mathrm{NS}}} &= \widetilde{n}_{ab;\pm}^{j} \\
\sum_{i\in I_\Omega^{(\mathrm{NS})}} B_{a,\pm}^{i,\,(\mathrm{NS})} B_{b,\mp}^{i,\,(\mathrm{NS})} S_{ij}^{\widetilde{\mathrm{NS}}\mathrm{R}} &= m_{ab;\pm}^{j} \\
\sum_{i\in I_\Omega^{(\mathrm{R})}} B_{a,\pm}^{i,\,(\mathrm{R})} B_{b,\mp}^{i,\,(\mathrm{R})} S_{ij}^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}} &= \widetilde{m}_{ab;\pm}^{j}\,.
\end{aligned} \tag{A.3.16}$$

We now want to find solutions to these constraints for some special modular invariant, in particular one that guarantees that we have as many Ishibashi states as there are primaries in the theory, so that the sums over i in the above equations run over all primaries. Notice also that the numbers $\widetilde{n}_{ab}^{j}, \widetilde{m}_{ab}^{j}$ are not multiplicities of some primary representation and therefore are not restricted to be positive, this allows us to use the Verlinde formulas for the $\widehat{M}$ coefficients derived in Appendix A.2. Using the properties[1]

$$\begin{aligned}
S_{lm;rs}^{\mathrm{NSNS}} &= \left(S_{r^+s^+;lm}^{\mathrm{NSNS}}\right)^* \\
S_{lm;rs}^{\widetilde{\mathrm{NS}}\mathrm{R}} &= \left(S_{lm;r^+s^+}^{\widetilde{\mathrm{NS}}\mathrm{R}}\right)^* = \left(S_{r^+s^+;lm}^{\mathrm{R}\widetilde{\mathrm{NS}}}\right)^* \\
S_{lm;rs}^{\mathrm{R}\widetilde{\mathrm{NS}}} &= \left(S_{lm;r^+s^+}^{\mathrm{R}\widetilde{\mathrm{NS}}}\right)^* = \left(S_{r^+s^+;lm}^{\widetilde{\mathrm{NS}}\mathrm{R}}\right)^*
\end{aligned} \tag{A.3.17}$$

we can write down the solutions

$$\begin{aligned}
B_{a_1a_2,+}^{lm,\,(\mathrm{NS})} &= \frac{S_{a_1a_2;lm}^{\mathrm{NSNS}}}{\sqrt{S_{00;lm}^{\mathrm{NSNS}}}} \qquad & B_{a_1a_2,-}^{lm,\,(\mathrm{NS})} &= \frac{S_{a_1a_2;lm}^{\mathrm{R}\widetilde{\mathrm{NS}}}}{\sqrt{S_{00;lm}^{\mathrm{NSNS}}}} \\
B_{a_1a_2,+}^{lm,\,(\mathrm{R})} &= \frac{S_{a_1a_2;lm}^{\widetilde{\mathrm{NS}}\mathrm{R}}}{\sqrt{S_{00;lm}^{\widetilde{\mathrm{NS}}\mathrm{R}}}} \qquad & B_{a_1a_2,-}^{lm,\,(\mathrm{R})} &= \frac{S_{a_1a_2;lm}^{\widetilde{\mathrm{R}}\widetilde{\mathrm{R}}}}{\sqrt{S_{00;lm}^{\widetilde{\mathrm{NS}}\mathrm{R}}}}\,.
\end{aligned} \tag{A.3.18}$$

The corresponding multiplicities are

$$\begin{aligned}
n_{a_1a_2;b_1b_2;+}^{rs} &= \widehat{N}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{NSNS}} \qquad & n_{a_1a_2;b_1b_2;-}^{rs} &= \widehat{N}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{RR}} \\
\widetilde{n}_{a_1a_2;b_1b_2;+}^{rs} &= \widehat{M}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{NSNS}} \qquad & \widetilde{n}_{a_1a_2;b_1b_2;-}^{rs} &= \widehat{M}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{RR}} \\
m_{a_1a_2;b_1b_2;+}^{rs} &= \widehat{N}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{NSR}} \qquad & m_{a_1a_2;b_1b_2;-}^{rs} &= \widehat{N}_{a_1a_2;b_1b_2}^{r^+s^+,\,\mathrm{RNS}} \\
\widetilde{m}_{a_1a_2;b_1b_2;+}^{rs} &= \widehat{M}_{a_1a_2;b_1b_2;}^{r^+s^+,\,\mathrm{NSR}} \qquad & \widetilde{m}_{a_1a_2;b_1b_2;-}^{rs} &= \widehat{M}_{a_1a_2;b_1b_2;}^{r^+s^+,\,\mathrm{RNS}}\,.
\end{aligned} \tag{A.3.19}$$

There is also another family of solutions that we can obtain reversing the signs of the R sector coefficients, since those affect only $\widetilde{n}_{ab}^{j}, \widetilde{m}_{ab}^{j}$ it is still a consistent family of solutions both within itself and with the family of solutions described above. All in all we found the physical boundary

[1] These derive from $S^{\widetilde{\mathrm{NS}}\mathrm{R}} S^{\mathrm{R}\widetilde{\mathrm{NS}}} = S^{\mathrm{NSNS}} S^{\mathrm{NSNS}} = C^{\mathrm{NSNS}}$ and $\left(S^{\widetilde{\mathrm{NS}}\mathrm{R}}\right)^T = S^{\mathrm{R}\widetilde{\mathrm{NS}}}$.

states

$$
\begin{aligned}
|B_{a_1,a_2;f,+}\rangle &= \sum_{(l,m)\in P_k} \left(\frac{S^{\mathrm{NSNS}}_{a_1a_2;lm}}{\sqrt{S^{\mathrm{NSNS}}_{00;lm}}} |\mathcal{B}^{(\mathrm{NS})}_{lm,+}\rangle\rangle + f\,\frac{S^{\mathrm{N\widetilde{S}R}}_{a_1a_2;lm}}{\sqrt{S^{\mathrm{N\widetilde{S}R}}_{00;lm}}} |\mathcal{B}^{(\mathrm{R})}_{lm,+}\rangle\rangle \right) \\
|B_{a_1,a_2;f,-}\rangle &= \sum_{(l,m)\in P_k} \left(\frac{S^{\mathrm{R\widetilde{N}S}}_{a_1a_2;lm}}{\sqrt{S^{\mathrm{NSNS}}_{00;lm}}} |\mathcal{B}^{(\mathrm{NS})}_{lm,-}\rangle\rangle + f\,\frac{S^{\mathrm{\widetilde{R}\widetilde{R}}}_{a_1a_2;lm}}{\sqrt{S^{\mathrm{N\widetilde{S}R}}_{00;lm}}} |\mathcal{B}^{(\mathrm{R})}_{lm,-}\rangle\rangle \right)
\end{aligned}
\tag{A.3.20}
$$

with $f = \pm 1$. Thus for pair $(a_1, a_2) \in P_k$ we can construct four boundary states, this means that we have two for each superconformal primary. For the untwisted B-type boundary conditions these solutions exist upon choosing the charge conjugation invariant partition function, while for the A-type boundary conditions, since the mirror map maps a representation in its charge conjugate, we need to pick the diagonal modular invariant.

A.3.1 Minimal Model Boundary States, Another Perspective

Perhaps a simpler way to study superconformal boundary states directly in the minimal model case is to employ the separation in half-families that proved useful for modular invariance. These half-families are labelled by $(a, c) \in Q_k$ and can be thought of as representations of the bosonic subalgebra. This subalgebra is really a subalgebra of the universal enveloping algebra and is obtained keeping all generators with even fermion number. This also includes products of an even number of fermion generators. In this set-up the study of conformal boundaries goes as in the bosonic case. To each of the subrepresentations $\mathbb{H}_{ac}$ we associate an Ishibashi state $|\mathcal{B}_{(a,c)}\rangle\rangle$ with components in $\mathbb{H}_{(a,c)} \oplus \mathbb{H}_{\omega((a^+,c^+))}$ such that

$$
\langle\langle \mathcal{B}_{(a',c')}| q^{L_0 - \frac{c}{24}} |\mathcal{B}_{(a,c)}\rangle\rangle = \delta_{a,a'}\delta_{c,c'}\chi_{ac}(q)\,. \tag{A.3.21}
$$

Then we expand the annulus partition function as

$$
Z_{\alpha\beta}(q) = \sum_{(a,c)\in Q_k} n^{ac}_{\alpha\beta}\chi_{ac}(q) \tag{A.3.22}
$$

while the boundary states as

$$
|B_\alpha\rangle = \sum_{(a,c)\in Q^\Omega_k} B^{ac}_\alpha |\mathcal{B}_{(a,c)}\rangle\rangle\,. \tag{A.3.23}
$$

The Cardy condition now simply reads

$$
n^{a'c'}_{\alpha\beta} = \sum_{(a,c)\in Q^\Omega_k} B^{ac}_\alpha B^{ac}_\beta S_{ac,a'c'} \tag{A.3.24}
$$

and, if $Q^\Omega_k = Q_k$, is solved by the Cardy states

$$
|B_{\alpha_1\alpha_2}\rangle = \sum_{(a,c)\in Q_k} \frac{S_{\alpha_1\alpha_2;ac}}{\sqrt{S_{00;ac}}} |\mathcal{B}_{ac}\rangle\rangle \tag{A.3.25}
$$

with multiplicities

$$
n^{ac}_{\alpha_1\alpha_2;\beta_1\beta_2} = N^{a^+c^+}_{\alpha_1\alpha_2;\beta_1\beta_2}\,. \tag{A.3.26}
$$

We can check that those boundary states are the same ones we found in the previous subsection, just written in another basis. First notice that, consistently, there is a boundary state for each half-family, since there are two of these for each superconformal primary the total number of boundary states agrees with what we found above. To match with the previous notation we have to divide Q_k in P'_k and the set of images under $P'_k \ni (a,c) \mapsto (k-a, c+k+2)$, we then define two families of boundary states

$$
\begin{aligned}
|X_{\alpha_1\alpha_2}\rangle &= \sum_{(a,c)\in P'_k} B^{ac}_{\alpha_1\alpha_2}\left(|\mathcal{B}_{ac}\rangle\rangle + (-1)^{a+c}|\mathcal{B}_{k-a,c+k+2}\rangle\rangle\right) \\
|Y_{\alpha_1\alpha_2}\rangle &= \sum_{(a,c)\in P'_k} (-1)^{a+c} B^{ac}_{\alpha_1\alpha_2}\left(|\mathcal{B}_{ac}\rangle\rangle + (-1)^{\alpha_1+\alpha_2}|\mathcal{B}_{k-a,c+k+2}\rangle\rangle\right)
\end{aligned}
\tag{A.3.27}
$$

where now $(\alpha_1, \alpha_2) \in P'_k$. Passing to the (l, m, λ) labeling it is now straightforward to verify that

$$
\begin{aligned}
|X_{lm0}\rangle &= |B_{lm;1,+}\rangle \qquad |X_{lm-1/2}\rangle = |B_{lm;1,-}\rangle \\
|Y_{lm0}\rangle &= |B_{lm;-1,+}\rangle \qquad |Y_{lm-1/2}\rangle = |B_{lm;-1,-}\rangle\,.
\end{aligned}
\tag{A.3.28}
$$

A.3.2 Folding Trick and Topological Lines

From a conformal boundary condition we can construct a topological defect line via the folding trick. A TDL is a topological interface between the CFT and itself, the topologicity condition being encoded in the vanishing commutator with both the holomorphic and anti-holomorphic energy momentum tensors

$$
[T(z), L] = [\bar{T}(\bar{z}), L] = 0 \tag{A.3.29}
$$

the operator L is supported on a closed curve and depends only on its homotopy class (which accounts also for other operator insertions). Locally a topological defect can always be interpreted as a topological interface separating two copies of the same CFT. Of course among the possible interfaces there's also the trivial one, for which the theories on the two sides are completely decoupled. The folding trick consists in folding the theory along the interface determined by the TDL, so that the defect is mapped to a boundary condition for the doubled theory CFT $\times$ $\overline{\text{CFT}}$. The bar in $\overline{\text{CFT}}$ represents the fact that folding acts by parity on the CFT and exchanges the holomorphic and anti-holomorphic sectors. Among the boundaries of the doubled CFT there are also those that are a direct product of a boundary for CFT and one for $\overline{\text{CFT}}$, these do not glue the stress energy tensor of CFT with that of $\overline{\text{CFT}}$ and hence, upon unfolding, the theories on the two sides of the interface are decoupled. Therefore non-trivial topological defect lines correspond to boundary conditions for the doubled theory that mix the two stress energy tensors, these are called permutation boundaries. In a generic CFT classifying TDL's is then as hard as classifying boundary conditions, the problem becomes tractable only in RCFTs requiring the boundary conditions to preserve the full chiral algebra. In this part of the appendix we shall consider a generic bosonic RCFT, and only later specify to the bosonic subalgebra of the $\mathcal{N} = 2$ superconformal symmetry.

TDLs from Untwisted Boundary Conditions

The TDLs we study will commute not only with Virasoro but also with all the other generators of the chiral algebra W^i

$$[W^i(z), L] = [\overline{W}^i(\bar{z}), L] = 0\,. \tag{A.3.30}$$

In the doubled theory these commutation relations become equations defining the permutation boundary states, indeed

$$\left(W^{i\,1}_n - (-1)^{h_i}\overline{W}^{i\,2}_{-n}\right)|B\rangle = \left(W^{i\,2}_n - (-1)^{h_i}\overline{W}^{i\,1}_{-n}\right)|B\rangle = 0 \tag{A.3.31}$$

where $1, 2$ refer to the two copies of the CFT and the overline for the anti-holomorphic sector. For instance the modes of the two stress energy tensors satisfy

$$\left(L^1_n - \overline{L}^2_{-n}\right)|B\rangle = \left(L^2_n - \overline{L}^1_{-n}\right)|B\rangle = 0 \tag{A.3.32}$$

or, in the open channel,

$$T^1(z) = \overline{T}^2(\bar{z}) \qquad T^2(z) = \overline{T}^1(\bar{z}) \quad \text{at} \quad z = \bar{z}\,. \tag{A.3.33}$$

For concreteness we work on the upper half plane, then unfolding the theory we identify T^1 and $\overline{T}^1$ with the holomorphic and anti-holomorphic tensors in the upper half plane while T^2 and $\overline{T}^2$ with the anti-holomorphic and holomorphic components of the energy momentum tensor on the lower half plane. In formulas

$$\begin{aligned} T^1(z) &= T(z) \qquad & \overline{T}^1(\bar{z}) &= \overline{T}(\bar{z}) \quad \mathrm{Im}(z) > 0 \\ T^2(z) &= \overline{T}(\bar{z}) \qquad & \overline{T}^2(\bar{z}) &= T(z) \quad \mathrm{Im}(z) < 0 \end{aligned} \tag{A.3.34}$$

then the gluing conditions ensure that both $T(z)$ and $\overline{T}(\bar{z})$ are continuous across the defect. Now, suppose we have chosen a certain modular invariant Hilbert space for the CFT

$$\mathbb{H} = \bigoplus_{i,j} M_{ij}\mathbb{H}_i \otimes \overline{\mathbb{H}}_j \tag{A.3.35}$$

in the doubled theory the total Hilbert space is then

$$\mathbb{H}^{(tot)} = \bigoplus_{i,j,k,l} M_{ij}M_{k,l}\mathbb{H}_i \otimes \overline{\mathbb{H}}_j \otimes \mathbb{H}_l \otimes \overline{\mathbb{H}}_k\,. \tag{A.3.36}$$

Again we use Ishibashi states $|\mathcal{B}_{ij}\rangle\rangle$ to express physical boundary states, these are now labelled by pairs of representations of the Chiral algebra $\mathcal{A}$ and have components in $\mathbb{H}_i \otimes \overline{\mathbb{H}}_{i^+} \otimes \mathbb{H}_j \otimes \overline{\mathbb{H}}_{j^+}$. Notice that the equations defining permutation boundary states require the representation content of the states to be equal for CFT_1 and CFT_2. We can see this from the fact that a generic physical boundary state

$$|B_{\alpha\beta}\rangle = \sum_{ij} B^{ij}_{\alpha\beta}|\mathcal{B}_{ij}\rangle\rangle \tag{A.3.37}$$

can satisfy both (A.3.31) only if

$$B^{ij}_{\alpha\beta} = \delta^{ij}B^i_{\alpha\beta}\,. \tag{A.3.38}$$

That is permutation boundary states are expanded only using the (available) diagonal Ishibashi states

$$|B_\alpha\rangle = \sum_i B^i_\alpha |\mathcal{B}_{ii}\rangle\rangle\,. \tag{A.3.39}$$

The overlap in the doubled theory is not difficult to compute. On the cylinder we have, for any boundary state satisfying (A.3.31)

$$\langle B|e^{-\frac{2\pi}{t}\left(L^1_0+L^2_0+\overline{L}^1_0+\overline{L}^2_0-\frac{c}{6}\right)}|B\rangle = \langle B|e^{-\frac{4\pi}{t}\left(L_0+\overline{L}_0-\frac{c}{12}\right)}|B\rangle = \langle B|q^{L_0-\frac{c}{24}}\bar{q}^{\overline{L_0}-\frac{c}{24}}|B\rangle \tag{A.3.40}$$

where we noticed that the central charge doubles in the folded theory and we used the boundary conditions. Then

$$\langle\langle \mathcal{B}_{ii}|q^{L_0-\frac{c}{24}}\bar{q}^{\overline{L_0}-\frac{c}{24}}|\mathcal{B}_{jj}\rangle\rangle = \delta_{ij}\chi_i(q)\chi_{i^+}(\bar{q})\,. \tag{A.3.41}$$

In the open channel, since the boundary conditions preserve the $\mathcal{A}\times\mathcal{A}$ symmetry, we expand the partition function as

$$\mathrm{Tr}_{\mathbb{H}_{\alpha\beta}}\tilde{q}^{L_0-\frac{c}{24}}\bar{\tilde{q}}^{\overline{L_0}-\frac{c}{24}} = \sum_{i,j} n^{ij}_{\alpha\beta}\chi_i(\tilde{q})\chi_j(\bar{\tilde{q}})\,. \tag{A.3.42}$$

Then Cardy's condition reads

$$\sum_i B^i_\alpha B^i_\beta S_{ij}S_{jk} = n^{jk}_{\alpha\beta}\,. \tag{A.3.43}$$

Since the sum over i runs over the available diagonal Ishibashi states it is difficult to provide a general solution. However when all Ishibashi states can be used we can set

$$B^i_\alpha = \frac{S_{i\alpha}}{S_{0i}} \tag{A.3.44}$$

using that the ratios S_{ai}/S_{0i} furnish a 1-dimensional representation of the fusion algebra togheter with the Verlinde formula it is not difficult to show that the resulting multiplicities in the open channel are integers. Another way of getting to this formula directly in the unfolded theory with a charge conjugation invariant HIlbert space is to notice that, since L commutes with all generators of $\mathcal{A}$ it must be proportional to the identity in every subspace $\mathbb{H}_i\otimes\mathbb{H}_{i^+}$ of the Hilbert space [191]. Therefore it can be written as a sum of projectors

$$L_\alpha = \sum_i B^i_\alpha \sum_{\mathbf{m},\mathbf{n}} |i,\mathbf{m};i^+,\mathbf{n}\rangle\otimes\langle i,\mathbf{m};i^+,\mathbf{n}| \tag{A.3.45}$$

where the sums over $\mathbf{m}$ and $\mathbf{n}$ run over descendants. Requiring that the Hilbert space twisted by L has a consistent Hilbert space interpretation one again gets (A.3.44). Clearly the coefficients B^i_α tell us how the TDL acts on local operators. We also see that the number of independent TDLs matches that of physical permutation boundaries in the doubled theory.

Untwisted case: minimal models .

This analysis carries over directly to the minimal model case with charge conjugation invariant partition function using the half-family basis. With this choice of partition function we have a physical B-type boundary state for each half-family. In the doubled theory we have a permutation Ishibashi state for each half-family from which we construct a TDL for each half family. On the circle Hilbert space of the unfolded theory this acts as the sum of projectors

$$
\begin{aligned}
L_{a_1 a_2} &= \sum_{(a,c)\in Q_k} \frac{S_{a_1,a_2;ac}}{S_{00;ac}} \sum_{\mathbf{m},\mathbf{n}} |(a,c),\mathbf{m};(a^+,c^+),\mathbf{n}\rangle \otimes \langle (a,c),\mathbf{m};(a^+,c^+),\mathbf{n}| \\
&= \sum_{(a,c)\in Q_k} \frac{S_{a_1,a_2;ac}}{S_{00;ac}} P_{ac}\,.
\end{aligned}
\tag{A.3.46}
$$

The primary states (of the bosonic subalgebra) appearing in the torus partition function can be labelled by a single half-family $|(a,c)\rangle \otimes |(a^+,c^+)\rangle = |\Phi_{ac}\rangle$ on which the TDLs act as

$$
L_{a_1,a_2}|\Phi_{ac}\rangle = \frac{S_{a_1,a_2;ac}}{S_{00;ac}}|\Phi_{ac}\rangle\,. \tag{A.3.47}
$$

We can work out the action on the full superconformal families simply changing basis. We have

$$
L_{a_1 a_2} = \sum_{(a,c)\in P'_k} \frac{S_{a_1,a_2;ac}}{S_{00;ac}} \left(P_{ac} + (-1)^{a_1+a_2} P_{k-a;c+k+2}\right) \tag{A.3.48}
$$

notice that the sum $P_{ac}+(-1)^{a_1+a_2}P_{k-a;c+k+2}$ projects on the full superconformal family labelled by $(a,c)\in P'_k$ if $[a_1+a_2]=0$, while if $[a_1+a_2]=1$ it still projects on the same superconformal family but with a minus sign for the states with odd fermion number. Reintroducing the (l,m,λ) parametrization we find the four types of TDLs

$$
\begin{aligned}
L_{lm;+,f} &= \sum_{(l',m')\in P_k} \left(\frac{S^{\text{NSNS}}_{lm;l'm'}}{S^{\text{NSNS}}_{00;l'm'}} P^{(\text{NS})}_{lm} + f\, \frac{S^{\widetilde{\text{NS}}\text{R}}_{lm;l'm'}}{S^{\text{NSNS}}_{00;l'm'}} P^{(\text{R})}_{lm} \right) \\
L_{lm;-,f} &= \sum_{(l',m')\in P_k} \left(\frac{S^{\text{R}\widetilde{\text{NS}}}_{lm;l'm'}}{S^{\text{R}\widetilde{\text{NS}}}_{00;l'm'}} \widetilde{P}^{(\text{NS})}_{lm} + f\, \frac{S^{\widetilde{\text{R}}\widetilde{\text{R}}}_{lm;l'm'}}{S^{\text{R}\widetilde{\text{NS}}}_{00;l'm'}} \widetilde{P}^{(\text{R})}_{lm} \right)
\end{aligned}
\tag{A.3.49}
$$

where $f=\pm 1$ and $P^{(X)}_{lm}$ and $\widetilde{P}^{(X)}_{lm}$ project on the (l,m) superconformal family in the X sector without or with signs for the odd fermion number states.

TDLs from Twisted Boundary Conditions

In general we have also twisted boundary conditions for the CFT which also induce permutation boundary states for the doubled theory

$$
\left(W^{i\,1}_n - (-1)^{h_i}\Omega\left(\overline{W}^{i\,2}_{-n}\right)\right)|B\rangle = \left(W^{i\,2}_n - (-1)^{h_i}\Omega\left(\overline{W}^{i\,1}_{-n}\right)\right)|B\rangle = 0\,. \tag{A.3.50}
$$

These corresponding TDLs in the unfolded theory commute with the generators of the chiral algebra only up to the automorphism Ω

$$
L_\Omega W(z) = \Omega(W(z))L_\Omega \tag{A.3.51}
$$

and similarly for the anti-holomorphic side. The Ishibashi states $|\mathcal{B}_{ij}\rangle_\Omega$ now have components in $\mathbb{H}_i\otimes\mathbb{H}_{\omega(i^+)}\otimes\mathbb{H}_j\otimes\mathbb{H}_{\omega(j^+)}$, and again only the diagonal ones with $i=j$ contribute to permutation boundaries. The analysis of the physical boundary states parallels that of the untwisted case as long as we use the twisted Ishibashi states and the solution

$$B^i_\alpha = \frac{S_{i\alpha}}{S_{0i}} \tag{A.3.52}$$

is available as long as we are allowed to use all Ishibashi states, that is $M_{ij} = \delta_{j,\omega(i^+)}$. The expression for the TDL as an operator on the Hilbert space of the unfolded theory is now

$$L_{\Omega,\alpha} = \sum_i \frac{S_{i\alpha}}{S_{0i}} \sum_{\mathbf{m},\mathbf{n}} |i, \mathbf{m}; \omega(i^+), \mathbf{n}\rangle \otimes \langle i, \mathbf{m}; \omega(i^+), \mathbf{n}| \tag{A.3.53}$$

Twisted case: minimal models We now pick a diagonal modular invariant partition function so that we have a physical A-type boundary state for each half-family, the corresponding topological lines are

$$\begin{aligned} L_{a_1 a_2} &= \sum_{(a,c)\in Q_k} \frac{S_{a_1,a_2;ac}}{S_{00;ac}} \sum_{\mathbf{m},\mathbf{n}} |(a,c), \mathbf{m}; (a,c), \mathbf{n}\rangle \otimes \langle (a,c), \mathbf{m}; (a,c), \mathbf{n}| \\ &= \sum_{(a,c)\in Q_k} \frac{S_{a_1,a_2;ac}}{S_{00;ac}} P_{ac} \,. \end{aligned} \tag{A.3.54}$$

In terms of the superconformal families we find again the expressions (A.3.49) where all projectors are now diagonal rather than charge conjugation invariant.

A.4 Toy model examples

In this appendix we test some formulas of the main text in simple solvable examples. We first discuss linear random couplings in free scalar theories and establish the validity of the generalized Ward identity (3.2.41) for 2-point functions both for the case of $h(x)$ (quenched disorder) and constant h (ensemble average). Subsequently we test the 't Hooft anomaly matching condition discussed in section 3.2.1 by working out a specific example.

A.4.1 Free scalar theories

We consider the toy example of a complex free scalar perturbed by a linear random coupling. The action is

$$S = \int d^d x \Big(|\partial\phi|^2 + m^2 |\phi|^2 + h\phi(x) + \bar{h}\bar{\phi}(x) \Big) \,. \tag{A.4.1}$$

The coupling to h explicitly breaks the $U(1)$ symmetry rotating ϕ. Here h can have or not a space dependence. In both cases we can write

$$Z[K, \bar{K}, h] = \exp\Big(\int d^d x d^d y (\bar{h} + \bar{K}(x)) G(x-y)(h + K(y)) \Big) \,, \tag{A.4.2}$$

where $G(x-y)$ is the massive scalar propagator in flat space and K, $\bar{K}$ are the external sources for ϕ and $\bar{\phi}$, respectively. We consider a Gaussian distribution with variance v and zero mean in order to simplify the expressions. In what follows we shall be sloppy with normalizations and overall constants which do not affect the main points we want to show.

Quenched disorder

It is convenient to introduce a compact notation

$$\begin{aligned}(hG)_x &:= \int d^d w\, h(w)G(w-x)\,, \qquad & (G\bar{h})_y &:= \int d^d w\, G(y-w)\bar{h}(w)\,, \\ G_{xy} &:= G(x-y)\,, & (GG)_{xy} &:= \int d^d w\, G(x-w)G(w-y)\,,\end{aligned} \tag{A.4.3}$$

so that, from (A.4.2), we get the one-point function

$$\langle \phi(x)\rangle = Z^{-1}\frac{\delta Z}{\delta K(x)}\bigg|_{K=0} = (G\bar{h})_x\,. \tag{A.4.4}$$

Since translation invariance is broken, this is not a constant. Similarly, for two point functions,

$$\begin{aligned}\langle \phi(x)\phi(y)\rangle &= Z^{-1}\frac{\delta^2 Z}{\delta K(x)\delta K(y)}\bigg|_{K=0} = (G\bar{h})_x(G\bar{h})_y\,, \\ \langle \bar{\phi}(x)\phi(y)\rangle &= Z^{-1}\frac{\delta^2 Z}{\delta \bar{K}(x)\delta K(y)}\bigg|_{K=0} = G_{xy} + (hG)_x(G\bar{h})_y\,.\end{aligned} \tag{A.4.5}$$

To take the average we simply Wick contract h and $\bar{h}$ with

$$\overline{h(x)\bar{h}(y)} = v\delta^{(d)}(x-y)\,. \tag{A.4.6}$$

Then

$$\overline{\langle \phi(x)\rangle} = \overline{\langle \phi(x)\phi(y)\rangle} = 0\,, \tag{A.4.7}$$

consistently with the $U(1)$ symmetry being recovered on average. The non vanishing two-point function is

$$\overline{\langle \bar{\phi}(x)\phi(y)\rangle} = G_{xy} + v(GG)_{xy}\,. \tag{A.4.8}$$

The explicitly broken Ward identities for a $U(1)$ transformation read

$$\begin{aligned}\langle \partial_\mu J^\mu(x)\phi(y)\bar{\phi}(z)\rangle =& \delta^{(d)}(x-y)\langle \phi(y)\bar{\phi}(z)\rangle - \delta^{(d)}(x-z)\langle \phi(y)\bar{\phi}(z)\rangle \\ & - h(x)\langle \phi(x)\phi(y)\bar{\phi}(z)\rangle + \bar{h}(x)\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle\,.\end{aligned} \tag{A.4.9}$$

The last two correlators equal

$$\begin{aligned}\langle \phi(x)\phi(y)\bar{\phi}(z)\rangle &= G_{xz}(G\bar{h})_y + G_{yz}(G\bar{h})_x + (hG)_z(G\bar{h})_y(G\bar{h})_x\,, \\ \langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle &= G_{xy}(hG)_z + G_{yz}(hG)_x + (hG)_z(G\bar{h})_y(hG)_x\,,\end{aligned} \tag{A.4.10}$$

so that

$$\begin{aligned}\overline{h(x)\langle \phi(x)\phi(y)\bar{\phi}(z)\rangle} &= vG_{xy}G_{xz} + vG_{yz}G(0) + v^2(GG)_{yz}G(0) + v^2(GG)_{xz}G_{xy}\,, \\ \overline{\bar{h}(x)\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle} &= vG_{xy}G_{xz} + vG_{yz}G(0) + v^2(GG)_{yz}G(0) + v^2(GG)_{xy}G_{xz}\,.\end{aligned} \tag{A.4.11}$$

The average of (A.4.9) reads then

$$\begin{aligned}\overline{\langle \partial_\mu J^\mu(x)\phi(y)\bar{\phi}(z)\rangle} =& \delta^{(d)}(x-y)\overline{\langle \phi(y)\bar{\phi}(z)\rangle} - \delta^{(d)}(x-z)\overline{\langle \phi(y)\bar{\phi}(z)\rangle} \\ & - v^2((GG)_{xz}G_{xy} - (GG)_{xy}G_{xz})\,.\end{aligned} \tag{A.4.12}$$

It is straightforward to check (A.4.12) by using the explicit form of $J_\mu = \bar\phi\partial_\mu\phi - \phi\partial_\mu\bar\phi$ and performing the Wick contractions. We can now explicitly check the disordered Ward identity (3.2.41). Using the equations of motion we have $\partial_\mu J^\mu = \big(\bar h(x)\bar\phi(x) - h(x)\phi(x)\big)$, so that

$$\langle \partial^\mu J_\mu(x)\rangle = \int d^dz\,\big(h(z)\bar h(x) - h(x)\bar h(z)\big)\,G_{xz}\,. \tag{A.4.13}$$

Equivalently we can directly compute

$$\langle J_\mu(x)\rangle = \int d^dw d^dz h(z)\bar h(w)\,\big(\partial_\mu^{(x)}G_{xz}G_{xw} - G_{xz}\partial_\mu^{(x)}G_{xw}\big) \tag{A.4.14}$$

and take a derivative. As expected from the recovery of translation invariance after the average we find $\overline{\langle\partial^\mu J_\mu\rangle} = 0$. However, due to the presence of h, inserting $\langle\partial^\mu J_\mu\rangle$ under the average modifies the correlators, in particular

$$\begin{aligned}\overline{\langle\partial^\mu J_\mu(x)\rangle\langle\phi(y)\bar\phi(z)\rangle} &= \int d^dw G_{xw}\overline{\big(h(w)\bar h(x) - h(x)\bar h(w)\big)\,\langle\phi(y)\bar\phi(z)\rangle}\\ &= -v^2\left((GG)_{xz}G_{xy} - (GG)_{xy}G_{xz}\right)\,.\end{aligned} \tag{A.4.15}$$

This precisely corresponds to the last term in the right hand side of (A.4.12). Therefore, using the improved current $\widetilde{J}_\mu := J_\mu - \langle J_\mu\rangle$, the Ward identity (A.4.12) becomes

$$\overline{\langle\partial_\mu\widetilde{J}^\mu(x)\phi(y)\bar\phi(z)\rangle} = \delta^{(d)}(x-y)\overline{\langle\phi(y)\bar\phi(z)\rangle} - \delta^{(d)}(x-z)\overline{\langle\phi(y)\bar\phi(z)\rangle}\,, \tag{A.4.16}$$

in agreement with (3.2.41) with $k = 2$ operators. From here one can reproduce the exponentiation procedure and determine the presence of a topological operator in the disordered theory.

Ensemble Average

When h is a constant every member of the ensemble is translation invariant. Indeed the one point function of the scalar field is now a constant:

$$\langle\phi(x)\rangle = \bar h\int d^dy\,G_{xy} = \frac{\bar h}{m^2}\,. \tag{A.4.17}$$

Note that the mass acts as a IR regulator. The two point functions are

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle &= \bar h^2\int d^dzd^dw\,G_{xz}G_{yw} = \frac{\bar h^2}{m^4}\,,\\ \langle\bar\phi(x)\phi(y)\rangle &= G_{xy} + |h|^2\int d^dzd^dw\,G_{xz}G_{yw} = G_{xy} + \frac{|h|^2}{m^4}\,.\end{aligned} \tag{A.4.18}$$

In agreement with the $U(1)$ average symmetry, the only non-vanishing average two point function is

$$\overline{\langle\bar\phi(x)\phi(y)\rangle} = G_{xy} + \frac{v}{m^4}\,. \tag{A.4.19}$$

The explicitly broken Ward identities are

$$\begin{aligned}\langle\partial_\mu J^\mu(x)\phi(y)\bar\phi(z)\rangle =&\delta^{(d)}(x-y)\langle\phi(y)\bar\phi(z)\rangle - \delta^{(d)}(x-z)\langle\phi(y)\bar\phi(z)\rangle\\ &- h\langle\phi(x)\phi(y)\bar\phi(z)\rangle + \bar h\langle\bar\phi(x)\phi(y)\bar\phi(z)\rangle\,.\end{aligned} \tag{A.4.20}$$

The operator

$$\partial^\mu J_\mu(x) + h\phi(x) - \bar{h}\bar{\phi}(x) \tag{A.4.21}$$

generates the Ward identities, and we can now explicitly check that it integrates to zero on the whole space. The left hand side of (A.4.20) vanishes when integrating x over the whole space. For the last two terms in the right hand side we get

$$\begin{aligned} \langle \phi(x)\phi(y)\bar{\phi}(z)\rangle &= \frac{\bar{h}}{m^2}\left(G_{xz}+G_{yz}\right)+\frac{h\bar{h}^2}{m^6}\,,\\ \langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle &= \frac{h}{m^2}\left(G_{xy}+G_{yz}\right)+\frac{\bar{h}h^2}{m^6}\,, \end{aligned} \tag{A.4.22}$$

so that

$$\overline{h\langle \phi(x)\phi(y)\bar{\phi}(z)\rangle} - \overline{\bar{h}\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle} = \frac{v}{m^2}\left(G_{xz}-G_{xy}\right)\,. \tag{A.4.23}$$

Then, by translation invariance, we have

$$\int d^dx \left(\overline{h\langle \phi(x)\phi(y)\bar{\phi}(z)\rangle} - \overline{\bar{h}\langle \bar{\phi}(x)\phi(y)\bar{\phi}(z)\rangle}\right) = \frac{v}{m^2}\int d^dx\left(G_{xz}-G_{xy}\right)=0\,, \tag{A.4.24}$$

where the support of the integral needs to be the entire space. In this simple example we have chosen a scalar deformation so that Poincaré invariance remains always unbroken, no tensor operator can get a vev, and all complications arising from non-vanishing vevs disappear. For example, specifying (A.4.13) to the case of constant h immediately gives $\langle \partial_\mu J^\mu\rangle = 0$.

We can also compute $\langle \bar{\phi}(x_1)\phi(x_2)\rangle$ when X is a disconnected space. For example, if $X^{(d)} = X_1^{(d)} \sqcup X_2^{(d)}$, $x_1 \in X_1^{(d)}$ an $x_2 \in X_2^{(d)}$, (A.4.2) reads

$$Z[K_{1,2},\bar{K}_{1,2},h] = \exp\Big(\sum_{i=1,2}\int_{X_i^{(d)}} d^dx_i d^dy_i (\bar{h}+\bar{K}_i(x_i))G(x_i-y_i)(h+K_i(y_i))\Big)\,, \tag{A.4.25}$$

and we get

$$\langle \bar{\phi}(x_1)\phi(x_2)\rangle_X = Z^{-1}\frac{\delta^2 Z}{\delta\bar{K}_1(x_1)\delta K_2(x_2)}\bigg|_{K=0} = \langle\bar{\phi}(x_1)\rangle_{X_1}\langle\phi(x_2)\rangle_{X_2} = \frac{|h|^2}{m^4}\,, \tag{A.4.26}$$

namely only the disconnected part of the correlator contributes. Averaging on h we have

$$\overline{\langle \bar{\phi}(x_1)\phi(x_2)\rangle}_X = \overline{\langle\bar{\phi}(x_1)\rangle_{X_1}\langle\phi(x_2)\rangle}_{X_2} = \frac{v}{m^4}\,. \tag{A.4.27}$$

We explicitly see that in both X_1 and X_2 the $U(1)$ symmetry is explicitly broken and conserved only globally over the entire space X.

A.4.2 't Hooft anomalies from replicas

We check the matching of t'Hooft anomalies between the pure and disordered theory in the simple example of the $U(1)$ chiral anomaly in $4d$. As well-known, a free massless Weyl fermion ψ in $4d$ suffers from a cubic 't Hooft anomaly, which in momentum space reads

$$p_1^\mu\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle = i\frac{k}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}\,p_2^\alpha\,p_3^\beta\,, \tag{A.4.28}$$

where $k=1$. We deform the theory with a space dependent complex mass term $m(x)$, which explicitly breaks the $U(1)$ symmetry down to fermion parity. However, if we sample $m(x)$ from a Gaussian distribution proportional to $\bar{m}(x)m(x)$, then the disordered theory recovers the $U(1)$ symmetry via the conserved current $\widetilde{J}_\mu$. Since $\langle \widetilde{J}_\mu \rangle = 0$ before averaging, we have

$$\overline{\langle \widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle} = \overline{\langle \widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle_c} = \overline{\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle_c}\,. \tag{A.4.29}$$

The last three-point function is most easily evaluated using the replica trick. The replicated theory has n Weyl fermions with a quartic deformation (spinor indices omitted)

$$S_{\rm rep} = \sum_{a=1}^{n} S_{0,a} + v^2 \sum_{a,b} \psi_a\psi_a\overline{\psi}_b\overline{\psi}_b\,, \tag{A.4.30}$$

which is invariant under the diagonal $U(1)_D$ symmetry, with conserved current

$$J^\mu_D = \sum_a J^\mu_a\,. \tag{A.4.31}$$

According to (3.2.79), we have

$$\overline{\langle J_\mu(p_1)J_\nu(p_2)J_\rho(p_3)\rangle_c} = \lim_{n\to 0}\frac{\partial}{\partial n}\langle J_{D,\mu}(p_1)J_{D,\nu}(p_2)J_{D,\rho}(p_3)\rangle^{\rm rep}\,. \tag{A.4.32}$$

The $U(1)_D$ in the replica theory also suffers from a a cubic 't Hooft anomaly

$$p_1^\mu\langle J_{D,\mu}(p_1)J_{D,\nu}(p_2)J_{D,\rho}(p_3)\rangle_{\rm rep} = \frac{ik}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta\,, \tag{A.4.33}$$

where $k=n$, since all n fermions rotate (with the same charge) under $U(1)_D$. We then get

$$p_1^\mu\overline{\langle \widetilde{J}_\mu(p_1)\widetilde{J}_\nu(p_2)\widetilde{J}_\rho(p_3)\rangle} = \lim_{n\to 0}\frac{\partial}{\partial n}\left(\frac{in}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta\right) = \frac{i}{16\pi^3}\epsilon_{\nu\rho\alpha\beta}p_2^\alpha p_3^\beta\,, \tag{A.4.34}$$

which shows that the anomaly of the pure theory persists after the quenched average and also affects the disordered symmetry, in agreement with the results in the main text.

A.5 Symmetry operators for averaged symmetries

In this appendix we prove the existence, and explicitly construct, an operator $\widehat{U}_g$ which implements the action of the group rather than the action of the corresponding Lie algebra for average symmetries. To this purpose we need to find an infinite set of operators $\widehat{Q}_n$ which have the same properties of $\widehat{Q}$ defined in (3.2.134) and which satisfy the identities

$$\langle \widehat{Q}_n\mathcal{O}_1\cdots\mathcal{O}_k\rangle = \chi^n(\Sigma^{(d-1)})\langle \mathcal{O}_1\cdots\mathcal{O}_k\rangle\,, \qquad \forall n\in\mathbb{N}\,, \tag{A.5.1}$$

where we recall that $\chi(\Sigma^{(d-1)})$ denotes the sum of the charges of the local operators which are inside the surface $\Sigma^{(d-1)}$. Note that (A.5.1) applies before ensemble averaging. We define $\widehat{Q}_0 = 1$ and $\widehat{Q}_1 = \widehat{Q}$. We find $\widehat{Q}_n[\Sigma^{(d-1)}, D^{(d)}]$ for $n>1$ iteratively. Suppose that there exists an operator $\widehat{Q}_{n-1}$ such that

$$\langle \widehat{Q}_{n-1}\Phi\rangle = \chi^{n-1}(\Sigma^{(d-1)})\langle\Phi\rangle \tag{A.5.2}$$

for any product of local operators Φ. We then compute

$$\begin{aligned}\langle \widehat{Q}_{n-1}\widehat{Q}_1\Phi\rangle &= \chi^{n-1}\langle Q\Phi\rangle + q_0\,(\chi+q_0)^{n-1}\,\langle h\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle - q_0\,(\chi-q_0)^{n-1}\,\langle \overline{h}\int_{D^{(d)}}\overline{\mathcal{O}_0}(x)\Phi\rangle \\ &= \chi^n\overline{\langle\Phi\rangle} + q_0\sum_{k=0}^{n-2}\binom{n-1}{k}\chi^k q_0^{n-1-k}\Big(\langle h\int_{D^{(d)}}\mathcal{O}_0(x)\Phi\rangle - (-1)^{n-k-1}\langle\overline{h}\int_{D^{(d)}}\overline{\mathcal{O}_0}(x)\Phi\rangle\Big)\,.\end{aligned} \tag{A.5.3}$$

Next we introduce operators Γ_l defined in such a way that

$$\langle \Gamma_l\int_{D^{(d)}}\,\mathcal{O}_0(x)\Phi\rangle = \chi^l\langle\int_{D^{(d)}}\,\mathcal{O}_0(x)\Phi\rangle\,. \tag{A.5.4}$$

Their existence follows from the (by now assumed) existence of the operators $\widehat{Q}_n$. In fact, it is easy to see that the Γ_l's satisfy the relation

$$\widehat{Q}_l = \sum_{s=0}^{l}\binom{l}{s}q_0^{l-s}\Gamma_s\,, \tag{A.5.5}$$

valid when inserted in (vacuum to vacuum) correlators of the form $\langle\int_{D^{(d)}}\,\mathcal{O}_0(x)\Phi\rangle$. Now consider the vectors $\widehat{\mathbf{Q}} = (\widehat{Q}_0,\widehat{Q}_1,\cdots,\widehat{Q}_N)$ and $\mathbf{\Gamma} = (\Gamma_0,\Gamma_1,\cdots,\Gamma_N)$, with $\Gamma_0 = 1$. These are related as $\widehat{\mathbf{Q}} = A\cdot\mathbf{\Gamma}$ where $A = \mathbb{1}+T$ and T is a strictly lower triangular matrix with non-vanishing entries

$$T_{l,s} = \binom{l}{s}q_0^{l-s}\,. \tag{A.5.6}$$

We can invert (A.5.5) as

$$\Gamma_l = \sum_{s=0}^{l}A^{-1}_{l,s}\widehat{Q}_s\,, \tag{A.5.7}$$

where we used that

$$A^{-1} = \mathbb{1}+\sum_{i=1}^{N}(-1)^iT^i \tag{A.5.8}$$

is again a lower triangular matrix. An analogous analysis can be carried out for the operators $\overline{\Gamma}_l$ defined by

$$\langle\overline{\Gamma}_l\overline{h}\int_{D^{(d)}}\,\overline{\mathcal{O}_0}(x)\Phi\rangle = \chi^l\langle\overline{h}\int_{D^{(d)}}\,\overline{\mathcal{O}}_0(x)\Phi\rangle\,, \tag{A.5.9}$$

by simply replacing q_0 with $-q_0$, and we define $\overline{A}$ as $\widehat{\mathbf{Q}} = \overline{A} \cdot \overline{\mathbf{\Gamma}}$. We rewrite (A.5.3) as

$$\begin{aligned}
&\langle \widehat{Q}_{n-1}\widehat{Q}_1\Phi\rangle = \langle \widehat{Q}_n\Phi\rangle \\
&+ q_0 \sum_{k=0}^{n-2} \binom{n-1}{k} q_0^{n-1-k} \left(\langle \Gamma_k h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - (-1)^{n-1-k} \langle \overline{\Gamma}_k \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right) \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \Bigg[\langle \widehat{Q}_{n-1} \left(h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x) \right) \Phi\rangle + \\
&+ \langle \Gamma_{n-1} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{\Gamma}_{n-1} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \Bigg] \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \Bigg[\langle \widehat{Q}_{n-1} \left(h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x) \right) \Phi\rangle + \\
&+ \sum_{k=0}^{n-1} \widehat{Q}_k \left(\langle A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right) \Bigg] \\
&= \langle \widehat{Q}_n\Phi\rangle + q_0 \sum_{k=0}^{n-2} \widehat{Q}_k \left[\langle A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x)\Phi\rangle - \langle \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0}(x)\Phi\rangle \right],
\end{aligned} \tag{A.5.10}$$

where we used that $A^{-1}_{k,k} = \overline{A}^{-1}_{k,k} = 1$. We then find the recursion relation

$$\widehat{Q}_n = \widehat{Q}_{n-1}\widehat{Q}_1 - q_0 \sum_{k=0}^{n-2} \widehat{Q}_k \left(A^{-1}_{n-1,k} h \int_{D^{(d)}} \mathcal{O}_0(x) - \overline{A}^{-1}_{n-1,k} \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right), \tag{A.5.11}$$

which proves the existence of $\widehat{Q}_n$ for every values of $n \in \mathbb{N}$.
As an example consider $N = 3$. We have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_0 & 1 & 0 & 0 \\ q_0^2 & 2q_0 & 1 & 0 \\ q_0^3 & 3q_0^2 & 3q_0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q_0 & 1 & 0 & 0 \\ q_0^2 & -2q_0 & 1 & 0 \\ -q_0^3 & 3q_0^2 & -3q_0 & 1 \end{pmatrix}, \tag{A.5.12}$$

and

$$\begin{aligned}
\widehat{Q}_2 &= \widehat{Q}_1^2 + q_0^2 \left(h \int_{D^{(d)}} \mathcal{O}_0(x) + \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right), \\
\widehat{Q}_3 &= \widehat{Q}_2\widehat{Q}_1 - q_0^3 \int_{D^{(d)}} \mathcal{D}(x) - 2q_0^2 \widehat{Q}_1 \left(h \int_{D^{(d)}} \mathcal{O}_0 + \overline{h} \int_{D^{(d)}} \overline{\mathcal{O}_0} \right).
\end{aligned} \tag{A.5.13}$$

We now crucially verify that the charges $\widehat{Q}_n$ vanish when $D^{(d)} = X^{(d)}$ *after* ensemble average in arbitrary local correlators. For this purpose we derive a further constraint on correlators involving arbitrary functions of h and $\overline{h}$. Consider

$$\int dh d\bar{h}\, P[\bar{h}h] f(h, \overline{h}) \frac{\int \mathcal{D}\mu e^{-S_0 - (h \int \mathcal{O}_0 + c.c.) + \int K_i \mathcal{O}_i}}{\int \mathcal{D}\mu e^{-S_0 - (h \int \mathcal{O}_0 + c.c.)}}, \tag{A.5.14}$$

where f is an arbitrary smooth function of h and $\bar{h}$. We shift $h \to h + \epsilon\delta h$, where $\delta h = -iq_0 h$. Using that $\delta h \mathcal{O}_0 = -h\delta\mathcal{O}_0$ and expanding to linear order in ϵ we get[2]

$$\overline{iq_0 f(h, \overline{h}) \langle \int_{X^{(d)}} \mathcal{D}(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle} = -\overline{\delta f(h, \overline{h}) \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle}, \tag{A.5.15}$$

[2] An extra term coming from the denominator of (A.5.14) vanishes because of (3.2.131).

where

$$\delta f(h,\overline{h}) = \partial f \delta h + \overline{\partial} f \delta \overline{h}\,. \tag{A.5.16}$$

Thanks to (A.5.15) we can now show that

$$\overline{\langle \widehat{Q}_n[\emptyset, X^{(d)}]\Phi\rangle} = 0\,, \qquad n > 0\,. \tag{A.5.17}$$

Let us explicitly work out the $n = 2, 3$ cases. For $n = 2$ it is enough to use (A.5.15) with $f = h$ and $f = \bar{h}$ to get the identity

$$\overline{\langle\Big(\int_{X^{(d)}} \mathcal{D}(x)\Big)^2 \Phi\rangle} + \overline{\langle\Big(h\int_{X^{(d)}} \mathcal{O}_0(x) + \overline{h}\int_{X^{(d)}} \overline{\mathcal{O}_0}\Big)\Phi\rangle} = 0\,. \tag{A.5.18}$$

We can plug this relation into $\widehat{Q}_2$ in (A.5.13) to immediately get (A.5.17) for $n = 2$. For $n = 3$ we use (A.5.15) with the functions $h^2, \overline{h}^2$ and $h\overline{h}$. In this way we get the relations

$$\overline{\langle\Big(\int_{X^{(d)}} \mathcal{D}(x)\Big)^3 \Phi\rangle} = 2\overline{\langle\Big(\int_{X^{(d)}} \mathcal{D}(x)\Big)\Big(h\int_{X^{(d)}} \mathcal{O}_0 + \overline{h}\int_{X^{(d)}} \overline{\mathcal{O}_0}\Big)\Phi\rangle} = 0\,, \tag{A.5.19}$$

which, pluggged in $\widehat{Q}_3$ in (A.5.13) allows us to get (A.5.17) for $n = 3$. We can then construct the non-genuine symmetry operator

$$\widehat{U}_g[\Sigma^{(d-1)}, D^{(d)}] = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \widehat{Q}_n[\Sigma^{(d-1)}, D^{(d)}]\,, \qquad g = e^{i\alpha}\,, \tag{A.5.20}$$

which, similarly to $\widehat{Q}[\Sigma^{(d-1)}, D^{(d)}]$, becomes quasi-genuine when $D^{(d)} = X^{(d)}$.

We have then shown the existence, and explicitly constructed, the operator $\widehat{U}_g$ which implements the selection rules imposed by the emergent symmetries.

Appendix B

Appendices for Chapter 4

B.1 Gauging in fusion categories

In this appendix we briefly review well known material about gauging in fusion categories and modular tensor categories (possibly extended by a 0-form symmetry). A complete review of the underlying formalism can be found for instance in [17] and [28], respectively.

Gauging and algebras. Gauging a generalized symmetry in two dimensions corresponds to the definition of a special symmetric Frobenius algebra $\mathcal{A} \subset \mathcal{C}$. This is described by a triplet:

$$\mathcal{A} \equiv (\mathcal{A},\, m\,, \eta)\,, \qquad m \in \mathrm{Hom}(\mathcal{A} \times \mathcal{A}, \mathcal{A})\,, \qquad \eta \in \mathrm{Hom}(\mathbb{1}, \mathcal{A})\,, \tag{B.1.1}$$

where $\mathcal{A} = \bigoplus_{x_i} Z_i(\mathcal{A})\, x_i$ is an object in $\mathcal{C}$, and we define $Z_i(\mathcal{A}) = \dim\big(\mathrm{Hom}(\mathcal{A}, x_i)\big)$. We use x_i to denote the simple objects in $\mathcal{C}$. The maps π_i are projectors $\pi_i : \mathcal{A} \to x_i$ onto the simple components of $\mathcal{A}$ and can be used to recast the commuting diagrams below as tensor-valued expressions. The algebra morphism m trivializes the associator: $m \circ (m \times \mathrm{id}_{\mathcal{A}}) = m \circ (\mathrm{id}_{\mathcal{A}} \times m)$. Furthermore $m \circ \eta = \mathrm{id}_{\mathcal{A}}$. We will henceforth suppress η for simplicity. The algebra also has a dual structure

$$(\Delta,\, \bar{\eta})\,, \qquad \Delta \in \mathrm{Hom}(\mathcal{A}, \mathcal{A} \times \mathcal{A})\,, \qquad \bar{\eta} \in \mathrm{Hom}(\mathcal{A}, \mathbb{1}) \tag{B.1.2}$$

satisfying $m \circ \Delta = \bar{\eta} \circ \Delta = \mathrm{id}_{\mathcal{A}}$. Furthermore Δ and m satisfy the so-called Frobenius condition, namely that the following diagram commutes, ensuring that crossing moves from any direction can be performed safely:

$$\begin{array}{ccccc} \mathcal{A} \times \mathcal{A} & & \xrightarrow{\Delta \times \mathrm{id}_{\mathcal{A}}} & & \mathcal{A} \times \mathcal{A} \times \mathcal{A} \\ & \searrow^{m} & & & \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}} \times \Delta} & & \mathcal{A} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}} \times m} \\ & & & \searrow^{\Delta} & \\ \mathcal{A} \times \mathcal{A} \times \mathcal{A} & & \xrightarrow{m \times \mathrm{id}_{\mathcal{A}}} & & \mathcal{A} \times \mathcal{A} \end{array} \tag{B.1.3}$$

In three dimensions an algebra must satisfy an additional condition which ensures that it is compatible with the braided structure:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \xrightarrow{b} & \mathcal{A} \times \mathcal{A} \\ & \searrow^{m} & \downarrow{\scriptstyle m} \\ & & \mathcal{A} \end{array} \tag{B.1.4}$$

Such an algebra is called *commutative*. The gauging of a symmetry $\mathcal{A}$ is implemented by inserting a network of $\mathcal{A}$ defects with morphisms m and Δ at three-valent junctions, along a graph that is dual to a triangulation of the spacetime manifold.

To understand the symmetry of the theory after gauging we must introduce the concept of modules. First, let us introduce the category of (left) $\mathcal{A}$-modules $\text{Mod}_{\mathcal{A}}$. Its elements are doublets (M, r_L) with M an object an $r_L \in \text{Hom}(\mathcal{A} \times M, M)$ a morphism allowing the algebra object to end on M. The morphism r_L must satisfy a natural compatibility condition:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} \times M & \xrightarrow{r_L} & \mathcal{A} \times M \\ \big\downarrow m & & \big\downarrow r_L \\ \mathcal{A} \times M & \xrightarrow{r_L} & M \end{array} \tag{B.1.5}$$

This equation allows us to interpret r_L as a sort of representation of the algebra $\mathcal{A}$ on M. Physically the category $\text{Mod}_{\mathcal{A}}$ describes an $\mathcal{A}$-invariant boundary condition. In two dimensions the category describing the symmetry after gauging $\mathcal{A}$ is the bimodule category $\text{Bimod}_{\mathcal{A}-\mathcal{A}}$ of $\mathcal{A}$-bimodules. A bimodule (B, r_L, r_R) is both a left and a right module for $\mathcal{A}$, such that the left and right actions commute:

$$\begin{array}{ccc} \mathcal{A} \times B \times \mathcal{A} & \xrightarrow{r_L} & B \times \mathcal{A} \\ \big\downarrow r_R & & \big\downarrow r_R \\ \mathcal{A} \times B & \xrightarrow{r_L} & B \end{array} \tag{B.1.6}$$

In three dimensions, instead, the category describing the symmetry after gauging $\mathcal{A}$ is that of local modules $\text{Mod}_{\mathcal{A}}^{\text{loc}}$ of the commutative algebra $\mathcal{A}$. These are modules which are compatible with braiding with $\mathcal{A}$. In particular, given a left-module morphism r_L, we define the right morphism r_R as

$$r_R = r_L \circ b \tag{B.1.7}$$

with the consistency condition:

$$\begin{array}{ccc} \mathcal{A} \times M & \xrightarrow{b \circ b} & \mathcal{A} \times M \\ & \searrow^{r_L} & \big\downarrow r_L \\ & & M \end{array} \tag{B.1.8}$$

This implements the intuition that the objects remaining after gauging $\mathcal{A}$ must braid trivially with $\mathcal{A}$. It is known that the dimension of the category of local modules is

$$\dim\big(\text{Mod}_{\mathcal{A}}^{\text{loc}}\big) = \frac{\dim(\mathcal{C})}{\dim(\mathcal{A})} \,, \qquad \dim(\mathcal{A}) \equiv \sum_{x_i \text{ simple}} Z_i(\mathcal{A}) \, \dim(x_i) \,. \tag{B.1.9}$$

Since the dimension of a fusion category must be ≥ 1, there is a notion of maximality in gauging commutative algebras, which implies that

$$\dim(\mathcal{A}) \leq \dim(\mathcal{C}) \,. \tag{B.1.10}$$

When the inequality is saturated the algebra $\mathcal{A}$ is called *Lagrangian* and is denoted by the letter $\mathcal{L}$.

There exist standard techniques to construct the category of modules, which employ the fact that the formal tensor product $\text{Ind}_{\mathcal{A}}(x_i) = \mathcal{A} \times x_i$ gives a (reducible) left $\mathcal{A}$-module. Such modules are called "induced" and the construction of $\text{Mod}_{\mathcal{A}}$ boils down to the decomposition of induced modules. The interested reader can consult [17, 20] for a review of these techniques.

Theories with a 0-form symmetry. Let us also recall some facts about 3d theories enriched with a 0-form symmetry G. These are the so-called G-crossed extensions and we refer to [28] for a complete review. A G-crossed extension is described by a graded tensor category

$$\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g \tag{B.1.11}$$

with $\mathcal{C}_g$ the g-twisted sector of the 0-form symmetry. This can be thought of as a 2-category $\Sigma\mathcal{C}$ with a single connected component, $\left|\pi_0(\Sigma\mathcal{C})\right| = 1$, in which the twist defects provide a basis for the homomorphisms $\sigma_g \in \mathrm{Hom}(U_g, \mathbb{1})$. We use x_i to denote simple objects in the untwisted sector, $\sigma_{g,i}$ to denote simple twist defects, and X_g to denote generic twist defects. The fusion product on twist defects is graded:

$$\mathcal{C}_g \times \mathcal{C}_h \subset \mathcal{C}_{gh} \; . \tag{B.1.12}$$

The 0-form symmetry naturally acts on the defects in $\mathcal{C}$ via an automorphism U of the fusion algebra: we write $U_g[X_h] = g(X)_{ghg^{-1}} \in \mathcal{C}_{ghg^{-1}}$. In the following we will restrict to Abelian 0-form symmetries G. The symmetry then acts on the junction spaces $V^{(gh,k)}_{(g,i),(h,j)}$ by (unitary) isomorphisms

$$\mathcal{U}_g : V^{(g_1g_2,k)}_{(g_1,i),(g_2,j)} \to V^{(g_1g_2,g(k))}_{(g_1,g(i)),(g_2,g(j))} \; , \tag{B.1.13}$$

while the G composition law is encoded in a morphism

$$\lambda_{x_i}(g,h) : \; g\big(h(x_i)\big) \to gh(x_i) \; . \tag{B.1.14}$$

The category comes with graded associator α and braiding isomorphism $b : X_g \times Y_h \to g(Y)_h \times X_g$, satisfying G-crossed versions of the pentagon and hexagon equations. The number of simple objects $\sigma_{g,i}$ in the g-twisted sector is equal to the number of g-invariant local lines $x_i \in \mathcal{C}_0$, such that $g(x_i) = x_i$. This follows from modularity of the Hilbert space on T^2 with G backgrounds. The dimension of each graded category $\mathcal{C}_g$ is the same, thus:

$$\dim(\mathcal{C}_g) = \dim(\mathcal{C}_0) \; , \qquad\qquad \dim(\mathcal{C}) = |G| \dim(\mathcal{C}_0) \; . \tag{B.1.15}$$

Gauging and equivariantization. There are two natural operations that can be introduced in this setting. The first one is gauging the 0-form symmetry G (or a subgroup thereof). This leads to a larger modular tensor category $\mathcal{C}/G$ which has dimension:

$$\dim(\mathcal{C}/G) = |G| \dim(\mathcal{C}) \; . \tag{B.1.16}$$

The category $\mathcal{C}/G$ has an anomaly-free 1-form symmetry $\mathrm{Rep}(G) = G^\vee$ that assigns charges $\in G$ to the liberated g-twisted sectors. The category after gauging is thus still graded by this charge:

$$\mathcal{C}/G = \bigoplus\nolimits_{g\in G} \mathcal{D}_g \; . \tag{B.1.17}$$

The way in which simple objects of $\mathcal{C}/G$ are constructed is familiar from the theory of orbifolds. A simple object $\sigma_{g,i}$ before gauging is equivariantized into an orbit $\Sigma_{g,i}$ after gauging:

$$\Sigma_{g,i} = \bigoplus_{h\in G/\mathrm{Stab}(\sigma_{g,i})} h(\sigma_{g,i}) \; , \tag{B.1.18}$$

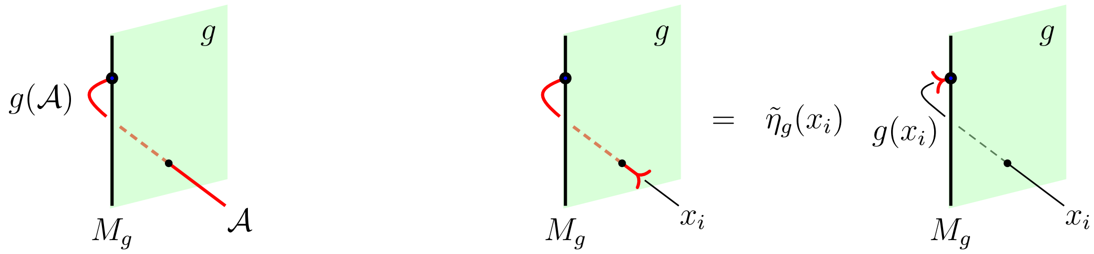


Figure B.1: Action of G on $\mathcal{A}$, both abstractly (left) and in components (right).

where $\mathrm{Stab}(X) = \{g \in G : g(X) = X\}$ is the stabilizer group of X. The object $\Sigma_{g,i}$ can furthermore be dressed by symmetry lines carrying a representation π of $\mathrm{Stab}(\sigma_{g,i})$. We thus get the lines $\Sigma^{\pi}_{g,i}$, whose number is $\left|\mathrm{Stab}(\sigma_{g,i})\right|$.

The second operation is gauging an algebra $\mathcal{A} \subset \mathcal{C}_0$. Let $H \subset G$ be the stabilizer of $\mathcal{A}$, namely $H = \{g \in G : g(\mathcal{A}) = \mathcal{A}\}$. We say that $\mathcal{A}$ preserves a subgroup H of the 0-form symmetry. In order to fully specify an H-invariant algebra, we must also associate a consistent H-action to the data (m, η). This constitutes an *equivariantization* of $\mathcal{A}$ and it is generally not unique nor it is guaranteed to exist. The required conditions are simple to summarize. First, we require the stabilizer H to leave the algebra morphism fixed:

$$m^{g(z)}_{g(x),\, g(y)} = m^{z}_{x,y} \ [\mathcal{U}_g]^{z}_{x,y} \ \frac{\tilde{\eta}_g(z)}{\tilde{\eta}_g(x)\, \tilde{\eta}_g(y)} \qquad \text{for all } g \in H. \tag{B.1.19}$$

In order to write this equation in components, one needs to define the projectors $\pi_x : \mathcal{A} \to x$ and the maps $\tilde{\eta}_g(x)$ that represent the action of H on the projectors:

$$\pi_{x_i} \to \tilde{\eta}_g(x_i)\, \pi_{g(x_i)} \ . \tag{B.1.20}$$

Furthermore, the maps $\tilde{\eta}_g(x)$ must compose nicely under the H action:

$$\tilde{\eta}_g(x_i)\ \tilde{\eta}_h\big(g(x_i)\big) = \tilde{\eta}_{gh}(x_i)\ \lambda_{x_i}(g,h) \ , \tag{B.1.21}$$

where the morphisms $\lambda_x(g,h)$ are the ones we defined in (B.1.14).

A solution to the equations (B.1.19)–(B.1.21) is not guaranteed to exist, and its existence is tied to the splitting of a certain short exact sequence [337]. Even if a solution exists, it must be modded out by the appropriate gauge transformations. Suppose that $\mathrm{Hom}(\mathcal{A}, x_i)$ is at most one-dimensional, then $\tilde{\eta}_g$ is a 1-cochain and we can redefine

$$\pi_{x_i} \to \mu(x_i)\, \pi_{x_i} \ , \qquad\qquad \tilde{\eta}_g(x_i) \to \tilde{\eta}_g(x_i)\, \frac{\mu\big(g(x_i)\big)}{\mu(x_i)} \ . \tag{B.1.22}$$

Once this is settled, gauging $\mathcal{A}$ preserves the subgroup H of the 0-form symmetry. The resulting category is: $\mathcal{C}/\mathcal{A} = \bigoplus_{h\in H} \mathcal{C}_h/\mathcal{A}$, and each entry has dimension $\dim(\mathcal{C}_h/\mathcal{A}) = \dim(\mathcal{C}_0)/\dim(\mathcal{A})$.

Lastly, let us describe the objects of the twisted category $\mathcal{C}/\mathcal{A}$. Since $\mathcal{A}$ has trivial grading, it is possible to define twisted module categories $\mathrm{Mod}^g_{\mathcal{A}}$ in terms of doublets (M_g, r_L) where $M_g \in \mathcal{C}_g$ and r_L is a left map $r_L : \mathcal{A} \times M_g \to M_g$. The interesting part of the construction involves making these modules local. In particular, the braiding map $b : M_g \times \mathcal{A} \to \mathcal{A} \times M_g$ induces a nontrivial action of g on the module morphism r_L that in components maps

$$r_L(x_i) \ \to \ \tilde{\eta}_g(x_i)\, r_L\big(g(x_i)\big) \ , \tag{B.1.23}$$

as in the pictures of Figure B.1. The local bimodule condition is encoded in the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A} \times M_g & \xrightarrow{b} & M_g \times \mathcal{A} & \xrightarrow{b} & g(\mathcal{A}) \times M_g \\ & \searrow^{r_L} & & & \downarrow g(r_L) \\ & & & & M_g \end{array} \tag{B.1.24}$$

or, in components,

$$r_L(x_i) = \tilde{\eta}_g(x_i)\; R_{x_i, M_g} \cdot R_{M_g, x_i} \cdot r_L\big(g(x_i)\big) \;. \tag{B.1.25}$$

Thus the specification of $\tilde{\eta}$ influences the structure of the H-twisted sectors after gauging $\mathcal{A}$.

B.2 Lagrangian algebras for DW theories

In this appendix we prove that any Lagrangian algebra of DW($\mathbb{A}$) is of the form

$$\mathcal{L}_{\mathbb{B},[\nu]} = \Big\{ \big(b, \beta\psi_\nu(b)\big) \;\Big|\; b \in \mathbb{B}\,,\; \beta \in N(\mathbb{B}) \Big\} \tag{B.2.1}$$

for some subgroup $\mathbb{B} \subset \mathbb{A}$ and a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$, and that the associated boundary condition corresponds to a theory obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$.

We denote by $\pi_{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A}$ and $\pi_{\mathbb{A}^\vee} : \mathbb{A} \times \mathbb{A}^\vee \to \mathbb{A}^\vee$ the projections on the two factors. Let $\mathcal{L} \subset \mathbb{A} \times \mathbb{A}^\vee$ be Lagrangian. We define a subgroup of $\mathbb{A}$

$$\mathbb{B} = \pi_{\mathbb{A}}(\mathcal{L}) \;\subset \mathbb{A} \;. \tag{B.2.2}$$

Notice that $(\mathbb{B}, 0)$ is not necessarily a subgroup of $\mathcal{L}$. On the other hand, any element of the form $(0, \beta)$ with $\beta \in N(\mathbb{B})$ has trivial braiding with any element of $\mathcal{L}$, and since $\mathcal{L}$ is maximal, it follows that $\big(0, N(\mathbb{B})\big)$ is a subgroup of $\mathcal{L}$ and thus $N(\mathbb{B}) \subset \pi_{\mathbb{A}^\vee}(\mathcal{L})$. Using the short exact sequence

$$1 \longrightarrow N(\mathbb{B}) \longrightarrow \mathbb{A}^\vee \longrightarrow \mathbb{B}^\vee \longrightarrow 1 \tag{B.2.3}$$

we realize any element of $\mathbb{A}^\vee$, and in particular of $\pi_{\mathbb{A}^\vee}(\mathcal{L})$, as a pair $\beta\omega$ with $\beta \in N(\mathbb{B})$ and $\omega \in \mathbb{B}^\vee$. All elements of $\mathcal{L}$ are then of the form $(b, \beta\omega)$ with $b \in \mathbb{B}$, $\beta \in N(\mathbb{B})$ and $\omega \in \mathbb{B}^\vee$, but since $|\mathcal{L}| = |\mathbb{A}| = |\mathbb{B}|\big|N(\mathbb{B})\big|$ there must exist a homomorphism $\psi : \mathbb{B} \to \mathbb{B}^\vee$ such that

$$\omega = \psi(b) \;. \tag{B.2.4}$$

The fact that $\mathcal{L}$ is Lagrangian and so all its elements have vanishing spin implies a constraint on ψ. Defining a bicharacter $\chi : \mathbb{B} \times \mathbb{B} \to U(1)$ as $\chi(b_1, b_2) = \psi(b_1)\, b_2$, and then imposing that $\big(b, \beta\psi(b)\big)$ has trivial spin, we obtain

$$1 = \theta_{(b,\, \beta\psi(b))} = \chi(b, b) \;. \tag{B.2.5}$$

Thus χ is alternating and it defines a class $[\nu] \in H^2\big(\mathbb{B}, U(1)\big)$, hence $\mathcal{L} = \mathcal{L}_{\mathbb{B},[\nu]}$.

Now we aim to prove that the boundary defined by $\mathcal{L}_{\mathbb{B},[\nu]}$, where the symmetry is

$$\mathcal{S} = \big(\mathbb{A} \times \mathbb{A}^\vee\big) / \mathcal{L}_{\mathbb{B},[\nu]} \;\cong\; \mathcal{L}^\vee_{\mathbb{B},[\nu]} \;, \tag{B.2.6}$$

is obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. First we notice that $\mathcal{L}_{\mathbb{B},[\nu]}$ is an extension of $\mathbb{B}$ by $N(\mathbb{B})$ determined as follows. Let $\tilde{c} \in H^2\big(\mathbb{B}^\vee, N(\mathbb{B})\big)$ be the class associated with the short exact sequence (B.2.3). This class is determined by Pontryagin duality from $1 \to \mathbb{B} \to \mathbb{A} \to \mathbb{A}/\mathbb{B} \to 1$, which is associated with a class $c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B})$.[1] The class $\tilde{c}$ enters in the composition rule of elements of $\mathbb{A}^\vee$ when they are represented as pairs $\beta\eta$, $\beta \in N(\mathbb{B})$, $\eta \in \mathbb{B}^\vee$:

$$\beta_1\eta_1 + \beta_2\eta_2 = \big(\beta_1 + \beta_2 - \tilde{c}(\eta_1, \eta_2)\big)\,(\eta_1 + \eta_2)\ . \tag{B.2.8}$$

The inverse is $-\beta\eta = \big(-\beta + \tilde{c}(\eta, -\eta)\big)(-\eta)$. The elements of $\mathcal{L}_{\mathbb{B},[\nu]}$ can be realized as pairs of $b \in \mathbb{B}$ and $\beta \in N(\mathbb{B})$ with $[b, \beta] \equiv \big(b,\ \beta\psi_\nu(b)\big)$, and their composition law is

$$[b_1, \beta_1] + [b_2, \beta_2] = \Big[b_1 + b_2\ ,\ \beta_1 + \beta_2 - \tilde{c}\big(\psi_\nu(b_1), \psi_\nu(b_2)\big)\Big]\ . \tag{B.2.9}$$

We conclude that $\mathcal{L}_{\mathbb{B},[\nu]}$ is an extension

$$1 \longrightarrow N(\mathbb{B}) \longrightarrow \mathcal{L}_{\mathbb{B},[\nu]} \longrightarrow \mathbb{B} \longrightarrow 1 \tag{B.2.10}$$

determined by the class $\psi_\nu^*(\tilde{c}) \in H^2\big(\mathbb{B}, N(\mathbb{B})\big)$. Taking the Pontryagin dual of (B.2.10) we get

$$1 \longrightarrow \mathbb{B}^\vee \xrightarrow{\ \iota\ } \mathcal{S} \xrightarrow{\ \pi\ } \mathbb{A}/\mathbb{B} \longrightarrow 1\ , \tag{B.2.11}$$

whose associated class is $\widehat{c} \equiv \psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$.

To show that this is the correct symmetry structure of the boundary theory obtained by gauging $\mathbb{B}$ with discrete torsion $[\nu]$, we consider its partition function coupled to a background

$$B = \iota(B_1) + s(B_2) \quad \in H^1(X_2, \mathcal{S})\ , \tag{B.2.12}$$

where $s : \mathbb{A}/\mathbb{B} \to \mathcal{S}$ is a section of π and B_1, B_2 are gauge fields valued in $\mathbb{B}^\vee$ and $\mathbb{A}/\mathbb{B}$, respectively. Closure $dB = 0$ implies that $dB_2 = 0$, whilst the differential of B_1 is equal to the pull-back through B_2 of the extension class $\widehat{c} \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$, namely $(dB_1)_{ijk} = \widehat{c}\,(B_{2ij}, B_{2jk}) \equiv (B_2^*\,\widehat{c}\,)_{ijk}$. On the other hand, the dynamical gauge field B' valued in $\mathbb{B}$ must satisfy $dB' = B_2^*\, c$ in the presence of a background B_2, and the partition function is thus

$$Z_{\mathbb{B},[\nu]} = \sum_{B' \text{ s.t. } dB' = B_2^* c} \exp\bigg[\int_{X_2} \Big(B'^*\nu + B_1 \cup B'\Big)\bigg]\ Z_{\mathrm{e}}[B', B_2]\ . \tag{B.2.13}$$

The exponent is not gauge invariant under $B' \to B' + d\rho$ unless B_1 satisfies

$$\psi_\nu(dB') - dB_1 = 0\ . \tag{B.2.14}$$

[1] Given an Abelian extension $1 \to \mathbb{A} \xrightarrow{i} \mathbb{B} \xrightarrow{\pi} \mathbb{C} \to 1$ with section $s : \mathbb{C} \to \mathbb{B}$, the class $[\epsilon] \in H^2(\mathbb{C}, \mathbb{A})$ has representative $i\big(\epsilon(c_1, c_2)\big) = s(c_1 + c_2) - s(c_1) - s(c_2)$ which is symmetric. For each $\alpha \in \mathbb{A}^\vee$, $\alpha\epsilon : \mathbb{C} \times \mathbb{C} \to U(1)$ is a symmetric 2-cocycle and is thus exact (see Sec. 4.1.2), therefore there exists $\beta : \mathbb{C} \times \mathbb{A}^\vee \to U(1)$ such that (in additive notation):

$$\alpha\epsilon(c_1, c_2) = \beta(c_1 + c_2, \alpha) - \beta(c_1, \alpha) - \beta(c_2, \alpha) \qquad \forall\ c_1, c_2 \in \mathbb{C},\ \ \alpha \in \mathbb{A}^\vee\ . \tag{B.2.7}$$

Construct $\Omega(c, \alpha_1, \alpha_2) = \beta(c, \alpha_1 + \alpha_2) - \beta(c, \alpha_1) - \beta(c, \alpha_2) \in U(1)$. One checks that this is linear in the first entry in $\mathbb{C}$, and thus it defines a map $\epsilon^\vee : \mathbb{A}^\vee \times \mathbb{A}^\vee \to \mathbb{C}^\vee$. This is the class of the Abelian extension $1 \to \mathbb{C}^\vee \to \mathbb{B}^\vee \to \mathbb{A}^\vee \to 1$, that reproduces the sum in (B.2.8) if we use the pairing $(\gamma, \alpha)(a, c) = \gamma(c) + \alpha(a) + \beta(c, \alpha)$.

This determines the modified cocycle condition for B_1 as

$$dB_1 = B_2^*(\psi_\nu \circ c) \, , \tag{B.2.15}$$

hence proving that $\mathcal{S}$ is the correct symmetry after gauging $\mathbb{B}$ with discrete torsion $[\nu]$.

B.3 General duality-invariant Lagrangian algebras

In this appendix we report technical details regarding Lagrangian algebras $\mathcal{L}_{\mathbb{B},[\nu]}$ for general $\mathbb{B}$. In particular, we give the conditions for their duality invariance and compute the mixed 't Hooft anomaly with the invertible duality symmetry in those cases, extending the discussion for $\mathbb{B} = \mathbb{A}$ given in the main text. For concreteness we look at the 2d/3d case, but the 4d/5d one is analogous.

B.3.1 Proof of duality invariance

In Section 4.1.2 after (4.1.64) we claimed that a Lagrangian algebra $\mathcal{L}_{\mathbb{B},[\nu]}$ as in (4.1.56) is duality invariant, namely the isomorphism Φ in (4.1.40) acts as $\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big) = \mathcal{L}_{\mathbb{B},[\nu]}$, if and only if

1. $\phi\big(\mathrm{Rad}(\nu)\big) = N(\mathbb{B})$;
2. the isomorphism $\sigma = \phi^{-1} \circ \psi_\nu$ acting on $\mathbb{B}/\,\mathrm{Rad}(\nu)$ satisfies $\sigma^2 = \mathbb{1}$.

Let us prove the claim. To prove it, we first notice that since $\mathcal{L}_{\mathbb{B},[\nu]}$ and $\Phi\big(\mathcal{L}_{\mathbb{B},[\nu]}\big)$ are both Lagrangian, they are equal if and only if all their lines are mutually transparent. In other words, if and only if

$$\phi(b')\, b \,\cdot\, \beta\big[\phi^{-1}\big(\beta'\psi_\nu(b')\big)\big] \,\cdot\, \psi_\nu(b)\big[\phi^{-1}\big(\beta'\psi_\nu(b')\big)\big] = 1 \tag{B.3.1}$$

for all $b, b' \in \mathbb{B}$ and $\beta, \beta' \in N(\mathbb{B})$.

First we prove that the two conditions above are necessary. Recall that $\mathrm{Rad}(\nu) = \mathrm{Ker}(\psi_\nu)$, and notice that $\phi(b)\, b' = \phi(b')\, b$ while $\psi_\nu(b)\, b' = \big[\psi_\nu(b')\, b\big]^{-1}$. Specializing (B.3.1) to $\beta = 1$ (in multiplicative notation) and $b \in \ker(\psi_\nu)$ we get $\phi(b) \in N(\mathbb{B})$ and thus $\phi\big(\mathrm{Ker}(\psi_\nu)\big) \subset N(\mathbb{B})$. Specializing (B.3.1) to $\beta = 1$ and $b' = 1$ we get

$$1 = \psi_\nu(b)\big(\phi^{-1}(\beta')\big) = \big[\psi_\nu\big(\phi^{-1}(\beta')\big)\, b\big]^{-1} \, , \tag{B.3.2}$$

thus $\phi^{-1}\big(N(\mathbb{B})\big) \subset \mathrm{Ker}(\psi_\nu)$. We conclude that $\phi\big(\mathrm{Ker}(\psi_\nu)\big) = N(\mathbb{B})$ which is condition 1. Specializing (B.3.1) to $\beta = \beta' = 1$ we get $\gamma(b', b) = \chi_\nu\big(\phi^{-1}\circ\psi_\nu(b'), b\big)$ for all $b, b' \in \mathbb{B}$. Assuming condition 1., both sides project consistently to $\mathbb{B}/\,\mathrm{Rad}(\nu)$, and thus $\phi(b') = \psi_\nu\big(\sigma(b')\big) \in \mathbb{B}/\,\mathrm{Rad}(\nu)$ for all $b' \in \mathbb{B}/\,\mathrm{Rad}(\nu)$. We conclude that $\sigma^2 = \mathbb{1}$, which is condition 2.

Conversely, we prove that the two conditions are also sufficient. From condition 1. it follows that $\phi^{-1}(\beta') \in \mathrm{Ker}(\psi_\nu) \subset \mathbb{B}$, therefore $\beta\big(\phi^{-1}(\beta')\big) = \psi_\nu\big(\phi^{-1}(\beta')\big) = 1$. Similarly $\beta\big(\phi^{-1} \circ \psi_\nu(b')\big) = 1$. Eqn. (B.3.1) then reduces to

$$\phi(b')\, b \,\cdot\, \psi_\nu(b)\big(\phi^{-1} \circ \psi_\nu(b')\big) = 1 \, , \tag{B.3.3}$$

that can be rewritten as $\gamma(b',b) = \chi_\nu\big(\sigma(b'),b\big) = \gamma\big(\sigma^2(b'),b\big)$ using the definition of σ. Both sides project consistently to $\mathbb{B}/\operatorname{Rad}(\nu)$, and the equation is satisfied using condition 2. This completes the proof.

It will be useful to discuss a few consequence of the theorem. Each of the commuting diagrams below expresses the fact that ϕ is a group isomorphism between the respective Abelian groups.

- Since $\phi\big(\operatorname{Rad}(\nu)\big) = N(\mathbb{B})$, then the short exact sequence $1 \to \operatorname{Rad}(\nu) \to \mathbb{A} \to \mathbb{A}/\operatorname{Rad}(\nu) \to 1$ is the image under ϕ^{-1} of $1 \to N(\mathbb{B}) \to \mathbb{A}^\vee \to \mathbb{B}^\vee \to 1$. In other words there is a commutative diagram:

$$\begin{array}{lccccccccc} S_1: & 1 & \longrightarrow & N(\mathbb{B}) & \longrightarrow & \mathbb{A}^\vee & \longrightarrow & \mathbb{B}^\vee & \longrightarrow & 1 \\ & & & \uparrow\phi & & \uparrow\phi & & \uparrow\phi & & \\ S_2: & 1 & \longrightarrow & \operatorname{Rad}(\nu) & \longrightarrow & \mathbb{A} & \longrightarrow & \mathbb{A}/\operatorname{Rad}(\nu) & \longrightarrow & 1 \end{array} \tag{B.3.4}$$

- Taking the Pontryagin dual of the diagram (B.3.4) and using the symmetry of ϕ, namely that $\phi^\vee = \phi$, we obtain an other commutative diagram:

$$\begin{array}{lccccccccc} S_3 = S_1^\vee: & 1 & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{A} & \longrightarrow & \mathbb{A}/\mathbb{B} & \longrightarrow & 1 \\ & & & \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ S_4 = S_2^\vee: & 1 & \longrightarrow & N\big(\operatorname{Rad}(\nu)\big) & \longrightarrow & \mathbb{A}^\vee & \longrightarrow & \operatorname{Rad}(\nu)^\vee & \longrightarrow & 1 \end{array} \tag{B.3.5}$$

- It is simple to prove that there is a canonical isomorphism $N\big(\operatorname{Rad}(\nu)\big)/N(\mathbb{B}) \cong \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee$. Then using that $\phi(\mathbb{B}) = N\big(\operatorname{Rad}(\nu)\big)$ and $\phi\big(\operatorname{Rad}(\nu)\big) = N(\mathbb{B})$, we find a commutative diagram:

$$\begin{array}{lccccccccc} S_5: & 1 & \longrightarrow & \operatorname{Rad}(\nu) & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{B}/\operatorname{Rad}(\nu) & \longrightarrow & 1 \\ & & & \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ S_6: & 1 & \longrightarrow & N(\mathbb{B}) & \longrightarrow & N\big(\operatorname{Rad}(\nu)\big) & \longrightarrow & \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee & \longrightarrow & 1 \end{array} \tag{B.3.6}$$

  as well as its Pontryagin dual:

$$\begin{array}{lccccccccc} S_7 = S_5^\vee: & 1 & \longrightarrow & \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee & \longrightarrow & \mathbb{B}^\vee & \longrightarrow & \operatorname{Rad}(\nu)^\vee & \longrightarrow & 1 \\ & & & \uparrow\phi & & \uparrow\phi & & \uparrow\phi & & \\ S_8 = S_6^\vee: & 1 & \longrightarrow & \mathbb{B}/\operatorname{Rad}(\nu) & \longrightarrow & \mathbb{A}/\operatorname{Rad}(\nu) & \longrightarrow & \mathbb{A}/\mathbb{B} & \longrightarrow & 1 \end{array} \tag{B.3.7}$$

B.3.2 Mixed anomaly in the general case

The discussion in this appendix is technical and it involves some notation. We will use several short exact sequences which we denote uniformly as

$$S_m: \quad 1 \longrightarrow \mathbb{B}_m \xrightarrow{\iota_m} \mathbb{A}_m \xrightarrow{\pi_m} \mathbb{A}_m/\mathbb{B}_m \longrightarrow 1\ , \tag{B.3.8}$$

where ι_m, π_m, s_m denote respectively the inclusion, the projection, and a section of π_m. Each sequence S_m induces an extension class $c_m \in H^2(\mathbb{A}_m/\mathbb{B}_m, \mathbb{B}_m)$. The sequences that will be used are the $S_1, \ldots, S_8$ introduced in Appendix B.3.1 above.

Moreover, we will systematically decompose gauge fields valued in $\mathbb{A}_m$ in terms of gauge fields values in the subgroup and the quotient according to

$$a_m = \iota_m(b_m) + s_m(b'_m) \ . \tag{B.3.9}$$

As discussed after (B.2.12), closure $da_m = 0$ of the gauge field implies that

$$db'_m = 0 \ , \qquad\qquad db_m = b'^*_m(c_m) \ . \tag{B.3.10}$$

All fields have a subscript labelling the corresponding short exact sequence, and both the field valued in the subgroup $\mathbb{B}_m$ and in the quotient are denoted by the same letter, but the one in the quotient is always primed. An important remark is in order. The relations (B.3.10) mean that in the presence of a non-trivial extension, the background b_m for the subgroup is the sum of an ordinary cohomology class and a particular co-chain solving the constraint (B.3.10), which depends on the background for the quotient. Hence all path integrals are intended to be done in order: one first integrates the cohomology part of the background b_m for the subgroup, and then the background b'_m for the quotient.

Let $\mathcal{L}_D$ be a duality-invariant algebra associated with $(\mathbb{B}, [\nu])$. We want to compute the mixed anomaly between $\mathcal{S} = \mathcal{Z}(\mathbb{A})/\mathcal{L}_D$ and the duality $G \cong \mathbb{Z}_2$ on the invertible boundary. This is obtained from the electric boundary by gauging $\mathbb{B}$ with discrete torsion $[\nu]$. A gauge field $A \in H^1(X, \mathbb{A})$ can be decomposed according to the sequence S_3 in (B.3.5) as

$$A = \iota_3(b_3) + s_3(b'_3) \ . \tag{B.3.11}$$

After gauging $\mathbb{B}$ with torsion, the dual symmetry $\mathcal{S}$ is an extension of $\mathbb{A}/\mathbb{B}$ by $\mathbb{B}^\vee$ with extension class $\widehat{c} = \psi_\nu \circ c \in H^2(\mathbb{A}/\mathbb{B}, \mathbb{B}^\vee)$ (see Appendix B.2), and a background field for $\mathcal{S}$ is described by a pair B, b'_3 valued in $\mathbb{B}^\vee$ and $\mathbb{A}/\mathbb{B}$, respectively, with $dB = b'^*_3(\widehat{c})$. The partition function on the invertible boundary is

$$Z_{\text{inv}}[B, b'_3] = \sum_{b_3} \exp\left[2\pi i \int \Big(b_3^*(\nu) + B \cup b_3\Big)\right] Z_{\text{e}}[b_3, b'_3] \ . \tag{B.3.12}$$

By acting with the duality on the electric boundary we get the magnetic one, corresponding to the gauging of $\mathbb{A}$ with trivial torsion:

$$\Phi \cdot Z_{\text{e}}[b_3, b'_3] = \sum_{a \,\in\, H^1(X, \mathbb{A})} \exp\left[2\pi i \int \phi(a) \cup \Big(\iota_3(b_3) + s_3(b'_3)\Big)\right] Z_{\text{e}}[a] \ . \tag{B.3.13}$$

We decompose the $\mathbb{A}$-valued field a according to the sequence S_3: $a = \iota_3(a_3) + s_3(a'_3)$. Because of the commutative diagram (B.3.5), $\phi(a) \in H^1(X, \mathbb{A}^\vee)$ has a decomposition using S_4:

$$\phi(a) = \iota_4(x_4) + s_4(x'_4) \qquad \text{with} \qquad x_4 = \phi(a_3) \ , \qquad x'_4 = \phi(a'_3) \ . \tag{B.3.14}$$

Furthermore, it is useful to decompose b_3 using S_5: $b_3 = \iota_5(y_5) + s_5(y'_5)$. Hence, using that ν vanishes on $\text{Rad}(\nu)$, we have

$$\begin{aligned} \Phi \cdot Z_{\text{inv}}[B, b'_3] = \sum_{y_5,\, y'_5,\, x_4,\, x'_4} \exp\Bigg[2\pi i \int \Bigg(& y'^*_5 \nu + B \cup \Big(\iota_5(y_5) + s_5(y'_5)\Big) + \\ & + \phi(a) \cup \Big(\iota_3 \iota_5(y_5) + \iota_3 s_5(y'_5) + s_3(b'_3)\Big)\Bigg)\Bigg] Z_{\text{e}}[a_3, a'_3] \ . \end{aligned} \tag{B.3.15}$$

We can perform the sum over y_5 and y'_5, in this order. Since y_5 appears linearly, the sum over it gives a delta function imposing

$$\pi_7\Big(B + \pi_1\big(\phi(a)\big)\Big) = 0 \qquad \Leftrightarrow \qquad B + \pi_1\big(\phi(a)\big) \in \iota_7\Big(\big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee\Big) . \tag{B.3.16}$$

We notice that $\pi_7\pi_1 = \pi_4$, and since $\phi(a) = \iota_4(x_4) + s_4(x'_4)$, it follows that (B.3.16) can be rewritten as $\pi_7(B) + x'_4 = 0$. This delta function will be resolved by the sum over x'_4, which however must be performed only after the sum over x_4. We can then integrate out y'_5. Since it appears quadratically, the sum over it can be performed by solving its equation of motion

$$\iota_7\big(\psi_\nu(y'_5)\big) + B + \pi_1\big(\phi(a)\big) = 0 . \tag{B.3.17}$$

This equation makes sense in virtue of (B.3.16). This equation can be inverted in virtue of (B.3.16) and using that $\psi_\nu : \mathbb{B}/\operatorname{Rad}(\nu) \to \big(\mathbb{B}/\operatorname{Rad}(\nu)\big)^\vee$ is invertible. Plugging the result back we get

$$\Phi \cdot Z_{\text{inv}}[B, b'_3] = \sum_{a_3,\, a'_3} \exp\left[2\pi i \int \left(\phi(a) \cup s_3(b'_3) - \Big(\psi_\nu^{-1}\big(B + \pi_1\, \phi(a)\big)\Big)^{\!*}\nu\right)\right] \delta\big(\pi_7\, B + x'_4\big)\, Z_{\text{e}}[a] . \tag{B.3.18}$$

We decompose B using S_7 as

$$B = \iota_7(B_7) + s_7(B'_7) , \tag{B.3.19}$$

and the duality maps

$$\Phi(B_7,\, B'_7,\, b'_3) = \big(\sigma^\vee(B_7),\, \phi(b'_3),\, \phi^{-1}(B'_7)\big) . \tag{B.3.20}$$

The sum over a'_3 resolves the delta function, while the one over a_3 reconstructs Z_{inv} up to a multiplicative factor which gives the anomaly:

$$\Phi \cdot Z_{\text{inv}}[B_7, B'_7, b'_3] = \exp\left[-\int \big(\phi^{-1}(\sigma^\vee B_7)\big)^*\nu\right] Z_{\text{inv}}\big[\sigma^\vee(B_7),\, \phi(b'_3),\, \phi^{-1}(B'_7)\big] . \tag{B.3.21}$$

B.4 Twisted cohomology and anomalies

Here we provide details on the topological actions that we use in 3d and 5d to cancel the mixed anomaly between the self-duality symmetry and the 0-form (in 2d) or the 1-form (in 4d) symmetry, when we go to the invariant boundary. In the 2d case this is an anomaly for a semi-direct product, while in 4d it is an anomaly for a split 2-group. In both cases we do not discuss the full anomaly, but only the piece linear in the gauge field $A \in H^d(X, G)$ for the self-duality symmetry.

B.4.1 Anomaly for a semi-direct product in 2d

We consider a semi-direct product $\mathbb{A} \rtimes_\rho G$ ($\mathbb{A}$ and G being both Abelian) with homomorphism $\rho : G \to \operatorname{Aut}(\mathbb{A})$. This is associated with a short exact sequence

$$1 \longrightarrow \mathbb{A} \xrightarrow{\ \iota\ } \mathbb{A} \rtimes_\rho G \xrightarrow{\ \pi\ } G \longrightarrow 1 \tag{B.4.1}$$

which splits, namely it admits a section $s : G \to \mathbb{A} \rtimes_\rho G$ which is a group homomorphism. Any element can be written uniquely as $\iota(a)\, s(g)$, $a \in \mathbb{A}$, $g \in G$, with product rule

$$\iota(a_1)\, s(g_1) \cdot \iota(a_2)\, s(g_2) = \iota\big(a_1 + \rho_{g_1}(a_2)\big)\, s(g_1 + g_2) \; . \tag{B.4.2}$$

In particular

$$s(g)\, \iota(a)\, s(g^{-1}) = \iota\big(\rho_g(a)\big) \; . \tag{B.4.3}$$

Semi-direct products are generically non-Abelian, and accordingly we only consider standard 1-form gauge fields. These are classes $\mathcal{A} \in H^1(X, \mathbb{A} \rtimes_\rho G)$, namely

$$(d\mathcal{A})_{ijk} = \mathcal{A}_{jk} \mathcal{A}_{ik}^{-1} \mathcal{A}_{ij} = 1 \; , \qquad\qquad \mathcal{A}_{ij} \sim \Lambda_i^{-1} \mathcal{A}_{ij} \Lambda_j \; , \tag{B.4.4}$$

where the order of multiplication matters. Since $\mathcal{A}_{ij} \in \mathbb{A} \rtimes_\rho G$, we can write

$$\mathcal{A}_{ij} = \iota(B_{ij})\, s(A_{ij}) \tag{B.4.5}$$

where $B \in C^1(X, \mathbb{A})$ and $A \in C^1(X, G)$. Using the commutation relation (B.4.3), the cocycle condition $(d\mathcal{A})_{ijk} = 1$ is equivalent to

$$\big(d_{\rho(A)} B\big)_{ijk} = \rho_{A_{ij}} B_{jk} - B_{ik} + B_{ij} = 0 \; , \qquad\qquad (dA)_{ijk} = A_{jk} - A_{ik} + A_{ij} = 0 \; . \tag{B.4.6}$$

The identification $\mathcal{A}_{ij} \sim \Lambda_i^{-1} \mathcal{A}_{ij} \Lambda_j$, upon decomposing $\Lambda_i = \iota(\theta_i)\, s(\lambda_i)$, becomes

$$B_{ij} \sim \rho_{\lambda_i}^{-1} \big(B_{ij} + \rho_{A_{ij}} \theta_j - \theta_i\big) = \rho_{\lambda_i}^{-1} \big(B + d_{\rho(A)} \theta\big)_{ij} \; , \qquad A_{ij} \sim A_{ij} + \lambda_j - \lambda_i = (A + d\lambda)_{ij} \; . \tag{B.4.7}$$

Hence A defines a class in the cohomology group $H^1(X, G)$, while B a class in the *twisted* cohomology group $H^1_\rho(X, \mathbb{A})$ — also called cohomology with local coefficients.

We are interested in the anomaly for $\mathbb{A} \rtimes_\rho G$ whose 3d inflow action is quadratic in B and "linear" in A. The word linear is in quotes since B is a twisted class, and thus A will appear not only linearly, but also in the twisting. This anomaly is identified by a characteristic class of $\mathbb{A} \rtimes_\rho G$ bundles, which lives in $H^1_\rho\big(G,\, H^2(\mathbb{A}, U(1))\big)$ [177]. Such a class can be thought of as a function μ on G with values in the group of alternating bicharacters over $\mathbb{A}$, satisfying (in additive notation):

$$\rho_g\, \mu(h) + \mu(g) = \mu(g + h) \; . \tag{B.4.8}$$

The G-action on bicharacters is given in (4.1.70). Besides, the function μ is subject to the identification

$$\mu(\,\cdot\,) \;\sim\; \mu(\,\cdot\,) + \rho_{(\cdot)} \xi - \xi \qquad\qquad \text{for any } \; \xi \in H^2\big(\mathbb{A}, U(1)\big) \; . \tag{B.4.9}$$

Notice that $\mu(0) = 0$, so that $\mu(-g) = -\rho_g^{-1} \mu(g)$.

Given $A \in H^1(X, G)$, we construct $\mu(A) \in C^1\big(X,\, H^2(\mathbb{A}, U(1))\big)$ (both notations $\mu(A)$ and $A^*\mu$ could be used). This is a cochain $\mu(A_{ij}) : \mathbb{A} \times \mathbb{A} \to U(1)$ satisfying the twisted cocycle condition:

$$\big(d_{\rho(A)} \mu(A)\big)_{ijk} \;\equiv\; \rho_{A_{ij}} \mu(A_{jk}) - \mu(A_{ik}) + \mu(A_{ij}) = 0 \; . \tag{B.4.10}$$

Moreover, under a gauge transformation $A \to A + d\lambda$, it changes by

$$\begin{aligned}\mu(A_{ij}) \;\to\; \mu\big(A_{ij} + \lambda_j - \lambda_i\big) &= \rho_{\lambda_i}^{-1}\, \mu\big(A_{ij} + \lambda_j\big) + \mu(-\lambda_i) \\ &= \rho_{\lambda_i}^{-1}\Big(\rho_{A_{ij}}\, \mu(\lambda_j) + \mu(A_{ij}) - \mu(\lambda_i)\Big) = \rho_{\lambda_i}^{-1}\Big(\mu(A_{ij}) + \big(d_{\rho(A)}\, \mu(\lambda)\big)_{ij}\Big) \;, \end{aligned} \tag{B.4.11}$$

hence $\mu(A) \in H^1_\rho\big(X,\, H^2(\mathbb{A}, U(1))\big)$.

Given $B \in H^1_\rho(X, \mathbb{A})$, we can form the cup product $\mu(A) \cup B \cup B \in H^3\big(X, U(1)\big)$ as:

$$\big(\, \mu(A) \cup B \cup B\,\big)_{ijkl} = \mu(A_{ij}) \Big(\rho_{A_{ij}} B_{jk} \,,\, \rho_{A_{ik}} B_{kl} \Big) \;, \tag{B.4.12}$$

see App. A of [178]. Under a gauge variation $A \to A + d\lambda$, $B \to \rho(\lambda)^{-1} B$ as in (B.4.7) we find:

$$\begin{aligned}\big(\, \mu(A) \cup B \cup B\,\big)_{ijkl} \;\to\; & \rho_{\lambda_i}^{-1}\Big(\mu(A_{ij}) + \big(d_{\rho(A)}\, \mu(\lambda)\big)_{ij}\Big)\Big(\rho_{\lambda_i}^{-1}\rho_{A_{ij}} B_{jk} \,,\, \rho_{\lambda_i}^{-1}\rho_{A_{ik}} B_{kl}\Big) \\ &= \big(\, \mu(A) \cup B \cup B\,\big)_{ijkl} + \big(d_{\rho(A)}\, \mu(\lambda)\big)_{ij}\Big(\rho_{A_{ij}} B_{jk} \,,\, \rho_{A_{ik}} B_{kl}\Big) \;. \end{aligned} \tag{B.4.13}$$

This means that we get a linear variation

$$\delta\Big(\mu(A) \cup B \cup B\Big) = \big(d_{\rho(A)}\, \mu(\lambda)\big) \cup B \cup B = d\Big(\mu(\lambda) \cup B \cup B\Big) \;. \tag{B.4.14}$$

We write the inflow action as

$$S_\mu = 2\pi i \int_{X_3} \mu(A) \cup B \cup B \;. \tag{B.4.15}$$

When X_3 is closed this is gauge invariant, however if $\partial X_3 = X_2$ we get a boundary term:

$$S_\mu \;\to\; S_\mu + 2\pi i \int_{X_2} \mu(\lambda) \cup B \cup B \;. \tag{B.4.16}$$

B.4.2 Anomaly for a split 2-group in 4d

In 4d we have an analog story, where $\mathbb{A}$ is now a 1-form symmetry. The full symmetry structure is a split 2-group, which is a higher categorical version of a semi-direct product. The definitions can be found in [357] and a more physical discussion is in [178, 356]. Here we simply use two facts which from our viewpoint can be motivated as being the straightforward generalization of the discussion on semi-direct products.

First, a background field for a split 2-group is made of an ordinary cohomology class $A \in H^1(X, G)$ and a twisted cohomology class $B \in H^2_\rho(X, \mathbb{A})$. The latter means that

$$\big(d_{\rho(A)} B\big)_{ijkl} = \rho_{A_{ij}} B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} = 0 \;, \tag{B.4.17}$$

and there is an identification (or gauge transformation)

$$B_{ijk} \;\sim\; \rho_{\lambda_i}^{-1}\big(B_{ijk} + \rho_{A_{ij}}\theta_{jk} - \theta_{ik} + \theta_{ij}\big) \;, \qquad\qquad A_{ij} \;\sim\; A_{ij} + \lambda_j - \lambda_i \;, \tag{B.4.18}$$

which are the obvious generalizations of (B.4.7).

Second, the piece of the anomaly for a split 2-group which is "linear" in A and quadratic in B is labelled by a characteristic class of 2-group gauge bundles:

$$\mu \in H^1_\rho\Big(G,\, H^4\big(B^2\mathbb{A}, U(1)\big)\Big) \;. \tag{B.4.19}$$

One can show [356] that $H^4\big(B^2\mathbb{A}, U(1)\big)$ is isomorphic to $\Gamma(\mathbb{A})^\vee$, the Pontryagin dual of the universal quadratic group of $\mathbb{A}$, which can be identified with the group of quadratic functions $q : \mathbb{A} \to U(1)$ (see [178, 356] for precise definitions and details, as well as the discussion around (4.1.164)). The G-action on them is naturally given by

$$(\rho_g q)(a) = q\big(\rho_g^{-1} a\big) \ . \tag{B.4.20}$$

The construction of the 5d anomaly inflow is very similar to the semi-direct product case, thus we skip many details. Given $A \in H^1(X, G)$, we construct $\mu(A)$ which satisfies (B.4.10) and (B.4.11), thus defining a class in $H^1_\rho\big(X, \Gamma(\mathbb{A})^\vee\big)$. Recall that $H^4\big(B^2\mathbb{A}, \Gamma(\mathbb{A})\big) \cong \mathrm{Hom}\big(\Gamma(\mathbb{A}), \Gamma(\mathbb{A})\big)$ has a distinguished element $\mathfrak{P}$ (the identity map) called the universal Pontryagin class, such that

$$B \in H^2_\rho(X, \mathbb{A}) \qquad \rightsquigarrow \qquad \mathfrak{P}_\rho(B) \equiv B^*\mathfrak{P} \in H^4_\rho\big(X, \Gamma(\mathbb{A})\big) \ . \tag{B.4.21}$$

The action of G on $\Gamma(\mathbb{A})$ is induced by the one on $\Gamma(\mathbb{A})^\vee$ in such a way to make the natural pairing $\langle\ ,\ \rangle : \Gamma(\mathbb{A}) \times \Gamma(\mathbb{A})^\vee \to U(1)$ invariant. Under $A \to A + d\lambda$ the latter transforms as

$$\mathfrak{P}_\rho(B)_{i_0,\ldots,i_4} \ \to\ \rho^{-1}_{\lambda_{i_0}}\, \mathfrak{P}_\rho(B)_{i_0,\ldots,i_4} \ . \tag{B.4.22}$$

Using the pairing between $\Gamma(\mathbb{A})$ and $\Gamma(\mathbb{A})^\vee$ we construct $\mu(A) \cup \mathfrak{P}_\rho(B) \in H^5\big(X, U(1)\big)$ as:

$$\big(\mu(A) \cup \mathfrak{P}_\rho(B)\big)_{i_0,\ldots,i_5} = \Big\langle \mu(A)_{i_0 i_1},\, \rho_{A_{i_0 i_1}} \mathfrak{P}_\rho(B)_{i_1,\ldots,i_4} \Big\rangle \ . \tag{B.4.23}$$

Under $A \to A + d\lambda$ we have

$$\mu(A) \cup \mathfrak{P}_\rho(B) \ \to\ \mu(A) \cup \mathfrak{P}_\rho(B) + d\Big(\mu(\lambda) \cup \mathfrak{P}_\rho(B)\Big) \ . \tag{B.4.24}$$

We conclude that the 5d inflow action is

$$S_\mu = 2\pi i \int_{X_5} \mu(A) \cup \mathfrak{P}_\rho(B) \ , \tag{B.4.25}$$

and its gauge variation on a manifold X_5 with boundary $X_4 = \partial X_5$ is

$$S_\mu \ \to\ S_\mu + 2\pi i \int_{X_4} \mu(\lambda) \cup \mathfrak{P}_\rho(B) \ . \tag{B.4.26}$$

B.5 Equivariantization for 2-algebras

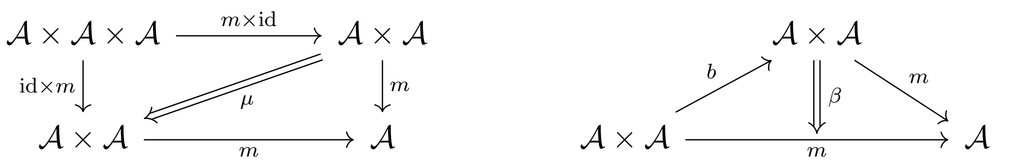
In this appendix we briefly review how the symmetry fractionalization datum η appears in the equivariantization of symmetric 2-algebras [43, 44]. We refer to those works for a complete introduction to the formalism, and here we limit ourselves to highlighting the main steps for our purposes. We assume that all objects are invertible for simplicity. We stress that a full definition of the equivariantization procedure is still an open problem.

In five dimensions, 2-categories are sylleptic [403], meaning that there exists a 2-morphism σ such that the braiding 1-morphism $b : X \times Y \to Y \times X$ satisfies the following commuting diagram:

$$\begin{array}{ccccc} X \times Y & & \xrightarrow{\quad \mathrm{id} \quad} & & X \times Y \\ & \searrow^{b} & \Downarrow \sigma & \nearrow^{b} & \\ & & Y \times X & & \end{array} \tag{B.5.1}$$

Physically $\sigma(X,Y)$ encodes the braiding data between 2d surfaces: $B_{X,Y} = \sigma(X,Y)/\sigma(Y,X)$. A symmetric 2-algebra $\mathcal{A}^{[2]}$ describes the gauging of a 2-categorical symmetry in five dimensions, and it is described by:[2]

- An object $\mathcal{A} = \bigoplus_{x\,\text{simple}} Z_x\, x \,\in \mathcal{C}$. We assume that $Z_x \in \{0,1\}$.
- A 1-morphism $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.
- Two 2-isomorphisms μ, β which uplift the associativity and commutativity relations for m to:



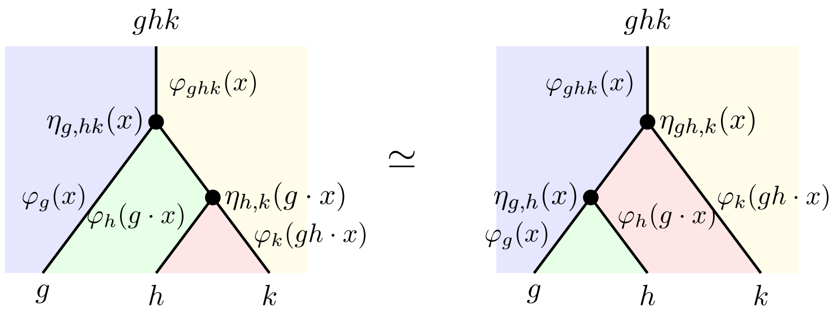
(B.5.2)

Besides, μ and β are subject to various higher algebraic identities that we do not report. Now, suppose that a 0-form symmetry G acts on the algebra $\mathcal{A}$, more specifically:

- To an object $x \in \mathcal{A}$ we associate a 1-isomorphism (an invertible line) $\varphi_g(x) : x \to g(x)$ localized on the surface U_g.
- To the algebra morphism m we associate a 2-morphism $\mu_g(x,y) : \varphi_g(x) \times \varphi_g(y) \to \varphi_g(x \times y)$, and similarly for the 2-morphism β.
- To an algebra 2-isomorphism we associate an identity for the equivariantization data.

All this data must be compatible with the natural multiplicative structure for G defects. In particular, we can consider the action of $ghk \simeq (gh)k \simeq g(hk) \in G$ on the algebra $\mathcal{A}$. Each three-valent junction $g \times h \to gh$ defines a 2-isomorphism $\eta_{g,h}(x) : \varphi_g(x) \times \varphi_h\big(g(x)\big) \to \varphi_{gh}(x)$. Clearly η is best though of as a 2-cochain $\eta(g,h)$ with values in the Pontryagin dual of $\mathcal{A}$. Consistency of η with the associativity of the G-defects implies (we draw below an horizontal section of the configuration):

$$\rho_g\eta(h,k)\;\eta(g,hk) = \eta(g,h)\;\eta(gh,k) \qquad \text{where} \qquad \rho_g\eta(h,k)[x] = \eta_{h,k}\big(g(x)\big)\;, \tag{B.5.3}$$



(B.5.4)

This means that $\eta \in H^2_\rho(G, \mathcal{A}^\vee)$. This datum matches the one in the symmetry fractionalization that we described in Section 4.1.2 for the 2d case.

[2]We omit the unit morphism and its higher morphisms, since they will not play a role in our presentation.

B.6 First obstruction and invariant TQFTs

In this last appendix we review the relationship between the first obstruction and the existence of duality-invariant TQFTs with symmetry $\mathbb{A}$. This was explained in [29] for the case of the Tambara-Yamagami category $\mathrm{TY}(\mathbb{A})_{\gamma,\epsilon}$ and in [78] for 4d theories with $\mathbb{Z}_n$ 1-form symmetry.

Self-dual TQFTs in 2d. A two-dimensional TQFT with symmetry $\mathbb{A}$ can be characterized by its unbroken subgroup $\mathbb{B}$ (in the sense of spontaneous symmetry breaking) and an SPT phase for $\mathbb{B}$ described by $\nu \in H^2\big(B\mathbb{B}, U(1)\big)$. The partition function is as in (4.1.32):

$$Z[B] = \begin{cases} \exp\big(2\pi i \int B^*\nu\big) & \text{if } \pi(B) = 0 \,, \\ 0 & \text{otherwise} \,, \end{cases} \tag{B.6.1}$$

where π is the projection in the short exact sequence $1 \to \mathbb{B} \xrightarrow{i} \mathbb{A} \xrightarrow{\pi} \mathbb{A}/\mathbb{B} \to 1$. The duality action on the TQFT is defined as

$$\mathcal{N} \cdot Z[B] = \frac{1}{\sqrt{|H^1(X, \mathbb{A})|}} \sum_{a \,\in\, H^1(X,\mathbb{A})} \exp\left(2\pi i \int_X a \cup_\gamma B\right) Z[a] \,. \tag{B.6.2}$$

Let us assume that ν satisfies (4.1.26)–(4.1.28). We evaluate the above equation on the torus T^2, however this can equivalently be done on any Riemann surface by choosing a decomposition of $H_1(\Sigma_g)$ in A- and B-cycles. We find:

$$\mathcal{N} \cdot Z[B_1,\, B_2] = \frac{1}{|\mathbb{A}|} \sum_{a_1, a_2 \,\in\, \mathbb{B}} \chi_\nu(a_1, a_2)\, \gamma(a_1, B_2)\, \gamma(a_2, B_1)^{-1} \,, \tag{B.6.3}$$

where we used that $Z[a] = 0$ if $\pi(a) \neq 0$ to restrict the sum. We first perform the sum over the subgroup $\mathrm{Rad}(\nu)$ of $\mathbb{B}$. This is zero unless:

$$B \in \phi^{-1}\left(N(\mathrm{Rad}(\nu))\right) \,, \tag{B.6.4}$$

this group coincides with $\mathbb{B}$ (and thus would give to correct delta function) if and only if $\phi(\mathrm{Rad}(\nu)) = N(\mathbb{B})$.[3] Thus (4.1.26) ensures that preserved subgroup is the same after the action of $\mathcal{N}$. Finally we perform the sum over (say) a_1 restricted to $\mathbb{B}/\mathrm{Rad}(\nu)$ which fixes $a_2 = \sigma B_2$ thus:

$$\mathcal{N} \cdot Z[B_1, B_2] = \gamma\big(\sigma B_2, B_1\big)^{-1}\, \delta\big(\pi(B_1)\big)\, \delta\big(\pi(B_2)\big) = Z[B_1, B_2] \,, \tag{B.6.5}$$

where (4.1.27) and (4.1.28) ensure that the SPT phase for $\mathbb{B}$ remains the same.

Self-dual TQFTs in 4d. A similar reasoning is valid in 4d. Without assuming whether the four manifold X is spin we can defined a 4d $\mathbb{A}^{(1)}$ TQFT by specifying the preserved subgroup $\mathbb{B}$ and an $\mathbb{B}$ SPT:

$$\nu \in H^2(B^2\mathbb{B},\, U(1)) \,, \tag{B.6.6}$$

[3]One uses the fact that, for any subgroup $\mathbb{B}$, $\phi^{-1}N(\phi^{-1}N(\mathbb{B})) = \mathbb{B}$.

and define

$$Z[B] = \begin{cases} \exp\left(2\pi i \int B^*\nu\right) \,, & \text{if } \pi(B) = 0 \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{B.6.7}$$

We then have[4]

$$\mathcal{N} \cdot Z[B] = \frac{1}{\sqrt{H^2(X, \mathbb{B})}} \sum_{a \,\in\, H^1(X, \mathbb{A})} \exp\left(2\pi i \int_X a \cup_\gamma B\right) \exp\left(2\pi i \int q_\nu(a)\right) \,. \tag{B.6.8}$$

First we perform the sum over $\text{Rad}(\nu)$. Since the bicharacter vanishes identically when evaluated on such elements it can be argued that its refinement q also does, independently on the choice of characteristic element. The sum then becomes linear and sets the result to zero unless

$$B \in \phi^{-1} N(\text{Rad}(\nu)) = \mathbb{B}\,. \tag{B.6.9}$$

The sum over the quotient $\mathbb{B}/\text{Rad}(\nu)$ is quadratic and we solve it by shifting $a \to a + \sigma(B)$, which decouples the two fields owning to

$$\chi_\nu(\sigma(a),\, b) = \gamma(a,\, b)^{-1}\,. \tag{B.6.10}$$

We are left with:

$$\begin{aligned} \mathcal{N} \cdot Z[B] \quad &= \begin{cases} G_\nu \, \exp\left(2\pi i \int B^* \left[2\nu + \nu \circ \sigma\right]\right) \,, & \text{if}\, \pi(B) = 0\,, \\ 0\,, & \text{otherwise} \end{cases} \,, \\ G_\nu \quad &= \frac{1}{\sqrt{H^2(X,\, \mathbb{B}/\text{Rad}(\nu))}} \sum_{a \,\in\, H^2(X, \mathbb{B}/\text{Rad}(\nu))} \exp\left(2\pi i \int a^* \nu\right) \,. \end{aligned} \tag{B.6.11}$$

If X is spin the quadratic refinement does not depend on the choice of characteristic element and we can lift identities for χ_ν to identities for q. Since

$$\chi_\nu(\sigma a,\, \sigma b) = \chi_\nu(a, b)^{-1}\,, \tag{B.6.12}$$

on spin manifolds $\nu \circ \sigma = -\nu$ as classes. Furthermore, it is possible [78] to prove that the Gauss sum G_ν is unity. We find that the TQFT $Z[B]$ is duality-invariant on spin manifolds. On the other hand, on non-spin manifolds, we need to impose the stronger condition:

$$q_\nu(\sigma B) = q_\nu(B)^{-1}\,, \quad B \in \mathbb{B}/\text{Rad}(\nu) \tag{B.6.13}$$

on the quadratic refinement of χ_ν. A similar story applies to triality defects with minimal modifications.

B.7 Anomalous boundary conditions

In this appendix we present an iterative procedure to consistently turn on a background for boundary theories with a $U(1)$ anomalous symmetry in generic even dimension. For the sake of

[4] This normalization makes sense on manifolds without torsion 2-cycles, on which we can trade it for the common one [57] by an Euler number counterterm.

concreteness we present this procedure in the simplest case of a $U(1)$ symmetry with anomaly, but the same idea can be used for higher groups and in the non-Abelian cases discussed in the main text. In general, the method presented here is necessary to determine consistent boundary conditions whenever the simple BF theory is modified by some non-Gaussian term containing derivatives.

Consider the TQFT with action

$$S = \frac{i}{2\pi} \int_{X_{d+1}} \left(b_{d-1} \wedge dA_1 + \kappa_d \, A_1 \wedge (dA_1)^{\frac{d}{2}} \right) , \qquad \kappa_d = \frac{k}{(2\pi)^{\frac{d}{2}-1} \left(\frac{d}{2}+1\right)!} , \tag{B.7.1}$$

and $k \in \mathbb{Z}$. In the presence of a boundary, the variation of the action produces a term

$$-\frac{i}{2\pi} \int_{\partial X_{d+1}} \left(b_{d-1} + \frac{d}{2} \kappa_d \, A_1 \wedge (dA_1)^{\frac{d}{2}-1} \right) \delta A_1 \, . \tag{B.7.2}$$

This can be cancelled by imposing the boundary condition

$$\star \, A_1 = -\frac{i}{R^2} \underbrace{\left(b_{d-1} + \frac{d}{2} \kappa_d \, A_1 \wedge (dA_1)^{\frac{d}{2}-1} \right)}_{\mathcal{T}_0} + \star \, \mathcal{A}_1 \tag{B.7.3}$$

and adding the boundary term

$$S_{\partial}^{(0)} = -\frac{1}{4\pi R^2} \int_{\partial X_{d+1}} \left(b_{d-1} + \frac{d}{2} \kappa_d \, A_1 \wedge (dA_1)^{\frac{d}{2}-1} \right) \wedge \star \left(b_{d-1} + \frac{d}{2} \kappa_d \, A_1 \wedge (dA_1)^{\frac{d}{2}-1} \right). \tag{B.7.4}$$

However, there is no gauge transformation of $\mathcal{A}_1$ that makes the boundary condition gauge invariant. The only way to have a gauge-invariant boundary condition is to add terms that mix $\mathcal{A}_1$ with the dynamical fields. The simplest such modification is to replace $\mathcal{T}_0$ in (B.7.3) with

$$\mathcal{T}_0' = \mathcal{T}_0 - \frac{d}{2} \kappa_d \, \mathcal{A}_1 \wedge (dA_1)^{\frac{d}{2}-1} \, . \tag{B.7.5}$$

Consequently we must modify the boundary term into

$$-\frac{1}{4\pi R^2} \int_{\partial X_{d+1}} \mathcal{T}_0' \wedge \star \, \mathcal{T}_0' \, . \tag{B.7.6}$$

However, since the boundary condition now imposes $\delta \mathcal{T}_0' = i R^2 \star \delta A_1$, we get an extra unwanted term in the variational principle:

$$-\frac{i}{2\pi} \int_{\partial X_{d+1}} \frac{d}{2} \kappa_d \, \mathcal{A}_1 \wedge (dA_1)^{\frac{d}{2}-1} \wedge \delta A_1 \, . \tag{B.7.7}$$

This can be cancelled by adding a topological term proportional to $\mathcal{A}_1 \wedge A_1 \wedge (dA_1)^{\frac{d}{2}-1}$ to the boundary term. Indeed

$$\int_{\partial X_{d+1}} \delta \Big(\mathcal{A}_1 \, A_1 \, (dA_1)^{\frac{d}{2}-1} \Big) = \int_{\partial X_{d+1}} \Big(\tfrac{d}{2} \, \mathcal{A}_1 \, (dA_1)^{\frac{d}{2}-1} \, \delta A_1 - \left(\tfrac{d}{2} - 1\right) d\mathcal{A}_1 \, A_1 \, (dA_1)^{\frac{d}{2}-2} \, \delta A_1 \Big) \, . \tag{B.7.8}$$

However, this also produces an extra term that must be cancelled. This is easily achieved by modifying both the boundary condition and the boundary term by the addition of this extra term to $\mathcal{T}_0'$. This produces

$$\mathcal{T}_1 = \mathcal{T}_0' + \kappa_d \left(\tfrac{d}{2} - 1\right) d\mathcal{A}_1\, A_1\, (dA_1)^{\frac{d}{2}-2}\,. \tag{B.7.9}$$

At the same time we modify the boundary term that, including the new topological term, becomes

$$S_\partial^{(1)} = -\frac{1}{4\pi R^2} \int_{\partial X_{d+1}} \mathcal{T}_1 \wedge \star \mathcal{T}_1 + \frac{i}{2\pi} \int_{\partial X_{d+1}} \kappa_d\, \mathcal{A}_1 \wedge A_1 \wedge (dA_1)^{\frac{d}{2}-1}\,. \tag{B.7.10}$$

These new boundary condition and boundary term give a consistent variational principle. However, the boundary condition is again non gauge invariant because of the last term we added to $\mathcal{T}_1$, and we have to repeat the procedure above.

At each step, the non-gauge-invariant piece in the boundary condition becomes of one lower degree in A_1 (and one higher in $\mathcal{A}_1$). Hence, the procedure stops when we reach a term linear in A_1: we can make the boundary condition gauge invariant by adding a term purely in $\mathcal{A}_1$, which does not modify the variational principle. The procedure stops after $(d/2-1)$ steps, yielding the boundary condition

$$\star \left(A_1 - \mathcal{A}_1\right) = -\frac{i}{R^2} \left(\Omega_{d-1} - \kappa_d\, \mathcal{A}_1\, (\mathcal{A}_1)^{\frac{d}{2}-1}\right) \tag{B.7.11}$$

where

$$\Omega_{d-1} = b_{d-1} + \kappa_d \sum_{r=0}^{\frac{d}{2}-2} \left(\tfrac{d}{2} - r\right) (d\mathcal{A}_1)^r \left(A_1 - \mathcal{A}_1\right) (dA_1)^{\frac{d}{2}-1-r} + \kappa_d\, (d\mathcal{A}_1)^{\frac{d}{2}-1} A_1\,. \tag{B.7.12}$$

The corresponding boundary term is

$$S_\partial = -\frac{1}{4\pi R^2} \int_{\partial X_{d+1}} \Omega_{d-1} \wedge \star\, \Omega_{d-1} + \frac{i\kappa_d}{2\pi} \sum_{r=0}^{\frac{d}{2}-2} \int_{\partial X_{d+1}} \mathcal{A}_1\, (d\mathcal{A}_1)^r A_1\, (dA_1)^{\frac{d}{2}-r-1}\,. \tag{B.7.13}$$

As a sanity check, we can verify that the boundary theory is anomalous under $U(1)$ gauge transformations. Under $\delta A_1 = \delta \mathcal{A}_1 = d\lambda_0$ the topological terms on the boundary produce

$$\begin{aligned}
&\frac{i\kappa_d}{2\pi} \sum_{r=0}^{\frac{d}{2}-2} \int_{\partial X_{d+1}} \left(d\lambda_0\, (d\mathcal{A}_1)^r A_1\, (dA_1)^{\frac{d}{2}-r-1} + \mathcal{A}_1\, (d\mathcal{A}_1)^r\, d\lambda_0\, (dA_1)^{\frac{d}{2}-r-1} \right) \\
&\quad = \frac{i\kappa_d}{2\pi} \sum_{r=0}^{\frac{d}{2}-2} \int_{\partial X_{d+1}} \lambda_0 \left((d\mathcal{A}_1)^{r+1} (dA_1)^{\frac{d}{2}-r-1} - (d\mathcal{A}_1)^r (dA_1)^{\frac{d}{2}-r} \right) \\
&\quad = \frac{i\kappa_d}{2\pi} \int_{\partial X_{d+1}} \left(\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} (dA_1) - \lambda_0\, (dA_1)^{\frac{d}{2}} \right).
\end{aligned} \tag{B.7.14}$$

Then, using the boundary condition,

$$\begin{aligned}
\delta S_\partial = \frac{i\kappa_d}{2\pi} \int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \left(A_1 - \mathcal{A}_1\right) - \frac{\kappa_d^2}{2\pi R^2} \int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \wedge \star \left((d\mathcal{A}_1)^{\frac{d}{2}-1} \mathcal{A}_1 \right) \\
- \frac{\kappa_d^2}{4\pi R^2} \int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \wedge \star \left(d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} \right) + \frac{i\kappa_d}{2\pi} \int_{\partial X_{d+1}} \left(\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1} (dA_1) - \lambda_0\, (dA_1)^{\frac{d}{2}} \right).
\end{aligned} \tag{B.7.15}$$

The bulk contributes with a term

$$\delta S = -\frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} d\lambda_0\, A_1\, (dA_1)^{\frac{d}{2}-1} \tag{B.7.16}$$

which, together with the last term in (B.7.14), combines to a total derivative (on the boundary) and can be neglected. We remain with

$$\delta S_{\rm tot} = -\frac{i\kappa_d}{2\pi}\int_{\partial X_{d+1}} d\lambda_0\, (d\mathcal{A}_1)^{\frac{d}{2}-1}\,\mathcal{A}_1 - \delta\left[\frac{\kappa_d^2}{4\pi R^2}\int_{\partial X_{d+1}} \Big(\mathcal{A}_1\,(d\mathcal{A}_1)^{\frac{d}{2}-1}\Big)\wedge\star\Big(\mathcal{A}_1\,(d\mathcal{A}_1)^{\frac{d}{2}-1}\Big)\right]. \tag{B.7.17}$$

We can isolate the anomalous variation adding a final counterterm

$$S_{\rm c.t.} = \frac{\kappa_d^2}{4\pi R^2}\int_{\partial X_{d+1}} \Big(\mathcal{A}_1\,(d\mathcal{A}_1)^{\frac{d}{2}-1}\Big)\wedge\star\Big(\mathcal{A}_1\,(d\mathcal{A}_1)^{\frac{d}{2}-1}\Big). \tag{B.7.18}$$

B.8 Non-compact TQFTs

In this appendix we provide a mathematical definition and details on the TQFTs with infinitely many operators introduced in [206, 210] and used as holographic duals. The main issue is defining the theory with cutting and gluing while avoiding infinities from inserting a complete basis of states. We argue that this is possible if all manifolds have at least one non-empty boundary component. On the other hand, the partition functions on closed manifolds will be generically infinite.

Review of standard TQFTs. Recall that standard TQFTs in d dimensions are defined by a symmetric monoidal functor $Z\colon \mathrm{Bord}_d^{\rm SO} \to \mathrm{Vec}_{\mathbb{C}}$ from the category of oriented bordisms to the category of complex vector spaces [203] (see *e.g.* [204] for a detailed review). A vector space $\mathcal{H}_{X_{d-1}} = Z(X_{d-1})$ is assigned to any closed codimension-one manifold and a linear map $Z(Y_d)\colon \mathcal{H}_{X_{d-1}} \to \mathcal{H}_{X'_{d-1}}$ to any bordism $Y_d\colon X_{d-1} \to X'_{d-1}$, namely an oriented manifold with boundary $\partial Y_d = X_{d-1} \sqcup \overline{X}'_{d-1}$ (here bar means orientation reversal) with *in* and *out* components given by X_{d-1} and X'_{d-1} respectively.[5] Functoriality implies that the vector space for a disjoint union is the tensor product, and gluing Y_d with Y'_d along a common boundary corresponds to composing linear maps.

In practice, it is convenient to work with an explicit basis. Hence, to concretely construct a TQFT we need the following ingredients:

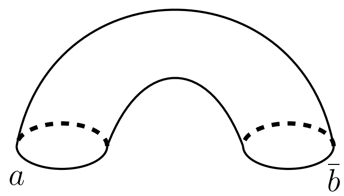
- Vector spaces $\mathcal{H}_{X_{d-1}}$ with a basis $|a\rangle$. We also denote by $|\bar{a}\rangle$ a basis of $\mathcal{H}_{\overline{X}_{d-1}}$.
- For any d-dimensional manifold Y_d with incoming and outgoing connected boundary components, respectively, $X^i_{d-1,\rm in}$, $i = 1, 2, \ldots$ and $X^j_{d-1,\rm out}$, $j = 1, 2, \ldots$ we assign a tensor $Z(Y_d)_{\{a_i\},\{b_j\}}$. This specifies the linear map $\bigotimes_i \mathcal{H}_{{\rm in},i} \to \bigotimes_j \mathcal{H}_{{\rm out},j}$

$$Z(Y_d)\Big(|a_1\rangle \otimes |a_2\rangle \otimes \cdots\Big) = \sum_{b_j} Z(Y_d)_{\{a_i\},\{b_j\}}\Big(|b_1\rangle \otimes |b_2\rangle \otimes \cdots\Big). \tag{B.8.1}$$

[5]The same manifold, with the same orientation, can be viewed as a bordism $\overline{X}'_{d-1} \to \overline{X}_{d-1}$.

Notice that the vector spaces $\mathcal{H}_{X_{d-1}}$ are not endowed with a scalar product as an extra datum: this simply arises from the composition of bordisms. To see this notice that, for any X_{d-1}, we can construct the cylinder $X_{d-1} \times [0,1]$ that can be viewed both as the *straight cylinder*, namely a bordism $X_{d-1} \to X_{d-1}$, or as the *horseshoe*, namely a bordism $X_{d-1} \otimes \overline{X}_{d-1} \to \emptyset$.[6] In the first case the functor Z associates the identity map $\mathrm{Id}_{\mathcal{H}_{X_{d-1}}} : \mathcal{H}_{X_{d-1}} \to \mathcal{H}_{X_{d-1}}$, while in the second case it gives a bilinear pairing $\eta(X_{d-1}) : \mathcal{H}_{X_{d-1}} \otimes \mathcal{H}_{\overline{X}_{d-1}} \to \mathbb{C}$. In components these read:

$\delta_{a,b} =$ 

$\eta(X_{d-1})_{a\bar{b}} =$ 

One can show that $\eta(X_{d-1})$ is a non-degenerate pairing that defines an isomorphism $\mathcal{H}_{\overline{X}_{d-1}} \cong \mathcal{H}^{\vee}_{X_{d-1}}$. This allows us to identify the basis $|\bar{a}\rangle$ of $\mathcal{H}_{\overline{X}_{d-1}}$ with the dual basis $\langle a|$ of $\mathcal{H}^{\vee}_{X_{d-1}}$ defined by $\langle a|b\rangle = \delta_{a,b}$:

$$|\bar{b}\rangle = \sum_a \eta_{a,\bar{b}} \, \langle a| \, . \tag{B.8.2}$$

With these pieces of data, it is clear how to glue various bordisms along common boundaries to generate others. The common boundaries must have opposite orientations. When one boundary is incoming and the other one is outgoing, the gluing is just the composition. On the other hand, if both are incoming (or both outgoing), we use $\eta(X_{d-1})_{a,\bar{b}}$. More concretely, let Y_d be a (possibly disconnected) bordism $\bigsqcup_i X^i_{d-1,\mathrm{in}} \to \bigsqcup_j X^j_{d-1,\mathrm{out}}$. If $X^1_{d-1,\mathrm{in}} = X^1_{d-1,\mathrm{out}}$ we can generate $\widetilde{Y}_d$ by gluing the two, and the associated tensor is

$$Z\big(\widetilde{Y}_d\big)_{\{a_2,\dots\},\{b_2,\dots\}} = \sum\nolimits_{a_1} Z(Y_d)_{\{a_1,a_2,\dots\},\{a_1,b_2,\dots\}} \, . \tag{B.8.3}$$

If instead $X^1_{d-1,\mathrm{in}} = \overline{X}^2_{d-1,\mathrm{in}}$ the tensor associated with the manifold obtained by gluing along these boundary components is

$$Z\big(\widetilde{Y}_d\big)_{\{a_3,\dots\},\{b_1\}} = \sum\nolimits_{a_1,a_2} Z(Y_d)_{\{a_1,a_2,a_3,\dots\},\{b_1,\dots\}} \, \eta\big(X^1_{d-1,\mathrm{in}}\big)_{a_1,a_2} \, . \tag{B.8.4}$$

Clearly, these pieces of data cannot be arbitrary: if the same manifold Y_d can be constructed in different ways by gluing smaller pieces, the results must coincide. Once these consistency conditions are satisfied, we can compute the tensor associated with any manifold starting from those associated with the more elementary pieces. By performing enough gluings to get a closed manifold, the result is a number: the partition function. For instance, gluing the outgoing and the incoming boundary of a cylinder $X_{d-1} \times [0,1]$ we get $X_{d-1} \times S^1$, hence

$$Z\big(X_{d-1} \times S^1\big) = \sum\nolimits_a \delta_{a,a} = \dim\big(\mathcal{H}_{X_{d-1}}\big) \, . \tag{B.8.5}$$

[6]The vector space associated with the empty $(d-1)$-dimensional manifold is $\mathcal{H}_\emptyset = \mathbb{C}$.

The non-compact case. Already the fact (B.8.5) suggests that in the non-compact case closed bordisms should not be included in the definition. We want to argue that, avoiding closed manifolds, there are classes of manifolds in which we can give a precise definition of the $U(1)/\mathbb{R}$ BF-like theories

$$S = \frac{i}{2\pi} \int_{\mathcal{M}_d} b_{d-p-1} \wedge dA_p \,. \tag{B.8.6}$$

As an illustration, we consider the case of $d = 2$ with $p = 1$. Hence $b_0 = \phi$ is a non-compact scalar, and A is a $U(1)$ gauge field. The Hilbert space $\mathcal{H}_{S^1}$ can be constructed by canonical quantization. We set $\mathcal{M}_2 = S^1 \times \mathbb{R}$, with $\mathbb{R}$ parametrized by t, and split $A = \widetilde{A} + A_0^t\, dt$. Then

$$S = -\frac{i}{2\pi} \int_{S^1\times\mathbb{R}} \Big(A_0^t\, \tilde{d}\phi \wedge dt + \phi\, \partial_t \widetilde{A} \wedge dt \Big) \,. \tag{B.8.7}$$

We choose the temporal gauge $A_0^t = 0$, and we need to impose the Gauss law $\tilde{d}\phi = 0$, namely $\phi = \phi(t)$ is independent of the spatial coordinate. Introducing

$$q(t) = \int_{S^1} \widetilde{A} \,, \qquad\qquad p(t) = \frac{1}{2\pi} \phi(t) \,, \tag{B.8.8}$$

we see that $q(t) \sim q(t) + 2\pi$ is a periodic variable, and the action becomes

$$S = -i \int_{\mathbb{R}} p\, \partial_t q \, dt \,. \tag{B.8.9}$$

This is a free infinitely-massive particle on a circle of radius 2π. The quantization is straightforward. We have the commutation relations

$$[\hat{q}, \hat{p}] = i \qquad\qquad \Rightarrow \qquad\qquad e^{i\alpha\hat{p}} \cdot e^{in\hat{q}} = e^{i\alpha n}\, e^{in\hat{q}} \cdot e^{i\alpha\hat{p}} \,. \tag{B.8.10}$$

Here $n \in \mathbb{Z}$ because of the periodicity of $\hat{q}$, while α is a generic real number. However the operator $e^{2\pi i\hat{p}}$ commutes with the whole operator algebra, hence it is a number that we can set to 1. Therefore the operators acting on the Hilbert space are

$$\widehat{\mathcal{O}}_\alpha = e^{i\alpha\hat{p}} \quad \text{with} \quad \alpha \in [0, 2\pi) \,, \qquad\qquad \widehat{W}_n = e^{in\hat{q}} \quad \text{with} \quad n \in \mathbb{Z} \,, \tag{B.8.11}$$

with algebra

$$\widehat{\mathcal{O}}_\alpha\, \widehat{\mathcal{O}}_\beta = \widehat{\mathcal{O}}_{\alpha+\beta \text{ (mod } 2\pi)} \,, \qquad \widehat{W}_n\, \widehat{W}_m = \widehat{W}_{n+m} \,, \qquad \widehat{\mathcal{O}}_\alpha\, \widehat{W}_n = e^{i\alpha n}\, \widehat{W}_n\, \widehat{\mathcal{O}}_\alpha \,. \tag{B.8.12}$$

Starting from a simultaneous eigenstate of the $\widehat{W}_n$'s such that

$$\widehat{W}_n \,|\theta\rangle = e^{in\theta}\, |\theta\rangle \,, \tag{B.8.13}$$

using the algebra we find

$$\widehat{\mathcal{O}}_\alpha \,|\theta\rangle = |\theta - \alpha\rangle \,. \tag{B.8.14}$$

Hence we get a basis labelled by a compact continuous variable $\theta \in U(1)$. We can also use a non-compact but countable basis, starting with an eigenstate of $\widehat{\mathcal{O}}_\alpha$:

$$\widehat{\mathcal{O}}_\alpha \,|k\rangle = e^{i\alpha k}\, |k\rangle \,. \tag{B.8.15}$$

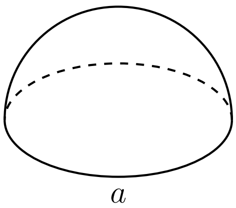
It must be $k \in \mathbb{Z}$ to respect the periodicity $\alpha \sim \alpha + 2\pi$. Then using the algebra we infer

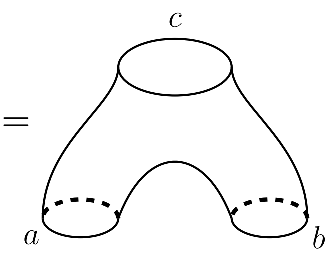
$$\widehat{W}_n \, |k\rangle = |k+n\rangle \, . \tag{B.8.16}$$

The relation between the two basis is

$$|k\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \, e^{ik\theta} \, |\theta\rangle \, , \qquad\qquad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{-ik\theta} \, |k\rangle \, . \tag{B.8.17}$$

Since the Hilbert space is infinite dimensional, the partition function on T^2 is infinite. Let us show that, on the other hand, we can consistently define a functor on the category of open oriented bordisms. In 2d the huge computational simplifications are that the only Hilbert space is $\mathcal{H}_{S^1}$, and that every 2d manifold has a pair of pants decomposition. Eventually, one also needs to *fill holes* by attaching a disk. Hence, on top of the horseshoe η_{ab}, the only other data one needs to assign are the disk and the pair of pants:

$h_a =$ 

$\mu^c_{ab} =$ 

The numbers h_a define a distinguished state $|HH\rangle = \sum_a h_a |a\rangle$, called the Hartle–Hawking state. These two data must satisfy the obvious condition that if we fill one of the two incoming holes of the pair of pants with the Hartle–Hawking state we get the cylinder:

$$\sum\nolimits_b \mu^c_{ab} \, h_b = \delta_{a,c} \, . \tag{B.8.18}$$

The only other consistency condition is the independence from the chosen pair of pants decomposition, that reduces to the Froboenius condition [205]:

$$\sum\nolimits_c \mu^c_{a,b} \, \mu^e_{c,d} = \sum\nolimits_c \mu^e_{a,c} \, \mu^c_{b,d} \, . \tag{B.8.19}$$

Let us use the continuous basis $|\theta\rangle$. The cylinder (identity) becomes a delta function $\delta(\theta_1 - \theta_2)$. Moreover, we define

$$h_\theta = \delta(\theta) \, , \qquad \eta_{\theta_1,\theta_2} = \delta(\theta_1 + \theta_2) \, , \qquad \mu^{\theta_3}_{\theta_1,\theta_2} = \delta(\theta_1 + \theta_2 - \theta_3) \, . \tag{B.8.20}$$

Also, all sums are replaced by integrals on $[0, 2\pi)$ in this basis. The condition (B.8.18) is obviously satisfied, while the Froboenius condition (B.8.19) reads

$$\int_0^{2\pi} d\theta \, \delta(\theta_1 + \theta_2 - \theta) \, \delta(\theta + \theta_3 - \theta_4) = \int_0^{2\pi} d\theta \, \delta(\theta_1 + \theta - \theta_4) \, \delta(\theta_2 + \theta_3 - \theta) \tag{B.8.21}$$

which is satisfied since both sides are equal to $\delta(\theta_1 + \theta_2 + \theta_3 - \theta_4)$. The choice of these data is motivated by the fact that the continuous basis $|\theta\rangle$ is related, by the state/operator correspondence, with the local operators $\mathcal{O}_\alpha(x) = e^{i\frac{\alpha}{2\pi}\phi(x)}$, and the pair of pants must reproduce their OPE $\mathcal{O}_\alpha \, \mathcal{O}_\beta = \mathcal{O}_{\alpha+\beta}$. Then the Hartle–Hawking state is fixed by (B.8.18).

With these pieces of data, we can compute the value of the functor for arbitrary bordisms with a non-empty boundary. The simplest nontrivial such manifold is the torus with a puncture. This can be obtained from the pair of pants by gluing one of the two incoming boundaries with the outgoing one. Denoting by θ the label of the puncture, namely the non-glued circle, the result is[7]

$$Z\big(\Sigma_1 \smallsetminus P_\theta\big) = \int_0^{2\pi} d\theta'\ \delta(\theta) = 2\pi\,\delta(\theta)\,. \tag{B.8.22}$$

This is a projector on the Hartle–Hawking state. Another simple example is the torus with two punctures that can be obtained from the previous result by gluing the remaining boundary to the outgoing boundary of another pair of pants. Hence, the result is

$$Z\Big(\Sigma_1 \smallsetminus \{P_{\theta_1}, P_{\theta_2}\}\Big) = \int_0^{2\pi} d\theta'\ \delta(\theta_1+\theta_2-\theta')\,2\pi\delta(\theta') = 2\pi\,\delta(\theta_1+\theta_2)\,. \tag{B.8.23}$$

We can now put these two examples together, gluing the boundary of a torus with one puncture to one of the two boundaries of the torus with two punctures, resulting in a genus-two surface with a puncture:

$$Z\big(\Sigma_2 \smallsetminus P_\theta\big) = \int_0^{2\pi} d\theta'\ 2\pi\delta(\theta+\theta')\,2\pi\delta(\theta') = (2\pi)^2\,\delta(\theta)\,. \tag{B.8.24}$$

Proceeding in this way it is not hard to prove the general result. The value of the functor an a genus g surface with n incoming boundaries labelled by $\theta_1,\dots,\theta_n$ and m outgoing boundaries labelled by $\theta'_1,\dots,\theta'_m$ is given by

$$Z\Big(\Sigma_g \smallsetminus \{P_{\theta_1},\dots,P_{\theta_n},P_{\theta'_1},\dots,P_{\theta'_m}\}\Big) = (2\pi)^g\ \delta\big(\theta_1+\ldots+\theta_n-\theta'_1-\ldots-\theta'_m\big)\,. \tag{B.8.25}$$

The important observation is that the partition function on compact Riemann surfaces is infinite. Indeed, a compact Riemann surface of genus g is obtained by closing the hole of a one-punctured Riemann surface $\Sigma_g \smallsetminus P_\theta$ by means of gluing the Hartle–Hawking state. The result is clearly infinite:

$$Z(\Sigma_g) = \int_0^{2\pi} d\theta\ (2\pi)^g\,\delta(\theta)\,\delta(\theta) = (2\pi)^g\,\delta(0)\,. \tag{B.8.26}$$

We conclude that the TQFT is well defined on the category of open oriented bordisms.

Let us remark that, given the Hilbert space we constructed, there is another set of data that can be formulated, which is essentially the same as the one we discussed but in the discrete basis $|k\rangle$:

$$h'_k = \delta_{k,0}\,,\qquad \eta'_{k_1,k_2} = \delta_{k_1,-k_2}\,,\qquad (\mu')^{k_3}_{k_1,k_2} = \delta_{k_1+k_2,k_3}\,. \tag{B.8.27}$$

With these data one gets infinite answers even on open manifolds, as soon as they have a non-trivial topology. It must be noticed that, indeed, these are not merely the data (B.8.20) written in a different basis: translating (B.8.20) in the discrete basis using (B.8.17) we get

$$h_k = \frac{1}{\sqrt{2\pi}}\,,\qquad \eta_{k_1,k_2} = \delta_{k_1,k_2}\,,\qquad \mu^{k_3}_{k_1,k_2} = \sqrt{2\pi}\ \delta_{k_1,k_2}\,\delta_{k_1,k_3}\,. \tag{B.8.28}$$

[7]We denote a genus g Riemann surface as Σ_g.

We conclude that (B.8.20) and (B.8.27) really define two different TQFTs.

How did we choose one instead of the other? As we already pointed out, in 2d TQFT the choice is really dictated by the fact that the pair of pants is related with the OPE of local operators. The data (B.8.27) would then be relevant for the TQFT with Lagrangian formulation

$$S' = \frac{i}{2\pi} \int_{\mathcal{M}_2} \Phi \, da_1 \,, \tag{B.8.29}$$

where $\Phi \sim \Phi + 2\pi$ is a compact scalar, while a_1 an $\mathbb{R}$ gauge field. Canonical quantization produces the same Hilbert space as the theory with non-compact scalar and $U(1)$ gauge field; however, here the local operators $\mathcal{O}_n(x) = e^{in\Phi(x)}$ are labeled by an integer, and hence are related with the discrete basis by the state/operator correspondence. For this reason, in contrast to the previous case, the quantization of this theory produces the data (B.8.27) in which the pair of pants gives the Abelian fusion algebra in the discrete basis.

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