SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

Physics Area
PhD course in Theoretical Particle Physics

Symmetries, Anomalies, and Phases of the Chiral Gross-Neveu Model

Candidate:
Riccardo Ciccone

Advisor:
Prof. Marco Serone
Co-advisor:
Prof. Lorenzo Di Pietro

Academic Year 2022-23

Abstract

The chiral Gross-Neveu model, in the large N limit and at finite temperature and density, displays spontaneous symmetry breaking of its axial $U(1)$ symmetry and of space translations to a diagonal subgroup, signalled by a complex condensate periodic in space - dubbed *chiral spiral*. In this thesis we analytically investigate the possibility of remnants of such condensate, both at zero and finite temperature, in the finite N setup. Using the tool of non-Abelian bosonization, with a careful treatment of certain global aspects, we are able to relate the model to a $SU(N) \times U(1)$ Wess-Zumino-Witten model with appropriate levels. At $T = 0$, we are able to rigorously prove that the chiral spiral configuration seen at large N persists at any finite $N \geq 2$. At finite T, instead, we study the two-point function of certain composite fermion operators and we are able to predict that at finite temperature spatially modulated structures still exist, provided that one puts periodic boundary conditions for fermions in the thermal cycle. This distinction is motivated by the existence of an 't Hooft anomaly for a D_8^F symmetry in the fermionic theory. In support of the finite N results we also rederive the large N chiral Gross-Neveu model phase diagram, assuming different boundary conditions for fermions, with a direct diagrammatic computation.

Declaration of authorship. I, Riccardo Ciccone, declare that this thesis and the work presented in it are my own. Wherever contributions of others are involved, every effort is made to indicate this clearly with due reference to the literature and acknowledgement of collaborative research. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma.
This thesis is based on the following two papers:

[1] R. Ciccone, L. Di Pietro, M. Serone. “On the Inhomogeneous Phase of the Chiral Gross-Neveu Model”. *Phys. Rev. Lett.* 129 (2022) 7, 071603, [hep-th/2203.07451]

[2] R. Ciccone, L. Di Pietro, M. Serone. “Anomalies and Persistent Order in the Chiral Gross-Neveu Model”. arXiv preprint, [hep-th/2312.13756]

Acknowledgments

Spesso le nostre giornate si complicano
mentre le perplessità rimangono qui
E ci si sposta lontano
in un orizzonte più strano
E i conti già fatti non tornano mai

First and foremost, I thank my advisors. I thank Marco Serone for his wise guidance, for his encouragement, and for the help he has given me to give well-defined shapes to my chaotic intuitions. He is to me one the most professional physicists.

I thank Lorenzo Di Pietro for always being available to listen, for his enthusiasm and curiosity, and his much-needed practical approach to problems. I also thank him for the shared passions beside Physics.

I thank all the people that I met during these years, both at SISSA and outside, and that have left a positive mark on me. From SISSA I particularly thank Fabiana, Francesco, Giorgia, Maria Vittoria, Mario, Matteo, Niloofar, Saman, Uriel, and Warren. From the outside world, I thank in particular Cosimo, Elisa, Emanuele, Jaime, Lorenzo, Peter, Roukaya, Simone, and Simone. I also thank all the people that I met that fight to make our society a better place despite the difficulties.

I thank my parents Attilio and Gelmina, that gave more than they possibly could to help me get where I am now. I thank my sister Valentina for always being a point of reference for me and for encouraging me to pursue a Ph.D. in Physics.

I thank Pietro for all the love, joy, and support that he brings into my life every day.

Contents

Introduction

Strongly-coupled systems comprise some of the most interesting and fundamental physical phenomena, ranging from the behavior of quark-gluon plasma in high-energy particle physics to the one of condensed matter systems like strongly-correlated electrons in materials. As most of their properties arise from the emergent collective behavior of their constituents, the specific features of such systems can be quantitatively described in terms of a small set of parameters, such as their temperature and the density of particles, or the values of certain couplings, and summarized in phase diagrams. By varying those parameters, one can encounter phase transitions, where the system undergoes a sudden qualitative change in its macroscopic properties.

Different phases are characterized by different realizations of the symmetries of the system. This perspective is neatly encoded in the Landau theory of conventional phases of matter [3] which, combined with Renormalization Group (RG) methods [4, 5], allows for a quantitive description of many phase transitions. However, only transitions between a symmetry-broken and a symmetry-preserved phase are well-captured by the original formulation of the Landau paradigm. More exotic scenarios, such as the confinement-deconfinement transition in non-Abelian gauge theories [6], integer [7] and fractional quantum Hall states [8] and other intrinsically topologically ordered phases [9, 10], topological insulators/superconductors [11] and other symmetry-protected topological phases [12], and Landau's Fermi-liquid theory itself [13], to name a few, require at the very least an appropriate extension of the concept of symmetry [14, 15] in order to be understood within this framework.

An essential ingredient for this extension is the incorporation of 't Hooft anomalies [16, 17] which, being preserved under RG flow, constrain how symmetries can act on the low-energy degrees of freedom. In particular, the presence of an 't Hooft anomaly for a given symmetry tells us that the system will sit in a phase that is either (i) gapless, (ii) symmetry-broken, or (iii) intrinsically topologically ordered [18–21]. In recent years, many works from the condensed matter and the high-energy theory communities have been deploying extensively this approach to rederive known results and infer new ones, see e.g. [22] for a modern review.

An interesting class of phases is the ones whose symmetry breaking pattern involves spacetime symmetries. This is indeed the normal situation in solid state physics, for instance in the change of the crystalline structure that relates different allotropes of metals, or at the fusion point of a crystal. In particular, the existence of inhomogeneous phases, where the ground state of the theory breaks a subset of the translational symmetries, has been found in dense systems. For example, certain Bardeen-Cooper-Schrieffer superconductors [23] subject to strong external magnetic fields display a so-called Fulde-

Ferrell-Larkin-Ovchinnikov phase, with a spatially non-uniform order parameter [24, 25]. In one-dimensional metals, the Peierls instability can induce a transition to a phase with a periodic fluctuation in the electron density, i.e. to chiral density wave states [26, 27]. Similar modulated structures can also be found for the spin density, for instance in several organic linear-chain compounds [28, 29].

Inhomogeneous phases have also been conjectured to form in a range of temperatures and matter densities also in relativistic theories, such as Quantum Chromodynamics (QCD). Before discussing them, let us review what is the current (at the time of writing) scientific consensus on the QCD phase diagram, as a function of the temperature T and of the chemical potential μ_B for the $U(1)_B$ baryon number symmetry (see e.g. [30] for a recent review). The strongly-coupled nature of QCD at low energies prevents a fully analytical treatment because, neglecting quark masses, this theory has no small fundamental parameters to be used for a perturbative expansion, the only independent intrinsic scale being the dynamically generated confinement scale $\Lambda_{\mathrm{QCD}} \sim 332$ MeV. Exact treatments are possible only in extreme situations such as in the high-temperature or high-density limit where the thermodynamics is dominated by the weakly-coupled short-distance physics, or in artificial regimes such as in a large N_c expansion [31]. For most practical purposes, one has to rely on a lattice approach, i.e. the idea of discretizing spacetime and computing physical observables from first-principles Monte-Carlo simulations. The lattice is particularly effective at probing the $\mu_B = 0$ axis: at low temperatures, there is a phase of massive neutral hadrons, separated from the high-temperature phase of deconfined quarks and gluons by a crossover phase transition. For $\mu_B > 0$, lattice computations suffer from the sign problem, and a Monte-Carlo approach is not reliable in this case. Nevertheless, it is predicted that the crossover phase transition becomes first order, with a second order critical point in between, and it is believed that the transition line curves to intersect the $T = 0$ axis at a finite μ_B^*. An analytic treatment predicts that at very large μ_B QCD is in a color-flavor-locked superconducting phase [32].

It is at intermediate values of the chemical potential (and at low temperatures) that inhomogeneous phases can possibly appear. Evidence that cold dense quark matter at large N_c might form standing chiral waves, i.e. configurations where only a linear combination of chiral symmetry and translations is linearly realized, has been first provided in [33]. Such chiral waves have been subsequently shown to be disfavored in actual QCD [34], but the possibility of other inhomogeneous phases in QCD has reemerged during the years, see e.g. [35–37]. If they exist, it has been conjectured that their signature might be detected by studying the properties of compact astrophysical objects such as neutron stars [38]. Modern evidence that neutron stars can have deconfined quark matter cores [39] leaves the door for this possibility open.

Let us also mention that also for QCD one can constrain the possible infrared phases by requiring 't Hooft anomaly matching, see e.g. [40] for notable efforts in this direction. Still, the anomaly matching condition provides only a consistency check, and it seems unlikely that this argument alone will lead to a definitive answer.

Due to the limited availability of analytical and numerical tools, it is difficult to rigorously assess the phase diagram of QCD in full glory. For this reason, it is interesting to understand the phase diagram of simpler theories that qualitatively resemble QCD, such as four-Fermi theories. Originally introduced to describe the weak interaction [41], four-Fermi theories in four spacetime dimensions have proven to be useful effective models for strongly-interacting matter at finite density [42, 43], despite their failure at describing features caused by gauge degrees of freedom, such as the confinement-deconfinement transition.

Another renowned route comes from studying certain four-Fermi theories in two spacetime dimensions [44]. These theories, like QCD, are UV-free and undergo dynamical mass generation in the IR. One version of these models, the Gross-Neveu model, has maximal $O(2N)$ vector symmetry, a mass gap, and features spontaneous breaking of a $\mathbb{Z}_2$ chiral symmetry. Another version of the theory, the chiral Gross-Neveu model, possesses a $U(N)$ vector symmetry and a $U(1)$ chiral symmetry instead. The latter theory is gapless in the IR, having in its spectrum a massless compact boson. Studies of the phase diagrams of both the ordinary and the chiral Gross-Neveu models at large N at finite temperature T and in the presence of a $U(1)$ baryon number chemical potential μ were initially conducted assuming translational invariance in [45] and [46], respectively. However, subsequent research has challenged this assumption, leading to revised large N phase diagrams that incorporate inhomogeneous phases [47–50]. These include a crystal phase in the ordinary Gross-Neveu model, with a spatially modulated order parameter at low T and sufficiently large μ, and a phase reminiscent of chiral density waves, dubbed "chiral spiral", in the chiral Gross-Neveu model, with a spatially periodic phase with μ-dependent period at low T for any $\mu > 0$. In recent years, the emergence of inhomogeneous phases in these models at finite N has started being investigated using numerical lattice methods, see e.g. [51, 52] for the ordinary and [53] for the chiral Gross-Neveu model.

In two spacetime dimensions, arguments relying on symmetries for the determination of phase diagrams are more powerful. The main reason is that the physics is much more constrained. On the one hand, breaking of a continuous symmetry is impossible due to the absence of massless Nambu-Goldstone bosons [54]. Correspondingly, there is no strictly ordered symmetry-breaking phase, but possibly only a symmetry-preserving quasi-long-range ordered phase, that is a phase in which the two-point function of the would-be order parameter decays at large distances only with a power-like behavior, separated from the usual disordered phase by a Berezinskii-Kosterlitz-Thouless transition [55–57]. Moreover, there are no bosonic phases with intrinsic topological order [58], the only non-trivial fermionic topologically ordered phase being the p-wave state of the Kitaev chain [59], thus diminishing the number of possible IR scenarios consistent with an 't Hooft anomaly. On the other hand, studying a $1 + 1d$ system at finite temperature is equivalent, via Wick rotation to (periodic) Euclidean time, to studying the same system at zero temperature and finite spatial length. Taking the temperature to be large, in this second picture the spatial volume shrinks to a point, and one is studying an effective quantum mechanical system. If there is an 't Hooft anomaly that survives compactification, then this will reflect on the nature of the vacuum in the quantum mechanics. For instance, this has been used to constrain the IR behavior of adjoint QCD in $1 + 1d$ with a single Majorana flavor [60, 61], where an important role for the persistence of the anomalies is played by the boundary conditions along the compactified direction.

In this thesis, we extend the analysis of the phase diagram of the chiral Gross-Neveu model to a finite number of Dirac flavors N.

Using non-Abelian bosonization [62], we relate the chiral Gross-Neveu model to a $SU(N)_1 \times U(1)_N$ WZW model deformed by current-current interactions. In the bosonized language, it is surprisingly simple to show that the chiral spiral phase of the large-N chiral Gross-Neveu model is present also at any finite N, at $T = 0$ for any $\mu > 0$ [1]. The chiral wave corresponds to a quasi long-range ordered gapless phase, in contrast to strict long-range order appearing at large N, with a free massless relativistic excitation. In this phase the would-be order parameter is neutral under IR spatial translations that are a linear combination of the original translations and the axial $U(1)$ symmetry.

In order to discuss phases at finite temperature [2], we are forced to review the non-Abelian bosonization procedure while taking additional care of global aspects. This is because global identifications by discrete symmetries have important effects when the theory is placed on a non-trivial manifold, such as $\mathbb{R} \times S^1$. In doing so we find that, in order to keep manifest a unitary flavor symmetry $SU(N) \times U(1)$ in the bosonized theory, the specification of the corresponding levels for the WZW models, as well as the precise set of global identifications, has a dependence on the parity of the number of Dirac flavors N which is not manifest on $\mathbb{R}^2$. Moreover, we are able to infer the existence of a certain 't Hooft anomaly for a D_8^F symmetry in the fermionic theory. This is a $\mathbb{Z}_2$-valued anomaly, that we detect from an anomaly for a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the $U(1)$ compact boson of the bosonized theory. This anomaly is activated in the presence of a non-trivial background for the fermion parity symmetry $\mathbb{Z}_2^F$, allowing us to predict the existence of persistent order when fermions are constrained to be periodic under the compact (Euclidean time) direction. We are able to check this claim explicitly by computing two-point functions of certain composite fermion operators in the interacting theory, both for thermal (antiperiodic) and periodic boundary conditions for fermions. We find that for thermal fermions a remnant of the spatial modulation is still present, but it has an exponentially decaying amplitude due to thermal effects. The exponential decay is absent instead for periodic boundary conditions, consistently with the anomaly argument, leading to persistent order.

As an additional byproduct, we have also developed a simple diagrammatic method for rederiving the large-N phase diagram that allows for a certain class of spatial inhomogeneities. This method leads to results consistent with the ones of [47, 50], which are based on finding ansätze that solve certain self-consistency equations. We also extend it to the case of periodic fermions, where we find again persistent order of the chiral spiral configuration, in agreement with the results at finite N.

Outline of the thesis. The work is organized as follows.
In Chapter 1 we review the known results about the large N phase diagrams of the ordinary and the chiral Gross-Neveu models, while setting the notation.
In Chapter 2, instead, we review some of the notions about two-dimensional Quantum Field Theory which we will use in the following Chapters. This concludes the review part.
In Chapter 3 we perform non-Abelian bosonization of the chiral Gross-Neveu model in full detail. While this procedure is not conceptually novel, we will perform it employing the full power of the modern tools we have presented in Chapter 2, and we will discover that the correct bosonization procedure depends on the parity of the number of flavors N.
In Chapter 4 we present the results for the allowed phases of the chiral Gross-Neveu model at finite N. At $T = 0$, we are able to rigorously prove that the chiral spiral configuration seen at large N persists at any finite $N \geq 2$. At finite T, instead, we study the two-point function of certain composite fermion operators and we are able to predict that at finite temperature spatially modulated structures still exist, provided that one puts periodic boundary conditions for fermions in the thermal cycle.

In Chapter 5 we rederive the phase diagram of the large N chiral Gross-Neveu model via a direct diagrammatic computations, assuming different periodicity conditions for fermions along the thermal cycle, finding agreement with previous results in the literature, as well as with the results of Chapter 4.

In Appendix A we discuss the free-field realization of the $J\bar{J}$-deformed $SU(N)_1$ model, with a special attention to the case of $N = 2$. We are able to prove that the classical potential has exactly N degenerate vacua, and for $N = 2$ we compute exactly $\langle \mathrm{Tr}(U) \rangle$ in the interacting theory, checking explicitly that it condenses.

In Appendix B we perform CFT computations that are useful in deriving the results of Section 4.3, for which also the identities reported in Appendix C are useful.

Chapter 1

Gross-Neveu Models

In this Chapter we review the physics of a special type of interacting theories of fermions in $d = 1+1$, that is the models put forward by Gross and Neveu [44], over which this thesis focuses. Being well-studied models, the literature that concerns them is quite vast. Here we distill only the knowledge that is useful to understand better the physical framework and motivation of this thesis work.

In order to simplify our understanding of such interacting theories, it will be convenient to distinguish them by which subset of the symmetries of the free theory is preserved. For this reason, and to introduce notation, we start by reviewing the global internal symmetries of free fermions in two dimensions in Section 1.1. We then present the ordinary and the chiral Gross-Neveu model in Sections 1.2 and 1.3, respectively. In both cases we also discuss the salient features that appear in the limit where the number of flavors is large, $N \to \infty$, namely chiral symmetry breaking and dynamical mass generation. In Section 1.4 we review the literature results about the thermodynamics of these models, both at large N and from the lattice.

1.1 Free fermions

Let us consider first a free theory of M massless Majorana fermions,

$$\mathcal{L}_0 = \frac{i}{2}\bar{\xi}_j \partial\!\!\!/ \xi^j \,, \qquad j = 1, \ldots, M \,. \tag{1.1}$$

Here ξ^j are two-components Majorana spinors,

$$\xi^j = \begin{pmatrix} \xi^j_+ \\ \xi^j_- \end{pmatrix} , \tag{1.2}$$

with $\xi^j_\pm$ being left- and right-moving Majorana-Weyl spinors, respectively. We work in a Euclidean setting with the following γ-matrices convention,

$$\gamma^1 = \sigma_1 \,, \qquad \gamma^2 = -i\sigma_2 \,, \qquad \gamma_* = \gamma^1\gamma^2 = \sigma_3 \,, \tag{1.3}$$

γ_* being the chirality matrix. This theory has global symmetry group

$$
\begin{aligned}
G(M\,\text{Majorana}) &= O(M)_L \times O(M)_R \\
&= \begin{cases} (SO(M)_L \times SO(M)_R) \times (\mathbb{Z}_2^{F_L} \times \mathbb{Z}_2^{F_R})\,, & M \text{ odd}\,, \\ (SO(2N)_L \times SO(2N)_R) \rtimes (\mathbb{Z}_2^{C_L} \times \mathbb{Z}_2^{C_R})\,, & M = 2N\,,\ N \text{ odd}\,, \\ (SO(2N)_L \times SO(2N)_R) \rtimes (\mathbb{Z}_2^{K_L} \times \mathbb{Z}_2^{K_R})\,, & M = 2N\,,\ N \text{ even}\,, \end{cases}
\end{aligned}
\tag{1.4}
$$

where we have also broken down $O(M)_{L,R}$ into their connected and disconnected parts. The action of $O(M)_{L,R}$ is linear on left- and right-moving Majorana-Weyl spinors, respectively,

$$
\begin{cases} O(2N)_L : & \xi_+^j \mapsto \xi_+^j\,, \qquad \xi_-^j \mapsto L^j{}_k \xi_-^k\,, \\ O(2N)_R : & \xi_+^j \mapsto R^j{}_k \xi_+^k\,, \qquad \xi_-^j \mapsto \xi_-^j\,. \end{cases}
\tag{1.5}
$$

For M odd, $-\mathbf{1} \in O(M) \setminus SO(M)$, and global fermion parity $\mathbb{Z}_2^F$ is the diagonal subgroup of $\mathbb{Z}_2^{F_L} \times \mathbb{Z}_2^{F_R}$.[1] The off-diagonal one, $\mathbb{Z}_2^\chi$, forbids the presence of a mass term for an odd number of Majorana fermions.

For $M = 2N$, the matrix $-\mathbf{1}$ belongs to $SO(2N)$, therefore fermion parity is connected to the identity, $\mathbb{Z}_2^F \subset Z(SO(2N)_L \times SO(2N)_R)$. Note that there are further differences between N even and odd. For N odd, we can define the matrix $C = \left(\begin{smallmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{smallmatrix}\right)$, that has determinant -1. C does not commute with $SO(2N)$ rotations, hence the semidirect product, leaving $SO(2N)$ as a normal subgroup of $O(2N)$. Its physical meaning is charge conjugation, and this is made manifest upon defining Dirac spinors,

$$
\Psi^a = \xi^a + i\xi^{a+N}\,, \qquad a = 1, \ldots, N\,.
\tag{1.6}
$$

For N even, instead, $C \in SO(2N)$. We can however define a reflection matrix K, for instance as the unit matrix with the first entry replaced by -1, that generates the $\mathbb{Z}_2^K$ factors in (1.4). For N odd, C is equivalent to K via an $SO(2N)$ rotation, so we can always use K to extend $SO(2N)$ to $O(2N)$.

1.2 Gross-Neveu Model

The Gross-Neveu model [44] is obtained from the free theory of N Dirac fermions by adding a scalar-scalar quartic interaction,

$$
\mathcal{L}_{GN} = i\bar{\Psi}_a \partial\!\!\!/ \Psi^a - \frac{\lambda_s}{2N}(\bar{\Psi}_a \Psi^a)^2\,, \qquad a = 1, \ldots, N\,.
\tag{1.7}
$$

The interaction explicitly breaks the symmetry group (1.4) to

$$
G(N\text{-flavor GN}) = O(2N)_V \times \mathbb{Z}_2^A\,,
\tag{1.8}
$$

where $O(2N)_V$ is the diagonal subgroup of $O(2N)_L \times O(2N)_R$, and $\mathbb{Z}_2^A$ is a chiral $\mathbb{Z}_2$ reflection,

$$
\mathbb{Z}_2^A : \quad \Psi^a \longmapsto \gamma_* \Psi^a\,.
\tag{1.9}
$$

[1] In this work we will always include fermion parity $\mathbb{Z}_2^F$ among the global internal symmetries of the theory, even though it is also a spacetime symmetry. The full global symmetry group of the theory will consist of both internal and spacetime symmetries, with $\mathbb{Z}_2^F$ being identified.

In our basis, a Dirac spinor Ψ^a splits into its Weyl components as

$$\Psi^a = \begin{pmatrix} \psi_+^a \\ \psi_-^a \end{pmatrix}\,, \tag{1.10}$$

so that the nontrivial $\mathbb{Z}_2^A$ element acts as $\psi_\pm^a \mapsto \pm\psi_\pm^a$.

Let us consider the theory in the $N \to \infty$ limit, obtained while keeping λ_s fixed. The Euclidean partition function,

$$Z = \int \mathrm{D}\Psi\, \mathrm{D}\bar{\Psi} e^{-S[\bar{\Psi},\,\Psi]}\,, \tag{1.11}$$

can be rewritten using the Hubbard-Stratonovich trick to get rid of the 4-fermi term by introducing an auxiliary scalar field σ. We write

$$1 = \int \mathrm{D}\sigma\; e^{-\frac{N}{2\lambda_s}\int \mathrm{d}^2x \left[\sigma + \frac{\lambda_s}{N}(\bar{\Psi}\Psi)\right]^2}\,, \tag{1.12}$$

In order to preserve the $\mathbb{Z}_2^A$ symmetry, we ask that the auxiliary field σ is odd under $\mathbb{Z}_2^A$. Plugging (1.12) in the partition function and integrating out the fermions one gets

$$Z = \int \mathrm{D}\sigma e^{-NS[\sigma]}\,, \qquad S[\sigma] = \int \mathrm{d}^2x \frac{\sigma^2}{2\lambda_s} - \mathrm{tr}\log(i\partial\!\!\!/ + \sigma)\,, \tag{1.13}$$

where the trace is over Dirac indices only (flavor indices have already been traced over).

The expression (1.13) is formally valid at any N, but in the large N limit the integral collapses to the contribution coming from the absolute minimum of $S[\sigma]$. Let us assume that such minimum is attained for some homogeneous configuration σ_*. Then, one obtains the so-called gap equation,

$$-\frac{\delta}{\delta\sigma}S[\sigma]\bigg|_{\sigma_*} = V'(\sigma_*) = \frac{\sigma_*}{\lambda_s} - \mathrm{tr}(i\partial\!\!\!/ + \sigma_*)^{-1} = 0\,. \tag{1.14}$$

This equation needs a regularization in order to be solved for σ_*. Let us introduce a UV cutoff Λ in the energy (but not in the spatial momenta). After the trace is performed, the gap equation reads

$$\frac{\sigma_*}{\lambda_s} = 2\sigma_* \int^\Lambda \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{p^2+\sigma_*^2}\,. \tag{1.15}$$

The integral is IR-divergent for fixed λ_s, Λ when $\sigma_* \to 0$, making $V''(0)$ large and negative. Therefore the trivial solution $\sigma_* = 0$ is a maximum, and σ aquires a nonzero expectation value, $\langle\sigma\rangle = \sigma_* \neq 0$. Explicitly,

$$\frac{1}{\lambda_s} = 2\int_0^\Lambda \frac{\mathrm{d}p_0}{2\pi}\frac{1}{\sqrt{p_0^2+\sigma_*^2}} = \frac{1}{\pi}\,\mathrm{arsinh}\,\frac{\Lambda}{|\sigma_*|}, \tag{1.16}$$

from which we read off, at large values of the cutoff Λ,

$$\sigma_* \simeq \pm 2\Lambda e^{-\pi/\lambda_s} \equiv \pm M. \tag{1.17}$$

Such solutions have to be minima since the potential grows at infinity. The theory is UV-free and M plays the role of the dynamically generated scale. Notice that if one reinstates fermions the latter acquire a non-zero mass equal to M. Since σ is $\mathbb{Z}_2^A$-odd, the $\mathbb{Z}_2^A$ symmetry gets spontaneously broken.

1.3 Chiral Gross-Neveu model

The Gross-Neveu model admits a generalization with a full $U(1)_A$ chiral symmetry, also called chiral Gross-Neveu model, where the scalar-scalar interaction is replaced as follows,

$$\mathcal{L} = i\bar{\Psi}_a \partial\!\!\!/ \Psi^a - \frac{\lambda_s}{2N}[(\bar{\Psi}_a\Psi^a)^2 + (\bar{\Psi}_a i\gamma_* \Psi^a)^2]\,. \tag{1.18}$$

Its global symmetry group reads

$$G(N\text{-flavor } cGN) = U(N)_V \times U(1)_A\,, \tag{1.19}$$

where $U(N)_V$ is the $U(N)$ subgroup of $O(2N)_V$ appearing in (1.8), and $U(1)_A$ acts as a chiral rotation of angle α,

$$U(1)_A : \Psi^a \longmapsto e^{i\alpha\gamma_*}\Psi^a\,, \qquad \alpha \sim \alpha + 2\pi\,. \tag{1.20}$$

The Lagrangian (1.18), however, is not the most complete one compatible with the symmetry (1.19), since the vector-vector interaction

$$\frac{\tilde{\lambda}_v}{2N}(\bar{\Psi}_a\gamma_\mu\Psi^a)^2\,, \tag{1.21}$$

also respects it. If not included, such quartic operator will be generated radiatively by the coupling λ_s. On the other hand, in the vector-like large-N expansion, one finds that the coupling associated to the quartic vector-vector operator has a vanishing β-function at leading order, and as such one can consistently set $\tilde{\lambda}_v = \lambda_v/N$ [63]. Therefore, we define

$$\mathcal{L}_{cGN} = i\bar{\Psi}_a \partial\!\!\!/ \Psi^a - \frac{\lambda_s}{2N}[(\bar{\Psi}_a\Psi^a)^2 + (\bar{\Psi}_a i\gamma_*\Psi^a)^2] + \frac{\lambda_v}{2N^2}(\bar{\Psi}_a\gamma_\mu\Psi^a)^2\,. \tag{1.22}$$

We can repeat the large-N analysis for the chiral version of the model. We need to introduce real auxiliary fields for each fermion bilinear appearing in the interaction,

$$1 = \int \mathrm{D}\sigma\,\mathrm{D}\pi\,\mathrm{D}v_\mu\; e^{-\frac{N}{2\lambda_s}\int \mathrm{d}^2x\left[\left(\sigma+\frac{\lambda}{N}\bar{\Psi}\Psi\right)^2+\left(\pi+\frac{\lambda}{N}\bar{\Psi}\mathrm{i}\gamma_*\Psi\right)^2\right]} e^{-\frac{N^2}{2\lambda_v}\int \mathrm{d}^2x(v_\mu+\frac{\lambda_v}{N^2}i\bar{\Psi}\gamma_\mu\Psi)^2}\,, \tag{1.23}$$

so that after integrating out the fermions one obtains

$$Z = \int \mathrm{D}\sigma\,\mathrm{D}\pi\,\mathrm{D}v_\mu e^{-N\left[\int \mathrm{d}^2x\frac{\sigma^2+\pi^2}{2\lambda_s}+\frac{N}{2\lambda_v}\int \mathrm{d}^2x v_\mu^2 - \mathrm{tr}\log(i\partial\!\!\!/+i v\!\!\!/+\sigma+\mathrm{i}\pi\gamma_*)\right]}\,. \tag{1.24}$$

At leading order in large N, the action is minimized for

$$v_*^\mu = 0\,, \qquad \frac{\sigma_*}{\lambda_s} = \int^\Lambda \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{2\sigma_*}{p^2+\sigma_*^2+\pi_*^2}, \qquad \frac{\pi_*}{\lambda_s} = \int^\Lambda \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{2\pi_*}{p^2+\sigma_*^2+\pi_*^2}. \tag{1.25}$$

Notice that we could pack the gap equations for σ, π into a single one for the complex scalar field $\Delta \equiv \sigma + \mathrm{i}\pi$,

$$\frac{\Delta_*}{\lambda_s} = \int^\Lambda \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{2\Delta_*}{p^2+|\Delta_*|^2}. \tag{1.26}$$

The field Δ is also charged under the chiral $U(1)_A$ symmetry, with charge 2,

$$U(1)_A : \qquad \Delta \longmapsto e^{2i\alpha}\Delta. \tag{1.27}$$

As before, the trivial vacuum is a local maximum, and therefore the true vacuum would sit at $|\Delta_*| = M \neq 0$. Any specific choice of the vacuum breaks the continuous $U(1)$ chiral symmetry. This would be in contrast with a theorem by Coleman-Mermin-Wagner [54, 64], that states that continuous global symmetries cannot be broken in two spacetime dimensions.

To understand what is happening, it is better to use a polar decomposition for the auxiliary fields [65],

$$\Delta = \sigma + i\pi \equiv \rho e^{i\theta} \,. \tag{1.28}$$

Now we can expand around a solution of the gap equation, $\rho_* = |\Delta_*|$, without breaking the axial $U(1)$ symmetry. However, one should not assume an expectation value for the phase field θ, which under $U(1)_A$ transforms as

$$U(1)_A: \qquad \theta \longmapsto \theta + 2\alpha \,, \qquad \alpha \sim \alpha + 2\pi \,. \tag{1.29}$$

Nevertheless, the presence of a nonzero expectation value for ρ gives nonzero mass to fermions.

Let us consider the effective action in terms of ρ, θ variables,

$$S[\rho, \theta] = \int \mathrm{d}^2 x \, \frac{\rho^2}{2\lambda_s} - \operatorname{tr}\log(i\partial\!\!\!/ + \rho e^{i\theta\gamma_*}) \,. \tag{1.30}$$

Expanding the effective action in terms of θ on top of the constant $\rho = M$ background, one finds that, at leading order in derivatives,

$$S_{\text{eff}}[\theta] = \frac{N}{8\pi} \int \mathrm{d}^2 x \, (\partial_\mu \theta)^2 \tag{1.31}$$

We can then interpret the auxiliary field θ as a compact scalar with radius N. With these variables, the breaking of $U(1)_A$ (or, the lack thereof) can be probed by computing the correlation function

$$\langle \bar{\Psi}(1+\gamma_5)\Psi(x)\bar{\Psi}(1-\gamma_5)\Psi(0)\rangle \sim \langle \rho e^{-\mathrm{i}\theta}(x) \rho e^{\mathrm{i}\theta}(0)\rangle \,, \tag{1.32}$$

If we are interested in the long-distance behavior of this correlator, we can trust the above effective action for θ and also treat ρ as a constant. Therefore the above is, up to constants,

$$\langle e^{-\mathrm{i}\theta(x)} e^{\mathrm{i}\theta(0)} \rangle \sim |x|^{-2/N}. \tag{1.33}$$

This correlator vanishes at large distances, as it should to ensure that the symmetry is preserved. But it goes to zero at a very slow rate at large N, and if one takes the large N limit before the large distance limit, the symmetry would look indeed broken, but $1/N$ corrections would then be IR-divergent, so we cannot trust the expansion away from the $N = \infty$ point. For any large, finite N, we find then "quasi-long-range" order, that is a phase in which the correlation length would look infinite, but the correlation function of charged operators still decays, albeit with a power-like behavior.

1.4 Thermodynamics at large N and from the lattice

It is fairly easy to generalize the gap equations (1.15), (1.26) to include the presence of an arbitrary temperature T and of a chemical potential μ for the $U(1)_V$ symmetry. The latter is dealt with by adding to the Lagrangian of the model the explicit coupling to the charge density,

$$\mathcal{L}_\mu = \mu \Psi_a^\dagger \Psi^a \,, \tag{1.34}$$

whereas the temperature is taken into account by replacing the integration over the energy coming from the fermion determinant with a sum over Matsubara modes,

$$\int \frac{\mathrm{d}p_0}{2\pi} f(p_0) \longrightarrow T \sum_{n\in\mathbb{Z}} f\left(2\pi(n+\tfrac{s}{2})T\right) \,, \tag{1.35}$$

where $s = 1$ for antiperiodic (i.e. thermal) boundary conditions for fermions around the compact Euclidean time direction, and $s = 0$ for periodic boundary conditions. Since literature focused mainly on the thermal case, in this section we work with $s = 1$.

With these modifications, (1.15) becomes

$$\frac{\sigma_*}{\lambda_s} = T \sum_{|p_0|<\Lambda} \int \frac{\mathrm{d}p_1}{2\pi} \frac{2\sigma_*}{(p_0+i\mu)^2 + p_1^2 + \sigma_*^2}. \tag{1.36}$$

A trivial solution to the gap equation is $\sigma_* = 0$. We look for nontrivial solutions. We can readily integrate over p_1,

$$\frac{1}{\lambda_s} = T \sum_{|p_0|<\Lambda} \frac{1}{\sqrt{(p_0+\mathrm{i}\mu)^2 + \sigma_*^2}}. \tag{1.37}$$

The right hand side is divergent. On the other hand, λ_s is the bare 't Hooft coupling, so it is also a divergent quantity. We remove UV divergences by renormalizing. In particular, we impose the $T = \mu = 0$ gap equation to write

$$\frac{1}{\lambda_s} \simeq \frac{1}{2\pi} \log \frac{2\Lambda}{M}. \tag{1.38}$$

Using the above relation to remove the cutoff Λ, we are led to

$$\log \frac{\pi T}{M} - \gamma = \sum_{n=0}^{\infty} \left(\mathrm{Re}\, \frac{1}{\sqrt{(n+1/2+\mathrm{i}\mu/(2\pi T))^2 + (\sigma_*/(2\pi T))^2}} - \frac{1}{n+1/2} \right), \tag{1.39}$$

where γ is the Euler-Mascheroni constant. The right hand side is finite, and the above gives a relation between physical quantities that does not renormalize. A formally identical relation holds for the chiral Gross-Neveu model, upon replacing σ_* with ρ_*.

By studying (1.39), a first proposal for the large-N phase diagram of the Gross-Neveu model was put forward in [45] and [46], see Figure 1.1. It displays two phases: an ordered phase at low T and low μ which features massive fermions, and a symmetric phase at high T and/or μ in which fermions are massless. The two phases are separated by a line of second-order phase transitions AB, which becomes a line of first-order phase transitions BD at the multicritical point B. We stress that the exact values of the (T, μ) coordinates of each point are renormalization scheme-dependent; the ones that appear in Figure 1.1 are derived using the renormalization condition (1.38).

However, in this derivation there has always been one tacit assumption, that is the spatial homogeneity of the condensate, $\langle\sigma_*(x)\rangle \equiv \langle\sigma_*\rangle$. There are good reasons to doubt

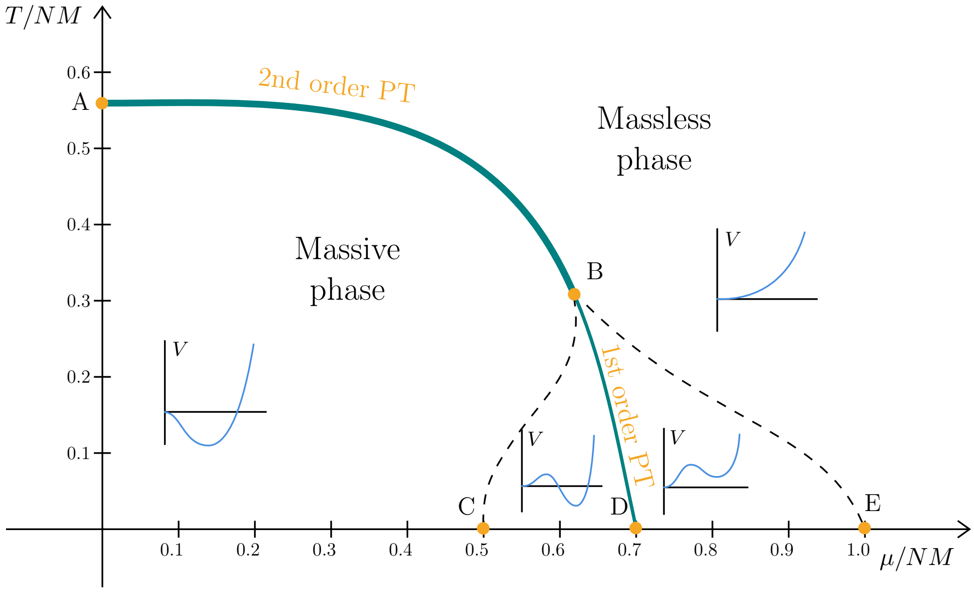


Figure 1.1: Large-N phase diagram of both the ordinary and the chiral Gross-Neveu models, derived assuming translational invariance. Adapted from [45, 66].

this assumption. At large N, the Gross-Neveu model possesses baryons, i.e. multi-fermion bound states [67]. In the case of the ordinary Gross-Neveu model, the mass of the lightest baryon is $M_B = 2NM/\pi$, whereas the chiral version of the model also has a massless baryon. Being baryons solitonic solutions of the field equations, they naturally break translational invariance. Given that they carry $U(1)_V$ charge, they are expected to have possibly non-zero density at finite chemical potential μ, and it is reasonable to expect that this can imply a loss of translational invariance.

In light of these considerations, the large-N phase diagram was rederived for both the ordinary Gross-Neveu model [48] and the chiral Gross-Neveu model [47, 50], this time without assuming translational invariance. The novelty is the appearance of a so-called "crystal" phase, that is a phase where the chiral condensate is periodic in space.

For the ordinary Gross-Neveu model, such phase appears at low temperatures and high enough chemical potential, see Figure 1.2. In this phase, the spatial profile of the condensate is determined from the following ansatz,

$$\langle \sigma_*(x) \rangle = A\kappa^2 \frac{\mathrm{sn}(Ax,\kappa)\,\mathrm{cn}(Ax,\kappa)}{\mathrm{dn}(Ax,\kappa)} \tag{1.40}$$

where $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ is the Jacobi elliptic functions, and $A \geq 0$ and $0 \leq \kappa \leq 1$ are parameters over which one minimizes the effective potential. Most importantly, the condensate is periodic, with period found to be $\ell = 2\mathbf{K}(\kappa)/A$, $\mathbf{K}(\kappa)$ being the quarter period, and it satisfies

$$\langle \sigma_*(x + \ell/2) \rangle = -\langle \sigma_*(x) \rangle\,, \tag{1.41}$$

that is, a shift by half-period $\ell/2$ corresponds to a chiral $\mathbb{Z}_2^A$ transformation. In other words, $\mathbb{Z}_2^A$ transformations combine with continuous spatial translations, which correspond to a non-compact $\mathbb{R}_{\mathrm{trans}}$ group, in such a way that only a diagonal $\mathbb{Z}_2 \times \mathbb{Z}_{\mathrm{trans}}$ subgroup is

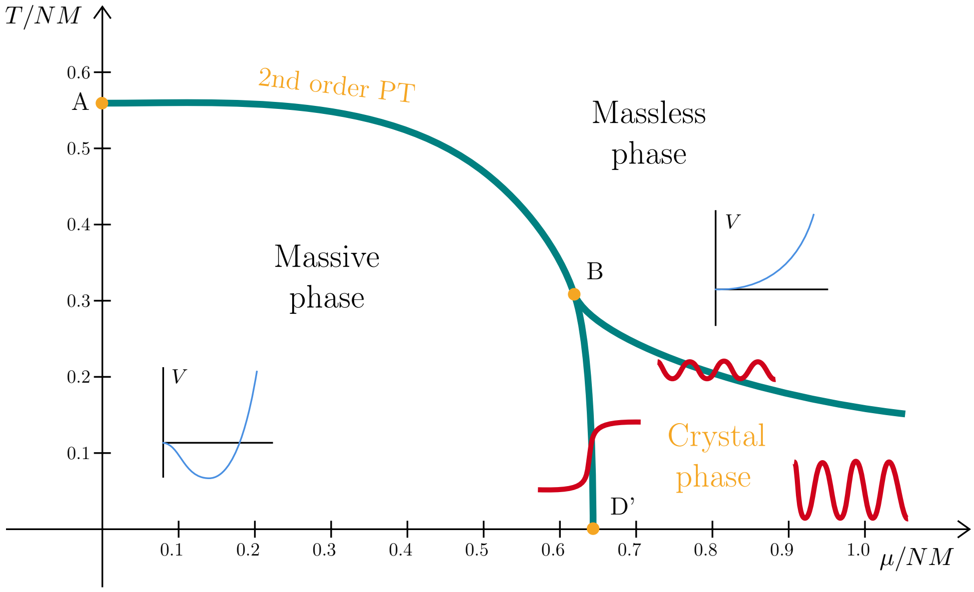


Figure 1.2: Large-N phase diagram of the ordinary Gross-Neveu model, derived without assuming translational invariance. In red we report the spatial profile of the condensate $\langle\sigma_*(x)\rangle$. Adapted from [66].

preserved, $\mathbb{Z}_{\text{trans}}$ being spatial translations by ℓ, $x \mapsto x + \ell$.

For the chiral Gross-Neveu model, the difference between the large-N phase diagrams derived with and without assuming translational invariance is even more remarkable, see Figure 1.3. In the latter case, the phase with homogeneous condensate completely disappears, and at low enough temperatures the system is always in a crystal phase, and the condensate

$$\langle\Delta_*(x)\rangle = M(T)e^{2i\mu x} , \tag{1.42}$$

where M is a non-negative parameter that depends only on the temperature, and has the interpretation of the thermal mass of the fermions. At $\mu \neq 0$, the condensate is manifestly periodic with period $\ell = \pi/|\mu|$, and

$$\langle\Delta_*(x + \alpha/\mu)\rangle = e^{2i\alpha}\langle\Delta_*(x)\rangle , \tag{1.43}$$

signalling that chiral $U(1)_A$ transformations and spatial translations $\mathbb{R}_{\text{trans}}$ get broken to a diagonal $U(1) \times \mathbb{Z}_{\text{trans}}$ subgroup.

On the one hand, the presence of such interesting structures at large N raises the question about whether these structures persist at finite number of flavors, and if yes to which extent. These models have been studied on the lattice, see e.g. [52] for a study of the phase diagram for the ordinary Gross-Neveu model at $N = 2, 8, 16$, and [53] for the phase diagram of the chiral Gross-Neveu model at $N = 2, 8$. In both cases, the numerics is compatible with the persistence of structures with spatial modulation. In the ordinary Gross-Neveu model, this feature is particularly robust and a phase diagram quite similar to the one in Figure 1.2 has been put forward. In the chiral Gross-Neveu model, it has

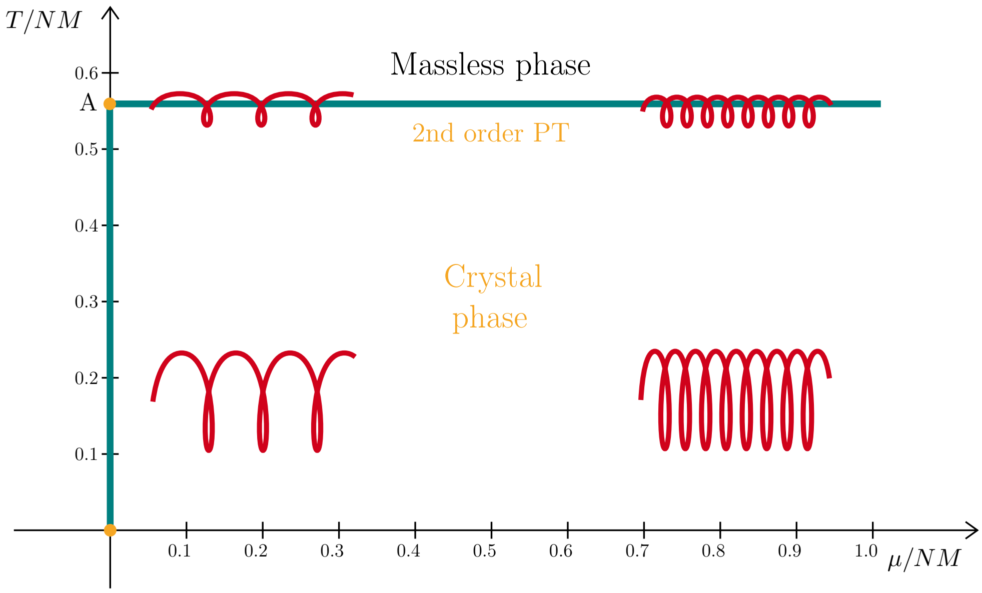


Figure 1.3: Large-N phase diagram of the chiral Gross-Neveu model, derived without assuming translational invariance. In red we report the spatial profile of the condensate $\langle \Delta_*(x) \rangle$. Adapted from [50].

instead deemed more likely that, at finite temperature, spatial modulation is detectable only at intermediate scales.

On the other hand, at finite N the lack of a small parameter has made most analytical approaches, up to the work presented in this thesis, prohibitive.

Chapter 2

Toolbox in $2d$ Field Theory

In the following Chapters, we will address the question of the fate of the inhomogeneous structures once again from the analytical side, focusing on the chiral Gross-Neveu model. To do so, we will need tools more refined than the large N expansion. The idea is to rephrase the model and the problem at hand in a dual set of variables, in particular through *bosonization.* At the free point, a set of M Majorana fermions is dual to a $Spin(M)_1$ Wess-Zumino-Witten (WZW) model [62]. The amount of information that is needed to make this statement precise is reviewed, in a distilled way, in this Chapter.

We begin by reminding in Section 2.1 the modern viewpoint on symmetries in Quantum Field theory, both for generic number of spacetime dimensions d and for the specific case of $d = 2$. In Section 2.2 we review the construction of Rational Conformal Field Theories (RCFTs) such as the WZW models and the compact scalar, as well as their relationship with Chern-Simons theories in three dimensions. In Section 2.3 we discuss in detail the set of well-known dualities that connect bosonic and fermionic Quantum Field Theories in $d = 2$.

2.1 Symmetries and Anomalies

The notion of symmetry is a crucial one in Physics and in Quantum Field Theory. It is also a notion whose definition has evolved considerably during the history of the field, and (at the time of writing) the most comprehensive one is the one that *defines* them as the set of topological operators of a given theory [14]. Historically, the relation went the other way around: topological operators were the ones defined from symmetries. In Subsection 2.1.1 we shall condense the basic concepts, and we will move to specific discussion and examples in $2d$ field theories in Subsection 2.1.2.

2.1.1 Symmetries $\equiv$ Topological Operators

Topological operators from symmetries. The first notion of symmetry that is usually encountered in textbooks is that of a continuous (zero-form) symmetry group G for a d-dimensional field theory, for which one can define a Noether current J that is conserved, $d * J = 0$, and correspondingly a conserved charge $Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} *J$, by integrating over a codimension one hypersurface Σ_{d-1}.[1] Conservation of the current implies Ward

[1] We assume for simplicity to be in an Euclidean setting, so Σ_{d-1} is always spacelike.

identities, which means that when inserted in correlation functions of charged operators, the charge $Q(\Sigma_{d-1})$ gives, schematically,

$$\langle Q(\Sigma_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \sum_{i=1}^{n}\Theta_{\Sigma_{d-1}}(x_i)\langle\mathcal{O}_1(x_1)\cdots\delta\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{2.1}$$

where $\Theta_{\Sigma_{d-1}}(x) = 1$ if the point x links with Σ_{d-1}, and 0 otherwise, and $\delta\mathcal{O}(x)$ is the symmetry variation of $\mathcal{O}(x)$. Since the action of $Q(\Sigma_{d-1})$ inserted in the correlation function $\langle\prod_i \mathcal{O}_i(x_i)\rangle$ does not depend on the specific shape or position of Σ_{d-1}, as long as in doing so Σ_{d-1} does not cross any of the insertion points x_i, $Q(\Sigma_{d-1})$ defines a topological operator.

Functions of $Q(\Sigma_{d-1})$ will also be topological. To fix ideas, let us consider the case of $G = U(1)$. Then $g \in G$ can be parametrized as $g = e^{i\alpha}$, $\alpha \sim \alpha + 2\pi$. Let us define then

$$U_{g=e^{i\alpha}}(\Sigma_{d-1}) = \exp\left(i\alpha Q(\Sigma_{d-1})\right)\,, \tag{2.2}$$

By exponentiating (2.1), we see that $U_{g=e^{i\alpha}}(\Sigma_{d-1})$ implements the $U(1)$ transformation of parameter α, as $\delta\mathcal{O}_i = i\alpha q_i\mathcal{O}_i$, with q_i being the $U(1)$ charge of $\mathcal{O}_i$.

More generally, one can define an unitary operator $U_g(\Sigma_{d-1})$ that implements the action of a group element $g \in G$, i.e. such that upon replacing $Q(\Sigma_{d-1})$ by $U_g(\Sigma_{d-1})$ in (2.1), one gets

$$\langle U_g(\Sigma_{d-1})\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \prod_{i=1}^{n}\langle\mathcal{O}_1(x_1)\cdots R_i(g)[\mathcal{O}_i(x_i)]\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{2.3}$$

if the operator $\mathcal{O}_i(x)$ transforms in the representation R_i of the symmetry group G, and for simplicity we have assumed that $\Theta_{\Sigma_{d-1}}(x_i) = 1$ for all i. This definition does not require the existence of a current and thus extends naturally also to discrete groups. By acting with two such operators in sequence, the group law is automatically implemented: if we define an ordering of the actions by saying that Σ^+_{d-1} is a slight deformation of Σ_{d-1}, such that operators supported on the former act after ones of the latter, then

$$U_g(\Sigma^+_{d-1})U_h(\Sigma_{d-1}) = U_{gh}(\Sigma_{d-1})\,, \quad U_g(\Sigma^+_{d-1})U_{g^{-1}}(\Sigma_{d-1}) = U_{g^{-1}}(\Sigma^+_{d-1})U_g(\Sigma_{d-1}) = 1\,, \tag{2.4}$$

which must be read as an identity holding inside correlation functions.

Symmetries from topological operators. Anytime a theory has a symmetry, it is possible to couple it to background fields. Again, let us consider for simplicity a $U(1)$ symmetry. This can be done via minimal coupling, i.e. by inserting a source term in correlators,

$$\langle\cdots\rangle \longrightarrow \langle e^{-\int A\wedge * J}\cdots\rangle\,, \tag{2.5}$$

with A denoting the background field, and the integral is over the whole spacetime. A background gauge transformation of parameter λ acts as $A \mapsto A + d\lambda$. If A is flat, $dA = 0$, then

$$[A] \in \frac{\ker d}{\operatorname{im} d} \equiv H^1(M_d, U(1))\,, \tag{2.6}$$

with $H^1(M_d, U(1))$ being the first singular cohomology group on spacetime M_d with coefficients in $U(1)$. By Poincaré duality, $[A] \in H^1(M_d, U(1))$ is dual to an element in homology,

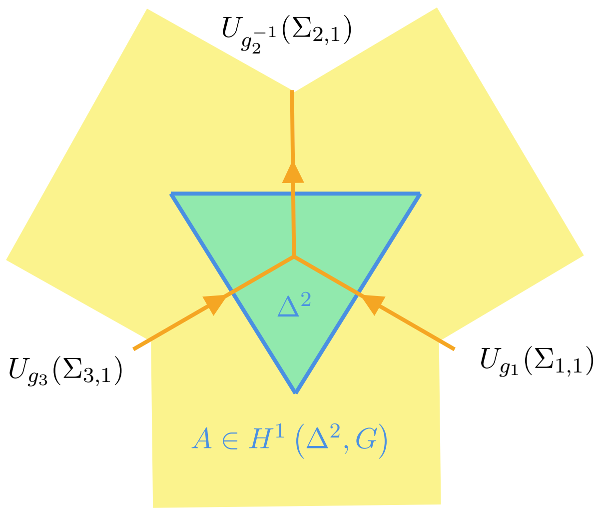


Figure 2.1: Example of Poincaré duality between a 0-form gauge field A in $d = 2$ and a network of topological operators. A is locally defined on simplices that triangulate the manifold (blue), and topological operators are placed along the edges of the dual triangulation (orange). The condition that the gauge field A is flat translates to the condition $g_1 g_2^{-1} g_3 = 1$. Note that the support of topological operators is oriented accordingly.

$PD(A) \in H_{d-1}(M_d, U(1))$, which can be interpreted as a (network of) codimension one submanifold, Σ_{d-1}. Gauge invariance ensures that the corresponding operator

$$\int A \wedge *J = \int_{PD(A)} *J \equiv Q(PD(A)), \tag{2.7}$$

is itself topological. Repeating the construction done above, one can see then that adding flat background field A is equivalent to inserting a specific network of topological operators, $U_{g_i}(\Sigma_{i,d-1})$, inside correlation functions. The flatness condition enforces the group law at the junction points of the network. This construction readily generalizes to any G Abelian.

p-form symmetries. The most immediate generalization is to higher-form symmetries [14] (see also [68] for a pedagogical review of the state-of-the-art). If $G^{(p)}$ is a continuous p-form symmetry group, i.e. such that its conserved current $J^{(p+1)}$ is a $(p+1)$-form, then its conserved charge will be defined as $Q(\Sigma_{d-p-1}) = \int_{\Sigma_{d-p-1}} *J^{(p+1)}$, with Σ_{d-p-1} being a codimension $(p+1)$ hypersurface. To generalize (2.1), we would need to work with operators whose support can possibly link with Σ_{d-p-1}: typically, these are extended p-dimensional objects, $\mathcal{O}(\gamma_p)$.[2] Then also (2.3) is readily generalized and extends also to G discrete. However, the topological nature of the action of the operators $U_g(\Sigma_{d-p-1})$ implies that the group G is necessarily Abelian: since Σ_{d-p-1} does not separate spacetime in two disconnected components, one can always move and deform hypersurfaces as to swap the role of Σ^+_{d-p-1} and Σ_{d-p-1} in the analog of (2.4),

$$U_g(\Sigma^+_{d-p-1}) U_h(\Sigma_{d-p-1}) = U_h(\Sigma^+_{d-p-1}) U_g(\Sigma_{d-p-1}) \iff gh = hg\,. \tag{2.8}$$

[2]In general, a p-form symmetry can be defined to act on objects of dimension *at least* p. We will not discuss the most general case here.

Similarly to the $p = 0$ case, a background for a (flat) $G^{(p)}$ symmetry, $A \in H^{p+1}(M_d, G)$, is dual to a network of topological operators, $U_{g_i}(\Sigma_{d-p-1})$.

Gauging and anomalies. Let us now restrict to the case of a theory $\mathcal{T}$ with a finite, abelian $G^{(p)}$ symmetry, which forces the gauge connection to be locally flat. Gauging $G^{(p)}$ means summing over all possible flat backgrounds $a \in H^{p+1}(M_d, G)$,[3]

$$Z_{\mathcal{T}/G^{(p)}} \sim \sum_{a \in H^{p+1}(M_d,G)} Z_{\mathcal{T}}[a]\,, \tag{2.9}$$

(let us disregard the overall normalization for now). Given our discussion above, this is equivalent to summing over all the networks of topological operators $\{U_{g_i}(\Sigma_{i,d-p-1})\}$ which correspond to elements $PD(a) \in H_{d-p-1}(M_d, G)$,

$$Z_{\mathcal{T}/G^{(p)}} \sim \sum_{PD(a) \in H_{d-p-1}(M_d,G)} Z_{\mathcal{T}}[PD(a)]\,, \tag{2.10}$$

As an example, let us consider $M_d = T^2$ for a 0-form symmetry G. For any G Abelian, $H^1(T^2, G) \simeq H_1(T^2, G) = G \oplus G$: a G-gauge field assigns to the two non-contractible cycles a, b elements $g_{\mathsf{a}}, g_{\mathsf{b}} \in G$, respectively. Thus gauging G can also be described as summing over all possible insertions of U_g, $g \in G$, along the non-contractible cycles of T^2.

Until now we have always tacitly assumed that the symmetry $G^{(p)}$ is anomaly-free. First we should define what does it mean that a discrete symmetry is anomalous. For a continuous symmetry, an 't Hooft anomaly is the lack of invariance under background gauge transformations, signalled at the partition function level by the appearance of an anomalous phase,

$$Z_{\mathcal{T}}[A + d\lambda] = e^{i\alpha(\lambda,A)} Z_{\mathcal{T}}[A]\,. \tag{2.11}$$

This can be used as a definition also in the case of discrete symmetries. Then, the background A is not defined anymore up to gauge redundancy, i.e. A does not define a class in $H^{p+1}(M_d, G)$, but only an element of $C^{p+1}(M_d, G)$. The lack of gauge invariance implies that the corresponding network of operators is not topological anymore, a change in topology corresponding to a background gauge transformation.

We can characterize the anomaly of $G^{(p)}$ in $\mathcal{T}$ by means of a so-called anomaly theory $\mathcal{A}$, a $(d + 1)$-dimensional theory which has the following properties [17, 69–72],

- $\mathcal{A}$ describes gapped phases with the same symmetry $G^{(p)}$ as $\mathcal{T}$,
- on any closed $(d + 1)$-dimensional manifold, the partition function in the presence of a background A for $G^{(p)}$ is well-defined and equal to a phase,

$$Z_{\mathcal{A}}[A] \in U(1)\,, \tag{2.12}$$

[3]One can also add a *discrete torsion* coefficient, that is a weighting phase $\epsilon(a)$, $\epsilon \in H^d(B^{p+1}G, U(1))$, to each term of the sum. Here $B^{p+1}G$ is the Eilenberg-Mac Lane space of G.

- on the other hand, on a manifold with $\partial M_{d+1} = M_d$, the partition function is not invariant under background gauge transformations, with

$$Z_{\mathcal{A}}[A + d\lambda] = e^{-i\alpha(\lambda, A)} Z_{\mathcal{A}}[A]\,, \tag{2.13}$$

where $\alpha(\lambda, A)$ is localized at the boundary, and thus cancels the phase in (2.11),

$$Z_{\mathcal{T}}[A + d\lambda] Z_{\mathcal{A}}[A + d\lambda] = Z_{\mathcal{T}}[A] Z_{\mathcal{A}}[A]\,, \tag{2.14}$$

- $\mathcal{A}$ depends on M_{d+1} only through its topology.

The gapness requirement implies that the Hilbert space of $\mathcal{A}$ on any closed d-dimensional manifold is one-dimensional at large distances: in other words, $\mathcal{A}$ is a $G^{(p)}$-symmetry protected topological (SPT) phase [17]. For bosonic symmetries, these are classified by

$$[\omega] \in H^{d+1}(B^{p+1}G, U(1))\,. \tag{2.15}$$

In passing, let us mention that we might also have situations where the background gauge invariance for a symmetry $G_1^{(p_1)}$ is spoiled only in the presence of a nontrivial background for another symmetry $G_2^{(p_2)}$. In that case, we say that there is a mixed 't Hooft anomaly between $G_1^{(p_1)}$ and $G_2^{(p_2)}$.

For discrete symmetries, gauging is an invertible operation. Any $G^{(p)}$ gauge theory has $(p+1)$-dimensional Wilson operators, parametrized by representations R of $G^{(p)}$. If $G^{(p)}$ is discrete, then these Wilson operators are topological since the $G^{(p)}$ gauge field a is flat in this case, and as such they generate a new symmetry in the gauged theory. If G is Abelian, then the symmetry group of the gauged theory is $\widehat{G}^{(d-p-2)}$, with

$$\widehat{G} = \mathrm{Hom}\,(G, U(1))\,, \tag{2.16}$$

the Pontryagin dual of G, being an Abelian group itself. Then, one can couple $\mathcal{T}/G^{(p)}$ to backgrounds $\widehat{B} \in H^{d-p-1}(M_d, \widehat{G})$ for the $\widehat{G}^{(d-p-2)}$ symmetry,[4]

$$Z_{\mathcal{T}/G^{(p)}}[\widehat{B}] \sim \sum_{a \in H^{p+1}(M_d, G)} Z_{\mathcal{T}}[a] e^{i\langle \widehat{B}, a\rangle}\,, \tag{2.17}$$

$\langle -, - \rangle$ being the intersection pairing. Since $\widehat{\widehat{G}} = G$ and $d - (d-p-2) - 2 = p$, by gauging $G^{(p)}$ and $\widehat{G}^{(d-p-2)}$ in sequence one reobtains a theory with a $G^{(p)}$ symmetry, which turns out to be the original theory $\mathcal{T}$ [14, 73].

Note that if G is not Abelian, then in general the representations of G do not form a group. However, gauging G is still an invertible procedure, but another generalization of the concept of symmetry must be invoked, which we now introduce.

[4]We adopt here a notation often used in the literature of denoting respectively by small and capital latin letters dynamical and non-dynamical gauge fields.

2.1.2 Symmetries and Anomalies in $2d$

Another direction in which we could have generalized the notion of symmetry is as follows. Nothing forbids to formally define a set of topological operators $U_a(\Sigma_{d-p-1})$, with a some arbitrary index, such that (2.4) generalizes to an expression of the kind

$$U_a(\Sigma^+_{d-p-1})U_b(\Sigma_{d-p-1}) = \sum_{c\in\mathcal{C}} U_c(\Sigma_{d-p-1})\,, \tag{2.18}$$

which, we stress again, must be read as an equality inside correlation functions, and the sum is over a given index range $\mathcal{C}$. This is the structure of a *non-invertible* (p-form) symmetry, namely a symmetry whose underlying structure is not necessarily group-like. The language in which non-invertible symmetries are better understood is that of (higher) categories. While these structures are extensively known in $2d$ literature since quite some time (see e.g. [15, 61, 74–81]), in the last years they have been studied also in $d \geq 3$, see for instance the seminal works [82–85]. In the following, we review part of the construction that applies to $d = 2$, with an emphasis on the implication for group-like symmetries and their anomalies. But first, let us show that this definition is non-empty.

Example 1: Verlinde Lines in RCFTs. Let us start with a well-known example where (2.18) appears. As we will see in Section 2.2, a RCFT is characterized by having a finite number of primary fields. In the case of WZW models, these primary operators are classified according to integrable representations $\hat{\lambda} \in P^k_+$ of an extended Kač-Moody symmetry algebra $\widehat{\mathfrak{g}}_k$. One can take tensor product of two such representations and decompose it,

$$\hat{\lambda} \otimes \hat{\mu} = \bigoplus_{\hat{\nu}\in P^k_+} \mathcal{N}^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}}\hat{\nu}\,, \tag{2.19}$$

where $\mathcal{N}^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} \in \mathbb{Z}_{\geq 0}$ are called fusion coefficients. The latter are related to the modular data of the RCFT by the Verlinde formula [74],

$$\mathcal{N}^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} = \sum_{\hat{\sigma}\in P^k_+} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}}\mathcal{S}_{\hat{\mu}\hat{\sigma}}\mathcal{S}^*_{\hat{\nu}\hat{\sigma}}}{\mathcal{S}_{\hat{0}\hat{\sigma}}}\,, \tag{2.20}$$

where $\hat{0}$ denotes the trivial representation.

Let us assume for simplicity that the corresponding RCFT is diagonal. Then, the Hilbert space of the RCFT is given by primary operators $|\hat{\lambda}, \hat{\bar{\lambda}}\rangle =: |\hat{\lambda}\rangle$ and their affine descendants. On the Hilbert space, we can define a set of abstract operators $\widehat{L}_{\hat{\mu}}$ that acts on primary operators as follows,

$$\widehat{L}_{\hat{\mu}}|\hat{\lambda}\rangle = \frac{\mathcal{S}_{\hat{\mu}\hat{\lambda}}}{\mathcal{S}_{\hat{0}\hat{\lambda}}}|\hat{\lambda}\rangle\,, \tag{2.21}$$

and such that its action commutes with the left- and right-moving chiral algebra, thus making it a symmetry of the theory. To this symmetry, we can associate topological operators $L_{\hat{\mu}}$ supported on codimension one surfaces, i.e. on lines. The lines $L_{\hat{\mu}}$ thus inherit the fusion rules from the chiral algebra,

$$L_{\hat{\lambda}}L_{\hat{\mu}} = \sum_{\hat{\nu}\in P^k_+} \mathcal{N}^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}}L_{\hat{\nu}}\,, \tag{2.22}$$

where we have used the unitarity of $\mathcal{S}$. The relation (2.22) is analog to (2.18) for a zero-form symmetry in $d = 2$.

Example 2: Ising model and Kramers-Wannier duality. The Ising model at the critical point is conformal, in particular it is a minimal model. As such, it has only finitely many Virasoro primaries, the identity $1_{(0,0)}$, the thermal operator $\epsilon_{(1/2,1/2)}$, and the order operator $\sigma_{(1/16,1/16)}$, whose OPEs with each other are given by

$$\epsilon \times \epsilon = 1\,, \quad \sigma \times \epsilon = \epsilon \times \sigma = \sigma\,, \quad \sigma \times \sigma = 1 + \epsilon\,. \tag{2.23}$$

Let us now take a different perspective than in the previous example. Let us find all consistency constraint that putative symmetries of the Ising model should satisfy. A symmetry should correspond to a line operator that commutes with the Virasoro algebra,

$$\widehat{L}|1\rangle = \ell_0|1\rangle\,, \quad \widehat{L}|\epsilon\rangle = \ell_{1/2}|\epsilon\rangle\,, \quad \widehat{L}|\sigma\rangle = \ell_{1/16}|\sigma\rangle\,. \tag{2.24}$$

On the torus, the insertion of the charge operator $\widehat{L}$ amounts to computing

$$\mathrm{tr}_{\mathcal{H}}(\widehat{L}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}) = \ell_0|\chi_0|^2 + \ell_{1/2}|\chi_{1/2}|^2 + \ell_{1/16}|\chi_{1/16}|^2\,. \tag{2.25}$$

Modular covariance requires that, up to a modular $\mathcal{S}$ transformation, we can reinterpret the insertion of L in the trace as tracing over a twisted Hilbert space,

$$\mathrm{tr}_{\mathcal{H}_L}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}) = \sum_{i,j} n_{ij}\chi_i\bar{\chi}_j\,, \quad n_{ij} \in \mathbb{Z}_{\geq 0}\,. \tag{2.26}$$

The integrality constraint is very stringent. We can use it to 'bootstrap' which are the allowed values of ℓ_i. We find three independent sets of solutions,

$$(\ell_0, \ell_{1/2}, \ell_{1/16}) = (1,1,1)\,,\ (1,1,-1)\,,\ (\sqrt{2}, -\sqrt{2}, 0)\,. \tag{2.27}$$

The first solution is trivial. The second one corresponds to the $\mathbb{Z}_2$ symmetry of the Ising model, whose corresponding line is often dubbed η in the literature,

$$\eta: \quad |1\rangle \mapsto |1\rangle\,, \quad |\epsilon\rangle \mapsto |\epsilon\rangle\,, \quad |\sigma\rangle \mapsto -|\sigma\rangle\,, \tag{2.28}$$

and correspondingly in the twisted Hilbert space one finds as twisted primaries the disorder operator $\mu_{(1/16,1/16)}$, which is $\mathbb{Z}_2$-even, and Majorana fermions $\psi_{(1/2,0)}, \bar{\psi}_{(0,1/2)}$, which are $\mathbb{Z}_2$-odd.

The second solution is new. Let us call the corresponding line $\mathcal{N}$. It cannot be an invertible symmetry because the corresponding operator has a zero eigenvalue. The most interesting thing is the fusion rules that η and $\mathcal{N}$ satisfy, which is readily deducible from (2.27),

$$\eta \times \eta = \mathbf{1}\,, \quad \mathcal{N} \times \eta = \eta \times \mathcal{N} = \mathcal{N}\,, \quad \mathcal{N} \times \mathcal{N} = \mathbf{1} + \eta\,. \tag{2.29}$$

These are the fusion rules of the so-called Tambara-Yamagami category $\mathrm{TY}_+(\mathbb{Z}_2)$.

By the fusion rules (2.29), we can understand what happens when a line $\mathcal{N}$ crosses the operator σ. Closing the $\mathcal{N}$ line on itself around the operator σ leaves a η line attached to it, and thus turns σ in the μ operator. For this reason, $\mathcal{N}$ is called the Kramers-Wannier duality defect: it exchanges σ and μ.

Symmetry Categories. While in the present thesis we will be concerned with group-like symmetries, for finite zero-form symmetries in $2d$ there is a general construction that encompasses both the invertible and the non-invertible cases. The notion is that of unitary fusion categories [86], also called symmetry categories in physics [15].

In a symmetry category $\mathcal{C}$, the objects are defined as the topological lines L_i generating the symmetry, and morphisms $m_{ij} \in \mathrm{Hom}\,(L_i, L_j)$ are defined as topological local operators that change the label of the topological line. The set of morphisms $\mathrm{Hom}\,(L_i, L_j)$ is required to be a vector space. To any line L_i, we can associate the partition function computed with the corresponding insertion of the operator $\widehat{L}_i$,

$$\langle \cdots \widehat{L_i}(\gamma) \cdots \rangle \,, \tag{2.30}$$

γ being an oriented path. Then the operators associated to morphisms m_{ij} live at the conjunction points of $\widehat{L}_i(\gamma_i)$ and $\widehat{L}_j(\gamma_j)$.

The category $\mathcal{C}$ should satisfy certain axioms. It should admit the existence of a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, compatible the existence of a unit object $\mathbf{1}$,[5]

$$\otimes : (L_i, L_j) \mapsto L_i \otimes L_j \,, \qquad \mathbf{1} \otimes L_i = L_i \otimes \mathbf{1} = L_i \,, \tag{2.31}$$

and that sends morphisms to morphisms,

$$\otimes : (m_{ik}, m_{jk}) \mapsto m_{ik} \otimes m_{jk} \in \mathrm{Hom}(L_i \otimes L_j, L_k) \,. \tag{2.32}$$

The space $\mathrm{Hom}(L_i \otimes L_j, L_k)$ is also called the space of trivalent junctions.
One must also require an associativity structure given by a set of morphisms

$$\omega_{i,j,k} \in \mathrm{Hom}((L_i \otimes L_j) \otimes L_k, L_i \otimes (L_j \otimes L_k)) \,, \tag{2.33}$$

subject to the pentagon identity,

$$\omega_{j,k,l} \circ \omega_{i,j\otimes k,l} \circ \omega_{i,j,k} = \omega_{i,j,k\otimes l} \circ \omega_{i\otimes j,k,l} \,. \tag{2.34}$$

The action of the morphism $\omega_{i,j,k}$, also called associator, is to change the order in which three lines are tensored, see Figure 2.2.

A symmetry category $\mathcal{C}$ must also have an additive structure $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Physically, the line $L_i \oplus L_j$ corresponds to the correlator

$$\langle \cdots \widehat{L_i \oplus L_j}(\gamma) \cdots \rangle = \langle \cdots \widehat{L}_i(\gamma) \cdots \rangle + \langle \cdots \widehat{L}_j(\gamma) \cdots \rangle \,. \tag{2.35}$$

Among the objects of $\mathcal{C}$, one can identify the set of indecomposable objects, i.e. those objects L_a who cannot be expressed as sum of any other two objects $L_i \oplus L_j$. It is usually assumed that all indecomposable objects L_a are also simple, that is they have $\mathrm{Hom}\,(L_a, L_a) = \mathbb{C}$ and $\mathrm{Hom}\,(L_a, L_b) = 0$ for $a \neq b$. In particular $\mathbf{1}$ is assumed to be simple. One then typically asks that all objects L_i can be decomposed as a finite sum $L_i =$

[5] The line $\mathbf{1}$ does nothing when inserted in correlation functions, and as such is called 'trivial'.

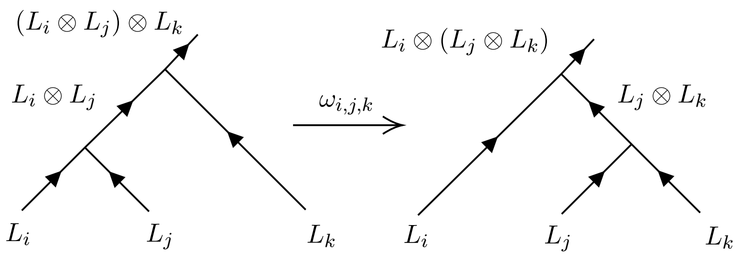


Figure 2.2: The action of the associator ω is equivalent to a local rearrangement of the network of lines.

$\bigoplus_a N_a L_a$, with L_a's simple, and $N_a \in \mathbb{Z}_{\geq 0}$ denoting $\dim(\mathrm{Hom}\,(L_i, L_a))$. In particular, given three simple objects L_a, L_b, L_c,

$$L_a \otimes L_b = \bigoplus_c N^c_{a,b} L_c \,, \tag{2.36}$$

with $N^c_{a,b} \in \mathbb{Z}_{\geq 0}$ being the dimension of the vector space of trivalent junctions $\mathrm{Hom}\,(L_a \otimes L_b, L_c)$.

Other structures are usually required. For every line L_i, there is also a dual line $L_i^\vee$ such that

$$\langle \cdots \widehat{L_i}(\gamma^*) \cdots \rangle = \langle \cdots \widehat{L_i^\vee}(\gamma) \cdots \rangle \,, \tag{2.37}$$

where γ^* denotes the orientation-reversal of γ. Additionally, one asks that there exist a unitary structure on the junction spaces, with a notion of Hermitian conjugate $m^\dagger_{ij} \in \mathrm{Hom}(L_j, L_i)$ of a morphism $m_{ij} \in \mathrm{Hom}(L_i, L_j)$, such that $m^\dagger_{ij} \circ m_{ij} \in \mathrm{Hom}(L_i, L_i)$ is a positive semi-definite linear operator.

The case of groups: the symmetry category $\mathcal{C}(G, \omega)$. Let us make contact with the usual notion of finite group-like symmetries. Consider a theory $\mathcal{T}$ with a group-like symmetry G. Each element $g \in G$ is associated to a simple object L_g. The tensor product structure is inherited by the group multiplication, $L_g \otimes L_h = L_{gh}$, and the dual line is just the inverse, $L_g^\vee = L_{g^{-1}}$. The interesting object is the associator $\omega_{g,h,k}$. We can think of the action of the associator as nucleating a closed loop $L_{h^{-1}}$ from the vacuum. Physically, $\omega_{g,h,k}$ is a map $\omega : G \times G \times G \to U(1)$ subject to the pentagon relation (2.34), which in the case of a group reads

$$\frac{\omega_{h,k,l}\,\omega_{g,hk,l}\,\omega_{g,h,k}}{\omega_{g,h,kl}\,\omega_{gh,k,l}} = 1 \,. \tag{2.38}$$

Additionally, at each junction point we can perform a change of basis in the vector spaces $\mathrm{Hom}(L_g \otimes L_h, L_k) \simeq \delta_{gh,k}\mathbb{C}$, i.e. associate to each junction a phase $\beta_{g,h} \in U(1)$. This means that $\omega_{g,h,k}$ is not uniquely determined: rather, there is a redundancy

$$\omega_{g,h,k} \sim \omega_{g,h,k} \frac{\beta_{g,h}\beta_{gh,k}}{\beta_{g,hk}\beta_{g,k}} \,. \tag{2.39}$$

It is not difficult to see that (2.38) and (2.39) combine to give $[\omega] \in H^3(G, U(1))$: the associator ω has a natural interpretation as the anomaly cocycle of G. By Poincaré duality, the configurations related by $\omega_{g,h,k}$ correspond to two different background field configurations related by a background gauge transformation. If ω is not trivial, then the transformation is anomalous. We label the symmetry category associated to the group symmetry G with associator/anomaly ω as $\mathcal{C}(G,\omega)$.

If ω is trivial this means that G can be gauged. The gauging operation is done by summing over G gauge fields $a \in H^1(M_2, G)$, or equivalently summing over the insertion of topological lines labeled by G elements, c.f. (2.9). From the categorical viewpoint, the latter sentence can be implemented as follows: we define the object

$$\mathcal{A} = \bigoplus_{g\in G} L_g \,, \tag{2.40}$$

and we sum over insertions of a fine-enough network of lines $\mathcal{A}$, where by fine-enough we mean that it is dual to a triangulation of the manifold. The vanishing of the anomaly, i.e. the triviality of the associator, ensures independence from the specific realization of the network.

When gauging G, as discussed in Section 2.1.1, in the theory $\mathcal{T}/G$ local operators are Wilson lines labeled by representations of G, $R \in \mathrm{Rep}(G)$. When G is finite Abelian, representations are encoded by characters, i.e. elements of $\mathrm{Hom}(G, U(1))$ which has a group structure: $\mathrm{Hom}(G, U(1)) = \widehat{G} \simeq G$. When G is non-Abelian, in general $\mathrm{Rep}(G)$, is not a group, as it can contain higher-dimensional representations, whose tensor product decomposition has generically multiple factors. This signals that in this case the symmetry of $\mathcal{T}/G$ is non-invertible. Indeed, $\mathcal{C}' = \mathrm{Rep}(G)$ is the symmetry category of $\mathcal{T}' = \mathcal{T}/G$.

Mixed anomalies and group extensions. Similarly to the discussion above, to gauge a subgroup $H \subset G$ one can simply restrict the sum in (2.40) to elements of H, $\mathcal{A}_H = \bigoplus_{h\in H} h$. In this case, it suffices to check that ω is trivial when restricted to elements of H, while in general it can be non-trivial on G. Depending on how H embeds in G, or whether H has a mixed 't Hooft anomaly with the rest of the symmetry group, some interesting features can arise. In the following we will convey the mathematical intuition, rather than providing a full proof, which would necessitate more advanced tools. We refer to [15, 87] for a full-fledged proof. We will assume that H is Abelian.

Let us assume that the anomaly is trivial for the whole G. The theory $\mathcal{T}/H$ has both a dual symmetry $\widehat{H} = \mathrm{Hom}(H, U(1))$ and a residual symmetry $K \simeq G/H$. For H Abelian, we can think of G as a central extension of K by H, i.e. there exists a short exact sequence

$$1 \to H \to G \to K = G/H \to 1 \,, \tag{2.41}$$

and an element $\kappa \in H^2(K, H)$, such that G can be obtained by equipping the set $H \times K$ with the following group operation,

$$(h_1, k_1)(h_2, k_2) = (h_1\, \varphi_{k_1}(h_2)\, \kappa_{k_1,k_2},\, k_1 k_2) \,, \tag{2.42}$$

where φ_k is an automorphism of H defined by conjugation by k in G.[6] As a concrete example, $G = \mathbb{Z}_4$ can be realized from $H = K = \mathbb{Z}_2$ in this way, by choosing $\kappa_{k_1,k_2} = 1$,

[6]The notion of group extension should not be confused with the notion of semidirect product $G = H \rtimes_\varphi K$, for which (2.42) is defined with a trivial κ. Moreover, note that for G Abelian any such semidirect product is necessarily direct, as then $\varphi_k(h) = k^{-1}hk = h$. Sometimes the notation $H \rtimes_\kappa K$ is used in the literature to denote the extension of K by H via (2.42).

the trivial element of $\mathbb{Z}_2$, if k_1 or k_2 is trivial, and $\kappa_{-1,-1} = -1$: then, $(h,k) = (1,-1)$ has order 4.
When gauging H, an element of the residual symmetry $k \in K$ can act on $\widehat{H}$, by sending the character $\chi \in \widehat{H}$ to $\chi \circ \varphi_{k^{-1}}$.[7] This defines a symmetry group which is a semidirect product $G' = \widehat{H} \rtimes_\varphi K$, and combining $\kappa \in H^2(K,H)$ with $\chi \in \mathrm{Hom}(H,U(1))$ we obtain the nontrivial associator

$$\omega_{(\chi_1,k_1),(\chi_2,k_2),(\chi_3,k_3)} = (\chi_3 \circ \varphi_{(k_1k_2)^{-1}})(\kappa_{k_1,k_2})\,. \tag{2.43}$$

Viceversa, let us assume for simplicity that G is of the form $G = H \times K$, and that G has an anomaly ω which trivializes when restricted to H, $\omega|_H = 1$, but is non-trivial when two of its arguments are in K, i.e. $[\omega] \in H^1(H,U(1)) \oplus H^2(K,U(1)) \subset H^3(H \times K, U(1))$. Now, let us gauge H. The theory $\mathcal{T}/H$ has both a dual symmetry $\widehat{H} = \mathrm{Hom}(H,U(1))$ and the symmetry K, of the form (2.41) with H replaced by $\widehat{H}$, and with $\kappa \in H^2(K,\widehat{H})$ defined by ω such that $\kappa_{k_1,k_2}(h) = \omega_{h,k_1,k_2}$.

Altogether, this suggests that under gauging there is the following exchange: mixed 't Hooft anomalies become group extensions, and nontrivial group extensions give rise to mixed 't Hooft anomalies. This is indeed the case [15, 87]. We will see this phenomenon at work in Section 4.2.

2.2 Rational CFTs

A special class of $2d$ Conformal Field Theories is the one of Rational CFTs (RCFTs). These are theories whose (possibly infinitely many) Virasoro primaries can be organized in a finite number of blocks under an extended chiral symmetry algebra. The classic examples of RCFTs, and the ones that we will deal with in thesis work, are the compact scalar and more generally G_k WZW models.

2.2.1 Compact scalar

Consider the compact scalar with radius R,

$$S = \frac{1}{8\pi}\int (\partial_\mu \phi)^2\, \mathrm{d}^2x\,, \qquad \phi \sim \phi + 2\pi R\,. \tag{2.44}$$

This can be equivalently described at the Lagrangian level by a compact scalar $\tilde{\phi}$ of radius $\tilde{R} = 2/R$.

Let us also introduce the holomorphic and anti-holomorphic components of ϕ,

$$\phi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})\,, \qquad \tilde{\phi}(z,\bar{z}) = \varphi(z) - \bar{\varphi}(\bar{z})\,. \tag{2.45}$$

Let us discuss the symmetries of the theory for irrational values of R^2. This theory has global symmetry group

$$G(R) = (U(1)_P \times U(1)_W) \rtimes \mathbb{Z}_2^C\,, \tag{2.46}$$

[7] This action is defined such that it leaves $\chi(h)$ invariant, $\chi(h) \mapsto (\chi \circ \varphi_{k^{-1}})(\varphi_k(h)) = \chi(h)$.

defined as

$$\begin{cases} U(1)_P: & \phi \mapsto \phi + \alpha_P R\,, & \tilde{\phi} \mapsto \tilde{\phi}\,, & \alpha_P \sim \alpha_P + 2\pi\,, \\ U(1)_W: & \phi \mapsto \phi\,, & \tilde{\phi} \mapsto \tilde{\phi} + 2\alpha_W/R\,, & \alpha_W \sim \alpha_W + 2\pi\,, \\ \mathbb{Z}_2^C: & \phi \mapsto -\phi\,, & \tilde{\phi} \mapsto -\tilde{\phi}\,. \end{cases} \tag{2.47}$$

At generic radius R there is no holomorphic (or anti-holomorphic) $U(1)$ symmetry. Rather, the holomorphic current $\partial\phi = \partial\varphi$ generates a non-compact $\mathbb{R}$ symmetry.

The torus partition function of a compact scalar with generic radius R is given by

$$Z_R(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{h_{e,m}} \bar{q}^{\bar{h}_{e,m}}\,, \tag{2.48}$$

where $q = e^{2\pi i \tau}$, and

$$h_{e,m} = \frac{1}{2}\left(\frac{e}{R} + \frac{mR}{2}\right)^2\,, \qquad \bar{h}_{e,m} = \frac{1}{2}\left(\frac{e}{R} - \frac{mR}{2}\right)^2\,, \tag{2.49}$$

are the conformal dimensions of the Virasoro primaries

$$\begin{aligned} \mathcal{V}_{e,m}(z,\bar{z}) &= \exp\left[ie\frac{\phi(z,\bar{z})}{R} + im\frac{R\tilde{\phi}(z,\bar{z})}{2}\right] \\ &= \exp\left[i\left(\frac{e}{R} + \frac{mR}{2}\right)\varphi(z) + i\left(\frac{e}{R} - \frac{mR}{2}\right)\bar{\varphi}(\bar{z})\right]. \end{aligned} \tag{2.50}$$

When $R^2 \in \mathbb{Q}$, the description of the theory greatly simplifies: in that case, the compact boson is a rational theory, there are additional symmetries, and its partition function can be rewritten in terms of a finite number of affine characters. Moreover, notice that $\partial\varphi$ is now a $U(1)$ current. In particular, letting

$$R^2 = \frac{2p'}{p}\,, \qquad \gcd(p, p') = 1\,, \tag{2.51}$$

we can recast the 'naive' symmetry of the theory as

$$G(R) = (U(1)_L \times U(1)_R) \rtimes \mathbb{Z}_2^C\,, \tag{2.52}$$

where

$$\begin{cases} U(1)_L: & \varphi \mapsto \varphi + \alpha_L\sqrt{2pp'}\,, & \bar{\varphi} \mapsto \bar{\varphi}\,, & \alpha_L \sim \alpha_L + 2\pi\,, \\ U(1)_R: & \varphi \mapsto \varphi\,, & \bar{\varphi} \mapsto \bar{\varphi} + \alpha_R\sqrt{2pp'}\,, & \alpha_R \sim \alpha_R + 2\pi\,, \\ \mathbb{Z}_2^C: & \varphi \mapsto -\varphi\,, & \bar{\varphi} \mapsto -\bar{\varphi}\,. \end{cases} \tag{2.53}$$

We can also write the action of $U(1)_L$, $U(1)_R$ on ϕ and $\tilde{\phi}$,

$$\begin{cases} U(1)_L: & \phi \mapsto \phi + \alpha_L pR\,, & \tilde{\phi} \mapsto \tilde{\phi} + \alpha_L p' \frac{2}{R}\,, \\ U(1)_R: & \phi \mapsto \phi + \alpha_R pR\,, & \tilde{\phi} \mapsto \tilde{\phi} - \alpha_R p' \frac{2}{R}\,, \end{cases} \qquad R^2 = \frac{2p'}{p}\,. \tag{2.54}$$

For generic values of R we cannot define chiral $U(1)$ symmetries, but we can still define chiral $\mathbb{Z}_n$ actions on fields as follows,

$$\begin{cases}\mathbb{Z}_n^L: & \phi\mapsto\phi+\frac{2\pi\ell}{n}pR\,, \qquad \tilde\phi\mapsto\tilde\phi+\frac{2\pi\ell}{n}p'\frac{2}{R}\,,\\ \mathbb{Z}_n^R: & \phi\mapsto\phi+\frac{2\pi\ell}{n}pR\,, \qquad \tilde\phi\mapsto\tilde\phi-\frac{2\pi\ell}{n}p'\frac{2}{R}\,,\end{cases} \qquad \ell=0,\,1,\,\dots\,n-1\,, \tag{2.55}$$

where p,p' are chosen such that at the rational point $R^2=2p'/p$ the actions in (2.55) reduce to $\mathbb{Z}_n$ subgroups of (2.54).

For rational R^2, on top of this manifest symmetry, there is an extended symmetry generated by higher spin currents, whose realization depends on the specific value of R^2. It is convenient to reorganize the operator spectrum not just in terms of electromagnetic primaries and descendants, but rather in terms of primaries and descendants of the extended algebra, which includes also the conserved chiral currents. This means that rather than considering the infinitely-many primaries labeled by two integers we only need to consider a finite number of primary vertex operators, out of which all others can be obtained by fusion with currents. This amounts to finding representatives of equivalence classes of vertex operators $\mathcal{V}_{e,m}$ under the relations $(e,m)\sim(e\pm p',m\pm p)\sim(e\pm p',m\mp p)$. It is not difficult to see that there are $2pp'$ such equivalence classes, and a choice of representatives is given by restricting to $e\in\mathbb{Z}_{2p'}$, $m\in\mathbb{Z}_p$ if $p\le p'$, or $e\in\mathbb{Z}_{p'}$, $m\in\mathbb{Z}_{2p}$ if $p>p'$.
One can then rewrite the torus partition function as a sum over affine characters,

$$Z_{\sqrt{2p'/p}}(\tau,\bar\tau)=\sum_{\lambda=0}^{2pp'-1}\chi_\lambda^{(2pp')}(\tau)\bar\chi_{\omega_0\lambda}^{(2pp')}(\bar\tau)\,, \tag{2.56}$$

where

$$\chi_\lambda^{(2pp')}(\tau)=\frac{1}{\eta(\tau)}\sum_{t\in\mathbb{Z}}q^{pp'\left(t+\frac{\lambda}{2pp'}\right)^2}\,, \tag{2.57}$$

and we have defined

$$\omega_0=pr_0+p's_0 \mod 2pp'\,, \tag{2.58}$$

with (r_0,s_0) being any Bézout pair for (p,p'), i.e. (positive) integer numbers such that

$$pr_0-p's_0=1\,. \tag{2.59}$$

A consequence of this is that $\omega_0^2=1 \mod 4pp'$: in fact,

$$\omega_0^2=(2pr_0-1)^2=1 \mod 4p, \qquad \omega_0^2=(2p's_0+1)^2=1 \mod 4p'\,, \tag{2.60}$$

from which the result follows using $\gcd(p,p')=1$.

Let us analyze the modular properties of (2.57). First of all, under a $\mathcal{T}$ transformation

$$\chi_\lambda^{(2pp')}(\tau+1)=e^{2\pi i\left(\frac{\lambda^2}{4pp'}-\frac{1}{24}\right)}\chi_\lambda^{(2pp')}(\tau)\,. \tag{2.61}$$

Notice that the combination appearing in (2.56) is $\mathcal{T}$-invariant because

$$h_\lambda=\frac{\lambda^2}{4pp'}, \qquad h_{\omega_0\lambda}=\frac{\omega_0^2\lambda^2}{4pp'}=h_\lambda \mod 1\,. \tag{2.62}$$

Under an $\mathcal{S}$ transformation, instead,

$$\chi_\lambda^{(2pp')}(-1/\tau) = \frac{1}{\sqrt{2pp'}} \sum_{\mu=0}^{2pp'-1} e^{-2\pi i \frac{\lambda\mu}{2pp'}} \chi_\mu^{(2pp')}(\tau), \tag{2.63}$$

and similarly

$$\begin{aligned}\chi_{\omega_0\lambda}^{(2pp')}(-1/\tau) &= \frac{1}{\sqrt{2pp'}} \sum_{\mu=0}^{2pp'-1} e^{-2\pi i \frac{(\omega_0\lambda)\mu}{2pp'}} \chi_\mu^{(2pp')}(\tau) \\ &= \frac{1}{\sqrt{2pp'}} \sum_{\nu=0}^{2pp'-1} e^{-2\pi i \frac{\lambda\nu}{2pp'}} \chi_{\omega_0\nu}^{(2pp')}(\tau)\,,\end{aligned} \tag{2.64}$$

where we have let $\omega_0\nu = \mu$ and used $\omega_0^2 = 1 \mod 4pp'$. Therefore the partition function (2.56) is indeed modular invariant.

Note also that the theory (2.56) is not a diagonal RCFT unless $\omega_0 = 1 \mod 2pp'$, which can happen only if either p or p' is equal to 1. Let us assume that it is the case, without loss of generality we let $p = 1$ (the case $p' = 1$ can be obtained by T-duality). Then,

$$Z_{\sqrt{2p'}}(\tau,\bar\tau) = \sum_{\lambda=0}^{2p'-1} |\chi_\lambda^{(2p')}(\tau)|^2\,. \tag{2.65}$$

This is the partition function of the diagonal $U(1)_{2p'}$ WZW model.[8]

When both $p, p' \neq 1$, the relation between the compact boson CFT and a $U(1)$ WZW model is more subtle. Let us see what happens with an example. Let us consider the compact boson with $R^2 = N$ odd, which has $p = 2$ and $p' = N$. There are $2pp' = 4N$ affine characters in this theory. A Bézout pair for $(p, p') = (2, N)$ is $(r_0, s_0) = (\frac{N+1}{2}, 1)$, which leads to $\omega_0 = 2N + 1$. The partition function (2.56) reads

$$\begin{aligned}Z_{\sqrt{N}}(\tau,\bar\tau) &= \sum_{\lambda=0}^{4N-1} \chi_\lambda^{(4N)}(\tau)\bar\chi^{(4N)}_{(2N+1)\lambda \bmod 4N}(\bar\tau) \\ &= \sum_{\substack{\lambda=0\\ \lambda\in 2\mathbb{Z}}}^{4N-1} |\chi_\lambda^{(4N)}(\tau)|^2 + \sum_{\substack{\lambda=0\\ \lambda\in 2\mathbb{Z}+1}}^{4N-1} \chi_\lambda^{(4N)}(\tau)\bar\chi^{(4N)}_{\lambda+2N \bmod 4N}(\bar\tau)\,.\end{aligned} \tag{2.66}$$

Let us compare this with another compact boson theory, with $R^2 = 4N$, for which $p = 1$ and $p' = 2N$. This theory also has the same $2pp' = 4N$ extended characters, but it is diagonal. Its partition function reads

$$Z_{2\sqrt{N}}(\tau,\bar\tau) = \sum_{\lambda=0}^{4N-1} |\chi_\lambda^{(4N-1)}(\tau)|^2\,. \tag{2.67}$$

It is immediate to realize that the theory with $R^2 = N$, N odd, is the $\mathbb{Z}_2$ orbifold of the theory with $R^2 = 4N$, where $\mathbb{Z}_2 = \mathbb{Z}_2^P \subset U(1)_P$ (this is consistent: the $\mathbb{Z}_n^P$ quotient of the compact boson with radius R is the theory with radius R/n).

[8] Since for $U(1)$ there is no Wess-Zumino term, the level has to be read off from the commutation relations of the affine symmetry currents, or equivalently from the properties of integrable representations.

In general, if the theory $R^2 = 2p'/p$ is non diagonal, in particular if it has $p \neq 1$, we can realize it as a $\mathbb{Z}_p$ orbifold of the diagonal theory $R^2 = 2q'/q$ with $q' = pp'$ and $q = 1$.

Since we will be interested only in theories with $R^2 \in \mathbb{Z}_{>0}$, it suffices to distinguish the cases $p = 1$ and $p = 2$, i.e. R^2 even or odd, respectively. In summary, the correspondence is

$$\begin{aligned} R^2 = 2n \text{ compact boson} \quad &\Longleftrightarrow \quad U(1)_{2n} \text{ WZW}\,, \\ R^2 = 2n+1 \text{ compact boson} \quad &\Longleftrightarrow \quad \frac{U(1)_{4(2n+1)} \text{ WZW}}{\mathbb{Z}_2^P}\,. \end{aligned} \tag{2.68}$$

We refer to the latter as the $U(1)_{2n+1}$ WZW model.

2.2.2 Wess-Zumino-Witten models

The two-dimensional G_k Wess-Zumino-Witten (WZW) model is a non-linear sigma model with target space a connected (but not necessarily simply connected) group G, whose action on a two-manifold M_2 reads

$$S_{G_k}[g] = \frac{k}{4\pi} \int_{M_2} \mathrm{d}^2 x \;\; \mathrm{tr}(g^{-1}\,\partial_\mu g g^{-1}\,\partial^\mu g) + \frac{ik}{2\pi} \int_{X_3} \mathrm{d}^3 y \;\epsilon^{\alpha\beta\gamma}\, \mathrm{tr}(\tilde{g}^{-1}\,\partial_\alpha \tilde{g}\tilde{g}^{-1}\,\partial_\beta \tilde{g}\tilde{g}^{-1}\,\partial_\gamma \tilde{g})\,, \tag{2.69}$$

where $g(x) \in G$. Here X_3 bounds M_2, and $\tilde{g}(y)$ is an extension of $g(x)$ to X_3. The requirement that the action is independent from the choice of extension imposes $k \in \mathbb{Z}_{>0}$. The coefficient of the kinetic term instead is fixed to give chiral and anti-chiral global symmetry currents. In complex coordinates, $z = x^1 + ix^2,\ \bar{z} = x^1 - ix^2$,

$$J(z) = \partial g g^{-1}\,, \qquad \bar{J}(\bar{z}) = g^{-1}\bar{\partial} g\,, \tag{2.70}$$

which satisfy

$$\bar{\partial} J(z) = 0\,, \qquad \partial \bar{J}(\bar{z}) = 0\,. \tag{2.71}$$

The chiral symmetry of the action is actually local, in the sense that the action is invariant under

$$g(z,\bar{z}) \mapsto g_L(z) g(z,\bar{z}) g_R^{-1}(\bar{z})\,, \tag{2.72}$$

for arbitrary (holomorphic and antiholomorphic) maps $g_L(z)$ and $g_R(\bar{z})$.

The G_k WZW model is conformal invariant at the quantum level. Global symmetry currents $J(z)$ and $\bar{J}(\bar{z})$ (or better, their mode expansions), satisfy a Kač-Moody algebra $\widehat{\mathfrak{g}}_k \oplus \bar{\widehat{\mathfrak{g}}}_k$.[9] Through the Sugawara construction, this leads to an energy-momentum tensor that satisfies the Virasoro algebra, with central charge

$$c = \bar{c} = \frac{k \dim \mathfrak{g}}{k + h^\vee}\,, \tag{2.73}$$

where $h^\vee$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$, that is (half) the quadratic Casimir of the adjoint representation.

[9] We will use $\bar{\widehat{\mathfrak{g}}}_k$ and $\widehat{\mathfrak{g}}_{-k}$ interchangeably to denote anti-holomorphic chiral algebras.

The G_k WZW model is also a rational CFT, as its operator spectrum can be organized in finitely many $\widehat{\mathfrak{g}}_k \oplus \widehat{\overline{\mathfrak{g}}}_k$ primaries and their descendants. Let us briefly recall the representation theory of $\mathfrak{g}$ and then of $\widehat{\mathfrak{g}}_k$. A $\mathfrak{g}$ highest weight representation V_λ is labeled by its highest weight

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \equiv (\lambda_1, \ldots, \lambda_r)\,, \tag{2.74}$$

where r is the rank of $\mathfrak{g}$, ω_i are the fundamental weights of $\mathfrak{g}$, and λ_i are Dynkin labels, $\lambda_i \in \mathbb{Z}_{\geq 0}$.[10] In a similar fashion, a $\widehat{\mathfrak{g}}_k$ highest weight representation $V_{\hat{\lambda}}$ is labeled by

$$\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \equiv (\lambda_0; \lambda_1, \ldots, \lambda_r)\,, \tag{2.75}$$

Here, $\hat{\omega}_i \equiv (\omega_i; a_i^\vee; 0)$ for $i \neq 0$, and $\hat{\omega}_0 \equiv (0; 1; 0)$ is the fundamental affine weight associated with the trivial $\mathfrak{g}$ representation (not to be confused with the trivial $\widehat{\mathfrak{g}}_k$ representation, which is associated to $k\hat{\omega}_0$), $a_i^\vee$ is the comark associated to the i-th simple coroot, $\alpha_i^\vee = 2\alpha_i/|\alpha_i|^2$. The extra Dynkin label λ_0 is defined as

$$\lambda_0 = k - \sum_{i=1}^{r} \lambda_i a_i^\vee\,, \tag{2.76}$$

and is thus an integer, since $k, \lambda_i, a_i^\vee \in \mathbb{Z}_{\geq 0}$. An affine highest weight representation $\hat{\lambda}$ is said to be integrable if

$$\lambda_0 \in \mathbb{Z}_{\geq 0}\,. \tag{2.77}$$

As a consequence, there are only finitely many affine integrable highest weight representations of $\hat{\mathfrak{g}}_k$ at any given level k. Let us denote by P_+^k the set of affine integrable weights.

The full heighest weight representation $V_{\hat{\lambda}}$ is obtained from an highest weight state $|\hat{\lambda}\rangle$, called *affine primary*, and its affine descendants, obtained by acting on them with negative modes of the global G-symmetry currents. The conformal dimension $h_{\hat{\lambda}}$ is equal to

$$h_{\hat{\lambda}} = \frac{C_2(\lambda)}{2(k + h^\vee)}, \tag{2.78}$$

where $C_2(\lambda)$ is the value of the quadratic Casimir C_2 of $\mathfrak{g}$ computed on the finite representation λ, obtained from $\hat{\lambda}$ by ignoring the Dynkin label λ_0.

For highest weight representations with $\hat{\lambda} \in P_+^k$, the highest weight state generates finite representations with respect to any finite $\mathfrak{su}(2)$ subalgebra of $\hat{\mathfrak{g}}_k$. On the other hand, non-integrable representations are made of null vectors, i.e. correlation functions involving fields in non-integrable representations vanish. From now, we will always assume highest weight representations to be integrable.

To affine primaries we can associate *affine characters*. For instance, on the torus of modular parameter τ,

$$\chi_{\hat{\lambda}}(\tau) = \mathrm{tr}_{\hat{\lambda}} q^{(L_0 - c/24)}, \tag{2.79}$$

where as usual $q = e^{2\pi i \tau}$, and the trace is overe the representation $V_{\hat{\lambda}}$. To distinguish between conjugate representations it is useful to introduce a dependence on fugacities,

$$\chi_{\hat{\lambda}}(z_i; \tau) = \mathrm{tr}_{\hat{\lambda}} q^{(L_0 - c/24)} e^{-2\pi i \sum_j z_j h_j}, \tag{2.80}$$

[10] In the case $\mathfrak{g} = \mathfrak{su}(N)$, we can associate to the weight λ the Young tableaux with λ_i columns of length i.

where h_j, $j = 1, \dots, r$, are the Cartan generators of $\mathfrak{g}$ in the Chevalley basis.

A strong constraint on any two-dimensional CFT is that of modular covariance. Setting $z_i = 0$ for simplicity, one can define the modular $\mathcal{T}$ and $\mathcal{S}$ matrices by

$$\chi_{\hat{\lambda}}(\tau+1) = \sum_{\hat{\mu} \in P_+^k} \mathcal{T}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\tau) , \qquad \chi_{\hat{\lambda}}(-1/\tau) = \sum_{\hat{\mu} \in P_+^k} \mathcal{S}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\tau) . \tag{2.81}$$

The general expression for $\mathcal{T}$ is particularly simple,

$$\mathcal{T}_{\hat{\lambda}\hat{\mu}} = \delta_{\hat{\lambda}\hat{\mu}} \exp\left[2\pi i (h_{\hat{\lambda}} - c/24)\right] . \tag{2.82}$$

Both $\mathcal{T}, \mathcal{S}$ must be unitary,

$$\mathcal{T}^{-1} = \mathcal{T}^\dagger, \qquad \mathcal{S}^{-1} = \mathcal{S}^\dagger . \tag{2.83}$$

Moreover,

$$\mathcal{S}^2 = \mathcal{C} , \qquad \mathcal{S}^4 = \mathcal{C}^2 = 1 , \tag{2.84}$$

where the charge conjugation matrix $\mathcal{C}$ acts on characters as $\mathcal{C}\chi_{\hat{\lambda}} = \chi_{\hat{\lambda}^*}$, $\hat{\lambda}^*$ being the conjugate weight of $\hat{\lambda}$.

One can build a candidate partition function by gluing holomorphic and anti-holomorphic characters in order to respect modular covariance,

$$Z(\tau, \bar{\tau}) = \sum_{\hat{\lambda}, \hat{\mu} \in P_+^k} \chi_{\hat{\lambda}}(\tau) \mathcal{M}_{\hat{\lambda}\hat{\mu}} \bar{\chi}_{\hat{\mu}}(\bar{\tau}) . \tag{2.85}$$

The *modular mass matrix* $\mathcal{M}_{\hat{\lambda}\hat{\mu}}$ can be interpreted as the multiplicity of the primary field which transforms in the $V_{\hat{\lambda}} \otimes \bar{V}_{\hat{\mu}}$ representation. In order to get a modular-invariant partition function, one needs

$$\mathcal{T}^\dagger \mathcal{M} \mathcal{T} = \mathcal{M} = \mathcal{S}^\dagger \mathcal{M} \mathcal{S} . \tag{2.86}$$

Moreover, $\mathcal{M}$ needs to be physically meaningful, i.e. that multiplicities are non-negative integers and that the vacuum representation is unique,

$$\mathcal{M}_{\hat{\lambda}\hat{\mu}} \in \mathbb{Z}_{\geq 0} \quad \forall\, \hat{\lambda}, \hat{\mu} \in P_+^k , \qquad \mathcal{M}_{\hat{0}\hat{0}} = 1 , \tag{2.87}$$

where $\hat{0} \equiv k\hat{\omega}_0$ is the integrable weight associated to the trivial $\widehat{\mathfrak{g}}_k$ representation.[11]

2.2.3 Chern-Simons/Wess-Zumino-Witten correspondence

Let us consider the G_k Chern-Simons (CS) theory on a a three-manifold M_3,

$$S[A] = \frac{k}{8\pi} \int_{M_3} \mathrm{d}^3 x \, \epsilon^{\mu\nu\rho} \, \mathrm{tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \tfrac{2}{3} A_\mu [A_\nu, A_\rho]\right) , \tag{2.88}$$

where 'tr' is the trace over the fundamental representation of the gaupe group G (which we assume to be compact, connected, and simply connected), and one must pick the level

[11] Such conditions are necessary, but in general are not sufficient. Another physical condition is that fusion coefficients of three primary operators (2.20) are nonnegative integers.

$k \in \mathbb{Z}$ for in order for the action to be gauge-invariant. A set of observables in this theory is given by the expectation values of Wilson lines. The latter are also both the symmetry operators and the charged objects for a one-form symmetry of the theory, whose extent depends on both G and k.

There is an intimate correspondence between the G_k Chern-Simons theory and the $\widehat{\mathfrak{g}}_k$ current algebra which, as we have seen, is used to define the G_k WZW model [88]. More precisely, let $M_3 = \mathbb{R} \times D_2$, with D_2 being the unit disk, and let us quantize the theory on D_2, with $\mathbb{R}$ regarded as time t. With complex coordinates $z, \bar{z}$ on Σ, the action (2.88) takes the form

$$S[A] = \frac{k}{4\pi} \int \mathrm{d}t \int_{D_2} \mathrm{d}^2 z \ \operatorname{tr}\left(A_z \frac{\partial A_{\bar{z}}}{\partial t} + A_t F_{z\bar{z}} \right) , \tag{2.89}$$

with F being the gauge curvature.

A_t acts as a Lagrange multiplier enforcing $F_{z\bar{z}} = 0$ identically. The quantization of the theory on D_2 is a quantization with a constraint. A way to deal with this is to perform canonical quantization ignoring the constraint $F_{z\bar{z}} = 0$, and only imposing the latter later on physical states. A_z and $A_{\bar{z}}$ are conjugate variables. The canonical commutation relations read

$$[A_z^a(z,\bar{z}), A_{\bar{z}}^b(w,\bar{w})] = \frac{\pi}{k} \delta^{ab} \delta(z-w)\delta(\bar{z}-\bar{w}) \, , \tag{2.90}$$

with $a, b = 1, \ldots, \dim G$. The Hilbert space $\mathcal{H}_{D_2}$ can be identified with the space of holomorphic wavefunctionals $\Psi[A_{\bar{z}}]$, and thanks to the canonical commutation relations we can interpret A_z as a differential operator on this space,

$$A_z^a = \frac{\pi}{k} \frac{\delta}{\delta A_{\bar{z}}^a} \, . \tag{2.91}$$

Then, the constraint $F_{z\bar{z}}\Psi[A_z] = 0$ reads

$$\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right] \right) \Psi[A_{\bar{z}}] = \frac{k}{\pi} \, \partial_z A_{\bar{z}} \Psi[A_{\bar{z}}] \, . \tag{2.92}$$

It is not difficult to see that these correspond to the Ward identity for the $\widehat{\mathfrak{g}}_k$ algebra of a G_k WZW model. Let us consider the latter coupled to a background G current,

$$Z_J[A_{\bar{z}}] = \langle e^{\frac{1}{\pi}\int_{D_2} \mathrm{d}^2 z A_{\bar{z}}^b J_z^b} \rangle_{G_k \, \mathrm{WZW}} \tag{2.93}$$

By taking a functional derivative with respect to $A_{\bar{z}}^a$, one easily proves that $Z_J[A_{\bar{z}}]$ satisfies the same relation as $\Psi[A_{\bar{z}}]$, i.e.

$$\left(\partial_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + \left[A_{\bar{z}}, \frac{\delta}{\delta A_{\bar{z}}} \right] \right) Z_J[A_{\bar{z}}] = \frac{k}{\pi} \, \partial_z A_{\bar{z}} Z_J[A_{\bar{z}}] \, . \tag{2.94}$$

The solutions of these equations are the conformal blocks of the chiral algebra. Therefore we conclude that states in the Hilbert space of the G_k CS theory on $M_3 = \mathbb{R} \times D_2$, canonically quantized on D_2, are in one-to-one correspondence with chiral $\widehat{\mathfrak{g}}_k$ conformal blocks of the G_k WZW model. Additionally, if we insert a Wilson line in an integrable representation $\hat{\lambda}$ along $\mathbb{R}$, this leads to a Hilbert space which is the integrable highest weight representation $V_{\hat{\lambda}}$ of the $\widehat{\mathfrak{g}}_k$ current algebra.

Alternatively, we can take M_3 to be a three-manifold with $\partial M_3 = \Sigma$, with Σ being a compact Riemann surface. The case $\Sigma = T^2$ is of particular interest. Upon imposing holomorphic boundary conditions for the CS gauge field, the states of a basis of the Hilbert space are in one-to-one correspondence with chiral affine characters of a G_k WZW model [89, 90]. Affine characters in the representation $V_{\hat{\lambda}}$ are obtained by inserting the corresponding Wilson line in the complementary direction.

In order to obtain the full-fledged G_k WZW model, however, a single chiral half is not enough. One needs to glue two chiral halves in such a way to obtain a modular invariant partition function, i.e. a physical modular mass matrix $\mathcal{M}$. This is done by considering the $G_k \times G_{-k}$ Chern-Simons theory on M_3, respectively with holomorphic and anti-holomorphic boundary conditions, and by gauging an appropriate one-form symmetry group in the bulk [91]. Different (yet consistent) choices of gauging lead to different modular invariants $\mathcal{M}$.

Example 1: the compact scalar at R^2 even. Let us consider the case of $G = U(1)$, and let us also take $k \in 2\mathbb{Z}$ [91]. The starting point is the $U(1)_k \times \widetilde{U(1)}_{-k}$ Chern-Simons theory,

$$S = \frac{ik}{4\pi} \int_{M_3} (a\, da - \tilde{a}\, d\tilde{a})\,. \tag{2.95}$$

The Wilson lines of the theory $W_{(\lambda,\tilde{\lambda})}[\gamma]$ are labelled by integrable representations of the $\widehat{\mathfrak{u}}(1)_k \oplus \widehat{\mathfrak{u}}(1)_{-k}$ chiral algebra, equivalently by $(\lambda, \tilde{\lambda}) \in \mathbb{Z}_k \times \mathbb{Z}_k$. The Wilson lines inherit the fusion rules of representations in the affine algebra, namely

$$W_{(\lambda_1,\tilde{\lambda}_1)}[\gamma] W_{(\lambda_2,\tilde{\lambda}_2)}[\gamma] = W_{(\lambda_1+\lambda_2,\tilde{\lambda}_1+\tilde{\lambda}_2)}[\gamma]\,. \tag{2.96}$$

The Wilson lines are topological, in the sense that depend only on the homotopy class of their support γ. Being topological extended objects, they generate a one-form symmetry group $G^{(1)} = \mathbb{Z}_k \times \widetilde{\mathbb{Z}}_k$, under which the lines themselves are the charged objects (see Section 2.1). For two Wilson lines whose supports are such that γ_1 is a closed loop winding λ_2 once,

$$W_{(\lambda_1,\tilde{\lambda}_1)}[\gamma_1] W_{(\lambda_2,\tilde{\lambda}_2)}[\gamma_2] = e^{2\pi i [B(\lambda_1,\lambda_2) - B(\tilde{\lambda}_1,\tilde{\lambda}_2)]} W_{(\lambda_2,\tilde{\lambda}_2)}[\gamma_2]\,, \tag{2.97}$$

where $B(\lambda, \mu)$ is the topological braiding of the lines,

$$B(\lambda, \mu) = h_{\lambda+\mu} - h_\lambda - h_\mu = \frac{\lambda\mu}{k}\,. \tag{2.98}$$

Here $h_\lambda = \lambda^2/(2k)$ is the chiral dimension of the corresponding primary in a $\widehat{u}(1)_k$ integrable representation λ.

On a manifold with boundary $\partial M_3 = \Sigma$, the action needs to be augmented by a boundary term in order to preserve gauge invariance,

$$S_\partial = \frac{k}{4\pi} \int_\Sigma \mathrm{d}^2 z\, \sqrt{g} g^{z\bar{z}} (a_z a_{\bar{z}} - \tilde{a}_z \tilde{a}_{\bar{z}})\,. \tag{2.99}$$

Again, let us consider the case $\Sigma = T^2$, that is $M_3 = S^1 \times D_2$ is a solid torus. As argued above, once we impose holomorphic (resp. anti-holomorphic) boundary conditions for the

$U(1)_k$ gauge field a (resp. for the $\widetilde{U(1)}_{-k}$ gauge field $\tilde{a}$), the expectation value for the $W_{(\lambda,\tilde{\lambda})}$ Wilson line inserted along the non-contractible cycle of M_3 evaluates to

$$\langle W_{(\lambda,\tilde{\lambda})}\rangle = \chi^{(k)}_{\lambda}(\tau)\bar{\chi}^{(k)}_{\tilde{\lambda}}(\bar{\tau})\,. \tag{2.100}$$

Here, $\chi^{(k)}_{\lambda}$ are the $U(1)_k$ characters defined in (2.57), with $p' = k/2$ and $p = 1$, who are the building blocks of the partition function of the compact scalar with radius $R^2 = k$ (and its $\mathbb{Z}^P_n$ quotients). In order to obtain a physical CFT partition function we should ensure modular invariance, but (2.100) is manifestly not modular invariant. When declaring $M_3 = S^1 \times D_2$ we have chosen which cycle of T^2 is contractible: this breaks modular invariance and leads to two inequivalent topologies. Since topology is detected by inserting Wilson lines on non-contractible cycles, in order not to be sensitive to the choice of topology we should get rid of Wilson lines in the bulk theory - and this can be achieved by gauging an appropriate one-form symmetry $\mathcal{A}$. In practice, the gauging procedure for an (Abelian) discrete one-form symmetry $\mathcal{A}$ amounts to summing over different gauge bundles, which are labelled by elements of $H^2(M_3,\mathcal{A})$. By Poincaré duality, this can also be expressed as a sum over elements over $H_1(M_3,\mathcal{A})$, which have the interpretation of summing over insertion of Wilson lines associated to $\mathcal{A}$.

Not all of the $\mathbb{Z}_k \times \widetilde{\mathbb{Z}}_k$ one-form symmetry group is gaugeable. For instance, both $\mathbb{Z}_k$ and $\widetilde{\mathbb{Z}}_k$ are anomalous [14]. A one-form symmetry group $\mathcal{A}$ can be gaugeable only if the corresponding Wilson lines $\{W_{(\lambda_i,\tilde{\lambda}_i)}\}$ mutually transparent - that is, they have trivial braiding $B(\lambda_i,\lambda_j) - B(\tilde{\lambda}_i,\tilde{\lambda}_j) = 0 \mod 1$ - and they all have integer spin, $s_{(\lambda,\tilde{\lambda})} \equiv h_\lambda - h_{\tilde{\lambda}} = 0 \mod 1$. A consistent choice is to gauge a subgroup $\mathcal{A} = \mathbb{Z}^{[\omega_0]}_k$, whose set of Wilson lines are of the form $\{W_{(\lambda_i,\tilde{\lambda}_i=\omega_0\lambda_i)}\}$ for some $\omega_0 \in \mathbb{Z}_k$, provided that $\omega_0^2 = 1 \mod 2k$. Then, in the gauged theory,

$$\sum_{\lambda=0}^{k-1}\langle W_{(\lambda,\tilde{\lambda}=\omega_0\lambda)}\rangle = \sum_{\lambda=0}^{k-1}\chi^{(k)}_{\lambda}(\tau)\bar{\chi}^{(k)}_{\omega_0\lambda}(\bar{\tau})\,. \tag{2.101}$$

If $\omega_0 = 1$, we recognize the partition function of the compact scalar with $R^2 = k$, cf. (2.65); choosing an $\omega_0 \neq 1$ one obtains the various orbifolds of the $R^2 = k$ compact boson, cf. (2.56).

Example 2: from $U(1)_8 \times U(1)_{-8}$ Chern-Simons to $U(1)_2$ WZW model. Let us consider $U(1)_8 \times U(1)_{-8}$ Chern-Simons on the solid torus $M_3 = S^1 \times D_2$, with boundary conditions as above. It has a $\mathbb{Z}_8 \times \mathbb{Z}_8$ one-form symmetry, and we label the corresponding lines as $(\lambda,\bar{\mu})\,,\lambda,\mu \in \mathbb{Z}_8$. We want to gauge the following set of lines,

$$\mathcal{A} = (0,\bar{0}) \oplus (0,\bar{4}) \oplus (4,\bar{0}) \oplus (4,\bar{4}) \oplus (2,\bar{2}) \oplus (2,\bar{6}) \oplus (6,\bar{2}) \oplus (6,\bar{6})\,. \tag{2.102}$$

All lines in $\mathcal{A}$ have integer spin, so $\mathcal{A}$ is gaugeable. Moreover, $\mathcal{A} \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ as a group. We can identify the following subsets,

$$\begin{aligned}
\mathcal{A}_1 &= (0,\bar{0}) \oplus (2,\bar{2}) \oplus (4,\bar{4}) \oplus (6,\bar{6}) \simeq \mathbb{Z}_4\,,\\
\mathcal{A}_2 &= (0,\bar{0}) \oplus (4,\bar{0}) \simeq \mathbb{Z}_2\,,\\
\mathcal{A}_3 &= (0,\bar{0}) \oplus (4,\bar{0}) \oplus (0,\bar{4}) \oplus (4,\bar{4}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\,.
\end{aligned} \tag{2.103}$$

Gauging $\mathcal{A}$ is equivalent to gauging first one of its subsets $\mathcal{A}_i$ and then the corresponding coset $\mathcal{A}/\mathcal{A}_i$,

$$\mathcal{A}/\mathcal{A}_1 \simeq \mathcal{A}_2\,, \quad \mathcal{A}/\mathcal{A}_2 \simeq \mathcal{A}_1\,, \quad \mathcal{A}/\mathcal{A}_3 = [(0,\bar{0})] \oplus [(2,\bar{2})] \simeq \mathbb{Z}_2\,. \tag{2.104}$$

We can proceed in different ways, but the end result must be the same.

1. Let us gauge $\mathcal{A}_1$ first. The requirement of $\mathcal{A}_1$-invariance projects onto lines $(\lambda,\bar{\mu})$ with $\lambda = \mu \mod 4$, and the 4 equivalence classes are given by the orbits $[(0,\bar{0})]$, $[(1,\bar{1})]$, $[(4,\bar{0})]$, $[(5,\bar{1})]$. The insertion of these lines in the solid torus leads to the following combination of characters on the bounding torus,

$$\begin{aligned}
\sum_{(\lambda,\bar{\mu})\in[(0,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= |\chi_0|^2+|\chi_2|^2+|\chi_4|^2+|\chi_6|^2 &&= \frac{1}{2}(Z_{\sqrt{8}}[0,0]+Z_{\sqrt{8}}[0,4])\,,\\
\sum_{(\lambda,\bar{\mu})\in[(1,\bar{1})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= |\chi_1|^2+|\chi_3|^2+|\chi_5|^2+|\chi_7|^2 &&= \frac{1}{2}(Z_{\sqrt{8}}[0,0]-Z_{\sqrt{8}}[0,4])\,,\\
\sum_{(\lambda,\bar{\mu})\in[(4,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= \chi_4\bar{\chi}_0+\chi_6\bar{\chi}_2+c.c. &&= \frac{1}{2}(Z_{\sqrt{8}}[4,0]+Z_{\sqrt{8}}[4,4])\,,\\
\sum_{(\lambda,\bar{\mu})\in[(5,\bar{1})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= \chi_5\bar{\chi}_1+\chi_7\bar{\chi}_3+c.c. &&= \frac{1}{2}(Z_{\sqrt{8}}[4,0]-Z_{\sqrt{8}}[4,4])\,.
\end{aligned} \tag{2.105}$$

We recognize here the combinations of characters that correspond to the twisted partition functions of the compact scalar with $R^2 = 8$ under a $\mathbb{Z}_2 \subset \mathbb{Z}_8^P$ symmetry. Now we gauge $\mathcal{A}/\mathcal{A}_1 \simeq \mathcal{A}_2$. This projects onto lines with $\lambda = 0 \mod 2$, and then there is only a single surviving equivalence class, $\{(0,\bar{0})\} = [(0,\bar{0})] \cup [(4,\bar{0})]$, and on the torus one gets

$$\begin{aligned}
\sum_{(\lambda,\bar{\mu})\in\{(0,\bar{0})\}} \langle W_{(\lambda,\bar{\mu})}\rangle &= \sum_{(\lambda,\bar{\mu})\in[(0,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle + \sum_{(\lambda,\bar{\mu})\in[(4,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle \\
&= \frac{1}{2}(Z_{\sqrt{8}}[0,0]+Z_{\sqrt{8}}[0,4]+Z_{\sqrt{8}}[4,0]+Z_{\sqrt{8}}[4,4]) \\
&= Z_{\sqrt{2}}\,,
\end{aligned} \tag{2.106}$$

i.e. the partition function of the compact scalar with $R^2 = 2$.

2. Alternatively, let us gauge $\mathcal{A}_3$ first. This projects onto the lines $(\lambda,\bar{\mu})$ with $\lambda = 0 \mod 2$ and $\mu = 0 \mod 2$. There are 4 equivalence classes, given by $[(0,\bar{0})]$, $[(2,\bar{0})]$, $[(0,\bar{2})]$, and $[(2,\bar{2})]$. On the torus,

$$\begin{aligned}
\sum_{(\lambda,\bar{\mu})\in[(0,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= |\chi_0+\chi_4|^2\,,\\
\sum_{(\lambda,\bar{\mu})\in[(2,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= \chi_2\bar{\chi}_0+\chi_6\bar{\chi}_4+c.c.\,,\\
\sum_{(\lambda,\bar{\mu})\in[(0,\bar{2})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= \chi_4\bar{\chi}_0+\chi_6\bar{\chi}_2+c.c.\,,\\
\sum_{(\lambda,\bar{\mu})\in[(2,\bar{2})]} \langle W_{(\lambda,\bar{\mu})}\rangle &= |\chi_2+\chi_6|^2\,.
\end{aligned} \tag{2.107}$$

Note that the combination of characters that appear in $[(0,\bar{0})]$ and $[(2,\bar{2})]$ is the (modulus squared of) the characters of $\widehat{\mathfrak{u}}(1)_2$, as can be checked explicitly using (2.57). Then we gauge $\mathcal{A}/\mathcal{A}_3$. This projects onto lines with $\lambda = \mu \mod 4$, and the surviving orbit is $\{(0,\bar{0})\} = [(0,\bar{0})] \cup [(2,\bar{2})]$. This, consistently, leads to the partition function of the diagonal $U(1)_2$ WZW model,

$$\begin{aligned}\sum_{(\lambda,\bar{\mu})\in\{(0,\bar{0})\}} \langle W_{(\lambda,\bar{\mu})}\rangle &= \sum_{(\lambda,\bar{\mu})\in[(0,\bar{0})]} \langle W_{(\lambda,\bar{\mu})}\rangle + \sum_{(\lambda,\bar{\mu})\in[(2,\bar{2})]} \langle W_{(\lambda,\bar{\mu})}\rangle \\ &= |\chi_0+\chi_4|^2 + |\chi_2+\chi_6|^2 \\ &= Z_{U(1)_2}\,.\end{aligned} \tag{2.108}$$

2.3 Bose-Fermi duality in $2d$

In this Section we review well-known relations among theories defined on a 2d closed manifold M_2 upon gauging a discrete symmetry G. We first review dualities among bosonic theories and then consider those turning a bosonic theory to a fermionic one, and viceversa. For simplicity, we restrict to Abelian cyclic groups, with $G = \mathbb{Z}_n$.

Let $\mathcal{T}$ be a bosonic theory on a 2-manifold M_2 with a non-anomalous symmetry $\mathbb{Z}_n$. As mentioned in section 2.1.1, the theory obtained by gauging $\mathbb{Z}_n$

$$\widehat{\mathcal{T}} = \mathcal{T}/\mathbb{Z}_n\,, \tag{2.109}$$

is guaranteed to have a non-anomalous "dual" symmetry $\widehat{G} = \mathrm{Hom}(\mathbb{Z}_n, U(1)) = \widehat{\mathbb{Z}}_n$ [73]. Since this is the case relevant for our discussion, let us now be more precise with normalizations. Its partition function in the presence of a background $\widehat{\mathbb{Z}}_n$ gauge field reads

$$Z_{\widehat{\mathcal{T}}}[\widehat{T}] = \frac{1}{n^g}\sum_{t\in H^1(M_2,\mathbb{Z}_n)} Z_{\mathcal{T}}[t]e^{\frac{2\pi i}{n}\int \widehat{T}\cup t}\,, \tag{2.110}$$

where $\widehat{T}$ denotes the $\widehat{\mathbb{Z}}_n$ background insertion, $\cup$ is the cup product in $H^1(M_2,\mathbb{Z}_n)$, g is the genus of M_2 and $n^g = \sqrt{|H^1(M_2,\mathbb{Z}_n)|}$. Gauging the dual symmetry gives back the original theory:

$$\widehat{\mathcal{T}}/\widehat{\mathbb{Z}}_n = (\mathcal{T}/\mathbb{Z}_n)/\widehat{\mathbb{Z}}_n = \mathcal{T}\,. \tag{2.111}$$

Explicitly,

$$\frac{1}{n^g}\sum_t Z_{\mathcal{T}}[t]\,\frac{1}{n^g}\sum_{\hat{t}} e^{\frac{2\pi i}{n}\int \hat{t}\cup t} e^{\frac{2\pi i}{n}\int T\cup\hat{t}} = \frac{1}{n^g}\sum_t Z_{\mathcal{T}}[t]n^g\delta_{t,T} = Z_{\mathcal{T}}[T]\,, \tag{2.112}$$

where we have used that

$$\frac{1}{n^g}\sum_v e^{\frac{2\pi i}{n}\int v\cup w} = n^g\delta_{w,0}\,. \tag{2.113}$$

Let us consider now fermionic theories, namely those theories where observables do depend on the choice of the spin structure ρ. These theories have a $\mathbb{Z}_2$ fermion parity symmetry $\mathbb{Z}_2^F$ for which we can think the spin structure ρ as a choice of background.[12] If, instead, the partition function of the theory is independent of the particular choice of spin

[12] It is tempting to take the correspondence literally, but one cannot identify a spin structure ρ with a background gauge field $S \in H^1(M_2,\mathbb{Z}_2)$ for $\mathbb{Z}_2^F$ in a natural way. Yet, we can add on top of ρ a $\mathbb{Z}_2^F$ connection S: this has the net effect of changing the spin structure from ρ to $S\cdot\rho$, which is defined by changing the periodicity of ρ around the cycles along which S has nontrivial holonomy, see (2.118).

structure ρ on M_2, we say that the theory is bosonic.

A standard way to get a bosonic theory out of a fermionic one $\mathcal{F}$ is to gauge $\mathbb{Z}_2^F$, which is equivalent to sum over the spin structures on M_2. In fact, we get two different bosonic theories $\mathcal{B}$ and $\mathcal{B}'$, depending on whether we consider $\mathcal{F}$ or we stack the Arf theory on top of it [92],

$$\begin{aligned}\mathcal{B} &= (\mathcal{F} \times \text{Arf})/\mathbb{Z}_2^F\,,\\ \mathcal{B}' &= \mathcal{F}/\mathbb{Z}_2^F\,.\end{aligned} \tag{2.114}$$

The Arf theory is one of the simplest non-trivial topological theories, the IR limit of the topologically non-trivial phase of the Kitaev chain [59]. Its partition function Z_{Arf} on M_2 is the Arf invariant, which is the index of the Dirac operator $\not{D}_\rho$ mod 2 [93]. We have

$$Z_{\text{Arf}}[\rho] = e^{i\pi \text{Arf}[\rho]}\,. \tag{2.115}$$

It is $+1$ or -1 respectively on even or odd spin structures.[13] For $M_2 = T^2$, the only odd spin structure is the one in which fermions are taken to be periodic in both cycles, i.e.

$$\text{Arf}[\rho] = \begin{cases} 0\,, & \rho = [\text{NS,NS}],\ [\text{NS,R}],\ [\text{R,NS}],\\ 1\,, & \rho = [\text{R,R}]\,.\end{cases} \tag{2.116}$$

Explicitly, we have

$$\begin{aligned} Z_{\mathcal{B}}[T,\rho] &= \frac{1}{2^g}\sum_s Z_{\mathcal{F}}[s\cdot\rho] Z_{\text{Arf}}[s\cdot\rho] e^{i\pi\int s\cup T}\,,\\ Z_{\mathcal{B}'}[T,\rho] &= \frac{1}{2^g}\sum_s Z_{\mathcal{F}}[s\cdot\rho] e^{i\pi\int s\cup T}\,,\end{aligned} \tag{2.117}$$

where in the last step in both relations we have used the fact that having a non-trivial background $\mathbb{Z}_2^F$ gauge field s is equivalent to changing the spin structure from ρ to $s\cdot\rho$, where for any one-cycle γ on M_2

$$(s\cdot\rho)(\gamma) = \rho(\gamma) + \int_\gamma s \mod 2\,. \tag{2.118}$$

In (2.117) T is the background gauge field for the $\mathbb{Z}_2$ symmetry dual to $\mathbb{Z}_2^F$. For any fixed ρ the sum over s can be traded for a sum over spin structures $\rho' = s\cdot\rho$, hence the theories $\mathcal{B}$ and $\mathcal{B}'$ do not depend on the choice of the fiducial spin structure ρ and are bosonic. Taking the fiducial spin structure ρ to be identically zero (that is, NS on all non-trivial cycles), it can be shown (see e.g. [94, 95] for details) that we can trade the sum over s for a sum over spin structures by

$$\sum_s e^{i\pi\int s\cup w} F[s\cdot\rho] \longmapsto \sum_{\rho'} e^{i\pi(\text{Arf}[w\cdot\rho'] - \text{Arf}[\rho'])} F[\rho']\,. \tag{2.119}$$

[13] A spin structure ρ is called even (odd) when a Majorana fermion with spin structure ρ has an even (odd) number of zero modes.

Therefore,

$$
\begin{aligned}
Z_{\mathcal{B}}[T] &= \frac{1}{2^g}\sum_{\rho} Z_{\mathcal{F}}[\rho] e^{i\pi \mathrm{Arf}[T\cdot\rho]}\,,\\
Z_{\mathcal{B}'}[T] &= \frac{1}{2^g}\sum_{\rho} Z_{\mathcal{F}}[\rho] e^{i\pi \mathrm{Arf}[T\cdot\rho]-i\pi\mathrm{Arf}[\rho]}\,.
\end{aligned}
\tag{2.120}
$$

We denote the $\mathbb{Z}_2$ symmetry of $\mathcal{B}$ dual to $\mathbb{Z}_2^F$ by $\mathbb{Z}_2^\vee$. It can be checked that the two bosonic theories $\mathcal{B}$ and $\mathcal{B}'$ are related by gauging, $\mathcal{B}' = \mathcal{B}/\mathbb{Z}_2^\vee$, and hence the dual $\mathbb{Z}_2$ symmetry of $\mathcal{B}'$ is $\widehat{\mathbb{Z}}_2^\vee$.

Conversely, to a given a bosonic theory $\mathcal{B}$ with a non-anomalous symmetry $\mathbb{Z}_2$ we can associate a fermionic theory $\mathcal{F}$ [96]. The latter is not unique as it depends on the choice of the specific $\mathbb{Z}_2$ symmetry of $\mathcal{B}$. $\mathcal{F}$ is obtained by stacking $\mathcal{B}$ with Arf, as to obtain a spin strucure dependence, and then gauging a diagonal $\mathbb{Z}_2$ symmetry between the two,

$$\mathcal{F} = (\mathcal{B}\times \mathrm{Arf})/\mathbb{Z}_2\,. \tag{2.121}$$

More precisely, we have

$$Z_{\mathcal{F}}[\rho] = \frac{1}{2^g}\sum_{t\in H^1(M_2,\mathbb{Z}_2)} Z_{\mathcal{B}}[t] e^{i\pi\mathrm{Arf}[t\cdot\rho]}\,, \tag{2.122}$$

where t is the dynamical gauge field associated to the $\mathbb{Z}_2$ symmetry. It can be shown that if we gauge $\mathbb{Z}_2^F$ in the fermionic theory $\mathcal{F}$ we have

$$\mathcal{F}/\mathbb{Z}_2^F = \mathcal{B}' = \mathcal{B}/\mathbb{Z}_2\,. \tag{2.123}$$

We can also define the fermionic theory $\mathcal{F}'$ as in (2.121), replacing $\mathcal{B}$ with $\mathcal{B}'$

$$\mathcal{F}' = (\mathcal{B}'\times \mathrm{Arf})/\mathbb{Z}_2'\,, \tag{2.124}$$

where $\mathbb{Z}_2' = \widehat{\mathbb{Z}}_2$ is the symmetry of $\mathcal{B}'$ dual to $\mathbb{Z}_2$ of $\mathcal{B}$, see (2.123). A simple computation shows that

$$Z_{\mathcal{F}'}[\rho] = Z_{\mathcal{F}}[\rho] e^{i\pi\mathrm{Arf}[\rho]} \tag{2.125}$$

and hence

$$\mathcal{F}' = \mathcal{F}\times\mathrm{Arf}\,. \tag{2.126}$$

To close the circle, we can get the original bosonic theory $\mathcal{B}$ from $\mathcal{F}'$ by inverting the above procedure,

$$\mathcal{B} = ((\mathcal{F}'\times\mathrm{Arf})\times\mathrm{Arf})/\mathbb{Z}_2^F = \mathcal{F}'/\mathbb{Z}_2^F\,, \tag{2.127}$$

that is,

$$Z_{\mathcal{B}}[T] = \frac{1}{2^g}\sum_{\rho} Z_{\mathcal{F}'}[\rho] e^{i\pi\mathrm{Arf}[T\cdot\rho]+i\pi\mathrm{Arf}[\rho]}\,, \tag{2.128}$$

which agrees with the expression in (2.120), given (2.125). We summarize all these results in the diagram in Figure 2.3.

It is instructive also to understand how the various gaugings map different states in the Hilbert space of the theory into each other, cf. Table 2.1. Since each gauging step is invertible, no state is lost in the process: for example, states of $\mathcal{B}$ which are odd under the gauged $\mathbb{Z}_2^{\mathcal{B}}$ symmetry become states in the twisted sector under the dual $\mathbb{Z}_2^{\mathcal{B}'}$ symmetry.

$$
\begin{tikzcd}
\mathcal{B} \arrow[rrr, shift left, "(\cdot\times\mathrm{Arf})/\mathrm{diag}(\mathbb{Z}_2^{\mathcal{B}},\mathbb{Z}_2^F)"] \arrow[d, shift left, "\cdot/\mathbb{Z}_2^{\mathcal{B}}"] & & & \mathcal{F} \arrow[lll, shift left, "(\cdot\times\mathrm{Arf})/\mathbb{Z}_2^F"] \arrow[d, leftrightarrow, "\times\mathrm{Arf}"] \\
\mathcal{B}' = \mathcal{B}/\mathbb{Z}_2^{\mathcal{B}} \arrow[u, shift left, "\cdot/\mathbb{Z}_2^{\mathcal{B}'}"] \arrow[rrr, shift left, "(\cdot\times\mathrm{Arf})/\mathrm{diag}(\mathbb{Z}_2^{\mathcal{B}'},\mathbb{Z}_2^F)"] & & & \mathcal{F}' = \mathcal{F}\times\mathrm{Arf} \arrow[lll, shift left, "(\cdot\times\mathrm{Arf})/\mathbb{Z}_2^F"]
\end{tikzcd}
$$

Figure 2.3: Diagram summarizing the relation through gaugings between the two bosonic $\mathcal{B}$, $\mathcal{B}'$ and fermionc $\mathcal{F}$, $\mathcal{F}'$ theories in two dimensions.

$\mathcal{B}$	$\mathbb{Z}_2^{\mathcal{B}} = +1$	$\mathbb{Z}_2^{\mathcal{B}} = -1$
$\mathcal{H}$	E	O
$\mathcal{H}_{tw}$	S	T

$\mathcal{F}$	$\mathbb{Z}_2^F = +1$	$\mathbb{Z}_2^F = -1$
$\mathcal{H}_{\mathrm{NS}}$	E	T
$\mathcal{H}_{\mathrm{R}}$	S	O

$\mathcal{B}'$	$\mathbb{Z}_2^{\mathcal{B}'} = +1$	$\mathbb{Z}_2^{\mathcal{B}'} = -1$
$\mathcal{H}'$	E	S
$\mathcal{H}'_{tw}$	O	T

$\mathcal{F}'$	$\mathbb{Z}_2^F = +1$	$\mathbb{Z}_2^F = -1$
$\mathcal{H}'_{\mathrm{NS}}$	E	T
$\mathcal{H}'_{\mathrm{R}}$	O	S

Table 2.1: Rearrangement of sectors under gauging $\mathbb{Z}_2$ symmetries as in Figure 2.3.

Example: compact scalar at the free fermion radius.

Here we check explicitly the duality between $\mathcal{F}' = 1$ Dirac and $\mathcal{B} = U(1)_4$,

$$
1\ \mathrm{Dirac} = \frac{U(1)_4 \times \mathrm{Arf}}{\mathbb{Z}_2^P} \times \mathrm{Arf}\,. \tag{2.129}
$$

Equivalently, using (2.124),

$$
1\ \mathrm{Dirac} = \frac{U(1)_1 \times \mathrm{Arf}}{\mathbb{Z}_2^{W\prime}}\,, \tag{2.130}
$$

where $\mathcal{B}' = U(1)_1$ is defined as in (2.68), and we will find $\mathbb{Z}_2^{W\prime}$ to be the $\mathbb{Z}_2$ subgroup of the winding $U(1)'_W$ symmetry for $U(1)_1$.

Let us start by discussing the $U(1)_4$ theory. Its momentum and winding symmetry are as in (2.47). Primary operators are of the form $\mathcal{V}_{e,m}$ defined in (2.50), their conformal dimensions $(h_{e,m}, \bar{h}_{e,m})$ being as in (2.49), with $R = 2$. To orbifold by the $\mathbb{Z}_2^P \subset U(1)_P$ symmetry, let us first classify the operator content of the theory according to it. This symmetry acts on primaries as

$$
\mathbb{Z}_2^P: \quad \mathcal{V}_{e,m} \mapsto (-1)^e\, \mathcal{V}_{e,m}\,. \tag{2.131}
$$

We quantize the theory on the circle and we classify operators according to their $\mathbb{Z}_2^P$ properties as in Table 2.2. To see this explicitly, let us consider the torus partition function. At $R = 2$, this reads

$$
Z_2 = \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{\frac{1}{2}(e/2+m)^2} \bar{q}^{\frac{1}{2}(e/2-m)^2}\,. \tag{2.132}
$$

This is the partition function without any $\mathbb{Z}_2^P$ background. Let us now add a $\mathbb{Z}_2^P$ background, labelled by $P = [p_a, p_b]$, $p_a, p_b \in \{0, 1\}$ denoting the p_a-twisted partition function

$U(1)_4$	$\mathbb{Z}_2^P=+1$	$\mathbb{Z}_2^P=-1$
$\mathcal{H}$	$\mathcal{V}_{2p,q}$	$\mathcal{V}_{2p+1,q}$
$\mathcal{H}_{tw}$	$\mathcal{V}_{2p,q+1/2}$	$\mathcal{V}_{2p+1,q+1/2}$
$U(1)_1$	$\mathbb{Z}_2^{W\prime}=+1$	$\mathbb{Z}_2^{W\prime}=-1$
$\mathcal{H}'$	$\mathcal{V}_{2p,q}$	$\mathcal{V}_{2p,q+1/2}$
$\mathcal{H}'_{tw}$	$\mathcal{V}_{2p+1,q}$	$\mathcal{V}_{2p+1,q+1/2}$

1 Dirac $\times$Arf	$\mathbb{Z}_2^F=+1$	$\mathbb{Z}_2^F=-1$
$\mathcal{H}_{\mathrm{NS}}$	$\mathcal{V}_{2p,q}$	$\mathcal{V}_{2p+1,q+1/2}$
$\mathcal{H}_{\mathrm{R}}$	$\mathcal{V}_{2p,q+1/2}$	$\mathcal{V}_{2p+1,q}$
1 Dirac	$\mathbb{Z}_2^F=+1$	$\mathbb{Z}_2^F=-1$
$\mathcal{H}'_{\mathrm{NS}}$	$\mathcal{V}_{2p,q}$	$\mathcal{V}_{2p+1,q+1/2}$
$\mathcal{H}'_{\mathrm{R}}$	$\mathcal{V}_{2p+1,q}$	$\mathcal{V}_{2p,q+1/2}$

Table 2.2: Primary spectrum of $U(1)_4$, $U(1)_1$, 1 Dirac, and 1 Dirac $\times$Arf, classified according to $\mathbb{Z}_2^P$, $\mathbb{Z}_2^{W\prime}$, $\mathbb{Z}_2^F$ symmetry properties, respectively. The primaries $\mathcal{V}_{e,m}$ are always intended to be computed at $R=2$, which makes it possible to follow how the operators get mapped across the various sectors under the gaugings.

with p_b insertions of the charge operator for $\mathbb{Z}_2^P$ in the trace,

$$\begin{aligned}
Z_2[0,0] &= Z_2\,,\\
Z_2[0,1] &= \frac{1}{|\eta(\tau)|^2}\sum_{e,m}(-1)^e q^{\frac{1}{2}(e/2+m)^2}\bar{q}^{\frac{1}{2}(e/2-m)^2}\,,\\
Z_2[1,0] &= \frac{1}{|\eta(\tau)|^2}\sum_{e,m} q^{\frac{1}{2}(e/2+m+1/2)^2}\bar{q}^{\frac{1}{2}(e/2-m-1/2)^2}\,,\\
Z_2[1,1] &= \frac{1}{|\eta(\tau)|^2}\sum_{e,m}(-1)^e q^{\frac{1}{2}(e/2+m+1/2)^2}\bar{q}^{\frac{1}{2}(e/2-m-1/2)^2}\,.
\end{aligned} \tag{2.133}$$

Now we perform the gauging of $\mathbb{Z}_2^P$, that is the sum over P, to get the $U(1)_1$ theory. Since we are orbifolding by a discrete symmetry, in the $U(1)_1$ theory there is a symmetry $\widehat{\mathbb{Z}}_2$ dual to the gauged $\mathbb{Z}_2^P$. We claim that it is $\mathbb{Z}_2^{W\prime}$, that is the $\mathbb{Z}_2$ subset of the winding symmetry $U(1)'_W$ of $U(1)_1$, as anticipated at the beginning of the Appendix. One way to see this is to compute the torus partition function of $U(1)_1$ twisted with respect to $\widehat{\mathbb{Z}}_2$, by means of (2.110). One obtains

$$\begin{aligned}
Z_1[0,0] &= \tfrac{1}{2}(Z_2[0,0]+Z_2[0,1]+Z_2[1,0]+Z_2[1,1]) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m} q^{\frac{1}{2}(e+\frac{m}{2})^2}\bar{q}^{\frac{1}{2}(e-\frac{m}{2})^2}\,,\\
Z_1[0,1] &= \tfrac{1}{2}(Z_2[0,0]+Z_2[0,1]-Z_2[1,0]-Z_2[1,1]) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m}(-1)^m q^{\frac{1}{2}(e+\frac{m}{2})^2}\bar{q}^{\frac{1}{2}(e-\frac{m}{2})^2}\,,\\
Z_1[1,0] &= \tfrac{1}{2}(Z_2[0,0]-Z_2[0,1]+Z_2[1,0]-Z_2[1,1]) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m} q^{\frac{1}{2}(e+\frac{m}{2}+\frac{1}{2})^2}\bar{q}^{\frac{1}{2}(e-\frac{m}{2}+\frac{1}{2})^2}\,,\\
Z_1[1,1] &= \tfrac{1}{2}(Z_2[0,0]-Z_2[0,1]-Z_2[1,0]+Z_2[1,1]) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m}(-1)^m q^{\frac{1}{2}(e+\frac{m}{2}+\frac{1}{2})^2}\bar{q}^{\frac{1}{2}(e-\frac{m}{2}+\frac{1}{2})^2}\,,
\end{aligned} \tag{2.134}$$

We can also use the twisted torus partition function to classify the operator content in the gauged $U(1)_1$ theory with respect to $\widehat{\mathbb{Z}}_2=\mathbb{Z}_2^{W\prime}$, cf. Table 2.2.

Moreover, recall that in $U(1)_4$ there is a mixed anomaly between $U(1)_P$ and $U(1)_W$, which is also there as a mixed anomaly between their $\mathbb{Z}_2$ subgroups. It is known that mixed anomalies get mapped to possibly nontrivial extensions of groups (see the corresponding discussion in section 2.1.2 for more details). In the orbifold theory, what used to be the winding symmetry in $U(1)_4$ gets extended by $\widehat{\mathbb{Z}}_2$. Indeed, a way to check this is to compute the partition function of the gauged theory in the presence of a twist $e^{i\theta Q_W}$ along the temporal cycle for the $U(1)_W$ symmetry of $U(1)_4$. One finds

$$Z_1[(0,0)_{\hat{P}}|(0,\theta)_W] = \frac{1}{|\eta(\tau)|^2} \sum_{e,m} e^{i\theta\frac{m}{2}} q^{\frac{1}{2}(e+\frac{m}{2})^2} \bar{q}^{\frac{1}{2}(e-\frac{m}{2})^2} . \tag{2.135}$$

This confirms that the radius of $U(1)'_W$ of $U(1)_1$, with respect to $U(1)_W$ of $U(1)_4$, is doubled upon gauging $\mathbb{Z}_2^P$, since the winding charges get halved. Hence the $U(1)_1$ theory has momentum and winding symmetry, in terms of the original scalar ϕ with $R = 2$,

$$\begin{cases} U(1)'_P : & \phi \mapsto \phi + 2\alpha\,, \qquad \alpha \sim \alpha + \pi\,, \\ U(1)'_W : & \tilde{\phi} \mapsto \tilde{\phi} + \tilde{\alpha}\,, \qquad \tilde{\alpha} \sim \tilde{\alpha} + 4\pi\,. \end{cases} \tag{2.136}$$

Note the different ranges of $\alpha, \tilde{\alpha}$ with respect to (2.47) with $R = 2$.

In order to fermionize the theory, we should choose a $\mathbb{Z}_2$ symmetry of $U(1)_1$ to fermionize, according to (2.121). We claim that the correct prescription is to choose it to be $\mathbb{Z}_2^{W\prime}$. On the torus we can check it using the relation (2.122),

$$\begin{aligned} Z_{\mathcal{F}'}[\mathrm{NS},\mathrm{NS}] &= \tfrac{1}{2}\left(Z_1[0,0] + Z_1[0,1] + Z_1[1,0] - Z_1[1,1]\right) = \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{NS},\mathrm{R}] &= \tfrac{1}{2}\left(Z_1[0,0] + Z_1[0,1] - Z_1[1,0] + Z_1[1,1]\right) = \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{R},\mathrm{NS}] &= \tfrac{1}{2}\left(Z_1[0,0] - Z_1[0,1] + Z_1[1,0] + Z_1[1,1]\right) = \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{R},\mathrm{R}] &= \tfrac{1}{2}\left(-Z_1[0,0] + Z_1[0,1] + Z_1[1,0] + Z_1[1,1]\right) = 0\,. \end{aligned} \tag{2.137}$$

We recognize the torus partition function for a single Dirac fermion in the different spin structures. We can also use (2.134) to write, equivalently,

$$\begin{aligned} Z_{\mathcal{F}'}[\mathrm{NS},\mathrm{NS}] &= \tfrac{1}{2}\left(Z_2[0,0] + Z_2[0,1] + Z_2[1,0] - Z_2[1,1]\right) = \left|\frac{\theta_3(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{NS},\mathrm{R}] &= \tfrac{1}{2}\left(Z_2[0,0] + Z_2[0,1] - Z_2[1,0] + Z_2[1,1]\right) = \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{R},\mathrm{NS}] &= \tfrac{1}{2}\left(Z_2[0,0] - Z_2[0,1] + Z_2[1,0] + Z_2[1,1]\right) = \left|\frac{\theta_2(\tau)}{\eta(\tau)}\right|^2 , \\ Z_{\mathcal{F}'}[\mathrm{R},\mathrm{R}] &= \tfrac{1}{2}\left(Z_2[0,0] - Z_2[0,1] - Z_2[1,0] - Z_2[1,1]\right) = 0\,. \end{aligned} \tag{2.138}$$

Indeed, this matches with $(U(1)_4 \times \mathrm{Arf})/\mathbb{Z}_2 \times \mathrm{Arf}$. This also matches with the analysis done in [97]. We use this knowledge to classify the operator content in the theory with respect to $\mathbb{Z}_2^F$, cf. Table 2.2.

Let us also mention that, for a free Dirac fermion, there is no difference in the value of $Z_{\mathcal{F}' = 1\,\mathrm{Dirac}}$ or $Z_{\mathcal{F} = 1\,\mathrm{Dirac}\,\times\,\mathrm{Arf}}$ on the torus. This is actually general: it is tricky to distinguish $\mathcal{F}$ and $\mathcal{F}'$. To say whether a fermionic theory is of type $\mathcal{F}$ or $\mathcal{F}'$, with respect to a given bosonic theory $\mathcal{B}$, we should look at its edge modes on manifolds with boundary [92].

Chapter 3

Non-Abelian Bosonization, Globally

In this Chapter, we perform non-Abelian bosonization of the chiral Gross-Neveu model in full detail. While this procedure is not conceptually novel, we will perform it employing the full power of the modern tools we have presented in the previous Chapter. After identifying in Section 3.1 that the chiral Gross-Neveu model, on a trivial manifold, is an appropriate current-current deformation of a $SU(N)_1 \times U(1)_N$ WZW model, we will discuss in Section 3.2 to which extent the mapping holds on a non-trivial manifold. This will force us to rediscuss the global aspects of non-Abelian bosonization for a set of N free Dirac fermions. In doing so we will discover that, if one wishes to keep manifest a unitary flavor symmetry, there are certain differences in the levels and in global identifications to perform between bosonic and fermionic theories, depending on the parity of the number of Dirac flavors N. A main technical result of this Chapter is equation (3.17), for which we provide a proof in Section 3.3.

3.1 The chiral Gross-Neveu model as $J\bar{J}$ deformations of WZW models

Let us consider again the chiral Gross-Neveu model presented in Section 1.3. For a finite N analysis, it is more convenient to express the Lagrangian of the theory as follows,

$$\mathcal{L}_{\rm cGN} = i\psi^\dagger_{+a}\,\partial_-\psi^a_+ - i\psi^\dagger_{-a}\,\partial_+\psi^a_- + \frac{\lambda_s}{N}|\psi^\dagger_{-a}\psi^a_+|^2 - \frac{\lambda_v}{N^2}(\psi^\dagger_{+a}\psi^a_+)(\psi^\dagger_{-b}\psi^b_-), \tag{3.1}$$

where $x^\pm = (x^1 \mp ix^2)/2$, $a = 1,\ldots,N$ are flavor indices, and $\psi^a_\pm$ are the two Weyl components of the N Dirac fermions. With a Fierzing, (3.1) can also be rewritten as

$$\mathcal{L}_{cGN} = i\psi^\dagger_{+a}\,\partial_-\psi^a_+ - i\psi^\dagger_{-a}\,\partial_+\psi^a_- + \frac{\lambda}{N}J^A_+J^A_- + \frac{\lambda'}{N^2}J_+J_-\,, \tag{3.2}$$

where

$$J^A_\pm = \pm\psi^\dagger_{a\pm}\left(T^A\right)^a{}_b\psi^b_\pm\,, \qquad J_\pm = \pm\psi^\dagger_{a\pm}\psi^a_\pm\,, \tag{3.3}$$

are the $SU(N)$ and $U(1)$ chiral currents, respectively, with T^A, $A = 1,\ldots,N^2-1$, being the generators of $SU(N)$ in the fundamental representation, chosen with the normalization $\mathrm{Tr}(T^AT^B) = \frac{1}{2}\delta^{AB}$. The couplings λ_s, λ_v, λ, λ' are related by

$$\lambda = 2\lambda_s\,, \qquad \lambda' = \lambda_v + \lambda_s\,. \tag{3.4}$$

The last term in (3.1) can be neglected at leading order in a large N analysis, as discussed in Section 1.3, but it should be kept at finite N. The $O(2N)_L \times O(2N)_R$ global symmetry of the N free Dirac fermions is broken by the interaction to

$$G(N\text{-flavor cGN}) = \frac{U(N)_V \times U(1)_A}{\mathbb{Z}_2} \rtimes \mathbb{Z}_2^C\,, \tag{3.5}$$

whose action on the fields is

$$\begin{cases} U(N)_V: & \psi_\pm^a \mapsto \mathcal{U}^a{}_b \psi_\pm^b\,, \qquad \mathcal{U} \in U(N)\,, \\ U(1)_A: & \psi_\pm^a \mapsto e^{\pm i\alpha} \psi_\pm^a\,, \qquad \alpha \sim \alpha + 2\pi\,. \\ \mathbb{Z}_2^C: & \psi_\pm^a \mapsto \psi^\dagger_{\pm a}\,, \end{cases} \tag{3.6}$$

Note that there is a diagonal $\mathbb{Z}_2$ in $U(N)_V \times U(1)_A$, with $\mathcal{U} = -\mathbf{1}$ and $\alpha = \pi$, whose action is trivial on fields, hence the quotient. The corresponding "off-diagonal" $\mathbb{Z}_2$ is identified with fermion parity $\mathbb{Z}_2^F$, $\psi_\pm^a \mapsto -\psi_\pm^a$.

Using "naïve" non-Abelian bosonization [62], the chiral Gross-Neveu model (3.1) can be (at least locally) expressed as a $J\bar{J}$ deformation of a $U(N)_1$ WZW model. The latter is described by an $SU(N)$ matrix U and a free compact scalar ϕ of radius $R_0 = \sqrt{N}$ parametrizing the $U(1)$ factor,

$$\mathcal{L}[U,\phi] = \mathcal{L}_0[U,\phi] + \frac{\lambda}{N} J_+^A J_-^A + \frac{\lambda'}{N^2} J_+ J_-\ , \tag{3.7}$$

where

$$\mathcal{L}_0[U,\phi] = \frac{1}{8\pi}\partial_+\phi\,\partial_-\phi + \frac{1}{8\pi}\mathrm{Tr}(\partial_+ U^\dagger \partial_- U + \partial_- U^\dagger \partial_+ U) + \mathcal{L}_{\mathrm{WZ}}^{SU(N)_1}\ , \tag{3.8}$$

is the Lagrangian of the undeformed theory, with $\mathcal{L}_{\mathrm{WZ}}^{SU(N)_1}$ being the level $k = 1$ $SU(N)$ Wess-Zumino term, and

$$J_+^A = \frac{i}{2\pi}\,\mathrm{Tr}\left(U^\dagger(\partial_+ U)T^A\right)\,, \qquad J_-^A = \frac{i}{2\pi}\,\mathrm{Tr}\left((\partial_- U)U^\dagger T^A\right)\,, \qquad J_\pm = -\frac{\sqrt{N}}{4\pi}\partial_\pm\phi\ , \tag{3.9}$$

are the bosonized $SU(N)$ and $U(1)$ currents. As we can see, the $J\bar{J}$ deformation does not mix the compact scalar ϕ with the $SU(N)_1$ sector.

Under the continuous global symmetries in (3.5) the fields transform as

$$U(N)_V: \begin{cases} U \mapsto \mathcal{U}^\dagger U \mathcal{U}\,, \\ \phi \mapsto \phi\,, \\ \tilde{\phi} \mapsto \tilde{\phi} + \frac{4}{\sqrt{N}}\arg\det(\mathcal{U})\,, \end{cases} \qquad U(1)_A: \begin{cases} U \mapsto U\,, \\ \phi \mapsto \phi - 2\sqrt{N}\alpha\,, \\ \tilde{\phi} \mapsto \tilde{\phi}\,, \end{cases} \tag{3.10}$$

while under charge conjugation we have

$$\mathbb{Z}_2^C: \begin{cases} U \mapsto U^*\,, \\ \phi \mapsto -\phi\,, \\ \tilde{\phi} \mapsto -\tilde{\phi}\,, \end{cases} \tag{3.11}$$

where $\tilde{\phi}$ is the scalar dual to ϕ.

The $J_+^A J_-^A$ deformation breaks the $(SU(N)_L \times SU(N)_R)/\mathbb{Z}_N^V$ symmetry of the $SU(N)_1$ WZW model,

$$SU(N)_L \times SU(N)_R : \qquad U \mapsto LUR^\dagger, \qquad L, R \in SU(N)\,, \tag{3.12}$$

to $(SU(N)_V/\mathbb{Z}_N^V) \times \mathbb{Z}_N^L$. Here $\mathbb{Z}_N^V$ denotes a transformation in the center of the diagonal group $SU(N)_V$ which leaves U invariant, while $\mathbb{Z}_N^L$ denotes a transformation in the center of $SU(N)_L$ and acts as a phase on U. Explicitly,

$$\begin{cases} SU(N)_V/\mathbb{Z}_N^V & : \quad U \mapsto VUV^\dagger, \quad V \in SU(N)\ , \\ \mathbb{Z}_N^L & : \quad U \mapsto e^{2\pi i k/N} U, \quad k \in \mathbb{Z}_N\ . \end{cases} \tag{3.13}$$

The operator $J_+ J_-$ is proportional to the kinetic term,

$$J_+ J_- = \frac{N}{16\pi^2} \partial_\mu \phi \partial^\mu \phi\,, \tag{3.14}$$

so the deformation rescales the radius of the compact scalar ϕ by a factor

$$R_0 \longrightarrow R = R_0 \sqrt{1 + \frac{\lambda'}{2\pi N}}\,, \tag{3.15}$$

where $R_0 = \sqrt{N}$ is the radius before the deformation.

Since $U(N) \simeq [SU(N) \times U(1)]/\mathbb{Z}_N$, to obtain the deformed $U(N)_1$ model from (3.7) we further need to gauge the diagonal $\mathbb{Z}_N$ symmetry between $\mathbb{Z}_N^L$ and a $\mathbb{Z}_N^P$ subgroup of the shift symmetry of the compact boson. A similar subtlety arises in the identification of the deformed $U(N)_1$ model with the original fermionic theory (3.1). To get a full equivalence, one needs to further gauge a $\mathbb{Z}_2$ symmetry.

The gauging of discrete symmetries does not affect the analysis on $\mathbb{R}^2$, where we do not have twisted sectors. On a non-trivial manifold, however, this treatment is only partial, as it does not take into account for certain subtle global aspects of the bosonization/fermionization duality. In the next Section we rediscuss non-Abelian bosonization from a modern point of view, paying more attention to such global aspects. These are important when discussing the chiral Gross-Neveu model, like any QFT with a gauged discrete symmetry or with a dependence on the spin structure, on non-trivial spaces. Most of the considerations we will make are independent of the $J\bar{J}$ deformation so we focus in what follows on the correspondence between free fermions and undeformed WZW models.

3.2 Non-Abelian Bosonization Revisited

It is well-known that $2N$ free massless Majorana fermions bosonize to the (diagonal) $Spin(2N)_1$ WZW model [62]. The precise correspondence, valid on arbitrary spin manifolds, requires some specifications. For example, a fermionic theory depends on the spin structure on M_2, while the $Spin(2N)_1$ WZW model, being a bosonic theory, cannot. The spin structure dependence of the fermionic theory is attached to the bosonic one by stacking the latter with the topological theory given by the Arf invariant. We have reviewed the basics of this construction in Section 2.3. In the notation of Section 2.3, if we denote by $\mathcal{B}$ the diagonal $Spin(2N)_1$ WZW model, we take as the free fermion theory $\mathcal{F}'$ and the two theories are related as [92][1]

$$\mathcal{F}' = (\mathcal{B} \times \mathrm{Arf})/\mathbb{Z}_2^L \times \mathrm{Arf}\,, \tag{3.16}$$

which is a combination of (2.121) and (2.126). The $\mathbb{Z}_2^L$ symmetry in (3.16) is the $\mathbb{Z}_2$ subgroup of the center Z of the $Spin(2N)_L$ subgroup of the global symmetry group of the $Spin(2N)_1$ WZW model whose action assigns charge -1 to operators transforming in the left-handed spinor representations, namely the diagonal of $\mathbb{Z}_2 \times \mathbb{Z}_2$ for even N, and the $\mathbb{Z}_2 \subset \mathbb{Z}_4$ for odd N. Its action on the Arf theory is given in (2.126). The inverse relation between $\mathcal{B}$ and $\mathcal{F}'$ is given by (2.127).

It is also well-known that there are different ways to perform bosonization of free fermions, according to the amount of global symmetry of the bosonized theory one wishes to keep manifest. While the bosonized theory $\mathcal{B}$ is unique, it can be described using different variables. For instance, in Abelian bosonization one bosonizes each Dirac fermion independently and keeps manifest only the Cartan subgroup $U(1)^N \subset Spin(2N)$. Alternatively, one can perform non-Abelian bosonization keeping manifest a $U(N) \subset Spin(2N)$ or the whole $Spin(2N)$ symmetry. Given that we will eventually restrict to the unbroken symmetries (3.5) left after the $J\bar{J}$ deformation, we focus on the $U(N)$ description. Nevertheless, the latter must be equivalent to the more general $Spin(2N)$ one. We argue that

$$\mathcal{B} = Spin(2N)_1 = \begin{cases} \dfrac{SU(N)_1 \times U(1)_N}{\mathbb{Z}^L_{N/2}} & N \text{ even}\,, \\ \dfrac{SU(N)_1 \times U(1)_{4N}}{\mathbb{Z}^L_N} & N \text{ odd}\,, \end{cases} \tag{3.17}$$

where $U(1)_{k'}$ denotes the compact boson with squared radius $R^2 = k'$.[2] For both N even and odd, the $\mathbb{Z}_n$ quotients are the diagonal ones between the one contained in $Z[SU(n)_L]$ and the $\mathbb{Z}_n \subset U(1)_L$.[3] The $\mathbb{Z}_2$ subgroup of $Z[Spin(2N)_L]$ that gets gauged according to (3.16) is mapped via (3.17) to $\mathbb{Z}_2 \subset U(1)_L$ for N odd. For N even, it is given by the diagonal $\mathbb{Z}_N$ generator of $Z[SU(N)_L]$ and $\mathbb{Z}_N \subset U(1)_L$, which in the $\mathbb{Z}^L_{N/2}$-gauged theory has order 2. The discrepancy in (3.17) between the even and odd case is due to the fact that the $R^2 = k'$ compact boson is not a diagonal RCFT when k' is odd, see Section 2.2.1 for details and Section 3.3 for a proof of (3.17).

The appearance of $U(1)_{4N}$ in place of $U(1)_N$ for N odd might sound at odds with the description on $\mathbb{R}^2$, but it is not. The relation (3.16) can also be written in a form where only $U(1)_N$ factors are involved in the bosonic theory, for any N. Indeed, when N is odd, we can gauge $\mathbb{Z}_2 \subset U(1)_L$ before $\mathbb{Z}^L_N$. Since the action of $\mathbb{Z}^L_N$ is identical to that of $\mathbb{Z}^P_N$ in

[1] In general, there are two fermionizations of a bosonic theory $\mathcal{B}$ with respect to a given $\mathbb{Z}_2$ symmetry, denoted by $\mathcal{F}$ and $\mathcal{F}' = \mathcal{F} \times \mathrm{Arf}$ in Section 2.3. To check which is the correct choice for a fixed $\mathcal{B}$ one can look at boundary states, as done in [92]. In our setup we deal with local operators on manifolds without boundary and there is no real distinction between $\mathcal{F}$ and $\mathcal{F}'$. However, certain $\mathbb{Z}_2$ anomalies distinguish the two theories. This will be discussed in detail in Section 4.2.

[2] We follow the conventions of Section 2.2.1, that is the self-T-dual radius for the compact boson is $R = \sqrt{2}$.

[3] One could alternatively use the action on the anti-holomorphic sector (R). The final result would be the same.

the $U(1)_{4N}$ theory, we can use (2.130) and (2.129) to get

$$\mathcal{F}' = \frac{\mathcal{B}' \times \mathrm{Arf}}{\widehat{\mathbb{Z}}_2^W}\,, \qquad \mathcal{B}' = \frac{SU(N)_1 \times U(1)_N}{\widehat{\mathbb{Z}}_N}\,, \qquad N \text{ odd}\,, \tag{3.18}$$

where $\widehat{\mathbb{Z}}_N$ is the diagonal between $Z[SU(N)_L]$ and $\widehat{\mathbb{Z}}_N^W \subset U(1)_W$. For N even, the action of $\mathbb{Z}_{N/2}^L$ is identical to that of $\mathbb{Z}_{N/2}^P$ in the $U(1)_N$ theory, so we can write

$$\mathcal{F}' = \left(\frac{SU(N)_1 \times U(1)_N}{\mathbb{Z}'_{N/2}} \times \mathrm{Arf} \right) \Big/ \mathbb{Z}_2^L \times \mathrm{Arf}\,, \quad N \text{ even}\,, \tag{3.19}$$

where $\mathbb{Z}'_{N/2}$ is the diagonal between $Z[SU(N)_L]$ and $\mathbb{Z}_{N/2}^P \subset U(1)_P$.

3.2.1 Global symmetries

The group-like symmetries of the $Spin(2N)_1$ WZW theory are given by

$$G(Spin(2N)_1) = \frac{Spin(2N)_L \times Spin(2N)_R}{Z(Spin(2N)_V)} \rtimes \mathbb{Z}_2^K\,, \tag{3.20}$$

where the subscript V denotes the diagonal $Spin(2N)_V \subset Spin(2N)_L \times Spin(2N)_R$. Explicitly, their action on the matrix field g is

$$\begin{cases} Spin(2N)_L: & g \mapsto Lg\,, \\ Spin(2N)_R: & g \mapsto gR^t\,, \\ Spin(2N)_V: & g \mapsto VgV^t\,. \end{cases} \tag{3.21}$$

The factor $\mathbb{Z}_2^K$ is a $\mathbb{Z}_2$ outer automorphism of the algebra that exchanges the spinor and the conjugate spinor representations. It does not act on the matrix field of the theory because it sits in a real representation of the symmetry algebra. For odd N $\mathbb{Z}_2^K$ can be identified with complex conjugation.

The symmetry group of $2N$ free Majorana fermions ξ^j ($j = 1, \ldots, 2N$) is, as discussed in Section 1.1,

$$G(2N\,\text{Majorana}) = O(2N)_L \times O(2N)_R\,, \tag{3.22}$$

with fermion parity $\mathbb{Z}_2^F$ being the diagonal subgroup of $\mathbb{Z}_2^{F_L} \times \mathbb{Z}_2^{F_R} = Z(SO(2N)_L \times SO(2N)_R)$. In order for the bosonization procedure to be consistent, upon gauging $\mathbb{Z}_2^F$ in the fermionic theory we should obtain the symmetry group (3.20). The symmetry $\mathbb{Z}_2^\vee$ dual of $\mathbb{Z}_2^F$ is the factor that extends $SO(2N)_L$ to $Spin(2N)_L$. This is easily seen by noting that the spinorial characters of $Spin(2N)_1$ are in the twisted sectors of the $\mathbb{Z}_2^F$ gauged theory. The symmetry group after the gauging is then

$$G\left(\frac{2N \text{ Majorana } \times \text{ Arf}}{\mathbb{Z}_2} \times \mathrm{Arf} \right) = \frac{Spin(2N)_L \times Spin(2N)_R}{Z(Spin(2N)_V)} \rtimes \mathbb{Z}_2^K\,, \tag{3.23}$$

where $\mathbb{Z}_2^K$ is the diagonal between $\mathbb{Z}_2^{K_L}$ and $\mathbb{Z}_2^{K_R}$,[4] and coincides with (3.20).

[4] The orthogonal combination becomes a non-invertible symmetry [61], which we neglect from now on.

It is useful to also report the manifest global symmetries in the $SU(N)$ description and its explicit realization in the fields. The global symmetry group of the $SU(N)_1$ WZW model is

$$G(SU(N)_1) = \begin{cases} \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2^V} & N = 2\,, \\ \dfrac{SU(N)_L \times SU(N)_R}{\mathbb{Z}_N^V} \rtimes \mathbb{Z}_2^C & N > 2\,, \end{cases} \tag{3.24}$$

where $\mathbb{Z}_N^V = Z(SU(N)_V)$ is the center of the diagonal $SU(N)_V \subset SU(N)_L \times SU(N)_R$ and $\mathbb{Z}_2^C$ is charge conjugation. Notice that for $N = 2$ there is no charge conjugation symmetry acting on the matrix field U because the latter is in the bifundamental representation of $SU(2)_L \times SU(2)_R$, which is pseudo-real. The global symmetries of the $U(1)_{k'}$ WZW model, that is, the compact boson with squared radius $R^2 = k' \equiv 2p'/p$, $\gcd(p, p') = 1$, are given by (cf. Section 2.2.1)

$$G(U(1)_{2p'/p}) = (U(1)_L \times U(1)_R) \rtimes \mathbb{Z}_2^C\,. \tag{3.25}$$

Their action on the $SU(N)$ field U and on the compact scalar ϕ and its dual $\tilde{\phi}$ are

$$\begin{cases} SU(N)_L: & U \mapsto LU\,, \\ SU(N)_R: & U \mapsto UR^\dagger\,, \\ SU(N)_V: & U \mapsto VUV^\dagger\,, \end{cases} \qquad \begin{cases} U(1)_L: & \phi \mapsto \alpha_L pR\,, & \tilde{\phi} \mapsto \tilde{\phi} + \alpha_L p' \frac{2}{R}\,, \\ U(1)_R: & \phi \mapsto \alpha_R pR\,, & \tilde{\phi} \mapsto \tilde{\phi} - \alpha_R p' \frac{2}{R}\,, \end{cases} \tag{3.26}$$

with $\alpha_{L,R} \sim \alpha_{L,R} + 2\pi$. Charge conjugation acts as in (3.11).

3.3 Proof of eq. (3.17)

In this Section we prove the relation (3.17). In order to do that, we study both theories on the torus. The torus partition functions of the three theories involved in the relation (3.17), both in twisted or un-twisted sectors, can be written as a sum of products of holomorphic/antiholomorphic affine characters of the corresponding affine algebras. It is in principle straightforward to check that, upon performing the sum that corresponds to gauging $\mathbb{Z}_N$, the sum of products of twisted/untwisted characters on the left-hand side reproduces the sum of squares of characters on the right-hand side. However it is rather cumbersome to perform this in practice, for generic N.

For this reason we follow a different route. We adopt a "holographic" viewpoint in which the holomorphic/antiholomorphic affine characters on the torus are realized as partition functions of Chern-Simons theory with positive/negative level on the solid torus, with lines wrapping the non-contractible cycle, and holomorphic/antiholomorphic boundary conditions [89, 90].

The strategy will be similar to the one adopted in the second example presented Section 2.2.3: we gauge a one-form symmetry in a Chern-Simons theory in two different ways, making manifest either the left- or the right-hand side of (3.17).

Let us briefly recall the procedure to gauge a subgroup $\mathcal{A}$ of a one-form symmetry [98, 99] (see also [100]). It amounts to sum over insertions of Wilson lines generating $\mathcal{A}$, which can either fuse or link with the Wilson lines of the theory, see figure 3.1. We focus on the case in which $\mathcal{A} = \mathbb{Z}_Q$ is an abelian subgroup of the one-form symmetry of a CS theory. For simplicity we describe the gauging procedure when the gauge group G is given by a

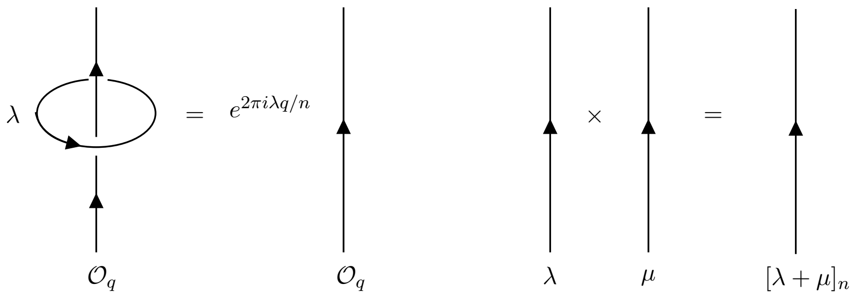


Figure 3.1: A $\mathbb{Z}_n$ one-form symmetry in $3d$ has topological lines labelled by $\lambda \in \mathbb{Z}_n$. They fuse according to the $\mathbb{Z}_n$ group law (right), and act on charged operators, which are also supported on lines, by linking (left). Here $[x]_n = x \mod n$ and q is the $\mathbb{Z}_n$ charge of the operator $\mathcal{O}_q$.

single factor, the generalizations to direct products of groups being straightforward. If $W_{\hat{\mu}}$ is the Wilson line generating $\mathbb{Z}_Q$, we have $W_{\hat{\mu}}^Q = 1$. The group $\mathcal{A}$ is gaugeable if and only if $W_{\hat{\mu}}$ has integer spin $h_{\hat{\mu}}$ and the lines in $\mathcal{A}$ are mutually transparent, i.e. they have trivial mutual braiding.[5] Given $\mathcal{A}$, we then select the Wilson lines which have trivial linking with lines in $\mathcal{A}$, and the lines of the gauged theory are given in terms of orbits under fusion with $\mathcal{A}$. Let P be the total number of Wilson lines in the CS theory. The Wilson lines $W_{\hat{\lambda}}$ with trivial linking with $W_{\hat{\mu}}^n$, $n = 1, \ldots, Q-1$ are those for which

$$B(\hat{\lambda}, \hat{\mu}) = 1\,. \tag{3.28}$$

There are in general P/Q Wilson lines which satisfy (3.28). Such lines organize into P/Q^2 gauge-invariant orbits of the form

$$O_W = \oplus_{q=0}^{Q-1} W_{\hat{\lambda}+q\hat{\mu}}\,, \tag{3.29}$$

which are the surviving lines in the gauged theory.[6] Inserting a Wilson line orbit of the gauged theory along the non-contractible cycle of M_3 amounts, on the boundary, to sum over the products of characters associated to the Wilson lines of the ungauged theory as given in (3.29). When the gauge group is of the form $G_k \times G_{-k}$, to each orbit we can associate a combination of left- and right-moving characters which we can interpret as a partition function for the associated 2d WZW model, provided the combination has the correct modular properties.

We prove (3.17) by showing that the left and right-hand side of this equation arise from gauging the same subgroup of one-form symmetry, which leads to a single orbit. The CS theories are

$$\begin{aligned} G_{\rm CS} &= SU(N)_1 \times U(1)_N \times SU(N)_{-1} \times U(1)_{-N}\,, \qquad N \text{ even} \\ G_{\rm CS} &= SU(N)_1 \times U(1)_{4N} \times SU(N)_{-1} \times U(1)_{-4N}\,, \qquad N \text{ odd}\,. \end{aligned} \tag{3.30}$$

[5]Recall that, given two Wilson lines $W_{\hat{\lambda}}$ and $W_{\hat{\mu}}$, their braiding $B(\hat{\lambda}, \hat{\mu})$ is given by

$$B(\hat{\lambda}, \hat{\mu}) = \exp\Big(2i\pi\big(h_{\hat{\lambda}+\hat{\mu}} - h_{\hat{\lambda}} - h_{\hat{\mu}}\big)\Big)\,, \tag{3.27}$$

and the spin of a Wilson line $W_{\hat{\lambda}}$ equals the chiral dimension h mod 1 of the corresponding $2d$ affine character $\chi_{\hat{\lambda}}$. Trivial braiding means $B = 1$.

[6]If the line $W_{\hat{\lambda}}$ is a fixed point under fusion with $W_{\hat{\mu}}^s$, s being a divisor of N, then there are N/s copies of the line $\hat{W}_{\hat{\lambda}}$ [100]. In our case the action is always free and no degeneracies occur.

We denote in the following by $(j,\lambda;\bar{k},\bar{\mu})$ a general Wilson line of the CS theories, where j and $\bar{k}$ are the ranks of the completely antisymmetric representations of $\widehat{\mathfrak{su}}(N)_1$ and $\widehat{\mathfrak{su}}(N)_{-1}$, while λ and $\bar{\nu}$ are the charges of $\widehat{\mathfrak{u}}(1)_Q$ and $\widehat{\mathfrak{u}}(1)_Q$. For N even, $Q=N$ and $\lambda,\mu\in\mathbb{Z}_N$, while for N odd $Q=4N$ and $\lambda,\mu\in\mathbb{Z}_{4N}$. The spin of $SU(N)_1$ and $U(1)_Q$ Wilson lines is

$$h_j=\frac{j(N-j)}{2N}\,,\qquad\qquad h_\lambda=\frac{\lambda^2}{2Q}\,. \tag{3.31}$$

3.3.1 Even N

For N even, the total one-form symmetry is $G^{(1)}=\mathbb{Z}_N^4$ and we have a total of N^4 Wilson lines. We want to gauge a subgroup $\mathcal{A}\subset G^{(1)}$, where $\mathcal{A}=\mathbb{Z}_{N/2}\times\mathbb{Z}_N\times\mathbb{Z}_2$ is generated by the following set of lines[7]

$$\mathcal{A}=\langle(2,2;\bar{0},\bar{0}),(0,0;\bar{2},\bar{2}),(1,1;\bar{1},\bar{1}),(0,N/2;\bar{0},\overline{N/2})\rangle\simeq\mathbb{Z}_{N/2}\times\mathbb{Z}_N\times\mathbb{Z}_2\,. \tag{3.32}$$

We define the subgroups

$$\begin{aligned}\mathcal{A}_1&=\langle(2,2;\bar{0},\bar{0}),(0,0;\bar{2},\bar{2})\rangle\simeq\mathbb{Z}_{N/2}^2\,,\\ \mathcal{A}_2&=\langle(0,N/2;\bar{0},\overline{N/2}),(1,1;\bar{1},\bar{1})\rangle\simeq\mathbb{Z}_N\times\mathbb{Z}_2\,.\end{aligned} \tag{3.33}$$

We also define the coset

$$\mathcal{A}_c\equiv\frac{\mathcal{A}}{\mathcal{A}_1}=\langle[(1,1;\bar{1},\bar{1})],[(0,N/2;\bar{0},\overline{N/2})]\rangle\simeq\mathbb{Z}_2^2\,. \tag{3.34}$$

We gauge $\mathcal{A}$ in steps, starting with $\mathcal{A}_1$. This acts independently on holomorphic and anti-holomorphic sector by projecting into lines with

$$j=\lambda\mod N/2\;,\qquad \bar{k}=\bar{\mu}\mod N/2\,. \tag{3.35}$$

We are left with $(2N)^2$ lines which forms 4^2 gauge-invariant orbits (each containing $(N/2)^2$ simple lines). In terms of characters, they are given by $\chi_a\bar{\chi}_b$, where $a=\mathbf{i},\mathbf{v},\mathbf{s},\mathbf{c}$, and

$$\begin{aligned}\chi_{\mathbf{i}}(\tau)&=\sum_{j,\lambda=0}^{N/2-1}\chi_{2j}(\tau)\chi_{2\lambda}(\tau)\delta_{j,\lambda}\,, &\chi_{\mathbf{v}}(\tau)&=\sum_{j,\lambda=0}^{N/2-1}\chi_{2j+1}(\tau)\chi_{2\lambda+1}(\tau)\delta_{j,\lambda}\,,\\ \chi_{\mathbf{s}}(\tau)&=\sum_{j,\lambda=0}^{N/2-1}\chi_{2j}(\tau)\chi_{2\lambda+N/2}(\tau)\delta_{j,\lambda}\,, &\chi_{\mathbf{c}}(\tau)&=\sum_{j,\lambda=0}^{N/2-1}\chi_{2j+1}(\tau)\chi_{2\lambda+1+N/2}(\tau)\delta_{j,\lambda}\,,\end{aligned} \tag{3.36}$$

and similarly for the right-moving characters. Note that the left-hand sides of (3.36) coincide with the four affine characters of the $\widehat{\mathfrak{spin}}(2N)_1$ chiral algebra, associated respectively to the identity, the vector and the two spinor representations of $Spin(2N)$, with highest conformal weights

$$h_{\mathbf{i}}=0\,,\qquad h_{\mathbf{v}}=\frac{1}{2}\,,\qquad h_{\mathbf{s}}=h_{\mathbf{c}}=\frac{N}{8}\,. \tag{3.37}$$

Their fusion rules can be computed using Verlinde's formula (2.20) and the modular matrices associated to the $\widehat{\mathfrak{su}}(N)_1$ and $\widehat{\mathfrak{u}}(1)_N$ algebras. We get as expected

$$\mathbf{v}\times\mathbf{v}=\mathbf{s}\times\mathbf{s}=\mathbf{c}\times\mathbf{c}=\mathbf{i}\,,\quad \mathbf{s}\times\mathbf{c}=\mathbf{v}\,,\quad \mathbf{v}\times\mathbf{s}=\mathbf{c}\,,\quad \mathbf{v}\times\mathbf{c}=\mathbf{s}\,. \tag{3.38}$$

[7]This set of generators is not minimal, as $(1,1;\bar{1},\bar{1})$ can be combined with either $(0,0;\bar{2},\bar{2})$ or $(2,2;\bar{0},\bar{0})$ to give the other one.

We now gauge the coset $\mathcal{A}_c$. This leaves 4 orbits which gives rise to a single orbit of orbits O_W (containing in total N^2 simple lines), the one obtained by summing the diagonal combination of the above characters. We then have

$$\langle O_W\rangle = |\chi_{\mathbf{i}}|^2 + |\chi_{\mathbf{v}}|^2 + |\chi_{\mathbf{s}}|^2 + |\chi_{\mathbf{c}}|^2 = Z_{Spin(2N)_1}\,. \tag{3.39}$$

We now discuss how to obtain the partition function of the $(SU(N)_1 \times U(1)_N)/\mathbb{Z}^L_{N/2}$ orbifold theory. To this purpose we can gauge $\mathcal{A}$ as follows. We define the subgroup

$$\mathcal{A}_3 = \langle (1,1;\bar{1},\bar{1}),(0,N/2;\bar{0},\overline{N/2})\rangle \simeq \mathbb{Z}_N \times \mathbb{Z}_2\,. \tag{3.40}$$

as well as the coset

$$\mathcal{A}'_c = \frac{\mathcal{A}}{\mathcal{A}_3} = \langle [(0,0;\bar{2},\bar{2})]\rangle \simeq \mathbb{Z}_{N/2}\,. \tag{3.41}$$

Gauging in succession $\mathcal{A}_3$ and $\mathcal{A}'_c$, we end up again with a single orbit O_W, which is automatically arranged in terms of simple lines as

$$O_W = \oplus_{\substack{k=0\\ k\,\text{even}}}^{N-2} O_W^{(k)}\,, \tag{3.42}$$

where

$$\begin{aligned} O_W^{(k)} = \oplus_{j=0}^{N/2-1} \Big[&(j,j;\overline{j+k},\overline{j+k}) \oplus \left(j,j+\tfrac{N}{2};\overline{j+k},\overline{j+k+\tfrac{N}{2}}\right) \\ &\oplus \left(j+\tfrac{N}{2},j;\overline{j+k+\tfrac{N}{2}},\overline{j+k}\right) \oplus \left(j+\tfrac{N}{2},j+\tfrac{N}{2};\overline{j+k+\tfrac{N}{2}},\overline{j+k+\tfrac{N}{2}}\right)\Big]\,. \end{aligned} \tag{3.43}$$

We show below that $O_W^{(k)}$ coincides with the k-twisted partition function of the orbifolded theory. The full partition function reads

$$Z_{\frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}} = \frac{2}{N}\sum_{\substack{k,\ell=0\\ k,\ell\,\text{even}}}^{N-2} Z_{SU(N)_1}[k,\ell] Z_{U(1)_N}[k,\ell] \equiv \sum_{\substack{k=0\\ k\,\text{even}}}^{N-2} Z^{(k)}_{\frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}}\,, \tag{3.44}$$

where $[k,\ell]$ denote the k-twisted sector with the insertion of ℓ $\mathbb{Z}^L_{N/2}$ charges of the individual $SU(N)$ and $U(1)$ sectors, while $Z^{(k)}$ is the partition function of the orbifolded theory restricted to the states with charge k under the dual symmetry $\widehat{\mathbb{Z}}^L_{N/2}$. The functions $Z_{SU(N)_1}[k,\ell]$ and $Z_{U(1)_N}[k,\ell]$ can be computed starting from the unwtwisted sector

$$Z_{SU(N)_1}[0,\ell] = \sum_{j=0}^{N-1} e^{-2\pi i\frac{j\ell}{N}}\chi_j\bar{\chi}_j\,, \quad Z_{U(1)_N}[0,\ell] = \sum_{\lambda=0}^{N-1} e^{2\pi i\frac{\lambda\ell}{N}}\chi_\lambda\bar{\chi}_\lambda\,, \tag{3.45}$$

and applying S and T modular transformations. We have

$$\begin{aligned} S\cdot Z_{SU(N)_1}[k,\ell] &= Z_{SU(N)_1}[N-\ell,k]\,, \qquad & S\cdot Z_{U(1)_N}[k,\ell] &= Z_{U(1)_N}[N-\ell,k]\,, \\ T\cdot Z_{SU(N)_1}[k,\ell] &= Z_{SU(N)_1}[k,k+\ell]\,, \qquad & T\cdot Z_{U(1)_N}[k,\ell] &= Z_{U(1)_N}[k,k+\ell]\,, \end{aligned} \tag{3.46}$$

where the action of S and T on the characters are given, e.g., in [101]. In particular, we have

$$Z_{SU(N)_1}[k,0] = \sum_{j=0}^{N-1}\chi_j\bar{\chi}_{j+k}\,, \quad Z_{U(1)_N}[k,0] = \sum_{\lambda=0}^{N-1}\chi_\lambda\bar{\chi}_{\lambda+k}\,. \tag{3.47}$$

Given the action of the $\mathbb{Z}^L_{N/2}$ charges on the characters, the partition functions $Z^{(k)}$ defined in the r.h.s. of (3.44) are obtained by projecting on the $\mathbb{Z}^L_{N/2}$ neutral states:

$$Z^{(k)}_{\frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}} = \frac{2}{N}\sum_{\substack{\ell=0\\ \ell\,\text{even}}}^{N-2}\sum_{j,\lambda=0}^{N-1} e^{-2\pi i\frac{(j-\lambda)\ell}{N}}\chi_j\bar{\chi}_{j+k}\chi_\lambda\bar{\chi}_{\lambda+k}\,. \tag{3.48}$$

It is now straightforward to see that the sum over ℓ in (3.48) gives rise to the combination of characters entering $O^{(k)}_W$, and hence

$$\langle O^{(k)}_W\rangle = Z^{(k)}_{\frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}}\,, \qquad \forall k = 0,2\ldots N-2\,. \tag{3.49}$$

Suming over k and using (3.39), (3.42) and (3.49), we immediately get

$$Z_{Spin(2N)_1} = Z_{\frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}}\,, \tag{3.50}$$

proving (3.17) for N even.

To later make contact with the fermionic theory we consider, on the solid torus with the O_W orbit inserted on the non contractible cycle, the insertion a further $\mathbb{Z}^\vee_2$ line with spin 1/2,

$$\psi \equiv (1,1;\bar{0},\bar{0})\,. \tag{3.51}$$

The line ψ has non-trivial braiding with some of the lines inside the orbit O_W, so if we insert ψ on the cycle that links with O_W it will result in a non-trivial action; ψ can also fuse with the lines inside O_W in the usual way. It is a genuine $\mathbb{Z}_2$ line because it squares to a generator of $\mathcal{A}$, which is gauged.

By looking for instance at the expressions (3.36), it is clear that under fusion it acts on the $\widehat{\mathfrak{spin}}(2N)_1$ representations as fusion with the vector representation $\mathbf{v}$, and that by linking it weights the left-moving characters $\chi_{\mathbf{s}}, \chi_{\mathbf{c}}$ with a -1 sign. Similarly, on (3.42) fusion with ψ shifts $O^{(k)}_W$ to $O^{(k+1)}_W$, and linking weights a simple line $(j,\lambda;\bar{k},\bar{\mu})$ with a $(-1)^{j+\lambda}$ sign: this amounts instead to shifting ℓ in the middle term of (3.44) by one. In other words, we further identify

$$\begin{aligned}Z_{Spin(2N)_1}[k_\psi,\ell_\psi] = \frac{1}{N/2}\sum_{\substack{\ell=0\\ \ell\,\text{even}}}^{N-2} & Z_{SU(N)_1}[k+k_\psi,\ell+\ell_\psi] \\ & \times Z_{U(1)_N}[k+k_\psi,\ell+\ell_\psi]\,, \quad N\text{ even,}\end{aligned} \tag{3.52}$$

with $k_\psi,\ell_\psi\in\mathbb{Z}_2$ identifying, respectively, the insertion of a ψ line along the non-contractible and the contractible cycle of the boundary torus. The left-hand side description matches with the $Spin(2N)_1$ partition function with a $\mathbb{Z}^\vee_2$ background.

3.3.2 Odd N

For N odd, the total one-form symmetry is $G^{(1)} = \mathbb{Z}^2_N\times\mathbb{Z}^2_{4N}$ and we have a total of $16N^4$ Wilson lines. We want to gauge a subgroup $\mathcal{A}\subset G^{(1)}$, where $\mathcal{A} = \mathbb{Z}^2_N\times\mathbb{Z}_4$ is generated by

$$\mathcal{A} = \langle (1,2N+2;\bar{0},\bar{0}),(0,0;\bar{1},\overline{2N+2}),(0,N;\bar{0},\bar{N})\rangle \simeq \mathbb{Z}^2_N\times\mathbb{Z}_4\,. \tag{3.53}$$

We identify the subgroups

$$\begin{aligned}\mathcal{A}_1 &= \langle (1, 2N+2; \bar{0}, \bar{0}), (0, 0; \bar{1}, \overline{2N+2}) \rangle \simeq \mathbb{Z}_N^2 \,, \\ \mathcal{A}_2 &= \langle (0, N; \bar{0}, \bar{N}) \rangle \simeq \mathbb{Z}_4 \,,\end{aligned} \tag{3.54}$$

and gauge $\mathcal{A}_1$ first. As before, this acts independently on holomorphic and anti-holomorphic sectors by projecting into lines with

$$2j = (N+1)\lambda \mod 2N \,, \qquad 2\bar{k} = (N+1)\bar{\mu} \mod 2N \,. \tag{3.55}$$

We are left with $(4N)^2$ lines which forms 4^2 gauge-invariant orbits, with N^2 elements. In terms of characters, they are given by $\chi_a \bar{\chi}_b$, where $a = \mathbf{i}, \mathbf{v}, \mathbf{s}, \mathbf{c}$, and

$$\begin{aligned}\chi_{\mathbf{i}}(\tau) &= \sum_{j=0}^{N-1} \sum_{\lambda=0}^{4N-1} \chi_j(\tau) \chi_\lambda(\tau) \delta_{[j(2N+2),\lambda]_{4N}} \,, \\ \chi_{\mathbf{v}}(\tau) &= \sum_{j=0}^{N-1} \sum_{\lambda=0}^{4N-1} \chi_j(\tau) \chi_{\lambda+2N}(\tau) \delta_{[j(2N+2),\lambda]_{4N}} \,, \\ \chi_{\mathbf{s}}(\tau) &= \sum_{j=0}^{N-1} \sum_{\lambda=0}^{4N-1} \chi_j(\tau) \chi_{\lambda+N}(\tau) \delta_{[j(2N+2),\lambda]_{4N}} \,, \\ \chi_{\mathbf{c}}(\tau) &= \sum_{j=0}^{N-1} \sum_{\lambda=0}^{4N-1} \chi_j(\tau) \chi_{\lambda+3N}(\tau) \delta_{[j(2N+2),\lambda]_{4N}} \,,\end{aligned} \tag{3.56}$$

where $\delta_{[a,b]_{4N}}$ stands for $a = b \mod 4N$. A similar result applies for the right-moving characters. The left hand sides of (3.56) again coincide with the four affine characters of the $Spin(2N)_1$ WZW model, associated respectively to the identity, the vector and the two spinor representations of $Spin(2N)$. Their highest conformal weights are as in the N even case (3.37), while their fusion rules read instead

$$\mathbf{v} \times \mathbf{v} = \mathbf{s} \times \mathbf{c} = \mathbf{i} \,, \quad \mathbf{s} \times \mathbf{s} = \mathbf{c} \times \mathbf{c} = \mathbf{v} \,, \quad \mathbf{v} \times \mathbf{s} = \mathbf{c} \,, \quad \mathbf{v} \times \mathbf{c} = \mathbf{s} \,. \tag{3.57}$$

We now gauge $\mathcal{A}_2$. This leaves 4 orbits which gives rise to a single orbit of orbits O_W (containing in total $4N^2$ simple lines), the one obtained by summing the diagonal combination of the above characters. We then have

$$\langle O_W \rangle = |\chi_{\mathbf{i}}|^2 + |\chi_{\mathbf{v}}|^2 + |\chi_{\mathbf{s}}|^2 + |\chi_{\mathbf{c}}|^2 = Z_{Spin(2N)_1} \,. \tag{3.58}$$

We now discuss how to obtain the partition function of the $(SU(N)_1 \times U(1)_{4N})/\mathbb{Z}_N^L$ orbifold theory. Like for the case of N even, we can gauge the whole group $\mathcal{A}$ at once. The form of the final orbit O_W in terms of simple lines can be written as

$$O_W = \oplus_{k=0}^{N-1} O_W^{(k)} \,, \tag{3.59}$$

where

$$O_W^{(k)} = \oplus_{j=0}^{N-1} \oplus_{s=0}^{3} \left(j, 2(N+1)j + sN; \overline{j+k}, \overline{2(N+1)(j+k) + sN} \right) . \tag{3.60}$$

We proceed as for the case of N even. The steps are almost identical, so we will be brief. The full partition function reads

$$Z_{\frac{SU(N)_1\times U(1)_{4N}}{\mathbb{Z}_N^L}} = \frac{1}{N}\sum_{k,\ell=0}^{N-1} Z_{SU(N)_1}[k,\ell] Z_{U(1)_{4N}}[2(N+1)k, 2(N+1)\ell] \equiv \sum_{k=0}^{N-1} Z^{(k)}_{\frac{SU(N)_1\times U(1)_{4N}}{\mathbb{Z}_N^L}} . \tag{3.61}$$

The functions $Z_{SU(N)_1}[k,\ell]$ and $Z_{U(1)_{4N}}[2(N+1)k, 2(N+1)\ell]$ can be computed starting from the unwtwisted sector

$$Z_{SU(N)_1}[0,\ell] = \sum_{j=0}^{N-1} e^{-2\pi i \frac{j\ell}{N}} \chi_j \bar{\chi}_j , \quad Z_{U(1)_{4N}}[0, 2(N+1)\ell] = \sum_{\lambda=0}^{4N-1} e^{2\pi i \frac{2(N+1)\lambda\ell}{4N}} \chi_\lambda \bar{\chi}_\lambda , \tag{3.62}$$

and applying S and T modular transformations. We have

$$\begin{aligned} S\cdot Z_{SU(N)_1}[k,\ell] &= Z_{SU(N)_1}[N-\ell, k] , \qquad & S\cdot Z_{U(1)_{4N}}[k,\ell] &= Z_{U(1)_{4N}}[4N-\ell, k] , \\ T\cdot Z_{SU(N)_1}[k,\ell] &= Z_{SU(N)_1}[k, k+\ell] , \qquad & T\cdot Z_{U(1)_{4N}}[k,\ell] &= Z_{U(1)_{4N}}[k, k+\ell] . \end{aligned} \tag{3.63}$$

In particular, we get

$$Z_{SU(N)_1}[k,0] = \sum_{j=0}^{N-1} \chi_j \bar{\chi}_{j+k} , \quad Z_{U(1)_{4N}}[2(N+1)k, 0] = \sum_{\lambda=0}^{4N-1} \chi_\lambda \bar{\chi}_{\lambda+2(N+1)k} , \tag{3.64}$$

and $Z^{(k)}$ in (3.61) read

$$Z^{(k)}_{\frac{SU(N)_1\times U(1)_{4N}}{\mathbb{Z}_N^L}} = \frac{1}{N}\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-1}\sum_{\lambda=0}^{4N-1} e^{-2\pi i \frac{(4j-2(N+1)\lambda)\ell}{4N}} \chi_j \bar{\chi}_{j+k} \chi_\lambda \bar{\chi}_{\lambda+2(N+1)k} . \tag{3.65}$$

The sum over ℓ in (3.65) again gives rise to the combination of characters entering $O_W^{(k)}$ in (3.60), and hence

$$\langle O_W^{(k)} \rangle = Z^{(k)}_{\frac{SU(N)_1\times U(1)_{4N}}{\mathbb{Z}_N^L}} , \qquad \forall k = 0, 1 \dots N-1 . \tag{3.66}$$

Summing over k and using (3.58), (3.59) and (3.66), we immediately get

$$Z_{Spin(2N)_1} = Z_{\frac{SU(N)_1\times U(1)_{4N}}{\mathbb{Z}_N^L}} , \tag{3.67}$$

proving (3.17) for N odd.

As before, to later make contact with the fermionic theory we also consider the insertion a further $\mathbb{Z}_2^\vee$ line with spin 1/2, which this time reads

$$\psi \equiv (0, 2N; \bar{0}, \bar{0}) . \tag{3.68}$$

Comparing with (3.56), it is clear that also this time under fusion ψ acts on the $\widehat{\mathfrak{spin}}(2N)_1$ representations as fusion with the vector representation $\mathbf{v}$, and that by linking it weights the left-moving characters $\chi_{\mathbf{s}}, \chi_{\mathbf{c}}$ with a -1 sign. Similarly, on (3.59) fusion with ψ shifts $O_W^{(k)}$ to $O_W^{(k+2N)}$, and linking weights the line $(j,\lambda;\bar{k},\bar{\mu})$ with a (-1) if $j = \lambda + N$

$SU(4)_1 \times U(1)_4$	$\mathbb{Z}_2^L$ even	$\mathbb{Z}_2^L$ odd
$\mathcal{H}$	$(0,0;\bar{0},\bar{0}), (2,2;\bar{2},\bar{2})$	$(0,1;\bar{0},\bar{1}), (2,3;\bar{2},\bar{3})$
	$(1,1;\bar{1},\bar{1}), (3,3;\bar{3},\bar{3})$	$(1,0;\bar{1},\bar{0}), (3,2;\bar{3},\bar{2})$
	$(0,2;\bar{0},\bar{2}), (2,0;\bar{2},\bar{0})$	$(0,3;\bar{0},\bar{3}), (2,1;\bar{2},\bar{1})$
	$(1,3;\bar{1},\bar{3}), (3,1;\bar{3},\bar{1})$	$(1,2;\bar{1},\bar{2}), (3,0;\bar{3},\bar{0})$
$\mathcal{H}_{tw}$	$(0,0;\bar{2},\bar{2}), (2,2;\bar{0},\bar{0})$	$(0,1;\bar{2},\bar{3}), (2,3;\bar{0},\bar{1})$
	$(1,1;\bar{3},\bar{3}), (3,3;\bar{1},\bar{1})$	$(1,2;\bar{3},\bar{0}), (3,0;\bar{1},\bar{2})$
	$(0,2;\bar{2},\bar{0}), (2,0;\bar{0},\bar{2})$	$(0,3;\bar{2},\bar{1}), (2,1;\bar{0},\bar{3})$
	$(1,3;\bar{3},\bar{1}), (3,1;\bar{1},\bar{3})$	$(1,0;\bar{3},\bar{2}), (3,2;\bar{1},\bar{0})$

$\frac{SU(4)_1 \times U(1)_4}{\mathbb{Z}_2}$	$\widehat{\mathbb{Z}}_2$ even	$\widehat{\mathbb{Z}}_2$ odd
$\widehat{\mathcal{H}}$	$(0,0;\bar{0},\bar{0}), (2,2;\bar{2},\bar{2})$	$(0,0;\bar{2},\bar{2}), (2,2;\bar{0},\bar{0})$
	$(1,1;\bar{1},\bar{1}), (3,3;\bar{3},\bar{3})$	$(1,1;\bar{3},\bar{3}), (3,3;\bar{1},\bar{1})$
	$(0,2;\bar{0},\bar{2}), (2,0;\bar{2},\bar{0})$	$(0,2;\bar{2},\bar{0}), (2,0;\bar{0},\bar{2})$
	$(1,3;\bar{1},\bar{3}), (3,1;\bar{3},\bar{1})$	$(1,3;\bar{3},\bar{1}), (3,1;\bar{1},\bar{3})$
$\widehat{\mathcal{H}}_{tw}$	$(0,1;\bar{0},\bar{1}), (2,3;\bar{2},\bar{3})$	$(0,1;\bar{2},\bar{3}), (2,3;\bar{0},\bar{1})$
	$(1,0;\bar{1},\bar{0}), (3,2;\bar{3},\bar{2})$	$(1,2;\bar{3},\bar{0}), (3,0;\bar{1},\bar{2})$
	$(0,3;\bar{0},\bar{3}), (2,1;\bar{2},\bar{1})$	$(0,3;\bar{2},\bar{1}), (2,1;\bar{0},\bar{3})$
	$(1,2;\bar{1},\bar{2}), (3,0;\bar{3},\bar{0})$	$(1,0;\bar{3},\bar{2}), (3,2;\bar{1},\bar{0})$

Table 3.1: Affine $SU(4)_1 \times U(1)_4$ primaries spectrum and their symmetry properties under the $\mathbb{Z}_2$ symmetry before gauging (left) and under its dual $\widehat{\mathbb{Z}}_2$ symmetry after gauging (right). We report in different colors the $SU(4)_1 \times U(1)_4$ primaries $(j, \lambda; \bar{k}, \bar{\mu})$ that combine together to form $Spin(8)_1$ primaries, namely identity, vector, spinor, conjugate spinor. There are also twisted primaries living at the end of a $\widehat{\mathbb{Z}}_2$ line defect, which do not correspond to any $\widehat{\mathfrak{spin}}(8)_1$ representation.

$Spin(8)_1$	$\mathbb{Z}_2^\vee$ even	$\mathbb{Z}_2^\vee$ odd
$\mathcal{H}$	$(\mathrm{i}, \bar{\mathrm{i}}), (\mathbf{v}, \bar{\mathbf{v}})$	$(\mathbf{s}, \bar{\mathbf{s}}), (\mathbf{c}, \bar{\mathbf{c}})$
$\mathcal{H}_{tw}$	$(\mathbf{s}, \bar{\mathbf{c}}), (\mathbf{c}, \bar{\mathbf{s}})$	$(\mathrm{i}, \bar{\mathbf{v}}), (\mathbf{v}, \bar{\mathrm{i}})$

4 Diracs	$\mathbb{Z}_2^F$ even	$\mathbb{Z}_2^F$ odd
$\mathcal{H}_{\rm NS}$	$(\mathrm{i}, \bar{\mathrm{i}}), (\mathbf{v}, \bar{\mathbf{v}})$	$(\mathrm{i}, \bar{\mathbf{v}}), (\mathbf{v}, \bar{\mathrm{i}})$
$\mathcal{H}_{\rm R}$	$(\mathbf{s}, \bar{\mathbf{s}}), (\mathbf{c}, \bar{\mathbf{c}})$	$(\mathbf{s}, \bar{\mathbf{c}}), (\mathbf{c}, \bar{\mathbf{s}})$

Table 3.2: Affine $Spin(8)_1$ primaries spectrum and their symmetry properties under the $\mathbb{Z}_2^\vee$ symmetry before fermionization (left) and under $\mathbb{Z}_2^F$ symmetry after fermionization (right). We use the same color coding as in Table 3.1 for affine representations and as in Table 2.1 for the mapping of sectors.

mod $2N$. This amounts to shifting the twists $2(N+1)k$ and/or $2(N+1)\ell$ for $Z_{U(1)_{4N}}$ in the middle term of (3.61) by $2N$. This allows to further identify

$$
\begin{aligned}
Z_{Spin(2N)_1}[k_\psi, \ell_\psi] = \frac{1}{N} \sum_{k,\ell \in \mathbb{Z}_N} & Z_{SU(N)_1}[k, \ell] \\
& \times Z_{U(1)_{4N}}[2(N+1)k + 2Nk_\psi, 2(N+1)\ell + 2N\ell_\psi] \,, \quad N \text{ odd.}
\end{aligned} \tag{3.69}
$$

3.3.3 Example

It is useful to see in some more detail how the $SU(N)_1$ and $U(1)_N$ (or $U(1)_{4N}$ for N odd) primary operators combine in $Spin(2N)_1$ ones. We consider N even and first we classify the spectrum of affine primaries of $SU(N)_1 \times U(1)_N$ under the $\mathbb{Z}^L_{N/2}$ symmetry. Like Wilson lines in the corresponding Chern-Simons theory, affine primaries of $SU(N)_1 \times U(1)_N$ are labeled by their representations under left- and right-moving chiral algebras, $(j, \lambda; \bar{k}, \bar{\mu})$, $j, \bar{k}, \lambda, \bar{\mu} = 0, \ldots, N-1$. Operators with

$$
\begin{aligned}
\lambda &\equiv j + Q \mod N/2 \,, \qquad & \bar{\mu} &\equiv \bar{k} + Q \mod N/2 \,, \\
\bar{k} &\equiv j + 2\ell \mod N \,, \qquad & \bar{\mu} &\equiv \lambda + 2\ell \mod N \,,
\end{aligned} \tag{3.70}
$$

belong to the charge Q subsector of the ℓ-twisted Hilbert space under $\mathbb{Z}_{N/2}$.

In order to understand the Hilbert space structure of the gauged $(SU(N)_1 \times U(1)_N)/\mathbb{Z}^L_{N/2}$ WZW theory, let us study first the one of the ungauged $(SU(N)_1 \times U(1)_N)$ WZW theory.

Upon gauging $\mathbb{Z}_{N/2}$, states in the Hilbert space get sorted according to the quantum $\widehat{\mathbb{Z}}_{N/2}$ symmetry: for instance, charge Q operators in the Hilbert space not twisted under

$\mathbb{Z}_{N/2}$ get mapped to neutral operators in the Hilbert space twisted Q times under $\mathbb{Z}_{N/2}$, and viceversa. Therefore all neutral states in the original theory are in the untwisted Hilbert space for the gauged theory. These $SU(N)_1 \times U(1)_N$ primaries then can be rearranged in such a way to give the diagonal modular invariant of the $Spin(2N)_1$ WZW model, as expressed in (3.36). On the other hand, operators in the gauged theory with $\widehat{\mathbb{Z}}_{N/2}$ background twists cannot be described in terms of $\widehat{\mathfrak{spin}}(2N)_1$ affine primaries. Moreover, the $\mathbb{Z}_2^\vee$ symmetry that gets fermionized is defined only in the $\widehat{\mathbb{Z}}_{N/2}$-untwisted sector.

It is useful to consider an explicit example, for instance $N = 4$. We report in Table 3.1 the symmetry properties of the affine primaries of $SU(4)_1 \times U(1)_4$. With different colors we highlight how they combine to form local $\widehat{\mathfrak{spin}}(8)_1$ primaries, in the absence of $\widehat{\mathbb{Z}}_2$ twists. There are also twisted primaries living at the end of a $\widehat{\mathbb{Z}}_2$ line defect, but these non-local operators are not $\widehat{\mathfrak{spin}}(8)_1$ affine primaries, and get projected out in the $\mathbb{Z}_2^L$ orbifold. We consider also the $\mathbb{Z}_2^\vee$ backgrounds for $Spin(8)_1 = (SU(4)_1 \times U(1)_4)/\mathbb{Z}_2^L$, and its fermionization to 4 Dirac fermions classifying the operator content according to the $\mathbb{Z}_2^F$ properties, in Table 3.2.

Chapter 4

Phases of the N-flavor chiral Gross-Neveu Model

In this Chapter we present our analysis and proposals for the phase diagram of the N-flavor chiral Gross-Neveu model, at finite N. While in the large N-limit one can evade the usual no-go theorems that forbid ordered phases in two spacetime dimensions, this is not the case at finite N. As we have discussed in Section 1.3 more complicated scenarios such as quasi-long-range ordered phases (at $T = 0$) are still viable. On the other hand, unless there is an 't Hooft anomaly that survives compactification, at $T > 0$ one would still not expect the persistence of any type of order.

In Section 4.1 we show that the chiral spiral configuration found at large N persists at finite N and $T = 0$ for any $\mu > 0$, using non-Abelian bosonization in its simplest formulation: with this description, the appearance of the inhomogeneous phase is surprisingly simple. To prepare the ground for the analysis of the $T > 0$ scenario, in Section 4.2 we discuss a set of 't Hooft anomalies for the model. In particular, via the bosonization/fermionization duality discussed in Section 3.2, we are able to argue for the existence of an 't Hooft anomaly for a discrete D_8^F symmetry subgroup of the fermionic theory, which is activated in the presence of a non-trivial $\mathbb{Z}_2^F$ background. Finally, in Section 4.3 we explicitly compute two-point functions of certain composite fermion operators in the chiral Gross-Neveu model at finite temperature, both for thermal (antiperiodic) and periodic fermions around the thermal cycle. We find that for thermal fermions the spatial modulation is still present, but has an exponentially decaying amplitude due to a finite thermal correlation length. This exponential decay is absent instead for periodic conditions. We interpret this as the effect of the D_8^F anomaly, that forbids tunneling effects.

4.1 Inhomogeneities at $T = 0$

In the simpler case $T = 0$ we can work on Euclidean $\mathbb{R}^2$, where global considerations do not matter, and we can work in the setting outlined in Section 3.1. Let us introduce a chemical potential for the $U(1)_V$ charge by adding to the Lagrangian (3.1) the term

$$\mathcal{L}_\mu = \mu(\psi^\dagger_{+a}\psi^a_+ + \psi^\dagger_{-a}\psi^a_-) \ . \tag{4.1}$$

Upon bosonization, this maps simply to

$$\mathcal{L}_\mu = -\mu\frac{\sqrt{N}}{2\pi}\partial_1\phi \ . \tag{4.2}$$

The term (4.2), which does not depend on the $SU(N)$ degrees of freedom, provides an expectation value for $\partial_1\phi$. By adding (4.2) to the Lagrangian (3.7) we get that the effective action for ϕ is minimized on configurations with

$$\langle\partial_1\phi\rangle = 2\mu\sqrt{N}\left(1+\frac{\lambda'}{2\pi N}\right)^{-1} \equiv 2\mu'\sqrt{N}\ . \tag{4.3}$$

The difference of the zero-temperature free energy density per flavour between the configuration with and without the expectation value for $\partial_1\phi$ is

$$\delta F = -\frac{\mu^2}{2\pi}\left(1+\frac{\lambda'}{2\pi N}\right)^{-1}\ , \tag{4.4}$$

showing that the former is favored on $\mathbb{R}^2$.

Using the bosonization identity $\psi^\dagger_{-a}\psi^b_+ = U_a{}^b e^{i\phi/\sqrt{N}}$ [62, 102, 103] (omitting a scheme-dependent renormalization mass scale), the two-point function of the fermion bilinear in terms of the $SU(N)$ and free scalar degrees of freedom is

$$\langle\psi^\dagger_{-a}\psi^a_+(x)\psi^\dagger_{+b}\psi^b_-(0)\rangle = \langle\mathrm{Tr}\,U(x)\,\mathrm{Tr}\,U^\dagger(0)\rangle e^{2i\mu' x^1}\langle e^{i\frac{\delta\phi(x)}{\sqrt{N}}}e^{-i\frac{\delta\phi(0)}{\sqrt{N}}}\rangle\,, \tag{4.5}$$

where $\delta\phi$ denotes excitations of ϕ around (4.3) and we have used the decoupling of the two sectors to factorize the correlator.

Let us now assume that the operator $\mathrm{Tr}\,U$ has a non-vanishing expectation value (at zero temperature) in the $SU(N)_1$ theory deformed by the current-current interaction. Then, in the limit $|x|\to\infty$, (4.5) approaches

$$|\langle\mathrm{Tr}\,U\rangle|^2 e^{2i\mu' x^1}|x|^{-\frac{2}{N(1+\lambda'/2\pi N)}}\,, \tag{4.6}$$

that is, it decays with power-like behavior times an oscillating factor. The latter is consequence of the chiral spiral configuration, whereas the former is the hallmark of quasi long-range order. Only a diagonal subgroup of the $U(1)_A$ symmetry and spatial translations preserves the would-be order parameter $\mathrm{Tr}\,U e^{2i\mu' x^1}$. Recalling that the $\psi^\dagger_{-a}\psi^a_+$ bilinear carries $U(1)_A$ charge $+2$, this subgroup is a $U(1)_A$ transformation of parameter α accompanied by a translation with parameter $\delta x^1 = -\alpha/\mu'$. The condensation of $\mathrm{Tr}\,U$ breaks completely the global center symmetry $\mathbb{Z}_N^L$ of the $SU(N)_1$ theory.

There are a number of arguments in favor of spontaneous breaking of $\mathbb{Z}_N^L$ on $\mathbb{R}^2$.[1] As we will discuss in the next Section, in the $SU(N)_1$ WZW model there is a mixed 't Hooft anomaly involving the $PSU(N)_V\times\mathbb{Z}_N^L$ symmetry (3.13), which is also the symmetry that is preserved by the $J\bar{J}$ deformation, hinting at the fact that the $\mathbb{Z}_N^L$ symmetry is spontaneously broken. Moreover, in Appendix A, we study the classical potential arising from the $J\bar{J}$ deformation in the free field realization of $SU(N)_1$, and indeed one finds there N degenerate minima with a spontaneously broken $\mathbb{Z}_N^L$ symmetry. Given the absence of tunneling effects in a QFT in infinite volume, one expects that this also holds at the quantum

[1] The full extent of those argument was not known to us at the time of publication of [1], for this reason one finds a more cautious discussion there, which we report after this paragraph.

level.

Even if the $\mathbb{Z}_N^L$ symmetry were unbroken in the $SU(N)_1$ theory (that is, $\langle \mathrm{Tr}\, U \rangle = 0$), one can find an operator of the cGN theory with a spatially modulated expectation value who contains only vertex operators of the form $e^{i\,kN\,\phi/\sqrt{N}}$, with $k \in \mathbb{Z}$. This ensures that the operator is invariant under $\mathbb{Z}_N^P$, without the need of a compensating factor charged under $\mathbb{Z}_N^L$ from the $SU(N)_1$ sector (recall that the diagonal subgroup of these two $\mathbb{Z}_N$ is gauged, so any physical operator must be neutral under it). For instance, the quasi-long range order could be detected in the two-point function of the composite fermion operator $\det(\psi^\dagger_{-a}\psi^b_+)$. In this case the combination of $U(1)_A$ and translation that preserves it will be different, namely a $U(1)_A$ transformation of parameter α needs to be accompanied by a translation with parameter $\delta x^1 = -N\alpha/\mu'$. There could also be intermediate possibilities in which only a nontrivial subgroup $\mathbb{Z}_{N'}^P \subset \mathbb{Z}_N^P$ is preserved, when N is a multiple of N'.

The vacuum is then in a so-called “chiral spiral" configuration, where only a linear combination of the $U(1)_A$ symmetry and of spatial translations preserves the would-be order parameter. Such combination depends on the realization of the $\mathbb{Z}_N^L$ symmetry in the vacuum of the deformed $SU(N)_1$ theory. Excitations on top of the chiral spiral are gapless and have a relativistic dispersion relation.

4.2 Anomalies and Persistent Order at $T > 0$

We have discussed in Section 3.2 the bosonization of the chiral Gross-Neveu model in absence of the $J\bar{J}$ deformation, i.e. of free fermions, on a non-trivial manifold. The deformation has a mild effect in the $U(1)$ sector, where it just changes the radius of the compact scalar, and a non-trivial effect to the $SU(N)$ sector, where it gives rise to a strongly coupled gapped theory [1]. In this Section we use 't Hooft anomaly matching conditions to determine key IR properties of the theory.

We start by discussing 't Hooft anomalies of the bosonic $SU(N)_k$ WZW bosonic model in isolation. It is known that $SU(N)_k$ WZW models have a mixed $\mathbb{Z}_N$ anomaly between $PSU(N)_V$ and $\mathbb{Z}_N^L$ symmetries for $k = 0 \mod N$ [104, 105].[2] In presence of a nontrivial $PSU(N)_V$ bundle P and a background gauge field A_L for the $\mathbb{Z}_N^L$ symmetry, the partition function is not background gauge invariant. Invariance can be restored by coupling the theory to a three-dimensional bulk topological theory defined on M_3 with $\partial M_3 = M_2$. This is a symmetry protected topological (SPT) phase with partition function

$$Z_{SPT} = \exp\left(\frac{2\pi i k}{N}\int_{M_3} A_L \cup u_2(P)\right) . \tag{4.7}$$

In (4.7) $u_2 \in H^2(BPSU(N), \mathbb{Z}_N) = H^2(PSU(N), \mathbb{Z}_N) = \mathbb{Z}_N$ is the Brauer class (for $N = 2$, it coincides with the more common Stiefel-Whitney class w_2). The SPT phase (4.7) is non-trivial unless $k = 0 \mod N$.

An alternative way of detecting such anomalies is to study QED with N Dirac fermions, i.e. gauging $U(1)_V$. This theory (upon summing over the spin structures) is expected to

[2] Another notable 't Hooft anomaly of $SU(N)_k$ WZW models is a $\mathbb{Z}_2$ mixed global-gravitational one present for N even and k odd [106–109]. This anomaly is at the root of the different treatment of even and odd N in Section 3.2.

flow in the IR to the $SU(N)_1$ WZW model. Using anomaly matching conditions, we can then determine the anomalies from the UV theory. The axial $U(1)_A$ symmetry is broken down to $\mathbb{Z}_{2N}$ by the ordinary chiral anomaly. The sum over the spin structures is performed by gauging $\mathbb{Z}_2^F \subset \mathbb{Z}_{2N}$, so in the bosonic theory we are left with a $\mathbb{Z}_N$ chiral symmetry. In presence of a non-trivial $PSU(N)$ bundle we can have fractional $U(1)$ fluxes,

$$\int_{M_2} \frac{F}{2\pi} = \frac{1}{N}\,, \tag{4.8}$$

where F is the field strength for the $U(1)_V$ gauge field. In this backgroud the fermion measure is not invariant under a chiral $\mathbb{Z}_N$ transformation and $\mathbb{Z}_N$ is completely broken. The IR manifestation of this anomaly is precisely the mixed anomaly (4.7).

Yet another way to get this anomaly is by using the correspondence between 3d G_k Chern-Simons on $M_2 \times S^1$ and the WZW coset G_k/G_k on M_2 (for G simply connected) obtained by dimensional reduction [61, 110]. The coset is implemented by gauging the vector G symmetry of the two-dimensional G_k WZW theory. For $G_k = SU(N)_k$, the 3d Chern-Simons theory has a global $\mathbb{Z}_N$ one-form symmetry $\widetilde{\mathbb{Z}}_N^{(1)}$, implemented by topological defect lines. This symmetry has an 't Hooft anomaly unless $k = 0 \bmod N$ [14]. Upon compactification to M_2, $\widetilde{\mathbb{Z}}_N^{(1)}$ gives both the $\mathbb{Z}_N^L$ global zero-form symmetry and the $\mathbb{Z}_N^{(1)}$ one-form symmetry of the $SU(N)$ two-dimensional gauge theory. These symmetries are implemented in the two-dimensional theory by topological defect lines and by topological local operators, respectively. The topological defect lines for $\mathbb{Z}_N^L$ are Wilson lines for the two-dimensional gauge theory, and are thus charged under $\mathbb{Z}_N^{(1)}$ unless they have vanishing N-ality. Conversely, the topological local operators that realize the one-form symmetry of the gauge theory are charged under $\mathbb{Z}_N^L$. We conclude that there is a mixed 't Hooft anomaly between $\mathbb{Z}_N^L$ and $\mathbb{Z}_N^{(1)}$ in the $SU(N)_1/SU(N)_1$ coset WZW model. On the other hand, in the limit of infinite gauge coupling a non-trivial $SU(N)$ gauge field configuration together with a non-trivial $\mathbb{Z}_N^{(1)}$ background form a (possibly non-trivial) $PSU(N)_V$ background, thus the result holds also for the non-gauged $SU(N)_1$ WZW model.

We now consider the $J\bar{J}$-deformed $SU(N)_1$ WZW model. As discussed in the previous chapter, the deformation preserves a $PSU(N)_V \times \mathbb{Z}_N^L$ subgroup of the symmetry group $(SU(N)_L \times SU(N)_R)/\mathbb{Z}_N^V$ of the undeformed theory. The above mixed $PSU(N)_V \times \mathbb{Z}_N^L$ anomaly forbids the $J\bar{J}$-deformed theory to be trivially gapped in the IR. The most natural possibility is to assume that the deformed $SU(N)_1$ theory has N vacua and displays spontaneous breaking of $\mathbb{Z}_N^L$. [3]

We can use (3.19) and the knowledge that the deformed $SU(N)_1$ flows in the IR to N gapped vacua connected by the $\mathbb{Z}_N^L$ symmetry to argue about the low energy effective theory of the cGN model. More specifically, we want to understand the fate of the vacuum at finite temperature. For this it will be important to have a closer look at the possible discrete 't Hooft anomalies associated to the free $U(1)$ compact scalar.

4.2.1 A mixed $\mathbb{Z}_2$ anomaly of the bosonic theory

In this subsection we discuss a certain $\mathbb{Z}_2$ 't Hooft anomaly of the bosonic theory $\mathcal{B}$ in (3.17). It involves the charge conjugation $\mathbb{Z}_2^C$ and other two $\mathbb{Z}_2$ symmetries which are not

[3] As further evidence, using the correspondence between modular invariants of 2d rational CFTs and topological interfaces in 3d TQFTs [111], it has been conjectured in [112] that the ground state of the $J\bar{J}$-deformed $SU(N)_1$ theory is exactly N-fold degenerate on $M_2 = T^2$.

broken by the $J\bar{J}$ deformations. The anomaly manifests itself as follows: when we gauge one of the two $\mathbb{Z}_2$ factors, the remaining global $\mathbb{Z}_2$ and $\mathbb{Z}_2^C$ are realized projectively in the twisted sector of the Hilbert space of the theory.

This anomaly is a generalization of an anomaly found in the compact boson in [113], so it is useful to first review the anomaly for the compact boson in isolation. In this case the $\mathbb{Z}_2$ symmetries involved are, in addition to $\mathbb{Z}_2^C$, the $\mathbb{Z}_2^P$ and $\mathbb{Z}_2^W$ subgroups of the momentum and winding symmetries (2.47). When we gauge, say, $\mathbb{Z}_2^W$, the vertex operators (2.50) with odd m are projected out, but new operators with half-integer e appear from the twisted sector. This phenomenon is a discrete remnant of the mixed $U(1)_P - U(1)_W$ anomaly. The physical vertex operators in the gauged theory are then $\mathcal{V}_{e+\ell/2,2m}$, $e, m \in \mathbb{Z}$, $\ell = 0, 1$. Let P and C be the topological lines implementing the $\mathbb{Z}_2^P$ and $\mathbb{Z}_2^C$ actions in the Hilbert space. On vertex operators $\mathcal{V}_{e+\frac{\ell}{2},2m}$ we have

$$P^2 = (-1)^\ell \,, \qquad CP = (-1)^\ell PC \,, \tag{4.9}$$

where we used the action of charge conjugation given by

$$\mathbb{Z}_2^C : \qquad \mathcal{V}_{p,q} \to \mathcal{V}_{-p,-q} \,, \tag{4.10}$$

for any fractional or integer p, q. We see that the symmetry $\mathbb{Z}_2^C \times \mathbb{Z}_2^P$ acts linearly in the untwisted sector, but projectively in the twisted sector of the orbifolded theory.

We can refine the analysis by considering the torus partition function in presence of nontrivial backgrounds $W = [W_a, W_b]$ and $S = [S_a, S_b]$ for the winding $\mathbb{Z}_2^W$ and its dual $\widehat{\mathbb{Z}}_2^W$ symmetry, where $T_a, T_b = 0, 1$ label the holonomy of the $\mathbb{Z}_2$ gauge field on the corresponding cycle of the torus. The partition function of the orbifolded theory is given by (2.110):

$$Z'_{2R}[S_a, S_b] = \frac{1}{2} \sum_w Z_R[w_a, w_b] e^{i\pi \int S \cup w} \,. \tag{4.11}$$

Eq.(4.9) implies that

$$P^2 = (-1)^{w_a} \,, \qquad PC = (-1)^{w_a} CP \,, \qquad w_a = 0, 1 \,, \tag{4.12}$$

from which we get

$$\begin{aligned}
Z_{2R}^{(PC)'}[S_a, S_b] &= \frac{1}{2} \sum_w Z_R^{(PC)}[w_a, w_b] e^{i\pi \int S \cup w} = \frac{1}{2} \sum_w (-1)^{w_a} Z_R^{(CP)}[w_a, w_b] e^{i\pi \int S \cup w} \\
&= \frac{1}{2} \sum_w Z_R^{(CP)}[w_a, w_b] e^{i\pi \int w \cup [S_a, S_b+1]} = Z_{2R}^{(CP)'}[S_a, S_b + 1] \,, \\
Z_{2R}^{(P^2)'}[S_a, S_b] &= \frac{1}{2} \sum_w Z_R^{(P^2)}[w_a, w_b] e^{i\pi \int S \cup w} = \frac{1}{2} \sum_w (-1)^{w_a} Z_R[w_a, w_b] e^{i\pi \int S \cup w} \\
&= \frac{1}{2} \sum_w Z_R[w_a, w_b] e^{i\pi \int w \cup [S_a, S_b+1]} = Z'_{2R}[S_a, S_b + 1] \,,
\end{aligned} \tag{4.13}$$

where $Z^{(A)}$ indicates the partition function with the insertion of the charge operator associated to the $\mathbb{Z}_2^A$ symmetry in the trace. While $\mathbb{Z}_2^P$ and $\mathbb{Z}_2^C$ acts projectively, the combination of $\mathbb{Z}_2^P$, $\mathbb{Z}_2^C$, and $\widehat{\mathbb{Z}}_2^W$ acts linearly, since from (4.13) we have

$$P^2 = \widehat{W} \,, \qquad PC = \widehat{W} CP \,. \tag{4.14}$$

$\mathcal{B}'$	$\mathbb{Z}_2^W = +1$	$\mathbb{Z}_2^W = -1$
$\mathcal{H}'$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N},m}$, $n+k=m \mod N$, $m=0 \mod 2\,,\ k\in\mathbb{Z}_N$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N},m}$ $n+k=m \mod N$, $m=1 \mod 2\,,\ k\in\mathbb{Z}_N$
$\mathcal{H}'_{tw}$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2}{2},m}$, $n+k=m \mod N$, $m=0 \mod 2\,,\ k\in\mathbb{Z}_N$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2}{2},m}$, $n+k=m \mod N$, $m=1 \mod 2\,,\ k\in\mathbb{Z}_N$

Table 4.1: Classification of the operators of the theory $\mathcal{B}' = \frac{SU(N)_1\times U(1)_N}{\mathbb{Z}_N}$, for N odd, under the $\mathbb{Z}_2^W$ symmetry (4.19).

We see that $\mathbb{Z}_2^P \times \mathbb{Z}_2^C$ is centrally extended by the dual symmetry $\widehat{\mathbb{Z}}_2^W$ to the group D_8, the dihedral group of order 8, defined as

$$D_8 = \langle P, C, \widehat{W} | P^4 = C^2 = 1,\, CPC = P^3,\, P^2 = \widehat{W}\rangle\,. \tag{4.15}$$

The D_8 symmetry group of the gauged theory is anomaly-free. Upon gauging its $\widehat{\mathbb{Z}}_2^W$ subgroup, we reobtain the original theory with the anomalous $\mathbb{Z}_2^P \times \mathbb{Z}_2^C \times \mathbb{Z}_2^W$ symmetry. The extension by $\widehat{\mathbb{Z}}_2^W$ and the anomaly involving $\mathbb{Z}_2^W$ get exchanged by their gauging [15]. If we gauge $\mathbb{Z}_2^P$ instead of $\mathbb{Z}_2^W$, all the considerations above apply with the replacement $P \leftrightarrow W$.

We are now ready to discuss a similar 't Hooft anomaly in our setup. We first discuss N odd and consider the theory $\mathcal{B}'$ in the formulation (3.18). The operator of $\mathcal{B}'$ are $\mathbb{Z}_N^L$ are gauge-invariant products of the operators of $\mathcal{S}$ in the $SU(N)_1$ sector and of the vertex operators $\mathcal{V}$ in the $U(1)_N$ sector.[4] As far as our considerations are concerned, it is enough to classify the $SU(N)_1$ operators according to their N-ality, so we denote by $\mathcal{S}_{n,\bar{n}}$ the $SU(N)_1$ operators with L and R N-ality n and $\bar{n}$ under $SU(N)_L$ and $SU(N)_R$, respectively ($n, \bar{n} = 0\ldots, N-1$, primary operators have $n = \bar{n}$). The diagonal $\mathbb{Z}_N$ symmetry acts on the $SU(N)$ and $U(1)$ operators as follows:

$$\begin{aligned} \mathbb{Z}_N^L &: \mathcal{S}_{n,\bar{n}} \to e^{-\frac{2i\pi n}{N}}\mathcal{S}_{n,\bar{n}}\,, \\ \mathbb{Z}_N^W &: \mathcal{V}_{e,m} \to e^{\frac{2i\pi m}{N}}\mathcal{V}_{e,m}\,. \end{aligned} \tag{4.16}$$

Charge conjugation acts on the $U(1)$ sector as in (4.10) and on $SU(N)_1$ operators as

$$\mathbb{Z}_2^C : \mathcal{S}_{n,\bar{n}} \to \mathcal{S}_{N-n,N-\bar{n}}\,. \tag{4.17}$$

After gauging the diagonal $\mathbb{Z}_N$ of (4.16), the physical operators of the theory are

$$\mathcal{O}^{(k)}_{e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N},m}\,, \qquad n+k = m \mod N\,, \tag{4.18}$$

where $k=0$ represents the unwisted sector and $k = 1, \ldots N-1$ the twisted sectors of the orbifolded theory. The second global symmetry involved is the $\mathbb{Z}_2^W$ appearing in (3.18). Its action on the fields is[5]

$$\mathbb{Z}_2^W : \mathcal{V}_{e+\frac{k}{N},m} \to e^{i\pi N^2 m}\mathcal{V}_{e+\frac{k}{N},m}\,, \qquad \mathcal{S}_{n+k,n} \to \mathcal{S}_{n+k,n}\,. \tag{4.19}$$

[4] In order to avoid clutter in the formulas, we remove the hat from $\mathbb{Z}_N^L$ and $\mathbb{Z}_2^W$ which appear in (3.18).

[5] Note that for N odd $e^{i\pi N^2} = e^{i\pi N}$, but the choice of shift in (4.19) ensures that the $\mathbb{Z}_2^W$ acts as a chiral action on the fields.

$\mathcal{B}$	$\mathbb{Z}_2^L = +1$	$\mathbb{Z}_2^L = -1$
$\mathcal{H}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+2k,n}\mathcal{V}_{e+k,m+\frac{2k}{N}}$, $n = e + \frac{N}{2}m \mod N$, $k \in \mathbb{Z}_{N/2}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+2k,n}\mathcal{V}_{e+k,m+\frac{2k}{N}}$, $n = e + \frac{N}{2}m + \frac{N}{2} \mod N$, $k \in \mathbb{Z}_{N/2}$
$\mathcal{H}_{tw}$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+2k+1,n}\mathcal{V}_{e+k+\frac{1}{2},m+\frac{2k+1}{N}}$, $n = e + \frac{N}{2}m \mod N$, $k \in \mathbb{Z}_{N/2}$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+2k+1,n}\mathcal{V}_{e+k+\frac{1}{2},m+\frac{2k+1}{N}}$, $n = e + \frac{N}{2}m + \frac{N}{2} \mod N$, $k \in \mathbb{Z}_{N/2}$

Table 4.2: Classification of the operators of the theory $\mathcal{B} = \frac{SU(N)_1\times U(1)_N}{\mathbb{Z}^L_{N/2}}$, for N even, under the $\mathbb{Z}_2^L$ symmetry (4.25).

When we gauge $\mathbb{Z}_2^W$, the physical operators read

$$\mathcal{O}^{(k,\ell)}_{e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2\ell}{2},2m}\,, \qquad n+k = 2m \mod N\,, \tag{4.20}$$

where $\ell = 0$ and $\ell = 1$ represents the $\mathbb{Z}_2^W$ untwisted and twisted sector, respectively. Finally, the third $\mathbb{Z}_2$ global symmetry involved is a $\mathbb{Z}_2^P \subset U(1)_P$ which acts as[6]

$$\mathbb{Z}_2^P : \mathcal{V}_{e+\frac{k}{N}+\frac{N^2\ell}{2},2m} \to e^{i\pi N(e+\frac{k}{N}+\frac{N^2\ell}{2})}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2\ell}{2},2m}\,, \qquad \mathcal{S}_{n+k,n} \to \mathcal{S}_{n+k,n}\,. \tag{4.21}$$

Using eqs.(4.10), (4.17) and (4.21), it is immediate to verify that on the operators (4.20),

$$P^2 = (-1)^\ell\,, \qquad CP = (-1)^\ell PC\,. \tag{4.22}$$

Similarly to the compact scalar case, $\mathbb{Z}_2^C \times \mathbb{Z}_2^P$ acts linearly in the $\ell = 0$ untwisted sector and projectively in the $\ell = 1$ twisted sector. The analysis from (4.11) until (4.15) applies with obvious changes. Due to the anomaly the $\mathbb{Z}_2^P \times \mathbb{Z}_2^C$ is centrally extended by the dual symmetry $\widehat{\mathbb{Z}}_2^W$ to the group D_8 in (4.15).

We now discuss the theory $\mathcal{B}$ with N even in (3.17). As we will see, the role played by $\mathbb{Z}_2^P$ and $\mathbb{Z}_2^W$ above will be respectively played by a $\mathbb{Z}_2^W$ and a $\mathbb{Z}_2^L$ symmetry. The action of the diagonal $\mathbb{Z}^L_{N/2}$ symmetry on the $SU(N)$ and $U(1)$ operators is as follows:[7]

$$\mathbb{Z}^L_{N/2} : \quad \mathcal{V}_{e,m} \to e^{\frac{4i\pi}{N}(e+\frac{N}{2}m)}\mathcal{V}_{e,m}\,, \qquad \mathcal{S}_{n,\bar n} \to e^{-\frac{4i\pi n}{N}}\mathcal{S}_{n,\bar n}\,. \tag{4.23}$$

Charge conjugation acts as before. After gauging $\mathbb{Z}^L_{N/2}$, the physical operators are

$$\mathcal{O}^{(k)}_{e,m} = \mathcal{S}_{n+2k,n}\mathcal{V}_{e+k,m+\frac{2k}{N}}\,, \qquad n = e \mod \frac{N}{2}\,, \tag{4.24}$$

where $k = 0$ represents the unwisted sector and $k = 1, \ldots N/2-1$ the twisted sectors of the orbifolded theory. The second global symmetry involved is the $\mathbb{Z}_2^L$ appearing in (3.16):

$$\mathbb{Z}_2^L : \quad \mathcal{V}_{e+k,m+\frac{2k}{N}} \to e^{\frac{2i\pi}{N}(e+k+\frac{N}{2}(m+\frac{2k}{N}))}\mathcal{V}_{e+k,m+\frac{2k}{N}}\,, \qquad \mathcal{S}_{n+2k,n} \to e^{-\frac{2i\pi(n+2k)}{N}}\mathcal{S}_{n+2k,n}\,. \tag{4.25}$$

Note that $\mathbb{Z}_2^L$ is not a $\mathbb{Z}_2$ action individually on $\mathcal{S}$ and $\mathcal{V}$, but only on the operators $\mathcal{O}^{(k)}_{e,m}$, where it acts as $\mathcal{O}^{(k)}_{e,m} \to (-)^m\mathcal{O}^{(k)}_{e,m}$. When we further gauge $\mathbb{Z}_2^L$, the physical operators read

$$\mathcal{O}^{(k,\ell)}_{e,m} = \mathcal{S}_{n+2k+\ell,n}\mathcal{V}_{e+k+\frac{\ell}{2},m+\frac{2k}{N}+\frac{\ell}{N}}\,, \qquad n = e + \frac{N}{2}m \mod N\,, \tag{4.26}$$

[6]This is obtained by taking $\alpha_P = N\pi$ in (2.47), since after gauging $\mathbb{Z}_N^W$ $\alpha_P \sim \alpha_P + 2\pi N$.

[7]Note that the action on the $U(1)_N$ sector is the same as a $\mathbb{Z}^P_{N/2}$ momentum shift.

where $\ell = 0$ and $\ell = 1$ represents the $\mathbb{Z}_2^L$ untwisted and twisted sector, respectively. Finally, the third $\mathbb{Z}_2$ global symmetry involved is a $\mathbb{Z}_2^W \subset U(1)_W$ which acts as

$$\mathbb{Z}_2^W: \quad \mathcal{V}_{e+k+\frac{\ell}{2},m+\frac{2k}{N}+\frac{\ell}{N}} \to e^{\frac{i\pi N}{2}(m+\frac{2k}{N}+\frac{\ell}{N})}\mathcal{V}_{e+k+\frac{\ell}{2},m+\frac{2k}{N}+\frac{\ell}{N}}\,, \qquad \mathcal{S}_{n+2k+\ell,n} \to \mathcal{S}_{n+2k+\ell,n}\,. \tag{4.27}$$

Using eqs.(4.10), (4.17) and (4.27), we get

$$W^2 = (-1)^\ell\,, \qquad CW = (-1)^\ell WC\,. \tag{4.28}$$

The $\mathbb{Z}_2^C \times \mathbb{Z}_2^W$ symmetry acts linearly in the $\ell = 0$ untwisted sector and projectively in the $\ell = 1$ twisted sector. The analysis from (4.11) until (4.15) applies again with obvious changes. Due to the anomaly the $\mathbb{Z}_2^W \times \mathbb{Z}_2^C$ symmetry is centrally extended by the dual symmetry $\widehat{\mathbb{Z}}_2^L$ to the group D_8 in (4.15).

Note that the mixed $\mathbb{Z}_2^P \times \mathbb{Z}_2^W \times \mathbb{Z}_2^C$ anomaly (4.9) reduces in 1d to the $\mathbb{Z}_2$ anomaly in quantum mechanics discussed in Appendix D of [114]. Indeed, the action for the compact scalar $\phi \sim \phi + 2\pi R$ with a background $U(1)_W$ gauge field A_μ on $\mathbb{R} \times S_L^1$ is

$$S_{2d}[\phi, A_\mu] = \frac{1}{8\pi}\int dt \int_0^L dx (\partial\phi)^2 - \frac{i}{2\pi R}\int dt \int_0^L dx \epsilon^{\mu\nu} A_\mu\, \partial_\nu \phi\,. \tag{4.29}$$

We choose A_μ to be a flat gauge field with a nontrivial holonomy θ around the compact S_L^1, $A_t = 0$, $A_x = \theta/L$, with $\theta \sim \theta + 2\pi$. If we neglect the 2d massive excitations and only keep the zero mode $\phi(x,t) \approx \phi_0(t)$ we get

$$S_{1d}[\phi_0, \theta] = \frac{L}{8\pi}\int dt\, \dot{\phi}_0^2 - \frac{i\theta}{2\pi R}\int dt\, \dot{\phi}_0\,. \tag{4.30}$$

The action (4.30) inherits the following symmetries from the 2d theory:

$$\begin{cases} \mathbb{Z}_2^P: & \phi_0 \mapsto \phi_0 + \pi R\,, \quad \text{for any } \theta, \\ \mathbb{Z}_2^C: & \phi_0 \mapsto -\phi_0\,, \qquad \text{for } \theta = 0, \pi. \end{cases} \tag{4.31}$$

We then recover the situation described in [114]: $\mathbb{Z}_2^P \times \mathbb{Z}_2^C$ is realized linearly at $\theta = 0$ and projectively at $\theta = \pi$. In the latter case we have a $\mathbb{Z}_2$ 't Hooft anomaly which forbids to have a unique gapped vacuum.

4.2.2 Fermionization and Persistent Order

In this Section we study the fate of the bosonic anomaly discussed above upon fermionization. It is useful to first look at the simplest case $N = 1$, i.e. the duality between a compact boson at $R = 1$ and a free Dirac fermion. The $\mathbb{Z}_2$ bosonic symmetry to be gauged is $\mathbb{Z}_2^W$ as in the relation (2.130). The fermion partition function is obtained from (2.122),

$$Z_{\mathcal{F}'}[\rho_a, \rho_b] = \frac{1}{2}\sum_w Z_1[w_a, w_b](-1)^{\text{Arf}[w\cdot\rho]}\,. \tag{4.32}$$

We determine the effect of (4.14) in the fermionic theory by adding the corresponding line operators for P and C in the partition function. We then get

$$Z_{\mathcal{F}'}^{(PC)}[\rho_a, \rho_b] = \frac{1}{2}\sum_w (-1)^{w_a} Z_1^{(CP)}[w_a, w_b] e^{i\pi \text{Arf}[w\cdot\rho]} = (-1)^{\rho_a} Z_{\mathcal{F}'}^{(CP)}[\rho_a, \rho_b + 1]\,, \tag{4.33}$$

where $\rho_b + 1$ means (with a slight abuse of notation) that we are exchanging $\rho_b = 0 = \mathrm{NS}$ and $\rho_b = 1 = \mathrm{R}$ spin structures on the b-cycle, and we have used the identity

$$\int w \cup T + \mathrm{Arf}[w \cdot \rho] = \mathrm{Arf}[w \cdot (T \cdot \rho)] + \mathrm{Arf}[T \cdot \rho] + \mathrm{Arf}[\rho] \tag{4.34}$$

specified for $T = [T_a, T_b] = [0, 1]$, i.e. for T having nontrivial holonomy only along the b-cycle. Similarly, we have

$$Z^{(P^2)}_{\mathcal{F}'}[\rho_a, \rho_b] = \frac{1}{2}\sum_w (-1)^{w_a} Z_1[w_a, w_b] e^{i\pi \mathrm{Arf}[w\cdot\rho]} = (-1)^{\rho_a} Z_{\mathcal{F}'}[\rho_a, \rho_b + 1]\,. \tag{4.35}$$

The $\mathbb{Z}_2^C \times \mathbb{Z}_2^P$ action is realized projectively in the fermionic theory by $(-1)^F$ due to the shift $\rho_b \to \rho_b + 1$:

$$P^2 = (-1)^F\,, \qquad PC = (-1)^F CP\,. \tag{4.36}$$

Like in the bosonic case discussed in the previous subsection, the resulting extended group is D_8^F, where D_8^F is the group (4.15) with $\widehat{W} \to (-1)^F$. Its action on the Weyl components of the free Dirac fermion is given by

$$D_8^F: \quad \begin{cases} P: & \psi_\pm \mapsto i\psi_\pm\,, \qquad \psi_\pm^\dagger \mapsto -i\psi_\pm^\dagger\,, \\ C: & \psi_\pm \mapsto \psi_\pm^\dagger\,, \qquad \psi_\pm^\dagger \mapsto \psi_\pm\,, \\ (-1)^F: & \psi_\pm \mapsto -\psi_\pm\,, \qquad \psi_\pm^\dagger \mapsto -\psi_\pm^\dagger\,. \end{cases} \tag{4.37}$$

As it can be seen, the symmetry (4.37) is not broken by the λ_s and λ_v deformations in (3.1). In fact, it is unbroken also under the $O(2N)$ deformation which defines the GN model. In the fermionized theory, P is a $\mathbb{Z}_4 \subset U(1)_V$ transformation whereas C is still charge conjugation. However, while D_8^F acts linearly in the NS sector, it acts projectively in the R sector because of an additional sign which arises from the $(-1)^{\rho_a}$ factor in (4.33) and (4.35). The projective action in the R sector is a manifestation of a $\mathbb{Z}_2$ anomaly involving the group D_8^F.

The same fermionic theory $\mathcal{F}'$ can be alternatively obtained by (2.129), namely starting from the compact boson with $R^2 = 4$ and gauging $\mathbb{Z}_2^P$. In this formulation, it is easy to see that the factor $(-1)^{\rho_a}$ disappears from the analogue of (4.33) and (4.35), being reabsorbed due to the presence of the additional Arf term. The opposite occurs to the fermion theory $\mathcal{F}$. We conclude that, depending on which projection we perform in the R sector (i.e. if we take $\mathcal{F}$ or $\mathcal{F}'$, see (2.126)), either a D_8^F involving $\mathbb{Z}_2^P$ (with P corresponding to a $\mathbb{Z}_4^V$ transformation) or one involving $\mathbb{Z}_2^W$ (with W corresponding to a $\mathbb{Z}_4^A$ transformation) is realized projectively in the R sector. No matter what we choose, there is no way to have both D_8^F's linearly realized and hence a $\mathbb{Z}_2$ anomaly persists. The $\mathbb{Z}_2^P$ or $\mathbb{Z}_2^W$ acts on fermions as discrete $U(1)_V$ or $U(1)_A$ rotations, so this phenomenon is nothing else than the discrete version of the well-known fact that we can move the mixed $U(1)_V - U(1)_A$ anomaly by counterterms, but there is no way to get rid of it altogether.

We now consider the fermionization of the bosonic theory (3.17). For N odd we start from the bosonic theory $\mathcal{B}'$ to get the fermionic theory $\mathcal{F}'$ using (3.18). The analysis is identical to the one performed for the $N = 1$ compact boson. It is enough to replace Z_1 from (4.33) to (4.35) with the partition function of the $\mathcal{B}'$ theory, to infer that in $\mathcal{F}'$ we have a D_8^F symmetry which is anomalous because it is realized projectively in the Ramond sector.

$\mathcal{F}'$	$\mathbb{Z}_2^F = +1$	$\mathbb{Z}_2^F = -1$
$\mathcal{H}_{\rm NS}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N},m}$, $n+k=m \mod N$, $m=0 \mod 2\,,\ k\in\mathbb{Z}_N$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2}{2},m}$, $n+k=m \mod N$, $m=1 \mod 2\,,\ k\in\mathbb{Z}_N$
$\mathcal{H}_{\rm R}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N},m}$ $n+k=m \mod N$, $m=1 \mod 2\,,\ k\in\mathbb{Z}_N$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+k,n}\mathcal{V}_{e+\frac{k}{N}+\frac{N^2}{2},m}$, $n+k=m \mod N$, $m=0 \mod 2\,,\ k\in\mathbb{Z}_N$

Table 4.3: Classification of the operators of the theory $\mathcal{F}'$, for N odd, under fermion parity $\mathbb{Z}_2^F$.

$\mathcal{F}$	$\mathbb{Z}_2^F = +1$	$\mathbb{Z}_2^F = -1$
$\mathcal{H}_{\rm NS}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+2k,n}\mathcal{V}_{e+k,m+\frac{2k}{N}}$, $n=e+\frac{N}{2}m \mod N\,,\ k\in\mathbb{Z}_{N/2}$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+2k+1,n}\mathcal{V}_{e+k+\frac{1}{2},m+\frac{2k+1}{N}}$, $n=e+\frac{N}{2}m+\frac{N}{2} \mod N\,,\ k\in\mathbb{Z}_{N/2}$
$\mathcal{H}_{\rm R}$	$\mathcal{O}^{(k,1)}_{n,e,m} = \mathcal{S}_{n+2k+1,n}\mathcal{V}_{e+k+\frac{1}{2},m+\frac{2k+1}{N}}$, $n=e+\frac{N}{2}m \mod N\,,\ k\in\mathbb{Z}_{N/2}$	$\mathcal{O}^{(k,0)}_{n,e,m} = \mathcal{S}_{n+2k,n}\mathcal{V}_{e+k,m+\frac{2k}{N}}$, $n=e+\frac{N}{2}m+\frac{N}{2} \mod N\,,\ k\in\mathbb{Z}_{N/2}$

Table 4.4: Classification of the operators of the theory $\mathcal{F}$, for N even, under fermion parity $\mathbb{Z}_2^F$.

For N even we use (3.19) but we look at the fermionic theory $\mathcal{F}$. We get

$$Z_{\mathcal{F}}^{(WC)}[\rho_a,\rho_b] = \frac{1}{2}\sum_l (-1)^{l_a} Z_{\mathcal{B}'}^{(CW)}[l_a,l_b] e^{i\pi \mathrm{Arf}[l\cdot\rho]} = (-1)^{\rho_a} Z_{\mathcal{F}}^{(CW)}[\rho_a,\rho_b+1]\,, \qquad (4.38)$$

where l_a and l_b are the holonomies of the $\mathbb{Z}_2^L$ gauge field around the two cycles a and b of T^2. The comment in the last paragraph holds also in this case, so we have chosen $\mathcal{F}'$ and $\mathcal{F}$ for N odd and even, respectively, because these are the theories where the above D_8^F's symmetries are projectively realized in the R sector.

As we will discuss in detail in Section 4.3, the above $\mathbb{Z}_2$ anomaly has an important physical consequence. The fate of the quasi long-range order discussed in [1] for the chiral Gross-Neveu model at finite temperature depends crucially on the fermion periodicity along the thermal circle. In the NS sector (antiperiodic fermions) we expect that as soon as $T \neq 0$ quasi long-range order disappears due to thermal fluctuations (as generally expected in 2d models due to quantum mechanical tunneling effects). On the other hand, in the R sector (periodic fermions) the presence of a $\mathbb{Z}_2$ anomaly forbids a trivially gapped spectrum and a form of order is expected. Interestingly enough, this ordered phase should persist at *arbitrarily high* temperatures.

4.3 Inhomogeneities at $T > 0$

In a two dimensional theory, quasi long-range ordered phases are not detected directly from the one-point function of an order parameter, but rather from the slow decay of the two-point function of a "would-be order parameter" [55, 56]. For this reason, in order to explore the existence of inhomogeneous phases at finite temperature, we now consider a two-point function in the fermionic theory, and in particular its long distance behavior when the temperature is turned on. The goal of this calculation is two-fold: firstly, we show the loss of the order at finite temperature in the theory with antiperiodic conditions, and the persistence of the order with periodic conditions; secondly, we show that the addition of a finite chemical potential μ for the $U(1)_V$ symmetry induces a spatial modulation of the would-be order parameter, i.e. the finite-temperature version of the chiral spiral behavior found at zero temperature in Section 4.1. We work at finite μ, but everything that we say in this Section can also be applied to the special case of vanishing chemical potential. The role of the chemical potential is simply to mix the $U(1)_A$ symmetry with translations.

What are the possible local operators that can play the role of the would-be order parameter? The minimal requirement is that the operator is charged under $U(1)_A$. The most obvious candidate is the lightest scalar operator in this category, namely $\psi^\dagger_{+,a}\psi^a_-$. This is the operator we used to detect the chiral spiral order in $\mathbb{R}^2$ in Section 4.1. However, upon bosonization, this operator is mapped to a primary operator of the form $\mathrm{tr}[\mathcal{S}_{1,1}]\mathcal{V}_{1,0} = \mathrm{tr}[U]e^{i\phi/R}$ for N even, or $\mathrm{tr}[\mathcal{S}_{1,1}]\mathcal{V}_{2,0} = \mathrm{tr}[U]e^{2i\phi/R}$ for N odd,[8] and its two-point function receives contributions from various correlation functions of the operator $\mathrm{tr}[U]$ in the deformed $SU(N)_1$ theory, which is gapped and strongly coupled at large distances. In $\mathbb{R}^2$ the only required information about the correlation function of $\mathrm{tr}[U]$ could be deduced from the spontaneous breaking of the $\mathbb{Z}_N^L$ center symmetry. At finite temperature, instead, we need to include a sum over the insertions of the charge operator along the Euclidean time-like circle, and this requires substantial more information about the $SU(N)_1$ sector which is hard to obtain with the present tools, unless one restricts to the special cases $N = 1$ or $N = 2$. To circumvent this problem we consider instead the baryon (or "determinant") operator in the fermionic theory

$$\mathcal{O}_F = \frac{1}{N!}\epsilon^{a_1\dots a_N}\epsilon_{b_1\dots b_N}\psi^\dagger_{+a_1}\psi_-^{b_1}\cdots\psi^\dagger_{+a_N}\psi_-^{b_N} \equiv \det(\psi^\dagger_+\psi_-)\,, \tag{4.39}$$

which under bosonization is mapped to

$$\begin{aligned} \mathcal{O}_B &= \mathcal{S}_{0,0}\mathcal{V}_{N,0} \equiv \mathbf{1}\,e^{Ni\phi/R}\,, && N \text{ even}\,, \\ \mathcal{O}_B &= \mathcal{S}_{0,0}\mathcal{V}_{2N,0} \equiv \mathbf{1}\,e^{2Ni\phi/R}\,, && N \text{ odd}\,. \end{aligned} \tag{4.40}$$

This choice has the advantage that the part of the correlator due to the $SU(N)_1$ sector greatly simplifies, reducing essentially to the contribution from (possibly twisted) partition function in the absence of nontrivial operator insertions, and most of the calculation can be performed in the free $U(1)_N$ or $U(1)_{4N}$ sector, where exact results can be obtained.

Let us clarify the type of long-distance behavior that we expect at finite temperature, and what we mean by the order being lost or persisting at finite temperature. In the derivation of the chiral spiral order in the zero temperature case (see Section 4.1) an important role was played by the spontaneous breaking of the $\mathbb{Z}_N^L$ symmetry in the strongly coupled sector of the bosonized theory. The long-distance behavior of the two-point function on $\mathbb{R}^2$ combined the approach to a constant value in the $SU(N)_1$ sector, associated to $\mathbb{Z}_N^L$

[8] In this Section, for N odd we use the description in terms of the ($J\bar{J}$ deformed) $U(1)_{4N}$ compact scalar, i.e. $\mathcal{B}$ and not $\mathcal{B}'$.

breaking, and the power-law decay of the free scalar sector, associated to the quasi long-range order. When we turn on the temperature, effectively at large distances we are in a one-dimensional system, and the order is expected to be destroyed due to tunelling. As a result, barring the presence of anomalies in the effective quantum mechanics, generically we expect both of the behaviors to turn into an exponential decay on a scale fixed by the temperature. We find that this expectation is met for antiperiodic conditions: in this case the spatially modulated chiral spiral has an exponentially decaying amplitude. However, this exponential decay is absent instead for periodic conditions. We interpret this as the presence of an obstruction to tunneling that makes the order persist. This is ultimately a manifestation of the anomaly presented in the previous Section, that survives at finite temperature in the presence of periodic conditions.

There are some general considerations about the tunneling induced by the temperature in the strongly-coupled sector of the bosonic theory, that will allow us to greatly simplify the study of the two-point function. Upon adding the $J\bar{J}$ deformation, the $SU(N)_1$ WZW model develops a mass gap. On $\mathbb{R}^2$, its $\mathbb{Z}_N^L$ center symmetry gets spontaneously broken, with N degenerate vacua. In 2d any non-vanishing temperature $T > 0$ leads generally (but not always, as we will see) to symmetry restoration with a unique symmetric ground state $|\Omega_0\rangle$ since there are kink solutions with a finite energy M_{kink} interpolating between the N distinct vacua. We can obtain a finite action bounce just by taking the time-independent kink solution and wrapping it around the Euclidean time circle. This leads to $S_{\text{bounce}} = \beta M_{\text{kink}}$. Of course, as in any other dimension $d > 2$, symmetry restoration with a unique symmetric ground state generally occurs also by taking the space volume to be finite (at any T, including $T = 0$), in which case the bounce is given by $S_{\text{bounce}} = LM_{\text{kink}}$, where L is the length of the spatial dimension in 2d.[9] In the following we will denote by M the mass gap of the deformed $SU(N)_1$ WZW model. We then write $M_{\text{kink}} = cM$, with c an order one coefficient. We will be interested in the limit of low temperatures $\beta M \gg 1$, in which case the prefactor $K = c'M$, with c' another order 1 coefficient.

We study the two-point function both in the limit of low temperatures and large spatial distances $|x|M \gg 1$. To perform the calculation, especially in the compact scalar sector, it is convenient to study the model on a torus T^2 with modulus $\tau = it$ and then take the limit $L \to \infty$ where the spatial dimension becomes non-compact.

4.3.1 N odd

The $SU(N)_1$ partition function with insertion of $\mathbb{Z}_N^L$ topological lines on both cycles, in the limit discussed above, behaves as follows,

$$Z_{SU(N)_1 + J\bar{J}}[t_a, t_b] \sim C \times \delta_{t_b,0} \, . \tag{4.41}$$

Here the first entry t_a denotes the insertion of topological lines wrapping the circle of length β, while t_b denotes lines wrapping the circle of length L (equivalently, t_a denotes the $\mathbb{Z}_N^L$ holonomy along the circle of length L and t_b the one along the circle of length β). The independence on t_a in our limit is the consequence of the $\mathbb{Z}_N^L$-invariance of the ground state $|\Omega_0\rangle$ in the quantization where the β circle is regarded as space. The fact that only $t_b = 0$ contributes is simply the statement that sectors with $t_b \neq 0$, as is manifest in this

[9] If on $\mathbb{R}^2$ we have a discrete $\mathbb{Z}_N$ symmetry breaking, we expect however $N-1$ almost-degenerate states $|\Omega_k\rangle$, $k = 1, \ldots, N-1$, whose gap with respect to $|\Omega_0\rangle$ is of order $\Delta E_k \sim Ke^{-S_{\text{bounce}}}$.

different quantization, have an energy gap above the ground state and decouple when we take $L \to \infty$. C is an unknown constant which we will not need to fix.

We can use the identities (3.16), (3.17), and (3.69) to express correlation functions in the fermionic theory in terms of $SU(N)_1$ and $U(1)_{4N}$ correlation functions. Thanks to (4.41), the partition function and the two-point function of the determinant operator further simplify to

$$\begin{aligned} Z_{\mathcal{F}'}[\rho] &\sim \frac{C}{2N} \sum_{s\in H^1(T^2,\mathbb{Z}_2)} (-1)^{\mathrm{Arf}[s\cdot\rho]-\mathrm{Arf}[\rho]} \sum_{t_a=0}^{N-1} Z_{U(1)_{4N}+J\bar{J}}[T(t_a,s)] \ , \\ Z_{\mathcal{F}'}[\rho]\langle \mathcal{O}_F(z,\bar{z})\mathcal{O}_F^\dagger(0,0)\rangle_\rho &\sim \frac{C}{2N} \sum_{s\in H^1(T^2,\mathbb{Z}_2)} (-1)^{\mathrm{Arf}[s\cdot\rho]-\mathrm{Arf}[\rho]} \times \\ &\times \sum_{t_a=0}^{N-1} Z_{U(1)_{4N}+J\bar{J}}[T(t_a,s)] \,\langle e^{2iN\phi/R}(z,\bar{z}) e^{-2iN\phi/R}(0,0)\rangle_{T(t_a,s)} \, . \end{aligned} \tag{4.42}$$

The subscript in the two-point function and the argument in square brackets in the partition function both denote that these quantities are computed with line insertions along the two cycles, corresponding to the two entries of the vector $T(t_a,s) \equiv 2(N+1)(t_a,0)+2Ns$. This is the background appearing in (3.69) with the renaming $[k_\psi,\ell_\psi] = s$ and $k = t_a$, $\ell = t_b$ which is set to 0. Thanks to (4.41) the gauging of the $\mathbb{Z}_N$ diagonal between the WZW and the compact boson is implemented through a single sum over the remaining label t_a. Finally the sum over the background s for $\mathbb{Z}_2^P$ implements the fermionization, and the choice of spin structure is denoted by ρ. Note that the $J\bar{J}$ deformation in the $U(1)_{4N}$ sector modifies the value of the radius R according to (3.15).

The quantities appearing in (4.42) can be computed exactly, since the summands are correlators in a free CFT. The computation is performed in full generality in Appendix B on the torus. We compute the expression on a rectangular torus of modular parameter $\tau = i\beta/L$. We will compute the two-point function of generic vertex operator of charge e, namely $e^{ie\phi/R}$, and then set $e=0$ to obtain $Z_{\mathcal{F}'}$ and $e=2N$ to obtain $Z_{\mathcal{F}'}\langle \mathcal{O}_F\mathcal{O}_F^\dagger\rangle$. Given that we are interested in the limit of high temperature $T=\beta^{-1}\gg 1$, it makes sense to consider correlation functions with operators inserted at equal times and a spatial distance x. The result is

$$\begin{aligned} &\sum_{t_a} Z_{U(1)_{4N}+J\bar{J}}[T(t_a,s)]\langle e^{ie\phi/R}(z,\bar{z})e^{-ie\phi/R}(0,0)\rangle_{T(t_a,s)} \\ &= \left|\frac{\theta_1'(0|iLT)}{\theta_1(-ixT|iLT)}\right|^{2e^2/R^2} \frac{N e^{4i\tilde{\mu}Nex/R^2}}{|\eta(iLT)|^2}\times \\ &\qquad \times F_1^s\left(\frac{2\tilde{\mu}N^2L}{\pi R^2} - \frac{2ieNxT}{R^2}\middle|\frac{2iLTN^2}{R^2}\right) F_2^s\left(0\middle|\frac{iLTR^2}{2}\right) \, . \end{aligned} \tag{4.43}$$

where

$$\begin{aligned} &F_1^{(0,0)} = F_2^{(0,0)} = \theta_3 \\ &F_1^{(0,1)} = \theta_3 \ , \quad F_2^{(0,1)} = \theta_2 \\ &F_1^{(1,0)} = \theta_4 \ , \quad F_2^{(1,0)} = \theta_3 \\ &F_1^{(1,1)} = \theta_4 \ , \quad F_2^{(1,1)} = \theta_2 \ , \end{aligned} \tag{4.44}$$

and $\tilde{\mu}$ is related to the chemical potential μ appearing in the action (4.1) by a rescaling,

$$\tilde{\mu} = \mu \sqrt{1 + \frac{\lambda'}{2\pi N}}\,, \tag{4.45}$$

such that $2\tilde{\mu}N$ is the chemical potential for the $U(1)_W$ current at radius R. The details of why the chemical potential induces a spatial modulation of the two-point function are very similar to the zero temperature case studied at $T = 0$ and they are spelled out in Appendix B.4.

The next step is to plug (4.43) back in (4.42) and perform the fermionization sum. We set $u \equiv \frac{2\tilde{\mu}N^2 L}{\pi R^2} - \frac{2ieNxT}{R^2}$. The result of the sum is

$$\begin{aligned} Z_F[\rho]\langle \mathcal{O}_F(z,\bar{z})\mathcal{O}_F^\dagger(0,0)\rangle_\rho \sim \frac{C}{2}\left|\frac{\theta_1'(0|iLT)}{\theta_1(-ixT|iLT)}\right|^{2e^2/R^2} \frac{e^{4i\tilde{\mu}Nex/R^2}}{|\eta(iLT)|^2}\times \\ \times\left[\theta_3\left(0\middle|\tfrac{iLTR^2}{2}\right)(\theta_3 + w_1^\rho\theta_4)\left(u\middle|\tfrac{2iLTN^2}{R^2}\right) + w_2^\rho\theta_2\left(0\middle|\tfrac{iLTR^2}{2}\right)(\theta_3 + w_3^\rho\theta_4)\left(u\middle|\tfrac{2iLTN^2}{R^2}\right)\right]\,. \end{aligned} \tag{4.46}$$

where $w^\rho_{1,2,3}$ are the following signs which depend on the spin structure[10]

$$\begin{aligned} w_1^{[\mathrm{NS},\,\mathrm{NS}]} &= +1\,,\ w_2^{[\mathrm{NS},\,\mathrm{NS}]} = +1\,,\ w_3^{[\mathrm{NS},\,\mathrm{NS}]} = -1\,, \\ w_1^{[\mathrm{R},\,\mathrm{NS}]} &= +1\,,\ \ w_2^{[\mathrm{R},\,\mathrm{NS}]} = -1\,,\ \ w_3^{[\mathrm{R},\,\mathrm{NS}]} = -1\,, \\ w_1^{[\mathrm{NS},\,\mathrm{R}]} &= -1\,,\ \ w_2^{[\mathrm{NS},\,\mathrm{R}]} = +1\,,\ \ w_3^{[\mathrm{NS},\,\mathrm{R}]} = +1\,, \\ w_1^{[\mathrm{R},\,\mathrm{R}]} &= -1\,,\ \ \ w_2^{[\mathrm{R},\,\mathrm{R}]} = -1\,,\ \ \ w_3^{[\mathrm{R},\,\mathrm{R}]} = -1\,. \end{aligned} \tag{4.47}$$

To take $L \to \infty$ we use the following asymptotic expansions for $t \to i\infty$,

$$\begin{aligned} \theta_1(u|it) &\underset{t\to i\infty}{\sim} 2e^{-\pi t/4}\sin(\pi u)\,, & \theta_2(u|it) &\underset{t\to i\infty}{\sim} 2e^{-\pi t/4}\cos(\pi u)\,, \\ \theta_3(u|it) &\underset{t\to i\infty}{\sim} 1 + 2e^{-\pi t}\cos(2\pi u)\,, & \theta_4(u|it) &\underset{t\to i\infty}{\sim} 1 - 2e^{-\pi t}\cos(2\pi u)\,, \\ \theta_1'(0|it) &\underset{t\to i\infty}{\sim} 2\pi e^{-\pi t/4}\,, & \eta(it) &\underset{t\to i\infty}{\sim} e^{-\pi t/12}\,. \end{aligned} \tag{4.48}$$

Plugging these asymptotics in the result (4.46), we obtain

$$\begin{aligned} Z_{\mathcal{F}'}[\rho]\langle \mathcal{O}_F(z,\bar{z})\mathcal{O}_F^\dagger(0,0)\rangle_\rho \sim C\left|\frac{\pi}{\sinh(\pi xT)}\right|^{2e^2/R^2} e^{4i\tilde{\mu}Nex/R^2}e^{\pi LT/6}\times \\ \times\begin{cases} 1 + \mathcal{O}(e^{-\beta M}), & \rho = [\text{any, NS}]\,, \\ 2e^{-2\pi LTN^2/R^2}\cosh(4\pi eNxT/R^2 + 4i\tilde{\mu}N^2L/R^2) + \mathcal{O}(e^{-\beta M}), & \rho = [\text{any, R}]\,. \end{cases} \end{aligned} \tag{4.49}$$

Note that upon taking $L \to \infty$ we lose dependence on the periodicity condition along the cycle of length L, but we retain dependence on the periodicity on the cycle of length $\beta = T^{-1}$. Evaluating for $e = 0$ this gives

$$Z_{\mathcal{F}'}[\rho] \sim Ce^{\pi LT/6}\begin{cases} 1 + O(e^{-\beta M}), & \rho = [\text{any, NS}]\,, \\ 2e^{-2\pi LTN^2/R^2}\cos(4\tilde{\mu}N^2L/R^2) + \mathcal{O}(e^{-\beta M}), & \rho = [\text{any, R}]\,, \end{cases} \tag{4.50}$$

[10]We remind that $\rho = [\rho_a, \rho_b]$ denotes the periodicity along the a cycle (of length L) and the b cycle (of length β), respectively.

and taking the ratio between the results for $e = 2N$ and $e = 0$ we get the result for the two-point function

$$\begin{aligned}
&\langle \mathcal{O}_F(z,\bar z)\mathcal{O}_F^\dagger(0,0)\rangle_\rho \\
&\underset{|x|T\to\infty}{\sim} (2\pi)^{8N^2/R^2} e^{8N^2 i\tilde\mu x/R^2} \begin{cases} e^{-8N^2\pi|x|T/R^2} + O(e^{-\beta M})\,, & \rho = [\text{any, NS}]\,, \\ \frac{1}{2} + O(e^{-\beta M} e^{-8N^2\pi|x|T/R^2})\,, & \rho = [\text{any, R}]\;. \end{cases}
\end{aligned} \tag{4.51}$$

In the last equation, in addition to the limit explained above, we also took the large-distance limit $|x|T \to \infty$.

Summarizing, we see the following behavior:

- for thermal fermions, i.e. fermions with antiperiodic conditions $\rho = [\text{any, NS}]$, the correlation function of the operator $\mathcal{O}_F = \det(\psi_+^\dagger \psi_-)$ vanishes at large $|x|$ exponentially with the temperature
$$\langle \mathcal{O}_F(z,\bar z)\mathcal{O}_F^\dagger(0,0)\rangle_{NS} \sim e^{-|x|/\xi_T} e^{2iN\mu' x}\,, \qquad \xi_T^{-1} = 2N\pi T\left(1 + \frac{\lambda'}{2\pi N}\right)\,, \tag{4.52}$$
where we used that $R = \sqrt{4N\left(1+\frac{\lambda'}{2\pi N}\right)}$ and we set $\mu' = \mu\left(1+\frac{\lambda'}{2\pi N}\right)^{-1/2}$, so the modulation is visible only at intermediate scales, provided that $N\mu' \gtrsim \pi\xi_T^{-1}$;
- for periodic fermions, i.e. with conditions $\rho = [\text{any, R}]$, the same correlation function does not decay, and the spatial modulation due to the chemical potential μ is visible at large distances even at finite temperature,
$$\langle \mathcal{O}_F(z,\bar z)\mathcal{O}_F^\dagger(0,0)\rangle_R \sim e^{2iN\mu' x}\,. \tag{4.53}$$

4.3.2 N even

The $SU(N)_1$ partition function with insertion of $\mathbb{Z}_{N/2}^L$ and possibly also $\mathbb{Z}_2^L$ topological lines on both cycles behaves as follows in the limit

$$\begin{aligned}
Z_{SU(N)_1+J\bar J}[2t_a, 2t_b] &\sim C \times \delta_{t_b,0}\;, \\
Z_{SU(N)_1+J\bar J}[2t_a+s_a, 2t_b+s_b] &\sim C \times \delta_{t_b,0} Z^{\pm}[s_a,s_b]\;.
\end{aligned} \tag{4.54}$$

where

$$Z^{\pm}[s_a,s_b] = (\pm 1)^{s_a s_b}\;, \tag{4.55}$$

is a sign that remains ambiguous due to the anomaly in $\mathbb{Z}_2^L$ in the $SU(N)_1$ theory. The independence from t_a and the projection to $t_b = 0$ have the same explanation as in the case of N odd, and again C denotes an undetermined constant.

Plugging (4.54) in the fermionization formula that can be derived from (3.16), (3.17), and (3.52), we obtain

$$\begin{aligned}
Z_{\mathcal{F}'}[\rho] &\sim \frac{C}{N} \sum_{s\in H^1(T^2,\mathbb{Z}_2)} (-1)^{\mathrm{Arf}[s\cdot\rho]-\mathrm{Arf}[\rho]} Z^{\pm}[s] \sum_{t_a=0}^{N/2-1} Z_{U(1)_N+J\bar J}[T(t_a,s)]\,, \\
Z_{\mathcal{F}'}[\rho]\langle \mathcal{O}_F(z,\bar z)\mathcal{O}_F^\dagger(0,0)\rangle_\rho &\sim \frac{C}{N} \sum_{s\in H^1(\mathbb{Z}_2)} (-1)^{\mathrm{Arf}[s\cdot\rho]-\mathrm{Arf}[\rho]} Z^{\pm}[s]\times \\
&\times \sum_{t_a=0}^{N/2-1} Z_{U(1)_N+J\bar J}[T(t_a,s)]\langle e^{iN\phi/R}(z,\bar z) e^{-iN\phi/R}(0,0)\rangle_{T(t_a,s)}\,.
\end{aligned} \tag{4.56}$$

where the notation follows the same conventions as in the case of N odd, explained under (4.42), and $T(t_a, s) = 2(t_a, 0) + s$ is the background appearing in (3.52) with the renaming $[k_\psi, \ell_\psi] = s$ and $k = 2t_a$, $\ell = t_b$ which is set to 0. Also for N even the radius R in the $U(1)_N$ sector is modified according to (3.15).

Like we did in the previous Section, we proceed by performing exactly the part of the calculation that involves the free scalar sector. Again, we compute for generic charge e of the vertex operator, and this will give the partition function for $e = 0$, and the product of the partition function with the two-point function for $e = N$. The result is

$$\begin{aligned}
&\sum_{t_a} Z_{U(1)_N + J\bar{J}}[T(t_a, s)] \langle e^{ie\phi/R}(z,\bar{z}) e^{-ie\phi/R}(0,0)\rangle_{T(t_a,s)} \\
&= \left| \frac{\theta_1'(0|iLT)}{\theta_1(-ixT|iLT)} \right|^{2e^2/R^2} \frac{N e^{2i\tilde{\mu}Nex/R^2}}{2|\eta(iLT)|^2} \times \\
&\qquad \times F_1^s \left(\frac{\tilde{\mu}N^2 L}{2\pi R^2} - \frac{ieNxT}{R^2} \middle| \frac{iLTN^2}{2R^2} \right) F_2^s \left(0 \middle| \frac{iLTR^2}{2} \right) ,
\end{aligned} \tag{4.57}$$

where

$$\begin{aligned}
F_1^{(0,0)} &= F_2^{(0,0)} = \theta_3 \ , \\
F_1^{(0,1)} &= F_2^{(0,1)} = \theta_2 \ , \\
F_1^{(1,0)} &= F_2^{(1,0)} = \theta_4 \ , \\
\pm F_1^{(1,1)} &= F_2^{(1,1)} = \theta_1 \ ,
\end{aligned} \tag{4.58}$$

and again $\tilde{\mu} = \mu\sqrt{1 + \frac{\lambda'}{2\pi N}}$. Note that the $s = (1,1)$ contribution, which seems to have an ambiguous sign, actually vanishes. This means that $Z^{\pm}[s]$ contributes always 1 and can be neglected.

Performing the fermionization sum over s we get

$$\begin{aligned}
Z_{\mathcal{F}'}[\rho] \langle \mathcal{O}_F(z,\bar{z}) \mathcal{O}_F^\dagger(0,0)\rangle_\rho &\sim \frac{C}{2} \left| \frac{\theta_1'(0|iLT)}{\theta_1(-ixT|iLT)} \right|^{2e^2/R^2} \frac{e^{2i\tilde{\mu}Nex/R^2}}{|\eta(iLT)|^2} \times \\
\times \Big[\theta_3 \left(0 \middle| \tfrac{iLTR^2}{2} \right) \theta_3 \left(u \middle| \tfrac{iLTN^2}{2R^2} \right) &+ w_1^\rho \theta_4 \left(0 \middle| \tfrac{iLTR^2}{2} \right) \theta_4 \left(u \middle| \tfrac{iLTN^2}{2R^2} \right) \\
&+ w_2^\rho \theta_2 \left(0 \middle| \tfrac{iLTR^2}{2} \right) \theta_2 \left(u \middle| \tfrac{iLTN^2}{2R^2} \right) \Big] \ ,
\end{aligned} \tag{4.59}$$

where we set $u \equiv \frac{\tilde{\mu}N^2 L}{2\pi R^2} - \frac{ieNxT}{R^2}$ and $w_{1,2}^\rho$ are the following signs that depend on the spin structure

$$\begin{aligned}
w_1^{[\mathrm{NS,\,NS}]} &= +1 \, , \; w_2^{[\mathrm{NS,\,NS}]} = +1 \ , \\
w_1^{[\mathrm{R,\,NS}]} &= +1 \, , \quad w_2^{[\mathrm{R,\,NS}]} = -1 \ , \\
w_1^{[\mathrm{NS,\,R}]} &= -1 \, , \quad w_2^{[\mathrm{NS,\,R}]} = +1 \ , \\
w_1^{[\mathrm{R,\,R}]} &= -1 \, , \quad w_2^{[\mathrm{R,\,R}]} = -1 \ .
\end{aligned} \tag{4.60}$$

We then use (4.48) to take the limit $L \to \infty$, obtaining

$$Z_{\mathcal{F}'}[\rho]\langle \mathcal{O}_F(z,\bar{z})\mathcal{O}_F^\dagger(0,0)\rangle_\rho \sim C\left|\frac{\pi}{\sinh(\pi x T)}\right|^{2e^2/R^2} e^{2i\tilde{\mu}Nex/R^2} e^{\pi LT/6}\times \\ \times \begin{cases} 1+\mathcal{O}(e^{-\beta M}), & \rho = [\text{any}\,,\ \text{NS}]\,, \\ 2e^{-\pi LTN^2/2R^2}\cosh(2\pi eNxT/R^2 + i\tilde{\mu}N^2L/R^2) + \mathcal{O}(e^{-\beta M}), & \rho = [\text{any}\,,\ \text{R}]\,. \end{cases} \tag{4.61}$$

Plugging $e = 0$, the result for the partition function is

$$Z_{\mathcal{F}'}[\rho] \sim Ce^{\pi LT/6}\begin{cases} 1+O(e^{-\beta M}), & \rho = [\text{any}\,,\ \text{NS}]\,, \\ 2e^{-\pi LTN^2/2R^2}\cos(\tilde{\mu}N^2L/R^2) + O(e^{-\beta M}), & \rho = [\text{any}\,,\ \text{R}]\,, \end{cases} \tag{4.62}$$

while taking the ratio between $e = N$ and $e = 0$ we obtain for the two-point function

$$\langle \mathcal{O}_F(z,\bar{z})\mathcal{O}_F^\dagger(0,0)\rangle_\rho \underset{|x|T\to\infty}{\sim} (2\pi)^{2N^2/R^2} e^{2i\tilde{\mu}N^2x/R^2}\times \\ \times \begin{cases} e^{-2\pi|x|N^2T/R^2} + \mathcal{O}(e^{-\beta M})\,, & \rho = [\text{any}\,,\ \text{NS}]\,, \\ \frac{1}{2} + \mathcal{O}(e^{-\beta M}e^{-2\pi|x|N^2T/R^2})\,. & \rho = [\text{any}\,,\ \text{R}]\ . \end{cases} \tag{4.63}$$

In the last equation, we also took $|x|T \to \infty$ and kept only the large-distance asymoptotic behavior. Upon substituting $R = \sqrt{4N\left(1+\frac{\lambda'}{2\pi N}\right)}$ and setting $\mu' = \mu\left(1+\frac{\lambda'}{2\pi N}\right)^{-1/2}$, we see that this is exactly the same behavior as in the case of N odd, summarized in equations (4.52)-(4.53).

Chapter 5

The $N = \infty$ Phase Diagram with Feynman Diagrams

The chiral Gross-Neveu model in the large N limit is conveniently studied by introducing a complex Hubbard-Stratonovich (HS) field Δ rather than reformulating the theory as a $J\bar{J}$ deformation of a WZW model. At large N the $\mathbb{Z}_N^L$ symmetry can be spontaneously broken even at finite T, because the usual no-go theorems do not apply in this limit. In this Chapter we derive the critical temperature T_c of the chiral Gross-Neveu model, reproducing the result of [47–50] with a different method, and we extend them also to the case of fermions with periodic boundary conditions along the thermal cycle. To this end, we compute the free energy density per flavour (which for simplicity is called free energy in what follows) both for a homogeneous condensate and for a inhomogeneous one, assumed to have the same chiral spiral form found at finite N for $T = 0$. We show that, for thermal fermions, at low temperatures $T < T_c$ the chiral spiral configuration minimizes the free energy also at large N; for $T > T_c$ the symmetry-preserving configuration, in which fermions are massless, is recovered. For periodic fermions, instead, we find again persistent order in the large-N limit, as hinted at in [45], consistently with our finite N analysis; in particular, we find that the chiral spiral configuration always minimizes the free energy.

5.1 Thermal Fermions

We take the 't Hooft limit $N \to \infty$ with λ_s, λ_v fixed in (1.22). To obtain the usual description of chiral Gross-Neveu model at large N we set $\lambda_v = 0$. The free energy of the chiral Gross-Neveu model at large N is given by

$$F(\Delta) = \frac{\overline{|\Delta|^2}}{\lambda_s} - \mathrm{Tr}\log\left(\partial\!\!\!/ + \Delta P_+ + \Delta^* P_-\right)\,, \tag{5.1}$$

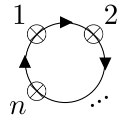
where $\overline{|\Delta|^2}$ denotes the spacetime average of the square modulus of the condensate. It is not known how to perform the minimization in full generality, and in practice one has to minimize within a given ansatz for the functional form of Δ.

Let us consider homogeneous configurations $\Delta(x) = M$, and let us minimize the free energy as a function of M. We can perform a perturbative expansion in the coupling λ_s, or equivalently in M. Neglecting irrelevant constant terms, we have

$$F(M) = \frac{M^2}{\lambda_s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}(\not{\partial}^{-1} M)^n \,. \tag{5.2}$$

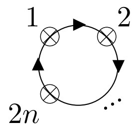
On the other hand,

$$- \mathrm{Tr}\left(\not{\partial}^{-1} M\right)^n = \quad , \tag{5.3}$$



where oriented lines denote massless fermion propagators and crosses $\otimes$ denote M insertions. By gamma matrix algebra the diagram is nonzero only when n is even. Resumming the insertions, at $T = \mu = 0$ one has

$$\sum_{n=1}^{\infty} \frac{1}{2n} \quad = \int \frac{d^2 p}{(2\pi)^2} \log\left(\frac{p^2 + M^2}{p^2}\right) \,. \tag{5.4}$$



At $T > 0$ we need to replace $\int \frac{dp_2}{2\pi} \to T \sum_{p_2}$, with $p_2 = \pi(2n+1)T$, and at $\mu \neq 0$ we have $p_2 \to p_2 + i\mu \equiv p_{2+}$. Performing the integral over p_1, we get

$$\begin{aligned} F(M) = M^2 &\left(\frac{1}{\lambda_s} - \frac{T}{2} \sum_{p_2} \frac{1}{\sqrt{p_{2+}^2}} \right) \\ &- T \sum_{p_2} \left(\sqrt{p_{2+}^2 + M^2} - \sqrt{p_{2+}^2} - \frac{M^2}{2\sqrt{p_{2+}^2}} \right) \,. \end{aligned} \tag{5.5}$$

The second sum is manifestly finite. Also the first sum is finite, because λ_s is the bare 't Hooft coupling. We can trade the bare coupling for a UV cutoff Λ via the $T = \mu = 0$ gap equation (1.26),

$$\frac{1}{\lambda_s} = \int_{-\Lambda}^{\Lambda} \frac{dp_2}{2\pi} \int \frac{dp_1}{2\pi} \frac{1}{p^2 + M_0^2} \approx \frac{1}{2\pi} \log\left(\frac{2\Lambda}{M_0}\right) \,, \tag{5.6}$$

where M_0 is the mass of the fermions at $T = \mu = 0$. We can define a renormalization scheme by putting a cutoff $p_{2,\max} = \pi(2n_{\max} - 1)T$ over the Matsubara frequencies and relate it to the UV cutoff as $\Lambda = 2\pi n_{\max} T$. In this way (5.5) is finite and we can safely remove both cut-offs by taking $n_{\max} \to \infty$. We then get

$$\begin{aligned} F(M) = \frac{M^2}{2\pi} &\left(\log\left(\frac{4\pi T}{M_0}\right) + \mathrm{Re}\,\psi\left(\frac{1}{2} + i\frac{\mu}{2\pi T}\right) \right) \\ &- T \sum_{p_2} \left(\sqrt{p_{2+}^2 + M^2} - \sqrt{p_{2+}^2} - \frac{M^2}{2\sqrt{p_{2+}^2}} \right) \,, \end{aligned} \tag{5.7}$$

where ψ is the digamma function. The minimization over M is performed as follows. Extremizing with respect to M, one obtains an implicit expression

$$\log\left(\frac{4\pi T}{M_0}\right) + \mathrm{Re}\,\psi\left(\frac{1}{2} + \frac{i\mu}{2\pi T}\right) - 2\pi T \mathrm{Re} \sum_{p_2 > 0} \left(\frac{1}{\sqrt{M^2 + p_{2+}^2}} - \frac{1}{\sqrt{p_{2+}^2}} \right) = 0 \,, \tag{5.8}$$

which is inverted numerically at fixed (μ, T). Then, the resulting values for M are plugged back into F to obtain $F_{\text{hom}} = \min_M F(M)$, which is plotted as a function of (μ, T) see Figure 5.1.

Let us now consider spiral configurations. Under a $U(1)_A$ transformation the HS field $\Delta = \rho\, e^{i\theta}$ transforms as (1.27),

$$U(1)_A : \qquad \rho \mapsto \rho \ , \qquad \theta \mapsto \theta + 2\alpha \ , \qquad \alpha \in \mathbb{Z} \ . \tag{5.9}$$

At large N we can then identify (cf. (3.10))

$$\theta = \frac{\phi}{\sqrt{N}} \ , \tag{5.10}$$

where ϕ is the compact scalar in the bosonization of the model at $T = 0$, c.f. (3.10). In terms of the HS field Δ, the chiral spiral configuration (1.42) reads

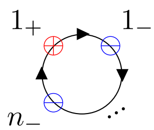
$$\Delta(x) = M e^{2iqx} \ , \tag{5.11}$$

where we expect $q = \mu$ at large N (at fixed λ_s and $\lambda_v = 0$), and we denote $x \equiv x^1$. Let us however keep q generic and minimize the free energy $F(M, q)$ over these configurations, to prove that indeed this is the case. We have

$$F(M, q) = \frac{M^2}{\lambda_s} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \times \mathrm{Tr}[\not{\partial}^{-1} M (e^{2iqx} P_+ + e^{-2iqx} P_-)]^n \ . \tag{5.12}$$

As before, we can interpret the traces diagrammatically, as each insertion of $P_\pm$, denoted by $\oplus, \ominus$ respectively, brings an insertion of mass M and spatial momentum $\mp 2q$. By momentum conservation in the loop there has to be equal number of $P_\pm$ insertions, which must be alternated because $P_\pm \gamma^\mu P_\pm = 0$. Therefore,

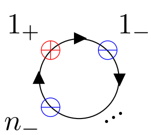
$$-\,\mathrm{Tr}\log\left[\not{\partial}^{-1} M (e^{2iqx} P_+ + e^{-2iqx} P_-)\right]^{2n} = 2$$



$$, \tag{5.13}$$

where the extra factor of 2 comes from $P_+ \leftrightarrow P_-$. Letting $p_{1\mp} = p_1 \mp 2q$, at $T = \mu = 0$ one has

$$\sum_{n=1}^{\infty} \frac{1}{n}$$



$$= \int \frac{d^2 p}{(2\pi)^2} \log\left(1 + \frac{M^2(p_1^2 + p_2^2 - q^2 + 2iqp_2)}{(p_{1-}^2 + p_2^2)(p_{1+}^2 + p_2^2)}\right) . \tag{5.14}$$

At nonzero T and μ, after p_1 integration we have

$$F(M, q) = \frac{M^2}{\lambda_s} - T \sum_{p_2} \left(\sqrt{M^2 + p_{2+}^2(q)} - \sqrt{p_{2+}^2(q)}\right) , \tag{5.15}$$

where $p_{2+}(q) = \pi(2n+1)T + i(\mu - q)$. It is immediate to verify that the configuration with

$$q = \mu \ , \tag{5.16}$$

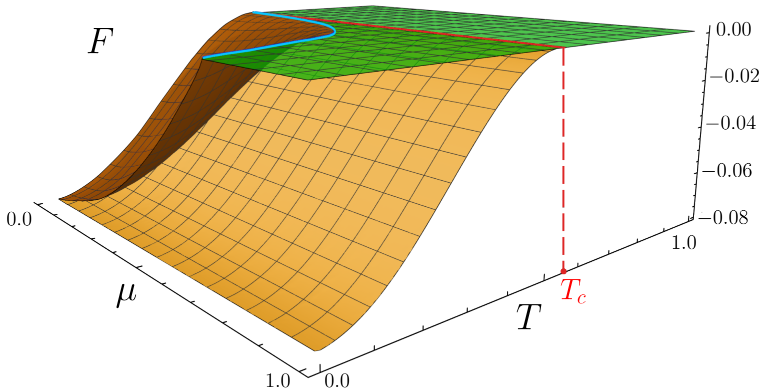


Figure 5.1: The large N free energy F as a function of μ and T, in units of M_0. At low T, the chiral spiral configuration (yellow) is favored with respect to the homogeneous configuration (brown) and the chirally symmetric phase (green). At $T = T_c$, there is a second order phase transition (red line) dividing the chiral spiral phase from the chirally symmetric phase. Assuming homogeneity one finds a different phase transition line (blue line).

achieves minimization of $F(M, q)$ with respect to q. Proceeding as in the homogeneous configuration, for $q = \mu$, we then get

$$F(M,\mu) = \left[\frac{M^2}{2\pi}\left(\log\left(\frac{\pi T}{M_0}\right) - \gamma\right) - T\sum_{p_2}\left(\sqrt{p_2^2 + M^2} - |p_2| - \frac{M^2}{2|p_2|}\right)\right], \qquad (5.17)$$

where γ is the Euler-Mascheroni constant. We see that the μ-dependence drops completely in $F(M,\mu)$, and that

$$F_{\text{cs}} = \min_{M,q} F(M,q) = F_{\text{hom}}(\mu = 0)\,. \qquad (5.18)$$

We compare the minimum of the free energy in the homogeneous and inhomogeneous configurations in Fig. 5.1. We see that the chiral spiral configuration $\Delta = M(T)e^{2i\mu x}$ is always favored with respect to the homogeneous configuration $\Delta = M(\mu, T)$; moreover, one has $M(T) = 0$, i.e. the symmetric massless phase is recovered, for

$$T \geq T_c = \frac{e^\gamma}{\pi} M_0\,, \qquad (5.19)$$

where a second order phase transition occurs.[1]

5.2 Periodic Fermions and Persistent Order

We can repeat the diagrammatic argument for the case of fermions with periodic boundary conditions along the thermal cycle. This is equivalent to choosing a different spin structure for the spacetime manifold. Let us start from the expression of the free energy density per flavor,

$$F(M,q) = \frac{M^2}{\lambda_s} - T\sum_{p_2}\left(\sqrt{M^2 + p_2^2(q)} - \sqrt{p_2^2(q)}\right), \qquad (5.20)$$

[1]Note that the value of T_c is renormalization scheme-dependent.

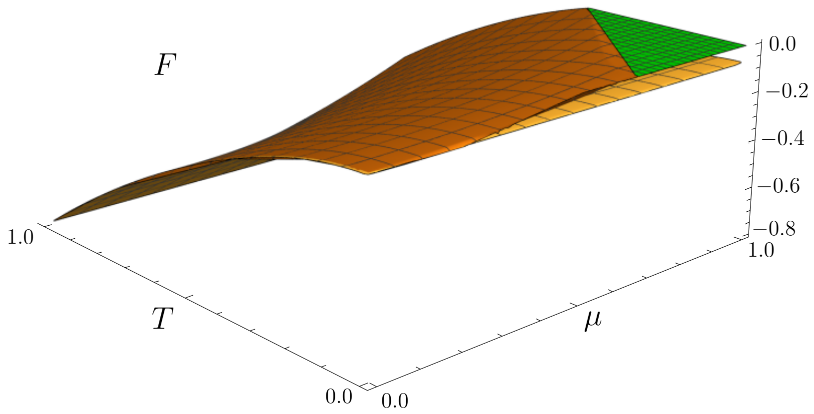


Figure 5.2: The large N free energy F (for periodic fermions) as a function of μ and T, in units of M_0. The massless configuration (green) is favored only at high μ with respect to the homogeneous configuration (orange), but the spiral configuration (yellow) is always favored with respect to both of them. As a consequence, symmetry is never restored at high temperature.

where now $p_2(q) = 2\pi nT + i(\mu - q)$, $n \in \mathbb{Z}$, in order to impose periodicity. As in the case of thermal fermions, this expression contains divergences that need to be regularized. It is convenient to separate the contribution from the zero-mode in the sum,

$$\begin{aligned} F(M,q) = \frac{M^2}{\lambda_s} &- T\left(\sqrt{M^2 - (\mu-q)^2} - \sqrt{-(\mu-q)^2}\right) \\ &- 2T\,\mathrm{Re}\sum_{p_2>0}\left(\sqrt{M^2 + p_2^2(q)} - \sqrt{p_2^2(q)}\right) . \end{aligned} \tag{5.21}$$

We proceed similarly as in the antiperiodic case. We use the $T = \mu = 0$ gap equation, to trade the dependence on the bare coupling λ_s for the UV cutoff Λ. Similarly, we regularize the sum by placing a cutoff $p_{2,\max} = 2\pi n_{\max} T$ over the Matsubara frequencies and relate it to the UV cutoff by letting $\Lambda = p_{2,\max}$. Removing the cutoff, we are left with

$$\begin{aligned} F(M,q) = \frac{M^2}{2\pi}&\left[\log\left(\frac{4\pi T}{M_0}\right) + \mathrm{Re}\,\psi\left(1 + \frac{i(\mu-q)}{2\pi T}\right)\right] \\ &- T\left(\sqrt{M^2 - (\mu-q)^2} - \sqrt{-(\mu-q)^2}\right) \\ &- 2T\,\mathrm{Re}\sum_{p_2>0}\left(\sqrt{M^2 + p_2^2(q)} - \sqrt{p_2^2(q)} - \frac{M^2}{2\sqrt{p_2^2(q)}}\right) . \end{aligned} \tag{5.22}$$

We now compare the minimum of F over the whole parameter space with the one computed assuming translational invariance ($q = 0$). In the former case, q-minimization is obtained for $q = \mu$, as in the antiperiodic setup. We can extremize with respect to M to get an implicit expression for M,

$$\log\left(\frac{4\pi T}{M_0}\right) + \mathrm{Re}\,\psi\left(1 + \frac{i(\mu - q)}{2\pi T}\right) - \pi T\,\mathrm{Re}\,\frac{1}{\sqrt{M^2 - (\mu - q)^2}} \\ - 2\pi T\,\mathrm{Re}\sum_{p_2>0}\left(\frac{1}{\sqrt{M^2 + p_2^2(q)}} - \frac{1}{\sqrt{p_2^2(q)}}\right) = 0\,, \tag{5.23}$$

and use this expression to get the value of the free energy numerically, both for $q = 0$ and for $q = \mu$. The results are plotted in Figure 5.2. We see that, as in the antiperiodic setup, the spiral configuration is always favored with respect to the homogeneous one. This time, however, the $M = 0$ configuration is never favored with respect to the spiral one at any value of T and μ, i.e. translational invariance is never restored.

Conclusions and Outlook

In this thesis, we have studied a specific interacting model of fermions in 1+1 dimensions, the chiral Gross-Neveu model. Our main accomplishments have been to rigorously derive the ground state of a strongly interacting theory, as well as having computed correlation functions of certain local operators in it. This determination has come from a mixture of 't Hooft anomaly matching arguments which have constrained the nature of the vacuum, either to spontaneously break a $\mathbb{Z}_N$ symmetry at zero temperature or to provide persistent order with periodic boundary conditions for fermions, and explicit CFT computations for the gapless excitation on top of the vacuum.

Thanks to our efforts, we are ready to draw the phase diagram of the chiral Gross-Neveu model at finite N. For fermions that are thermal, i.e. antiperiodic around the Euclidean time direction, the quasi-long-range ordered crystalline phase at $T = 0$ is the only remnant at finite N of the large-N crystal phase, with an amplitude of the two-point function of the would-be order parameter that decays power-like with the distance. At $T > 0$, thermal fluctuations destroy any asymptotic large-distance behavior with a finite correlation length, and the oscillating behavior in the two-point function is visible only at finite scales, cf. Figure 5.3. For periodic fermions, instead, we have found that the crystal phase persists at arbitrarily high temperatures, and the amplitude of the two-point function approaches a finite value at large distances, see Figure 5.4.

The obvious question that is left open is to which extent the methods and the results of this thesis can be extended to the physics of the ordinary Gross-Neveu model. The main difference between the two is in the amount of symmetry preserved by the deformation, which is orthogonal, $G_{GN} = O(2N)_V \times \mathbb{Z}_2^A$, rather than unitary, $G_{cGN} = U(N)_V \times U(1)_A$. This implies that its bosonization has a more natural description as a deformed $Spin(2N)_1$ WZW model, since the marginally relevant $(\bar{\psi}\psi)^2$ operator has a local realization in terms of $SO(2N)$ currents. Nevertheless, it is possible that an expression in terms of a compact scalar degree of freedom can still be obtained, but for sure it will not be free. Moreover, we expect that the D_8^F anomaly derived in the free theory will still be present with the Gross-Neveu deformation, pointing towards the existence of persistent order for periodic fermions also in that setup.

There is another direction from which we could have studied this problem, completely orthogonal to the one present in this thesis, and that is integrability. Both the ordinary and the chiral Gross-Neveu model are integrable field theories. Their symmetries are so constraining that one can solve the Yang-Baxter equation for the three-body scattering and determine their exact spectrum of asymptotic states and S-matrix elements [115–118].

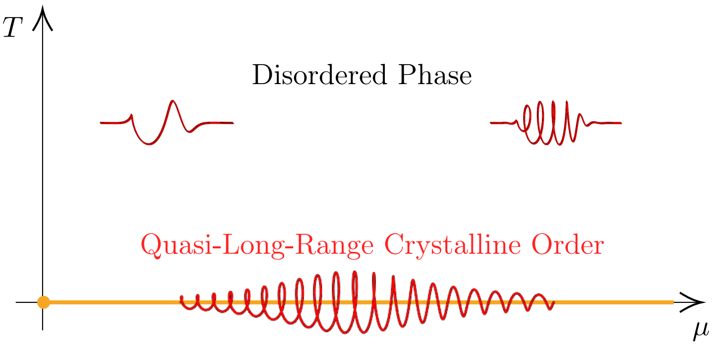


Figure 5.3: (T, μ) phase diagram for the finite N chiral Gross-Neveu model, with thermal fermions. The drawings sketch the shape of the two-point function of the would-be order parameters (4.5) and (4.52).

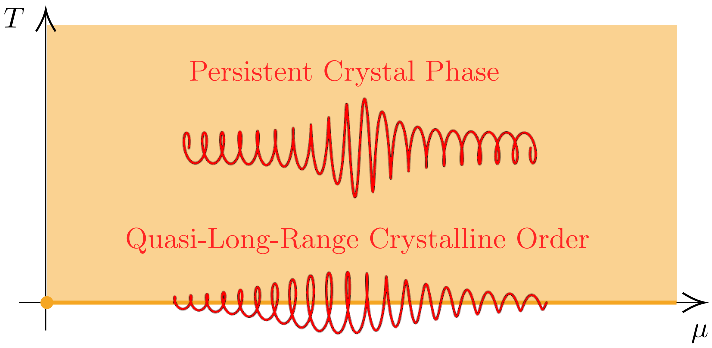


Figure 5.4: (T, μ) phase diagram for the finite N chiral Gross-Neveu model, with periodic fermions around the Euclidean time direction. The drawings sketch the shape of the two-point function of the would-be order parameters (4.5) and (4.53).

With Thermodynamic Bethe Ansatz techniques, one can compute essentially any information about the ground state properties in the thermodynamic limit as a function of the thermodynamic parameters, see e.g. [119] for a review. This approach is extremely powerful as it is completely free from assumptions, except the underlying symmetries of the system. It has one downside, namely that its computational complexity increases rapidly with the number of flavors. Even for N as small as 2, though, it would be interesting to have our predictions checked explicitly by the Bethe Ansatz.

Our thesis contains a discussion about a certain D_8^F anomaly of a system of free Dirac fermions. Anomalies of fermionic systems are classified in terms of spin group cobordisms, rather than group cohomology. The cobordism groups for the discrete D_8 symmetry are not discussed as extensively in literature as simpler cases (see however [120–123]). Our construction has the advantage of making the appearence of the anomaly manifest in a concrete way from a parent $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ anomaly of a compact scalar under the bosonization/fermionization duality. It would be interesting to rederive this result within the formalism of discrete fermionic anomalies.

In two spacetime dimensions, the dualities presented in Section 2.3 are exact as they are obtained by gauging discrete symmetries. In three spacetime dimensions, a similar set of dualities exists, but they are not exact dualities, rather statements about emerging properties in the infrared [124, 125]. As such, they are useful for describing quantum critical points of condensed matter systems, in particular those who fall outside the Landau paradigm. Our work is very similar in spirit to this approach, but from a more high-energy perspective.

Summarizing, we have used dualities to find equivalent descriptions of a strongly-interacting system, in which our question – namely, the nature of the phase diagram of the chiral Gross-Neveu model at finite N – became easier to answer. The techniques in this thesis can in principle be extended to many other interacting theories in 1+1 dimensions. In particular it would be nice to study models of fermions with more general deformations.

Appendix A

Free field realization at N flavors

The $SU(N)_1$ WZW model can also be described in terms of $N-1$ compact scalars, in what is known as free field realization [126]. This description makes manifest only the $U(1)^{N-1}$ Cartan subalgebra of $SU(N)_1$. However, it is useful here because it allows us to write in an alternative form the current-current deformation. Let $\theta_\ell\,, \ell = 1, \ldots, N-1$, be scalars with radius $r = \sqrt{2}$. In these variables, the undeformed $SU(N)_1$ theory reads simply

$$\mathcal{L}_0 = \sum_{\ell=1}^{N-1} \frac{2}{8\pi}\,\partial_+\theta_\ell\,\partial_-\theta_\ell\,, \tag{A.1}$$

where we have made explicit the radius in the normalization of the kinetic term, so that with these conventions $\theta_\ell \sim \theta_\ell + 2\pi$. The $SU(N)$ currents are given by

$$J^\ell_\pm = -\frac{1}{4\pi}\,\partial_\pm\theta_\ell\,, \qquad J^\alpha_+ = \frac{i}{2\pi}\exp(i\alpha\cdot\vartheta)\,, \qquad J^\alpha_- = \frac{i}{2\pi}\exp(-i\alpha\cdot\bar{\vartheta})\,, \tag{A.2}$$

where ϑ_ℓ and $\bar{\vartheta}_\ell$ are the holomorphic and antiholomorphic components of θ_l, and $\alpha \in \Delta^+$ are the positive roots of the $\mathfrak{su}(N)$ algebra. We then have

$$\sum_A J^A_+ J^A_- = \frac{1}{16\pi^2}\,\partial_+\theta_\ell\,\partial_-\theta_\ell - \frac{1}{2\pi^2}\sum_{\alpha\in\Delta^+}\cos(\alpha\cdot\theta)\,. \tag{A.3}$$

The scalar potential can be rewritten in a more explicit form as

$$\sum_{\alpha\in\Delta^+}\cos(\alpha\cdot\theta) = \sum_{p=1}^{N-1}\;\sum_{i_1>i_2>\cdots>i_p=1}^{N-1}\cos\left(\sum_{\ell=1}^{N-1}(A_{i_1\ell}+A_{i_2\ell}+\cdots+A_{i_p\ell})\theta_\ell\right)\,, \tag{A.4}$$

where $A_{ij} = 2\delta_{i,j} - \delta_{|i-j|,1}$ is the $\mathfrak{su}(N)$ Cartan matrix. Using (A.4) it is not difficult to see that the potential has exactly N classical minima, attained at

$$\theta_\ell = \Theta^{(k)}_\ell \equiv \frac{2\pi\ell}{N}k\,, \qquad k = 0, 1, \ldots, N-1\,. \tag{A.5}$$

It is instructive to look at the symmetries preserved by the $J\bar{J}$ deformation in this language. Of the full $PSU(N)_V \times \mathbb{Z}^L_N$ symmetry, the ones that are explicit are only a $\mathbb{Z}^K_N \times \mathbb{Z}^S_N \times \mathbb{Z}^L_N$ subgroup, with $\mathbb{Z}^K_N \times \mathbb{Z}^S_N \subset PSU(N)_V$ being 'clock' and 'shift' transformations.

The action on the matrix field U is as follows,

$$\begin{cases} \mathbb{Z}_N^L: & U \mapsto e^{\frac{2\pi i}{N}} U\,, \\ \mathbb{Z}_N^K: & U \mapsto M_K U M_K^\dagger\,, \qquad (M_K)_{ab} = e^{\frac{2\pi i(a-1)}{N}} \delta_{a,b}\,, \\ \mathbb{Z}_N^S: & U \mapsto M_S U M_S^\dagger\,, \qquad (M_S)_{ab} = \delta_{a,b-1 \mod N}\,, \end{cases} \tag{A.6}$$

which translates to the following action on the (anti-)holomorphic components of θ_ℓ,

$$\begin{cases} \mathbb{Z}_N^L: & \vartheta_\ell \mapsto \vartheta_\ell + \frac{2\pi}{N}\ell\,, \qquad \bar\vartheta_\ell \mapsto \bar\vartheta_\ell\,, \\ \mathbb{Z}_N^K: & \vartheta_\ell \mapsto \vartheta_\ell + \frac{2\pi}{N}\ell\,, \qquad \bar\vartheta_\ell \mapsto \bar\vartheta_\ell - \frac{2\pi}{N}\ell\,, \\ \mathbb{Z}_N^S: & \vartheta_\ell \mapsto \vartheta_{\ell+1} - \vartheta_1\,, \qquad \bar\vartheta_\ell \mapsto \bar\vartheta_{\ell+1} - \bar\vartheta_1\,, \qquad \vartheta_N = \bar\vartheta_N \equiv 0\,, \end{cases} \tag{A.7}$$

The configurations $\Theta_\ell^{(k)}$ spontaneously break the $\mathbb{Z}_N^L$ symmetry: under $\mathbb{Z}_N^L$, $\Theta_\ell^{(k)} \mapsto \Theta_\ell^{(k+1)}$; on the other hand, they preserve the $\mathbb{Z}_N^K \times \mathbb{Z}_N^S$ symmetry.

A.1 Condensation for $N = 2$

For $N = 2$, the deformed theory is a sine-Gordon model, for which $\langle \mathrm{Tr}\, U \rangle$ can be exactly computed and shown to be non-vanishing. The non-Abelian $J_+^A J_-^A$ deformation modifies the free $SU(2)_1 = U(1)_2$ WZW model,

$$\mathcal{L} = \frac{2}{8\pi}\,\partial_+\theta\,\partial_-\theta + \frac{\lambda}{16\pi^2}\,\partial_+\theta\,\partial_-\theta - \frac{\lambda}{4\pi^2}\cos(2\theta)\,, \tag{A.8}$$

with $\theta \sim \theta + 2\pi$ in this normalization. This is also known as the 2-folded sine-Gordon model introduced in [127]. The k-folded sine-Gordon model differs from the original sine-Gordon model as the target space for the scalar is wrapped into a circle in order to have exactly k minima of the potential. This spontaneously breaks the $U(1)$ translational symmetry of the scalar to its $\mathbb{Z}_k$ subgroup. We have two classical degenerate minima of the potential, at $\theta = 0$ and $\theta = \pi$.

In the undeformed $SU(2)_1$ theory U is a $(h, \bar h) = (1/4, 1/4)$ field transforming in the bifundamental representation of $SU(2)$. In terms of the compact scalar θ, up to unitary transformations, we have

$$U^a{}_b = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta} & -e^{i\tilde\theta} \\ e^{-i\tilde\theta} & e^{-i\theta} \end{pmatrix}, \tag{A.9}$$

where $\tilde\theta$ is the T-dual of θ. Note that $\cos(\theta) = \mathrm{Tr}\, U/\sqrt{2}$ is a local operator and takes the classical value $+1$ at $\theta = 0$ and -1 at $\theta = \pi$. The exact quantum vacuum expectation value of $\langle \mathrm{Tr}\, U \rangle$ can be computed using the results of [128], where a formula for one-point functions of vertex operators in the sine-Gordon model is derived. Using eq.(20) of [128] we get

$$\langle \mathrm{Tr}\, U \rangle = \sqrt{2} \left[\frac{2M\Gamma\left(\frac{1+\xi}{2}\right)}{4\pi^{\frac{3}{2}}\Gamma\left(\frac{\xi}{2}\right)} \right]^{\frac{\xi}{2(1+\xi)}} \exp\Big(A(\xi)\Big)\,, \tag{A.10}$$

where

$$A(\xi) = \int_0^\infty \frac{dt}{t} \left[\frac{\sinh^2\left(\frac{\xi\, t}{\xi+1}\right)}{2\sinh\left(\frac{\xi\, t}{\xi+1}\right)\sinh(t)\cosh\left(\frac{1}{\xi+1}t\right)} - \frac{\xi e^{-2t}}{2(1+\xi)} \right], \qquad \xi = \frac{4\pi}{\lambda}\,, \tag{A.11}$$

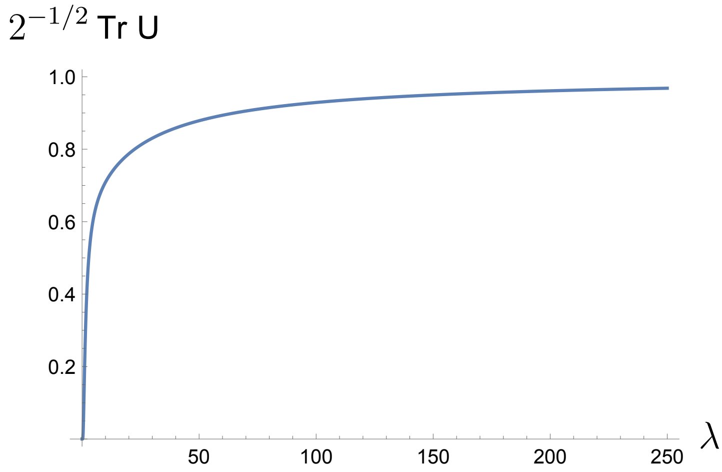


Figure A.1: The value of $\langle \mathrm{Tr}\, U\rangle/\sqrt{2}$ as a function of λ.

and M is the mass of the sine-Gordon soliton, which is exactly determined in terms of ξ:

$$M = \frac{2^{\frac{1-\xi}{2}}\Gamma\left(\frac{\xi}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1+\xi}{2}\right)}\left(\frac{\Gamma\left(\frac{1}{1+\xi}\right)}{\xi\,\Gamma\left(\frac{\xi}{1+\xi}\right)}\right)^{\frac{1+\xi}{2}} . \tag{A.12}$$

We report in figure A.1 the value of $\mathrm{Tr}U/\sqrt{2}$ as a function of λ. As can be seen, it is non-vanishing for any value of $\lambda > 0$. For large λ we enter a semi-classical regime. The soliton mass (A.12) reads

$$M \approx \frac{\sqrt{2}\lambda}{2\pi^2}(1+\mathcal{O}(\lambda^{-1}))\,, \tag{A.13}$$

and agrees with its semi-classical value. Consistently, $\mathrm{Tr}\,U$ approaches the classical value $\sqrt{2}$.

A.1.1 Symmetry breaking from the mixed $\mathbb{Z}_2$ anomaly

We could have argued for the same result without computations, as the 2-folded sine-Gordon model presents the mixed 't Hooft anomaly (4.9) between $\mathbb{Z}_2^P - \mathbb{Z}_2^W - \mathbb{Z}_2^C$ inherited from the free theory. As a consequence, it cannot have a unique symmetric ground state. Noticeably, the $J\bar{J}$-deformed $SU(2)_1$ WZW model has a mixed $PSU(2)_V - \mathbb{Z}_2^L$ anomaly (4.7) leading to the same conclusion. We show here that these anomalies are in fact the same, in the sense that the anomaly (4.9) embeds into the anomaly (4.7) for $N = 2$.

Let us recall how $\mathbb{Z}_2^P$, $\mathbb{Z}_2^W$, and $\mathbb{Z}_2^C$ act on the scalar fields,

$$\begin{cases} \mathbb{Z}_2^P : & \theta \mapsto \theta + \pi\,, & \tilde{\theta} \mapsto \tilde{\theta}\,, \\ \mathbb{Z}_2^W : & \theta \mapsto \theta\,, & \tilde{\theta} \mapsto \tilde{\theta} + 2\pi\,, \\ \mathbb{Z}_2^C : & \theta \mapsto -\theta\,, & \tilde{\theta} \mapsto -\tilde{\theta}\,, \end{cases} \tag{A.14}$$

where we have also included the action on the T-dual field $\tilde{\theta}$. On the $SU(2)_1$ side, let us work with "clock" and "shift" matrices, cf. (A.6). Using the expression (A.9), it is not difficult to see that

$$K(U) \equiv M_K U M_K^{-1} = W(U)\,, \quad S(U) \equiv M_S U M_S^{-1} = WC(U)\,, \quad L(U) \equiv -U = PW(U)\,, \tag{A.15}$$

where P, W, C are the generators of $\mathbb{Z}_2^P$, $\mathbb{Z}_2^W$, and $\mathbb{Z}_2^C$, respectively.

Thanks to this correspondence, we can translate the triple anomaly of the compact boson (4.9) to a special case of the $PSU(2)_V - \mathbb{Z}_2^L$ mixed anomaly (4.7). We can largely repeat the argument for $N = 1$ presented in Section 4.2.1 in an almost identical way. The symmetry operators $K = W$ and $S = WC$ will not commute on the $L = PW$-twisted sector. Indeed, on $U(1)_2$ primary operators,

$$\begin{cases} \mathbb{Z}_2^L : & \mathcal{V}_{e,m} \mapsto (-1)^{e+m} \mathcal{V}_{e,m}\,, \\ \mathbb{Z}_2^K : & \mathcal{V}_{e,m} \mapsto (-1)^{m} \mathcal{V}_{e,m}\,, \\ \mathbb{Z}_2^S : & \mathcal{V}_{e,m} \mapsto (-1)^{-m} \mathcal{V}_{-e,-m}\,, \end{cases} \tag{A.16}$$

and since in the L-twisted sector $e, m \in \mathbb{Z} + 1/2$, it is evident that K and S will not commute there, but rather anticommute. Similarly, $K^2 = -1$ on the L-twisted sector.

Appendix B

Two-point function of vertex operators on T^2

B.1 Free case

Let ϕ be a free compact boson of radius R, $\phi \sim \phi + 2\pi R$, living on the torus with parameter τ. We are interested in computing the two-point functions of primary operators on the torus. Because of electric and magnetic neutrality, the only non-vanishing two-point functions are of the kind

$$\langle \mathcal{V}_{e,m}(z_1, \bar{z}_1) \mathcal{V}_{-e,-m}(z_2, \bar{z}_2) \rangle \ . \tag{B.1}$$

This has been computed e.g. in [101], to which we refer for the full computation.[1] Let us report only the key steps that we wish to generalize. Since the boson is compact, periodicity of ϕ on the torus can be satisfied up to $2\pi R\,\mathbb{Z}$. The two-point function (B.1) gets contributions from each periodicity sector (n, n'), where the field ϕ satisfies the following boundary conditions,

$$\begin{aligned} \phi(z+1, \bar{z}+1) &= \phi(z, \bar{z}) + 2\pi R n \,, \\ \phi(z+\tau, \bar{z}+\bar{\tau}) &= \phi(z, \bar{z}) + 2\pi R n' \,. \end{aligned} \tag{B.2}$$

We can decompose ϕ such that it satisfies the boundary conditions (B.2) and that it has a discontinuity of $2\pi R m$ along the line that connects z_1 and z_2,

$$\phi = \Phi^{\rm cl}_{n,n'} + \Phi^{\rm cl}_{m} + \phi_0 \,, \tag{B.3}$$

with

$$\Phi^{\rm cl}_{n,n'}(z, \bar{z}) = 2\pi R \frac{\operatorname{Im}\left(z(n' - n\bar{\tau})\right)}{\operatorname{Im}\tau} \,, \tag{B.4}$$

$$\Phi^{\rm cl}_{m}(z, \bar{z}) = mR \operatorname{Im}\left[\log \frac{\theta_1(z - z_1|\tau)}{\theta_1(z - z_2|\tau)} - \frac{2\pi}{\operatorname{Im}\tau} z \operatorname{Re} z_{12}\right] \,, \tag{B.5}$$

and ϕ_0 the free part, with propagator

$$\langle \phi_0(z, \bar{z}) \phi_0(0, 0) \rangle = -\log \left| \frac{\theta_1(z|\tau)}{\partial_z \theta_1(0|\tau)} e^{-\pi \frac{(\operatorname{Im} z)^2}{\operatorname{Im}\tau}} \right|^2 . \tag{B.6}$$

[1] Please note that the corresponding expression and its derivation as appear in [101] contain some typographical errors. Here we report the correct expression.

The final result reads

$$
\begin{aligned}
Z_R\langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\rangle = \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\frac{\overline{\partial_z\theta_1(0|\tau)}}{\overline{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} \times \\
\times \frac{1}{|\eta(\tau)|^2}\sum_{e',m'} q^{h_{e',m'}}\bar{q}^{\bar{h}_{e',m'}} e^{4\pi i\left[\alpha_{e',m'}\alpha_{e,m}z_{12}-\bar{\alpha}_{e',m'}\bar{\alpha}_{e,m}\bar{z}_{12}\right]},
\end{aligned}
\tag{B.7}
$$

where

$$
\alpha_{e,m} = \frac{1}{\sqrt{2}}\left(\frac{e}{R}+\frac{mR}{2}\right), \qquad \bar{\alpha}_{e,m} = \frac{1}{\sqrt{2}}\left(\frac{e}{R}-\frac{mR}{2}\right). \tag{B.8}
$$

B.2 Nontrivial $\mathbb{Z}_N^P$ background

We repeat the above computation in the presence of a nontrivial $\mathbb{Z}_N^P \subset U(1)_P$ background on the torus. The insertion of a $\mathbb{Z}_N^P$ defect alters the boundary conditions of the scalar field. Let $a, b \in \mathbb{Z}_N$ denote twists by $\mathbb{Z}_N^P$ on the two cycles. Then the new boundary conditions in each (n, n') sector become

$$
\begin{aligned}
\phi(z+1,\bar{z}+1) &= \phi(z,\bar{z}) + 2\pi R(n+a/N), \\
\phi(z+\tau,\bar{z}+\bar{\tau}) &= \phi(z,\bar{z}) + 2\pi R(n'+b/N).
\end{aligned}
\tag{B.9}
$$

One easily computes the partition function in the presence of defects,

$$
Z_R[a,b] = \frac{1}{|\eta(\tau)|^2}\sum_{e,m} q^{h_{e,m+a/N}}\bar{q}^{\bar{h}_{e,m+a/N}} e^{-2\pi i e b/N}, \tag{B.10}
$$

and similarly for the (unnormalized) correlation function one obtains the following result,

$$
\begin{aligned}
&Z_R[a,b]\,\langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\rangle_{[a,b]} \\
&= \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\frac{\overline{\partial_z\theta_1(0|\tau)}}{\overline{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} \frac{1}{|\eta(\tau)|^2}\times \\
&\times \sum_{e',m'} q^{h_{e',m'+a/N}}\bar{q}^{\bar{h}_{e',m'+a/N}} e^{-2\pi i e' b/N} e^{4\pi i\left[\alpha_{e',m'+a/N}\alpha_{e,m}z_{12}-\bar{\alpha}_{e',m'+a/N}\bar{\alpha}_{e,m}\bar{z}_{12}\right]}.
\end{aligned}
\tag{B.11}
$$

B.3 Nontrivial $\mathbb{Z}_2^L$ background

We would like to turn on a background for the symmetry $\mathbb{Z}_2^L$ as in (2.55),

$$
\mathbb{Z}_2^L: \qquad \phi \mapsto \phi + \frac{2\pi\ell}{2}pR, \qquad \tilde{\phi} + \frac{2\pi\ell}{2}p'\frac{2}{R}, \qquad \ell = 0, 1. \tag{B.12}
$$

Here, p and p' are coprime positive integers such that $\mathbb{Z}_2^L$ matches the $\mathbb{Z}_2$ subgroup of the $U(1)_L$ symmetry at the rational value $R^2 = 2p'/p$. In doing so we lose modular covariance due to the chiral anomaly. To restore modular covariance, we should consider a product theory of the compact scalar with another theory with equal (and opposite) anomaly for a $\mathbb{Z}_2^L$ symmetry. Then turning on the diagonal $\mathbb{Z}_2^L$ background in the product theory is equivalent to turning on the same $\mathbb{Z}_2^L$ background for both factors. This is what we will be doing, and for this reason we are allowed to ignore the issue of modular covariance.

The partition function for general $\mathbb{Z}_2^L$ background $[l_a, l_b]$, $l_a, l_b \in \{0,1\}$, can be computed with the same method as above, and one obtains

$$Z_R[l_a, l_b] = \frac{1}{|\eta(\tau)|^2} \sum_{e,m\in\mathbb{Z}} (-1)^{l_b(pe+p'm)} q^{h_{e+p'l_a/2, m+pl_a/2}} \bar{q}^{\bar{h}_{e+p'l_a/2, m+pl_a/2}} \,, \tag{B.13}$$

The correlation functions between vertex operators is also easily generalized from the one in the absence of background to

$$\begin{aligned} &Z_R[l_a,l_b] \left\langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\right\rangle_{R[l_a,l_b]} \\ &\quad = \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\overline{\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} \frac{1}{|\eta(\tau)|^2} \times \\ &\qquad \times \sum_{e'\in\mathbb{Z}+p'l_a/2} \sum_{m'\in\mathbb{Z}+pl_a/2} q^{h_{e',m'}} \bar{q}^{\bar{h}_{e',m'}} e^{4\pi i\left[\alpha_{e',m'}\alpha_{e,m}z_{12} - \bar{\alpha}_{e',m'}\bar{\alpha}_{e,m}\bar{z}_{12}\right]} \,. \end{aligned} \tag{B.14}$$

B.4 Chemical potential for $U(1)_W$

In the presence of a chemical potential for $U(1)_W$, the action reads

$$S = \frac{1}{8\pi}\int (\partial\phi)^2\, \mathrm{d}^2x + \frac{\mu}{2\pi R}\int \partial_1\phi\, \mathrm{d}^2x \,, \tag{B.15}$$

The classical minimum of this action is

$$\Phi_\mu^{\mathrm{cl}}(z,\bar{z}) = 2\mu\, \mathrm{Re}\, z/R \,. \tag{B.16}$$

This means that ϕ will in general satisfy the boundary condition

$$\begin{aligned} \phi(z+1,\bar{z}+1) &= \phi(z,\bar{z}) + 2\pi R(n + \mu/(\pi R^2)) \,, \\ \phi(z+\tau,\bar{z}+\bar{\tau}) &= \phi(z,\bar{z}) + 2\pi R(n' + \mu\, \mathrm{Re}\,\tau/(\pi R^2)) \,. \end{aligned} \tag{B.17}$$

Proceeding as usual, one gets the following expression for the partition function,

$$Z_{R,\mu} = \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{h_{e,m+\mu N/\pi R^2}} \bar{q}^{\bar{h}_{e,m+\mu N/\pi R^2}} e^{-2ie\mu N\, \mathrm{Re}\,\tau} \,. \tag{B.18}$$

Let us now compute the two-point function. To the decomposition (B.3) we should add the term (B.16). By doing so, one obtains

$$\begin{aligned} &Z_{R,\mu} \left\langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\right\rangle_\mu \\ &\quad = \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\overline{\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} \times e^{2ie\mu\, \mathrm{Re}\, z_{12}/R^2 - 2m\mu\, \mathrm{Im}\, z_{12}/R^2} \times \\ &\qquad \times \frac{1}{|\eta(\tau)|^2} \sum_{e',m'} \Big[q^{h_{e',m'+\mu/\pi R^2}} \bar{q}^{\bar{h}_{e',m'+\mu/\pi R^2}} e^{-2ie'\mu\, \mathrm{Re}\,\tau} \times \\ &\qquad\qquad \times e^{4\pi i\left(\alpha_{e',m'+\mu/\pi R^2}\alpha_{e,m}z_{12} - \bar{\alpha}_{e',m'+\mu/\pi R^2}\bar{\alpha}_{e,m}\bar{z}_{12}\right)} \Big] \,. \end{aligned} \tag{B.19}$$

Let us also add a nontrivial $\mathbb{Z}_N^P$ background. Then for the partition function one gets

$$Z_{R,\mu}[a,b] = \frac{1}{|\eta(\tau)|^2} \sum_{e,m} q^{h_{e,m+a/N+\mu/\pi R^2}} \bar{q}^{\bar{h}_{e,m+a/N+\mu/\pi R^2}} e^{-2\pi i e\left(\frac{b}{N}+\frac{\mu}{\pi R^2}\operatorname{Re}\tau\right)}, \tag{B.20}$$

while the correlator reads

$$\begin{aligned} &Z_{R,\mu}[a,b]\, \langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\rangle_{\mu,[a,b]} \\ &= \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\overline{\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} e^{2ie\mu \operatorname{Re} z_{12}/R^2 - 2m\mu \operatorname{Im} z_{12}/R^2} \frac{1}{|\eta(\tau)|^2} \times \\ &\times \sum_{e',m'} \Big[q^{h_{e',m'+a/N+\mu/\pi R^2}} \bar{q}^{\bar{h}_{e',m'+a/N+\mu/\pi R^2}} e^{-2\pi i e'\left(\frac{b}{N}+\frac{\mu}{\pi R^2}\operatorname{Re}\tau\right)} \times \\ &\qquad\qquad \times e^{4\pi i\left(\alpha_{e',m'+a/N+\mu/\pi R^2}\alpha_{e,m} z_{12} - \bar{\alpha}_{e',m'+a/N+\mu/\pi R^2}\bar{\alpha}_{e,m}\bar{z}_{12}\right)} \Big]. \end{aligned} \tag{B.21}$$

For a nontrivial $\mathbb{Z}_2^L$ background, similar expressions hold,

$$Z_{R,\mu}[l_a,l_b] = \frac{1}{|\eta(\tau)|^2} \sum_{e\in\mathbb{Z}+p'\frac{l_a}{2}} \sum_{m\in\mathbb{Z}+p\frac{l_a}{2}} q^{h_{e,m+\mu/\pi R^2}} \bar{q}^{\bar{h}_{e,m+\mu/\pi R^2}} e^{-2ie\mu \operatorname{Re}\tau} (-1)^{l_b(pe+p'm)}, \tag{B.22}$$

and

$$\begin{aligned} &Z_{R,\mu}[l_a,l_b]\, \langle \mathcal{V}_{e,m}(z_1,\bar{z}_1)\mathcal{V}_{-e,-m}(z_2,\bar{z}_2)\rangle_{\mu,[l_a,l_b]} \\ &\qquad = \left(\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}\right)^{2h_{e,m}} \left(\overline{\frac{\partial_z\theta_1(0|\tau)}{\theta_1(z_{12}|\tau)}}\right)^{2\bar{h}_{e,m}} e^{2ie\mu N \operatorname{Re} z_{12}/R^2 - 2m\mu N \operatorname{Im} z_{12}/R^2} \frac{1}{|\eta(\tau)|^2} \times \\ &\qquad \times \sum_{e'\in\mathbb{Z}+p'l_a/2} \sum_{m'\in\mathbb{Z}+pl_a/2} \Big[q^{h_{e',m'+\mu/\pi R^2}} \bar{q}^{\bar{h}_{e',m'+\mu/\pi R^2}} e^{-2ie'\mu \operatorname{Re}\tau} (-1)^{l_b(pe'+p'm')} \\ &\qquad\qquad \times e^{4\pi i\left(\alpha_{e',m'+\mu/\pi R^2}\alpha_{e,m} z_{12} - \bar{\alpha}_{e',m'+\mu/\pi R^2}\bar{\alpha}_{e,m}\bar{z}_{12}\right)} \Big]. \end{aligned} \tag{B.23}$$

Appendix C

Useful identities for elliptic θ-functions

In dealing with CFT computations on the torus one systematically encounters infinite sums that can be expressed in terms of elliptic θ-functions. This class of functions has a huge number of identities that can be used to further simplify expressions. We follow [129].

The elliptic θ-function of argument $u \in \mathbb{C}$ and modular parameter $\tau \in \mathbb{C}$, with characteristics $a, b \in \mathbb{R}$, is defined as

$$\theta_{a,b}(u|\tau) = \sum_{k\in\mathbb{Z}} \exp\left\{\pi i\tau(k+a)^2 + 2\pi i(k+a)(u+b)\right\} . \tag{C.1}$$

This series converges absolutely for any $u \in \mathbb{C}$ if $\operatorname{Im}\tau > 0$. It has the following properties,

$$\begin{aligned} \theta_{a,b}\left(u+\tau a'+b'|\tau\right) &= e^{-2\pi i a'(u+b+b'+a'\tau/2)}\theta_{a+a',b+b'}(u|\tau)\,, \\ \theta_{a+1,b}(u|\tau) &= \theta_{a,b}(u|\tau)\,, \\ \theta_{a,b+1}(u|\tau) &= e^{2\pi i a}\theta_{a,b}(u|\tau)\,. \end{aligned} \tag{C.2}$$

A special class of θ-functions is given by the ones with a, b either integer or half-integer, which can be reconduced to the following four basic θ-functions,

$$\begin{aligned} \theta_1(u|\tau) &= -\theta_{\frac{1}{2},\frac{1}{2}}(u|\tau) = -i\sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{1}{2}\left(k+\frac{1}{2}\right)^2} e^{\pi i(2k+1)u}\,, \\ \theta_2(u|\tau) &= \theta_{\frac{1}{2},0}(u|\tau) = \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}\left(k+\frac{1}{2}\right)^2} e^{\pi i(2k+1)u}\,, \\ \theta_3(u|\tau) &= \theta_{0,0}(u|\tau) = \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{2}} e^{2\pi iku}\,, \\ \theta_4(u|\tau) &= \theta_{0,\frac{1}{2}}(u|\tau) = \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k^2}{2}} e^{2\pi iku}\,, \end{aligned} \tag{C.3}$$

with[1]

$$q = e^{2\pi i\tau}\,. \tag{C.4}$$

[1] Here we use the most common convention in physics. The most common convention in mathematics is $q = e^{\pi i\tau}$.

Additionally, another typically encountered function is $\theta_1'(u|\tau) = \partial_u\theta_1(u|\tau)$, as well as the Dedekind η-function,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{C.5}$$

Elliptic θ-functions have particularly simple transformation laws under a shift of the argument u by one of the two periods of the torus at fixed τ,

$$\begin{aligned}
\theta_1(u+1|\tau) &= -\theta_1(u|\tau)\,, & \theta_1(u+\tau|\tau) &= -e^{-\pi i(2u+\tau)}\theta_1(u|\tau)\,,\\
\theta_2(u+1|\tau) &= -\theta_2(u|\tau)\,, & \theta_2(u+\tau) &= e^{-\pi i(2u+\tau)}\theta_2(u|\tau)\,,\\
\theta_3(u+1|\tau) &= \theta_3(u|\tau)\,, & \theta_3(u+\tau) &= e^{-\pi i(2u+\tau)}\theta_3(u|\tau)\,,\\
\theta_4(u+1|\tau) &= \theta_4(u|\tau)\,. & \theta_4(u+\tau) &= -e^{-\pi i(2u+\tau)}\theta_4(u|\tau)\,,
\end{aligned} \tag{C.6}$$

as well as under shifts of u by half-periods,

$$\begin{aligned}
\theta_1\left(u+\tfrac{1}{2}|\tau\right) &= \theta_2(u|\tau)\,, & \theta_1\left(u+\tfrac{\tau}{2}|\tau\right) &= ie^{-\pi i(u+\tau/4)}\theta_4(u|\tau)\,,\\
\theta_2\left(u+\tfrac{1}{2}|\tau\right) &= -\theta_1(u|\tau)\,, & \theta_2\left(u+\tfrac{\tau}{2}|\tau\right) &= e^{-\pi i(u+\tau/4)}\theta_3(u|\tau)\,,\\
\theta_3\left(u+\tfrac{1}{2}|\tau\right) &= \theta_4(u|\tau)\,, & \theta_3\left(u+\tfrac{\tau}{2}|\tau\right) &= e^{-\pi i(u+\tau/4)}\theta_2(u|\tau)\,,\\
\theta_4\left(u+\tfrac{1}{2}|\tau\right) &= \theta_3(u|\tau)\,, & \theta_4\left(u+\tfrac{\tau}{2}|\tau\right) &= ie^{-\pi i(u+\tau/4)}\theta_1(u|\tau)\,.
\end{aligned} \tag{C.7}$$

However, their most relevant relations are the ones that involve modular transformations. Under a modular $\mathcal{T}$ transformation,

$$\begin{aligned}
\theta_1(u|\tau+1) &= e^{\frac{\pi i}{4}}\theta_1(u|\tau)\,,\\
\theta_2(u|\tau+1) &= e^{\frac{\pi i}{4}}\theta_2(u|\tau)\,,\\
\theta_3(u|\tau+1) &= \theta_4(u|\tau)\,,\\
\theta_4(u|\tau+1) &= \theta_3(u|\tau)\,,\\
\eta(\tau+1) &= e^{i\pi/12}\eta(\tau)\,,
\end{aligned} \tag{C.8}$$

whereas under a modular $\mathcal{S}$ transformation,

$$\begin{aligned}
\theta_1(u/\tau|-1/\tau) &= -i\sqrt{-i\tau}e^{\pi iu^2/\tau}\theta_1(u|\tau)\,,\\
\theta_2(u/\tau|-1/\tau) &= \sqrt{-i\tau}e^{\pi iu^2/\tau}\theta_4(u|\tau)\,,\\
\theta_3(u/\tau|-1/\tau) &= \sqrt{-i\tau}e^{\pi iu^2/\tau}\theta_3(u|\tau)\,,\\
\theta_4(u/\tau|-1/\tau) &= \sqrt{-i\tau}e^{\pi iu^2/\tau}\theta_2(u|\tau)\,,\\
\eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau)\,.
\end{aligned} \tag{C.9}$$

Oftentimes one also finds useful also the following addition formulas,

$$\begin{aligned}
\theta_1(u|\tau)\theta_1(v|\tau) &= \theta_3(u+v|2\tau)\theta_2(u-v|2\tau) - \theta_2(u+v|2\tau)\theta_3(u-v|2\tau)\,,\\
\theta_1(u|\tau)\theta_2(v|\tau) &= \theta_1(u+v|2\tau)\theta_4(u-v|2\tau) + \theta_4(u+v|2\tau)\theta_1(u-v|2\tau)\,,\\
\theta_2(u|\tau)\theta_2(v|\tau) &= \theta_2(u+v|2\tau)\theta_3(u-v|2\tau) + \theta_3(u+v|2\tau)\theta_2(u-v|2\tau)\,,\\
\theta_3(u|\tau)\theta_3(v|\tau) &= \theta_3(u+v|2\tau)\theta_3(u-v|2\tau) + \theta_2(u+v|2\tau)\theta_2(u-v|2\tau)\,,\\
\theta_3(u|\tau)\theta_4(v|\tau) &= \theta_4(u+v|2\tau)\theta_4(u-v|2\tau) - \theta_1(u+v|2\tau)\theta_1(u-v|2\tau)\,,\\
\theta_4(u|\tau)\theta_4(v|\tau) &= \theta_3(u+v|2\tau)\theta_3(u-v|2\tau) - \theta_2(u+v|2\tau)\theta_2(u-v|2\tau)\,,
\end{aligned} \tag{C.10}$$

or equivalently,

$$
\begin{aligned}
2\theta_1(u+v|2\tau)\theta_1(u-v|2\tau) &= \theta_4(u|\tau)\theta_3(v|\tau) - \theta_3(u|\tau)\theta_4(v|\tau)\,, \\
2\theta_1(u+v|2\tau)\theta_4(u-v|2\tau) &= \theta_1(u|\tau)\theta_2(v|\tau) + \theta_2(u|\tau)\theta_1(v|\tau)\,, \\
2\theta_2(u+v|2\tau)\theta_2(u-v|2\tau) &= \theta_3(u|\tau)\theta_3(v|\tau) - \theta_4(u|\tau)\theta_4(v|\tau)\,, \\
2\theta_2(u+v|2\tau)\theta_3(u-v|2\tau) &= \theta_2(u|\tau)\theta_2(v|\tau) - \theta_1(u|\tau)\theta_1(v|\tau)\,, \\
2\theta_3(u+v|2\tau)\theta_3(u-v|2\tau) &= \theta_3(u|\tau)\theta_3(v|\tau) + \theta_4(u|\tau)\theta_4(v|\tau)\,, \\
2\theta_4(u+v|2\tau)\theta_4(u-v|2\tau) &= \theta_3(u|\tau)\theta_4(v|\tau) + \theta_4(u|\tau)\theta_3(v|\tau)\,,
\end{aligned}
\tag{C.11}
$$

which, as a corollary, have the famous relation

$$
\theta_1^4(u|\tau) + \theta_3^4(u|\tau) = \theta_2^4(u|\tau) + \theta_4^4(u|\tau)\,. \tag{C.12}
$$

Bibliography

[1] Riccardo Ciccone, Lorenzo Di Pietro, and Marco Serone. "Inhomogeneous Phase of the Chiral Gross-Neveu Model". In: *Phys. Rev. Lett.* 129.7 (2022), p. 071603. DOI: 10.1103/PhysRevLett.129.071603. arXiv: 2203.07451 [hep-th].

[2] Riccardo Ciccone, Lorenzo Di Pietro, and Marco Serone. "Anomalies and Persistent Order in the Chiral Gross-Neveu Model". In: (2023).

[3] L. D. Landau. "On the theory of phase transitions". In: *Zh. Eksp. Teor. Fiz.* 7 (1937). Ed. by D. ter Haar, pp. 19–32. DOI: 10.1016/B978-0-08-010586-4.50034-1.

[4] K. G. Wilson and John B. Kogut. "The Renormalization group and the epsilon expansion". In: *Phys. Rept.* 12 (1974), pp. 75–199. DOI: 10.1016/0370-1573(74)90023-4.

[5] Michael E. Fisher. "The renormalization group in the theory of critical behavior". In: *Rev. Mod. Phys.* 46 (1974). [Erratum: Rev.Mod.Phys. 47, 543–543 (1975)], pp. 597–616. DOI: 10.1103/RevModPhys.46.597.

[6] Alexander M. Polyakov. "Thermal Properties of Gauge Fields and Quark Liberation". In: *Phys. Lett. B* 72 (1978), pp. 477–480. DOI: 10.1016/0370-2693(78)90737-2.

[7] K. von Klitzing, G. Dorda, and M. Pepper. "New method for high accuracy determination of the fine structure constant based on quantized Hall resistance". In: *Phys. Rev. Lett.* 45 (1980), pp. 494–497. DOI: 10.1103/PhysRevLett.45.494.

[8] R. B. Laughlin. "Anomalous quantum Hall effect: An Incompressible quantum fluid with fractionallycharged excitations". In: *Phys. Rev. Lett.* 50 (1983), p. 1395. DOI: 10.1103/PhysRevLett.50.1395.

[9] F. J. Wegner. "Duality in Generalized Ising Models and Phase Transitions Without Local Order Parameters". In: *J. Math. Phys.* 12 (1971), pp. 2259–2272. DOI: 10.1063/1.1665530.

[10] X. G. Wen. "Topological Order in Rigid States". In: *Int. J. Mod. Phys. B* 4 (1990), p. 239. DOI: 10.1142/S0217979290000139.

[11] F. D. M. Haldane. "Nonlinear field theory of large spin Heisenberg antiferromagnets. Semiclassically quantized solitons of the one-dimensional easy Axis Neel state". In: *Phys. Rev. Lett.* 50 (1983), pp. 1153–1156. DOI: 10.1103/PhysRevLett.50.1153.

[12] Xie Chen et al. "Symmetry-Protected Topological Orders in Interacting Bosonic Systems". In: *Science* 338.6114 (2012), pp. 1604–1606. DOI: 10.1126/science.1227224. eprint: https://www.science.org/doi/pdf/10.1126/science.1227224. URL: https://www.science.org/doi/abs/10.1126/science.1227224.

[13] L. D. Landau. “The Theory of a Fermi Liquid”. In: *Zh. Eksp. Teor. Fiz.* 30.6 (1956), p. 1058.

[14] Davide Gaiotto et al. “Generalized Global Symmetries”. In: *JHEP* 02 (2015), p. 172. DOI: 10.1007/JHEP02(2015)172. arXiv: 1412.5148 [hep-th].

[15] Lakshya Bhardwaj and Yuji Tachikawa. “On finite symmetries and their gauging in two dimensions”. In: *JHEP* 03 (2018), p. 189. DOI: 10.1007/JHEP03(2018)189. arXiv: 1704.02330 [hep-th].

[16] Gerard ’t Hooft. “Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking”. In: *NATO Sci. Ser. B* 59 (1980). Ed. by Gerard ’t Hooft et al., pp. 135–157. DOI: 10.1007/978-1-4684-7571-5_9.

[17] Anton Kapustin. “Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology”. In: (Mar. 2014). arXiv: 1403.1467 [cond-mat.str-el].

[18] Elliott H. Lieb, Theodore Schultz, and Daniel Mattis. “Two soluble models of an antiferromagnetic chain”. In: *Annals Phys.* 16 (1961), pp. 407–466. DOI: 10.1016/0003-4916(61)90115-4.

[19] Masaki Oshikawa. “Topological Approach to Luttinger’s Theorem and the Fermi Surface of a Kondo Lattice”. In: *Phys. Rev. Lett.* 84 (15 Apr. 2000), pp. 3370–3373. DOI: 10.1103/PhysRevLett.84.3370. URL: https://link.aps.org/doi/10.1103/PhysRevLett.84.3370.

[20] M. B. Hastings. “Lieb-Schultz-Mattis in higher dimensions”. In: *Phys. Rev. B* 69 (2004), p. 104431. DOI: 10.1103/PhysRevB.69.104431. arXiv: cond-mat/0305505.

[21] Ashvin Vishwanath and T. Senthil. “Physics of three dimensional bosonic topological insulators: Surface Deconfined Criticality and Quantized Magnetoelectric Effect”. In: *Phys. Rev. X* 3.1 (2013), p. 011016. DOI: 10.1103/PhysRevX.3.011016. arXiv: 1209.3058 [cond-mat.str-el].

[22] John McGreevy. “Generalized Symmetries in Condensed Matter”. In: (Apr. 2022). DOI: 10.1146/annurev-conmatphys-040721-021029. arXiv: 2204.03045 [cond-mat.str-el].

[23] John Bardeen, L. N. Cooper, and J. R. Schrieffer. “Theory of superconductivity”. In: *Phys. Rev.* 108 (1957), pp. 1175–1204. DOI: 10.1103/PhysRev.108.1175.

[24] Peter Fulde and Richard A. Ferrell. “Superconductivity in a Strong Spin-Exchange Field”. In: *Phys. Rev.* 135 (1964), A550–A563. DOI: 10.1103/PhysRev.135.A550.

[25] A. I. Larkin and Y. N. Ovchinnikov. “Nonuniform state of superconductors”. In: *Zh. Eksp. Teor. Fiz.* 47 (1964), pp. 1136–1146.

[26] R. E Peierls. “Quantum theory of solids”. In: *Ecole d’Eté de Physique Théorique.* 1953.

[27] G. Gruner. “The dynamics of charge-density waves”. In: *Rev. Mod. Phys.* 60 (1988), pp. 1129–1181. DOI: 10.1103/RevModPhys.60.1129.

[28] A. W. Overhauser. “Giant Spin Density Waves”. In: *Phys. Rev. Lett.* 4 (1960), pp. 462–465. DOI: 10.1103/PhysRevLett.4.462.

[29] G. Gruner. “The dynamics of spin-density waves”. In: *Rev. Mod. Phys.* 66 (1994), pp. 1–24. DOI: 10.1103/RevModPhys.66.1.

[30] Jana N. Guenther. "Overview of the QCD phase diagram: Recent progress from the lattice". In: *Eur. Phys. J. A* 57.4 (2021), p. 136. DOI: 10.1140/epja/s10050-021-00354-6. arXiv: 2010.15503 [hep-lat].

[31] Larry McLerran and Robert D. Pisarski. "Phases of cold, dense quarks at large N(c)". In: *Nucl. Phys. A* 796 (2007), pp. 83–100. DOI: 10.1016/j.nuclphysa.2007.08.013. arXiv: 0706.2191 [hep-ph].

[32] Mark G. Alford, Krishna Rajagopal, and Frank Wilczek. "Color flavor locking and chiral symmetry breaking in high density QCD". In: *Nucl. Phys. B* 537 (1999), pp. 443–458. DOI: 10.1016/S0550-3213(98)00668-3. arXiv: hep-ph/9804403.

[33] D. V. Deryagin, Dmitri Yu. Grigoriev, and V. A. Rubakov. "Standing wave ground state in high density, zero temperature QCD at large N(c)". In: *Int. J. Mod. Phys. A* 7 (1992), pp. 659–681. DOI: 10.1142/S0217751X92000302.

[34] E. Shuster and D. T. Son. "On finite density QCD at large N(c)". In: *Nucl. Phys. B* 573 (2000), pp. 434–446. DOI: 10.1016/S0550-3213(99)00615-X. arXiv: hep-ph/9905448.

[35] Toru Kojo et al. "Quarkyonic Chiral Spirals". In: *Nucl. Phys. A* 843 (2010), pp. 37–58. DOI: 10.1016/j.nuclphysa.2010.05.053. arXiv: 0912.3800 [hep-ph].

[36] Roberto Anglani et al. "Crystalline color superconductors". In: *Rev. Mod. Phys.* 86 (2014), pp. 509–561. DOI: 10.1103/RevModPhys.86.509. arXiv: 1302.4264 [hep-ph].

[37] Michael Buballa and Stefano Carignano. "Inhomogeneous chiral condensates". In: *Prog. Part. Nucl. Phys.* 81 (2015), pp. 39–96. DOI: 10.1016/j.ppnp.2014.11.001. arXiv: 1406.1367 [hep-ph].

[38] Michael Buballa and Stefano Carignano. "Inhomogeneous chiral symmetry breaking in dense neutron-star matter". In: *Eur. Phys. J. A* 52.3 (2016), p. 57. DOI: 10.1140/epja/i2016-16057-6. arXiv: 1508.04361 [nucl-th].

[39] Eemeli Annala et al. "Evidence for quark-matter cores in massive neutron stars". In: *Nature Phys.* 16.9 (2020), pp. 907–910. DOI: 10.1038/s41567-020-0914-9. arXiv: 1903.09121 [astro-ph.HE].

[40] Zheyan Wan and Juven Wang. "Higher anomalies, higher symmetries, and cobordisms III: QCD matter phases anew". In: *Nucl. Phys. B* 957 (2020), p. 115016. DOI: 10.1016/j.nuclphysb.2020.115016. arXiv: 1912.13514 [hep-th].

[41] E. Fermi. "Tentativo di una Teoria Dei Raggi β". In: *Nuovo Cim.* 11 (1934), pp. 1–19. DOI: 10.1007/BF02959820.

[42] U. Vogl and W. Weise. "The Nambu and Jona Lasinio model: Its implications for hadrons and nuclei". In: *Prog. Part. Nucl. Phys.* 27 (1991), pp. 195–272. DOI: 10.1016/0146-6410(91)90005-9.

[43] S. P. Klevansky. "The Nambu-Jona-Lasinio model of quantum chromodynamics". In: *Rev. Mod. Phys.* 64 (1992), pp. 649–708. DOI: 10.1103/RevModPhys.64.649.

[44] David J. Gross and Andre Neveu. "Dynamical Symmetry Breaking in Asymptotically Free Field Theories". In: *Phys. Rev. D* 10 (1974), p. 3235. DOI: 10.1103/PhysRevD.10.3235.

[45] U. Wolff. “The phase diagram of the infinite N Gross-Neveu Model at finite temperature and chemical potential”. In: *Phys. Lett. B* 157 (1985), pp. 303–308. DOI: 10.1016/0370-2693(85)90671-9.

[46] A. Barducci et al. “Thermodynamics of the massive Gross-Neveu model”. In: *Phys. Rev. D* 51 (1995), pp. 3042–3060. DOI: 10.1103/PhysRevD.51.3042. arXiv: hep-th/9406117.

[47] Verena Schon and Michael Thies. “Emergence of Skyrme crystal in Gross-Neveu and ’t Hooft models at finite density”. In: *Phys. Rev. D* 62 (2000), p. 096002. DOI: 10.1103/PhysRevD.62.096002. arXiv: hep-th/0003195.

[48] Michael Thies and Konrad Urlichs. “Revised phase diagram of the Gross-Neveu model”. In: *Phys. Rev. D* 67 (2003), p. 125015. DOI: 10.1103/PhysRevD.67.125015. arXiv: hep-th/0302092.

[49] Oliver Schnetz, Michael Thies, and Konrad Urlichs. “Phase diagram of the Gross-Neveu model: Exact results and condensed matter precursors”. In: *Annals Phys.* 314 (2004), pp. 425–447. DOI: 10.1016/j.aop.2004.06.009. arXiv: hep-th/0402014.

[50] Gokce Basar, Gerald V. Dunne, and Michael Thies. “Inhomogeneous Condensates in the Thermodynamics of the Chiral NJL(2) model”. In: *Phys. Rev. D* 79 (2009), p. 105012. DOI: 10.1103/PhysRevD.79.105012. arXiv: 0903.1868 [hep-th].

[51] Rajamani Narayanan. “Phase diagram of the large N Gross-Neveu model in a finite periodic box”. In: *Phys. Rev. D* 101.9 (2020), p. 096001. DOI: 10.1103/PhysRevD.101.096001. arXiv: 2001.09200 [hep-th].

[52] Julian Lenz et al. “Inhomogeneous phases in the Gross-Neveu model in 1+1 dimensions at finite number of flavors”. In: *Phys. Rev. D* 101.9 (2020), p. 094512. DOI: 10.1103/PhysRevD.101.094512. arXiv: 2004.00295 [hep-lat].

[53] Julian J. Lenz, Michael Mandl, and Andreas Wipf. “Inhomogeneities in the two-flavor chiral Gross-Neveu model”. In: *Phys. Rev. D* 105.3 (2022), p. 034512. DOI: 10.1103/PhysRevD.105.034512. arXiv: 2109.05525 [hep-lat].

[54] Sidney R. Coleman. “There are no Goldstone bosons in two-dimensions”. In: *Commun. Math. Phys.* 31 (1973), pp. 259–264. DOI: 10.1007/BF01646487.

[55] V.L. Berezinsky. “Destruction of long range order in one-dimensional and two-dimensional systems having a continuous symmetry group. I. Classical systems”. In: *Sov. Phys. JETP* 32 (1971), pp. 493–500.

[56] V.L. Berezinsky. “Destruction of Long-range Order in One-dimensional and Two-dimensional Systems Possessing a Continuous Symmetry Group. II. Quantum Systems.” In: *Sov. Phys. JETP* 34.3 (1972), p. 610.

[57] J.M. Kosterlitz and D.J. Thouless. “Ordering, metastability and phase transitions in two-dimensional systems”. In: *J. Phys. C* 6 (1973), pp. 1181–1203. DOI: 10.1088/0022-3719/6/7/010.

[58] F. Verstraete et al. “Renormalization group transformations on quantum states”. In: *Phys. Rev. Lett.* 94 (2005), p. 140601. DOI: 10.1103/PhysRevLett.94.140601. arXiv: quant-ph/0410227.

[59] A. Yu Kitaev. “Unpaired Majorana fermions in quantum wires”. In: *Phys. Usp.* 44.10S (2001), pp. 131–136. DOI: 10.1070/1063-7869/44/10S/S29. arXiv: cond-mat/0010440.

[60] Aleksey Cherman et al. "Anomalies, a mod 2 index, and dynamics of 2d adjoint QCD". In: *SciPost Phys.* 8.5 (2020), p. 072. DOI: 10.21468/SciPostPhys.8.5.072. arXiv: 1908.09858 [hep-th].

[61] Zohar Komargodski et al. "Symmetries and strings of adjoint QCD_2". In: *JHEP* 03 (2021), p. 103. DOI: 10.1007/JHEP03(2021)103. arXiv: 2008.07567 [hep-th].

[62] Edward Witten. "Nonabelian Bosonization in Two-Dimensions". In: *Commun. Math. Phys.* 92 (1984). Ed. by M. Stone, pp. 455–472. DOI: 10.1007/BF01215276.

[63] Alessandro Bondi et al. "Metric and Central Charge in the Perturbative Approach to Two-dimensional Fermionic Models". In: *Annals Phys.* 199 (1990), p. 268. DOI: 10.1016/0003-4916(90)90380-7.

[64] N. D. Mermin and H. Wagner. "Absence of ferromagnetism or antiferromagnetism in one-dimensional or two-dimensional isotropic Heisenberg models". In: *Phys. Rev. Lett.* 17 (1966), pp. 1133–1136. DOI: 10.1103/PhysRevLett.17.1133.

[65] Edward Witten. "Chiral Symmetry, the 1/n Expansion, and the SU(N) Thirring Model". In: *Nucl. Phys. B* 145 (1978), pp. 110–118. DOI: 10.1016/0550-3213(78)90416-9.

[66] Michael Thies. "From relativistic quantum fields to condensed matter and back again: Updating the Gross-Neveu phase diagram". In: *J. Phys. A* 39 (2006), pp. 12707–12734. DOI: 10.1088/0305-4470/39/41/S04. arXiv: hep-th/0601049.

[67] Roger F. Dashen and Y. Frishman. "Thirring model with u(n) symmetry - scale invariant only for fixed values of a coupling constant". In: *Phys. Lett. B* 46 (1973), pp. 439–442. DOI: 10.1016/0370-2693(73)90161-5.

[68] Lakshya Bhardwaj et al. "Lectures on Generalized Symmetries". In: (July 2023). arXiv: 2307.07547 [hep-th].

[69] Anton Kapustin et al. "Fermionic Symmetry Protected Topological Phases and Cobordisms". In: *JHEP* 12 (2015), p. 052. DOI: 10.1007/JHEP12(2015)052. arXiv: 1406.7329 [cond-mat.str-el].

[70] Xiao-Gang Wen. "Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders". In: *Phys. Rev. D* 88.4 (2013), p. 045013. DOI: 10.1103/PhysRevD.88.045013. arXiv: 1303.1803 [hep-th].

[71] Daniel S. Freed and Constantin Teleman. "Relative quantum field theory". In: *Commun. Math. Phys.* 326 (2014), pp. 459–476. DOI: 10.1007/s00220-013-1880-1. arXiv: 1212.1692 [hep-th].

[72] Anton Kapustin and Ryan Thorngren. "Anomalies of discrete symmetries in three dimensions and group cohomology". In: *Phys. Rev. Lett.* 112.23 (2014), p. 231602. DOI: 10.1103/PhysRevLett.112.231602. arXiv: 1403.0617 [hep-th].

[73] Cumrun Vafa. "Quantum Symmetries of String Vacua". In: *Mod. Phys. Lett. A* 4 (1989), p. 1615. DOI: 10.1142/S0217732389001842.

[74] Erik P. Verlinde. "Fusion Rules and Modular Transformations in 2D Conformal Field Theory". In: *Nucl. Phys. B* 300 (1988), pp. 360–376. DOI: 10.1016/0550-3213(88)90603-7.

[75] V. B. Petkova and J. B. Zuber. "Generalized twisted partition functions". In: *Phys. Lett. B* 504 (2001), pp. 157–164. DOI: 10.1016/S0370-2693(01)00276-3. arXiv: hep-th/0011021.

[76] Jurg Frohlich et al. "Kramers-Wannier duality from conformal defects". In: *Phys. Rev. Lett.* 93 (2004), p. 070601. DOI: 10.1103/PhysRevLett.93.070601. arXiv: cond-mat/0404051.

[77] Jurg Frohlich et al. "Duality and defects in rational conformal field theory". In: *Nucl. Phys. B* 763 (2007), pp. 354–430. DOI: 10.1016/j.nuclphysb.2006.11.017. arXiv: hep-th/0607247.

[78] Jurgen Fuchs et al. "Topological defects for the free boson CFT". In: *J. Phys. A* 40 (2007), p. 11403. DOI: 10.1088/1751-8113/40/37/016. arXiv: 0705.3129 [hep-th].

[79] Chi-Ming Chang et al. "Topological Defect Lines and Renormalization Group Flows in Two Dimensions". In: *JHEP* 01 (2019), p. 026. DOI: 10.1007/JHEP01(2019)026. arXiv: 1802.04445 [hep-th].

[80] Ryan Thorngren and Yifan Wang. "Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases". In: (Dec. 2019). arXiv: 1912.02817 [hep-th].

[81] Ryan Thorngren and Yifan Wang. "Fusion Category Symmetry II: Categoriosities at $c = 1$ and Beyond". In: (June 2021). arXiv: 2106.12577 [hep-th].

[82] Ben Heidenreich et al. "Non-invertible global symmetries and completeness of the spectrum". In: *JHEP* 09 (2021), p. 203. DOI: 10.1007/JHEP09(2021)203. arXiv: 2104.07036 [hep-th].

[83] Justin Kaidi, Kantaro Ohmori, and Yunqin Zheng. "Kramers-Wannier-like Duality Defects in (3+1)D Gauge Theories". In: *Phys. Rev. Lett.* 128.11 (2022), p. 111601. DOI: 10.1103/PhysRevLett.128.111601. arXiv: 2111.01141 [hep-th].

[84] Yichul Choi et al. "Noninvertible duality defects in 3+1 dimensions". In: *Phys. Rev. D* 105.12 (2022), p. 125016. DOI: 10.1103/PhysRevD.105.125016. arXiv: 2111.01139 [hep-th].

[85] Lakshya Bhardwaj et al. "Non-invertible higher-categorical symmetries". In: *SciPost Phys.* 14.1 (2023), p. 007. DOI: 10.21468/SciPostPhys.14.1.007. arXiv: 2204.06564 [hep-th].

[86] P. Etingof et al. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, 2016. ISBN: 9781470434410.

[87] Yuji Tachikawa. "On gauging finite subgroups". In: *SciPost Phys.* 8.1 (2020), p. 015. DOI: 10.21468/SciPostPhys.8.1.015. arXiv: 1712.09542 [hep-th].

[88] Edward Witten. "Quantum Field Theory and the Jones Polynomial". In: *Commun. Math. Phys.* 121 (1989). Ed. by Asoke N. Mitra, pp. 351–399. DOI: 10.1007/BF01217730.

[89] Shmuel Elitzur et al. "Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory". In: *Nucl. Phys. B* 326 (1989), pp. 108–134. DOI: 10.1016/0550-3213(89)90436-7.

[90] J. M. F. Labastida and A. V. Ramallo. "Operator Formalism for Chern-Simons Theories". In: *Phys. Lett. B* 227 (1989), pp. 92–102. DOI: 10.1016/0370-2693(89)91289-6.

[91] Francesco Benini, Christian Copetti, and Lorenzo Di Pietro. "Factorization and global symmetries in holography". In: *SciPost Phys.* 14.2 (2023), p. 019. DOI: 10.21468/SciPostPhys.14.2.019. arXiv: 2203.09537 [hep-th].

[92] Yoshiki Fukusumi, Yuji Tachikawa, and Yunqin Zheng. "Fermionization and boundary states in 1+1 dimensions". In: *SciPost Phys.* 11 (2021), p. 082. DOI: 10.21468/SciPostPhys.11.4.082. arXiv: 2103.00746 [hep-th].

[93] Michael Francis Atiyah. "Riemann surfaces and spin structures". In: *Annales Scientifiques De L Ecole Normale Superieure* 4 (1971), pp. 47–62.

[94] Andreas Karch, David Tong, and Carl Turner. "A Web of 2d Dualities: $\mathbf{Z}_2$ Gauge Fields and Arf Invariants". In: *SciPost Phys.* 7 (2019), p. 007. DOI: 10.21468/SciPostPhys.7.1.007. arXiv: 1902.05550 [hep-th].

[95] Yuji Tachikawa. *TASI Lectures on Anomalies and Topological Phases.* 2019. URL: https://member.ipmu.jp/yuji.tachikawa/lectures/2019-top-anom/.

[96] Pascual Jordan and Eugene P. Wigner. "About the Pauli exclusion principle". In: *Z. Phys.* 47 (1928), pp. 631–651. DOI: 10.1007/BF01331938.

[97] Wenjie Ji, Shu-Heng Shao, and Xiao-Gang Wen. "Topological Transition on the Conformal Manifold". In: *Phys. Rev. Res.* 2.3 (2020), p. 033317. DOI: 10.1103/PhysRevResearch.2.033317. arXiv: 1909.01425 [cond-mat.str-el].

[98] Gregory W. Moore and Nathan Seiberg. "Naturality in Conformal Field Theory". In: *Nucl. Phys. B* 313 (1989), pp. 16–40. DOI: 10.1016/0550-3213(89)90511-7.

[99] Gregory W. Moore and Nathan Seiberg. "Taming the Conformal Zoo". In: *Phys. Lett. B* 220 (1989), pp. 422–430. DOI: 10.1016/0370-2693(89)90897-6.

[100] Po-Shen Hsin, Ho Tat Lam, and Nathan Seiberg. "Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d". In: *SciPost Phys.* 6.3 (2019), p. 039. DOI: 10.21468/SciPostPhys.6.3.039. arXiv: 1812.04716 [hep-th].

[101] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory.* Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3, 978-1-4612-7475-9. DOI: 10.1007/978-1-4612-2256-9.

[102] Ian Affleck. "On the Realization of Chiral Symmetry in (1+1)-dimensions". In: *Nucl. Phys. B* 265 (1986), pp. 448–468. DOI: 10.1016/0550-3213(86)90168-9.

[103] Ian Affleck. "Exact Critical Exponents for Quantum Spin Chains, Nonlinear Sigma Models at Theta = pi and the Quantum Hall Effect". In: *Nucl. Phys. B* 265 (1986), pp. 409–447. DOI: 10.1016/0550-3213(86)90167-7.

[104] Yuya Tanizaki and Tin Sulejmanpasic. "Anomaly and global inconsistency matching: θ-angles, $SU(3)/U(1)^2$ nonlinear sigma model, $SU(3)$ chains and its generalizations". In: *Phys. Rev. B* 98.11 (2018), p. 115126. DOI: 10.1103/PhysRevB.98.115126. arXiv: 1805.11423 [cond-mat.str-el].

[105] Yuan Yao, Chang-Tse Hsieh, and Masaki Oshikawa. "Anomaly matching and symmetry-protected critical phases in $SU(N)$ spin systems in 1+1 dimensions". In: *Phys. Rev. Lett.* 123.18 (2019), p. 180201. DOI: 10.1103/PhysRevLett.123.180201. arXiv: 1805.06885 [cond-mat.str-el].

[106] Doron Gepner and Edward Witten. "String Theory on Group Manifolds". In: *Nucl. Phys. B* 278 (1986), pp. 493–549. DOI: 10.1016/0550-3213(86)90051-9.

[107] Shunsuke C. Furuya and Masaki Oshikawa. "Symmetry Protection of Critical Phases and a Global Anomaly in $1 + 1$ Dimensions". In: *Phys. Rev. Lett.* 118.2 (2017), p. 021601. DOI: 10.1103/PhysRevLett.118.021601. arXiv: 1503.07292 [cond-mat.stat-mech].

[108] Tokiro Numasawa and Satoshi Yamaguchi. "Mixed global anomalies and boundary conformal field theories". In: *JHEP* 11 (2018), p. 202. DOI: 10.1007/JHEP11(2018)202. arXiv: 1712.09361 [hep-th].

[109] Ying-Hsuan Lin and Shu-Heng Shao. "$\mathbb{Z}_N$ symmetries, anomalies, and the modular bootstrap". In: *Phys. Rev. D* 103.12 (2021), p. 125001. DOI: 10.1103/PhysRevD.103.125001. arXiv: 2101.08343 [hep-th].

[110] Matthias Blau and George Thompson. "Derivation of the Verlinde formula from Chern-Simons theory and the G/G model". In: *Nucl. Phys. B* 408 (1993), pp. 345–390. DOI: 10.1016/0550-3213(93)90538-Z. arXiv: hep-th/9305010.

[111] Anton Kapustin and Natalia Saulina. "Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory". In: (Dec. 2010). Ed. by Hisham Sati and Urs Schreiber, pp. 175–198. arXiv: 1012.0911 [hep-th].

[112] Davide Gaiotto and Justin Kulp. "Orbifold groupoids". In: *JHEP* 02 (2021), p. 132. DOI: 10.1007/JHEP02(2021)132. arXiv: 2008.05960 [hep-th].

[113] Yahya Alavirad and Maissam Barkeshli. "Anomalies and unusual stability of multicomponent Luttinger liquids in Zn×Zn spin chains". In: *Phys. Rev. B* 104.4 (2021), p. 045151. DOI: 10.1103/PhysRevB.104.045151. arXiv: 1910.00589 [cond-mat.str-el].

[114] Davide Gaiotto et al. "Theta, Time Reversal, and Temperature". In: *JHEP* 05 (2017), p. 091. DOI: 10.1007/JHEP05(2017)091. arXiv: 1703.00501 [hep-th].

[115] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. "Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models". In: *Annals Phys.* 120 (1979). Ed. by I. M. Khalatnikov and V. P. Mineev, pp. 253–291. DOI: 10.1016/0003-4916(79)90391-9.

[116] B. Berg et al. "Factorized U(n) Symmetric s Matrices in Two-Dimensions". In: *Nucl. Phys. B* 134 (1978), pp. 125–132. DOI: 10.1016/0550-3213(78)90489-3.

[117] V. Kurak and J. A. Swieca. "Anti-particles as Bound States of Particles in the Factorized S Matrix Framework". In: *Phys. Lett. B* 82 (1979), pp. 289–291. DOI: 10.1016/0370-2693(79)90758-5.

[118] N. Andrei and J. H. Lowenstein. "Diagonalization of the Chiral Invariant Gross-Neveu Hamiltonian". In: *Phys. Rev. Lett.* 43 (1979), p. 1698. DOI: 10.1103/PhysRevLett.43.1698.

[119] Stijn J. van Tongeren. "Introduction to the thermodynamic Bethe ansatz". In: *J. Phys. A* 49.32 (2016), p. 323005. DOI: 10.1088/1751-8113/49/32/323005. arXiv: 1606.02951 [hep-th].

[120] Juven Wang. "C-P-T fractionalization". In: *Phys. Rev. D* 106.10 (2022), p. 105009. DOI: 10.1103/PhysRevD.106.105009. arXiv: 2109.15320 [hep-th].

[121] Maissam Barkeshli et al. "Classification of (2+1)D invertible fermionic topological phases with symmetry". In: *Phys. Rev. B* 105.23 (2022), p. 235143. DOI: 10.1103/PhysRevB.105.235143. arXiv: 2109.11039 [cond-mat.str-el].

[122] Naren Manjunath, Vladimir Calvera, and Maissam Barkeshli. "Nonperturbative constraints from symmetry and chirality on Majorana zero modes and defect quantum numbers in (2+1) dimensions". In: *Phys. Rev. B* 107.16 (2023), p. 165126. DOI: 10.1103/PhysRevB.107.165126. arXiv: 2210.02452 [cond-mat.str-el].

[123] Arun Debray et al. "The Chronicles of IIBordia: Dualities, Bordisms, and the Swampland". In: (Jan. 2023). arXiv: 2302.00007 [hep-th].

[124] Nathan Seiberg et al. "A Duality Web in 2+1 Dimensions and Condensed Matter Physics". In: *Annals Phys.* 374 (2016), pp. 395–433. DOI: 10.1016/j.aop.2016.08.007. arXiv: 1606.01989 [hep-th].

[125] T. Senthil et al. "Duality between $(2+1)d$ Quantum Critical Points". In: *Phys. Rept.* 827 (2019), pp. 1–48. DOI: 10.1016/j.physrep.2019.09.001. arXiv: 1810.05174 [cond-mat.str-el].

[126] Tom Banks, D. Horn, and H. Neuberger. "Bosonization of the SU(N) Thirring Models". In: *Nucl. Phys. B* 108 (1976). Ed. by M. Stone, p. 119. DOI: 10.1016/0550-3213(76)90127-9.

[127] Z. Bajnok et al. "The k-folded sine-Gordon model in finite volume". In: *Nucl. Phys. B* 587 (2000), pp. 585–618. DOI: 10.1016/S0550-3213(00)00441-7. arXiv: hep-th/0004181.

[128] Sergei L. Lukyanov and Alexander B. Zamolodchikov. "Exact expectation values of local fields in quantum sine-Gordon model". In: *Nucl. Phys. B* 493 (1997), pp. 571–587. DOI: 10.1016/S0550-3213(97)00123-5. arXiv: hep-th/9611238.

[129] S. Kharchev and A. Zabrodin. "Theta vocabulary I". In: *Journal of Geometry and Physics* 94 (Aug. 2015), pp. 19–31. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2015.03.010. URL: http://dx.doi.org/10.1016/j.geomphys.2015.03.010.