

# Convergence of metric measure spaces satisfying the CD condition for negative values of the dimension parameter



Mattia Magnabosco<sup>a</sup>, Chiara Rigoni<sup>b,\*</sup>, Gerardo Sosa

<sup>a</sup> *Institut für Angewandte Mathematik, Universität Bonn, Germany*

<sup>b</sup> *Faculty of Mathematics, University of Vienna, Austria*

## ARTICLE INFO

### Article history:

Received 20 December 2022

Accepted 9 August 2023

Communicated by Enrico Valdinoci

### Keywords:

Metric measure space

Curvature-dimension condition

Negative dimension

Convergence

Stability result

## ABSTRACT

We study the problem of whether the curvature-dimension condition with negative values of the generalized dimension parameter is stable under a suitable notion of convergence. To this purpose, first of all we propose an appropriate setting to introduce the  $CD(K, N)$  condition for  $N < 0$ , allowing metric measure structures in which the reference measure is quasi-Radon. Then in this class of spaces we define the distance  $d_{iKRW}$ , which extends the already existing notions of distance between metric measure spaces. Finally, we prove that if a sequence of metric measure spaces satisfying the  $CD(K, N)$  condition with  $N < 0$  is converging with respect to the distance  $d_{iKRW}$  to some metric measure space, then this limit structure is still a  $CD(K, N)$  space.

© 2023 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

In the last years, the class of metric measure spaces satisfying the synthetic curvature-dimension condition has been a central object of investigation. These spaces, in which a lower bound on the curvature formulated in terms of optimal transport holds, have been introduced by Sturm in [1,2] and independently by Lott and Villani in [3]. For a metric measure space  $(X, d, \mathbf{m})$  the curvature-dimension condition  $CD(K, N)$  depends on two parameters  $K \in \mathbb{R}$  and  $N \in [1, \infty]$  and it relies on a suitable convexity property of the entropy functional defined on the space of probability measures on  $X$ : the  $CD(K, N)$  condition for finite  $N$  is an appropriate reformulation of the  $CD(K, \infty)$  one introduced as the  $K$ -convexity of the relative entropy with respect to  $\mathbf{m}$ . Spaces satisfying the curvature-dimension condition are Riemannian manifolds [1,2], Finsler spaces [4] and Alexandrov spaces [5,6]. In particular, in the case of a weighted Riemannian manifold, namely a Riemannian manifold  $(M, g)$  equipped with a weighted measure  $\mathbf{m} = e^{-\psi} \text{vol}_g$  which leads to a weighted Ricci curvature tensor  $\text{Ric}_N$ , being a  $CD(K, N)$  space is equivalent to the condition  $\text{Ric}_N \geq K$  that can be regarded as the combination of a lower bound by  $K$  on the curvature and an upper bound by  $N$  on the

\* Corresponding author.

E-mail addresses: [magnabosco@iam.uni-bonn.de](mailto:magnabosco@iam.uni-bonn.de) (M. Magnabosco), [chiara.rigoni@univie.ac.at](mailto:chiara.rigoni@univie.ac.at) (C. Rigoni).

dimension. Moreover, in the setting of Riemannian manifolds, it turns out that for  $N > 0$  it is possible to characterize the  $CD(K, N)$  condition in terms of a property of the relative entropy: the required property is the  $(K, N)$ -convexity introduced in [7]. This notion reinforces the one of  $K$ -convexity and can be generalized to the case of metric measure spaces.

In the Euclidean setting a direct application of the results in [8] ensures that given any convex measure  $\mu$  with full dimensional convex support and  $C^2$  density  $\Psi$ , the space  $(\mathbb{R}^n, d_{Eucl.}, \mu)$  satisfies the  $CD(0, N)$  condition for  $1/N \in (-\infty, 1/n]$  (i.e.  $N \in (-\infty, 0) \cup [n, \infty]$ ), in the sense that  $Ric_N \geq 0$ . This class of measures, introduced by Borell in [9], extends the set of the so-called log-concave measures and it has been largely studied for example in [10–12]. In particular, following the terminology adopted by Bobkov, the case  $N \in (-\infty, 0)$  corresponds to the “heavy-tailed measures” (see also [13]), identified by the condition that  $1/\Psi^{1/(n-N)}$  is convex. An explicit example of these measures is given by the family of Cauchy probability measures on  $(\mathbb{R}^n, d_{Eucl.})$

$$\mu^{n,\alpha} = \frac{c_{n,\alpha}}{(1 + |x|^2)^{\frac{n+\alpha}{2}}} dx, \quad \alpha > 0, \tag{1.1}$$

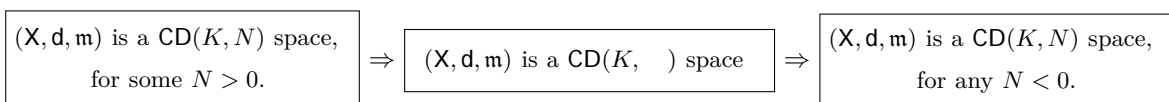
where  $c_{n,\alpha} > 0$  is a normalization constant. It then follows that  $(\mathbb{R}^n, d_{Eucl.}, \mu^{n,\alpha})$  is a  $CD(0, -\alpha)$  space.

Admitting  $N < 0$  may sound strange if one thinks of  $N$  as an upper bound on the dimension; however, as explained in [14,15], in the case of weighted Riemannian manifolds it is useful to consider a generalization of the entropy, called  $m$ -relative entropy  $H_m(\cdot|\nu)$ ,  $m \in \mathbb{R} \setminus \{1\}$ , stemming from the Bregman divergence in information geometry, which is closely related to the Rényi entropies in statistical mechanics. More precisely, in these papers Ohta and Takatsu prove that if  $(M, \omega)$  is a weighted Riemannian manifold and  $\nu = \exp_m(\Psi)\omega$  is a conformal deformation of  $\omega$  in terms of the  $m$ -exponential function, then the fact that  $H_m(\cdot|\nu) \geq K$  in the Wasserstein space  $(P_2(M), W_2)$  is equivalent to the fact that  $Hess \Psi \geq K$  and  $Ric_N \geq 0$  with  $N = 1/(1-m)$ , where  $Ric_N$  is the weighted Ricci curvature tensor associated with  $(M, \omega)$ . In this setting, depending on the choice of the particular entropy, i.e., the value of  $m$ , the value of the dimension  $N$  can be negative. Hence they show that the bounds  $Hess \Psi \geq K$  and  $Ric_N \geq 0$  imply appropriate variants of the Talagrand, HWI, logarithmic Sobolev and the global Poincaré inequalities as well as the concentration of measures. Moreover, using similar techniques as in [16–18], they prove that the gradient flow of  $H_m(\cdot|\nu)$  produces a weak solution to the porous medium equation (for  $m > 1$ ) or the fast diffusion equation (for  $m < 1$ ) of the form

$$\frac{\partial \rho}{\partial t} = \frac{1}{m} \Delta^\omega(\rho^m) + \text{div}_\omega(\rho \nabla \Psi), \tag{1.2}$$

$\Delta^\omega$  and  $\text{div}_\omega$  being the Laplacian and the divergence associated with the measure  $\omega$ . This result was demonstrated also by Otto [19] in the case in which the reference measure  $\nu$  in the  $m$ -relative entropy  $H_m(\cdot|\nu)$  is given by the family of  $m$ -Gaussian measures, which is in turn closely related to the Barenblatt solution to (1.2) without drift (see [20,21]).

In [22], the author extends the range of admissible “dimension parameters” to negative values of  $N$  in the theories of  $(K, N)$ -convex functions, of tensors  $Ric_N$  and of the  $CD(K, N)$  condition in the more general setting of metric measure spaces. In particular, it is proved that the  $(K, N)$ -convexity for  $N < 0$  is weaker than the  $K$ -convexity, thus it covers a wider class of functions. This means that the class of metric measure spaces satisfying the  $CD(K, N)$  condition for negative values of  $N$  includes all  $CD(K, \infty)$  ones; in particular, since a metric measure space which satisfies the  $CD(K, N)$  condition for some  $N > 0$  is also a  $CD(K, \infty)$  space, it follows that:



The curvature-dimension condition for negative values of the dimension has not been largely studied up to now. In the setting of metric measure spaces, the only paper devoted to the study of this notion is the aforementioned work by Ohta [22]. Therein, many direct consequences are extracted from the definitions as in the case of the standard curvature-dimension bounds theory and a number of results valid in the case of  $N > 0$  are generalized to these spaces, including the Brunn–Minkowski inequality and some other functional ones.

Nevertheless, most of the results on this topic are obtained in the case of weighted Riemannian manifolds. A first example of a model space is provided in [23]: it is therein proved that the  $n$ -dimensional unit sphere equipped with the harmonic measure, namely the hitting distribution by the Brownian motion started at  $x \in \mathbb{S}^n$ ,  $|x| < 1$  (which can be equivalently described as the probability measure whose density is proportional to  $\mathbb{S}^n \ni y \rightarrow 1/|y - x|^{n+1}$ ) is a  $\text{CD}(n - 1 - (n + 1)/4, -1)$  space. More generally, Milman provides an equivalent to the family (1.1) of Cauchy measures in  $\mathbb{R}^{n+1}$ , showing that the family of probability measures on the  $n$ -dimensional unit sphere having density proportional to

$$\mathbb{S}^n \ni y \mapsto \frac{1}{|y - x|^{n+\alpha}}$$

satisfies the curvature-dimension condition  $\text{CD}(n - 1 - \frac{n+\alpha}{4}, -\alpha)$  for all  $|x| < 1$ ,  $\alpha \geq -n$  and  $n \geq 2$ . In [24] the author studies the isoperimetric, functional and concentration properties of  $n$ -dimensional weighted Riemannian manifolds satisfying a uniform bound from below on the tensor  $\text{Ric}_N$ , when  $N \in (-\infty, 1)$ , providing a new one-dimensional model-space under an additional diameter upper bound (namely, a positively curved sphere of possibly negative dimension). In this setting, many other rigidity results have been obtained (see for example [25,26]). As for the Lorentzian splitting theorem in this setting, we would cite Wylie–Woolgar’s paper [27]. Other interesting geometric results have been proved when the tensor  $\text{Ric}_N$  for  $N \in (-\infty, 0]$  is uniformly bounded from below: for example, in the paper [28], Kolesnikov and Milman prove various Poincaré-type inequalities on the manifolds and their boundaries (making use of the Bochner’s inequality and of the Reilly formula, when the boundary is nonempty).

Finally, let us underline that Bochner’s inequality, generalized to the setting of weighted Riemannian manifolds satisfying the  $\text{CD}(K, N)$  condition for  $N < 0$  in [22] and in [28], independently, does not yet have a corresponding in the nonsmooth setting of metric measure spaces. We recall that for  $N > 0$ , this important inequality has been extended to the setting of singular spaces in a series of works, precisely in [29] for Finsler manifolds, in [6,30] for Alexandrov spaces and in [31] for  $\text{RCD}(K, \infty)$  spaces.

Despite the progress made in [22], some fundamental questions remain open. The objective of this paper is to address the question of whether the curvature-dimension condition with negative value of generalized dimension is stable under convergence in a suitable topology. Special attention has to be paid to establishing an appropriate setting. In fact, inspired by some of the results found in [22], we prove that for any  $N < -1$  the interval  $I := [-\pi/2, \pi/2]$  equipped with the Euclidean distance and the weighted measure  $\text{dm}(x) := \cos^N(x)d\mathcal{L}^1(x)$ ,  $\mathcal{L}^1$  being the 1-dimensional Lebesgue measure on  $I$ , is a  $\text{CD}(N, N)$  space. This fundamental example shows that the natural setting to introduce this curvature-dimension condition cannot be the one of complete and separable (Polish, in short) metric spaces equipped with Radon measures as in the case of  $\text{CD}(K, N)$  spaces with  $N > 0$ , but rather the one of Polish spaces endowed with quasi-Radon measures, i.e., measures which are Radon outside a negligible set. In fact, roughly speaking, the information that the weighted measure  $\cos^N(x)d\mathcal{L}^1(x)$  is the right one to consider in order to have a space with negative dimension comes from the theory of  $(K, N)$ -convex functions (see [22, Section 2]). However, despite the fact that the “natural” domain for the function  $\cos^N(x)$  with  $N < 0$  would be the open interval  $(-\pi/2, \pi/2)$ , the theory of optimal transport forces us to consider the underlying metric space to be complete and separable, in order to ensure that also the Wasserstein space  $(\mathcal{P}_2(X), W_2)$  enjoys the same properties. Furthermore, we prove that also the space obtained by gluing together  $n$ -copies of the interval  $(I, \text{d}_{\text{Eucl.}}, \text{m})$  introduced above still satisfies the  $\text{CD}(N, N)$  condition: this in particular shows that the negligible set of points in which the reference measure explodes is not just appearing in the “boundary” of our space, but also in the interior of it.

In this new and more general setting, a sequence of spaces satisfying the curvature-dimension condition for negative dimension parameter may fail to be stable under the standard measured Gromov–Hausdorff convergence of metric measure spaces. For example, it can be the case that a well-defined limit of a sequence of  $CD(K, N)$  spaces does not exist due to failure of convergence of the metric or the measure. We present some examples of metric measure spaces whose reference measures are quasi-Radon and presenting this kind of behavior:

- (1) ( *$\sigma$ -finiteness lost in limit*) Consider the sequence of compact metric measure spaces given by  $\{([0, 1], d_{\text{Eucl.}}, \mathbf{m}_n := x^{-n} dx)\}_{n \geq 2}$ . Each one of these spaces satisfy the  $CD(0, -n)$  condition, but since all these measures are unbounded, it is not clear a priori in which way we want the measures to converge. One possibility, however, is the following: note that for every neighborhood  $U$  of 0 the measure  $\mathbf{m}_n|_{[0,1] \setminus U}$  is finite, therefore, up to being cautious with boundaries, one could ask for the weak\*-convergence of the restricted finite measures  $\mathbf{m}_n|_{X \setminus U}$  to some measure  $\mathbf{m}_\infty^U$ , for every neighborhood  $U$  of the origin. Then using an extension theorem we would obtain a unique measure  $\mathbf{m}_\infty$  defined on the whole interval  $[0, 1]$ . However this construction leads to a measure  $\mathbf{m}_\infty$  which is infinite for every measurable subset  $A$  of  $[0, 1]$  with  $\mathcal{L}^1(A) > 0$ , losing thus any regularity.
- (2) (*Bounded measures to unbounded measures*) Consider now the sequence of metric measure spaces given by  $\{([2^{-n}, +\infty), |\cdot|, x^N \mathcal{L}^1)\}_{n \in \mathbb{N}}$  for some fixed  $N < -1$ . For each  $n \in \mathbb{N}$  the space  $([2^{-n}, +\infty), |\cdot|, x^N \mathcal{L}^1)$  is  $CD(0, N)$  and its reference measure is Radon, but the limit measure is such that every neighborhood of 0 has infinite mass.

We recall that in the setting of metric measure spaces, a suitable notion of convergence, called measured Gromov–Hausdorff convergence, was introduced by Fukaya in [32] as a natural variant of the purely metric Gromov–Hausdorff one. Then the stability of the  $CD(K, N)$  condition for  $N \in [1, \infty)$  was proved following these two approaches:

- Lott and Villani proved that the  $CD(K, N)$  condition is stable under pointed measured Gromov–Hausdorff convergence in the class of proper pointed metric measure spaces. Roughly speaking, this means that for any  $R > 0$  there is a measured Gromov–Hausdorff convergence of balls of radius  $R$  around the given points of the spaces;
- Sturm worked in the setting of Polish spaces equipped with probability measures with finite second moment as reference measures. In this class of spaces he defined a distance  $\mathbb{D}$  by putting

$$\mathbb{D}\left((X_1, d_1, \mathbf{m}_1), (X_2, d_2, \mathbf{m}_2)\right) := \inf W_2((\iota_1)_\# \mathbf{m}_1, (\iota_2)_\# \mathbf{m}_2),$$

the infimum being taken among all complete and separable metric spaces  $(X, d)$  and all the isometric embeddings  $\iota_i : (\text{supp}(\mathbf{m}_i), d_i) \rightarrow (X, d)$ ,  $i = 1, 2$ . He then showed that the curvature-dimension condition is stable with respect to this  $\mathbb{D}$ -convergence.

In particular, these two techniques produce the same convergence in the case of compact and doubling metric measure spaces. Then, in [33] Gigli, Mondino and Savaré introduced a notion of convergence of metric measure spaces, called pointed measured Gromov convergence, which works without any compactness assumptions on the metric structure and for more general Radon measures which are finite on bounded sets. Moreover, they prove that lower Ricci bounds are stable with respect to this convergence.

As the first achievement of this paper we propose a suitable setting to introduce the curvature-dimension condition for negative values of the dimension parameter, extending and complementing the work by Ohta [22]. We then propose an appropriate notion of distance, that we call intrinsic pointed Kantorovich–Rubinstein–Wasserstein distance  $d_{iKRW}$ , and we prove that the curvature-dimension bounds with negative values of the dimension are stable with respect to the  $d_{iKRW}$ -convergence. In particular, this distance extends the one introduced in [33] to the set of equivalence classes of metric measure spaces with more general  $\sigma$ -finite

measures, allowing us to analyze sequences of metric measure spaces in which the reference measures may “explode” in some points and are not necessarily finite on bounded sets (we underline that also in this setting we do not require the local compactness assumption on the metric structure).

More specifically, the structures we work with are isomorphism classes of *pointed generalized metric measure spaces*  $(X, d, m, C, p)$  where:

- $(X, d)$  is a complete separable metric space,
- $m \in \mathcal{M}^{qR}(X)$  is a quasi-Radon measure,  $m \neq 0$ ,
- $C \subset X$  is a closed set with empty interior and  $m(C) = 0$ ,
- $p \in \text{supp}(m) \subset X$  is a distinguished point,

and  $(X_1, d_1, m_1, C_1, p_1)$  is said to be isomorphic to  $(X_2, d_2, m_2, C_2, p_2)$  if there exists

$$\text{an isometric embedding } i: \text{supp}(m_1) \rightarrow X_2 \text{ such that } i(C_1) = C_2, i_{\#}m_1 = m_2 \text{ and } i(p_1) = p_2.$$

Intuitively, here for “quasi-Radon measure”  $m$  on  $(X, d)$  (following the terminology introduced in [34, Volume 4]) we mean a complete  $\sigma$ -finite measure with the following properties:

- there exists a closed negligible set with empty interior  $S_m \subset X$  such that  $m(U) = \infty$  for every open neighborhood  $U$  of  $x \in S_m$
- the restricted measure  $m|_{X \setminus S_m}$  is Radon on the open set  $X \setminus S_m$ .

In this class of spaces we then introduce the intrinsic distance  $d_{iKRW}$ . This is constructed by taking partitions of the space: each element of the partition has finite measure and can be then renormalized; hence, we measure the intrinsic Kantorovich–Rubinstein–Wasserstein distance between these renormalized elements. In doing so, we take inspiration from the ideas behind the construction of the distance  $pG_W$  in [33]. However, in contrast to their setting, the lack of regularity of the measure becomes an obstacle to find a canonical and appropriate manner to partition the metric measure space. In particular, it turns out that a control on the Hausdorff distance of the singular sets in the definition of the  $d_{iKRW}$ -distance is actually necessary in order to provide an extrinsic realization of the distance given as an intrinsic one.

Then in this setting the  $CD(K, N)$  condition for negative values of  $N$  is introduced requiring a suitable convexity property of the extended Rényi entropy functional defined on the space of probability measures on  $X$ , as in the case  $N > 0$ .

We prove that this notion is stable with respect to the  $d_{iKRW}$ -distance: our main result ([Theorem 4.1](#)) shows that if a sequence of pointed generalized metric measure spaces  $\{(X_n, d_n, m_n, C_n, p_n)\}_{n \in \mathbb{N}}$  satisfying the  $CD(K, N)$  condition for some  $N < 0$  (and some other technical assumptions) is converging in the  $d_{iKRW}$ -distance to some generalized metric measure space  $(X_\infty, d_\infty, m_\infty, C_\infty, p_\infty)$ , then this limit structure is still a  $CD(K, N)$  space.

This result in the case  $N > 0$  strongly relies on the fact that the (standard) Rényi entropy functional is lower semicontinuous with respect to the weak topology in  $P_2(X)$ . Unfortunately, the same property does not hold for the extended Rényi entropy functional  $S_{N,m}$  when the reference measure  $m$  is quasi-Radon. Therefore we provide a new argument to prove this stability, which extends the proofs of Lott–Villani and Sturm when  $N > 0$  and the one of Gigli–Mondino–Savaré when  $N = \infty$  (in all these classes of spaces the reference measure  $m$  is Radon, namely  $S_m = \emptyset$ ). We show that  $S_{N,m}$  is weakly lower semicontinuous on the space

$$P^{S_m}(X) := \{\mu \in P_2(X) : \mu(S_m) = 0\}$$

and that this will be enough to prove the desired stability result, provided that each one of the spaces in the converging sequence  $\{(X_n, d_n, m_n, C_n, p_n)\}_{n \in \mathbb{N}}$  is not accumulating “too much mass” around any of the points in  $S_{m_n}$  in a uniform way.

Finally, in [Theorem 4.2](#) we manage to adapt this *Stability Result* also in the case in which we only have at our disposal a distance which is not explicitly dependent on the behavior of the  $\mathfrak{m}$ -singular sets. Intuitively, one of the examples we would like to include in our theory consists in approximating the  $\text{CD}(0, N + 1)$  space  $([0, \infty), |\cdot|, x^N \mathcal{L}^1)$ , where  $N < -1$ , making use of the sequence of metric measure spaces  $([2^{-n}, +\infty), |\cdot|, x^N \mathcal{L}^1)$ : clearly each space in this sequence is still a  $\text{CD}(0, N + 1)$  space for  $N < -1$  but now the singularity of the measure is ruled out from the domain, meaning that each metric space  $([2^{-n}, +\infty), |\cdot|)$  is actually equipped with a Radon measure. Hence, we rely on an extrinsic approach to convergence which does not require any control on the Hausdorff distance between  $\mathfrak{m}$ -singular sets in the definition of the  $d_{\text{KRW}}$ -distance.

## 2. Metric spaces equipped with quasi-Radon measures

### 2.1. Measure theory background

#### 2.1.1. Quasi-Radon measures

We begin this section by introducing some notation and concepts from measure theory. Let  $X$  be a set,  $\mathcal{T}, \Sigma$  be, respectively, a topology and a  $\sigma$ -algebra on  $X$ , and  $\mathfrak{m}$  be a positive measure defined on  $\Sigma$  such that  $\mathfrak{m}(X) \neq 0$ . Whenever  $\mathcal{T} \subseteq \Sigma$  we say that the quadruple  $(X, \mathcal{T}, \Sigma, \mathfrak{m})$  is a *topological measure space*.

**Definition 2.1.** Let  $(X, \mathcal{T}, \Sigma, \mathfrak{m})$  be a topological measure space. We say that the measure  $\mathfrak{m}$  is:

- (i) *locally finite* if for every  $x \in X$  there exists a neighborhood  $U \in \mathcal{T}$  with  $\mathfrak{m}(U) < \infty$ ;
- (ii) *effectively locally finite* if for every  $A \in \Sigma$  with  $\mathfrak{m}(A) > 0$ , there exists an open set  $U \in \mathcal{T}$  with finite measure such that  $\mathfrak{m}(A \cap U) > 0$ ;
- (iii)  *$\sigma$ -finite* if there exists  $\{A_i\}_{i \in \mathbb{N}} \subset \Sigma$  with  $\mathfrak{m}(A_i) < \infty$  for any  $i \in \mathbb{N}$  such that  $X = \cup_{i \in \mathbb{N}} A_i$ ;
- (iv) *inner regular with respect to a family of sets  $\mathcal{F} \subset \Sigma$*  if for any  $E \in \Sigma$  it holds

$$\mathfrak{m}(E) = \sup\{\mathfrak{m}(A) : A \in \mathcal{F} \text{ and } A \subset E\}.$$

In particular when  $\mathcal{F}$  is the family of compact sets of  $X$ , this property is called *tightness*.

We denote by  $\mathcal{B}(X)$  the *Borel  $\sigma$ -algebra* of  $(X, \mathcal{T})$ , namely the smallest  $\sigma$ -algebra containing all open sets of a topological space  $(X, \mathcal{T})$ . A measure defined on  $\mathcal{B}(X)$  is referred to as a *Borel measure*. In the following we will always consider measures  $\mathfrak{m}$  on  $X$  which are Borel.

For the purpose of this paper we can restrict our study to the case in which the Borel measures are defined on a complete and separable metric space  $(X, d)$ , in short, on a *Polish space*. In particular it is useful to recall that every metric space is Hausdorff, which ensures that every compact subset is closed and in  $\mathcal{B}(X)$ .

It can be proven that every effectively locally finite Borel measure which is defined on a metric space is actually inner regular with respect to closed sets (see [[34](#), Volume 4, Theorem 412E]). Moreover, every finite Borel measure on a Polish space is tight (see [[35](#), Volume 2, Theorem 7.1.7]). In this regard, we recall the following useful characterization of tight measures valid in the setting of Polish spaces (see [[34](#), Volume 4, Corollary 412B]):

**Proposition 2.2.** *Let  $(X, d)$  be a Polish space and  $\mathfrak{m}$  be a Borel measure on  $X$  which is effectively locally finite. Then the following are equivalent:*

- (i) *the measure  $\mathfrak{m}$  is tight,*
- (ii) *for every  $A \in \mathcal{B}(X)$  with  $\mathfrak{m}(A) > 0$  there exist a measurable compact set  $K \subset A$  with the property that  $\mathfrak{m}(K) > 0$ .*



This result in particular allows us to prove an important property of effectively locally finite measures defined on a Polish space:

**Lemma 2.3.** *Let  $(X, d)$  be a Polish space and  $\mathbf{m}$  be a Borel measure on  $X$  which is effectively locally finite. Then  $\mathbf{m}$  is tight.*

**Proof.** Let us fix any subset  $A \in \mathcal{B}(X)$  with  $\mathbf{m}(A) > 0$ . Since  $\mathbf{m}$  is effectively locally finite, there exists an open set  $U$  such that  $0 < \mathbf{m}(A \cap U) < \infty$ . Moreover, being  $\mathbf{m}$  defined on a metric space, the inner regularity with respect to closed sets guarantees the existence of a closed set  $C \subset A \cap U$  with the property that  $0 < \mathbf{m}(C) < \infty$ . At this point the elementary observation that every closed subset of a Polish space is still a Polish space together with the fact that a finite Borel measure on a Polish space is tight ensure the existence of a measurable compact set  $K \subset A$  with the property that  $\mathbf{m}(K) > 0$ . Thanks to [Proposition 2.2](#) we can conclude that  $\mathbf{m}$  is tight.

We can now introduce the following classes of Borel measures, which are of central interest to us. We borrow the terminology proposed in [[34](#), Volume 4, Definitions 411H(a), (b)], where the classes of Radon and quasi Radon measures are defined in the more general setting of topological measure spaces, specializing these characterizations to the setting of Polish spaces.

**Definition 2.4** (*Radon and Quasi-Radon Measures*). Let  $(X, d)$  be a Polish space. We say that a complete Borel measure  $\mathbf{m}$  is

- (i) *Radon* if it is locally finite;
- (ii) *quasi-Radon* if it is effectively locally finite.

**Remark 2.5** (*Assumptions on Inner Regularity*). We remark that if the metric space  $(X, d)$  is just separable but not complete, an additional assumption on the inner regularity of  $\mathbf{m}$  is needed: in fact in this case a Radon measure has to be also inner regular with respect to compact sets, while a quasi-Radon measure is required to be inner regular with respect to closed sets. However in our setting of Polish spaces both locally finite and effectively locally finite measures are tight, in view of [Lemma 2.3](#), and for Hausdorff spaces tight measures are inner regular with respect to closed sets.

Let us now list some properties that these classes of measures satisfy. Before stating and proving these results, we recall that a topological space  $Y$  is called *Lindelöf* if for every open cover of  $Y$ , there exists a countable sub-cover. Moreover,  $Y$  is called *hereditary Lindelöf* if the same property holds for every subset  $\mathcal{V} \subset Y$ . Now we note that a Polish space  $(X, d)$  is second countable, since it is separable, and we recall that second countable topological spaces are Lindelöf. Moreover, since second countability is an hereditary property we have that actually any separable metric space is hereditary Lindelöf.

**Proposition 2.6.** *Let  $(X, d)$  be a Polish space equipped with a complete Borel measure  $\mathbf{m}$ . Then it holds:*

- (i) *if  $\mathbf{m}$  is a Radon measure, then  $\mathbf{m}$  is a quasi-Radon measure;*
- (ii) *if  $\mathbf{m}$  is a quasi-Radon measure, then  $\mathbf{m}$  is  $\sigma$ -finite;*
- (iii) *if  $\mathbf{m}$  is quasi-Radon, then there exists a closed set  $S_{\mathbf{m}}$  with empty interior and  $\mathbf{m}(S_{\mathbf{m}}) = 0$  such that  $\mathbf{m}|_{X \setminus S_{\mathbf{m}}}$  is a Radon measure on the open set  $X \setminus S_{\mathbf{m}}$ .*

**Proof.** The first point (i) follows from the fact that on a separable metric space  $(X, d)$  a locally finite tight measure is essentially locally finite (see [[34](#), Volume 4, 416A] for a proof of this result).

Let us then prove (ii) by showing the existence of a countable collection of open sets  $\{U_i\}_{i \in \mathbb{N}}$  such that  $\mathbf{m}(U_i) < \infty$  for every  $i \in \mathbb{N}$  and that  $\mathbf{m}(X \setminus \bigcup_{i \in \mathbb{N}} U_i) = 0$ . The effective local finiteness property of the measure  $\mathbf{m}$  ensures the existence of a family  $\mathcal{V}$  of open sets  $V \subset X$  with  $\mathbf{m}(V) < \infty$ . Using the hereditary Lindelöf property of  $X$ , we can extract a countable sub-cover  $\{U_i\}_{i \in \mathbb{N}}$  of the family  $\mathcal{V}$ , still satisfying the property  $\mathbf{m}(U_i) < \infty$  for every  $i \in \mathbb{N}$ . Finally, we observe that  $X \setminus \bigcup_{i \in \mathbb{N}} U_i$  is a closed set with  $\mathbf{m}(X \setminus \bigcup_{i \in \mathbb{N}} U_i) = 0$ , since the intersection of  $X \setminus \bigcup_{i \in \mathbb{N}} U_i$  with each open set of finite measure is empty.

At this point, the proof of (iii) is straightforward. In fact, we can take as  $\mathcal{S}_m$  the set  $X \setminus \bigcup_{i \in \mathbb{N}} U_i$ : as already remarked this is a closed set with  $\mathbf{m}(\mathcal{S}_m) = 0$ . The fact that  $\mathcal{S}_m$  has empty interior is guaranteed by the fact that  $\mathbf{m}$  is effectively locally finite, while to conclude that  $\mathbf{m}|_{X \setminus \mathcal{S}_m}$  is a Radon measure follows from the fact that it is locally finite.

Finally, we show the validity of the Radon–Nikodym Theorem for quasi-Radon measures. With this aim, we first introduce another concept which is a strengthening of absolute continuity between measures.

**Definition 2.7.** Let  $(X, \Sigma, \mathbf{m})$  be a measurable space and  $\mu$  be a measure on  $\Sigma$ . We say that a measure  $\mu$  is truly continuous with respect to  $\mathbf{m}$  if:

- (i)  $\mu$  is absolutely continuous with respect to  $\mathbf{m}$
- (ii) for any  $E \in \Sigma$  with  $\mu(E) > 0$  there is  $F \in \Sigma$  such that  $\mathbf{m}(F) < \infty$  and  $\mu(E \cap F) > 0$ .

We refer to [34, Volume 2, Section 232] for a proof of the following result:

**Theorem 2.8** (Radon–Nikodym Theorem on Measurable Spaces). Let  $(X, \Sigma, \mathbf{m})$  be a measurable space equipped with a quasi-Radon measure, and  $\mu$  be a measure on  $X$  which is truly continuous with respect to  $\mathbf{m}$ . Then there exists a measurable function  $f$  on  $X$  such that for any  $B \in \mathcal{B}(X)$  it holds

$$\mu(B) = \int_B f \, d\mathbf{m}.$$

**Lemma 2.9.** Let  $(X, \Sigma, \mathbf{m})$  be a measurable space equipped with a quasi-Radon measure which is  $\sigma$ -finite. Then  $\mu$  is truly continuous with respect to  $\mathbf{m}$  if and only if it is absolutely continuous.

**Proof.** Directly from the definition, we have that a measure  $\mu$  which is truly continuous with respect to  $\mathbf{m}$  is also absolutely continuous with respect to  $\mathbf{m}$ . In order to get the conclusion, we have just to show that if  $\mu$  is an absolutely continuous measure with respect to a  $\sigma$ -finite measure  $\mathbf{m}$ , then point (ii) in Definition 2.7 is automatically satisfied. To show this, let  $\{X_n\}_{n \in \mathbb{N}}$  be a non-decreasing sequence of sets of finite measure covering  $X$ , and  $\mu$  absolutely continuous with respect to  $\mathbf{m}$ . For any  $E \in \Sigma$  such that  $\mu(E) > 0$ , we have that  $\lim_{n \rightarrow \infty} \mu(E \cap X_n) > 0$ , which means that there exists a  $\bar{n} \in \mathbb{N}$  with  $\mu(E \cap X_{\bar{n}}) > 0$ .

**Theorem 2.10** (Radon–Nikodym Theorem on Polish Spaces). Let  $(X, d, \mathbf{m})$  be a Polish space equipped with a quasi-Radon measure, and  $\mu$  be a measure on  $X$  which is absolutely continuous with respect to  $\mathbf{m}$ . There exists a measurable function  $f$  on  $X$  such that for any  $B \in \mathcal{B}(X)$  it holds

$$\mu(B) = \int_B f \, d\mathbf{m}.$$

**Proof.** Since in this setting Proposition 2.6 ensures that the measure  $\mathbf{m}$  is  $\sigma$ -finite, we can conclude just applying Lemma 2.9 and Theorem 2.8.



2.1.2. Convergence of quasi-Radon measures

Let  $(X, d)$  be a Polish space and let us define

$$P(X) := \{m : m \text{ is a probability measure on } X\};$$

$$P_2(X) := \left\{ m \in P(X) : \int d^2(x, x_0) dm(x) < \infty \text{ for some, and thus any, } x_0 \in X \right\}.$$

On the space  $P_2(X)$  we introduce the 2-Wasserstein distance

$$W_2^2(\mu, \nu) := \inf_{\gamma \in \text{Adm}(\mu, \nu)} \int_{X \times X} d(x, y)^2 d\gamma(x, y), \tag{2.1}$$

where  $\text{Adm}(\mu, \nu) := \{ \gamma \in P(X \times X) \mid \pi_1^* \gamma = \mu \text{ and } \pi_2^* \gamma = \nu \}$ ,  $\pi^{1,2}: X \times X \rightarrow X$  being the natural projection onto the first and the second coordinate respectively.

It is important to recall that the infimum in (2.1) is always realized and the plans  $\gamma \in \text{Adm}(\mu, \nu)$  such that  $\int d(x, y)^2 d\gamma(x, y) = W_2^2(\mu, \nu)$  are called optimal couplings, or optimal transport plans. The set that contains them all is denoted by  $\text{Opt}(\mu, \nu)$ . It is well known that  $W_2$  is a complete and separable distance on  $P_2(X)$ .

Now let  $S \subset X$ . We say that  $U \in \mathcal{B}(X)$  is a neighborhood of  $S$ , if there exists an open  $V \in \mathcal{B}(X)$ , such that  $S \subset V \subset U$  and we write  $\mathcal{N}_S$  for the set of all the neighborhoods of  $S$  in  $X$ . Let us fix a closed set with empty interior  $S \subset X$  to introduce the following classes of measures:

$$M(X) := \{m : m \text{ is a finite measure on } X\};$$

$$M_{loc}^R(X) := \{m : m \text{ is a Radon measure on } X \text{ s.t. } m(B) < \infty, \forall B \subset X \text{ bounded}\};$$

$$M_S(X) := \left\{ m : m \text{ is a quasi-Radon measure on } X, S \text{ is an } m\text{-null set and} \right.$$

$$\left. m|_{X \setminus U} \in M_{loc}^R(X), \text{ for every } U \in \mathcal{N}_S \right\}.$$

The next class of measures is of central importance in our work,

$$M^{qR}(X) := \left\{ m : m \text{ is a quasi-Radon measure on } X \text{ for which there exists } S \subset X \right.$$

$$\left. \text{closed with the property that } m(S) = 0 \text{ and } m \in M_S(X) \right\}.$$

Notice that we have the following chain of inclusions:  $P(X) \subset M(X) \subset M_{loc}^R(X) \subset M^{qR}(X)$ .

The adequate study of quasi-Radon measures will require us to monitor their singularities. Intuitively said, given a closed set  $S \subset X$  with empty interior, in the definition above we isolate the set of singular points of a quasi-Radon measure inside  $S$ . Thus one should regard  $M_S(X)$  as the set of quasi-Radon measures which are locally finite and concentrated in  $X \setminus S$ . Recall that the effective local finiteness implies that all singular sets  $S$  of quasi-Radon measures have empty interior, that is,  $M_S(X) = \emptyset$  if  $\text{int}(S) \neq \emptyset$ . Moreover, Proposition 2.6 proves that for every  $m \in M^{qR}(X)$  there exists a singular set  $S_m \subset X$ , closed with empty interior, providing that  $m \in M_{S_m}(X)$ . Finally, note that, in particular,  $M_{loc}^R(X) \subset M^{qR}(X) \cap M_\emptyset(X)$ .

Let us now introduce the following sets of functions

$$C_{bs}(X) := \{ \text{bounded continuous functions with bounded support on } X \},$$

$$C_b(X) := \{ \text{bounded continuous functions on } X \},$$

$$C_S(X) := \{ \text{continuous functions on } X \text{ which vanish on some neighborhood of } S \},$$

where  $S$  is a closed set with empty interior, and proceed to define a convergence on  $M_S(X)$  in duality with functions in  $C_{bs}(X) \cap C_S(X)$ . In detail, we say that

**Definition 2.11** (Weak Convergence for Quasi-Radon Measures). We say that a sequence of measures  $\{m_n\}_{n \in \mathbb{N}} \subset M_S(X)$  converges weakly to  $m_\infty \in M_S(X)$ , and we write  $m_n \rightharpoonup m_\infty$ , if

$$\lim_{n \rightarrow \infty} \int f dm_n = \int f dm_\infty \quad \text{for every } f \in C_{bs}(X) \cap C_S(X). \tag{2.2}$$

We wish to emphasize that many useful properties enjoyed by Radon measures are not necessarily valid in the setting of quasi-Radon measures. For example, it is well known that in a complete and separable metric space  $(X, d)$  equipped with a Radon measure  $\mathbf{m}$ , the set  $C_b(X) \cap L^1(\mathbf{m})$  is dense in  $L^1(\mathbf{m})$  (see [36, Theorem 3.14]), while this result is no more true when  $\mathbf{m}$  is a quasi-Radon measure.

The following proposition substantiates our choice of convergence.

**Proposition 2.12.** *Let  $(X, d)$  be a Polish space,  $S \subset X$  be a closed set with empty interior, and  $\mathbf{m}, \mathbf{n} \in M_S(X)$  two quasi-Radon measures on  $X$  such that  $\int f d\mathbf{m} = \int f d\mathbf{n}$  for every function  $f \in C_{bs}(X) \cap C_S(X)$ . Then  $\mathbf{m} = \mathbf{n}$ .*

**Proof.** According to [34, Volume 4, Proposition 415I], if  $\mathbf{m}, \mathbf{n} \in M^{qR}(X)$  are such that  $\int f d\mathbf{m} = \int f d\mathbf{n}$ , for every function  $f \in C_b(X) \cap L^1(\mathbf{m}) \cap L^1(\mathbf{n})$ , then  $\mathbf{m} = \mathbf{n}$ . In particular this is valid for measures  $\mathbf{m}, \mathbf{n} \in M_S(X) \subset M^{qR}(X)$ . The conclusion is attained using an approximating argument.

Let  $x_0 \in X \setminus S$  and, for any  $n \in \mathbb{N}$ , consider a sequence of Lipschitz functions  $g_n : X \rightarrow [0, 1]$  with the property that

$$g_n = \begin{cases} 1 & \text{on } B_{2^n}(x_0) \cap \{x \in X : d(x, S) \geq 2^{-n}\}, \\ 0 & \text{on } X \setminus B_{2^{n+1}}(x_0) \cap \{x \in X : d(x, S) \leq 2^{-(n+1)}\}. \end{cases} \tag{2.3}$$

Now, for every  $f \in C_b(X) \cap L^1(\mathbf{m}) \cap L^1(\mathbf{n})$ , the sequence  $f_n := g_n f$  is such that

$$\{f_n\}_{n \in \mathbb{N}} \subset C_{bs}(X) \cap C_S(X), \quad \lim_{n \rightarrow \infty} f_n = f \quad \mathbf{m}, \mathbf{n}\text{-a.e.}, \quad \text{and } |f_n| \leq |f|,$$

since  $\mathbf{m}(S) = \mathbf{n}(S) = 0$ . We can then conclude applying the dominated convergence theorem.

Our definition of weak convergence for quasi-Radon measures turns out to be well-fitted for our purposes. Indeed, we have tailored it precisely with this goal. So let us then conclude this Subsection giving some observations regarding the corresponding topology.

**Remark 2.13.**

- (i) For our purposes we would like to have at disposal a notion of convergence for quasi-Radon measures without making any a priori assumption on the uniformity of singular sets. However this seems out of reach: in fact, without having any control on the singular sets of a given sequence, we would be able to generate an unfavorable limiting singular set and thus, for instance, obtain that  $C_S(X) = \{f \equiv 0\}$ . In this case, the weak convergence is trivial. As an example, consider a dense and countable collection of points  $P = \{p_m\}_{m \in \mathbb{N}} \subset X$  in a complete and separable space, and non-atomic measures  $\nu_n \in M^{qR}(X)$ ,  $\forall n \in \mathbb{N}$ , such that for any neighborhood  $U^n \subset X$  of the set of the first  $n$ -points  $P_n := \{p_1, \dots, p_n\}$ ,  $\nu_n(U^n) = \infty$  while  $\nu_n(X \setminus U^n) < \infty$ . Letting  $n \rightarrow \infty$ , we would expect a limit measure having  $P$  as a singular set but, for the reason given above, convergence defined against any meaningful subclass of continuous functions turns out to be trivial. Furthermore, note that such a limit measure would fall outside the realm of quasi-Radon measures.
- (ii) *Consistency.* Let us underline that by considering  $S = \emptyset$  and by restricting the topology to  $M_{loc}^R(X)$  the above definition coincides with the weak\* topology (induced in duality with  $C_{bs}(X)$ ); by further restricting the topology to  $M(X)$ , the weak topology agrees with the narrow topology (defined in duality with  $C_b(X)$ ).

## 2.2. Pointed generalized metric measure spaces and their convergence

### 2.2.1. Metric spaces equipped with quasi-Radon measures

In the following we say that  $(X, d, m)$  is a *metric measure space* if  $(X, d)$  is a Polish space equipped with a quasi-Radon measure  $m$ . We will refer to a *generalized metric measure space* meaning a structure  $(X, d, m, C)$  where:

- $(X, d)$  is a complete separable metric space,
- $m \in \mathcal{M}^{qR}(X)$  is a quasi-Radon measure,  $m \neq 0$ ,
- $C \subset X$  is a closed set with empty interior and  $m(C) = 0$ .

A *pointed generalized metric measure space* is then the structure  $(X, d, m, C, p)$  consisting of a generalized metric measure space with a distinguished point  $p \in \text{supp}(m) \subset X$ .

Two generalized metric measure spaces  $(X_i, d_i, m_i, C_i)$ ,  $i = 1, 2$  are called *isomorphic* if there exists

$$\text{an isometric embedding } i: \text{supp}(m_1) \rightarrow X_2 \text{ such that } i(C_1) = C_2 \text{ and } i_{\#}m_1 = m_2$$

and, in the case of pointed metric measure spaces  $(X_i, d_i, m_i, C_i, p_i)$ ,  $i = 1, 2$ , we further require that  $i(p_1) = p_2$ . Any such  $i$  is called an *isomorphism* from  $X_1$  to  $X_2$ .

We denote by  $\mathbb{X} := [X, d, m, C, p]$  the equivalence class of the given pointed generalized metric measure space  $(X, d, m, C, p)$  and by  $\mathcal{M}^{qR}$  the collection of all equivalence classes of pointed generalized metric measure spaces.

In particular, the portion of the space outside the support of the measure can be neglected since  $(X, d, m, C)$  (resp.  $(X, d, m, C, p)$ ) is isomorphic to  $(\text{supp}(m), d, m, C)$  (resp.  $(\text{supp}(m), d, m, C, p)$ ). Hence, we will assume that  $\text{supp}(m) = X$ , except when considering the associated *kth cuts*,  $\mathbb{X}^k$ , of a metric measure space, which we now turn to define.

For a quasi-Radon measure  $m \in \mathcal{M}^{qR}(X)$ , let  $S_m \subset X$  be the *m-singular set*, or singular set in short, namely the set of all points in  $X$  for which every open neighborhood has infinite measure

$$S_m := \{x \in X : m(U) = \infty \text{ for every open neighborhood } U \text{ of } x\}. \tag{2.4}$$

Recall that from [Proposition 2.6](#) we have that  $S_m$  is a closed set with  $m(S_m) = 0$ . Moreover  $S_m = \emptyset$  if and only if the measure  $m$  is Radon. In particular, to any metric measure space  $(X, d, m)$  we can associate a generalized metric measure space in a canonical way by considering  $(X, d, m, S_m)$ . Now we fix once and for all a cut-off Lipschitz function  $f_{\text{cut}}: [0, \infty) \rightarrow [0, 1]$  such that

$$\begin{cases} f_{\text{cut}}(x) = 1 & \text{for } 0 \leq x \leq 1, \\ f_{\text{cut}}(x) \in (0, 1) & \text{for } 1 < x < 2, \\ f_{\text{cut}}(x) = 0 & \text{for } 2 \leq x \end{cases}$$

and for  $k \in \mathbb{N}$  we define the *kth cut* of  $\mathbb{X}$  as the generalized metric measure space  $\mathbb{X}^k := (X, d, m^k, C, p)$  where the measure is given by

$$m^k := f^k m, \text{ where } f^k(x) := \begin{cases} f_{\text{cut}}(d(x, p)2^{-k})(1 - f_{\text{cut}}(d(x, S_m)2^k)) & \text{if } S_m \neq \emptyset, \\ f_{\text{cut}}(d(x, p)2^{-k}) & \text{if } S_m = \emptyset. \end{cases} \tag{2.5}$$

Intuitively, the *kth cut*  $(X, d, m^k, C, p)$  resembles more  $\mathbb{X}$  as  $k$  grows (see [Remark 2.17](#) below).

**Remark 2.14** (*Regularity of the Measure m*). We point out that since we are considering metric measure spaces  $(X, d, m)$  endowed with measures  $m \in \mathcal{M}^{qR}(X)$ , it holds that

- $m^k(X) < \infty$  for any  $k \in \mathbb{N}$ , and that
- there exists a  $\tilde{k} \in \mathbb{N}$  such that for any  $k \geq \tilde{k}$  it holds  $m^k(X) > 0$ .

We say that  $(X, d, m)$  is a metric measure space with  $m$ -regularity parameter  $\tilde{k}$  if the aforementioned condition is satisfied for  $\tilde{k} \in \mathbb{N}$ .

Finally, for a metric measure space  $(X, d, m)$ , we define its  $k$ th  $m$ -regular set, or  $k$ -regular set in short, as

$$\mathcal{R}^k := B_{2^{k+1}}(p) \setminus \mathcal{N}_{2^{-k}}(\mathcal{S}_m) \text{ for any } k \in \mathbb{N}, \tag{2.6}$$

where  $\mathcal{N}_{2^{-k}}(\mathcal{S}_m) := \cup_{x \in \mathcal{S}_m} B_{2^{-k}}(x)$ . Observe that  $m|_{\mathcal{R}^k}$  is a finite measure and that  $\text{supp}(m^k) = \mathcal{R}^k$ .

### 2.2.2. Convergence of pointed metric measure spaces

First of all, we recall what is the intrinsic Kantorovich–Rubinstein–Wasserstein (iKRW, in short) distance between two metric measure spaces of finite mass. For this aim, we start fixing a cost function  $c$ , that is,

$$c \in C([0, \infty)) \text{ is non-constant and concave with } c(0) = 0, c(d) > 0 \text{ for } d > 0 \text{ and } \lim_{d \rightarrow \infty} c(d) < \infty \tag{2.7}$$

(e.g.,  $c(d) = \tanh(d)$  or  $c(d) = d \wedge 1$ ). Then the iKRW-distance between two probability measures  $m, n \in \mathcal{P}(X)$  on a complete and separable metric space  $(X, d)$  is given by

$$W_c(m, n) := \inf_{\gamma \in \text{Adm}(m, n)} \int_{X \times X} c(d(x, y)) \, d\gamma(x, y). \tag{2.8}$$

Observe that the distance  $W_c$  allows us to deal with all measures in  $\mathcal{P}(X)$ , rather than with the ones in the restricted set  $\mathcal{P}_2(X)$ . Moreover, regardless of the choice of  $c$  as in (2.7),  $(\mathcal{P}(X), W_c)$  is a complete and separable metric space and the convergence with respect to the weak topology of probability measures is equivalent to the convergence provided by the  $W_c$ -distance (see [37, Chapter 6]); the last claim is a consequence of the fact that  $c \circ d$  defines a bounded complete distance on  $X$ , whose induced topology coincides with the one induced by  $d$ .

In the same spirit as Sturm’s  $\mathbb{D}$  distance, the iKRW-distance is used to define an intrinsic complete separable distance  $d_{\text{iKRW}}^{fm}$  between pointed metric measure spaces with finite mass [33]. Let  $\mathbb{X}_1 := (X_1, d_1, m_1, \mathcal{C}_1, p_1)$ ,  $\mathbb{X}_2 := (X_2, d_2, m_2, \mathcal{C}_2, p_2) \in \mathcal{M}^{qR}$  be generalized metric measure spaces with finite mass, then we set

$$d_{\text{iKRW}}^{fm}(\mathbb{X}_1, \mathbb{X}_2) := \left| \log \left( \frac{m_1(X_1)}{m_2(X_2)} \right) \right| + \inf \left\{ d(i_1(p_1), i_2(p_2)) + d_H(i_1(\mathcal{C}_1), i_2(\mathcal{C}_2)) + W_c((i_1)_\# \bar{m}_1, (i_2)_\# \bar{m}_2) \right\}, \tag{2.9}$$

where the infimum is taken over all isometric embeddings  $i_j : (X_j, d_j) \rightarrow (X, d)$  into a complete separable metric space,  $\bar{m}_j := \frac{m_j}{m_j(X_j)}$  is a normalization of the measure  $m_j$ , for  $j \in \{1, 2\}$  and  $d_H$  is the Hausdorff distance between the two closed sets  $i_1(\mathcal{C}_1)$  and  $i_2(\mathcal{C}_2)$ . In the following we set  $d_H(\emptyset, A) := +\infty$  if  $A \neq \emptyset$  while  $d_H(\emptyset, \emptyset) := 0$ .

Notice that the distance  $d_{\text{iKRW}}^{fm}$  is defined only in the case in which the total mass of the two measures  $m_1$  and  $m_2$  is finite (and strictly positive). Therefore, in order to define a distance between two generalized metric measure spaces in  $\mathcal{M}^{qR}$ , we cover the spaces making use of the  $k$ -cuts and we sum up the contributions given by the  $d_{\text{iKRW}}^{fm}$ -distance between them.

In particular, we need the mass of the  $k$ -cuts to be strictly positive: for that purpose, given any  $\bar{k} \in \mathbb{N}$ , we introduce the following class of spaces

$$\mathcal{M}_k^{qR} := \left\{ (X, d, m, \mathcal{C}, p) \in \mathcal{M}^{qR} : m^{\bar{k}}(X) > 0 \right\}$$

Let us observe that for any finite family of generalized metric measure spaces in  $\mathcal{M}^{qR}$ , there exists a  $\bar{k} \in \mathbb{N}$  such that the whole family is contained in  $\mathcal{M}_{\bar{k}}^{qR}$  (in particular, it is sufficient to take  $\bar{k} := \max \tilde{k}_i$ , where  $\tilde{k}_i$  is the regularity parameter of the  $i$ th space). Nevertheless, for a sequence in  $\mathcal{M}^{qR}$  it is necessary to assume the existence of a common regularity parameter in order to introduce a meaningful distance. Hence, in the following, we will restrict ourselves to the class  $\mathcal{M}_{\bar{k}}^{qR}$  for some  $\bar{k} \in \mathbb{N}$ .

**Definition 2.15** (*Intrinsic Pointed Kantorovich–Rubinstein–Wasserstein Distance*). For any couple of metric measure spaces  $\mathbb{X}_i := (\mathbf{X}_i, \mathbf{d}_i, \mathbf{m}_i, \mathcal{C}_i, p_i) \in \mathcal{M}_{\bar{k}}^{qR}$ ,  $i \in \{1, 2\}$ ,  $\bar{k} \in \mathbb{N}$ , we define the pointed iKRW-distance as

$$d_{iKRW}(\mathbb{X}_1, \mathbb{X}_2) := \sum_{k \geq \bar{k}} \frac{1}{2^k} \min \left\{ 1, d_{iKRW}^{fm}(\mathbb{X}_1^k, \mathbb{X}_2^k) \right\},$$

where  $\mathbb{X}_i^k = (\mathbf{X}_i, \mathbf{d}_i, \mathbf{m}_i^k, \mathcal{C}_i, p_i)$  is the  $k$ th cut of  $\mathbb{X}_i$ , for  $i \in \{1, 2\}$ .

Notice that the distance  $d_{iKRW}$  depends on the common regularity parameter  $\bar{k}$ , but we drop this dependence, since it will be clear from the context.

**Definition 2.16** (*Converging Sequence of Pointed Generalized Metric Measure Spaces*). We say that a sequence of pointed generalized metric measure spaces  $\{\mathbb{X}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\bar{k}}^{qR}$ , for some  $\bar{k} \in \mathbb{N}$ , is iKRW-converging to  $\mathbb{X}_\infty \in \mathcal{M}_{\bar{k}}^{qR}$  if

$$\lim_{n \rightarrow \infty} d_{iKRW}(\mathbb{X}_n, \mathbb{X}_\infty) = 0.$$

Observe that the fact that  $d_{iKRW}^{fm}$  is a distance function guarantees that also  $d_{iKRW}: \mathcal{M}_{\bar{k}}^{qR} \rightarrow \mathbb{R}^+ \cup \{0\}$  defines a finite distance function.

**Remark 2.17.** Directly from the definitions of  $d_{iKRW}$  and  $d_{iKRW}^{fm}$ , it follows that

$$\lim_{n \rightarrow \infty} d_{iKRW}(\mathbb{X}_n, \mathbb{X}_\infty) = 0 \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} d_{iKRW}^{fm}(\mathbb{X}_n^k, \mathbb{X}_\infty^k) = 0 \quad \text{for every } k \geq \bar{k}, \tag{2.10}$$

where  $\bar{k}$  is the common regularity parameter associated to the converging sequence.

In the next result we prove an *extrinsic approach to convergence*. From now on we assume that the generalized metric measure space  $(\mathbf{X}, \mathbf{d}, \mathbf{m}, \mathcal{C})$  is the canonical one associated to  $(\mathbf{X}, \mathbf{d}, \mathbf{m})$ , namely  $\mathcal{C} = \mathcal{S}_\mathbf{m}$  is the  $\mathbf{m}$ -singular set.

**Proposition 2.18.** *Let  $\{\mathbb{X}_n\}_{n \in \mathbb{N} \cup \{\infty\}} \subset \mathcal{M}_{\bar{k}}^{qR}$ ,  $\mathbb{X}_n = (\mathbf{X}_n, \mathbf{d}_n, \mathbf{m}_n, \mathcal{S}_{\mathbf{m}_n}, p_n)$  be a sequence of pointed generalized metric measure spaces,  $\bar{k} \in \mathbb{N}$ . Then the following statements are equivalent:*

- (i)  $\lim_{n \rightarrow \infty} d_{iKRW}(\mathbb{X}_n, \mathbb{X}_\infty) = 0$ ,
- (ii) *there exist a complete and separable metric space  $(Z, \mathbf{d}_Z)$  and a sequence of isometric embeddings  $\{i_n: \mathbf{X}_n \rightarrow Z\}_{n \in \mathbb{N}}$  for which*

$$\begin{aligned} & \left| \log \left( \frac{\mathbf{m}_n^k(\mathbf{X}_n)}{\mathbf{m}_\infty^k(\mathbf{X}_\infty)} \right) \right| + \mathbf{d}_Z(i_n(p_n), i_\infty(p_\infty)) + (\mathbf{d}_Z)_H(i_n(\mathcal{S}_{\mathbf{m}_n}), i_\infty(\mathcal{S}_\mathbf{m})) \\ & + W_c((i_n)_\# \bar{\mathbf{m}}_n^k, (i_\infty)_\# \bar{\mathbf{m}}_\infty^k) \xrightarrow{n \rightarrow \infty} 0, \end{aligned} \tag{2.11}$$

for any  $k \geq \bar{k}$ .

We refer to  $\left( (Z, \mathbf{d}_Z), \{i_n\}_{n \in \mathbb{N}} \right)$  as an effective realization for the convergence of  $\{\mathbb{X}_n\}_{n \in \mathbb{N}}$  to  $\mathbb{X}_\infty$ .

**Proof.** (i)  $\Rightarrow$  (ii) We start assuming that  $d_{i\text{KRW}}(\mathbb{X}_n, \mathbb{X}_\infty) \rightarrow 0$ . In this case, the metric space  $(Z, d_Z)$  as well as the isometric embeddings  $\{i_n\}_{n \in \mathbb{N}}$  are constructed relying on a twofold gluing argument. Roughly speaking, the strategy is the following: for any fixed  $k \geq \bar{k}$  we use a “gluing” procedure to construct a common space  $Z^k$  equipped with the metric that makes all the  $k$ th cuts  $\{\mathbb{X}_n^k\}_{n \in \mathbb{N} \cup \{\infty\}}$  be isometrically embedded. Next, we show that a certain compatibility condition holds between the spaces  $\{Z^k\}_{k \in \mathbb{N}}$ : this allows us to “glue” one more time, and obtain the desired common complete and separable metric space  $(Z, d_Z)$  in which we will embed the sequence  $\{\mathbb{X}_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ . In the following we present the detailed argument, which is a suitable adaptation of [33, Theorem 3.15].

For every fixed  $k \geq \bar{k}$ , (2.10) ensures the existence of a sequence of complete and separable metric spaces  $\{(Z_n^k, d_{Z_n^k})\}_{n \in \mathbb{N}}$ , and of two sequences of isometric embeddings  $\{i_n^k: \mathcal{R}_n^k \rightarrow Z_n^k\}_{n \in \mathbb{N}}$  and  $\{i_{\infty,n}^k: \mathcal{R}_\infty^k \rightarrow Z_n^k\}_{n \in \mathbb{N}}$ , where  $\mathcal{R}_n^k = \text{supp}(\mathbf{m}_n^k)$  and  $\mathcal{R}_\infty^k = \text{supp}(\mathbf{m}_\infty^k)$ , with the property that

$$\begin{aligned} & \left| \log \left( \frac{\mathbf{m}_n^k(\mathbb{X}_n)}{\mathbf{m}_\infty^k(\mathbb{X}_\infty)} \right) \right| + d_{Z_n^k}(i_n^k(p_n), i_{\infty,n}^k(p_\infty)) + (d_{Z_n^k})_H(i_n^k(\mathcal{S}_{\mathbf{m}_n}), i_{\infty,n}^k(\mathcal{S}_{\mathbf{m}})) \\ & \quad + W_c((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_{\infty,n}^k)_\# \bar{\mathbf{m}}_\infty^k) \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \tag{2.12}$$

We then define the set  $Z^k = \sqcup_{n \in \mathbb{N}} Z_n^k$  and the function  $d_{Z^k}: Z^k \times Z^k \rightarrow [0, \infty)$  by setting

$$d_{Z^k}(x, y) := \begin{cases} d_{Z_n^k}(x, y) & \text{if } (x, y) \in Z_n^k \times Z_n^k, \exists n \in \mathbb{N}, \\ \inf_{w \in X^k} d_{Z_n^k}(x, i_{\infty,n}^k(w)) + d_{Z_m^k}(i_{\infty,m}^k(w), y) & \text{if } (x, y) \in Z_n^k \times Z_m^k, \exists n \neq m. \end{cases} \tag{2.13}$$

Thus, we can define an equivalence relation  $\sim$  on  $Z^k$  saying that  $v \sim w$  if and only if  $d_{Z^k}(v, w) = 0$ , for  $v, w \in Z^k$ : we take the quotient of  $Z^k$  by this relation and then its completion. We denote by  $\tilde{Z}^k$  the resulting space. Note that  $d_{Z^k}$  canonically induces a distance function on  $\tilde{Z}^k \times \tilde{Z}^k$ , which we still denote by  $d_{Z^k}$ , and that the operations made so far preserve the separability of the space. Thus, the pair  $(\tilde{Z}^k, d_{Z^k})$  is a complete and separable metric space. By construction, for  $n \in \mathbb{N}$ , the composition

$$i_n^k := p^k \circ j_n^k \circ i_n^k: \mathcal{R}_n^k \rightarrow \tilde{Z}^k \tag{2.14}$$

is an isometric embedding, where  $j_n^k: Z_n^k \rightarrow Z^k$  is the canonical inclusion and  $p^k: Z^k \rightarrow \tilde{Z}^k$  the projection map. Moreover, the fact that for every  $m, n \in \mathbb{N}$  the set  $j_n^k(i_{\infty,n}^k(\mathcal{R}_\infty^k))$  is identified under the equivalence relation with  $j_m^k(i_{\infty,m}^k(\mathcal{R}_\infty^k))$  implies that the maps

$$p^k \circ j_n^k \circ i_{\infty,n}^k: \mathcal{R}_\infty^k \rightarrow \tilde{Z}^k \quad \text{and} \quad p^k \circ j_m^k \circ i_{\infty,m}^k: \mathcal{R}_\infty^k \rightarrow \tilde{Z}^k$$

coincide for every  $n, m \in \mathbb{N}$ . In this manner, we see that also  $p^k \circ j_m^k \circ i_{\infty,m}^k: \mathcal{R}_\infty^k \rightarrow \tilde{Z}^k$  is an isometric embedding, which is independent of  $m$ . Let us denote it by  $i_\infty^k$ . The convergence in (2.12) yields

$$\left| \log \left( \frac{\mathbf{m}_n^k(\mathbb{X}_n)}{\mathbf{m}_\infty^k(\mathbb{X}_\infty)} \right) \right| + d_{Z^k}(i_n^k(p_n), i_\infty^k(p_\infty)) + (d_{Z^k})_H(i_n^k(\mathcal{S}_{\mathbf{m}_n}), i_\infty^k(\mathcal{S}_{\mathbf{m}})) \xrightarrow{n \rightarrow \infty} 0. \tag{2.15}$$

To finish the first step of the argument we note that the pushforward of a coupling under the map  $(p^k \circ j_n^k) \times (p^k \circ j_n^k): Z_n^k \times Z_n^k \rightarrow \tilde{Z}^k \times \tilde{Z}^k$ , is again a coupling between the pushforward of the original marginal measures, namely

$$\text{if } \pi \in \text{Adm}((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_{\infty,n}^k)_\# \bar{\mathbf{m}}_\infty^k), \text{ then } \tilde{\pi} := ((p^k \circ j_n^k)^2)_\# \pi \in \text{Adm}((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_\infty^k)_\# \bar{\mathbf{m}}_\infty^k).$$

Therefore, if we choose  $\pi \in \text{Opt}((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_{\infty,n}^k)_\# \bar{\mathbf{m}}_\infty^k)$ , we get

$$W_c^{\tilde{Z}^k}((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_\infty^k)_\# \bar{\mathbf{m}}_\infty^k) \leq W_c^{Z_n^k}((i_n^k)_\# \bar{\mathbf{m}}_n^k, (i_{\infty,n}^k)_\# \bar{\mathbf{m}}_\infty^k),$$



since  $p^k \circ j_n^k : Z_n^k \rightarrow \tilde{Z}^k$  is an isometry. Jointly with the last term in (2.12), this inequality implies the convergence  $(i_n^k)_\# \bar{m}_n^k \rightarrow (i_\infty^k)_\# \bar{m}_\infty^k$  in  $(P(\tilde{Z}^k), W_c^{\tilde{Z}^k})$ . We have hereby shown the existence of a complete and separable metric space  $(\tilde{Z}^k, d_{Z^k})$  and a sequence of isometric embeddings  $\{i_n^k : \mathcal{R}_n^k \rightarrow \tilde{Z}^k\}_{n \in \mathbb{N} \cup \{\infty\}}$  which provide a realization of the convergence  $\mathbb{X}_n^k \rightarrow \mathbb{X}_\infty^k$  for any  $k \geq \bar{k}$ .

For the second part of the argument, first of all we prove that for any  $\bar{k} \leq j < k - 1$ , the embeddings  $i_n^k : \mathcal{R}_n^k \rightarrow \tilde{Z}^k$  serve as an effective realization for the convergence  $\mathbb{X}_n^j \rightarrow \mathbb{X}_\infty^j$ . To that purpose, let us consider the function

$$g_n^{j,k} : \tilde{Z}^k \rightarrow [0, 1]$$

$$y \mapsto \begin{cases} f_{\text{cut}}(d_{Z^k}(y, i_n^k(p_n)) 2^{-(j+1)}) (1 - f_{\text{cut}}(d_{Z^k}(y, i_n^k(\mathcal{S}_{m_n})) 2^{j+1})) & \text{if } \mathcal{S}_{m_n} \neq \emptyset, \\ f_{\text{cut}}(d_{Z^k}(y, i_n^k(p_n)) 2^{-(j+1)}) & \text{if } \mathcal{S}_{m_n} = \emptyset. \end{cases}$$

The Lipschitz continuity of the cut-off function  $f_{\text{cut}}$ , together with the convergence of  $\{i_n^k(p_n)\}_{n \in \mathbb{N}}$  to  $i_\infty^k(p_\infty)$  by (2.12), ensures that the sequence  $f_{\text{cut}}(d_{Z^k}(y, i_n^k(p_n)) 2^{-(j+1)})$  is uniformly converging to  $f_{\text{cut}}(d_{Z^k}(y, i_\infty^k(p_\infty)) 2^{-(j+1)})$  as  $n \rightarrow \infty$ . In the same way, the triangular inequality ensures that

$$\left| d_{Z^k}(y, i_n^k(\mathcal{S}_{m_n})) - d_{Z^k}(y, i_\infty^k(\mathcal{S}_{m_\infty})) \right| \leq (d_{Z^k})_H(i_n^k(\mathcal{S}_{m_n}), i_\infty^k(\mathcal{S}_{m_\infty})).$$

and the convergence (2.12) guarantees that the sequence  $f_{\text{cut}}(d_{Z^k}(y, i_n^k(\mathcal{S}_{m_n})) 2^{j+1})$  uniformly converges to  $f_{\text{cut}}(d_{Z^k}(y, i_\infty^k(\mathcal{S}_{m_\infty})) 2^{j+1})$  as  $n \rightarrow \infty$ . Hence, for every  $y \in \tilde{Z}^k$  the sequence  $\{g_n^{j,k}(y)\}_{n \in \mathbb{N}}$  uniformly converges as  $n \rightarrow \infty$  to

$$g^{j,k}(y) := \begin{cases} f_{\text{cut}}(d_{Z^k}(y, i_\infty^k(p_\infty)) 2^{-(j+1)}) (1 - f_{\text{cut}}(d_{Z^k}(y, i_\infty^k(\mathcal{S}_{m_\infty})) 2^{j+1})) & \text{if } \mathcal{S}_{m_\infty} \neq \emptyset, \\ f_{\text{cut}}(d_{Z^k}(y, i_\infty^k(p_\infty)) 2^{-(j+1)}) & \text{if } \mathcal{S}_{m_\infty} = \emptyset. \end{cases}$$

This in particular implies the weak convergence of the sequence of measures

$$(i_n^k)_\# \bar{m}_n^j = (g_n^{j,k} \circ i_n^k)_\# \bar{m}_n^k \xrightarrow{n \rightarrow \infty} (i_\infty^k)_\# \bar{m}_\infty^j = (g^{j,k} \circ i_\infty^k)_\# \bar{m}_\infty^k$$

(note that we ask for  $\bar{k} \leq j < k - 1$ ). The former convergence, together with (2.15) and the fact that  $\left| \log\left(\frac{m_n^j(X_n)}{m^j(X)}\right) \right| \rightarrow 0$ , shows that the embeddings  $i_n^k : \mathcal{R}_n^k \rightarrow \tilde{Z}^k$  realize the convergence of the sequence of  $j$ -cuts, for every  $\bar{k} \leq j < k - 1$ .

At this point an analogous ‘‘gluing’’ argument can be applied to the sequence  $(\tilde{Z}^k, \{i_n^k\}_{n \in \mathbb{N} \cup \{\infty\}})$  when  $k \geq \bar{k}$ : we can in fact construct a common space  $Z := \sqcup_{k \geq \bar{k}} \tilde{Z}^k$ , which, endowed with the distance  $d_Z$  defined analogously as in (2.13), is a complete and separable metric space, and a sequence of embeddings  $i_n : X_n \rightarrow Z$  for  $n \in \mathbb{N} \cup \{\infty\}$  as in (2.14). The pair  $((Z, d_Z), \{i_n\}_{n \in \mathbb{N} \cup \{\infty\}})$  is the desired effective realization of the iKRW-convergence  $\mathbb{X}_n \rightarrow \mathbb{X}_\infty$ .

(ii)  $\Rightarrow$  (i) Note that the existence of an effective realization of the convergence  $\mathbb{X}_n \xrightarrow{n \rightarrow \infty} \mathbb{X}_\infty$  implies that  $d_{\text{iKRW}}^{fm}(\mathbb{X}_n^k, \mathbb{X}_\infty^k) \xrightarrow{n \rightarrow \infty} 0$ , for all  $k \geq \bar{k}$ . Then, we can conclude by using (2.10).

In some situations, it would be practical to have at our disposal a metric which is not explicitly dependent on the behavior of the  $\mathbf{m}$ -singular sets. For instance, we could gain flexibility by not asking for a control on the Hausdorff distance between  $\mathbf{m}$ -singular sets in the definition of the  $d_{\text{iKRW}}$ -distance. However, as we just saw, this term is necessary to provide an extrinsic realization of the distance given as an intrinsic one. Therefore, the following definition turns out to be useful.

**Definition 2.19 (Extrinsic Convergence).** We say that a sequence of pointed generalized metric measure spaces  $\{\mathbb{X}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_k^{qR}$  converges extrinsically to  $\mathbb{X}_\infty \in \mathcal{M}_k^{qR}$ ,  $\bar{k} \in \mathbb{N}$ , if there exist a complete and separable

metric space  $(Z, d_Z)$  and a sequence of isometric embeddings  $\{i_n : X_n \rightarrow Z\}_{n \in \mathbb{N}}$  for which

$$\left| \log \left( \frac{m_n^k(X_n)}{m_\infty^k(X_\infty)} \right) \right| + d_Z(i_n(p_n), i_\infty(p_\infty)) + W_c((i_n)_\# \bar{m}_n^k, (i_\infty)_\# \bar{m}_\infty^k) \xrightarrow{n \rightarrow \infty} 0, \tag{2.16}$$

for any  $k \geq \bar{k}$ .

Note that we dropped the assumption on the Hausdorff distance between singular sets at the cost of presenting ourselves a space providing the extrinsic realization. Furthermore, by Proposition 2.18 we know that a iKRW-converging sequence converges also in the extrinsic manner.

We finish the Section with some remarks.

**Remark 2.20.** We observe that the convergence with respect to the Wasserstein distance have the following characterization (cf. [38, Section 7.1]):

$$\mu_n \xrightarrow{W_2} \mu \iff \mu_n \rightharpoonup \mu \text{ and } \int d(x_0, x)^2 d\mu_n \rightarrow \int d(x_0, x)^2 d\mu \quad \forall x_0 \in X.$$

This description shows in particular that for a sequence of probability measures with uniformly bounded support, the  $W_2$ -convergence is equivalent to the weak one and, consequently, to the  $W_c$ -convergence. Hence, in this case, (2.11) in Proposition 2.18 and Eq. (2.16) in Definition 2.19 remain valid when we replace the  $W_c$ -distance with the  $W_2$ -one.

**Remark 2.21** (*Connection to Gigli–Mondino–Savaré’s  $pG_W$  Distance*). In [33] the authors define a distance between  $\mathbb{X}_1$  and  $\mathbb{X}_2$  metric measure spaces endowed with Radon measures giving finite mass to bounded sets. This is what inspired us to propose the definition of  $d_{iKRW}$ : in fact, in this case the  $m$ -singular set of a metric measure space in such a family is the empty set and thus our definition coincides with theirs.

**Remark 2.22.** We recall that in [33, Theorem 3.17] the authors prove that the class of all metric measure spaces equipped with Radon measures is complete with respect to the  $pG_W$  distance. It is worth to underline that in our context we cannot hope for a similar completeness result. The main reason is that the set of all closed sets with empty interior is not closed for the Hausdorff distance. Hence, intuitively, we cannot prevent a sequence of quasi-Radon measures from converging to a measure which is not quasi-Radon.

### 3. CD condition for negative generalized dimension

#### 3.1. Basic definitions and properties

We introduce the Rényi entropy  $S_{N,m}$  for  $N < 0$  with respect to the reference measure  $m$  as the functional defined on  $\mathcal{P}(X)$  by posing

$$S_{N,m}(\mu) := \begin{cases} \int_X \rho(x)^{\frac{N-1}{N}} dm(x) & \text{if } \mu \ll m, \mu = \rho m, \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\rho = d\mu/dm$  is the Radon–Nikodym derivative of  $\mu$  with respect to  $m$ , whose existence is guaranteed by Theorem 2.10. In the following we will denote by  $\mathcal{P}^{ac}(X, m)$  the set of probability measures in  $\mathcal{P}_2(X)$  that are absolutely continuous with respect to the reference measure  $m$ .

**Remark 3.1.** We can already point out two major differences between the case  $N \geq 1$  and  $N < 0$ .

- when  $N \geq 1$  the Rényi entropy is defined as  $S_{N,m}(\rho\mathbf{m}) := - \int_{\mathbf{X}} \rho(x)^{\frac{N-1}{N}} d\mathbf{m}(x)$ , while when  $N < 0$ , the minus sign have to be left out to get the convexity of the function  $h(s) = s^{(N-1)/N}$ .
- for  $N \geq 1$ , the Rényi entropy is defined on Polish spaces equipped with Radon reference measures. In this case, under a volume growth condition on the reference measure  $\mathbf{m}$ , the functional  $S_{N,m}(\cdot)$  is lower semicontinuous with respect to the weak topology and, in particular, it is also lower semicontinuous with respect to the 2-Wasserstein convergence in  $\mathcal{P}_2(\mathbf{X})$ . Unfortunately, the same property is not necessarily true for negative values of  $N < 0$  and quasi-Radon reference measures  $\mathbf{m}$ . However, what we prove in Proposition 4.8 is that  $S_{N,m}(\cdot)$  is lower semicontinuous with respect to the weak convergence on the subspace

$$\mathcal{P}^{S_m}(\mathbf{X}) := \{ \mu \in \mathcal{P}_2(\mathbf{X}) : \mu(S_m) = 0 \text{ where } S_m \text{ is the } \mathbf{m}\text{-singular set} \}.$$

In fact, we show a more general result stating that the Rényi entropy functional  $S_{N,n}(\nu)$  is a lower semicontinuous function of  $(\mathbf{n}, \nu) \in M_{S_m}(\mathbf{X}) \times \mathcal{P}^{S_m}(\mathbf{X})$ , where the convergence of the first coordinate is intended to be the weak convergence of quasi-Radon measures.

In order to give the definition of curvature-dimension bounds, we need also to introduce the following distortion coefficients for  $K \in \mathbb{R}$  and  $N < 0$ :

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \leq N\pi^2, \\ \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 < K\theta^2 < 0, \\ t & \text{if } K\theta^2 = 0, \\ \frac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 > 0 \end{cases} \tag{3.1}$$

and

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}, \tag{3.2}$$

for every  $\theta \in [0, \infty)$  and  $t \in [0, 1]$ .

**Definition 3.2.** For any couple of measures  $\mu_0, \mu_1 \in \mathcal{P}^{ac}(\mathbf{X}, \mathbf{m})$ ,  $\mu_i = \rho_i \mathbf{m}$ , we denote by  $\pi \in \mathcal{P}(\mathbf{X} \times \mathbf{X})$  a coupling between them, and by  $T_{K,N}^t(\pi|\mathbf{m})$  the functional defined by

$$T_{K,N}^{(t)}(\pi|\mathbf{m}) := \int_{\mathbf{X} \times \mathbf{X}} \left[ \tau_{K,N}^{(1-t)}(\mathbf{d}(x,y)) \rho_0(x)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(\mathbf{d}(x,y)) \rho_1(y)^{-\frac{1}{N}} \right] d\pi(x,y).$$

We are ready to introduce the definition of metric measure spaces satisfying a curvature-dimension condition for negative values of the dimensional parameter.

**Definition 3.3 (CD Condition).** For fixed  $K \in \mathbb{R}, N \in (-\infty, 0)$ , we say that a metric measure space  $(\mathbf{X}, \mathbf{d}, \mathbf{m})$  satisfies the  $CD(K, N)$  condition if, for each pair  $\mu_0 = \rho_0 \mathbf{m}, \mu_1 = \rho_1 \mathbf{m} \in \mathcal{P}^{ac}(\mathbf{X}, \mathbf{m})$ , there exists an optimal coupling  $\pi \in \text{Opt}(\mu_0, \mu_1)$  and a  $W_2$ -geodesic  $\{\mu_t\}_{t \in [0,1]} \subset \mathcal{P}_2(\mathbf{X})$  such that

$$S_{N, \cdot}(\mu_t) \leq T_{K,N}^{(t)}(\pi|\mathbf{m}) \tag{3.3}$$

holds, for every  $t \in [0, 1]$ , and every  $N' \in [N, 0)$ , provided that  $S_{N', \cdot}(\mu_0), S_{N', \cdot}(\mu_1) < \infty$ .

We mention that a notion of  $CD(K, N)$  condition with  $N = 0$  was also introduced by Ohta in [39].

**Remark 3.4.** Note that the CD-inequality becomes trivial when  $K < 0$  and

$$\pi\left(\{(x, y) \in X \times X : d(x, y) \geq \pi\sqrt{(N' - 1)/K}\}\right) > 0.$$

and that the coefficients  $\tau_{K,N}^{(t)}(\cdot)$  are bounded whenever  $K \geq 0$  or  $\text{diam}(X) < \sqrt{\pi(N - 1)/K}$  for  $K < 0$ . Furthermore observe that Jensen’s inequality guarantees that  $S_{N,m}(\mu_0), S_{N,m}(\mu_1)$  are finite, provided that the entropies  $S_{N,m}(\mu_0), S_{N,m}(\mu_1)$  are finite for some  $N' \in [N, 0)$ . In this case the  $\text{CD}(K, N)$  condition guarantees that the Wasserstein geodesics along which the inequality (3.3) holds are absolutely continuous with respect to  $\mathbf{m}$ .

**Remark 3.5.** It is worth to underline that Definition 3.3 requires the finiteness of the entropies at the marginal measures, restricting the domain where inequality (3.3) have to be verified to the set  $\mathcal{D}(S_{N,m}) := \{\mu : S_{N,m}(\mu) < \infty\}$ . This is consistent with the standard definition of curvature-dimension condition for positive values of  $N$ : indeed, in the classical theory of curvature-dimension bounds for  $N \geq 1$ , the assumption on the finiteness of the entropy is not necessary, since the Rényi entropy is bounded on any absolutely continuous measure in  $P_2(X)$  as a consequence of the fact that  $\text{CD}(K, N)$  spaces possess reference measures with a controlled volume growth. A proof of the finiteness of some entropy functionals, in particular the Rényi one, under suitable volume growth assumptions can be found for example in [3, Proposition E.17].

Moreover, in the case in which the terminal marginals have bounded supports, Definition 3.3 coincides with the one introduced by Ohta in [22]. In fact, if the supports of  $\mu_0$  and  $\mu_1$  are bounded in  $(X, d)$ , then the coefficients  $\tau_{K,N}^{(1-t)}(\cdot)$  are bounded below away from 0 on the support of any coupling  $\pi \in \text{Opt}(\mu_0, \mu_1)$  for fixed  $0 < t < 1$ . Thus, if for some  $N' \in [N, 0)$  one of the terminal measures has unbounded entropy  $S_{N',m}$ , then  $T_{K,N}^{(t)}(\pi|\mathbf{m}) = \infty$ , for any  $t \in (0, 1)$ , and inequality (3.3) is always satisfied.

We underline that, as in the case  $N \geq 1$ , the definition of curvature-dimension condition is invariant under standard transformations of metric measure structures. Precisely, the CD condition is stable under isomorphisms, scalings, and restrictions to convex subsets of metric measure spaces (this can be proved using the same techniques as in [1, Propositions 4.12, 4.13 and 4.15]) and in [2, Proposition 1.4]. We also point out, that the “hierarchy property” of  $\text{CD}(K, N)$  spaces, with  $N < 0$ , remains valid. Specifically,

**Proposition 3.6.** *If  $(X, d, \mathbf{m})$  satisfies the curvature-dimension condition  $\text{CD}(K, N)$  for some  $K \in \mathbb{R}, N < 0$ , then it also satisfies the curvature-dimension condition  $\text{CD}(K', N')$  for any  $K' \leq K$  and  $N' \in [N, 0)$ .*

**Proof.** The monotonicity in  $N$  follows directly from Definition 3.3, while the monotonicity in  $K$  follows from the fact that the coefficient  $\sigma_\kappa^{(t)}(\theta)$  is non-decreasing in  $\kappa$  once  $t$  and  $\theta$  are fixed (see [40, Remark 2.2]).

Let us conclude by recalling that the  $\text{CD}(K, N)$  condition is weaker than the  $\text{CD}(K, \infty)$  one (see [22]) and it follows that  $\text{CD}(K, \infty)$  spaces are also  $\text{CD}(K, N)$  for every  $N < 0$ .

### 3.2. Examples

In this section we present some examples of negative dimensional CD spaces, referring to [23,24,28] for other model spaces satisfying the  $\text{CD}(K, N)$  condition with  $N < 0$ . Moreover, we show that singular points of the reference measure in negative dimensional CD spaces can appear as inner points of geodesics. This fact motivates us to present the definitions of *approximate CD condition* and  *$\omega$ -uniform convexity*, objects of Section 3.3, which will enable us to deal with this kind of behavior in the proof of our Stability Theorem.

A fundamental notion in the presentation of the examples is the one of  $(K, N)$ -convexity of a function on a metric space, for a negative value of  $N$ . This definition is the natural counterpart of the one with positive  $N$ , and it was introduced by Ohta in [22].

**Definition 3.7** ( $(K, N)$ -Convexity). In a metric space  $(X, d)$ , for every fixed  $K \in \mathbb{R}$  and  $N \in (-\infty, 0)$ , a function  $f : X \rightarrow \bar{\mathbb{R}}$  is said to be  $(K, N)$ -convex if for every  $x_0, x_1 \in \{f < +\infty\}$ , with  $d := d(x_0, x_1) < \pi\sqrt{N/K}$  when  $K < 0$ , there exists a constant speed geodesic  $\gamma$  connecting  $x_0$  and  $x_1$ , such that

$$f_N(\gamma_t) \leq \sigma_{K,N}^{(1-t)}(d)f_N(x_0) + \sigma_{K,N}^{(t)}(d)f_N(x_1) \quad \forall t \in [0, 1], \tag{3.4}$$

where  $f_N(x) = e^{-f(x)/N}$ .

The following result ([22, Corollary 4.12]) is used to produce examples of  $CD(K, N)$  spaces with negative values of the generalized dimension.

**Proposition 3.8.** *Let  $M$  be a  $n$ -dimensional complete Riemannian manifold with Riemannian distance  $d_g$  and Riemannian volume  $vol_g$ . Let us then consider a weighted volume measure  $\mathbf{m} = e^{-\psi} vol_g$ , for some function  $\psi : M \rightarrow \mathbb{R}$ , and let numbers  $K_1, K_2 \in \mathbb{R}$ ,  $N_2 \geq n$  and  $N_1 < -N_2$  be given.*

*Then if  $(M, d_g, \mathbf{m})$  satisfies the  $CD(K_2, N_2)$  condition, the weighted space  $(M, d_g, e^{-\psi} \mathbf{m})$  satisfies the  $CD(K_1 + K_2, N_1 + N_2)$  condition provided that  $\Psi \in C^2(M)$  is  $(K_1, N_1)$ -convex.*

**Example 3.9** (1-Dimensional Models). In the following we will denote by  $|\cdot|$  the Euclidean distance and by  $\mathcal{L}^1$  the 1-dimensional Lebesgue measure.

(i) For any pair of real numbers  $K > 0, N < -1$  the weighted space  $(\mathbb{R}, |\cdot|, V\mathcal{L}^1)$  with

$$V(x) = \cosh\left(x\sqrt{-\frac{K}{N}}\right)^N$$

satisfies the curvature-dimension condition  $CD(K, N + 1)$  with no singular set, i.e.  $\mathcal{S}_{V\mathcal{L}^1} = \emptyset$ .

(ii) For any pair of real numbers  $K > 0, N < -1$  also the weighted space  $([0, \infty), |\cdot|, V\mathcal{L}^1)$  with

$$V(x) = \sinh\left(x\sqrt{-\frac{K}{N}}\right)^N$$

satisfies the curvature-dimension condition  $CD(K, N + 1)$  with singular set  $\mathcal{S}_{V\mathcal{L}^1} = \{0\}$ .

(iii) For any  $N < -1$  the space  $([0, \infty), |\cdot|, x^N\mathcal{L}^1)$  is a  $CD(0, N + 1)$  space with singular set  $\mathcal{S}_{x^N\mathcal{L}^1} = \{0\}$ .

(iv) For any pair of real numbers  $K < 0, N < -1$  the weighted space

$$\left( \left[ -\frac{\pi}{2}\sqrt{\frac{N}{K}}, \frac{\pi}{2}\sqrt{\frac{N}{K}} \right], |\cdot|, \cos\left(x\sqrt{\frac{K}{N}}\right)^N \mathcal{L}^1 \right)$$

satisfies the curvature-dimension condition  $CD(K, N + 1)$  with singular set given by

$$\mathcal{S}_{\cos(x\sqrt{K/N})^N \mathcal{L}^1} = \left\{ -\frac{\pi}{2}\sqrt{\frac{N}{K}}, \frac{\pi}{2}\sqrt{\frac{N}{K}} \right\}.$$

Example 3.9 provides negative dimensional CD spaces, whose set of singular points is a subset of their topological boundary. Unfortunately, this is not a general behavior and we proceed now to show this. With this goal in mind, we will rely on a modification of Proposition 3.8, whose proof needs a preliminary result.

**Lemma 3.10.** Let  $f : I \rightarrow \bar{\mathbb{R}}$  be a function on the interval  $I := [a, b] \subset \mathbb{R}$ . Assume that there exists  $c \in (a, b) \cap \{f < +\infty\}$  such that  $f|_{[a,c]}$  and  $f|_{[c,b]}$  are  $(K, N)$ -convex and for every  $x_0 \in [a, c)$ ,  $x_1 \in (c, b]$  it holds that

$$f_N(c) \leq \sigma_{K,N}^{\left(\frac{x_1-c}{x_1-x_0}\right)}(x_1-x_0)f_N(x_0) + \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0)f_N(x_1). \tag{3.5}$$

Then  $f$  is  $(K, N)$ -convex.

**Proof.** We have to prove the convexity inequality (3.4) for every  $x_0, x_1 \in I$  in the domain  $\{f < \infty\}$ . However, this holds by hypothesis, if  $x_0, x_1 \in [a, c]$  or  $x_0, x_1 \in [c, b]$ , thus it is sufficient to consider the case where  $x_0 \in [a, c)$  and  $x_1 \in (c, b]$ . Without loss of generality, we can assume that  $x_t \in [a, c)$ , then the  $(K, N)$ -convexity of  $f|_{[a,c]}$  yields that

$$f_N(x_t) \leq \sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0)f_N(x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)f_N(c).$$

Combining this last inequality with (3.5) we obtain

$$f_N(x_t) \leq \left[ \sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) \sigma_{K,N}^{\left(\frac{x_1-c}{x_1-x_0}\right)}(x_1-x_0) \right] f_N(x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) f_N(x_1). \tag{3.6}$$

On the other hand it is easy to realize that

$$\sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) = \sigma_{K,N}^{\left(\frac{x_t-x_0}{x_1-x_0}\right)}(x_1-x_0).$$

As an example, we consider  $K < 0$ : it holds that

$$\begin{aligned} & \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) = \\ & \frac{\sin(\sqrt{K/N}(x_t-x_0))}{\sin(\sqrt{K/N}(c-x_0))} \cdot \frac{\sin(\sqrt{K/N}(c-x_0))}{\sin(\sqrt{K/N}(x_1-x_0))} = \frac{\sin(\sqrt{K/N}(x_t-x_0))}{\sin(\sqrt{K/N}(x_1-x_0))} = \\ & \sigma_{K,N}^{\left(\frac{x_t-x_0}{x_1-x_0}\right)}(x_1-x_0). \end{aligned}$$

Moreover, with an explicit computation, using the sum-to-product trigonometric formulas, it is also possible to prove that

$$\sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) \sigma_{K,N}^{\left(\frac{x_1-c}{x_1-x_0}\right)}(x_1-x_0) = \sigma_{K,N}^{\left(\frac{x_1-x_t}{x_1-x_0}\right)}(x_1-x_0).$$

Combining the previous trigonometric identities with inequality (3.6) we obtain the  $(K, N)$ -convexity inequality.

An immediate corollary follows from the fact that  $f_N(c) = 0$  if  $f(c) = -\infty$ .

**Corollary 3.11.** Let  $f : I \rightarrow \bar{\mathbb{R}}$  be a function on the interval  $I := [a, b]$ . Assume that there exists  $c \in (a, b)$  such that  $f|_{[a,c]}$  and  $f|_{[c,b]}$  are  $(K, N)$ -convex and that  $f(c) = -\infty$ , then  $f$  is  $(K, N)$ -convex.

Now, we present an alternative version of Proposition 3.8, in which we do not need to assume regularity of the weight function  $\psi$  at the price of restricting to the case  $M = \mathbb{R}$ .



**Proposition 3.12.** *Let  $\psi : \mathbb{R} \rightarrow [-\infty, \infty)$  be  $(K, N)$ -convex with  $N < -1$ , such that  $\mathcal{L}^1(\{\psi = -\infty\}) = 0$ . Then the metric measure space  $(\mathbb{R}, |\cdot|, e^{-\psi}\mathcal{L}^1)$  is a  $CD(K, N + 1)$  space.*

**Proof.** In this proof we denote with  $\mathbf{m}$  the reference measure  $e^{-\psi}\mathcal{L}^1$ . In order to prove the CD condition we fix two absolutely continuous measures  $\mu_0 = \rho_0\mathbf{m}, \mu_1 = \rho_1\mathbf{m} \in \mathcal{P}_2(\mathbb{R})$ . Notice that the assumption  $\mathcal{L}^1(\{\psi = -\infty\}) = 0$  ensures that  $\mu_0, \mu_1 \ll \mathcal{L}^1$  and we denote by  $\tilde{\rho}_0$  and  $\tilde{\rho}_1$  (respectively) their densities, that is  $\mu_0 = \tilde{\rho}_0\mathcal{L}^1$  and  $\mu_1 = \tilde{\rho}_1\mathcal{L}^1$ . Now, Brenier’s theorem ensures that there exists a unique optimal transport plan between  $\mu_0$  and  $\mu_1$ , and it is induced by a map  $T$ , which is differentiable  $\mu_0$ -almost everywhere. It is also well known that the map  $T$  is increasing, thus  $T'(x)$  will be non-negative when defined. Moreover, the unique Wasserstein geodesic connecting  $\mu_0$  and  $\mu_1$  is given by  $\mu_t = (T_t)_\# \mu_0$ , where  $T_t = (1 - t)\text{id} + tT$ . Then, calling  $\tilde{\rho}_t$  the density of  $\mu_t$  with respect to the Lebesgue measure  $\mathcal{L}^1$ , the Jacobi equation holds and gives that

$$\tilde{\rho}_0(x) = \tilde{\rho}_t(T_t(x))T_t'(x) = \tilde{\rho}_t(T_t(x))(1 + t(T'(x) - 1)),$$

for  $\mu_0$ -almost every  $x$ . On the one hand it is obvious that  $\tilde{\rho}_t = e^{-\psi}\rho_t$  for every  $t \in [0, 1]$ , therefore

$$e^{-\psi(x)}\rho_0(x) = e^{-\psi(T_t(x))}\rho_t(T_t(x))(1 + t(T'(x) - 1)) \tag{3.7}$$

for every  $t \in [0, 1]$  and  $\mu_0$ -almost every  $x$ . On the other hand notice that for every  $N' < -1$

$$\begin{aligned} S_{N'+1}(\mu_t) &= \int \rho_t^{-\frac{1}{N'+1}} d\mu_t = \int \rho_t^{-\frac{1}{N'+1}} d[(T_t)_\# \mu_0] = \int \rho_t(T_t(x))^{-\frac{1}{N'+1}} d\mu_0(x) \\ &= \int (e^{-\psi(x)}\rho_0(x))^{-\frac{1}{N'+1}} (e^{-\psi(T_t(x))}(1 + t(T'(x) - 1)))^{\frac{1}{N'+1}} d\mu_0(x). \end{aligned} \tag{3.8}$$

We can then prove the convexity pointwise, using the  $(K, N)$ -convexity of  $\psi$ . In particular,  $\psi$  is  $(K, N')$ -convex for every  $N' \in [N, 0)$  (cf. [22, Lemma 2.9]). Therefore, calling  $A(x) = T'(x) - 1$  in order to ease the notation, for every  $N' \in [N, -1)$ , it holds that

$$\begin{aligned} (e^{-\psi(T_t(x))}(1 + tA(x)))^{\frac{1}{N'+1}} &= [e^{-\psi(T_t(x))/N}]^{\frac{N}{N'+1}} (1 + tA(x))^{\frac{1}{N'+1}} \\ &\leq (1 + tA(x))^{\frac{1}{N'+1}} \left[ \sigma_{K,N}^{(1-t)}(|T(x) - x|)e^{-\psi(x)/N} + \sigma_{K,N}^{(t)}(|T(x) - x|)e^{-\psi(T(x))/N} \right]^{\frac{N}{N'+1}}. \end{aligned}$$

Then, by rewriting the last term, we obtain

$$\begin{aligned} &\left[ (1 + tA(x))^{\frac{1}{N}} \sigma_{K,N}^{(1-t)}(|T(x) - x|)e^{-\psi(x)/N} + (1 + tA(x))^{\frac{1}{N}} \sigma_{K,N}^{(t)}(|T(x) - x|)e^{-\psi(T(x))/N} \right]^{\frac{N}{N'+1}} \\ &= \left[ \frac{1 - t}{1 + tA(x)} \cdot \frac{(1 + tA(x))^{\frac{N+1}{N}}}{1 - t} \sigma_{K,N}^{(1-t)}(|T(x) - x|)e^{-\psi(x)/N} \right. \\ &\quad \left. + \frac{t(1 + A(x))}{1 + tA(x)} \cdot \frac{(1 + tA(x))^{\frac{N+1}{N}}}{t(1 + A(x))} \sigma_{K,N}^{(t)}(|T(x) - x|)e^{-\psi(T(x))/N} \right]^{\frac{N}{N'+1}} \\ &\leq \tau_{K,(N+1)}^{(1-t)}(|T(x) - x|)e^{-\psi(x)/(N+1)} + \tau_{K,(N+1)}^{(t)}(|T(x) - x|)(1 + A(x))^{\frac{1}{N+1}} e^{-\psi(T(x))/(N+1)}, \end{aligned}$$

where the last inequality follows from the definition of  $\tau_{K,N}^{(t)}$  (see (3.2)) and from the convexity inequality, applied to the function  $(\cdot)^{\frac{N}{N'+1}}$  with coefficients  $\frac{1-t}{1+tA(x)}$  and  $\frac{t(1+A(x))}{1+tA(x)}$ .

Substituting the above inequality into (3.8) and using (3.7) with  $t = 1$ , we obtain that

$$\begin{aligned} S_{N+1}(\mu_t) &\leq \int \tau_{K,(N+1)}^{(1-t)} (|T(x) - x|) \rho_0(x)^{-\frac{1}{N+1}} d\mu_0(x) \\ &\quad + \int \tau_{K,(N+1)}^{(t)} (|T(x) - x|) (1 + A(x))^{\frac{1}{N+1}} (e^{-\psi(x)} \rho_0(x))^{-\frac{1}{N+1}} e^{-\psi(T(x))/(N+1)} d\mu_0(x) \\ &= \int \tau_{K,(N+1)}^{(1-t)} (|T(x) - x|) \rho_0(x)^{-\frac{1}{N+1}} d\mu_0(x) + \int \tau_{K,(N+1)}^{(t)} (|T(x) - x|) \rho_1(T(x))^{-\frac{1}{N+1}} d\mu_0(x) \\ &= \int \left[ \tau_{K,(N+1)}^{(1-t)} (|y - x|) \rho_0(x)^{-\frac{1}{N+1}} + \tau_{K,(N+1)}^{(t)} (|y - x|) \rho_1(y)^{-\frac{1}{N+1}} \right] d[(\text{id} \times T)_{\#} \mu_0](x, y), \end{aligned}$$

for every  $N' \in [N, -1]$ , which is the desired inequality.

A direct application of the previous result leads to the following refinement of Example 3.9:

**Example 3.13.** (ii') For any pair of real numbers  $K > 0, N < -1$ , the weighted space

$$\left( \mathbb{R}, |\cdot|, V \mathcal{L}^1 \right) \quad \text{with } V(x) = \sinh \left( x \sqrt{-\frac{K}{N}} \right)^N,$$

obtained gluing two copies of the half-line space in Example 3.9-(ii), satisfies the curvature-dimension condition  $\text{CD}(K, N + 1)$  with singular set  $\mathcal{S}_{V \mathcal{L}^1} = \{0\}$ .

(iii') Similarly, for any  $N < -1$  the space  $(\mathbb{R}, |\cdot|, |x|^N \mathcal{L}^1)$  is a  $\text{CD}(0, N + 1)$  space with singular set  $\mathcal{S}_{|x|^N \mathcal{L}^1} = \{0\}$ .

(iv') For any pair of real numbers  $K < 0, N < -1$  the space which is obtained gluing  $J$ -copies of the interval in Example 3.9-(iv), for example by considering  $\left( \bigcup_{j=1}^J I_j, |\cdot|, V \mathcal{L}^1 \right)$  with

$$I_j := \left[ \frac{(2j-1)\pi}{2} \sqrt{\frac{N}{K}}, \frac{(2j+1)\pi}{2} \sqrt{\frac{N}{K}} \right] \quad \text{and} \quad V := \sum_{j=1}^J \mathbb{1}_{I_j} \cdot \cos \left( (x - x_j) \sqrt{\frac{K}{N}} \right)^N, \quad x_j := j\pi \sqrt{\frac{N}{K}},$$

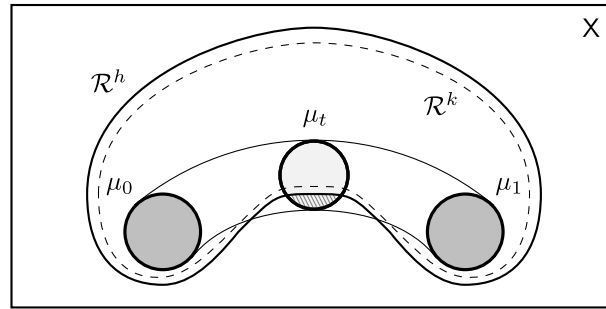
satisfies the curvature-dimension condition  $\text{CD}(K, N + 1)$  with singular set given by

$$\mathcal{S}_{V \mathcal{L}^1} = \left\{ \frac{(2j-1)\pi}{2} \sqrt{\frac{N}{K}} : j = 1, \dots, J + 1 \right\}.$$

We end this section by pointing out that there exist unbounded CD spaces with negative dimension for every value of the curvature. In particular, unlike to what happens for positive dimensional CD spaces, it is never possible to obtain a bound on the diameter of the space. Actually this not only happens for singular spaces, as in Example 3.13 (iv'): for example, also the hyperbolic plane satisfies the  $\text{CD}(-1, N)$  condition for any  $N < 0$  (recall that every  $\text{CD}(K, N)$  space with  $N \geq 1$  is automatically a  $\text{CD}(K, N)$  space for any  $N < 0$ ). Therefore, there exists no counterpart of the generalized Bonnet-Myers theorem [2, Corollary 2.6] for negative dimensional CD spaces. This is not completely surprising, since there exist also  $\text{CD}(K, \infty)$  spaces with  $K > 0$  (such as Gaussian spaces), having infinite diameter.

### 3.3. Approximate CD condition and regularity assumptions

Examples 3.9 and 3.13 exhibit that  $m$ -singular sets associated to  $\text{CD}(K, N)$  spaces are not necessarily empty sets. Moreover, as already noticed, the geometric behavior in these examples can be extremely different: in opposition to Examples 3.9, where points in  $\mathcal{S}_m$  appear only as terminal points of geodesics in the space, singular points in Examples 3.13 occur as inner points of geodesics. This observation shows



**Fig. 1.** A visual representation of the property of  $\omega$ -uniform convexity. In particular the shaded set has  $\mu_t$ -mass bounded above by  $\omega(k, h, M)$ , if  $S_{N,m}(\mu_0), S_{N,m}(\mu_1) \leq M$ .

that the  $k$ th regular set  $\mathcal{R}^k$  of a space  $X$ , which was introduced in (2.6), is not necessarily geodesically convex and this turns out to produce major difficulties in the proof of our main result. Indeed, we would like to approximate metric spaces equipped with purely quasi-Radon measures  $\mathfrak{m}$  (meaning that  $\mathcal{S}_{\mathfrak{m}} \neq \emptyset$ ) by considering their  $k$ th regular sets, but the CD condition is precisely a condition made on geodesics: the problems arising from the non-geodesic convexity of the regular sets will be the main challenge to overcome in the proof of the Stability Theorem 4.1. For this reason, we introduce the following two definitions.

**Definition 3.14 (Approximate CD Condition).** We say that a metric measure space  $(X, d, \mathfrak{m})$  satisfies the *approximate curvature-dimension condition*  $CD^a(K, N)$  if the CD condition in Definition 3.3 is satisfied by further requiring that the supports of the measures  $\mu_0, \mu_1 \in P^{ac}(X, \mathfrak{m})$  satisfy  $\text{supp}(\mu_0), \text{supp}(\mu_1) \subset \mathcal{R}^k$ , for some  $k \in \mathbb{N}$ .

Note that in the definition above  $k$  is not fixed.

As discussed above, we need to carefully approach the topic of the non-geodesic convexity of the regular sets  $\mathcal{R}^k$ , for some  $k \in \mathbb{N}$ . The following concept directs us in this direction by quantifying in terms of masses – and thus, controlling – to what extent convexity of the  $k$ th cuts is unsatisfied.

**Definition 3.15 ( $\omega$ -Uniform Convexity).** A metric measure space  $(X, d, \mathfrak{m})$  is  $\omega$ -uniformly convex if there exists a function  $\omega: \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \rightarrow [0, 1]$  with the following properties:

- for any  $\mu_0, \mu_1 \in P_2(X)$ , with  $S_{N,m}(\mu_0), S_{N,m}(\mu_1) \leq M$  and  $\text{supp}(\mu_0), \text{supp}(\mu_1) \subseteq \mathcal{R}^k$ , every  $t$ -middle point of any geodesic  $\{\mu_t\}_{t \in [0,1]} \subset P_2(X)$  between  $\mu_0$  and  $\mu_1$  satisfies

$$\mu_t(\mathcal{R}^h) \geq 1 - \omega(k, h, M) \text{ for any } h \in \mathbb{N};$$

- for any  $k \in \mathbb{N}, M \in \mathbb{R}^+$

$$\lim_{h \rightarrow \infty} \omega(k, h, M) = 0. \tag{3.9}$$

Fig. 1 shows a schematic representation of the  $\omega$ -uniform convexity.

**Remark 3.16.** We illustrate with some examples the concept of  $\omega$ -uniform convexity.

- \* If the  $k$ th regular set  $\mathcal{R}^k$  is geodesically convex, then we can choose  $\omega(\ell, h, M) = 0$ , for all entropy bounds  $M > 0$  and  $\ell \leq k \leq h$ . This is the case, for example, of metric measure spaces with empty  $\mathfrak{m}$ -singular set and the spaces presented in Example 3.9.

\* Conversely, if  $\mu_0, \mu_1 \in P_2(X)$  are supported in  $\mathcal{R}^k$  with bounded entropies and the support of  $\mu_t$ , a geodesic joining  $\mu_0$  and  $\mu_1$  evaluated at time  $t$ , is contained in the complement of  $\mathcal{R}^h$ , for some  $t \in (0, 1)$  and  $h \in \mathbb{N}$ , then  $\omega(k, h, M) = 1$ .

In particular, the metric measure space  $([-1, 1], |\cdot|, \mathbf{m})$ , with  $d\mathbf{m} = \delta_{-1} + \delta_1 + 1/x^2 d\mathcal{L}^1$  serves as an example of a metric measure space which is not  $\omega$ -uniformly convex. Indeed, at time  $t = 1/2$ , the support of the unique 2-Wasserstein geodesic  $(\delta_{2t-1})_{t \in [0,1]}$  is contained inside  $X \setminus \mathcal{R}^h$  for any  $h \in \mathbb{N}$ , since  $S_m = \{0\}$ , while its terminal points have entropy equal to 1.

\* Lastly, a more interesting behavior occurs in a convex subset of Example 3.13 given by

$$\left( [0, \pi], |\cdot|, \mathbb{1}_{[0, \frac{\pi}{2}]} \cos(x)^{-2} \mathcal{L}^1 + \mathbb{1}_{[\frac{\pi}{2}, \pi]} \cos(x - \pi)^{-2} \mathcal{L}^1 \right),$$

whose singular set is  $S_m = \{\frac{\pi}{2}\}$ . This is a  $CD(-2, -1)$  space as well as an  $\omega$ -uniformly convex space for a non-trivial function  $\omega(k, h, M)$ . Note that since there exist Wasserstein geodesics which are, at some time  $t$ , entirely contained in the complement of  $\mathcal{R}^k$ , there are actually some values  $k, h \in \mathbb{N}$ ,  $M \in \mathbb{R}^+$  for which  $\omega(k, h, M) = 1$ . Also, for fixed values of  $k \in \mathbb{N}$  and  $M \in \mathbb{R}^+$ ,  $\omega(k, h, M) \rightarrow 0$  as  $h \rightarrow \infty$ , since  $W_2$ -geodesics are absolutely continuous and  $V\mathcal{L}^1(X \setminus \mathcal{R}^h) \rightarrow 0$  as  $h \rightarrow \infty$ . Indeed, the key observation here is that we cannot force arbitrarily large amounts of mass to transit through  $X \setminus \mathcal{R}^h$ , at a given time, without losing the upper bound on the entropy of the terminal points. Intuitively, to produce such geodesics, we would have to consider measures with arbitrarily small supports or which accumulate arbitrarily large masses around a point. However, these type of measures have large entropy.

A concrete and useful property which  $\omega$ -uniformly convex metric measure spaces enjoy is that we are able to quantify interpolated mass outside the set  $\mathcal{R}^h$ , even if the marginals are not necessarily supported on  $\mathcal{R}^k$ , granted they supply sufficient mass to the  $k$ th regular sets. In the following, we will denote by  $\text{Geo}(X)$  the set of all the constant speed geodesics in the Polish space  $(X, d)$ .

**Proposition 3.17.** *Let  $(X, d, \mathbf{m})$  be an  $\omega$ -uniformly convex space. Then there exists a function  $\Omega : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \times [0, 1] \rightarrow \mathbb{R}$  such that:*

- (i) for any  $\mu_0, \mu_1 \in P_2(X)$  with  $S_{N, \mathbf{m}}(\mu_0), S_{N, \mathbf{m}}(\mu_1) \leq M$  and  $\mu_0(\mathcal{R}^k), \mu_1(\mathcal{R}^k) \geq 1 - \delta$ , any  $t$ -middle point of the geodesic  $\{\mu_t\}_{t \in [0,1]}$  satisfies  $\mu_t(\mathcal{R}^h) \geq 1 - \Omega(k, h, M, \delta)$  (see Fig. 2),
- (ii) for every  $0 \leq \delta < \frac{1}{4}$  it holds that

$$\limsup_{h \rightarrow \infty} \Omega(k, h, M, \delta) \leq 2\delta, \tag{3.10}$$

for every fixed  $k \in \mathbb{N}$  and  $M \in \mathbb{R}^+$ .

**Proof.** Notice that we can limit ourselves to the case when  $0 \leq \delta < \frac{1}{4}$ , because we can simply put  $\Omega(k, h, M, \delta) = 1$  if  $\delta \geq \frac{1}{4}$ . Fixed  $\mu_0, \mu_1 \in P_2(X)$  such that  $S_{N, \mathbf{m}}(\mu_0), S_{N, \mathbf{m}}(\mu_1) \leq M$  and  $\mu_0(\mathcal{R}^k), \mu_1(\mathcal{R}^k) \geq 1 - \delta$ , consider a  $t$ -middle point of a geodesic  $\{\mu_t\}_{t \in [0,1]}$ , connecting  $\mu_0$  and  $\mu_1$ . Let  $\eta \in P(\text{Geo}(X))$  be a representation of  $\{\mu_t\}_{t \in [0,1]}$  and define

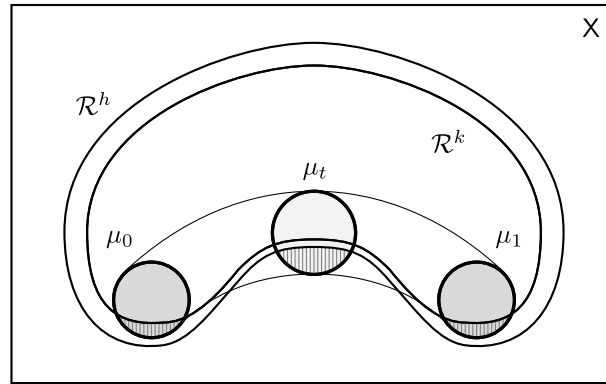
$$\tilde{\eta} := \frac{1}{\eta(G)} \cdot \eta|_G \in P(\text{Geo}(X)),$$

where

$$G := \{\gamma \in \text{Geo}(X) : \gamma(0), \gamma(1) \in \mathcal{R}^k\}.$$

Notice that  $\tilde{\eta}$  is actually well-defined, since our condition on  $\mu_0$  and  $\mu_1$  ensures that  $\eta(G) \geq 1 - 2\delta > 0$ . Moreover,

$$\eta = \eta(G) \cdot \tilde{\eta} + \bar{\eta} \quad \text{for some } \bar{\eta} \in \mathcal{M}(\text{Geo}(X)) \text{ with } \bar{\eta}(\text{Geo}(X)) \leq 2\delta. \tag{3.11}$$



**Fig. 2.** A visual representation of the property provided by the function  $\Omega$ , that is (i) in Proposition 3.17. In particular the shaded set in the center has  $\mu_t$ -mass bounded above by  $\Omega(k, h, M, \delta)$ , if the shaded set on the left and the one on the right have  $\mu_0$ -mass and  $\mu_1$ -mass (respectively) less than  $\delta$  and  $S_{N,m}(\mu_0), S_{N,m}(\mu_1) \leq M$ .

Observe that  $\{\tilde{\mu}_t = (e_t)_{\#}\tilde{\eta}\}_{t \in [0,1]}$  is a Wasserstein geodesic connecting two measures  $\tilde{\mu}_0$  and  $\tilde{\mu}_1$ , which are supported on  $\mathcal{R}^k$  and satisfy

$$\begin{aligned} \max\{S_{N,m}(\tilde{\mu}_0), S_{N,m}(\tilde{\mu}_1)\} &\leq \left[\frac{1}{\eta(G)}\right]^{1-\frac{1}{N}} \max\{S_{N,m}(\mu_0), S_{N,m}(\mu_1)\} \\ &\leq \left[\frac{1}{1-2\delta}\right]^{1-\frac{1}{N}} M \leq 2^{1-\frac{1}{N}} M. \end{aligned}$$

Then, the  $\omega$ -uniform convexity of  $(X, d, m)$  ensures that, for every  $h$ ,

$$\tilde{\mu}_t(\mathcal{R}^h) \geq 1 - \omega(k, h, 2^{1-\frac{1}{N}} M).$$

Moreover, taking into account (3.11), we can conclude that

$$\mu_t(\mathcal{R}^h) \geq (1 - 2\delta) \cdot \tilde{\mu}_t(\mathcal{R}^h) \geq 1 - \omega(k, h, 2^{1-\frac{1}{N}} M) - 2\delta.$$

Therefore, to satisfy (i), we can set

$$\Omega(k, h, M, \delta) := \omega(k, h, 2^{1-\frac{1}{N}} M) + 2\delta.$$

With this definition, (ii) is a straightforward consequence of the condition (3.9) on  $\omega(k, h, M)$ .

### 4. Stability of CD condition

In this last section we present the proof of our main result.

**Theorem 4.1 (Stability).** *Let  $K \in \mathbb{R}$ ,  $N \in (-\infty, 0)$ , and  $\{(X_n, d_n, m_n, S_{m_n}, p_n)\}_{n \in \mathbb{N}} \subset \mathcal{M}_k^{qR}$  be a sequence of pointed generalized metric measure spaces converging to  $(X_\infty, d_\infty, m_\infty, S_{m_\infty}, p_\infty) \in \mathcal{M}_k^{qR}$  in the iKRW-distance, for some  $\bar{k} \in \mathbb{N}$ . Assume further that:*

- (i)  $(X_n, d_n, m_n)$  is a  $CD(K, N)$  space for every  $n \in \mathbb{N}$ ;
- (ii) there exists  $\omega : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \rightarrow [0, 1]$ , for which  $(X_n, d_n, m_n)$  is  $\omega$ -uniformly convex, for every  $n \in \mathbb{N}$ ;
- (iii)  $\sup_{n \in \mathbb{N} \cup \{\infty\}} \text{diam}(X_n, d_n) < \pi \sqrt{\frac{1}{|K|}}$ , if  $K < 0$ .

Then  $(X_\infty, d_\infty, m_\infty)$  is a  $CD(K, N)$  space.

As a matter of fact, [Theorem 4.1](#) is concluded from the slightly more general statement below, since [Proposition 2.18](#) provides an effective realization for an iKRW-converging sequence of metric measure spaces. Recall that the extrinsic convergence of metric measure spaces is presented in [Definition 2.19](#).

**Theorem 4.2 (Extrinsic Stability).** *Let  $K \in \mathbb{R}$ ,  $N \in (-\infty, 0)$ . Then the  $CD(K, N)$  condition is stable under the extrinsic convergence of metric measure spaces, granted conditions (i)-(iii) from [Theorem 4.1](#) are satisfied by the converging sequence.*

The following is an immediate result of [Theorem 4.2](#).

**Corollary 4.3.** *Let  $K \in \mathbb{R}$ ,  $N \in (-\infty, 0)$ , and  $\{(X_n, d_n, m_n, S_{m_n}, p_n)\}_{n \in \mathbb{N}} \subset \mathcal{M}_k^{qR}$  be a sequence converging to  $(X_\infty, d_\infty, m_\infty, S_{m_\infty}, p_\infty) \in \mathcal{M}_{\bar{k}}^{qR}$ , in the extrinsic or intrinsic manner. Assume that the regular sets  $\mathcal{R}_n^k$  are geodesically convex, for all  $k, n \in \mathbb{N}$ . If for every  $n \in \mathbb{N}$  the space  $(X_n, d_n, m_n)$  satisfies the  $CD(K, N)$  condition (with  $\sup_{n \in \mathbb{N} \cup \{\infty\}} \text{diam}(X_n, d_n) < \pi|K|^{-1/2}$ , if  $K < 0$ ), then also  $(X_\infty, d_\infty, m_\infty)$  is a  $CD(K, N)$  space.*

When compared with [Stability Theorem 4.1](#), the advantage of [Extrinsic Stability Theorem 4.2](#), is that no assumptions have to be made, regarding the limiting behavior of singular sets along the sequence. This contrasts with the [Stability Theorem](#), which is stated in terms of the intrinsic  $d_{iKRW}$ -convergence, since the  $d_{iKRW}$ -distance controls the Hausdorff distance between singular sets. Therefore, with the latter Theorem one gains some flexibility to study the aforementioned sets; nevertheless, there is a price to pay in exchange. Namely, it is necessary to be in possession of an effective realization for the convergence. In this sense, we find that both results complement very well each other.

We fix some notation prior to outlining the argument in the proof of [Theorem 4.2](#).

First of all, since by assumption we have a realization of the convergence in a complete and separable metric space  $(Z, d_Z)$ , for simplicity we will identify all objects with their embedded version. In particular, for every  $n \in \mathbb{N} \cup \{\infty\}$  we will call  $X_n$  the embedded set  $i_n(X_n)$ ,  $m_n$  the push-forward measure  $(i_n)_\# m_n$  (and the same for its restricted and normalized versions),  $p_n$  the reference point  $i_n(p_n)$ . Moreover, since the embeddings are isometries, it will suffice to work with the distance  $d_Z$ , which from now on will be denoted by  $d$ , for sake of simplicity. With this identification, our extrinsically convergent sequence of pointed generalized metric measure spaces  $\{X_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_k^{qR}$ , which converges to  $X_\infty \in \mathcal{M}_{\bar{k}}^{qR}$ , satisfies

$$\left| \log \left( \frac{m_n^k(X_n)}{m_\infty^k(X_\infty)} \right) \right| + d(p_n, p_\infty) + W_2(\bar{m}_n^k, \bar{m}_\infty^k) \xrightarrow{n \rightarrow \infty} 0, \tag{4.1}$$

for any  $k \geq \bar{k}$ . Notice that it was possible to put the Wasserstein distance  $W_2$  in [\(4.1\)](#), according to [Remark 2.20](#).

In the remainder, we use the adjective *horizontal* to refer to the approximations we construct inside a fixed space  $X_n$ , for  $n \in \mathbb{N}$ . Respectively, we denote as *vertical* approximations those approximations made over the sequence  $X_n \rightarrow X_\infty$ , when we let  $n \rightarrow \infty$ . Our objective is, naturally, to demonstrate, for every pair of measures  $\mu_0, \mu_1 \in P^{ac}(X_\infty, m_\infty)$ , the existence of a 2-Wasserstein geodesic  $\{\mu_t\}_{t \in [0,1]} \subset P_2(X_\infty)$  and an optimal plan  $q \in \text{Opt}(\mu_0, \mu_1)$ , for which the curvature-dimension inequality [\(3.3\)](#) is satisfied. We accomplish this by following the next steps.

1. We assume that  $\text{supp}(\mu_i) \subseteq \mathcal{R}_\infty^k$ , for  $i \in \{0, 1\}$  and fixed  $k \in \mathbb{N}$ , and construct a geodesic  $(\mu_t)_{t \in [0,1]} \subset P_2(X_\infty)$  between  $\mu_0$  and  $\mu_1$  and an optimal plan  $q \in \text{Opt}(\mu_0, \mu_1)$ , for which the  $CD$ -inequality [\(3.3\)](#) is fulfilled, relying on the following vertical approximation argument. (Above, and in the following, we write  $\mathcal{R}_n^k \subset X_n$  to denote the  $k$ th  $m_n$ -regular set of  $X_n$ ,  $k$ -regular set in short, for  $n \in \mathbb{N} \cup \{\infty\}$ .)



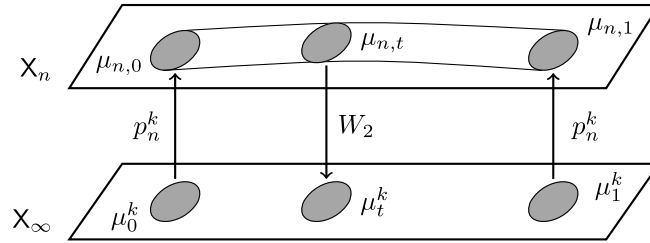


Fig. 3. Approximation procedure for the midpoints.

The assumption on the supports allows us to approximate vertically the marginal measures  $\mu_0$  and  $\mu_1$ , by employing a canonical map between Wasserstein spaces  $P_n^k : P^{ac}(X_\infty, \mathbf{m}_\infty^k) \rightarrow P^{ac}(X_n, \mathbf{m}_n^k)$ , induced via an optimal coupling of the normalized reference measures  $p_n^k \in \text{Opt}(\bar{\mathbf{m}}_\infty^k, \bar{\mathbf{m}}_n^k)$ . Let us denote these approximations by  $(\mu_{n,i})_{n \in \mathbb{N}}$ , for  $i \in \{0, 1\}$ .

At this point we construct the pair  $(\mu_t, q)$  as the vertical limits of a sequence of geodesics  $(\mu_{n,t})_{t \in [0,1]} \subset P_2(X_n)$ , between  $\mu_{n,0}$  and  $\mu_{n,1}$ , and a sequence of optimal plans  $q_n \in \text{Opt}(\mu_{n,0}, \mu_{n,1})$ , both indexed by  $n \in \mathbb{N}$ . Furthermore, we provide these sequences using the CD-hypothesis on  $(X_n, d_n, \mathbf{m}_n)$ , so in particular we can guarantee that, for  $n \in \mathbb{N}$ , each pair  $(\mu_{n,t}, q_n)$  satisfies the CD-inequality, for every  $t \in [0, 1]$ .

After demonstrating the lower semicontinuity  $S_{N, \mathbf{m}}(\mu_t) \leq \liminf_{n \rightarrow \infty} S_{N, \mathbf{m}_n}(\mu_{n,t})$  and the upper semicontinuity  $\limsup_{n \rightarrow \infty} T_{K,N}^{(t)}(q_n | \mathbf{m}_n) \leq T_{K,N}^{(t)}(q | \mathbf{m}_\infty)$ , along our sequences as  $n \rightarrow \infty$ , we conclude the validity of the CD-inequality (3.3) for  $(\mu_t, q)$ , for every  $t \in [0, 1]$ .

(Look at Fig. 3 for a schematic representation)

2. Additionally, we produce, for  $i \in \{0, 1\}$ , favorable horizontal approximations  $(\mu_i^k)_{k \in \mathbb{N}} \subset P^{ac}(X_\infty, \mathbf{m}_\infty)$ ,  $W_2$ -converging to  $\mu_i$ , whose supports satisfy  $\text{supp}(\mu_i^k) \subseteq \mathcal{R}_\infty^k$ , for every  $k \in \mathbb{N}$ . Subsequently, by approximating with the pairs constructed in Step 1, we show the existence of the sought geodesic  $\{\mu_t\}_{t \in [0,1]} \subset P_2(X_\infty)$  and optimal plan  $q \in \text{Opt}(\mu_0, \mu_1)$ . After showing that the appropriate semicontinuity of the functionals  $S_{N, \mathbf{m}}(\cdot)$  and  $T_{K,N}^{(t)}(\cdot | \mathbf{m}_\infty)$  hold, we are able to verify the CD condition and conclude.

The rough idea behind a proof of geometric stability in Wasserstein spaces is well known: for well-behaved measures, as a first step one shows that  $P(X_\infty)$  inherits the CD-convexity property directly from the stability of the geometry of  $P(X_n)$  under vertical approximations. Hence one can conclude the same property for more general measures by approximating them horizontally using a sequence of well-behaved measures. To pursue this plan, the properties of semicontinuity and precompactness play a crucial role.

Inspired by the techniques used in [3], we manage to provide a Legendre-type representation formula for the entropy, which handles one of the functionals in question. Therefore a suitable generalization of the arguments in [2] allows to conclude the upper semicontinuity of  $T_{K,N}^{(t)}(\cdot | \mathbf{m}_\infty)$ .

The very challenging obstacles in the proof of the stability theorem appear when we approach the problem of the existence of limits and of the convergence of inner points of geodesics. We empathize that the general class of metric measure structures under consideration is not even locally compact, while the “wildness” of quasi-Radon measures prohibit us to control the reference measures in any uniform way, preventing in particular to recover any tightness results from them. Thus, to overcome these problems, we propose some original arguments.

The crucial ingredient to get back into track will be the control of the mass given by Wasserstein geodesics to  $\mathbf{m}$ -singular sets when taking the limits and this can be extracted from the  $\omega$ -uniform convexity.

We advance to the presentation of some auxiliary results in the next Section. The vertical approximation argument is presented afterwards in Section 4.2, while Step 2. above is discussed in the final Section 4.3.

### 4.1. Auxiliary results

We collect in this section the preliminary results needed to prove [Theorem 4.1](#). In particular, in the first part we present the tools which turn out to be useful in approximating  $t$ -midpoints of geodesics, while in the consecutive subsection we deal with the required semicontinuity results.

#### 4.1.1. Approximation and compactness results

We start by exhibiting the existence of well-behaved horizontal approximations to measures. Recall that for a reference measure  $\mathbf{m} \in M^{qR}(\mathbf{X})$ ,  $\mathbf{m}^k$  is its  $k$ -cut defined by [\(2.5\)](#).

**Lemma 4.4.** *Let  $(\mathbf{X}, d, \mathbf{m})$  be a metric measure space,  $\mathbf{m} \in M^{qR}(\mathbf{X})$ , and  $\mu \in P^{ac}(\mathbf{X}, \mathbf{m})$ . Then:*

- (1) *The sequence of measures  $\{\mathbf{m}^k\}_{k \in \mathbb{N}}$  approximates  $\mathbf{m}$ , in the sense of quasi-Radon measures:*

$$\mathbf{m}^k \rightharpoonup \mathbf{m}.$$

- (2) *There exists a sequence of measures  $\{\mu^k\}_{k \in \mathbb{N}} \subset P^{ac}(\mathbf{X}, \mathbf{m})$ ,  $\mu^k \ll \mathbf{m}^k$  for any  $k \in \mathbb{N}$ , converging to  $\mu$  in the  $W_2$ -distance. In particular, for every  $k \in \mathbb{N}$ , we have that  $\text{supp}(\mu^k) \subseteq \mathcal{R}^k := B_{2^{k+1}}(p) \setminus \mathcal{N}_{2^{-(k+1)}}(\mathcal{S}_m)$ , thus these measures have bounded support.*

**Proof.** We start noticing that (1) follows directly from the definition of weak convergence because  $\text{supp}(f) \subseteq \mathcal{R}^k$  holds eventually, for any function  $f \in C_{bs}(\mathbf{X}) \cap C_{\mathcal{S}_m}(\mathbf{X})$ .

As for (2), let us consider  $\mu^k := c^k f^k \mu$ , where  $f^k$  is the cut-off function defined in [\(2.5\)](#),  $c^k$  is the normalization constant providing  $\mu^k(\mathbf{X}) = 1$ , and  $k$  is a sufficiently large number, the estimate of which will be determined along the proof. Clearly,  $\mu^k \ll \mathbf{m}^k$ . At this point we recall that  $\text{supp}(f^k) \subset \mathcal{R}^k$  with  $0 \leq f^k \leq 1$  for any  $k \in \mathbb{N}$ , and that  $f^k \rightarrow 1$  pointwise  $\mathbf{m}$ -almost everywhere as  $k \rightarrow \infty$ . As a consequence,  $f^k \rightarrow 1$  pointwise  $\mu$ -almost everywhere, and  $\text{supp}(\mu^k)$  is bounded since  $\text{supp}(\mu^k) \subset \text{supp}(\mathbf{m}^k) \subset \mathcal{R}^k$ .

By choosing  $k_0$  sufficiently large, we can assume that  $\text{supp}(\mu) \cap \mathcal{R}^k \neq \emptyset$ , for all  $k \geq k_0$ . (Although the particular choice of  $k_0$  does depend on  $\mu$ , there is no loss of generality, since such a bound exists for every measure  $\mu$  and we are interested exclusively in the limit behavior of  $\mu^k$ .) Let  $c^k := (\int_{\mathbf{X}} f^k d\mu)^{-1}$ : in view of the previous remarks,  $c^k$  is well-defined and monotone decreasing in  $k \in \mathbb{N}$  and  $\lim_{k \rightarrow \infty} c^k = 1$ .

We can then conclude using the dominated convergence theorem, recalling that a sequence of measures is  $W_2$ -convergent if and only if it is weakly convergent and the sequence of its second moments is also convergent.

Take two complete and separable metric measure spaces  $(\mathbf{X}, d_X, \mathbf{m}_X)$  and  $(\mathbf{Y}, d_Y, \mathbf{m}_Y)$  embedded in  $Z$ , such that  $\mathbf{m}_X$  and  $\mathbf{m}_Y$  are probability measures. Recall that, given a coupling  $p \in \text{Adm}(\mathbf{m}_X, \mathbf{m}_Y)$ , we can consider a canonical map between their Wasserstein spaces  $P : P^{ac}(\mathbf{X}, \mathbf{m}_X) \rightarrow P^{ac}(\mathbf{Y}, \mathbf{m}_Y)$ , which is induced by pushing forward weighted versions of the coupling  $p$ . We refer to these maps as *Weighted Marginalizations* and we will use them to produce vertical approximations.

In detail, for each  $n, k \in \mathbb{N}$  we consider (and fix) an optimal coupling  $p_n^k \in \text{Opt}(\bar{\mathbf{m}}_\infty^k, \bar{\mathbf{m}}_n^k)$ . Here  $\bar{\mathbf{n}} = \mathbf{n}(\mathbf{Y})^{-1} \mathbf{n}$  denotes the normalization of a finite measure  $\mathbf{n} \in M(\mathbf{Y})$ . We then write  $\{P_{n,k}(x)\}_{x \in \mathbf{X}} \subset P(\{x\} \times \mathbf{X}_n) \approx P(\mathbf{X}_n)$  and  $\{P'_{n,k}(y)\}_{y \in \mathbf{X}_n} \subset P(\mathbf{X}_\infty \times \{y\}) \approx P(\mathbf{X}_\infty)$  the disintegration kernels of the coupling  $p_n^k$  with respect to the projection maps  $\mathbf{p}_1 : \mathbf{X}_\infty \times \mathbf{X}_n \rightarrow \mathbf{X}_\infty$  and  $\mathbf{p}_2 : \mathbf{X}_\infty \times \mathbf{X}_n \rightarrow \mathbf{X}_n$ , respectively. More precisely, for  $\bar{\mathbf{m}}_\infty^k$ -a.e.  $x \in \mathbf{X}_\infty$ , and  $\bar{\mathbf{m}}_n^k$ -a.e.  $y \in \mathbf{X}_n$ , we let  $P_{n,k}(x)$  and  $P'_{n,k}(y)$  be the measures given by the Disintegration Theorem, which are characterized by

$$p_n^k(A) = \int_{\mathbf{X}} P_{n,k}(x)(A_x) d\bar{\mathbf{m}}_\infty^k(x) = \int_{\mathbf{X}_n} P'_{n,k}(y)(A_y) d\bar{\mathbf{m}}_n^k(y).$$

for every measurable  $A \subset X_\infty \times X_n$ , where  $A_x = \{y : (x, y) \in A\}$  and  $A_y = \{x : (x, y) \in A\}$ . Furthermore, we can define the Weighted Marginalization maps between Wasserstein spaces via the push forward along the coordinate projections of the weighted couplings  $\rho p_n^k$ . With a slight abuse of notation, we denote again these maps as  $P'_{n,k}$  and  $P_{n,k}$ . Specifically, let

$$\begin{aligned} P'_{n,k} : P^{ac}(X_\infty, \mathbf{m}_\infty^k) &\rightarrow P^{ac}(X_n, \mathbf{m}_n^k) \\ \mu = \rho \bar{\mathbf{m}}_\infty^k &\mapsto P'_{n,k}(\mu) := (\mathfrak{p}_2)_\# \rho p_n^k = \rho' \bar{\mathbf{m}}_n^k, \\ &\text{with } \rho'(y) = \int_X \rho(x) P'_{n,k}(y)(dx). \end{aligned} \tag{4.2}$$

The map  $P_{n,k} : P^{ac}(X_n, \mathbf{m}_n^k) \rightarrow P^{ac}(X_\infty, \mathbf{m}_\infty^k)$  is defined in an analogous manner. Note that, in particular,  $\rho p_n^k \in \text{Adm}(\mu, P'_{n,k}(\mu))$ . The following lemma shows that the well-known properties of the Weighted Marginalization map  $P'_{n,k}$  are still valid in our framework.

**Lemma 4.5.** *Let  $\mu = \rho \mathbf{m}_\infty^k \in P_2(X_\infty)$ , then  $P'_{n,k}$  satisfies the following properties:*

(i) *For every  $N < 0$  the functional  $S_{N,\cdot}(\cdot)$  satisfies the contraction property:*

$$\begin{aligned} S_{N,\mathbf{m}_n^k}(P'_{n,k}(\mu)) &= \mathbf{m}_n^k(X_n)^{\frac{1}{N}} S_{N,\bar{\mathbf{m}}_n^k}(P'_{n,k}(\mu)) \\ &\leq \mathbf{m}_n^k(X_n)^{\frac{1}{N}} S_{N,\bar{\mathbf{m}}_\infty^k}(\mu) = \left[ \frac{\mathbf{m}_n^k(X_n)}{\mathbf{m}_\infty^k(X_\infty)} \right]^{\frac{1}{N}} S_{N,\mathbf{m}_\infty^k}(\mu). \end{aligned} \tag{4.3}$$

(ii) *If the density  $\rho$  of  $\mu$  is bounded, then the Wasserstein convergence holds:*

$$W_2^2(\mu, P'_{n,k}(\mu)) \leq \int d^2(x, y) \tilde{\rho}(x) dp_n^k(x, y) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where  $\tilde{\rho} = \mathbf{m}_\infty^k(X_\infty) \rho$  is the density of  $\mu$  with respect to the normalized measure  $\bar{\mathbf{m}}_\infty^k$ .

**Proof.** Observe that the two equalities in (4.3) are obvious, then we just have to prove the inequality. Consequently (i) follows directly from Jensen’s inequality applied to the convex function  $\psi(r) := r^{1-\frac{1}{N}}$ . Indeed,

$$\begin{aligned} S_{N,\bar{\mathbf{m}}_n^k}(P'_{n,k}(\mu)) &= \int_{X_n} \left[ \int_X \tilde{\rho}(x) P'_{n,k}(y)(dx) \right]^{1-\frac{1}{N}} d\bar{\mathbf{m}}_n^k(y) \\ &\leq \int_{X_n} \int_X \tilde{\rho}(x)^{1-\frac{1}{N}} P'_{n,k}(y)(dx) d\bar{\mathbf{m}}_n^k(y) = S_{N,\bar{\mathbf{m}}_\infty^k}(\mu). \end{aligned}$$

Regarding (ii), notice that, since  $\rho$  is bounded, the same holds for  $\tilde{\rho}$ . Moreover, we have that  $\tilde{\rho} p_n^k \in \text{Adm}(\mu, P'_{n,k}(\mu))$ , and consequently

$$W_2^2(\mu, P'_{n,k}(\mu)) \leq \int d^2(x, y) \tilde{\rho}(x) dp_n^k(x, y) \leq \|\tilde{\rho}\|_{L^1(\mathbf{m}_n^k)} W_2^2(\bar{\mathbf{m}}_\infty^k, \bar{\mathbf{m}}_n^k) \rightarrow 0.$$

The last result we are going to prove in this subsection is useful to conclude tightness for a sequence of measures, provided that we have a uniform bound on their Rényi entropies, and a tightness condition on the reference measures. The analogous result stated for the relative entropy functional was proven in [33, Proposition 4.1], and this proof can be easily adapted.

**Lemma 4.6.** *Let  $\{\mathbf{n}_n\}_{n \in \mathbb{N}}, \{\mu_n\}_{n \in \mathbb{N}} \subset P(Z)$  be two sequences of measures such that  $\{\mathbf{n}_n\}_{n \in \mathbb{N}}$  is tight and  $\sup_{n \in \mathbb{N}} S_{N,\mathbf{n}_n}(\mu_n) < \infty$ . Then  $\{\mu_n\}_{n \in \mathbb{N}}$  is tight.*

**Proof.** First of all we observe that, being the entropy bounded, we can write  $\mu_n = \rho_n \mathbf{n}_n$ . Thus, a direct application of Jensen’s inequality gives that, for every  $n \in \mathbb{N}$  and for every measurable set  $E \subset Z$ ,

$$\frac{\mu_n(E)^{1-\frac{1}{N}}}{\mathbf{n}_n(E)^{1-\frac{1}{N}}} \leq \frac{1}{\mathbf{n}_n(E)} \int_E \rho_n^{1-\frac{1}{N}} d\mathbf{n}_n \leq \frac{S_{N,\mathbf{n}_n}(\mu_n)}{\mathbf{n}_n(E)}.$$

The tightness of  $\{\mathbf{n}_n\}_{n \in \mathbb{N}}$  assures the existence of a sequence of compact sets  $\{D_l\}_{l \in \mathbb{N}}$  such that  $\sup_{n \in \mathbb{N}} \mathbf{n}_n(Z \setminus D_l) \rightarrow 0$  as  $l \rightarrow \infty$ . We write  $E_l = Z \setminus D_l$  and we conclude from the above inequality that  $\{\mu_n\}_{n \in \mathbb{N}}$  is tight, since

$$\sup_{n \in \mathbb{N}} \mu_n(E_l)^{1-\frac{1}{N}} \leq \sup_{n \in \mathbb{N}} \mathbf{n}_n(E_l)^{-\frac{1}{N}} \sup_{n \in \mathbb{N}} S_{N,\mathbf{n}_n}(\mu_n) \xrightarrow{l \rightarrow \infty} 0.$$

This result can be applied to our extrinsic converging sequence  $\{\mathbb{X}_n\}_{n \in \mathbb{N} \cup \{\infty\}} \subset \mathcal{M}_{\frac{q}{k}}^{qR}$ , with a straightforward normalization argument by recalling that  $\mathbf{m}_n^k(\mathbb{X}_n)$  approaches  $\mathbf{m}_\infty^k(\mathbb{X}_\infty)$  as  $n \rightarrow \infty$ , for every suitable  $k$ .

**Corollary 4.7.** *Given a fixed  $k \geq \bar{k}$ , and  $\{\mu_n\}_{n \in \mathbb{N}} \subset P(Z)$  a sequence of probability measures, such that  $\sup_{n \in \mathbb{N}} S_{N,\mathbf{m}_n^k}(\mu_n) < \infty$ , then  $\{\mu_n\}_{n \in \mathbb{N}}$  is tight.*

#### 4.1.2. Semicontinuity properties

We present semicontinuity properties of  $S_{N,\cdot}(\cdot)$  and  $T_{K,N}^{(t)}(\cdot|\cdot)$  conditioned to their domain of definition. We start by proving the lower semicontinuity of  $S_{N,\cdot}(\cdot)$  that we anticipated in Section 3.1. This property is well known in classical frameworks, that is, for positive values of  $N$  and well-behaved reference measures. For example, lower semicontinuity for a big class of functionals, in which the Rényi entropy is included, was proved in [3] for locally compact spaces endowed with reference measures having uniformly bounded volume growth. Inspired by the techniques used in [3], we provide below a Legendre-type representation formula for the entropy to attain our result. With this aim, let us write

$$P^{\mathcal{S}}(\mathbb{X}) := \{\mu \in P_2(\mathbb{X}) : \mu(\mathcal{S}) = 0\}.$$

**Proposition 4.8.** *Let  $(\mathbb{X}, d, p)$  be a pointed Polish space and  $\mathcal{S} \subset \mathbb{X}$  a closed subset with empty interior. Then the Rényi entropy functional  $S_{N,\mathbf{n}}(\nu)$  is a lower semicontinuous function of  $(\mathbf{n}, \nu) \in M_{\mathcal{S}}(\mathbb{X}) \times P^{\mathcal{S}}(\mathbb{X})$ . Specifically, for sequences*

$$(\mathbf{n}_n)_{n \in \mathbb{N} \cup \{\infty\}} \subset M_{\mathcal{S}}(\mathbb{X}) \text{ and } (\nu_n)_{n \in \mathbb{N} \cup \{\infty\}} \subset P^{\mathcal{S}}(\mathbb{X}),$$

such that  $\mathbf{n}_n \rightharpoonup \mathbf{n}_\infty$  as quasi-Radon measures and  $\nu_n \rightharpoonup \nu_\infty$ , we have that,

$$S_{N,\mathbf{n}}(\nu_\infty) \leq \liminf_{n \rightarrow \infty} S_{N,\mathbf{n}_n}(\nu_n).$$

In particular, the conclusion remains valid under  $W_2$ -convergence in the second argument.

**Proof.** The semicontinuity of the Rényi entropy functional is verified by exhibiting  $S_{N,\mathbf{m}}$  as the supremum of a set of continuous functions on  $M_{\mathcal{S}}(\mathbb{X}) \times P^{\mathcal{S}}(\mathbb{X})$  endowed with the corresponding product convergence. In particular we define  $\mathcal{R}^k := B_{2^{k+1}}(p) \setminus \mathcal{N}_{2^{-(k+1)}}(\mathcal{S})$  and we show that, for every pair  $(\mathbf{m}, \mu) \in M_{\mathcal{S}}(\mathbb{X}) \times P^{\mathcal{S}}(\mathbb{X})$ ,

$$S_{N,\mathbf{m}}(\mu) = \sup \left\{ \int F d\mu - \int f^*(F) d\mathbf{m} : F \in C_b(\mathbb{X}) \text{ supported in } \mathcal{R}^k, \text{ for some } k \in \mathbb{N} \right\}, \tag{4.4}$$

where  $f^*$  is the convex conjugate function of  $f(x) = |x|^{1-\frac{1}{N}}$ , for  $x \in \mathbb{R}$ . That is,

$$f^* : \mathbb{R} \rightarrow [0, \infty)$$

$$y \mapsto f^*(y) := \sup_{x \in \mathbb{R}} (yx - f(x)) = -\frac{1}{N} \left( \frac{N}{N-1} \right)^{1-N} |y|^{1-N}.$$

Proving (4.4) will be enough to deduce the lower semicontinuity of the functional  $S_{N, \cdot}(\cdot)$ , in fact, for every  $F \in C_b(X)$  supported in some  $\mathcal{R}^k$ , the functional

$$(\mu, \mathbf{m}) \mapsto \int F d\mu - \int f^*(F) d\mathbf{m}$$

is indeed (weakly) continuous in  $M_S(X) \times P^S(X)$ , since  $f^*(F) \in C_{bs}(X) \cap C_S(X)$ . As a result, the functional  $S_{N, \cdot}(\cdot)$  will be the supremum of (weakly) continuous functionals, and thus it will be (weakly) lower semicontinuous. For simplicity, let us denote by  $\mathfrak{S}_{N, \mathbf{m}}(\mu)$  the expression on the right-hand side of (4.4).

We first verify that  $S_{N, \mathbf{m}}(\mu) \geq \mathfrak{S}_{N, \mathbf{m}}(\mu)$ , for every  $(\mathbf{m}, \mu) \in M_S(X) \times P^S(X)$ . We assume that  $\mu = \rho\mathbf{m}$  since the aforementioned inequality is trivially satisfied when  $\mu \ll \mathbf{m}$ . We get the desired result, integrating the expression  $f(z) \geq zy^* - f^*(y^*)$  with respect to  $\mathbf{m}$  which, by definition of  $f^*$ , holds for any  $z, y^* \in \mathbb{R}$ , after replacing  $z = \rho(x)$  and  $y^* = F(x)$ , for  $F \in C_b(X)$  with support inside some  $\mathcal{R}^k$ .

Before proceeding with the converse inequality let us point out that  $\mathfrak{S}_{N, \mathbf{m}}(\mu) = \infty$  granted  $\mu \not\ll \mathbf{m}$ . Indeed, in this case, there exists a Borel set  $A \subset X$  with  $\mathbf{m}(A) = 0$  and  $\mu(A) > 0$ . At this point, recall that every Borel finite measure in a Polish spaces is inner regular with respect to compact sets and outer regular with respect to open sets (see for instance [35, Theorem 7.1.7]). For this reason, since  $\mu \in P^S(X)$  and  $A$  is a compact set, it is possible to assume that  $A \cap S = \emptyset$ . Observe also that compactness grants the existence of  $k \in \mathbb{N}$  for which  $A \subset \mathcal{R}^k$ . Since  $\mathbf{m}$  and  $\mu$  restricted to  $\mathcal{R}^{k+1}$  are finite measures, there exist a sequence of compact sets  $(K_n)_{n \in \mathbb{N}}$  and one of open sets  $(A_n)_{n \in \mathbb{N}}$  such that  $K_n \subset A \subset A_n \subset \mathcal{R}^{k+1}$  and  $(\mu + \mathbf{m})(A_n \setminus K_n) < 1/n$ , for any  $n \in \mathbb{N}$ . Then for  $M > 0$  Tietze's Theorem ensures the existence of a sequence of approximating functions  $(F_n^M)_{n \in \mathbb{N}} \subset C_b(X)$  satisfying:  $0 \leq F_n^M \leq M$ ,  $F_n^M = M$  on  $K_n$ , and  $F_n^M = 0$  on  $X \setminus A_n$ . Therefore, as  $n \rightarrow \infty$ , the functions  $F_n^M$  converge, in  $L^1(\mu + \mathbf{m})$ , to the scaled characteristic function  $M \cdot \mathbb{1}_A$ . On the other hand, since for every  $n$ ,  $F_n^M$  is an admissible function for the supremum in  $\mathfrak{S}_{N, \mathbf{m}}$ , we have that

$$\int F_n^M d\mu - \int f^*(F_n^M) d\mathbf{m} \leq \mathfrak{S}_{N, \mathbf{m}}(\mu), \tag{4.5}$$

for any  $n \in \mathbb{N}$ . Passing now to the limit as  $n$  goes to infinity in (4.5), we obtain that  $M \cdot \mu(A) \leq \mathfrak{S}_{N, \mathbf{m}}(\mu)$ . The arbitrariness of  $M$  implies then that  $\mathfrak{S}_{N, \mathbf{m}}(\mu) = \infty$ .

We proceed now to prove that  $S_{N, \mathbf{m}}(\mu) \leq \mathfrak{S}_{N, \mathbf{m}}(\mu)$  and we will assume that  $\mathfrak{S}_{N, \mathbf{m}}(\mu) < +\infty$ , since otherwise there is nothing to prove. The paragraph above enables us to write  $\mu = \rho\mathbf{m}$ . We then have the following expression for  $\mathfrak{S}_{N, \mathbf{m}}(\mu)$ :

$$\mathfrak{S}_{N, \mathbf{m}}(\mu) = \sup \left\{ \int F \rho - f^*(F) d\mathbf{m} : F \in C_b(X) \text{ supported in } \mathcal{R}^k, \text{ for some } k \in \mathbb{N} \right\}.$$

And, recalling that  $f(x) = (f^*)^*(x) = \sup_{y \in \mathbb{R}} \{xy - f^*(y)\}$  because  $f$  is finite, convex and continuous, it follows that

$$S_{N, \mathbf{m}}(\mu) = \int \sup_{s \in \mathbb{Q}} \{\rho(x)s^* - f^*(s^*)\} d\mathbf{m}(x).$$

By fixing  $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$ , an enumeration of rational numbers with  $q_0 = 0$ , we introduce the family of approximating functionals

$$\left\{ S_{N, \mathbf{m}}^h(\mu) := \int \sup_{s \in \{q_0, \dots, q_h\}} \{\rho(x)s^* - f^*(s^*)\} d\mathbf{m}(x) \right\}_{h \in \mathbb{N}}.$$

Observe that the integrands are monotone increasing in  $h$  and that  $0 = \rho(x)q_0 - f^*(q_0)$ . In particular, Beppo Levi’s Theorem ensures that  $S_{N,m}^h(\mu) \rightarrow S_{N,m}(\mu)$ , as  $h \rightarrow \infty$ . Therefore, it suffices to show that  $S_{N,m}^h \leq \check{S}_{N,m}$ , for any fixed  $h \in \mathbb{N}$ . To this aim, one confirms directly that

$$S_{N,m}^h(\mu) = \sup \left\{ \int (\rho F - f^*(F)) d\mathbf{m} : F \text{ is a step function with values in } \{q_0, \dots, q_h\} \right\}. \tag{4.6}$$

Note that the fact that  $S$  is an  $\mathbf{m}$ -null set guarantees that we can further require that the aforementioned functions are supported in  $\mathcal{R}^k$  for some  $k \in \mathbb{N}$  without modifying the supremum, as an approximation argument using the Monotone Convergence Theorem shows. Finally, since  $\mathbf{m}$  and  $\mu$  are finite measures when restricted to  $\mathcal{R}^{k+1}$ , every step function with support in  $\mathcal{R}^k$  can be obtained as the  $L^1(\mu + \mathbf{m})$ -limit of continuous and uniformly bounded functions implying that  $S_{N,m}^h \leq \check{S}_{N,m}$ , which concludes the proof.

Applying this proposition to our extrinsic converging sequence  $\{\mathbb{X}_n\}_{n \in \mathbb{N} \cup \{\infty\}} \subset \mathcal{M}_k^{qR}$ , we can extract a useful corollary with a couple of observations. From (4.1) we easily deduce that, for any  $k \geq \bar{k}$ , the sequence  $(\mathbf{m}_n^k)_{n \in \mathbb{N}}$  converges to  $\mathbf{m}_\infty^k \in M(Z) \subset M_\emptyset(Z)$  in the weak convergence. Moreover, Remark 2.13 states that, granted we restrict ourselves to the set  $M(Z)$ , then weak convergence in the sense of quasi-Radon measures coincides with the usual weak one.

**Corollary 4.9.** *Given a fixed  $k \in \mathbb{N}$  and  $\{\mu_n\}_{n \in \mathbb{N}} \subset P_2(Z)$  a sequence converging weakly to  $\mu \in P_2(Z)$ , it holds that*

$$S_{N,m^k}(\mu) \leq \liminf_{n \rightarrow \infty} S_{N,m_n^k}(\mu_n). \tag{4.7}$$

We conclude the section with a corresponding continuity result for the functional  $T_{K,N}^{(t)}$ . We stress that although, it would be sufficient for the proof of Theorem 4.1 to have the upper semicontinuity of  $T_{K,N}^{(t)}$ , we prefer to present a more general statement.

**Proposition 4.10.** *Let  $K \geq 0$  and  $N < 0$  and  $(X, \mathbf{d}, \mathbf{m})$  be a metric measure space. Furthermore, set  $\mu_0 = \rho_0 \mathbf{m}, \mu_1 = \rho_1 \mathbf{m} \in P(X)$  to be absolutely continuous with respect to the quasi-Radon reference measure  $\mathbf{m}$ , with  $S_{N,m}(\mu_0), S_{N,m}(\mu_1) < \infty$ . Consider a sequence  $(\pi_n)_{n \in \mathbb{N}} \subset P(X \times X)$ , weakly converging to  $\pi \in P(X \times X)$  and such that*

$$(\mathbf{p}_1)_{\#} \pi_n = \mu_0 \quad \text{and} \quad (\mathbf{p}_2)_{\#} \pi_n = \mu_1 \quad \text{for every } n \in \mathbb{N}.$$

Then, for  $t \in [0, 1]$ , it holds that

$$\lim_{n \rightarrow \infty} T_{K,N}^{(t)}(\pi_n | \mathbf{m}) = T_{K,N}^{(t)}(\pi | \mathbf{m}).$$

Additionally, the conclusion remains valid for  $K < 0$ , granted  $\text{diam}(X) < \pi \sqrt{\frac{N-1}{K}}$ .

**Proof.** Let us fix any  $t \in (0, 1)$ , since the statement is clearly true for the remaining values. We want to prove that

$$\lim_{n \rightarrow \infty} \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi_n(x, y) = \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi(x, y), \tag{4.8}$$

since the other term of  $T_{K,N}^{(t)}(\cdot | \mathbf{m})$  can be treated analogously. Notice that being  $S_{N,m}(\mu_0) < \infty, \rho_0(x)^{-1/N} \in L^1(\mu_0)$  thus by the density of  $C_b(X) \cap L^1(\mathbf{m})$  in  $L^1(\mathbf{m})$ , for every fixed  $\varepsilon > 0$  there exists  $f^\varepsilon \in C_b(X)$  such that  $\|\rho_0^{-1/N} - f^\varepsilon\|_{L^1(\mu_0)} < \varepsilon$ . Moreover, notice that the coefficients  $\tau_{K,N}^{(1-t)}(\cdot)$  are bounded and continuous.

Indeed, this is always the case for  $K \geq 0$ , and since  $\text{diam}(X) < \pi \sqrt{\frac{N-1}{K}}$  is bounded by our assumptions, this holds as well for  $K < 0$ . Therefore,

$$\tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x)$$

is itself bounded and continuous. Consequently, the weak convergence  $(\pi_n)_n \rightharpoonup \pi$  shows that,

$$\lim_{n \rightarrow \infty} \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) d\pi_n(x, y) = \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) d\pi(x, y).$$

Furthermore, the boundedness of  $\tau_{K,N}^{(1-t)}$  allows to deduce the following estimate

$$\begin{aligned} \limsup_n \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi_n(x, y) &= \lim_n \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) d\pi_n(x, y) + \varepsilon \tau_{K,N}^{(1-t)} L^\infty \\ &= \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) d\pi(x, y) + \varepsilon \tau_{K,N}^{(1-t)} L^\infty \\ &\quad + \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi(x, y) + 2\varepsilon \tau_{K,N}^{(1-t)} L^\infty. \end{aligned}$$

Analogously, it follows that

$$\liminf_n \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi_n(x, y) = \int_{X \times X} \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} d\pi(x, y) - 2\varepsilon \tau_{K,N}^{(1-t)} L^\infty,$$

and since  $\varepsilon > 0$  can be chosen arbitrarily, Eq. (4.8) holds true. We conclude by recalling the arbitrariness of  $t$ .

#### 4.2. Proof of the approximate CD condition

The objective of this section is to prove the next partial result

**Theorem 4.11.** *Let  $K \in \mathbb{R}$ ,  $N \in (-\infty, 0)$ , and  $\{(X_n, \mathbf{d}_n, \mathbf{m}_n, \mathcal{S}_{\mathbf{m}_n}, p_n)\}_{n \in \mathbb{N} \cup \{\infty\}} \subset \mathcal{M}_{\bar{k}}^{qR}$  be a sequence of metric measure spaces satisfying the assumptions of the Stability Theorem 4.1, for some  $\bar{k} \in \mathbb{N}$ . Then  $(X_\infty, \mathbf{d}_\infty, \mathbf{m}_\infty)$  is a  $CD^a(K, N)$  space.*

We recall the discussion before Eq. (4.1) to work directly in the realization space  $(Z, \mathbf{d})$ , where we will consider the embedded versions of the converging sequence of pointed metric measure spaces.

We follow the plan explained above and argue using vertical approximations. Specifically, the following Step 1 and Step 2 serve the purpose of constructing useful approximations of the marginals  $\mu_0$  and  $\mu_1$ . Next, we follow the argumentation of Sturm in [2] in Steps 3 to 6, to exhibit the upper semicontinuity of  $T_{K,N}^{(t)}$  along a sequence of optimal couplings, provided by the curvature-dimension assumption. Additionally, we demonstrate the existence of a favorable limiting optimal coupling. Step 7 focuses on proving the convergence of inner points of a vertical sequence of Wasserstein geodesics, as well as, the lower semicontinuity of the Rényi entropy along this sequence.

Let us fix first the notation. Set  $k \geq \bar{k}$  and assume that  $\mu_0, \mu_1 \in P^{ac}(X_\infty, \mathbf{m}_\infty)$  have supports satisfying  $\text{supp}(\mu_0), \text{supp}(\mu_1) \subset \mathcal{R}_\infty^{k-1}$ . We denote by  $\rho_i$  the density of  $\mu_i$  with respect to  $\mathbf{m}_\infty$ , for  $i = \{0, 1\}$ . Define the set,

$$I := \{N' \in [N, 0) : S_{N', \mathbf{m}}(\mu_0), S_{N', \mathbf{m}}(\mu_1) < \infty\},$$

and observe that  $I$  is an interval, as a consequence of Jensen's inequality. Then, surely, we are able to assume that (and set)

$$(M :=) \max \{S_{N, \mathbf{m}}(\mu_0), S_{N, \mathbf{m}}(\mu_1)\} < \infty,$$

and that, for every  $q \in \text{Opt}(\mu_0, \mu_1)$ ,

$$T_{K,N}^{(t)}(q|\mathbf{m}_\infty) < \infty, \tag{4.9}$$



since the CD condition is trivial in failure of any of these inequalities. Therefore, the arbitrariness of  $k$  and initial measures shows that, in order to demonstrate [Theorem 4.11](#), we are required to validate the CD-inequality [\(3.3\)](#), for every  $N' \in I$ , and every  $t \in [0, 1]$ .

Additionally, we fix as before an optimal coupling  $p_n \in \text{Opt}(\bar{\mathbf{m}}_\infty^k, \bar{\mathbf{m}}_n^k)$  between the normalized  $k$ -cuts of the reference measures, for every  $n \in \mathbb{N}$ . And, as defined in [Section 4.1](#), we consider  $\{P_n(x)\}_{x \in X} \subset P(X_n)$  and  $\{P'_n(y)\}_{y \in X_n} \subset P(X_\infty)$  the disintegration kernels of  $p_n$  with respect to the projections  $\mathbf{p}_1$  and  $\mathbf{p}_2$  respectively, and consider the map  $P'_n: P^{ac}(X_\infty, \mathbf{m}_\infty^k) \rightarrow P^{ac}(X_n, \mathbf{m}_n^k)$ . Note that, in contrast to [Section 4.1](#), here we have omitted the dependence on the number  $k$ , since it is fixed for now.

**STEP 1: Horizontal approximation with bounded densities**

During the argument it proves useful to work with bounded-density measures. Therefore we construct here a horizontal approximation of  $\mu_0$  and  $\mu_1$ , for which its elements enjoy this property.

For the construction, we fix an arbitrary optimal coupling  $\tilde{q} \in \text{Opt}(\mu_0, \mu_1)$  and define, for every  $r > 0$ ,

$$E_r := \{(x_0, x_1) \in X_\infty \times X_\infty : \rho_0(x_0) < r, \rho_1(x_1) < r\} \tag{4.10}$$

and consequently, for sufficiently large  $r$ ,

$$\tilde{q}^{(r)} := \alpha_r^{-1} \tilde{q}(\cdot \cap E_r),$$

where  $\alpha_r := \tilde{q}(E_r)$ . The measure  $\tilde{q}^{(r)} \in P(X_\infty \times X_\infty)$  has marginals given by

$$\mu_0^{(r)} := (\mathbf{p}_1)_\# \tilde{q}^{(r)} \quad \text{and} \quad \mu_1^{(r)} := (\mathbf{p}_2)_\# \tilde{q}^{(r)}. \tag{4.11}$$

Notice that both  $\mu_0^{(r)}$  and  $\mu_1^{(r)}$  have bounded densities and that  $\mu_i^{(r)}$  converges to  $\mu_i$  in  $(P^2(X_\infty), W_2)$ , for  $i = 0, 1$ . Moreover, notice that  $S_{N, \mathbf{m}}(\mu_i^{(r)}) \rightarrow S_{N, \mathbf{m}}(\mu_i)$  as  $r \rightarrow \infty$ , for  $i = 0, 1$ . Then we fix  $\varepsilon > 0$  and find  $r = r(\varepsilon)$  such that  $\alpha_r \geq 1 - \varepsilon$  and that the following estimates hold:

$$\max_{i \in \{0,1\}} W_2(\mu_i, \mu_i^{(r)}) \leq \varepsilon \quad \text{and} \quad \max_{i \in \{0,1\}} S_{N, \mathbf{m}^k}(\mu_i^{(r)}) = \max_{i \in \{0,1\}} S_{N, \mathbf{m}}(\mu_i^{(r)}) \leq M + \frac{1}{2}. \tag{4.12}$$

We point out that the parameter  $r$  depends on  $\varepsilon$ , but we won't be explicit on this dependence for the sake of the presentation.

**STEP 2: Vertical approximation**

Once we have identified the horizontal approximations  $\mu_0^{(r)}$  and  $\mu_1^{(r)}$ , we may proceed to their vertical approximation. First of all, observe that  $\mu_0^{(r)}$  and  $\mu_1^{(r)}$  are absolutely continuous with respect to the normalized reference measure  $\bar{\mathbf{m}}_\infty^k$ , so we denote by  $\tilde{\rho}_0^{(r)}$  and  $\tilde{\rho}_1^{(r)}$  their bounded densities. Then, for every  $n \in \mathbb{N}$ , we define  $\mu_{0,n}, \mu_{1,n} \in P_2(X_n, \mathbf{d}_n, \mathbf{m}_n^k)$  as

$$\mu_{i,n} := P'_n(\mu_i^{(r)}) = \rho_{i,n} \bar{\mathbf{m}}_n^k, \tag{4.13}$$

where  $\rho_{i,n}(y) = \int \tilde{\rho}_i^{(r)}(x) P'_n(y)(dx)$ . Notice that  $\mu_{0,n}$  and  $\mu_{1,n}$  depend on  $r$  (and ultimately on  $\varepsilon$ ), but, once again, we prefer not to make this dependence explicit, in order to maintain an easy notation in the following. Anyway, we invite the reader to keep in mind that every object we are going to define depends only on  $\varepsilon$ . Now, since  $\mathbf{m}_n^k(X_n) \rightarrow \mathbf{m}_\infty^k(X_\infty)$ , observe that [Lemma 4.5](#) guarantees the existence of an  $\bar{n} \in \mathbb{N}$ , such that if  $n \geq \bar{n}$  it holds that

$$\max_{i \in \{0,1\}} W_2(\mu_i^{(r)}, \mu_{i,n}) \leq \varepsilon. \tag{4.14}$$

and that

$$\max_{i \in \{0,1\}} S_{N, \mathbf{m}_n}(\mu_{i,n}) \leq \max_{i \in \{0,1\}} S_{N, \mathbf{m}_n^k}(\mu_{i,n}) \leq M + 1. \tag{4.15}$$

Moreover, according to Lemma 4.5, for every  $N' \in I$ , it holds that

$$S_{N', \mathbf{m}_n}(\mu_{i,n}) \leq S_{N', \mathbf{m}_n^k}(\mu_{i,n}) \leq \left[ \frac{\mathbf{m}_n^k(\mathbf{X}_n)}{\mathbf{m}_\infty^k(\mathbf{X}_\infty)} \right]^{\frac{1}{N'}} S_{N', \mathbf{m}^k}(\mu_i^{(r)}) < \infty.$$

Therefore, for every  $n \in \mathbb{N}$  large enough, since  $(\mathbf{X}_n, \mathbf{d}_n, \mathbf{m}_n)$  is a  $\text{CD}(K, N)$  space, there exist an optimal plan  $\pi_n \in \text{Opt}(\mu_{0,n}, \mu_{1,n})$  and a 2-Wasserstein geodesic  $(\mu_{t,n})_{t \in [0,1]} \subset \mathcal{P}_2(\mathbf{X}_n)$  connecting  $\mu_{0,n}$  and  $\mu_{1,n}$ , for which,

$$S_{N', \mathbf{m}_n}(\mu_{t,n}) \leq T_{K,N}^{(t)}(\pi_n | \mathbf{m}_n) \tag{4.16}$$

holds, for every  $t \in [0, 1]$  and every  $N' \in I$ . Note that Remark 3.4 together with the assumption that  $\sup_{i \in \mathbb{N} \cup \{\infty\}} \text{diam}(\mathbf{X}_i, \mathbf{d}_i) < \pi \sqrt{\frac{1}{|K|}}$ , if  $K < 0$ , assures that the geodesic  $\mu_{t,n}$  is absolutely continuous with respect to  $\mathbf{m}_n$ .

**STEP 3: Estimate for  $T_{K,N}^{(t)}$**

In this step we start the proof of the upper semicontinuity of the functional  $T_{K,N}^{(t)}$ . In particular, we fix  $N' \in [N, 0)$  and a time  $t \in [0, 1]$  and we call  $Q_n$  and  $Q'_n$  be the disintegrations of  $\pi_n$  with respect to  $\mu_{0,n}$  and  $\mu_{1,n}$  respectively. Then we define the following two functions

$$v_0(y_0) = \int_{\mathbf{X}_n} \tau_{K,N}^{(1-t)}(\mathbf{d}(y_0, y_1)) Q_n(y_0, dy_1)$$

and

$$v_1(y_1) = \int_{\mathbf{X}_n} \tau_{K,N}^{(t)}(\mathbf{d}(y_0, y_1)) Q'_n(y_1, dy_0).$$

A direct application of Jensen's theorem leads to

$$\begin{aligned} T_{K,N}^{(t)}(\pi_n | \bar{\mathbf{m}}_n^k) &= \sum_{i=0}^1 \int_{\mathbf{X}_n} \rho_{i,n}(y_i)^{1-1/N} \cdot v_i(y_i) \, d\bar{\mathbf{m}}_n^k(y_i) \\ &= \sum_{i=0}^1 \int_{\mathbf{X}_n} \left[ \int_{\mathbf{X}} \tilde{\rho}_i^{(r)}(x_i) P'_n(y_i, dx_i) \right]^{1-1/N} \cdot v_i(y_i) \, d\bar{\mathbf{m}}_n^k(y_i) \\ &\leq \sum_{i=0}^1 \int_{\mathbf{X}_n} \int_{\mathbf{X}} \tilde{\rho}_i^{(r)}(x_i)^{1-1/N} P'_n(y_i, dx_i) \cdot v_i(y_i) \, d\bar{\mathbf{m}}_n^k(y_i) \\ &= \sum_{i=0}^1 \int_{\mathbf{X}} \tilde{\rho}_i^{(r)}(x_i)^{1-1/N} \left[ \int_{\mathbf{X}_n} v_i(y_i) P_n(x_i, dy_i) \right] \, d\bar{\mathbf{m}}_\infty^k(x_i). \end{aligned}$$

At this point we see that

$$\begin{aligned} \int_{\mathbf{X}_n} v_0(y_0) P_n(x_0, dy_0) &= \int_{\mathbf{X}_n \times \mathbf{X}_n} \tau_{K,N}^{(1-t)}(\mathbf{d}(y_0, y_1)) Q_n(y_0, dy_1) P_n(x_0, dy_0) \\ &= \int_{\mathbf{X}_n \times \mathbf{X}_n \times \mathbf{X}} \tau_{K,N}^{(1-t)}(\mathbf{d}(y_0, y_1)) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) \\ &\leq \int_{\mathbf{X}_n \times \mathbf{X}_n \times \mathbf{X}} [\tau_{K,N}^{(1-t)}(\mathbf{d}(x_0, x_1)) + C \cdot |\mathbf{d}(y_0, y_1) - \mathbf{d}(x_0, x_1)|] \\ &\quad \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) \\ &\leq \int_{\mathbf{X}_n \times \mathbf{X}_n \times \mathbf{X}} [\tau_{K,N}^{(1-t)}(\mathbf{d}(x_0, x_1)) + C \cdot (\mathbf{d}(x_0, y_0) + \mathbf{d}(x_1, y_1))] \\ &\quad \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0), \end{aligned}$$

and analogously that

$$\int_{X_n} v_1(y_1)P_n(x_1, dy_1) \leq \int_{X_n \times X_n \times X} [\tau_{K,N}^{(t)}(d(x_0, x_1)) + C \cdot (d(x_0, y_0) + d(x_1, y_1))] \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) Q'_n(y_1, dy_0) P_n(x_1, dy_1),$$

where  $C := \max_{\theta \in [0, \Theta], s \in [0, 1]} \frac{\partial}{\partial \theta} \tau_{K,N}^{(s)}(\theta)$  and  $\Theta$  is the maximum between  $d(x_0, x_1)$  and  $d(y_0, y_1)$ . Observe that the constant  $C$  is indeed finite because we know that  $\sup_{n \in \mathbb{N} \cup \{\infty\}} \text{diam}(X_n, d_n) < \pi \sqrt{\frac{1}{|K|}}$ , if  $K < 0$ , from assumption (iii) in Theorem 4.1.

Moreover we notice that

$$\begin{aligned} \int_X \tilde{\rho}_0^{(r)}(x_0)^{1-1/N} \int_{X_n \times X_n \times X} d(x_0, y_0) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) d\bar{m}_\infty^k(x_0) \\ = \int_X \tilde{\rho}_0^{(r)}(x_0)^{1-1/N} \int_{X_n} d(x_0, y_0) P_n(x_0, dy_0) d\bar{m}_\infty^k(x_0) \\ \leq r^{1-1/N} \int_{X_n \times X} d(x_0, y_0) dp_n(x_0, y_0) \leq r^{1-1/N} W_2(\bar{m}_n^k, \bar{m}_\infty^k) \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \int_X \tilde{\rho}_0^{(r)}(x_0)^{1-1/N} \int_{X_n \times X_n \times X} d(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) d\bar{m}_\infty^k(x_0) \\ \leq r^{-1/N} \int_{X_n \times X_n \times X} d(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) \tilde{\rho}_0^{(r)}(x_0) d\bar{m}_\infty^k(x_0) \\ = r^{-1/N} \int_{X_n \times X_n \times X} d(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) \mu_{0,n}(dy_0) \\ = r^{-1/N} \int_{X_n \times X} d(x_1, y_1) \tilde{\rho}_1^{(r)}(x_1) P'_n(y_1, dx_1) d\bar{m}_n^k(y_1) \\ \leq r^{1-1/N} \int_{X_n \times X} d(x_1, y_1) dp_n(x_1, y_1) \leq r^{1-1/N} W_2(\bar{m}_n^k, \bar{m}_\infty^k), \end{aligned} \tag{4.18}$$

where the last inequality in both chains follows by the Jensen’s inequality. Consequently, for every  $n \in \mathbb{N}$ , we define a – not necessarily optimal – coupling  $\bar{q}_n^{(r)} \in \text{Adm}(\mu_0^{(r)}, \mu_1^{(r)})$  by imposing that

$$\begin{aligned} d\bar{q}_n^{(r)}(x_0, x_1) &= \int_{X_n \times X_n} \frac{\tilde{\rho}_0^{(r)}(x_0) \tilde{\rho}_1^{(r)}(x_1)}{\rho_{0,n}(y_0) \rho_{1,n}(y_1)} P'_n(y_1, dx_1) P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ &= \int_{X_n \times X_n} \frac{\tilde{\rho}_0^{(r)}(x_0) \tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) Q_n(y_0, dy_1) P_n(x_0, dy_0) d\bar{m}_\infty^k(x_0) \\ &= \int_{X_n \times X_n} \frac{\tilde{\rho}_0^{(r)}(x_0) \tilde{\rho}_1^{(r)}(x_1)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) Q'_n(y_1, dy_0) P_n(x_1, dy_1) d\bar{m}_\infty^k(x_1). \end{aligned}$$

With this definition of  $\bar{q}_n^{(r)}$  and keeping in mind (4.17) and (4.18), we end up with

$$T_{K,N}^{(t)}(\pi_n | \bar{m}_n^k) \leq T_{K,N}^{(t)}(\bar{q}_n^{(r)} | \bar{m}_\infty^k) + 4Cr^{1-1/N} W_2(\bar{m}_\infty^k, \bar{m}_n^k).$$

Now, up to taking a greater  $\bar{n}$ , we can require that for every  $n \geq \bar{n}$  it holds that

$$W_2(\bar{m}_\infty^k, \bar{m}_n^k) \leq \frac{\varepsilon}{4Cr \frac{N-1}{N}},$$

for every  $N' \in [N, \varepsilon)$ . As a consequence we obtain that

$$T_{K,N}^{(t)}(\pi_n | \bar{m}_n^k) \leq T_{K,N}^{(t)}(\bar{q}_n^{(r)} | \bar{m}_\infty^k) + \varepsilon, \tag{4.19}$$

for every  $n \geq \bar{n}$  and every  $N' \in [N, \varepsilon)$ .

**STEP 4:  $\bar{q}_n^{(r)}$  converges to an optimal plan**

The objective now is to prove that

$$\int d^2(x_0, x_1) d\bar{q}_n^{(r)}(x_0, x_1) \rightarrow W_2^2(\mu_0^{(r)}, \mu_1^{(r)}) \quad \text{as } n \rightarrow \infty. \tag{4.20}$$

First of all notice that, since every  $\bar{q}_n^{(r)}$  is an admissible plan between  $\mu_0^{(r)}$  and  $\mu_1^{(r)}$ , then for every  $n \in \mathbb{N}$  it holds that

$$\int d^2(x_0, x_1) d\bar{q}_n^{(r)}(x_0, x_1) \geq W_2^2(\mu_0^{(r)}, \mu_1^{(r)}). \tag{4.21}$$

On the other hand the triangular inequality ensures that

$$d(x_0, x_1) \leq d(x_0, y_0) + d(y_0, y_1) + d(x_1, y_1)$$

and consequently, since  $d(y_0, y_1) < \text{diam}(\mathcal{R}_n^k) \leq 2^{k+2}$  for  $\pi_n$ -almost every pair  $(y_0, y_1)$ , we have that

$$d^2(x_0, x_1) - d^2(y_0, y_1) \leq 2d^2(x_0, y_0) + 2d^2(x_1, y_1) + 2^{k+3}d(x_0, y_0) + 2^{k+3}d(x_1, y_1)$$

for  $\pi_n$ -almost every pair  $(y_0, y_1)$ . It is then possible to perform the following estimate

$$\begin{aligned} & \int_{\mathcal{X} \times \mathcal{X}} d^2(x_0, x_1) d\bar{q}_n^{(r)}(x_0, x_1) \\ &= \int_{\mathcal{X} \times \mathcal{X}} d^2(x_0, x_1) \int_{\mathcal{X}_n \times \mathcal{X}_n} \frac{\tilde{\rho}_0^{(r)}(x_0)\tilde{\rho}_1^{(r)}(x_1)}{\rho_{0,n}(y_0)\rho_{1,n}(y_1)} P'_n(y_1, dx_1) P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ &\leq \int_{\mathcal{X}_n \times \mathcal{X}_n} d^2(y_0, y_1) d\pi_n(y_0, y_1) \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2d^2(x_0, y_0) \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2^{k+3}d(x_0, y_0) \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2d^2(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) d\pi_n(y_0, y_1) \\ &\quad + \int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2^{k+3}d(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) d\pi_n(y_0, y_1) \end{aligned}$$

We can now consider one term at a time and start by noticing that, according to [Lemma 4.5](#),

$$\begin{aligned} & \int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2d^2(x_0, y_0) \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ &= \int_{\mathcal{X}} \int_{\mathcal{X}_n} 2d^2(x_0, y_0) \tilde{\rho}_0^{(r)}(x_0) P'_n(y_0, dx_0) d\bar{m}_n^k(y_0) \\ &= \int_{\mathcal{X}} \int_{\mathcal{X}_n} 2d^2(x_0, y_0) \tilde{\rho}_0^{(r)}(x_0) dp_n(x_0, y_0) \rightarrow 0, \end{aligned}$$

and similarly

$$\int_{\mathcal{X}} \int_{\mathcal{X}_n \times \mathcal{X}_n} 2d^2(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) d\pi_n(y_0, y_1) \rightarrow 0.$$

Moreover Hölder’s inequality ensures that

$$\begin{aligned} & \int_{\mathbb{X}} \int_{\mathbb{X}_n \times \mathbb{X}_n} 2^{k+3} d(x_0, y_0) \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \\ & \leq 2^{k+3} \left[ \int_{\mathbb{X}} \int_{\mathbb{X}_n \times \mathbb{X}_n} d^2(x_0, y_0) \frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)} P'_n(y_0, dx_0) d\pi_n(y_0, y_1) \right]^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

and analogously that

$$\int_{\mathbb{X}} \int_{\mathbb{X}_n \times \mathbb{X}_n} 2^{k+3} d(x_1, y_1) \frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)} P'_n(y_1, dx_1) d\pi_n(y_0, y_1) \rightarrow 0.$$

Therefore, putting together the estimates on every term, we can conclude that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int d^2(x_0, x_1) d\bar{q}_n^{(r)}(x_0, x_1) & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{X}_n \times \mathbb{X}_n} d^2(y_0, y_1) d\pi_n(y_0, y_1) \\ & = \limsup_{n \rightarrow \infty} W_2^2(\mu_{0,n}, \mu_{1,n}) = W_2^2(\mu_0^{(r)}, \mu_1^{(r)}), \end{aligned}$$

where we used that  $\pi_n$  is an optimal plan and that  $\mu_{0,n} \xrightarrow{W_2} \mu_0^{(r)}$ ,  $\mu_{1,n} \xrightarrow{W_2} \mu_1^{(r)}$ . This last inequality, combined with (4.21), allows us to conclude (4.20).

**STEP 5: Definition of approximating plan with fixed marginals**

We have shown the existence of  $n(\varepsilon) \geq \bar{n}$  such that

$$\left| \int d^2(x_0, x_1) d\bar{q}_{n(\varepsilon)}^{(r)}(x_0, x_1) - W_2^2(\mu_0^{(r)}, \mu_1^{(r)}) \right| < \varepsilon. \tag{4.22}$$

Recalling the properties of  $\bar{n}$  proven in the previous steps, we also know that

$$T_{K,N}^{(t)}(\pi_n | \bar{m}_n^k) \leq T_{K,N}^{(t)}(\bar{q}_n^{(r)} | \bar{m}_\infty^k) + \varepsilon, \tag{4.23}$$

for every  $n \geq \bar{n}$  and every  $N' \in [N, \varepsilon)$ . At this point, using  $\bar{q}_{n(\varepsilon)}^{(r)}$ , we define a coupling  $q^\varepsilon$  between  $\mu_0$  and  $\mu_1$  by

$$q^\varepsilon(\cdot) := \alpha_r \bar{q}_{n(\varepsilon)}^{(r)} + \tilde{q}(\cdot \cap (\mathbb{X}_\infty^2 \setminus E_r)). \tag{4.24}$$

First of all, notice that

$$\left| \int d^2(x_0, x_1) d\bar{q}_{n(\varepsilon)}^{(r)}(x_0, x_1) - \int d^2(x_0, x_1) dq^\varepsilon(x_0, x_1) \right| \leq 2(1 - \alpha_r) \text{diam}(\mathcal{R}_\infty^{k-1})^2 \leq \varepsilon 2^{2k+3}.$$

Consequently, putting together this last estimate with (4.12) and (4.22), we can conclude that

$$\int d^2(x_0, x_1) dq^\varepsilon(x_0, x_1) = W_2^2(\mu_0, \mu_1) + O(\varepsilon). \tag{4.25}$$

On the other hand, it is immediate from the definition of  $q^\varepsilon$  that

$$(1 - \varepsilon)^{1-1/N} T_{K,N}^{(t)}(\bar{q}_{n(\varepsilon)}^{(r)} | \bar{m}_\infty^k) \leq \alpha_r^{1-1/N} T_{K,N}^{(t)}(\bar{q}_{n(\varepsilon)}^{(r)} | \bar{m}_\infty^k) \leq T_{K,N}^{(t)}(q^\varepsilon | \bar{m}_\infty^k). \tag{4.26}$$

**STEP 6: Convergence of plans**

We turn now to prove the weak convergence of the plans  $q^\varepsilon$  introduced in the previous step, as  $\varepsilon \rightarrow 0$ , and consequently the upper semicontinuity of  $T_{K,N}^{(t)}$ .

We first note that, since for every  $\varepsilon > 0$ , it holds that  $q^\varepsilon \in \text{Adm}(\mu_0, \mu_1)$ , then the family  $(q^\varepsilon)_{\varepsilon > 0}$  is tight and Prokhorov Theorem ensures the existence of a sequence  $(\varepsilon_m)_{m \in \mathbb{N}}$  converging to 0 such that

$q^{\varepsilon_m} \rightharpoonup q \in \text{Adm}(\mu_0, \mu_1)$ . Eq. (4.25) ensures the optimality of  $q \in \text{Opt}(\mu_0, \mu_1)$ . Furthermore, putting together the estimates (4.23) (that holds definitely for every  $N' \in [N, 0)$ ) and (4.26), we conclude that, for every  $N' \in [N, 0)$  and  $t \in [0, 1]$ ,

$$\begin{aligned} \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\pi_{n(\varepsilon_m)} | \mathbf{m}_{n(\varepsilon_m)}) &\leq \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\pi_{n(\varepsilon_m)} | \mathbf{m}_{n(\varepsilon_m)}^k) \\ &= \limsup_{m \rightarrow \infty} \frac{1}{\mathbf{m}_{n(\varepsilon_m)}^k (\mathbf{X}_{n(\varepsilon_m)})^{-1/N}} \cdot T_{K,N}^{(t)}(\pi_{n(\varepsilon_m)} | \bar{\mathbf{m}}_{n(\varepsilon_m)}^k) \\ &= \frac{1}{\mathbf{m}_{\infty}^k (\mathbf{X}_{\infty})^{-1/N}} \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\pi_{n(\varepsilon_m)} | \bar{\mathbf{m}}_{n(\varepsilon_m)}^k) \\ &\leq \frac{1}{\mathbf{m}_{\infty}^k (\mathbf{X}_{\infty})^{-1/N}} \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\bar{q}_{n(\varepsilon_m)}^{(r)} | \bar{\mathbf{m}}_{\infty}^k) + \varepsilon_m \tag{4.27} \\ &= \frac{1}{\mathbf{m}_{\infty}^k (\mathbf{X}_{\infty})^{-1/N}} \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\bar{q}_{n(\varepsilon_m)}^{(r)} | \bar{\mathbf{m}}_{\infty}^k) \\ &\leq \frac{1}{\mathbf{m}_{\infty}^k (\mathbf{X}_{\infty})^{-1/N}} \limsup_{m \rightarrow \infty} \frac{1}{(1 - \varepsilon_m)^{1-1/N}} T_{K,N}^{(t)}(q^{\varepsilon_m} | \bar{\mathbf{m}}_{\infty}^k) \\ &= \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(q^{\varepsilon_m} | \mathbf{m}_{\infty}^k). \end{aligned}$$

Now, notice that every  $q^{\varepsilon_m}$  has as marginals  $\mu_0$  and  $\mu_1$ , which are supported in  $\mathcal{R}_{\infty}^{k-1}$  and therefore

$$T_{K,N}^{(t)}(q^{\varepsilon_m} | \mathbf{m}_{\infty}^k) = T_{K,N}^{(t)}(q^{\varepsilon_m} | \mathbf{m}_{\infty}).$$

Thus we can apply Proposition 4.10 to  $T_{K,N}^{(t)}(q^{\varepsilon_m} | \mathbf{m}_{\infty})$ , for  $N' \in I$ , which together with the above estimate guarantees that

$$\limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(\pi_{n(\varepsilon_m)} | \mathbf{m}_{n(\varepsilon_m)}) \leq \limsup_{m \rightarrow \infty} T_{K,N}^{(t)}(q^{\varepsilon_m} | \mathbf{m}_{\infty}) = T_{K,N}^{(t)}(q | \mathbf{m}_{\infty}) \tag{4.28}$$

holds for every  $N' \in I$  and  $t \in [0, 1]$ .

**STEP 7: Convergence of midpoints**

The goal of this step is to show the existence of a limit geodesic  $\{\mu_t\}_{t \in [0,1]}$ , such that for any  $t \in [0, 1]$ ,  $\mu_{t,n(\varepsilon_m)}$   $W_2$ -converges (up to subsequences) to  $\mu_t$ , as  $m \rightarrow \infty$ . Furthermore, we are going to prove a suitable lower semicontinuity of the Rényi entropies that will allow us to pass to the limit in the CD inequality. In order to ease the notation we will denote the Rényi entropy  $S_{N,\mathbf{m}_{n(\varepsilon_m)}}$  by  $S_{N,n(\varepsilon_m)}$ .

**Claim 1.** *For every fixed  $t \in [0, 1]$ , the sequence  $(\mu_{t,n(\varepsilon_m)})_{m \in \mathbb{N}}$  converges (up to subsequences) to a measure  $\mu_t \in \mathcal{P}(\mathbf{X}_{\infty})$ .*

First of all, notice that estimate (4.28), the CD condition (4.16) and assumption (4.9) together ensure that the entropies  $S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)})$  are uniformly bounded above by a constant  $M'$ , for  $m \in \mathbb{N}$ . Moreover, for every  $n(\varepsilon_m)$ , the approximation Lemma 4.4 provides the existence of a sequence  $(\mu_{t,n(\varepsilon_m)}^l)_{l \in \mathbb{N}}$ , that  $W_2$ -converges to  $\mu_{t,n(\varepsilon_m)}$ , as  $l \rightarrow \infty$ , and such that  $\text{supp}(\mu_{t,n(\varepsilon_m)}^l) \subseteq \mathcal{R}_{n(\varepsilon_m)}^l$ . From the proof of Lemma 4.4 we recall that  $\mu_{t,n(\varepsilon_m)}^l = c^l f^l \mu_{t,n(\varepsilon_m)}$  so, we can easily notice that,

$$(c^l)^{-1} = \int f^l d\mu_{t,n(\varepsilon_m)} \geq \mu_{t,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^{l-1}) \geq 1 - \omega(k, l - 1, M + 1).$$

Consequently, for sufficiently large  $l$  it holds that, as measures,

$$\mu_{t,n(\varepsilon_m)}^l \leq \frac{1}{1 - \omega(k, l - 1, M + 1)} \cdot \mu_{t,n(\varepsilon_m)}. \tag{4.29}$$

Notice that we took into account that  $\text{supp}(\mu_{0,n(\varepsilon_m)}), \text{supp}(\mu_{1,n(\varepsilon_m)}) \subseteq \mathcal{R}_{n(\varepsilon_m)}^k$  and we have used the  $\omega$ -uniform convexity assumption, keeping in mind that  $M + 1$  bounds from above the terminal entropies (4.15). In turn, inequality (4.29) implies that,

$$S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) \leq \frac{1}{(1 - \omega(k, l - 1, M + 1))^{1-1/N}} S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}). \tag{4.30}$$

Moreover, the fact that  $\text{supp}(\mu_{0,n(\varepsilon_m)}), \text{supp}(\mu_{1,n(\varepsilon_m)}) \subseteq \mathcal{R}_{n(\varepsilon_m)}^k$  shows that the measure  $\mu_{t,n(\varepsilon_m)}$  has bounded support. In particular, for every  $l \in \mathbb{N}$  (and every  $m \in \mathbb{N}$ )

$$\text{supp}(\mu_{t,n(\varepsilon_m)}^l) \subseteq \text{supp}(\mu_{t,n(\varepsilon_m)}) \subseteq B(p_{n(\varepsilon_m)}, 2^{k+2}).$$

As a consequence of this bound, it is easy to deduce that,

$$W_2^2(\mu_{t,n(\varepsilon_m)}^l, \mu_{t,n(\varepsilon_m)}) \leq (2 \cdot 2^{k+2})^2 \omega(k, l - 1, M + 1), \tag{4.31}$$

because  $\mu_{t,n(\varepsilon_m)} \leq \mu_{t,n(\varepsilon_m)}^l$  when restricted to  $\mathcal{R}_{n(\varepsilon_m)}^{l-1}$ , and  $\mu_{t,n(\varepsilon_m)}(X \setminus \mathcal{R}_{n(\varepsilon_m)}^{l-1}) \leq \omega(k, l - 1, M + 1)$ , by  $\omega$ -uniform convexity. Now, for every fixed  $l$  sufficiently large, such that  $\omega(k, l - 1, M + 1) < 1$ , observe that, according to (4.30), the entropies  $S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l)$  are uniformly bounded above by the constant

$$\frac{1}{(1 - \omega(k, l - 1, M + 1))^{1-1/N}} M',$$

for all  $m \in \mathbb{N}$ . Notice also that, since  $\mu_{t,n(\varepsilon_m)}^l$  is supported in  $\mathcal{R}_{n(\varepsilon_m)}^l$ , it holds that

$$S_{N,m^{l+1}}^{n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) = S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l).$$

Therefore, Corollary 4.7 shows that  $\mu_{t,n(\varepsilon_m)}^l$  weakly converges to some  $\mu_t^l \in \mathcal{P}(X_\infty)$  as  $m \rightarrow \infty$ , for some choice of a subsequence. Moreover, we extract the bound  $S_{N,m^{l+1}}(\mu_t^l) < \infty$  from Corollary 4.9, which guarantees the lower semicontinuity of  $S_{N,\cdot}(\cdot)$  along our sequence. Consequently, this implies that the support of  $\mu_t^l$  is contained in  $\mathcal{R}_\infty^{l+1}$ . Finally, note that then the sequence of measures  $(\mu_{t,n(\varepsilon_m)}^l)_{m \in \mathbb{N}}$  is supported in a uniformly bounded set, since  $\text{supp}(\mu_{t,n(\varepsilon_m)}^l) \subseteq B(p_{n(\varepsilon_m)}, 2^{k+2})$  and  $p_{n(\varepsilon_m)} \rightarrow p_\infty$ , as  $m \rightarrow \infty$ . Thus, we are able to conclude, up to picking again subsequence, that for every sufficiently large  $l \in \mathbb{N}$

$$\mu_{t,n(\varepsilon_m)}^l \xrightarrow{W_2} \mu_t^l \quad \text{as } m \rightarrow \infty.$$

As a matter of fact, we can show using inequality (4.31) that,

$$W_2^2(\mu_t^i, \mu_t^j) \leq 2^{2k+7} [\omega(k, i - 1, M + 1) + \omega(k, j - 1, M + 1)],$$

for every (large enough)  $i, j \in \mathbb{N}$ . Then, our assumption on  $\omega$  ensures that  $(\mu_t^l)_{l \in \mathbb{N}}$  is a Cauchy sequence, which therefore  $W_2$ -converges to  $\mu_t \in \mathcal{P}(X_\infty)$ . We conclude by noting that the uniform estimate (4.31) guarantees that  $\mu_{t,n(\varepsilon_m)} \rightarrow \mu_t$ .

**Claim 2.** For every  $t \in [0, 1]$  the measure  $\mu_t$  does not give mass to the set  $S$  of singular points.

For every  $m \in \mathbb{N}$  and every  $l \in \mathbb{N}$  sufficiently large, let us introduce the measures

$$\tilde{\mu}_{t,n(\varepsilon_m)}^l = [1 - \omega(k, l - 1, M + 1)] \mu_{t,n(\varepsilon_m)}^l,$$

and notice that, for every  $l \in \mathbb{N}$  sufficiently large,

$$\tilde{\mu}_{t,n(\varepsilon_m)}^l \rightarrow \tilde{\mu}_t^l := [1 - \omega(k, l - 1, M + 1)] \mu_t^l.$$



Observe also that all measures  $\tilde{\mu}_{t,n(\varepsilon_m)}^l$  have total mass equal to  $[1 - \omega(k, l - 1, M + 1)]$ , as  $m$  varies. Thus,  $\tilde{\mu}_t^l$  also has total mass equal to  $[1 - \omega(k, l - 1, M + 1)]$ . On the other hand it follows from the uniform convexity properties (and in particular from (4.29)) that for every  $m \in \mathbb{N}$  and every  $l \in \mathbb{N}$  sufficiently large, there exists a positive measure  $\bar{\mu}_{t,n(\varepsilon_m)}^l$  such that

$$\mu_{t,n(\varepsilon_m)} = \tilde{\mu}_{t,n(\varepsilon_m)}^l + \bar{\mu}_{t,n(\varepsilon_m)}^l.$$

Notice that, since the sequences  $\mu_{t,n(\varepsilon_m)}$  and  $(\tilde{\mu}_{t,n(\varepsilon_m)}^l)_{m \in \mathbb{N}}$  are weakly converging, the sequence  $\bar{\mu}_{t,n(\varepsilon_m)}^l$  is also weakly converging to a (positive) measure  $\bar{\mu}_t^l$ , such that

$$\mu_t = \tilde{\mu}_t^l + \bar{\mu}_t^l.$$

As pointed out before  $\mu_t^l$  is supported in  $\mathcal{R}_\infty^{l+1}$ , thus the same holds for  $\bar{\mu}_t^l$ , and therefore

$$\mu_t(\mathcal{R}_\infty^{l+1}) \geq 1 - \omega(k, l - 1, M + 1).$$

Finally observe that this is sufficient to prove the claim, because of the arbitrariness of  $l$ .

**Claim 3.** *The lower semicontinuity of the Rényi entropies holds, that is for every  $N' \in [N, 0)$*

$$S_{N',m}(\mu_t) \leq \liminf_{m \rightarrow \infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}). \tag{4.32}$$

First of all notice that, the result of Claim 2 combined with Proposition 4.8 yields that

$$S_{N',m}(\mu_t) \leq \liminf_{l \rightarrow \infty} S_{N',m}(\mu_t^l). \tag{4.33}$$

On the other hand Corollary 4.9 ensures that for every  $l \in \mathbb{N}$  large enough

$$S_{N',m}(\mu_t^l) = S_{N',m^{l+2}}(\mu_t^l) \leq \liminf_{m \rightarrow \infty} S_{N',m^{l+2},n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) = \liminf_{m \rightarrow \infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l). \tag{4.34}$$

Moreover, we deduce as in Claim 1 the following estimate for every  $N' \in [N, 0)$

$$S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) \leq \frac{1}{(1 - \omega(k, l - 1, M + 1))^{1-1/N}} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)})$$

and consequently for every  $l \in \mathbb{N}$

$$\begin{aligned} \liminf_{m \rightarrow \infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) &\geq \liminf_{m \rightarrow \infty} (1 - \omega(k, l - 1, M + 1))^{1-1/N} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}^l) \\ &\geq (1 - \omega(k, l - 1, M + 1))^{1-1/N} S_{N',m}(\mu_t^l), \end{aligned}$$

where the last passage follows from (4.34). Then, since this last inequality holds for every  $l \in \mathbb{N}$ , we can conclude that

$$\begin{aligned} \liminf_{m \rightarrow \infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}) &\geq \liminf_{l \rightarrow \infty} (1 - \omega(k, l - 1, M + 1))^{1-1/N} S_{N',m}(\mu_t^l) \\ &\geq S_{N',m}(\mu_t), \end{aligned}$$

where we used (4.33). This is exactly what we wanted to prove.

### Conclusion

So far we were able to prove that for every fixed  $t \in [0, 1]$ , the sequence  $(\mu_{t,n(\varepsilon_m)})_{m \in \mathbb{N}}$  converges (up to subsequences) to a measure  $\mu_t \in \mathcal{P}_2(\mathbb{X}_\infty)$  and

$$S_{N',m}(\mu_t) \leq \liminf_{m \rightarrow \infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}), \tag{4.35}$$

for every  $N' \in [N, 0)$ . Now, a diagonal argument ensures that, by selecting a suitable subsequence (that we do not rename for sake of simplicity),

$$(\mu_{t,n(\varepsilon_m)}) \xrightarrow{W_2} \mu_t,$$

and that estimate (4.35) holds for every  $t \in [0, 1] \cap \mathbb{Q}$ . Our approximation ensures also that  $\mu_{0,n(\varepsilon_m)} \xrightarrow{W_2} \mu_0$  and  $\mu_{1,n(\varepsilon_m)} \xrightarrow{W_2} \mu_1$  therefore, since  $\mu_{t,n(\varepsilon_m)}$  is a  $t$ -midpoint of  $\mu_{0,n(\varepsilon_m)}$  and  $\mu_{1,n(\varepsilon_m)}$ , for every  $t \in [0, 1] \cap \mathbb{Q}$  the limit point  $\mu_t$  is a  $t$ -midpoint of  $\mu_0$  and  $\mu_1$ . Now it is easy to realize that we can extend by continuity  $\mu_t$  to a Wasserstein geodesic (connecting  $\mu_0$  and  $\mu_1$ ) on the whole interval  $[0, 1]$ , obtaining also that

$$(\mu_{t,n(\varepsilon_m)}) \xrightarrow{W_2} \mu_t, \quad \text{for every } t \in [0, 1].$$

This argument will be explained with more details in Lemma 4.12 below. Moreover, we know from the proof of Claim 2, that for every  $l \in \mathbb{N}$

$$\mu_t(\mathcal{R}_\infty^{l+1}) \geq 1 - \omega(k, l - 1, M + 1),$$

for every  $t \in [0, 1] \cap \mathbb{Q}$ . Then by continuity we can conclude the same inequality for every  $t \in [0, 1]$ , and consequently we know that  $\mu_t$  gives no mass to the set of singular points.

Finally, inequality (4.35), combined with (4.28), allows to pass to the limit as  $m \rightarrow \infty$  of the inequality (4.16) at every rational time and obtaining that

$$S_{N',m}(\mu_t) \leq T_{K,N}^{(t)}(q|\mathbf{m}_\infty)$$

holds for every  $t \in [0, 1] \cap \mathbb{Q}$  and every  $N' \in I$ . Finally, the lower semicontinuity of the entropy (ensured by the fact that  $\mu_t$  gives no mass to the set of singular points) and the continuity of  $T_{K,N}^{(t)}(q|\mathbf{m}_\infty)$  in  $t$  (which is a straightforward consequence of the dominated convergence theorem), allow to extend this last inequality to every  $t \in [0, 1]$ , concluding the proof of the approximate CD condition.

#### 4.3. Proof of the CD condition

This final section is devoted to the proof of our main result Theorem 4.2. As already mentioned, the proof of the approximate CD condition and the approximation argument that made it possible are exactly the fundamental tools to prove the CD condition. As the reader will notice, we are going to use basically the same techniques, but refining them further to achieve a more general result. We specify that we could prove the CD condition directly, but we preferred to divide the proof in order to be clearer.

Before going on, we prove a preliminary lemma, that will help us later in this section: a result of this type is now needed because the marginals may not have bounded support.

**Lemma 4.12.** *Given a metric space  $(X, d)$ , for every  $n \in \mathbb{N}$  let  $(\nu_t^n)_{t \in [0,1]} \subset \mathcal{P}_2(X)$  be a Wasserstein geodesic. Assume that for every  $t \in [0, 1]$  the family  $(\nu_t^n)_{n \in \mathbb{N}}$  is tight and that there exist  $\nu_0, \nu_1 \in \mathcal{P}_2(X)$  such that*

$$\nu_0^n \xrightarrow{W_2} \nu_0 \quad \text{and} \quad \nu_1^n \xrightarrow{W_2} \nu_1 \quad \text{as } n \rightarrow \infty.$$

*Then there exists a Wasserstein geodesic  $(\nu_t)_{t \in [0,1]} \subset \mathcal{P}_2(X)$  connecting  $\nu_0$  and  $\nu_1$  such that, up to subsequences,*

$$\nu_t^n \rightarrow \nu_t \quad \text{as } n \rightarrow \infty, \text{ for every } t \in [0, 1] \cap \mathbb{Q}.$$

**Proof.** First of all, notice that applying Prokhorov theorem we deduce that, for every fixed  $t$ , the sequence  $(\nu_t^n)_{n \in \mathbb{N}}$  is weakly convergent, up to subsequences. Thus the diagonal argument ensures that, up to taking a suitable subsequence which we do not recall for simplicity, for every  $t \in [0, 1] \cap \mathbb{Q}$  there exists  $\nu_t \in P_2(X)$  such that

$$\nu_t^n \rightharpoonup \nu_t \quad \text{as } n \rightarrow \infty, \text{ for every } t \in [0, 1] \cap \mathbb{Q}.$$

It is well-known that the Wasserstein distance is lower semicontinuous with respect to the weak convergence (see for example Proposition 7.1.3 in [38]), then

$$W_2(\nu_0, \nu_t) \leq \liminf_{n \rightarrow \infty} W_2(\nu_0^n, \nu_t^n) = \liminf_{n \rightarrow \infty} t \cdot W_2(\nu_0^n, \nu_1^n) = t \cdot W_2(\nu_0, \nu_1)$$

and analogously

$$W_2(\nu_t, \nu_1) \leq (1 - t) \cdot W_2(\nu_0, \nu_1).$$

Combining this two inequalities with the triangular inequality we deduce that

$$W_2(\nu_0, \nu_t) = t \cdot W_2(\nu_0, \nu_1) \quad \text{and} \quad W_2(\nu_t, \nu_1) = (1 - t) \cdot W_2(\nu_0, \nu_1),$$

which means that  $\nu_t$  is a  $t$ -midpoint of  $\nu_0$  and  $\nu_1$ . The lower semicontinuity of the Wasserstein distance also ensures that for every  $s, t \in [0, 1] \cap \mathbb{Q}$  it holds that

$$W_2(\nu_t, \nu_s) \leq \liminf_{n \rightarrow \infty} W_2(\nu_t^n, \nu_s^n) = |t - s| \cdot \liminf_{n \rightarrow \infty} W_2(\nu_0^n, \nu_1^n) = |t - s| \cdot W_2(\nu_0, \nu_1).$$

Finally, since for every  $r \in [0, 1] \cap \mathbb{Q}$   $\nu_r$  is an  $r$ -midpoint of  $\nu_0$  and  $\nu_1$ , the triangular inequality allow us conclude that

$$W_2(\nu_t, \nu_s) = |t - s| \cdot W_2(\nu_0, \nu_1), \quad \text{for every } s, t \in [0, 1] \cap \mathbb{Q},$$

then we can extend  $\nu_t$  to the whole interval  $[0, 1]$ , finding a Wasserstein geodesic  $(\nu_t)_{t \in [0, 1]}$  connecting  $\nu_0$  and  $\nu_1$ .

Now that we have this last result at our disposal we can proceed to the proof of [Theorem 4.2](#). To this aim, we fix  $\mu_0, \mu_1 \in P^{ac}(X_\infty, \mathfrak{m}_\infty)$ . In analogy with the previous section, we can assume that  $S_{N, m}(\mu_0), S_{N, m}(\mu_1) < \infty$  and introduce the constant

$$M := \max\{S_{N, m}(\mu_0), S_{N, m}(\mu_1)\}.$$

We can also define the interval

$$I := \{N' \in [N, 0) : S_{N', m}(\mu_0), S_{N', m}(\mu_1) < \infty\},$$

in particular we will need to prove [\(3.3\)](#) for every  $N' \in I$  and every  $t \in [0, 1]$ . Now, according to [Lemma 4.4](#) there exist two sequences  $(\mu_0^l)_{l \in \mathbb{N}}$  and  $(\mu_1^l)_{l \in \mathbb{N}}$ ,  $W_2$ -converging to  $\mu_0$  and  $\mu_1$  respectively and such that

$$\text{supp}(\mu_0^l), \text{supp}(\mu_1^l) \subseteq \mathcal{R}_\infty^{l-1} \quad \text{for every } l \in \mathbb{N}.$$

Moreover, keeping in mind the definition of  $\mu_0^l$  and  $\mu_1^l$  (see [Lemma 4.4](#)), it is easy to realize that for  $l$  sufficiently large

$$S_{N', m}(\mu_0^l), S_{N', m}(\mu_1^l) < \infty \quad \text{for every } N' \in I$$

and that the dominated convergence theorem ensures that

$$\lim_{l \rightarrow \infty} S_{N', m}(\mu_0^l) = S_{N', m}(\mu_0) \quad \text{and} \quad \lim_{l \rightarrow \infty} S_{N', m}(\mu_1^l) = S_{N', m}(\mu_1).$$

Thus, for every  $l$  large enough

$$S_{N,m}(\mu_0^l), S_{N,m}(\mu_1^l) \leq \max\{S_{N,m}(\mu_0), S_{N,m}(\mu_1)\} + 1 = M + 1$$

and then we can apply the argument presented in the last section and deduce the existence of an optimal plan  $q^l \in \text{Opt}(\mu_0^l, \mu_1^l)$  and of a Wasserstein geodesic  $(\mu_t^l)_{t \in [0,1]}$  connecting  $\mu_0^l$  and  $\mu_1^l$ , such that

$$S_{N,m}(\mu_t^l) \leq T_{K,N}^{(t)}(q^l | \mathbf{m}_\infty) \tag{4.36}$$

holds for every  $t \in [0, 1]$  and every  $N' \in I$ . Now, we divide the proof into two steps, the first dedicated to the convergence of the plans  $(q^l)_{l \in \mathbb{N}}$  and to the upper semicontinuity of  $T_{K,N}^{(t)}$ , the second dedicated to the convergence of the measures  $(\mu_t^l)_{l \in \mathbb{N}}$  and the lower semicontinuity of  $S_{N,m}$ .

**Step 1: Upper semicontinuity for  $T_{K,N}^{(t)}$**

Notice that  $(q^l)_{l \in \mathbb{N}}$  is a sequence of probability measures having as marginals two sequences of converging, and thus tight, probability measures. As a consequence the sequence  $(q^l)_{l \in \mathbb{N}}$  is itself tight, then up to subsequences it weakly converges to a plan  $q \in \mathcal{P}(X_\infty \times X_\infty)$ . It is well known and easy to prove that  $q \in \text{Opt}(\mu_0, \mu_1)$ . We are now going to prove that

$$\limsup_{l \rightarrow \infty} T_{K,N}^{(t)}(q^l | \mathbf{m}_\infty) \leq T_{K,N}^{(t)}(q | \mathbf{m}_\infty) \tag{4.37}$$

for every  $t \in [0, 1]$  and every  $N' \in I$ . The argument we are going to use is essentially the same as the one explained in the proof of Proposition 4.10, nevertheless we briefly recall it for the sake of completeness, avoiding to repeat all the details.

In particular, for every  $l \in \mathbb{N}$  let us call  $\rho_0^l$  and  $\rho_1^l$  the densities of  $\mu_0^l$  and  $\mu_1^l$  with respect to the reference measure  $\mathbf{m}_\infty$ , we just need to prove that

$$\limsup_{l \rightarrow \infty} \int \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0^l(x)^{-\frac{1}{N}} dq^l \leq \int \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) \rho_0(x)^{-\frac{1}{N}} dq.$$

Notice that, the particular definition of  $\mu_0^l$  (check Lemma 4.4), ensures that the density  $\rho_0^l$  is a suitable renormalization of  $f^l \rho_0$ , then for a fixed  $\varepsilon > 0$  we can find  $\bar{l} \in \mathbb{N}$  such that

$$\|(\rho_0^l)^{-1/N} - \rho_0^{-1/N}\|_{L^1(\mu_0^l)} < \varepsilon \text{ for every } l \geq \bar{l}.$$

Furthermore, recalling that  $C_b(X) \cap L^1(\mathbf{m})$  is dense in  $L^1(\mathbf{m})$ , for the same reason (up to possibly changing  $\bar{l}$ ) we can find  $f^\varepsilon \in C_b(X)$  such that

$$\|\rho_0^{-1/N} - f^\varepsilon\|_{L^1(\mu_0)} < \varepsilon \quad \text{and} \quad \|\rho_0^{-1/N} - f^\varepsilon\|_{L^1(\mu_0^l)} < \varepsilon \text{ for every } l \geq \bar{l}.$$

Putting together this last two estimates we end up proving that

$$\|(\rho_0^l)^{-1/N} - f^\varepsilon\|_{L^1(\mu_0^l)} < 2\varepsilon \text{ for every } l \geq \bar{l}.$$

On the other hand, since the function

$$\tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x)$$

is bounded and continuous, the weak convergence of  $(q^l)_l$  to  $q$  yields that

$$\lim_{l \rightarrow \infty} \int \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) dq^l = \int \tau_{K,N}^{(1-t)}(\mathbf{d}(x, y)) f^\varepsilon(x) dq.$$

Then, since definitely  $l \geq \bar{l}$ , we can deduce the following estimate

$$\begin{aligned} \limsup_{l \rightarrow \infty} \int \tau_{K,N}^{(1-t)}(d(x, y)) \rho_0^l(x)^{-\frac{1}{N}} dq^l &\leq \lim_{l \rightarrow \infty} \int \tau_{K,N}^{(1-t)}(d(x, y)) f^\varepsilon(x) dq^l + 2\varepsilon \|\tau_{K,N}^{(1-t)}\|_L \\ &= \int \tau_{K,N}^{(1-t)}(d(x, y)) f^\varepsilon(x) dq + 2\varepsilon \|\tau_{K,N}^{(1-t)}\|_L \\ &\leq \int \tau_{K,N}^{(1-t)}(d(x, y)) \rho_0(x)^{-\frac{1}{N}} dq + 3\varepsilon \|\tau_{K,N}^{(1-t)}\|_L \end{aligned}$$

and since  $\varepsilon > 0$  can be chosen arbitrarily, (4.37) holds true.

**Step 2: Lower semicontinuity for  $S_{N,m}$**

In this second step we prove an additional property on  $\mu_t^l$ , which is fundamental to prove the CD condition. Let us start with a preliminary lemma.

**Lemma 4.13.** Fix  $k \leq h \in \mathbb{N}$  and let  $\nu \in P^{ac}(\mathbf{X}_\infty, \mathbf{m}_\infty)$  with bounded density be such that  $\text{supp}(\nu) \subseteq \mathcal{R}_\infty^{k-1}$ . Then, for every  $\epsilon > 0$ , there exists  $\tilde{n} \in \mathbb{N}$  large enough such that,  $P'_{n,h}(\nu)(\mathcal{R}_n^{k+1}) \geq 1 - \epsilon$  for every  $n \geq \tilde{n}$ .

**Proof.** Notice that, according to Lemma 4.5, both the sequences  $(P'_{n,k}(\nu))_{n \in \mathbb{N}}$  and  $(P'_{n,h}(\nu))_{n \in \mathbb{N}}$   $W_2$ -converge to  $\nu$ . Assume that  $P'_{m,h}(\nu)(\mathcal{R}_m^{k+1}) < 1 - \epsilon$  for some arbitrarily large  $m \in \mathbb{N}$ . Observe that

$$\inf \{d(x, y) : x \in \mathcal{R}_m^k, y \in (\mathcal{R}_m^{k+1})^c\} \geq 2^{-(k+2)},$$

as a consequence, since  $P'_{m,k}(\nu)$  is supported in  $\mathcal{R}_m^k$ , we obtain that

$$W_2^2(P'_{m,k}(\nu), P'_{m,h}(\nu)) \geq \epsilon \cdot 2^{-(2k+4)}. \tag{4.38}$$

On the other hand, since the sequences  $(P'_{n,k}(\nu))_{n \in \mathbb{N}}$  and  $(P'_{n,h}(\nu))_{n \in \mathbb{N}}$  have the same limit, it holds that

$$W_2^2(P'_{n,k}(\nu), P'_{n,h}(\nu)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then definitely (4.38) cannot hold, proving the desired result.

Fix  $\epsilon > 0$  and take  $k(\epsilon) \in \mathbb{N}$  such that  $\mu_0(\mathcal{R}_\infty^{k(\epsilon)-1}), \mu_1(\mathcal{R}_\infty^{k(\epsilon)-1}) > 1 - \frac{\epsilon}{2}$ . Then we take  $l > k(\epsilon)$  and repeat the argument of the previous section on  $\mu_0^l$  and  $\mu_1^l$ . We are also going to use the same notation, forgetting for the moment the dependence on  $l$ . It is easy to realize that there exist two measures  $\nu_0^{k(\epsilon)}$  and  $\nu_1^{k(\epsilon)}$  with  $\text{supp}(\nu_0^{k(\epsilon)}), \text{supp}(\nu_1^{k(\epsilon)}) \subseteq \mathcal{R}_\infty^{k(\epsilon)-1}$  and  $\nu_0^{k(\epsilon)}(\mathbf{X}_\infty), \nu_1^{k(\epsilon)}(\mathbf{X}_\infty) > 1 - \epsilon$ , such that for  $r$  sufficiently large (and thus for  $\epsilon$  sufficiently small)  $\mu_0^{(r)} \geq \nu_0^{k(\epsilon)}$  and  $\mu_1^{(r)} \geq \nu_1^{k(\epsilon)}$  (in particular this tells us that  $\nu_0^{k(\epsilon)}$  and  $\nu_1^{k(\epsilon)}$  have bounded density). Then we can apply Lemma 4.13 to the probability measures

$$\frac{1}{\nu_0^{k(\epsilon)}(\mathbf{X}_\infty)} \nu_0^{k(\epsilon)} \quad \text{and} \quad \frac{1}{\nu_1^{k(\epsilon)}(\mathbf{X}_\infty)} \nu_1^{k(\epsilon)},$$

obtaining that for  $m$  sufficiently large (in particular such that  $n(\varepsilon_m) \geq \tilde{n}$ ) it holds that

$$\mu_{0,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^{k(\epsilon)+1}), \mu_{1,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^{k(\epsilon)+1}) \geq (1 - \epsilon)^2 \geq 1 - 2\epsilon.$$

Consequently our uniform convexity assumption ensures that

$$\mu_{t,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^h) \geq 1 - \Omega(k(\epsilon) + 1, h, M + 2, 2\epsilon),$$

for every  $t \in [0, 1]$  and every  $h \in \mathbb{N}$ . Proceeding as in Step 7 of the previous section (see in particular Claim 2), we can actually conclude that

$$\mu_t^l(\mathcal{R}_\infty^{h+1}) \geq 1 - \Omega(k(\epsilon) + 1, h - 1, M + 2, 2\epsilon), \tag{4.39}$$

for every  $t \in [0, 1]$  and every  $h \in \mathbb{N}$  sufficiently large.

**Claim 4.** For a fixed  $t > 0$ , the family  $(\mu_t^l)_{l \in \mathbb{N}}$  is tight.

Given a fixed  $\delta > 0$ , we have to find a compact set  $K_\delta$ , such that  $\mu_t^l(K_\delta) \geq 1 - \delta$  for every  $l \in \mathbb{N}$ . To this aim we take suitable  $\epsilon$  and  $h$  such that (4.39) ensures that

$$\mu_t^l(\mathcal{R}_\infty^{h+1}) \geq 1 - \frac{\delta}{2}. \tag{4.40}$$

Moreover, combining the result of Step 1 (that is (4.37)) with (4.36), we conclude that  $S_{N, \mathfrak{m}}(\mu_t^l)$  is definitely bounded. Then, since  $\mathfrak{m}_\infty|_{\mathcal{R}^{h+1}}$  is a finite Radon measure, we can argue as in the proof of Lemma 4.6 and prove the tightness of the family of measures  $(\mu_t^l|_{\mathcal{R}^{h+1}})_{l \in \mathbb{N}}$ . As a consequence, keeping in mind (4.40), there exists a compact set  $K_\delta$  such that

$$\mu_t^l(K_\delta) \geq \mu_t^l|_{\mathcal{R}^{h+1}}(K_\delta) \geq 1 - \delta,$$

proving the claim.

Now, we can apply Lemma 4.12 and find a Wasserstein geodesic  $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}(\mathbb{X}_\infty)$  connecting  $\mu_0$  and  $\mu_1$  such that (up to subsequences)

$$\mu_t^l \rightharpoonup \mu_t \in \mathcal{P}(\mathbb{X}_\infty) \quad \text{as } l \rightarrow \infty \text{ for every } t \in [0, 1] \cap \mathbb{Q}$$

Then, since the bound (4.39) is uniform in  $l$ , we can conclude that

$$\mu_t(\mathcal{R}_\infty^{h+1}) \geq 1 - \Omega(k(\epsilon) + 1, h - 1, M + 2, 2\epsilon), \tag{4.41}$$

for every  $t \in [0, 1] \cap \mathbb{Q}$  and every  $h \in \mathbb{N}$  sufficiently large. Moreover, by continuity we can deduce (4.41) for every time  $t \in [0, 1]$  (and every  $h \in \mathbb{N}$  sufficiently large). This is sufficient to conclude that  $\mu_t$  gives no mass to the set  $\mathcal{S}$  of singular points, for every  $t \in [0, 1]$ . In fact, assume by contradiction that  $\mu_t(\mathcal{S}) = \delta > 0$ , then condition (3.10) on  $\Omega$  ensures that there exist  $\epsilon$  and  $h \in \mathbb{N}$  such that

$$\Omega(k(\epsilon) + 1, h - 1, M + 2, 2\epsilon) < \delta,$$

and consequently

$$\mu_t(\mathcal{R}_\infty^{h+1}) \geq 1 - \delta,$$

which contradicts  $\mu_t(\mathcal{S}) = \delta$ . At this point we know from Proposition 4.8 that

$$\liminf_{l \rightarrow \infty} S_{N, \mathfrak{m}}(\mu_t^l) \geq S_{N, \mathfrak{m}}(\mu_t), \tag{4.42}$$

for every  $t \in [0, 1] \cap \mathbb{Q}$  and  $N' \in [N, 0)$ .

Finally, we can use (4.37) and (4.42) to pass to the limit as  $l \rightarrow \infty$  of the inequality (4.36) and deduce that

$$S_{N, \mathfrak{m}}(\mu_t) \leq T_{K, N}^{(t)}(q|\mathfrak{m}_\infty) \tag{4.43}$$

holds for every  $t \in [0, 1] \cap \mathbb{Q}$  and every  $N' \in I$ . Then the lower semicontinuity of  $S_{N, \mathfrak{m}}$  (granted by (4.41)) and the continuity of  $T_{K, N}^{(t)}(q|\mathfrak{m}_\infty)$  in  $t$  (which is a straightforward consequence of the dominated convergence theorem), allow to conclude (4.43) for every  $t \in [0, 1]$ , finishing the proof.

## Acknowledgments

The authors thank Professor Karl-Theodor Sturm for suggesting them the problem and for many valuable discussions. Many thanks are due to Lorenzo Dello Schiavo, for many enlightening conversations on measure theory. Finally, we thank the anonymous reviewer for their careful reading of our manuscript and their many insightful comments and suggestions.

The second author gratefully acknowledges support by the European Union through the ERC-AdG 694405 RicciBounds. This research was partially funded by the Austrian Science Fund (FWF) [F65, ESP 224]. A large part of this work was written while the third author was employed in the group of Professor Karl-Theodor Sturm and furthermore he very gratefully acknowledges support by the European Union through the ERC-AdG 694405 RicciBounds. For the purpose of open access, the authors have applied a CC BY public copyright licence to any Author Accepted Manuscript version arising from this submission.

## References

- [1] K.-T. Sturm, On the geometry of metric measure spaces. I, *Acta Math.* 196 (1) (2006) 65–131.
- [2] K.-T. Sturm, On the geometry of metric measure spaces. II, *Acta Math.* 196 (1) (2006) 133–177.
- [3] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, *Ann. of Math.* (2) 169 (3) (2009) 903–991.
- [4] S.-I. Ohta, Finsler interpolation inequalities, *Calc. Var. Partial Differential Equations* 36 (2) (2009) 211–249.
- [5] A. Petrunin, Alexandrov meets Lott-Villani-Sturm, *Münster J. Math.* 4 (2011) 53–64.
- [6] H.-C. Zhang, X.-P. Zhu, Ricci curvature on alexandrov spaces and rigidity theorems, *Comm. Anal. Geom.* 18 (3) (2010) 503–553.
- [7] M. Erbar, K. Kuwada, K.-T. Sturm, On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces, *Invent. Math.* 201 (3) (2014) 1–79.
- [8] H. Brascamp, E. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* (22) (1976) 366–389.
- [9] C. Borell, Convex set functions in  $d$ -space, *Period. Math. Hungar.* (6) (1975) 111–136.
- [10] S.G. Bobkov, Large deviations and isoperimetry over convex probability measures with heavy tails, *Electron. J. Probab.* (12) (2007) 1072–1100.
- [11] S.G. Bobkov, M. Ledoux, Weighted Poincaré -type inequalities for Cauchy and other convex measures, *Ann. Probab.* (37) (2009) 403–427.
- [12] A.V. Kolesnikov, Hessian metrics,  $CD(K,N)$ -spaces, and optimal transportation of log-concave measures, *Am. Inst. Math. Sci.* 34 (4) (2014) 1511–1532.
- [13] F. Barthe, P. Cattiaux, C. Roberto, Concentration for independent random variables with heavy tails, *AMRX Appl. Math. Res. eXpress* (2) (2005) 39–60.
- [14] S.-I. Ohta, A. Takatsu, Displacement convexity of generalized relative entropies I, *Adv. Math.* 228 (2011) 1742–1787.
- [15] S.-I. Ohta, A. Takatsu, Displacement convexity of generalized relative entropies II, *Comm. Anal. Geom.* 21 (2013) 687–785.
- [16] R. Jordan, D. Kinderlehrer, F. Otto, The variational formulation of the Fokker-Planck equation, *SIAM J. Math. Anal.* 29 (1) (1998) 1–17.
- [17] C. Villani, *Topics in Optimal Transportation*, American Mathematical Society, Providence, RI, 2003.
- [18] S.-i. Ohta, K.-T. Sturm, Heat flow on finsler manifolds, *Comm. Pure Appl. Math.* 62 (2009) 1386–1433.
- [19] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, *Comm. Partial Differential Equations* 26 (1–2) (2001) 101–174.
- [20] A. Ohara, T. Wada, Information geometry of  $q$ -Gaussian densities and behaviors of solutions to related diffusion equations, *J. Phys. A* 43 (035002) (2010) 18 pp..
- [21] A. Takatsu, Wasserstein geometry of porous medium equation, *Ann. l’Inst. Henri Poincaré C, Anal. non linéaire* 29 (2) (2012) 217–232.
- [22] S.-I. Ohta,  $(K,N)$ -convexity and the curvature-dimension condition for negative  $N$ , *J. Geom. Anal.* 26 (2016) 2067–2096.
- [23] E. Milman, Harmonic measures on the sphere via curvature-dimension, *Ann. Fac. Sc. Toulouse* 26 (2) (2017) 437–449.
- [24] E. Milman, Beyond traditional curvature-dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension, *Trans. Amer. Math. Soc.* 369 (5) (2017) 3605–3637.
- [25] Y. Sakurai, Rigidity phenomena in manifolds with boundary under a lower weighted Ricci curvature bound, *J. Geom. Anal.* (29) (2019) 1–32.
- [26] Y. Sakurai, Comparison geometry of manifolds with boundary under a lower weighted Ricci curvature bound, *Canad. J. Math.* 72 (1) (2020) 243–280.
- [27] E. Woolgar, W. Wylie, Curvature-dimension bounds for Lorentzian splitting theorems, *J. Geom. Phys.* (132) (2018).
- [28] A.V. Kolesnikov, E. Milman, Poincaré and Brunn-Minkowski inequalities on the boundary of weighted Riemannian manifolds, *Amer. J. Math.* 140 (5) (2018) 1147–1185.



- [29] S.-I. Ohta, K.-T. Sturm, Bochner-Weitzenböck formula and li-yau estimates on finsler manifolds, *Adv. Math.* 252 (2014) 429–448.
- [30] N. Gigli, K. Kuwada, S.-i. Ohta, Heat flow on alexandrov spaces, *Comm. Pure Appl. Math.* 66 (3) (2013) 307–331.
- [31] L. Ambrosio, N. Gigli, G. Savaré, Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds, *Ann. Probab.* 43 (1) (2015) 339–404.
- [32] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of the Laplace operator, *Invent. Math.* (87) (1987) 517–547.
- [33] N. Gigli, A. Mondino, G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, *Proc. Lond. Math. Soc.* (2013).
- [34] D. Fremlin, *Measure Theory*, Torres Fremlin, 25 Ireton Road, Colchester CO3 3AT, England, 2001.
- [35] V. Bogachev, *Measure Theory. Vol. I, II*, Springer-Verlag, Berlin, 2007, Vol. I: xviii+500 pp., Vol. II: xiv+575.
- [36] W. Rudin, *Real and Complex Analysis*, third ed., McGraw-Hill Book Co., New York, 1987.
- [37] C. Villani, *Optimal Transport. Old and New*, in: *Grundlehren der Mathematischen Wissenschaften*, vol. 338, Springer-Verlag, Berlin, 2009, p. xxii+973.
- [38] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, second ed., in: *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel, 2008, p. x+334.
- [39] S.-I. Ohta, Needle decompositions and isoperimetric inequalities in finsler geometry, *J. Math. Soc. Japan* 70 (2) (2018) 651–693.
- [40] K. Bacher, K.-T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, *J. Funct. Anal.* 259 (1) (2010) 28–56.