
The ABCD's of Mirror Symmetry

Novel Dualities in Three Dimensions

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Abstract

This thesis investigates mirror symmetry in three-dimensional quantum field theories with reduced supersymmetry, focusing on theories with $\mathcal{N} \leq 2$. While mirror symmetry for $\mathcal{N} = 4$ theories admits a clear geometric interpretation in terms of Type IIB brane constructions and $SL(2; \mathbb{Z})$ duality, this intuition breaks down once supersymmetry is reduced since the moduli space may be lifted, and canonical brane realizations are often unavailable.

Despite the breakdown of the geometric intuition afforded to us by $\mathcal{N} = 4$ supersymmetry, we show that mirror-like dualities persist beyond the conventional $\mathcal{N} = 4$ setting. Guided by symmetry considerations – most notably the exchange of flavor and topological symmetries – we develop a systematic framework to construct mirror duals of non-Abelian $\mathcal{N} = 2$ Chern-Simons-matter theories. We analyze unitary and symplectic gauge groups, study their renormalization group flows under real mass deformations, and map gauge-invariant operators across dual descriptions. The framework further admits extensions to non-supersymmetric theories, revealing a broader notion of three-dimensional mirror symmetry as a duality between order and disorder operators.

(Woefully Incomplete)

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“...Life itself likewise always turns towards one an infinitely meaningless, discouraging, dispiriting blank side on which there is nothing, any more than on a blank canvas.

But, however meaningless and vain, however dead life appears, the man of faith, of energy, of warmth, and who knows something, doesn't let himself be fobbed off like that. He steps in and does something... ”

Vincent van Gogh, to Theo van Gogh (October 2, 1884)

ब्रह्मार्पणं ब्रह्म हविर्ब्रह्माग्नौ ब्रह्मणा हुतम् ।
ब्रह्मैव तेन गन्तव्यं ब्रह्मकर्मसमाधिना ॥

“The offering is Brahman, the oblation is Brahman, offered by Brahman into the fire of Brahman; Brahman alone is to be attained by one who is absorbed in action as Brahman.”

Bhagavad Gītā 4.24

Publications

This thesis includes work by the author that has been published or accepted for publication. These publications are the own work of the author of this thesis, and the author has the permission of the publishers to reproduce the contents of these publications for academic purposes.

In particular, some data, ideas, opinions and figures presented in this thesis have previously appeared or may appear shortly after the submission of this thesis as follows:

1. S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota, **A. Shri**, “*Planar Abelian Mirror Duals of $\mathcal{N} = 2$ SQCD₃*”, in *Phys. Rev. D* 112.10 (2025) [2].
2. S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota, **A. Shri** “*A Chiral-Planar Dualization Algorithm for 3d $\mathcal{N} = 2$ Chern-Simons-Matter Theories*”, in *JHEP* (10) (2025) [3].
3. S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota, **A. Shri** “*Planar Abelian Duals of Chern-Simons QCD*”, in *JHEP* (07) (2026) [4].
4. S. Benvenuti, R. Comi, G. Pedde Ungureanu, S. Rota, **A. Shri** “*Universal Planar Abelian Duals for 3d $\mathcal{N} = 2$ Unitary CS-SQCD*”, arXiv: 2603.08842 [5], accepted for publication in *JHEP* (2026).
5. S. Benvenuti, V. Cagioni, S. Rota, **A. Shri** “*Universal Planar Abelian Duals for 3d $\mathcal{N} = 2$ Symplectic CS-SQCD*”, arXiv: 2605.06776 [6].

Except where reference is made in the text of the thesis, this thesis contains no other material published elsewhere or extracted in whole or in part from a thesis accepted for the award of any other degree or diploma. No other person’s work has been used without due acknowledgment in the main text of the thesis. This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

Captatio Benevolentiae

Motivation

A key feature at the heart of our understanding of supersymmetric gauge theories is the idea of *universality* [7–13]: two vastly different theories may furnish the same macroscopic description upon flowing to the same infrared fixed point (IRFP) [14–18]. Consequently, when the renormalization group (RG) flow renders one theory non-perturbative due to strong coupling, one may still extract useful information about the system by studying a more perturbatively amenable dual theory.

With the advent of the Second Superstring Revolution, this notion of universality has become pervasive in contemporary physics. Embedding a quantum field theory in String Theory [19] sheds light on properties of the former that would not be immediately visible otherwise; chief among these are various infrared dualities which are more easily uncovered by exploiting the geometric properties of brane setups.

Nowhere is this perspective more striking than in three-dimensional supersymmetric gauge theories, which often admit two very distinct kinds of dual descriptions. In addition to the more familiar *Aharony-like* dualities [18, 20, 21] which retain the global symmetry structure, $3d$ gauge theories also display *mirror symmetry* [22] which exchanges topological and flavor symmetries.

$3d$ $\mathcal{N} = 4$ mirror symmetry can be immediately understood in terms of Type IIB Hanany-Witten brane setups [19, 23] as a consequence of the $\mathcal{S}$ -duality of the latter (inherited from the $SL(2; \mathbb{Z})$ invariance of IIB String Theory), which exchanges $NS5$ and $D5$ branes. Thus, $3d$ $\mathcal{N} = 4$ mirror symmetry relates pairs of theories with the classically exact Higgs and quantum-corrected Coulomb branches of the moduli space exchanged.

It was believed that a large amount of supersymmetry is necessary to observe this kind of IR duality. While generalizations to $3d$ $\mathcal{N} = 2$ Abelian theories were possible [24–26], their non-Abelian counterparts proved to be far more elusive and were only recently discovered by invoking generalized bifundamentals as building blocks [27, 28].

$\mathcal{N} \leq 2$ theories have far richer dynamics than their $\mathcal{N} = 4$ counterparts since they admit the inclusion of Chern-Simons (CS) terms [29–33]. The subject of mirror symmetry in the context of $\mathcal{N} \leq 2$ Abelian CS theories was first studied in [25, 26, 34, 35] & [36–46] respectively, but generalizations to non-Abelian $\mathcal{N} \leq 2$ theories are not as straightforward and merit renewed attention.

Rather than viewing reduced supersymmetry as a limitation, we treat it as an opportunity: Chern-Simons interactions and chiral matter introduce new infrared behaviors that are invisible in $\mathcal{N} = 4$ settings, but can still be organized into precise duality frameworks. Motivated by this, we study the infrared dynamics of $3d$ $\mathcal{N} \leq 2$ unitary and symplectic

gauge theories with CS interactions. The central theme of this thesis is that mirror symmetry, even in the absence of strong geometric control, can be systematically engineered using symmetry principles and real-mass flows, yielding large families of dualities across supersymmetric and non-supersymmetric regimes.

Organizational Note

The thesis is organized into five parts, each addressing a distinct aspect of mirror symmetry in three-dimensional quantum field theories, while building progressively toward less supersymmetric and more general settings.

1. Part I: Introduction

This part provides the theoretical background and sets the stage for the rest of the thesis. We begin with a concise review of supersymmetric quantum field theories, with particular emphasis on $2 + 1$ -dimensional dynamics. We then introduce brane constructions in Type IIB string theory, focusing on their role in the geometric engineering of three-dimensional gauge theories. The part concludes with a pedagogical review of $3d \mathcal{N} = 4$ mirror symmetry, which serves as the conceptual and technical anchor for the subsequent developments.

2. Part II: $\mathcal{N} = 2$ Mirror Symmetry in Unitary Gauge Theories

In this part, we develop systematic methods for constructing mirror duals of $3d \mathcal{N} = 2$ unitary Chern-Simons-matter gauge theories. We introduce an algorithmic procedure based on real mass deformations to generate large classes of $\mathcal{N} = 2$ dual pairs. Particular attention is devoted to the renormalization group flows, the mapping of global and topological symmetries, and the identification of gauge-invariant operators across the duality. This part incorporates the main results of the published works on unitary theories [2, 3], as well as their extensions to massive deformations [5].

3. Part III: $\mathcal{N} = 2$ Mirror Symmetry in Symplectic Gauge Theories

This part extends the techniques developed for unitary gauge groups to symplectic gauge theories. We construct mirror duals of $3d \mathcal{N} = 2$ theories with symplectic gauge symmetry, highlighting both the similarities and the new features that arise relative to the unitary case. Renormalization group flows and massive deformations are analyzed in detail, providing further evidence for the proposed dualities. This part incorporates the results of [6].

4. Part IV: Non-Supersymmetric Planar Mirror Symmetry

In this part, we explore extensions of the mirror symmetry framework to non-supersymmetric quantum field theories. Guided by insights from the supersymmetric constructions, we propose planar Abelian dual descriptions for classes of non-supersymmetric Chern-Simons-matter theories. We discuss which structural aspects of mirror symmetry persist in the absence of supersymmetry, which features are modified, and which remain conjectural, thereby delineating the scope and limitations of these non-supersymmetric dualities. This part is based on the results of [4].

5. Part V: Conclusions and Outlook

The thesis concludes with a summary of the main results and a discussion of open

problems and future directions. We outline possible generalizations of the dualization algorithms developed in this work, including the implementation of an $SL(2, \mathbb{Z})$ -based construction for $\mathcal{N} = 2$ theories, extensions to orthogonal and orthosymplectic gauge groups, and connections to two-dimensional $\mathcal{N} = (2, 2)$ theories. These directions highlight the broader applicability of the methods introduced in this thesis and point toward a more unified understanding of infrared dualities in low-dimensional quantum field theories.

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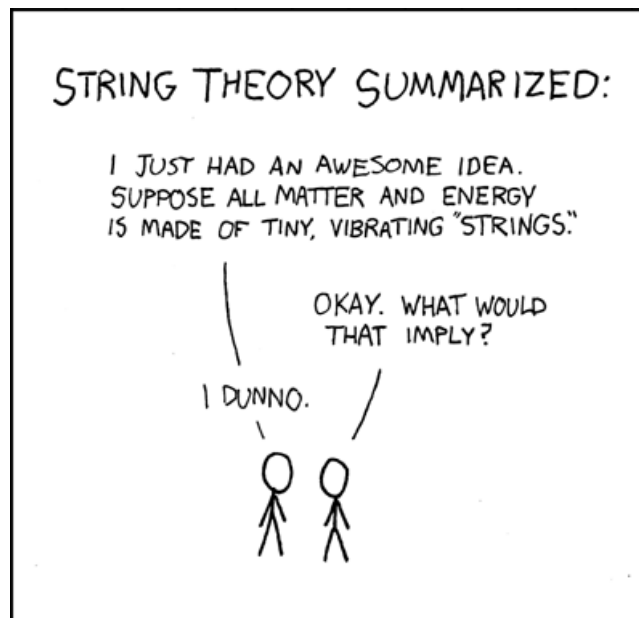
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Part I: Introduction



Chapter 1

D-Brane Engineering of Three-Dimensional Supersymmetric Gauge Theories

Supersymmetry has played a central role in shaping our modern understanding of quantum field theory beyond perturbation theory. By severely constraining the structure of quantum corrections, supersymmetric theories admit exact results that would otherwise be inaccessible, allowing for the controlled study of strongly coupled dynamics. Among the most striking consequences of these constraints is the existence of nonperturbative dualities: distinct microscopic descriptions that flow to the same infrared fixed point.

String theory provides a natural framework in which many such dualities acquire a geometric origin. In particular, extended objects such as D-branes furnish concrete realizations of supersymmetric gauge theories, with their low-energy dynamics captured by world-volume field theories. Operations that are highly nontrivial from a purely field-theoretic perspective—such as strong–weak coupling dualities, symmetry enhancements, or infrared equivalences—often translate into simple geometric manipulations of brane configurations.

Three-dimensional supersymmetric gauge theories offer a particularly rich arena for these ideas. Unlike their four-dimensional counterparts, they exhibit intrinsically nonperturbative operators, such as monopoles, which play a central role in determining the infrared dynamics. At the same time, three-dimensional theories admit a wide class of dualities whose origin is most transparent when viewed through their string-theoretic embeddings. Chief among these is *mirror symmetry*, an infrared duality that exchanges the Higgs and Coulomb branches of $\mathcal{N} = 4$ theories and interchanges perturbative and nonperturbative degrees of freedom.

In this chapter, we review the interplay between supersymmetry, brane constructions, and mirror symmetry in three dimensions. We begin with a brief overview of supersymmetry and D-branes and their low-energy worldvolume descriptions, emphasizing the emergence of supersymmetric gauge theories from string theory. We then introduce the Hanany–Witten construction, which provides a geometric realization of three-dimensional gauge dynamics and their dualities. Within this framework, mirror symmetry arises naturally as the action of Type IIB $\mathcal{S}$ -duality, offering a clear geometric interpretation of the exchange between Coulomb and Higgs branch physics.

This review serves two purposes. First, it establishes the standard framework for mirror

symmetry in $\mathcal{N} = 4$ theories, which will be used extensively throughout this thesis. Second, it motivates the central question addressed in later chapters: how much of this structure survives when supersymmetry is reduced, brane realizations are no longer available, and the classical notions of Higgs and Coulomb branches cease to exist. The techniques and perspectives introduced here will provide the foundation for extending mirror symmetry beyond its traditional setting.

1.1 The View From Field Theory

Supersymmetry provides a powerful theoretical framework in which strongly coupled quantum field theories can be studied with a remarkable degree of analytical control. In three spacetime dimensions, supersymmetric gauge theories exhibit a rich interplay between gauge dynamics, global symmetries, and quantum effects that has led to deep insights into infrared physics, including the existence of nontrivial fixed points and intricate patterns of duality. The purpose of this section is to review the aspects of supersymmetric field theories that will be essential throughout this thesis, with an emphasis on structures that are robust under renormalization group flow and therefore characterize the infrared theory. To that end, we mainly collect and review some key results from [47–56] (see also [57–70] and the references therein).

Rather than aiming for a comprehensive introduction to supersymmetry, we focus on the ingredients that play a central role in our study of three-dimensional dynamics. This material will serve both to fix conventions and to establish a common language for discussing infrared dualities, mass deformations, and symmetry realizations in later chapters. In this sense, the section is intended as a conceptual and technical foundation, highlighting those features of supersymmetric gauge theories that persist beyond a specific ultraviolet description and govern the universal infrared behavior.

1.1.1 An Overture: Supersymmetry in Diverse Dimensions

The history of supersymmetry is rather peculiar, even when compared to that of most other topics in theoretical physics. Originally proposed in the early 1970s, supersymmetry was progressively developed into a remarkably rich and elegant mathematical framework. It unifies particles of different spins into common symmetry multiplets and has far-reaching implications for both quantum field theory and fundamental physics [64].

Early attempts to extend the symmetry structure of relativistic quantum field theories were motivated by the idea of combining spacetime symmetries with internal gauge symmetries. In particular, it appeared natural to seek a unified framework incorporating the Poincaré group in four dimensions, $ISO(1, 3)$, together with the gauge group of the Standard Model, $SU(3)_c \times SU(2)_L \times U(1)_Y$. Concrete proposals along these lines were explored, including attempts to embed these symmetries into a larger group such as $SU(6)$ [64]. However, these ideas were ultimately ruled out by a series of no-go theorems, the most notable of which is the **Coleman–Mandula theorem**:

Theorem 1 (Coleman–Mandula Theorem) *The most general bosonic symmetry of scattering amplitudes is a direct product of the Poincaré group and an internal symmetry [71]*

$$\mathfrak{g} = \mathfrak{iso}(1, 3) \times \mathfrak{g}_{\text{internal}}.$$

In other words, under the assumptions of the theorem, there exists *no non-trivial way of combining spacetime and internal symmetries* within a purely bosonic framework.

However, like all good no-go theorems, the Coleman–Mandula result admits a crucial loophole: it assumes that all symmetry generators are bosonic. Shortly after its formulation, this limitation was exploited with the emergence of supersymmetry. In 1975, Haag, Łopuszański, and Sohnius extended the theorem to include fermionic symmetry generators, thereby identifying supersymmetry as the most general consistent extension of spacetime symmetries:

Theorem 2 (Haag–Łopuszański–Sohnius Theorem) *The most general symmetry of the S -matrix consists of the Poincaré group, internal symmetries, and additional fermionic generators that form a non-trivial extension of the Poincaré algebra [72].*

The resulting super-Poincaré algebra is a $\mathbb{Z}_2$ -graded¹ algebra [74–76]: its even part coincides with the ordinary Poincaré algebra, while its odd part is generated by spinorial charges. At a physical level, supersymmetry transformations exchange bosonic and fermionic degrees of freedom, mapping particles of integer spin to particles of half-integer spin and vice versa. In this sense, supersymmetry constitutes a natural and minimal extension of the known symmetries of relativistic quantum field theory.

Remarkably, supersymmetry is the *only* known symmetry that can relate bosons and fermions in a quantum-consistent way. More generally, it is the only symmetry of the S -matrix capable of connecting states with different spin-statistics. Classical symmetries relating fields of different spin are typically spoiled by quantum effects, unless they are protected by supersymmetry. As a consequence, sufficiently extended supersymmetry is often required for classical geometric and algebraic structures of field theory to survive quantization and retain a precise mathematical meaning [61].

The exact form of the Poincaré algebra depends on the zoology of spinors in various dimensions and the corresponding space-time signatures. However, the general structure is fairly universal, and we comment on that before focusing on supersymmetry in 3 spacetime dimensions.

Spinors in D Dimensions

A spinor is an element of a complex vector space $\mathbf{S}$ on which the universal Dirac matrices Γ_μ act. In D spacetime dimensions, the complex dimension of $\mathbf{S}$ is

$$\dim_{\mathbb{C}} \mathbf{S} = 2^{\lfloor D/2 \rfloor}.$$

The space $\mathbf{S}$ furnishes a representation of the complexified spin group $Spin(D; \mathbb{C})$.

In even dimensions, $D \equiv 0 \pmod{2}$, the spinor representation is reducible and decomposes into two irreducible chiral representations,

$$\mathbf{S} = \mathbf{S}_+ \oplus \mathbf{S}_-,$$

corresponding to positive and negative chirality. Elements of $\mathbf{S}_\pm$ are called *Weyl spinors*: the *Weyl condition* is a chirality projection of the form $\Gamma_* \Psi = \pm \Psi$.

¹A *graded vector space* is a direct sum over vector spaces $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}_i$ over a field $\mathbb{K}$ of characteristic zero. The $\mathfrak{a}_i$ are components of $\mathfrak{a}$ of degree i , and the degree of $a \in \mathfrak{a}$ is denoted $|a|$. A graded vector space with degree shifted by n is denoted by $\mathfrak{a}[n] = \bigoplus_{i \in \mathbb{Z}} (\mathfrak{a}[n])_i$ with $(\mathfrak{a}[n])_i = \mathfrak{a}_{i+n}$. A *graded Lie Algebra* $\mathfrak{a}$ of degree n is a graded vector space $\mathfrak{a}$ endowed with a graded Lie bracket on $\mathfrak{a}[n]$, that is, a bilinear operation $\{.,.\} : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}[-n]$ satisfying graded antisymmetry and graded Jacobi relations [73]

$$\begin{aligned} \{a, b\} &= -(-1)^{(|a|+n)(|b|+n)} \{b, a\}, \\ \{a, \{b, c\}\} &= \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)} \{b, \{a, c\}\}. \end{aligned}$$

The properties of spinors depend crucially on the spacetime signature (p, q) ². In particular, the spinor space $\mathbf{S}$ may admit a real, complex, or quaternionic structure compatible with the action of $Spin(p, q)$. This determines the possible reality conditions that can be imposed on spinors.

The group $Spin(p, q)$ admits a minimal realization [79] in terms of $2^\alpha \times 2^\alpha$ matrices with entries in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on the value of $p - q \pmod{8}$, as summarized in Table 1.1.

Table 1.1: $Spin(p, q)$ groups and their minimal matrix realizations

$p - q \pmod{8}$	Algebra	2α
0	$\mathbb{R}$	$p + q - 2$
± 1	$\mathbb{R}$	$p + q - 1$
± 2	$\mathbb{C}$	$p + q - 2$
± 3	$\mathbb{H}$	$p + q - 3$
4	$\mathbb{H}$	$p + q - 4$

We say that in D dimensions with signature (p, q) there exist *Majorana spinors*³, respectively *symplectic–Majorana spinors*, if the spinor space $\mathbf{S}$ admits an invariant real, respectively quaternionic, structure under the action of $Spin(p, q)$. In this case, the corresponding (pseudo)real elements of $\mathbf{S}$ are called Majorana or symplectic–Majorana spinors.

If, in addition, the chiral subspaces $\mathbf{S}_\pm$ admit an invariant real or quaternionic structure, the corresponding spinors are called *Majorana–Weyl* or *symplectic–Majorana–Weyl* spinors, respectively. Hence, in Minkowski signature $(D-1, 1)$, the minimal spinor is as in Table 1.2.

Supercharges and Superalgebras

To make manifest the automorphism group of the supersymmetry algebra, it is convenient to write the supercharges in terms of minimal spinors. When the supersymmetry automorphism is a physical symmetry of the theory, we refer to it as the R-symmetry of the theory.

In supersymmetric theories, $\mathcal{N}$ denotes the number of independent supersymmetry generators, counted in units of the minimal spinor appropriate to the spacetime dimension and signature. Equivalently, $\mathcal{N}$ measures how many copies of the minimal supersymmetry

²In turn, this structure descends from the properties of the Clifford algebra and its Bott periodicity [77]. In general, the **Clifford algebra** $\mathfrak{Cl}(V; Q)$ over a vector space V equipped with a quadratic form $Q : V \rightarrow \mathbb{R}$ is defined by the quotient:

$$\mathfrak{Cl}(V; Q) = \frac{\bigoplus_{k=0}^{\infty} V^{\otimes k}}{(x \otimes x - Q(x))}.$$

Typically, $Q(x_1, \dots, x_k) := -\sum x_i^2$, and we define $\mathfrak{Cl}_k := \mathfrak{Cl}(\mathbb{R}^k; Q_k)$ which can be regarded as $\mathbb{Z}_2$ -graded algebras.

We can now define $A_k := M(\mathfrak{Cl}_k)/i^*M(\mathfrak{Cl}_{k+1})$, where $M(A)$ denotes the *Grothendieck group of graded representations* of A , and $i^* : M(\mathfrak{Cl}_{k+1}) \rightarrow M(\mathfrak{Cl}_k)$. Bott periodicity [78] establishes the existence of an isomorphism between A_k and A_{k+8} , as seen in Table 1.1.

³The Majorana condition is a reality condition of the form $\Psi^* = B\Psi$, $B \sim C$, compatible with the Lorentz action and C is the charge-conjugation matrix.

Table 1.2: Minimal spinors in Minkowski space

<i>Type of Spinor</i>	<i>D</i>
<i>Majorana</i>	$8k + 1, 8k + 3$
<i>Symplectic-Majorana</i>	$8k + 5, 8k + 7$
<i>Weyl</i>	$8k, 8k + 4$
<i>Majorana-Weyl</i>	$8k + 2$
<i>Symplectic-Majorana-Weyl</i>	$8k + 6$

algebra are realized in the theory. Since the dimension and reality properties of spinors depend on the spacetime dimension, a fixed value of $\mathcal{N}$ corresponds to a **different total number of real supercharges in different dimensions** - more plainly, the invariant quantity is the number of real supercharges and not $\mathcal{N}$. In even dimensions, where chiral spinors exist, supersymmetry may further decompose into independent left- and right-moving sectors, characterized by a pair $(\mathcal{N}_R, \mathcal{N}_L)$. Table 1.3 summarizes the values of $\mathcal{N}$ (or $(\mathcal{N}_R, \mathcal{N}_L)$) that are physically allowed in various spacetime dimensions, subject to spinorial constraints and consistency of the supersymmetry algebra.

Table 1.3: Physically allowed $\mathcal{N}$ s in various dimensions. Note that in two, six, and ten spacetime dimensions, we have independent left/right supersymmetries (see also Table 1.4). Then we have distinct right and left SUSY extension numbers $\mathcal{N}_R, \mathcal{N}_L$. Bold face numbers correspond to supersymmetric extensions allowed only in the local case (supergravity).

<i>D</i>	Allowed extensions $\mathcal{N}$ (resp. $(\mathcal{N}_R, \mathcal{N}_L)$)
3	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16
4	1, 2, 3, 4, 5, 6, 8
5	1, 2, 4, 6, 8
6	(1, 0), (1, 1), (2, 0), (2, 2), (4, 0), (4, 2), (6, 0), (4, 4), (6, 2), (8, 0)
7	1, 2, 4
8	1, 2
9	1, 2
10	(1, 0), (1, 1), (2, 0)
11	1

In Table 1.4, we summarize the supersymmetry generators, the Poincaré algebras and automorphism groups for different dimensions. In Table 1.4, ‘CC’ refers to “central charges”, $\mathcal{C}$ denotes the charge-conjugation matrix, $B := \mathcal{C}\Gamma^0$, and the “(*)” denotes a property that holds in a Majorana representation ⁴.

⁴Refer to [80–83] for a more detailed exposition.

Table 1.4: Supercharges and superalgebras in diverse dimensions

D	3	4	5	6
<i>Supercharges</i>	Majorana	Weyl	Symplectic–Majorana	Symplectic–Majorana–Weyl
<i>Reality</i>	$(Q_\alpha^a)^\dagger = Q_\alpha^a \ (*)$	$(Q_\alpha^a)^\dagger = \bar{Q}_{\dot{\alpha}a}$	$(Q_\alpha^a)^\dagger = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$	$(Q_\alpha^a)^\dagger = \bar{Q}_{\alpha a} = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$
<i>Superalgebra</i>	$\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Gamma^\mu \Gamma_0)_{\alpha\beta} P_\mu + \text{CC}$	$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = 2\delta^a{}_b(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu$ $\{Q_\alpha^a, Q_\beta^b\} = \epsilon_{\alpha\beta} Z^{ab}$	$\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu + \text{CC}$	$\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Sigma^\mu)_{\alpha\beta} P_\mu$ $\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = \mathcal{C}_{\alpha\dot{\beta}} Z^{ab}$
<i>Automorphism</i>	$SO(\mathcal{N})$	$U(\mathcal{N})$	$Sp(\mathcal{N})$	$Sp(\mathcal{N}_R) \times Sp(\mathcal{N}_L)$
∞ <i>Centralizer</i>	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$
D	7	8	9	10
<i>Supercharges</i>	Symplectic–Majorana	Weyl	Majorana	Majorana–Weyl
<i>Reality</i>	$(Q_\alpha^a)^\dagger = \Omega_{ab} B_\alpha{}^\beta Q_\beta^b$	$(Q_\alpha^a)^\dagger = \bar{Q}_{\dot{\alpha}a}$	$(Q^a)^\dagger = \mathcal{C}(\bar{Q}^a)^t$	$(Q_\alpha^a)^\dagger = Q_\alpha^a$
<i>Superalgebra</i>	$\{Q_\alpha^a, Q_\beta^b\} = 2\Omega^{ab}(\Gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu + \text{CC}$	$\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = 2\delta^a{}_b(\Sigma^\mu)_{\alpha\dot{\beta}} P_\mu$ $\{Q_\alpha^a, Q_\beta^b\} = \mathcal{C}_{\alpha\beta} Z^{ab}$	$\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Gamma^\mu \mathcal{C})_{\alpha\beta} P_\mu + \text{CC}$	$\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\Sigma^\mu)_{\alpha\beta} P_\mu$ $\{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} = \text{CC}$
<i>Automorphism</i>	$Sp(\mathcal{N})$	$U(\mathcal{N})$	$SO(\mathcal{N})$	$SO(\mathcal{N}_R) \times SO(\mathcal{N}_L)$
<i>Centralizer</i>	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$

Central Charges and the BPS Bound

The positivity of the anticommutator $\{Q, Q\}$ is not a formality: it is a bound on the mass of every state, and the states that saturate it are the protected spectrum the constructions compute [69, 84–86]. Acting on a one-particle state of mass M and central charge Z , the supersymmetry algebra reduces, in the relevant sector, to a Hermitian block built from the rest mass and the central charge,

$$A = \begin{pmatrix} M & Z \\ \bar{Z} & M \end{pmatrix}; \quad \text{eigenvalues } M \pm |Z|.$$

Because $A = \{Q, Q\} \geq 0$ is a sum of squares, both eigenvalues are nonnegative, which is the BPS bound

$$M \geq |Z|.$$

A state saturating the bound, $M = |Z|$, makes $\det A = M^2 - |Z|^2$ vanish: the block drops rank, and half of the supercharges (the kernel of A) annihilate the state. These preserved supercharges are the ones for which $M - |Z| = 0$; the remaining broken supercharges, with eigenvalue $2M$, act non-trivially and build a multiplet that is shortened, with half the states of a generic massive multiplet. The shortening is what protects BPS quantities - a short multiplet has fewer states than a long one, so its mass cannot drift off $M = |Z|$ under a deformation that keeps the multiplet short. **The protection is robust but not absolute:** what the shortening protects is the algebraic mass formula for a state that stays in that short representation. It does not, by itself, guarantee that the state survives as a stable one-particle state everywhere in moduli space. Even strongly shortened BPS sectors can meet walls of marginal stability, singular loci, or decay channels. What is protected is the shortening relation and the appropriate charge or index data in its chamber.

The protection is graded by the supersymmetry fraction: a low fraction, such as a quarter-BPS dyon degeneracy, is more susceptible to recombination and to wall-crossing, where the protected number jumps between chambers; but the caveat is not limited to the low fractions, since a short multiplet can recombine into a long one whenever the spectrum supplies the missing states. This graded protection is what lets supersymmetric field theory compute BPS masses, indices, and central charges through controlled chambers and deformations, surviving deformations that would otherwise leave perturbative control.

1.1.2 Aspects of Supersymmetry in Three Dimensions

The results presented in this thesis are rooted in the study of three-dimensional gauge theories. Having established the general structure of the supersymmetry algebra, we now turn to a review of several features that are specific to supersymmetric field theories in three spacetime dimensions. These theories exhibit a number of distinctive properties — both technical and conceptual — that set them apart from their four-dimensional counterparts and make them particularly rich and tractable objects of study⁵.

Elements of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ Gauge Theories

The Supersymmetry Algebra. The supersymmetry algebra of $(2 + 1)$ -dimensional $\mathcal{N} = 4$ theories is built upon the *bosonic* Poincaré algebra

$$\mathfrak{iso}(1, 2) \cong \mathbb{R}^{1,2} \rtimes \mathfrak{so}(1, 2),$$

⁵As [61] eloquently remarks, “3d supersymmetry is to 4d supersymmetry what Diet Coke is to Coke — no sugar, no caffeine, all the flavor.”

generated by translations P_μ and Lorentz transformations $M_{\mu\nu}$ (we adopt the same conventions for the basis of the algebra as [87])⁶. These generators obey

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}), \\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu). \end{aligned}$$

The supersymmetric extension introduces four *Majorana spinor* supercharges Q^a ($a = 1, \dots, 4$), satisfying

$$\begin{aligned} \{Q_\alpha^a, Q_\beta^b\} &= 2\delta^{ab}(\gamma^\mu P_\mu)_{\alpha\beta} + 2i\epsilon_{\alpha\beta}Z^{[ab]}, \\ [Q_\alpha^a, P_\mu] &= 0, \\ [M_{\mu\nu}, Q_\alpha^a] &= (\gamma_{\mu\nu})_\alpha^\beta Q_\beta^a, \end{aligned}$$

where α, β are spinor indices, $\mu = 0, 1, 2$ are Lorentz indices, and $Z^{[ab]}$ is an antisymmetric matrix of central charges. The Dirac matrices γ^μ furnish a representation of the Clifford algebra [93]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \gamma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

The automorphism group of the algebra is

$$SO(4) \cong SU(2)_H \times SU(2)_C,$$

which acts as the R-symmetry group.

For $\mathcal{N} = 4$ *superconformal field theories* (SCFTs), the symmetry algebra enhances to $\mathfrak{osp}(4|4)$. Its bosonic subalgebra contains the three-dimensional conformal algebra $\mathfrak{so}(2, 3)$, generated by dilatations D and special conformal transformations K_μ , with commutation relations⁷

$$\begin{aligned} [M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\ [D, P_\mu] &= -iP_\mu, \\ [D, K_\mu] &= iK_\mu, \\ [P_\mu, K_\nu] &= 2i(M_{\mu\nu} + \eta_{\mu\nu}D). \end{aligned}$$

The R-symmetry algebra $\mathfrak{so}(4)_R$ satisfies

$$[R_{rs}, R_{ut}] = i(\delta_{ru}R_{st} + \delta_{st}R_{ru} - \delta_{rt}R_{su} - \delta_{su}R_{rt}).$$

⁶See [60, 88–92] and the references therein for a more detailed exposition, albeit with vastly different conventions and normalizations.

⁷Note that we only mention nonzero relations for brevity.

The fermionic generators are extended by conformal supercharges S_α^a , and the full superconformal algebra takes the form

$$\begin{aligned}
\{S_\alpha^a, S_\beta^b\} &= 2\delta^{ab}(\gamma^\mu K_\mu)_{\alpha\beta}, \\
\{Q_\alpha^a, S_\beta^b\} &= -i\delta^{ab}(-2i(\gamma^{\mu\nu} M_{\mu\nu})_{\alpha\beta} + 2\epsilon_{\alpha\beta} D) + 2i\epsilon_{\alpha\beta} R^{ab}, \\
[K_\mu, Q_\alpha^a] &= -(\gamma_\mu)_\alpha^\beta S_\beta^a, \\
[P_\mu, S_\alpha^a] &= -(\gamma_\mu)_\alpha^\beta Q_\beta^a, \\
[M_{\mu\nu}, S_\alpha^a] &= (\gamma_{\mu\nu})_\alpha^\beta S_\beta^a, \\
[D, Q_\alpha^a] &= -\frac{i}{2} Q_\alpha^a, \\
[D, S_\alpha^a] &= \frac{i}{2} S_\alpha^a, \\
[R_{rs}, Q_\alpha^t] &= i(\delta_r^t Q_{\alpha s} - \delta_s^t Q_{\alpha r}), \\
[R_{rs}, S_\alpha^t] &= i(\delta_r^t S_{\alpha s} - \delta_s^t S_{\alpha r}).
\end{aligned}$$

Here S_α^a are conformal supercharges, i.e. the fermionic superpartners of K_μ . Consistency of the algebra requires that the R-symmetry be included explicitly. More generally, for a theory with $\mathcal{N}$ supercharges, the superconformal algebra satisfies

$$\mathfrak{so}(2, 3)_{\text{conf}} \oplus \mathfrak{so}(\mathcal{N})_R \subset \mathfrak{osp}(\mathcal{N}|4).$$

Representation Theory of the Supersymmetry Algebra. The elementary representations of the $\mathcal{N} = 4$ supersymmetry algebra are the vector multiplet and the hypermultiplet. Since the theories studied in this thesis preserve at most $\mathcal{N} = 2$ supersymmetry, it is convenient to express these multiplets in terms of $\mathcal{N} = 2$ representations.

The basic building blocks of a three-dimensional $\mathcal{N} = 2$ theory are

$$\text{chiral multiplet:} \quad Q = (\phi, \psi_\alpha, F),$$

$$\text{vector multiplet:} \quad V = (A_\mu, \lambda_\alpha, \sigma, D).$$

Here ϕ and F are complex scalars, ψ_α and λ_α are two-component fermions, and σ ⁸ and D are real scalars. The auxiliary fields F and D enter algebraically and determine the moduli space of supersymmetric vacua through their equations of motion. Chiral multiplets carry holomorphic data while vector multiplets carry gauge redundancies.

In terms of $\mathcal{N} = 2$ multiplets, an $\mathcal{N} = 4$ theory decomposes as

$$\text{hypermultiplet:} \quad \mathcal{H} = Q_{\mathcal{N}=2} \oplus \tilde{Q}_{\mathcal{N}=2},$$

$$\text{vector multiplet:} \quad \mathcal{V} = V_{\mathcal{N}=2} \oplus \Phi_{\mathcal{N}=2},$$

where Q and $\tilde{Q}$ transform in conjugate representations R and $\bar{R}$, and Φ is an adjoint chiral multiplet. Hypermultiplets may be seen as carrying paired holomorphic data from a lesser supersymmetric perspective.

⁸While the σ field is necessary for closure of the supersymmetry algebra, its existence can also be motivated by studying the compactification of the $4d$ vector multiplet. Upon dimensional reduction, the $4d$ vector field A_M splits into a $3d$ gauge field A_μ and a leftover component A_3 which carries no gauge index and is, therefore, identified with the real scalar σ .

Supersymmetric Lagrangians. As with four-dimensional $\mathcal{N} = 2$ theories, three-dimensional $\mathcal{N} = 4$ gauge theories are highly constrained: once the gauge group and matter content are specified, the interactions are essentially fixed. The Yang–Mills action is

$$\mathcal{S}_{YM} = \frac{1}{g_{YM}^2} \int d^3x d^2\theta \operatorname{Tr} \left(\mathcal{W}_\alpha^2 - \Phi^\dagger e^{2V} \Phi \right) + \text{c.c.},$$

where $\mathcal{W}_\alpha := -\frac{1}{4} \overline{D} \overline{D} D_\alpha V$ is the gauge field strength superfield, and D_α are supersymmetry-covariant derivatives. The Yang–Mills coupling has scaling dimension $\frac{1}{2}$, implying that even Abelian theories flow to strong coupling in the infrared. The action of the $\mathcal{N} = 4$ R-symmetry is as follows - $SU(2)_C$ acts on the vector multiplet, rotating the three scalars as a triplet, while $SU(2)_H$ acts on hypermultiplets, rotating the complex scalars $Q, \tilde{Q}^\dagger$ as a doublet.

Supersymmetry also uniquely fixes the superpotential,

$$\mathcal{W}_{\mathcal{N}=4} = \tilde{Q} \Phi Q. \quad (1.1)$$

While $\mathcal{N} = 4$ theories admit no additional supersymmetric interactions, theories with $\mathcal{N} \leq 3$ allow for *Chern–Simons* (CS) terms⁹. These topological interactions are parametrized by integer-valued levels. For an $\mathcal{N} = 2$ $U(N)_{(k_1, k_2)}$ theory, the CS Lagrangian reads [97]

$$\begin{aligned} \mathcal{L}_{CS} = & -\frac{ik_1}{4\pi} \operatorname{Tr} \left[\epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) + 2i D\sigma + \bar{\lambda}\lambda \right] \\ & - \frac{i(k_2 - k_1)}{4\pi N} \epsilon^{\mu\nu\rho} (\operatorname{Tr} A_\mu) \partial_\nu (\operatorname{Tr} A_\rho). \end{aligned} \quad (1.2)$$

Here k_1 is the level for the $SU(N)$ factor, while $\ell = \frac{k_2 - k_1}{N}$ governs the Abelian $U(1)$ interaction. One may also introduce mixed BF couplings between Abelian factors at level k_{IJ} ,

$$\mathcal{L}_{BF}^{I \neq J} = -\frac{ik_{IJ}}{8\pi} (\epsilon^{\mu\nu\rho} A_\mu^I \partial_\nu A_\rho^J - D_I \sigma_J - D_J \sigma_I + i\bar{\lambda}_I \lambda_J + i\bar{\lambda}_J \lambda_I). \quad (1.3)$$

For completeness, we display the $\mathcal{N} = 3$ CS interaction, which must be added to the $\mathcal{N} = 2$ CS term to preserve the additional supersymmetry [31]

$$\mathcal{L}_{CS}^{\mathcal{N}=3} = \frac{ik}{8\pi} \int d^2\theta d^2\bar{\theta} \operatorname{Tr}(\Phi^2 + \text{c.c.}). \quad (1.4)$$

A 0-form Topological Symmetry. A distinctive feature of three-dimensional gauge theories is the presence of an additional global 0-form symmetry [98, 99], known as the *topological* or *magnetic* symmetry¹⁰. For a $U(1)$ gauge theory with field strength $F = dA$, the Bianchi identity $dF = 0$ implies conservation of the current [101]

$$\mathcal{J}_T = \frac{\star F}{2\pi},$$

which defines a global $U(1)_T$ symmetry. Since this conservation law holds identically independent of the equations of motion, the symmetry is topological.

⁹Following [94–96], it is possible to construct $\mathcal{N} \geq 4$ quiver gauge theories with CS interactions. While interesting in their own right, their analysis (and subsequent reduction to $\mathcal{N} = 2$ supersymmetry) is beyond the scope of this work.

¹⁰Also referred to as the Nielsen–Olesen–Abrikosov symmetry [100] by our friendly neighborhood condensed matter theorists.

Locally, one may introduce a compact scalar γ , the *dual photon*, such that¹¹

$$\star F = \frac{g^2}{2\pi} d\gamma,$$

with periodicity fixed by the Dirac quantization condition [102, 103]. The shift symmetry $\gamma \rightarrow \gamma + 2\pi$ realizes $U(1)_T$. The associated charge

$$Q_T = \frac{1}{2\pi} \int_{\Sigma_2} F \in \mathbb{Z}$$

measures total magnetic flux, and operators charged under $U(1)_T$ are monopole (disorder) operators defined by prescribed magnetic singularities.¹²

Another essential ingredient of three-dimensional supersymmetric gauge theories is the presence of *Fayet–Iliopoulos* (FI) terms. For each Abelian gauge factor $U(1) \subset G$, one may introduce an FI parameter η ,

$$\mathcal{L}_{\text{FI}} = \frac{\eta}{2\pi} \int d^2\theta d^2\bar{\theta} V = \frac{\eta}{2\pi} D,$$

which modifies the equations of motion (and is, therefore, equivalently viewed as a parameter in a moment-map equation). In three dimensions, FI parameters may be interpreted as background real masses for the topological $U(1)_T$ symmetry, a fact that plays a central role in dualities. In $\mathcal{N} = 4$ theories, FI parameters assemble into an $SU(2)_H$ triplet, while only a single real FI parameter survives upon breaking to $\mathcal{N} = 2$.

Real Mass Deformations and Radiative Chern–Simons Terms. Three-dimensional theories admit two inequivalent notions of mass. A *complex mass* arises from a superpotential deformation in vector-like theories,

$$\mathcal{W} \supset m_{\mathbb{C}} Q\tilde{Q}, \quad (1.5)$$

with $m_{\mathbb{C}} \in \mathbb{C}$.

A genuinely three-dimensional deformation is provided by *real masses*, which originate from expectation values of the scalar σ in a background vector multiplet. Setting $\langle \sigma \rangle \neq 0$ induces

$$Qe^V Q^\dagger \xrightarrow{\langle \sigma \rangle \neq 0} \frac{m_{\mathbb{R}}^2}{2} |\phi|^2 + im_{\mathbb{R}} \epsilon^{\alpha\beta} \bar{\psi}_\alpha \psi_\beta. \quad (1.6)$$

Unlike complex masses, real masses do not require vector-like matter and are therefore available in chiral theories. In general, fields acquire real masses proportional to their charges under the corresponding global symmetry. Real mass deformations play a central role in triggering RG flows and, crucially, in generating Chern–Simons and BF couplings.

In three dimensions, conformal invariance allows parity-odd contact terms in two-point functions of conserved currents. In particular,

$$\langle J_\mu(x) J_\nu(0) \rangle \supset k \epsilon_{\mu\nu\rho} \partial^\rho \delta^{(3)}(x),$$

¹¹Notice this is a consequence of the fact that a three-dimensional vector and a scalar have only one on-shell degree of freedom.

¹²The presence of a 0-form topological symmetry can be alternately motivated as follows: the magnetic flux through a two-sphere $\mathbf{S}^2$ around a point is labelled by $\pi_1(G)$, the fundamental group of the gauge group. For a simply connected gauge group, $\pi_1(G) = 0$, and there is no topological symmetry. For a non-simply connected gauge group, $\pi_1(G) \neq 0$, and the topological symmetry is nontrivial - specifically, for a $U(1)$ factor, $\pi_1(U(1)) = \mathbb{Z}$, and cannot change continuously, so it is a conserved charge.

where the coefficient k is a physical observable. In a UV Lagrangian description, such contact terms are encoded by CS interactions for the background gauge fields coupled to the conserved currents [104–106]. The effective interaction takes the schematic form

$$\mathcal{L}_{\text{CS}} \sim k \text{Tr}(A \wedge F).$$

These interactions modify Gauss' law. When $k \neq 0$, magnetic flux contributes to the electric charge, so that magnetic configurations acquire an effective electric charge proportional to the CS level. In particular, for a $U(1)$ symmetry,

$$q_{\text{el}}^{\text{CS}} = k q_{\text{top}}, \quad (1.7)$$

so any operator carrying topological charge acquires an induced electric charge. This has important consequences for the spectrum of gauge-invariant operators, especially monopole operators, and therefore for the infrared dynamics of the theory.

Even if the classical CS level vanishes, such terms may be generated radiatively [107, 108]. Along the vacuum, configurations with $\langle \sigma \rangle \neq 0$ render charged fields massive,

$$m_i(\sigma) = n_i \langle \sigma \rangle,$$

where n_i is the charge of the field. Integrating out these massive fermions induces an effective CS term through the parity anomaly.

Considering the vacuum polarization tensor, the effective action for the gauge field is

$$\mathcal{L}_{\text{eff}}[A] = \frac{1}{2} n_i^2 \int \frac{d^3 p}{(2\pi)^3} A_j(-p) A_k(p) \left[-i \Pi^{jk}(p) \right],$$

where

$$\Pi^{jk}(p) = -ie^2 \int \frac{d^3 k}{(2\pi)^3} \text{Tr} \left[\gamma^j \frac{-i}{\not{p} + \not{k} - m(\sigma)} \gamma^k \frac{-i}{\not{k} - m(\sigma)} \right].$$

For large $m(\sigma)$, this reduces to

$$\Pi^{jk}(p) \longrightarrow -i \epsilon^{jkl} p_l \text{sgn}[m(\sigma)] + (\text{divergent terms});$$

when the large-mass limit of the vacuum polarization tensor is substituted in the effective action for the gauge field, we see that each fermion shifts the Chern-Simons level by $\frac{1}{2} n_i^2 \text{sgn}[n_i \sigma]$.

The effective CS level, therefore, becomes

$$k_{\text{eff}} = k + \frac{1}{2} \sum_i n_i^2 \text{sgn}[n_i \sigma]. \quad (1.8)$$

For non-Abelian gauge groups, this generalizes to

$$k_{\text{eff}} = k + \frac{1}{2} \sum_i S_2(R_{\psi_i}) \text{sgn}[n_i \sigma], \quad (1.9)$$

where $S_2(R)$ denotes the Dynkin index of the fermion representation.

As a consequence, the electric charge of operators carrying topological charge is modified to

$$q_{\text{el}} = \left(k + \frac{1}{2} \sum_i n_i^2 \text{sgn}[n_i \sigma] \right) q_{\text{top}}. \quad (1.10)$$

The same reasoning applies in the presence of mixed CS or BF interactions, which are likewise shifted by integrating out massive matter.

Monopole Operators

It has become increasingly clear in recent years that the set of local operators in a quantum field theory is not exhausted by operators constructed as polynomials of the microscopic fields appearing in the Lagrangian [109, 110]. In addition to these conventional *order* operators, one must also include so-called *disorder* (or defect) operators, which are defined by prescribing suitable singular boundary conditions in the path integral.

At first sight, local disorder operators appear to be fundamentally different from standard local operators, since their definition involves modifying the functional integration rather than inserting a local composite operator. However, this distinction is largely an artifact of the particular choice of field variables used to describe the quantum field theory. In the full quantum theory, all local operators are on the same footing. This is reflected in the existence of numerous dualities that map order operators to disorder operators, and vice versa¹³.

In three-dimensional gauge theories, the relevant local disorder operators are *monopole operators* [115]. These operators are naturally defined in the Euclidean formulation of the theory and can be viewed as arising from dimensional reduction of 't Hooft loop operators in four dimensions. Operationally, inserting a monopole operator $\mathfrak{M}$ into a correlation function amounts to integrating over gauge field configurations that exhibit a Dirac monopole singularity at the insertion point. More precisely, for a monopole operator of magnetic charge m , one defines [116–119]

$$\langle \mathfrak{M}^m(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \rangle = \int_{\int_{\Sigma_2} F = 2\pi m} \mathcal{D}A, \left(\prod_{i=1}^n \mathcal{O}_i(x_i) \right) e^{i\mathcal{S}},$$

where Σ_2 is a small two-sphere surrounding the insertion point x .

In an $\mathcal{N} = 2$ supersymmetric theory, monopole operators can be made supersymmetric by imposing singular boundary conditions not only for the gauge field but for all bosonic fields in the $\mathcal{N} = 2$ vector multiplet. Introducing spherical coordinates (r, θ, φ) centered at the insertion point x , a $\frac{1}{2}$ -BPS *bare* monopole operator is defined by the following asymptotic behaviour as $r \rightarrow 0$:

$$\begin{aligned} A_{\pm} &\sim \frac{m}{2}(\pm 1 - \cos \theta) d\varphi, \\ \sigma &\sim \frac{m}{2r}. \end{aligned}$$

Here, the $\pm$ labels distinguish the gauge field configurations on the northern and southern hemispheres of the two-sphere Σ_2 surrounding the singularity.

Consistency of the gauge bundle requires the magnetic charge m to satisfy a generalized Dirac quantization condition [120, 121],

$$e^{2\pi i m} = \mathbb{I}_G \quad \implies \quad m \in \Gamma_{G^V} / \mathcal{W},$$

where Γ_{G^V} denotes the weight lattice of the Langlands dual group G^V of the gauge group G [122], and $\mathcal{W}$ is its Weyl group.

In the absence of Chern–Simons interactions, a *bare monopole*—that is, a chiral operator $\mathfrak{M}^m$ defined solely in terms of the vector multiplet—can be made gauge invariant by averaging over the action of the Weyl group.

¹³These include sine-Gordon–massive Thirring duality [111, 112], T-duality and mirror symmetry [113, 114] in two spacetime dimensions, and mirror symmetry and Aharony duality in three spacetime dimensions.

More generally, the singular boundary conditions imposed on $A_{\pm}$ and σ in the definition of a monopole operator break the gauge group G to a residual subgroup

$$H \subset G,$$

defined as the stabilizer of the magnetic flux m , namely the subgroup satisfying $H \cdot m = 0$. In the monopole background, the vector multiplets associated with H remain massless, while those corresponding to the coset G/H become massive. Upon integrating out these massive modes, quantum corrections are induced that modify the global and gauge charges of the monopole operator.

A similar mechanism applies to matter fields. In the presence of a monopole background, matter fields that are neutral under the unbroken $U(1) \subset G \times G_F$ may acquire expectation values without breaking supersymmetry. These fields can therefore be used to *dress* the bare monopole operator. Concretely, one constructs dressed monopole operators by multiplying the bare, generally gauge-variant monopole by gauge-invariant polynomials of the residual massless matter fields, yielding operators invariant under the unbroken subgroup H .

Finally, full G -invariance is restored by averaging over the coset of Weyl groups $\mathcal{W}_G/\mathcal{W}_H$.

The presentation of monopole operators as disorder operators reflects a more modern viewpoint on $\mathcal{N} = 2$ gauge theories; historically, this was not the perspective adopted in the earliest studies of mirror symmetry [30]. In the original interpretation, monopole operators were instead motivated by a semiclassical analysis of the low-energy degrees of freedom of the theory.

To illustrate this viewpoint, consider a $U(1)$ gauge theory in three dimensions. As discussed earlier, the gauge field can be dualized to a compact scalar γ , known as the *dual photon*. Together with the real scalar σ in the $\mathcal{N} = 2$ vector multiplet, one may form the complex field

$$\Phi \simeq \sigma + i\gamma.$$

In a semiclassical regime where the expectation value of Φ is large and the theory is weakly coupled, monopole operators admit a simple effective description. In this limit, a monopole operator carrying magnetic charge m can be identified with the exponential operator¹⁴

$$\mathfrak{M}^m \sim e^{m\Phi},$$

which creates a configuration with quantized magnetic flux and carries charge under the topological $U(1)_T$ symmetry [29].

From this perspective, monopole operators appear as ordinary local operators constructed from the effective fields of the theory, rather than as disorder operators defined by singular boundary conditions. The modern definition of monopole operators as disorder operators can be understood as a fully non-perturbative formulation of this semiclassical picture, valid throughout the quantum theory and in particular at strong coupling.

A Technical Aside: The non-supersymmetric Monopole. Before reviewing the properties of supersymmetric monopoles, we collect some results regarding non-supersymmetric monopoles here.

¹⁴Note that this identification can only be done for Abelian theories.

Chern–Simons Interactions Coupled to Fermions: Consider a $U(1)$ gauge theory with N_f massless Dirac fermions ψ_j ,

$$\mathcal{L} = \sum_{j=1}^{N_f} i\bar{\psi}_j \not{D}_A \psi_j + \frac{k_0}{4\pi} AdA, \quad (1.11)$$

where A is a dynamical $U(1)$ gauge field and $k_0 \in \mathbb{Z}$ is the *bare* Chern–Simons level.

Parity anomaly and regularization. The simultaneous presence of fermions and a CS interaction leads to a subtle anomaly in time-reversal symmetry $\mathcal{T}$ [123, 124]. Upon quantization, defining the fermion determinant requires a regulator that necessarily breaks $\mathcal{T}$ invariance (for instance Pauli–Villars regularization). The regulated partition function acquires a phase

$$Z_{\text{fermion}} = |\det \not{D}_A| \exp\left(-\frac{i\pi}{2}\eta(A)\right), \quad (1.12)$$

where $\eta(A)$ is the Atiyah–Patodi–Singer (APS) η -invariant [101, 123, 125–127].

For practical purposes, this phase can be interpreted as inducing:

- a shift of the bare CS level,

$$k_0 \longrightarrow k_0 - \frac{N_f}{2},$$

- and the addition of a mixed gravitational Chern–Simons term

$$-N_f CS_g,$$

which ensures the absence of Dirac string singularities [124].

It is therefore convenient to define the renormalized level

$$k = k_0 - \frac{N_f}{2}, \quad (1.13)$$

so that the theory will be denoted $U(1)_k$. Notice that k may be half-integer.

Mass deformation and infrared CS levels. Consider now a real mass deformation m for the fermions. For a single charged fermion,

$$i\bar{\psi} \not{D}_A \psi \xrightarrow{m\bar{\psi}\psi} \begin{cases} 0, & m > 0, \\ -\frac{1}{4\pi} AdA - 2 CS_g, & m < 0. \end{cases} \quad (1.14)$$

Thus integrating out a massive fermion generates a level $-\frac{1}{2}$ CS term for A together with a gravitational CS contribution.

For N_f fermions of equal mass m , the low-energy effective CS level becomes

$$k_{\text{IR}} = k_0 - \frac{N_f}{2} + \frac{N_f}{2} \text{sign}(m), \quad (1.15)$$

which is always an integer, as required for a well-defined infrared topological field theory. In addition, a gravitational CS term

$$N_f CS_g \text{sign}(m)$$

is generated.

Of particular interest is the case where $k = 0$. When a positive mass ($m > 0$) is turned on for all fermions, the flow leads to a trivial theory with a vanishing Lagrangian. Conversely, a negative mass ($m < 0$) results in:

$$-\frac{N_f}{4\pi} AdA - 2N_f CS_g.$$

The gravitational term $-2N_f CS_g$ is sometimes denoted as $U(N_f)_1$ in the literature. Time reversal acts on a fermion as:

$$\mathcal{T} : i\bar{\psi}\not{D}_A\psi \rightarrow i\bar{\psi}\not{D}_A\psi + \frac{1}{4\pi}AdA + 2CS_g$$

Monopoles in Quiver Gauge Theories. In the present work, a major role is played by quiver gauge theories, wherein fermions can be charged under multiple gauge groups.

Consider then a $U(1)^{(a)} \times U(1)^{(b)}$ gauge theory with a fermion ψ of charge $+1$ and -1 under the two gauge groups, bare CS levels $k_{0,a}$ and $k_{0,b}$ and mixed CS level $k_{0,ab}$:

$$\mathcal{L} = i\bar{\psi}\not{D}_{a-b}\psi + \frac{k_{0,a}}{4\pi}ada + \frac{k_{0,b}}{4\pi}bdb + \frac{k_{0,ab}}{4\pi}adb, \quad (1.16)$$

the regularization of the fermion determinant produces a $-\frac{1}{2}$ shift for the gauge field $a - b$:

$$-\frac{1}{8\pi}(a-b)d(a-b) = -\frac{1}{8\pi}ada - \frac{1}{8\pi}bdb + \frac{1}{4\pi}adb$$

resulting in the shifted levels:

$$k_a = k_{0,a} - \frac{1}{2}, \quad k_b = k_{0,b} - \frac{1}{2}, \quad k_{ab} = k_{0,ab} + 1.$$

Furthermore, a mass deformation for the fermion produces an additional $\pm\frac{1}{2}$ shift in the CS levels k_a , k_b , and a ∓ 1 shift in the mixed CS level k_{ab} . In the quiver notation used throughout this work, the theory (1.16) and its mass deformations are denoted as follows¹⁵
¹⁶:

$$\begin{array}{c} \textcircled{1} \xrightarrow[k_a]{k_{ab}} \textcircled{1} \xrightarrow[k_b]{} \end{array} \xrightarrow{m\bar{\psi}\psi} \begin{cases} \left(\textcircled{1} \xrightarrow[k_a + \frac{1}{2}]{k_{ab}-1} \textcircled{1} \xrightarrow[k_b + \frac{1}{2}]{} \right), & m > 0 \\ \left(\textcircled{1} \xrightarrow[k_a - \frac{1}{2}]{k_{ab}+1} \textcircled{1} \xrightarrow[k_b - \frac{1}{2}]{} \right), & m < 0 \end{cases} \quad (1.17)$$

Quantum Numbers of Monopole Operators: A monopole operator of topological charge $+1$ is defined by inserting a Dirac singularity at a point $p \in \mathbb{R}^3$, such that one unit of magnetic flux passes through any $\mathbf{S}^2$ surrounding p . Equivalently, the path integral is evaluated over gauge field configurations satisfying

$$\frac{1}{2\pi} \int_{\mathbf{S}^2} F = 1.$$

The resulting local operator $O(x)$ has the characteristic operator product expansion with the current,

$$J^\mu(x)O(0) \sim \frac{1}{4\pi} \frac{x^\mu}{|x|^3} O(0) + \text{less singular terms.} \quad (1.18)$$

¹⁵We do not keep track of background gravitational CS levels CS_g .

¹⁶We expand on our conventions for quiver gauge theories in later sections.

Gauge charge from fermionic zero modes. The quantum numbers of monopole operators are most efficiently computed in radial quantization, where the theory is placed on $\mathbf{S}^2 \times \mathbb{R}$ with magnetic flux through $\mathbf{S}^2$ [116, 128].

For each charged fermion, the spectrum in a unit-flux background contains zero modes. Their contribution to the Gauss law constraint can be encoded as an effective shift of $\frac{1}{2}$ to the CS level. Consequently, in a $U(1)_k$ theory with N_f fermions, a bare monopole with GNO flux $+1$ carries gauge charge

$$Q_{\text{gauge}} = k + \frac{N_f}{2}.$$

Using $k = k_0 - \frac{N_f}{2}$, this equals the bare CS level k_0 . Gauge-invariant operators are therefore obtained by dressing the bare monopole with appropriate powers of charged matter fields (and derivatives).

Extensions to Multiple Gauge Nodes. The analysis generalizes straightforwardly to product gauge groups. As an illustrative example, consider a two-node abelian theory (1.16),

$$\textcircled{1} \xrightarrow[k_a]{k_{ab}} \textcircled{1} \xrightarrow[k_b]{} \quad (1.19)$$

with a bifundamental fermion of charge $(+1, -1)$.

In a monopole background, the fermionic zero modes induce:

- a $+\frac{1}{2}$ shift of k_a ,
- a $+\frac{1}{2}$ shift of k_b ,
- a -1 shift of the mixed CS coupling k_{ab} .

Let $\mathfrak{M}^{(n,m)}$ denote a bare monopole with flux n under $U(1)_a$ and m under $U(1)_b$. Gauss' law implies that its gauge charges are

$$\begin{aligned} Q[\mathfrak{M}^{(n,m)}] &= nQ[\mathfrak{M}^{(+,0)}] + mQ[\mathfrak{M}^{(0,+)}] \\ &= n\left(k_a + \frac{1}{2}, \frac{k_{ab}-1}{2}\right) + m\left(\frac{k_{ab}-1}{2}, k_b + \frac{1}{2}\right) \\ &= \left(n(k_a + \frac{1}{2}) + m\frac{(k_{ab}-1)}{2}, m(k_b + \frac{1}{2}) + n\frac{(k_{ab}-1)}{2}\right) \end{aligned} \quad (1.20)$$

Dressed Monopole Operators. Gauge-invariant states can be constructed by dressing with modes of ψ (or $\bar{\psi}$) in very specific cases. Since ψ has charges $(+1, -1)$, a monopole dressed p times with ψ and q times with $\bar{\psi}$ has charge:

$$Q[\mathfrak{M}^{(n,m)}\psi^p\bar{\psi}^q] = \left(n(k_a + \frac{1}{2}) + m\frac{(k_{ab}-1)}{2} + p - q, m(k_b + \frac{1}{2}) + n\frac{(k_{ab}-1)}{2} + q - p\right) \quad (1.21)$$

As an example, for $k_a = k_b = \frac{1}{2}$ and $k_{ab} = -1$ the monopoles $\mathfrak{M}^{(+,0)}\partial_{\bullet}\bar{\psi}$ and $\mathfrak{M}^{(0,+)}\partial_{\bullet}\psi$ are gauge invariant¹⁷:

$$\textcircled{1} \xrightarrow[\frac{1}{2}]{-1} \textcircled{1} \xrightarrow[\frac{1}{2}]{} \rightarrow \begin{aligned} Q[\mathfrak{M}^{(+,0)}\partial_{\bullet}\bar{\psi}] &= (0, 0), \\ Q[\mathfrak{M}^{(0,+)}\partial_{\bullet}\psi] &= (0, 0) \end{aligned} \quad (1.22)$$

¹⁷Here $\partial_{\bullet}$ denotes a generic set of derivatives.

Throughout this work, we write dressed monopoles by combining the flux assignment and the dressing in the following way:

$$\mathfrak{M}^{(+,0)}_{\bar{\psi}} \equiv \mathfrak{M}^{+ \leftarrow 0} \quad (1.23)$$

$$\mathfrak{M}^{(0,+)}_{\psi} \equiv \mathfrak{M}^{0 \rightarrow +} \quad (1.24)$$

Statistical Transmutation via Flux Attachment: In the presence of a monopole background, the spin of fundamental fields in the ground state undergoes a shift — a phenomenon known as *statistical transmutation* [129] (see also [130–135]). This shift depends on the GNO charge (or flux) of the monopole, the intrinsic spin of the particle, and its gauge charge.

In what follows, we summarize the effects of statistical transmutation for a particle of spin s , and examine how this behavior is modified in the presence of multiple $U(1)$ gauge factors, particularly in the context of the quiver gauge theories discussed in the main text.

Single Gauge Group: Consider a spin s field φ with charge q under a $U(1)$ gauge group. In the background of a monopole with flux m , the ground state of the field φ has the following spin channels:

$$|\frac{qm}{2} - s|, |\frac{qm}{2} - s| + 1, \dots, \frac{qm}{2} + s,$$

For example, in the background of a monopole with flux $m = 1$ a fermion has a spin-0 and a spin-1 channel, and a scalar has a spin- $\frac{1}{2}$ channel.

Bifundamental Matter: Consider a spin- s field φ with charge $(+1, -1)q$ under a $U(1) \times U(1)$ gauge symmetry, shown explicitly in Equation (1.25):

$$\textcircled{1} \xrightarrow{\varphi} \textcircled{1} \quad (1.25)$$

The dressed operator $\mathfrak{M}^{(m,n)}\varphi$ will have the following spin channels:

$$|\frac{q}{2}(m - n) - s|, |\frac{q}{2}(m - n) - s| + 1, \dots, \frac{q}{2}(m - n) + s,$$

which depends on the difference of the GNO fluxes of the two gauge groups. In particular in the background of a monopole $\mathfrak{M}^{(+,+)}$ the bifundamental field φ “sees” a vanishing effective flux, and its spin is not transmuted.

Absence of Transmutation from Distant Fluxes: Finally, we consider the quiver gauge theory shown in Equation (1.26):

$$\textcircled{1} \xrightarrow{\varphi_1} \textcircled{1} \xrightarrow{\varphi_2} \textcircled{1} \quad (1.26)$$

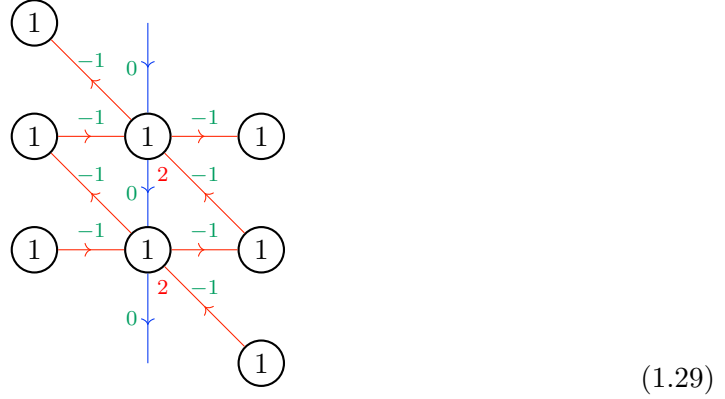
where φ_i are bifundamental spin- s_i fields. On the background of a monopole $\mathfrak{M}^{[0,0,m]}$ the field φ_1 does not “see” the magnetic flux, and its spin remains s_1 . On the other hand the field φ_2 “sees” an effective flux $-m$, and has spin channels shifted by $-m$:

$$\begin{aligned} \varphi_1 : & \quad | -s_1 |, | -s_1 | + 1, \dots, | s_1 | \\ \varphi_2 : & \quad | -s_2 - \frac{m}{2} |, | -s_2 - \frac{m}{2} | + 1, \dots, | s_2 - \frac{m}{2} | \end{aligned} \quad \text{on } \mathfrak{M}^{[0,0,m]} \quad (1.27)$$

Generally, consider a field φ of spin- s and charges q_i under some $U(1)_i$ gauge groups, with $i = 1, \dots, M$. Then in the background of a monopole with GNO fluxes m_i the field φ has the same spin channels as a spin- s field with charge 1 in the background of an “effective” GNO flux m_{eff} :

$$m_{eff} = \sum_{i=1}^M q_i m_i \quad (1.28)$$

As an example, we consider the following theory, which may be thought of as part of one of the quiver theories considered in Chapter 5:



(1.29)

where we only reported CS and mixed CS terms relevant for the computations that follow. We can consider the monopole with GNO flux $+1/-1$ under the gauge nodes in the central column. Its gauge charges are:

$$Q \left[\mathfrak{M} \left(\begin{pmatrix} 0 & & \\ 0 & + & 0 \\ 0 & - & 0 \\ & & 0 \end{pmatrix} \right) \right] = \begin{pmatrix} -1 & & \\ 0 & 4 & -1 \\ 1 & -4 & 0 \\ & & 1 \end{pmatrix} \quad (1.30)$$

therefore it can be made gauge invariant with the following dressing:

$$\mathfrak{M} \left(\begin{pmatrix} 0 & & \\ 0 & \text{red arrows} & + & \text{red arrows} & 0 \\ 0 & \text{red arrows} & - & \text{red arrows} & 0 \\ & & & & 0 \end{pmatrix} \right) \quad (1.31)$$

Each fermion involved in this dressing “sees” a GNO flux ± 1 , therefore it has a spin-0 zero mode. Therefore we can dress this monopole to obtain a gauge invariant spin-0 monopole operator. In the theories considered in Chapter 5, this is one of the monopoles entering the

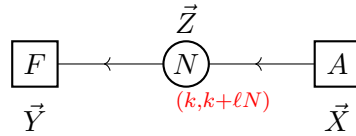
$\mathcal{V}_{monopole}$ interaction. Similarly, we can consider the monopole operators involved in the $\mathcal{V}_{monopole}$ interaction that have GNO flux in the “bosonic” part of the quivers discussed in Section 5.3. There we find that the following monopole can be dressed to be gauge invariant and a Lorentz scalar:

$$\rightarrow \mathfrak{M} \begin{pmatrix} 0 & & \\ \text{blue arrow } 0 & + & 0 \\ 0 & - & 0 \\ & \text{blue arrow } 0 & 0 \end{pmatrix} \quad (1.32)$$

Notice in particular that the middle scalar involved (twice) in the dressing “sees” a GNO flux 2, while the other two scalars “see” no flux, therefore all scalars have a spin-0 mode in the background of the monopole and we obtain a spin-0 gauge invariant operator. Similarly, one can show that all the monopoles involved in the interactions $\mathcal{V}_{monopole}$ throughout Chapter 5 can be dressed to obtain a gauge invariant operator which is a Lorentz scalar.

Another Technical Aside: The Fate of the Supersymmetric Monopole. We now review several properties of monopole operators in three-dimensional supersymmetric gauge theories that are repeatedly used in the main text ¹⁸.

Monopoles in $U(N)$ Chern-Simons Theories: We begin by considering a generic $U(N)_{(k,k+\ell N)}$ gauge theory with F fundamental and A anti-fundamental chiral multiplets. This is shown in the quiver gauge theory below:



Here $\vec{X}$ and $\vec{Y}$ denote fugacities for the $S[U(A) \times U(F)]$ flavor symmetry, while $\vec{Z}$ are the gauge fugacities associated with the Cartan of $U(N)$.

Quantum Numbers of Supersymmetric Monopoles. A bare monopole operator $\mathfrak{M}^{\vec{m}}$, labeled by a magnetic flux vector $\vec{m} = (m_1, \dots, m_N)$, has R-charge given by [30, 118, 136]

$$R[\mathfrak{M}^{\vec{m}}] = \Lambda \sum_{i=1}^N m_i - \frac{1}{2} \left[\sum_{i \neq j} |m_i - m_j| + F(r_f - 1) \sum_{i=1}^N |m_i| + A(r_a - 1) \sum_{i=1}^N |m_i| \right], \quad (1.33)$$

where Λ parametrizes the mixing between the R-symmetry and the topological symmetry, and r_f, r_a are the R-charges of the fundamental and anti-fundamental chiral multiplets, respectively.

¹⁸It was pointed out to the author by Vittorio Cagioni that there are four possible conventions for parametrizing quantum numbers of supersymmetric monopoles, and we used all four in [2–6]. Good luck.

The representation of the monopole under the gauge symmetry is encoded in the polynomial

$$Q_{\text{gauge}}[\mathfrak{M}^{\vec{m}}] = - \sum_{i=1}^N \left(kZ_i + \ell \sum_{j=1}^N Z_j \right) m_i + \frac{1}{2} \sum_{i=1}^N (F - A) Z_i |m_i|, \quad (1.34)$$

which determines the highest weight of the gauge representation carried by the bare monopole.

A generic magnetic flux $\vec{m}$ breaks the gauge symmetry as

$$U(N) \longrightarrow \prod_{p=1}^L U(M_p), \quad \sum_{p=1}^L M_p = N,$$

where the decomposition corresponds to blocks of equal magnetic flux. The Cartan generators are accordingly grouped as

$$Z_{1,\dots,M_1}, Z_{M_1+1,\dots,M_1+M_2}, \dots$$

An Example: To illustrate these points, we consider the example of $U(3)_{(2,2)}$ SQCD₃ with $[6, 2]$ chiral multiplets. Using Equation 1.34, we calculate the gauge charge of the monopole $\mathfrak{M}^{(+,0,0)}$

$$\begin{aligned} Q_{\text{gauge}}[\mathfrak{M}^{(+,0,0)}] &= -(2z_1) + \frac{1}{2}(6-2)z_1 \\ &= 0, \end{aligned}$$

indicating gauge invariance. Assuming no mixing between the $U(1)_R$ symmetry and $U(1)_{\text{top}}$ symmetries, we now calculate the R-charge of this monopole using Equation 1.33

$$\begin{aligned} R[\mathfrak{M}^{(+,0,0)}] &= -\frac{1}{2} \left[6(r_f - 1) + 2(r_a - 1) + 4 \right] \\ &= 2 - 3r_f - r_a. \end{aligned}$$

Assigning a common trial R-charge $r_f = r_a = \frac{2}{5}$, we find

$$R[\mathfrak{M}^{(+,0,0)}] = \frac{2}{5},$$

implying the superconformal index will capture the contribution of a gauge invariant 1/2-BPS monopole at order $x^{2/5}$.

By Weyl symmetry, the GNO fluxes $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are indistinguishable, and the above analysis applies equally to each of them. Introducing a topological fugacity $\eta^{m_1+m_2+m_3}$, the superconformal index takes the form

$$\mathcal{I} = 1 + \eta x^{2/5} + (\eta^2 + 12) x^{4/5} + 12\eta x^{6/5} + \mathcal{O}(x^{8/5}). \quad (1.35)$$

As expected, the term at order x^0 corresponds to the vacuum. At order $x^{2/5}$ we find the gauge-invariant monopole operator, while at order $x^{4/5}$ we recover both its square and the 12 mesonic operators $Q_i \tilde{Q}_j$. Finally, the terms at order $x^{6/5}$ arise from monopole-meson composite operators, and higher-order contributions follow similarly.

Monopoles in Abelian Quiver Gauge Theories: In the main text, an important role is played by Abelian quiver gauge theories with mixed Chern–Simons interactions and matter in bifundamental representations. We therefore summarize here the corresponding monopole charge formulas.

We consider an Abelian quiver theory specified by:

- n_g Abelian gauge nodes with gauge parameters u_i , $i = 1, \dots, n_g$;
- n_f flavor nodes;
- A global symmetry $U(1)^r$, excluding topological symmetries, with fugacities y_a , $a = 1, \dots, r$;
- Chern-Simons levels k_i , BF couplings k_{ij} , and FI parameters λ_i , as described in Section 2.1;
- Bifundamental chiral fields α_{ij} , of charge -1 under node i and $+1$ under node j ;
- Fundamental fields β_{ai} and anti-fundamental fields $\tilde{\beta}_{ia}$.

We consider a monopole operator with magnetic flux $\vec{m} = (m_1, \dots, m_{n_g})$ under the gauge nodes and vanishing flux under global symmetries.

The charges of the monopole under gauge and global symmetries are compactly encoded in the polynomial

$$\begin{aligned}
 \mathcal{M}(\vec{m}) = & -\frac{1}{2} \left[\sum_{\alpha_{ij}} |m_i - m_j| \left((R[\alpha_{ij}] - 1) \frac{iQ}{2} - u_i + u_j + \sum_{a=1}^r Q_a[\alpha_{ij}] y_a \right) \right. \\
 & + \sum_{\tilde{\beta}_{ia}} |m_i| \left((R[\tilde{\beta}_{ia}] - 1) \frac{iQ}{2} - u_i + \sum_{a=1}^r Q_a[\tilde{\beta}_{ia}] y_a \right) \\
 & \left. + \sum_{\beta_{ai}} |m_i| \left((R[\beta_{ai}] - 1) \frac{iQ}{2} + u_i + \sum_{a=1}^r Q_a[\beta_{ai}] y_a \right) \right] \\
 & - \sum_{i=1}^{n_g} k_i m_i u_i + \sum_{i=1}^{n_g} \lambda_i m_i - \frac{1}{2} \sum_{i < j} k_{ij} (m_i u_j + m_j u_i).
 \end{aligned} \tag{1.36}$$

The first line captures contributions from bifundamental fields, the second and third from anti-fundamental and fundamental matter, respectively. The final line accounts for Chern–Simons terms, FI couplings, and BF interactions. The parameters λ_i encode the mixing between topological symmetries and the R-symmetry.

The charges of the monopole under the various symmetries are read off as the coefficients of the corresponding fugacities. The R-charge is given by the coefficient of $\frac{iQ}{2}$ in $\mathcal{M}(\vec{m})$.

Moduli Space of Vacua

An important and characteristic feature of supersymmetric gauge theories, in three dimensions as well as in higher dimensions, is the existence of continuous families of degenerate vacuum solutions. These arise when the equations of motion admit configurations in which the vacuum expectation values (VEVs) of scalar fields are not uniquely fixed, but instead vary continuously without costing energy. The space parametrized by these VEVs is known as the *moduli space of vacua* [137].

Moduli spaces play a central role in both the physical and mathematical study of supersymmetric field theories. Physically, they govern the low-energy dynamics, encode patterns of spontaneous symmetry breaking, and determine the spectrum of massless degrees of freedom around a given vacuum. Mathematically, they are often realized as geometric spaces with rich structure—such as Kähler or hyperkähler manifolds—whose properties are tightly constrained by supersymmetry. The existence and structure of moduli spaces therefore provide one of the most powerful tools for analyzing supersymmetric gauge theories.

In a supersymmetric gauge theory, a moduli space of vacua arises whenever the scalar potential admits continuous families of zero-energy configurations. In supersymmetric theories, the scalar potential takes the universal form [138]

$$\mathcal{V} = \sum_i |F_i|^2 + \frac{g^2}{2} \sum_\alpha |D^\alpha|^2,$$

and is manifestly non-negative. Supersymmetric vacua are therefore characterized by the simultaneous vanishing of all F -terms and D -terms.

The F -terms are determined by the superpotential $\mathcal{W}$ and are given by

$$F_i = \frac{\partial \mathcal{W}}{\partial \phi_i},$$

where ϕ_i denote the scalar components of the chiral multiplets. The D -terms arise from the gauge interactions and depend on the matter content and possible Fayet–Iliopoulos parameters

$$D^\alpha = \bar{\varphi}_i (T^\alpha)_j^i \varphi^j + \eta^\alpha,$$

where φ_i is a generic field, T^α is the generator of the algebra in the corresponding representation of φ and η^α is a possible FI parameter.

Whenever the equations

$$F_i = 0, \quad D^\alpha = 0$$

admit solutions that are not isolated but instead form continuous families, the space of such solutions—modulo gauge transformations—defines the moduli space of supersymmetric vacua.

In three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories, the space of supersymmetric vacua generically decomposes into distinct branches [32, 139]:

- **Higgs Branch:** The Higgs branch is defined as the space of vacua in which the scalar fields in the matter (hypermultiplet) sector acquire non-vanishing expectation values, while the scalar components of the vector multiplet vanish. Supersymmetry then requires the vanishing of both F -terms and D -terms, and the moduli space is parametrized by gauge-invariant composite operators, such as mesons and baryons, subject to classical constraints. At the classical level, the Higgs branch is described exactly by the solutions to these constraints. In $\mathcal{N} = 4$ theories, the Higgs branch is fully protected against quantum corrections by the enhanced R -symmetry and remains exact at the quantum level [140]. The Higgs branch of $\mathcal{N} = 4$ theories is a hyperkähler manifold, obtained as a hyperkähler quotient by the gauge group [141, 142].
- **Coulomb Branch:** The Coulomb branch is defined as the space of vacua in which the scalar fields in the matter multiplets have vanishing expectation values, while the

scalar components of the vector multiplet acquire non-zero expectation values. As a result, the gauge symmetry is generically broken by adjoint Higgsing to a subgroup containing Abelian factors, with special loci where non-Abelian symmetry may be restored.

As mentioned before, in three dimensions, each unbroken $U(1)$ factor gives rise to a dual photon γ , a compact scalar defined by $\star F = d\gamma$. The natural holomorphic coordinates on the Coulomb branch are provided by monopole operators, which can be viewed semiclassically as exponentials of the complex combinations $\sigma + i\gamma$. While the Coulomb branch exists as a moduli space at the quantum level, its metric and operator dimensions generally receive quantum corrections [143–145]. The Coulomb branch of $\mathcal{N} = 4$ theories is also a hyperkähler manifold due to the existence of the dual photon.

As the $\mathcal{N} = 4$ action contains no mixed terms involving the scalars of the matter and vector multiplets, the moduli space of $\mathcal{N} = 4$ theories is a product $\mathcal{M} = \mathcal{M}_C \times \mathcal{M}_H$.

A First Look at Infrared Dualities

As mentioned previously, local *order* and *disorder* operators in three-dimensional gauge theories are on an equal footing in the quantum theory, and any apparent distinction between them is largely an artifact of the choice of microscopic field variables used to define the Lagrangian. This observation has profound consequences for the infrared (IR) dynamics of such theories and is reflected in the existence of numerous nontrivial *infrared dualities*.

Infrared dualities relate pairs of ultraviolet (UV) theories with seemingly different gauge groups, matter content, and interactions, which nevertheless flow to the same interacting fixed point at long distances. At the IR fixed point, the two descriptions become physically indistinguishable: they share the same global symmetries, spectrum of local operators, correlation functions, and moduli spaces of vacua. Crucially, duality maps gauge-invariant operators in one theory to gauge-invariant operators in the dual description, often exchanging elementary fields with composite operators and, in three dimensions, order operators with disorder (monopole) operators.

Among the most well-studied examples of three-dimensional infrared dualities are the so-called *Aharony-like dualities* [18, 20, 21, 146]. These dualities relate $\mathcal{N} = 2$ supersymmetric gauge theories to dual theories with different gauge groups and matter content, supplemented by additional gauge-singlet fields and superpotential couplings.

A defining feature of Aharony-like dualities is the exact matching of global symmetries across the dual pair, including both flavor symmetries and topological symmetries. Operators charged under these symmetries are mapped consistently between the two descriptions. In particular, monopole operators parametrizing the Coulomb branch of one theory are mapped to elementary or composite gauge-singlet fields in the dual theory, while Higgs-branch operators are mapped to Higgs-branch operators. The duality therefore provides distinct ultraviolet realizations of the same infrared fixed point, without exchanging the roles of the Higgs and Coulomb branches.

We now provide a brief summary of the various IR duals of $U(N_c)_k$ SQCD. All known Aharony-like dualities can be obtained as mass deformations of the *seed duality*:

- **Aharony duality (the seed duality):** The most fundamental infrared duality is

$$U(N_c)_0, \quad N_F (\square \oplus \bar{\square}) \longleftrightarrow U(N_F - N_c)_0, \quad N_F (\square \oplus \bar{\square}), \quad (M, \mathcal{F}[\mathfrak{M}_\mathfrak{d}^\pm]). \quad (1.37)$$

Here, the *electric* theory consists of a $U(N_c)$ gauge group at zero CS level with N_F fundamental chiral multiplets Q and N_F anti-fundamental chiral multiplets $\tilde{Q}$.

The *magnetic* theory consists of a $U(N_F - N_c)$ gauge group at zero CS level with the same matter content with N_F dual quarks φ , $\tilde{\varphi}$, supplemented by the superpotential interaction:

$$\mathcal{W} = \tilde{\varphi}^\alpha M_\alpha^\beta \varphi_\beta + \text{Flip}[\mathfrak{M}_\mathfrak{d}^+] + \text{Flip}[\mathfrak{M}_\mathfrak{d}^-]$$

where $M_\alpha^\beta = Q_\alpha \tilde{Q}^\beta$, $\mathfrak{M}_\mathfrak{d}$ are the monopoles of the magnetic theory, and $\mathcal{F}[\mathfrak{M}_\mathfrak{d}^\pm]$ ¹⁹ are gauge singlets (identified with the monopoles of the electric theory) which flip the monopoles of the magnetic theory.

- **Minimally chiral duality:** When $2|k| > |N_F - N_A|$, the dual of $U(N_c)_k$ with N_F fundamental chiral fields and N_A anti-fundamental chiral fields is:

$$U(N_c)_k, \quad (N_F \square \oplus N_A \bar{\square}) \longleftrightarrow U\left(\frac{N_F + N_A}{2} + |k| - N_c\right)_{-k}, \quad (N_A \square \oplus N_F \bar{\square}), \quad (M). \quad (1.38)$$

- **Marginally chiral duality:** When $2|k| = |N_F - N_A|$, the dual of $U(N_c)_k$ with N_F fundamental chiral fields and N_A anti-fundamental chiral fields is:

$$U(N_c)_k, \quad (N_F \square \oplus N_A \bar{\square}) \longleftrightarrow U(\max(N_F, N_A) - N_c)_{-k}, \quad (N_A \square \oplus N_F \bar{\square}), \quad (M, \mathcal{F}[\mathfrak{M}_\mathfrak{d}^\epsilon]), \quad (1.39)$$

where $\epsilon = \pm$ corresponding to $2k = N_F - N_A$ or $2k = -N_F + N_A$.

- **Maximally chiral duality:** When $2|k| < |N_F - N_A|$, the dual of $U(N_c)_k$ with N_F fundamental chiral fields and N_A anti-fundamental chiral fields is:

$$U(N_c)_k, \quad (N_F \square \oplus N_A \bar{\square}) \longleftrightarrow U(\max(N_F, N_A) - N_c)_{-k}, \quad (N_A \square \oplus N_F \bar{\square}), \quad (M). \quad (1.40)$$

More general dualities for $U(N_c)_{(k, k+\ell N)}$ gauge theories with N_F fundamental chiral multiplets and N_A anti-fundamental chiral multiplets were reported in [147] by using Witten's prescription for gauging and ungauging topological symmetries [106].

1.1.3 Supersymmetric Localization and Observables

Supersymmetric localization [148–153] has emerged as one of the most powerful tools for the exact study of observables in supersymmetric quantum field theories. Its central insight is that, in the presence of sufficient supersymmetry, certain observables can be computed exactly, receiving contributions only from a restricted set of field configurations.

¹⁹A useful operation that frequently appears in three-dimensional supersymmetric dualities is the *flip* of a chiral operator. Given a gauge-invariant chiral operator $\mathcal{O}$, one introduces a new gauge-singlet chiral multiplet $\mathcal{F}[\mathcal{O}]$ and adds a superpotential coupling

$$\text{Flip}[\mathcal{O}] := \mathcal{F}[\mathcal{O}] \mathcal{O}.$$

The resulting F-term condition sets $\mathcal{O} = 0$ in the chiral ring, thereby removing it as an independent operator and replacing it with the elementary field $\mathcal{F}[\mathcal{O}]$, which carries the same global quantum numbers.

In general, the full dynamical content of a quantum field theory is encoded in its Feynman path integral, an infinite-dimensional integral over the space of field configurations. While this object is conceptually fundamental, it is typically intractable to evaluate exactly, except in perturbative regimes. Supersymmetric localization provides a remarkable mechanism by which this infinite-dimensional integral can be reduced to a much simpler expression, often amenable to exact evaluation.

The key ingredient underlying localization is supersymmetry. For supersymmetric theories—either topologically twisted or defined on curved manifolds admitting rigid supersymmetry—one can deform the action by a supersymmetry-exact term without changing the value of the partition function or of suitably protected observables. By tuning the coefficient of this deformation, the path integral localizes onto the locus of field configurations that are fixed under the chosen supercharge, known as the *localization locus*. Fluctuations around this locus are captured exactly by one-loop determinants.

From a mathematical perspective, supersymmetric localization can be viewed as a field-theoretic generalization of equivariant localization for finite-dimensional integrals, where integrals invariant under a symmetry group receive contributions only from the fixed points of the group action. In the quantum field theory setting, this principle leads to a dramatic reduction in complexity: a path integral over D -dimensional fields may reduce to an effective path integral over d -dimensional fields, with $d < D$. In particularly favorable cases—such as when the localization locus consists of constant field configurations—this further collapses to a finite-dimensional integral, or even to a discrete sum over fixed points.

Physically, localization may be understood as a situation in which the semiclassical (saddle-point or one-loop) approximation becomes exact. Importantly, this exactness does not arise from taking the original coupling $\hbar$ to zero, but instead from introducing an auxiliary deformation parameter whose vanishing limit enforces localization without altering the value of the path integral.

Supersymmetric localization therefore yields non-perturbative, exact expressions for partition functions and protected observables. These results provide powerful probes of the infrared dynamics of supersymmetric gauge theories, enable precision tests of infrared dualities, and often admit elegant mathematical interpretations. In the following, we will not attempt a systematic review of localization techniques; rather, we will directly present the specific partition functions and observables that are relevant for the applications discussed in this thesis.

An Intermezzo: Some Special Functions

Before introducing the supersymmetric observables that will be studied throughout this thesis, we first collect the definitions and basic properties of a class of special functions that serve as fundamental building blocks for these quantities.

We begin with the *multiple Gamma function* $\Gamma_r(z \mid \vec{\omega})$, defined in terms of the multiple zeta function as

$$\Gamma_r(z \mid \vec{\omega}) := \exp \left(\frac{\partial}{\partial s} \zeta_r(z, s \mid \vec{\omega}) \Big|_{s=0} \right),$$

where the multiple zeta function ζ_r is given by

$$\zeta_r(z, s \mid \vec{\omega}) := \sum_{n_1, \dots, n_r=0}^{\infty} \frac{1}{(z + \sum_{i=1}^r n_i \omega_i)^s}.$$

Using the multiple Gamma function, one defines the *multiple sine function* $S_r(z \mid \vec{\omega})$ [154–156] as

$$S_r(z \mid \vec{\omega}) := \Gamma_r(z \mid \vec{\omega})^{-1} \Gamma_r(|\vec{\omega}| - z \mid \vec{\omega})^{(-1)^r},$$

where we have introduced the shorthand

$$|\vec{\omega}| := \sum_{i=1}^r \omega_i.$$

The multiple sine function enjoys several properties that will be repeatedly used in the evaluation of supersymmetric partition functions. In particular:

- **Periodicity:**

$$S_r(z + \omega_j \mid \vec{\omega}) = \frac{S_r(z \mid \vec{\omega})}{S_r(z \mid \vec{\omega}/\omega_j)},$$

- **Reflection:**

$$S_r(z \mid \vec{\omega}) S_r(|\vec{\omega}| - z \mid \vec{\omega})^{(-1)^r} = 1.$$

Observables of three-dimensional theories are written in terms of $S_2(z) = S_2(|\vec{\omega}| - z \mid \omega_1, \omega_2)^{-1}$, which is often recast into the *double sine function*:

$$s_b(x) := S_2\left(\frac{Q}{2} - ix \mid b, b^{-1}\right),$$

where $Q := b + \frac{1}{b}$.

$\mathbf{S}^2 \times \mathbf{S}^1$ partition function

We now briefly review some basic properties of the partition function localized on $\mathbf{S}^2 \times \mathbf{S}^1$, commonly referred to as the three-dimensional *supersymmetric index* [119, 157–161]. When computed using the superconformal R -symmetry, this partition function coincides with the *superconformal index*.

The index is defined as a trace over the Hilbert space of the theory quantized on $\mathbf{S}^2 \times \mathbb{R}$,

$$\mathcal{I}(x, \vec{\mu}) := \text{Tr} \left[(-1)^{2J_3} x^{\Delta + J_3} \prod_i \mu_i^{T_i} \right],$$

where Δ denotes the energy of the state (measured in units of the inverse radius of $\mathbf{S}^2$), J_3 is the Cartan generator of the $SO(3)$ isometry group of $\mathbf{S}^2$, and T_i are the charges under non- R global symmetries, with corresponding fugacities μ_i .

A crucial property of the supersymmetric index is that it is invariant under continuous deformations of the theory that preserve supersymmetry. This follows from the fact that only states annihilated by a chosen supercharge and its Hermitian conjugate contribute to the trace. Equivalently, the index only receives contributions from states satisfying the BPS condition

$$\Delta - J_3 - R = 0.$$

As a result, the index is protected from quantum corrections and is an RG-invariant observable, making it a powerful tool for probing infrared dynamics and testing proposed infrared dualities.

The supersymmetric index admits an exact integral representation obtained via supersymmetric localization,

$$\mathcal{I}(x; \{\vec{\mu}, \vec{n}\}) := \sum_{\vec{m}} \frac{1}{|\mathcal{W}_{\vec{m}}|} \oint_{\mathbb{T}^{\text{rk}(G)}} \prod_{\alpha=1}^{\text{rk}(G)} \frac{dz_{\alpha}}{2\pi i z_{\alpha}} Z_{\text{cl}} Z_{\text{vec}} Z_{\text{mat}}. \quad (1.41)$$

Here $\vec{z} = (z_1, \dots, z_{\text{rk}(G)})$ parametrizes the maximal torus of the gauge group G , while $\vec{m}$ labels the GNO-quantized magnetic fluxes on $\mathbf{S}^2$. The integration contour for each z_{α} is the unit circle $\mathbb{T}$. The factor $|\mathcal{W}_{\vec{m}}|$ denotes the order of the Weyl group of the residual gauge symmetry preserved in the monopole background specified by $\vec{m}$.

The integrand factorizes into classical, vector multiplet, and matter multiplet contributions, denoted respectively by Z_{cl} , Z_{vec} , and Z_{mat} . We have also collectively denoted by $\{\vec{\mu}, \vec{n}\}$ the fugacities and background magnetic fluxes associated with global symmetries. These different contributions are defined as follows:

- the classical contribution is

$$Z_{\text{cl}} := \prod_{i=1}^{\text{rk}(G)} (-)^{km_i} \omega^{-m_i} z_i^{-km_i - \mathbf{n}},$$

where k is the CS level, and ω and $\mathbf{n}$ are the fugacities and flux for the topological symmetry, respectively.

A mixed CS interaction at level k_{ij} is captured by the term

$$Z_{\text{cl}} \supset (z_i^{-m_j} z_j^{-m_i})^{k_{ij}}.$$

Thus, the total classical contribution of a $U(N)_{(k, k+\ell N)}$ gauge group to the index is

$$Z_{\text{cl}} = \omega^{-\sum_{i=1}^N m_i} \left(\prod_{i=1}^N (-)^{km_i} z_i^{-km_i} \right) \left(\prod_{i=1}^N z_i \right)^{-\ell \sum_{i=1}^N m_i}.$$

- the vector contribution is

$$Z_{\text{vec}} := \prod_{\alpha \in \mathfrak{g}} x^{-\frac{|\alpha(\vec{m})|}{2}} (1 - \bar{z}^{\alpha} x^{|\alpha(\vec{m})|}),$$

where α are the roots in the algebra $\mathfrak{g}$ and we introduce a shorthand notation as follows

$$\bar{z}^{\alpha} := \prod_{i=1}^{\text{rk}(G)} z_i^{\alpha_i}, \quad \alpha(\vec{m}) = \sum_{i=1}^{\text{rk}(G)} \alpha_i m_i.$$

- the contribution of an $\mathcal{N} = 2$ chiral field transforming as $(\mathcal{R}, \mathcal{R}_F) \sim (G, G_F)$ of the gauge and global symmetries with R-charge r is

$$Z_{\text{mat}} := \prod_{\rho \in \mathcal{R}} \prod_{\rho_F \in \mathcal{R}_F} (\bar{z}^{\rho} \bar{\mu}^{\rho_F} x^{r-1})^{-\frac{|\rho(\vec{m}) + \rho_F(\vec{n})|}{2}} \times \frac{((-1)^{\rho(\vec{m}) + \rho_F(\vec{n})} \bar{z}^{-\rho} \bar{\mu}^{-\rho_F} x^{2-r+|\rho(\vec{m}) + \rho_F(\vec{n})|}; x^2)_{\infty}}{((-1)^{\rho(\vec{m}) + \rho_F(\vec{n})} \bar{z}^{\rho} \bar{\mu}^{\rho_F} x^{r+|\rho(\vec{m}) + \rho_F(\vec{n})|}; x^2)_{\infty}},$$

where ρ (ρ_F) are the weights of $\mathcal{R}$ ($\mathcal{R}_F$) respectively, and the q -Pochhammer symbol has the form

$$(a; q)_{\infty} := \prod_{k=1}^{\infty} (1 - aq^k).$$

The index captures information about the spectrum of short superconformal multiplets, up to possible recombination into long multiplets.

In practice, the index is most conveniently evaluated perturbatively by expanding the integrand as a power series in the fugacity x and performing the gauge projection

$$\oint \frac{dz}{2\pi iz}$$

order by order in the expansion. Upon setting the background magnetic fluxes for global symmetries to zero, the resulting expression takes the form of a power series in x , whose coefficients encode the contributions of gauge-invariant short operators.

S_b^3 partition function

We now briefly review some properties of the partition function localized on the squashed 3-sphere S_b^3 [162–166]. The squashed 3-sphere is parametrized as

$$b^2|z_1|^2 + \frac{1}{b^2}|z_2|^2 = 1, \quad (z_1, z_2) \in \mathbb{C}^2.$$

The partition function admits an integral representation obtained via localization,

$$Z(\vec{\mu}, \eta, k) := \frac{1}{|\mathcal{W}|} \int_{-\infty}^{+\infty} \prod_{\alpha=1}^{\text{rk}(G)} dz_{\alpha} Z_{\text{cl}} Z_{\text{vec}} Z_{\text{mat}}. \quad (1.42)$$

The various contributions to the integral are defined as follows:

- the classical contribution is

$$Z_{\text{cl}} := e^{2\pi i \eta \sum_{\alpha=1}^{\text{rk}(G)} z_{\alpha} - k\pi i \sum_{\alpha=1}^{\text{rk}(G)} z_{\alpha}^2},$$

where k is the CS level and η is the FI parameter.

- the vector contribution is

$$Z_{\text{vec}} := \frac{1}{\prod_{\alpha>0} s_b\left(\frac{iQ}{2} + \alpha(z)\right)},$$

where $Q = b + \frac{1}{b}$, α are the positive roots in the algebra and we introduce the shorthand notation

$$\alpha(z) := \sum_{\beta=1}^{\text{rk}(G)} \alpha_{\beta} z_{\beta}.$$

- the contribution of an $\mathcal{N} = 2$ chiral field transforming as $(\mathcal{R}, \mathcal{R}_F) \sim (G, G_F)$ of the gauge and global symmetries with R-charge r is

$$Z_{\text{mat}} := \prod_{\rho \in \mathcal{R}} \prod_{\rho_F \in \mathcal{R}_F} s_b\left(\frac{iQ}{2}(1-r) - \rho(z) - \rho_F(\mu)\right).$$

The partition function is also an RG flow invariant, and is the main tool used to study dualities in this thesis.

In three-dimensional $\mathcal{N} = 2$ superconformal field theories, the exact superconformal R-symmetry in the infrared is not fixed *a priori*, as it may mix with other Abelian global symmetries:

$$U(1)_{R_{\text{sc}}} := U(1)_{R_{\text{trial}}} + \sum_I \Delta^I U(1)_I,$$

where $U(1)_{R_{\text{trial}}}$ is a trial R-symmetry and Δ^I parametrizes the mixing with additional $U(1)_I$ global symmetries. In such a situation, the $\mathbf{S}_b^3$ partition function depends on the mixing parameters Δ .

A powerful tool to determine the exact R-symmetry $U(1)_{R_{\text{sc}}}$ is *F-maximization* [167, 168], which extremizes the free energy

$$F = -\ln |Z_{\mathbf{S}_b^3}|$$

with respect to all possible mixing parameters. More precisely, in terms of the real part of the free energy,

$$\Re[F(\Delta)] = -\Re[\ln |Z_{\mathbf{S}_b^3}(\Delta)|],$$

the superconformal R-symmetry corresponds to a local maximum of $\Re[F(\Delta)]$, so that

$$\left. \frac{\partial}{\partial \Delta^I} \Re[F(\Delta)] \right|_{\Delta=\Delta_{\text{sc}}} = 0.$$

The value of F at the extremum is a physical observable, invariant along the renormalization group flow, and plays a role analogous to the central charge in two dimensions or the a -anomaly in four dimensions²⁰, as shown in [105, 170]:

$$\left. \frac{\partial^2}{(\partial \Delta^I)^2} \Re[F(\Delta)] \right|_{\Delta=\Delta_{\text{sc}}} = -\frac{\pi^2}{2} C_I,$$

where C_I is the central charge associated with the $U(1)_I$ symmetry (see [171] for a more pedagogical discussion).

1.2 Brane Dynamics and Gauge Theories

Brane configurations in string theory provide a powerful geometric framework for engineering and analyzing supersymmetric quantum field theories. By realizing gauge degrees of freedom as the low-energy dynamics of D-branes and matter fields as arising from open strings stretched between them, one obtains a concrete and often intuitive description of strongly coupled phenomena that are otherwise difficult to access using purely field-theoretic methods. In this approach, properties such as moduli spaces of vacua, global symmetries, and infrared dualities admit a direct geometric interpretation in terms of the relative positions and orientations of branes.

In this section, we review the basic brane constructions relevant for three-dimensional supersymmetric gauge theories (without justifying their string origins in much detail), focusing primarily on Type IIB Hanany–Witten setups. These configurations naturally realize $\mathcal{N} = 4$ and $\mathcal{N} = 2$ gauge theories in three dimensions and provide a unified framework for understanding mirror symmetry, real mass and Fayet–Iliopoulos deformations, and a variety of infrared dualities. The results reviewed here will serve as the geometric underpinning for the mirror symmetry and duality constructions discussed in the following chapters. This section collects results from [172–179].

²⁰ An important criterion for $3d$ $\mathcal{N} = 4$ SCFTs is whether they are *good*, *bad* or *ugly* [169], which indicates whether the UV and IR R-charges coincide. For $\mathcal{N} = 4$ gauge theories, a single gauge group G with N_f hypermultiplets is considered *good* if

$$\text{for } U(N): N_f \geq 2N, \quad \text{for } SO(N): N_f \geq N - 1, \quad \text{for } USp(2N): N_f \geq 2N + 1.$$

Good theories are those in which all monopole operators obey the unitarity bound, and the theory flows to an interacting SCFT in the infrared. *Ugly theories* contain some monopole operators that saturate the unitarity bound, and the theory flows to an interacting SCFT with some free multiplets at the IR FP. Monopole operators in *bad theories* violate the unitarity bound, and the infrared physics of these theories is unclear.

1.2.1 The Zoology of Dp -Branes

In addition to the fundamental string, string theory contains extended p -dimensional objects known as p -branes. At weak string coupling g_s , these objects can be broadly classified according to how their tension scales with g_s . *Solitonic*, or *Neveu–Schwarz* (NS), branes have tension scaling as $T \sim g_s^{-2}$, while *Dirichlet* (D) branes have tension scaling as $T \sim g_s^{-1}$. This distinction is central to the non-perturbative structure of string theory and underlies many duality relations.

In the low-energy supergravity description, p -branes appear as extended objects electrically charged under $(p+1)$ -form gauge fields. We follow the notation of [173]. The $(9+1)$ -dimensional spacetime of string theory is labeled by coordinates $(x^0, x^1, \dots, x^9)$, with associated Dirac matrices Γ^M ($M = 0, \dots, 9$).

Type IIA string theory possesses $(1,1)$ spacetime supersymmetry, with left- and right-moving supercharges satisfying

$$\begin{aligned}\Gamma^0 \dots \Gamma^9 Q_L &= +Q_L, \\ \Gamma^0 \dots \Gamma^9 Q_R &= -Q_R.\end{aligned}$$

Type IIB string theory, by contrast, has $(2,0)$ spacetime supersymmetry, with both supercharges obeying

$$\begin{aligned}\Gamma^0 \dots \Gamma^9 Q_L &= +Q_L, \\ \Gamma^0 \dots \Gamma^9 Q_R &= +Q_R.\end{aligned}$$

As a consequence, Type IIA string theory is non-chiral, whereas Type IIB string theory is chiral.

The spectrum of string theory consists of a finite number of light degrees of freedom together with an infinite tower of massive excitations, whose masses are of order the string scale or higher. To make contact with low-energy physics, it is therefore convenient to focus on the dynamics of the light modes. This is achieved by integrating out the massive string excitations, yielding an effective field theory governing the low-energy degrees of freedom.

Applying this procedure to Type II string theory leads to two distinct ten-dimensional low-energy effective theories: Type IIA and Type IIB supergravity. In both cases, the field content decomposes into a Neveu–Schwarz–Neveu–Schwarz (NS–NS) sector, common to the two theories, containing the spacetime metric g_{MN} , the Kalb–Ramond two-form B_{MN} , and the dilaton Φ . The Ramond–Ramond (R–R) sector differs between the two theories: Type IIA supergravity contains odd-degree form potentials, while Type IIB supergravity contains even-degree form potentials. Correspondingly, the brane content of the two theories is

- **Type IIA:** Dp -branes with even p , namely D0, D2, D4, D6, and D8 branes.
- **Type IIB:** Dp -branes with odd p , namely D(−1), D1, D3, D5, and D7 branes. The D(−1) brane is special, as it corresponds to a D-instanton [180].

From a microscopic perspective, D-branes are defined as hypersurfaces on which fundamental strings can end [172]. A Dp -brane extended along the $(x^1, \dots, x^p)$ directions and localized at a point in $(x^{p+1}, \dots, x^9)$ is characterized by Neumann boundary conditions on the open-string endpoints along $(x^1, \dots, x^p)$ and Dirichlet boundary conditions along $(x^{p+1}, \dots, x^9)$.

The tension of a Dp -brane is given by

$$T_p = \frac{1}{g_s l_s^{p+1}},$$

where l_s denotes the fundamental string length. The tension of a Dp -brane is equal to its R–R charge. D-branes are $\frac{1}{2}$ -BPS objects, meaning that they preserve half of the thirty-two supercharges of Type II string theory. Concretely, a Dp -brane extended along the $(x^1, \dots, x^p)$ directions preserves supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ subject to

$$\epsilon_L = \Gamma^0 \Gamma^1 \dots \Gamma^p \epsilon_R.$$

An anti- Dp -brane carries the opposite R–R charge and preserves the complementary set of supercharges.

Dp -branes are dynamical objects in their own right, as is evident from their coupling to Ramond–Ramond gauge fields. The minimal electric coupling of a $(p+1)$ -form potential to a $(p+1)$ -dimensional worldvolume is

$$\mathcal{S}_{p+1} = \frac{T_p}{(p+1)!} \int_{\Sigma^{(p+1)}} A_{\mu_1 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}.$$

From this perspective, classical supergravity solutions carrying non-trivial charge under A_{p+1} correspond precisely to Dp -branes. By Hodge duality in ten dimensions, there exists a dual potential $\tilde{A}_{7-p}$ satisfying

$$d\tilde{A}_{7-p} = \star dA_{p+1}.$$

We now turn to the analysis of the worldvolume action describing the dynamics of a single Dp -brane. On general grounds of target-space Lorentz invariance and worldvolume reparametrization invariance, the bosonic action must reduce, in the absence of background fields, to a higher-dimensional generalization of the Nambu–Goto action,

$$\mathcal{S}_{Dp} = -T_p \int d^{p+1}\sigma e^{-\Phi} \sqrt{\det g_{ab}},$$

where σ^a ($a = 0, \dots, p$) are worldvolume coordinates and g_{ab} is the induced metric

$$g_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \eta_{\mu\nu}.$$

In the presence of background NS–NS fields, the Kalb–Ramond two-form $B_{\mu\nu}$ enters through the gauge-invariant combination

$$\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab},$$

where B_{ab} is the pullback of $B_{\mu\nu}$ and F_{ab} is the field strength of the worldvolume $U(1)$ gauge field arising from open strings ending on the brane. Including these effects yields the Dirac–Born–Infeld (DBI) action [181],

$$\mathcal{S}_{\text{DBI}} = -T_p \int d^{p+1}\sigma e^{-\Phi} \sqrt{\det(g_{ab} + \mathcal{F}_{ab})}. \quad (1.43)$$

Consistency further requires couplings to higher R–R fields, encoded in the Chern–Simons action [182, 183], which we report for completeness

$$\mathcal{S}_{\text{R–R}} = \mu_p \int C \wedge \text{ch}(\mathcal{F}) \wedge \frac{\sqrt{\hat{A}(\text{TW})}}{\sqrt{\hat{A}(\text{NW})}}, \quad (1.44)$$

where μ_p is the R–R charge of the brane, $C = \sum_q C_q$ denotes the formal sum of R–R potentials, and $\hat{A}(TW)$ ($\hat{A}(NW)$) is the $\hat{A}$ -genus of the tangent (normal) bundle of the worldvolume [101].

Both $\mathcal{S}_{\text{DBI}}$ and $\mathcal{S}_{\text{R-R}}$ admit generalizations to a stack of N coincident branes [184, 185]. In this case, the worldvolume fields become $U(N)$ -valued, and to leading order in α' the action reduces to

$$-T_p \int d^{p+1}\sigma e^{-\Phi} \text{Tr} \left(F^{ab} F_{ab} + 2D_a X^i D^a X_i + [X_i, X_j]^2 \right), \quad (1.45)$$

describing a $U(N)$ gauge theory with coupling $g_{\text{YM}}^2 \sim g_s T_p^{-1} \alpha'^{-2}$.

The observation that a stack of N coincident Dp -branes gives rise to a $U(N)$ super–Yang–Mills theory with sixteen supercharges in $(p+1)$ dimensions lies at the heart of the geometric engineering of gauge theories. In this framework, gauge symmetries, matter content, and coupling constants acquire a direct geometric interpretation in terms of brane configurations and background fields. As a result, intrinsically non-perturbative phenomena in quantum field theory—such as infrared dualities, symmetry enhancement, and strong-coupling dynamics—can often be analyzed using classical or semiclassical string-theoretic tools. Geometric engineering thus provides a powerful and unifying perspective on the structure of supersymmetric gauge theories. In what follows, we will study three-dimensional gauge theories from this perspective.

1.2.2 D-Brane Realizations of Three-Dimensional Results

Picking up from the discussion of D-branes and their worldvolume dynamics, we now turn to the brane engineering of three-dimensional gauge theories in string theory. The central idea of geometric engineering is that the low-energy dynamics of a quantum field theory can be realized as the effective description on the worldvolume of branes embedded in a suitable string-theoretic background. By appropriately choosing the type of branes, their dimensionality, and their relative orientations, one can engineer supersymmetric gauge theories with a prescribed amount of supersymmetry, matter content, and global symmetries.

Three-dimensional gauge theories are particularly well suited to this approach. Many of their characteristic features—such as monopole operators, topological symmetries, Chern–Simons interactions, and intricate webs of infrared dualities—admit a natural and often geometric interpretation in terms of brane configurations.

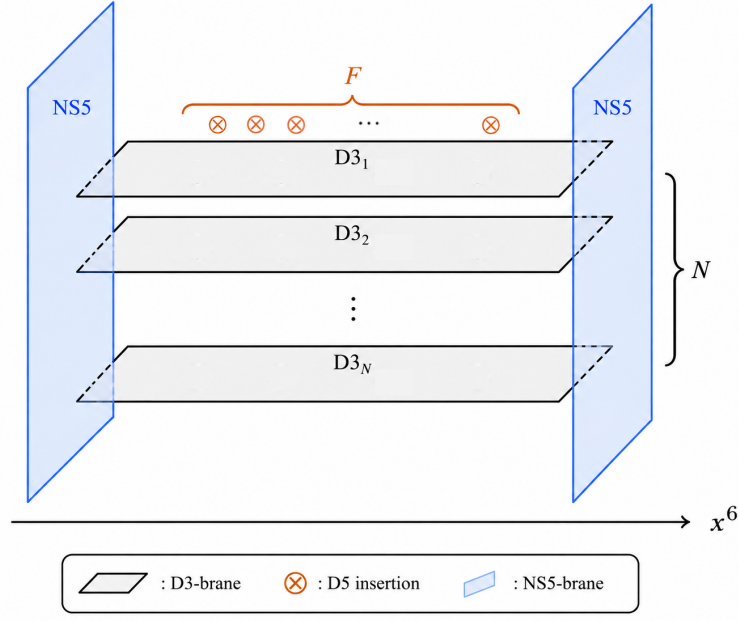
$\mathcal{N} = 4$ Theories

To study three-dimensional gauge dynamics, we consider, following [19], configurations of N $D3$ -branes suspended between $NS5$ -branes and intersecting F $D5$ -branes, arranged as in Table 1.5. This system realizes a large class of three-dimensional supersymmetric gauge theories as the low-energy effective theories living on the $D3$ -branes.

We consider two infinite $NS5$ -branes separated by a distance L_6 in the x^6 direction, located at the same point in (x^7, x^8, x^9) . N $D3$ -branes are suspended between them in the x^6 direction. This configuration is shown schematically in Figure 1.46.

Brane	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	✓	✓	✓	✓	✓	✓	–	–	–	–
D5	✓	✓	✓	–	–	–	–	✓	✓	✓
D3	✓	✓	✓	–	–	–	✓	–	–	–

Table 1.5: Standard Hanany–Witten brane configuration in Type IIB string theory. A checkmark indicates that the brane extends along the corresponding spacetime direction, while a dash indicates that it is localized in that direction. Such a setup preserves 8 supercharges, and is schematically shown in Figure 1.46.



(1.46)

The presence of the branes breaks the ten-dimensional Lorentz symmetry $SO(1, 9)$ according to

$$SO(1, 9) \longrightarrow SO(1, 2) \times SO(3) \times SO(3),$$

where $SO(1, 2)$ acts on the common directions (x^0, x^1, x^2) , one $SO(3)$ factor acts on the directions (x^3, x^4, x^5) , and the second $SO(3)$ factor acts on (x^7, x^8, x^9) . The $SO(1, 2)$ factor is interpreted as the Lorentz symmetry of the $(2 + 1)$ -dimensional worldvolume theory on the $D3$ -branes, while the two $SO(3)$ factors are global symmetries. In the low-energy theory, these are identified with the $SU(2)_C \times SU(2)_H$ R-symmetry of $\mathcal{N} = 4$ supersymmetry.

To analyze the low-energy physics of this configuration, it is useful to distinguish between three classes of excitations:

- bulk modes propagating in the ten-dimensional spacetime;
- modes localized on the $NS5$ -branes;
- modes localized on the $D3$ -branes.

For an observer restricted to the $(2 + 1)$ -dimensional directions (x^0, x^1, x^2) , the bulk modes and the excitations living on the $NS5$ -branes are effectively frozen at low energies in the weak string coupling limit. By contrast, the degrees of freedom living on the N $D3$ -branes remain dynamical, since the $D3$ -branes have finite extent along the x^6 direction.

The worldvolume theory on the $D3$ -branes is therefore naturally viewed as a four-dimensional theory on $\mathbb{R}^{1,2} \times I$, where I is an interval of finite length L_6 along the x^6 direction, bounded by $NS5$ -branes. At energy scales much smaller than $1/L_6$, the theory reduces effectively to a $(2+1)$ -dimensional quantum field theory.

The light degrees of freedom on a stack of N coincident $D3$ -branes consist of a four-dimensional $U(N)$ gauge field A_μ , with $\mu = 0, 1, 2, 6$, together with six real scalar fields transforming in the adjoint representation of $U(N)$. These scalars correspond to transverse fluctuations of the $D3$ -branes in the directions

$$(X^3, X^4, X^5, X^7, X^8, X^9).$$

The presence of the $NS5$ -branes imposes boundary conditions on the fields at the ends of the interval I . In particular, the fluctuations of the $D3$ -branes in the (x^7, x^8, x^9) directions are frozen at the $NS5$ -brane boundaries, while the scalars (X^3, X^4, X^5) remain dynamical. As a result, Kaluza–Klein modes with masses of order $1/L_6$ decouple in the low-energy limit, and the remaining massless degrees of freedom organize themselves into a three-dimensional $\mathcal{N} = 4$ *vector multiplet*, consisting of the $U(N)$ gauge field A_μ ($\mu = 0, 1, 2$) and the three adjoint scalars (X^3, X^4, X^5) , with gauge coupling $g_{YM}^2 \sim \frac{g_s}{L_6}$.

Additional matter fields arise from open strings stretching between the $D3$ - and $D5$ -branes. Each $D5$ -brane intersecting the $D3$ -branes contributes a three-dimensional $\mathcal{N} = 4$ *hypermultiplet* transforming in the fundamental representation of $U(N)$. Thus, a configuration with F $D5$ -branes realizes a $U(N)$ gauge theory with F fundamental hypermultiplets.

Geometric parameters of the brane configuration are naturally mapped to field theory couplings. Relative positions of $D5$ -branes in the (x^3, x^4, x^5) directions correspond to real mass parameters for the hypermultiplets. Similarly, separations of $NS5$ -branes in the (x^7, x^8, x^9) directions give rise to Fayet–Iliopoulos parameters.

The positions of the N $D3$ -branes stretched between the $NS5$ -branes in the transverse directions (x^3, x^4, x^5) , together with the N dual photons arising from the unbroken $U(1)^N$ gauge symmetry, parametrize the $4N$ -dimensional Coulomb branch of the three-dimensional $\mathcal{N} = 4$ gauge theory. These moduli correspond to vacuum expectation values of the scalars in the vector multiplet and their dual photons.

From the perspective of the fivebrane worldvolume theory, however, this moduli space admits a different and highly instructive interpretation. The theory living on the pair of $NS5$ -branes is a six-dimensional $\mathcal{N} = (1, 1)$ supersymmetric gauge theory with gauge group $U(2)$, which is spontaneously broken to $U(1) \times U(1)$ by the separation of the fivebranes. In this description, $D3$ -branes stretched between the $NS5$ -branes are identified with magnetic monopole strings of the broken $SU(2)$ gauge theory [186].

Consequently, the $4N$ -dimensional Coulomb branch of the three-dimensional theory is naturally identified, from the fivebrane viewpoint, with the moduli space of N BPS monopoles.

$\mathcal{N} = 2$ Theories

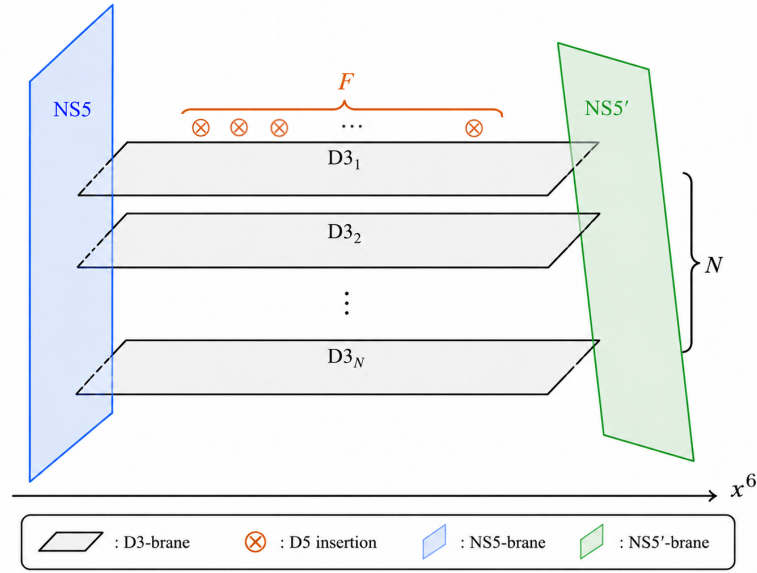
Following [173], we now consider a configuration of N $D3$ -branes suspended between an $NS5$ - and $NS5'$ -brane, with insertions of F $D5$ -branes, arranged as in Table 1.6 and schematically shown in Figure 1.47.

The presence of both $NS5$ and $NS5'$ fivebranes breaks the rotational symmetry in the transverse directions and reduces the preserved supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

Brane	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	✓	✓	✓	✓	✓	✓	–	–	–	–
NS5'	✓	✓	✓	✓	–	–	–	–	✓	✓
D5	✓	✓	✓	–	–	–	–	✓	✓	✓
D3	✓	✓	✓	–	–	–	✓	–	–	–

Table 1.6: Type IIB brane configuration engineering three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The presence of both *NS5* and rotated *NS5'* fivebranes breaks supersymmetry to four supercharges, schematically shown in Figure 1.47.

The relative orientation of the fivebranes lifts the adjoint chiral multiplet present in $\mathcal{N} = 4$ theories, leaving only a single real scalar in the vector multiplet, as expected for $\mathcal{N} = 2$ supersymmetry in three dimensions.



(1.47)

The Hanany-Witten Transition. The Hanany-Witten transition [187] is a phenomenon exhibited by branes in Type II string theory. In these brane configurations, moving an *NS5*-brane across a $D(2p + \sigma)$ -brane creates a $D(2p - 2 + \sigma)$ -brane²¹ stretching between the two. This is a consequence of charge conservation and supersymmetry preservation in the brane setup. This is schematically shown in Figure 1.1 [188, 189].

It was suggested by [190] that the Hanany-Witten effect may be understood entirely as a consequence of the Bianchi identity for the RR-field-fluxes $F_{8-2p-\sigma}$ sourced by the D-branes, which is twisted by the B-field-flux H_3 sourced by *NS5*-branes:

$$dF_{8-2p-\sigma} = H_3 \wedge F_{6-2p-\sigma},$$

although we defer the interested reader to [190, 191] for a more detailed discussion of this point.

Aharony Duality from Brane Realizations. Aharony duality admits a brane realization in the Hanany-Witten construction of three-dimensional $\mathcal{N} = 2$ supersymmetric

²¹ $\sigma = 0(1)$ for IIA (IIB) setups.

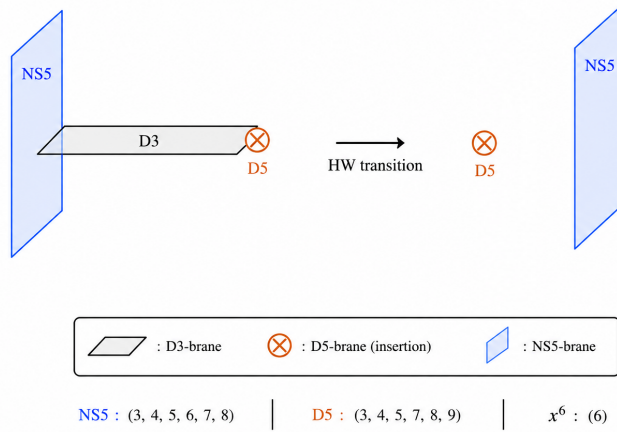


Figure 1.1: An example of the Hanany-Witten transition commonly encountered in the context of brane configurations relevant for $3d$ mirror theories is shown here. The $D5$ -brane is moved across an $NS5$ -brane, resulting in the annihilation of the $D3$ -brane stretching between the two.

gauge theories, although, as we shall see, the realization is not purely classical. We recover the duality here from the $\mathcal{N} = 2$ setup of Table 1.6, following [173].

A key ingredient in the brane realization of the duality is the *Hanany-Witten effect* [19]. We remind the reader that when the $NS5'$ -brane is adiabatically moved past a $D5$ -brane along the x^6 direction, a $D3$ -brane is created stretching between the $NS5'$ -brane and the $D5$ -brane in order to preserve charge conservation and supersymmetry. As the $NS5'$ -brane crosses all N_f $D5$ -branes, a total of N_f $D3$ -branes are created. Taking into account the original N_c color $D3$ -branes, the resulting configuration contains $N_f - N_c$ $D3$ -branes suspended between the $NS5$ and $NS5'$ branes, realizing a dual $U(N_f - N_c)$ gauge theory at low energies.

At the classical level, this is as far as the brane move takes us: the configuration reproduces the magnetic gauge group $U(N_f - N_c)$ together with the magnetic quarks and the N_f^2 singlet mesons M , coupled through $\mathcal{W} = Mq\tilde{q}$. The chiral multiplets $V^\pm$ dual to the electric monopole operators, and their couplings, are absent at this level. This is to be contrasted with the Giveon-Kutasov dualities at $k > 0$, where the brane construction is manifest. The resolution [173] is that the equivalence of the electric and magnetic descriptions is a quantum statement, whereas the analysis of the brane configuration sketched above is classical: the fields $V^\pm$ are present in the magnetic configuration, but non-geometrically, and are recovered only upon studying the quantum vacuum structure of the two brane configurations.

With that caveat in place, the geometric counterparts of these degrees of freedom can be identified. In addition to the color branes, the brane crossing produces semi-infinite $D3$ -branes stretching between the $NS5'$ -brane and the $D5$ -branes, which from the three-dimensional perspective are non-dynamical gauge singlets. Monopole operators correspond to $D1$ -strings stretched between the $NS5$ and $NS5'$ branes, and the couplings between these $D1$ -strings and the semi-infinite $D3$ -branes are what reproduce, in the quantum treatment, the superpotential interactions that flip the monopoles of the dual theory.

An important consistency condition in this construction is the *s-rule* (also see [192]),

which states that at most a single supersymmetric $D3$ -brane can stretch between an $NS5$ -brane and a $D5$ -brane while preserving supersymmetry. In the present context, the s -rule ensures that the number of color $D3$ -branes suspended between the $NS5$ and $NS5'$ branes is bounded by N_f , and in particular that a supersymmetric vacuum exists only when $N_f \geq N_c$. This constraint has a direct field-theoretic interpretation: it mirrors the condition for the existence of a stable supersymmetric vacuum in three-dimensional $\mathcal{N} = 2$ SQCD and guarantees that the brane configuration correctly encodes the infrared dynamics of the theory.

Embedding these theories in string theory therefore relates quantum field theories in ways which are far from obvious from the field-theoretic perspective, though the example above shows that the dictionary is not always purely geometric: some features of the infrared duality are visible in the brane picture only after the quantum dynamics has been taken into account.

The Hanany–Witten construction not only provides a geometric realization of Aharony duality, but also furnishes a unifying framework in which a wide class of three-dimensional dualities arise from simple operations on brane configurations. Among these, *mirror symmetry* occupies a distinguished position, admitting a particularly transparent geometric interpretation in Type IIB string theory. Mirror symmetry is the central theme of this thesis, and we will review its basic structure and physical implications in greater detail in the following section.

An Interlude: Quiver Gauge Theories

Quiver diagrams ([193, 194]) are central to our work, and we take a brief pause to outline our conventions regarding the same here.

The low-energy effective field theory associated with an HW setup is often conveniently encoded in terms of *quiver diagrams*. A quiver diagram is an oriented diagram built with the following ingredients (following [3, 21, 28, 158, 162, 195]):

- **Nodes** (or vertices) encode the symmetries of the theory. Gauge symmetries are represented by circular nodes, while global symmetries are denoted by squares. In what follows, we consider classical Lie groups and summarize their respective contributions to the $\mathbf{S}_b^3$ partition function below:

- the $\mathcal{N} = 2$ vector multiplet associated with a **unitary**²² symmetry is reported as follows:

$$\begin{array}{c} \eta \\ \textcircled{N} \\ (k, k+\ell N) \end{array} \quad \begin{array}{c} Z_{\mathbf{S}_b^3} \\ := \end{array} \quad \int \prod_{\alpha=1}^N \frac{du_{\alpha} e^{-\pi i \left(k \sum_{\alpha} u_{\alpha}^2 + \ell (\sum_{\alpha} u_{\alpha})^2 \right)} e^{2\pi i \eta \sum_{\alpha} u_{\alpha}}}{N! \prod_{\alpha > \beta} s_b \left(\frac{iQ}{2} \pm (u_{\alpha} - u_{\beta}) \right)} \quad (1.49)$$

where $\{u_{\alpha}\}_{\alpha=1}^N$ are the fugacities associated with the generators of the Cartan subalgebra of the $U(N)$ gauge symmetry, the Chern-Simons level is indicated in red below the node, and the fugacity of the topological $U(1)_T \in U(N)$ (denoted by η) is indicated in orange above the node.

- the $\mathcal{N} = 2$ vector multiplet associated with a **symplectic** symmetry is reported as follows:

$$\begin{array}{c} \textcircled{C_N} \\ k \end{array} \quad \begin{array}{c} Z_{\mathbf{S}_b^3} \\ := \end{array} \quad \int \prod_{\alpha=1}^N \frac{du_{\alpha} e^{-\pi i k \sum_{\alpha} u_{\alpha}^2}}{N! 2^N \prod_{\alpha, \beta} s_b \left(\frac{iQ}{2} \pm u_{\alpha} \pm u_{\beta} \right)} \quad (1.50)$$

where $\{u_{\alpha}\}_{\alpha=1}^N$ are the fugacities associated with the generators of the Cartan subalgebra of the $USp(2N)$ gauge symmetry, and the Chern-Simons level is indicated in red below the node.

- the $\mathcal{N} = 2$ vector multiplet associated with a **special orthogonal** symmetry is reported as follows:

$$\begin{array}{c} \textcircled{B_N} \\ k \end{array} \quad \begin{array}{c} Z_{\mathbf{S}_b^3} \\ := \end{array} \quad \int \prod_{\alpha=1}^N \frac{du_{\alpha} e^{-\pi i k \sum_{\alpha} u_{\alpha}^2}}{N! 2^N \prod_{\alpha < \beta} s_b \left(\frac{iQ}{2} \pm u_{\alpha} \pm u_{\beta} \right) s_b \left(\frac{iQ}{2} \pm u_{\alpha} \right)} \quad (1.51)$$

$$\begin{array}{c} \textcircled{D_N} \\ k \end{array} \quad \begin{array}{c} Z_{\mathbf{S}_b^3} \\ := \end{array} \quad \int \prod_{\alpha=1}^N \frac{du_{\alpha} e^{-\pi i k \sum_{\alpha} u_{\alpha}^2}}{N! 2^{N-1} \prod_{\alpha < \beta} s_b \left(\frac{iQ}{2} \pm u_{\alpha} \pm u_{\beta} \right)} \quad (1.52)$$

where $\{u_{\alpha}\}_{\alpha=1}^N$ are the fugacities associated with the generators of the Cartan subalgebra of the $SO(2N+1)$ and $SO(2N)$ gauge symmetries, and the Chern-Simons level is indicated in red below the node.

- **Edges** represent matter fields of the theory.

- The bifundamental $\mathcal{Q}$ with R-charge R transforming under various symmetry groups contributes to the $\mathbf{S}_b^3$ partition function as follows:

$$\boxed{M} \leftarrow \boxed{N} \quad \begin{array}{c} Z_{\mathbf{S}_b^3} \\ := \end{array} \quad \prod_{j=1}^N \prod_{k=1}^M s_b \left(\frac{iQ}{2} (1-R) - X_j + Y_k \right) \quad (1.53)$$

²²The contribution of the $\mathcal{N} = 2$ vector multiplet associated with a **special unitary** symmetry can be obtained by a straightforward modification of the integration measure of the unitary symmetry as follows:

$$\int \prod_{\alpha=1}^N du_{\alpha} \rightarrow \int \prod_{\alpha=1}^N du_{\alpha} \delta \left(\sum_{\alpha=1}^N u_{\alpha} \right). \quad (1.48)$$

An $SU(N)$ gauge symmetry will be reported as a double circled node labeled as N .

$$\boxed{M} \leftarrow \boxed{C_N} \quad \stackrel{Z_{\mathbf{S}_b^3}}{:=} \quad \prod_{j=1}^N \prod_{k=1}^M s_b \left(\frac{iQ}{2} (1-R) \pm X_j + Y_k \right) \quad (1.54)$$

$$\boxed{C_M} \longrightarrow \boxed{B_N} \quad \stackrel{Z_{\mathbf{S}_b^3}}{:=} \quad \prod_{j=1}^N \prod_{k=1}^M s_b \left(\frac{iQ}{2} (1-R) \pm X_j \pm Y_k \right) \\ \prod_{k=1}^M s_b \left(\frac{iQ}{2} (1-R) \pm Y_k \right) \quad (1.55)$$

$$\boxed{C_M} \longrightarrow \boxed{D_N} \quad \stackrel{Z_{\mathbf{S}_b^3}}{:=} \quad \prod_{j=1}^N \prod_{k=1}^M s_b \left(\frac{iQ}{2} (1-R) \pm X_j \pm Y_k \right) \quad (1.56)$$

Note that an edge directed into (outside) a node denotes an anti-fundamental (fundamental) representation.

- The presence of an adjoint superfield in the theory is indicated by a closed loop:

$$\boxed{G_N} \quad (1.57)$$

Their contributions to the $\mathbf{S}_b^3$ partition function can be inferred from those of the respective vector multiplets.

- The presence of a field with R-charge R in the rank-2 antisymmetric representation of a unitary symmetry (and its conjugate) is indicated as follows:

$$\boxed{N} \rightleftarrows \boxed{1} \quad \stackrel{Z_{\mathbf{S}_b^3}}{:=} \quad \prod_{i>j}^N s_b \left(\frac{iQ}{2} (1-R) \pm (X_i + X_j - s) \right) \quad (1.58)$$

where $\{X_i\}_{i=1}^N$ are the fugacities associated with the generators of the Cartan subalgebra of the $U(N)$ symmetry, and s is the mass of the antisymmetric field.

- The presence of a mixed CS interaction at level k_{XY} between the $U(1) \subset U(N)_{\vec{X}}$ and $U(1) \subset U(M)_{\vec{Y}}$ is reported as follows:

$$\boxed{M} \cdots \boxed{N} \quad \stackrel{Z_{\mathbf{S}_b^3}}{:=} \quad e^{-\pi i k_{XY} (\sum_j X_j) (\sum_k Y_k)} \quad (1.59)$$

- There is a superpotential interaction associated with each **cycle** of the quiver diagram. For instance, consider the following quiver diagram:

$$\begin{array}{c} \text{Quiver Diagram: } \begin{array}{ccc} & \boxed{G_N} & \\ x \swarrow & & \searrow y \\ \boxed{G_F} & & \boxed{G_A} \\ & \xrightarrow{z} & \end{array} \end{array} \longrightarrow \mathcal{W} = XYZ \quad (1.60)$$

In what follows, such interactions will be collectively denoted as $\mathcal{W}_{\text{planar}}$.

Quiver diagrams are central to this work, as they provide a compact encoding of all matter content and interactions.

1.3 Mirror Symmetry in Three Dimensions

Unlike ordinary perturbative symmetries, which are visible order by order in a weak-coupling expansion, a *nonperturbative quantum symmetry* [196] is not manifest at the perturbative level. In general, establishing such a symmetry would require complete control over the full quantum theory, including its strongly coupled regime. What makes duality symmetries particularly powerful is that, in supersymmetric theories, they can often be inferred using only weak-coupling and semiclassical data, without detailed knowledge of the full nonperturbative dynamics.

Supersymmetry imposes strong constraints on the structure of quantum corrections. In theories with sufficient supersymmetry, many semiclassical quantities receive no quantum corrections—neither perturbative nor nonperturbative—and are therefore exact. Such protected quantities can be computed reliably in the weak-coupling regime and then analytically continued to strong coupling. If two theories are related by a duality, all nonrenormalized observables must agree. The requirement that these quantities match provides highly nontrivial consistency checks, which are often compelling enough to establish the existence of a duality even in the absence of a rigorous proof [197].

Type IIB string theory provides a canonical example of this phenomenon. In addition to discrete $\mathbb{Z}_2$ perturbative symmetries, it enjoys a nonperturbative $SL(2; \mathbb{Z})$ duality symmetry, which is visible at the level of the equations of motion governing the axio-dilaton field. The modular group $SL(2; \mathbb{Z})$ is generated by the elements

$$\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Provided that strong-coupling effects do not break supersymmetry, consistency implies that Type IIB string theory can only be dual to itself under this symmetry.

There are two classes of semiclassical quantities that are expected to be exact and must therefore exhibit the $SL(2; \mathbb{Z})$ duality group:

1. the massless spectrum and the corresponding equations of motion;
2. the spectrum of BPS-saturated states.

Consequently, one expects the nonperturbative $SL(2; \mathbb{Z})$ invariance of Type IIB string theory to manifest itself as a duality symmetry in Type IIB Hanany–Witten brane configurations.

We focus on the $\mathcal{S} \in SL(2; \mathbb{Z})$ element, which implements a strong–weak duality by acting on the Type IIB coupling as $g_s \mapsto 1/g_s$. In brane language, the action of $\mathcal{S}$ on the objects appearing in an $\mathcal{N} = 4$ Hanany–Witten setup is

$$D3 \longrightarrow D3, \quad NS5 \longrightarrow D5, \quad D5 \longrightarrow -NS5.$$

Here the minus sign indicates a reversal of orientation, which is irrelevant for most field-theoretic considerations but becomes important when keeping track of Chern–Simons couplings and linking numbers.

Since the $D3$ -branes are self-dual under $\mathcal{S}$, while $NS5$ - and $D5$ -branes are exchanged, the action of $\mathcal{S}$ effectively interchanges the directions (x^3, x^4, x^5) and (x^7, x^8, x^9) in the standard Hanany–Witten configuration (Table 1.5). At the level of the three-dimensional worldvolume theory, this induces an automorphism of the $\mathcal{N} = 4$ supersymmetry algebra

that exchanges the two $SU(2)$ factors of the R-symmetry,

$$SU(2)_C \longleftrightarrow SU(2)_H.$$

More precisely, in the Hanany–Witten construction, $NS5$ -branes engineer vector multiplet degrees of freedom on the $D3$ -branes, while $D5$ -branes engineer hypermultiplet matter. The $\mathcal{S}$ transformation therefore exchanges the sectors of the theory associated with vector multiplets and hypermultiplets. At the level of the moduli space of vacua, this corresponds to an exact exchange of the Coulomb and Higgs branches (both of which are hyperkähler manifolds [198, 199]) at the infrared fixed point.

The central statement of $\mathcal{N} = 4$ **mirror symmetry** [19, 24, 200–202] is thus that it maps perturbative Higgs-branch operators, such as mesons and baryons, to intrinsically nonperturbative Coulomb-branch operators, namely monopole operators. From this perspective, mirror symmetry can be understood as an infrared duality *that exchanges flavor symmetries with topological symmetries* of the three-dimensional theory.

It is important to stress that this mechanism is qualitatively different from Aharony-like dualities in three dimensions. While Aharony duality relates distinct ultraviolet gauge theories that flow to the same infrared fixed point, it does not exchange topological and flavor symmetries. In contrast, $\mathcal{N} = 4$ mirror symmetry precisely interchanges these symmetries, mapping flavor currents on one side to topological currents on the other. Mirror symmetry is therefore a distinct type of infrared duality, characterized by an exact exchange of Higgs and Coulomb branch data rather than a rearrangement of matter content within a fixed global symmetry structure.

$\mathcal{N} = 4$ Unitary Mirror Symmetry

We now make these ideas concrete by considering three-dimensional $\mathcal{N} = 4$ SQCD with gauge group $U(N)$ and F hypermultiplets. This theory admits a particularly simple realization in Type IIB string theory via the Hanany–Witten brane configuration summarized in Table 1.5. The action of Type IIB $\mathcal{S}$ -duality on this setup naturally leads to mirror pairs of $\mathcal{N} = 4$ theories, as shown in Figure 1.2.

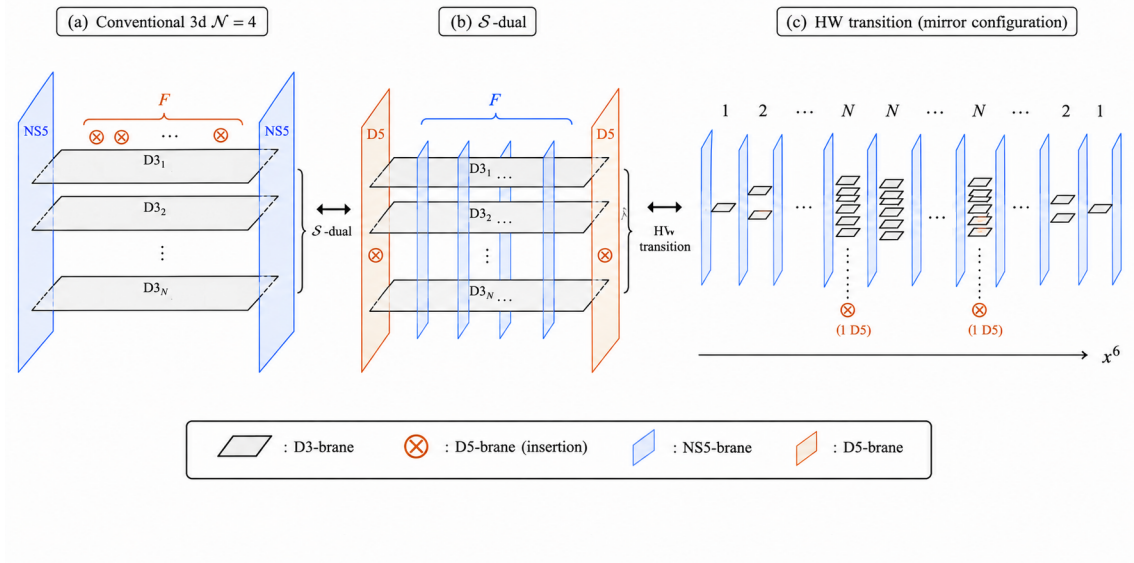


Figure 1.2: Action of $\mathcal{S}$ -duality on a Type IIB Hanany–Witten configuration. The left panel shows the original setup realizing $\mathcal{N} = 4$ $U(N)$ SQCD with F hypermultiplets: N $D3$ -branes are suspended between two $NS5$ -branes, with F $D5$ -branes providing the flavors of the theory. Applying the $\mathcal{S}$ element of $SL(2; \mathbb{Z})$ exchanges $NS5$ - and $D5$ -branes, yielding the configuration shown in the middle panel. This intermediate configuration violates the s -rule and therefore does not preserve supersymmetry. To restore supersymmetry, one must perform a sequence of Hanany–Witten moves (see Figure 1.1 and the accompanying discussion), during which $D3$ -branes are created or annihilated as fivebranes cross. The resulting supersymmetric configuration, shown in the right panel, realizes the mirror dual theory.

The quiver diagrams of the resulting mirror pair can be read from the brane setup, and are shown schematically in Figure 1.3, while the field content of the two dual theories is summarized in (1.61). Although both theories preserve $\mathcal{N} = 4$ supersymmetry, it is convenient to describe them using $\mathcal{N} = 2$ language. In this notation, the $\mathcal{N} = 4$ R-symmetry

$$SU(2)_C \times SU(2)_H$$

is reorganized into two abelian factors,

$$U(1)_R = U(1)_{C+H}, \quad U(1)_\tau = U(1)_{C-H},$$

where $U(1)_C \subset SU(2)_C$ and $U(1)_H \subset SU(2)_H$. From the $\mathcal{N} = 2$ perspective, $U(1)_R$ plays the role of the R-symmetry, while $U(1)_\tau$ should be regarded as a global symmetry.

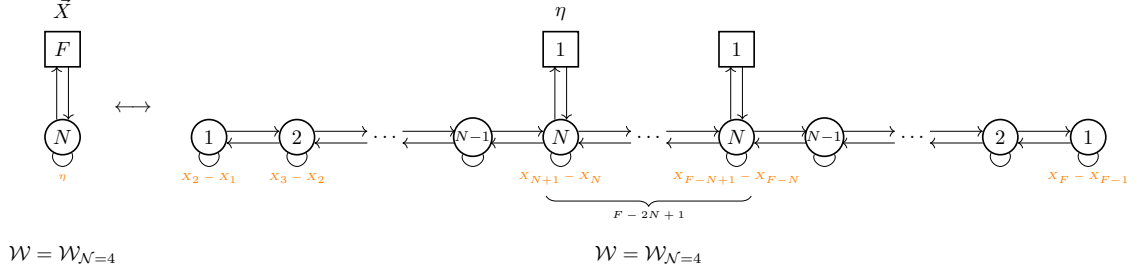


Figure 1.3: Mirror duality for $\mathcal{N} = 4$ SQCD with gauge group $U(N)$ and F flavors, depicted in $\mathcal{N} = 2$ notation. Circular nodes denote gauge groups and square nodes denote flavor symmetry groups, while arrows represent chiral multiplets charged under the corresponding nodes. Black labels indicate fugacities associated with flavor symmetries, while orange labels denote FI parameters for the gauge groups.

We do not fix the exact superconformal R-charges. Instead, we assign trial charges such that the fundamental and anti-fundamental chiral multiplets have R-charge 1 and $U(1)_\tau$ charge $-\frac{1}{2}$, while adjoint chiral multiplets have R-charge 0 and $U(1)_\tau$ charge 1. On the electric side, we denote by Φ , Q , and $\tilde{Q}$ the adjoint and fundamental/anti-fundamental chiral multiplets of SQCD. On the mirror side, we use the notation Φ_J , $b_{J,J+1}$, and $\tilde{b}_{J+1,J}$ for the adjoint and bifundamental chiral multiplets of the linear quiver, together with fundamental chiral multiplets $q, \tilde{q}$ and $p, \tilde{p}$ at the two ends of the quiver.

	$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$		$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$
Φ	1	1	0	0	Φ_J	1	-1	2	0
Q	$\bar{\square}$	$-\frac{1}{2}$	1	0	$b_{J,J+1}, \tilde{b}_{J,J+1}$	1	$\frac{1}{2}$	0	0
$\tilde{Q}$	$\square$	$-\frac{1}{2}$	1	0	$q, \tilde{q}$	1	$\frac{1}{2}$	0	∓ 1
					$p, \tilde{p}$	1	$\frac{1}{2}$	0	0

(1.61)

Global symmetries. The global symmetry of the $U(N)$ SQCD with F flavors, depicted in the left panel of Figure 1.3, is

$$SU(F) \times U(1)_\eta \times U(1)_\tau,$$

where $U(1)_\eta$ denotes the topological symmetry.²³

On the mirror side, the manifest global symmetry is

$$\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_\eta \times U(1)_\tau,$$

where the first factor corresponds to the collection of topological symmetries associated with the $F - 1$ gauge nodes of the linear quiver, while $U(1)_\eta$ is realized as a flavor symmetry.²⁴ A highly nontrivial prediction of mirror symmetry is that the topological symmetries

²³The faithful global symmetry group is $PSU(F) \times U(1)_\eta \times U(1)_\tau$. We will not discuss the role of discrete factors. Moreover, the $U(1)_\eta$ symmetry enhances to $SU(2)$ in the infrared when $F = 2N$, a feature that is more manifest on the mirror side.

²⁴Although two $U(1)$ flavor nodes appear in the quiver, the $\mathcal{N} = 4$ superpotential ensures that only one independent combination, denoted by η , is realized faithfully.

of the mirror theory enhance in the infrared to an $SU(F)$ flavor symmetry, matching exactly the global symmetry of the SQCD theory.

This enhancement has been verified using a variety of complementary methods: matching the infrared metrics on Higgs and Coulomb branches²⁵ [201], comparing $\mathbf{S}_b^3$ partition functions [203], and studying correlation functions and operator products of Coulomb branch operators in the one-dimensional topological sector [204].

Mapping of gauge-invariant operators. Mirror symmetry also provides a precise correspondence between gauge-invariant operators of the two theories [118, 144]. In particular,

- The fundamental monopole operators $\mathfrak{M}^\pm$ of the SQCD theory are mapped to long mesonic operators on the mirror quiver side.
- The electric mesons, transforming in the traceless adjoint representation of $SU(F)$, are mapped to a collection of monopole operators in the mirror theory. These assemble into an adjoint-valued matrix of $SU(F)$ constructed from the traces of the adjoint fields Φ_J together with monopole operators carrying GNO flux under consecutive gauge nodes: monopoles with flux ± 1 under a single node, monopoles with flux ± 1 under two adjacent nodes, and so on, up to monopoles carrying flux ± 1 under all $F - 1$ gauge nodes.

Mirror symmetry has a natural interpretation for theories with $\mathcal{N} = 4$ supersymmetry, and is seen as a consequence of $\mathcal{S}$ -duality inherited from IIB string theory. As such, it is an infrared duality that relates theories with swapped flavor and topological symmetries.

Insights from Field Theory. While $\mathcal{N} = 4$ mirror symmetry has a definitive “stringy” origin, it is possible to realize these results in a purely field-theoretic framework. This was pioneered by [24], who demonstrated that $\mathcal{N} = 4$ Abelian mirror symmetry could be equivalently realised using an algorithmic strategy grounded in field theory. Their approach is straightforward:

- take an Abelian $\mathcal{N} = 4$ theory and decompose it into bifundamental hypermultiplets or flavor hypermultiplets after freezing gauge interactions;
- each hypermultiplet can subsequently be dualized by exploiting the relation between a free hypermultiplet and a $U(1)$ gauge theory with a single flavor; while this relation is transparent in the HW-brane setup, it is equivalently encoded as a field-theoretic identity establishing the equivalence of $\mathbf{S}_b^3$ partition functions [158, 162, 205]:

$$\int dz e^{2\pi i(\eta_1 - \eta_2)z} s_b\left(-\frac{iQ}{2} + \tau\right) s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm z\right) = s_b(\tau \pm (\eta_1 - \eta_2));$$

where τ is the fugacity of the $U(1)_\tau \subset SO(4)_R$ flavor symmetry. The basic Abelian mirror duality, therefore, is the self-Fourier-transform property of the hypermultiplet $\mathbf{S}_b^3$ partition function.

- Notice that, for Abelian $\mathcal{N} = 4$ theories, the $\mathcal{S}$ operation is encoded as a mixed CS interaction in the partition function (of the form $e^{2\pi iXY}$). Thus, the relation

²⁵This result was stated in a more mathematically precise language by [22], and we repeat it here: the Higgs branch of Kronheimer [141] gauge theories is an *asymptotically locally Euclidean* (ALE) space, while the Coulomb branch gives the moduli space of a small ADE instanton. In this sense, mirror symmetry shows ALE gravitational instantons and the moduli space of ADE instantons are naturally dual to each other.

$\mathcal{S}\mathcal{S}^{-1} = \mathbb{I}$ can be encoded as:

$$\int dz e^{2\pi i(\eta_1 - \eta_2)z} s_b\left(-\frac{iQ}{2} + \tau\right) = \frac{\delta(\eta_1 - \eta_2)}{s_b\left(-\frac{iQ}{2} + \tau\right)}.$$

In practice, the functional identity indicates that composing two Abelian $\mathcal{S}$ -operations produces the bad $U(1)$ $\mathcal{N} = 4$ SYM theory. The UV spectrum of the theory contains two monopoles, one of which acquires an expectation value and breaks the 0-form topological symmetry of the theory (reflected by the delta function).

- After local dualization, gauging is restored and the blocks are glued back using the $\mathcal{S}\mathcal{S}^{-1} = \mathbb{I}$ identity to obtain the dual theory.

While this approach allows us to realise Abelian $\mathcal{N} = 4$ mirror symmetry purely in field theoretic terms, it was extended to non-Abelian theories, starting from [206]. As such, $\mathcal{N} = 4$ mirror symmetry has a firm field-theoretic origin as well, and one need not invoke its string-theoretic origin to make physical predictions.

What makes theories with $\mathcal{N} < 4$ so intractable? The subject of this thesis is the study of mirror symmetry in three-dimensional supersymmetric gauge theories with $\mathcal{N} < 4$ supersymmetry. This regime is substantially more subtle than the well-understood $\mathcal{N} = 4$ case. As we had mentioned before, $\mathcal{N} = 4$ mirror symmetry may be intuitively seen as an automorphism on the supersymmetry algebra that exchanges the two $SU(2)$ factors of the R-symmetry. When the amount of supersymmetry is reduced, this intuition fails, as the R-symmetry does not have identical factors (e.g.: the R-symmetry of $\mathcal{N} = 3$ theories is $SO(3) \cong SU(2)$, while that of $\mathcal{N} = 2$ theories is $SO(2) \cong U(1)$) that can be exchanged.

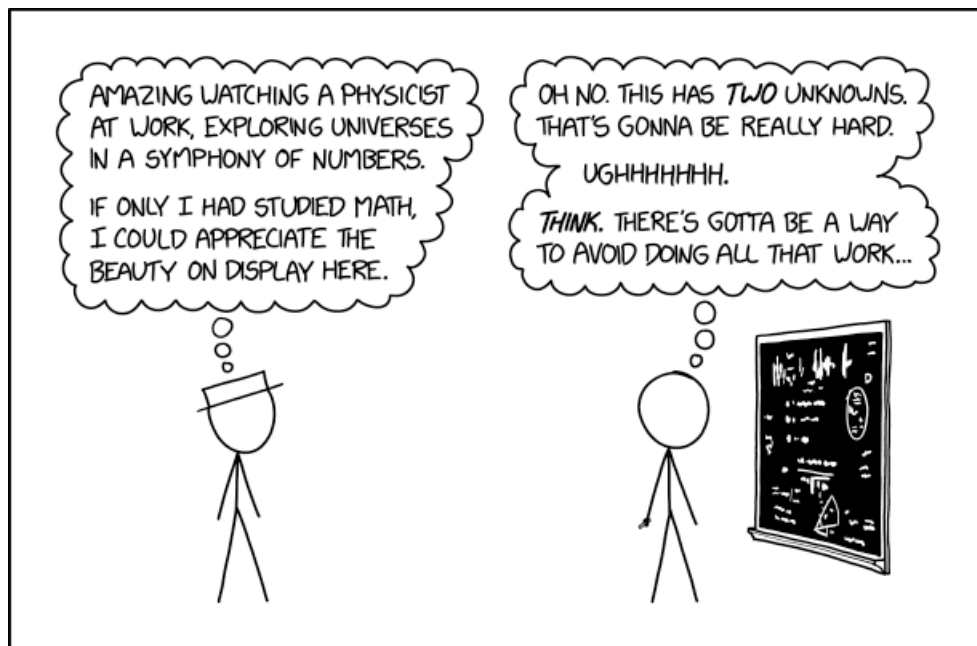
More precisely, when the amount of supersymmetry is reduced, the Higgs and Coulomb branches may not factorize into distinct product manifolds, or may be partially or completely lifted, and the canonical Type IIB brane realizations that underlie mirror symmetry are often no longer available²⁶.

As a consequence, mirror duals of $\mathcal{N} < 4$ theories are not known in general and cannot be systematically derived from brane constructions. Their identification therefore requires intrinsically field-theoretic tools. The aim of this thesis is to develop such methods and to propose and test mirror-like dualities in broad classes of three-dimensional theories with reduced supersymmetry.

Throughout this work, we are guided by the defining feature of mirror symmetry: **it is a duality that exchanges topological and flavor symmetries, mapping order and disorder operators to each other**. Thus, even when geometric engineering techniques no longer exist, constraints from arguments based on symmetries are strong enough for us to generalize this notion to theories with minimal supersymmetry.

²⁶It was shown in [27] that applying $\mathcal{S}$ -duality on $\mathcal{N} = 2$ brane setups (like the one shown in Figure 1.47) is possible, but the resulting dual description is inherently non-Lagrangian since it is built with superconformal matter. Invoking superconformal matter as the building blocks is necessary to match the global symmetries in both theories.

Part II: $\mathcal{N} = 2$ Mirror Symmetry in Unitary Gauge Theories



Chapter 2

Mirror Duals of $\mathcal{N} = 2$ Unitary Chern-Simons-Matter Theories

In this chapter, we develop a systematic framework to derive a new class of three-dimensional $\mathcal{N} = 2$ dualities, which we refer to as *chiral-planar mirror dualities*¹ [2, 3]. These dualities relate non-abelian chiral gauge theories with Chern-Simons interactions to purely abelian, planar quiver gauge theories, and arise as infrared limits of real mass deformations of known $\mathcal{N} = 4$ mirror pairs [19, 22].

Our starting point is the observation that many $\mathcal{N} = 2$ chiral theories of interest can be obtained from $\mathcal{N} = 4$ gauge theories by turning on a real mass deformation for the axial symmetry

$$U(1)_\tau = U(1)_{C-H},$$

which breaks supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ [26, 30].

A prototypical example is provided by three-dimensional $\mathcal{N} = 4$ $U(N)$ SQCD with F hypermultiplets. By combining the axial real mass deformation with appropriate shifts along the Coulomb branch, this theory flows to a chiral $\mathcal{N} = 2$ $U(N)$ SQCD with F fundamental chiral multiplets and nontrivial Chern-Simons levels. Since the original $\mathcal{N} = 4$ theory admits a mirror description in terms of a linear quiver [19], one may attempt to perform the same deformation on the dual side. However, the mirror theory contains many gauge nodes and hence many Coulomb branch directions, leading to a large set of candidate vacua. Identifying the vacuum that correctly reproduces the dual of the chiral SQCD is therefore highly nontrivial.

A powerful tool to address this problem is provided by the exact partition function on the squashed three-sphere $\mathbf{S}_b^3$ [158, 162]. As shown in [21, 195], real mass deformations correspond to controlled large-mass limits of the $\mathbf{S}_b^3$ partition function. In such limits, the partition function factorizes into a finite contribution—associated with the interacting infrared theory—and a highly oscillatory phase. Crucially, this phase depends on how the real mass limit is combined with shifts of the gauge fugacities, and therefore encodes the choice of vacuum on the Coulomb branch. As a consequence, matching the oscillatory phases on the two sides of a known $\mathcal{N} = 4$ mirror identity provides a sharp criterion to identify the pair of infrared theories that remain dual after the deformation.

In this chapter, we apply this strategy in detail to the case of $U(N)$ SQCD. Starting from the equality of the $\mathbf{S}_b^3$ partition functions of the $\mathcal{N} = 4$ mirror pair, we perform a

¹This was historically called the *Dorito Duality* by the author. Never forget what was taken from us.

systematic real mass deformation and show how to isolate the vacuum that reproduces the chiral $\mathcal{N} = 2$ SQCD on the electric side. Remarkably, we find that the corresponding magnetic theory is a *planar*² *abelian quiver*, obtained by fully Higgsing each $U(k)$ gauge node of the $\mathcal{N} = 4$ mirror theory down to its Cartan subgroup. The resulting dual theory exhibits nontrivial mixed Chern–Simons couplings and a rich pattern of topological symmetry enhancement, inherited from the parent $\mathcal{N} = 4$ mirror symmetry.

Beyond establishing this specific duality, the analysis presented here highlights a general structural feature: real mass deformations of $\mathcal{N} = 4$ mirror pairs naturally lead to chiral–planar mirror dualities, in which mesonic symmetries are mapped to topological symmetries and abelian gauge dynamics emerge in a planar arrangement. While the deformation procedure itself is technically involved, it provides valuable guidance for identifying the correct infrared dual and motivates the search for a more systematic construction.

This motivation leads naturally to the algorithmic approach developed in the subsequent half of the chapter, where we extend the mirror dualization algorithm [206, 213] to the chiral–planar setting. The detailed partition-function analysis presented here thus serves a dual purpose: it establishes the validity of the chiral–planar mirror duality in a concrete example, and it provides the conceptual and technical foundation for the algorithmic framework that follows.

2.1 General Yoga of Planar Abelianization

We pick up where we left off in the previous chapter while reviewing 3d $\mathcal{N} = 4$ mirror symmetry in $U(N)$ gauge theories with F hypermultiplets, shown in the top row of Figure 2.1.

We remind the reader we are working in the $\mathcal{N} = 2^*$ set-up [26] to effectively study supersymmetry breaking real mass deformations. Thus, we set $U(1)_R = U(1)_{C+H}$ as the R-symmetry group, where $U(1)_C \in SU(2)_C$ and $U(1)_H \in SU(2)_H$ of the non-abelian $\mathcal{N} = 4$ R-symmetry $SU(2)_H \times SU(2)_C$. The commutant of $U(1)_R$, which is $U(1)_\tau = U(1)_{C-H}$, is a global symmetry from the point of view of the $\mathcal{N} = 2$ theory. Thus, the global symmetry of both theories is $SU(F) \times U(1)_{top} \times U(1)_\tau$.

We now consider a supersymmetry breaking mass deformation. This choice is not arbitrary, and we do so by turning on a non-zero vacuum expectation value (VEV) for the real scalar component of the background vector multiplet associated with the $U(1)_\tau$ symmetry. The VEV induces a mass term for every chiral superfield proportional to its charge under the symmetry. Recall that $U(1)_\tau$ is a flavor subgroup of the $\mathcal{N} = 4$ R-symmetry; therefore, the deformation performed has the effect of breaking the R-symmetry from $SU(2)_C \times SU(2)_H \rightarrow U(1)_R$, and the supersymmetry is broken from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

In the most generic vacuum, the resulting IR theory would be a topological quantum field theory. However, we follow the flow to distinguished points on the Coulomb branch where the theory is an interacting one instead. While non-trivial, this can be done effectively at the level of the $\mathbf{S}_b^3$ partition function [158, 162] and the equations of motion [32, 214]. However, before doing this, we will attempt to qualitatively motivate this picture first.

We first consider the “*electric*” $\mathcal{N} = 4$ SQCD theory. We move on the Coulomb branch in such a way to reach an interacting vacuum where all the fundamental chiral fields Q remain

²The dual theory is planar, in the sense that it is a quiver drawn on a plane, with a cubic superpotential term for each closed oriented loop, as in the 4d periodic quivers associated to the dimers of [207, 208], or in the minimally supersymmetric $\text{AdS}_5/\text{CFT}_4$ correspondence studied in [209–212].

massless while the adjoint Φ and the F anti-fundamental chiral fields $\tilde{Q}$ acquire a mass, negative and positive respectively, and are integrated out³. The procedure of integrating out chiral fields produces CS interactions due to fermionic modes becoming massive. The resulting theory is the $\mathcal{N} = 2$ $U(N)_{-\frac{F}{2}+N, -\frac{F}{2}}$ SQCD depicted on the bottom-left panel of Figure 2.1. Here, $U(N)_{k, k+\ell N}$ ⁴ denotes a $U(N)$ gauge group with CS terms:

$$-i\frac{k}{4\pi} \int \text{tr}(A \wedge dA) - i\frac{\ell}{4\pi} \int \text{tr}(A) \wedge \text{tr}(dA) + \text{SUSY completion} \quad (2.1)$$

where A is the $U(N)$ gauge field. Each massive anti-fundamental multiplet, with negative mass, contributes as $(-\frac{1}{2}, -\frac{1}{2})$ to the CS level, and the massive adjoint chiral, with positive mass, contributes $(N, 0)$. Notice in particular that we have $\ell = -1$.

In the magnetic theory (top right panel of Figure 2.1), under the same deformation, something interesting happens - we propose that the magnetic vacuum corresponding to the electric theory is such that each gauge node $U(k)$ is Higgsed to its maximal torus $U(1)^k$ producing a column of k black $U(1)$ gauge nodes, while some chiral fields remain massless (as shown in the bottom right panel of Figure 2.1).

We present the duality (with refined fugacities) in Figure 2.2.

A few comments regarding the planar dual in the right panel of Figure 2.2 are in order:

- There is a cubic superpotential term for every triangular loop. There is one term with -1 ($+1$) coefficient for each clock-wise (anti-clock-wise) closed triangle. The cubic terms in the superpotential are a remnant of the cubic $\mathcal{N} = 4$ superpotential that are preserved by the deformation. We denote these superpotential terms in short as $\mathcal{W}_{\text{planar}}$. The diagonal/vertical chiral multiplets have trial R-charges r , $2 - 2r$, respectively, compatible with the superpotential.
- $\mathcal{W}_{\text{monopole}}$ are the monopole terms in the superpotential generated by the Polyakov mechanism [215] due to the Higgsing of a $U(k)$ gauge symmetry to $U(1)^k$. The exact collection of terms can be obtained with the following schematic rule. For each vertical arrow, there is a linear superpotential for the monopole with GNO flux -1 and $+1$ under the nodes connected by the arrow, from top to bottom. As a consequence of these monopole superpotential terms, the topological symmetry is broken to $U(1)^{F-1}$ which is expected to enhance in the infrared to $SU(F)$, thus matching the global symmetry on the electric side. The FI terms of the $U(1)$ gauge nodes are compatible with the monopole superpotential terms. With the choice of trial R-symmetry described above, the α^{th} gauge node in the I^{th} column of the planar quiver ($\alpha = 1, \dots, |G^{(I)}|$, and $|G^{(I)}|$ is the rank of the corresponding gauge node in the $\mathcal{N} = 4$ quiver, or equivalently, the number of $U(1)$ gauge nodes in the I^{th} column of the $\mathcal{N} = 2$ quiver) has FI terms equal to $X_{I+1} - X_I + \frac{iQ}{4}(\delta_{\alpha,1} - \delta_{\alpha,|G^{(I)}|})$. In Figure 2.2 we only report the X_I -dependent part of the FIs.
- CS interactions are present. Self and mixed-interactions can be encoded in a matrix k_{ab} such that:

$$-i\frac{k_{ab}}{4\pi} \int A_a \wedge dA_b + \text{SUSY completion} \quad (2.2)$$

³One can also consider different interacting vacua where a certain number of fundamental and anti-fundamental fields remain massless. While this problem is tractable, we will tackle the case of SQCD theories with more general matter content in a different, algorithmic way in Section 2.4.

⁴We follow the standard notation for Chern-Simons levels: $U(N)_{(k_1, k_2)} = \frac{SU(N)_{k_1} \times U(1)_{Nk_2}}{\mathbb{Z}_N}$. To avoid anomalies $k_1 \in \mathbb{Z}$ and the $U(1)$ level must be of the form $k_2 = k_1 + \ell N$, with $\ell \in \mathbb{Z}$.

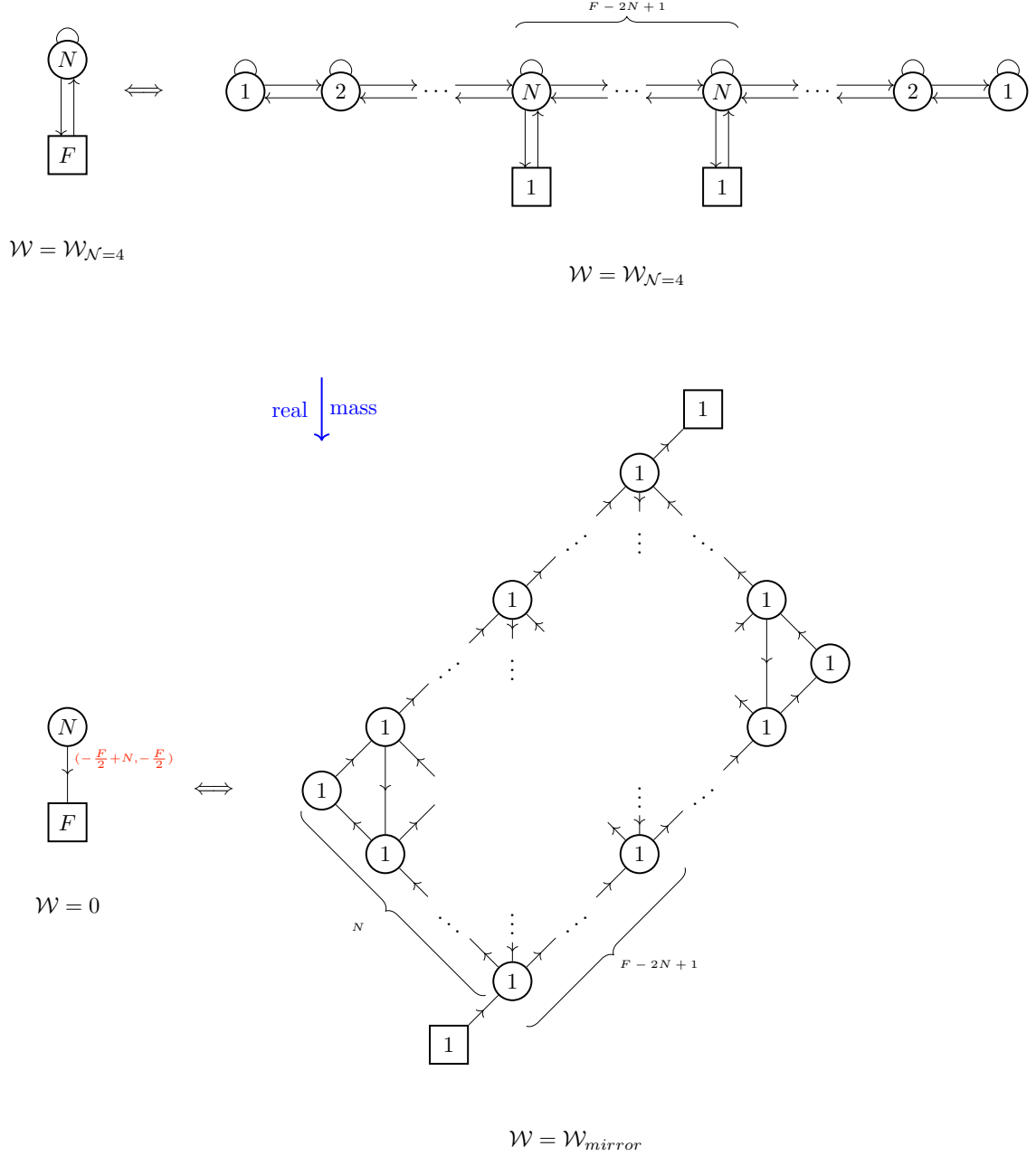


Figure 2.1: Schematic depiction of the $\mathcal{N} = 2$ mirror duality (bottom row) observed between $U(N)$ SQCD with $[F, 0]$ fundamental fields and a planar, Abelian quiver gauge theory obtained by adding a suitably chosen real mass deformation to $\mathcal{N} = 4$ $U(N)$ SQCD with $[F, F]$ flavors and its mirror dual (top row). The electric $\mathcal{N} = 2$ non-Abelian SQCD has CS level $(-\frac{F}{2} + N, -\frac{F}{2})$ and a vanishing superpotential, whereas the mirror $\mathcal{N} = 2$ Abelian, planar quiver has CS and mixed CS interactions, a “planar” superpotential inherited from the $\mathcal{N} = 4$ parent theory, and a monopole superpotential generated via the Polyakov Mechanism [215] (captured by $\mathcal{W}_{mirror}$). Flavor and topological symmetries are exchanged by the duality, such that disorder operators (monopoles) are mapped to more familiar “mesonic” operators.

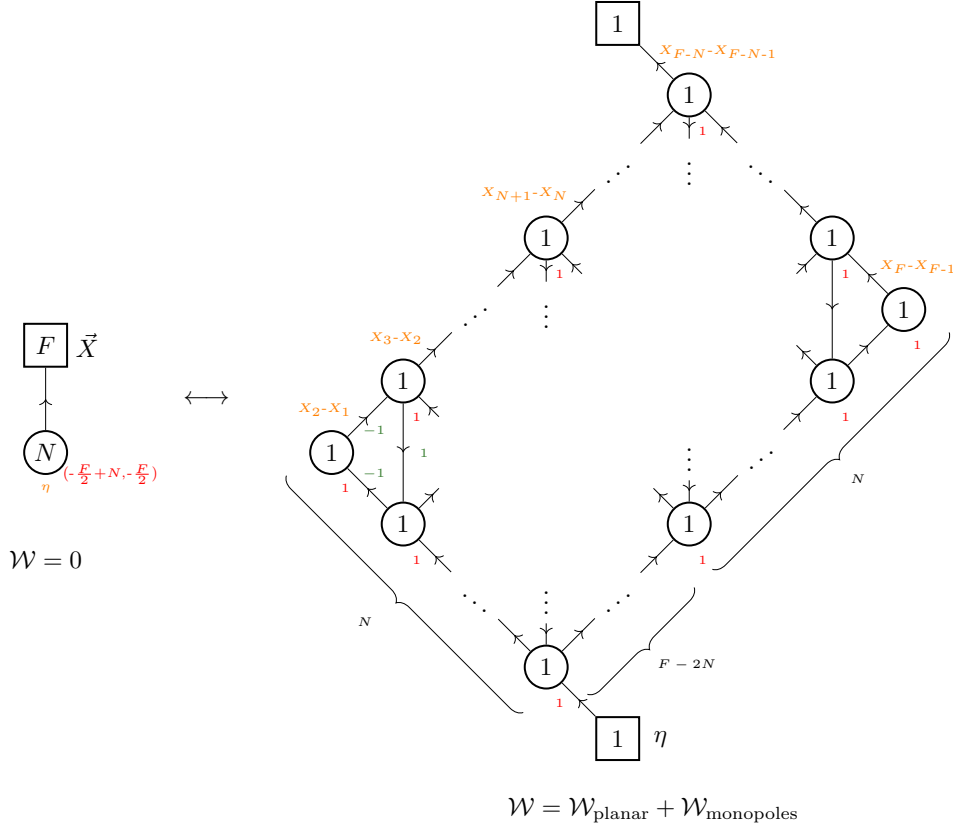


Figure 2.2: On the left hand side, we present the $\mathcal{N} = 2$ $U(N)_{-\frac{F}{2}+N, -\frac{F}{2}}$ SQCD with F fundamental chiral multiplets, obtained by a real mass deformation of the parent $\mathcal{N} = 4$ SQCD. On the right, we show its planar, mirror-like dual description. Each gauge node carries Chern-Simons level $+1$ (indicated by red labels). Mixed Chern-Simons interactions (green labels) are present for every pair of gauge nodes connected by a link, with level -1 for diagonal links and $+1$ for vertical links. FI parameters are indicated in orange. On the SQCD side, the fundamental chiral multiplets are assigned trial R-charge 1. On the planar side, diagonal chiral fields have trial R-charge 0, while vertical chiral fields have trial R-charge 2. Diagonal chirals oriented from SE to NW connect N gauge nodes, whereas diagonal chirals oriented from SW to NE connect $F - N$ gauge nodes.

N	F	r	Index
2	4	1/5	$1 + \eta x^{4/5} + \eta^2 x^{8/5} - 16x^2 + \eta^3 x^{12/5} + \eta^4 x^{16/5} + (\eta^5 + 88)x^4 + O(x^{21/5})$
2	5	1/5	$1 + \eta x + (\eta^2 - 25)x^2 + \eta^3 x^3 + (\eta^4 + 250)x^4 + (\eta^5 - 100\eta)x^5 + O(x^6)$
2	6	1/5	$1 + \eta x^{6/5} - 36x^2 + \eta^2 x^{12/5} + \eta^3 x^{18/5} + 558x^4 + \eta^4 x^{24/5} + O(x^{26/5})$
3	6	1/7	$1 + \eta x^{6/7} + \eta^2 x^{12/7} - 36x^2 + \eta^3 x^{18/7} + \eta^4 x^{24/7} + 558x^4$ $+ \eta^5 x^{30/7} - 225\eta x^{34/7} + O(x^{36/7})$

Table 2.1: Values of the Superconformal Index of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2})}$ SQCD with F fundamental chiral multiplets for some values of N, F . The chiral multiplets are assigned trial R-charge $1 - r$. We checked that the indices match with the corresponding mirror abelian quiver in Figure 2.2.

where A_a is the vector field of the a -th $U(1)$ gauge field. Indeed, by construction k_{ab} is symmetric. For the theory in Figure 2.2 the rule is that each gauge node has (self) CS interaction with level 1, denoted by the red labels in Figure 2.2. Nodes connected by diagonal/vertical lines have a mixed-CS interaction with level $-1/+1$, that is $k_{ab} = \mp \frac{1}{2}$. Mixed CS levels are denoted by the green labels along the corresponding line in Figure 2.2.

Global Symmetries and Gauge Invariant Operators

We now comment on the features of the duality described above. As a preliminary check, we compute the **Superconformal Index** [159, 216]. This guarantees that a number of properties are correctly reproduced by the dual theory such as - the presence of conserved currents associated to the correct global symmetry, and the properties of the operators in the chiral ring. We do this for $N = 2, F = 4, 5, 6$ and $N = 3, F = 6$, and report the results in Table 2.1. Notice that in order to perform the matching, it is important to have the mirror theory parameterized as in Figure 2.2. In particular the mixing between the R-symmetry and the topological symmetries is crucial for the enhancement of the global symmetry to be seen from the Superconformal Index.

Matching Global Symmetries The global symmetry of the SQCD theory is, up to discrete factors, $SU(F) \times U(1)_\eta$. On the mirror side, we can match the rank of the global symmetry as follows - in the absence of the monopole superpotential there is a $U(1)$ topological symmetry for each $U(1)$ gauge node. The monopole superpotential breaks the topological symmetries associated to the nodes in a column of the quiver to a diagonal $U(1)$. Therefore there is a $U(1)^{F-1}$ unbroken topological symmetry associated to the $F - 1$ columns of the planar quiver. Furthermore, there is a single $U(1)_\eta$ flavor symmetry which is preserved by the planar superpotential. In total, the rank of the global symmetry is F , matching the SQCD side.

The mirror-like duality implies that the $U(1)^F$ symmetry of the quiver theory is enhanced to $SU(F) \times U(1)$, as can be seen from the index calculations as well.

Mapping Chiral Ring Operators The chiral ring of the planar theory is generated by a single mesonic operator, constructed by contracting the F diagonal bifundamental fields along a path starting from the $U(1)_\eta$ flavor node and ending at the opposite flavor node. This operator carries trial R-charge 0 and $U(1)_\eta$ charge 1. Its uniqueness follows

from the F-term relations imposed by the planar superpotential $\mathcal{W}_{\text{planar}}$.

From the superconformal index, we expect this mesonic operator to be mapped to a monopole operator generating the chiral ring of the SQCD. The bare monopole operators $\mathfrak{M}^{\pm,0,\dots,0}$ are gauge variant, but they can be dressed to form gauge-invariant operators, as reviewed in Chapter 1. We now argue that the relevant dressing involves the gaugino fields.

Turning on GNO fluxes induces a monopole background in which the gauge group is partially broken. For a fundamental monopole, the breaking pattern is

$$U(N) \longrightarrow U(1) \times U(N-1).$$

Accordingly, all gauge-variant operators decompose into representations of the residual gauge group. The branching rules for irreducible representations under $U(N) \supset U(N-1) \times U(1)$ are [217]

$$\begin{aligned} \square_N &= (\square_{N-1}, 1) \oplus (\mathbf{1}, 1-N) \\ \square_N \otimes \bar{\square}_N &= (\square_{N-1} \otimes \bar{\square}_{N-1}, 0) \oplus (\square_{N-1}, N) \oplus (\bar{\square}_{N-1}, -N) \oplus (\mathbf{1}, 0), \end{aligned} \quad (2.3)$$

where $\square_K$ ($\bar{\square}_K$) denotes the (anti-)fundamental representation of $U(K)$, and $(\cdot, \cdot)$ labels representations of $U(N-1) \times U(1)$.

In particular, the $U(N)$ gaugino decomposes into the gaugino of $U(N-1)$, a singlet, and additional components transforming in the fundamental and anti-fundamental representations of $U(N-1)$. These latter components can be used to dress monopole operators.

More generally, in a $U(N)$ gauge theory with F fundamentals and Chern–Simons levels $(k, k + \ell N)$, the bare monopoles $\mathfrak{M}^{\pm,0,\dots,0}$ transform in a representation of $U(N-1)$ with $(N-1)$ -ality $\mp \ell(N-1)$ and carry $U(1)$ charge

$$-\frac{F}{2} \mp k \mp \ell,$$

as discussed in [136]. Dressing the monopole with $N-1$ powers of the gaugino component transforming in the fundamental of $U(N-1)$ compensates these charges, yielding

$$Q_{U(1)}(\mathfrak{M}^{\pm,0,\dots,0}) = (N-1) - \left(\frac{F}{2} \pm k \pm \ell \right). \quad (2.4)$$

In the theory considered here, we have $k = -\frac{F}{2} + N$ and $\ell = -1$. As a result, the monopole $\mathfrak{M}^{+,0,\dots,0}$ can be dressed $N-1$ times with gaugino components in the fundamental representation of $U(N-1)$ to form a gauge-invariant operator⁵. We identify this operator as the generator of the chiral ring of $U(N)$ SQCD with Chern–Simons levels $(-\frac{F}{2} + N, -\frac{F}{2})$ and F fundamental chirals, and propose that it is mapped to the planar meson described above.

This identification is supported by an R-charge computation. The fundamental monopole $\mathfrak{M}^{+,0,\dots,0}$ has trial R-charge

$$R(\mathfrak{M}^{+,0,\dots,0}) = (1 - R_Q) \frac{F}{2} - (N-1) = -(N-1), \quad (2.5)$$

⁵For $k = \frac{F}{2} - N$ and $\ell = 1$, instead, the gauge-invariant operator is $\mathfrak{M}^{-,0,\dots,0}$ dressed $N-1$ times with the gaugino.

where we have assigned trial R-charge $R_Q = 1$ to the fundamental chiral multiplets. The negative contribution arises from the gaugino zero modes in the monopole background. Dressing the monopole with $N - 1$ gaugino fields contributes an additional factor of $N - 1$ to the R-charge, yielding

$$R(\mathfrak{M}^{+,0,\dots,0}\lambda^{N-1}) = 0. \quad (2.6)$$

This exactly matches the R-charge of the gauge-invariant mesonic operator of the planar theory, providing further evidence for the proposed mapping.

The View from the $\mathbf{S}_b^3$ Partition Function

We now show how to derive the chiral–planar mirror duality for SQCD, depicted in Figure 2.2, by analyzing the $\mathbf{S}_b^3$ partition function [158, 162], following the strategy of [21, 195].

We begin with the $\mathbf{S}_b^3$ partition functions of the two $\mathcal{N} = 4$ mirror theories shown in Figure 1.3. The partition function of the SQCD is

$$Z(\vec{X}, \eta, \tau) = \int \prod_{\alpha=1}^N du_{\alpha} \Delta_{(N)}(\vec{u}, \tau) e^{2\pi i \eta \sum_{\alpha=1}^N u_{\alpha}} \prod_{j=1}^F s_b\left(\frac{\tau}{2} \pm (u_{\alpha} - X_j)\right). \quad (2.7)$$

The partition function of its mirror dual reads

$$\begin{aligned} \hat{Z}(\eta, \vec{X}, \tau) &= e^{2\pi i \eta \sum_{j=1}^N X_j} \int \prod_{I=1}^{F-1} \left[\prod_{\alpha=1}^{|G^{(I)}|} dz_{\alpha}^{(I)} e^{2\pi i z_{\alpha}^{(I)} (X_{I+1} - X_I)} \Delta_{(|G^{(I)}|)}(\vec{z}^{(I)}, iQ - \tau) \right] \\ &\times \prod_{I=1}^{F-2} \prod_{\alpha=1}^{|G^{(I)}|} \prod_{\beta=1}^{|G^{(I+1)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_{\alpha}^{(I)} - z_{\beta}^{(I+1)})\right) \\ &\times \prod_{\alpha=1}^{|G^{(N)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_{\alpha}^{(N)} - \eta)\right) \prod_{\alpha=1}^{|G^{(F-N)}|} s_b\left(\frac{iQ}{2} - \frac{\tau}{2} \pm z_{\alpha}^{(F-N)}\right). \end{aligned} \quad (2.8)$$

Here $\vec{z}^{(I)}$ denotes the gauge fugacities of the I -th gauge node, whose rank is $|G^{(I)}|$. We use the shorthand notation for the $\mathcal{N} = 4$ vector multiplet measure

$$\Delta_{(N)}(\vec{z}, \tau) = \frac{1}{N!} \frac{\prod_{\alpha, \beta=1}^N s_b\left(\frac{iQ}{2} - \tau + z_{\alpha} - z_{\beta}\right)}{\prod_{\beta < \alpha}^N s_b\left(\frac{iQ}{2} \pm (z_{\alpha} - z_{\beta})\right)}, \quad (2.9)$$

following the conventions of [158, 162, 167]. In particular, a chiral multiplet of R -charge r and charge $+1$ under a $U(1)_x$ symmetry contributes $s_b\left(\frac{iQ}{2}(1-r) - x\right)$, with $Q = b + b^{-1}$.

Since the two theories are mirror dual, their partition functions must agree,

$$Z(\vec{X}, \eta, \tau) = \hat{Z}(\eta, \vec{X}, \tau). \quad (2.10)$$

The parameters $\vec{X}$ are real masses for the $SU(F)$ flavor symmetry of the SQCD, satisfying $\sum_{j=1}^F X_j = 0$. From the mirror perspective, they parametrize the $F - 1$ topological symmetries. The parameter η is the real mass associated with the topological symmetry of the SQCD (or equivalently a flavor symmetry of the mirror), while τ is the real mass for the $U(1)$ flavor symmetry inside the $\mathcal{N} = 4$ R-symmetry.

Real mass deformation of the Electric Theory. At the level of the partition function, the real mass deformation is implemented by taking the limit $\tau \rightarrow +\infty$ together with suitable shifts of the gauge fugacities [21, 195]. On the SQCD side, we perform

$$\vec{u} \rightarrow \vec{u} + \frac{\tau}{2}, \quad \eta \rightarrow \eta - \frac{F}{2}\tau. \quad (2.11)$$

This selects a non-trivial Coulomb branch vacuum in which the fundamental chiral multiplets remain massless.

The limit is evaluated using the asymptotic behavior of the double sine function,

$$\lim_{\xi \rightarrow \pm\infty} s_b(x + \xi) \sim e^{\pm \frac{\pi i}{2}(x+\xi)^2}. \quad (2.12)$$

When x is a global parameter, the limit produces background Chern–Simons terms and finite phases. When x is a gauge fugacity, it generates effective Chern–Simons interactions and divergent FI terms. The redefinition of η above precisely cancels all divergent FI contributions.

As a result, the partition function factorizes as

$$Z(\vec{X}, \eta, \tau) \xrightarrow{\tau \rightarrow +\infty} e^{\frac{i\pi}{2}\Phi(\tau, \eta)} Z(\vec{X}, \eta), \quad (2.13)$$

where the oscillating phase is

$$\Phi(\tau, \eta) = -N^2\tau^2 + 2N\tau \left(\eta + \frac{iQ}{2}N \right), \quad (2.14)$$

and the resulting $\mathcal{N} = 2$ partition function is

$$\begin{aligned} Z(\vec{X}, \eta) &= e^{-\frac{N\pi i}{2} \sum_{j=1}^F X_j^2} \int \prod_{\alpha=1}^N du_{\alpha} \frac{1}{\prod_{\alpha < \beta} s_b\left(\frac{iQ}{2} \pm (u_{\alpha} - u_{\beta})\right)} \\ &\times e^{2\pi i \eta \sum_{\alpha} u_{\alpha} + \pi i (\sum_{\alpha} u_{\alpha})^2 + \pi i \left(\frac{F}{2} - N\right) \sum_{\alpha} u_{\alpha}^2} \prod_{j=1}^F s_b(X_j - u_{\alpha}). \end{aligned} \quad (2.15)$$

The oscillating phase plays a crucial role: vacua related by mirror symmetry must produce identical phases under the deformation [21, 195]. Different shifts of gauge fugacities correspond to distinct Coulomb branch vacua, and matching the phases is a necessary criterion to identify dual vacua.

Mirror Analysis. We now perform the same real mass deformation on the mirror partition function, taking $\tau \rightarrow +\infty$ with $\eta \rightarrow -\frac{F}{2}\tau + \eta$. Among the many possible shifts of the gauge fugacities $z_{\alpha}^{(I)}$, we search for those reproducing the same oscillating phase as on the SQCD side. This requirement uniquely selects the shifts

$$\begin{aligned} z_{\alpha}^{(I)} - z_{\alpha+1}^{(I)} &\rightarrow z_{\alpha}^{(I)} - z_{\alpha+1}^{(I)} - \tau, \\ z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} &\rightarrow z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} + \frac{\tau}{2}, \quad I < F - N, \\ z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} &\rightarrow z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} - \frac{\tau}{2}, \quad I \geq F - N, \\ z_1^{(N)} &\rightarrow z_1^{(N)} - \frac{F-1}{2}\tau. \end{aligned} \quad (2.16)$$

These shifts fully Higgs each gauge node to its Cartan subgroup. Most adjoint and bi-fundamental fields become massive and are integrated out, leaving precisely the $\mathcal{N} = 2$ planar quiver shown on the right-hand side of Figure 2.2.

The shifts (2.16) are not Weyl-invariant and this is essential. As noted above, they are uniquely selected by the requirement that the oscillating phase match the one produced on the SQCD side: a shift with two coincident components would yield a gauge-variant phase, and hence would not furnish a candidate mirror at all. The admissible shifts therefore have pairwise distinct components, are fixed by no non-trivial Weyl element, and fully Higgs each gauge node to its Cartan subgroup, leaving no residual non-abelian factor. This is why the planar duals are Abelian throughout.

The cancellation of the Weyl factors then proceeds as follows, and we stress that it is exact rather than a matter of discarding an overall constant. Since the integrand is Weyl-invariant, the integration over $\mathbb{R}^{|G^{(I)}|}$ may be replaced by $|G^{(I)}|!$ copies of the integration over a single Weyl chamber; the $1/|G^{(I)}|!$ of the $\mathcal{N} = 4$ measure is thereby consumed, with no approximation. Equivalently, the Weyl-related shifts correspond to distinct Coulomb branch vacua yielding identical oscillating phases, and summing over them produces exactly the compensating factor. Because the shifts are regular, the resulting chamber walls are themselves τ -dependent, and in the $\tau \rightarrow +\infty$ limit they recede to infinity: the integration domain opens back out to the whole of $\mathbb{R}^{|G^{(I)}|}$, now with no prefactor.

Finally, taking the $\tau \rightarrow +\infty$ limit yields the partition function (2.17), which coincides with the $\mathbf{S}_b^3$ partition function of the planar mirror theory, up to background terms encoded in the first line of (2.17):

$$\begin{aligned} \hat{Z}(\eta, \vec{X}) = & e^{2\pi i \eta \left(-(2N-1)\frac{iQ}{4} + \sum_{i=1}^N X_i \right)} e^{-\frac{i\pi}{2} \eta^2} \int \prod_{I=1}^{F-1} \prod_{\alpha=1}^{|G^{(I)}|} dz_{\alpha}^{(I)} e^{2\pi i z_{\alpha}^{(I)} (X_{I+1} - X_I)} e^{-i\pi (z_{\alpha}^{(I)})^2} \\ & \prod_{I=1}^{F-1} \prod_{\alpha} s_b \left(-\frac{iQ}{2} + z_{\alpha+1}^{(I)} - z_{\alpha}^{(I)} \right) e^{-i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I)}} \\ & \prod_{I=1}^{N-1} \prod_{\alpha} s_b \left(\frac{iQ}{2} + z_{\alpha}^{(I)} - z_{\alpha}^{(I+1)} \right) s_b \left(\frac{iQ}{2} - z_{\alpha}^{(I)} + z_{\alpha+1}^{(I+1)} \right) e^{i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I+1)}} \\ & \prod_{I=N}^{F-1} \prod_{\alpha} s_b \left(\frac{iQ}{2} - z_{\alpha}^{(I)} + z_{\alpha}^{(I+1)} \right) s_b \left(\frac{iQ}{2} + z_{\alpha}^{(I)} - z_{\alpha+1}^{(I+1)} \right) e^{i\pi z_{\alpha}^{(I)} z_{\alpha+1}^{(I+1)}} \\ & s_b \left(\frac{iQ}{2} - \eta + z_1^{(N)} \right) s_b \left(\frac{iQ}{2} - z_N^{(F-N)} \right) e^{i\pi z_1^{(N)} \eta} \end{aligned} \tag{2.17}$$

The View from the Equations of Motion

We now show that the flow from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ mirror dualities described in Figure 2.1 is consistent with the equations of motion of the parent $\mathcal{N} = 4$ theories. In particular, we demonstrate that the vacua corresponding to the shifts in gauge fugacities satisfy both the F-term and D-term equations [32, 146, 214]. Moreover, we show that fluctuations around these vacua reproduce the equations of motion of the resulting $\mathcal{N} = 2$ theories⁶. This provides a non-trivial consistency check of our proposal.

⁶Following the analysis of [139], the solutions to the equations of motion come in four possible families:

- **Higgs vacua**, where only the scalar components of the chiral multiplets acquire a VEV, and span a GLSM with a *flag manifold* as the target space.
- **Topological vacua**, where the real scalars in the vector multiplet acquire a VEV giving a mass to

Electric theory. We begin by considering the $\mathcal{N} = 4$ $U(N)$ SQCD with F flavors. When viewed as an $\mathcal{N} = 2$ theory, its field content consists of F fundamental and anti-fundamental chiral multiplets Q and $\tilde{Q}$, together with an adjoint chiral multiplet Φ . We also turn on a real FI parameter η .

After diagonalizing the real scalar σ in the vector multiplet, the F-term and D-term equations take the following form [32]:

$$\begin{aligned}
\sum_{i=1}^F Q_\alpha^i \tilde{Q}_i^\beta &= 0, & \sum_{\alpha=1}^N Q_\alpha^i \Phi_\beta^\alpha &= 0, & \sum_{\alpha=1}^N \Phi_\alpha^\beta \tilde{Q}_i^\alpha &= 0, \\
(\sigma_\alpha - X_i - \frac{\tau}{2}) Q_\alpha^i &= 0, & (X_i - \sigma_\alpha - \frac{\tau}{2}) \tilde{Q}_i^\alpha &= 0, & [\sigma, \Phi]_\alpha^\beta + \tau \Phi_\alpha^\beta &= 0, \\
[\Phi, \Phi^\dagger]_\alpha^\beta + \sum_{i=1}^F (Q_\alpha^{i,\dagger} Q_\beta^i - \tilde{Q}_i^\alpha \tilde{Q}_i^{\beta,\dagger}) &= \frac{\delta_\alpha^\beta}{2\pi} \left[-\eta + \frac{1}{2} \sum_{i=1}^F \left(|\sigma_\alpha - X_i - \frac{\tau}{2}| - |X_i - \sigma_\alpha - \frac{\tau}{2}| \right) \right. \\
&\quad \left. + \frac{1}{2} \sum_{\gamma=1}^N \left(|\sigma_\alpha - \sigma_\gamma + \tau| - |\sigma_\gamma - \sigma_\alpha + \tau| \right) \right].
\end{aligned} \tag{2.18}$$

Here Q_α^i , $\tilde{Q}_i^\alpha$, and Φ_β^α denote the vacuum expectation values of the scalar components of the corresponding chiral multiplets. The real scalar σ can always be diagonalized by a gauge transformation, and we write $\sigma_\alpha^\beta = \delta_\alpha^\beta \sigma_\alpha$. The real parameters X_i , τ , and η are the real masses associated with the $SU(F)$, $U(1)_\tau$, and $U(1)_\eta$ global symmetries, respectively, with the constraint $\sum_{i=1}^F X_i = 0$.

The equations in the first line of (2.18) arise from the $\mathcal{N} = 4$ superpotential

$$\mathcal{W} = \sum_{i=1}^F \sum_{\alpha,\beta=1}^N Q_\alpha^i \Phi_\beta^\alpha \tilde{Q}_i^\beta,$$

those in the second line correspond to real mass terms, and the last line is the one-loop corrected D-term equation.

Large- τ deformation and vacuum. We now introduce a real mass deformation for the axial symmetry $U(1)_\tau$ by taking τ to be large and positive. Rather than solving the vacuum equations in full generality, we focus on vacua in which the fundamental fields remain massless. To this end, we set $X_i = 0$ and take $\tau \gg 0$.

The effective mass of Q_α^i is $\sigma_\alpha - \frac{\tau}{2}$, so we choose

$$\sigma_\alpha = \frac{\tau}{2}.$$

The F-term equations then imply $\tilde{Q}_i^\alpha = \Phi_\beta^\alpha = 0$. In addition, we set $Q_\alpha^i = 0$ so that these fields do not acquire expectation values and remain light degrees of freedom.

The D-term equation reduces to

$$0 = \frac{1}{2\pi} \left(-\eta - \frac{F}{2} \tau \right) \implies \eta = -\frac{F}{2} \tau. \tag{2.19}$$

the remaining matter content. The resulting theory in the infrared is a TQFT.

- **Hybrid vacua**, which are a mixture of the above two cases. Typically, these are products of TQFTs and GLSMs on flag manifolds.
- **Strongly coupled vacua** arise only for specific values of the CS levels. These are encountered near the origin of the classical Coulomb branch once the semiclassical approximation breaks down.

Thus, the deformation involves both $U(1)_\tau$ and $U(1)_\eta$, while keeping fixed the linear combination $\frac{F}{2}U(1)_\tau + U(1)_\eta$, in agreement with the deformation discussed previously.

Fluctuations and effective $\mathcal{N} = 2$ theory. To identify the low-energy theory in this vacuum, we study small fluctuations:

$$\begin{aligned} Q_\alpha^i &= \check{Q}_\alpha^i, & X_i &= \check{X}_i, \\ \sigma_\alpha &= \frac{\tau}{2} + \check{\sigma}_\alpha, & \eta &= -\frac{F}{2}\tau + \check{\eta}. \end{aligned} \quad (2.20)$$

The equations of motion imply that $\tilde{Q}$ and Φ do not fluctuate, as expected since they are massive in this vacuum.

Expanding (2.18) in the light fields yields

$$\begin{aligned} (\check{\sigma}_\alpha - \check{X}_i)\check{Q}_\alpha^i &= 0, \\ \sum_{i=1}^F \check{Q}_\alpha^{i,\dagger} \check{Q}_i^\beta &= \frac{\delta_{\alpha\beta}}{2\pi} \left[-\underbrace{\check{\eta}}_{\eta_{\text{eff}}} + \underbrace{\left(-\frac{F}{2} + N \right) \check{\sigma}_\alpha - \sum_{\gamma=1}^N \check{\sigma}_\gamma + \frac{1}{2} \sum_{i=1}^F |\check{\sigma}_\alpha - \check{X}_i|}_{\text{CS interactions}} \right]. \end{aligned} \quad (2.21)$$

This precisely matches the equations of motion of a $U(N)$ gauge theory with F massless fundamentals and Chern–Simons levels $(-\frac{F}{2} + N, -\frac{F}{2})$, upon comparison with the generic form

$$\sum_{i=1}^F Q_\alpha^{i,\dagger} Q_i^\beta = \frac{\delta_{\alpha\beta}}{2\pi} \left[-\eta_{\text{eff}} + k_{\text{eff}} \sigma_\alpha + \ell_{\text{eff}} \sum_{\gamma=1}^N \sigma_\gamma + \frac{1}{2} \sum_{i=1}^F |\sigma_\alpha - \check{X}_i| \right], \quad (2.22)$$

where the CS levels are $(k, k + \ell N)$.

Mirror theory. We now turn to the mirror of the $\mathcal{N} = 4$ $U(N)$ SQCD, shown in the top right panel of Figure 2.1. Solving its equations of motion directly is a highly non-trivial task, due both to the large number of equations and to the existence of multiple possible vacua for fixed real mass parameters.

Instead, we exploit mirror symmetry to determine how the real mass parameters are mapped. The relevant mass-term equations are

$$\begin{aligned} (\sigma_i^{(a)} - \sigma_j^{(a+1)} + \frac{\tau}{2}) b_{i,j}^{(a,a+1)} &= 0, & (-\sigma_i^{(a)} + \sigma_j^{(a+1)} + \frac{\tau}{2}) \tilde{b}_{i,j}^{(a,a+1)} &= 0, \\ (\sigma_i^{(N)} - \eta + \frac{\tau}{2}) q_i &= 0, & (-\sigma_i^{(N)} + \eta + \frac{\tau}{2}) \tilde{q}_i &= 0, \\ (\sigma_i^{(F-N)} + \frac{\tau}{2}) p_i &= 0, & (-\sigma_i^{(F-N)} + \frac{\tau}{2}) \tilde{p}_i &= 0, \\ [\sigma^{(a)}, \Phi^{(a)}]_i^j - \tau \Phi^{(a)}_i^j &= 0. \end{aligned} \quad (2.23)$$

All σ 's are taken to be diagonal. The D-term equations read

$$[\Phi^{(a)}, \Phi^{(a),\dagger}]_{i,j} + \dots = \frac{\delta_{i,j}}{2\pi} F_i^{(a)}, \quad (2.24)$$

with $F_i^{(a)}$ defined as follows:

$$\begin{aligned}
F_i^{(a)} := & X_{a+1} - X_a + \sum_{k=1}^{a+1} \left(|\sigma_i^{(a)} - \sigma_k^{(a+1)} + \frac{\tau}{2}| - |-\sigma_i^{(a)} + \sigma_k^{(a+1)} + \frac{\tau}{2}| \right) \\
& + \sum_{k=1}^{a-1} \left(|-\sigma_k^{(a-1)} + \sigma_i^{(a)} + \frac{\tau}{2}| - |\sigma_k^{(a-1)} - \sigma_i^{(a)} + \frac{\tau}{2}| \right) \\
& + \sum_{a=1}^k \left(|\sigma_i^{(a)} - \sigma_k^{(a)} - \tau| - |-\sigma_i^{(a)} + \sigma_k^{(a)} - \tau| \right).
\end{aligned}$$

Fixing $\tau > 0$, $\eta = -\frac{F}{2}\tau$, and $X_a = 0$, one finds a solution

$$\begin{aligned}
\sigma_\alpha^{(a)} - \sigma_{\alpha+1}^{(a)} &= -\tau, \\
\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} &= \frac{\tau}{2}, \quad a < F - N, \\
\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} &= -\frac{\tau}{2}, \quad a \geq F - N, \\
\sigma_1^{(N)} &= -\frac{F-1}{2}\tau.
\end{aligned} \tag{2.25}$$

This reproduces the vacuum described in (2.16). The resulting light degrees of freedom, determined from (2.23), coincide with those obtained from the $\mathbf{S}_b^3$ partition function analysis. Expanding the equations of motion around this vacuum also reproduces the correct self and mixed Chern–Simons interactions of the planar theory in Figure 2.2. The planar superpotential arises from the surviving F-term constraints, while the monopole superpotential is not directly accessible from this analysis.

As a technical aside, we comment on the similarities and differences between the analysis based on the $\mathbf{S}_b^3$ partition function and that based on the equations of motion.

It can be verified that, under a real mass deformation parametrized by τ , the requirement that the $\mathbf{S}_b^3$ partition function factorizes into a divergent oscillating phase times a finite integral is equivalent to solving the D-term equations up to $\mathcal{O}(\tau^0)$. This correspondence arises because large shifts of the gauge fugacities in the $\mathbf{S}_b^3$ partition function precisely capture motion far along the Coulomb branch of the moduli space.

By contrast, the $\mathbf{S}_b^3$ partition function is insensitive to motion along the Higgs branch. As a consequence, solving the remaining equations of motion provides an independent and non-trivial check of the consistency of the real mass deformation considered above.

Finally, when multiple vacua are present, the analysis of the equations of motion alone does not yield a systematic prescription for identifying how vacua are mapped across the duality. In this context, the divergent phase extracted from the $\mathbf{S}_b^3$ partition function plays a crucial role: it is sensitive to the spectrum of massive fields in a given vacuum and therefore provides a powerful constraint on the correct matching of vacua under the duality.

Shifting ℓ via Witten’s $SL(2; \mathbb{Z})$ Action

In this chapter, we will primarily focus on dualities involving unitary gauge theories of the form $U(N)_{(k, k-N)}$, namely theories in which the abelian Chern–Simons level ℓ is fixed to $\ell = -1$. Nevertheless, it is often useful to relate these theories to their special-unitary counterparts. In particular, an $SU(N)_k$ Chern–Simons theory can be obtained from a $U(N)$ theory by removing the diagonal $U(1)$ factor. This is achieved by introducing a

background BF coupling between the topological symmetry $U(1)_Y$ and a new abelian symmetry $U(1)_B$, and subsequently gauging the topological symmetry.

This procedure is commonly referred to as the *ungauging* of the diagonal $U(1) \subset U(N)$. Within the conventions adopted throughout this thesis, the ungauging operation is implemented as

$$\int dY e^{2\pi i Y B N} \begin{array}{c} \textcircled{N} \\ \downarrow (k, k + N\ell) \\ \boxed{F} \vec{X} \\ \xi \end{array} = \begin{array}{c} \textcircled{\textcircled{N}} \\ \downarrow k \\ \boxed{F} \vec{X} \\ -\Delta Y \frac{N}{F} \left(\xi, \xi + F \left(\frac{Nk + N^2\ell - \xi F}{F^2} \right) \right) \end{array}, \quad \tilde{X}_a = X_a + B \quad (2.26)$$

where the double circle denotes an $SU(N)$ gauge node. In the $U(N)$ SQCD description the non-abelian flavor symmetry is $SU(F)$, and therefore the real mass parameters satisfy $\sum_{j=1}^F X_j = 0$. The parameter ξ represents a possible background Chern–Simons coupling for the flavor symmetry. After ungauging, the resulting $SU(N)$ theory exhibits a baryonic $U(1)_B$ symmetry, and the real masses obey $\sum_{j=1}^F \tilde{X}_j = FB$.

Conversely, starting from an $SU(N)$ SQCD theory, one can recover a unitary gauge group by gauging the baryonic symmetry $U(1)_B$. This inverse operation is implemented as

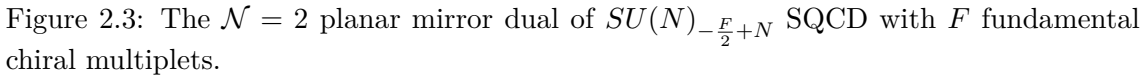
$$\int d(BN) e^{-2\pi i \eta B N} \begin{array}{c} \textcircled{N} \\ \downarrow k \\ \boxed{F} \vec{X} \\ (\xi_1, \xi_1 + F\xi_2) \\ -\Delta\eta \frac{N}{F} \end{array} = \begin{array}{c} \textcircled{N} \\ \downarrow \left(k, k + N \left(\frac{F\xi_1 + F^2\xi_2 - Nk}{N^2} \right) \right) \\ \boxed{F} \vec{X} \\ \xi_1 \end{array}, \quad \tilde{X}_a = X_a + B \quad (2.27)$$

where ξ_1 and ξ_2 denote possible background Chern–Simons levels for the $U(F)$ global symmetry of the SQCD on the right-hand side, and the term $-\Delta\eta \frac{N}{F}$ corresponds to a background FI coupling. One may verify explicitly that applying the ungauging operation (2.26) followed by the gauging operation (2.27) maps a theory back to itself, as expected.

In what follows, we will make use of the ungauging operation (2.26) to derive the $SU(N)$ version of the chiral–planar mirror duality for SQCD, starting from the unitary duality presented in Figure 2.2. On the electric side, this procedure yields $\mathcal{N} = 2$ $SU(N)_{-\frac{F}{2}+N}$ SQCD. On the mirror side, the topological symmetry of the unitary theory is realized as a flavor symmetry acting on the bottom flavor node of the planar quiver. Consequently, the ungauging operation described above translates into gauging this flavor node.

To implement this gauging consistently, it is essential to keep track of the background Chern–Simons and BF terms that are not explicitly shown in Figure 2.2, but are instead encoded in the $\mathbf{S}_b^3$ partition function (2.17). Taking these contributions into account, one arrives at the chiral–planar mirror duality depicted in Figure 2.3. This duality was first presented in [2]⁷.

⁷The newly introduced gauge node is a $U(1)_{\frac{1}{2}}$ theory with a single chiral multiplet and therefore undergoes confinement. As a result, one obtains a chiral field that is neutral under all gauge symmetries and is coupled to the theory as a flipper for a gauge-invariant monopole operator. This does not lead to a cascade of sequential confinements, except in the special case $N = 2$.



The combined effect of gauging and ungauging operations, together with the introduction of background Chern–Simons terms, realizes Witten’s $SL(2; \mathbb{Z})$ action [106] on the topological symmetry of the $U(N)$ theory.

For $\Delta\ell = 0, \pm 1$, the planar quiver in Figure 2.4 can be further simplified by explicitly integrating out the additional gauge node.

For $\Delta\ell = \pm 1$, the additional gauge node is a topological $U(1)_{\pm 1}$ sector. This theory is almost trivial, and its path integral can be performed exactly using standard results for abelian Chern–Simons theories [24, 106]. Diagrammatically, this integration corresponds

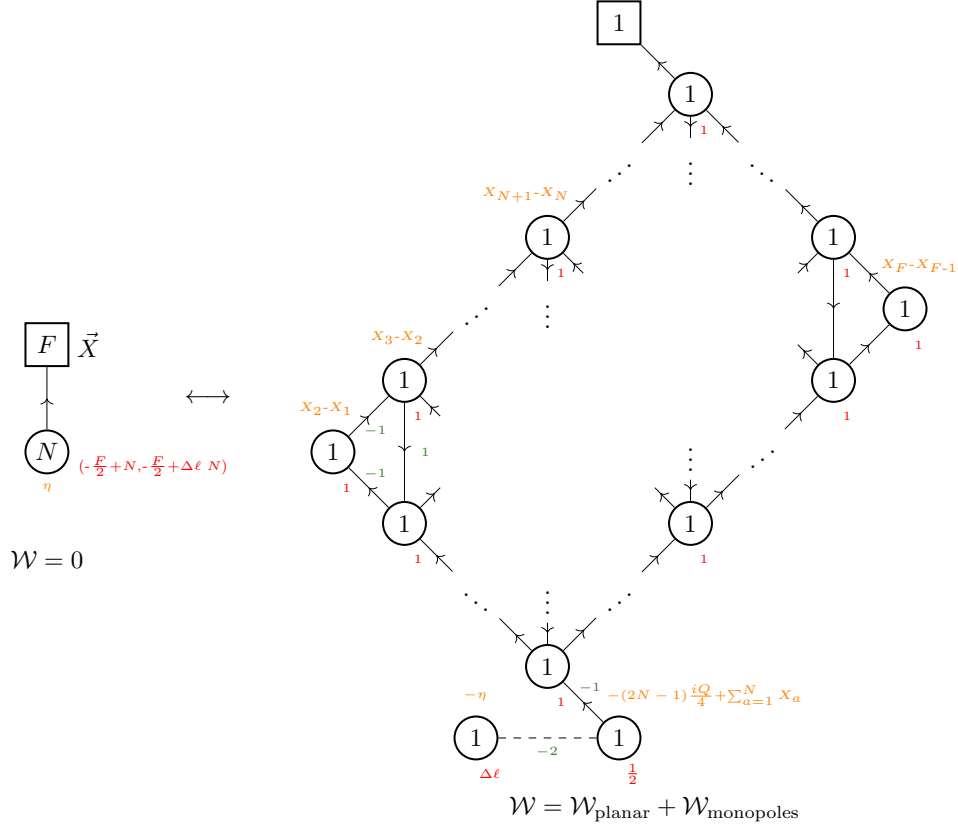


Figure 2.4: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2}+N\Delta\ell)}$ SQCD with F fundamental chiral multiplets.

to

$$\begin{array}{c} \lambda_1 \\ \textcircled{1} \\ \pm 1 \end{array} \text{---}^{k_{12}} \text{---} \begin{array}{c} \lambda_2 \\ \textcircled{1} \\ k_2 \end{array} \cdots = \begin{array}{c} \lambda_2 \mp \frac{\lambda_1 k_{12}}{2} \\ \textcircled{1} \\ k_2 \mp \frac{k_1^2}{4} \end{array} \cdots \times e^{\pm i\pi \lambda_1^2} \quad (2.28)$$

In particular, for $\Delta\ell = 1$ we obtain a planar abelian dual of $U(N)_{(k,k)}$ SQCD. Integrating out the additional $U(1)_1$ node shifts the Chern–Simons level of the adjacent gauge node by -1 . As a consequence, the bottom-most node becomes a $U(1)_{-1/2}$ gauge theory with a single fundamental chiral multiplet. This sector can be locally dualized into a free chiral field.

Planar duals of $U(N)_{(k,k)}$ will play a key role in the study of massive deformations of these theories (Chapter 3), while planar duals of $SU(N)_k$ will serve as the starting point of the analysis of non-supersymmetric dualities (Chapter 5).

2.2 Planar Abelian Duals of A_N Quiver Gauge Theories

In this section, we focus on applying our findings to more general linear quiver gauge theories. To that end, we pick the example of the $FT[U(N)]$ quiver gauge theory [218] as a starting point. This theory is closely related to the well-known $T[SU(N)]$ theory introduced in [169], differing only by the presence of an additional singlet chiral field transforming in the adjoint representation of a global $SU(N)$ symmetry and flipping the Higgs-branch moment map. The $FT[U(N)]$ theory plays a distinguished role in our analysis because it is *self-mirror*, a property that will be central to what follows. We will make these ideas more precise below.

The $FT[U(N)]$ theory is a strongly interacting SCFT with global symmetry $SU(N)_{\vec{X}} \times SU(N)_{\vec{Y}} \times U(1)_\tau$ ⁸. We denote this theory compactly:

$$\boxed{N}_{\vec{X}} \text{---} \text{---} \boxed{N}_{\vec{Y}} \quad (2.29)$$

where $\vec{X}, \vec{Y}$ are fugacities parametrizing the global symmetries and the label “—” is included for later convenience (see Section 2.3).

The operator spectrum of the $FT[U(N)]$ SCFT contains two operators, denoted $A_{\vec{X}}$ and $A_{\vec{Y}}$, transforming in the adjoint representations of $SU(N)_{\vec{X}}$ and $SU(N)_{\vec{Y}}$, respectively. These operators should be interpreted as the moment maps associated with the two non-Abelian global symmetry factors.

The theory admits a Lagrangian UV completion in the form of a linear quiver gauge theory, shown in Figure 2.5.

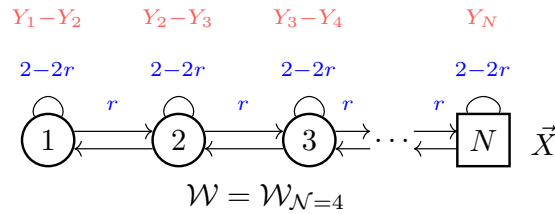


Figure 2.5: The UV completion of the $FT[U(N)]$ SCFT is shown here. Fugacities for FI terms are shown in yellow, while trial R-charges of the respective fields are highlighted in blue.

In this description, the UV global symmetry is

$$SU(N)_{\vec{X}} \times \prod_{j=1}^{N-1} U(1)_{Y_j - Y_{j+1}} \times U(1)_\tau, \quad (2.30)$$

which enhances in the infrared to the full

$$SU(N)_{\vec{X}} \times SU(N)_{\vec{Y}} \times U(1)_\tau. \quad (2.31)$$

In this UV completion, the operator $A_{\vec{X}}$ is realized perturbatively as the traceless adjoint singlet field a_N associated with the last node of the quiver.

⁸Notice that we work in an off-shell parametrization wherein the manifest symmetry is $U(N)$ and not $SU(N)$, as in [27, 28].



Figure 2.6: The $FT[U(N)]$ quiver gauge theory is self-mirror under the exchange of the manifest and topological symmetries. We also indicate the assignment of trial R-charges for the bifundamental and adjoint chiral fields. Notice that the bifundamental fields have trial R-charge r and adjoint chiral fields have trial R-charge $2 - 2r$ on both sides of the duality. This is a special feature of the $FT[U(N)]$ theory. As a consequence, the two moment maps $A_{\vec{X}}, A_{\vec{Y}}$ have both trial R-charge $2 - 2r$. Note that we have set $U(1)_\tau$ fugacity to zero in the figure for simplicity; however, the analysis can be easily extended to include the fugacity.

By contrast, the operator $A_{\vec{Y}}$ emerges non-perturbatively. It is constructed from a collection of gauge-invariant monopole operators with various GNO charges, together with the traces of the adjoint chiral fields associated with the gauge nodes. More precisely, the off-diagonal components of $A_{\vec{Y}}$ are identified with monopole operators carrying magnetic fluxes of the form $\mathfrak{M}^{+0\dots}, \mathfrak{M}^{++0\dots}, \dots, \mathfrak{M}^{0+0\dots}, \mathfrak{M}^{0++0\dots}, \dots$, together with their adjoint (with negative GNO charges), and the $N - 1$ traces $\text{Tr}(a_i)$ with $i = 1, \dots, N - 1$. For example, for $N = 4$ we have:

$$A_{\vec{Y}} = \begin{bmatrix} 0 & \mathfrak{M}^{+00} & \mathfrak{M}^{++0} & \mathfrak{M}^{+++} \\ \mathfrak{M}^{-00} & 0 & \mathfrak{M}^{0+0} & \mathfrak{M}^{0++} \\ \mathfrak{M}^{--0} & \mathfrak{M}^{0-0} & 0 & \mathfrak{M}^{00+} \\ \mathfrak{M}^{---} & \mathfrak{M}^{0--} & \mathfrak{M}^{00-} & 0 \end{bmatrix} + \sum_{i=1}^3 \text{Tra}_i \mathcal{D}_i \quad (2.32)$$

where $\mathcal{D}_i$ are the traceless diagonal generators of $SU(N)_{\vec{Y}}$.

The $FT[U(N)]$ admits a second UV completion, as shown in Figure 2.6. In this alternative description, the roles of the two $SU(N)$ global symmetries are exchanged. In each completion, only one of the two global $SU(N)$ symmetries is realized manifestly as the flavor symmetry rotating the last bifundamental. The second $SU(N)$ global symmetry instead enhances from the $N - 1$ topological symmetries due to all the nodes being balanced. The two UV completions are related by mirror duality - in this case, these theories are self-mirror since they describe the same IR fixed point.

This self-mirror property is reflected in the structure of the $\mathbf{S}_b^3$ partition function. The partition function of the first UV completion can be defined recursively as:

$$Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) := e^{2\pi i Y_N \sum_j^N X_j} \prod_{j,k=1}^N s_b \left(\tau - \frac{iQ}{2} (1 - 2r) + X_j - X_k \right) \\ \int \frac{\prod_{\alpha=1}^{N-1} du_\alpha e^{-2\pi i Y_N u_\alpha}}{(N-1)! \prod_{\alpha < \beta} s_b \left(\frac{iQ}{2} \pm (u_\alpha - u_\beta) \right)} s_b \left(\frac{iQ}{2} (1 - r) - \frac{\tau}{2} \pm (u_\alpha - X_j) \right) \\ Z_{FT[U(N-1)],I}(\vec{u}, \{Y_i\}_{i=1}^{N-1}, \tau). \quad (2.33)$$

with the base of recursion:

$$Z_{FT[U(1)],I}(X, Y, \tau) := e^{2\pi i XY} s_b \left(\tau - \frac{iQ}{2} (1 - 2r) \right). \quad (2.34)$$

Since both UV descriptions flow to the same SCFT, their partition functions must coincide. We therefore define

$$Z_{FT[U(N)]}(\vec{X}, \vec{Y}, \tau) := Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) = Z_{FT[U(N)],II}(\vec{Y}, \vec{X}, \tau). \quad (2.35)$$

In the first UV completion, the first slot corresponds to the manifest $U(N)$ symmetry, and the second to the emergent one, while in the second UV completion, this assignment is reversed.

Finally, the partition function satisfies the identity

$$Z_{FT[U(N)]}(\vec{X}, \vec{Y}, \tau) = Z_{FT[U(N)]}(-\vec{X}, -\vec{Y}, \tau), \quad (2.36)$$

which follows from a redefinition of the integration variables.

In the remainder of this work, we will treat the global symmetry as $U(N)_{\vec{X}} \times U(N)_{\vec{Y}}$ rather than strictly imposing the tracelessness constraints $\sum_i X_i = \sum_i Y_i = 0$. In practice, this corresponds to relaxing these constraints in the $\mathbf{S}_b^3$ partition function and does not affect any of the physical conclusions.

The Flow to $\mathcal{N} = 2$: The $G[U(N)]$ Theory

Starting from the self-mirror pair relating the two UV completions of the $FT[U(N)]$ SCFT in Figure 2.6, we will now consider a real mass deformation that produces a new duality relating a chiral quiver to a planar quiver. The resulting RG flow is depicted in Figure 2.7.

As in the previous section, the real mass deformation on the electric side is chosen to flow to a chiral, non-Abelian quiver gauge theory. On the mirror side, matching the same vacuum requires moving onto a point of the Coulomb branch where all gauge symmetries are broken to their maximal torus. As a result, the IR description on the dual side is given by a planar Abelian quiver.

The two theories appearing at the bottom of Figure 2.7 are IR dual to one another and can be regarded as two distinct UV completions of a single interacting SCFT, which we denote by $G[U(N)]$ ⁹. This theory has global symmetry $SU(N) \times U(1)^{N-1}$ and will play an important role in the subsequent analysis. We represent it compactly as

$$G[\vec{X}, \vec{Y}] = \boxed{N}_{\vec{X}} \text{-----} \boxed{1^N}_{\vec{Y}} \quad (2.37)$$

We emphasize that, in our notation, the first argument of $G[\cdot; \cdot]$ corresponds to fugacities for the $SU(N)$ global symmetry, while the second argument parametrizes the $U(1)^{N-1}$ global symmetry.

We now describe this deformation in detail, starting from the first UV completion of the $FT[U(N)]$ theory, which we refer to as the electric description, and in which the $U(N)_{\vec{X}}$ global symmetry is manifest using the machinery of the $\mathbf{S}_b^3$ partition function.

We implement a real mass deformation by introducing a large background expectation value for the scalar in the $U(1)_{\tau}$ vector multiplet. At the level of fugacities in the $\mathbf{S}_b^3$

⁹Named so after Gabriel Pedde Ungureanu – at the time, the author did not realise how important these theories were and how much confusion could be avoided if he had just called them an $\mathcal{S}$ -wall like their $\mathcal{N} = 4$ counterparts.

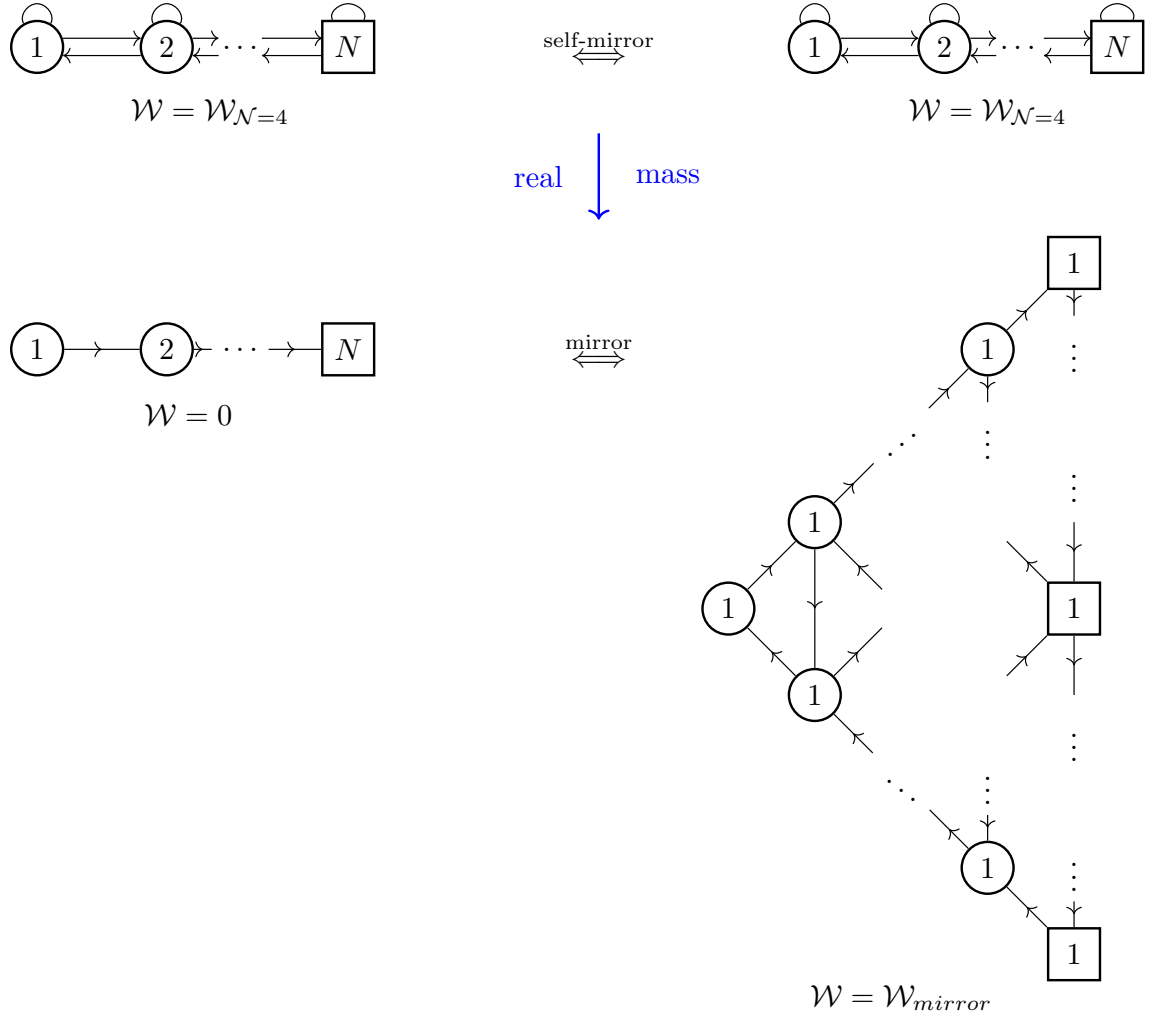


Figure 2.7: Starting from the two UV completions of self-mirror $FT[U(N)]$ quiver gauge theory (top row), we study the effect of adding a real mass deformation that breaks the $U(1)_\tau$ global symmetry. The resulting duality relates a chiral, non-Abelian, linear quiver gauge theory to a planar, Abelian quiver gauge theory. The details of the global symmetries and Chern-Simons interactions are expanded upon in the text.

partition function, this deformation is realized as

$$\begin{cases} Y_J \longrightarrow Y_J + \frac{\tau}{2}(2J - N - 1), & J = 1, \dots, N, \\ \vec{X} \longrightarrow \vec{X} + \frac{\tau}{2}, \end{cases} \quad \tau \longrightarrow +\infty, \quad (2.38)$$

which is paired with the following shifts in gauge fugacities:

$$\vec{u}^{(J)} \rightarrow \vec{u}^{(J)} - \frac{\tau}{2}(N - 1 - J). \quad (2.39)$$

where $\vec{u}^{(J)}$ denotes the fugacities associated with the J -th gauge node of the quiver, ordered from left to right.

After integrating out the massive fields generated by this deformation, the theory flows in the infrared to an $\mathcal{N} = 2$ chiral quiver gauge theory shown in Figure 2.8.

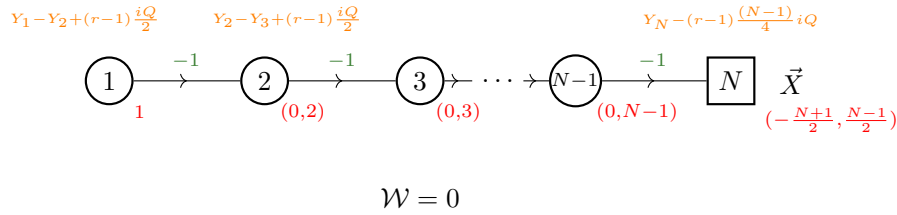


Figure 2.8: The chiral UV completion of the $G[U(N)]$ theory is shown here. CS and mixed CS interactions are highlighted in red and green, respectively, while FI terms (including the mixing of the topological and $U(1)_R$ symmetries) are shown in yellow. The UV global symmetries of this theory are $SU(N)_{\vec{X}} \times U(1)_{\vec{Y}}^{N-1}$.

The $\mathbf{S}_b^3$ partition function in the large mass limit $\tau \rightarrow +\infty$ limit is given by:

$$\lim_{\tau \rightarrow +\infty} Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau \rightarrow +\infty} e^{\frac{\pi i}{2} H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau)} Z_{G[U(N)],I}(\vec{X}, \vec{Y}) \quad (2.40)$$

where $Z_{G[U(N)],I}(\vec{X}, \vec{Y})$ is the partition function of the first UV completion of the $G[U(N)]$ theory in Figure 2.8 and:

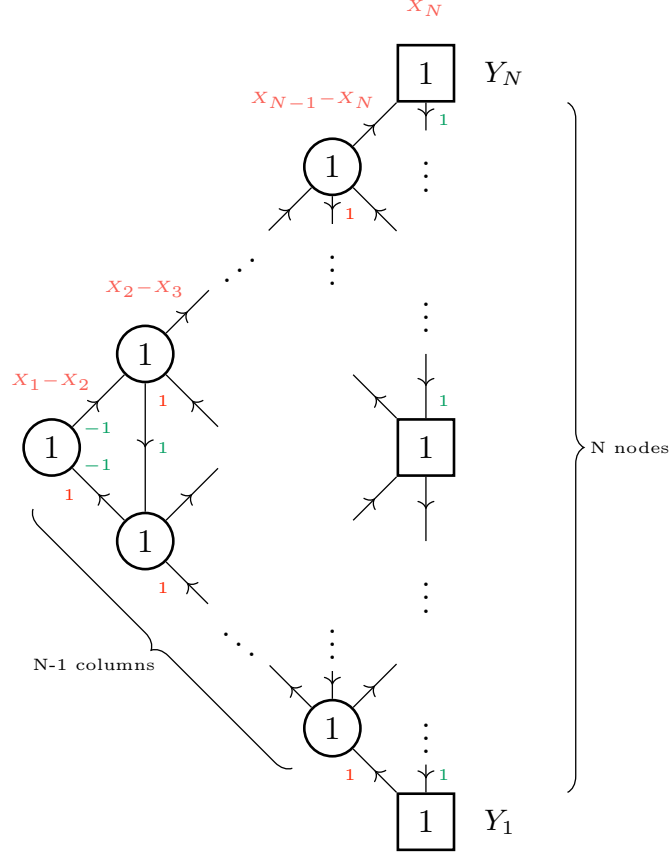
$$H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) = H_{\text{div}}^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) + H_{\text{res}}^{\text{chiral}}, \quad (2.41)$$

$$\begin{aligned} H_{\text{div}}^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) = & \frac{\tau^2}{4} \left(\frac{4}{3} N^3 + 2N^2 + \frac{2}{3} N \right) \\ & + \frac{\tau}{2} \left(8 \sum_{J=1}^N J Y_J - 4 \sum_{J=1}^N Y_J - iQ(N + N^2) + iQr \left(\frac{2}{3} N^3 + 2N^2 + \frac{4}{3} N \right) \right), \end{aligned} \quad (2.42)$$

and $H_{\text{res}}^{\text{chiral}}$ is a finite term independent of the global parameters $\vec{X}$ and $\vec{Y}$, and only depending on the squashing parameter Q .

We now map the effects of this deformation to the second UV completion of the $FT[U(N)]$ theory.

As in the case of the planar mirror of the chiral SQCD discussed in the previous section, in addition to the real mass deformation (2.38), flowing to the vacuum dual to the chiral



$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopole}$$

Figure 2.9: The planar UV completion of the $G[U(N)]$ theory is shown here. CS and mixed CS interactions are highlighted in red and green, respectively, while the FI terms are shown in yellow. The UV global symmetries of this theory are $U(1)_{\vec{Y}}^{N-1} \times SU(N)_{\vec{X}}$.

quiver requires an appropriate choice of Coulomb branch vacuum on the mirror side. This is achieved by performing the following shifts of the gauge fugacities:

$$\omega_{\alpha}^{(J)} \longrightarrow \omega_{\alpha}^{(J)} + \tau \left(\alpha - \frac{J+1}{2} \right), \quad (2.43)$$

and subsequently taking the limit $\tau \rightarrow +\infty$.

Here $\omega_{\alpha}^{(J)}$ denotes the α -th Cartan fugacity of the J -th gauge group in the quiver, ordered from left to right. This deformation completely breaks the non-Abelian gauge symmetry to its maximal torus, allowing the massive fields to be consistently integrated out.

After integrating out the massive degrees of freedom, the theory flows in the infrared to an $\mathcal{N} = 2$ planar Abelian quiver gauge theory, shown in Figure 2.9. This planar quiver is IR dual to the chiral quiver depicted in Figure 2.8, and together they furnish two distinct UV completions of the same interacting SCFT, which we denote by $G[U(N)]$.

A few comments regarding the structure of the superpotential and Chern–Simons interactions of the planar quiver in Figure 2.9 are in order:

- For every closed triangular cycle in the planar quiver, there is a cubic superpotential term. These interactions descend directly from the cubic $\mathcal{N} = 4$ superpotential terms that are preserved by the real mass deformation. We collectively denote these contributions by $\mathcal{W}_{\text{planar}}$.

The assignment of trial R-charges is as follows: diagonal chiral fields are assigned R-charge r , while vertical chiral fields are assigned R-charge $2 - 2r$. This assignment is consistent with the cubic planar superpotential. Moreover, the FI parameters depend explicitly on the value of r , reflecting a mixing between the topological symmetries and the R-symmetry.

- Due to the adjoint Higgsing of the gauge nodes, monopole operators appear in the effective description. In particular, there is a linear monopole superpotential term for each monopole carrying charges $+1$ and -1 under the upper and lower gauge nodes connected by a vertical edge [215]. We denote these contributions collectively by $\mathcal{W}_{\text{monopole}}$.

These monopole superpotential terms play a crucial role in the infrared dynamics: they are essential for the enhancement of the topological $U(1)^{N-1}$ symmetry to the non-Abelian $SU(N)$ global symmetry, as will be discussed later.

- The planar theory also includes both self and mixed Chern-Simons interactions. Each gauge node carries a Chern-Simons level $+1$. In addition, for each pair of gauge nodes connected by a diagonal (respectively vertical) chiral multiplet, there is a mixed Chern-Simons interaction with level -1 (respectively $+1$). All such Chern-Simons terms arise from integrating out fermionic modes that become massive along the real mass deformation.
- Finally, there are shifts in the FI parameters induced by the mixing between the topological symmetries of the gauge nodes and the R-symmetry. The FI parameters for the gauge nodes in the J -th column from the right take the form

$$\text{FIs: } X_J - X_{J+1} - \frac{iQr}{4}, X_J - X_{J+1}, \dots, X_J - X_{J+1} + \frac{iQr}{4}, \quad (2.44)$$

ordered from top to bottom.

Similarly, the background FI parameters associated to the j -th flavor node labeled by Y_j are given by

$$\text{FIs: } X_N + \frac{iQr}{2}(2j - N - 1) + \delta_{j,1} \frac{iQ}{2} - \delta_{j,N} \frac{iQ}{2}, \quad (2.45)$$

where $j = 1$ corresponds to the top flavor node in Figure 2.9, and $j = N$ to the bottom one. In Figure 2.9 we have suppressed these shifts, displaying only the contributions proportional to the parameters X_i .

The $\mathbf{S}_b^3$ partition function in the $\tau \rightarrow +\infty$ limit takes the form

$$\lim_{\tau \rightarrow +\infty} Z_{FT[U(N)],II}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau \rightarrow +\infty} e^{\frac{\pi i}{2} H^{\text{planar}}(\vec{X}, \vec{Y}, \tau)} Z_{G[U(N)],II}(\vec{X}, \vec{Y}), \quad (2.46)$$

where $Z_{G[U(N)],II}(\vec{X}, \vec{Y})$ denotes the $\mathbf{S}_b^3$ partition function of the second UV completion of the $G[U(N)]$ theory shown in Figure 2.9. The phase H^{planar} admits the decomposition

$$H^{\text{planar}}(\vec{X}, \vec{Y}, \tau) = H_{\text{div}}^{\text{planar}}(\vec{X}, \vec{Y}, \tau) + H_{\text{res}}^{\text{planar}}, \quad (2.47)$$

where $H_{\text{res}}^{\text{planar}}$ is a constant phase independent of the global parameters $\vec{X}$ and $\vec{Y}$, depending only on the squashing parameter Q .

The divergent contribution $H_{\text{div}}^{\text{planar}}$ exactly matches the corresponding divergent phase $H_{\text{div}}^{\text{chiral}}$ arising on the chiral side. This provides a strong consistency check of the proposed duality between the theories in Figure 2.9 and Figure 2.8. Moreover, we find that the residual phases also agree,

$$H_{\text{res}}^{\text{planar}} = H_{\text{res}}^{\text{chiral}}. \quad (2.48)$$

As a consequence, after removing the universal phase factor in the $\tau \rightarrow +\infty$ limit, we conclude that the partition functions of the two UV completions coincide:

$$Z_{G[U(N)],I}(\vec{X},\vec{Y}) = Z_{G[U(N)],II}(\vec{X},\vec{Y}). \quad (2.49)$$

Mapping Operators and Global Symmetries

We now comment on the duality map between global symmetries and the chiral rings of the two theories.

Let us begin with the *global symmetries*. As stated at the beginning of this section, the $G[U(N)]$ SCFT has a $U(1)^{N-1} \times SU(N)$ global symmetry. In the chiral UV completion (Figure 2.8), both factors of this symmetry are realized manifestly in the ultraviolet: the $SU(N)$ symmetry appears as a flavor symmetry acting on the matter fields, while the $U(1)^{N-1}$ symmetries correspond to the topological symmetries associated with the $N-1$ gauge nodes.

In contrast, in the planar UV completion shown in Figure 2.9, the roles of these symmetries are exchanged. The $U(1)^{N-1}$ symmetry is realized as a flavor symmetry that is preserved by the planar superpotential, up to gauge transformations. The $SU(N)$ global symmetry, on the other hand, emerges in the infrared from the enhancement of the topological symmetries associated with the gauge nodes.

We recall that the presence of the monopole superpotential $\mathcal{W}_{\text{monopole}}$ plays a crucial role in this enhancement mechanism. In particular, $\mathcal{W}_{\text{monopole}}$ breaks the $U(1)^k$ topological symmetries associated with a column containing k gauge nodes down to a single diagonal $U(1)$. As a result, each column of gauge nodes effectively contributes one independent topological $U(1)$ symmetry. The enhancement of these remaining topological symmetries in the infrared gives rise to the non-Abelian $SU(N)$ global symmetry of the $G[U(N)]$ SCFT.

Let us now move to the *operator map*. The chiral ring of the planar $G[U(N)]$ theory shown in Figure 2.9 is generated by the $N-1$ vertical bifundamental chiral fields connecting adjacent vertical flavor nodes. Up to F-term relations imposed by the planar superpotential $\mathcal{W}_{\text{planar}}$, these fields can be taken as the only independent generators of the chiral ring.

We expect these operators to map, under the duality, to monopole operators in the chiral $G[U(N)]$ theory in Figure 2.8. This expectation is correct, with the important caveat that the fundamental monopole operators are gauge variant and must be appropriately dressed by powers of fundamental chiral multiplets in order to construct gauge-invariant operators.

To illustrate this, let us focus on a local segment of the chiral $G[U(N)]$ quiver,

$$\begin{array}{ccccccc} \rightarrow & \textcircled{k-1} & \xrightarrow{-1} & \textcircled{k} & \xrightarrow{-1} & \textcircled{k+1} & \rightarrow \\ & (0, k-1) & Q & (0, k) & P & (0, k+1) & \end{array} \quad (2.50)$$

where the k -th gauge node is a $U(k)$ gauge group. The fundamental monopole operator associated with this node carries non-trivial gauge charge and therefore must be dressed by $(k-1)$ powers of the bifundamental field Q in order to form a gauge-invariant operator.

In this way, one constructs $N-1$ dressed fundamental monopole operators, one for each gauge node of the chiral $G[U(N)]$ theory. These dressed monopoles generate the chiral ring of the theory.

We now verify that these operators have the correct quantum numbers to be mapped to the $N-1$ vertical bifundamental chiral fields connecting adjacent vertical flavor nodes in the planar mirror theory. This identification can also be independently checked using the superconformal index.

Assume that each bifundamental chiral multiplet in the chiral theory (such as Q , P , and so on) has trial R-charge r . Consequently, the vertical bifundamental chiral fields in the planar mirror theory, which are mapped to these monopole operators, have R-charge $2-2r$.

Using the general monopole R-charge formula (1.33), the R-charge of the dressed monopole operator associated with the k -th gauge node,

$$\text{tr}\left(Q^{k-1} \mathfrak{M}^{0,\dots,0,-,0,\dots,0}\right),$$

is given by

$$\begin{aligned} R\left(Q^{k-1} \mathfrak{M}^{0,\dots,0,-,0,\dots,0}\right) &= -(r-1) + \frac{1}{2}(k-1)(1-r) + \frac{1}{2}(k+1)(1-r) \\ &\quad - (k-1) + (k-1)r \\ &= (1-r) + k(1-r) - (k-1) + (k-1)r \\ &= 2 - 2r. \end{aligned} \tag{2.51}$$

The first term arises from the mixing between the topological symmetry and the R-symmetry¹⁰. The second and third terms correspond to the contributions of the bifundamental fermions, the term $-(k-1)$ arises from the gaugini, and the final term $(k-1)r$ accounts for the dressing by Q^{k-1} .

We thus conclude that the $N-1$ dressed fundamental monopole operators of the chiral $G[U(N)]$ theory map precisely to the $N-1$ vertical bifundamental chiral fields connecting adjacent vertical flavor nodes in the planar mirror theory.

In the remainder of this chapter, we denote these gauge-invariant dressed monopole operators collectively as $\mathfrak{M}_d$, omitting the explicit dressing fields for notational simplicity.

Thus, we see that longer quiver theories also admit planar mirror dual descriptions.

2.3 Towards an Algorithmic Approach

It is clear that we can derive a duality between $\mathcal{N} = 2$ $U(N)$ SQCD with $[F, 0]$ chiral fields and a planar Abelian quiver gauge theory by adding suitably chosen real masses to the $\mathcal{N} = 4$ parent theories. Notice that the $\mathcal{N} = 2$ theory inherits the $SU(F)$ flavor symmetry of the $\mathcal{N} = 4$ theories. In a similar vein, we can also consider a slight variation on this

¹⁰This mixing can be read off from the coefficient of $\frac{iQ}{2}$ in the FI parameter of the k -th gauge node, $Y_k - Y_{k+1} + \frac{iQ}{2}(r-1)$. The contribution to the R-charge is obtained by multiplying this coefficient by the monopole's topological charge.

such that the deformation now partially breaks the $SU(F)$ global symmetry and following the RG flow results in $U(N)$ SQCD with n_f fundamental and n_a anti-fundamental chiral multiplets on the electric side:

$$\begin{array}{ccc} \mathcal{N} = 4 & U(N) & \\ F \text{ hypermultiplets} & \rightarrow & \mathcal{N} = 2 \quad U(N)_{-\frac{F}{2}+N, -\frac{F}{2}} \\ & & [n_f, n_a] \text{ chiral fields, } F = n_f + n_a \end{array} \quad (2.52)$$

The number of (anti-) fundamentals n_f and n_a satisfy $F = n_f + n_a$ but are otherwise unconstrained and depend on the specific mass deformation that we consider. On the other hand, the choice of CS levels **is not arbitrary** and is fixed by the parent $\mathcal{N} = 4$ theory: each massive (anti)fundamental multiplet contributes as $(-\frac{1}{2}, -\frac{1}{2})$ to the CS level and the massive adjoint chiral contributes as $(N, 0)$. In particular, we always have $\ell = -1$. These features persist in all non-Abelian linear quiver theories analyzed in this chapter.

One can then consider the same limit on the mirror side and try to identify the dual vacuum. In general, this is a non-trivial exercise as there are multiple potential vacua corresponding to possible ways of moving on the Coulomb branch of the $\mathcal{N} = 4$ mirror (which, in turn, match how we shift gauge fugacities in the $\mathbf{S}_b^3$ partition function).

More generally, one can start on the electric side with a linear unitary $\mathcal{N} = 4$ quiver with gauge group $\prod_{N_i} U(N_i)$. We consider a real mass deformation, paired with a VEV for the scalar in the $\mathcal{N} = 2$ vector multiplets, such that the adjoint chiral fields and one $\mathcal{N} = 2$ chiral multiplet in each matter hypermultiplet acquire a positive real mass. Integrating out the massive field we obtain a chiral linear quiver:

$$(2.53)$$

where the CS levels k_i are fixed by the parent theory:

$$k_i = -\frac{N_{i-1} + N_{i+1} + F_i}{2} + N_i \quad (2.54)$$

As before, one can consider the mirror dual quiver theory and implement the same limit to reach the dual vacuum. This remains a non-trivial exercise.

Starting from the equality of the partition functions of pairs of $\mathcal{N} = 4$ mirror theories is a hugely successful, albeit tedious, paradigm to find $\mathcal{N} = 2$ mirror duals. Clearly, it is desirable to have a more systematic way to determine the mirror dual of a generic flavored

$\mathcal{N} = 2$ quiver without the need to resort to $\mathbf{S}_b^3$ partition functions. We do so by adapting the idea of the local dualization algorithm of [206, 213] in this section.

2.3.1 Lightning Recap: The $\mathcal{N} = 4$ Dualization Algorithm

The dualization algorithm for $\mathcal{N} = 4$ linear quiver theories provides a purely field-theoretical proof of mirror duality, assuming only basic Seiberg-like dualities, and allows us to access the full map of symmetries and protected gauge invariant operators.

It is well known that $\mathcal{N} = 4$ linear quiver theories can be realized on Hanany-Witten brane set-ups and inherit mirror dualities from the $\mathcal{S}$ -duality of Type IIB set-ups [19]. The idea of the algorithm originates from the observation [31, 219, 220] that on linear or circular brane setups, $\mathcal{S}$ -duality can act locally on each 5-brane creating an $\mathcal{S}$ -duality wall on its right and an $\mathcal{S}^{-1}$ -duality wall on its left: $D5 = \mathcal{S} \cdot NS5 \cdot \mathcal{S}^{-1}$ and $\overline{NS5} = \mathcal{S} \cdot D5 \cdot \mathcal{S}^{-1}$. The dualization algorithm implements this local action of $\mathcal{S}$ -duality in field theory.

The dualization algorithm is based on two fundamental ingredients:

1. a realization of the generator $\mathcal{S}$ of the duality group $SL(2; \mathbb{Z})$ that satisfies the fusion to identity $\mathcal{S}\mathcal{S}^{-1} = \mathbb{I}$. For $\mathcal{N} = 4$ theories, this can be defined from the $FT[U(N)]$ SCFT [213]:

$$\begin{aligned} \mathcal{S}\text{-wall} : FT[\vec{X}, -\vec{Y}] &= \boxed{N}_{\vec{X}} \cdots \overset{+}{\tau} \cdots \boxed{N}_{\vec{Y}} \\ \mathcal{S}^{-1}\text{-wall} : FT[\vec{X}, \vec{Y}] &= \boxed{N}_{\vec{X}} \cdots \overset{-}{\tau} \cdots \boxed{N}_{\vec{Y}} \end{aligned} \quad (2.55)$$

As discussed in Section 2.2, the $FT[U(N)]$ theory admits a Lagrangian UV completion (Figure 2.6) and is self-mirror under the exchange of emergent and manifest $SU(N)$ symmetries. The $\mathcal{S}$ -wall also satisfies the fusion to identity encoded in a partition function identity:

$$\int d\vec{W} \Delta_{(N)}(\vec{W}; \tau) Z_{\mathcal{S}}^{(N)}(\vec{W}; \vec{X}; \tau) Z_{\mathcal{S}^{-1}}^{(N)}(\vec{W}; \vec{Y}; \tau) =_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau) \quad (2.56)$$

This identity can pictorially be represented as follows:

$$\boxed{N}_{\vec{X}} \cdots \overset{+}{\tau} \cdots \boxed{N}_{\vec{W}} \cdots \overset{-}{\tau} \cdots \boxed{N}_{\vec{Y}} \longleftrightarrow {}_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau) \quad (2.57)$$

and the identity wall is defined below:

$${}_{\vec{X}} \mathbb{I}_{\vec{Y}}(\tau) = \frac{1}{N! \Delta_{(N)}(\vec{X}, \tau)} \sum_{\sigma \in S_N} \prod_{j=1}^N \delta(X_j - Y_{\sigma(j)}), \quad (2.58)$$

with

$$\Delta_{(N)}(\vec{z}, \tau) = \frac{1}{N!} \frac{\prod_{\alpha, \beta=1}^N s_b(\frac{iQ}{2} - \tau + z_{\alpha} - z_{\beta})}{\prod_{\beta < \alpha}^N s_b(\frac{iQ}{2} \pm (z_{\alpha} - z_{\beta}))} \quad (2.59)$$

2. inspired by HW brane setups, we identify the following basic QFT blocks associated to the different 5-brane configurations:

$$\begin{array}{c} \boxed{1} \\ \updownarrow \\ \boxed{N} \end{array} \quad \begin{array}{c} \text{N D3} \quad \text{D5} \quad \text{N D3} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \end{array} \otimes \quad \begin{array}{c} \text{N D3} \quad \text{NS5} \quad \text{N D3} \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \end{array} \quad \begin{array}{c} \boxed{N} \quad \boxed{N} \\ \vec{X} \quad \vec{Y} \end{array} \quad (2.60)$$

which are denoted the flavor block and the bifundamental block respectively.

We can now define the action of the duality walls on the basic QFT blocks:

1. The first basic move consists of the $\mathcal{S}$ -dualization of a flavor matter block into a bifundamental block (which is the field-theoretic analog of $\overline{NS5} = \mathcal{S} \cdot D5 \cdot \mathcal{S}^{-1}$):

$$\begin{array}{c} \boxed{1}^\eta \\ \updownarrow^{\frac{\tau}{2}} \\ \boxed{N} \end{array} \quad \begin{array}{c} + \\ - \end{array} \quad \begin{array}{c} \boxed{N} \quad \boxed{N} \\ \vec{X} \quad \vec{Y} \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \boxed{N} \quad \boxed{N} \\ \vec{X} \quad \vec{Y} \end{array} \quad (2.61)$$

$\mathcal{W} = a(A_L + A_R) + \tilde{f}(A_L - A_R)f$
 $\mathcal{W} = 0$

wherein we have explicitly reported the $U(1)_\tau$ charge of the various fields in the theory. This corresponds to the partition function identity:

$$\begin{aligned} \int d\vec{W} \Delta_{(N)}(\vec{W}, \tau) Z_S^{(N)}(\vec{W}, \vec{X}, \tau) Z_{S^{-1}}^{(N)}(\vec{W}, \vec{Y}, \tau) \prod_{\alpha=1}^N s_b \left(\frac{iQ}{2} - \frac{\tau}{2} \pm (W_\alpha - \eta) \right) \\ = e^{-2\pi i \eta \sum_{j=1}^N (X_j - Y_j)} \prod_{j,k=1}^N s_b \left(\frac{\tau}{2} \pm (X_j - Y_k) \right). \end{aligned} \quad (2.62)$$

where the integration measure $\Delta_{(N)}(\vec{W}, \tau)$ is defined in Equation (2.59).

2. The inverse basic move consists of the $\mathcal{S}$ -dualization of a fundamental block into a bifundamental block (which is the field-theoretic counterpart of $\mathcal{S} \cdot NS5 \cdot \mathcal{S}^{-1} = D5$):

$$\begin{array}{c} \boxed{N} \quad \boxed{N} \\ \vec{X} \quad \vec{Y} \end{array} \quad \begin{array}{c} + \\ - \end{array} \quad \begin{array}{c} \boxed{N} \quad \boxed{N} \\ \vec{X} \quad \vec{Y} \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \boxed{1}^\eta \\ \updownarrow^{1-\frac{\tau}{2}} \\ \boxed{N} \end{array} \quad \begin{array}{c} \vec{X} \quad \vec{Y} \end{array} \quad (2.63)$$

This inverse basic move is encoded in the following partition function identity:

$$\begin{aligned} \prod_{j=1}^N s_b \left(\frac{\tau}{2} \pm (X_j - \eta) \right) \vec{X} \vec{Y}(\tau) = \\ = \int d\vec{U} d\vec{V} \frac{e^{2\pi i \eta \sum_{a=1}^N (U_a - V_a)}}{\prod_{j,k=1}^N s_b \left(\frac{iQ}{2} \pm (U_j - U_k) \right) s_b \left(\frac{iQ}{2} \pm (V_j - V_k) \right)} \\ Z_{\mathcal{S}_{N=4}}^{(N)}(\vec{X}, \vec{U}; \tau) Z_{\mathcal{S}_{N=4}^{-1}}^{(N)}(\vec{V}, \vec{Y}; \tau) \prod_{a,b=1}^N s_b \left(\frac{iQ}{2} - \frac{\tau}{2} \pm (U_a - V_b) \right). \end{aligned} \quad (2.64)$$

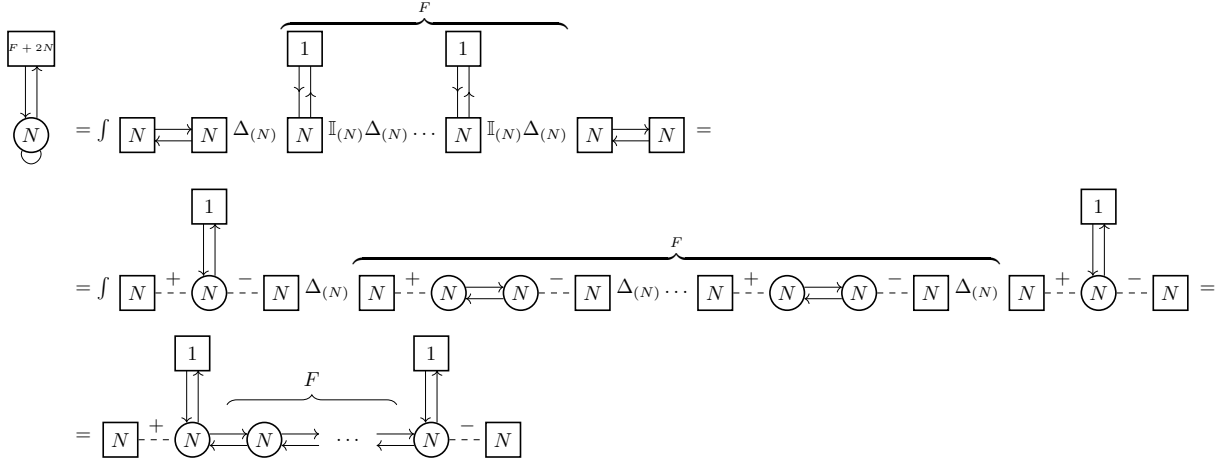


Figure 2.10: Applying the $\mathcal{N} = 4$ dualization to $U(N)$ SQCD with $[F + 2N, F + 2N]$ flavors proceeds as follows - in the first line, the quiver is broken into its constituent QFT blocks and each block is sandwiched between an $\mathcal{S}^{-1}$ and $\mathcal{S}$ -wall. We have explicitly kept the integral and integrating measures to remind the reader that care must be taken when the gauge nodes are ungauged to implement the algorithm. The resulting QFT blocks obtained after the local action are written in the second line, and are glued back together by exploiting the fusion to identity property of the $\mathcal{S}$ -wall to obtain the final theory in the third line. At this stage, an explicit Lagrangian description may be written using the UV completion of the $FT[U(N)]$ theory, which is indeed the mirror description one would get by utilizing conventional HW brane setups. Note that we would need to incorporate HW transitions to dualize more general quiver theories.

We point out that this move is slightly different than naively sandwiching the previous move between an $\mathcal{S}^{-1}$ and $\mathcal{S}$ wall. Observe that the gluing on the RHS **does not** include the addition of $U(N)$ adjoint multiplets coupling linearly to the moment map of the glued $FT[U(N)]$ theory.

We consider the case of $\mathcal{N} = 4$ $U(N)$ SQCD with $F + 2N$ flavors to demonstrate the algorithm at work (Figure 2.10).

When decomposing the theory into QFT blocks, it is important to keep in mind that each block effectively plays the role of a matrix element, with two sets of indices corresponding to the two N -dimensional vectors of parameters associated with the two $U(N)$ global symmetries. In particular, note that flavor blocks are accompanied by an identity wall, reflecting this matrix structure.

It was shown that, armed with the fundamental blocks and duality moves described above, one can construct the mirror of a vast set of $\mathcal{N} = 4$ Lagrangian theories [206, 213, 221]. In [27], the local dualization algorithm paradigm has been extended to a class of generalized $\mathcal{N} = 2$ quiver theories that can be realized in set-ups involving $NS, NS', D5, D5$ branes preserving four supercharges. Unlike the $\mathcal{N} = 4$ case, where brane setups directly inform the structure of the low-energy field theory, here the situation is reversed: the algorithmic field theory approach yields an exact dual description, while the corresponding brane configurations are difficult to interpret directly. This inversion of logic highlights the power of the algorithmic method — not only does it provide precise duals, but it also offers a new lens through which to understand and possibly reconstruct the brane setups themselves. Motivated by this, our aim is now to develop a similar algorithm tailored to chiral-planar $\mathcal{N} = 2$ theories in the remainder of this section.

2.3.2 The $\mathcal{N} = 2$ Toolbox

We now extend the $\mathcal{N} = 4$ framework to $\mathcal{N} = 2$ by considering appropriate real mass deformations of the $\mathcal{N} = 4$ basic blocks and corresponding identities here.

The $\mathcal{S}$ -wall, Fusion to Identity and Gluing Rules

In Section 2.2, we consider the chiral-planar limits of the $FT[U(N)]$ theory (which is identified as the $\mathcal{N} = 4$ $\mathcal{S}$ -wall). We identify the resulting $\mathcal{N} = 2$ chiral-planar mirror pair as the $\mathcal{N} = 2$ $\mathcal{S}$ -wall:

$$\begin{aligned} \mathcal{S}\text{-wall : } \mathcal{S}[\vec{X}, \vec{Y}] &= G[-\vec{X}, \vec{Y}] = \boxed{N}_{\vec{X}} \overset{+}{\text{-----}} \boxed{1^N}_{\vec{Y}} \\ \mathcal{S}^{-1}\text{-wall : } \mathcal{S}^{-}[\vec{X}, \vec{Y}] &= G[\vec{X}, \vec{Y}] = \boxed{N}_{\vec{X}} \overset{-}{\text{-----}} \boxed{1^N}_{\vec{Y}} \end{aligned} \quad (2.65)$$

Both theories admit (at least) two UV completions whose difference lies in the direction of the arrows and the sign of FIs. For convenience, the UV completions of both theories are reported in Figures 2.11 and 2.12.

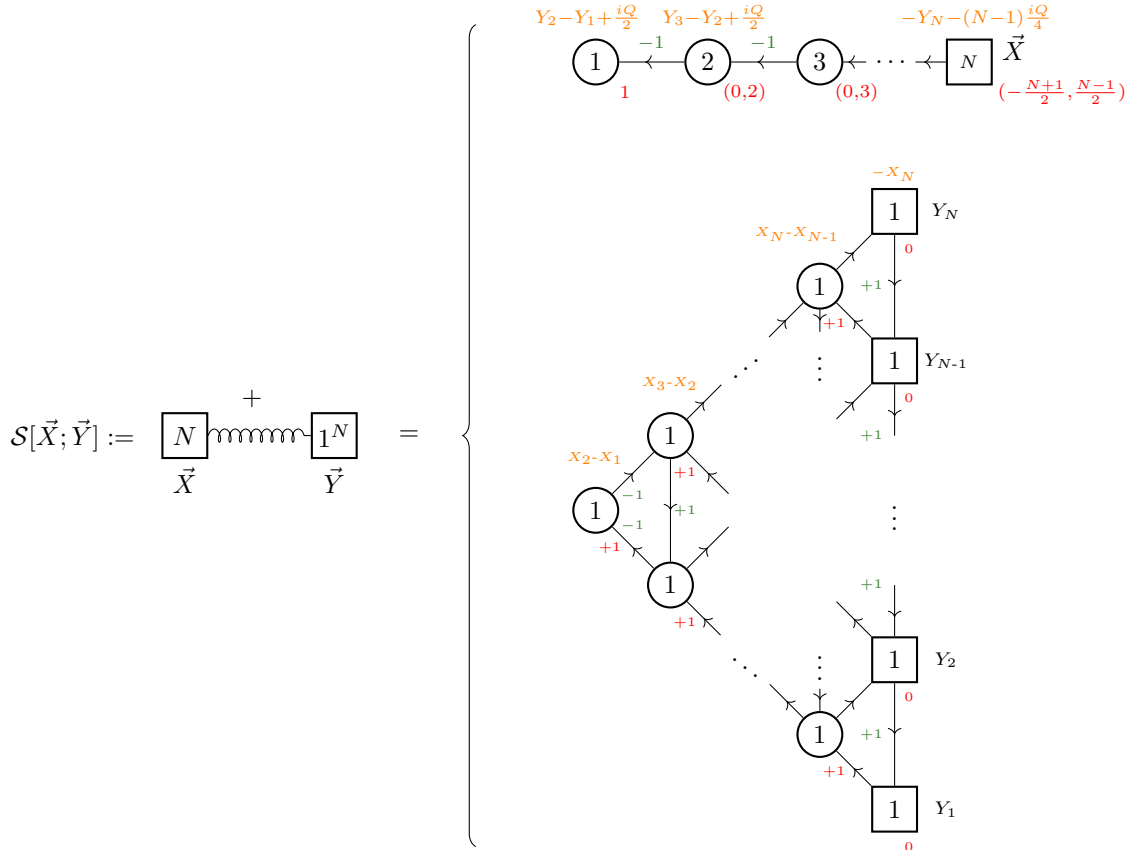


Figure 2.11: The planar and chiral UV completions of the $\mathcal{N} = 2$ chiral-planar $\mathcal{S}$ -wall coinciding with $G[-\vec{X}; \vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity.

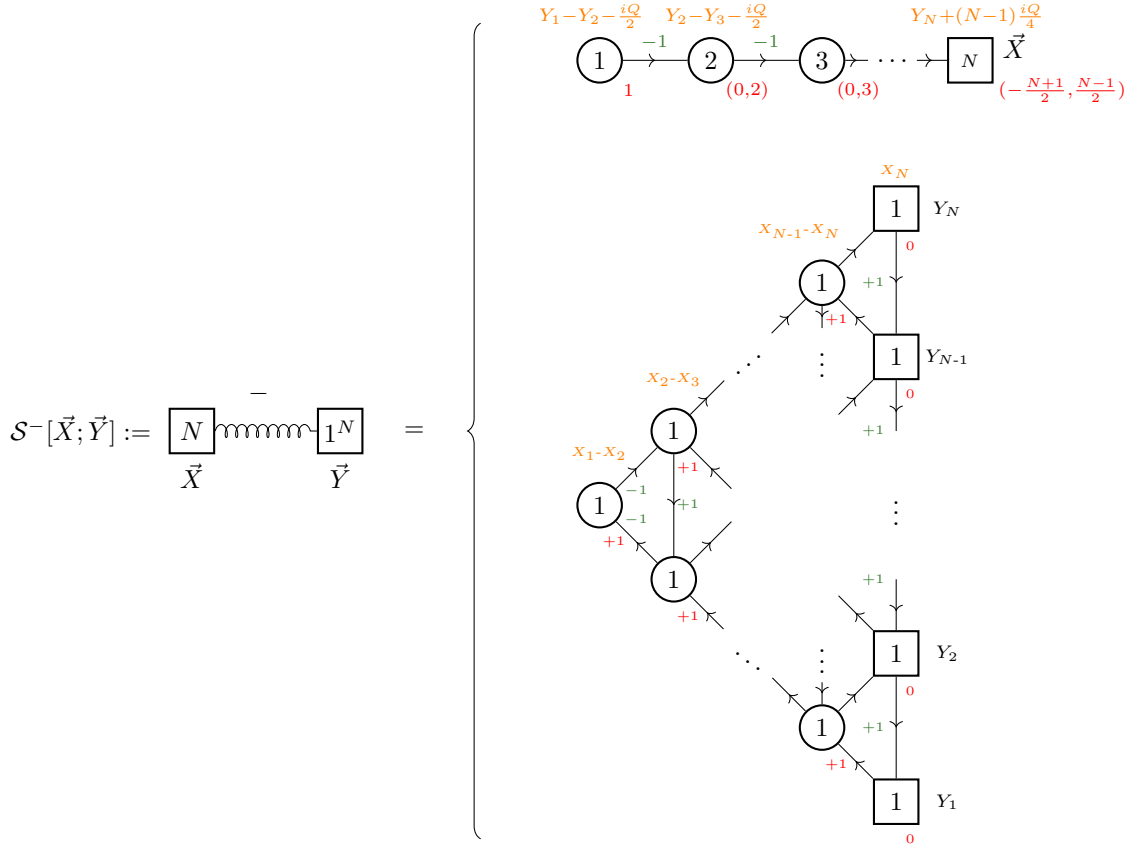


Figure 2.12: The planar and chiral UV completions of the $\mathcal{N} = 2$ $\mathcal{S}$ -wall coinciding with $G[-\vec{X}; \vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity.

In our notation, the first argument of the $Z_{\mathcal{S}^\pm}[\cdot; \cdot]$ theories corresponds to fugacities for an $U(N)$ global symmetry, while the second argument corresponds to a $U(1)^{N-1}$ global symmetry.

We must now verify that the $\mathcal{N} = 2$ $\mathcal{S}$ -wall satisfies the fusion to identity $\mathcal{S}\mathcal{S}^{-1} = \mathbb{I}$. To that end, we consider real mass deformations at the level of the $\mathbf{S}_b^3$ partition function of the $\mathcal{N} = 4$ fusion to identity (Equation 2.56).

Integration Measures. Starting from the $\mathcal{N} = 4$ integration measure (Equation 2.59), we consider real mass deformations that break the $U(1)_\tau$ global symmetry. Two possible limits can be taken, which correspond to two different resulting integration measures:

- The **planar measure** is obtained as a large mass limit of the $\mathcal{N} = 4$ integration measure (Equation 2.59):

$$\lim_{\tau \rightarrow +\infty} \Delta_{(N)}(\vec{Z}, \tau) \Big|_{Z_i \rightarrow Z_i + \frac{\tau}{2}(2i-N-1)} = \Delta^{[1^N]}(\vec{Z}) \times (\text{divergent phase}), \quad (2.66)$$

from which we obtain:

$$\Delta^{[1^N]}(\vec{Z}) = e^{i\pi H^{[1^N]}(\vec{Z})} \prod_{i=1}^{N-1} s_b \left(\frac{iQ}{2} - Z_i + Z_{i+1} \right) \quad (2.67)$$

with

$$H^{[1^N]}(\vec{Z}) := \frac{iQ}{2}(Z_1 - Z_N) - \frac{1}{2}(Z_1^2 + Z_N^2) - \sum_{i=2}^{N-1} Z_i^2 + \sum_{j=1}^{N-1} Z_j Z_{j+1}. \quad (2.68)$$

Some key observations are in order:

- The contribution of the vector multiplet of the $U(N)$ symmetry has disappeared, in agreement with the Higgsing of the $U(N) \rightarrow U(1)^N$.
- Conversely, the same limit for the adjoint hypermultiplet of the $U(N)$ symmetry reduces to a string of chiral multiplets connecting adjacent $U(1)$ nodes.
- The quadratic form $H^{[1^N]}(\vec{Z})$ encodes the CS levels and the BF couplings and FI shifts for the middle column of $U(1)^N$ gauge nodes.
- The **chiral measure** is obtained as another large mass limit of the $\mathcal{N} = 4$ integration measure (Equation 2.59):

$$\lim_{\tau \rightarrow +\infty} \Delta_{(N)}(\vec{Z}, \tau) \Big|_{Z_i \rightarrow Z_i + N \frac{\tau}{2}} = \Delta^{[N]}(\vec{Z}) \times (\text{divergent phase}), \quad (2.69)$$

from which we obtain:

$$\Delta^{[N]}(\vec{X}) = e^{i\pi H^{[N]}(\vec{X})} \prod_{\alpha=1}^N \prod_{\beta < \alpha} \frac{1}{s_b(\frac{iQ}{2} \pm (X_\alpha - X_\beta))}; \quad (2.70)$$

with

$$\begin{aligned} H^{[N]}(\vec{X}) &:= -(N-1) \left(\sum_{i=1}^N X_i^2 \right) + 2 \sum_{i < j}^N X_i X_j = \\ &= -N \sum_{i=1}^N X_i^2 + \left(\sum_{i=1}^N X_i \right)^2, \end{aligned} \quad (2.71)$$

which corresponds to a CS coupling at level $(N, 0)$.

Chiral Fusion to Identity. We consider the following shifts in fugacities which have the effect of breaking the $U(N)_{\vec{X}, \vec{Y}}$ global symmetries to their Cartan subalgebra:

$$\begin{cases} \vec{u}^{(I)} \rightarrow \vec{u}^{(I)} - \frac{\tau}{2} I, & \text{for } I = 1, \dots, 2N-1 \\ X_j \rightarrow X_j + \frac{\tau}{2} (2j - N), & \text{for } j = 1, \dots, N \\ Y_j \rightarrow Y_j + \frac{\tau}{2} (2j - N), & \text{for } j = 1, \dots, N \end{cases} \quad (2.72)$$

In this asymptotic limit, we flow to the $\mathcal{N} = 2$ chiral fusion to identity, which admits the following UV quiver description:

$$\begin{array}{c} \textcolor{brown}{X_2 - X_1 + \frac{iQ}{2}} \qquad \qquad \qquad \textcolor{brown}{Y_N - X_N} \qquad \qquad \qquad \textcolor{brown}{Y_1 - Y_2 - \frac{iQ}{2}} \\ \textcircled{1} \text{---} \textcircled{2} \text{---} \dots \text{---} \textcircled{N} \text{---} \dots \text{---} \textcircled{2} \text{---} \textcircled{1} \quad \longleftrightarrow \quad \mathbb{I}_{\vec{X} \vec{Y}}^{[1^N]} \\ \textcolor{red}{1} \qquad \textcolor{red}{(0,2)} \qquad \qquad \qquad \textcolor{red}{(-1, N-1)} \qquad \qquad \qquad \textcolor{red}{(0,2)} \qquad \textcolor{red}{1} \end{array} \quad (2.73)$$

The corresponding $\mathbf{S}_b^3$ partition function identity is given below:

$$\int d\vec{Z} \Delta^{[N]}(\vec{Z}) Z_S(\vec{Z}, \vec{X}) Z_{S^-}(\vec{Z}, \vec{Y}) = \frac{1}{\Delta^{[1^N]}(\vec{X})} \prod_{i=1}^N \delta(X_i - Y_i) \equiv_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]}. \quad (2.74)$$

Notice the integration is performed over the non-Abelian $U(N)_{\vec{Z}}$ symmetry and we have identified the $\mathcal{N} = 2$ **planar identity wall**:

$$\mathbb{I}_{\vec{Y}}^{[1^N]} \equiv \frac{1}{\Delta^{[1^N]}(\vec{X})} \prod_{i=1}^N \delta(X_i - Y_i), \quad (2.75)$$

Notice now the integration is performed over the Abelian $U(1)_{\vec{Z}}^N$ global symmetries, and we have identified the $\mathcal{N} = 2$ chiral identity wall:

$$\vec{X} \mathbb{I}_{\vec{Y}}^{[N]} \equiv \frac{1}{N! \Delta^{[N]}(\vec{X})} \sum_{\sigma \in S_N} \delta(X_i - Y_{\sigma(i)}) \quad (2.81)$$

which has the effect of identifying the $U(N)_X$ and $U(N)_Y$ global symmetries. We schematically depict this planar fusion to identity as:

$$\begin{array}{c} \begin{array}{ccc} \boxed{N} & \overset{+}{\text{---}} & \boxed{1^N} & \overset{-}{\text{---}} & \boxed{N} \\ \vec{X} & & \left(\begin{array}{c} \frac{1}{2} \\ 1 \\ \vdots \\ 1 \\ \frac{1}{2} \end{array} \right) & & \vec{Y} \end{array} & \longleftrightarrow & \vec{X} \mathbb{I}_{\vec{Y}}^{[N]} \end{array} \quad (2.82)$$

wherein the vector in red indicates the CS levels for the middle column of $U(1)^N$ nodes in (2.79) and BF interactions have been suppressed.

QFT Building Blocks

In analogy with the $\mathcal{N} = 4$ basic QFT blocks, we now want to study the $\mathcal{N} = 2$ basic QFT blocks. As before, this is achieved by considering appropriate chiral-planar real mass deformations of the $\mathcal{N} = 4$ ancestors.

The Chiral Flavor Block. We consider the following large mass limit for the $\mathcal{N} = 4$ flavor block:

$$\begin{cases} X_j, Y_j \rightarrow X_j, Y_j \\ \eta \rightarrow \eta + \frac{\tau}{2} \end{cases} \quad (2.83)$$

The resulting $\mathbf{S}_b^3$ partition function identity is given below:

$$Z_{\text{flavor, fund}}^{[N]}(\vec{X}, \eta) = e^{\frac{i\pi}{2} [N\eta^2 + \sum_{j=1}^N (X_j^2 - 2\eta X_j)]} \prod_{j=1}^N s_b(\eta - X_j)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[N]}. \quad (2.84)$$

An analogous limit can be considered, which would now yield:

$$Z_{\text{flavor, antif}}^{[N]}(\vec{X}, \eta) = e^{\frac{i\pi}{2} [N\eta^2 + \sum_{j=1}^N (X_j^2 - 2\eta X_j)]} \prod_{j=1}^N s_b(-\eta + X_j)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[N]}. \quad (2.85)$$

Taken together, we identify the $\mathcal{N} = 2$ **chiral flavor block**:

$$\begin{array}{ccc} \begin{array}{c} \eta \boxed{1} \\ \uparrow \text{---} \frac{-N}{2} \\ \boxed{N} \\ \uparrow \text{---} 1 \\ \vec{X} \end{array} & \begin{array}{c} \vec{X} \mathbb{I}_{\vec{Y}}^{[N]} \\ \text{---} \frac{(-\frac{1}{2}, -\frac{1}{2})}{\vec{X}} \end{array} & \begin{array}{c} \eta \boxed{1} \\ \downarrow \text{---} \frac{-N}{2} \\ \boxed{N} \\ \downarrow \text{---} 1 \\ \vec{X} \end{array} \\ Z_{\text{flavor, fund}}^{[N]}(\vec{X}; \eta) := & & Z_{\text{flavor, antif}}^{[N]}(\vec{X}; \eta) := \end{array} \quad (2.86)$$

The Planar Flavor Block. We consider the following large mass limit for the $\mathcal{N} = 4$ flavor block:

$$\begin{cases} X_j \rightarrow X_j - (2N - 2j + 1)\frac{\tau}{2} \\ Y_j \rightarrow Y_j - (2N - 2j + 1)\frac{\tau}{2} \\ \eta \rightarrow \eta \end{cases} \quad (2.87)$$

which yields the following $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{flavor,fund}}^{[1^N]}(\vec{X}; \eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_1 + X_1^2)} s_b(\eta - X_1)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]}. \quad (2.88)$$

An analogous limit can be considered, which would now yield:

$$Z_{\text{flavor,antif}}^{[1^N]}(\vec{X}; \eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_N + X_N^2)} s_b(X_N - \eta)_{\vec{X}} \mathbb{I}_{\vec{Y}}^{[1^N]}. \quad (2.89)$$

Taken together, we identify the $\mathcal{N} = 2$ planar flavor block shown in compact notation:

$$Z_{\text{flavor,fund}}^{[1^N]}(\vec{X}; \eta) := \begin{array}{c} \vec{X} \boxed{1^N} \vec{X} \mathbb{I}_{\vec{Y}}^{[1^N]} \\ \downarrow 1 \\ \eta \boxed{1} \end{array} := \begin{array}{c} X_N \boxed{1} \begin{array}{l} \text{orange } \frac{iQ}{4} \\ \text{red } -\frac{1}{2} \end{array} \\ \downarrow \text{green } 1 \\ \boxed{1} \begin{array}{l} \text{orange } 0 \\ \text{red } -1 \end{array} \\ \vdots \\ \downarrow \text{green } 1 \\ X_1 \boxed{1} \begin{array}{l} \text{orange } -\frac{iQ}{4} \\ \text{red } -1 \end{array} \\ \downarrow \text{green } 1 \\ \eta \boxed{1} \begin{array}{l} \text{orange } 0 \\ \text{red } -\frac{1}{2} \end{array} \end{array} \prod_{i=1}^N \delta(X_i - Y_i) \quad (2.90)$$

$$Z_{\text{flavor,antif}}^{[1^N]}(\vec{X}, \eta) := \begin{array}{c} \eta \boxed{1} \\ \downarrow \text{green } 1 \\ \eta \boxed{1} \downarrow \text{green } 1 \\ \vec{X} \boxed{1^N} \vec{X} \mathbb{I}_{\vec{Y}}^{[1^N]} \end{array} := \begin{array}{c} \eta \boxed{1} \begin{array}{l} \text{red } -\frac{1}{2} \\ \text{green } 1 \end{array} \\ \downarrow \text{green } 1 \\ X_N \boxed{1} \begin{array}{l} \text{orange } \frac{iQ}{4} \\ \text{red } -1 \end{array} \\ \downarrow \text{green } 1 \\ \boxed{1} \begin{array}{l} \text{orange } 0 \\ \text{red } -1 \end{array} \\ \vdots \\ \downarrow \text{green } 1 \\ X_1 \boxed{1} \begin{array}{l} \text{orange } -\frac{iQ}{4} \\ \text{red } -\frac{1}{2} \end{array} \end{array} \prod_{i=1}^N \delta(X_i - Y_i) \quad (2.91)$$

The Chiral Bifundamental Block. We consider the following large mass limit for the $\mathcal{N} = 4$ bifundamental block:

$$\begin{cases} \vec{X} \rightarrow \vec{X} + N\frac{\tau}{2} \\ \vec{Y} \rightarrow \vec{Y} + (N+1)\frac{\tau}{2} \\ \eta \rightarrow \eta - \frac{N}{2}\tau \end{cases} \quad (2.92)$$

The resulting $\mathbf{S}_b^3$ partition function is given below:

$$Z_{\text{bif, left}}^{[N]}(\vec{X}, \vec{Y}, \eta; r) := e^{2\pi i \sum_j X_j(-\eta - N(1-r)iQ)} e^{2\pi i \sum_j Y_j(\eta + N(1-r)iQ)} e^{\frac{N\pi i}{2} \sum_j (X_j^2 + Y_j^2) - \pi i \sum_{j,k} X_j Y_k} \prod_{j,k=1}^N s_b\left(\frac{iQ}{2}(1-r) + X_k - Y_j\right). \quad (2.93)$$

This describes the $\mathcal{N} = 2$ **left-pointing chiral bifundamental block**:

$$Z_{\text{bif, left}}^{[N]}(\vec{X}, \vec{Y}, \eta; r) := \begin{array}{ccc} & \xrightarrow{-\eta - N(1-r)\frac{iQ}{4}} & \xleftarrow{\eta + N(1-r)\frac{iQ}{4}} \\ \boxed{N} & \xleftarrow{1} & \boxed{N} \\ \textcolor{red}{(-\frac{N}{2}, -\frac{N}{2})} & & \textcolor{red}{(-\frac{N}{2}, -\frac{N}{2})} \\ \vec{X} & & \vec{Y} \end{array} \quad (2.94)$$

Similarly, we can define the **right-pointing bifundamental block**, whose $\mathbf{S}_b^3$ partition function is given by:

$$Z_{\text{bif, right}}^{[N]}(\vec{X}, \vec{Y}, \eta; r) := Z_{\text{bif, left}}^{[N]}(-\vec{X}, -\vec{Y}, -\eta; r). \quad (2.95)$$

The Planar Bifundamental Block. We consider the following large mass limit for the $\mathcal{N} = 4$ bifundamental block:

$$\begin{cases} X_j \rightarrow X_j + (2j - N)\frac{\tau}{2} \\ Y_j \rightarrow Y_j + \frac{\tau}{2}(2j - N + 1) \\ \eta \rightarrow \eta - \frac{N+1}{2}\tau. \end{cases} \quad (2.96)$$

The resulting $\mathbf{S}_b^3$ partition function is given below:

$$Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) := e^{\frac{\pi i}{2}(X_1^2 + Y_N^2)} e^{\pi i(\sum_{j=2}^N X_j^2 + \sum_{j=1}^{N-1} Y_j^2)} e^{2\pi i \eta \sum_j (Y_j - X_j)} e^{-\pi i \sum_j Y_j (X_j + X_{j+1})} e^{2\pi i \sum_j X_j (r-1)(N+2-2J)\frac{iQ}{2}} e^{-2\pi i \sum_j Y_j (r-1)(N-2J)\frac{iQ}{2}} e^{2\pi i (Y_N \frac{iQ}{4} - X_1 \frac{iQ}{4})} \prod_{j=1}^N s_b\left(\frac{iQ}{2} - r\frac{iQ}{2} + X_j - Y_j\right) \prod_{j=1}^{N-1} s_b\left(\frac{iQ}{2} - r\frac{iQ}{2} + Y_j - X_{j+1}\right). \quad (2.97)$$

This describes the $\mathcal{N} = 2$ **left-pointing planar bifundamental block**:

$$\begin{aligned}
 Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) := & \quad \begin{array}{c}
 \begin{array}{c}
 \text{---} \eta \\
 + (r-1) \left(N \frac{iQ}{2} - \frac{iQ}{4} \right) \\
 X_1 \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{+1} \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_1 \\
 \text{---} \frac{1}{2} \\
 + (r-1)(N-2) \frac{iQ}{2} \\
 X_2 \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{-1} \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_1 \\
 \vdots \\
 - (r-1)(N-2) \frac{iQ}{2} \\
 X_N \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow{-1} \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_{N-1} \\
 \text{---} \frac{1}{2} \\
 - (r-1) \left(N \frac{iQ}{2} - \frac{iQ}{4} \right) \\
 \begin{array}{|c|} \hline 1 \\ \hline \end{array} Y_N \\
 \text{---} \frac{1}{2}
 \end{array} \\
 \\
 \begin{array}{c}
 \text{---} \eta \\
 \vec{X} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \xrightarrow{+1} \begin{array}{|c|} \hline 1^N \\ \hline \end{array} \vec{Y} \\
 \text{---} \frac{1}{2} \\
 \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \vdots \\ -1 \end{pmatrix} \\
 \begin{pmatrix} -1 \\ \vdots \\ -1 \\ -\frac{1}{2} \end{pmatrix}
 \end{array}
 \end{array}
 \quad (2.98)
 \end{aligned}$$

Similarly, we can define a **right-pointing planar bifundamental block**, whose $\mathbf{S}_b^3$ partition function is given by:

$$Z_{\text{bif, right}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; r) := Z_{\text{bif, left}}^{[1^N]}(\vec{Y}, \vec{X}, -\eta; r) \quad (2.99)$$

Basic Duality Moves

Having identified the $\mathcal{S} \in SL(2; \mathbb{Z})$ generator and QFT building blocks for $\mathcal{N} = 2$ chiral-planar dual pairs, we are finally ready to study the action of the duality walls on these blocks.

For convenience, we repeat the UV quiver Lagrangian description of the $\mathcal{N} = 4$ duality moves in Figure 2.13.

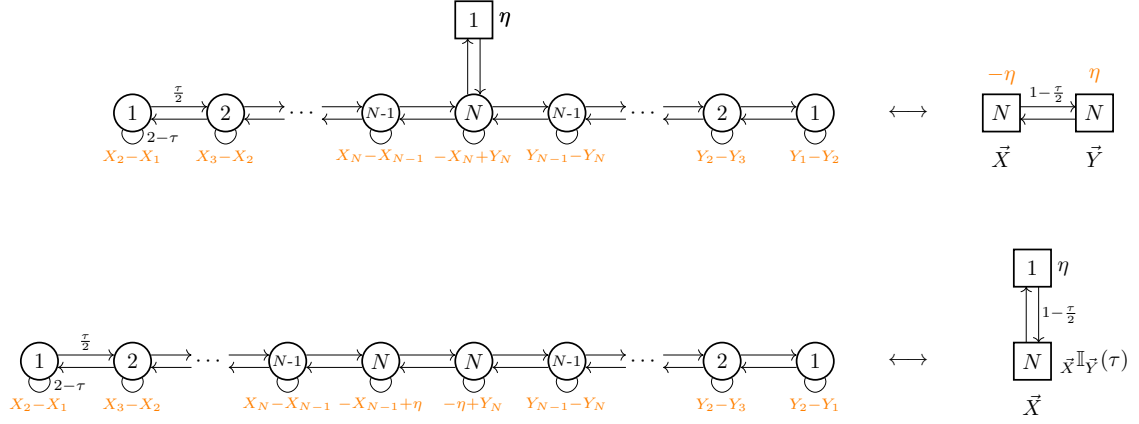
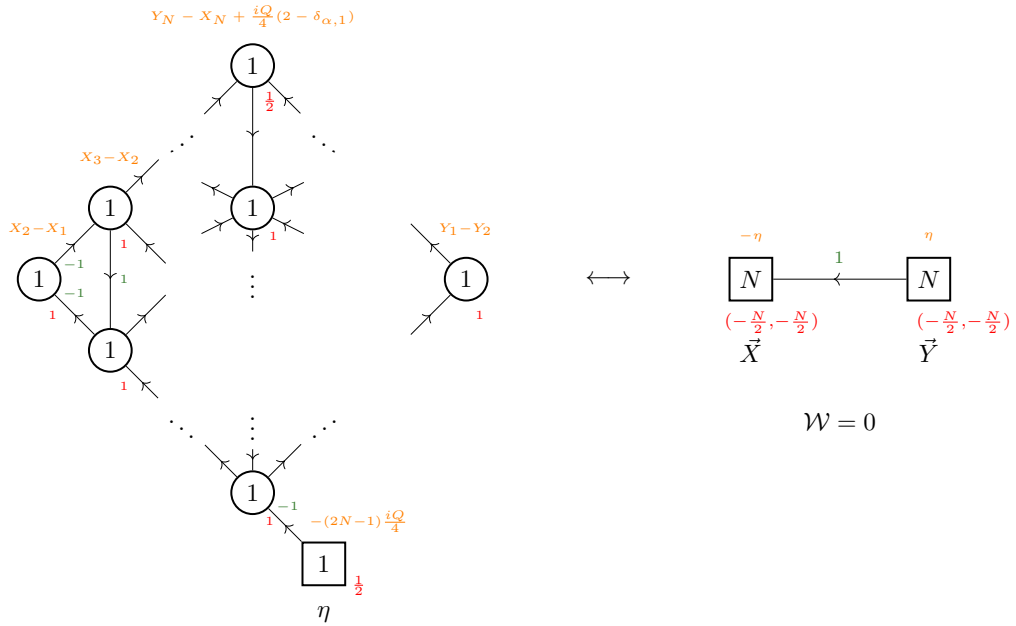


Figure 2.13: The $\mathcal{N} = 4$ basic duality moves are shown here. Fugacities of topological symmetries are highlighted in yellow, and global fugacities and charges under $U(1)_\tau$ have been specified for completeness.

$\mathcal{S}$ -dualization of the Chiral Bifundamental. Starting from the first row of Figure 2.13, we consider the following large mass limit of the $\mathbf{S}_b^3$ partition function identity (Equation 2.62):

$$\begin{cases} \vec{X} \rightarrow \vec{X} + N\frac{\tau}{2} \\ \vec{Y} \rightarrow \vec{Y} + (N+1)\frac{\tau}{2} \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I - 1), & \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 1), & \text{for } I = N+1, \dots, 2N-1 \\ \eta \rightarrow \eta - \frac{N}{2}\tau \end{cases} \quad (2.100)$$

where $u_\alpha^{(I)}$ are gauge fugacities of the I^{th} gauge node (from the left). The resulting flow is shown in Equation 2.101.

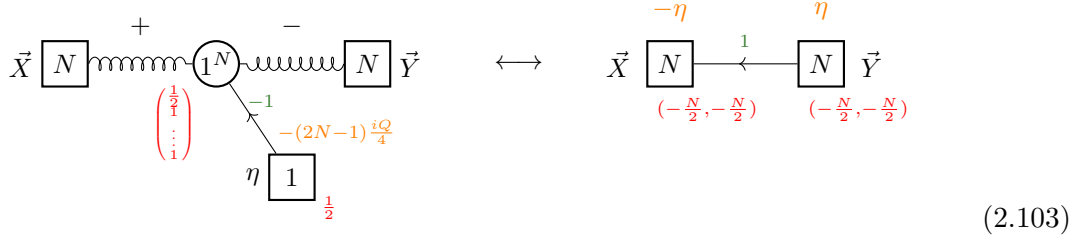


$$(2.101)$$

This corresponds to the following $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{bif, left}}^{[N]}(\vec{X}, \vec{Y}, \eta; 1) = \int d\vec{Z} \Delta^{[1^N]}(\vec{Z}) e^{-\frac{i\pi}{2}(\eta^2 - 2\eta Z_1 + Z_1^2 + iQ[(2N-1)\eta - Z_1 - 2\sum_{j=2}^N Z_j])} Z_{\mathcal{S}}(\vec{X}, \vec{Z})_{s_b} \left(\frac{iQ}{2} + Z_1 - \eta \right) Z_{\mathcal{S}^-}(\vec{Y}, \vec{Z}). \quad (2.102)$$

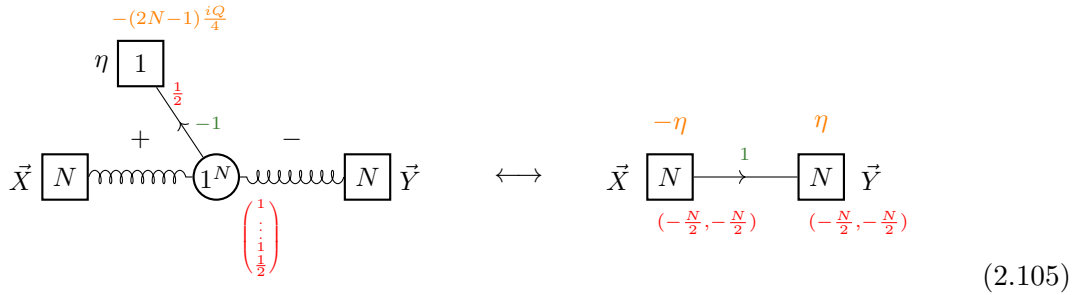
This is compactly encoded as follows:



In a similar vein, we can follow the flow to the $\mathcal{S}$ -dual of the right-pointing chiral bifundamental block. This corresponds to the $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{bif, right}}^{[N]}(\vec{X}, \vec{Y}, \eta; 1) = \int d\vec{Z} \Delta^{[1^N]}(\vec{Z}) e^{-\frac{i\pi}{2}(\eta^2 - 2\eta Z_N + Z_N^2 + iQ[-(2N-1)\eta + Z_1 + 2\sum_{j=2}^N Z_j])} Z_{\mathcal{S}}(\vec{X}, \vec{Z})_{s_b} \left(\frac{iQ}{2} - Z_N + \eta \right) Z_{\mathcal{S}^-}(\vec{Y}, \vec{Z}) \quad (2.104)$$

which is compactly encoded as:



$\mathcal{S}$ -dualization of the Planar Bifundamental. Starting from the first row of Figure 2.13, we consider the following large mass limit of the $\mathbf{S}_b^3$ partition function identity (Equation 2.62):

$$\begin{cases} X_j \rightarrow X_j + (2j - N) \frac{\tau}{2} \\ Y_j \rightarrow Y_j + \frac{\tau}{2} (2j - N + 1) \\ \vec{u}^{(I)} \rightarrow \vec{u}^{(I)} - \frac{\tau}{2} I, \quad \text{for } I = 1, \dots, 2N - 1 \\ \eta \rightarrow \eta - \frac{N+1}{2} \tau. \end{cases} \quad (2.106)$$

The resulting flow is shown in Figure 2.107.

$$\begin{aligned}
 & \text{Left side: } \text{Planar block with nodes } 1, 2, \dots, N, 2, 1. \text{ Central node } N \text{ is connected to a box labeled } 1 \text{ with a vertical arrow labeled } \eta \text{ and } \frac{N}{2}. \\
 & \text{Right side: } \text{Tree-like structure with nodes } X_1, X_2, \dots, X_N \text{ and } Y_1, Y_2, \dots, Y_N. \\
 & \text{Diagram is labeled } W=0.
 \end{aligned}
 \tag{2.107}$$

This corresponds to the following $\mathbf{S}_b^3$ partition function identity:

$$\begin{aligned}
 Z_{\text{bif, left}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; 1) &= \int d\vec{Z} \Delta^{[N]}(\vec{Z}) e^{-\frac{i\pi}{2} [N\eta^2 + iQ(N\eta - \sum_{j=1}^N Z_j) + \sum_{j=1}^N (Z_j^2 - 2\eta Z_j)]} \\
 & Z_S(\vec{Z}, \vec{X}) \prod_{j=1}^N s_b\left(\frac{iQ}{2} + Z_j - \eta\right) Z_{S^-}(\vec{Z}, \vec{Y}),
 \end{aligned}
 \tag{2.108}$$

which is compactly encoded as follows:

$$\begin{aligned}
 & \text{Left side: } \text{Planar block with nodes } X, 1^N, N, 1^N, Y. \text{ Central node } N \text{ is connected to a box labeled } 1 \text{ with a vertical arrow labeled } \eta \text{ and } \frac{N}{2}. \\
 & \text{Right side: } \text{Tree-like structure with nodes } X \text{ and } Y. \\
 & \text{Diagram is labeled } (2.109).
 \end{aligned}$$

In a similar vein, we can follow the flow to the $\mathcal{S}$ -dual of the right-pointing planar bifundamental block. This corresponds to the $\mathbf{S}_b^3$ partition function identity:

$$\begin{aligned}
 Z_{\text{bif, right}}^{[1^N]}(\vec{X}, \vec{Y}, \eta; 1) &= \int d\vec{Z} \Delta^{[N]}(\vec{Z}) e^{-\frac{i\pi}{2} [N\eta^2 + iQ(-N\eta + \sum_{j=1}^N Z_j) + \sum_{j=1}^N (Z_j^2 - 2\eta Z_j)]} \\
 & Z_S(\vec{Z}, \vec{X}) \prod_{j=1}^N s_b\left(\frac{iQ}{2} - Z_j + \eta\right) Z_{S^-}(\vec{Z}, \vec{Y}),
 \end{aligned}
 \tag{2.110}$$

which is shown in compact notation as follows:

$$\begin{aligned}
 & \text{Left side: } \text{Planar block with nodes } X, 1^N, N, 1^N, Y. \text{ Central node } N \text{ is connected to a box labeled } 1 \text{ with a vertical arrow labeled } \eta \text{ and } \frac{N}{2}. \\
 & \text{Right side: } \text{Tree-like structure with nodes } X \text{ and } Y. \\
 & \text{Diagram is labeled } (2.111).
 \end{aligned}$$

$\mathcal{S}$ -dualization of the Chiral Flavor. Starting from the second row of Figure 2.13, we consider the following large mass limit of the $\mathbf{S}_b^3$ partition function identity (Equation

2.64):

$$\begin{cases} X_j, Y_j \rightarrow X_j, Y_j \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I), \quad \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \rightarrow u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 2), \quad \text{for } I = N + 1, \dots, 2N \\ \eta \rightarrow \eta + \frac{\tau}{2} \end{cases} \quad (2.112)$$

The resulting flow is shown in Figure 2.113.

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}} \quad (2.113)$$

This corresponds to the following $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{flavor, fund}}^{[N]}(\vec{X}, \eta) = \int d\vec{Z} d\vec{W} Z_S(\vec{X}, \vec{Z}) Z_{\text{bif, right}}^{[1^N]}(\vec{Z}, \vec{W}, -\eta; 0) Z_{S^-}(\vec{Y}, \vec{W}), \quad (2.114)$$

which can be compactly encoded as follows:

$$\quad (2.115)$$

In a similar vein, we can follow the flow to the $\mathcal{S}$ -dual of the chiral antifundamental flavor block. This corresponds to the $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{flavor, antif}}^{[N]}(\vec{X}, \eta) = \int d\vec{Z} d\vec{W} Z_S(\vec{X}, \vec{Z}) Z_{\text{bif, left}}^{[1^N]}(\vec{Z}, \vec{W}, -\eta; 0) Z_{S^-}(\vec{Y}, \vec{W}). \quad (2.116)$$

As before, we can report this identity more compactly:

$$\begin{array}{c}
\eta \begin{array}{|c|} \hline 1 \\ \hline \end{array} \xrightarrow[-\frac{N}{2}]{1} \begin{array}{|c|} \hline N \\ \hline \end{array} \vec{X}^{\mathbb{I}\vec{Y}^{[N]}} \\
(-\frac{1}{2}, -\frac{1}{2})
\end{array}
\longleftrightarrow
\begin{array}{c}
\begin{array}{|c|} \hline N \\ \hline \end{array} \vec{X} \xrightarrow{+} \begin{array}{c} \eta \\ \textcircled{1^N} \\ \left(\begin{smallmatrix} \frac{1}{2} \\ 1 \\ \vdots \\ 1 \end{smallmatrix} \right) \end{array} \xrightarrow[-1]{-} \begin{array}{c} -\eta \\ \textcircled{1^N} \\ \left(\begin{smallmatrix} 1 \\ \vdots \\ 1 \\ \frac{1}{2} \end{smallmatrix} \right) \end{array} \xrightarrow{-} \begin{array}{|c|} \hline N \\ \hline \end{array} \vec{Y}
\end{array}
\quad (2.117)$$

S-dualization of the Planar Flavor. Starting from the second row of Figure 2.13, we consider the following large mass limit of the $\mathbf{S}_b^3$ partition function identity (Equation 2.64):

$$\begin{cases} X_j \rightarrow X_j - (2N - 2j + 1)\frac{\pi}{2} \\ Y_j \rightarrow Y_j - (2N - 2j + 1)\frac{\pi}{2} \\ \vec{u}^{(I)} \rightarrow \vec{u}^{(I)} + (2N - I + 1)\frac{\pi}{2}, \quad \text{for } I = 1, \dots, 2N \\ \eta \rightarrow \eta \end{cases} \quad (2.118)$$

The resulting flow is shown in Figure 2.119.

$$\begin{array}{c}
\eta \quad \boxed{1} \\
\swarrow \begin{array}{l} -\frac{1}{2} \\ 1 \end{array} \\
X_N \quad \boxed{1} \quad \begin{array}{l} -\frac{iQ}{4} \\ -1 \end{array} \\
\downarrow \begin{array}{l} 1 \\ 1 \end{array} \\
\boxed{1} \quad \begin{array}{l} 0 \\ -1 \end{array} \\
\vdots \\
X_1 \quad \boxed{1} \quad \begin{array}{l} \frac{iQ}{4} \\ -\frac{1}{2} \end{array}
\end{array}
\prod_{i=1}^N \delta(X_i - Y_i)
\longleftrightarrow
\begin{array}{c}
\begin{array}{ccccccc}
X_2 - X_1 + \frac{iQ}{2} & & -X_N + \eta + \frac{iQ}{2} & & Y_1 - Y_2 - \frac{iQ}{4} \\
\textcolor{red}{1} \quad \textcolor{red}{1} \quad \textcolor{red}{2} & \leftarrow \textcolor{red}{-1} & \cdots & \leftarrow \textcolor{red}{-1} & \cdots & \leftarrow \textcolor{red}{-1} & \textcolor{red}{2} \quad \textcolor{red}{1} \\
& & (0,2) & & (-\frac{1}{2}, N - \frac{1}{2}) & & (0,2) \\
& & & & (-\frac{1}{2}, N - \frac{1}{2}) & &
\end{array}
\end{array}
\quad (2.119)$$

This corresponds to the following $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{flavor, antif}}^{[1^N]}(\vec{X}; \eta) = \int d\vec{Z} d\vec{W} Z_S[\vec{Z}; \vec{X}] Z_{\text{bif, left}}^{[N]}(\vec{Z}; \vec{W}; -\eta; 0) Z_{S^-}[\vec{W}; \vec{Y}] \quad (2.120)$$

and can be represented more compactly as:

$$\begin{array}{c} \eta \\ \boxed{1} \end{array} \xrightarrow[\textcolor{green}{1}]{-\frac{1}{2}} \vec{X} \boxed{1^N} \bar{X}^\parallel \bar{Y}^{[1^N]} \longleftrightarrow \vec{X} \boxed{1^N} + \overset{\textcolor{brown}{\eta}}{(N)}_{(-\frac{1}{2}, \frac{2N-1}{2})} \xleftarrow[\textcolor{green}{-1}]{} \overset{-\eta}{(N)}_{(-\frac{1}{2}, \frac{2N-1}{2})} \xrightarrow[-]{1^N} \vec{Y} \quad (2.121)$$

In a similar vein, we can follow the flow to the $\mathcal{S}$ -dual of the planar fundamental flavor block. This corresponds to the $\mathbf{S}_b^3$ partition function identity:

$$Z_{\text{flavor, fund}}^{[1^N]}(\vec{X}; \eta) = \int d\vec{Z} \, d\vec{W} \, Z_S[\vec{Z}; \vec{X}] \, Z_{\text{bif, right}}^{[N]}(\vec{Z}; \vec{W}; -\eta; 0) \, Z_{S^-}[\vec{W}; \vec{Y}]. \quad (2.122)$$

We have now assembled the machinery necessary to dualize $\mathcal{N} = 2$ chiral-planar dual pairs systematically via our algorithmic prescription.

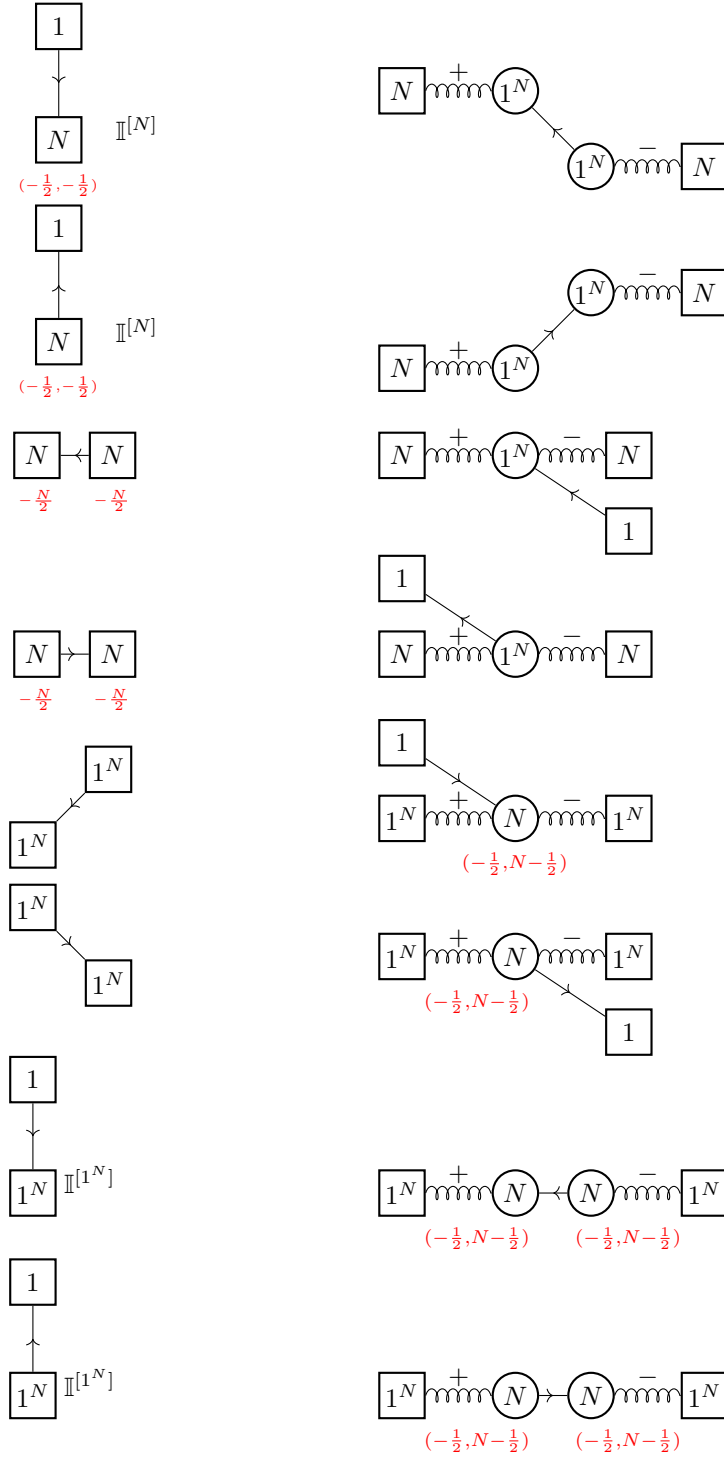
We are finally ready to define a dualization algorithm to dualize $\mathcal{N} = 2$ quiver gauge theories.

- **Step 1:** The quiver is chopped into its basic QFT blocks that can be either bifundamentals or (anti-)fundamental chiral fields. This step is performed by freezing the gauge interactions that will be reintroduced later.
- **Step 2:** Each QFT block is dualized via the basic duality moves described. These identities, or *basic moves* are summarized for reference in Table 2.124.
- **Step 3:** The dualized blocks are glued together via the $\mathcal{N} = 2$ chiral/planar fusion to identity when the original gauging is restored.

We will use the algorithm to dualize non-Abelian chiral $\mathcal{N} = 2$ quivers into abelian-planar quivers. Although the same algorithm can be applied in reverse — starting from a planar abelian quiver to obtain its chiral dual — we do not discuss this case here.

QFT Block

$\mathcal{S}$ -dual



(2.124)

2.4.1 An Example: CS-SQCD₃ with Fundamental and Anti-Fundamental Matter

As a first example of the application of the algorithm, we consider the case of $U(N)_{(-\frac{F}{2}+N, -\frac{F}{2})}$ SQCD with $[n_f, n_a]$ flavors, constrained by the relation $F = n_f + n_a$. As we had mentioned in Section 2.1, these theories can be obtained from $\mathcal{N} = 4$ $U(N)$ SQCD with F flavors by a positive real axial mass deformation at the level of the $\mathbf{S}_b^3$ partition function. However, instead of a brute force approach, we shall do this by using the algorithmic prescription.

Our algorithm can be employed to obtain the mirror-like duals of these theories in the following cases:

$$[n_f, n_a] = \begin{cases} [N + F_1, N + F_2] \\ [2N + F_1, F_2] \\ [F_1, 2N + F_2] \end{cases} \quad F_1, F_2 \geq 0 \quad (2.125)$$

$U(N)$ CS-SQCD with $[F_1 + N, F_2 + N]$ Chiral Multiplets

We start from the first case in (2.125) and consider $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + N, F_2 + N]$ chiral multiplets:

$$(2.126)$$

To apply our algorithm, we need to define this theory using the basic ingredients defined in Section 2.3.2: the fundamental blocks can be used to account for the matter content, and the gluing rules account for the gauging. The matter content of (2.126) can be accounted for by considering two chiral bifundamental blocks (2.94), F_1 chiral fundamental blocks (2.86) and F_2 chiral antifundamental blocks. Each (anti)fundamental block includes an identity operator $\mathbb{I}^{[N]}$ which has the effect of identifying two $U(N)$ gauge groups. Therefore, in order to engineer the SQCD theory, where there is one $U(N)$ gauge group, we need $F_1 + F_2 + 1$ gluing operations. These ingredients can be combined by joining the matter blocks and gluing two adjacent blocks through one of their $U(N)$ symmetries with the glue operation. Schematically the SQCD theory (2.126) can be written as:

$$(2.127)$$

In this schematic picture, multiplication between two consecutive blocks is understood. Notice the presence of the chiral measure $\Delta^{[N]}$ defined in (2.70): as we ungauged the $U(N)$ gauge symmetry and decompose the theory into fundamental QFT blocks, it becomes essential to keep track of the integration measures to recover the original theory (2.126). We have suppressed global fugacity labels for brevity.

One can verify that we do recover the correct CS levels for the SQCD theories upon gauging. Recall that each factor of $\Delta^{[N]}$ contributes as $(N, 0)$ to the CS level. Each flavor block consists of a (anti-)fundamental chiral field, with a CS level $(-\frac{1}{2}, -\frac{1}{2})$, and an identity wall, contributing as $(N, 0)$, for a total contribution of $(-N - \frac{1}{2}, -\frac{1}{2})$. Lastly, a

bifundamental chiral contributes with a CS level of $(-\frac{N}{2}, -\frac{N}{2})$. Therefore, the total CS level is:

$$\underbrace{2(-\frac{N}{2}, -\frac{N}{2})}_{\text{bifundamental blocks}} + \underbrace{(F_1 + F_2)(-N - \frac{1}{2}, -\frac{1}{2})}_{\text{flavor blocks}} + \underbrace{(F_1 + F_2 + 1)(N, 0)}_{\text{chiral measure}} = (-\frac{F_1 + F_2}{2}, -\frac{F_1 + F_2}{2} - N) \quad (2.128)$$

which matches the CS level of the SQCD theory we are considering.

Now we dualize each block by exploiting the basic moves derived in Sections 2.3.2 and we obtain:

$$\begin{aligned}
& \int \left[\text{Diagram 1} \right] \Delta^{[N]} \dots \left[\text{Diagram 2} \right] \Delta^{[N]} \times \\
& \times \left[\text{Diagram 3} \right] \Delta^{[N]} \dots \left[\text{Diagram 4} \right] \Delta^{[N]} \left[\text{Diagram 5} \right] \Delta^{[N]} \times
\end{aligned}
\tag{2.129}$$

Note, once again, the presence of the chiral measure. The presence of the chiral measure is needed for diagonal gauging of adjacent $U(N)$ nodes to obtain connected generalized quivers. Once adjacent $U(N)$ nodes are gauged, we can dualize those subquivers by using the fusion to identity to obtain an identity wall:

$$\cdots \textcircled{1^N} \overset{+}{\text{---}} \textcircled{N} \overset{-}{\text{---}} \textcircled{1^N} \cdots \quad \rightarrow \quad \cdots \textcircled{1^N} \cdots \quad (2.130)$$

where each of the $U(1)$ gauge nodes on the left, from top to bottom, is identified with the corresponding $U(1)$ gauge node on the right, from top to bottom, as dictated by the fusion to identity rule.

The resulting generalized quiver is shown below:

(2.131)

The generalized quiver in (2.131) can finally be written in the explicit Lagrangian form by using the planar UV completion of the $\mathcal{S}$ -walls and the definition of the planar bifundamental and fundamental blocks. The result is depicted in Figure 2.14.

We now comment on several features of the duality shown in Figure 2.14.

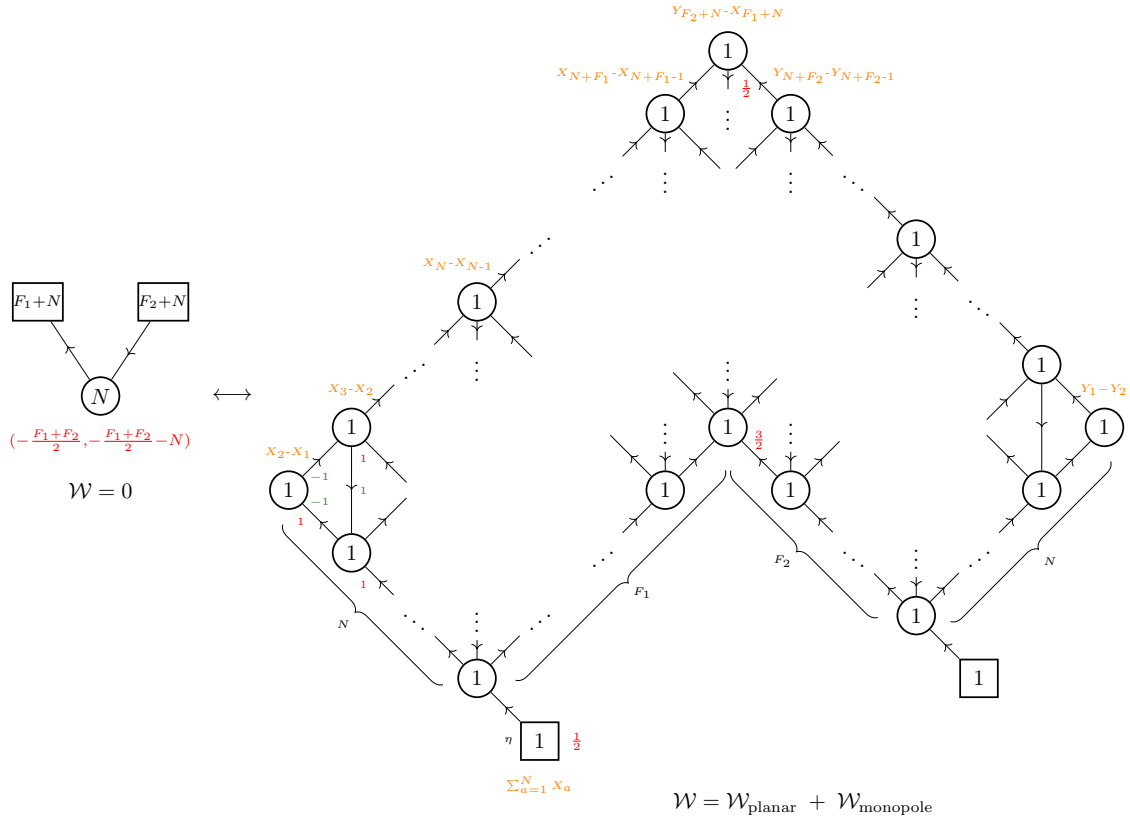


Figure 2.14: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + N, F_2 + N]$ chiral multiplets. The superpotential terms and (mixed) Chern-Simons interactions are as described in Section 2.1.

The global symmetry of the SQCD theory is, in general,

$$S[U(F_1 + N) \times U(F_2 + N)] \times U(1),$$

where the first factor is the flavor symmetry acting on the fundamental and anti-fundamental fields, while the second factor is the topological symmetry.¹¹ In the planar mirror dual, after taking into account gauge transformations and superpotential constraints, the only manifest flavor symmetry is a single $U(1)$.

The monopole superpotential $\mathcal{W}_{\text{monopole}}$ contains terms that are linear in monopole operators carrying magnetic flux -1 and $+1$ under pairs of gauge nodes connected by vertical links. These terms explicitly break the individual topological symmetries associated with the nodes in each column down to a single diagonal $U(1)$. As a result, the UV topological symmetry of the planar quiver is

$$U(1)^{F_1 + F_2 + 2N - 1},$$

which enhances in the IR to

$$S[U(F_1 + N) \times U(F_2 + N)].$$

This enhancement ensures that the global symmetry of the mirror quiver matches that of the SQCD theory.

In the SQCD theory, the chiral ring is generated by mesonic operators constructed from the fundamental fields Q_i and anti-fundamentals $\tilde{Q}_j$, which transform in the bifundamental representation of

$$SU(F_1 + N) \times SU(F_2 + N).$$

Under the duality, these mesons are mapped to gauge-invariant monopole operators in the planar Abelian theory. For instance, the meson $Q_{F_1 + N} \tilde{Q}_{F_2 + N}$ is mapped to the monopole operator with magnetic flux $+1$ under the topmost gauge node of the Abelian quiver. More generally, the remaining mesons are mapped to monopole operators characterized by two strings of $+1$ magnetic fluxes, originating at the topmost node and propagating along the two upper diagonals of the planar quiver. There are precisely $(F_1 + N)(F_2 + N)$ such monopole operators, reproducing the bifundamental representation of the enhanced $SU(F_1 + N) \times SU(F_2 + N)$ global symmetry.

Finally, we note that in this class of dualities, the planar Abelian theory does not exhibit any mesonic chiral ring generators. This implies that, for all values of F_1 and F_2 , the SQCD theory does not admit dressed gauge-invariant monopole operators that would parametrize a non-compact moduli space.

$U(N)$ CS-SQCD with $[F_1 + 2N, F_2]$ Chiral Multiplets

We now consider the second case in (2.125) (which is analogous to the third one), the $U(N)_{(-\frac{F_1 + F_2}{2}, -\frac{F_1 + F_2}{2} - N)}$ SQCD with $[F_1 + 2N, F_2]$ chiral multiplets:

$$(2.132)$$

¹¹In the special case $F_1 = F_2 = 0$, the topological symmetry enhances to $SU(2)$. This enhancement can also be understood by considering the Aharony-like dual of the theory [147], which is Abelian, and then taking its mirror dual [26].

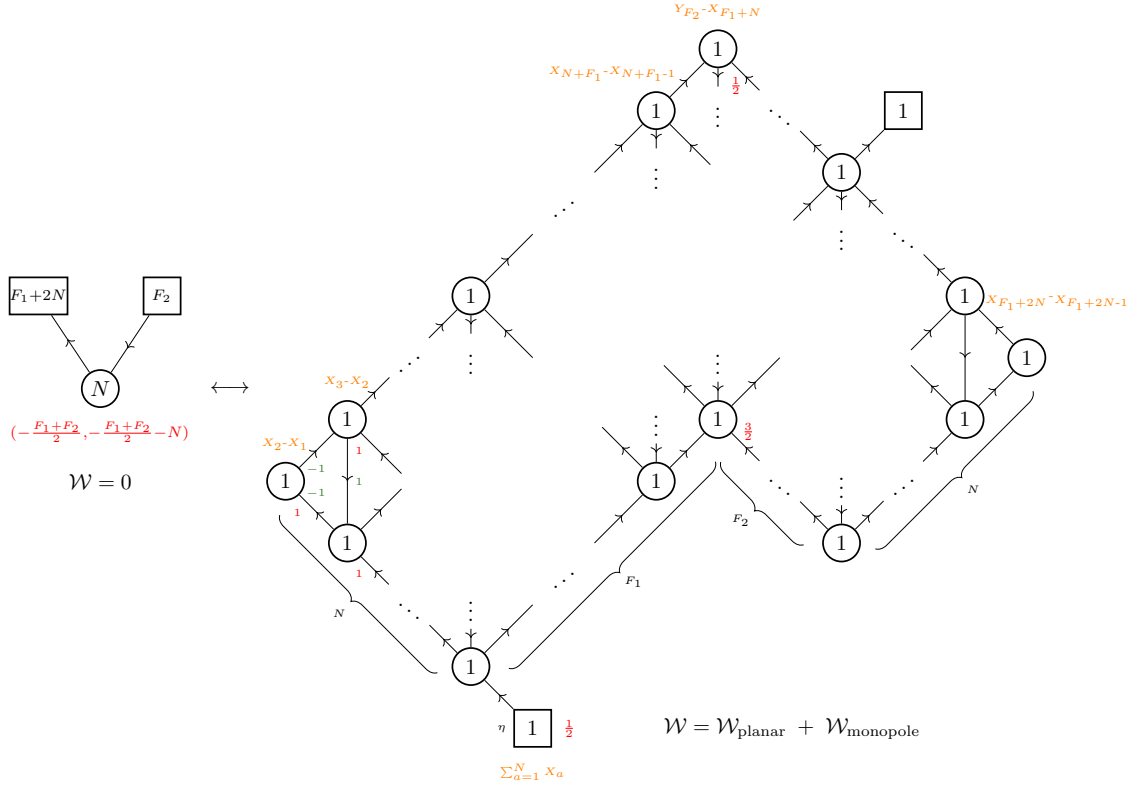


Figure 2.15: The $\mathcal{N} = 2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1 + 2N, F_2]$ chiral multiplets. The superpotential terms and (mixed) Chern-Simons interactions are as described in Section 2.1.

It is possible to derive planar abelian duals for SQCD theories with $SU(N)$ gauge group or with $U(N)$ gauge group and CS levels $(k, k + \ell N)$ for any ℓ by applying Witten's $SL(2, \mathbb{Z})$ action [106] on both sides of the dualities described above.

2.4.2 An Example: The Planar Limit of “Electric” CS-SQCD₃

In Section 2.1, we started from the $\mathcal{N} = 4$ mirror pair of the $U(N)$ SQCD and considered the real mass deformation leading to the $\mathcal{N} = 2$ mirror duality relating the chiral $U(N)$ SQCD to its planar quiver dual pair, depicted in Figure 2.2. A natural question is whether we could exchange the two operations, i.e.: if we could take the planar limit of the SQCD theory.

It is possible to directly take the planar limit of the SQCD theory, and follow the real mass and Higgsing at the level of the $\mathbf{S}_b^3$ partition function. However, since these quivers are (generically) overbalanced, this is a tedious exercise for general values of N and F . The dualization algorithm allows us to reach the planar limit of such theories and is the subject of this section. To achieve this, we consider the chiral limit of the mirror of SQCD, namely:

$$(2.136)$$

We decompose this theory into fundamental blocks, dualize each of them, and then glue back the result by implementing the Identity-walls. These steps are depicted in the following figure:

$$(2.137)$$

Notice that the quiver considered in (2.136) actually does not follow the rule prescribed at the beginning of the section since it contains two tails built with asymmetric $U(N) \times U(M)$ bifundamentals. However, we can bypass the problem of dualizing asymmetric bifundamental blocks by noticing that the two tails should be regarded as $\mathcal{S}$ and $\mathcal{S}^{-1}$ -walls.

These walls do not need to be dualized, and in the last step, they fuse to Identity-walls with two other walls coming from the dualization of the flavor blocks.

The resulting quiver can then be written explicitly in Lagrangian form, providing the planar mirror description for the original linear quiver gauge theory (2.136), as shown in Figure 2.16.

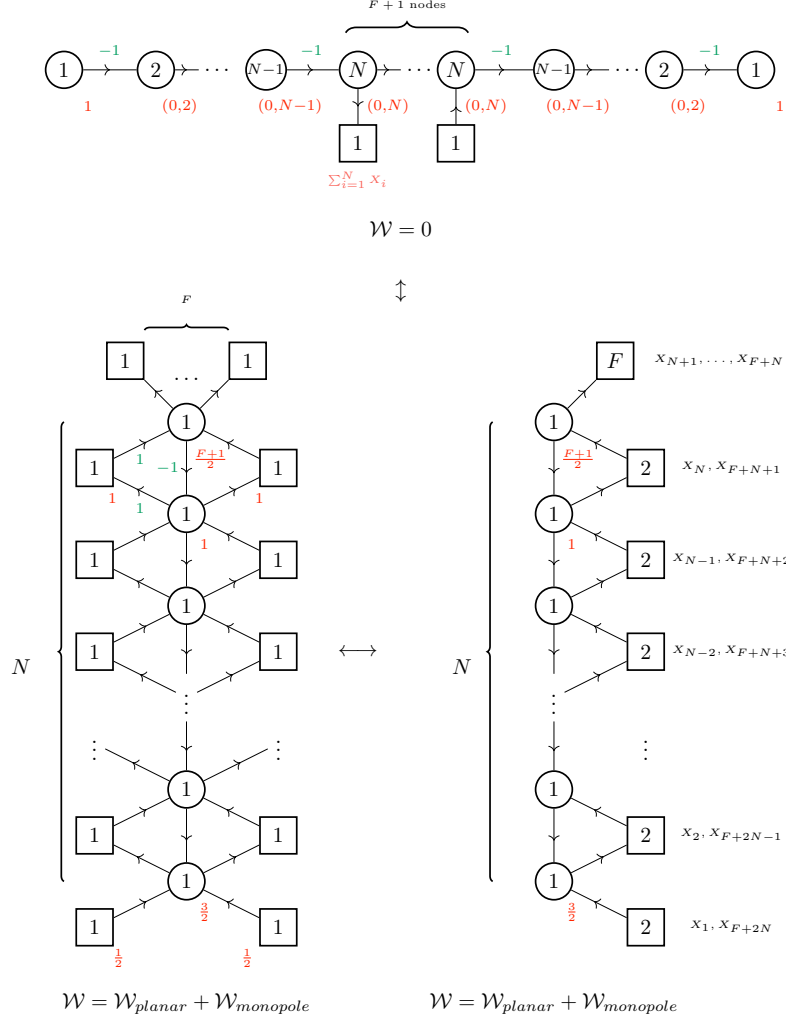


Figure 2.16: Top: the mirror dual of the planar limit of the $\mathcal{N} = 4$ SQCD, as obtained from (2.136). Bottom: the planar limit of the $\mathcal{N} = 4$ SQCD itself. For brevity, we have suppressed the FI parameters and all background terms. The FI parameters for the gauge nodes are $\eta + \frac{iQ}{4}(F\delta_{1,\alpha} - \delta_{N,\alpha})$. On the right-hand side, the same theory is shown after making the flavor symmetry $S[U(F) \times U(2)^N]$ manifest and writing it explicitly.

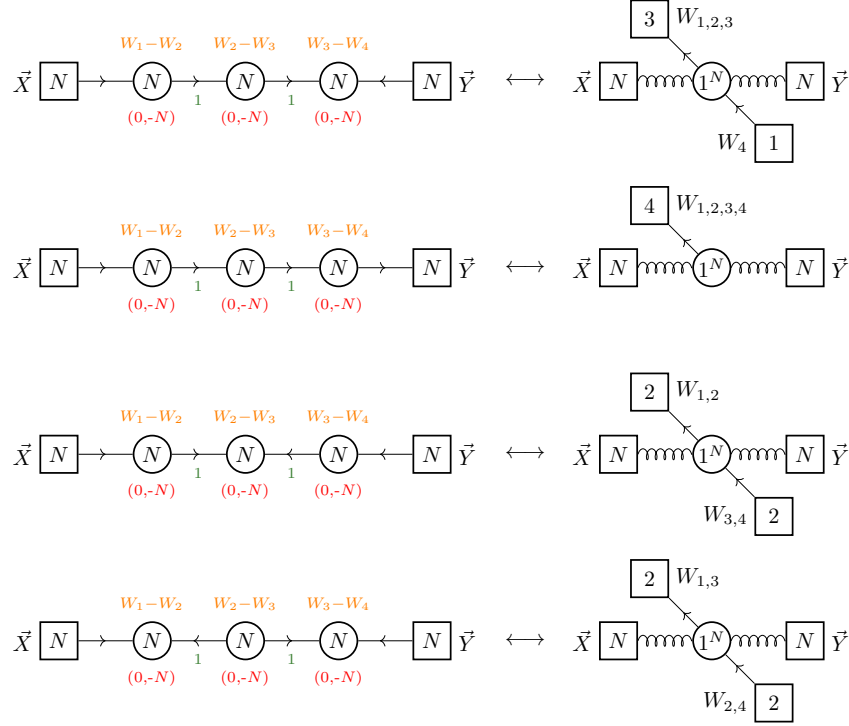
Interestingly, in the planar dual the flavor symmetry is not the naive $U(1)^{F+2N}/U(1)$, but instead enhances to

$$S[U(F) \times U(2)^N],$$

as shown in Figure 2.16. This provides a particularly striking example of topological symmetry enhancement.

The emergence of the $U(F)$ factor admits a natural explanation: the central portion of the quiver consists of a chain of $U(N)$ gauge nodes, each coupled to $[N, N]$ flavors, with bi-fundamental fields oriented uniformly along the quiver - the resulting mirror dual displays a non-trivial enhancement of flavor symmetry. This pattern of symmetry enhancement is

made transparent by using the dualization algorithm and is summarized below:



By contrast, the appearance of the multiple $U(2)$ factors is a genuinely new feature of this duality. This enhancement involves topological symmetries associated with *non-adjacent* gauge nodes and is therefore intrinsically non-local, unlike the local mechanism responsible for the $U(F)$ enhancement.

Remarkably, both types of enhancement are simultaneously realized and made manifest in the planar dual description.

2.4.3 An Example: Circular Quivers

Our final example is that of an $\mathcal{N} = 2$ circular quiver (*à la* [222–225]). Since the $SL(2; \mathbb{Z})$ action is defined locally, much of the machinery from the linear case carries over unchanged to circular quivers. As discussed in [222], circular quivers can be “chopped” open, transforming them into linear quivers, with the understanding that the circular structure is recovered when the ends are identified. While the algorithm applies unchanged from this point onwards, circular quivers display an additional $U(1)$ symmetry due to their global topology. We do not keep track of this symmetry here and only focus on applications of the chiral-planar dualization algorithm to these quivers; nevertheless, [225] develops a systematic approach to do the same for $\mathcal{N} = 4$ quivers at the level of the $\mathbf{S}_b^3$ partition function, and we expect a similar approach would hold for $\mathcal{N} = 2$ chiral-planar quiver duals.

Concretely, we consider an $\mathcal{N} = 4$ circular quiver of length M and its corresponding mirror dual (of length $\sum_{i=1}^M F_i$), shown in Figure 2.17.

Upon considering a suitable real mass deformation, we flow to the $\mathcal{N} = 2$ mirror dual pair, shown in Figure 2.18. Alternatively, one may arrive at the duality starting directly from the $\mathcal{N} = 2$ circular quiver and applying the algorithm directly.

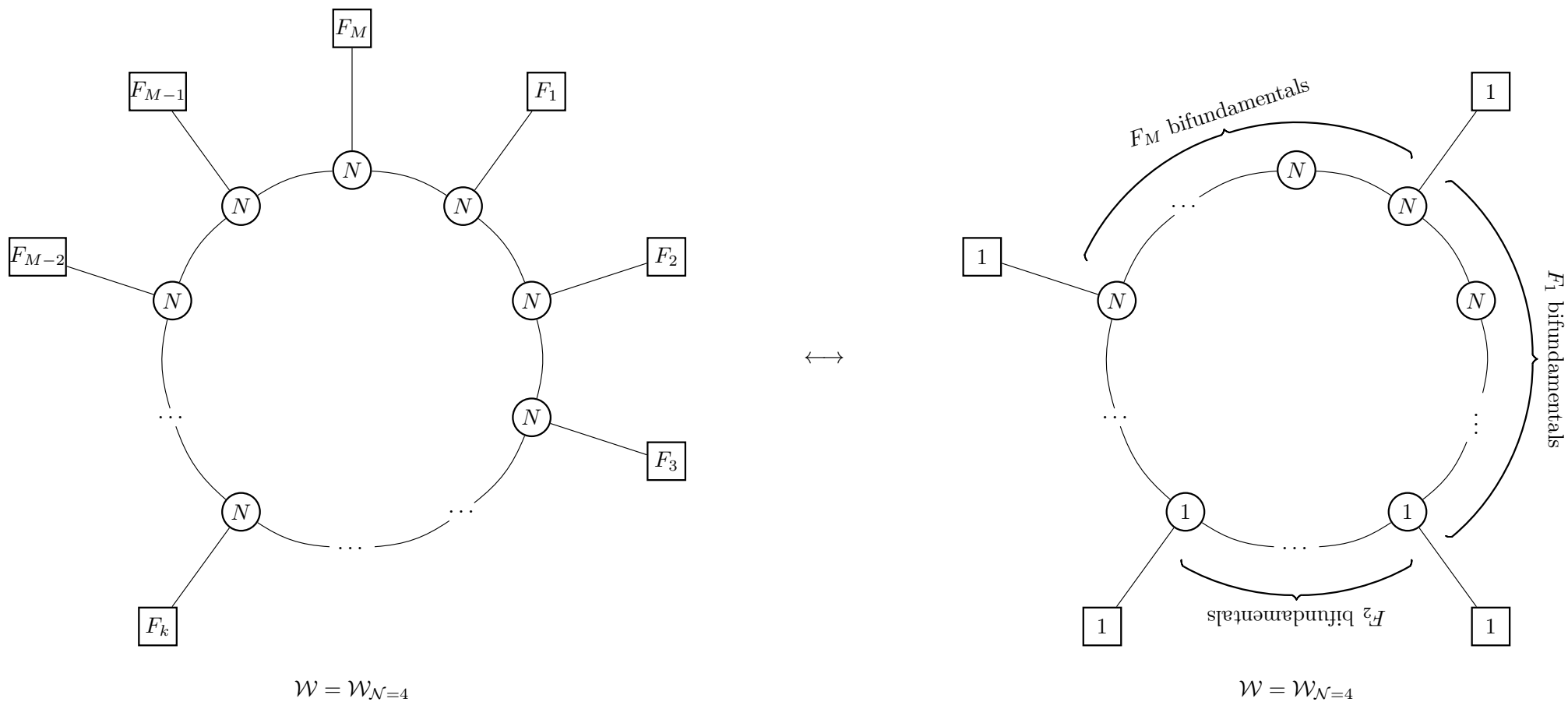


Figure 2.17: An $\mathcal{N} = 4$ circular quiver of length M and its mirror dual are shown here. Note that all the matter content is $\mathcal{N} = 4$ hypermultiplets, consisting of an $\mathcal{N} = 2$ chiral and anti-chiral multiplet.

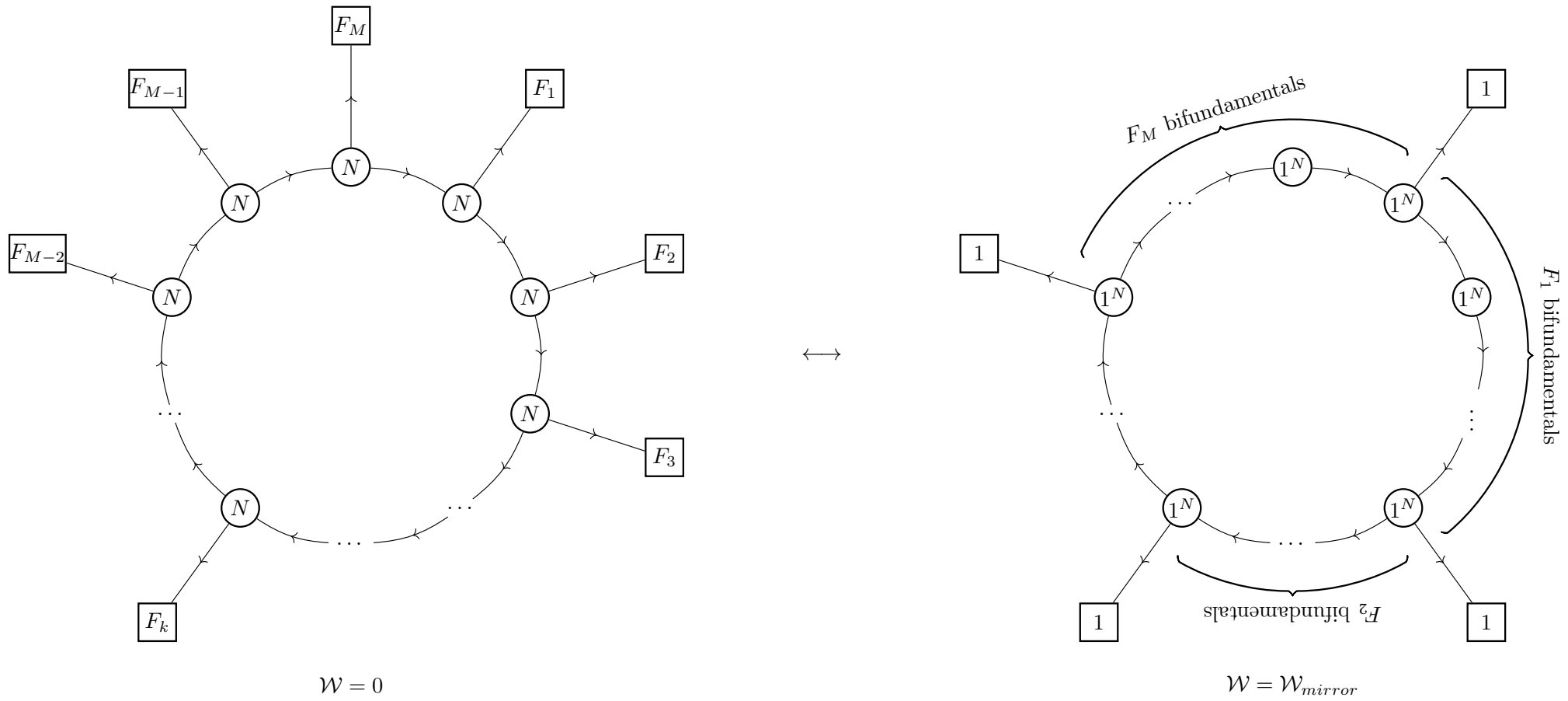
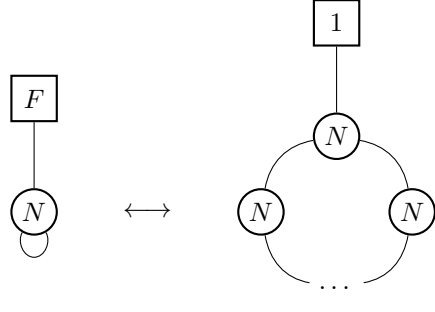


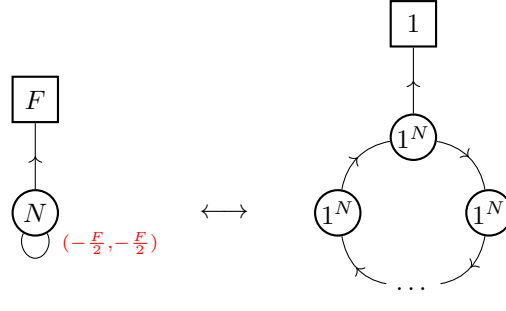
Figure 2.18: The result of applying the dualization algorithm to an $\mathcal{N} = 2$ circular quiver of length M is shown here. While the mirror dual admits a Lagrangian description, the resulting quiver is written in terms of the compact notation. In fact, the geometry of the quiver suggests the underlying $\mathcal{N} = 2$ HW setup would involve brane tilings [226]. We have suppressed the CS interactions for brevity.

Extensions to Adjoint CS-SQCD₃

As a limiting case of $\mathcal{N} = 4$ mirror dualities of circular quivers, we consider the duality for adjoint $U(N)$ SQCD:


(2.138)

We would flow to the following $\mathcal{N} = 2$ duality by considering an appropriate real mass deformation. Alternatively, we would arrive at this duality by applying the dualization algorithm to $\mathcal{N} = 2$ adjoint $U(N)_{(-\frac{F}{2}, -\frac{F}{2})}$ SQCD with $[F, 0]$ chiral fields.


(2.139)

It would be interesting to see how planar Abelianization persists in $\mathcal{N} = 2$ theories with tensor matter.

Going Beyond the Algorithm

The results of this chapter strongly suggest that planar Abelian dual descriptions exist for a broad class of three-dimensional $\mathcal{N} = 2$ quiver gauge theories. While we do not provide a formal proof that all $\mathcal{N} = 2$ quivers admit such a description, our findings indicate that this possibility is far more general than might have been expected *a priori*.

In particular, the strategy illustrated in Section 2.1—based on real mass deformations of known $\mathcal{N} = 4$ mirror pairs—appears to be applicable whenever the parent $\mathcal{N} = 4$ theories admit a Lagrangian description. In all cases analyzed, this procedure leads to a consistent $\mathcal{N} = 2$ chiral-planar mirror duality.¹³

From this perspective, the assumptions required to implement the algorithm in the examples studied in this work should be viewed as primarily technical rather than fundamental. Indeed, many of these restrictions can already be relaxed by performing a direct real mass analysis, as in Section 2.1, or may be overcome by developing a more refined version of the algorithm. In the case of the latter, the natural starting point would be accomplished by implementing the supersymmetry-breaking real mass deformation for the more general algorithmic prescription presented in [213]. As such, many generalizations appear possible.

¹³This argument relies on the availability of a Lagrangian description of the $\mathcal{N} = 4$ mirror theories. For many interesting $\mathcal{N} = 4$ SCFTs such a description is not known, and in those cases neither the partition function nor the equations of motion are under sufficient control to directly apply the present strategy.

Chapter 3

The Zoology of $\mathcal{N} = 2$ Unitary CS-SQCD₃ and their Planar Abelian Duals

In Chapter 2 (based on [2, 3]), we showed that a large class of $\mathcal{N} = 2$ Chern-Simons-matter theories admit planar, Abelian dual descriptions. These planar Abelian duals arise from suitable axial mass deformations of known $\mathcal{N} = 4$ mirror pairs [19, 22]. Because they descend from $\mathcal{N} = 4$ theories, these $\mathcal{N} = 2$ duals exist only on a special locus of parameter space — the $\mathcal{N} = 4$ descendant line

$$F = 2|k| + 2N$$

where the Chern–Simons level k is fixed by the number of flavors F and gauge rank N of the parent theory. Away from this locus, the parameters (F, k, N) vary independently, and the structure of a planar Abelian dual is, *a priori*, unclear. Extending the paradigm of planar Abelianization beyond the $\mathcal{N} = 4$ locus is, therefore, essential for obtaining a unified understanding of the landscape of $\mathcal{N} = 2$ SQCD.

The key observation of [5] is that **real mass deformations provide a systematic bridge** between the known planar duals on the $\mathcal{N} = 4$ descendant line and generic $\mathcal{N} = 2$ theories. Real masses shift Chern–Simons levels, integrate out matter fields, and move the theory through the (F, k, N) parameter space along controlled RG-flow trajectories. By tracking these deformations simultaneously in the non-Abelian theory and its planar mirror, we are able to study the duality away from the $\mathcal{N} = 4$ locus and construct the planar Abelian dual for any $U(N)_k$ ¹ SQCD with F fundamental fields.

3.1 The Taxonomy of $\mathcal{N} = 2$ CS-SQCD₃ with Fundamental Matter

Three-dimensional $\mathcal{N} = 2$ $U(N)_k$ SQCD theories with F fundamental chiral multiplets can be organized in a two-dimensional parameter space spanned by (k, F) . Each point in this plane corresponds to a distinct theory. While we exclusively focus on $k \leq 0$, analogous generalizations can be seen for $k > 0$.

¹While the subject of Chapter 2 are $U(N)_{(k, k+N\ell)}$ theories, we focus on $U(N)_k$ here instead to connect our findings to known Aharony-like dualities more transparently. The results obtained for $U(N)_k$ can be generalized to theories with $\ell \neq 0$ using the standard gauging/ungauging prescription.

This organization, summarized in Figure 3.1, reveals five broad regimes distinguished by the relative magnitude of F , k & N , which can be further understood in terms of their low-energy behavior, corresponding to their Aharony-dual phase [18, 21] (which defines the maximally, marginally and minimally chiral regions of parameter space).

We identify the supersymmetry breaking region defined by:

$$\mathcal{SUSY} : 2|k| < 2N - F \cap F < N. \quad (3.1)$$

On the boundary of this region, for $(F = N) \cap (2|k| < N)$ and $(F < N) \cap (2|k| = 2N - F)$, the theory is trivial in the IR, except for $F = N$ and $2|k| = N$, which corresponds to a single free chiral multiplet.

A special role is played by the $\mathcal{N} = 4$ -descendant line in this landscape,

$$F = 2|k| + 2N,$$

highlighted in orange in Figure 3.1. Theories on this line are precisely those that can be obtained as axial mass deformations of $\mathcal{N} = 4$ SQCD₃ *a la* [2, 3]. In [5], we show that the planar Abelian dual of more general SQCD theories is qualitatively different depending on whether the CS level is larger or smaller compared to the $\mathcal{N} = 4$ -descendant, namely whether the theory lies above or below the orange line in Figure 3.1.

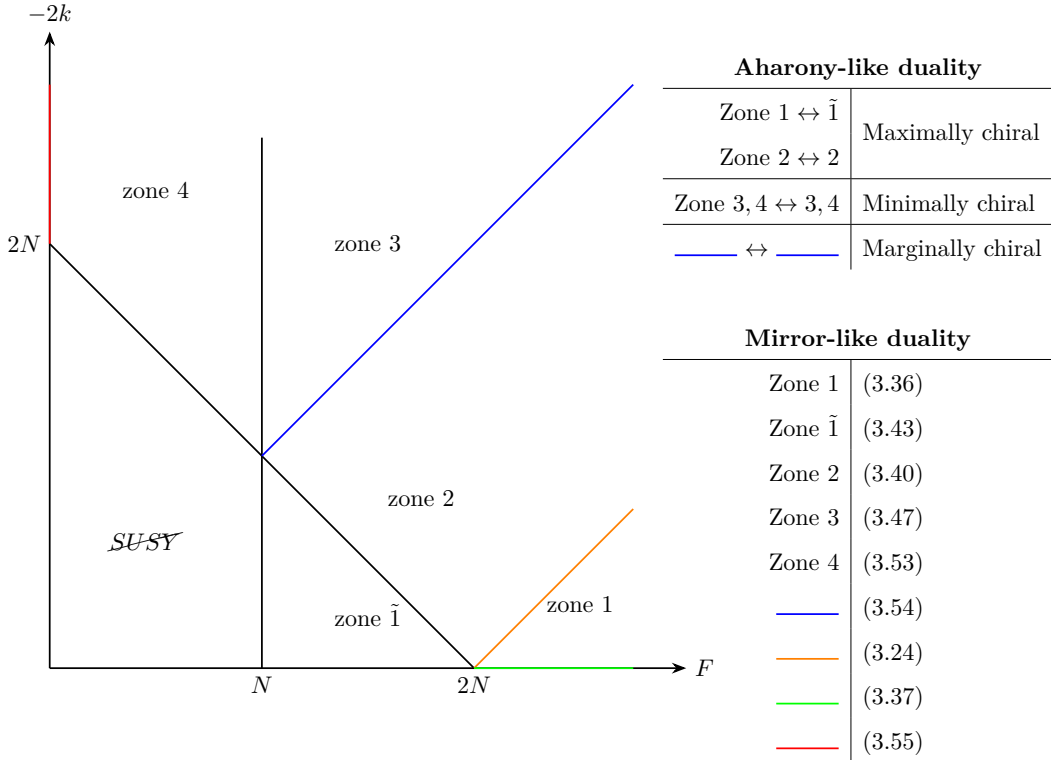


Figure 3.1: Phase diagram of $U(N)_k$ SQCD with F fundamental fields in the (F, k) -plane. The diagram is partitioned into various zones, within which the planar Abelian duals take qualitatively distinct forms. On the right, the action of Aharony-like duality [18, 20, 21, 146] and the mirror-like dual of the SQCD theories in the various zones. Theories on the red line flow to CS TQFTs, theories on the blue line correspond to marginally chiral SQCD and theories on the orange line are obtained from an axial mass deformation of $\mathcal{N} = 4$ SQCD and their mirror-like dual was proposed in [2, 3].

Motion in the k/F – Plane

A convenient way to explore this parameter space is through real mass deformations of the matter fields. For each fundamental chiral multiplet, there are four basic possibilities:

1. **Non-Higgsing Mass Flows:** Assigning a positive or negative real mass $\pm m$ removes one chiral field from the spectrum and shifts the CS level by $\pm \frac{1}{2}$ without affecting the rank of the gauge group. On the (k, F) –plane these correspond to diagonal moves toward the left:

$$\begin{array}{c} -m \\ \nwarrow \\ \bullet \text{ SQCD} \\ \swarrow \\ +m \end{array} \quad (3.2)$$

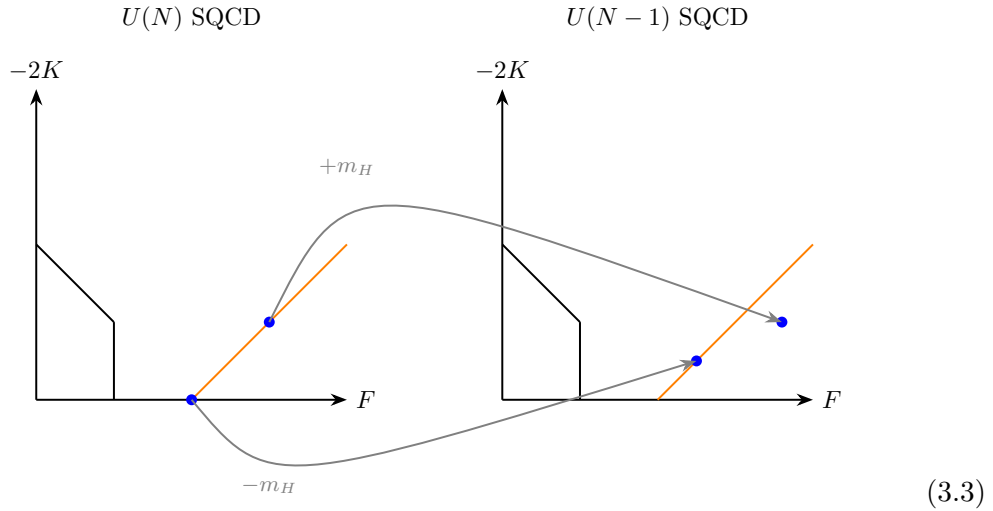
which can be stated more succinctly:

$$(F, k) \xrightarrow{\pm m} (F - 1, k \pm \tfrac{1}{2}).$$

2. **Higgsing Mass Flows:** Assigning a positive or negative real mass $\pm m_H$ removes one chiral field from the spectrum, shifts the CS level by $\pm \frac{1}{2}$ and reduces the rank of the gauge group by 1. On the (k, F) –plane these correspond to diagonal moves toward the right, which can be stated more succinctly:

$$(N, F, k) \xrightarrow{\pm m_H} (N - 1, F - 1, k \pm \tfrac{1}{2}).$$

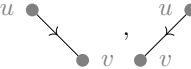
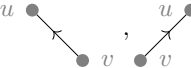
As an example, starting from $F = 2N$, $k = 0$, the deformation $-m_H$ produces a theory with $F' > 2N'$, which sits on the $\mathcal{N} = 4$ -descendant line to the right of the pseudo-self-dual point $F' = 2N'$. The deformation $+m_H$ reduces a theory sitting on the $\mathcal{N} = 4$ -descendant line to a theory which lies in Zone 1:



Taken together, these deformations allow us to probe the (k, F) –plane completely, starting from a given theory.

An Intermezzo: Compact Quiver Notation

For brevity, we adopt a uniform, compact notation for the planar Abelian quivers studied in this section:

Quiver notation	Field theory	$\mathbf{S}_b^3$
u 	$U(1)$ $\mathcal{N} = 2$ vector multiplet	$\int du$
X 	$U(1)$ Flavor node	—
	bifundamental chiral with R-charge r and mixed CS level $+1$	$s_b \left(\frac{iQ}{2} (1-r) - u + v \right) e^{-\pi i u v}$
	bifundamental chiral with R-charge $2r$ and mixed CS level $+1$	$s_b \left(\frac{iQ}{2} (1-2r) - u + v \right) e^{-\pi i u v}$
	bifundamental chiral with R-charge $1-r$ and mixed CS level -1	$s_b \left(\frac{iQ}{2} r + u - v \right) e^{\pi i u v}$
	bifundamental chiral with R-charge $2-2r$ and mixed CS level -1	$s_b \left(\frac{iQ}{2} (2r-1) + u - v \right) e^{\pi i u v}$
	Mixed CS term at level -2	$e^{2\pi i u v}$

(3.4)

Each gauge or flavor node $U(1)_i$ carries a Chern–Simons term at level k_i , determined by its mixed CS couplings through

$$k_i = -\frac{1}{2} \sum_{j \neq i} k_{ij}, \quad (3.5)$$

which contributes to the localized partition function as

$$e^{-\pi i k_i u_i^2},$$

where u_i denotes the corresponding gauge or background fugacity. This sum runs over *all* nodes in the quiver, including flavor nodes.

There is a monopole superpotential compatible with the R-charges and topological symmetries of each column.

3.1.1 Mass Deformations of CS-SQCD₃ with Fundamental Fields

In this section, we study mass deformations in $\mathcal{N} = 2$ $U(N)$ SQCD with CS levels $(k, k + \ell N)$ and F fundamental chiral fields using the equations of motion [32] of the theory. We introduce a large real mass deformation for one of the fundamentals, implemented through a real mass associated with a $U(1)$ subgroup in the Cartan of the $SU(F)$ flavor symmetry.

We parameterize the real masses associated to the Cartan of $SU(F)$ as X_i , $i = 1, \dots, F$ subject to the following constraint:

$$\sum_{i=1}^F X_i = 0 \quad (3.6)$$

In the presence of these real mass deformations as well as an FI term η , the equations of motion are:

$$(\sigma_\alpha - X_i) Q_\alpha^i = 0, \quad \alpha = 1, \dots, N, \quad i = 1, \dots, F \quad (3.7)$$

$$\sum_{i=1}^F Q_\alpha^{i\dagger} Q_i^\beta = \frac{\delta_{\alpha\beta}}{2\pi} \left[-\eta + k\sigma_\alpha + \ell \sum_{\gamma=1}^N \sigma_\gamma + \frac{1}{2} \sum_{i=1}^F |\sigma_\alpha - X_i| \right], \quad \alpha, \beta = 1, \dots, N \quad (3.8)$$

where Q_α^i are the quark VEVs, and we use gauge freedom to rotate the VEVs for the scalar σ in the $\mathcal{N} = 2$ vector multiplet to a diagonal matrix.

Mass Flows without Higgsing: We first turn on a real mass deformation parametrized by the shift:

$$\begin{cases} X_1 \rightarrow X_1 \mp (F-1)m, \\ X_i \rightarrow X_i \pm m, \\ \eta \rightarrow \eta + \left(\frac{F}{2} \pm (k + \ell N)\right)m. \end{cases} \quad 2 \leq i \leq F, \quad (3.9)$$

Since the quarks transform in the antifundamental representation of the $SU(F)$ flavor group, these shifts correspond to a positive mass (upper sign) and a negative mass (lower sign) for the first quark, respectively. We consider a vacuum wherein the gauge group is not Higgsed and the last $F-1$ quarks remain massless in the limit $m \rightarrow \infty$. This is achieved by giving a large VEV to the real scalars σ in the $\mathcal{N} = 2$ vector multiplet:

$$\sigma \rightarrow \pm \text{diag}(m, m, \dots, m) + \sigma. \quad (3.10)$$

The effective mass for the quarks Q_α^i with $i \geq 2$, given by $(\sigma_\alpha - X_i)$ vanish at leading order in m , while the effective mass for the first quark Q_α^1 is:

$$m_1 = \pm mF + \mathcal{O}(m^0) \quad (3.11)$$

rendering this quark massive in the $m \rightarrow \infty$ limit and, hence, can be integrated out. Up to $\mathcal{O}(m^0)$ terms, the equations of motion (3.7) reduce to the equations of motion for $U(N)$ with $(F-1)$ fundamentals and CS levels $k \pm \frac{1}{2}, k \pm \frac{1}{2} + \ell N$. The real mass deformations (3.9), together with the choice of vacuum (3.10), produce the following RG flows:

$$\begin{array}{c} U(N)_{k-\frac{1}{2}, k-\frac{1}{2}+N\ell} \text{ with } F-1 \square \\ \swarrow -m \\ U(N)_{k, k+N\ell} \text{ with } F \square \\ \searrow +m \\ U(N)_{k+\frac{1}{2}, k+\frac{1}{2}+N\ell} \text{ with } F-1 \square \end{array} \quad (3.12)$$

We denote these RG flows “+m” and “-m” in the remainder of this chapter.

Mass Flows with Higgsing: We now consider real mass deformations that simultaneously give mass to one quark and Higgs the gauge group to $U(N-1)$. We consider the following shifts of real mass parameters for large m_H :

$$\begin{cases} X_1 \rightarrow X_1 \mp (F-1)m_H, \\ X_i \rightarrow X_i \pm m_H, \\ \eta \rightarrow \eta + \left(\frac{F}{2} \pm (k + \ell N - \ell F)\right)m_H. \end{cases} \quad 2 \leq i \leq F \quad (3.13)$$

and we combine it with a large VEV for the real scalars σ :

$$\sigma \rightarrow \pm \begin{pmatrix} -(F-1) & 0 \\ 0 & \mathbf{1}_{N-1 \times N-1} \end{pmatrix} m_H + \sigma \quad (3.14)$$

which Higgses the gauge group to $U(N-1) \times U(1)$. The fundamentals Q_α^i with $\alpha \geq 2$ and $i \geq 2$ remain massless and form $F-1$ fundamentals for the $U(N-1)$ factor, while the component Q_1^1 is massless as well and carries a charge +1 under the $U(1)$ factor. The other quarks are massive and are integrated out. The D-terms for the $U(N-1)$ sector are satisfied up to $\mathcal{O}(m_H^0)$ order and reduce to the equations for $U(N-1)$ with $F-1$ fundamentals. The D-term for the $U(1)$ gauge factor, namely (3.8) with $\alpha = \beta = 1$, read:

$$\begin{aligned}
 2\pi|Q_1^1|^2 &= -\eta - \left(\frac{F}{2} \pm (k + \ell N - \ell F)\right) m_H + k\sigma_1 \mp k(F-1)m_H \\
 &\quad + \ell \sum_{\gamma=1}^N \sigma_\gamma \pm \ell(-F+N)m_H + \frac{1}{2} \sum_{i=2}^F |\sigma_1 - X_i \pm (-F)m_H| + \frac{1}{2} |\sigma_1 - X_1| \\
 &= \left(\frac{1}{2}F(F-1) - \frac{1}{2}F \pm (-k - \ell N + \ell F - k(F-1) + \ell(N-F))\right) m_H + \mathcal{O}(m_H^0) \\
 &= \left(\frac{1}{2}F(F-2) \mp kF\right) m_H + \mathcal{O}(m_H^0)
 \end{aligned} \tag{3.15}$$

This can be solved by giving a non-zero VEV to Q_1^1 such that²:

$$|Q_1^1|^2 = \frac{m_H}{2\pi} \left(\frac{1}{2}F(F-2) \mp kF\right) \geq 0 \tag{3.16}$$

Positivity of this expression determines the range of validity for the two possible deformations:

$$\begin{aligned}
 m_H : \quad &F \geq 2(k+1), \\
 -m_H : \quad &F \geq 2(-k+1).
 \end{aligned} \tag{3.17}$$

These regions of parameter space are illustrated in Figure 3.2.

When the deformation is allowed, the $U(1)$ factor is fully Higgsed, and the resulting theory is $U(N-1)$ SQCD with $F-1$ fundamentals and CS levels $(k \pm \frac{1}{2}, k \pm \frac{1}{2} + \ell(N-1))$. Together with the previous non-Higgsing deformations, we obtain four distinct flows from $U(N)$ SQCD with F fundamentals to theories with $F-1$ fundamentals:

$$\begin{array}{ccc}
 U(N)_{k-\frac{1}{2}, k-\frac{1}{2}+N\ell} \text{ with } F-1 \square & & U(N-1)_{k-\frac{1}{2}, k-\frac{1}{2}+(N-1)\ell} \text{ with } F-1 \square \\
 \swarrow -m & & \nearrow -m_H \\
 & U(N)_{k, k+N\ell} \text{ with } F \square & \\
 \swarrow +m & & \searrow +m_H \\
 U(N)_{k+\frac{1}{2}, k+\frac{1}{2}+N\ell} \text{ with } F-1 \square & & U(N-1)_{k+\frac{1}{2}, k+\frac{1}{2}+(N-1)\ell} \text{ with } F-1 \square
 \end{array} \tag{3.18}$$

²When $\frac{F}{2} - 1 \mp k = 0$ the D-terms are satisfied with vanishing VEV for the chirals and Q_1^1 remains a massless d.o.f. charged under the $U(1)$ factor of the gauge group. This is the case when the SQCD obtained after the mass deformation is marginally chiral.

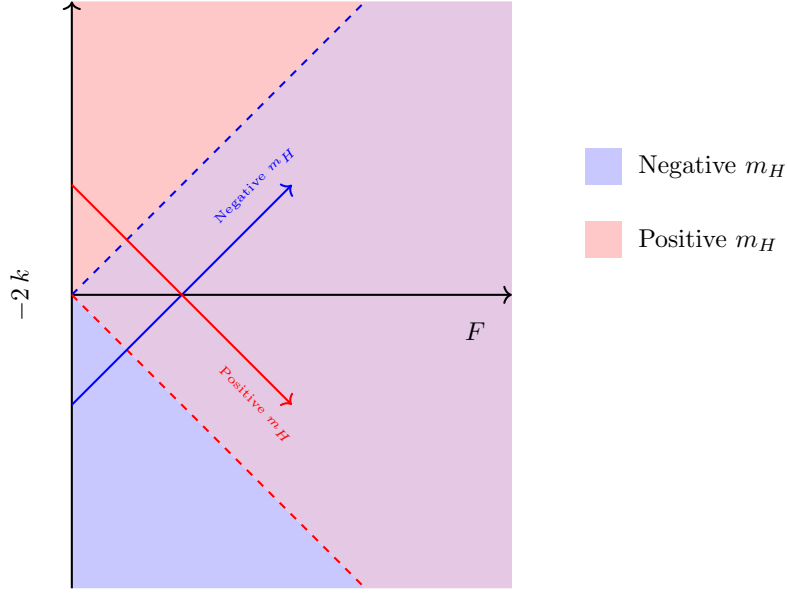


Figure 3.2: RG-flow trajectories induced by real mass deformations accompanied with Higgsing in the (F, k) -plane.

3.2 Mapping Mass Flows Across Aharony-like Duality

We remind the reader that SQCD with $U(N)$ gauge group and CS levels $k, k + \ell N$ and F fundamentals is mapped by Aharony-like duality to $U(\tilde{N})$ SQCD with CS levels $-k, -k + \ell \tilde{N}$ and F antifundamentals [18, 21] (see also [20, 146, 227] for a comprehensive review).

For simplicity, we focus on the case $\ell = 0$, corresponding to equal CS levels for the two $U(1)$ factors. The general case with $\ell \neq 0$ can be obtained by applying Witten's $SL(2, \mathbb{Z})$ action on both sides of the duality [106].

The dual rank $\tilde{N}$ depends on the relative values of F and k , dividing the parameter space into three regimes — minimally chiral, marginally chiral, and maximally chiral — characterized as follows:

$$\begin{aligned}
 \text{minimally chiral: } F < |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N \\
 \text{marginally chiral: } F = |2k|, & \quad \tilde{N} = \frac{F}{2} + |k| - N = F - N \\
 \text{maximally chiral: } F > |2k|, & \quad \tilde{N} = F - N
 \end{aligned} \tag{3.19}$$

In the marginally chiral case there is also an additional gauge singlet in the magnetic theory that flips a monopole:

$$\mathcal{W}_{mag}^{\frac{F}{2}=|k|} = T\mathfrak{M}^{\mp} \tag{3.20}$$

where the upper (lower) sign correspond to $k > 0$ ($k < 0$).

One can flow between different Aharony-like dualities by consistently turning on real masses on both sides of the duality. We outline the mapping of mass deformations $\pm m$ and $\pm m_H$ across Aharony-like duality; a detailed derivation is provided in the Appendix of [5].

In the *maximally chiral case*, all four deformations in (3.18) are available, as can be checked from Figure 3.2. Aharony duality maps positive real masses to negative real

masses. Consistency with the value of $\tilde{N}$ implies that mass deformations are mapped as follows:

$$\left\{ \begin{array}{l} +m \\ -m \\ +m_H \\ -m_H \end{array} \right\} \xleftrightarrow[\text{maximally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -\widetilde{m_H} \\ +\widetilde{m_H} \\ -\tilde{m} \\ +\tilde{m} \end{array} \right\} \quad (3.21)$$

where $\pm\tilde{m}$ and $\pm\widetilde{m_H}$ denote mass deformations with or without Higgsing for $U(N)$ SQCD with only antifundamentals, analogous to the deformations discussed above for SQCD with only fundamentals.

In the *minimally* and *marginally chiral* cases, only three of the four deformations are available. The two non-Higgsing deformations, $+m$ and $-m$, are always accessible, whereas only one of the two Higgsing deformations ($\pm m_H$) is allowed, depending on the sign of the Chern–Simons level k . For concreteness, we set $k < 0$, where the electric theory admits the m_H deformation. The Aharony dual theory has CS level $-k > 0$ and one can check that the deformation $-\widetilde{m_H}$ is available. The mapping of mass deformations is given by:

$$k < 0 : \quad \left\{ \begin{array}{l} +m \\ -m \\ +m_H \end{array} \right\} \xleftrightarrow[\text{marginally or minimally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -\widetilde{m_H} \\ +\tilde{m} \\ -\tilde{m} \end{array} \right\} \quad (3.22)$$

similarly for $k > 0$:

$$k > 0 : \quad \left\{ \begin{array}{l} +m \\ -m \\ -m_H \end{array} \right\} \xleftrightarrow[\text{marginally or minimally chiral}]{\text{Aharony}} \left\{ \begin{array}{l} -m \\ -\widetilde{m_H} \\ +m \end{array} \right\} \quad (3.23)$$

3.3 Mapping Mass Flows Across Mirror Duals

In this section, we analyze how real-mass deformations of $\mathcal{N} = 2$ SQCD are realized on their planar Abelian duals. We do this by comparing the equations of motion on the electric and mirror sides in the large-mass limit. We then focus on a pedagogical example to elaborate on some features of our proposal. At each stage, we indicate the zone of the k/F -plane (Figure 3.1) to which the theory belongs.

3.3.1 Deformations of the $\mathcal{N} = 4$ Descendant SQCD₃

As a starting point, we consider the planar Abelian dual of $U(N)_{N-\frac{F}{2}}$ SQCD with F fundamental fields, shown in Figure 3.24. We remind the reader that this dual is the one we would obtain directly from $\mathcal{N} = 4$ mirror symmetry.

$$U(N)_{N - \frac{F}{2}} \text{ with } F \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Quiver Diagram} \\ \text{with } F - N \text{ nodes} \end{array} \quad (3.24)$$

Such theories lie in the maximally chiral regime, where all four deformations $\pm m$ and $\pm m_H$ are available. Across the duality, a real mass for a quark in SQCD maps to an FI term for the gauge nodes contained in a single column of the planar Abelian quiver, while the FI parameter of the SQCD maps to a real mass for one of the two flavor nodes of the Abelian theory.

Consequently, giving a real mass to the i -th quark corresponds to turning on a large FI parameter for one (or two) adjacent columns of the Abelian quiver. Although different choices of i are equivalent at the level of the IR-SCFT—being related by Weyl transformations of the $SU(F)$ flavor symmetry—this symmetry is not manifest in the planar quiver representation. As a result, the UV analysis of the deformation depends on the specific choice of i , even though the corresponding IR deformations describe the same flow.

For concreteness, we adopt the following conventions for the four possible deformations:

$$\begin{aligned} +m, +m_H : & \quad \text{large mass for the last quark } Q^F \leftrightarrow X_F \rightarrow -\infty \\ -m_H, -m : & \quad \text{large mass for the first quark } Q^1 \leftrightarrow X_1 \rightarrow +\infty \end{aligned} \quad (3.25)$$

Therefore, the deformations $+m, +m_H$ correspond to evaluating a large FI limit for the rightmost node in the quiver, while the deformations $-m, -m_H$ correspond to a large FI limit for the leftmost node.

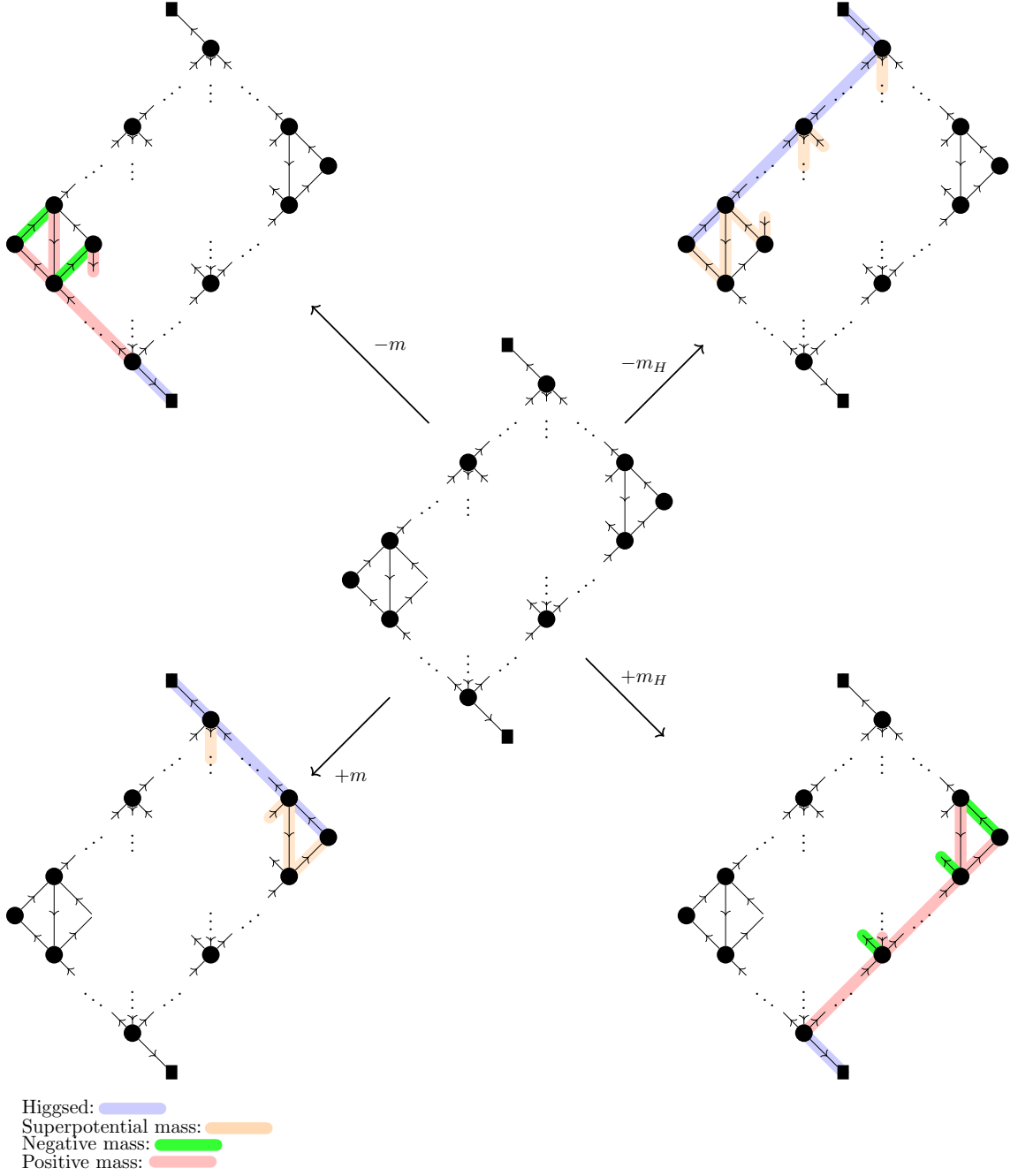


Figure 3.3: Deformations of the planar Abelian quiver dual to $U(N)_{(N-\frac{F}{2})}$ SQCD with F fundamentals. Chiral fields highlighted in pink acquire a large positive mass, fields highlighted in green acquire a large negative mass, and fields highlighted in blue acquire a VEV, thereby Higgsing the nodes that they connect to a diagonal $U(1)$. The quivers resulting after integrating out the massive fields and performing the Higgsing are shown in Figure 3.4.

We propose the following vacua for the four mass deformations:

- $+m$: In this deformation, the chiral fields along the top-right diagonal and the chiral field attached to the topmost gauge node acquire non-zero VEVs. These fields are highlighted in blue in the bottom-left panel of Figure 3.3. The Higgsing identifies all gauge nodes along the top-right diagonal with the adjacent flavor node. Conse-

quently, the planar superpotential reduces to a set of mass terms, allowing one to integrate out the $2(N-1)$ chiral fields highlighted in orange in Figure 3.3, as shown in Equation 3.26.

$$\mathcal{W} \supset \mathcal{W}_{\text{planar}} \xrightarrow{\text{Higgs}} \mathcal{W} \supset \sum_{i=1}^{2N-3} q_i q_{i+1} \xrightarrow{\text{Integrate out}} \mathcal{W} \supset \mathcal{W}_{\text{planar}} \quad (3.26)$$

– m : In this deformation, the real scalars in the vector multiplets associated with the gauge groups along the bottom-left diagonal acquire large VEVs:

$$(\sigma_1^{\text{bottom}}, \sigma_2^{\text{bottom}}, \dots, \sigma_{N-1}^{\text{bottom}}) \propto -((N-1)Fm, (N-2)Fm, \dots, Fm) \quad (3.27)$$

where σ_i^{bottom} is the real scalar of the gauge group associated to the bottom-most node in the i -th column. As a consequence, the chiral multiplets connecting these nodes acquire large positive real masses. Furthermore, vertical chiral multiplets also acquire a large positive mass, while the remaining fields (highlighted in green in Figure 3.3) acquire large negative masses. Moreover, the chiral field attached to the bottom flavor node is Higgsed.

Integrating out the massive fields yields the quiver shown below:

$$U(N)_{N-\frac{F}{2}-\frac{1}{2}} \text{ with } F-1 \text{ } \square \longleftrightarrow$$

(3.28)

This intermediate quiver contains a tail of $U(1)$ gauge nodes that interact with the rest of the quiver only through mixed CS couplings. Such a tail can be systematically contracted by sequentially integrating out the last node, which corresponds to an almost trivial $U(1)_1$ Chern–Simons theory³.

³Background gravitational CS terms (CS_g) are omitted from the quiver notation.

The resulting duality is then represented schematically as:

$$U(N)_{N-\frac{F}{2}-\frac{1}{2}} \text{ with } F-1 \text{ } \square \longleftrightarrow \text{Diagram} \quad (3.29)$$

$+m_H$: In this deformation, the real scalars in the vector multiplets of the gauge groups in the right-bottom-most diagonal acquire non-zero VEVs:

$$(\sigma_{F-1}^{bottom}, \sigma_{F-2}^{bottom}, \dots, \sigma_{N+1}^{bottom}) \propto -((F-N-1)Fm, (F-N-2)Fm, \dots, m). \quad (3.30)$$

Consequently, the chiral fields connecting these nodes acquire a large positive mass. Furthermore, vertical chiral multiplets acquire large positive masses, while the remaining chiral multiplets (highlighted in green in the bottom-right panel of Figure 3.3) acquire a negative mass.

Furthermore, the chiral field connected to the bottom flavor node is Higgsed. Integrating out the massive fields and contracting the resulting tail, analogous to the $+m$ case, results in the following duality:

$$U(N-1)_{N-\frac{F}{2}+\frac{1}{2}} \text{ with } F-1 \square \longleftrightarrow \text{Quiver Diagram} \quad (3.31)$$

$-m_H$: In this deformation, the chiral fields in the top-left diagonal and the chiral field connected to the top-most gauge node acquire a VEV. These fields are highlighted in blue in the top-right panel of Figure 3.3. As a result, all the gauge nodes in the top-right diagonal are identified with the flavor node. The planar superpotential reduces to mass terms which we use to integrate out $2(N-1)$ chiral fields, analogously to (3.26).

We summarize these mass flows in Figure 3.4. Furthermore, in Figure 3.4, in addition to the conventions for nodes and arrows introduced in Section 3.1, we have also included dashed lines representing mixed CS interactions. Specifically, a dashed line connecting two nodes labeled $U(1)_u$ and $U(1)_v$ denotes a mixed CS term at level -2 between these gauge groups.

After performing either a $+m$ or a $-m_H$ deformation, the resulting quiver retains the same rectangular structure as the original planar Abelian dual. In fact, the resulting theory is precisely the planar Abelian dual of the SQCD model obtained after the corresponding mass deformation on the electric side. These two deformations therefore provide a **nontrivial self-consistency check** of the duality originally proposed in [2, 3].

In contrast, the deformations $-m$ and $+m_H$ yield quivers with a modified geometry, leading to different planar structures. Successive applications of these deformations require a separate analysis, which we undertake in the next section, where we construct and study the planar Abelian duals of SQCD theories obtained through multiple $-m$ or $+m_H$ mass deformations.

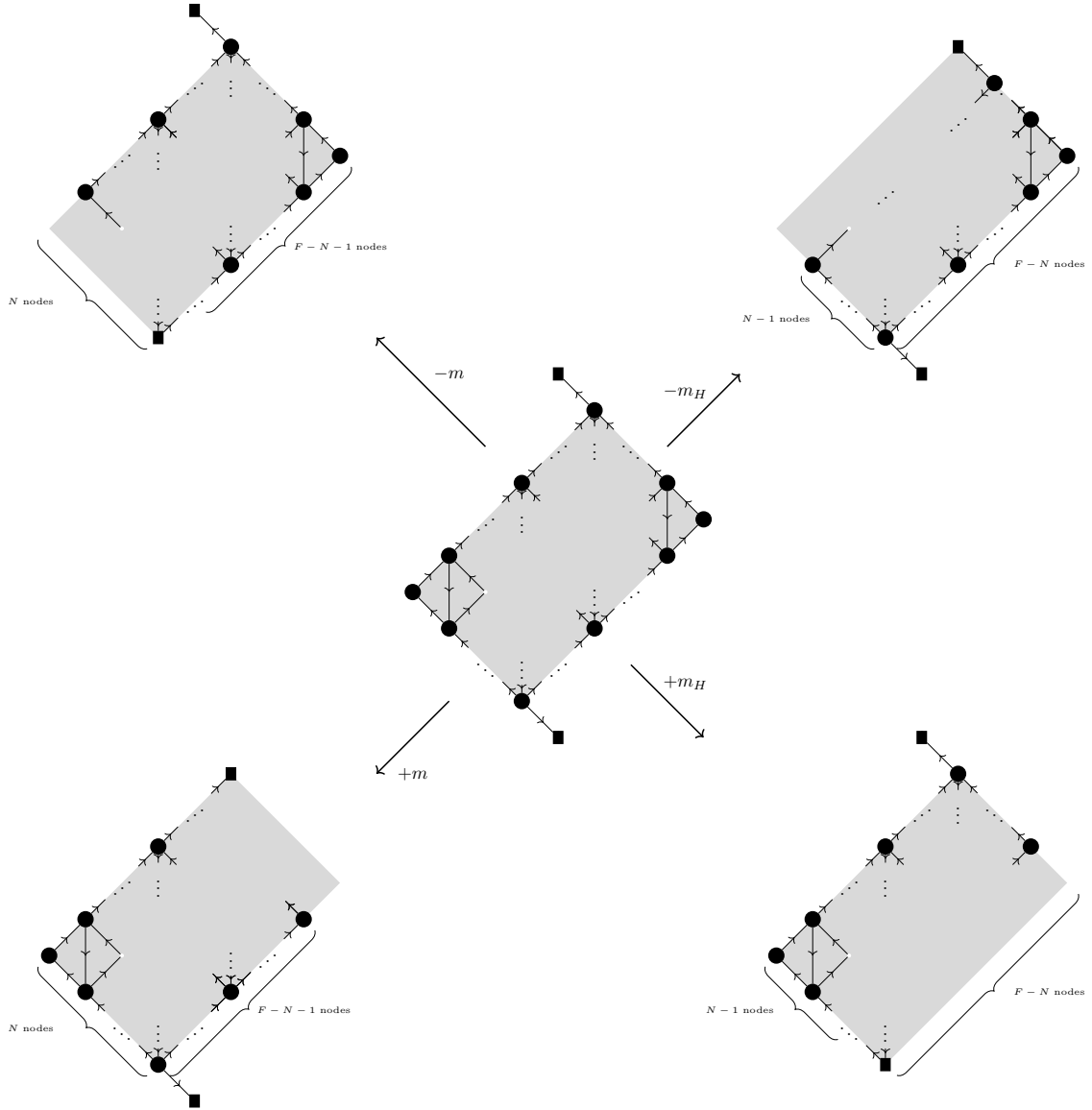


Figure 3.4: Deformations of the planar quivers dual to the SQCD mass deformations in (3.18). The grey shading highlights which parts of the quiver are modified by each mass deformation and makes explicit how the overall quiver geometry changes relative to the initial configuration.

Exploring Zone 1 via Multiple $+m_H$ Mass Deformations

In the previous section, we discussed the mapping of a single positive mass deformation accompanied with Higgsing on the planar Abelian dual of SQCD on the $\mathcal{N} = 4$ -descendant line. This resulted in a duality for SQCD theories with F fundamentals and $k = N - \frac{F}{2} + 1$ CS level, which lie in Zone 1 in the k/F plane (see Figure 3.1), just below the $\mathcal{N} = 4$ descendant.

To explore theories in Zone 1 more systematically, we must map the effect of adding multiple positive masses accompanied with Higgsing ($+m_H$) on the planar theory.

We perform the $+m_H$ deformation associated with the F -th quark, which is implemented by the real mass deformations in (3.13).

$$\begin{aligned}
 & U(N)_{N - \frac{F}{2} + 1} \text{ with } F \text{ } \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Diagram of a planar Abelian dual of SQCD} \\ \text{with } N \text{ nodes and } F - N \text{ nodes} \end{array} \\
 & \begin{array}{l} \text{Higgsed: } \text{blue bar} \\ \text{Superpotential mass: } \text{orange bar} \\ \text{Negative mass: } \text{green bar} \\ \text{Positive mass: } \text{red bar} \end{array}
 \end{aligned} \tag{3.32}$$

We propose that the corresponding deformation on the magnetic side is the following:

- The vertical bifundamental chiral connected to the bottom-most flavor node is Higgsed.
- The scalars in the vector multiplets of the gauge node in the bottom-right diagonal acquire a VEV:

$$(\sigma_{F-1}^{bottom}, \sigma_{F-2}^{bottom}, \dots, \sigma_{N+1}^{bottom}) \propto -((F-N-1)Fm, (F-N-2)Fm, \dots, Fm) \tag{3.33}$$

We checked that the RG flow induced by these deformations can be followed on the $\mathbf{S}_b^3$

partition function, and the resulting divergent phases cancel between the electric and magnetic theories. Similar to the first mass deformation $+m_H$ considered in Section 3.3.1, the gauge groups in the bottom-right diagonal become a tail of CS theories that only interact via mixed CS levels after integrating out the massive fields. This tail can be sequentially contracted, resulting in the following duality:

$$U(N-1)_{N-\frac{F}{2}+\frac{3}{2}} \text{ with } F-1 \square \longleftrightarrow \text{Diagram} \quad (3.34)$$

Further positive mass deformations accompanied by Higgsing are mapped in an analogous manner. In this case, the real scalars associated with the gauge nodes along the bottom-right diagonal acquire non-zero vacuum expectation values, which in turn generate large masses for the fields charged under them. Simultaneously, the vertical chiral multiplet attached to the bottom flavor node develops a Higgs VEV. Performing this sequence of deformations yields the planar Abelian dual quiver corresponding to a generic SQCD theory in Zone 1:

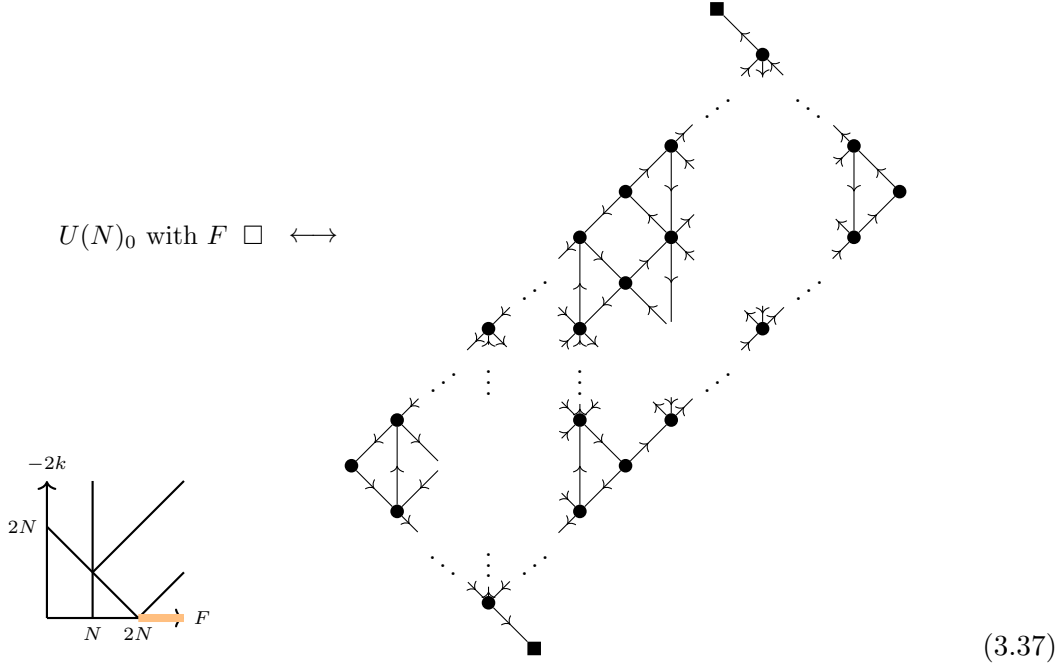
$$U(N)_k \text{ with } F \square \longleftrightarrow \text{Diagram} \quad (3.35)$$

The quiver on the left-hand side of (3.35) can be systematically simplified by applying local Aharony-like dualities, similar to the sequence shown in Section 3.4. Specifically, we dualize the quiver column by column, proceeding from left to right and halting at the $(\frac{F}{2} - k)$ -th column. The gauge nodes within this column that lie outside the rectangle of dimensions $N \times (F - N)$ confine. We then repeat the process, restarting from the left and stopping one column earlier each time, until only the leftmost gauge node is dualized. At every iteration, the nodes of the final dualized column that fall outside the $N \times (F - N)$ rectangle again confine.

After performing this complete sequence of local dualizations, we obtain the duality describing SQCD theories with $F > 2N + 2|k|$, namely those lying in **Zone 1** of Figure 3.1:

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \iff \\
 & (F \geq 2|k| + 2N)
 \end{aligned}
 \quad \longleftrightarrow \quad
 \begin{array}{c}
 \begin{array}{c}
 \text{Quiver diagram showing a sequence of dualizations. The diagram is a diamond shape with nodes and arrows. The leftmost column is labeled } \frac{F}{2} - |k| \text{ nodes. The rightmost column is labeled } \frac{F}{2} + |k| \text{ nodes.} \\
 \dots \\
 \text{Quiver diagram showing a sequence of dualizations. The diagram is a diamond shape with nodes and arrows. The leftmost column is labeled } \frac{F}{2} - |k| \text{ nodes. The rightmost column is labeled } \frac{F}{2} + |k| \text{ nodes.} \\
 \dots
 \end{array}
 \end{array}
 \quad (3.36)$$

As a limiting case, we show the dual of $U(N)_0$ with F fundamental chiral multiplets, for which the dual has a particularly simple, symmetric form:



$$U(N)_0 \text{ with } F \square \longleftrightarrow \text{Quiver Diagram} \quad (3.37)$$

Some additional comments regarding the $k = 0$ duality are in order:

- Beyond the CS and mixed CS levels encoded in the quiver, there are background contributions to the $\mathbf{S}_b^3$ partition function of the form $\exp\left(\frac{\pi i}{2}\phi^{(mag)}\right)$, with

$$\phi^{(mag)} = N \left(\sum_{i=1}^{\frac{F}{2}} (X_i^2 - iQ(1-3r)X_i) - \sum_{j=\frac{F}{2}+1}^F (X_j^2 + iQ(1-r)X_j) \right) + \mathcal{O}(Q^2). \quad (3.38)$$

Here, the quadratic terms encode the CS levels for the UV global symmetries $U(1)_{X_i}$, while the linear terms correspond to mixed CS couplings between $U(1)_{X_i}$ and the R-symmetry.

- We also identify the $U(1)_\Lambda$ flavor symmetry associated with the bottom node as

$$U(1)_\Lambda := U(1)_\eta - \sum_{i=1}^{\frac{F}{2}} U(1)_{X_i} + \frac{F}{2}(3r-1)U(1)_R,$$

where $U(1)_\eta$ is the topological symmetry of the SQCD. This assignment is consistent with our notion of mirror symmetry, which exchanges topological and flavor symmetries.

- There is an additional background interaction between the $U(1)_\Lambda$ flavor symmetry and the global $U(1)_{\vec{X}}$ and $U(1)_R$ symmetries. This may be regarded as an FI term for the $U(1)_\Lambda$ symmetry, and has the form

$$\sum_{i=1}^N X_i - \frac{iQ}{4}(r-1+2Nr).$$

- We encode the mapping in the FIs for the gauge nodes in the i -th column, which are, from top to bottom:
 - l.h.s. of the quiver: $X_{i+1} - X_i - \frac{iQ}{4}r$, $X_{i+1} - X_i$, $\dots$, $X_{i+1} - X_i$, $X_{i+1} - X_i + \frac{iQ}{4}r$
 - central column: $X_{\frac{F}{2}+1} - X_{\frac{F}{2}} + \frac{iQ}{4}(r-1)$, $X_{\frac{F}{2}+1} - X_{\frac{F}{2}}$, $\dots$, $X_{\frac{F}{2}+1} - X_{\frac{F}{2}}$, $X_{\frac{F}{2}+1} - X_{\frac{F}{2}} + \frac{iQ}{4}r$
 - r.h.s. of the quiver: $X_{i+1} - X_i + \frac{iQ}{4}(r-1)$, $X_{i+1} - X_i$, $\dots$, $X_{i+1} - X_i$, $X_{i+1} - X_i - \frac{iQ}{4}(r-1)$

Starting from this fully specified duality, including all background terms, one can in principle invert the logic of our construction: by tracking the RG flow at the level of the $\mathbf{S}_b^3$ partition function, one can recover the $\mathcal{N} = 4$ parent theory *à la* [5]. In the thesis, we follow the opposite route—starting from the $\mathcal{N} = 4$ descendants and performing deformations to obtain the general planar Abelian duals—but this observation highlights that the $k = 0$ duality contains all the necessary data for the reverse reconstruction. Moreover, the partition function provides an independent consistency check for our proposal and allows us to recover results for $\mathcal{N} = 4$ descendants.

We point out that the planar duals of theories in **Zones 1**, $\tilde{\mathbf{1}}$ naturally encode Aharony-like duality, rather than being derived from it. In these regions of parameter space, a theory generically admits a planar dual whose quiver has dimensions $N \times (F - N)$, manifestly symmetric under the exchange $N \leftrightarrow F - N$.

As a consequence, theories related by Aharony-like duality flow to the same planar description. For instance, the Aharony-like dual pair consisting of $U(3)_{-\frac{1}{2}}$ SQCD with 7 fundamental chiral multiplets and $U(4)_{\frac{1}{2}}$ SQCD with 7 antifundamental chiral multiplets gives rise to an identical planar dual, shown in Equation 3.57. We expand on this in Section 3.4.

Exploring Zones $\tilde{\mathbf{1}}, 2, 3$ and 4 via Multiple $-m$ Mass Deformations

In the previous section, we also discussed the mapping of a single negative mass $-m$ deformation on the planar Abelian dual of SQCD on the $\mathcal{N} = 4$ -descendant line. We now systematically study the effect of multiple deformations $-m$.

Zones 2: At each step, the $-m$ mass deformation is implemented by giving vacuum expectation values to the real scalars in the vector multiplets associated with the gauge nodes along the bottom-left diagonal:

$$\left(\sigma_1^{\text{bottom}}, \sigma_2^{\text{bottom}}, \dots, \sigma_{\frac{F}{2}-|k|-1}^{\text{bottom}} \right) \propto \left(\left(\frac{F}{2} - |k| - 1 \right) Fm, \left(\frac{F}{2} - |k| - 2 \right) Fm, \dots, Fm \right). \quad (3.39)$$

In addition, the vertical chiral multiplet connected to the bottom flavor node acquires a Higgs VEV.

Starting from the $\mathcal{N} = 4$ descendant line, this mapping remains valid for the first N successive negative real mass deformations. This sequence of deformations allows us to fully explore the parameter regions corresponding to Zone 2 in the k/F -plane. The resulting duality is shown in Figure 3.40.

$$\begin{aligned}
 & U(N)_k \text{ with } F \square \longleftrightarrow \\
 & 2|k| > |F - 2N| \cap F > 2|k|
 \end{aligned}
 \tag{3.40}$$

Notice that Zone 2 includes, on its boundary, the case of $U(N)_{-\frac{N}{2}}$ SQCD with N flavors, which corresponds to the cusp of the supersymmetry-breaking region. The planar dual of this theory cannot be obtained as a simple limiting case of the dualities in (3.40) and (3.54). Indeed, in this limit, certain parameters controlling the size of the quiver degenerate, indicating that the planar dual takes a qualitatively different form.

Nevertheless, the dual description can be accessed by following real mass deformations in the Abelian theory. For instance, starting from the dual of $U(N)_{-\frac{N}{2}+\frac{1}{2}}$ with $N+1$ flavors and integrating out one flavor, we are led to the duality shown in (3.41).

$$\begin{aligned}
 & U(N)_{-\frac{N}{2}+\frac{1}{2}} \text{ with } N+1 \square \xrightarrow{-m} U(N)_{-\frac{N}{2}} \text{ with } N \square
 \end{aligned}
 \tag{3.41}$$

By applying a sequence of Aharony dualities on the gauge nodes of the Abelian quiver, one finds that the theory confines to a single free chiral multiplet. This reproduces the

expected infrared duality

$$U(N)_{-\frac{N}{2}} \text{ with } N \square \longleftrightarrow \text{free chiral}, \quad (3.42)$$

where the free chiral is identified with the gauge-invariant monopole operator of the SQCD theory⁴.

Zone 1: This region of parameter space can be accessed only through a combination of positive and negative real mass deformations, $+m$ and $-m$ respectively. As discussed in Section 3.3.1, the $+m$ deformations shrink the quiver along the SW-NE direction, while the $-m$ deformations act analogously to those considered for theories in Zone 2 in (3.39). The resulting duality is shown in Figure 3.43.

$$U(N)_k \text{ with } F \square \longleftrightarrow (2N - 2|k| > F > N) \quad (3.43)$$

On the boundary between Zone 1 and the SUSY-breaking region, corresponding to $F = N$ and $2|k| \leq 2N - F$, the theory is expected to become trivial. This behavior cannot be directly inferred from (3.43), since reaching this point would require setting one of the quiver size parameters to zero, leading to a degenerate description.

Nevertheless, this limit can be accessed by considering theories that flow to this boundary via a single real mass deformation. For example, one may start from theories along the

⁴By following the standard $SL(2; \mathbb{Z})$ gauging/ungauging prescription, we obtain a duality between $U(N)_{(-\frac{N}{2}, -\frac{3N}{2})}$ SQCD with $N \square$ and $U(1)_{\frac{3}{2}}$ SQED with a single $\square$ chiral multiplet. This theory exhibits an enhancement to $\mathcal{N} = 4$ supersymmetry, as noted in [228]. From the perspective of the planar Abelian dual, this enhancement becomes particularly transparent, as it can be directly inferred from the topology of the quiver diagram. In particular, the triangular subquiver in the planar mirror can be confined to a single free chiral multiplet via local dualization sequences, after which the resulting quiver reduces to $U(1)_{\frac{3}{2}}$ SQED with a single chiral multiplet.

vertical line with $F = N + 1$ and $2|k| \leq 2N - F$, and follow the deformation. In this way, one finds that the planar Abelian dual reduces to a quiver that fully confines to a $U(1)_1$ theory, which is an almost-trivial theory.

In contrast, the boundary between Zone $\tilde{1}$ and Zone 2 can be straightforwardly obtained by setting $2|k| = 2N - F$ in either (3.40) or (3.43), which is a well-defined operation at the level of the quiver.

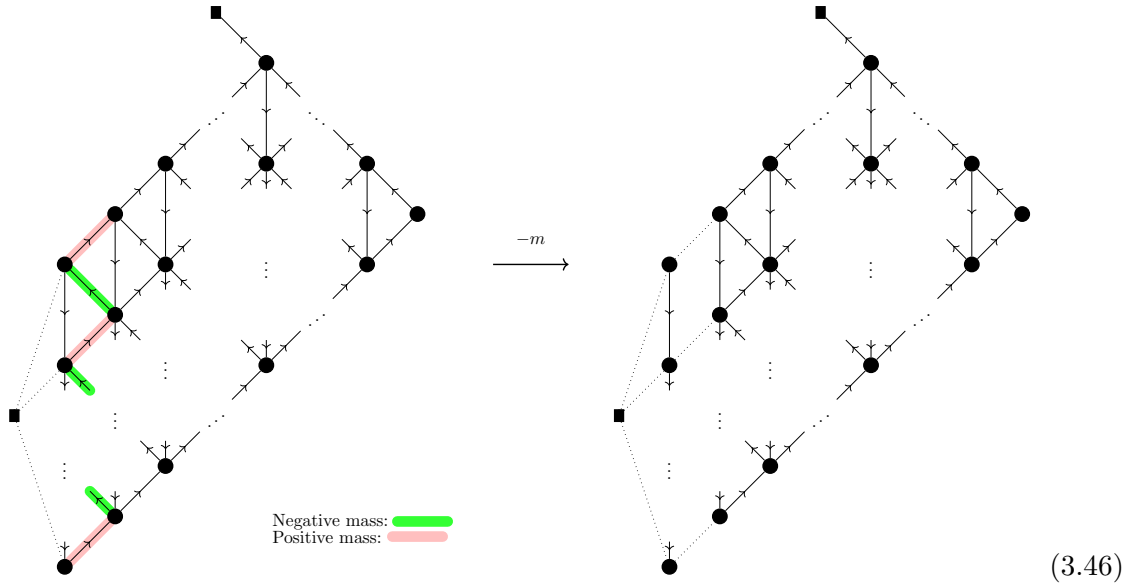
Similarly, theories along the horizontal line $k = 0$, with $N < F < 2N$, can be obtained by directly setting $k = 0$ in (3.43). The special case $k = 0$ and $F = 2N$ is instead described by (3.37). This point is not captured by a naive limit of (3.43), as expected, since it cannot be reached via positive real mass deformations starting from theories in Zone 2.

Zones 3 and 4: Each additional mass deformation $-m$ corresponds to giving a VEV to the real scalars in the $(|k| - \frac{F}{2})$ -th column—namely, the leftmost column still connected to the rest of the quiver by diagonal chiral fields:

$$\sigma_{|k| - \frac{F}{2}}^i = \sigma_{|k| - \frac{F}{2}}^j \quad \forall i, j \quad (3.44)$$

$$\sigma_{|k| - \frac{F}{2}}^i - \sigma_{|k| - \frac{F}{2} + 1}^i \propto Fm \quad (3.45)$$

The vertical chirals in the $(|k| - \frac{F}{2})$ -th column remain massless, whereas the diagonal chirals connecting this column to the next one become massive. In particular, the diagonals oriented toward the right acquire positive masses, while those pointing left acquire negative masses, as shown in Figure 3.46.



This mapping remains valid as long as the quiver contains columns of diagonal chiral fields, which corresponds to the condition $F \geq N$, shown in Figure 3.47.

$$\begin{aligned}
 &U(N)_k \text{ with } F \square \quad \longleftrightarrow \quad \text{Quiver Diagram} \\
 &(2|k| > F \geq N) \\
 &\text{Plot of } -2k \text{ vs } F \text{ showing a shaded region for } F \geq N.
 \end{aligned}
 \tag{3.47}$$

In Zone 4 with $F < N$ (Figure 3.53), the negative mass deformation $-m$ instead corresponds to giving a VEV to the vertical chiral at the top of the F^{th} column from the right, together with VEVs for the real scalars in the top-right diagonal:

$$\left(\sigma_1^{\text{top}}, \dots, \sigma_{F-1}^{\text{top}} \right) \propto ((F-1)Fm, (F-2)Fm, \dots, Fm) \tag{3.48}$$

where σ_i^{top} is the real scalar associated to the top node of the i -th column from the right.

On the boundary between Zone 4 and the SUSY-breaking region, namely for $2|k| = 2N - F$ with $F < N$, the planar quiver takes a qualitatively different form and cannot be obtained as a well-defined zero-size limit of (3.53).

However, using the fact that the theory at $F = |2k| = N$ reduces to a single free chiral multiplet, one can immediately infer the behavior along this boundary. Indeed, a negative real mass deformation of $U(N)_{-\frac{N}{2}}$ SQCD with N flavors maps, in the dual description, to a mass term for the free chiral. Integrating out this field yields a trivial theory, implying that the boundary theory is trivial.

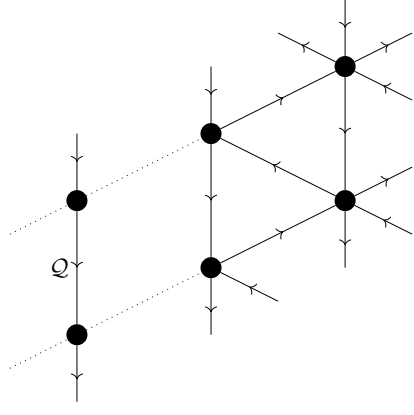
Before concluding our discussion on the planar theories in Zones 3 and 4, we briefly comment on the form of the monopole superpotential in these theories. There is a monopole superpotential term for each pair of consecutive nodes in a column with GNO flux ∓ 1 under the top (bottom) nodes. In the “bulk” of the quiver, where there are diagonal arrows on both sides of the column, these monopoles are gauge-invariant BPS operators and, hence, enter the superpotential as described before.

- In the column that separates the “bulk” of the quiver and the region with BF interactions, as shown in Equation 3.50, these monopoles must be dressed with the chiral field $\mathcal{Q}$.

The FI parameters of the nodes in this type of column are, from top to bottom:

$$\text{FIs: } X_i - \frac{iQ}{4}, \quad X_i - r\frac{iQ}{2}, \quad \dots, X_i - r\frac{iQ}{2}, \quad X_i + \frac{iQ}{4}(1 - 3r) \quad (3.49)$$

where X_i is a Cartan of the $SU(F)$ flavor symmetry, and the coefficient in front of $\frac{iQ}{2}$ encodes the mixing between the topological and the R-symmetry. We remind the reader that we assign a trial R-charge r to the chiral multiplets in the *electric* SQCD theory.

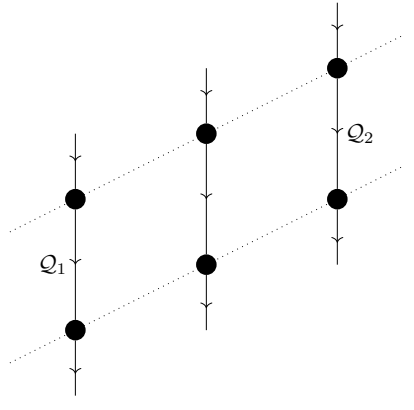


$$\mathcal{W}_{\text{monopole}} \supset \mathcal{QM} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.50)$$

- In the columns with BF interactions on both sides, as shown in Equation 3.52, these monopoles must be dressed with both $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

The FI parameters of the nodes in this type of column are, from top to bottom:

$$\text{FIs: } \frac{iQ}{4}(2r - 1), \quad 0, \quad \dots, 0, \quad \frac{iQ}{4}(1 - 2r) \quad (3.51)$$



$$\mathcal{W}_{\text{monopole}} \supset \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{M} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.52)$$

- We further notice that in a column with BF interactions on both sides, the $N - 1$ monopole superpotentials completely break the topological symmetries. Therefore, only the columns in the “bulk” of the quiver and at the edge of the “bulk” contribute to the Cartan subalgebra of the global symmetry, matching the global symmetry of the dual SQCD.

$U(N)_k$ with F $\square$ $\longleftrightarrow$

$(N > F > 2N - 2|k|)$

$|k| + \frac{F}{2} - N - 1$ nodes

$F - N$ nodes

N nodes

$-2k$

$2N$

N

$2N$

F

(3.53)

Dualities for Marginally Chiral SQCD: Turning on exactly N negative real masses for the planar quiver shown in Figure 3.40 results in a planar Abelian dual for marginally chiral SQCD:

Figure 1: A diagram illustrating the correspondence between a Dynkin diagram and a quiver. On the left, a Dynkin diagram for $U(N)_{-F/2}$ is shown, with a thick orange line segment on the branch between nodes N and $2N$. On the right, a quiver is shown with a thick pink line segment on the branch between the $F-N$ nodes and the N nodes. The quiver has a central node connected to $F-N$ nodes and N nodes, with arrows indicating the flow of the quiver.

In this case, we observe that the planar Abelian quiver contains a long gauge-invariant mesonic operator (highlighted in maroon) in the chiral ring that parametrizes a non-compact moduli space. Consistently, the dual SQCD features a gauge-invariant monopole operator parametrizing a $\mathbb{C}$ -moduli space of vacua.

Dualities for TQFTs: Upon integrating out all the flavors in the SQCD theory, we find a planar Abelian dual for $\mathcal{N} = 2$ $U(N)_k$ CS-YM theory:

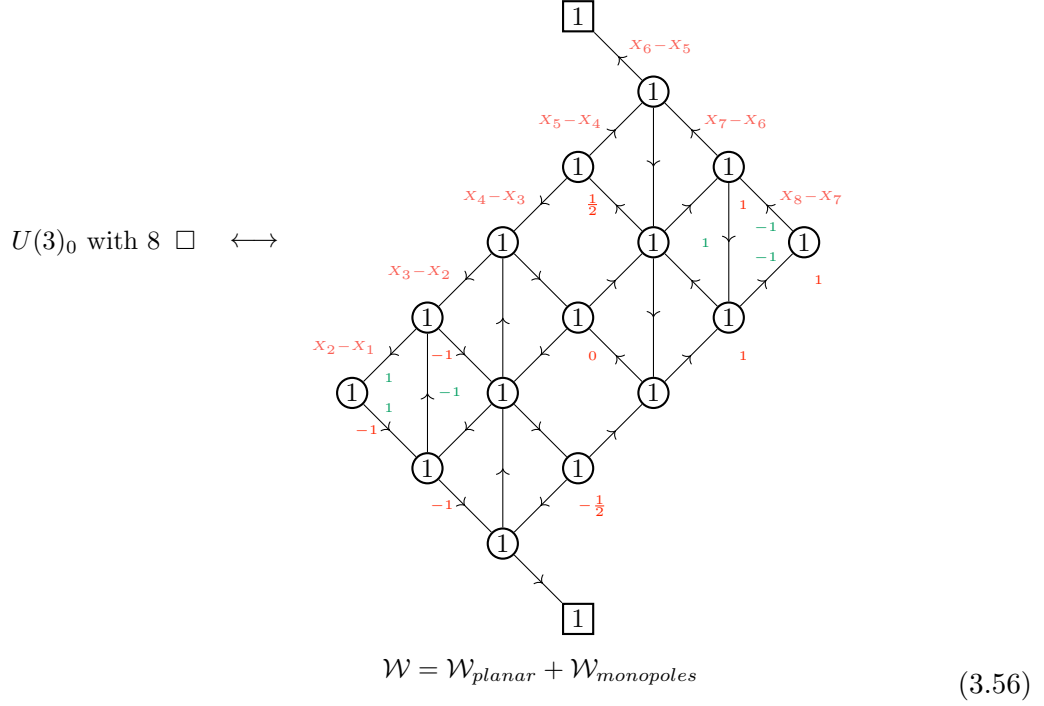
$$\begin{array}{c}
 \mathcal{N} = 2 \ U(N)_k \\
 \updownarrow \\
 \mathcal{N} = 0 \ U(N)_{(k+N,k)}
 \end{array}
 \longleftrightarrow
 \begin{array}{c}
 \begin{array}{c}
 |k| - N - 1 \text{ nodes} \\
 \vdots \\
 \text{Quiver Diagram} \\
 \vdots \\
 N \text{ nodes}
 \end{array}
 \end{array}
 \quad (3.55)$$

Interestingly, the resulting planar dual is not manifestly topological, as it still contains massless charged chiral fields. As such, the duality would imply that the planar Abelian quiver shown in (3.55) is in fact a gapped theory, flowing in the infrared to a $U(N)$ Chern–Simons theory with levels $(k + N, k)$.

We stress, however, that what we have established is a matching of $\mathbf{S}_b^3$ partition functions between the two sides, and that this falls short of demonstrating the above. We do not have a mechanism for the decoupling of the charged matter in the infrared, and we have not verified that the resulting topological data agree. The latter point is not a formality: a topological field theory is characterized by its line operators and their braiding, and the topological lines of Abelian theories behave rather differently from those of non-Abelian ones. The supersymmetric partition function is insensitive to precisely this data, so the check that is required here is the computation of defect indices [160, 229–233] on both sides, which we have not carried out. We therefore record (3.55) as a conjectural statement, well supported at the level of the partition function, and regard the infrared mechanism and the matching of the line spectra as an open direction.

3.3.2 An Example: Non-Higgsing Mass Deformations of $U(3)_{-\frac{1}{2}}$ SQCD₃ with 7 $\square$

Having established the framework of mapping mass deformations across mirror duals, we now focus on a pedagogical example to illustrate our proposal. For concreteness, we begin with the planar Abelian dual of $U(3)_0$ with 8 fundamental chiral multiplets, shown in Figure 3.56.



We now demonstrate the effect of a negative mass deformation on the duality shown in Figure 3.56. The resulting duality between $U(3)_{-\frac{1}{2}}$ SQCD with 7 fundamental chiral multiplets is shown in Figure 3.57.

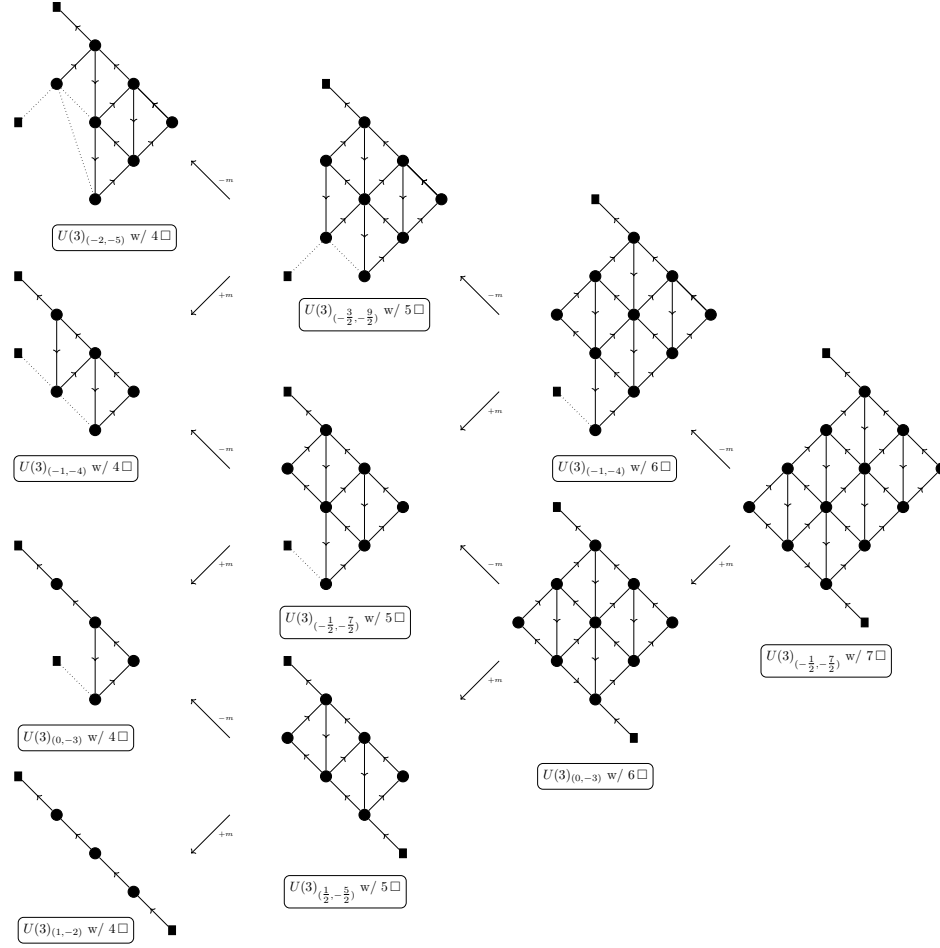


Figure 3.5: The planar mirror dual of $U(3)_{(-\frac{1}{2}, -\frac{7}{2})}$ with $[7, 0]$ chiral multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by real mass deformations $\pm m$. In this figure, we display the first three levels of the deformation sequence. Specifically, we begin with an *electric* theory containing $[7, 0]$ chiral multiplets and follow the flow to theories with $[4, 0]$ chiral multiplets, accompanied by various CS levels. We only show the global topology of the resulting quiver diagram, while omitting details of the CS levels and FI terms of the various planar quivers; these can be obtained by following the same flow on the $\mathbf{S}_b^3$ partition function, or with the equations of motion of the mirror dual.

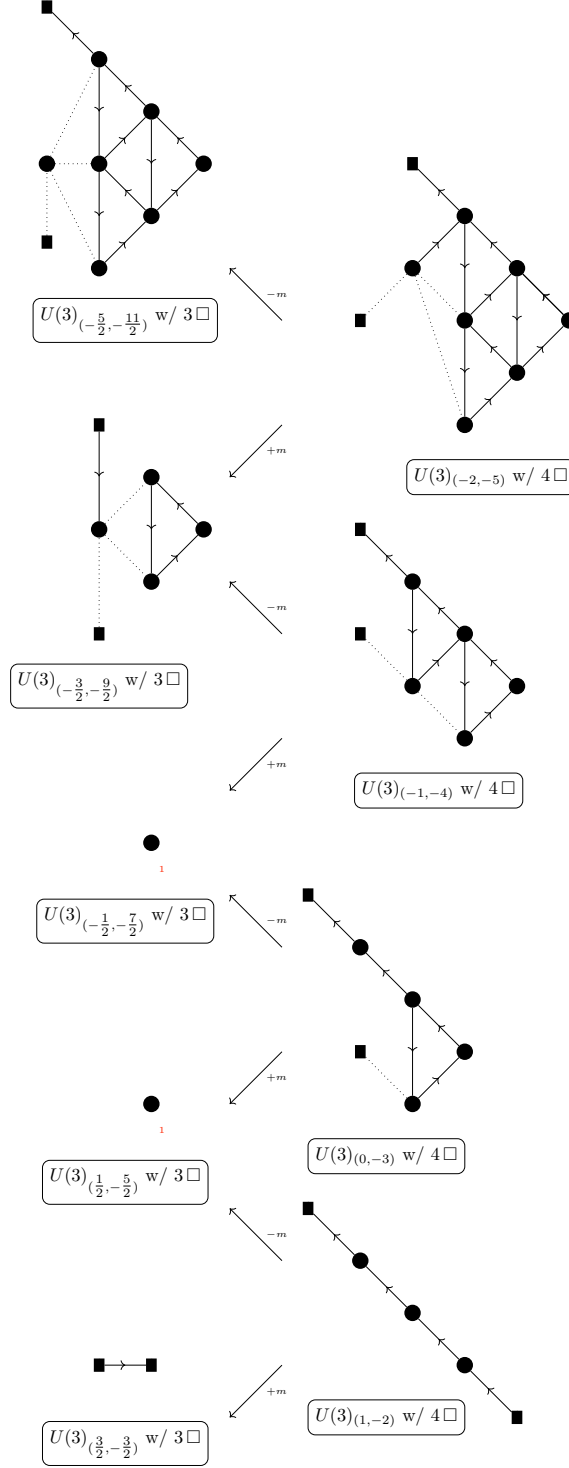


Figure 3.6: We continue analyzing the RG flow illustrated in Figure 3.5. At Level 3, only two theories—the planar mirrors of $U(3)_{(-2, -5)}$ and $U(3)_{(-1, -4)}$ with $[4, 0]$ chiral multiplets—admit non-trivial flows under $\pm m$ deformations. The remaining theories flow to nearly trivial infrared fixed points: either a $U(1)_1$ TQFT, obtained by sequentially confining gauge nodes connected via BF couplings, or a free chiral multiplet. Determining that the corresponding electric theories reduce to TQFTs ($U(3)_{(-\frac{1}{2}, -\frac{7}{2})}$ and $U(3)_{(\frac{1}{2}, -\frac{5}{2})}$ with $[3, 0]$ chiral multiplets) or free chirals ($U(3)_{(\frac{3}{2}, -\frac{3}{2})}$ with $[3, 0]$ chiral multiplets) is non-trivial; however, this conclusion becomes transparent upon analyzing their planar mirror descriptions. As before, we only show the global topology of the resulting quiver diagram, while omitting details of the CS levels and FI terms of the various planar quivers.

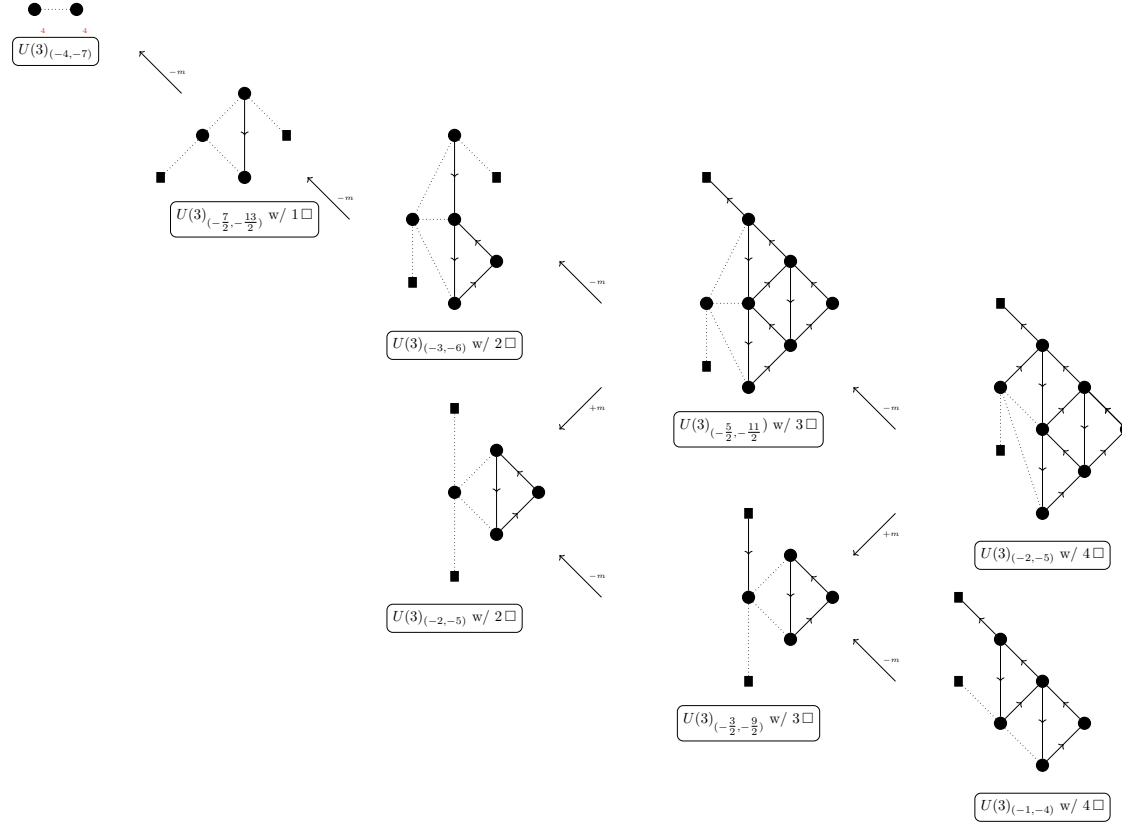


Figure 3.7: We continue analyzing the RG flow illustrated in Figure 3.6. We study the RG flow trajectories induced by $\pm m$ deformations on $U(3)_{(-\frac{5}{2}, -\frac{11}{2})}$ and $U(3)_{(-\frac{3}{2}, -\frac{9}{2})}$ with $[3,0]$ chiral multiplets here. Determining the TQFT nature of $U(3)_{(-\frac{5}{2}, -\frac{11}{2})}$ with $[1,0]$ chiral multiplets is non-trivial in the electric theory, but very transparent in the dual description. As before, we only show the global topology of the resulting quiver diagram, while omitting details of the CS levels and FI terms of the various planar quivers; these can be obtained by following the same flow on the $\mathbf{S}_b^3$ partition function, or with the equations of motion of the mirror dual.

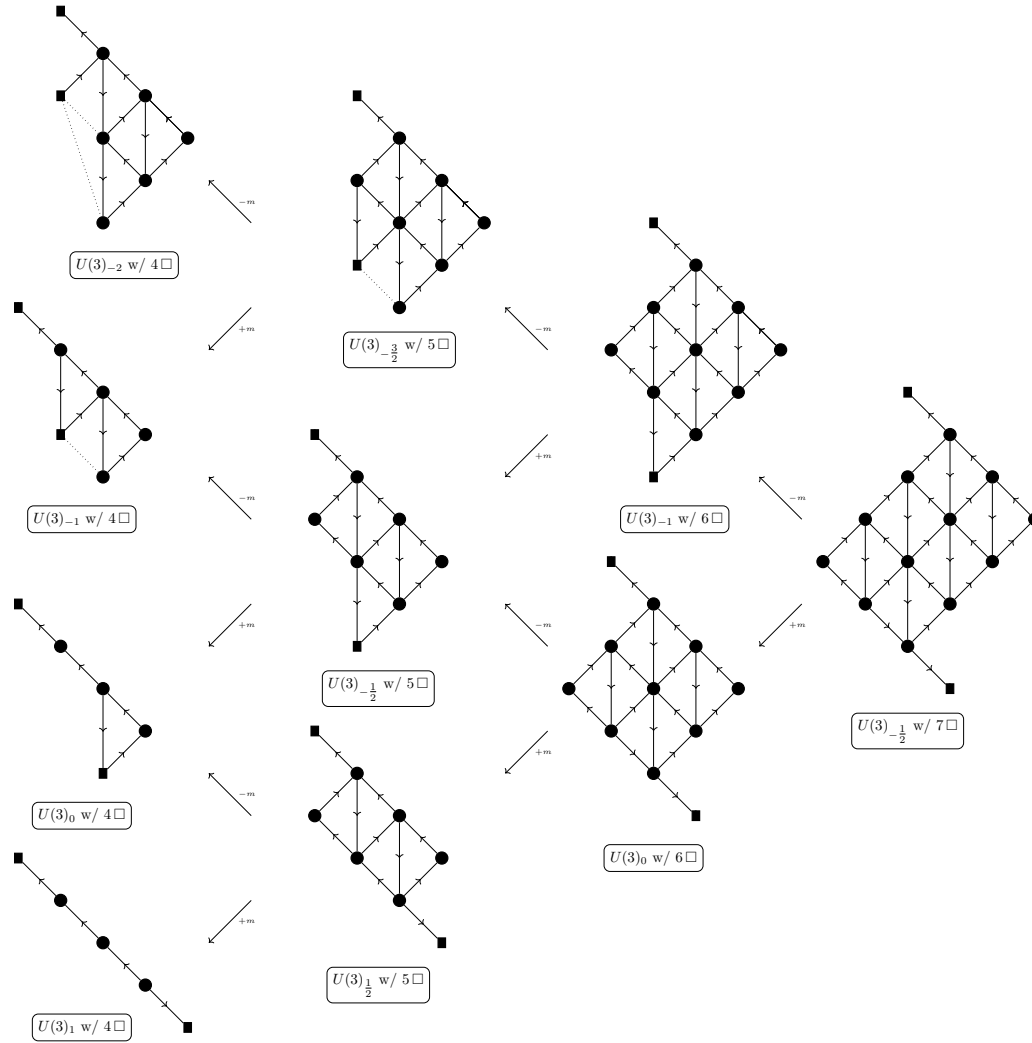


Figure 3.8: The planar mirror dual of $U(3)_{-1/2}$ SQCD with 7 fundamental chiral multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by $\pm m$ real mass deformations. In this figure, we display the first three levels of this deformation sequence. Specifically, we begin with an electric theory containing $[7, 0]$ chiral multiplets and follow the flow to theories with $[4, 0]$ chiral multiplets. We only display the global topology of the resulting quiver diagrams; details of the CS levels and FI terms can be recovered following our usual conventions.

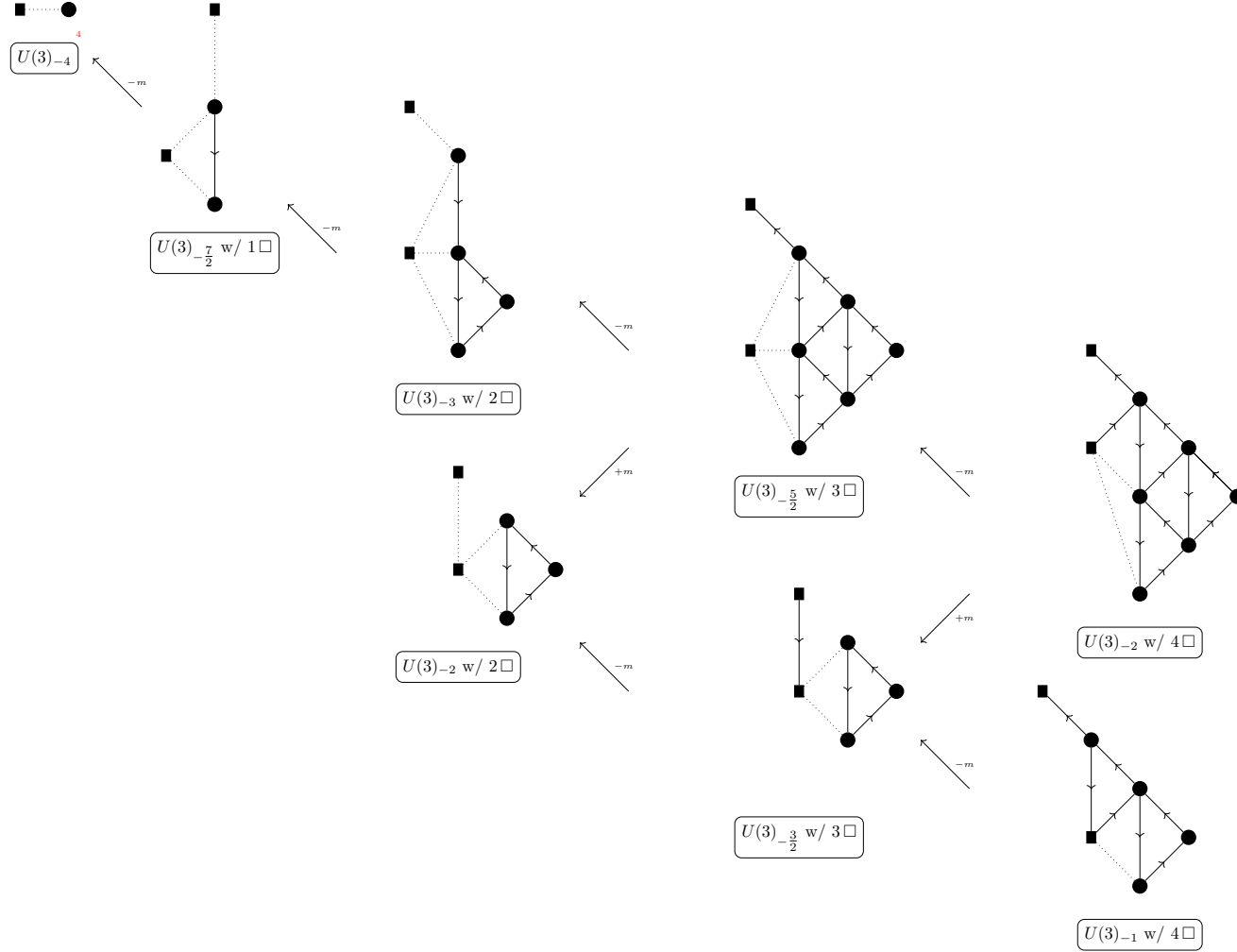


Figure 3.9: We continue the analysis of RG flow trajectories induced by $\pm m$ deformations, extending those considered in Figure 3.8. Among the theories appearing in these flows, only $U(3)_{-2}$ SQCD and $U(3)_{-1}$ SQCD with $[4, 0]$ chiral multiplets admit non-trivial RG flows under $\pm m$ deformations. By contrast, $U(3)_{-\frac{1}{2}}$, $U(3)_{\frac{1}{2}}$, and $U(3)_{\frac{3}{2}}$ SQCD with $[3, 0]$ chiral multiplets are dual to trivial theories in the infrared. Accordingly, we focus on the RG flow trajectories of $U(3)_{-\frac{5}{2}}$ and $U(3)_{-\frac{3}{2}}$ SQCD with $[3, 0]$ chiral multiplets and follow these flows until all matter is integrated out. As before, we display only the global topology of the resulting quiver diagrams, suppressing the detailed Chern–Simons levels and FI terms.

3.4 Relating Planar Abelian Duals and Aharony-like Duals

The $\mathcal{N} = 2$ $U(N)_k$ SQCD theory enjoys an Aharony-like duality for arbitrary parameters (N, k, F) (see [146] and references therein). In this section, we restrict to theories with only fundamental matter fields, whose dual descriptions involve only anti-fundamental fields. As discussed previously, the structure of these dualities depends qualitatively on the region of the (k, F) plane at fixed N (see Figure 3.37).

Our goal is to clarify how the planar Abelian dualities proposed in this chapter relate to Aharony duality. We argue that, assuming the validity of the planar Abelian dualities together with certain special instances of Aharony-like duality—namely those involving Abelian gauge groups in both the electric and magnetic phases (see Appendix B)—one can derive the general Aharony-like dualities for $U(N)_k$ SQCD with F fundamental chiral fields. In this sense, Aharony duality emerges from more elementary building blocks and is effectively proven within this framework. An immediate consequence is that Aharony duality for SQCD with fundamentals can be understood as a consequence of mirror symmetry.

To make this concrete, we consider SQCD in Zone $\tilde{1}$ with $F < 2N$ fundamentals and Chern–Simons level k , satisfying $2|k| < 2N - F$, corresponding to a maximally chiral regime [21]. This theory is Aharony-dual to $U(F - N)$ SQCD with F anti-fundamentals and Chern–Simons level $-k$, which lies in (the anti-fundamental analogue of) Zone 1:

$$U(N)_k \text{ with } F < 2N \quad \longleftrightarrow \quad U(F - N)_{-k} \text{ with } F > 2(F - N) \quad (3.59)$$

Across the duality, the topological symmetry is mapped to a linear combination of the R-symmetry and minus the topological symmetry of the magnetic gauge group:

$$U(1)_{\text{top}} = k U(1)_R - U(1)_{\text{top}}. \quad (3.60)$$

We now show that the planar Abelian duals of these SQCD theories—referred to as the *electric* and *magnetic* theories—are equivalent up to a sequence of local dualities. These local dualities relate SQED theories to SQED theories and can be understood as limiting cases of mirror symmetry or Aharony duality. Assuming the validity of this local dualization procedure, it follows that Aharony-like dualities for SQCD with fundamentals arise as a consequence of mirror symmetry. We summarize this structure schematically in Figure 3.10.

The representations of the fields under the $SU(F) \times U(1)_\eta \times U(1)_R$ symmetries are:

	$U(N)$	$U(F - N)$	$SU(F)$	$U(1)_R$	$U(1)_\eta$
Q	$\square$		$\bar{\square}$	r	0
$\tilde{q}$		$\bar{\square}$	$\square$	$1 - r$	0

(3.61)

Here, Q denotes the fundamental chiral fields of the electric theory, and $\tilde{q}$ denotes the anti-fundamental chiral fields of the magnetic theory. No additional gauge-singlet fields are required in this duality. The only operators charged under $U(1)_\eta$ are the monopole operators in the respective phases.

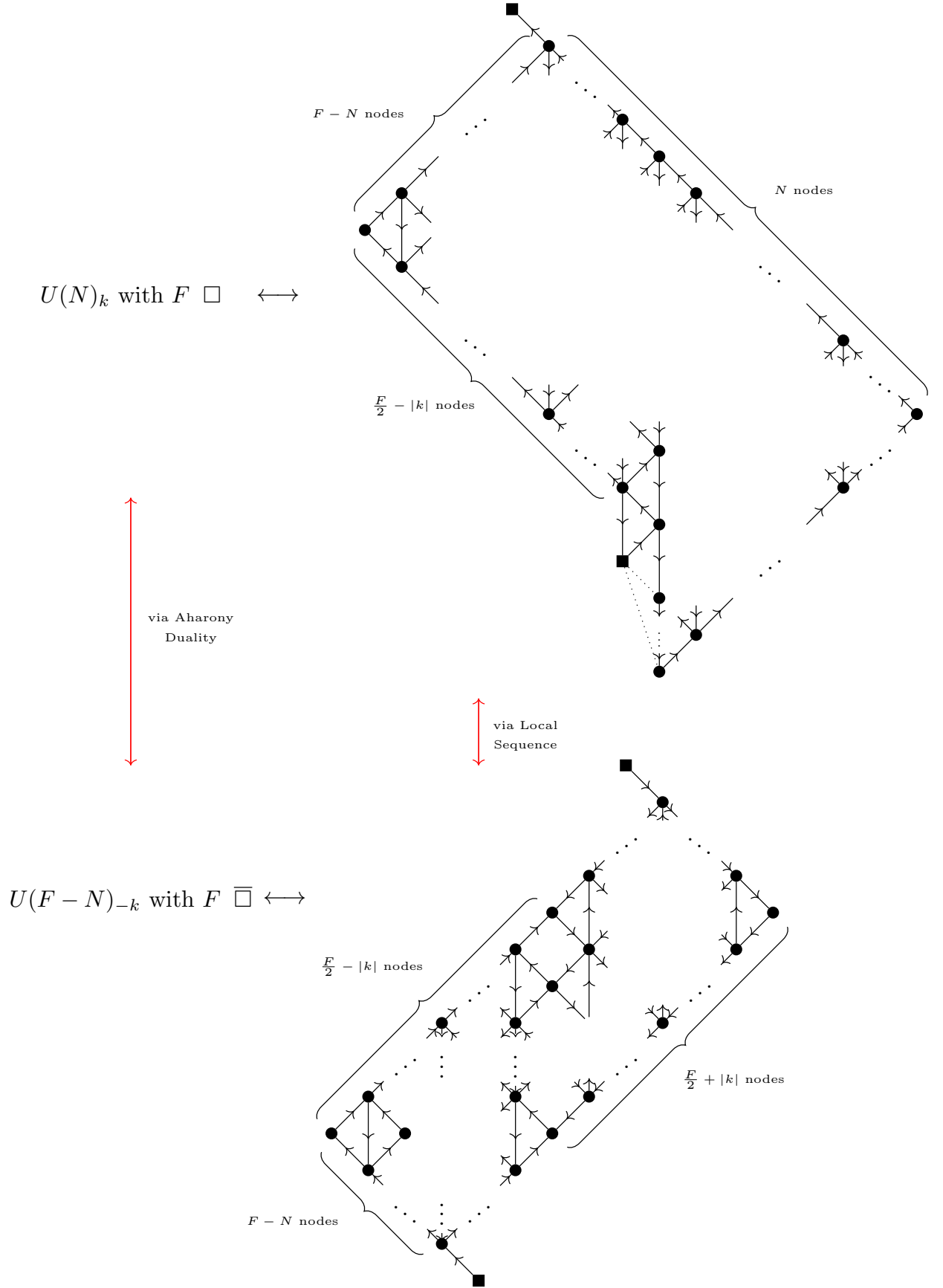


Figure 3.10: Schematic depiction of the duality web between a theory in Zone $\tilde{1}$ and its Aharony dual in Zone 1.

Electric Theory. We begin by rewriting the planar Abelian dual of the electric theory (3.43), characterized by gauge group $U(N)$, Chern–Simons level k , FI parameter η , and trial R-charge r . We then perform a sequence of dualizations analogous to the planar flip–flip duality discussed in [3], referring the reader to that work for the details of the individual steps.

The dualization proceeds column by column, starting from the rightmost column and stopping at the column immediately to the right of the bottom flavor node. The result of this first pass is shown in the central quiver of Figure 3.11. After this step, a column of square nodes appears at position $\frac{F}{2} - |k|$, the flavor node shifts one column to the right, and the bottom-right diagonal shortens by one node.

We then repeat this procedure iteratively. Each pass starts from the right and ends at the column adjacent to the bottom flavor node, shortening the bottom-right diagonal by one node and reducing the number of columns involved by one. The result after the second pass is shown on the right of Figure 3.11.

After a finite number of iterations, we arrive at the quiver shown on the left of Figure 3.12, which has the form of a rectangular lattice of size $N \times (F - N)$. Finally, we apply a flip–flip duality [3] to the rightmost triangular subquiver, obtaining the quiver displayed on the right of Figure 3.12.

This provides our final expression for the electric theory and can be viewed as an alternative presentation of the planar Abelian dual in Zone $\tilde{1}$.

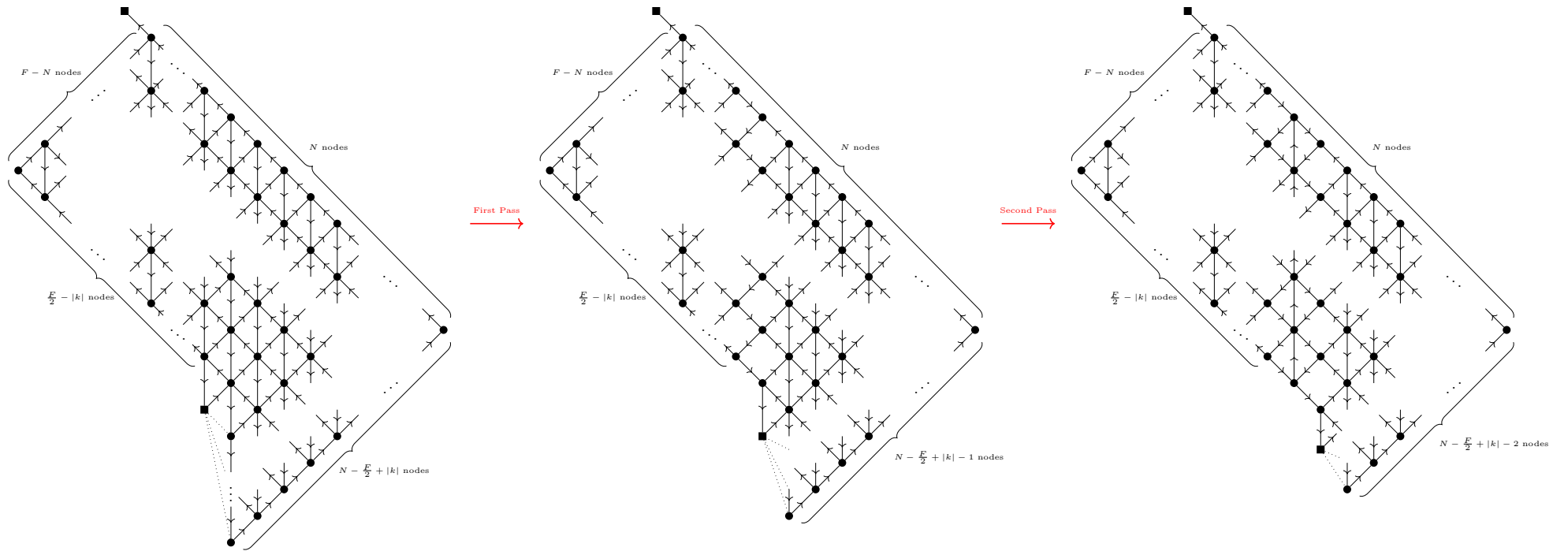


Figure 3.11: The first two iterations in the dualization sequence of the planar dual of a theory in Zone $\tilde{1}$.

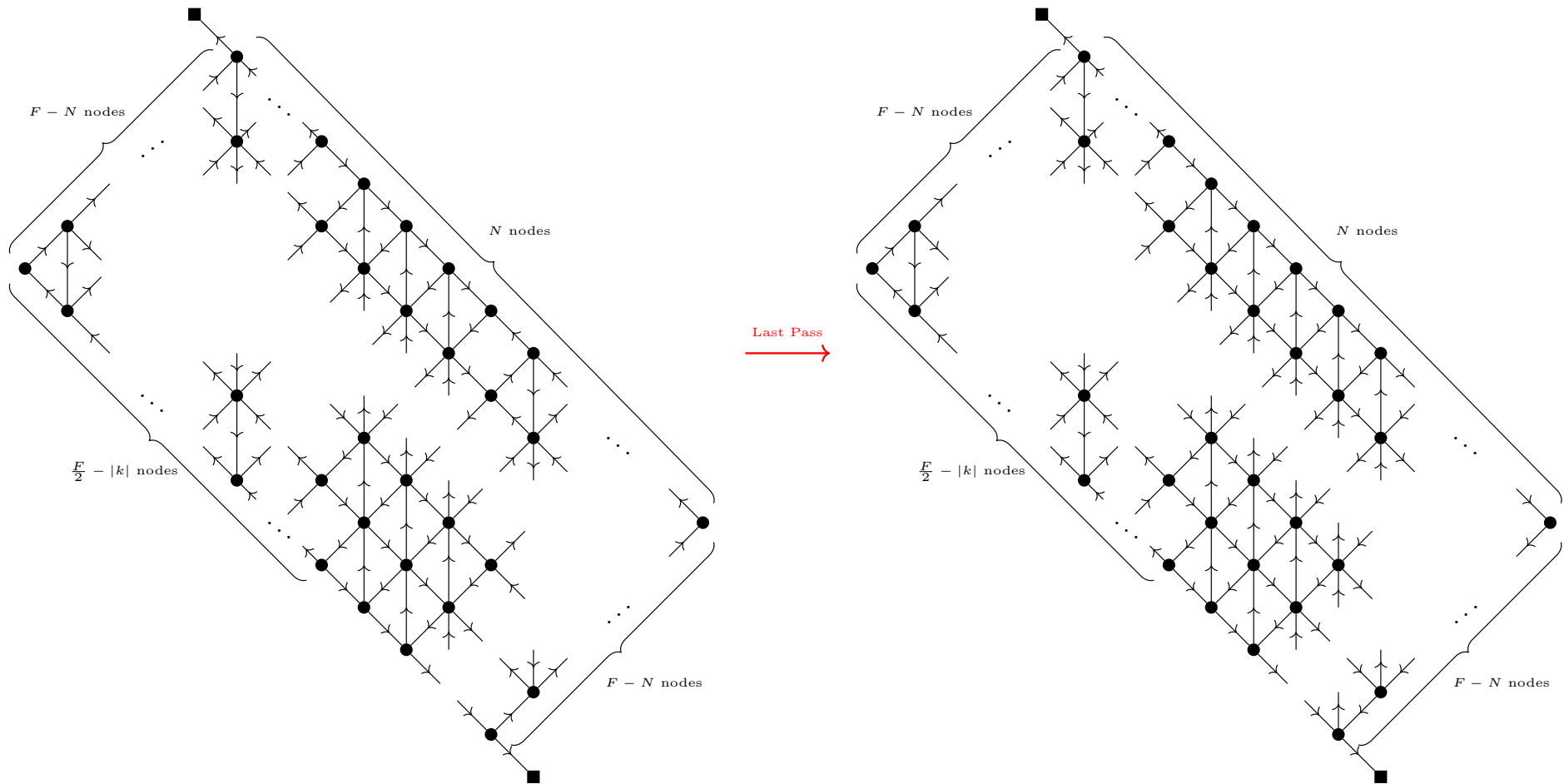


Figure 3.12: The last two iterations in the dualization sequence of the planar dual of a theory in Zone $\tilde{1}$.

Magnetic Theory. The planar Abelian dual of SQCD with only anti-fundamental fields can be obtained by applying parity ($\mathcal{P}$) and charge conjugation ($\mathcal{C}$) to both sides of the dualities discussed in the rest of the chapter. We start from the duality for SQCD in Zone 1 (3.36), with gauge group $U(F - N)$, Chern–Simons level k , FI parameter $\eta - k\frac{iQ}{2}$, and trial R-charge $1 - r$. Up to background terms, this corresponds to the theory appearing on the right-hand side of the Aharony-like duality (3.59).

Applying the combined $\mathcal{CP}$ transformation to the left-hand side yields SQCD with gauge group $U(F - N)$, F anti-fundamentals, Chern–Simons level $-k$, and FI parameter $k\frac{iQ}{2} - \eta$. In the planar quiver, the $\mathcal{CP}$ action reverses the orientation of all arrows and flips the sign of all Chern–Simons and mixed Chern–Simons couplings, including background terms⁵.

At the level of the $\mathbf{S}_b^3$ partition function, the actions of $\mathcal{C}$ and $\mathcal{P}$ are given by:

$$\mathcal{C} : Z(\vec{k}, \vec{\mu}) \longrightarrow Z(\vec{k}, -\vec{\mu}), \quad (3.62)$$

$$\mathcal{P} : Z(\vec{k}, \vec{\mu}) \longrightarrow \overline{Z(\vec{k}, \vec{\mu})}, \quad (3.63)$$

where $\vec{k}$ denotes the set of all Chern–Simons and mixed Chern–Simons levels, including background contributions.

The resulting duality takes the form shown in Equation 3.64.

$k\omega - \eta$

$U(F - N)_{-k}$ with F $\longleftrightarrow$

$\frac{F}{2} - |k|$ nodes

$F - N$ nodes

$\frac{F}{2} + |k|$ nodes

(3.64)

The Chern–Simons couplings, mixed Chern–Simons terms, and trial R-charges can be determined using the rules outlined in Section 3.1.

We begin by applying the “flip–flip” dualization sequence [3] to the triangular subquivers at the ends of the quiver. Next, we dualize the column of nodes corresponding to the

⁵FI terms can be interpreted as mixed Chern–Simons couplings between gauge and topological symmetries and are therefore odd under parity. Furthermore, the fugacity of the $U(1)_R$ symmetry $\frac{iQ}{2}$ should be regarded as a real parameter.

column of squares in the bulk of the quiver, located at position $\frac{F}{2} - |k|$. We then proceed iteratively, dualizing columns from right to left, starting with the adjacent column and continuing until we reach the column attached to the flavor node, thereby completing the first pass.

We then repeat the procedure, now starting from the column at position $\frac{F}{2} - |k| + 1$, which has become a column of squares. Again, we dualize columns from right to left until we reach the opposite column of squares, completing the second pass. Iterating this process eventually brings us to the rightmost flavor node.

The resulting quiver takes the general form shown in Equation 3.65.

(3.65)

The final step consists of manipulating the two chiral fields connected to the flavor nodes. This is achieved by performing a sequence of Witten's $SL(2; \mathbb{Z})$ transformations [106] acting on the $U(1)_\eta$ symmetry of the quiver, which combine to an overall trivial action. The non-trivial effect arises from the fact that there is a single flavor $U(1)$ symmetry in the planar mirror, which can be associated with either the left or the right flavor node; these two choices are gauge equivalent.

Concretely, we perform a $\mathcal{T}^{-1}\mathcal{S}\mathcal{T}^{-1}$ transformation on the left flavor node and a $\mathcal{T}\mathcal{S}^{-1}\mathcal{T}$ transformation on the right flavor node. Although these combine into a trivial $SL(2; \mathbb{Z})$ action, the intermediate steps induce a non-trivial reorganization of the theory, which remains IR dual to the original quiver. The $\mathcal{T}$ transformation shifts the Chern–Simons level by +1, while the $\mathcal{S}$ transformation gauges the symmetry and introduces a new $U(1)$ topological symmetry⁶.

⁶At the level of the $\mathbf{S}_b^3$ partition function, the $\mathcal{T}$ and $\mathcal{S}$ transformations act as

$$\begin{aligned} \mathcal{T}: Z(\mu) &\rightarrow e^{-i\pi\mu^2} Z(\mu), \\ \mathcal{S}: Z(\mu) &\rightarrow \int d\mu e^{-2i\pi\mu\lambda} Z(\mu), \end{aligned} \tag{3.66}$$

After these transformations, both nodes are confined to a chiral field, corresponding to a monopole operator of the associated gauge node.

The resulting quiver is shown in Equation 3.67.

(3.67)

Notice that the planar quiver obtained this way is equivalent to the quiver obtained by locally dualizing a Zone $\tilde{1}$ mirror (last panel of Figure 3.12) up to a horizontal reflection.

This concludes the argument that Aharony duality for $U(N)_k$ SQCD with F fundamental chiral fields follows from the planar Abelian dualities discussed in this chapter, together with their massive RG flows and the assumption of Aharony duality for Abelian gauge theories. We expect that a similar argument extends to more general $U(N)_k$ theories with both fundamental and anti-fundamental matter, although we leave this extension for future work.

An Example: $U(2)_{\frac{1}{2}}$ SQCD₃ with $9 \overline{\square}$: To illustrate the manipulations underlying the relation between Aharony duality and planar Abelian dualities, we consider the example of $U(2)_{\frac{1}{2}}$ SQCD₃ with $[0, 9]$ chiral fields. We display the full sequence of local dualizations in Figure 3.13.

where μ and λ are the fugacities associated with the original and emergent $U(1)$ symmetries, respectively.

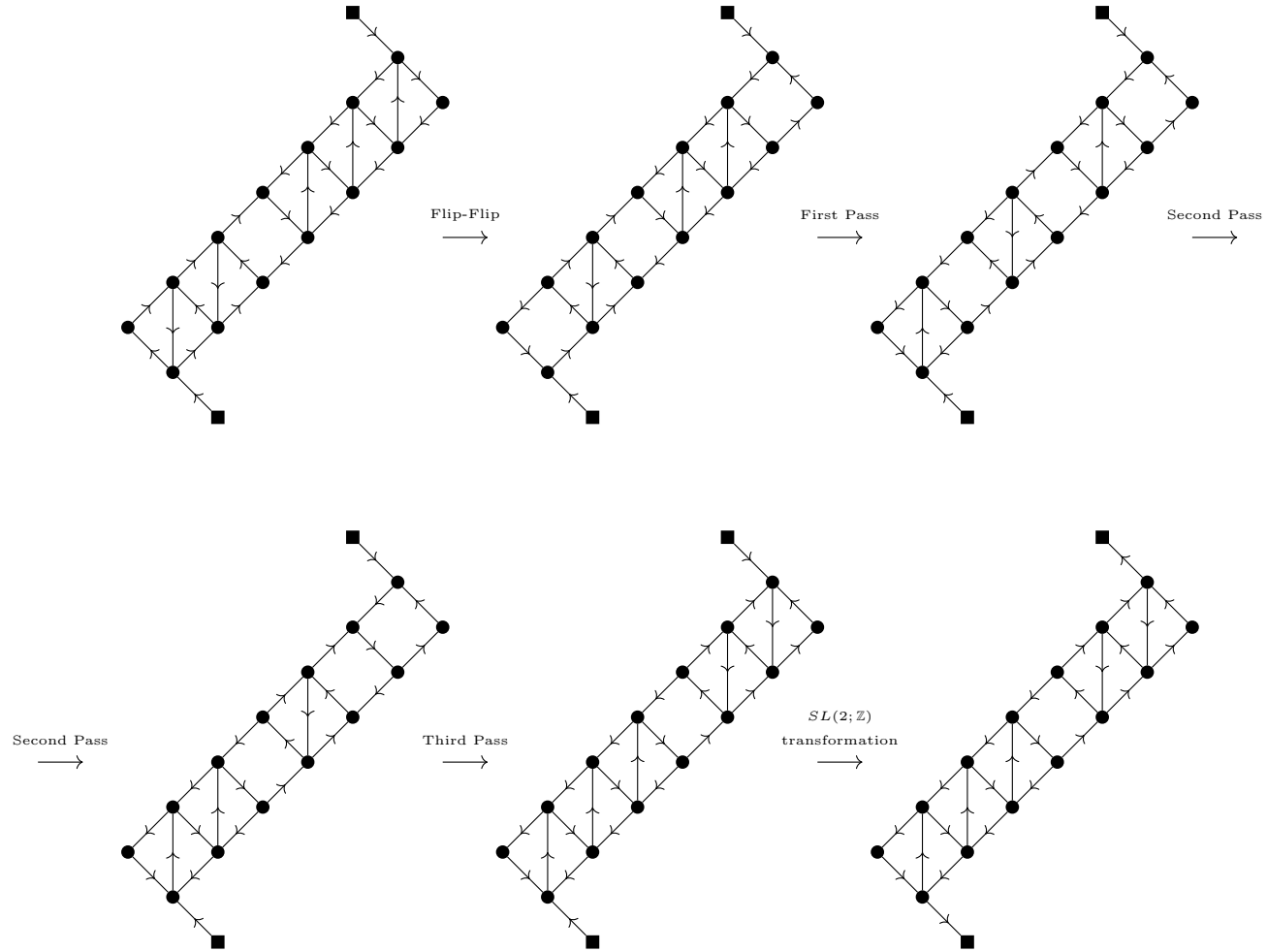


Figure 3.13: Sequence of local dualizations relating the planar mirror dual of $U(2)_{\frac{1}{2}}$ SQCD₃ with $[0, 9]$ chiral fields to the planar mirror dual of $U(7)_{-\frac{1}{2}}$ SQCD₃ with $[9, 0]$ chiral fields (see the discussion before Figures 3.11 and 3.12). This example provides an explicit realization of how Aharony duality emerges from the planar Abelian duality framework.

Scope and Future Directions

In this chapter, we have restricted our attention to $U(N)_k$ CS-SQCD with F fundamental chiral multiplets. We conclude by briefly commenting on natural extensions of this analysis.

Real Mass Deformations of CS-SQCD₃ with both Fundamental and Anti-Fundamental Fields. Planar Abelian duals were originally derived for theories containing only a single type of chiral multiplet—either all fundamentals or all anti-fundamentals. Planar duals for CS-SQCD₃ with both F fundamental and A anti-fundamental fields can also be constructed, though a full treatment of such general chiral configurations lies beyond the scope of this chapter. Nevertheless, real mass deformations provide a systematic bridge from the $\mathcal{N} = 4$ descendant locus to generic $\mathcal{N} = 2$ theories with asymmetric matter content.

A natural starting point for analyzing these theories is the class of $U(N)_k$ SQCD models with $[F, F]$ flavors satisfying the constraint $2|k| = 2F - 2N$. As an illustrative example, consider $U(2)_{(-1, -3)}$ SQCD with $[3, 3]$ chiral multiplets, which arises as an $\mathcal{N} = 4$ descendant. Its planar mirror dual can be constructed via the *chiral-planar* dualization algorithm (see Equation 2.14 and the accompanying discussion), or by adding suitable $\mathcal{N} = 4$ supersymmetry-breaking real-mass deformations. Starting from this theory, one can explore the full three-dimensional parameter space (k, F, A) by turning on real mass deformations and tracking their effects in the planar Abelian dual. We emphasize that all these dualities can be viewed, in a broad sense, as instances of mirror symmetry.

We now turn on real mass deformations for selected chiral multiplets (see Figure 3.14 for a summary):

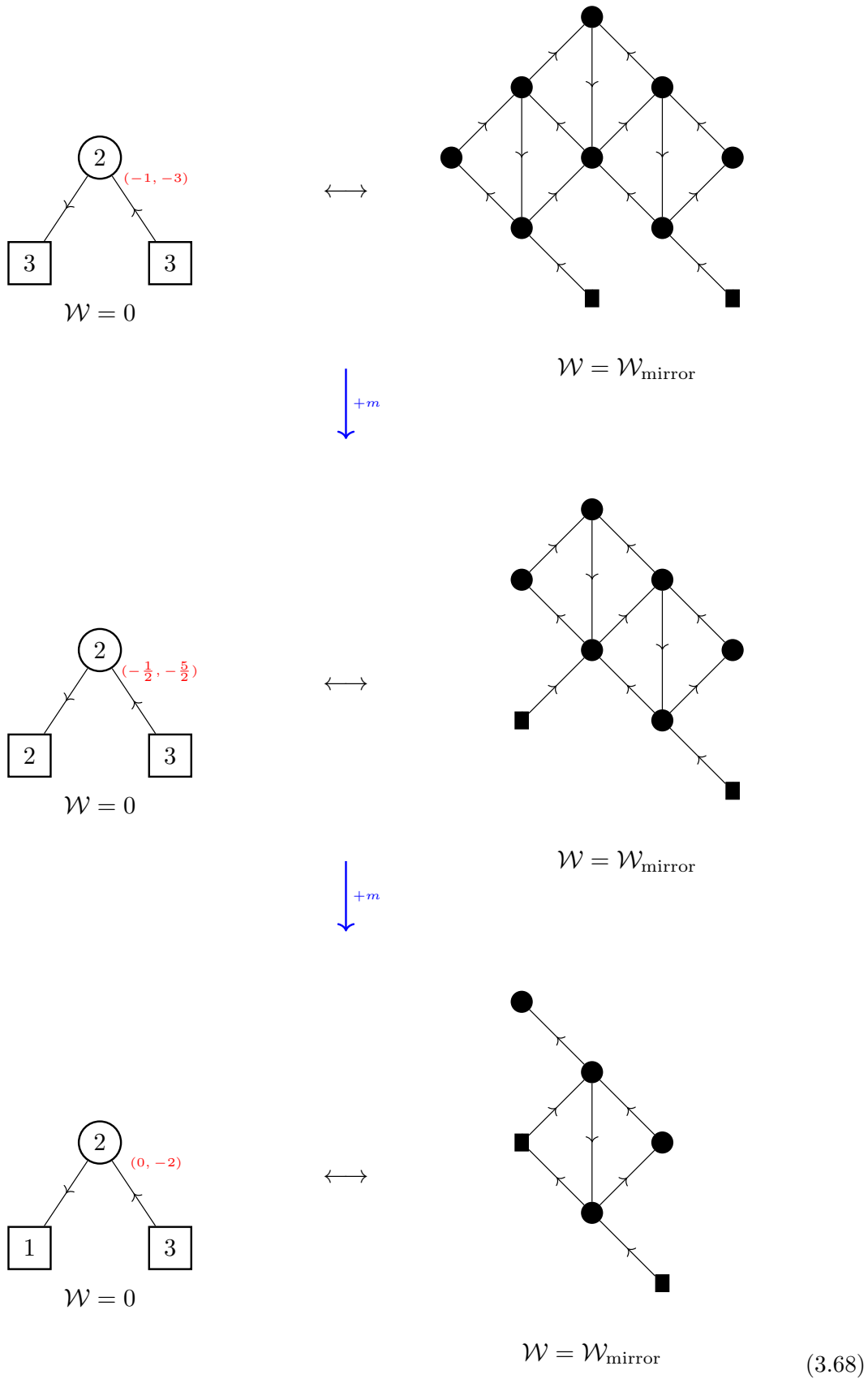
- Assigning positive real masses, $+m$, to two of the fundamental chirals and integrating them out shifts the Chern–Simons levels appropriately. The theory then flows to $U(2)_{(0, -2)}$ with $[1, 3]$ chirals. Although this configuration lies outside the direct domain of the dualization algorithm, its planar dual can be systematically determined by following the controlled RG flow. The corresponding trajectory is shown in Equation 3.68.
- Conversely, assigning negative real masses, $-m$, to two fundamental chirals leads to a distinct RG-flow trajectory, depicted in Equation 3.69. Again, the resulting theory falls outside the original algorithmic regime but remains accessible through the same deformation logic.
- We can also assign a positive real mass $+m$ to one fundamental chiral and a negative real mass $-m$ to a second chiral. The resulting RG flow trajectory is shown in Equation 3.70. Again, the resulting theory falls outside the original algorithmic regime but remains accessible through the same deformation logic.
- One can also turn on a negative mass $-m_F$ for one fundamental and a positive mass $+m_A$ for one anti-fundamental. The resulting RG flow is shown in Equation 3.71. Again, the resulting theory falls outside the original algorithmic regime but remains accessible through the same deformation logic.
- Similarly, a positive mass $+m_F$ for one fundamental together with a negative mass $-m_A$ for one anti-fundamental produces the trajectory in Equation 3.72. This configuration is in the domain of the dualization algorithm and provides a non-trivial consistency check of this paradigm.

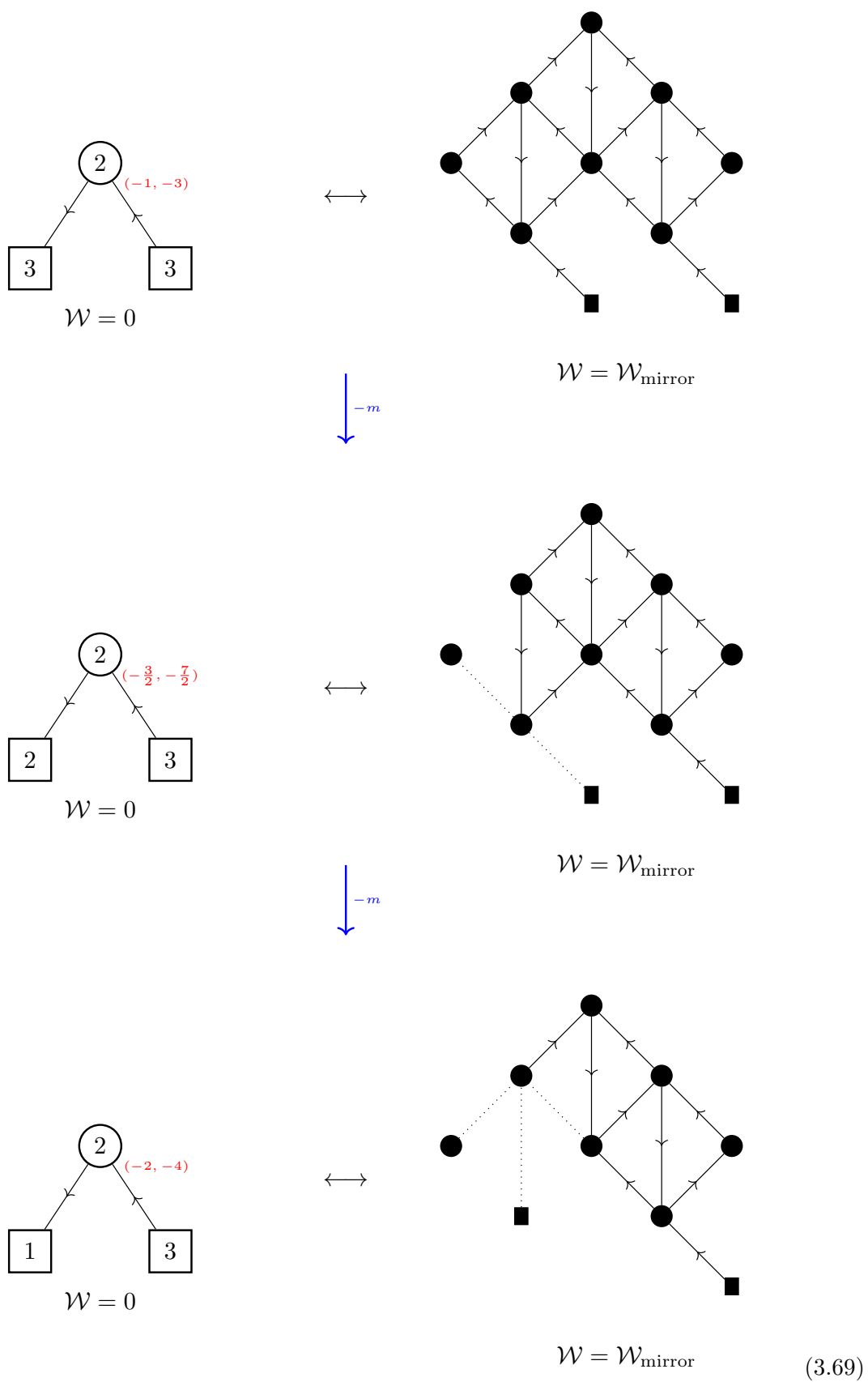
- One can also turn on a negative mass $-m_F$ for one fundamental and a negative mass $-m_A$ for one anti-fundamental. The resulting RG flow is shown in Equation 3.73. Again, the resulting theory falls outside the original algorithmic regime but remains accessible through the same deformation logic.
- Similarly, one can also turn on a positive mass $+m_F$ for one fundamental and a positive mass $+m_A$ for one anti-fundamental. The resulting RG flow is shown in Equation 3.74. Again, the resulting theory falls outside the original algorithmic regime but remains accessible through the same deformation logic. Notice the qualitative similarity in the resulting theory with the one obtained in Equation 3.73 – this is in line with our expectations.

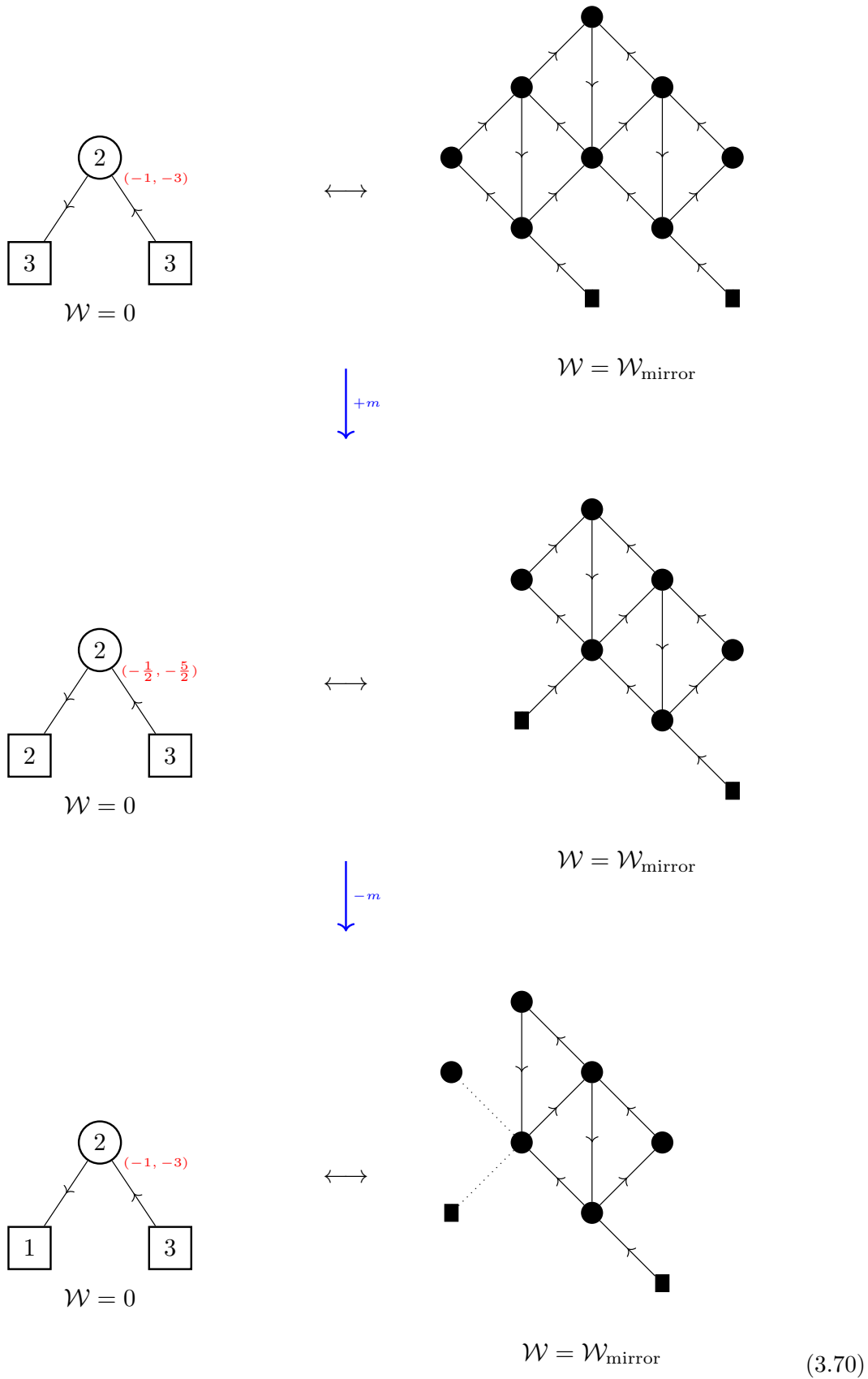
To emphasize the qualitative differences in the RG flows induced by the same mass deformations for different values of ℓ , we also study the effect of the same on $U(2)_{-1}$ SQCD with $[3, 3]$ chiral multiplets in Figure 3.15. We can recover the results of Figure 3.14 at each step consistently by following the gauging/ungauging prescription. Notice that SQCD theories with $|F - A| = 2|k|$ are *marginally chiral* and possess an additional gauge invariant monopole that parametrizes a $\mathbb{C}$ -moduli space of vacua. Consistently, the planar dual of these theories possesses an additional gauge-invariant mesonic operator.

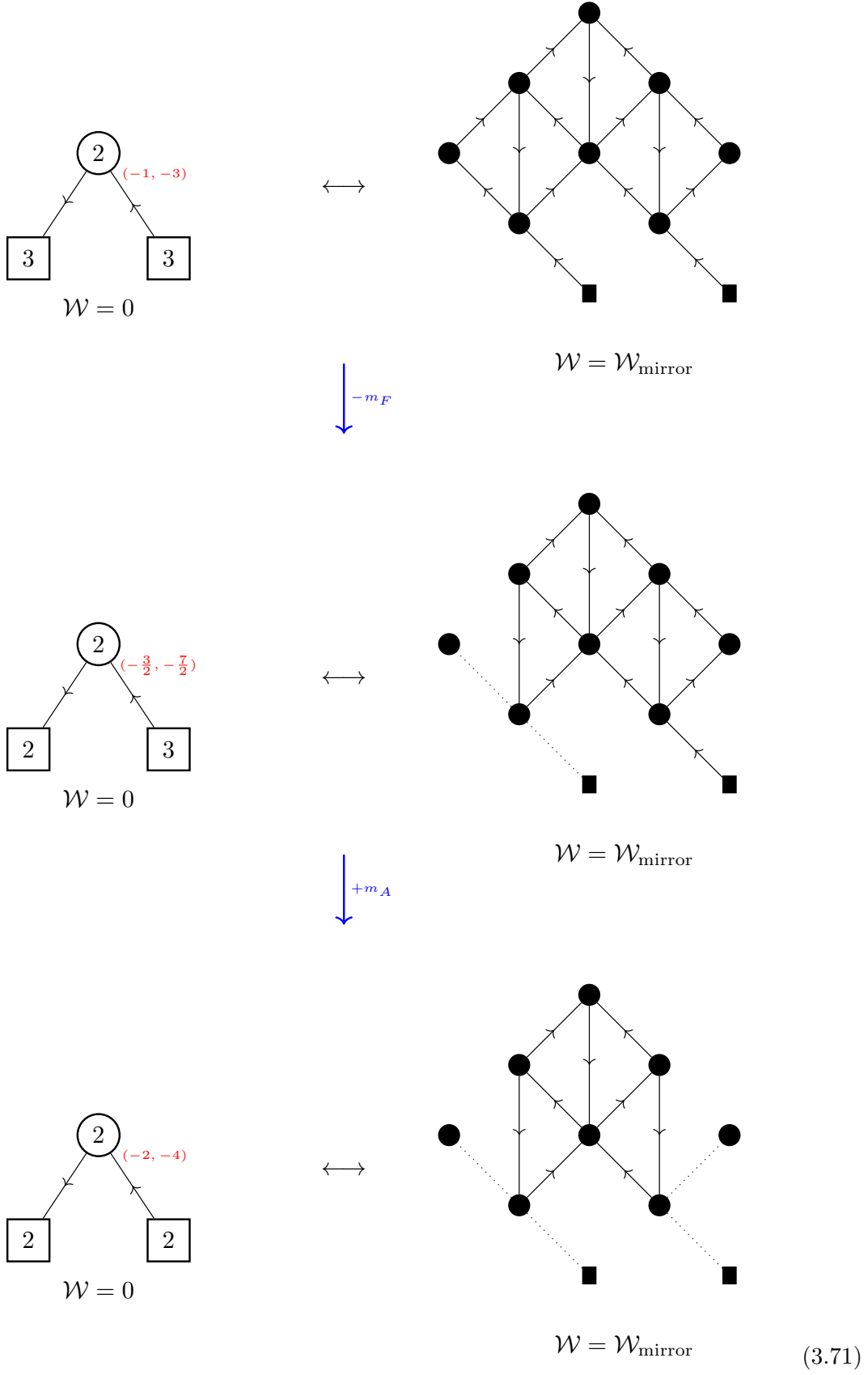
These examples illustrate how real mass deformations enlarge the effective domain of planar Abelianization, allowing one to probe chiral CS-SQCD₃ theories beyond the $\mathcal{N} = 4$ descendant locus. Their taxonomy, however, is more involved, and is deferred to future work.

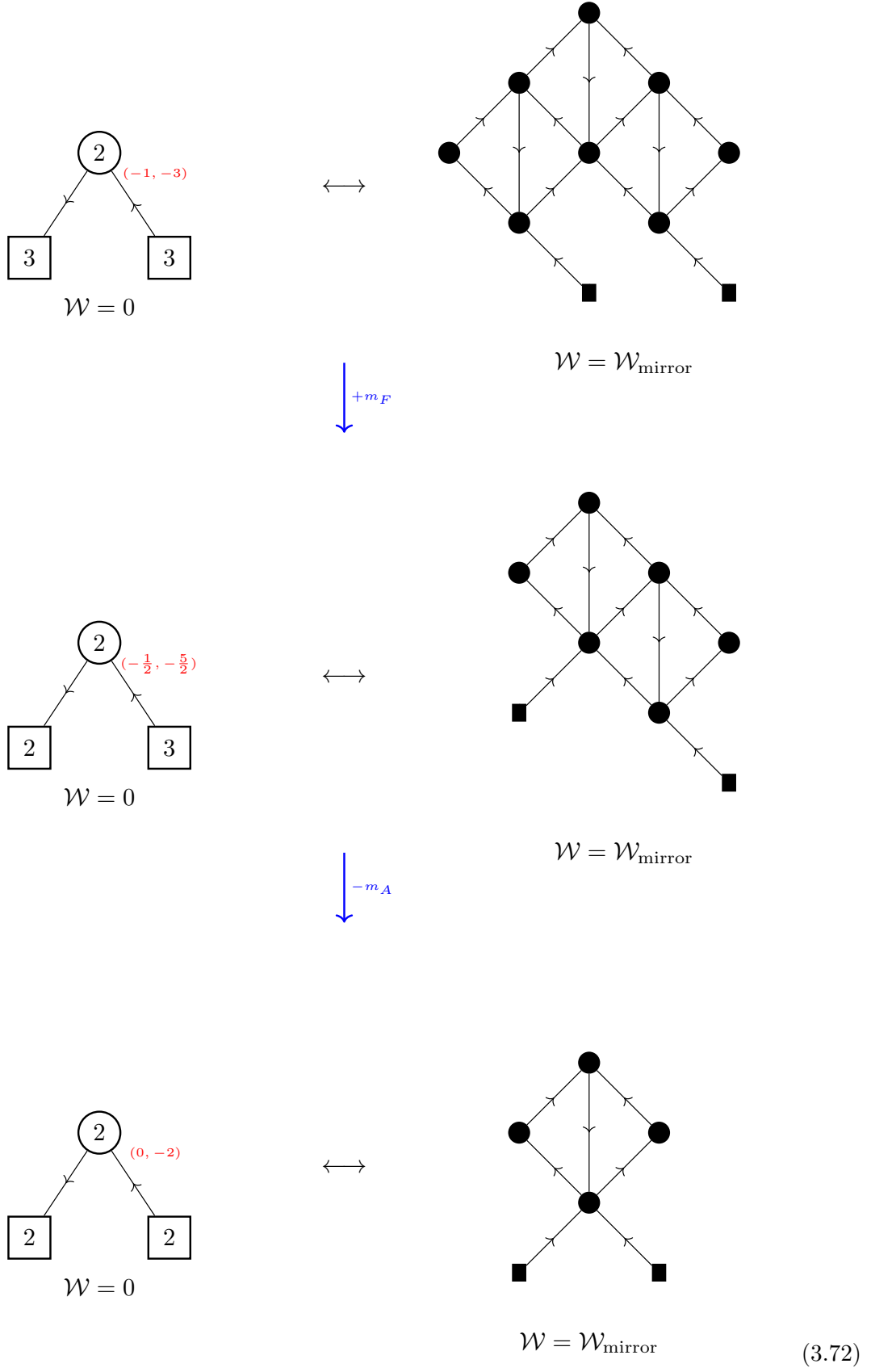
Dualities for TQFTs. Furthermore, the RG-flow trajectories we analyze naturally yield planar dual descriptions for pure $U(N)_k$ supersymmetric Yang–Mills theories. It would be interesting to investigate these dualities in greater depth, particularly through the lens of generalized global symmetries and their associated topological operators.

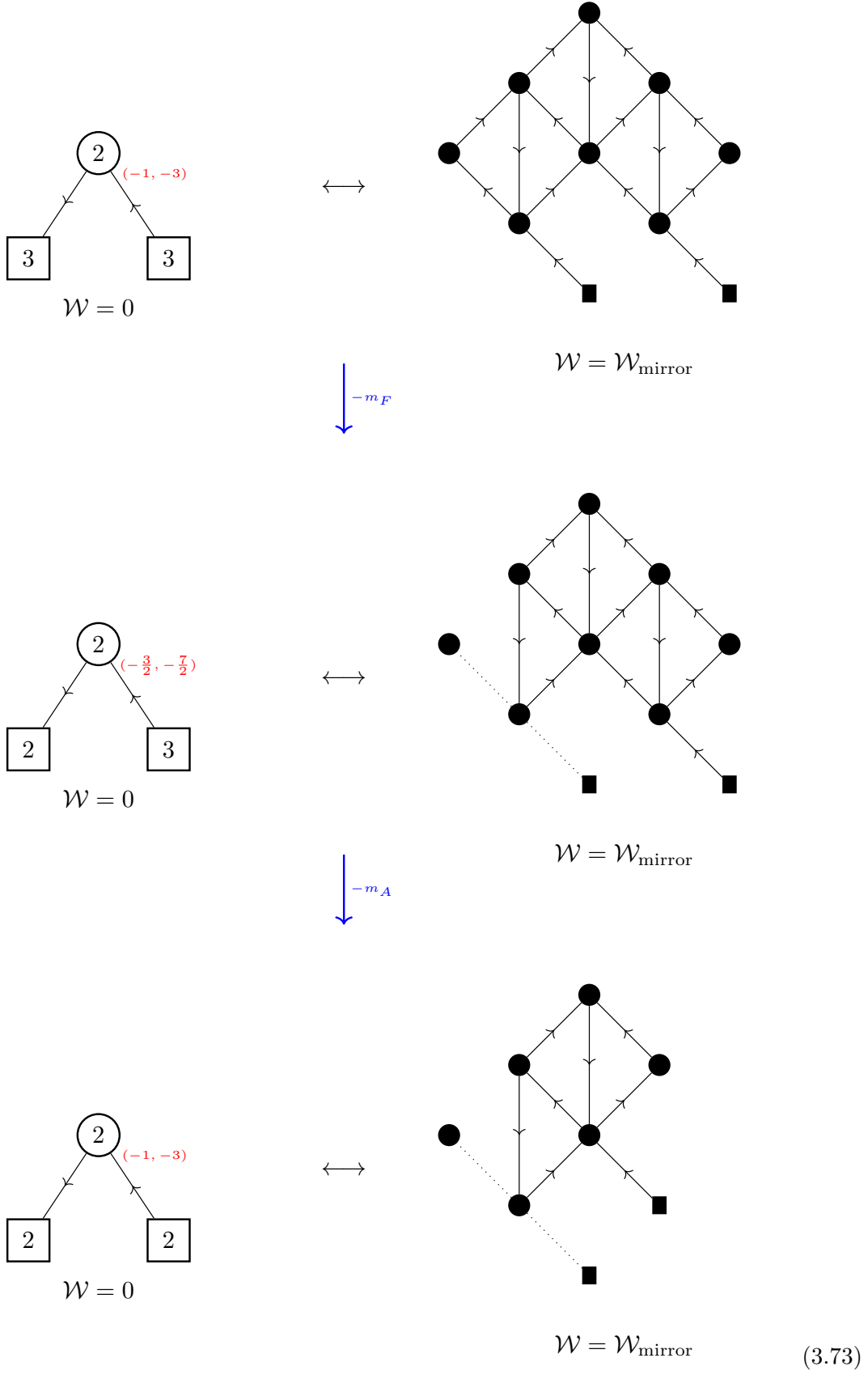


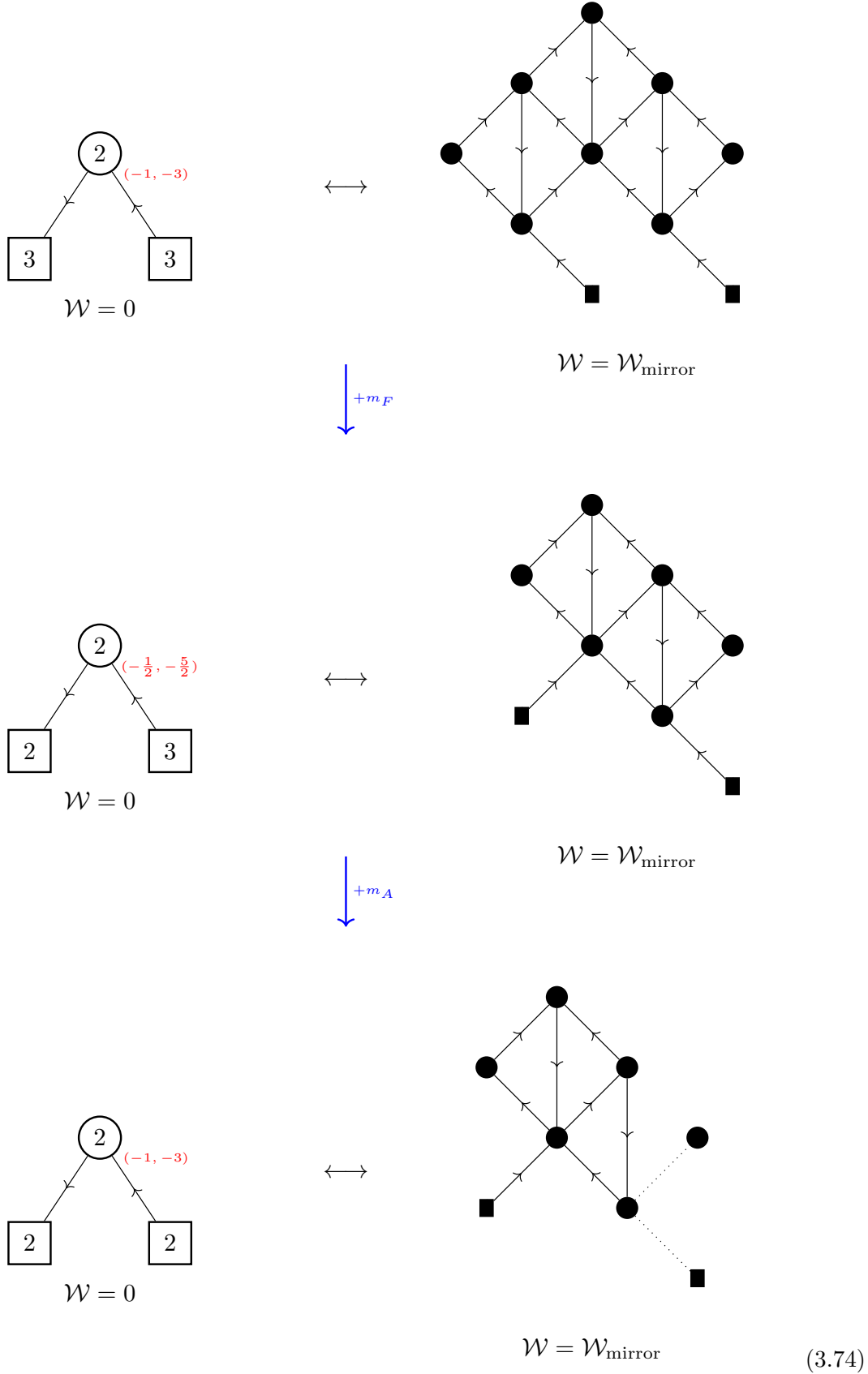












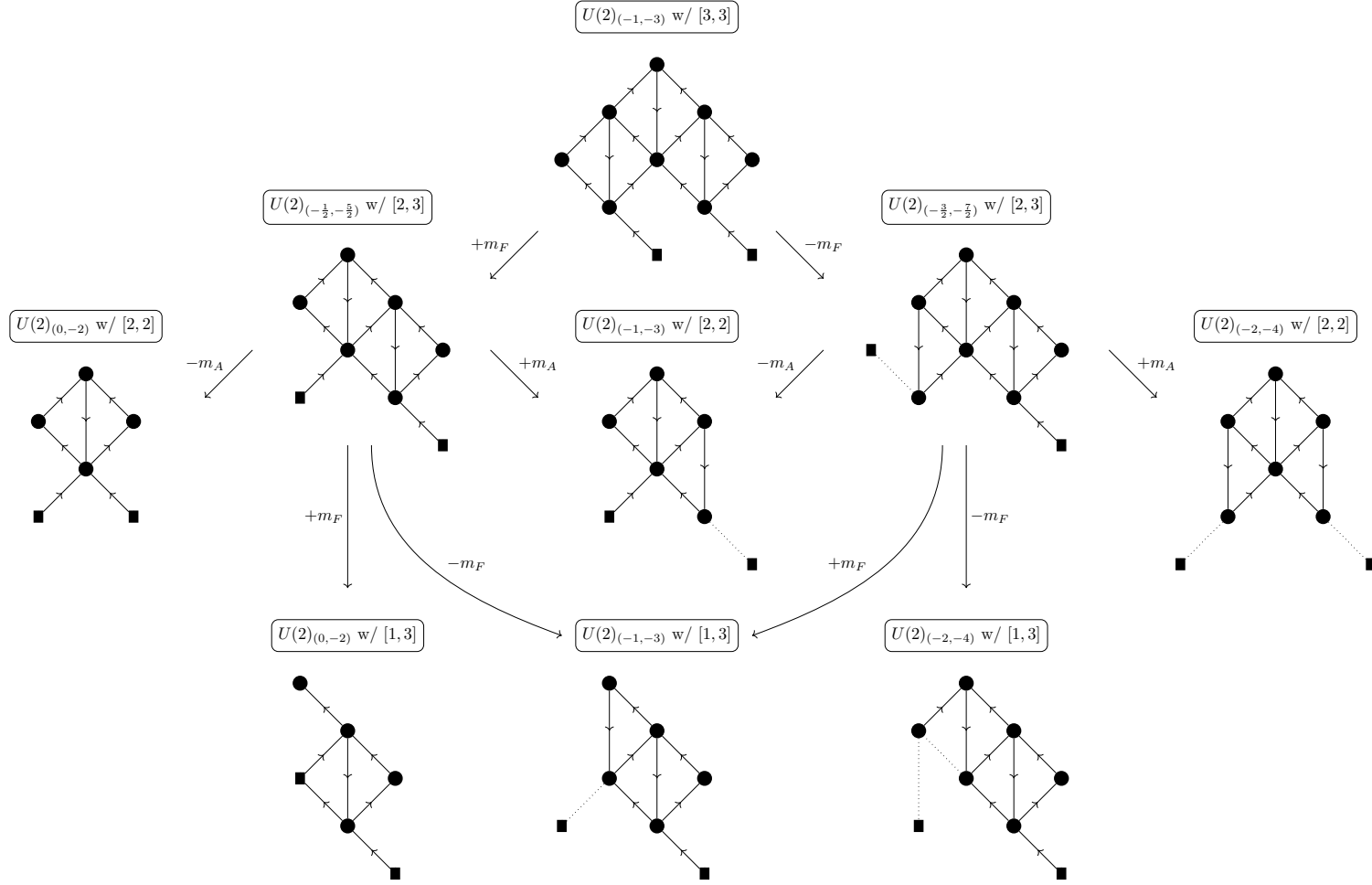


Figure 3.14: The planar mirror dual of $U(2)_{(-1,-3)}$ with $[3,3]$ chiral multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by two real mass deformations to probe theories in the regime $4 \leq F + A \leq 6$ with different CS levels. A positive (negative) real mass for a fundamental chiral multiplet shifts the CS level by $\pm \frac{1}{2}$, while a positive (negative) real mass for an anti-fundamental shifts it by $\mp \frac{1}{2}$. We only show the global topology of the resulting planar quiver diagrams; all other details can be obtained by following the same flow on the $\mathbf{S}_b^3$ partition function, or with the equations of motion.

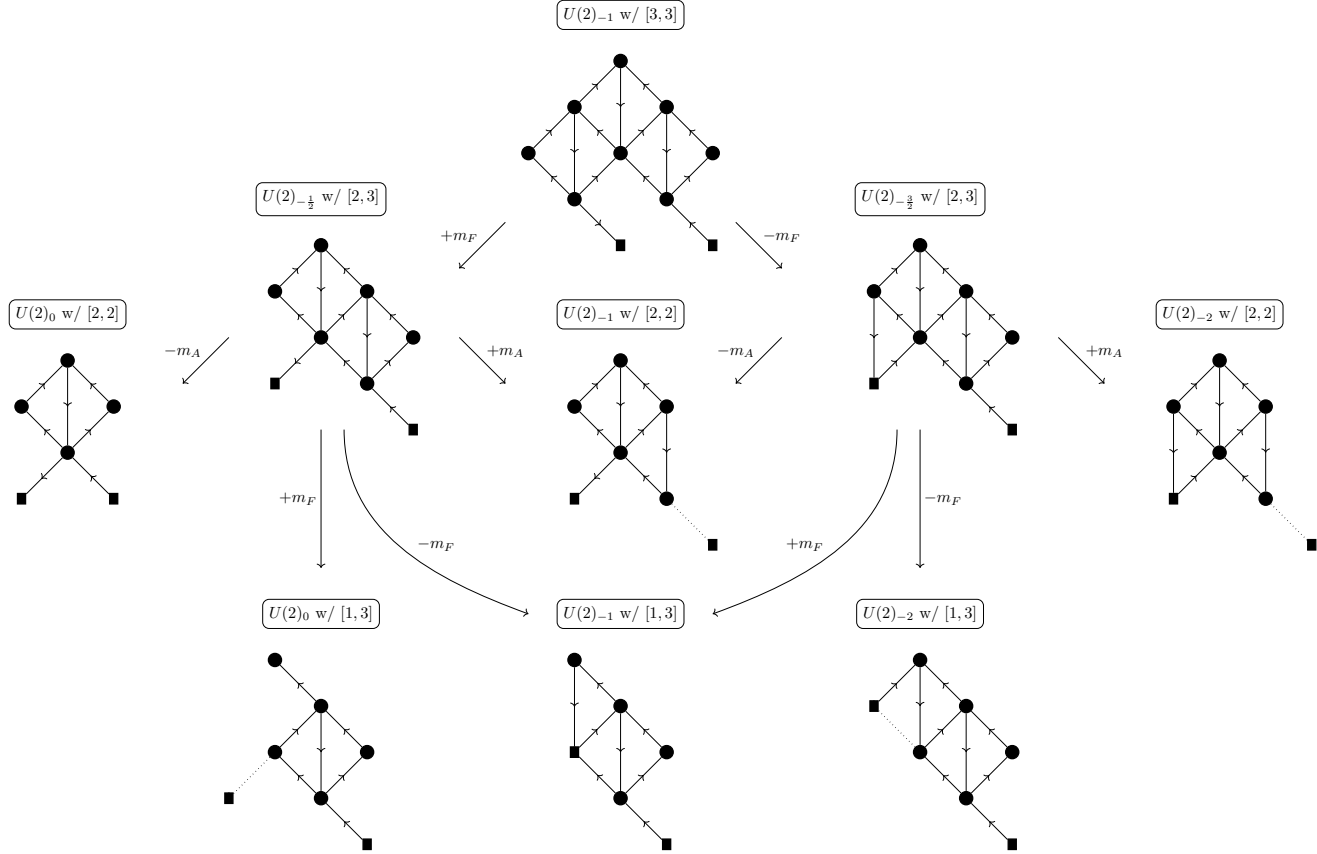
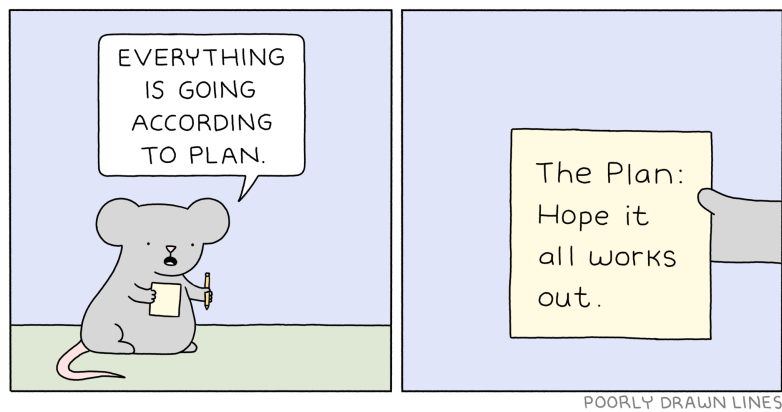


Figure 3.15: The planar mirror dual of $U(2)_{-1}$ with $[3, 3]$ chiral multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by two real mass deformations to probe theories in the regime $4 \leq F + A \leq 6$ with different CS levels. A positive (negative) real mass for a fundamental chiral multiplet shifts the CS level by $\pm \frac{1}{2}$, while a positive (negative) real mass for an anti-fundamental shifts it by $\mp \frac{1}{2}$. We only show the global topology of the resulting planar quiver diagrams; all other details can be obtained by following the same flow on the $\mathbf{S}_b^3$ partition function, or with the equations of motion. Notice that $U(2)_{-\frac{1}{2}}$ SQCD with $[2, 3]$ chiral multiplets, $U(2)_0$ SQCD with $[2, 2]$ chiral multiplets and $U(2)_{-1}$ SQCD with $[1, 3]$ chiral multiplets are marginally chiral theories; hence, their planar mirror duals contain a mesonic operator which is dual to a gauge invariant monopole.

Part III: $\mathcal{N} = 2$ Mirror Symmetry in Symplectic Gauge Theories



Chapter 4

Mirror Duals of $\mathcal{N} = 2$ Symplectic Chern-Simons-Matter Theories

The constructions of Chapters 2 and 3 (based on [2, 3]) focused on theories with unitary gauge groups. In this chapter, we extend the paradigm of planar Abelianization to theories with symplectic gauge symmetries. Three-dimensional $\mathcal{N} = 4$ symplectic gauge theories are distinguished by the existence of two distinct mirror descriptions. These mirrors typically take the form of either unitary D -type quiver gauge theories [173, 234–237] or quivers involving orthosymplectic gauge groups [238], which can be engineered via orientifolding Hanany–Witten brane setups.

While both classes of mirrors capture the same infrared physics [239, 240], incorporating orthosymplectic quivers within the paradigm of planar Abelianization presents a significant challenge. In such descriptions, not all properties of the infrared SCFT are manifest in the ultraviolet Lagrangian, notably hidden FI parameters and the associated symmetry enhancement [144, 236, 241]. By contrast, unitary D -type quivers provide a fully Lagrangian and transparent framework in which these issues do not arise.

Accordingly, in this chapter (based on [6]), we focus on planar Abelian duals of USp gauge theories, obtained from $\mathcal{N} = 4$ USp SQCD and their unitary D -type quiver mirrors. This setting allows us to construct explicit $\mathcal{N} = 2$ Abelian mirror duals for symplectic gauge theories and to extend the paradigm of planar Abelianization beyond the unitary case in a controlled and systematic manner.

4.1 Planar Abelian Duals of $U(2N)$ CS-SQCD₃ with a Tensor

We begin from a somewhat unconventional vantage point: the mirror dual of an $\mathcal{N} = 4$ $U(2N)$ gauge theory with F hypermultiplets and a rank-2 antisymmetric tensor [237, 242, 243]. This setup provides a natural and instructive arena in which to explore how the paradigm of planar Abelianization extends to gauge theories with tensor matter.

Beyond its intrinsic interest, this theory furnishes a nontrivial consistency check across different gauge groups. Indeed, **both unitary and symplectic theories** can be recovered as appropriate limits of this construction: the former by integrating out the antisymmetric tensor, and the latter by giving it a vacuum expectation value¹. From this perspective,

¹The full analysis of these RG flows at the level of the $\mathbf{S}_b^3$ partition function is beyond the scope of this

the $\mathcal{N} = 2$ chiral-planar mirror duals obtained in this section should be viewed as common ancestors of the unitary chiral-planar mirror duals studied in [2, 3, 5], as well as of the symplectic chiral-planar mirror duals constructed in later sections of this work.

The $\mathcal{N} = 4$ Mirror Pair

We show the mirror dual pair of an $\mathcal{N} = 4$ $U(2N)$ gauge theory with $F (\geq 2N + 2)$ hypermultiplets and a rank-2 antisymmetric tensor in Equation 4.1².

(4.1)

The FI terms of the mirror dual are shown explicitly, with the shorthand $X_{ij} := X_i - X_j$ in Equation 4.1.

Although both theories preserve $\mathcal{N} = 4$ supersymmetry, it is convenient to describe them in the $\mathcal{N} = 2^*$ setup [26]. In this notation, the $\mathcal{N} = 4$ R-symmetry

$$SU(2)_C \times SU(2)_H$$

is reorganized into two abelian factors,

$$U(1)_R = U(1)_{C+H}, \quad U(1)_\tau = U(1)_{C-H},$$

where $U(1)_C \subset SU(2)_C$ and $U(1)_H \subset SU(2)_H$. From the $\mathcal{N} = 2$ perspective, $U(1)_R$ plays the role of the R-symmetry, while $U(1)_\tau$ should be regarded as a global symmetry.

We do not fix the exact superconformal R-charges. Instead, we assign trial charges such that the chiral and anti-chiral multiplets have R-charge $1 - R$ and $U(1)_\tau$ charge $-\frac{1}{2}$, while adjoint chiral multiplets have R-charge $2R$ and $U(1)_\tau$ charge 1. On the electric side, we denote by Φ , Q , $\tilde{Q}$, λ , and $\tilde{\lambda}$ the adjoint, fundamental/anti-fundamental chiral multiplets and the chiral/anti-chiral components of the rank-2 antisymmetric field of SQCD. On the mirror side, we use the notation Φ_J , $b_{J, J+1}$, and $\tilde{b}_{J+1, J}$ for the adjoint and bifundamental chiral multiplets of the linear quiver, together with fundamental chiral multiplets $q, \tilde{q}$ and $p, \tilde{p}$ at the two ends of the quiver.

work. However, it is intuitively evident that if the antisymmetric field acquires a VEV proportional to an antisymmetric matrix Ω , the residual gauge group is made of those matrices satisfying $U^T \Omega U = \Omega$, which is the defining feature of $USp(2N)$ symmetries.

²To avoid ambiguity, we denote an $SU(F)$ flavor symmetry as A_{F-1} in the quiver diagrams featured in this chapter.

	$SU(F)$	$U(1)_s$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$		$SU(F)$	$U(1)_s$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$
Φ	1	0	1	$2R$	0	Φ_J	1	0	-1	$2 - 2R$	0
Q	$\square$	0	$-\frac{1}{2}$	$1 - R$	0	$b_{J,J+1}, \tilde{b}_{J,J+1}$	1	0	$\frac{1}{2}$	R	0
$\tilde{Q}$	$\square$	0	$-\frac{1}{2}$	$1 - R$	0	$q, \tilde{q}$	1	0	$\frac{1}{2}$	R	∓ 1
λ	1	-1	$-\frac{1}{2}$	$1 - R$	0	$p, \tilde{p}$	1	0	$\frac{1}{2}$	R	0
$\tilde{\lambda}$	1	1	$-\frac{1}{2}$	$1 - R$	0						

(4.2)

The assignment of $U(1)_R$ and $U(1)_\tau$ charges is consistent with the $\mathcal{N} = 4$ superpotential.

Global Symmetries. The global symmetry of the *electric* SQCD theory, depicted in the left panel of Equation 4.1, is

$$SU(F) \times U(1)_s \times U(1)_\eta \times U(1)_\tau,$$

while the UV global symmetry of the *mirror* flavored D-quiver theory, depicted in the right panel of Equation 4.1, is

$$\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_{-X_F-X_{F-1}+s} \times U(1)_\eta \times U(1)_\tau.$$

Mirror symmetry predicts the enhancement of the topological symmetries $\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_{-X_F-X_{F-1}+s}$ to $SU(F) \times U(1)_s$, which was verified by calculating the superconformal index of the two theories for small values of N & F , providing a strong check in favor of the duality.

Mapping Gauge-Invariant Operators. Mirror symmetry provides a precise map between the order and disorder operators of the two theories:

- The fundamental monopoles $\mathfrak{M}^\pm$ of the *electric* SQCD theory are mapped to the mesonic operators in the *mirror theory* constructed by contracting the fields connecting the two $U(1)$ flavor nodes.
- The composite operator $\lambda \tilde{Q} \tilde{Q}$ (and its conjugate $\tilde{\lambda} Q Q$) are mapped to a collection of monopole operators of the mirror theory. More precisely, these monopoles carry GNO fluxes ∓ 1 under consecutive gauge nodes starting from the gauge node with topological symmetry $U(1)_{-X_F-X_{F-1}+s}$.

The Flow to the $\mathcal{N} = 2$ Chiral-Planar Mirror Duals

We now consider a supersymmetry-breaking mass deformation, implemented by turning on a nonzero vacuum expectation value for the real scalar in the background vector multiplet associated with the $U(1)_\tau$ symmetry. Since $U(1)_\tau$ is a flavor subgroup of the $\mathcal{N} = 4$ R-symmetry, this deformation breaks $SU(2)_C \times SU(2)_H \rightarrow U(1)_R$ and reduces supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

For generic values of the Coulomb branch parameters, the infrared theory is a topological quantum field theory. However, there exist distinguished loci on the Coulomb branch where the flow instead leads to an interacting $\mathcal{N} = 2$ theory. The identification of these vacua and the resulting effective descriptions can be carried out efficiently at the level of

the $\mathbf{S}_b^3$ partition function, following the methods developed in [2, 3]. We do not repeat these analyses and limit ourselves to a qualitative summary of the outcome.

We first consider the “*electric*” $\mathcal{N} = 4$ SQCD theory. We move on the Coulomb branch in such a way to reach an interacting vacuum where all the fundamental chiral fields Q and the anti-chiral component of the antisymmetric tensor $\tilde{\lambda}$ remain massless, while the adjoint Φ , the F anti-chiral fields $\tilde{Q}$ and the chiral component of the antisymmetric tensor λ acquire a mass and are integrated out. Integrating out these fields generates CS interactions, resulting in an $\mathcal{N} = 2$ $U(2N)$ theory with F fundamentals Q and an anti-chiral antisymmetric field $\tilde{\lambda}$ with CS level fixed by the matter content

$$\underbrace{\left(-\frac{F}{2}, -\frac{F}{2}\right)}_{F \text{ fundamentals}} + \underbrace{(2N, 0)}_{\text{adjoint}} + \underbrace{(1-N, 1-2N)}_{\text{antisymmetric}} = \left(1 - \frac{F}{2} + N, 1 - 2N - \frac{F}{2}\right);$$

notice, in particular, we have $\ell = -\frac{3}{2}$.

In direct analogy with the analysis of SQCD without tensor matter, the corresponding $\mathcal{N} = 2$ mirror dual is obtained by Higgsing each $U(k)$ gauge node to its maximal torus $U(1)^k$. The resulting theory is a planar Abelian quiver, organized into columns of k $U(1)$ gauge nodes. We show the resulting duality in Figure 4.1.

The planar dual shown in Figure 4.1 retains most features of the planar theory dual to SQCD without tensor matter, and we only comment on the distinguishing features of this theory:

- There is no gauge-invariant mesonic operator in the chiral ring of the planar mirror dual. Consequently, we conclude that there is no gauge-invariant monopole operator in the chiral ring of the electric theory.
- The monopole superpotential $\mathcal{W}_{monopole}$ contains additional monopole terms with generic GNO flux assignment shown in Equation 4.3. These monopoles also have an R-charge of +2, and are necessary to ensure the correct pattern of symmetry enhancement in the deep IR:

$$\left(\begin{array}{ccc} & & - \\ & - & \\ \dots & & 0 \\ & + & \\ & & + \\ & \vdots & \vdots \end{array} \right), \left(\begin{array}{ccc} & & 0 \\ & 0 & \\ & - & \\ \dots & - & \\ & & 0 \\ & + & \\ & & + \\ & \vdots & \vdots \end{array} \right), \dots \quad (4.3)$$

Matching Global Symmetries. The global symmetry of the *electric* SQCD theory, shown in the left panel of Figure 4.1, is

$$SU(F) \times U(1)_s \times U(1)_\eta.$$

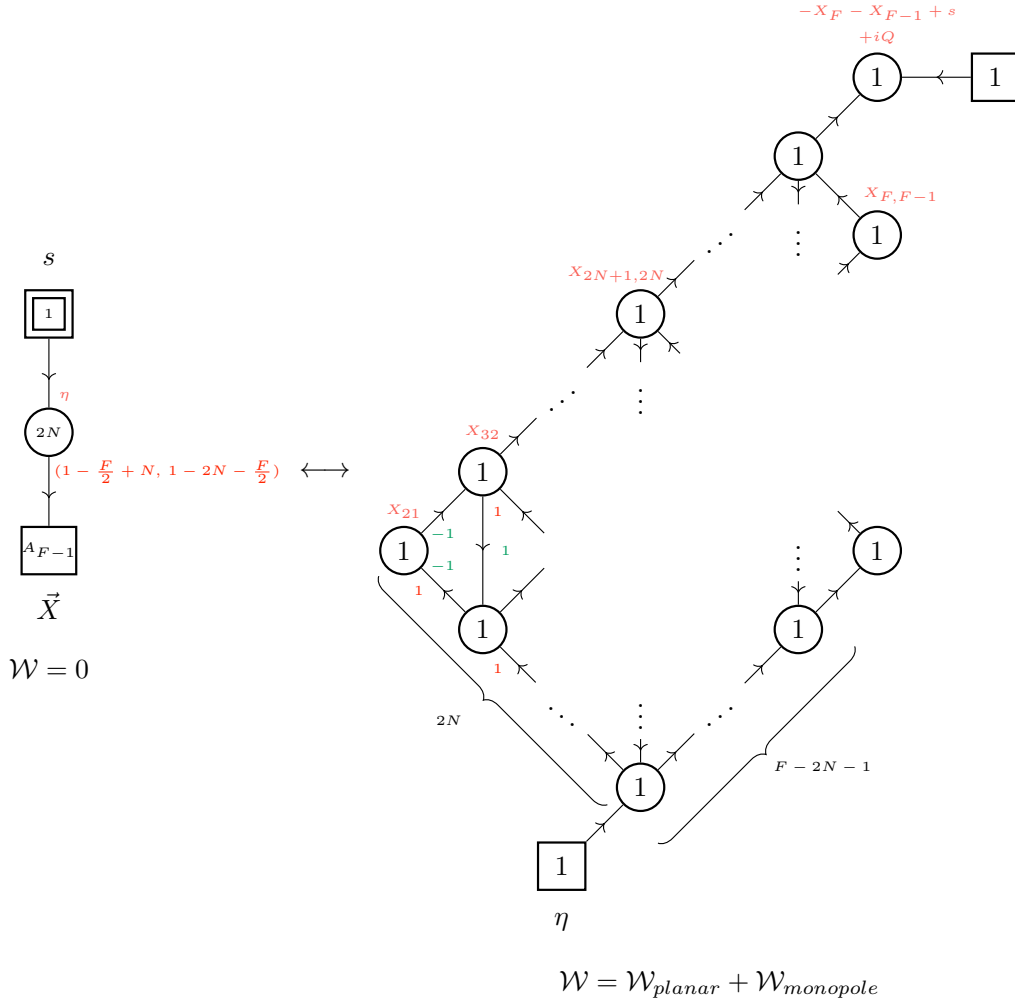


Figure 4.1: The $\mathcal{N} = 2$ mirror dual of $U(2N)_{(1-\frac{F}{2}+N, 1-2N-\frac{F}{2})}$ SQCD with F fundamental multiplets and an antisymmetric tensor is shown here. In our parametrization, the fields in the electric theory are assigned a trial R-charge of $1 - R$, while the diagonal chiral fields have trial R-charge R and the vertical ones $2 - 2R$ in the planar dual. The final column contains $2N$ $U(1)$ gauge nodes that are **not** connected by any matter or topological interactions to each other, and whose FI terms alternate between $-X_F - X_{F-1} + s + iQ$ and $X_F - X_{F-1}$ from top to bottom. Unless otherwise stated, we set $R = 0$ without loss of generality.

N	k	F	r	Superconformal Index
1	0	4	3/5	$1 + 6x^{6/5} - 17x^2 + 20x^{12/5} - 64x^{16/5} + \mathcal{O}(x^{18/5})$
1	1/2	5	3/5	$1 + 10x^{6/5} - 26x^2 + 50x^{12/5} - 175x^{16/5} + \mathcal{O}(x^{18/5})$
1	1	6	3/5	$1 + 15x^{6/5} - 37x^2 + 105x^{12/5} - 384x^{16/5} + \mathcal{O}(x^{18/5})$

Table 4.1: Values of the unrefined superconformal index of $\mathcal{N} = 2$ $U(2N)_{(-k, -k-3N)}$ SQCD with F anti-fundamental fields is reported here. The chiral multiplets are assigned a trial R-charge $1-r$. We verify that the index matches that of the corresponding Abelian mirror quiver (Figure 4.1).

The UV global symmetry of the *mirror* theory, depicted in the right panel of Figure 4.1, is instead

$$\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_{-X_F-X_{F-1}+s} \times U(1)_\eta.$$

In the absence of a monopole superpotential, each $U(1)$ gauge node would give rise to an independent topological $U(1)$ symmetry. The monopole superpotential breaks the topological symmetries associated with the nodes in each column of the planar quiver to a single diagonal $U(1)$. Consequently, the theory admits a $U(1)^{F-2}$ topological symmetry associated with the $F-2$ columns of the planar quiver, together with an additional $U(1)^2$ topological symmetry associated with the final column, due to the presence of the extra monopole operators in the superpotential (4.3). Consistently, we see symmetry enhancement to $SU(F) \times U(1)$ in the deep IR, as can be seen from index calculations in Table 4.1.

Mapping Gauge Invariant Operators. The chiral ring of the *electric* theory is generated by the $\binom{F}{2}$ composite mesonic operators $\mathbf{M}_{ij} := \tilde{\lambda}_{\alpha\beta} Q_i^\alpha Q_j^\beta$ (where α, β are gauge indices, and i, j are flavor indices) with R-charge $3-3R$.

We now provide details of the monopole operators dual to the composite mesons.

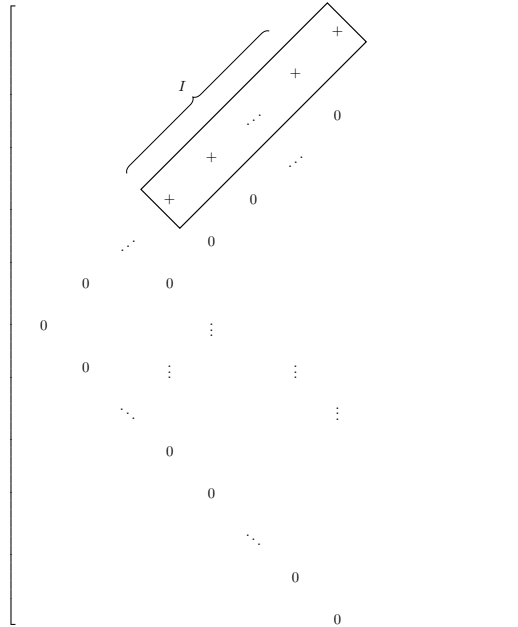
- The simplest operator is

$$\mathbf{M}_{F,F-1} = \tilde{\lambda}_{\alpha\beta} Q_F^\alpha Q_{F-1}^\beta.$$

In the mirror theory, this operator is mapped to a monopole carrying +1 GNO flux under the top gauge node of the final column:

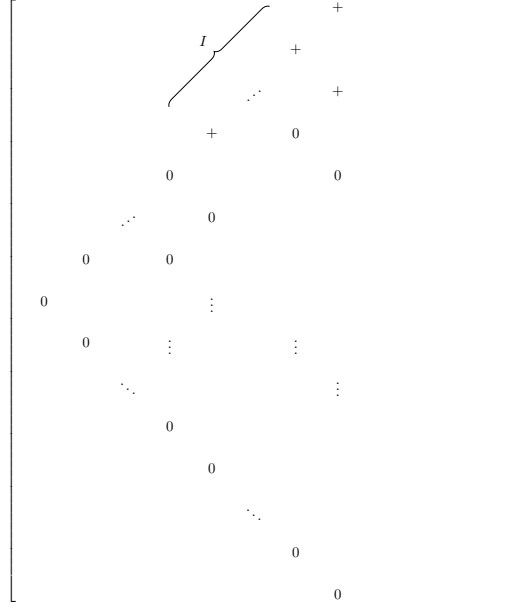


- More generally, the charge assignment propagates contiguously along the diagonal of the planar quiver. For example, the meson $\mathbf{M}_{F,F-1-I}$ is dual to a monopole with the flux configuration

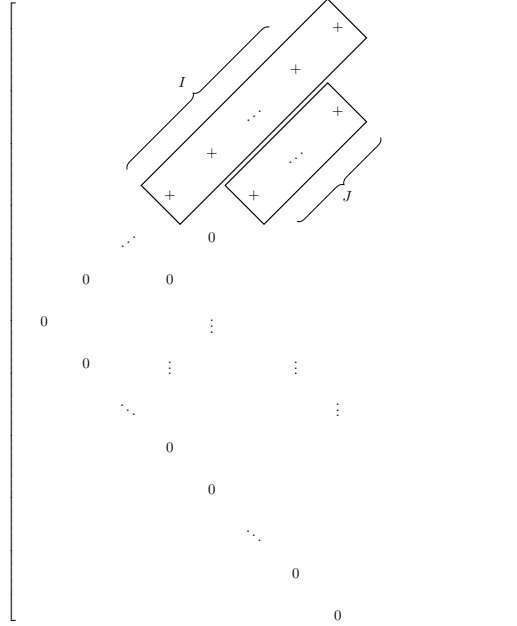


- Each column can host **at most** two nonzero GNO fluxes. This restriction follows from representation-theoretic constraints: the dual gauge-invariant operator transforms as a rank-two tensor, and therefore cannot accommodate more than two independent flux insertions in a given column. However, a +1 GNO flux may be reassigned to the I^{th} column only if there is already a flux in the $(I-1)^{\text{th}}$ column. For example, the meson $\mathbf{M}_{F-1,F-1-I}$ is dual to a monopole with the flux

configuration



Thus, a generic operator $\mathbf{M}_{F-J, F-1-I}$ is dual to a flux configuration with $I + 1$ GNO fluxes in the topmost diagonal, and $J (< I) + 1$ GNO fluxes in the second diagonal



It would be interesting to further investigate the duality between $SU(2N)_{1-\frac{F}{2}+N}$ SQCD with F chiral multiplets and a rank-2 antisymmetric tensor, and its corresponding planar mirror dual. This duality can be obtained in a straightforward manner by gauging the topological symmetry of the $U(2N)$ theory [106], which on the mirror side corresponds to gauging one of the two $U(1)$ flavor symmetries.

This case is particularly rich, as the chiral ring is generated by a larger class of operators, including composite mesons of the form $\tilde{\lambda} Q Q$, baryonic operators Q^{2N} , and composite operators $(\tilde{\lambda}\tilde{\lambda})^k$ arising from higher powers of the antisymmetric tensor.

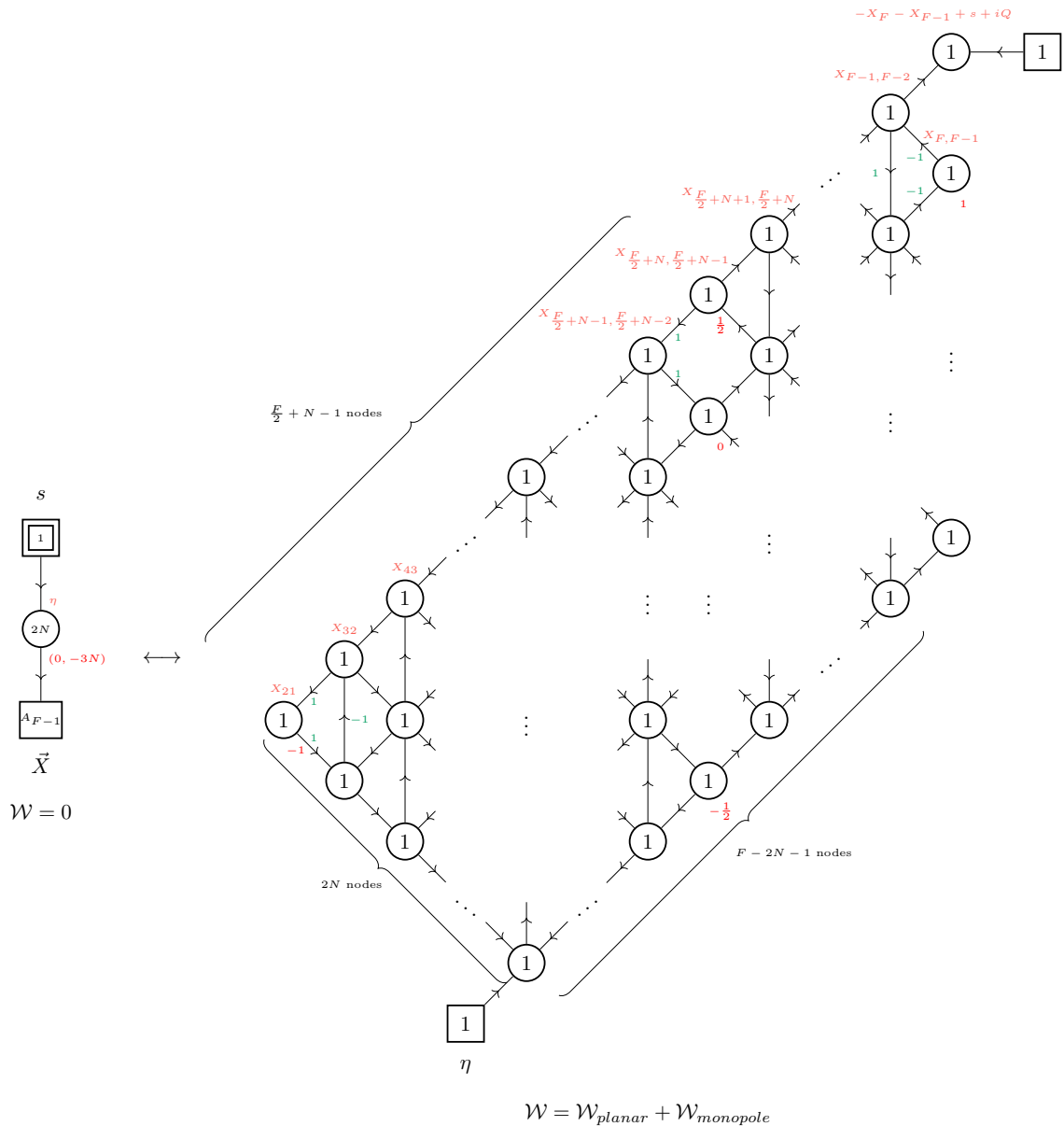
The study of this duality is, however, complicated by an ambiguity in gauging the $U(1)$

flavor symmetries on the mirror side: one may choose to gauge either of the two $U(1)$ factors, leading to inequivalent presentations of the gauge-invariant monopole operators. In particular, one choice yields a description in which the composite operators $(\tilde{\lambda}\tilde{\lambda})^k$ are mapped to monopoles with higher GNO flux, whereas the alternative choice leads to a presentation in which the composite mesons and baryonic operators are mapped to monopoles with higher GNO flux. As such, we defer the resolution of this ambiguity and a more detailed study of these theories to future work.

A Dual for $U(2N)_{(0,-3N)}$ CS-SQCD₃ with $F \geq 2N + 2$ $\square$ and $\overline{\square}$

We conclude by proposing an Abelian quiver gauge theory that is infrared dual to $U(2N)_{(0,-3N)}$ SQCD with F fundamental chiral multiplets and a rank-2 antisymmetric tensor, with $F \geq 2N + 2$ in Equation 4.4. Consistency of Chern–Simons level quantization requires F to be even.

This proposal is motivated by three complementary considerations. First, its structure closely parallels the planar Abelian duals of $U(N)_0$ SQCD with F fundamental chiral multiplets proposed in [5]. Second, we have verified agreement of superconformal indices for several low-rank and low-flavor examples, providing nontrivial evidence in favor of the duality. Finally, under appropriate real mass deformations, the proposed Abelian quiver flows to the known $\mathcal{N} = 4$ descendant theories discussed before. A more detailed *a posteriori* justification of this proposal will be given once we construct the analogous dual theory for $USp(2N)_0$ with F fundamental multiplets (see the discussion before Equation 4.47), since both constructions share many qualitative features.



(4.4)

The planar dual shown in Equation 4.4 retains most of the features of the planar theory shown in Figure 4.1; we only comment on the qualitative differences between the two theories here.

- $\mathcal{W}_{\text{planar}}$ contains one cubic (quartic) term for each triangular (square) face of the quiver, with sign -1 ($+1$) if the arrows run clockwise (anticlockwise) around the face.
- $\mathcal{W}_{\text{monopole}}$ includes linear monopole superpotentials involving monopoles with GNO fluxes under two adjacent nodes within the same column. Specifically there is a

monopole $\mathfrak{M} \begin{pmatrix} - \\ + \end{pmatrix}$ for each pair of adjacent nodes.

In addition, $\mathcal{W}_{\text{monopole}}$ also contains additional monopole terms of the form described in Equation 4.3.

The FI terms of the $U(1)$ gauge nodes are compatible with the monopole superpotential terms and are **not arbitrary**.

- There is a mixed CS term between each pair of nodes connected by an arrow. For downward-pointing arrows ($\searrow$, $\swarrow$, and $\downarrow$), the mixed CS level is $+1$, while for upward-pointing arrows ($\swarrow$, $\searrow$, and $\uparrow$), it is -1 .
- The CS level of a $U(1)_i$ gauge node is determined by its mixed CS couplings:

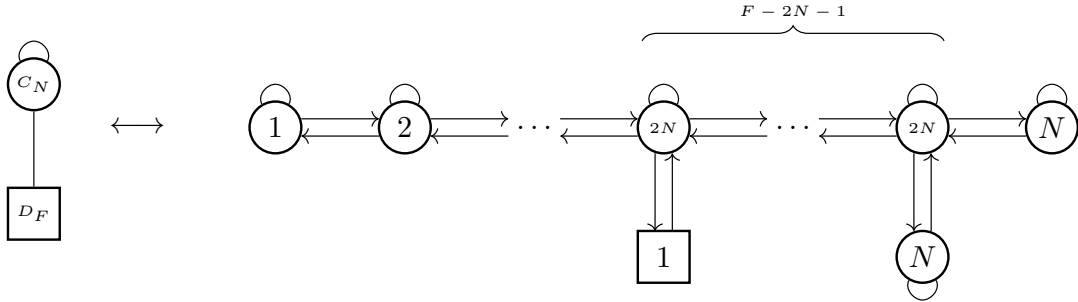
$$k_i = -\frac{1}{2} \sum_{j \neq i} k_{ij}, \quad (4.5)$$

where k_{ij} denotes the level of the mixed CS term between $U(1)_i$ and $U(1)_j$.

4.2 Planar Abelian Duals of Symplectic CS-SQCD₃

Three-dimensional $\mathcal{N} = 4$ gauge theories with symplectic gauge group are known to admit two distinct mirror descriptions. The first is a unitary quiver with the structure of a D -type Dynkin diagram (which we will refer to as a D -type quiver) [173, 234–237]. The second description is a linear quiver with alternating orthogonal and symplectic gauge groups [238], which can be obtained from Hanany–Witten brane constructions in the presence of $O3$ planes (often denoted Feng–Hanany setups).

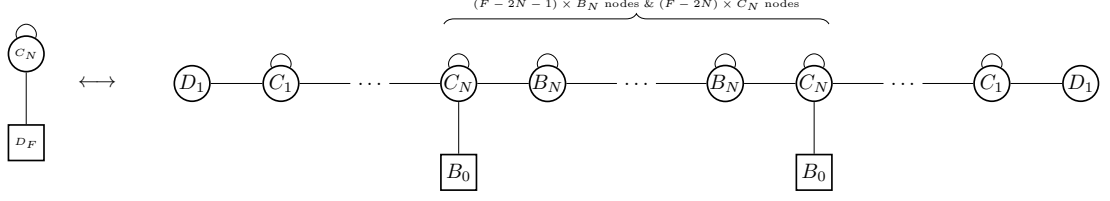
In this work, we restrict our attention to the first mirror dual, postponing the analysis of orthosymplectic mirror duals to future work. The D -type mirror dual of $USp(2N)$ SQCD is shown in Figure 4.2, while the orthosymplectic mirror dual is displayed for completeness in Figure 4.3.



$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$$

$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$$

Figure 4.2: The $\mathcal{N} = 4$ mirror duality between $USp(2N)$ SQCD and $F \geq 2N + 2$ hypermultiplets, and a D -type unitary quiver gauge theory is shown here.



$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$$

$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$$

Figure 4.3: The $\mathcal{N} = 4$ mirror duality between $USp(2N)$ SQCD and $F (\geq 2N + 2)$ hypermultiplets and a linear, orthosymplectic quiver gauge theory is shown here.

Both, Figures 4.2 and 4.3 are presented in $\mathcal{N} = 2$ language. In the orthosymplectic mirror of $USp(2N)$ (Figure 4.3), the flavor nodes labeled by B_0 represent half-hypermultiplets transforming in the fundamental representation of C_N .

A Technical Aside: The $\mathcal{N} = 4$ Superpotential. The $\mathcal{N} = 4$ superpotential of theories with a symplectic gauge symmetry takes the form [244]:

$$\mathcal{W}_{\mathcal{N}=4} = \tilde{Q}_\alpha^i \Omega^{\alpha\beta} \Omega_{\beta\gamma} \Phi_\delta^\gamma Q_i^\delta, \quad (4.6)$$

where Ω is the $2N \times 2N$ matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_N \\ -\mathbb{I}_N & 0 \end{pmatrix},$$

satisfying $M^T \Omega M = \Omega$ for $M \in Sp(2N; \mathbb{R})$, so that $\Omega^{-1} = \Omega^T = -\Omega$, and $\alpha, \beta, \dots$ are gauge indices, and $i, j, \dots$ are flavor indices.

Since the fundamental representation of $USp(2N)$ is pseudoreal, the generators $(T^A)_\beta^\alpha$ of the fundamental representation satisfy

$$(\Omega T^A)_{\alpha\beta} = (\Omega T^A)_{\beta\alpha}.$$

Consequently, if a field Q_i^α transforms in the fundamental representation of $USp(2N)$, then so does $\Omega^{\alpha\beta} \tilde{Q}_\beta^i$. This allows us to define the vector

$$q_i^\alpha := (Q_1^\alpha, \dots, Q_F^\alpha, \Omega^{\alpha\beta} \tilde{Q}_\beta^1, \dots, \Omega^{\alpha\beta} \tilde{Q}_\beta^F),$$

which enables us to rewrite the superpotential as

$$\mathcal{W}_{\mathcal{N}=4} = -\frac{1}{2} q_i^\alpha (\Omega \Phi)_{\alpha\beta} q_j^\beta \eta_{ij}, \quad (4.7)$$

where η is a symmetric $F \times F$ matrix of the form

$$\eta = \begin{pmatrix} 0 & \mathbb{I}_F \\ \mathbb{I}_F & 0 \end{pmatrix}.$$

In this form, the symmetry group acting on $\mathcal{W}$ is the set of matrices that preserve η . To make this symmetry explicit, we perform a suitable change of basis:

$$\begin{aligned} i\mathbf{v}_1^\alpha &:= \frac{1}{\sqrt{2}}(Q_1^\alpha + \Omega^{\alpha\beta} \tilde{Q}_\beta^1) & \mathbf{v}_2^\alpha &:= \frac{1}{\sqrt{2}}(Q_1^\alpha - \Omega^{\alpha\beta} \tilde{Q}_\beta^1); \\ &\dots & & \\ i\mathbf{v}_{2F-1}^\alpha &:= \frac{1}{\sqrt{2}}(Q_F^\alpha + \Omega^{\alpha\beta} \tilde{Q}_\beta^F) & \mathbf{v}_{2F}^\alpha &:= \frac{1}{\sqrt{2}}(Q_F^\alpha - \Omega^{\alpha\beta} \tilde{Q}_\beta^F); \end{aligned}$$

the resulting superpotential is

$$\mathcal{W}_{\mathcal{N}=4} = \frac{1}{2} \delta^{ij} \mathbf{v}_i^\alpha (\Omega \Phi)_{\alpha\beta} \mathbf{v}_j^\beta, \quad (4.8)$$

which is manifestly $SO(2F)$ invariant. Hence, the flavor symmetry of $\mathcal{N} = 4$ $USp(2N)$ SQCD theory with F hypermultiplets is $SO(2F)$ instead of $U(F)$.

In the remainder of this section, however, we focus on $USp(2N)$ theories dual to D -type quiver gauge theories and study $\mathcal{N} = 2$ deformations of the mirror duality shown in Figure 4.2.

We begin by analyzing the *chiral limit* of the SQCD theory. More precisely, we perform a supersymmetry-breaking deformation from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ in $USp(2N)$ SQCD that preserves the gauge group. At the level of the $\mathbf{S}_b^3$ partition function, this deformation follows the same steps outlined in Section 2.1 ([3]). The adjoint and the F chiral multiplets³ acquire a mass and are integrated out, generating a Chern–Simons level equal to $2N+2-F$. The resulting theory is an $\mathcal{N} = 2$ gauge theory with gauge group $USp(2N)_{2N+2-F}$ and F chiral multiplets in the fundamental representation.

The deformation can be mapped to the mirror side of Figure 4.2, yielding the planar Abelian quiver shown in Figure 4.4. This mapping admits an algorithmic generalization closely analogous to that of [3], as further discussed in [245].

³Since the representations of $USp(2N)$ are pseudoreal, there is no intrinsic distinction between fundamental and antifundamental chiral multiplets. When we refer to a field as transforming in the fundamental representation of $USp(2N)$, we therefore mean that it transforms in the antifundamental representation of the $U(F)$ flavor symmetry, or vice versa.

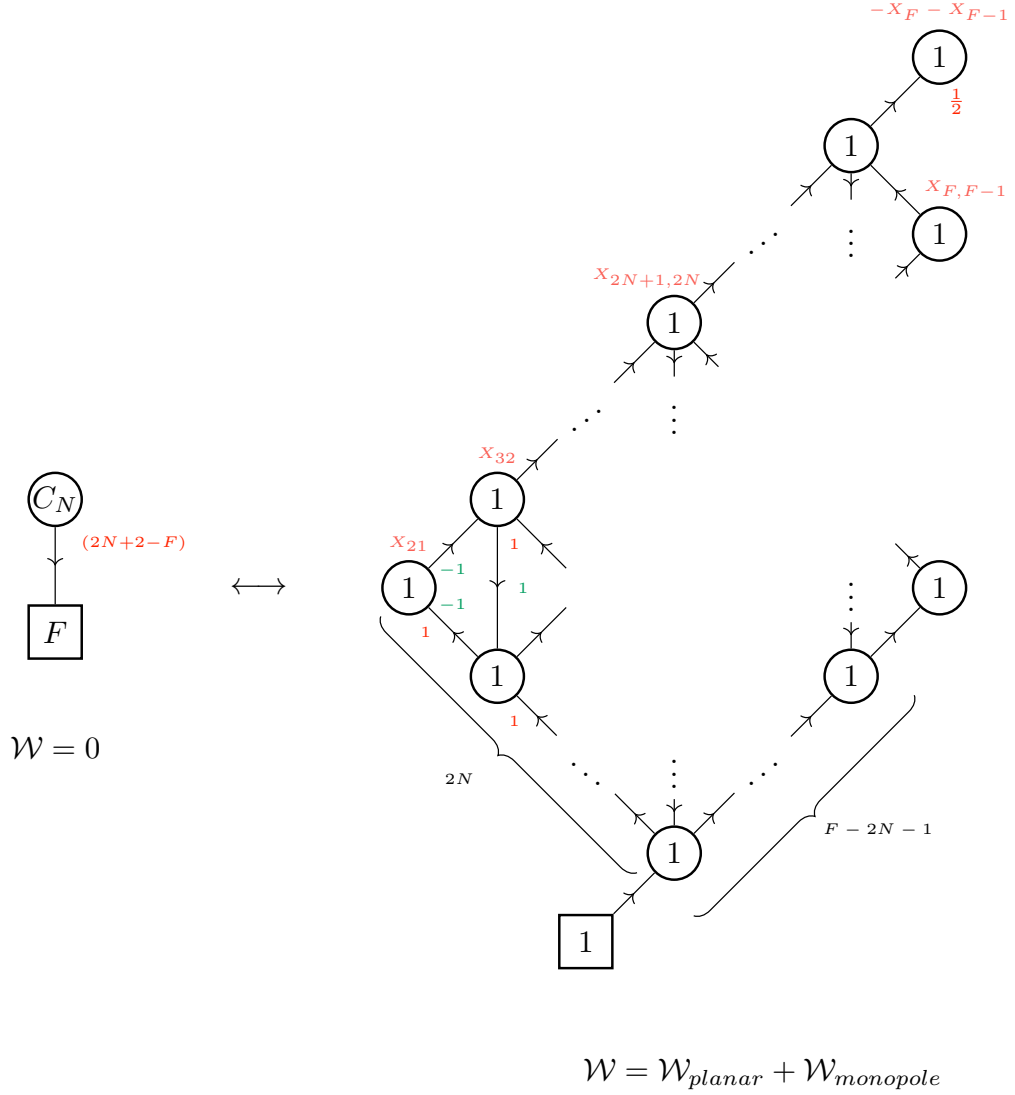


Figure 4.4: Schematic depiction of the $\mathcal{N} = 2$ mirror duality between $USp(2N)$ CS-SQCD₃ with F fundamental multiplets, and a planar Abelian quiver gauge theory. The FI terms of the mirror dual are shown explicitly (their mixing with the $U(1)_R$ symmetry has been suppressed for brevity), with the shorthand $X_{ij} := X_i - X_j$. The final column contains $2N$ $U(1)$ gauge nodes with no vertical lines or BF interactions, and whose FI terms alternate between $-X_F - X_{F-1}$ and $X_F - X_{F-1}$ from top to bottom.

We conclude with a few remarks highlighting the main differences between the planar dual of $USp(2N)$ and the planar dual of $U(N)$ shown in Section 2.1:

- There is no flavor node attached to the topmost gauge node. As a consequence, no long mesons can be formed on the planar side, which in turn implies the absence of gauge-invariant monopole operators on the $USp(2N)$ side. The sole exception to this is encountered for $F = 2N + 2$ fundamental multiplets, which we discuss via an example in Section 4.4.3.
- In the last column, the full FI parameter of the topmost gauge node is $-X_F - X_{F-1} + \frac{3}{4}iQ(1-r)$. The second-highest node has FI parameter $X_F - X_{F-1}$, while the third-highest node has FI parameter $-X_F - X_{F-1} + iQ(1-r)$. The fourth-highest node again has FI parameter $X_F - X_{F-1}$, and this pattern continues down the column.

Apart from the topmost node, which exhibits a different mixing, the structure is therefore determined by the second- and third-highest nodes.

- The monopole superpotential $\mathcal{W}_{\text{monopole}}$ contains additional terms whose GNO fluxes are

$$\begin{pmatrix} & & - \\ & - & \\ \dots & & 0 \\ & + & \\ & & + \\ & \vdots & \vdots \end{pmatrix}, \begin{pmatrix} & & 0 \\ & 0 & \\ & & - \\ \dots & - & \\ & & 0 \\ & + & \\ & & + \\ & \vdots & \vdots \end{pmatrix}, \dots \quad (4.9)$$

Owing to the R-charge mixing discussed above, these monopole operators have R-charge equal to +2 and therefore appear in the superpotential. This feature is essential in order to recover the correct symmetry enhancement.

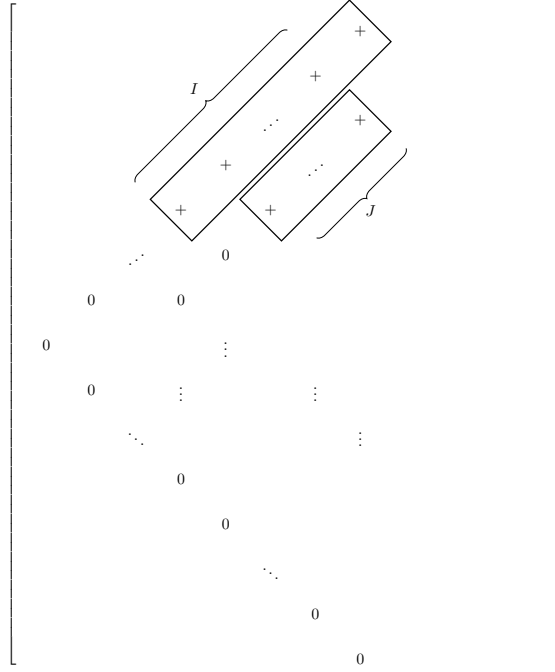
Matching Global Symmetries. The global symmetry of the *electric* SQCD theory, shown in the left panel of Figure 4.4, is $U(F)$.

The UV global symmetry of the *mirror* theory, depicted in the right panel of Figure 4.4, is instead

$$\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_{-X_F-X_{F-1}}.$$

Analogous to the $\mathcal{N} = 2$ chiral-planar mirror duals studied in Section 4.1, the theory admits a $U(1)^{F-2}$ topological symmetry associated with the $F - 2$ columns of the planar quiver, together with an additional $U(1)^2$ topological symmetry associated with the final column, due to the presence of the extra monopole operators in the superpotential. As before, we consistently see symmetry enhancement to $U(F)$ in the deep IR, as can be seen from index calculations as well.

Mapping Gauge Invariant Operators. The chiral ring of the *electric* theory is generated by the $\binom{F}{2}$ composite mesonic operators $\mathbf{M}_{ij} := \Omega_{\alpha\beta} Q_i^\alpha Q_j^\beta$ with R-charge $2 - 2R$. The dual flux configurations are identical to the ones studied in Section 4.1. As such, a generic operator $\mathbf{M}_{F-J, F-1-I}$ is dual to a flux configuration with $I + 1$ GNO fluxes in the topmost diagonal, and $J (< I) + 1$ GNO fluxes in the second diagonal



4.3 Planar Abelian Duals of D_F Quiver Gauge Theories

The second deformation of the duality in Figure 4.2 is the *planar limit* of the SQCD theory, which corresponds to a *chiral limit* of the D -type mirror. This deformation preserves the non-abelian structure of all gauge nodes in the mirror theory. As a result, one obtains an $\mathcal{N} = 2$ D -type unitary quiver on the right-hand side and a planar abelian quiver on the left-hand side.

Because the resulting $\mathcal{N} = 2$ electric theory exhibits several peculiar features, it is useful to first analyze an explicit example before presenting the general proposal.

An Example: The Planar Limit of $USp(2)$ SQCD₃ with F fundamental multiplets We start from the duality shown in Figure 4.2, focusing on the case of $\mathcal{N} = 4$ $USp(2)$ SQCD with F hypermultiplets. We then perform a real mass deformation that breaks supersymmetry to $\mathcal{N} = 2$ and simultaneously Higgses the gauge group to $U(1)$. The resulting massless spectrum is summarized in Table 4.10, while the full $\mathcal{N} = 2$ duality is presented in Figure 4.5.

	$U(1)_{gauge}$	$U(F - 4)$	$U(2)$	$SO(2) \times SO(2)$	$U(1)_R$	
φ	+2	1	1	1	$2R$	
q	+1	1	$\overline{\square}$	1	$1 - R$	(4.10)
$Q, \tilde{Q}$	-1	1	1	$(\square, 1), (1, \square)$	$1 - R$	
ϖ	+1	$\square$	1	1	$1 - R$	

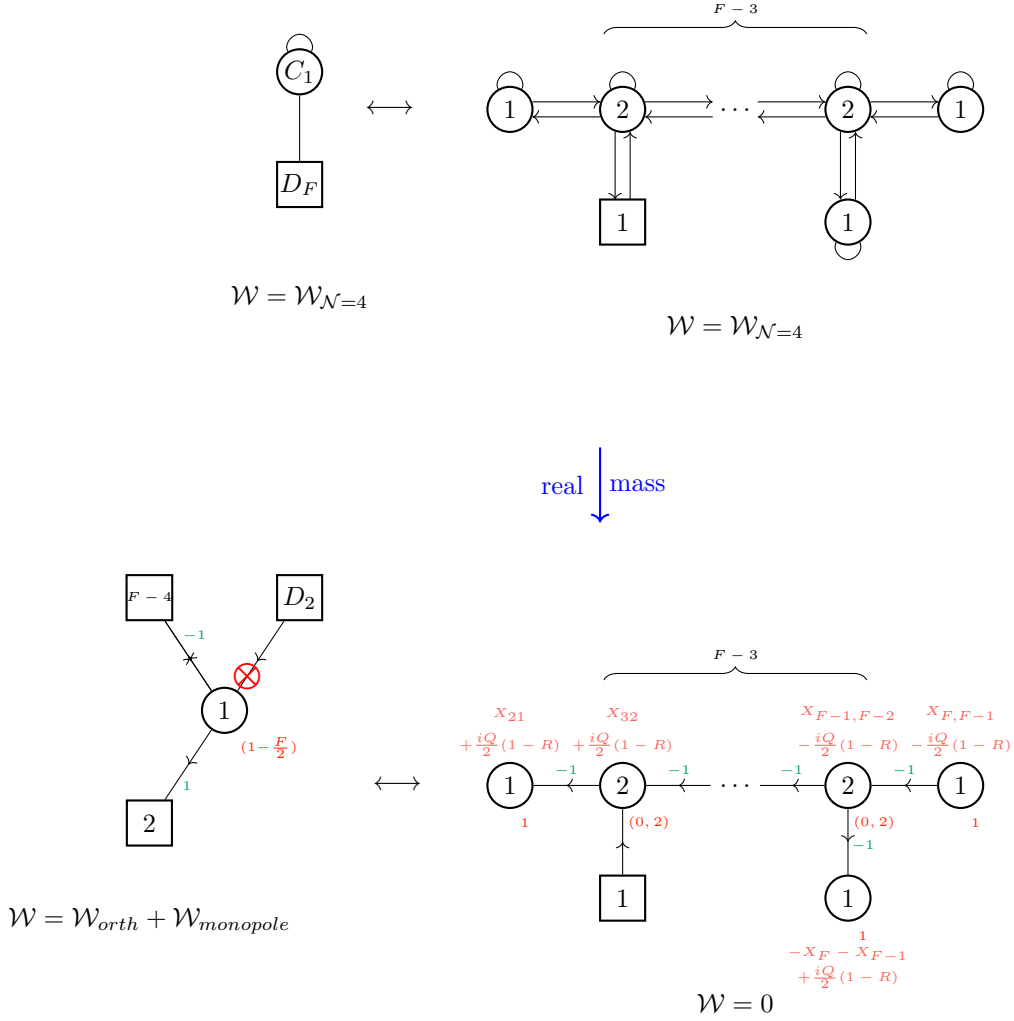


Figure 4.5: Schematic depiction of the $\mathcal{N} = 2$ mirror duality obtained by introducing a real mass deformation that Higgses the $USp(2)$ gauge symmetry to $U(1)$, while leaving the non-Abelian gauge symmetries of the mirror theory intact (bottom row). The spectrum of the *electric* $U(1)$ theory is given in Equation 4.10; accordingly, the field φ is represented by $\otimes$, while the remaining fields follow our standard conventions.

Some comments on the duality shown in Figure 4.5 are in order:

- As shown in Equation 4.10, the quiver contains a field of charge $+2$. This field originates from the adjoint representation of $USp(2)$ and survives the Higgsing to $U(1)$.
- In Equation 4.10, the fields Q and $\tilde{Q}$ are written as transforming under an $SO(2) \times SO(2)$ flavor symmetry. At the level of the $\mathbf{S}_b^3$ partition function, however, there is no distinction between a field transforming in the fundamental representation of $SO(2)$ and one transforming in the fundamental of $SU(2)$, since the two representations have identical weights. From a UV perspective, the starting point is an $\mathcal{N} = 4$ theory with $SO(2F)$ flavor symmetry, and it is therefore natural to expect a symmetry-breaking pattern of the form

$$SO(2F) \rightarrow SO(2) \times SO(2) \times \cdots.$$

Nevertheless, a computation of the superconformal index reveals that the symme-

try is instead compatible with the adjoint representation of $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$.

- The fields charged under $U(F-4)$ have charge +1 and transform in the fundamental representation of $U(F-4)$. According to our conventions, they are therefore not bifundamental fields, which explains why they are depicted by arrows with arrowheads in both directions. To make them genuinely bifundamental, one may flip the signs of the corresponding fugacities in the $\mathbf{S}_b^3$ partition function, applying the same redefinition on the mirror side. After this redefinition, the $U(F-4)$ and $U(2)$ fields recombine into a single $U(F-2)$ symmetry group⁴. Up to discrete factors, the resulting symmetry-breaking pattern induced by the $\mathcal{N} = 2$ deformation is therefore

$$SO(2F) \rightarrow U(F-2) \times SO(4).$$

Although different chiral limits may be realized on the planar side, depending on which fields acquire a mass, this symmetry-breaking pattern is independent of these choices.

- The mixing between the R-symmetry and the topological symmetry has not been displayed explicitly; its coefficient is given by $1 + \frac{1}{2}(2-F)R$. This value ensures that the monopole operator $\mathfrak{M}^+$ has R-charge equal to +2. As a consequence, the monopole operator $\mathfrak{M}^+$ must appear in the superpotential. This is expected, since by the Polyakov mechanism [215], higgsing $USp(2)$ to $U(1)$ necessarily generates a monopole potential.
- In the electric planarized theory, there is a superpotential inherited from the original $\mathcal{N} = 4$ theory,

$$\mathcal{W}_{\text{orth}} = \tilde{Q}_i^1 \varphi Q_2^i. \quad (4.11)$$

This superpotential is consistent with the gauge charge assignments listed in Equation 4.10.

To illustrate the features discussed above, we consider a simple example of the proposed duality corresponding to the case $F = 5$. The full $\mathcal{N} = 2$ duality for this case is depicted in Equation 4.12:

$\mathcal{W} = \mathcal{W}_{\text{orth}} + \mathcal{W}_{\text{monopole}}$

$\longleftrightarrow$

$\mathcal{W} = 0$

(4.12)

⁴Care must be taken in this procedure, since such a redefinition also modifies the FI terms of the mirror dual.

The field content entering the $\mathbf{S}_b^3$ partition function is:

$$\mathcal{Z}_{\mathbf{S}_b^3} \supset s_b \left(\frac{iQ}{2} R + u \pm X_2 \right) s_b \left(\frac{iQ}{2} R + u \pm X_4 \right) s_b \left(\frac{iQ}{2} (1 - 2R) - 2u \right) \\ s_b \left(\frac{iQ}{2} R - u - X_3 \right) s_b \left(\frac{iQ}{2} R - u + X_1 \right) s_b \left(\frac{iQ}{2} R - u + X_5 \right). \quad (4.13)$$

Referring to Equation 4.12, the variable u denotes the gauge fugacity. The parameters X_2 and X_4 are the fugacities associated with the D_2 fields in the top-right node, while X_3 is the fugacity of the top-left node (note the relative minus sign of the BF coupling of the X_3 node). Finally, X_1 and X_5 are the fugacities of the $U(2)$ fields in the bottom-left node.

Further insight into the symmetry structure of the theory can be obtained by computing the superconformal index on both sides of the duality. If all fugacities are turned off except for the topological fugacity η , one finds

$$\mathcal{I} = 1 + 12x^{8/7} + \left(\frac{1}{\eta} - 16 \right) x^2 + 54x^{16/7} + \mathcal{O}(x^{22/7}), \quad (4.14)$$

where we have chosen $R = 3/7$. We explicitly verify that the index agrees on the two sides of the duality. As expected, the monopole operator with flux -1 appears at order x^2 . Due to the constraint imposed by the monopole superpotential $\mathcal{W} = \mathfrak{M}^+$, one is forced to set $\eta = 1$, thereby eliminating one of the 16 conserved currents that would otherwise be associated with the topological symmetry.

If one wishes to keep track of all global symmetries, it is convenient to introduce exponentiated fugacities for all flavor symmetries by defining $\phi_j := e^{iX_j}$. The coefficient of the $-x^2$ term, which encodes the conserved currents, then takes the form

$$\mathcal{I}|_{-x^2} = \left(\phi_2 \phi_1 + \frac{\phi_1}{\phi_2} + \frac{\phi_2}{\phi_1} + \frac{1}{\phi_2 \phi_1} + 2 \right) + \left(\phi_3 \phi_4 + \phi_3 \phi_5 + \frac{\phi_5}{\phi_4} + \frac{1}{\phi_3 \phi_4} + \frac{\phi_4}{\phi_5} + \frac{1}{\phi_3 \phi_5} + 3 \right). \quad (4.15)$$

The first contribution can be identified with the character of the adjoint representation of the Lie algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This identification becomes manifest after performing the change of variables

$$X_2 \rightarrow X_2 + X_4, \quad X_4 \rightarrow X_2 - X_4,$$

which corresponds to the redefinition of exponentiated fugacities

$$\phi_2 \rightarrow \phi_2 \phi_4, \quad \phi_4 \rightarrow \phi_2 / \phi_4.$$

With this choice, the two $\mathfrak{su}(2)$ factors are disentangled and appear explicitly.

The second contribution instead reproduces the character of the adjoint representation of $\mathfrak{u}(3)$, once one sends $X_3 \rightarrow -X_3$, or equivalently $\phi_3 \rightarrow 1/\phi_3$.

The Planar Limit of $USp(2N)$ SQCD₃ with F fundamental multiplets The results of the previous example generalize straightforwardly to $USp(2N)$. After performing a mass deformation analogous to the one described above, we obtain the $\mathcal{N} = 2$ duality shown in Figure 4.6.

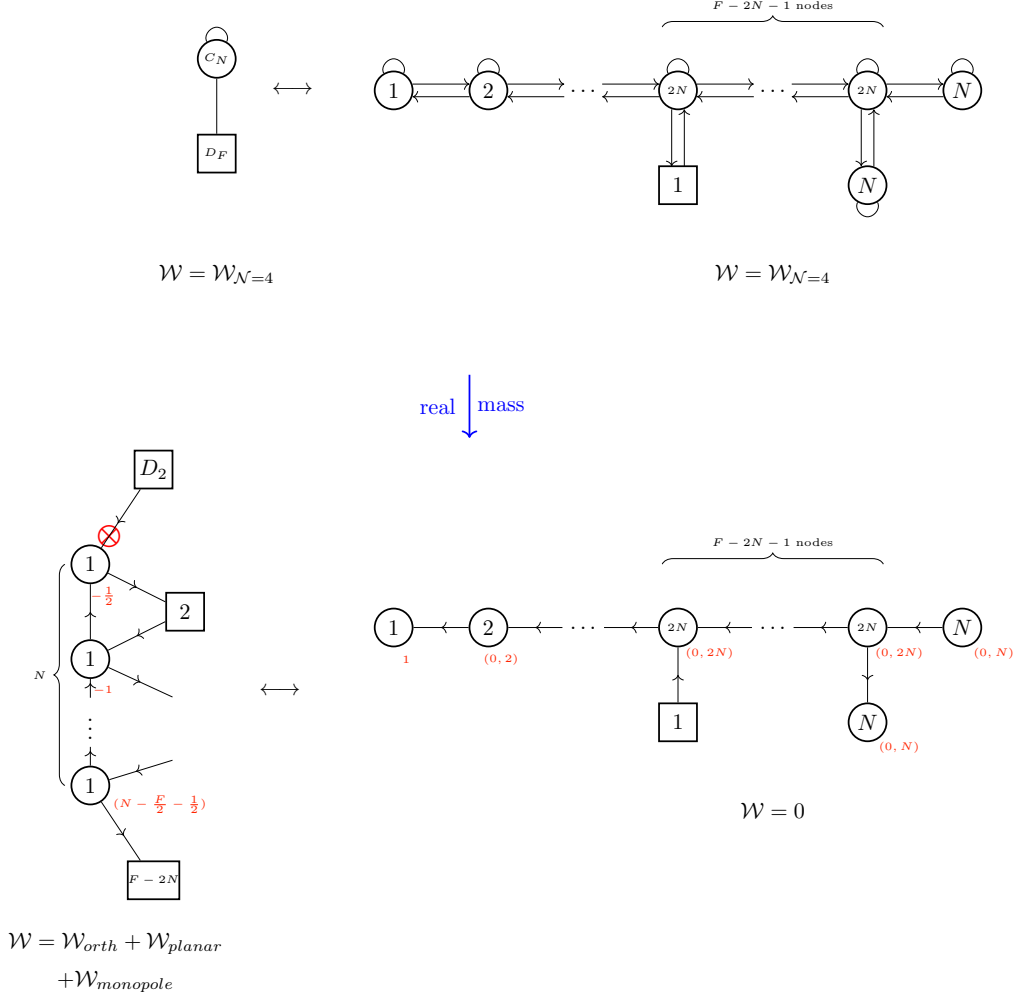


Figure 4.6: Schematic depiction of the $\mathcal{N} = 2$ mirror duality obtained by introducing a real mass deformation that Higgses the $USp(2N)$ gauge symmetry to $U(1)^N$, while leaving the non-Abelian gauge symmetries of the mirror theory intact (bottom row). For clarity, FI terms and mixed CS interactions have been suppressed.

Similarly to the $USp(2)$ case, there is an “orthogonal” superpotential $\mathcal{W}_{\text{orth}}$ that preserves the $SO(4)$ symmetry. In addition, the planar superpotential $\mathcal{W}_{\text{planar}}$ contains a term for each closed triangle in the quiver. Both of these superpotentials are remnants of the original $\mathcal{N} = 4$ superpotential. The monopole superpotential $\mathcal{W}_{\text{monopole}}$ is also modified accordingly, ensuring that *all* topological symmetries in the electric theory are broken.

We emphasize that multiple chiral limits of the mirror theory are possible, depending on which fundamental and anti-fundamental fields acquire a mass and are integrated out. These correspond to analogous deformations on the electric side that Higgs the gauge group. Importantly, the resulting symmetry-breaking pattern is independent of the specific fields that survive the deformation.

4.4 Mapping Mass Flows Across Mirror Duals

The planar Abelian duals of symplectic SQCD constructed in Section 4.2 arise from specific axial mass deformations of known $\mathcal{N} = 4$ mirror pairs. Since these $\mathcal{N} = 2$ duals descend

from $\mathcal{N} = 4$ theories, they exist only for a restricted range of parameters:

$$F = 2N + 2 + |k|,$$

where the CS level k is fixed in terms of the rank and flavor content of the $\mathcal{N} = 4$ parent. For values of (F, k, N) that do not satisfy this relation, it is not *a priori* guaranteed that the dual description will remain a planar Abelian quiver. Supersymmetry is broken for

$$F < 2N + 2 - |k|;$$

this bound coincides with the regime in which the Aharony-like dual gauge group has a negative rank, signaling the absence of a supersymmetric vacuum [21].

The key observation of [5] is that **real mass deformations provide a systematic bridge** between $\mathcal{N} = 4$ descendants and more general $\mathcal{N} = 2$ theories. Real masses shift Chern–Simons levels, integrate out matter fields, and move the theory along controlled RG-flow trajectories in the (F, k, N) parameter space. By tracking these deformations simultaneously in the non-Abelian theory and its planar mirror, we can follow how the quiver structure, CS interactions, and monopole superpotentials evolve, thereby extending planar Abelian duality beyond the original $\mathcal{N} = 4$ descendants.

Analogous to [5], we can organize $\mathcal{N} = 2$ $USp(2N)_k$ SQCD₃ with F chiral multiplets in a two-dimensional parameter space spanned by $(|k|, F)$. Each point in this plane corresponds to a distinct theory. In this chapter, we focus on SQCD with fundamentals, with CS level $k \leq 0$. The case $k > 0$ is analogous. We show this organization in Figure 4.7.

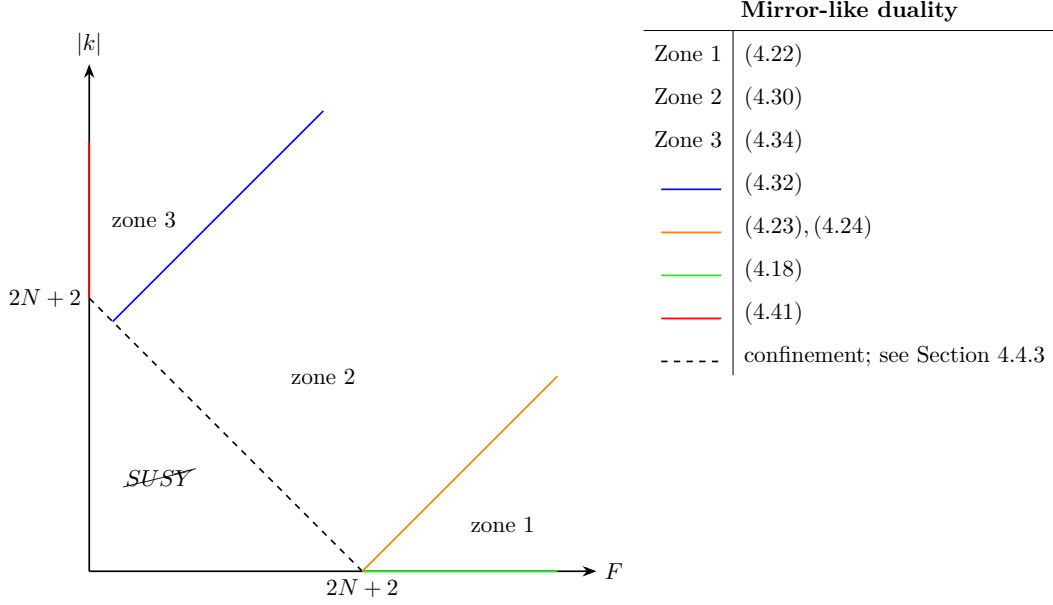


Figure 4.7: Phase diagram of $USp(2N)_k$ SQCD with F fundamental fields in the $(F, |k|)$ -plane for fixed N . The diagram is partitioned into various zones, within which the planar Abelian duals take qualitatively distinct forms.

We extend this approach here to study $\mathcal{N} = 2$ symplectic SQCD for generic ranks, flavors, and Chern–Simons levels, and then focus on a representative example to highlight the features and subtleties of this construction. Our main tool in this exercise is the $\mathbf{S}_b^3$ partition function - at each step, we are able to consistently match the asymptotic phases generated by the integrating out procedure, which provides strong evidence in favor of the dualities generated along the RG flow trajectories.

For the remainder of this section, we adopt the compact notation for the Abelian planar quivers studied here (refer to Equation 4.16)⁵.

Quiver notation	Field theory	$\mathbf{S}_b^3$
$u \bullet$	$U(1)$ $\mathcal{N} = 2$ vector multiplet	$\int du$
$X \blacksquare$	$U(1)$ Flavor node	—
$u \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} v, u \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} v$	bifundamental chiral with R-charge $1 - R$ and mixed CS level +1	$s_b \left(\frac{iQ}{2} R - u + v \right) e^{-\pi i u v}$
$u \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} v$	bifundamental chiral with R-charge $2 - 2R$ and mixed CS level +1	$s_b \left(\frac{iQ}{2} (2R - 1) - u + v \right) e^{-\pi i u v}$
$u \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} v, u \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} v$	bifundamental chiral with R-charge R and mixed CS level -1	$s_b \left(\frac{iQ}{2} (1 - R) + u - v \right) e^{\pi i u v}$
$u \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} v$	bifundamental chiral with R-charge $2R$ and mixed CS level -1	$s_b \left(\frac{iQ}{2} (1 - 2R) + u - v \right) e^{\pi i u v}$
$u \bullet \text{---} \bullet v$	Mixed CS term at level -2	$e^{2\pi i u v}$

(4.16)

For symplectic theories, assigning a positive or negative real mass $\pm m$ removes one chiral field from the spectrum and shifts the CS level by ± 1 without affecting the rank of the

⁵There is a minor change in notation between Chapter 3 and Chapter 4, since we assign a trial R-charge of $1 - R$ to the fundamental fields in the *electric* SQCD theory here. We update the compact quiver notation to reflect the same.

gauge group. On the (k, F) -plane these correspond to diagonal moves toward the left:

$$\begin{array}{c}
 -m \\
 \swarrow \\
 \bullet \text{ SQCD} \\
 \searrow \\
 +m
 \end{array} \tag{4.17}$$

which can be stated more succinctly:

$$(F, k) \xrightarrow{\pm m} (F - 1, k \pm 1).$$

As such, for the remainder of this section, we will study the effect of these $\pm m$ deformations on the electric theory and map their effect to the planar dual.

4.4.1 A Dual for $\text{USp}(2N)_0$ CS-SQCD₃ with $F \geq 2N + 2$ $\square$ and its Deformations

We begin by proposing an Abelian quiver gauge theory that is infrared dual to $\text{USp}(2N)_0$ SQCD with F fundamental multiplets, where $F \geq 2N + 2$. Consistency of Chern–Simons level quantization requires F to be even.

This proposal is motivated by three complementary considerations. First, its structure closely parallels the planar Abelian duals of $U(N)_0$ SQCD with F fundamental chiral multiplets proposed in [5]. Second, we have verified agreement of superconformal indices for several low-rank and low-flavor examples, providing nontrivial evidence for the duality. Finally, upon performing appropriate real mass deformations, the proposed Abelian quiver correctly flows to the known $\mathcal{N} = 4$ descendant theories discussed in earlier sections.

We propose the dual shown in Equation 4.18.

$$\begin{aligned}
 & USp(2N)_0 \text{ with } F \square \quad \longleftrightarrow \quad \text{Quiver Diagram} \\
 & (F \geq 2N + 2)
 \end{aligned}$$

(4.18)

Across the duality, flavor and topological symmetries are exchanged, which is a characteristic feature of mirror-like dualities. We comment on some additional properties of these duals:

- $\mathcal{W}_{\text{planar}}$ contains one cubic (quartic) term for each triangular (square) face of the quiver, with sign -1 ($+1$) if the arrows run clockwise (anticlockwise) around the face.
- $\mathcal{W}_{\text{monopole}}$ includes linear monopole superpotentials involving monopoles with GNO fluxes under two adjacent nodes within the same column. Specifically there is a

monopole $\mathfrak{M} \begin{pmatrix} - \\ + \end{pmatrix}$ for each pair of adjacent nodes.

There are additional monopole interactions of the form considered in Equation 4.3.

The FI terms of the $U(1)$ gauge nodes are compatible with the monopole superpo-

tential terms and are **not arbitrary**.

- The assignment of R-charges and CS levels follows the general conventions established in Equation 4.16.

Overall, the information required to fully specify the planar Abelian theory is completely encoded in the quiver diagram.

This duality may be regarded as the *seed duality* for $USp(2N)$ SQCD, since all subsequent constructions can be obtained by suitably deforming this dual pair and following the RG flow at the level of the $\mathbf{S}_b^3$ partition function.

Mapping Multiple Negative Mass Deformations

Starting from the duality between $USp(2N)_0$ SQCD with F fundamental multiplets and its planar Abelian dual (see Equation 4.18), we now introduce negative real mass deformations and study the corresponding RG flows in both descriptions.

On the “electric” side, this deformation drives the theory to $USp(2N)_{-1}$ SQCD with $F - 1$ fundamental multiplets. The analysis of the RG trajectory in the planar Abelian dual is more subtle. We propose that the corresponding deformation is implemented by Higgsing all the chiral fields along the bottom-left diagonal, while the chiral field connecting the bottom-most gauge and flavor nodes acquires a positive mass. This Higgsing identifies all gauge nodes along the diagonal with the final node, and the planar superpotential correspondingly reduces to mass terms.

In quiver notation, we propose the duality shown in Equation 4.19.

Higgsed: —

Superpotential mass: —

Positive mass: —

$USp(2N)_{-1}$ with $F-1$ $\square \longleftrightarrow$

$\frac{E}{2} + N - 1$ nodes

$2N$ nodes

$F - 2N - 1$ nodes

(4.19)

Here, the chiral fields shown in blue are those that acquire Higgs VEVs, while those in orange obtain complex masses via the planar superpotential. The chiral field highlighted in pink acquires a positive mass.

After this Higgsing, the planar superpotential reduces to mass terms for the fields, as illustrated in Equation 4.20:

$\mathcal{W} \supset \mathcal{W}_{\text{planar}}$
 $\xrightarrow{\text{Higgs}}$
 $\mathcal{W} \supset \sum_{i=1}^{4N-3} q_i q_{i+1}$
 $\xrightarrow{\text{Integrate out}}$
 $\mathcal{W} \supset \mathcal{W}_{\text{planar}}$

(4.20)

Upon integrating out the massive fields, the $U(1)$ gauge node at the bottom of the quiver becomes s -confining. We can therefore apply a local duality to confine this node, yielding an equivalent description.

As a result, we obtain the planar Abelian dual of $USp(2N)_{-1}$ SQCD with $F - 1$ fundamental multiplets, shown in Equation 4.21.

$$\begin{aligned}
 & USp(2N)_{-1} \text{ with } F - 1 \square \longleftrightarrow \\
 & (F > 2N + 2)
 \end{aligned}
 \quad
 \begin{aligned}
 & \begin{array}{c}
 \text{Quiver Diagram} \\
 \text{with nodes and arrows} \\
 \text{and labels: } \frac{F}{2} + N - 2 \text{ nodes, } 2N \text{ nodes, } F - 2N - 2 \text{ nodes}
 \end{array}
 \end{aligned}
 \tag{4.21}$$

The resulting dual quiver closely resembles the Abelian dual before the mass deformation. The only differences are the number of columns and the position of the “column of squares”.

Further negative mass deformations are mapped analogously, leading to the duality for $USp(2N)_k$ SQCD with F fundamental multiplets shown in Equation 4.22.

$USp(2N)_k$ with $F \square$

$(F \geq 2N + 2 + |k|)$

↔

(4.22)

In this way, we can follow the RG flow trajectories induced by $\frac{F}{2} - N - 1$ negative mass deformations. The limiting case $-USp(2N)_{N+1-\frac{F}{2}}$ SQCD with $\frac{F}{2} + N + 1$ fundamental multiplets – has the form shown in Equation 4.23.

$USp(2N)_{2N+2-F}$ with $F \square$

$\longleftrightarrow$

(4.23)

We remind the reader that this theory is an $\mathcal{N} = 4$ descendant. The equivalence of the presentation of this quiver with the standard form of $\mathcal{N} = 4$ descendant theories (see Figure 4.4) can be shown via a sequence of local dualizations following the procedure outlined in Appendix B of that work. Specifically, starting from the quiver in Equation 4.23, one dualizes each column from left to right, stopping at the $(2N - 1)$ -th column, then repeats the process up to the $(2N - 2)$ -th column, and so on. After performing this sequence $2N - 1$ times, the resulting quiver matches the planar Abelian dual of the $\mathcal{N} = 4$ descendant SQCD.

Thus, our analysis of $USp(2N)$ theories naturally sheds light on $U(2N)$ gauge theories with a rank-2 antisymmetric tensor. Understanding how to construct planar mirror duals for $USp(2N)_k$ SQCD with arbitrary F naturally informs the corresponding construction for $U(2N)_{(\tilde{k}, \tilde{k}-3N)}$ SQCD with F fundamentals and a rank-2 antisymmetric tensor. This provides an *a posteriori* justification for our choice to begin from the $U(2N)$ theory in Section 4.1.

4.4.2 Deformations of the $\mathcal{N} = 4$ Descendant SQCD₃

We now consider massive deformations of the planar dual of $USp(2N)_{(2+2N-F)}$ SQCD with F fundamental fields, shown in Equation 4.24. We remind the reader that this is an $\mathcal{N} = 4$ descendant theory.

$USp(2N)_{2+2N-F}$ with F $\square$ $\longleftrightarrow$

(4.24)

We have the possibility of studying the RG flow trajectories induced by both $\pm m$ deformations for these theories. Across the duality, a real mass for a quark in SQCD maps to an FI term for the gauge nodes contained in a single column of the planar Abelian quiver, while the FI parameter of the SQCD maps to a real mass for one of the two flavor nodes of the Abelian theory.

Consequently, giving a real mass to the i -th quark corresponds to turning on a large FI parameter for one (or two) adjacent columns of the Abelian quiver. Although different choices of i are equivalent at the level of the IR-SCFT—being related by Weyl transformations of the $U(F)$ flavor symmetry—this symmetry is not manifest in the planar quiver representation. As a result, the UV analysis of the deformation depends on the specific choice of i , even though the corresponding IR deformations describe the same flow.

For concreteness, we adopt the following conventions for the two possible deformations:

$$\begin{aligned}
 +m : & \quad \text{large mass for the last quark } Q^F \leftrightarrow X_F \rightarrow -\infty \\
 -m : & \quad \text{large mass for the first quark } Q^1 \leftrightarrow X_1 \rightarrow +\infty
 \end{aligned}
 \tag{4.25}$$

Therefore, the deformation $+m$ corresponds to evaluating a large FI limit for the rightmost node in the quiver, while the deformation $-m$ corresponds to a large FI limit for the leftmost node. We show these flows in Figure 4.8.

We identify the following vacua corresponding to these deformations:

- $+m$: The $+m$ deformation has the effect of Higgsing the “saw”-like structure at the end of the quiver. The chiral fields in blue acquire Higgs VEVs, while the planar superpotential reduces to mass terms for the other fields (refer to Equation 3.26).
- $-m$: In this deformation, the real scalars in the vector multiplets associated with the gauge groups along the bottom-left diagonal acquire large VEVs. As a consequence, the chiral multiplets connecting these nodes acquire large positive real masses, while those connecting them to the rest of the quiver acquire negative real masses. These fields are highlighted in green in Figure 4.8.

Integrating out the massive fields yields the quiver shown in Equation 4.26.

$$\begin{array}{c}
 \text{Diagram of a quiver structure with nodes and arrows, showing a saw-tooth pattern.} \\
 \text{Labels: } 2N \text{ nodes, } F - 2N - 1 \text{ nodes} \\
 \text{Equation label: } (4.26)
 \end{array}$$

$USp(2N)_{1+2N-F} \text{ with } F-1 \square \longleftrightarrow$

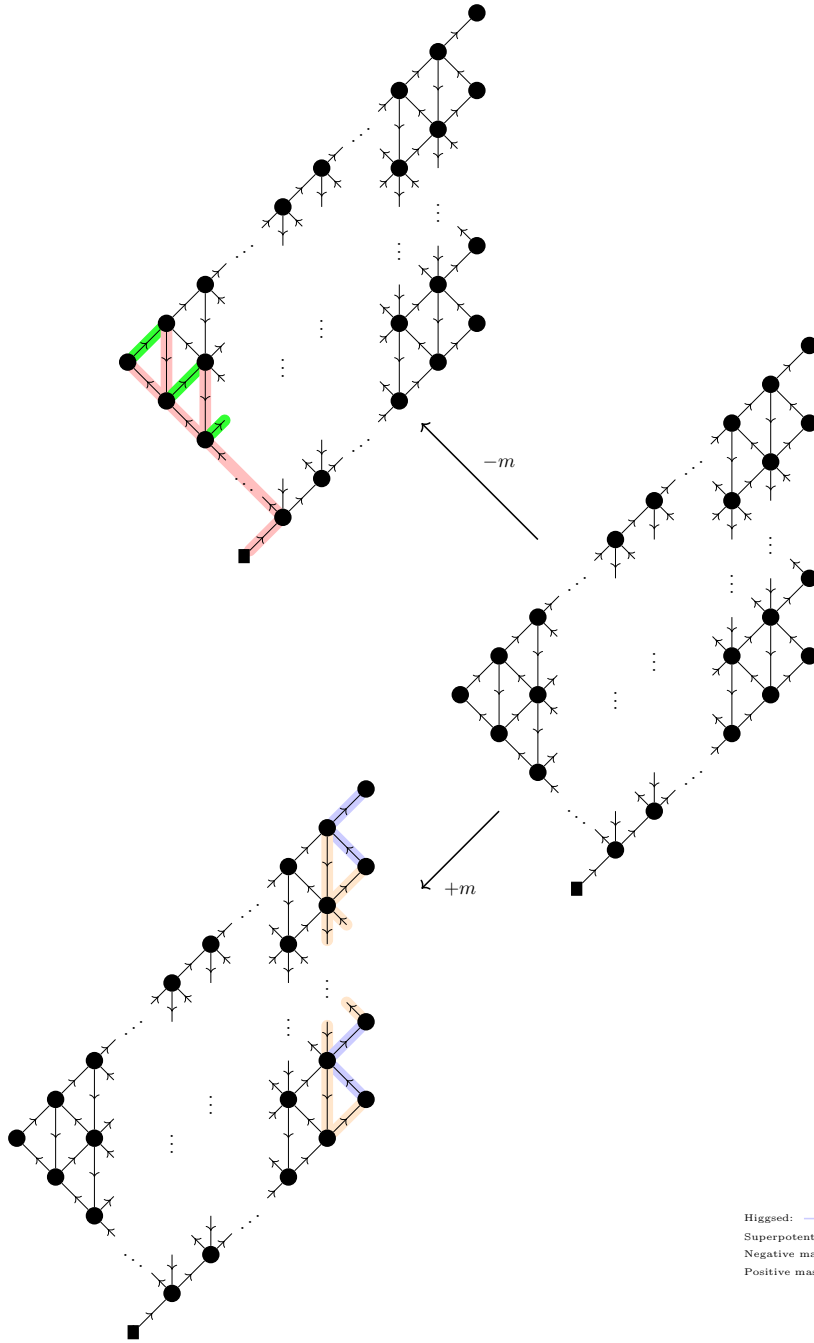


Figure 4.8: Deformations of the planar Abelian quiver dual to $USp(2N)_{(2+2N-F)}$ SQCD with F fundamentals. Chiral fields highlighted in pink acquire a large positive mass, fields highlighted in green acquire a large negative mass, and fields highlighted in blue acquire a VEV, thereby Higgsing the nodes that they connect to a diagonal $U(1)$.

This intermediate quiver contains a tail of $U(1)$ gauge nodes that interact with the rest of the quiver only through mixed CS couplings. Such a tail can be systematically contracted by sequentially integrating out the last node.

The resulting quiver diagram is shown in Equation 4.27.

$$\begin{array}{c}
 \text{USp}(2N)_{1+2N-F} \text{ with } F-1 \square \quad \longleftrightarrow \quad \begin{array}{c} \text{Quiver Diagram} \\ \text{2N-1 nodes} \quad \text{F-2N-1 nodes} \end{array}
 \end{array} \tag{4.27}$$

We summarize these mass flows in Figure 4.9. Notice that a $+m$ deformation produces another $\mathcal{N} = 4$ descendant since the resulting quiver has the same global topology as the original dual. Hence, this deformation provides a **nontrivial self-consistency check** of the duality originally proposed in Section 4.2. Thus, if we start from an $\mathcal{N} = 4$ descendant, successive $+m$ deformations will produce the corresponding $\mathcal{N} = 4$ descendant.

In contrast, the deformation $-m$ yields quivers with a modified geometry, leading to different planar structures. We will now systematically study the effect of multiple $-m$ deformations.

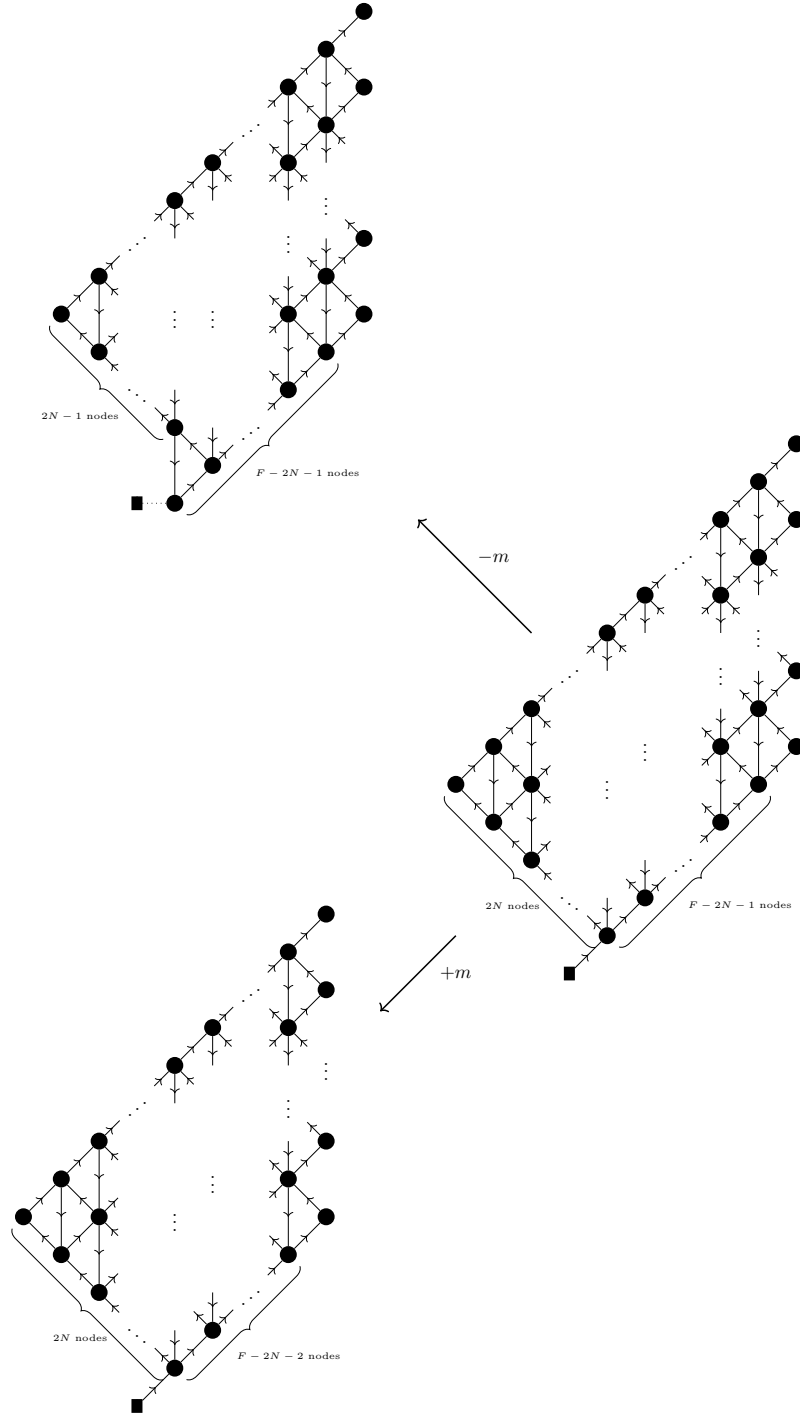


Figure 4.9: Deformations of the planar quivers dual to the SQCD mass deformations $\pm m$ are shown here.

We now consider the RG trajectory induced by another $-m$ deformation on the dual pair shown in Equation 4.27. The effect of this deformation is shown in Equation 4.28.

$$USp(2N)_{2N-F} \text{ with } F-2 \square \longleftrightarrow \text{Quiver Diagram} \quad (4.28)$$

Integrating out the massive fields yields the quiver shown in Equation 4.29.

$$USp(2N)_{2N-F} \text{ with } F-2 \square \longleftrightarrow \text{Quiver Diagram} \quad (4.29)$$

In this way, we can follow the RG flow trajectory induced by the first $k (< 2N) - m$ deformations. The resulting quiver is shown in Equation 4.30.

$$\begin{aligned}
 & \text{USp}(2N)_k \text{ with } F \square \\
 & (-k - 2N < F < 2N + 2 - k) \\
 & \longleftrightarrow \text{Quiver Diagram} \\
 & \text{Quiver Diagram Labels:} \\
 & \quad N + \frac{k}{2} + \frac{F}{2} - 1 \text{ nodes} \\
 & \quad N - \frac{F}{2} - \frac{k}{2} \text{ BF interactions} \\
 & \quad \frac{F}{2} - \frac{k}{2} - N - 1 \text{ nodes} \\
 & \quad \text{Orange shaded region in } (F, |k|) \text{ plane} \\
 & \quad \text{Orange region vertices: } (0, 2N+2), (2N+2, 0), (2N+2, 2N+2) \\
 & \quad \text{Dashed line: } |k| = 2N+2 - F
 \end{aligned}
 \tag{4.30}$$

The RG flow induced by the $2N^{\text{th}} - m$ deformation results in the quiver shown in Equation 4.31.

The diagram illustrates a quiver structure. On the left, a node is connected to a square symbol, with the label $USp(2N)_{2-2N-F}$ with F $\square$ next to it. A double-headed arrow points from this node to a central node. From this central node, several lines radiate outwards. One line connects to a node in a vertical chain of nodes. Another line connects to a node in a triangular arrangement of nodes. A bracket labeled $2N - 1$ BF interactions groups a set of lines connecting the central node to a vertical chain of nodes. Another bracket labeled $F - 2$ nodes groups a set of lines connecting the central node to a triangular arrangement of nodes. The diagram is labeled (4.31) on the right.

The RG flow induced by the $(2N + 1)^{\text{th}}$ $-m$ deformation results in the quiver shown in Equation 4.32.

$USp(2N)_{-F-2N}$ with $F \square$

$\longleftrightarrow$

(4.32)

Successive $(F - 2N - 3) - m$ deformations act on the bulk of the quiver. We remind the reader of the action of a single $-m$ deformation on the bulk of the quiver in Equation 4.33.

Negative mass: —
Positive mass: —

$\xrightarrow{-m}$

(4.33)

The resulting quiver is shown in Equation 4.34.

$USp(2N)_k$ with $F \square$
 $(k < -F - 2N)$

$\longleftrightarrow$

(4.34)

The mapping of the $-m$ deformation on the “saw” is similar to the previous case, resulting in the duality of $USp(2N)$ CS-SQCD with 1 fundamental chiral shown in Equation 4.35.

$USp(2N)_k$ with 1 $\square$ $\longleftrightarrow$

(4.35)

We remind the reader of the form of the monopole superpotential in these theories:

- In the column that separates the “bulk” of the quiver and the region with BF interactions, as shown in Equation 4.37, these monopoles must be dressed with the chiral field Q .

The FI parameters of the nodes in this type of column are, from top to bottom:

$$\text{FI}s: X_i - \frac{iQ}{4}, X_i - (1-R)\frac{iQ}{2}, \dots, X_i - (1-R)\frac{iQ}{2}, X_i + \frac{iQ}{4}(3R-2) \quad (4.36)$$

where X_i is a Cartan of the $SU(F)$ flavor symmetry, and the coefficient in front of $\frac{iQ}{2}$ encodes the mixing between the topological and the R-symmetry. We remind the reader that we assign a trial R-charge $1-R$ to the chiral multiplets in the *electric* SQCD theory.

$\mathcal{W}_{\text{monopole}} \supset \mathcal{QM} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(4.37)

- In the columns with BF interactions on both sides, as shown in Equation 4.39, these monopoles must be dressed with both $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

The FI parameters of the nodes in this type of column are, from top to bottom:

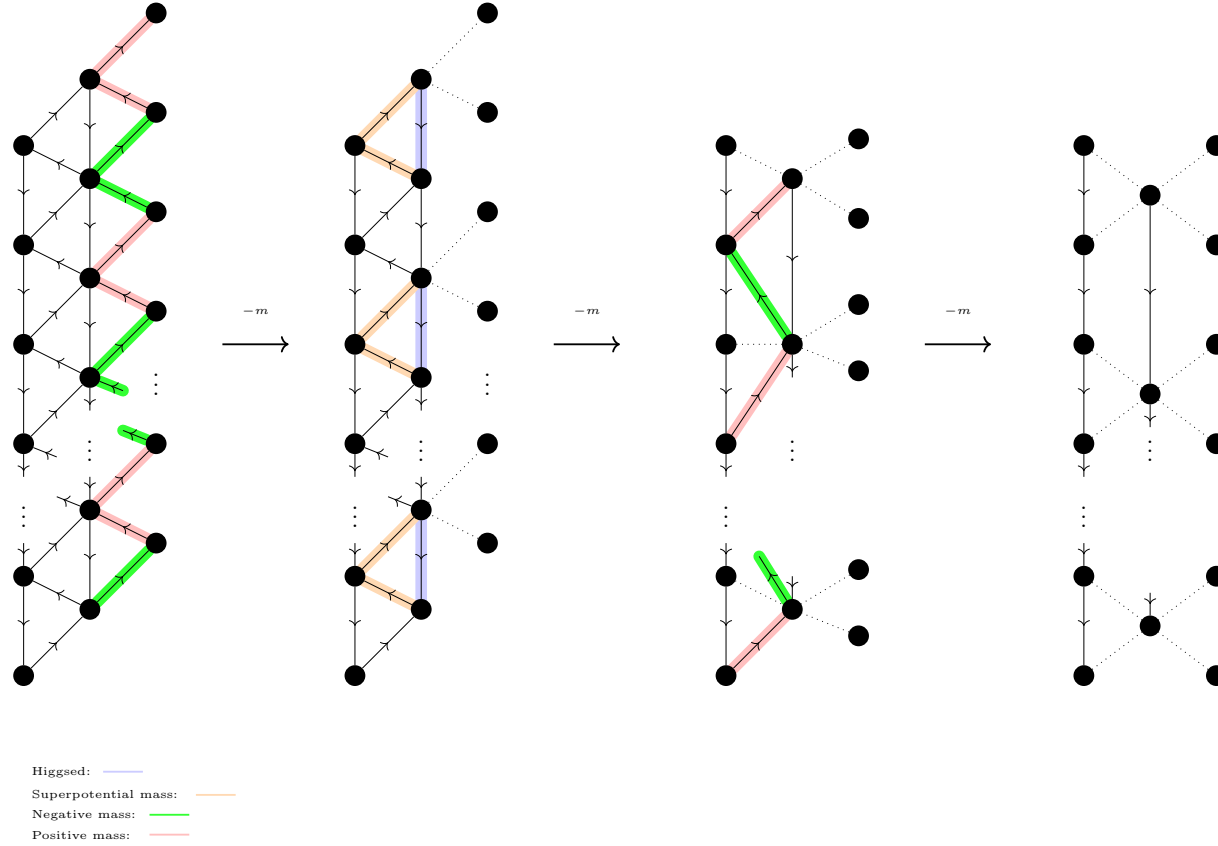
$$\text{FIs: } \frac{iQ}{4}(1-2R), \quad 0, \quad \dots, 0, \quad \frac{iQ}{4}(2R-1) \quad (4.38)$$

$$\mathcal{W}_{\text{monopole}} \supset \mathcal{Q}_1 \mathcal{Q}_2 \mathfrak{M} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(4.39)

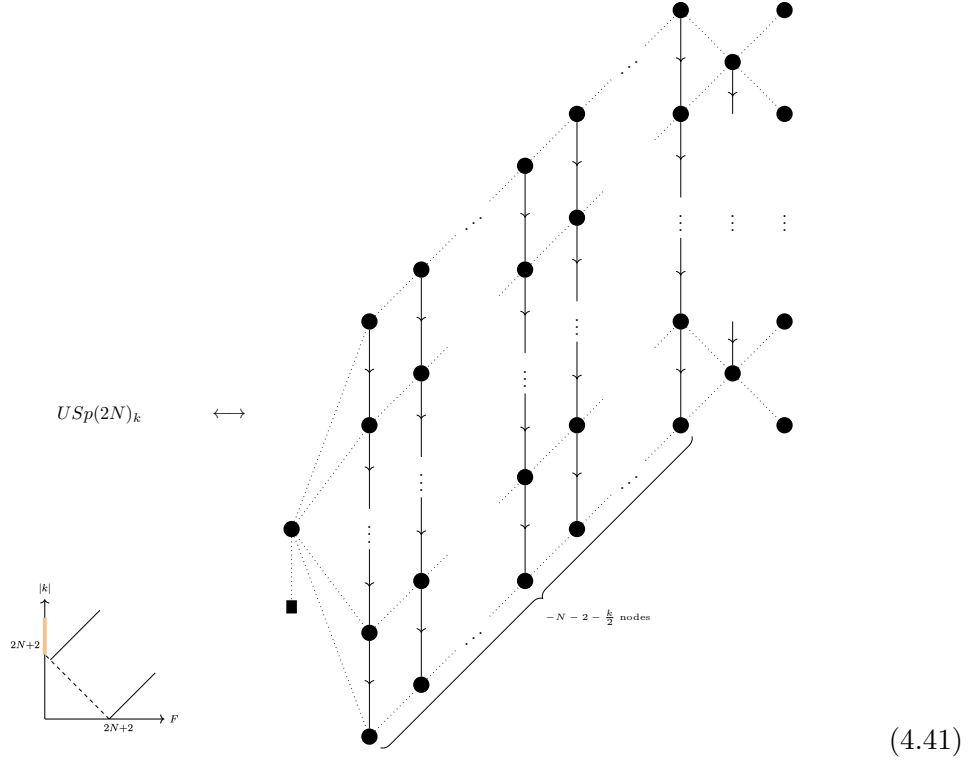
- We further notice that in a column with BF interactions on both sides, the $N-1$ monopole superpotentials completely break the topological symmetries. Therefore, only the columns in the “bulk” of the quiver and at the edge of the “bulk” contribute to the Cartan subalgebra of the global symmetry, matching the global symmetry of the dual SQCD.

In our conventions, a $-m$ deformation corresponds to acting on the planar quiver from the left. However, due to the Weyl invariance of the $U(F)$ flavor symmetry, we could also act on the “saw” at the end of the quiver. We demonstrate the effect of three $-m$ deformations at the end of the quiver in Equation 4.40. Due to the difference in the assignment of the fugacities of topological symmetries, the action of a single $-m$ deformation is different from the one shown in Equation 4.33. Nevertheless, it can be verified that the IR behavior of the resulting SCFT is independent of the order in which the X_i fugacities are set to $\pm\infty$.



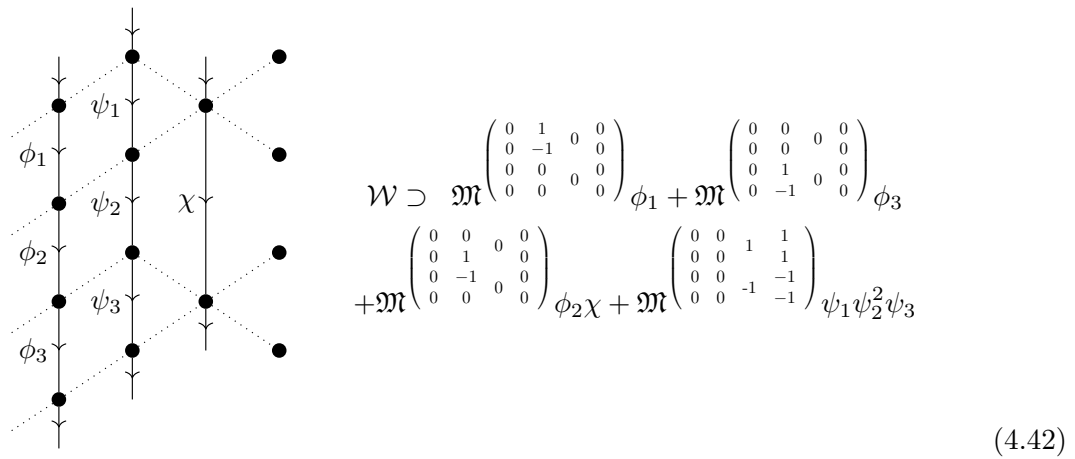
(4.40)

Upon integrating out the remaining chiral field, we obtain a planar Abelian dual for $USp(2N)_{2+2N-2F}$ CS-SYM theory, shown in Equation 4.41. The dual description is not manifestly topological as it contains multiple massless charged fields.



(4.41)

Note that the height of the $-N-2-\frac{k}{2}$ columns is $2N$, while that of the final column is N . Furthermore, the vertical chiral fields in the final column have R-charge $4-4R$, in contrast to the other vertical chiral fields, which have R-charge $2-2R$. The monopole superpotential terms associated with the last two columns contain the following (dressed) monopoles:



$$\begin{aligned}
 \mathcal{W} \supset & \mathfrak{M} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi_1 + \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \phi_3 \\
 & + \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \phi_2 \chi + \mathfrak{M} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \psi_1 \psi_2^2 \psi_3
 \end{aligned}$$

(4.42)

It would be interesting to study this duality further - particularly its possible connection with level-rank duality. We defer this to future work.

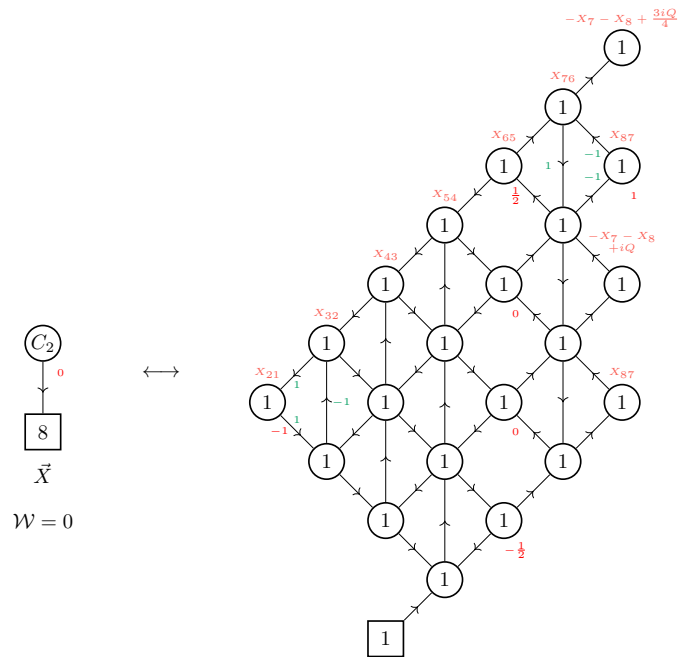
4.4.3 An Example: Massive Deformations of $\mathrm{USp}(4)_{-1}$ SQCD₃ with 7 $\square$

We now analyze a representative example of such RG flows induced by $\pm m$ deformations, highlighting how the electric and mirror descriptions encode these deformations.

For concreteness, we consider the planar mirror dual of $U(4)_{(0,-6)}$ with 8 fundamental chiral multiplets and a rank-2 antisymmetric tensor in Equation 4.43. This will serve as a prototype for the remainder of this section.

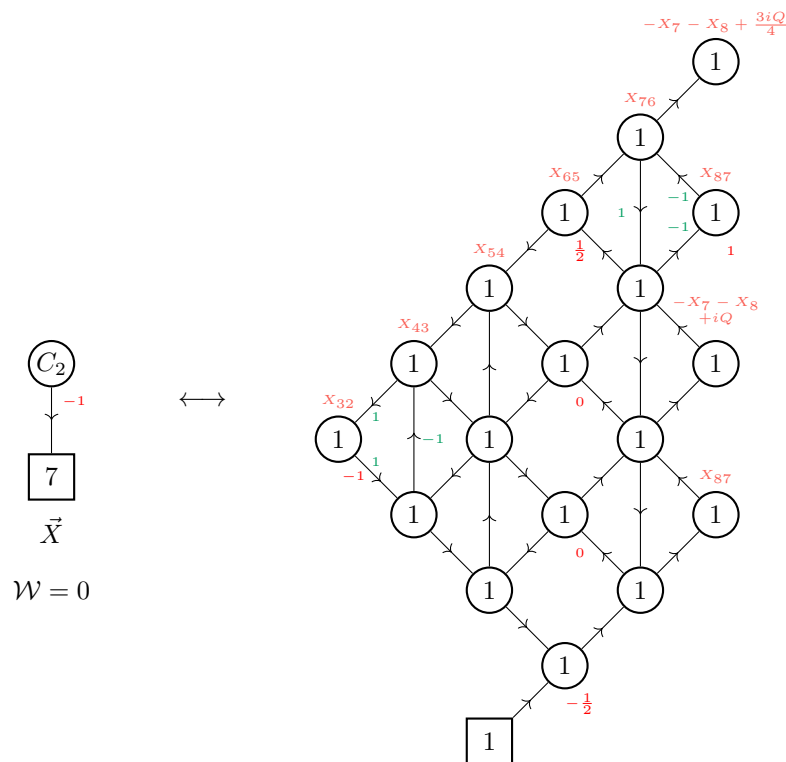
$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$ (4.43)

Once the rank-2 antisymmetric tensor acquires an expectation value, the theory is Higgsed to $USp(4)_0$ with 8 fundamental chiral multiplets, shown in Equation 4.44.



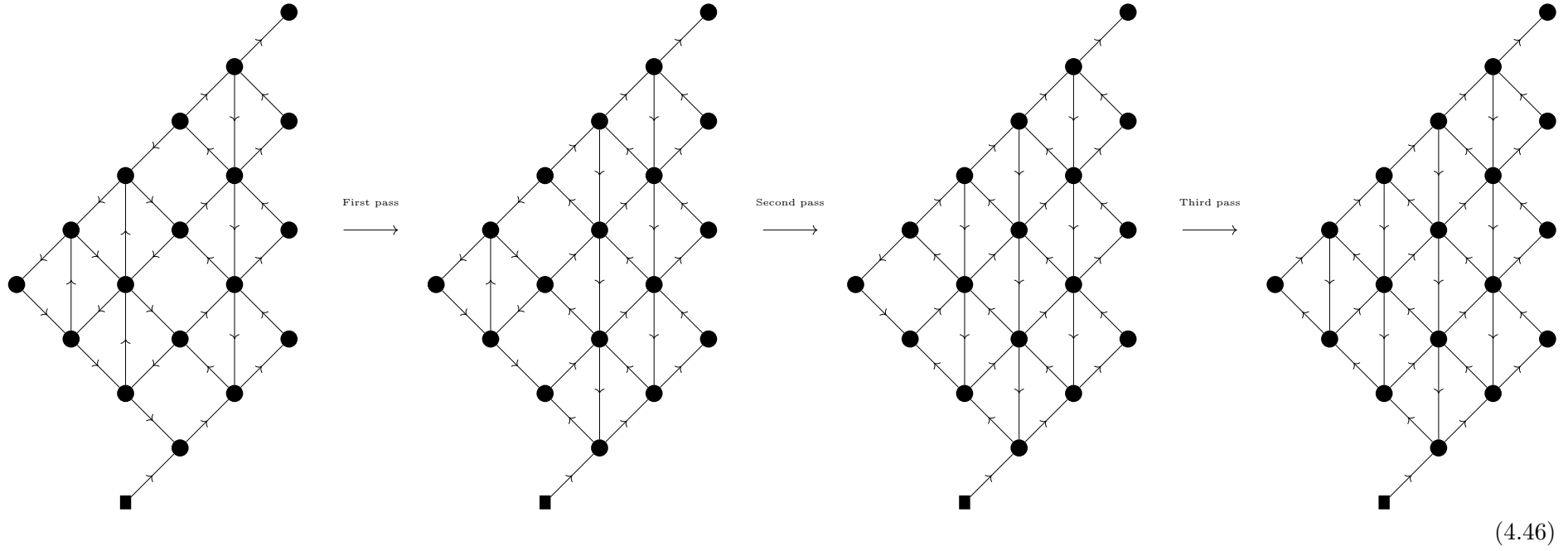
$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopole} \quad (4.44)$$

We demonstrate the effect of a negative mass deformation on the duality shown in Equation 4.44. The resulting duality between $USp(4)_{-1}$ SQCD with 7 fundamental fields is shown in Equation 4.45.

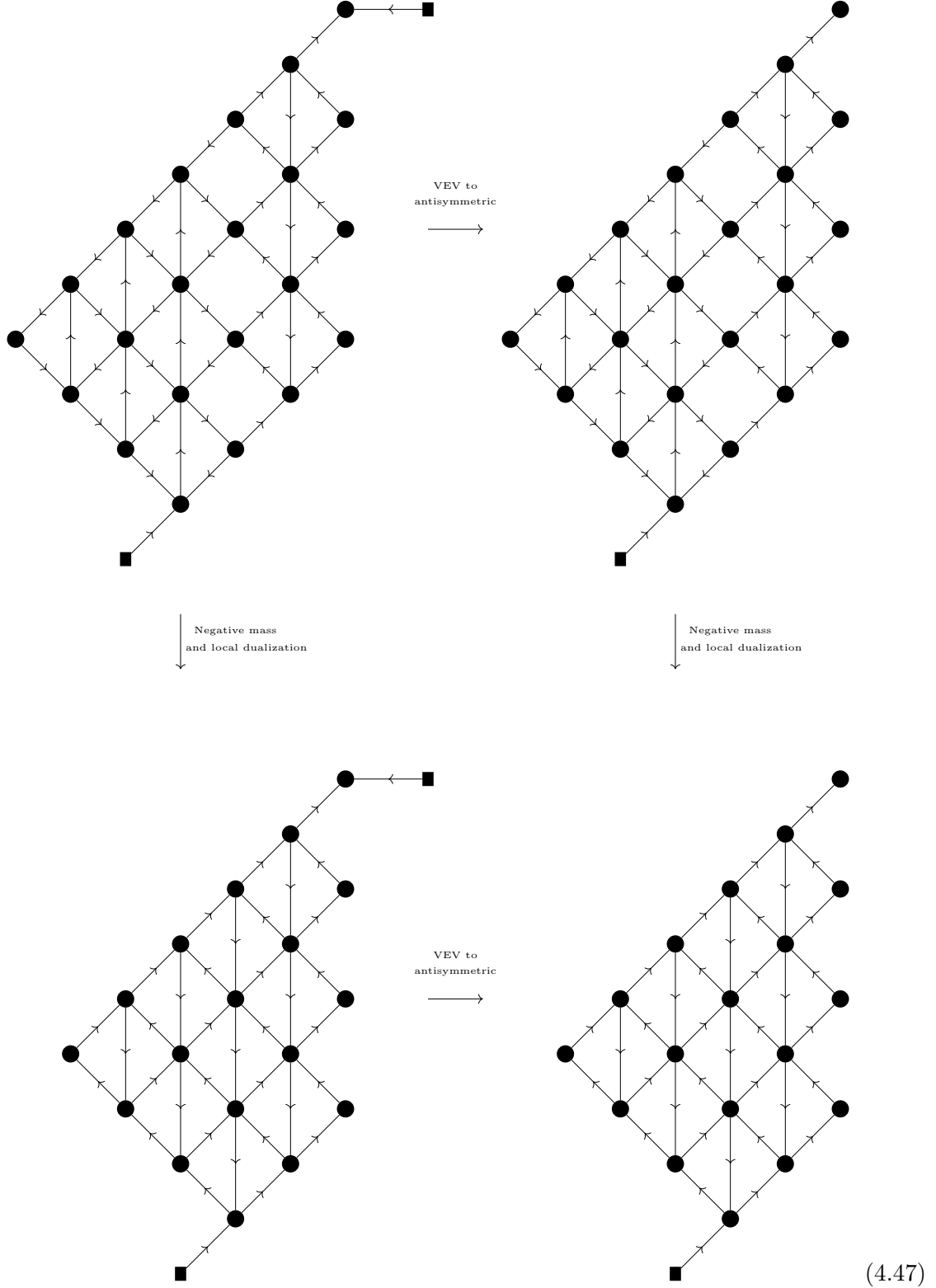


$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopole} \quad (4.45)$$

We recognize the dual pair shown in Equation 4.45 as an $\mathcal{N} = 4$ descendant pair; as such, we can recast it in a more familiar presentation via a sequence of local dualizations schematically summarized in Equation 4.46.



We summarize the connection between the planar mirror dual of $U(4)_{(0,-6)}$ SQCD with a rank-2 antisymmetric tensor and 8 fundamental chiral multiplets, and $USp(4)_0$ SQCD with 8 fundamental multiplets in Equation 4.47.



Dualities with One Real Mass Deformation

Following the general discussion of Section 4.4.2, we study the RG flow induced by $\pm m$ deformations on the planar dual theory.

$+m$ deformations: $USp(4)_0$ with $6\Box$: We consider the large mass limit $X_7 \rightarrow -\infty$ in the *electric* SQCD theory. The theory flows to $USp(4)_0$ with 6 fundamental multiplets in the infrared. At the level of the $\mathbf{S}_b^3$ partition function, the divergent asymptotic phase generated along this flow is $-2\pi QRX_7$, and we must match this phase once the deformation is mapped to the mirror dual [21, 195].

We now consider the large mass limit $X_7 \rightarrow -\infty$ in the *mirror* partition function. To obtain the right vacuum, corresponding to the divergent asymptotic phase $-2\pi QRX_7$, we consider shifts in the mirror partition function that implement a Higgs VEV for the fields in the “saw”-like structure of the quiver (analogous to the setup shown in the bottom left panel of Figure 4.8). After integrating out the massive fields, we obtain the planar dual of $USp(4)_0$ with 6 fundamental fields, shown in Equation 4.48. We remind the reader that this is also an $\mathcal{N} = 4$ descendant, and this is a non-trivial consistency check of our proposal.

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}} \quad (4.48)$$

The Aharony-like dual of $USp(4)_0$ SQCD with 6 fundamental multiplets is a Wess–Zumino model with superpotential

$$\mathcal{W} = \gamma \text{Pf}(M),$$

where $M_{ij} = Q_i Q_j$ are the $\binom{6}{2} = 15$ mesons of the electric theory, and the singlet field γ is dual to the gauge-invariant monopole operator $\mathfrak{M}$.

This description predicts the existence of a corresponding gauge-invariant monopole operator in the planar mirror theory, dual to γ . Explicitly, one finds the operator map

$$\mathfrak{M} \leftrightarrow \gamma \leftrightarrow \mathfrak{M} \begin{pmatrix} & & & - \\ & & - & \\ & - & & - \\ - & - & - & -2 \\ & - & -2 & \\ & - & & - \\ & & 0 & \end{pmatrix};$$

We generically anticipate the existence of such gauge invariant higher GNO flux monopoles in the planar mirrors of $USp(2N)_0$ with $2N + 2$ fundamental multiplets.

$-m$ deformations: $USp(4)_{-2}$ with $6\Box$: We consider the large mass limit $X_1 \rightarrow +\infty$ in the *electric* SQCD theory. The theory flows to $USp(4)_{-2}$ with 6 fundamental multiplets in the infrared. At the level of the $\mathbf{S}_b^3$ partition function, the divergent asymptotic phase generated along this flow is $+i\pi X_1^2$, and we must match this phase once the deformation is mapped to the mirror dual.

We now consider the large mass limit $X_1 \rightarrow +\infty$ in the *mirror* partition function. To obtain the right vacuum, corresponding to the divergent asymptotic phase $+i\pi X_1^2$, we consider shifts in the mirror partition function that implement large VEVs for the gauge groups along the bottom-left diagonal of the quiver (analogous to the setup shown in the bottom left panel of Figure 4.8). After integrating out the heavy fields, we obtain the planar dual of $USp(4)_{-2}$ with 6 fundamental fields, shown in Equation 4.49. The tail of gauge nodes connected by BF couplings can be confined; however, we do not do this here to highlight the difference in global structure of the resulting theory from the $\mathcal{N} = 4$ descendant.

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}} \quad (4.49)$$

Dualities with Multiple Real Mass Deformations

Having established the general framework of how $\pm m$ deformations are mapped to the planar mirror dual, we now catalog the various dualities generated by $\pm m$ mass flows. We do not comment on the CS levels and FI terms, which can be recovered from the respective $\mathbf{S}_b^3$ partition functions, and only display the global topology of the resulting quiver diagrams in Figures 4.10 and 4.11.

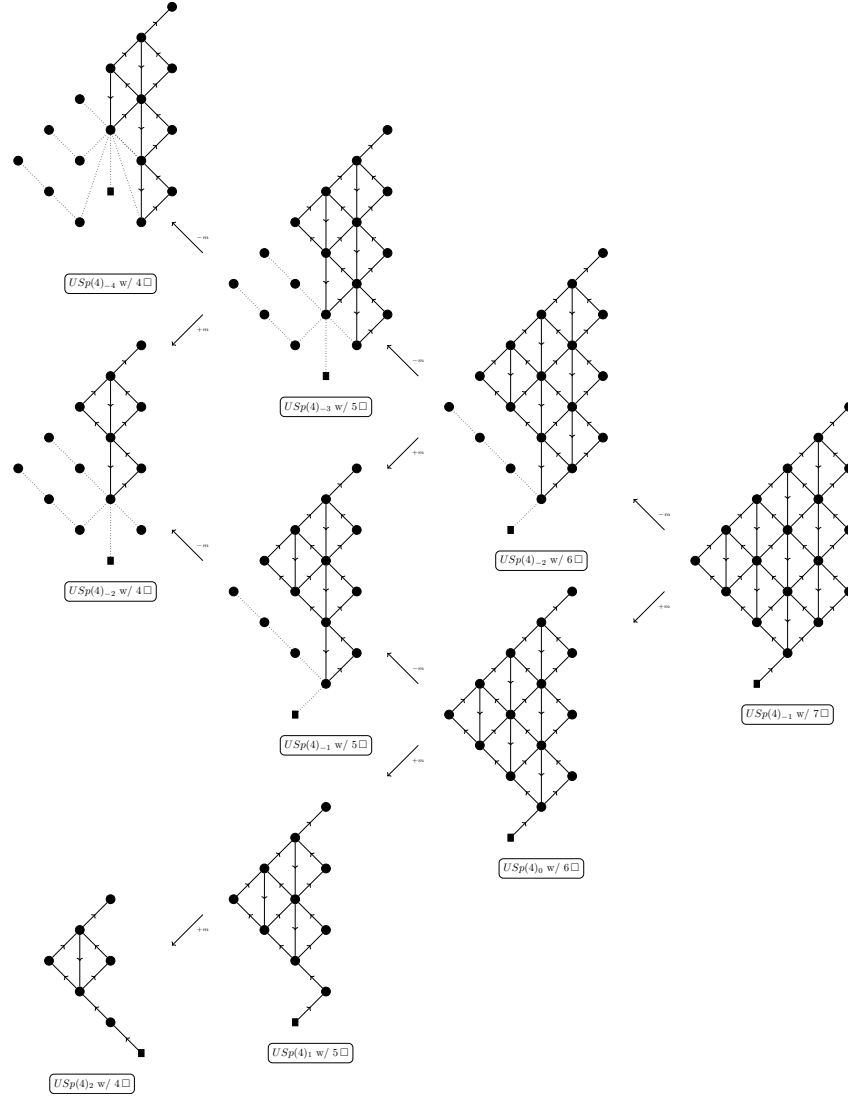


Figure 4.10: The planar mirror dual of $USp(4)_{-1}$ SQCD with 7 fundamental multiplets is an $\mathcal{N} = 4$ descendant theory. Starting from this UV theory, we study the RG flow trajectories induced by $\pm m$ real mass deformations. In this figure, we display the first three levels of this deformation sequence. Specifically, we begin with an electric theory containing [7] fundamental multiplets and follow the flow to theories with [4] fundamental multiplets. We only display the global topology of the resulting quiver diagrams; details of the CS levels and FI terms can be recovered following our usual conventions.

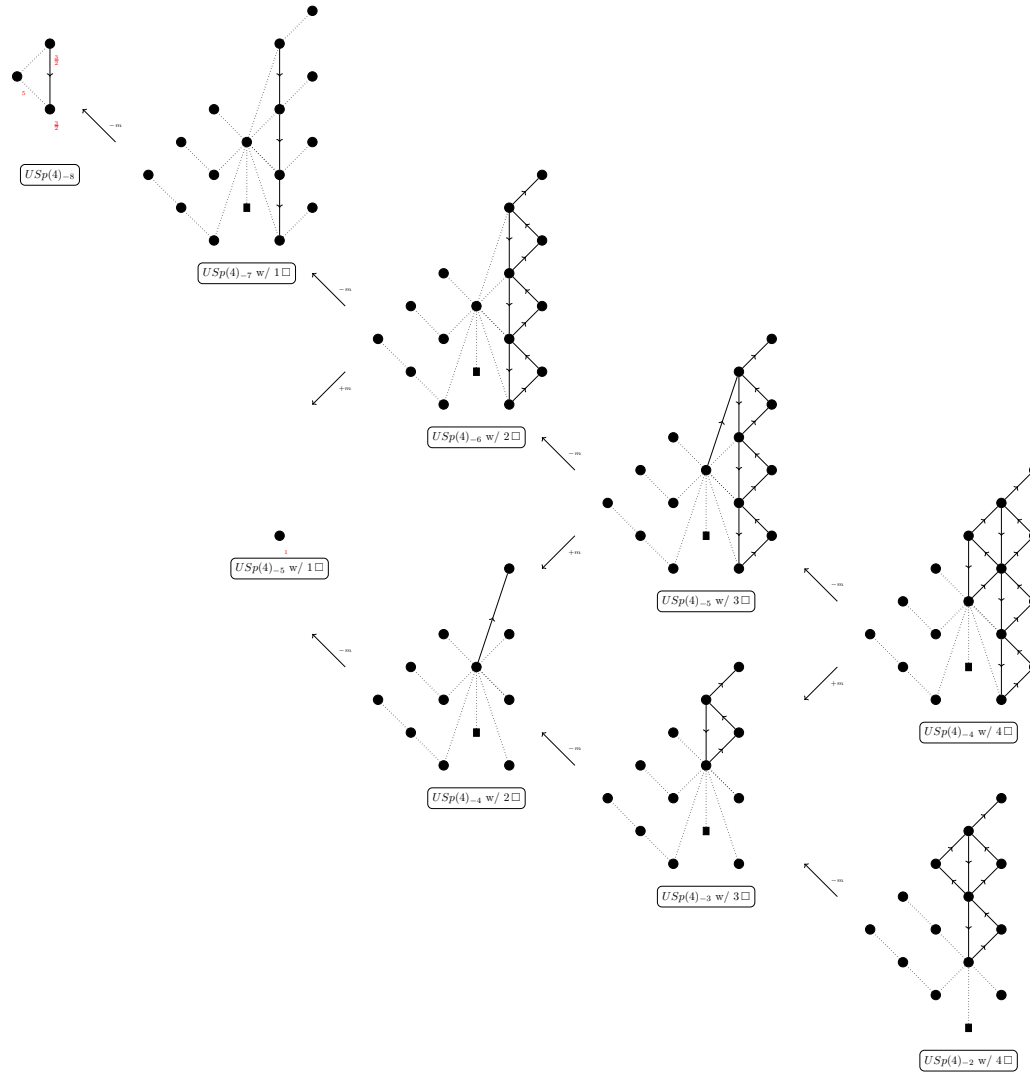


Figure 4.11: We continue the analysis of the RG flow trajectories induced by $\pm m$ mass deformations, extending those considered in Figure 4.10. Among the theories appearing in these flows, only $USp(4)_{-4}$ SQCD and $USp(4)_{-2}$ SQCD with $[4]$ fundamental multiplets admit non-trivial RG flows under $\pm m$ deformations. Accordingly, we only focus on $USp(4)_{-5}$ SQCD and $USp(4)_{-3}$ SQCD with $[3]$ fundamental multiplets, and follow the flow until all matter is integrated out. As before, details of the CS levels and FI terms can be recovered following our usual conventions.

Confinement along the RG Flow

Recall that the Aharony-like dual of *electric* $USp(2N)_k$ SQCD with F fundamental multiplets is a *magnetic* $USp(2\tilde{N})_{-k}$ SQCD with F antifundamental multiplets, together with $\binom{F}{2}$ gauge-singlet chiral fields. In the region of parameter space defined by

$$(F < 2N + 2) \cap (-F + 2N + 2 = |k|),$$

the rank of the magnetic gauge group vanishes, $\tilde{N} = 0$. As a consequence, the theory *confines*⁶ in the deep infrared, and the low-energy spectrum consists solely of $\binom{F}{2}$ gauge-singlet chiral multiplets. This behavior is completely transparent in the Aharony-like magnetic description.

In the context of Figure 4.10, this implies that $USp(4)_{\pm 1}$ SQCD with 5 fundamental multiplets and $USp(4)_{\pm 2}$ SQCD with 4 fundamental multiplets should both *confine*. We verify that this expectation is indeed borne out: the corresponding planar duals of these theories can be made to *confine* through an appropriate sequence of local dualizations.

We point out that in the corresponding $\mathcal{N} = 4$ parent theories, the deep infrared is characterized by symmetry enhancement that is not visible in the UV Lagrangian [144, 239–241, 246]. By contrast, in the planar $\mathcal{N} = 2$ theories obtained along the RG flow, no such enhancement takes place: the Abelianized description already accounts for the full infrared spectrum, including confinement and the correct counting of gauge-singlet degrees of freedom.

Insights from Index Spectroscopy. It is instructive to examine more closely the superconformal index of the *confining* theories discussed above. The index of a theory consisting of F free chiral multiplets has a particularly distinctive structure. Assigning a trial R -charge r and a fugacity ϕ_i to each free field, the index takes the form

$$\begin{aligned} \mathcal{I}_{\text{free}} &= 1 + x^r \left(\sum_{i=1}^F \phi_i \right) - x^{2-r} \left(\sum_{i=1}^F \phi_i^{-1} \right) - x^2 \left(\sum_{i,j=1}^F \frac{\phi_i}{\phi_j} \right) + \dots \\ &= 1 + x^r \chi_{\square}(\phi) - x^{2-r} \chi_{\overline{\square}}(\phi) - x^2 \chi_{\text{adj}}(\phi) + \dots \end{aligned} \quad (4.50)$$

Beyond the vacuum contribution, the term of order x^r arises from the bosonic primaries of each chiral multiplet and corresponds to the character of the fundamental representation of $U(F)$. The contribution at order x^{2-r} originates from the fermionic primaries and transforms in the antifundamental representation of $U(F)$. Finally, the term of order x^2 counts the conserved currents of the theory⁷ and correctly reproduces the character of the adjoint representation of $U(F)$.

The salient features of the superconformal index of a free theory can therefore be summarized as follows:

- There is a bosonic positive contribution at order x^r and a fermionic negative contribution at order x^{2-r} ; these two terms transform in conjugate representations of the flavor symmetry group.

⁶We draw a distinction between *s-confinement* and *confinement* here. We say a theory *s-confines* if there are holomorphic relations between some (or all) chiral fields in the theory, imposed in the form of a superpotential; by contrast, we say a theory *confines* if the action describes free chiral fields.

⁷The contribution at order x^2 may in principle receive additional contributions from composite operators; throughout this discussion, we assume that the trial R -charge r is chosen so as to avoid this possibility.

k	Superconformal Index
1	$1 + x^{2r} \chi_{\Lambda^2}(\phi) - x^{2-2r} \chi_{\bar{\Lambda}^2}(\phi) - x^2 \chi_{\Lambda^2}(\phi) \chi_{\bar{\Lambda}^2}(\phi) + \dots$
2	$1 + x^{2r} \chi_{\Lambda^2}(\phi) - x^{3-3r} \chi_{\bar{\Lambda}^3}(\phi) - x^2 \chi_{adj}(\phi) + \dots$
3	$1 + x^{2r} \chi_{\Lambda^2}(\phi) - x^{4-4r} \chi_{\bar{\Lambda}^4}(\phi) - x^2 \chi_{adj}(\phi) + \dots$
4	$1 + x^{2r} \chi_{\Lambda^2}(\phi) - x^{5-5r} \chi_{\bar{\Lambda}^5}(\phi) - x^2 \chi_{adj}(\phi) + \dots$

Table 4.2: Values of the superconformal index for $USp(4)_{\pm k}$ SQCD₃ with five chiral fields are shown here. All fields are assigned a trial R -charge r , and ϕ denotes the fugacities of the UV flavor symmetry group $U(5)$. The notation Λ^n refers to the n -index antisymmetric representation of $U(5)$, while $\bar{\Lambda}^n$ denotes its complex conjugate.

- The contribution at order x^2 is precisely given by the product of the first bosonic and fermionic terms.

These properties must persist even when the full infrared symmetry group is not manifest from a ultraviolet perspective. This is precisely what occurs in *confining* theories. As an illustrative example, consider the $USp(4)_1$ theory with five chiral multiplets, which is known to confine [21]. In the ultraviolet, the symmetry group is $U(5)$, acting on the five chiral fields, and the ten gauge-invariant mesons transform in the two-index antisymmetric representation Λ^2 of this group. In the infrared, however, the symmetry is enhanced to $U(10)$. Since the confined theory is free, the two structural properties of the index discussed above must be realized, even though the index itself is written solely in terms of ultraviolet fugacities.

The mechanism underlying this symmetry enhancement is highly non-trivial and relies on delicate cancellations among different contributions to the superconformal index, as can be verified by explicit computations. While the mesons and the 25 conserved currents of $U(5)$ are captured by the zero GNO flux sector, the fermionic contribution at order $-x^{2-2r}$, together with the remaining conserved currents, arises from the +1 GNO flux sector⁸. Note that the tensor product $\Lambda^2 \otimes \bar{\Lambda}^2$ always contains the adjoint representation, corresponding to the classical contribution to the current algebra.

It is worth emphasizing that it is quite remarkable that contributions from different GNO flux sectors recombine to reproduce the product structure characteristic of a free theory. This phenomenon results from the interplay between the matter and vector multiplet contributions and the specific value of the Chern–Simons level that ensures confinement. It is therefore natural to ask how this structure is modified as the Chern–Simons level is varied. In Table 4.2, we report the values of the superconformal index for several choices of the level k .

As we vary k , the rank of the representation appearing in the fermionic contribution changes by one unit, while its R -charge shifts by $1 - r$. Confinement is reached when the representation becomes $\bar{\Lambda}^2$. At this point, the coefficient of the x^2 term factorizes automatically, in agreement with the expected confining behavior.

⁸The Weyl group of $USp(2N)$ is $(\mathbb{Z}_2)^N \rtimes S_N$, acting on the GNO flux vector $(m_1, \dots, m_N)$ by permutations and sign inversions. As a result, the +1 sector is uniquely characterized by the GNO configuration $(+1, 0, \dots, 0)$.

Scope and Future Directions

In this chapter, we have restricted attention to $USp(2N)$ CS-SQCD₃ with fundamental matter and their real mass deformations. Within this class, we were able to track how real mass deformations shift the Chern–Simons levels and follow the induced RG flows in both the electric and planar Abelian mirror descriptions. Understanding these RG flows also allows us to track analogous flows in $U(2N)$ CS-SQCD₃ with fundamental matter and a rank-2 antisymmetric tensor and their mirror duals, since these two families of theories share many qualitative features.

A natural extension of this framework is to generalize the planar Abelianization prescription to $SO(N)$ CS-SQCD₃. For $\mathcal{N} = 4$ theories, these gauge groups can be realized as $\mathbb{Z}_2$ quotients of a $U(2N)$ gauge symmetry [169, 238], and we anticipate that a similar mechanism operates in the $\mathcal{N} = 2$ setting considered here. Another promising direction is the investigation of Aharony duality [21] in the *symplectic* setting. In the *unitary* case, [5] showed that Aharony duality can be realized as a consequence of mirror duality, and it would be interesting to uncover how an analogous structure manifests in $USp(2N)$ CS-SQCD₃.

In this chapter, we establish that the phenomenon of planar Abelianization persists for unitary theories with a rank-2 antisymmetric tensor. A natural extension of this work is to investigate whether planar Abelianization persists in theories with higher rank tensors. Much work has already been done in this direction, and we highlight a few relevant references below:

- [238] consider orientifolded brane setups for $\mathcal{N} = 4$ theories with multiple rank-2 antisymmetric tensors;
- [247] and the references therein establish dual pairs for $\mathcal{N} = 4$ unitary SQCD with a symmetric tensor (which are partly non-Lagrangian quiver gauge theories);
- [248] establishes $\mathbf{S}_b^3$ partition function identities relating $\mathcal{N} = 4$ unitary SQCD with a rank-2 antisymmetric tensor to $\mathcal{N} = 4$ unitary SQCD with a rank-2 symmetric tensor;
- [225] considers $\mathcal{N} = 4$ unitary SQCD with a rank-2 antisymmetric tensor and a rank-2 symmetric tensor, and establishes the dual pairs for these theories are, in fact, circular quivers.

As such, we anticipate that the planar Abelianization prescription can be extended to unitary theories with tensor matter, but we defer a systematic investigation of this question to future work.

Part IV: Non-Supersymmetric Planar Mirror Symmetry



Chapter 5

Non-Supersymmetric Unitary Dualities

Over the past fifteen years, the study of IR *bosonization* dualities in three-dimensional CS gauge theories has become a vibrant and rapidly developing area of theoretical physics. These dualities relate theories with fundamentally different microscopic degrees of freedom — *bosonic* versus *fermionic* matter — yet identical long-distance physics. Beginning with large- N analyses and later extending to finite-rank gauge groups, this program has led to a detailed understanding of strongly coupled dynamics in three dimensions [36–44, 46].

A particularly fruitful idea has been to derive non-supersymmetric dualities as mass deformations of known supersymmetric ones [45, 249–253]. Thus, non-supersymmetric 3D Chern-Simons dualities have become a central tenet within contemporary research on QFTs, with broad implications across particle physics, condensed matter physics, and mathematics.

In [4], we proposed a new class of dualities relating non-Abelian theories—such as $SU(N)$ Chern-Simons QCD_3 with both bosonic and fermionic matter—to Abelian theories with multiple $U(1)$ gauge groups. A defining feature of the Abelian duals is that both gauge and matter content are naturally encoded in *planar quiver diagrams*: scalar fields are represented by blue links, fermions by red links, and each face of the planar embedding corresponds to an interaction term in the effective action. These dualities closely parallel recently established $\mathcal{N} = 2$ mirror dualities¹ [2, 3], which themselves descend from mass-deformed $\mathcal{N} = 4$ mirror pairs [19, 22].

A simple rule governs the quiver structure: adding a bosonic flavor on the QCD side corresponds to inserting a fermionic column in the quiver (red), while adding a fermionic flavor adds a bosonic column (blue). In this way, our dualities can be viewed as generalizations of bosonization dualities [254–256], now extended to non-Abelian theories with richer matter content.

We provide several nontrivial consistency checks of these dualities. First, we establish an explicit operator map relating gauge-invariant mesons, baryons, and conserved currents of the non-Abelian theory to appropriately dressed monopole operators in the Abelian quiver. Second, we study mass deformations of individual bosonic and fermionic fields

¹We expect these non-supersymmetric dualities to arise from supersymmetry-breaking mass deformations of the $\mathcal{N} = 2$ dualities of [2, 3]. A detailed analysis of the RG flows connecting the supersymmetric and non-supersymmetric theories lies beyond the scope of this chapter.

and follow the resulting renormalization group flows on both sides of the duality. These flows generate new dual pairs within the same framework and provide stringent tests of the proposed correspondence.

Taken together, these results suggest that the dualities developed in this chapter may be interpreted as a form of *non-Abelian mirror symmetry without supersymmetry*, extending the conceptual reach of mirror symmetry beyond the supersymmetric setting ².

5.1 A Primer on Bosonization and Fermionization in CS-QED₃

In this section, we review a duality for Abelian Chern-Simons QED₃ with both bosonic and fermionic matter. This duality is a mild generalization of the bosonization dualities originally discussed in [257], and serves as a useful starting point for the more general non-Abelian constructions developed later in this chapter.

Concretely, the duality relates a $U(1)_{-N_s - \frac{N_f}{2} - k}$ QED theory with N_s complex scalars and N_f complex fermions to an Abelian linear quiver theory with $N_s + N_f + k - 1$ $U(1)$ gauge nodes. This construction can be regarded as the $N = 1$ limit of the general dualities for $U(N)$ and $SU(N)$ gauge theories with bosonic and fermionic matter discussed in the remainder of the chapter.

The matter content consists of complex fermions ψ , complex scalars ϕ , and real scalars σ . Throughout this section, we will focus on the infrared dynamics of these theories, omitting irrelevant interactions when they are not essential to the discussion.

Elementary Dualities

The simplest realizations of this duality framework are provided by the well-known Abelian bosonization and fermionization dualities. These can be written schematically as follows:³

$$U(1)_{-\frac{1}{2}} + 1\psi \quad \leftrightarrow \quad 1\phi \quad (5.1)$$

$$U(1)_{-1} + 1\phi \quad \leftrightarrow \quad 1\psi \quad (5.2)$$

For later convenience, and in view of the generalizations discussed below, it is useful to represent the dualities in (5.1) and (5.2) using quiver notation. In this language, the bosonization duality can be written as

$$\text{bosonization:} \quad \begin{array}{c} \textcircled{1} \xrightarrow[\textcolor{red]{-\frac{1}{2}}]{\textcolor{red}{\psi}} \boxed{1} \end{array} \quad \leftrightarrow \quad \begin{array}{c} \boxed{1} \xrightarrow[\textcolor{blue]{\mathcal{V} = |\phi|^4}]{\textcolor{blue}{\phi}} \boxed{1} \end{array} \quad (5.3)$$

²While Chapter 2 focused on $U(N)_{(k,k+\ell N)}$ gauge theories with fundamental matter, we instead concentrate here on $SU(N)_k$, whose richer operator spectrum allows us to illustrate the subtleties of our proposal more concretely.

³When presenting dualities in a compact form, such as in (5.1), (5.2), and in the quiver-like notation introduced below, we use the slightly imprecise but convenient convention of identifying the phase $\frac{i\pi}{2}\eta(A)$ with a $\frac{1}{2}$ shift in the Chern-Simons level of A . Throughout, we also suppress background gravitational Chern-Simons terms.

while the fermionization duality takes the form

$$\text{fermionization: } \begin{array}{c} \textcircled{1} \xrightarrow[\textcolor{red}{-1}]{\textcolor{blue}{\phi}} \boxed{1} \\ \mathcal{V} = |\phi|^4 \end{array} \leftrightarrow \begin{array}{c} \boxed{1} \xrightarrow[\textcolor{red}{\psi}]{\textcolor{red}{\psi}} \boxed{1} \end{array} \quad (5.4)$$

General Abelian Duality

We now propose a generalization to QED₃ with N_f fermions ψ , N_s complex scalars ϕ , and Chern-Simons level $-N_s - \frac{N_f}{2} - k$, with $k \geq 0$, shown in Figure 5.1. This proposal refines and extends the Abelian dualities studied in [250, 257].

The dual theory is a $U(1)^{N_s+N_f+k-1}$ gauge theory with N_f complex scalars and N_s complex fermions, each charged under two adjacent $U(1)$ gauge groups with opposite charges. Moreover, for each $U(1)$ gauge node coupled to at least two fermions, a real scalar σ_j is present, interacting with the fermions through Yukawa couplings.

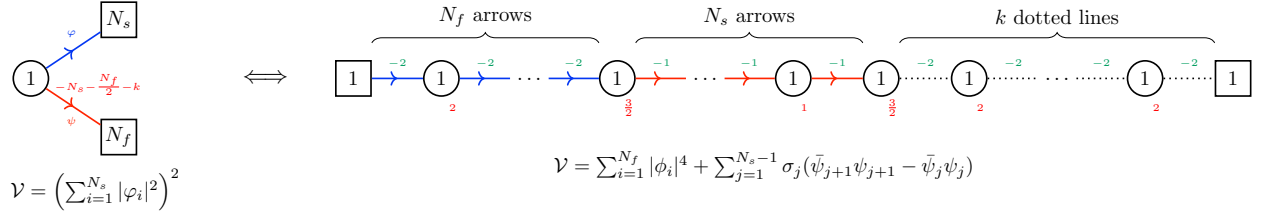


Figure 5.1: The proposed dual for QED₃ with N_s scalars, N_f fermions and CS level $-N_s - \frac{N_f}{2} - k$ with $k > 0$. Round nodes denote $U(1)$ gauge groups, square nodes denote flavor groups. Red and blue arrows denote fermions and scalars, respectively, charged under the nodes that they connect. Red labels denote CS levels and green labels denote mixed CS levels.

Operator Mapping

We now describe the mapping between gauge-invariant operators in the QED theory and operators in the dual quiver description. In particular, off-diagonal mesonic operators in QED are mapped to dressed monopole operators in the quiver theory.

Consider first the spin-0 and spin-1 fermionic mesons $\bar{\psi}_i \psi_j$ with $1 \leq i < j \leq N_f$. These operators are mapped to monopole operators carrying GNO flux +1 under all gauge groups from the i -th to the $(j-1)$ -th node in the quiver:

$$\bar{\psi}_i \psi_j \leftrightarrow \mathfrak{M} \left(\begin{array}{ccccccc} & & & i & & j-1 & \\ 0 & \dots & 0 & \xrightarrow{\textcolor{blue}{}} & + & \dots & + \xleftarrow{\textcolor{blue}{}} 0 & \dots \end{array} \right), \quad 1 \leq i < j \leq N_f. \quad (5.5)$$

Here, the two arrows indicate dressing by the corresponding scalar fields, while the left-pointing arrow denotes dressing with the complex conjugate scalar. Each boson in the monopole background experiences a GNO flux +1 and therefore contributes spin $\frac{1}{2}$, resulting in operators of spin 0 and 1.

Similarly, the scalar and mixed mesons $\varphi_i^* \varphi_j$ and $\bar{\psi}_i \varphi_j$ are mapped to dressed monopole operators as follows:

$$\varphi_i^* \varphi_j \quad \leftrightarrow \quad \mathfrak{M} \left(0 \quad \dots \quad 0 \xrightarrow{i+N_f} + \quad \dots \quad + \xleftarrow{j-1+N_f} 0 \quad \dots \right), \quad 1 \leq i < j \leq N_s, \quad (5.6)$$

$$\bar{\psi}_i \varphi_j \quad \leftrightarrow \quad \mathfrak{M} \left(0 \quad \dots \quad 0 \xrightarrow{i} + \quad \dots \quad + \xleftarrow{j-1+N_f} 0 \quad \dots \right), \quad (5.7)$$

which correspond to operators of spin 0 and spin $\frac{1}{2}$, respectively.

For generic values of k , monopole operators in the QED theory transform in non-trivial representations of the flavor symmetry $S(U(N_s) \times U(N_f))$. These operators are expected to map to monopole operators in the dual quiver theory. We will not discuss the detailed mapping of such operators here.

A particularly interesting simplification occurs when $k = 0$. In this case, the QED theory admits monopole operators that are neutral under the flavor symmetry $S(U(N_s) \times U(N_f))$. These are obtained by dressing the bare monopole $\mathfrak{M}^+$, which carries gauge charge $-N_s$, with N_s scalars φ (and their derivatives). The scalar fields are antisymmetrized in flavor indices, resulting in an operator that is charged under the topological symmetry but neutral under $S(U(N_s) \times U(N_f))$.

On the dual side, the quiver theory now admits long mesonic operators connecting the two flavor nodes, constructed from (derivatives of) all matter fields. These long mesons are mapped to the dressed monopole operators described above:

$$\mathfrak{M}^+ \overbrace{\partial_{\bullet} \varphi_{i_1} \cdots \partial_{\bullet} \varphi_{i_{N_s}}}^{N_s} \quad \leftrightarrow \quad \partial_{\star} \phi_1 \cdots \partial_{\star} \phi_{N_f} \partial_{\star} \psi_1 \cdots \partial_{\star} \psi_{N_s}. \quad (5.8)$$

Here, $\partial_{\bullet}$ and $\partial_{\star}$ denote generic sets of derivatives. On the left-hand side, Lorentz indices are contracted so as to obtain a non-vanishing operator. While it would be interesting to understand the precise mapping between the derivative structures $\partial_{\bullet}$ and $\partial_{\star}$, we will not attempt to do so here. Although the spin of these operators depends on the specific choice of derivatives, the integrality of the spin is fixed to be $\frac{N_s}{2} \bmod 1$ on both sides of the duality.

Consistency Checks and Special Limits

A non-trivial consistency check of the proposal shown in Figure 5.1 is that, for special values of the non-negative parameters N_f , N_s , and k , it reproduces several known Abelian dualities in the literature.

Single-species limits. For $k = 0$ and a single type of matter field, the proposal reduces to the Abelian bosonic and fermionic QED dualities discussed in [257]. In particular, for

fermionic QED with N_f flavors, one finds

$$\begin{array}{ccc}
 \text{Quiver 1} & \leftrightarrow & \text{Quiver 2} \\
 \text{Node 1: } \textcircled{1} \xrightarrow[\text{red } -\frac{N_f}{2}]{\text{red } \psi} \text{Node 2: } \boxed{N_f} & & \text{Node 1: } \boxed{1} \xrightarrow[\text{blue } 2]{\text{blue } \phi} \text{Node 2: } \textcircled{1} \xrightarrow[\text{blue } -2]{\text{blue } \phi} \dots \xrightarrow[\text{blue } -2]{\text{blue } \phi} \text{Node } n: \textcircled{1} \xrightarrow[\text{blue } 2]{\text{blue } \phi} \text{Node } n+1: \boxed{1} \\
 & & V = \sum_{i=1}^{N_f} |\phi_i|^4
 \end{array} \quad (5.9)$$

while the purely bosonic theory is dual to

$$\begin{array}{ccc}
 \text{Quiver 1} & \leftrightarrow & \text{Quiver 2} \\
 \text{Node 1: } \textcircled{1} \xrightarrow[\text{blue } -N_s]{\text{blue } \varphi} \text{Node 2: } \boxed{N_s} & & \text{Node 1: } \boxed{1} \xrightarrow[\text{red } 1]{\text{red } \psi} \text{Node 2: } \textcircled{1} \xrightarrow[\text{red } -1]{\text{red } \psi} \dots \xrightarrow[\text{red } -1]{\text{red } \psi} \text{Node } n: \textcircled{1} \xrightarrow[\text{red } 1]{\text{red } \psi} \text{Node } n+1: \boxed{1} \\
 \mathcal{V} = \left(\sum_{i=1}^{N_s} |\varphi_i|^2 \right)^2 & & V = \sum_{i=1}^{N_s-1} \sigma_i (\bar{\psi}_{i+1} \psi_{i+1} - \bar{\psi}_i \psi_i)
 \end{array} \quad (5.10)$$

In particular, for $k = 0$ and $N_f + N_s = 1$, the duality reduces to the original bosonization and fermionization dualities in Eqs. (5.1) and (5.2). For $k = 0$ and $N_f + N_s = 2$, the proposal yields dualities between different QED theories. The case $N_f = N_s = 1$ reproduces the duality studied in [46] for a $U(1)_{-\frac{3}{2}}$ theory with one scalar and one fermion.

Sigma-model limits. Further evidence for the duality arises from the behavior under relevant deformations. In the bosonic QED theory in Eq. (5.10), one may turn on a negative mass squared for all scalars in a way that preserves the global $SU(N_s)$ symmetry. The scalars condense, Higgsing the gauge group and driving the theory to a $\mathbb{CP}^{N_s-1}$ nonlinear sigma model in the infrared [217].

On the dual side, the corresponding deformation gives negative masses to the fermions, leading to a $(U(1)_0)^{N_s-1}$ gauge theory. We conjecture that the $N_s - 1$ dual photons combine with the $N_s - 1$ real scalars σ_i to form a complex $(N_s - 1)$ -dimensional target space, reproducing the expected sigma-model description on the original side.

Global Symmetries

We also discuss the global symmetry structure of the duality shown in Figure 5.1.

On the QED side, the continuous global symmetry is, up to discrete factors,

$$U(1)_{\text{top}} \times S(U(N_s) \times U(N_f)),$$

where $U(1)_{\text{top}}$ is the topological symmetry associated with the Abelian gauge field, while the special unitary factors act on the scalar and fermionic matter fields, respectively.

On the quiver side, the symmetry structure depends on the value of the parameter k . For $k > 0$, the theory contains $N_f + N_s$ independent $U(1)$ topological symmetries, one for each gauge node. When $k = 0$, one linear combination of these topological symmetries is absent, leaving $N_f + N_s - 1$ topological symmetries together with an additional $U(1)$ flavor symmetry rotating the matter fields. In both cases, the total rank of the continuous global symmetry matches that of the QED theory.

The duality therefore predicts a non-trivial infrared enhancement of symmetry on the quiver side: although the UV description exhibits an Abelian $U(1)^{N_f+N_s}$ global symmetry, this symmetry is expected to reorganize into the non-Abelian group

$$S(U(N_s) \times U(N_f))$$

in the infrared, in precise agreement with the global symmetry of the QED theory.

Mass Deformations

Mass deformations of the matter fields in the QED theory admit a simple and intuitive mapping to mass deformations in the dual quiver theory. Schematically, the correspondence takes the form

$$\text{LHS} \quad \begin{array}{c} |\phi|^2 \\ \bar{\psi}\psi \end{array} \quad \leftrightarrow \quad \begin{array}{c} \bar{\psi}\psi \\ -|\phi|^2 \end{array} \quad \text{RHS}. \quad (5.11)$$

In other words, scalar mass deformations on one side of the duality are mapped to fermionic mass deformations on the other side, and vice versa, with a relative sign dictated by the structure of the duality.

One can explicitly verify that turning on masses for the fermions and positive masses for the scalars reproduces the same duality shown in Figure 5.1, albeit with different values of the parameters k , N_f , and N_s . This behavior provides a non-trivial consistency check of our proposal, as it confirms that the duality is stable under continuous deformations of the microscopic parameters.

Turning on a negative mass for one of the scalars instead triggers a Higgs mechanism on the QED side, partially breaking the gauge symmetry. On the dual side, the corresponding deformation gives a positive mass to one of the fermions, which can then be integrated out. This initiates a sequential confinement process along the quiver, ultimately resulting in a theory of free fields. The infrared description obtained in this way precisely matches that of the Higgsed electric theory, providing further support for the proposed duality.

5.2 Planar Abelianization in Bosonic CS-QCD₃

In this section, we construct a quiver gauge theory dual to non-Abelian QCD₃ with fundamental matter. Our proposal builds upon recent results [2–4] that established a duality between $\mathcal{N} = 2$ supersymmetric non-Abelian linear CS quiver gauge theories and planar Abelian quiver gauge theories with mixed CS interactions. Motivated by these supersymmetric dualities, we conjecture an analogous correspondence in the non-supersymmetric case: non-Abelian CS QCD₃ with fundamental fermionic and bosonic matter should be dual to a planar Abelian quiver theory containing both bosons and fermions. This expectation follows from the observation that our proposed non-supersymmetric dualities should emerge as mass deformations of their supersymmetric counterparts.

5.2.1 General Proposal

We propose the planar Abelian dual for $SU(N)$ QCD₃ with N_s critical scalars at Chern-Simons level $2N - N_s - k$ ($k \geq 0$) shown in Figure 5.2. This duality maps flavor symmetries to topological symmetries and many baryonic and mesonic operators of one theory are mapped to gauge invariant disorder operators (monopoles) of its mirror dual. In this section, we discuss some simple examples explicitly to justify our proposal. The proposed duality is believed to hold in the following range of parameters:

$$N_s \geq 2N, \quad k \geq 0.$$

In the planar quiver shown in Figure 5.2, the parameter k counts the number of “empty” columns — columns that are connected to the rest of the quiver solely through horizontal BF couplings.

We now comment on the potential $\mathcal{V}$ shown in Figure 5.2 for the planar mirror theory:

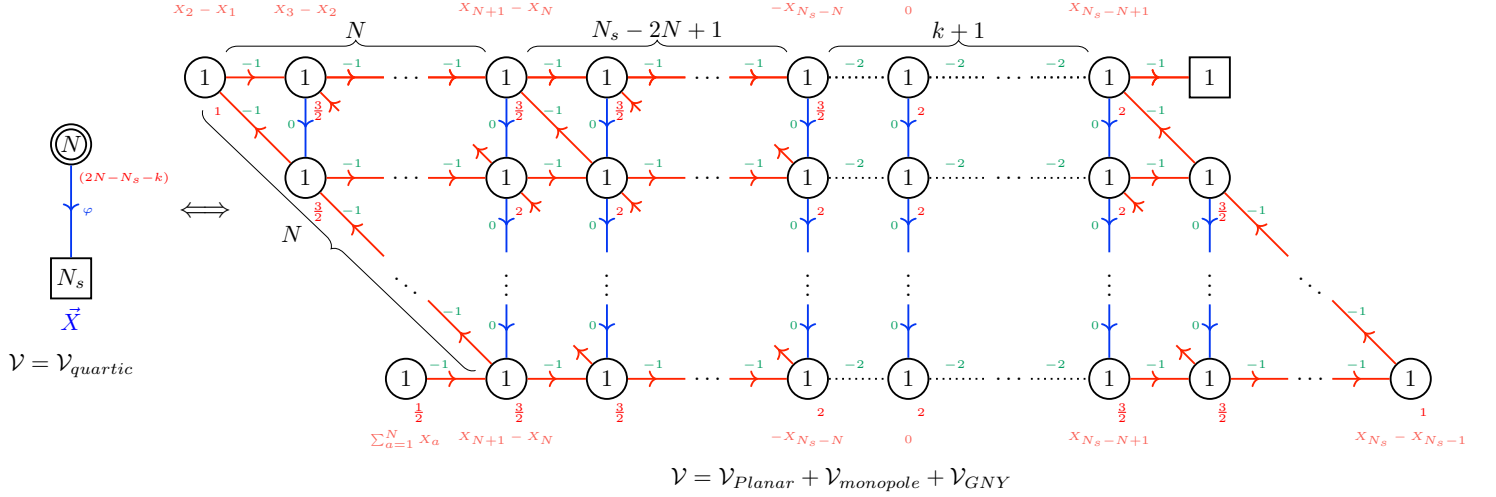
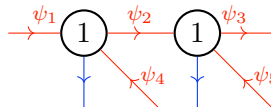


Figure 5.2: The proposed dual for $SU(N)$ QCD₃ with N_s scalars at Chern-Simons level $2N - N_s - k$ is shown here. Single/double circles (squares) correspond to U/SU gauge (flavor) symmetries. We describe scalar (fermion) bifundamental fields in blue (red). Chern-Simons levels of gauge nodes are indicated in maroon, while mixed CS couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. BF couplings between adjacent gauge nodes connected by a fermionic field are -1 , BF couplings between gauge nodes connected by a bosonic field are 0 , while BF couplings between adjacent gauge nodes connected by dotted lines alone are -2 . We also indicate the fugacities of the topological symmetries of each column in orange. The orientation of arrows indicates the representation of the field under the symmetry group, i.e.: an arrow pointing out of (into) a node transforms in the (anti-)fundamental representation of the symmetry group.

- $\mathcal{V}_{Planar}$ contains a cubic Yukawa interaction ($\phi_i \psi_j \psi_k + c.c.$) for every closed planar triangle. There is a term with a coefficient of $+1$ (-1) for each triangle closed clockwise (anticlockwise).
- $\mathcal{V}_{monopole}$ captures the monopole terms in the potential. For each vertical bosonic bifundamental field, there is a monopole potential with GNO flux $+1$ and -1 under the nodes connected by the arrow, from top to bottom ($\mathfrak{M}^{(+)} + c.c.$). As explained in Chapter 1 these monopoles can be dressed to obtain a gauge invariant spin-0 operator. This breaks the topological symmetry of the gauge nodes in a column of the quiver to a single, diagonal combination. We label the unbroken $U(1)$ symmetry with the orange label at the top (or bottom) of the corresponding column. The label encodes the mapping of the $U(1)$ group to a subgroup $U(1) \subset U(N_s)$ of the electric flavor symmetry, as explained below.
- $\mathcal{V}_{GNY}$ captures additional cubic interactions between a real scalar field associated with the α^{th} gauge node in the I^{th} column, and the fermions charged under this gauge group (an arrow pointing out of (into) a node has charge $+1$ (-1) under the gauge symmetry). For example, consider a gauge theory represented by the quiver diagram in (5.12).



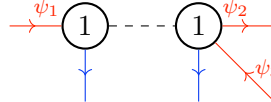
(5.12)

$\mathcal{V}_{\text{GNY}}$ for this theory has the form:

$$\mathcal{V}_{\text{GNY}} = \sigma_1(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_4\psi_4) + \sigma_2(\bar{\psi}_3\psi_3 - \bar{\psi}_2\psi_2 - \bar{\psi}_5\psi_5). \quad (5.13)$$

Each gauge node **except** for the bottom-most gauge node has a GNY potential accompanying the Yukawa interactions.

By contrast, if we consider the following gauge theory (the dashed line indicates the absence of any fermionic bifundamental field, and the two nodes may share a bosonic bifundamental field or only mixed CS interactions):


(5.14)

we expect the identification of σ_1 and σ_2 so the resulting $\mathcal{V}_{\text{GNY}}$ has the form:

$$\mathcal{V}_{\text{GNY}} = \sigma(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_3\psi_3). \quad (5.15)$$

In the quiver theory in Figure 5.2 there are $N(N_s - N)$ real scalars σ .

In the QCD₃ theory, the quartic scalar potential is $\mathcal{V}_{\text{quartic}}$ ⁴:

$$\mathcal{V}_{\text{quartic}} = \sum_{\alpha, \beta=1}^N \sum_{I, J=1}^{N_s} [\varphi^{\alpha I} \bar{\varphi}_{\alpha J} \varphi^{\beta J} \bar{\varphi}_{\beta I} + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\varphi^{\beta J} \bar{\varphi}_{\beta J})] \quad (5.16)$$

which preserves the full $U(N_s)$ global symmetry.

In the planar mirror theory the presence of the interactions $\mathcal{V}_{\text{monopole}}$ and $\mathcal{V}_{\text{Planar}}$ are essential to match the rank of the global symmetry on the electric side. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)_{\text{top}}^{N_s}$, where each $U(1)$ factor corresponds to the topological symmetries of a column of the quiver, broken to a diagonal combination by $\mathcal{V}_{\text{monopole}}$ ⁵. We further claim that the symmetry enhances in the IR to:

$$U(1)_{\text{top}}^{N_s} \rightarrow SU(N_s) \times U(1) \quad (5.17)$$

reproducing the $SU(N_s) \times U(1)$ flavor and baryonic symmetries of the electric theory respectively. This is most transparent in our choice of parametrization of the fugacities of the topological symmetries of each column in Figure 5.2. Explicitly, the diagonal combination of the topological symmetries in the i -th column, associated to $X_{i+1} - X_i$, is mapped to the combination $U(1)_{X_{i+1}} - U(1)_{X_i}$ in the maximal torus of $U(N_s)$. Similarly, the topological symmetry of the bottom left node, associated to $\sum_{a=1}^N X_a$, is mapped to the linear combination $\sum_{i=1}^N U(1)_{X_i}$. Therefore the fugacities denoted with orange labels in 5.2 encode the explicit embedding of $U(1)_{\text{top}}^{N_s}$ in the enhanced $U(N_s)$ global symmetry group.

⁴N.B.: Latin indices $i, j = 1, \dots, F$ run over flavor indices, and Greek indices $\alpha, \beta = 1, \dots, N$ run over gauge indices.

⁵There are $N_s - 1$ columns which contribute $N_s - 1$ $U(1)$ factors, and the additional $U(1)$ factor is given by the topological symmetry of the bottom-most gauge node. The topological symmetries associated to the columns within the “empty” region of the quiver can be reabsorbed by a gauge transformation and do not contribute to the global symmetry.

Moreover, the faithful symmetry of the QCD theory is [46, 258]:

$$\frac{U(N_s)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^C \quad (5.18)$$

where $\mathbb{Z}_2^C$ is charge conjugation, acting as complex conjugation on the scalars for $N \geq 2^6$. $\mathbb{Z}_N$ is generated by the following element of the maximal torus of $U(N_s)$:

$$\left\{ e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi i}{N}} \right\} \in \{U(1)_{X_1}, \dots, U(1)_{X_{N_s}}\} \quad (5.20)$$

All the gauge invariant dressed monopoles in the dual quiver transform trivially under this element, and are therefore compatible with being components of representations of the enhanced $\frac{U(N_s)}{\mathbb{Z}_N}$.

The fact that there are no mesonic $U(1)$ symmetries in the planar theory is a non-trivial statement and can be checked as follows by looking at the planar quiver:

$$\begin{aligned} \# \text{mesonic } U(1)s &= \# \text{lines} - \# \text{gauge nodes} - \# \text{triangles} = \\ &= \{3N(N_s - N) - 2N_s + 3\} - \{N(N_s - N) + 1\} = \\ &\quad - \{2N(N_s - N) - 2(N_s - 1)\} \\ &= 0 \end{aligned} \quad (5.21)$$

where we assume that there is a mesonic $U(1)$ rotating each boson/fermion in the mirror, and a cubic potential term associated to each triangle appearing in the quiver⁷.

The presence of the Gross-Neveu-Yukawa potential $\mathcal{V}_{GNY}$ also plays an important role in the duality and can be motivated as follows. On adding a negative mass deformation for the N_s scalars in the electric theory, the gauge group is completely Higgsed and the IR phase of the theory is a sigma model with the target manifold identified as the coset space [101, 259]:

$$\frac{SU(N_s)}{SU(N_s - N) \times SU(N)} \quad (5.22)$$

which has $2NN_s - 2N^2 + 1$ real dimensions. Upon mapping the same deformation in the mirror theory, we expect to flow to $N(N_s - N) + 1$ copies of $U(1)_0$, whose dual photons parameterize $(\mathbf{S}^1)^{N(N_s - N) + 1}$ in the deep IR. This can be achieved by giving a negative mass to all the fermions and a positive mass to all the scalars in the quiver. The dual photons are expected to combine with the $N(N_s - N)$ real scalars σ , resulting in an NLSM with real dimension $2N(N_s - N) + 1$, consistent with the QCD theory.

In the rest of this section, we will make these ideas more precise by considering some explicit examples. Through these examples, we illustrate the effects of increasing the Chern-Simons level k and the rank of the gauge symmetries on the respective planar duals.

⁶For $N = 2$, as $SU(2) \cong USp(2)$, the global symmetry of the electric $SU(2)$ QCD₃ theory is:

$$\frac{USp(2N_s)}{\mathbb{Z}_2}; \quad (5.19)$$

which is broken by $\mathcal{V}_{quartic}$ to $\frac{U(N_s)}{\mathbb{Z}_2} \rtimes \mathbb{Z}_2^C$ [46].

⁷Note that this straightforward computation holds when the parameter $k = 0$. For $k > 0$, matching the $U(1)$ symmetries becomes more subtle. Nonetheless, the matching can be verified in this case as well.

5.2.2 Example: $SU(2)_{-1}$ QCD₃ with $N_s = 5$ Scalars

Let us begin with the example of $SU(2)_{-1}$ with 5 scalars and its planar dual description. Since this is our first example, we are pedantic and give many details to illustrate our proposal. We propose the duality shown in Figure 5.3.

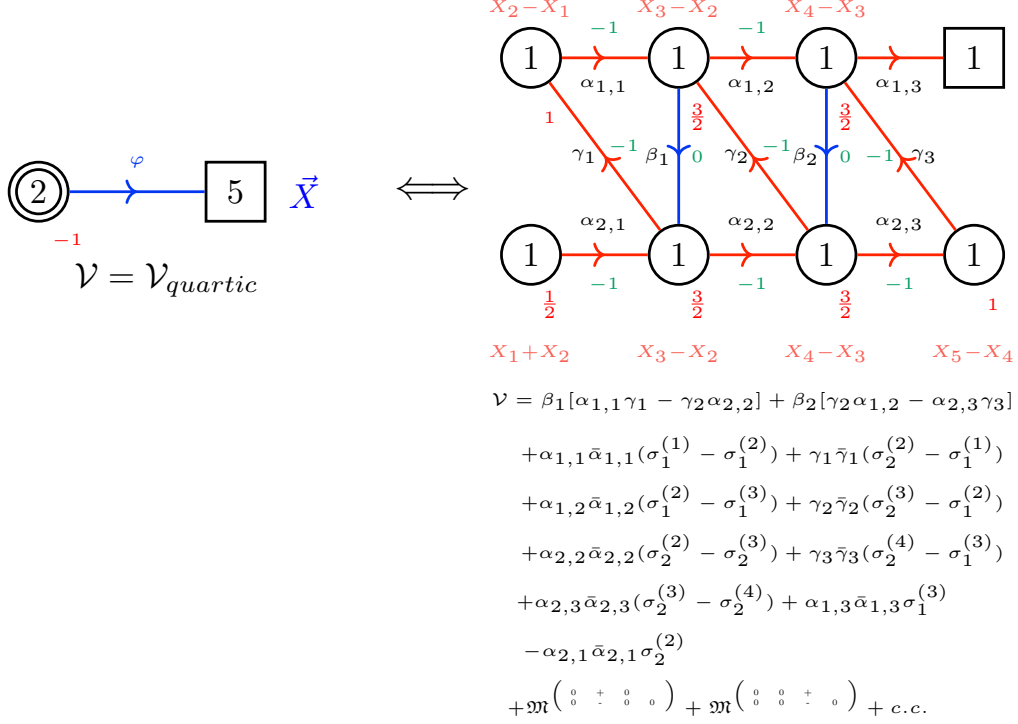


Figure 5.3: The dual for $SU(2)_{-1}$ QCD₃ with 5 scalars is shown here. All labelling conventions follow those of Figure 5.2. The superscript (subscript) for the real scalar σ indicates its column (position in the column), e.g.: $\sigma_1^{(3)}$ corresponds to the 1st gauge node in the 3rd column.

Henceforth, we refer to the $SU(2)_{-1}$ QCD₃ with 5 scalars as the “electric” theory and the planar theory as its “mirror”. We are mainly concerned with the mapping of the following gauge-invariant operators in the electric theory:

1. **25 mesons** of the form $\varphi_i\bar{\varphi}_j$ transforming in the adjoint representation of the flavor symmetry group $U(5)$. Hence, with respect to the maximal torus $U(1)_{X_i}^5 \subset U(5)$, they are charged under $U(1)_{X_i} - U(1)_{X_j}$. The off-diagonal mesons (those with $i \neq j$) are then mapped to monopole operators of the dual quiver theory. The proposed mapping for the mesonic operators is shown in Table 5.1. The diagonal mesons $|\varphi_i|^2$ are neutral under the $U(1)_{X_i}$ and are mapped to non-monopole operators of the dual theory. The four traceless diagonal mesons are mapped to the real scalars σ and the traceful diagonal meson is mapped to a combination of $\bar{\psi}_i\psi_i$ mesonic operators in the quiver.
2. **25 – 5 = 20 conserved currents** associated to the global symmetry enhancement $U(1)^5 \rightarrow U(5)$. These are spin-1 operators charged under $U(1)_{X_i} - U(1)_{X_j}$ with $i \neq j$ and are expected to be mapped to monopole operators, as illustrated below.
3. **$\binom{5}{2} = 10$ baryons** constructed by anti-symmetrizing the color indices with the help of the Levi-Civita tensor $\epsilon_{\alpha\beta}\varphi_i^\alpha\varphi_j^\beta$. Together, these transform in the 2-index anti-symmetric representation of the flavor symmetry group $SU(5)$ and are charged under

$U(1)_{X_i} + U(1)_{X_j}$, with $i \neq j$. The proposed mapping for the baryonic operators is shown in Table 5.2.

The off-diagonal mesons, off-diagonal currents and the baryons are charged under the maximal torus $U(1)_{X_i}^5 \subset U(5)$, therefore they should be mapped to monopole operators in the dual quiver theory. In what follows, we propose a mapping for these operators, starting from the off-diagonal mesons and the baryons. Thus, the task at hand is to determine gauge invariant monopole operators that are **Lorentz scalars** and are charged under $U(1)_{X_i} \mp U(1)_{X_j}$, which are mapped to the topological symmetries of the quiver theory as described above. We make this statement precise in the next paragraphs by considering some examples. The techniques needed to compute the gauge charges of monopole operators, the dressing needed to construct gauge invariant monopoles and the spin of the resulting operator are reviewed in Chapter 1.

“Mesonic” Monopole Operators We study monopole operators with the right quantum numbers to be mapped to the mesons of the electric theory here. As an example, we consider the following monopole operator:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.23)$$

This operator has charge +1 under the topological symmetries of the first two gauge nodes; hence, it is charged under $U(1)_{X_3} - U(1)_{X_1}$, and transforms as a component of the adjoint representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 3 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix}. \quad (5.24)$$

Clearly, it is not gauge invariant and must be dressed appropriately. We choose the minimal dressing⁸:

$$\begin{pmatrix} + & 2 & - \\ 0 & - & - & 0 \end{pmatrix}, \quad (5.25)$$

hence, we see that the operator:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\gamma_1 \gamma_2 \bar{\alpha}_{1,2}} \quad (5.26)$$

is gauge invariant. We denote this operator by combining the GNO flux assignment and the dressing in the following way:

$$\mathfrak{M} \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\gamma_1 \gamma_2 \bar{\alpha}_{1,2}} \equiv \begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.27)$$

⁸By this, we mean that we use the minimal number of fields to dress the bare monopole and construct a gauge-invariant operator. A non-minimal dressing—i.e., using additional fields—would yield an operator with a higher scaling dimension. In this work, we focus exclusively on mapping the operators with the lowest scaling dimensions.

As each of the three fermionic fields is quantized in a background of total magnetic flux 1, their spin statistics changes to that of spin-0 bosons⁹. We conclude that the dressed monopole operator has Lorentz spin 0, and is the right operator to be mapped to $\varphi_3\bar{\varphi}_1$.

We can readily identify the off-diagonal mesonic operators and summarize their dual descriptions—including necessary dressings—in Table 5.1. The mapping of diagonal mesonic operators $|\varphi_i|^2$ is less straightforward because these operators are neutral under the maximal torus of $U(5)$, visible in the quiver UV Lagrangian. We expect that these operators are mapped to suitable linear combinations of mesonic operators of the quiver, compatible with the mass deformations discussed in Section 5.5. We defer the study of the mapping of diagonal mesonic operators to future work.

Table 5.1: Dressed monopole operators of the planar mirror (Figure 5.3) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	“Electric” Meson
$\begin{pmatrix} + & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	$\varphi_2\bar{\varphi}_1$
$\begin{pmatrix} + & + & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & 2 & 0 \\ 0 & - & 0 \end{pmatrix}$	0	$\varphi_3\bar{\varphi}_1$
$\begin{pmatrix} + & + & + \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & + & 2 \\ 0 & - & - \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_1$
$\begin{pmatrix} + & + & + \\ 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} + & + & + \\ 0 & - & -2 \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_1$
$\begin{pmatrix} 0 & + & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	$\varphi_3\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 2 & 2 \\ 0 & 0 & - \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + \\ 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} - & 2 & + \\ 0 & 0 & -2 \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_2$
$\begin{pmatrix} 0 & 0 & + \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & - & 3 \\ 0 & 0 & 0 \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & + \\ 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} 0 & - & 2 \\ 0 & 0 & - \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & - \\ 0 & 0 & -2 \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_4$

Let us comment on a particular feature of the monopole operators in the quiver theories considered in this chapter, which is a consequence of the presence of the monopole interaction $\mathcal{V}_{monopole}$. When looking for a candidate for the dual of an electric meson, say $\varphi_4\bar{\varphi}_3$,

⁹In principle, they also have a spin-1 channel. Unless otherwise stated we dress the monopoles with the zero modes of the fermion fields, resulting in a spin-0 operator [116].

we may also consider the monopole:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.28)$$

Indeed, this is a spin-0 gauge invariant operator with the right charges to be mapped to $\varphi_4 \bar{\varphi}_3$, similarly to the monopole listed in Table 5.1. This is possible due to the presence of the monopole potential $\mathcal{V}_{monopole}$: even though these two monopoles have different GNO flux assignments they have the same charges under the linear combination of topological symmetries preserved by $\mathcal{V}_{monopole}$. We expect these two monopoles to mix, therefore the electric meson $\varphi_4 \bar{\varphi}_3$ will be mapped to some linear combination of the two:

$$\alpha \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \varphi_4 \bar{\varphi}_3, \quad (|\alpha|^2 + |\beta|^2 = 1). \quad (5.29)$$

This phenomenon is ubiquitous in the quiver theories studied in this chapter and the amount of monopoles that can mix between each other increases as N increases due to the higher number of monopole interactions in $\mathcal{V}_{monopole}$. In Table 5.1 and in the rest of this section we only report one dressed monopole that has the correct charges to be mapped to a given operator in the QCD theory¹⁰. It is understood that, in most cases, there are other monopole operators with the same global charges but different GNO flux assignment and dressing, and the mapping will involve a suitable linear combination of these operators. It would be interesting to analyze this mixing systematically, for example by extending the techniques of [116, 128, 260] in the presence of monopole interactions, but we defer this to future work.

Conserved Currents We now examine how the $U(5)$ flavor symmetry currents - including the baryonic $U(1)$ subgroup - map between the electric and dual theories. These spin-1 conserved currents take the form:

$$J_{jk}^\mu = i(\bar{\varphi}_j \partial^\mu \varphi_k - \partial^\mu \bar{\varphi}_j \varphi_k) \quad (5.30)$$

where $j, k = 1, \dots, 5$, transforming in the adjoint representation of $U(5)$. Note that under charge conjugation: $(J_{jk}^\mu)^* = J_{kj}^\mu$.

The off-diagonal currents map to dressed spin-1 monopole operators in the dual theory. For example, J_{21}^μ corresponds to:

$$J_{21}^\mu \leftrightarrow \mathfrak{M} \begin{pmatrix} + & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.31)$$

where we dress with the spin-1 mode of one fermion and the spin-0 mode of the other fermion, resulting in a spin-1 operator. Under charge conjugation, this maps to J_{12}^μ :

¹⁰Notice that also on the QCD side there are in general many operators with the same spin and global charges. For example any operator can be multiplied by a spin-0 flavor singlet or it can be acted on by contracted derivatives.

$$J_{12}^\mu \leftrightarrow \mathfrak{M} \begin{pmatrix} - & \xrightarrow{\text{red}} 0 & 0 \\ 0 & \xrightarrow{\text{red}} 0 & 0 & 0 \end{pmatrix} \quad (5.32)$$

where, again, we dress with a spin-0 and a spin-1 mode.

The other off-diagonal spin 1 currents map in a similar way. In particular, the (off-diagonal) current J_{jk}^μ is mapped to the same bare monopole as the meson $\varphi_j \bar{\varphi}_k$, the difference being that the former is dressed such that it has spin-1 while the latter has spin-0.

The five diagonal currents J_{jj}^μ map to topological $U(1)$ currents in the dual theory:

$$J_{jj}^\mu \longleftrightarrow \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu A_\rho^{(j)} \quad (5.33)$$

with the gauge field combinations:

$$A_\mu^{(j)} = \begin{cases} A_\mu^{(1,1)} & j = 1 \\ A_\mu^{(2,1)} + A_\mu^{(2,2)} & j = 2 \\ A_\mu^{(3,1)} + A_\mu^{(3,2)} & j = 3 \\ A_\mu^{(4,1)} & j = 4 \\ A_\mu^{(1,2)} & j = 5 \end{cases} \quad (5.34)$$

Here $A_\mu^{(j,k)}$ denotes the gauge field for the $U(1)$ node at row k and column j in quiver (5.3). The mapping (5.33) reproduces the mapping of the maximal torus of the $U(5)$ global symmetry across the duality.

“Baryonic” Monopole Operators We follow the same line of reasoning as before to study monopole operators with the right quantum numbers to be mapped to the baryons of the electric theory here. As an example, we consider the following *bare* monopole operator:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \\ + & + & 0 & 0 \end{pmatrix} \quad (5.35)$$

This operator has charge +1 under the topological symmetries of the bottom two gauge nodes; hence, it is charged under $U(1)_{X_3} + U(1)_{X_1}$, and therefore can be identified with a component of the two-index antisymmetric representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix}. \quad (5.36)$$

Clearly, it is not gauge invariant and must be dressed appropriately. We again choose the minimal dressing that cancels the gauge charges of the bare monopole:

$$\begin{pmatrix} - & 0 & 0 \\ 0 & \text{---} 2 \text{---} & - \\ & & 0 \end{pmatrix}, \quad (5.37)$$

Therefore the dressed monopole operator:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \\ + & + & 0 & 0 \end{pmatrix} \bar{\gamma}_1 \bar{\alpha}_{2,2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ + & \text{---} + \text{---} 0 & 0 \\ & & 0 \end{pmatrix}; \quad (5.38)$$

is a gauge invariant Lorentz scalar and has the correct charges to be mapped to $\varphi_3\varphi_2$ (color indices have been suppressed). Similarly, we can identify the 10 baryonic operators, reported with the appropriate dressing in Table 5.2.

Table 5.2: Dressed monopole operators of the planar mirror (Figure 5.3) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	“Electric” Baryon
$\begin{pmatrix} 0 & 0 & 0 \\ + \text{---} 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ + \text{---} - & 0 & 0 \end{pmatrix}$	0	$\varphi_2\varphi_1$
$\begin{pmatrix} 0 & 0 & 0 \\ + \text{---} + \text{---} 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 \text{---} 2 \text{---} - & 0 & 0 \end{pmatrix}$	0	$\varphi_3\varphi_1$
$\begin{pmatrix} 0 & 0 & 0 \\ + \text{---} + \text{---} + \text{---} 0 \end{pmatrix}$	$\begin{pmatrix} 0 & - & 0 \\ 0 \text{---} + \text{---} 2 \text{---} - \end{pmatrix}$	0	$\varphi_4\varphi_1$
$\begin{pmatrix} 0 & 0 & 0 \\ + \text{---} + \text{---} + \text{---} + \end{pmatrix}$	$\begin{pmatrix} 0 & - & - \\ 0 \text{---} + \text{---} + \text{---} + \end{pmatrix}$	0	$\varphi_5\varphi_1$
$\begin{pmatrix} + \text{---} 0 & 0 \\ + & + \text{---} 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + \text{---} - & 0 \\ 0 & + \text{---} - & 0 \end{pmatrix}$	0	$\varphi_3\varphi_2$
$\begin{pmatrix} + \text{---} 0 & 0 \\ + & + \text{---} + \text{---} 0 \end{pmatrix}$	$\begin{pmatrix} + \text{---} -2 & 0 \\ 0 & 0 \text{---} 2 \text{---} - \end{pmatrix}$	0	$\varphi_4\varphi_2$
$\begin{pmatrix} + \text{---} 0 & 0 \\ + & + \text{---} + \text{---} + \end{pmatrix}$	$\begin{pmatrix} + \text{---} -2 & - \\ 0 & 0 \text{---} + \text{---} + \end{pmatrix}$	0	$\varphi_5\varphi_2$
$\begin{pmatrix} + & + \text{---} 0 \\ + & + & + \text{---} + \end{pmatrix}$	$\begin{pmatrix} 0 & + \text{---} -2 \\ 0 & 0 & 0 \text{---} + \end{pmatrix}$	0	$\varphi_5\varphi_3$
$\begin{pmatrix} + & + & + \text{---} + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & + \text{---} 0 \end{pmatrix}$	0	$\varphi_5\varphi_4$

We reiterate that we only consider dressed minimal flux operators in the mapping.

5.2.3 Example: $SU(2)_{-2}$ QCD₃ with $N_s = 5$ Scalars

We now consider the example of $SU(2)_{-2}$ with 5 scalars. Notice that it has a lower Chern-Simons level than that of the theory considered in Section 5.2.2; this minor difference will greatly alter the operator map. We propose the duality shown in Figure 5.4.

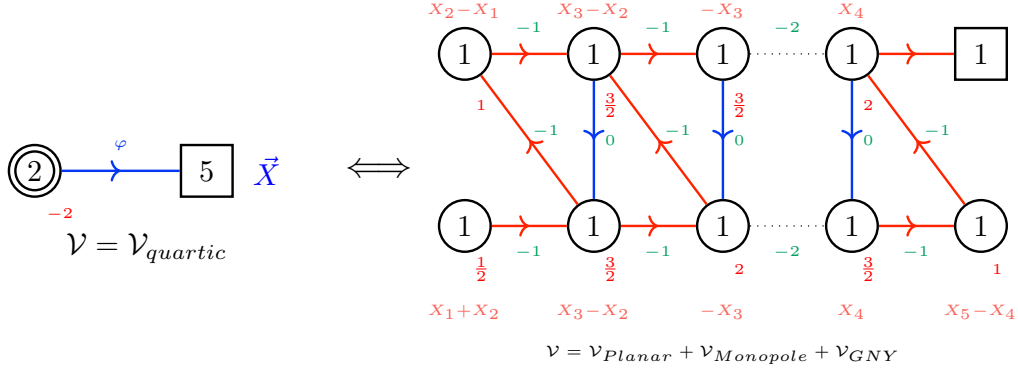


Figure 5.4: The dual for $SU(2)_{-2}$ QCD₃ with 5 scalars is shown here. All labelling conventions follow those of Figure 5.2.

As before, we are mainly concerned with determining the monopole operators that are dual to the 20 off-diagonal **mesons** and 10 **baryons**. We do not show the explicit calculation and state the results in Tables 5.3 and 5.4 respectively.

Table 5.3: Dressed monopole operators of the planar mirror (Figure 5.4) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & \text{---} & 0 & 0 & 0 \\ 0 & \text{---} & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{---} & + & \text{---} & 0 & 0 & 0 \\ 0 & \text{---} & 0 & \text{---} & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_1$
$\begin{pmatrix} + & \text{---} & + & \text{---} & + & + & 0 \\ 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{---} & + & \text{---} & + & + & + \\ 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \text{---} & + & \text{---} & 0 & 0 & 0 \\ 0 & \text{---} & 0 & \text{---} & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{---} & + & \text{---} & + & + & 0 \\ 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \text{---} & + & \text{---} & + & + & + \\ 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 & \text{---} & + & + & + & 0 \\ 0 & 0 & \text{---} & 0 & 0 & \text{---} & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & \text{---} & + & + & + & + \\ 0 & 0 & \text{---} & 0 & 0 & \text{---} & + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \text{---} & + & + \\ 0 & 0 & 0 & 0 & \text{---} & 0 & + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_4$

Table 5.4: Dressed monopole operators of the planar mirror (Figure 5.4) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & \text{---} & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \varphi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & \text{---} & + & \text{---} & 0 & 0 \end{pmatrix}$	$\varphi_3 \varphi_1$
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & \text{---} & + & \text{---} & + & + \end{pmatrix}$	$\varphi_4 \varphi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & \text{---} & + & \text{---} & + & + \end{pmatrix}$	$\varphi_5 \varphi_1$

Continued on next page

Table 5.4 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & + & \xrightarrow{\text{red}} 0 & 0 \end{pmatrix}$	$\varphi_3\varphi_2$	$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & + & \xrightarrow{\text{red}} + & + \end{pmatrix}$	$\varphi_4\varphi_2$
$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & + & \xrightarrow{\text{red}} + & + \end{pmatrix}$	$\varphi_5\varphi_2$	$\begin{pmatrix} + & + & \xrightarrow{\text{red}} 0 & 0 \\ + & + & + & \xrightarrow{\text{red}} 0 \end{pmatrix}$	$\varphi_4\varphi_3$
$\begin{pmatrix} + & + & \xrightarrow{\text{red}} 0 & 0 \\ + & + & + & \xrightarrow{\text{red}} + \end{pmatrix}$	$\varphi_5\varphi_3$	$\begin{pmatrix} + & + & + & + \\ + & + & + & + \end{pmatrix}$	$\varphi_5\varphi_4$

5.2.4 Mapping Gauge Invariant Operators

Before we conclude, we briefly comment on the general trend of mapping gauge invariant operators between the dual theories. We remind the reader that the proposed duality exchanges flavor and topological symmetries; hence, mesons, conserved currents and baryons of the electric theory will be mapped to suitably dressed disorder operators (monopoles) of the mirror theory. We ask the reader to keep Figure 5.2 in mind for the rest of the discussion. We also take this opportunity to remind the reader that a specific choice of fugacities of the topological symmetry parametrizes each column.

Mesons and Conserved Currents

Mesons and conserved currents are Lorentz scalars and vectors, respectively, that transform in the adjoint representation of the flavor symmetry group $SU(N_s)$. We now specify the general pattern of mapping these operators to monopoles of the mirror theory (Figure 5.2). We focus on the mapping of the $\frac{1}{2}N_s(N_s - 1)$ “off-diagonal” mesons $\varphi_I\bar{\varphi}_J$ with $J < I$. The case $J > I$ can be obtained by taking the Hermitian conjugate of these operators, which map to monopoles with negative GNO fluxes.

A generic minimal flux mesonic operator **cannot** be charged under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram. This is due to the parameterization of topological fugacities and representation-theoretic constraints. Modulo a subtlety arising from the presence of a monopole interaction in the dual quiver—already discussed in Section 5.2.1—we propose that *off-diagonal* mesonic operators are mapped to monopoles carrying GNO flux under the gauge nodes in the top row and/or along the rightmost diagonal of the quiver.

To describe this mapping systematically, we find it convenient to organize the mesons $\varphi_I\bar{\varphi}_J$ into three distinct classes, based on the values of the flavor indices I and J :

- First, we consider the case where both I and J are less than $N_s - N$.
- Next, we study the regime where $J < N_s - N \leq I$.
- Finally, we address the case in which both I and J are greater than or equal to $N_s - N$.

$\varphi_I\bar{\varphi}_J$, **where** $J < I < N_s - N$: The simplest case is that of the meson $\varphi_{I+1}\bar{\varphi}_I$ with $(1 \leq I \leq N_s - N - 1)$. In the mirror theory, this is mapped to a monopole with +1 GNO flux under the top node of the I^{th} column:

$$\varphi_{I+1}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots \boxed{+} \dots 0 \ 0 \ 0 \dots 0 \ 0}^{N_s - N \quad k} \\ \uparrow \\ I \\ 0 \text{ GNO Flux} \end{array} \begin{array}{c} \nearrow^N \\ 0 \\ \vdots \\ 0 \end{array} \right) \quad (5.39)$$

Thereafter, +1 charges must propagate contiguously from left to right. A meson of the form $\varphi_{I+1+A}\bar{\varphi}_I$ ($A \neq 0$) is dual to a monopole with the following charge assignment:

$$\varphi_{I+1+A}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + \dots + 0 \ 0 \dots 0 \ 0}^{N_s - N \quad k} \\ \underbrace{\hspace{1.5cm}}_A \\ 0 \text{ GNO Flux} \end{array} \begin{array}{c} \nearrow^N \\ 0 \\ \vdots \\ 0 \end{array} \right) \quad (5.40)$$

$\varphi_I\bar{\varphi}_J$, **where** $J < N_s - N \leq I$: In this case, the dual monopole has a contiguous string of +1 GNO fluxes starting from the J^{th} column and ending on the $(I + k - 1)^{\text{th}}$ column. In particular, each of the k nodes inside the region of the quiver connected by BF coupling has GNO flux +1. The first case of this type is the meson $\varphi_{N_s-N+1}\bar{\varphi}_I$, which is mapped to a monopole of the form:

$$\varphi_{N_s-N+1}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + \dots + + \dots + +}^{N_s - N \quad k} \\ 0 \text{ GNO Flux} \end{array} \begin{array}{c} \nearrow^N \\ 0 \\ \vdots \\ 0 \end{array} \right) \quad (5.41)$$

Once a charge has been assigned to a column, no additional charges can be assigned to the same column in lower rows. The propagation of fluxes for lower rows follows a strict “staircase” pattern, i.e.: charges in the lower rows can only be assigned to the rightmost column. Thus, a generic mesonic monopole mapped to $\varphi_{N_s-N+1+J}\bar{\varphi}_I$ ($1 \leq J \leq N$) will have the following flux assignment:

$$\varphi_{N_s-N+1+J}\bar{\varphi}_I \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots + + + \dots + +}^{N_s - N \quad k} \\ 0 \text{ GNO Flux} \end{array} \begin{array}{c} \nearrow^J \\ + \\ \vdots \\ + \\ 0 \end{array} \right) \quad (5.42)$$

$\varphi_I \bar{\varphi}_J$, **where** $N_s - N \leq J < I$: Mesons corresponding to values of $I \geq N_s - N + 1$ will instead have their +1 flux assigned to the rightmost column of the lower rows, and subsequent charge assignment will follow the usual staircase assignment. Thus, the meson $\varphi_I \bar{\varphi}_J$ ($N_s - N + 1 \leq J < I \leq N_s$) will be mapped to a monopole of the form:

$$\varphi_I \bar{\varphi}_J \leftrightarrow \left(\begin{array}{c} \overbrace{0 \dots 0}^{N_s - N} \overbrace{0 \dots 0}^k \\ \hline 0 \text{ GNO Flux} \end{array} \begin{array}{c} + \\ \vdots \\ + \\ 0 \end{array} \begin{array}{c} \swarrow \\ \vdots \\ \searrow \end{array} \begin{array}{c} I - J \\ \vdots \\ 0 \end{array} \right) \quad (5.43)$$

Generically, all the monopoles considered above will have spin-0 and spin-1 channels after they have been dressed - the spin-0 operators correspond to the off-diagonal elements of the meson matrix, while the spin-1 operators correspond to conserved currents in the original electric theory. We remind the reader that the diagonal elements $|\varphi_i|^2$ are mapped to mass deformations (refer Section 5.5 for a more detailed discussion on mass deformations) in the dual theory, while the diagonal currents map to topological $U(1)$ currents (refer Section 5.2.2 for an example of the dressing) in the mirror.

Baryons

Baryons are Lorentz scalars that transform in the rank- N_s antisymmetric representation of the flavor symmetry group $SU(N_s)$. We describe how these operators map to minimal flux operators in the planar mirror theory. We focus on the mapping of $\binom{N_s}{N}$ baryons to monopoles in the dual theory. Anti-baryons can be obtained by taking the Hermitian conjugate of these operators, which would map to monopoles with negative GNO fluxes.

$(\prod_{j=1}^{N-1} \varphi_j) \varphi_{N+I}$, **where** $0 \leq I \leq N_s + k - N - 1$: A generic minimal flux baryonic monopole operator **must** carry charge +1 under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram¹¹.

Accordingly, the baryon $\prod_{j=1}^N \varphi_j = \varphi_1 \dots \varphi_N$ is dual to the minimal flux configuration:

$$\prod_{j=1}^N \varphi_j \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline + \end{array} \right) \quad (5.44)$$

Thereafter, the charge assignment must propagate contiguously from left to right in a row. A baryon of the form $\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N+I} = \varphi_1 \dots \varphi_{N-1} \varphi_{N+I}$ is dual to the flux

¹¹The topological symmetry associated to this node is parametrized by $\sum_{k=1}^N X_k$, whereas all other nodes have topological symmetries parametrized by $X_i - X_{i+1}$. Thus, the group theoretic properties of a monopole acquiring charge +1 under this specific node follow from representation theory.

configuration:

$$\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N+I} \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline \underbrace{+ \ + \ \dots \ +}_I \end{array} \right) \quad (5.45)$$

As before, care must be taken if there is a BF-coupled region (of length k), and each of the k nodes inside the region of the quiver connected only by BF couplings must have GNO flux $+1$. For instance, the baryon $\varphi_1 \dots \varphi_{N-1} \varphi_{N_s-N+1}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-1} \varphi_j \right) \varphi_{N_s-N+1} \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline \underbrace{+ \ + \ \dots \ +}_{N_s-N+1} \underbrace{+ \ \dots \ +}_k \end{array} \right) \quad (5.46)$$

$(\prod_{k=1}^{N-J} \varphi_j)(\prod_{k=N-J+2}^{N+1} \varphi_k)$, **where** $1 \leq J \leq N-1$: The flux assignment in higher rows follows a “staircase” pattern. For instance, the baryon $\left(\prod_{j=1}^{N-2} \varphi_j \right) \varphi_N \varphi_{N+1} = \varphi_1 \dots \varphi_{N-2} \varphi_N \varphi_{N+1}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-2} \varphi_j \right) \varphi_N \varphi_{N+1} \leftrightarrow \left(\begin{array}{c} 0 \text{ GNO Flux} \\ \hline \begin{array}{ccccccc} & & & & & & \\ & + & & & & & \\ + & & & & & & \\ + & + & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array} \end{array} \right) \quad (5.47)$$

$\underbrace{\hspace{10em}}_{N_s-N+1} \quad \underbrace{\hspace{2em}}_k \quad \underbrace{\hspace{2em}}_N$

Thus, the baryon $\left(\prod_{k=1}^{N-J} \varphi_k \right) \left(\prod_{k=N-J+2}^{N+1} \varphi_k \right) = \varphi_1 \dots \varphi_{N-J} \varphi_{N-J+2} \dots \varphi_{N+1}$ is dual to the flux configuration:

$$\left(\prod_{k=1}^{N-J} \varphi_k \right) \left(\prod_{k=N-J+2}^{N+1} \varphi_k \right) \leftrightarrow \left(\begin{array}{c} 0 \hspace{10em} 0 \text{ GNO Flux} \\ \hline \begin{array}{ccccccc} & & & & & & \\ & \ddots & & & & & \\ & & + & & & & \\ & & & \ddots & & & \\ & & & & + & & \\ & & & & & \ddots & \\ & & & & & & + \\ & & & & & & + & + & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array} \end{array} \right) \quad (5.48)$$

$\underbrace{\hspace{10em}}_{N_s-N+1} \quad \underbrace{\hspace{2em}}_k \quad \underbrace{\hspace{2em}}_N$

Mapping a generic baryon: Unlike mesons, baryons allow column reuse: a single column may host fluxes in multiple rows, provided the staircase rule is respected. Thus, a generic baryonic monopole will have the following flux assignment:

$$\left(\prod_{m=1}^{N-J-1} \varphi_m \right) \left(\prod_{l=1}^J \varphi_{N+b_l-l} \right) \varphi_{N+I} \quad \leftrightarrow \quad \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & \overbrace{+ \dots +}^{b_J} & 0 & \dots & \\ & & \swarrow & & & \\ & & & \overbrace{+ \dots +}^{b_1} & 0 & \dots & \\ & & & \swarrow & & & \\ & & & & \underbrace{+ \dots +}_{I} & \dots & 0 & \dots \end{pmatrix} \quad (5.49)$$

We note that $k \geq 0$ does not change the counting of the mesonic and baryonic operators.

5.3 Planar Abelianization in CS-QCD₃ with Fermions and Bosons

In this section, we generalize the proposal of the previous section to non-Abelian theories with fundamental scalars and fermions. We anticipate that adding a fermionic flavor in the electric theory corresponds to “bosonizing” a column in the planar theory. We provide several examples of this here.

5.3.1 General Proposal

We propose the following planar, Abelian dual for $SU(N)$ QCD₃ with N_s scalars and N_f fermions at Chern-Simons level $2N - N_s - \frac{N_f}{2} - k$ in Figure 5.5. The proposed duality is believed to hold in the following range of parameters¹²:

$$N_s + N_f \geq 2N, \quad N_s \geq 2N, \quad k \geq 0 \quad (5.50)$$

This proposal should be seen as a generalization of the one considered for scalar QCD₃ obtained by replacing N_f scalars with N_f fermions and replacing the corresponding fermionic columns in the planar mirror with purely bosonic columns.

We briefly comment on the potential $\mathcal{V}$ shown in Figure 5.5. In addition to quartic scalar interactions (Equation 5.16), mixed scalar-fermion interactions are also generated radiatively in the electric QCD₃ theory and are captured by the electric potential $\mathcal{V}_{quartic}$:

$$\begin{aligned} \mathcal{V}_{quartic} = & \sum_{\alpha, \beta=1}^N \sum_{I, J=1}^{N_s} \sum_{A, B=1}^{N_f} [\varphi^{\alpha I} \bar{\varphi}_{\alpha J} \varphi^{\beta J} \bar{\varphi}_{\beta I} + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\varphi^{\beta J} \bar{\varphi}_{\beta J}) + \varphi^{\alpha I} \bar{\varphi}_{\beta I} \bar{\psi}_{\alpha A} \psi^{\beta A} \\ & + (\varphi^{\alpha I} \bar{\varphi}_{\alpha I})(\psi^{\beta A} \bar{\psi}_{\beta A})]. \end{aligned} \quad (5.51)$$

On the other hand, in addition to the usual Yukawa interactions, the monopole potential, and the GNY potential, an extra contribution is captured by $\mathcal{V}_{Planar}$ in the mirror

¹²We consider some examples with $N_s < 2N$ in Sections 5.6.1 and 5.6.2.

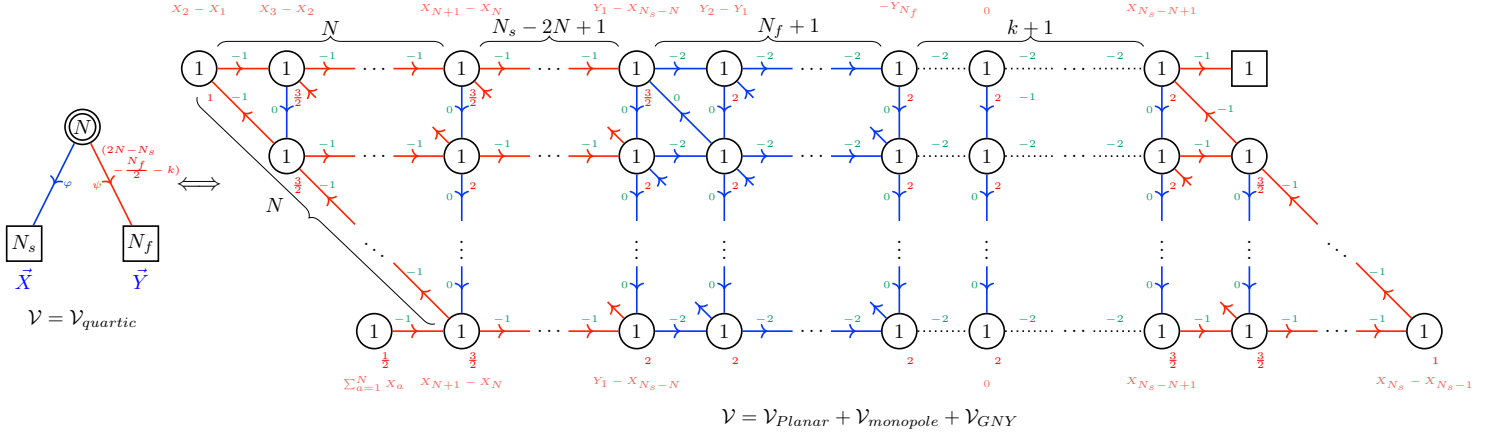


Figure 5.5: The proposed dual for $SU(N)$ QCD₃ with N_s scalars and N_f fermions at Chern-Simons level $2N - N_s - \frac{N_f}{2} - k$ is shown here. As before, single/double circles (squares) correspond to U/SU gauge (flavor) symmetries, scalar (fermion) bifundamental fields are indicated in blue (red), Chern-Simons levels of gauge nodes are indicated in maroon, while mixed BF couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. Note that the Chern-Simons level of each gauge node is equal to minus half the sum of all BF couplings with the surrounding nodes. We also indicate the fugacities of the topological symmetries of each column in orange.

theory. $\mathcal{V}_{Planar}$ contains a cubic scalar interaction ($\phi_i \phi_j \phi_k + c.c.$) for every closed planar triangle. There is a term with a coefficient of $+1$ (-1) for each triangle closed clockwise (anticlockwise). $\mathcal{V}_{monopole}$ involves every monopole with GNO fluxes $+1/-1$ under two nodes connected by a vertical scalar, analogous to the case discussed in Section 5.2, and are dressed as explained in Chapter 1. As before, we claim $\mathcal{V}_{monopole}$ and $\mathcal{V}_{Planar}$ are essential to match the rank of the global symmetry on the electric side. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)_{top}^{N_s} \times U(1)_{top}^{N_f}$, where each $U(1)$ factor corresponds to the topological symmetry of the $N_s + N_f - 1$ columns of the quiver, and the additional $U(1)$ factor is the topological symmetry of the bottom-most gauge node. We also claim the symmetry enhances in the IR:

$$U(1)_{top}^{N_s+N_f} \rightarrow U(N_s) \times U(N_f) \quad (5.52)$$

which is mapped precisely to the flavor symmetries of the electric theory. Furthermore, we expect the faithful symmetry to be [46]:

$$\frac{U(N_s) \times U(N_f)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^C \quad (5.53)$$

where $\mathbb{Z}_N$ is the center of $SU(N)$ and $\mathbb{Z}_2^C$ is a charge conjugation symmetry.

In the rest of this section, we make these ideas more precise by considering some examples.

5.3.2 Example: $SU(3)_{-\frac{1}{2}}$ QCD₃ with $N_s = 6$ Scalars and $N_f = 1$ Fermion

Let us begin with the example of $SU(3)_{-\frac{1}{2}}$ with 6 scalars and 1 fermion, and its planar dual description. We propose the duality shown in Figure 5.6.

We are mainly concerned with mapping the following gauge invariant operators of the electric theory:

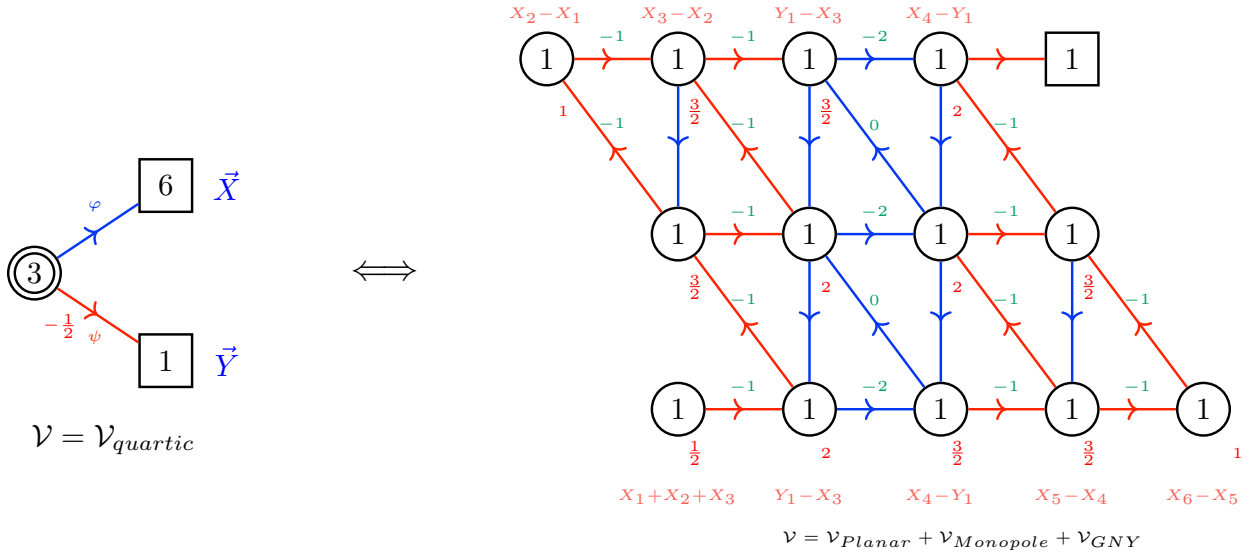


Figure 5.6: The dual for $SU(3)_{-\frac{1}{2}}$ QCD₃ with 6 scalars and 1 fermion is shown here. All labelling conventions are the same as Figure 5.5, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

1. There are 48 **mesonic operators** of which:

- (a) 35 spin-0 mesons of the form $\varphi_i \bar{\varphi}_j$, identified by $X_i - X_j$ (including their charge conjugates).
- (b) 12 spin- $\frac{1}{2}$ mesons of the form $\varphi_i \bar{\psi}$, identified by $X_i - Y_1$ (including their charge conjugates).
- (c) 1 meson with a spin-0 and spin-1 channel $\psi \bar{\psi}$.

As already discussed in the previous cases the mapping of diagonal mesons, which are neutral under the maximal torus of the global symmetry, is less straightforward, and we leave it to future work. The mapping of these off-diagonal mesons is given in Table 5.5.

2. There are many **baryonic operators**:

- (a) $\binom{6}{3} = 20$ spin-0 baryons of the form $\varphi_i \varphi_j \varphi_k$, identified by $X_i + X_j + X_k$.
- (b) 6 baryons with a spin-1 channel of the form $\psi \psi \phi_i$, identified by $2Y_1 + X_i$.
- (c) $\binom{6}{2} = 15$ spin- $\frac{1}{2}$ baryons of the form $\psi \varphi_i \varphi_j$, identified by $Y_1 + X_i + X_j$.
- (d) 1 baryon with spin- $\frac{3}{2}$ channels, identified by $3Y_1$, constructed by symmetrizing ψ in the flavor and spinor indices.

The mapping of baryons is given in Table 5.6.

Table 5.5: Dressed monopole operators of the planar mirror (Figure 5.6) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + \text{---} 0 & 0 & 0 & 0 \\ \text{---} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$	$\begin{pmatrix} + \text{---} + \text{---} 0 & 0 & 0 & 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_1$
$\begin{pmatrix} + \text{---} + \text{---} + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_1$	$\begin{pmatrix} + \text{---} + \text{---} + \text{---} 0 & + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_1$
$\begin{pmatrix} + \text{---} + \text{---} + \text{---} + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5 \bar{\varphi}_1$	$\begin{pmatrix} + \text{---} + \text{---} + \text{---} + \text{---} 0 & + \text{---} 0 & + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_6 \bar{\varphi}_1$
$\begin{pmatrix} 0 \text{---} + \text{---} 0 & 0 & 0 & 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_2$	$\begin{pmatrix} 0 \text{---} + \text{---} + \text{---} 0 & 0 & 0 & 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_2$
$\begin{pmatrix} 0 \text{---} + \text{---} + \text{---} + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_2$	$\begin{pmatrix} 0 \text{---} + \text{---} + \text{---} + \text{---} 0 & + \text{---} 0 & + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5 \bar{\varphi}_2$
$\begin{pmatrix} 0 \text{---} + \text{---} + \text{---} + \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_6 \bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 \text{---} + \text{---} 0 & 0 & 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 \text{---} + \text{---} + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 \text{---} + \text{---} + \text{---} 0 & + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 \text{---} + \text{---} + \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_6 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 \text{---} + \text{---} 0 & 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 & 0 \text{---} + \text{---} 0 \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5 \bar{\varphi}_4$	$\begin{pmatrix} 0 & 0 & 0 & 0 \text{---} + \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_6 \bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 \text{---} - \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_5$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_6 \bar{\varphi}_5$
$\begin{pmatrix} 0 & 0 & 0 \text{---} - \text{---} + \\ \text{---} 0 & \text{---} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi \bar{\varphi}_6$		

Continued on next page

Table 5.6 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_2\varphi_3\psi$	$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & \xrightarrow{\text{red}} + & + & \xrightarrow{\text{red}} 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \end{pmatrix}$	$\varphi_2\varphi_4\varphi_6$
$\begin{pmatrix} + & + & \xrightarrow{\text{red}} 0 & 0 \\ + & + & + & \xrightarrow{\text{red}} 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \end{pmatrix}$	$\varphi_3\varphi_4\varphi_6$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{red}} 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \end{pmatrix}$	$\varphi_4\varphi_6\psi$
$\begin{pmatrix} 0 & \xrightarrow{\text{red}} 0 & 0 & 0 & \xrightarrow{\text{red}} + \\ + & \xrightarrow{\text{red}} + & + & \xrightarrow{\text{red}} + & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_1\varphi_5\varphi_6$	$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 & \xrightarrow{\text{red}} + \\ + & \xrightarrow{\text{red}} + & + & \xrightarrow{\text{red}} + & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_2\varphi_5\varphi_6$
$\begin{pmatrix} + & + & \xrightarrow{\text{red}} 0 & 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_3\varphi_5\varphi_6$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \\ + & + & + & + & + \end{pmatrix}$	$\psi\varphi_5\varphi_6$
$\begin{pmatrix} + & + & + & + & \xrightarrow{\text{red}} + \\ + & + & + & + & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_4\varphi_5\varphi_6$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	ψ^3
$\begin{pmatrix} 0 & \xrightarrow{\text{red}} 0 & 0 & 0 & 0 \\ + & \xrightarrow{\text{red}} + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_1$	$\begin{pmatrix} + & \xrightarrow{\text{red}} 0 & 0 & 0 & 0 \\ + & \xrightarrow{\text{red}} + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_2$
$\begin{pmatrix} + & + & \xrightarrow{\text{red}} 0 & 0 & 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_3$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & + & + & \xrightarrow{\text{red}} 0 & 0 \end{pmatrix}$	$\psi^2\varphi_4$
$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \xrightarrow{\text{red}} + & \xrightarrow{\text{red}} 0 \end{pmatrix}$	$\psi^2\varphi_5$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 & 0 & \xrightarrow{\text{red}} + \\ + & + & + & + & \xrightarrow{\text{red}} + & \xrightarrow{\text{red}} + \end{pmatrix}$	$\psi^2\varphi_6$

5.3.3 Example: $SU(2)_{-\frac{5}{2}}$ QCD₃ with $N_s = 5$ Scalars and $N_f = 1$ Fermion

We now consider the example of $SU(2)_{-\frac{5}{2}}$ with 5 scalars and 1 fermion. Notice that this theory has $k = 1$. We propose the duality shown in Figure 5.7.

We are mainly concerned with the mapping of $\binom{5}{2} = 10$ spin-0 mesons, 5 spin- $\frac{1}{2}$ mesons, and $\binom{5}{2} = 10$ spin-0 baryons, 5 spin- $\frac{1}{2}$ baryons and 1 baryon with spin-1 channel. Since the details of the dressings are identical to the discussions preceding this section, we leave the complete operator map to the reader and provide the GNO fluxes of the bare monopoles are a representative example of the general trends of identifying these operators.

The mapping of the off-diagonal spin-0 mesons is provided in Equation 5.54 (the remaining mesons can be identified by charge conjugation):

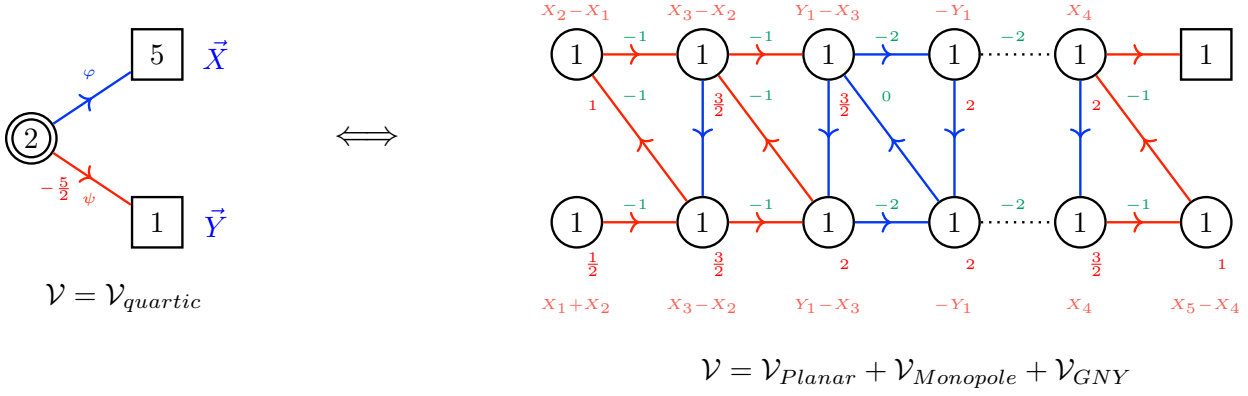


Figure 5.7: The dual for $SU(2)_{-\frac{5}{2}}$ QCD₃ with 5 scalars and 1 fermion is shown here. All labelling conventions are the same as Figure 5.5, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

$$\begin{aligned}
 \mathbf{M}|_{\text{spin-0}} \leftrightarrow & \left\{ \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\
 & \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
 & \mathfrak{M} \begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}, \\
 & \left. \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix} \right\}.
 \end{aligned} \tag{5.54}$$

A generic spin-0 mesonic monopole operator will be dressed as follows¹³:

$$\mathfrak{M} \begin{pmatrix} 0 \rightarrow + \leftarrow 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 \rightarrow + \leftarrow \cdots + \leftarrow 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.55}$$

The mapping of the off-diagonal spin- $\frac{1}{2}$ mesons is provided in Equation 5.56 (the remaining mesons can be identified by charge conjugation):

¹³We indicate the presence of only BF couplings in the second example by keeping a dotted line explicit between the two nodes.

$$\mathbf{M}|_{\text{spin}-\frac{1}{2}} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} + & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix} \right\}. \quad (5.56)$$

A generic spin- $\frac{1}{2}$ meson will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} 0 & 0 \xrightarrow{\text{red}} + \xleftarrow{\text{blue}} 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 \xrightarrow{\text{blue}} + \cdots + \xleftarrow{\text{red}} 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.57)$$

The mapping of the spin-0 baryons is provided in Equation 5.58 (the remaining antibaryons can be identified by charge conjugation):

$$\mathbf{B}|_{\text{spin}-0} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}, \right. \\ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}, \\ \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & + & + \\ + & + & + & + & + & + \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}, \\ \left. \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix} \right\}. \quad (5.58)$$

A generic spin-0 baryon will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} + \xleftarrow{\text{red}} 0 & 0 & 0 & 0 \\ + & + \xleftarrow{\text{red}} 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} 0 \xleftarrow{\text{red}} 0 & 0 \xleftarrow{\text{red}} 0 & 0 & 0 & 0 \\ + & + & + & + \cdots + \xleftarrow{\text{red}} 0 \end{pmatrix} \quad (5.59)$$

The mapping of the spin- $\frac{1}{2}$ baryons is provided in Equation 5.60 (the remaining antibaryons can be identified by charge conjugation):

$$\mathbf{B}|_{\text{spin-}\frac{1}{2}} \leftrightarrow \left\{ \mathfrak{M} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \mathfrak{M} \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix} \right\}. \quad (5.60)$$

A generic spin- $\frac{1}{2}$ baryon will be dressed as follows:

$$\mathfrak{M} \begin{pmatrix} + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 \\ + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}, \mathfrak{M} \begin{pmatrix} + & + & + \xleftarrow{\text{blue}} 0 & 0 \\ + & + & + & + \cdots + \xrightarrow{\text{red}} 0 \end{pmatrix} \quad (5.61)$$

Finally, the baryon ψ^2 obtained by symmetrizing the flavor and spin indices is mapped to the dressed monopole:

$$\mathfrak{M} \begin{pmatrix} + & + & + \xleftarrow{\text{blue}} 0 & 0 \\ + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix} \quad (5.62)$$

5.3.4 Mapping Gauge Invariant Operators

We take a moment to comment on higher-spin baryonic operators and their dual flux configurations here. Consider a generic baryon of the electric theory:

$$(\chi_1)^{i_1} (\chi_2)^{i_2} \cdots (\chi_F)^{i_F} \quad (5.63)$$

where each χ_i is either a boson or fermion, and the powers satisfy $\sum_{k=1}^F i_k = N$ ¹⁴. This baryon maps to the following monopole operator in the mirror theory:

$$\left(\begin{array}{ccccccc} \overbrace{+ \ + \ \cdots}^{j_1} & \overbrace{+ \ +}^{j_2} & \cdots & \overbrace{+ \ +}^{j_N} & \cdots & & \\ & + & \cdots & + & + & + & \\ & & & + & + & \cdots & \cdots \ 0 \ 0 \\ & & & \overbrace{+ \ +}^{j_{F-1}} & \cdots & \cdots \ 0 \ 0 \end{array} \right) \quad (5.64)$$

¹⁴N.B.: If multiple copies of the same scalar appear, derivatives must be inserted; otherwise, the anti-symmetric contraction vanishes.

Here, j_k denotes the number of $+$ in column k (excluding the base corner $+$ which is always present). The j_k are related to the baryonic charges i_k via:

$$\begin{cases} j_1 &= 1 - i_1 \\ j_2 &= 2 - i_1 - i_2 \\ \vdots & \\ j_N &= N - \sum_{k=1}^N i_k \\ j_{N+1} &= N - \sum_{k=1}^{N+1} i_k \\ \vdots & \\ j_{F-1} &= N - \sum_{k=1}^{F-1} i_k \end{cases} \quad (5.65)$$

some j_k may be negative—for instance, if $i_1 > 1$, then $j_1 < 0$. In such cases, negative fluxes (denoted by “ $-$ ” symbols) must appear in the corresponding column. The fluxes within each column should be distributed across rows in a way that ensures the resulting monopole is *minimally* dressed. We will see some explicit examples of this now.

Finally, we note that the number of spin- J baryons that can be built without derivatives in $SU(N)$ QCD₃ with N_s scalars and N_f fermions is $\binom{N_f+2J-1}{2J} \times \binom{N_s}{N-2J}$, constructed by (anti-)symmetrizing the (bosonic) fermionic fields. Consider in particular the spin- J baryons:

$$\psi_{\{i_1} \psi_{i_2} \dots \psi_{i_{2J}}\} \varphi_{[j_1} \varphi_{j_2} \dots \varphi_{j_{N-2J}}] \quad (5.66)$$

where the flavor indices of scalars j_n are antisymmetrized and the flavor indices of fermions i_n are symmetrized. The fermions are contracted to give a spin- J operator. This operator transform in the $2J$ -index symmetric representation of $SU(N_f)$ and in the $(N-2J)$ -index antisymmetric representation of $SU(N_s)$. The operator in the highest weight:

$$(\psi_1)^{2J} \varphi_1 \varphi_2 \dots \varphi_{N-2J} \quad (5.67)$$

is mapped to the following monopole on the mirror side:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & & & \\ & + & + & + & \dots & + \\ & & + & + & \dots & + \\ & & + & + & \dots & + \end{pmatrix} \left. \begin{matrix} \\ \\ \\ \end{matrix} \right\}^J \dots \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.68)$$

$\underbrace{\hspace{10em}}_{N_s - N}$

5.4 Planar Abelian Duals of Unitary CS-QCD₃ with Bosons and Fermions

We have so far focused on constructing duals of $SU(N)$ QCD₃ with mixed matter content. These results, however, extend naturally to the case of *unitary* QCD₃ by gauging the baryonic symmetry. From the discussion in Section 5.2, we know that the baryonic symmetry of the $SU(N)$ QCD₃ maps to the topological symmetry of the bottom-left gauge node of the planar-abelian quiver (Figure 5.2). Gauging of this topological symmetry with zero CS level introduces a $U(1)_0$ sector with no charged matter whose path integral results in a delta-function, which effectively ungauges the $U(1)$ bottom-left gauge symmetry. While

we did not keep track of background CS levels when presenting the duality in Figure 5.2, we expect the duality to involve a non-zero CS level for the baryonic symmetry. In analogy with the SUSY result of [2, 3], we expect a background CS at level -2 for the baryonic symmetry on the QCD side of 5.2. Therefore we propose that the dual description for $U(N)$ QCD₃ with N_s critical scalars and N_f fermions, where the theory is defined at Chern-Simons level $(2N - N_s - \frac{N_f}{2} - k, -N_s - \frac{N_f}{2} - k)$ is the planar-abelian quiver depicted in Figure 5.8.

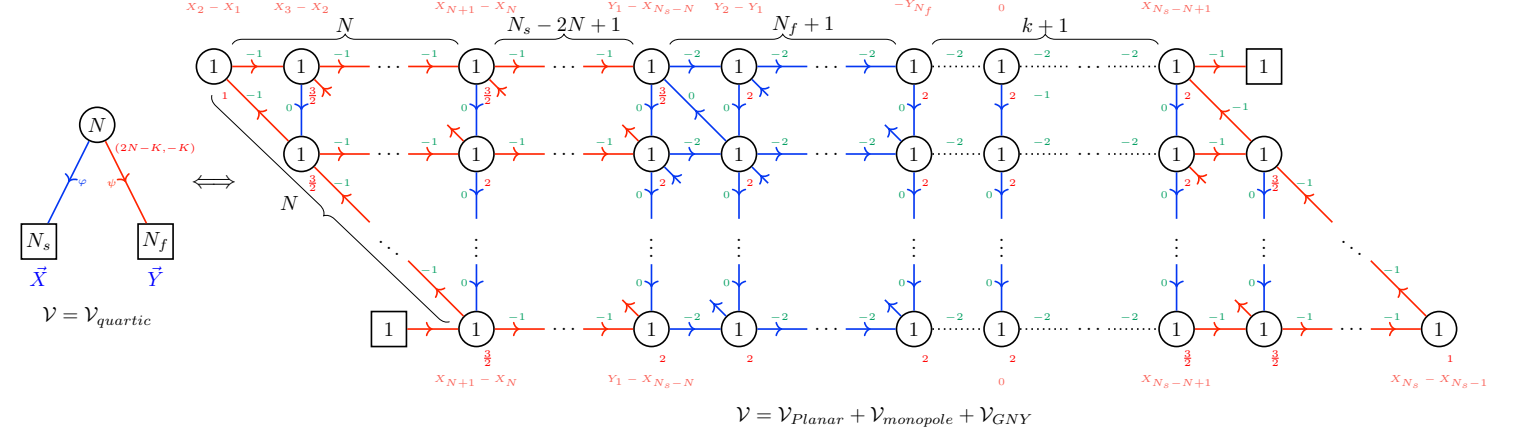


Figure 5.8: The dual of $U(N)$ QCD₃ with N_s scalars and N_f fermions at Chern-Simons level $(2N - K, -K)$, where for clarity we define $K = N_s + \frac{N_f}{2} + k$, is shown here. All labelling conventions are the same as Figure 5.5.

A particularly interesting special case arises when $k = 0$, shown in Figure 5.9. In this setting, the planar mirror theory contains a gauge-invariant mesonic operator built from bifundamental fields that connect the two $U(1)$ flavor nodes. This meson must map to a monopole operator in the electric $U(N)$ QCD₃ theory that is neutral under the $S(U(N_s) \times U(N_f))$ global symmetry. Consider the bare monopole $\mathfrak{M}$ with GNO fluxes $(+, 0, \dots, 0)$ in the $U(N)_{2N-N_s-\frac{N_f}{2}, -N_s-\frac{N_f}{2}}$ QCD₃ that transforms as the highest weight of the $(N_s - 2N)$ -index symmetric conjugate representation of $SU(N)$ and has charge N_s under the diagonal $U(1)$. This monopole can be made gauge invariant by dressing with N_s modes of the scalars φ , where two sets of N scalars are antisymmetrized in gauge indices and the other $N_s - 2N$ are symmetrized in gauge indices. By further symmetrizing all the scalars in $SU(N_s)$ indices we obtain operators that are neutral under $S(U(N_s) \times U(N_f))$ and have charge -1 under the topological symmetry, which are mapped to the long mesons described above. Therefore, we have the mapping:

$$\mathfrak{M}^+ \overbrace{\partial_\bullet \varphi_1^{[1]} \partial_\bullet \varphi_1^2 \dots \partial_\bullet \varphi_1^{N_s-2N}}^{N_s-2N} \overbrace{\varphi_1^{N_s-2N+1} \varphi_2^{N_s-2N+2} \dots \varphi_N^{N_s-N}}^N \overbrace{\varphi_1^{N_s-N+1} \varphi_2^{N_s-N+2} \dots \varphi_N^{N_s}}^N \leftrightarrow \text{long mesons} \quad (5.69)$$

where lower indices are $SU(N)$ gauge indices and upper indices are $SU(N_s)$ flavor indices, which are completely antisymmetrized. The modes with gauge index 1 “see” the GNO flux and have half-integer spin, while the other modes have spin-0. Then the spin of this operator mod 1 is given by $\frac{1}{2}(N_s - 2N + 2) \bmod 1$. This matches the integrality of the spin of the long mesons, because each meson contains an even (odd) number of fermions if N_s is even (odd).

Prior to gauging the baryonic $U(1)$ symmetry, one can also introduce a Chern-Simons

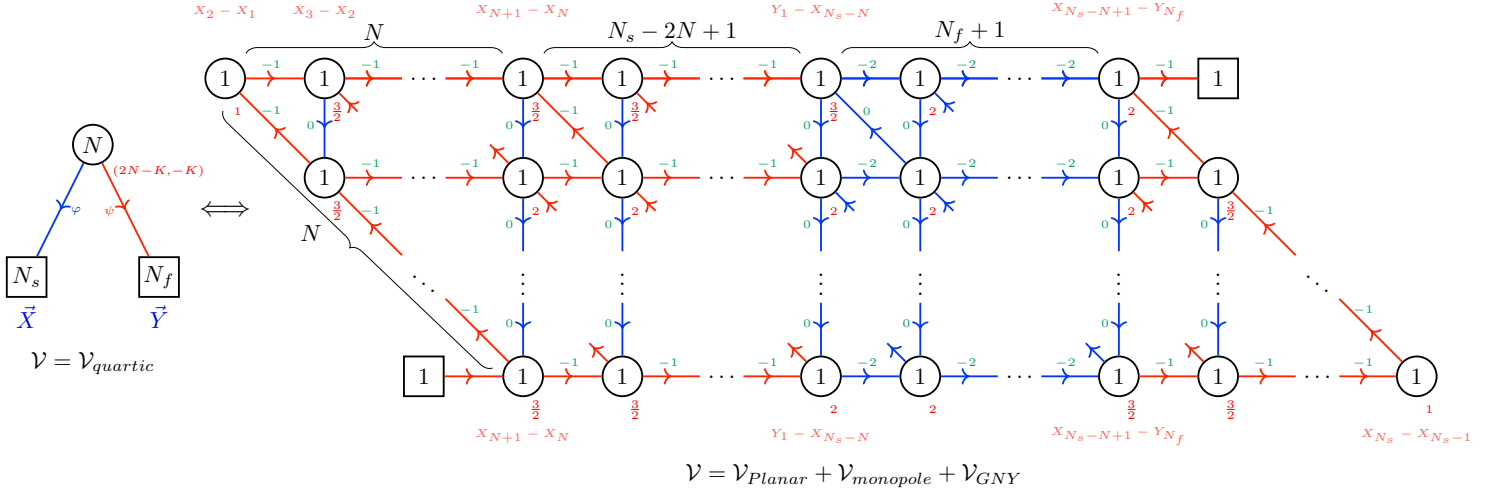


Figure 5.9: The dual of $U(N)$ QCD₃ with N_s scalars and N_f fermions at Chern-Simons level $(2N - K, -K)$, with $K = N_s + \frac{N_f}{2}$, is shown here. All labeling conventions follow those of Figure 5.5.

term for it at level Δ_ℓ , following the discussion in [106]. This leads to a $U(N)$ gauge theory with CS level $(2N - N_s - \frac{N_f}{2} - k, \Delta_\ell N - N_s - \frac{N_f}{2} - k)$. In the mirror dual, this corresponds to the addition of a new $U(1)_{\Delta_\ell}$ gauge node, coupled to the rest of the quiver via BF interactions. The resulting duality is depicted in Figure 5.10.

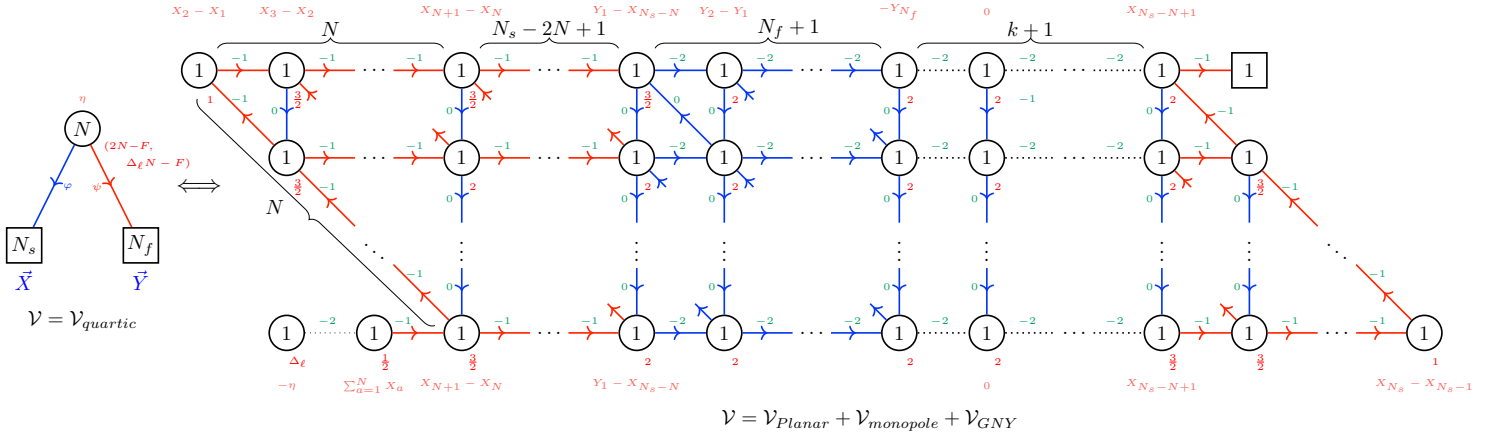


Figure 5.10: The dual of $U(N)$ QCD₃ with N_s scalars and N_f fermions at Chern-Simons level $(2N - F, \Delta_\ell N - F)$, with $F = N_s + \frac{N_f}{2} + k$, is shown here. All labeling conventions follow those of Figure 5.5.

5.5 Consistency Checks via RG Flow Trajectories

In this section, we focus on studying the mapping of mass deformations across dualities, which serves as an important consistency check for our proposed dualities. We consider mass deformations for the fundamental fields in the QCD theory and propose a corresponding deformation in the Abelian dual that is partly inspired by the structure of the supersymmetric dualities of [3, 4] and is a natural generalization of the Abelian case.

5.5.1 Mass for a Single Field

A simple deformation that can be studied in the QCD_3 theory on the LHS of Figure 5.5, consists of giving a mass to a boson or to a fermion that can be either positive or negative. One could think of it as a linear potential term for a mesonic operator of the form $|\varphi|^2$ or $\bar{\psi}\psi$. In principle, knowing the map of these operators across the duality 5.5 gives the knowledge of the deformation that is turned on in the dual theory. However, as we partially comment in Subsection 5.2.4, the map for these operators is not trivial¹⁵. Partially inspired by the SUSY ancestor of the duality, for which the deformations will be analyzed in [4], we consider examples to discuss massive deformations of the duality 5.5.

An Electric Scalar With A Positive Mass

Concretely, consider $SU(2)$ at CS level $4 - F$ with F scalars and its planar mirror dual (top row of Figure 5.11). Suppose we turn on a positive mass term $m^2|\phi_3|^2$ in the electric QCD_3 theory for one of the scalar fields (ϕ_3). We claim that this deformation maps to real mass terms for the fermions in one of the columns of the dual theory as:

$$m^2|\phi_3|^2 \quad \leftrightarrow \quad m(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3), \quad m > 0 \quad (5.70)$$

where ψ_1 and ψ_3 are the horizontal fermions in the second column and ψ_2 is the diagonal fermion in the second column. Upon integrating out the massive scalar field, the electric theory flows to $SU(2)$ QCD_3 with $F - 1$ scalars at CS level $4 - F$. On the mirror side, we integrate out the fermions in the second column, leaving behind only BF couplings between the horizontal gauge nodes, as shown in the bottom row of Figure 5.11. The resulting duality is exactly the one we propose for $SU(N)$ with $F - 1$ scalars at CS level $2N - k - (F - 1)$, where $N = 2$ and $k = 1$ in this case. Notice that this is a natural generalization of the Abelian case in Figure 5.1, where giving a positive mass to a boson on the QED side corresponds to giving a positive mass to a fermion in the quiver side. The choice of signs for the fermionic masses in (5.70) reproduces the BF couplings expected from the general proposal 5.2.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric scalar with a positive mass results in the integration of a column of fermions in the bulk of its planar mirror, leaving behind only horizontal BF couplings between gauge nodes.

One can immediately realize that the number of columns composed of fermions in the bulk of the dual quiver is $N_s - 2N$, where N_s is the number of scalars. Therefore, we can use this strategy to integrate out only $N_s - 2N$ bosons out of N_s . We can not use this strategy to flow to QCD_3 with $N_s < 2N$, which indeed is beyond the range of validity of the duality 5.5. In principle, one could study the effect of further massive deformations to go beyond the range $N_s \geq 2N$, this is a quite non-trivial exercise and we leave it for future work. However, we propose examples of dualities with $N_s < 2N$ in Section 5.6.

One could also consider a negative mass for an electric scalar, which would condense and Higgs the gauge group $SU(N)_{2N-N_s-k} \rightarrow SU(N-1)_{2(N-1)-(N_s-1)-k'}$, where $k' = k - 1$. We stress that the map of the deformation in (5.70) works only for $m > 0$. It would be interesting to see how this may be seen in the planar mirror theory, but it is beyond the scope of this work.

¹⁵We recall that the reason behind the difficulty of this map is that these operators are not charged under the Cartan subgroup of the global symmetry. Indeed, in the dual theory, there are a large number of operators, carrying the correct spin, that are not charged under the global symmetry, and in principle any linear combination of them is a valid candidate as the target of the map.

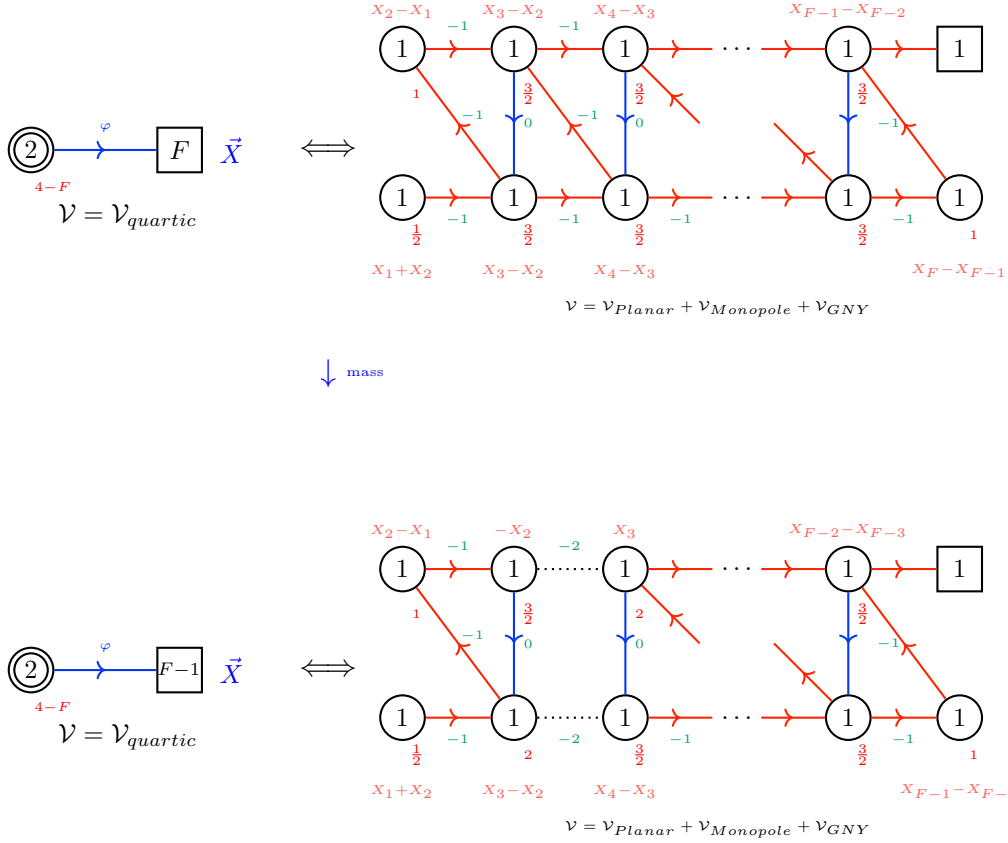


Figure 5.11: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single scalar field in $SU(2)_{4-F}$ QCD₃ with F scalar fields and the corresponding fermionic column in its planar dual (top row). Notice that CS and BF interactions have been generated according to the sign of the corresponding fermion masses. The resulting duality is exactly what we propose for $SU(2)$ with $F - 1$ scalars at CS level $4 - F$.

An Electric Fermion With A Positive Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 5.12). Suppose we turn on a positive mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a negative mass term for the horizontal bosonic fields in a column of the mirror theory:

$$m\bar{\psi}_{N_f}\psi_{N_f} \leftrightarrow -m^2(|\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2). \quad (5.71)$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD₃ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$. On the mirror side, the deformation causes the horizontal scalars to take a negative mass and thus a VEV, Higgsing pairs of $U(1)$ gauge groups connected by a horizontal scalar to the diagonal $U(1)$. This effectively identifies the two columns of $U(1)$ gauge groups to the left and right of the column of scalars. Furthermore, some of the cubic interactions for the scalars turn into effective masses, which allow one to integrate out the 2 diagonal scalars and 2 of the vertical scalars. The resulting quiver is shown in the bottom row of Figure 5.12. The resulting duality coincides with our proposed dual for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2}$, where $N = 3$ in this case.

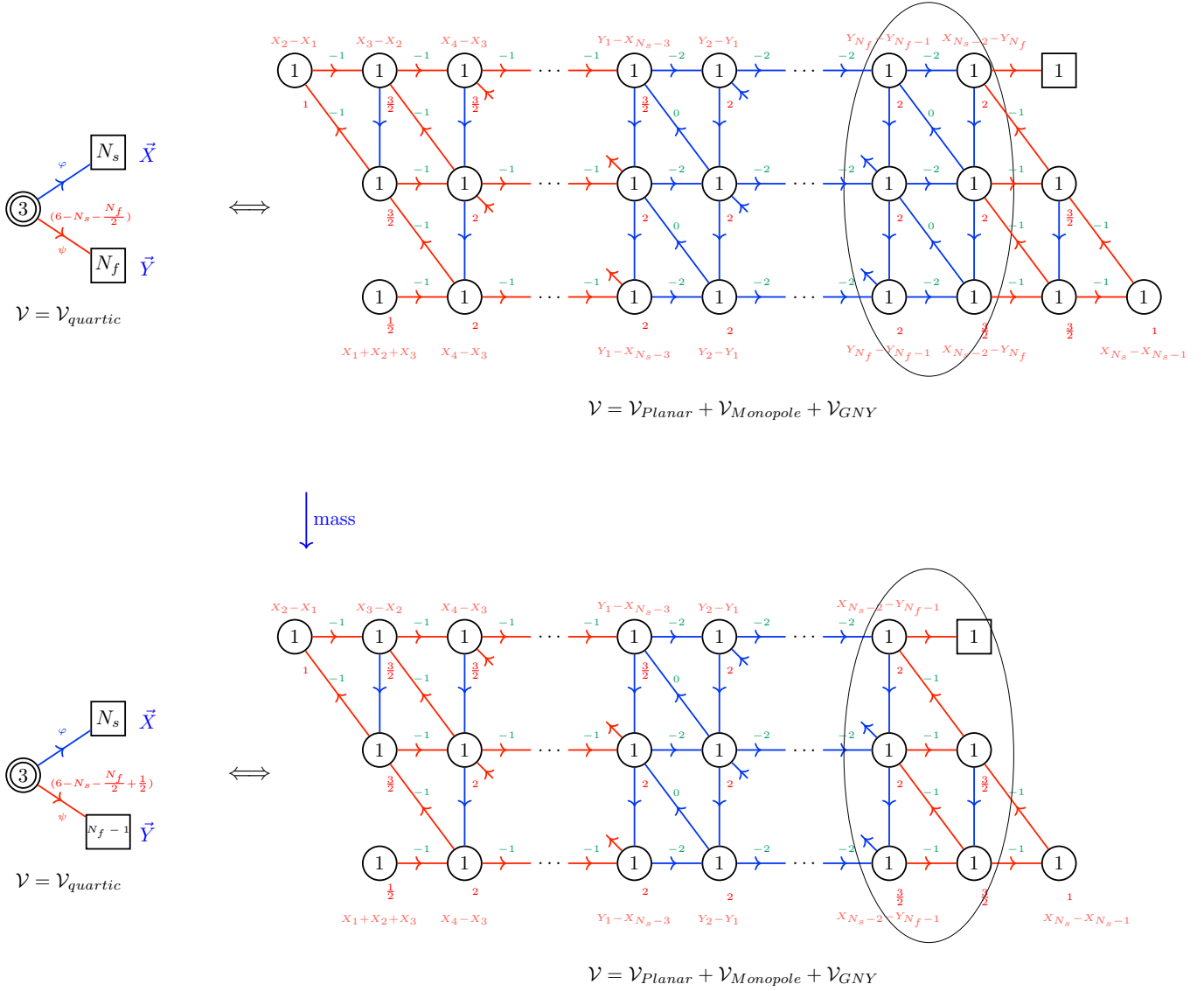


Figure 5.12: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with a positive mass in $SU(3)_{6-N_s-\frac{N_f}{2}} \text{QCD}_3$ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a positive mass results in Higgsing the horizontal bosons in a column in the bulk of its planar mirror, thereby shortening the quiver. One can check that CS and BF coupling of the resulting theory agree with our proposal (Figure 5.5), providing a consistency check of our results.

An Electric Fermion With A Negative Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 5.13). Suppose we turn on a negative mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a positive mass term for the bosonic fields in a column of the mirror theory:

$$-m\bar{\psi}_{N_f}\psi_{N_f} \quad \leftrightarrow \quad m^2(|\phi_{diag}^{(1)}|^2 + |\phi_{diag}^{(2)}|^2 + |\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2). \quad (5.72)$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD₃ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$. On the mirror side, the same deformations lead to the integration of a column of bosons, leaving behind only BF couplings between horizontal gauge nodes, as shown in the bottom row of Figure 5.13. The resulting duality is exactly the one we propose for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2} - k$, where $N = 3$ and $k = 1$ in this case.

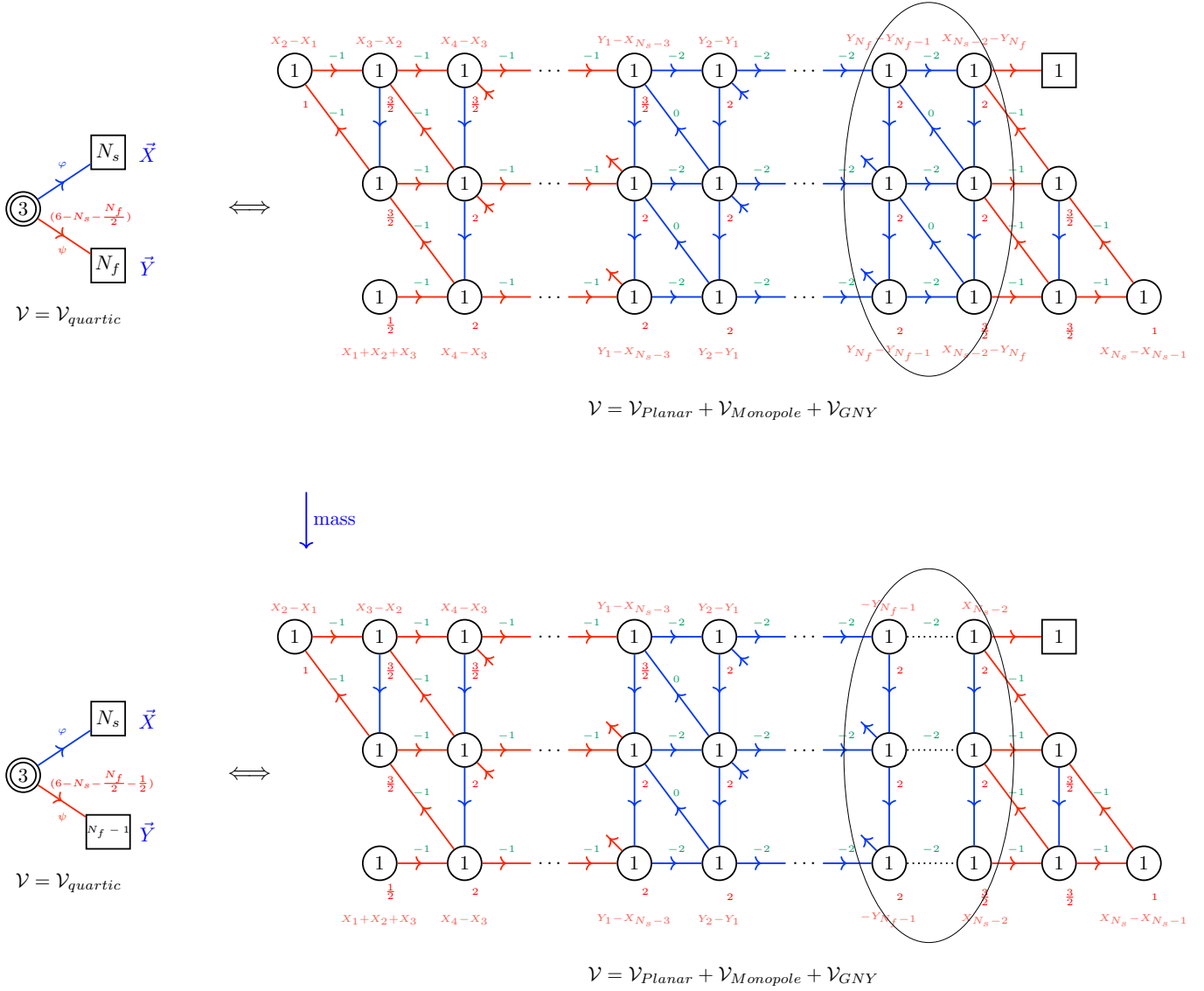


Figure 5.13: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with negative mass in $SU(3)_{6-N_s-\frac{N_f}{2}}$ QCD₃ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a negative mass results in the integration of a column of bosons in the bulk of its planar mirror, leaving behind only BF couplings between horizontal gauge nodes.

We conclude by mentioning that in the duality 5.5, each fermion in the QCD_3 corresponds to a column of bosons in the dual theory. Therefore, the results summarized in Figure 5.12 and 5.13 can be used to study the effect on the dual theory of a mass for any number of fermions in the QCD_3 .

5.5.2 Duals of Topological Quantum Field Theories

Instead of studying mass deformations of individual bosonic or fermionic fields, one may consider turning on masses for all matter fields simultaneously. In particular, we can give positive masses to all scalars and a homogeneous mass to all fermions in the QCD_3 theory. Such deformations generically gap out the matter sector and drive the theory to a purely topological infrared phase, described by a TQFT.

Since the original non-supersymmetric duality relates two theories that are expected to flow to the same IR fixed point, the corresponding mass-deformed theories should also flow to dual TQFTs. However, a direct analysis of these flows in the non-supersymmetric setting rapidly becomes complicated. A more tractable strategy is therefore to exploit the existence of supersymmetric “ancestor” dualities: one can study gapped phases of the corresponding $\mathcal{N} = 2$ theories, where the renormalization group flows are better controlled, and then infer the resulting TQFT dualities. We expect that the TQFTs obtained in this way coincide with those arising directly from mass-deforming the non-supersymmetric dual pairs.

A detailed analysis of these supersymmetric flows is presented in [5]. Here, we focus on the outcome in two representative cases, corresponding to SQCD theories whose Abelian planar duals contain either one or zero “bulk” columns.

The case of $SU(N)_{-1}$ with $2N$ scalars

We first consider the duality for $SU(N)_{-1}$ QCD with $N_s = 2N$ scalars and no fermions, shown in Figure 5.2. Assigning positive masses to all scalars gaps the matter fields and leaves behind a pure $SU(N)_{-1}$ Chern-Simons theory in the infrared.

On the Abelian side, the planar quiver dual contains a single empty bulk column (see Figure 5.2). This structure plays a crucial role in determining the resulting topological theory. More generally, the analysis described here applies to any QCD theory whose Abelian dual contains exactly one bulk column.

Rather than following the non-supersymmetric flow directly, we consider a chiral-planar $\mathcal{N} = 2$ duality studied in [3], in which the electric theory flows to the same $SU(N)_{-1}$ TQFT after appropriate mass deformations. In the supersymmetric setting, these deformations can be explicitly tracked on the planar Abelian side [4], leading to an Abelian Chern-Simons theory with gauge group $U(1)^2$ and Chern-Simons matrix

$$\mathbf{K} = \begin{pmatrix} N & -N \\ -N & N+1 \end{pmatrix}. \quad (5.73)$$

This TQFT duality is represented diagrammatically in Figure 5.14. We expect that the

same Abelian TQFT is obtained by mass-deforming the non-supersymmetric duality for $SU(N)_{-1}$ with $2N$ scalars and no fermions.

$$\begin{array}{c} \textcircled{N-1} \times \textcircled{N} \\ \textcolor{red}{1} \quad \textcolor{red}{-1} \end{array} \quad \textcolor{red}{\longleftrightarrow} \quad \begin{array}{c} \textcircled{1} \times \textcircled{1} \\ \textcolor{red}{1} \quad \textcolor{red}{N} \end{array} \quad \textcolor{green}{\longleftrightarrow} \quad \begin{array}{c} \textcircled{1} \cdots \textcircled{1} \\ \textcolor{red}{N} \quad \textcolor{red}{N+1} \end{array} \quad \textcolor{green}{-2N}$$

Figure 5.14: Duality between $SU(N)_{-1} \times U(N-1)_1$ CS theory and the Abelian CS theory associated to the CS matrix (5.73). (Mixed) Chern-Simons interactions are indicated in (green) red. The red arrow indicates level/rank duality for $SU(N)_{-1}$, while the double arrow indicates a duality between Abelian TQFTs described in the main body.

In the remainder of this section, we provide an independent check of this claim by deriving the duality using known TQFT equivalences.

TQFT matching and level/rank duality

The Abelian Chern–Simons theory associated with the matrix $\mathbf{K}$ has a $\mathbb{Z}_N^{(1)}$ one-form symmetry, as can be determined from the Smith normal form of $\mathbf{K}$ [261]. This matches the one-form symmetry of the $SU(N)_{-1}$ theory on the electric side.

The equivalence between the Abelian theory defined by $\mathbf{K}$ and $SU(N)_{-1}$ can be established in two steps. First, one applies level/rank duality to $SU(N)_{-1}$, as indicated by the red arrow in Figure 5.14. Second, one relates the resulting Abelian theories, represented by the double arrow in the same figure.

This second step requires constructing integer-valued 2×2 matrices $\mathbf{Q}$ and $\mathbf{P}$, following the framework of [261], such that

$$\mathbf{Q}^T \mathbf{K}_L^{-1} \mathbf{Q} - \mathbf{K}^{-1} = \mathbf{P}, \quad (5.74)$$

where $\mathbf{K}_L^{-1} = \text{diag}(1, 1/N)$ is the inverse Chern–Simons matrix of the intermediate Abelian theory obtained after level/rank duality, and $\mathbf{K}^{-1}$ is the inverse of (5.73).

One choice satisfying these conditions is

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad (5.75)$$

where the diagonal entries of $\mathbf{P}$ are even, ensuring that the duality preserves the spin structure.

The matrix $\mathbf{Q}$ specifies a bijective mapping between the $2|\det(\mathbf{K})|$ independent Wilson lines of the two theories. Explicitly, if $\vec{\alpha}$ and $\vec{\alpha}_L$ denote the charge vectors of Wilson lines on the dual and electric sides respectively, they are related by

$$\vec{\alpha}_L = \mathbf{Q} \vec{\alpha}. \quad (5.76)$$

The sequence of dualities shown in Figure 5.14 also requires the inclusion of an additional invertible $U(N-1)_1$ TQFT on the left-hand side. It would be interesting to track the precise invertible backgrounds generated by deforming the quiver theory directly, in order to fully characterize the corresponding symmetry-protected topological (SPT) factors. We leave this analysis to future work.

The case of $SU(N)_0$ with $2N$ scalars

A similar analysis applies to $SU(N)_0$ QCD with $N_s = 2N$ scalars. Giving positive masses to all scalars drives the theory to pure $SU(N)_0$ Yang–Mills in the infrared. The corresponding supersymmetric analysis, together with its planar Abelian dual, predicts that the dual theory flows to a $U(1)_1$ Chern–Simons theory. This is an almost trivial TQFT, in agreement with the expected infrared behavior of the electric theory.

5.6 Probing the $N_s < 2N$ Regime

In this section, we attempt to study theories where the number of scalars N_s is less than $2N$. We find that we are able to consistently *bosonize* two columns of height $N - 1$ outside of the bulk of the quiver gauge theory, which means that in the duality (5.5) we replace the tallest column in the two triangular sections with a column made only by bosons. Each replacement of columns using this strategy consists of decreasing/increasing the number of bosons/fermions by 1 in duality (5.5). We emphasize that this can be done to reduce the number of bosons by 2 at most.

This strategy allows us to propose a quiver which is dual to a QCD_3 theory with $N_s = 2N - 1$ or $N_s = 2N - 2$ scalars and N_f fermions such that $N_s + N_f \geq 2N$, generalizing the dualities discussed in Section 5.3. Extending this procedure to even lower N_s by considering quivers with more bosonic columns in the outer triangle would require a generalization of our prescription to construct monopole interactions. We leave this to future work.

We illustrate this proposal using two examples, one with $N_s = 2N - 1$ and one with $N_s = 2N - 2$. In particular, we notice that the $SU(2)$ QCD_3 when $N_s = 2$ and $N_f \geq 2$ is dual to a completely bosonic planar theory.

5.6.1 Example: $SU(3)_0$ QCD_3 with $N_s = 5$ Scalars and $N_f = 2$ Fermions

We consider the example of $SU(3)_0$ with 5 scalars and 2 fermions, and its planar dual description. We propose the duality shown in Figure 5.15. Notice that the second-to-last column has been “bosonized” in the sense that it is made only of scalar fields, similar to the column in the bulk.

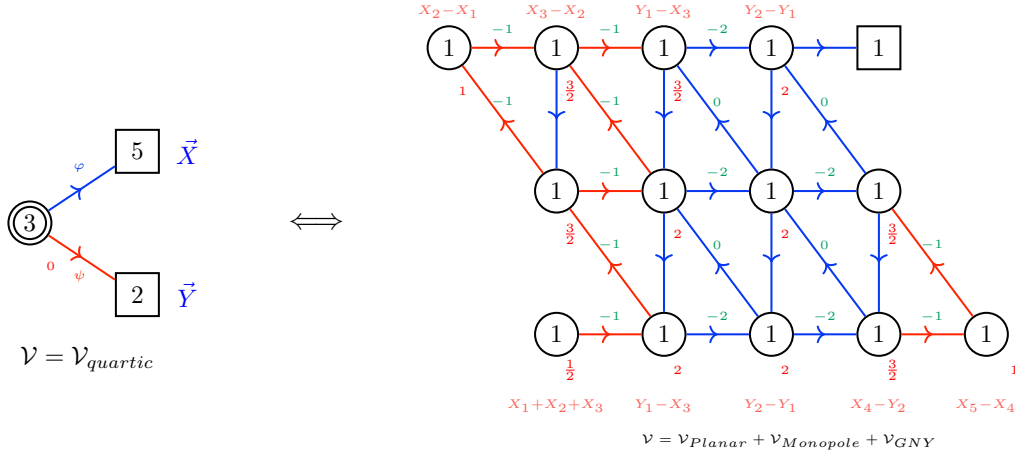


Figure 5.15: The dual for $SU(3)_0$ QCD₃ with 5 scalars and 2 fermions is shown here. All labelling conventions are the same as Figure 5.5, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

Following the spirit of Sections 5.2 and 5.3 we propose a map between mesons and baryons with monopole operators, which is reported in Tables 5.7 and 5.8 respectively. We reiterate that the monopoles reported there are representatives of all monopole operators with given global charge and spin.

Table 5.7: Dressed monopole operators of the planar mirror (Figure 5.15) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_1$
$\begin{pmatrix} + & \text{---} & + & + & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_1 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{---} & + & + & + & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_2 \bar{\varphi}_1$
$\begin{pmatrix} + & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & + & + & + & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_1$	$\begin{pmatrix} + & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & + & + & + & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \text{---} & + & 0 & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{---} & + & + & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_1 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \text{---} & + & + & + & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_2 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & + & + & + & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4 \bar{\varphi}_2$
$\begin{pmatrix} 0 & \text{---} & 0 & 0 & 0 & 0 \\ & \text{---} & + & + & 0 & 0 \\ & & 0 & 0 & + & + \end{pmatrix}$	$\varphi_5 \bar{\varphi}_2$	$\begin{pmatrix} 0 & \text{---} & 0 & + & 0 & 0 \\ & \text{---} & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_1 \bar{\varphi}_3$

Continued on next page

Table 5.7 – continued from previous page

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} 0 & 0 \xrightarrow{\text{red}} + & + \xleftarrow{\text{blue}} & & \\ & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_2 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 \xrightarrow{\text{red}} 0 & 0 & & \\ & 0 \xrightarrow{\text{red}} + & + & + \xrightarrow{\text{red}} 0 & \\ & 0 & 0 & 0 & \end{pmatrix}$	$\varphi_4 \bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 \xrightarrow{\text{red}} 0 & 0 & 0 \xrightarrow{\text{red}} + & \\ & 0 \xrightarrow{\text{red}} + & + & + & \end{pmatrix}$	$\varphi_5 \bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 \xleftarrow{\text{blue}} - \xrightarrow{\text{blue}} & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_1 \bar{\psi}_2$
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 \xleftarrow{\text{blue}} - & 0 \xrightarrow{\text{red}} 0 & \\ & 0 & 0 & 0 & 0 & \end{pmatrix}$	$\psi_2 \bar{\varphi}_4$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 \xleftarrow{\text{blue}} - & & \\ & 0 & 0 & 0 & 0 \xrightarrow{\text{red}} - & \end{pmatrix}$	$\psi_2 \bar{\varphi}_5$
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 \xleftarrow{\text{blue}} - & - & 0 \xrightarrow{\text{red}} 0 & \\ & 0 & 0 & 0 & 0 & \end{pmatrix}$	$\psi_1 \bar{\varphi}_4$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 \xrightarrow{\text{red}} 0 & \\ & 0 & 0 & 0 & 0 \xrightarrow{\text{red}} + & \end{pmatrix}$	$\varphi_5 \bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 \xrightarrow{\text{red}} - & \\ & 0 & 0 \xleftarrow{\text{blue}} - & - & 0 \xrightarrow{\text{red}} - & \end{pmatrix}$	$\psi_1 \bar{\varphi}_5$		

Table 5.8: Dressed monopole operators of the planar mirror (Figure 5.15) and their putative mapping to baryonic operators of the electric theory (all indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \varphi_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \psi_1$
$\begin{pmatrix} 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_3 \psi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_2 \psi_2$
$\begin{pmatrix} 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \varphi_3 \psi_2$	$\begin{pmatrix} 0 \xrightarrow{\text{red}} 0 & 0 \xrightarrow{\text{red}} 0 & 0 & 0 & & \\ & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \psi_1 \psi_2$
$\begin{pmatrix} + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \varphi_3 \psi_2$	$\begin{pmatrix} + \xrightarrow{\text{red}} 0 & 0 \xrightarrow{\text{red}} 0 & 0 & 0 & & \\ & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \psi_1 \psi_2$
$\begin{pmatrix} + & + \xrightarrow{\text{red}} 0 & 0 & 0 & & \\ & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 & \\ & + & + & + \xleftarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_3 \psi_1 \psi_2$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + & + \xrightarrow{\text{red}} 0 & \end{pmatrix}$	$\varphi_1 \varphi_2 \varphi_4$
$\begin{pmatrix} 0 \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + & + \xrightarrow{\text{red}} 0 & \end{pmatrix}$	$\varphi_1 \varphi_3 \varphi_4$	$\begin{pmatrix} + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & & \\ & + \xrightarrow{\text{red}} 0 & 0 & 0 & 0 & \\ & + & + & + & + \xrightarrow{\text{red}} 0 & \end{pmatrix}$	$\varphi_2 \varphi_3 \varphi_4$

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Continued on next page

Table 5.8 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & + \xrightarrow{\text{red}} 0 & 0 & & \\ & + & + \xrightarrow{\text{blue}} 0 & 0 & \\ & + & + \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\psi_1^2 \varphi_3$	$\begin{pmatrix} + & + & + \xrightarrow{\text{blue}} 0 & & \\ & + & + \xrightarrow{\text{blue}} 0 & 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_1^2 \psi_2$
$\begin{pmatrix} + & + & + \xrightarrow{\text{blue}} 0 & & \\ & + & + \xrightarrow{\text{blue}} 0 & 0 & \\ & + & + & + \xrightarrow{\text{red}} 0 & \end{pmatrix}$	$\psi_1^2 \varphi_4$	$\begin{pmatrix} + & + & + \xrightarrow{\text{blue}} 0 & & \\ & + & + \xrightarrow{\text{blue}} 0 & 0 \xrightarrow{\text{red}} + & \\ & + & + & + & \end{pmatrix}$	$\psi_1^2 \varphi_5$
$\begin{pmatrix} + & + & + & + \xrightarrow{\text{blue}} & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	ψ_2^3	$\begin{pmatrix} 0 \xrightarrow{\text{red}} & 0 \xrightarrow{\text{red}} & 0 & 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_2^2 \varphi_1$
$\begin{pmatrix} + \xrightarrow{\text{red}} 0 & 0 & 0 & & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_2^2 \varphi_2$	$\begin{pmatrix} + & + \xrightarrow{\text{red}} 0 & 0 & & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_2^2 \varphi_3$
$\begin{pmatrix} + & + & + \xrightarrow{\text{blue}} 0 & & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_2^2 \psi_1$	$\begin{pmatrix} + & + & + & + \xrightarrow{\text{blue}} & \\ & + & + & + \xrightarrow{\text{blue}} 0 & \\ & + & + & + & \xrightarrow{\text{red}} 0 \end{pmatrix}$	$\psi_2^2 \varphi_4$
$\begin{pmatrix} + & + & + & + \xrightarrow{\text{blue}} & \\ & + & + & + \xrightarrow{\text{blue}} 0 \xrightarrow{\text{red}} + & \\ & + & + & + & \end{pmatrix}$	$\psi_2^2 \varphi_5$		

5.6.2 Example: $SU(2)_{\frac{1}{2}}$ QCD₃ with $N_s = 2$ Scalars and $N_f = 3$ Fermions

We finally consider the example of $SU(2)_{\frac{1}{2}}$ with 2 scalars and 3 fermions, and its planar dual description. We propose the duality shown in Figure 5.16. Notice that the planar dual is completely bosonic.

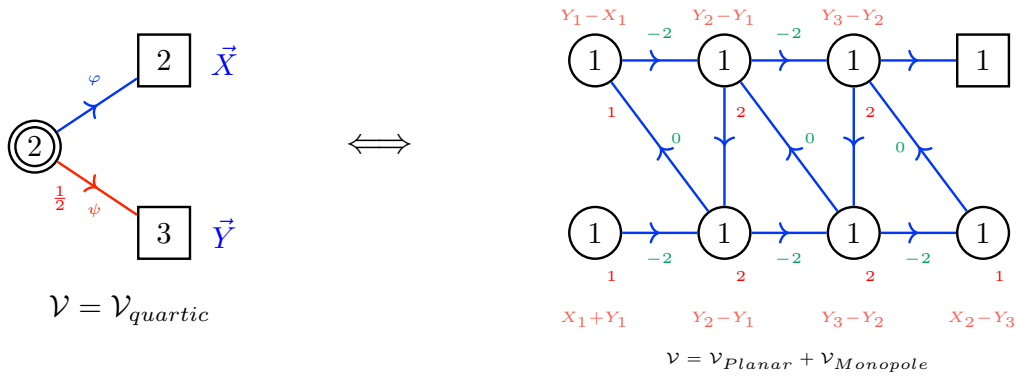


Figure 5.16: The dual for $SU(2)_{\frac{1}{2}}$ QCD₃ with 2 scalars and 3 fermions is shown here. All labelling conventions are the same as Figure 5.5, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

As before, we are concerned with determining the monopole operators dual to the mesons and baryons, and we report the results in Tables 5.9 and 5.10 respectively.

Table 5.9: Dressed monopole operators of the planar mirror (Figure 5.16) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_1 \bar{\varphi}_1$	$\begin{pmatrix} + & + & \xrightarrow{\text{blue}} 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_2 \bar{\varphi}_1$
$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_3 \bar{\varphi}_1$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} + \\ 0 & 0 & 0 & \xrightarrow{\text{blue}} + \end{pmatrix}$	$\varphi_2 \bar{\varphi}_1$
$\begin{pmatrix} 0 & \xrightarrow{\text{blue}} + & \xrightarrow{\text{blue}} 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_2 \bar{\psi}_1$	$\begin{pmatrix} 0 & \xrightarrow{\text{blue}} + & + & \xrightarrow{\text{blue}} 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_3 \bar{\psi}_1$
$\begin{pmatrix} 0 & 0 & 0 & + \\ 0 & \xrightarrow{\text{blue}} + & + & + \end{pmatrix}$	$\varphi_2 \bar{\psi}_1$	$\begin{pmatrix} 0 & 0 & \xrightarrow{\text{blue}} + & \xrightarrow{\text{blue}} 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi_3 \bar{\psi}_2$
$\begin{pmatrix} 0 & 0 & 0 & + \\ 0 & 0 & \xrightarrow{\text{blue}} + & + \end{pmatrix}$	$\varphi_2 \bar{\psi}_2$	$\begin{pmatrix} 0 & 0 & 0 & \xrightarrow{\text{blue}} + \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2 \bar{\psi}_3$

Table 5.10: Dressed monopole operators of the planar mirror (Figure 5.16) and their putative mapping to baryonic operators of the electric theory. The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	$\varphi_1 \psi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\varphi_1 \psi_2$
$\begin{pmatrix} 0 & 0 & 0 & \xrightarrow{\text{blue}} 0 \\ + & + & + & + \end{pmatrix}$	$\varphi_1 \psi_3$	$\begin{pmatrix} 0 & 0 & 0 & + \\ + & + & + & + \end{pmatrix}$	$\varphi_1 \varphi_2$
$\begin{pmatrix} + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & + & \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	$\psi_1 \psi_2$	$\begin{pmatrix} + & \xrightarrow{\text{blue}} 0 & 0 & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 \end{pmatrix}$	$\psi_1 \psi_3$
$\begin{pmatrix} + & \xrightarrow{\text{blue}} 0 & 0 & + \\ + & + & + & + \end{pmatrix}$	$\varphi_2 \psi_1$	$\begin{pmatrix} + & + & \xrightarrow{\text{blue}} 0 & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 \end{pmatrix}$	$\psi_2 \psi_3$
$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & + \end{pmatrix}$	$\varphi_2 \psi_2$	$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} + \\ + & + & + & + \end{pmatrix}$	$\varphi_2 \psi_3$
$\begin{pmatrix} + & \xrightarrow{\text{blue}} 0 & 0 & 0 \\ + & \xrightarrow{\text{blue}} 0 & 0 & 0 \end{pmatrix}$	ψ_1^2	$\begin{pmatrix} + & + & \xrightarrow{\text{blue}} 0 & 0 \\ + & + & \xrightarrow{\text{blue}} 0 & 0 \end{pmatrix}$	ψ_2^2
$\begin{pmatrix} + & + & + & \xrightarrow{\text{blue}} 0 \\ + & + & + & \xrightarrow{\text{blue}} 0 \end{pmatrix}$	ψ_3^2		

Notice that some of the monopole operators in Table 5.10 seem to factorize, in the sense that they can be thought of as the product of two independent gauge invariant monopoles. We recall, however, that there are additional monopoles with the same global charges and spin, which will, in general, mix. The monopoles reported in Table 5.10 are representatives

of such linear combinations.

In line with our observation in the last section, we can consistently “bosonize” the entire quiver. This is a special feature of $SU(2)$, and we anticipate that the planar dual of $SU(2)_{2-\frac{N_f}{2}-k}$ ($k \geq 0$) with 2 scalars and N_f fermions will be a fully bosonic Abelian quiver gauge theory.

Scope and Future Directions

We restrict ourselves to the study of $SU(N)$ QCD₃ with at least $N_s = 2N - 2$ bosonic fundamental flavors and any number of fermionic fundamental flavors. Extending our proposal to study duals for smaller values of N_s would be the next step. Preliminary results indicate this can be achieved by utilizing the machinery of [5] for the $\mathcal{N} = 2$ supersymmetric case and subsequently considering the $\mathcal{N} = 0$ limit. Due to this, we can safely say the current non-supersymmetric paradigm generalizes to unitary as well as symplectic CS-QCD₃; however, undertaking a systematic study to catalog these dualities is beyond the scope of the current work.

Another promising direction of research would be to develop an algorithmic paradigm to construct these bosonization dualities, in a vein similar to their $\mathcal{N} = 2$ [3, 27] and $\mathcal{N} = 4$ [213] counterparts. Such an approach would help construct planar Abelian duals for broader classes of non-Abelian gauge theories.

Part V: Conclusions and Outlook



Chapter 6

Conclusions and Outlook

Mirror symmetry for $\mathcal{N} = 4$ theories admits a compelling and intuitive interpretation in terms of Type IIB string theory and Hanany–Witten brane constructions. In this setting, the exchange of Higgs and Coulomb branches, as well as the corresponding interchange of flavor and topological symmetries, is geometrically manifest. However, this intuition ceases to apply once the amount of supersymmetry is reduced. For theories with $\mathcal{N} < 4$, the Higgs and Coulomb branches may be lifted or collapse to isolated vacua, and the canonical brane realizations underlying mirror symmetry are often absent. As a result, *geometric engineering alone is no longer a reliable guide*.

The primary goal of this thesis has been to investigate whether mirror-like dualities can nevertheless persist in theories with $\mathcal{N} = 2$, and even in the absence of supersymmetry. Despite the breakdown of string-theoretic intuition, symmetry considerations remain sufficiently powerful to uncover and test such dualities. In particular, we find that the defining structural feature of mirror symmetry — namely, the *exchange of topological and flavor symmetries* — continues to provide a robust organizing principle beyond the $\mathcal{N} = 4$ setting.

From this perspective, mirror symmetry should be viewed more generally as a duality that maps *perturbative, order-type operators* to *non-perturbative, disorder-type operators*, such as monopole operators, rather than as a consequence of a specific brane construction. This viewpoint allows mirror symmetry to be formulated and tested intrinsically within quantum field theory, opening the door to systematic explorations of dualities in theories with reduced, or no, supersymmetry.

A central result of this thesis is the development of the paradigm of **Planar Abelianization**, which proves especially compelling for several reasons:

- The resulting $\mathcal{N} = 2$ mirror duals are Abelian *and* admit a Lagrangian description — two highly nontrivial and *a priori* independent properties.
- The construction admits straightforward generalizations to A_N and D_N quivers, theories with symplectic and special unitary gauge groups, circular quivers, and models with rich matter content, including adjoint fields, rank-2 antisymmetric tensors, as well as the presence of both fundamental and anti-fundamental representations [3, 6].
- The framework enables systematic deformations, yielding theories with more general Chern–Simons levels and flavor content [5]. For special choices of parameters,

the resulting theories display a rich infrared phenomenology, such as s -confinement, chiral symmetry breaking, and the emergence of non-Abelian TQFTs.

- The paradigm further admits extensions beyond supersymmetry, encompassing non-supersymmetric dualities [4].

In addition to the future directions discussed at the end of each chapter, we now briefly outline several further avenues of investigation suggested by the results of this thesis.¹

The Road to Orthosymplectic Mirror Symmetry

Mirror duals of $\mathcal{N} = 4$ orthogonal and symplectic SQCD can be engineered using *Feng-Hanany* brane setups [238], which incorporate orientifold planes in addition to the standard Hanany–Witten configurations. These constructions typically give rise to quivers with alternating orthogonal and symplectic gauge groups. Even in the presence of $\mathcal{N} = 4$ supersymmetry, analyzing these theories remains highly nontrivial. This complexity arises from the emergence of various symmetries acting on the moduli space in the deep infrared, which are absent in the UV Lagrangian and hence invisible to supersymmetric localization.

A number of tools, such as the *magnetic quivers* paradigm, have been developed to study these theories [239, 240, 262, 263], but a complete and detailed analysis remains intricate.

Both $\mathcal{N} = 4$ $U(N)$ and $USp(2N)$ SQCD with F hypermultiplets admit RG flows under supersymmetry-breaking deformations to $\mathcal{N} = 2$ vacua in which the gauge symmetries are Higgsed to their maximal tori, $U(1)^N$. Finding these vacua is nontrivial, but they form the building blocks of algorithmic planar dual constructions, which proceed by gluing together “*planarized*” $U(N)$ [3] or $USp(2N)$ [245] blocks.

A natural question is whether $SO(2N)$ and $SO(2N + 1)$ SQCD with F hypermultiplets also admit such flows. The answer is affirmative: these theories likewise reach $\mathcal{N} = 2$ vacua in which the gauge symmetries are Higgsed to $U(1)^N$, as illustrated in Equation 6.1. This observation is important because, even if a fully planar Abelianization of orthosymplectic quivers is not yet known, the existence of these planar limits suggests that a proto-algorithm for constructing orthosymplectic planar duals may still be feasible. These may also be regarded as candidates for the RG-walls [264] of SYM theories based on orthogonal and symplectic gauge groups.

¹These are broadly based on projects started and subsequently abandoned by the author.

$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$	$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$	$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$	$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$
real ↓ mass	real ↓ mass	real ↓ mass	real ↓ mass
$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$	$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$	$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$	$\mathcal{W} = \mathcal{W}_{\text{orth}} + \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$

(6.1)

Towards non-Abelian $2d \mathcal{N} = (2, 2)$ Mirrors

Mirror symmetry also admits a rich realization in two dimensions. In contrast to its three-dimensional avatar [199, 265], *two-dimensional mirror symmetry* does not involve monopole operators or topological symmetries. Instead, it relates seemingly different descriptions of $\mathcal{N} = (2, 2)$ supersymmetric quantum field theories. In its simplest form, it maps a non-linear sigma model with a Kähler target space to a Landau–Ginzburg theory with a superpotential, exchanging geometric data with purely holomorphic information [113, 114, 187, 266–268]. From a physical perspective, this duality exchanges momentum and winding modes and interchanges complex-structure and Kähler moduli, providing a powerful non-perturbative equivalence between theories with very different ultraviolet descriptions.

It is natural to expect a deeper connection between three-dimensional and two-dimensional mirror symmetry [34]. However, the reduction of three-dimensional dualities to two-dimensional ones is a delicate matter [269], for several reasons.

- Supersymmetric theories with four supercharges often possess a classical moduli space of vacua (at least for suitable values of their parameters), and two-dimensional theories are no exception. This moduli space can be modified or lifted by quantum corrections. In higher dimensions, each point on the quantum moduli space corresponds to a distinct vacuum lying in a different superselection sector, and the low-energy theory at a generic point is free, while interacting dynamics arise only

near singular loci. In contrast, in two dimensions quantum fluctuations are strong enough that the ground state (and, more generally, all states) explore the entire target space or “moduli space.”

- The sigma-model metric is classically marginal in two dimensions and is generically renormalized. As a result, it cannot be neglected in the analysis of infrared dynamics. This issue is particularly acute for non-compact target spaces or in the presence of runaway directions, where supersymmetric configurations may lie at infinite distance. In such cases, there may be no normalizable ground state, and instead a continuum of states can extend down to zero energy, depending on the asymptotic behavior of the Kähler potential. Such continua can contribute to supersymmetric indices in ways that differ qualitatively from the contributions of isolated vacua.
- A phenomenon unique to two dimensions is the possibility that a single ultraviolet supersymmetric theory can flow to more than one superconformal field theory in the infrared. More precisely, the infrared limit may be a direct sum of decoupled SCFTs, so that the Hilbert space is a sum rather than a tensor product of the individual Hilbert spaces.

These features become particularly significant when compactifying three-dimensional $\mathcal{N} = 2$ theories on a circle. In this context, different ways of taking the radius of the circle to zero can lead to distinct two-dimensional conformal field theories lying on the same conformal manifold.

Constructing mirror duals of two-dimensional $\mathcal{N} = (2, 2)$ theories thus remains a formidable challenge. A variety of complementary approaches have been developed, including analyses based on A- and B-twists, constraints from K-theory [270–273], and the study of topologically twisted indices [146, 274].

The approach we advocate builds on the results of [275–278]. Upon supersymmetric localization of three-dimensional theories on $\mathbf{S}^2 \times \mathbf{S}^1$ or $\mathbf{D}^2 \times \mathbf{S}^1$, the resulting partition functions admit distinct asymptotic limits as the radius of $\mathbf{S}^1$ is taken to zero. The precise infrared limit depends on how real mass parameters and Fayet–Iliopoulos terms are scaled in this process.

Concretely, two inequivalent limits can be defined:

$$\textbf{Higgs limit:} \quad \lim_{q \rightarrow 1} \frac{(q^x; q)_\infty}{(q^y; q)_\infty} = \frac{\Gamma(y)}{\Gamma(x)} (1 - q)^{y-x},$$

$$\textbf{Coulomb limit:} \quad \lim_{q \rightarrow 1} \frac{(zq^x; q)_\infty}{(zq^y; q)_\infty} = (1 - z)^{y-x}.$$

In the *Higgs limit*, matter fields remain light and the effective theory localizes onto the Higgs branch. The resulting partition function reduces to that of a two-dimensional theory on $\mathbf{S}^2$ [279–282] or on $\mathbf{D}^2$ [230, 283, 284], depending on the geometry.

By contrast, in the *Coulomb limit* all matter fields become heavy and are integrated out. The resulting expression can be identified with a Dotsenko–Fateev conformal block [285–289], and, *with some significant mathematical gymnastics*, can be recast into the twisted superpotential of a Landau–Ginzburg model with a logarithmic twisted superpotential.

For a pair of three-dimensional mirror dual theories, the two limits described above are not independent. Mirror symmetry exchanges the Higgs and Coulomb branches, and therefore taking the Coulomb limit of one theory necessarily forces the Higgs limit of its mirror dual,

and vice versa. As a result, mirror symmetry in three dimensions descends to a nontrivial equivalence between distinct two-dimensional limits of the same parent theory.

It is therefore natural to apply this framework to the chiral–planar mirror dualities studied in this work. By taking the Coulomb limit on one side of the duality and the Higgs limit on the other, one obtains pairs of $\mathcal{N} = (2, 2)$ partition functions which are expected to coincide. Studying these limits provides a concrete and quantitative probe of mirror symmetry in two dimensions and offers a promising route toward identifying and testing mirror duals of non-Abelian $\mathcal{N} = (2, 2)$ theories.²

Tetrality in Non-Abelian (p, q) -Webs

Brane configurations for five-dimensional $\mathcal{N} = 1$ gauge theories were first constructed in [177], following the general framework pioneered in [19], using Type IIB $D5$ - and $NS5$ -branes. In Type IIB string theory, fivebranes transform as a (p, q) multiplet under the $SL(2; \mathbb{Z})$ duality group, where the $D5$ -brane corresponds to a $(0, 1)$ fivebrane and the $NS5$ -brane to a $(1, 0)$ fivebrane.

At the classical level, a configuration in which a $D5$ -brane ends on an $NS5$ -brane preserves eight supercharges. Quantum mechanically, however, this picture is modified: the $NS5$ -brane bends due to a Coulomb-like interaction sourced by the attached $D5$ -brane [173, 296, 297]. A more accurate description is that charge conservation forbids a pure $D5$ -brane from ending on a pure $NS5$ -brane. Instead, at the junction the two combine into a bound state carrying both charges, known as a $(1, 1)$ *fivebrane*.

More generally, bound states of (p, q) fivebranes can be formed whenever the (p, q) charges are conserved. The resulting webs of (p, q) fivebranes provide a powerful geometric framework for engineering five-dimensional $\mathcal{N} = 1$ gauge theories. Analyzing the interacting five-dimensional worldvolume theories supported on these configurations is an interesting and formidable problem, which we will not mention beyond this.

(p, q) fivebranes also play an important role in the geometric engineering of three-dimensional gauge theories, where they naturally give rise to Chern–Simons interactions. From this perspective, it is natural to expect that the *chiral–planar mirror duals* studied in this thesis may admit a Type IIB realization in terms of brane tilings constructed from (p, q) fivebranes. Such a description would provide a geometric underpinning for these dualities and merits a renewed and systematic investigation of their properties.

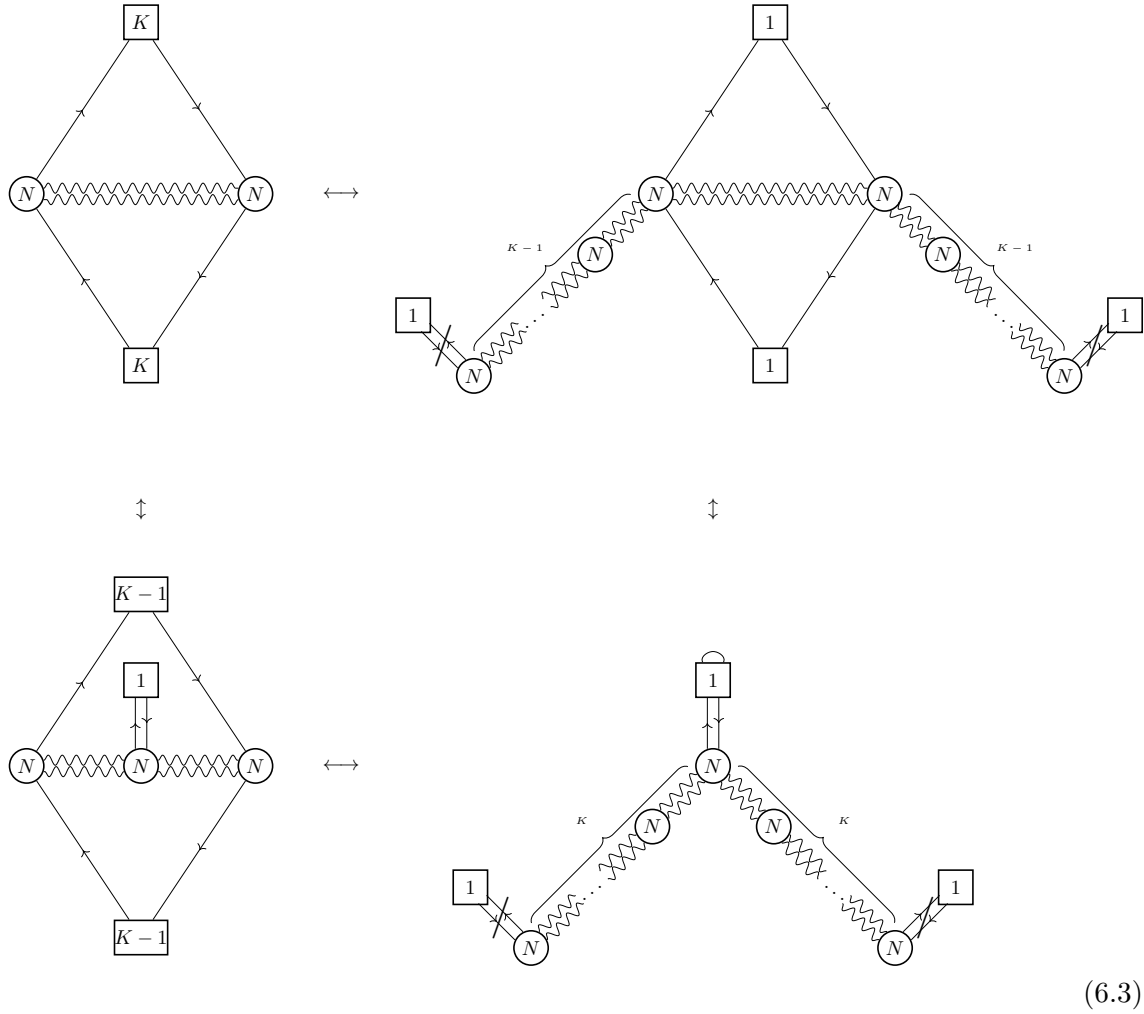
[205] studied the low-energy effective theory on the worldvolume of two $D3$ -branes ending on a $(1, K)$ -web on each side. Such $\mathcal{N} = 2$ theories display a non-trivial *tetrality*, shown in Equation 6.2, where we have suppressed the details of the superpotential for brevity.

²These problems are interesting mathematical questions in their own right and may provide insight into toric fibrations over Grassmannian manifolds and into the interplay between branes, mirror symmetry, and toric geometry [290–295].

(6.2)

By invoking the $FM[U(N)]$ *generalized bifundamental block*³[27, 28], it is possible to generalize this tetrality to $\mathcal{N} = 2$ non-Abelian gauge theories, shown in Equation 6.3. We have once again suppressed the details of the superpotential for brevity.

³The $FM[U(N)]$ theory is a (famously self-mirror) 3d $\mathcal{N} = 2$ SCFT which displays an $SU(N) \times SU(N) \times U(1)^2$ global symmetry at its infrared fixed point. The gauge invariant spectrum of this theory includes two adjoint operators and two bifundamental operators charged under the two $SU(N)$ global symmetries. Theories built with *superconformal matter* as building blocks are inherently non-Lagrangian, and these theories are no exception.



It would be interesting to evaluate the connection of such constructions to our *chiral-planar mirror dual pairs*, and to study the low-energy effective theory of a more general (p, q) -web suspended between N $D3$ -branes.

Towards a Chiral-Planar Realization of the Dualization Algorithm

We remarked earlier that Type IIB string theory enjoys an $SL(2; \mathbb{Z})$ duality symmetry, and that mirror symmetry for $\mathcal{N} = 4$ theories admits a natural interpretation in terms of the action of the $\mathcal{S}$ generator. For $\mathcal{N} = 4$ gauge theories, the $\mathcal{S}$ transformation is implemented by coupling to the $FT[U(N)]$ theory.

The $\mathcal{T}$ generator of $SL(2; \mathbb{Z})$, on the other hand, corresponds to a shift of the Chern–Simons level of a background gauge field [106]. As in the case of $\mathcal{S}$, the action of $\mathcal{T}$ admits a local interpretation on brane configurations.

Building on this observation, a systematic $\mathcal{N} = 4$ dualization algorithm was introduced in [213], in which general duality transformations are constructed as sequences of $\mathcal{S}$ and $\mathcal{T}$ operations acting locally on brane configurations. This framework reveals a large web of three-dimensional dualities unified by the action of a single $SL(2; \mathbb{Z})$ structure.

We emphasize at the outset that the $SL(2; \mathbb{Z})$ action on three-dimensional $\mathcal{N} = 2$ theories

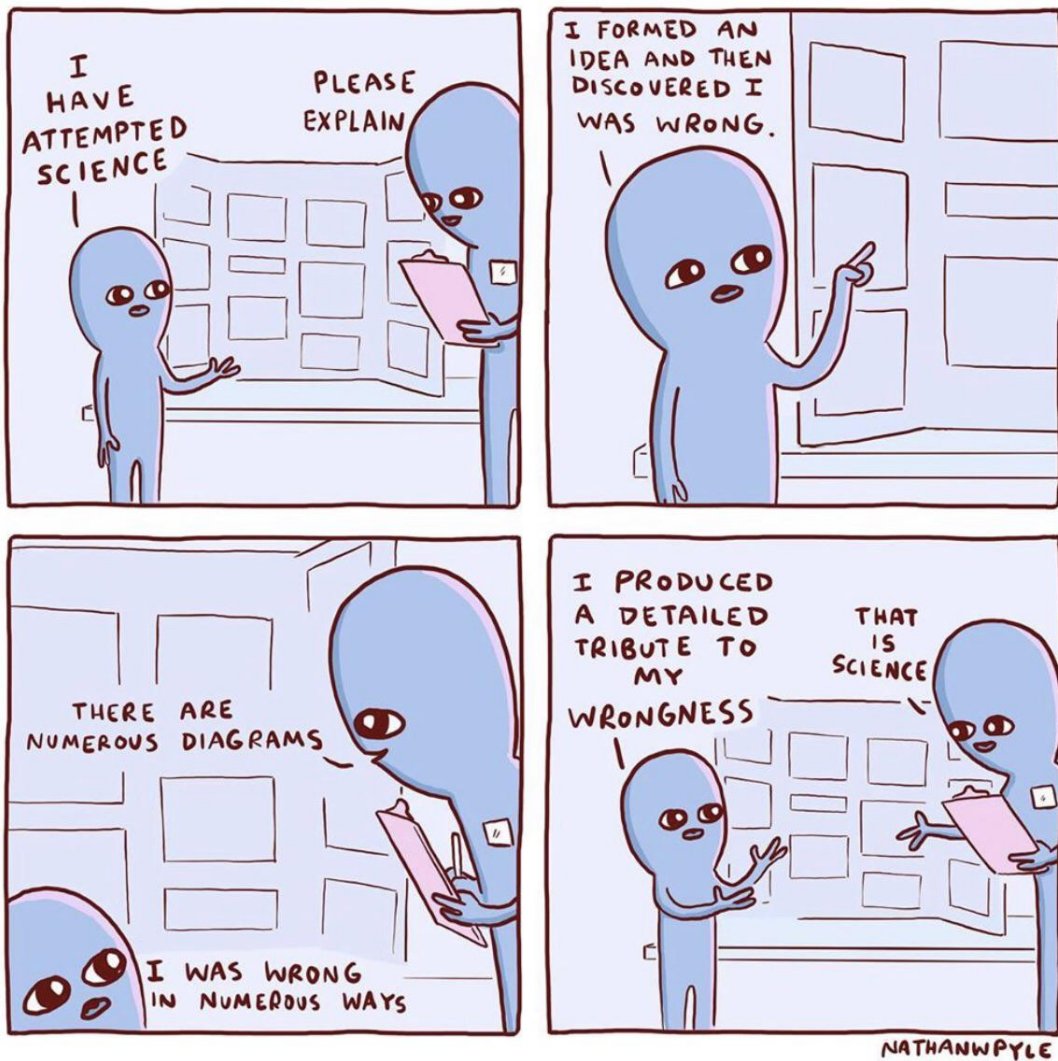
is of course already well understood in the sense of [106], and has been used to generate dualities in a number of constructions [146, 298, 299]. What we have in mind here is narrower: not the $SL(2; \mathbb{Z})$ action itself, but a *chiral-planar realization of the generators* appearing in the $\mathcal{N} = 4$ dualization algorithm of [213], obtained by taking the chiral-planar limit of that construction.

We have already encountered the action of the $\mathcal{S}$ generator in the context of $\mathcal{N} = 2$ theories in this thesis. It is therefore natural to ask whether the paradigm of planar Abelianization can be extended to construct $\mathcal{N} = 2$ analogues of the $\mathcal{T}$ generator.⁴

It should be stressed that this falls short of establishing an $\mathcal{N} = 2$ dualization algorithm proper. There is no *a priori* reason for the full $SL(2; \mathbb{Z})$ group to persist with reduced supersymmetry, and verifying the defining relation $(\mathcal{ST})^3 = \mathbb{I}$ by direct computation has so far proved prohibitively involved. It nevertheless seems reasonable to expect that a non-trivial subgroup survives and continues to organize these theories. Should this be the case, such a framework would generate large classes of dualities among three-dimensional theories with reduced supersymmetry, and could connect our $\mathcal{N} = 2$ *chiral-planar mirror duals* to other existing constructions [27, 177, 202, 298–300].

⁴We have found an $\mathcal{N} = 2$ realization of the $\mathcal{T}$ operation in terms of chiral and planar blocks, and work in progress by Vittorio Cagioni has made further progress towards a chiral-planar description of $\mathcal{T}^T$.

Part VI: Appendices



Appendix A

Higher Dimensional Constructions

Many of the structures and dualities encountered in lower-dimensional supersymmetric gauge theories admit a unifying interpretation when viewed from a higher-dimensional perspective. In particular, a large class of four- and three-dimensional supersymmetric quantum field theories can be obtained by compactifying the six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theory on suitably chosen manifolds. This viewpoint underlies the construction of *Class S* theories, as well as the AGT correspondence relating supersymmetric gauge theories to two-dimensional conformal field theories.

The purpose of this appendix is to provide a brief and self-contained review of these ideas, with an emphasis on their conceptual role rather than on technical details. We summarize the essential features of the six-dimensional $\mathcal{N} = (2, 0)$ theory, its compactification on Riemann surfaces leading to *Class S* constructions, and the basic statement of the AGT correspondence. This discussion is intended solely to place the results of the main text in a broader theoretical context, and we will not attempt an exhaustive or pedagogical treatment of these subjects.

A.1 SCFTs in Six Dimensions

Six-dimensional superconformal field theories play a distinguished role in the landscape of supersymmetric quantum field theories, often serving as *master theories* from which many lower-dimensional constructions descend. This privileged status stems from the fact that superconformal algebras do not exist in spacetime dimensions $D > 6$ [52]. The obstruction arises from the additional constraints imposed by superconformality on Lie superalgebras: the bosonic subalgebra $\mathfrak{g}_0$ must act on the fermionic generators $\mathfrak{g}_1$ in a spinorial representation [88]. Remarkably, this requirement is satisfied only marginally in six dimensions. The triality properties of the complexified conformal group $SO(6, 2; \mathbb{C})$ allow vector representations of $\mathfrak{g}_0$ to be interchanged with spinorial representations of $\mathfrak{g}_1$, thereby permitting the existence of superconformal symmetry. No such construction is possible in higher dimensions, rendering six dimensions the upper critical dimension for interacting SCFTs.

The on-shell representations of six-dimensional supersymmetry are well known and consist of the following multiplets [301, 302]:

- the **hypermultiplet**, containing four real scalars and two spin- $\frac{1}{2}$ fermions;
- the **vector multiplet**, containing a vector field and two gaugini;

- the **tensor multiplet**, containing a real scalar, a spin- $\frac{1}{2}$ fermion, and a two-form gauge field with anti-self-dual field strength;
- the **gravity multiplet**, containing the graviton, two gravitini, and a chiral two-form gauge field with self-dual field strength.

The requirement that the theory admit a conserved stress-energy tensor places strong restrictions on the allowed amount of supersymmetry. In six dimensions, this implies that interacting superconformal theories can have at most sixteen supercharges, corresponding to $\mathcal{N} = (1, 0)$, $\mathcal{N} = (2, 0)$, or $\mathcal{N} = (1, 1)$ supersymmetry.

Six-dimensional SCFTs are intrinsically non-gravitational theories. As such, they are built from combinations of hypermultiplets, vector multiplets, and tensor multiplets, but do not include dynamical gravity. A crucial and somewhat counterintuitive feature of these theories is that, unlike in lower dimensions, interacting conformal field theories in six dimensions do not generally admit a Lorentz-covariant Lagrangian description. Indeed, only trivial examples—such as the free scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2$$

—possess a conventional local action.

In particular, $\mathcal{N} = (2, 0)$ superconformal theories necessarily contain tensor multiplets, whose anti-self-dual two-form fields obstruct the existence of a manifestly Lorentz-covariant action. As a result, these theories are best understood through indirect means, such as their operator content, symmetry structure, and protected observables, rather than through a microscopic Lagrangian formulation.

This conceptual impasse was resolved during the Second String Revolution, which revealed a geometric and string-theoretic origin for six-dimensional SCFTs [303]. The key insight is that such theories can arise as decoupling limits of string or M-theory compactifications: by compactifying extra dimensions, decoupling gravity, and tuning parameters to special loci in moduli space, one encounters regimes in which effective strings become tensionless. The resulting interacting fixed points furnish precisely the six-dimensional superconformal field theories that lack a conventional Lagrangian description.

A remarkable property of six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theories is that interacting theories are classified by simply-laced Lie algebras of ADE type. That is, there exists a family of $\mathcal{N} = (2, 0)$ SCFTs labeled by $\mathfrak{g} = A_{N-1}, D_N, E_6, E_7$, and E_8 , with no known interacting theories outside this classification.

This ADE classification admits a natural geometric interpretation in string and M-theory. $\mathcal{N} = (2, 0)$ theories of type $\mathfrak{g}$ arise as the low-energy worldvolume theories of coincident M5-branes, or equivalently from Type IIB string theory compactified on ALE spaces with ADE singularities. In this picture, the appearance of ADE algebras is tied to the classification of isolated $\mathbb{C}^2/\Gamma_{\text{ADE}}$ singularities, and the tensionless strings of the six-dimensional theory originate from M2-branes wrapping vanishing two-cycles at the singular locus.

Although six-dimensional $\mathcal{N} = (2, 0)$ SCFTs do not admit a conventional Lagrangian description, their physical content becomes accessible upon compactification. Compactifying these theories on a circle produces maximally supersymmetric five-dimensional Yang–Mills theories, while compactification on lower-dimensional manifolds gives rise to a rich web of supersymmetric quantum field theories in four and three dimensions. In particular, many dualities and non-perturbative phenomena observed in lower-dimensional supersymmetric gauge theories can be understood as descending from geometric operations—such as degen-

erations or dualities of the compactification manifold—at the level of the six-dimensional parent theory.

For this reason, six-dimensional SCFTs provide a unifying framework for a wide class of supersymmetric field theories and dualities.

A.2 Class $\mathcal{S}$ Constructions

As discussed above, interacting six-dimensional superconformal field theories do not admit a conventional Lagrangian description. Nevertheless, a powerful handle on their dynamics is obtained by compactifying them on lower-dimensional manifolds. In particular, one may consider compactifying a six-dimensional $\mathcal{N} = (2, 0)$ SCFT on a Riemann surface $\mathcal{C}$, performing a partial topological twist to preserve supersymmetry. The resulting lower-dimensional theories inherit much of their structure from the geometry of $\mathcal{C}$ and from the choice of defects inserted along it.

The four-dimensional $\mathcal{N} = 2$ theories obtained in this way are known as *Class $\mathcal{S}$* theories¹. These constructions provide a geometric framework for understanding a vast class of supersymmetric quantum field theories, encoding their matter content, couplings, and dualities in terms of the topology and complex structure of the underlying Riemann surface [244, 304, 305].

Seiberg–Witten Geometry. As a motivating example, we focus on *Seiberg–Witten geometry* in four-dimensional $\mathcal{N} = 2$ gauge theories [306]. From a purely four-dimensional perspective, the appearance of the Seiberg–Witten curve and its relation to the low-energy prepotential may seem somewhat ad hoc.

This structure, however, becomes much more natural when viewed through the lens of Class $\mathcal{S}$ constructions. In this framework, the Seiberg–Witten curve is not an auxiliary object introduced to solve the theory, but rather has a direct geometric origin: it is precisely the Riemann surface on which the parent six-dimensional $\mathcal{N} = (2, 0)$ theory is compactified. The Seiberg–Witten geometry of the four-dimensional theory is therefore inherited directly from the compactification geometry of the underlying six-dimensional SCFT.

The AGT Correspondence. Another important duality uncovered using these constructions is the *$4d/2d$ -correspondence*, or the *Alday–Gaiotto–Tachikawa* (AGT) correspondence [153, 307, 308].

The basic idea can be summarized as follows. One starts from a six-dimensional SCFT $\chi(\mathfrak{g})$ and considers its partition function $\mathcal{Z}$ on a product manifold of the form $\mathcal{W}_4 \times \mathcal{C}_2$, where $\mathcal{W}_4$ is a four-dimensional manifold and $\mathcal{C}_2$ is a two-dimensional Riemann surface. The resulting quantity $\mathcal{Z}[\mathcal{W}_4 \times \mathcal{C}_2]$ depends in a controlled way on the geometric and topological properties of both factors.

Crucially, the same partition function can be evaluated in two equivalent ways, depending on which factor is regarded as the compactification manifold:

$$\mathcal{Z}[\mathcal{W}_4 \times \mathcal{C}_2] \equiv \mathcal{Z}[\mathcal{W}_4]|_{\mathcal{C}_2} = \mathcal{Z}[\mathcal{C}_2]|_{\mathcal{W}_4}.$$

That is, one may either interpret the result as a two-dimensional theory living on $\mathcal{C}_2$, obtained by compactifying the six-dimensional theory on $\mathcal{W}_4$, or as a four-dimensional

¹The notation $\mathcal{S}$, much to the author’s disappointment, stands for “six”.

theory on $\mathcal{W}_4$, obtained by compactification on $\mathcal{C}_2$. In terms of observables, the AGT correspondence relates $4d$ partition functions to $2d$ conformal blocks.

The 3d/3d Correspondence. A closely related example of this philosophy is the *3d/3d correspondence*, which relates supersymmetric quantum field theories in three dimensions to geometric and topological data of three-manifolds [309, 310]. Starting from the six-dimensional $\mathcal{N} = (2, 0)$ theory of type $\mathfrak{g}$, one considers its compactification on a three-manifold M_3 , possibly with a partial topological twist. This construction gives rise to a three-dimensional $\mathcal{N} = 2$ theory, often denoted $T[M_3]$, whose protected observables capture invariants of M_3 .

As in the AGT correspondence, the same six-dimensional partition function may be evaluated by exchanging the roles of spacetime and internal geometry. In particular, supersymmetric partition functions of $T[M_3]$ on curved three-manifolds—such as $\mathbf{S}_b^3$ or $\mathbf{S}^2 \times \mathbf{S}^1$ —are related to topological invariants of M_3 , including Chern–Simons theory and quantum invariants of knots and three-manifolds. The *3d/3d* correspondence thus provides a powerful bridge between three-dimensional supersymmetric field theories, low-dimensional topology, and quantum geometry.

It would be interesting to study the chiral-planar dualities presented in this thesis with the machinery of Class $\mathcal{S}$. While it seems possible to recover $2d$ $\mathcal{N} = (2, 2)$ mirror symmetry by compactifying the chiral-planar mirror pairs on $\mathbf{S}^1$, it may also be possible to study the holomorphic blocks of the resulting theory to arrive at the putative partition function of a $4d$ $\mathcal{N} = 1$ theory (*à la* [311]). In this way, we may realise a rich web of dualities across dimensions.

Appendix B

From the basic $\mathcal{N} = 4$ Abelian confining dualities to the $\mathcal{N} = 2$ Abelian dualities

In this appendix, we show how to prove the self-dualities for $U(1)_0$ with $[2, 2]$ flavors, for $U(1)_{-\frac{1}{2}}$ with $[2, 1]$ flavors and $U(1)_{-1}$ with $[1, 1]$ flavors assuming only that $U(1)_0$ with $[1, 1]$ flavors confines.

Starting from the basic $\mathcal{N} = 4$ Abelian mirror duality, we present the Aharony-like dualities needed to dualize the planar quivers presented in the thesis. We start from the basic $\mathcal{N} = 4$ Abelian mirror dual pair (Equation B.1) which relates a $U(1)$ gauge theory with one hypermultiplet to a free hypermultiplet.

$$\begin{array}{ccc}
 \begin{array}{c} \textcircled{1} \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{\tilde{Q}} \end{array} \boxed{1} \\ \mathcal{W} = \text{Flip}[Q\tilde{Q}] \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \begin{array}{c} \xleftarrow{X} \\ \xrightarrow{\tilde{X}} \end{array} \boxed{1} \\ \mathcal{W} = 0 \end{array}
 \end{array} \quad (\text{B.1})$$

The operator map is shown below:

$$\begin{aligned}
 \mathfrak{M}^+ &\leftrightarrow X \\
 \mathfrak{M}^- &\leftrightarrow \tilde{X}.
 \end{aligned} \quad (\text{B.2})$$

$U(1)_0$ with $[1, 1]$ chiral fields:

Analogous to the $\mathcal{N} = 4$ mirror pair (Equation B.1), the fundamental $\mathcal{N} = 2$ mirror pair is shown in Equation B.3:

$$\begin{array}{ccc}
 \begin{array}{c} \overset{\lambda}{\textcircled{1}} \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{\tilde{Q}} \end{array} \boxed{1} \\ \underset{0}{\mathcal{W}} = 0 \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \begin{array}{c} \xleftarrow{X} \\ \xrightarrow{\tilde{X}} \end{array} \boxed{1} \\ \mathcal{W} = \text{Flip}[X\tilde{X}] \end{array}
 \end{array} \quad (\text{B.3})$$

with the operator map:

$$\begin{aligned}
 Q\tilde{Q} &\leftrightarrow \mathcal{F}[X\tilde{X}] \\
 \mathfrak{M}^+ &\leftrightarrow X \\
 \mathfrak{M}^- &\leftrightarrow \tilde{X}.
 \end{aligned} \quad (\text{B.4})$$

Self-Dual SQED

By composing the duality operation shown in Equation B.1, one can derive mirror dual pairs of SQED with a generic number of hypermultiplets *a la* Kapustin-Strassler [24]. We provide a brief summary of the piecewise dualization procedure needed to find the mirror of $\mathcal{N} = 2$ SQED with F flavors [205]. Each flavor is dualized using Equation B.3 to obtain a $U(1)^{F+1}$ with $2F$ charged fields, F *flippers* (and a corresponding superpotential) and $2F$ mixed BF interactions connecting the original $U(1)$ gauge group with the $2F$ new gauge factors. At this point, the original $U(1)$ gauge group has no flavors and we can perform the functional integral over it obtaining a functional delta function. Implementing the delta reduces the number of gauge groups to $F - 1$, which is indeed the correct mirror dual.

To dualize the planar quivers in this thesis, we need some additional self-dualities of $\mathcal{N} = 2$ SQED with $F = 2$. The starting point of this is the $\mathcal{N} = 2$ self-dual (*modulo* flips) pair shown in Equation B.5:

$$\begin{array}{ccc}
 \begin{array}{c} \boxed{1} \xleftrightarrow{\tilde{Q}_1} \textcircled{1} \xleftrightarrow{Q_2} \boxed{1} \\ \xleftarrow{Q_1} \xleftarrow{\tilde{Q}_2} \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \xleftrightarrow{\tilde{X}_1} \textcircled{1} \xleftrightarrow{X_2} \boxed{1} \\ \xleftarrow{X_1} \xleftarrow{\tilde{X}_2} \end{array} \\
 \mathcal{W} = 0 & & \mathcal{W} = \text{Flip}[X_1\tilde{X}_1, X_2\tilde{X}_2]
 \end{array} \tag{B.5}$$

with the operator map:

$$\begin{array}{ccc}
 \begin{pmatrix} Q_1\tilde{Q}_1 & Q_1\tilde{Q}_2 \\ Q_2\tilde{Q}_1 & Q_2\tilde{Q}_2 \end{pmatrix} & \leftrightarrow & \begin{pmatrix} \mathcal{F}[X_1\tilde{X}_1] & \mathfrak{M}^+ \\ \mathfrak{M}^- & \mathcal{F}[X_2\tilde{X}_2] \end{pmatrix} \\
 \mathfrak{M}^+ \leftrightarrow X_2\tilde{X}_1 & & \\
 \mathfrak{M}^- \leftrightarrow X_1\tilde{X}_2 & &
 \end{array} \tag{B.6}$$

We can apply the same duality with a different choice of mesons that are flipped on the dual side, which amounts to a relabeling of the fields $X_{1,2}$ and $\tilde{X}_{1,2}$. This provides a third mirror dual frame (Equation B.7):

$$\begin{array}{ccc}
 \begin{array}{c} \boxed{1} \xleftrightarrow{\tilde{X}_1} \textcircled{1} \xleftrightarrow{X_2} \boxed{1} \\ \xleftarrow{X_1} \xleftarrow{\tilde{X}_2} \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \xleftrightarrow{\tilde{Q}_1} \textcircled{1} \xleftrightarrow{Q_2} \boxed{1} \\ \xleftarrow{Q_1} \xleftarrow{\tilde{Q}_2} \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \xleftrightarrow{\tilde{A}_1} \textcircled{1} \xleftrightarrow{A_2} \boxed{1} \\ \xleftarrow{A_1} \xleftarrow{\tilde{A}_2} \end{array} \\
 \mathcal{W} = \sum_{i=1}^2 \text{Flip}[X_i\tilde{X}_i] & & \mathcal{W} = 0 & & \mathcal{W} = \sum_{i \neq j}^2 \text{Flip}[A_i\tilde{A}_j]
 \end{array} \tag{B.7}$$

where the mapping of operators across the triality is given by:

$$\begin{array}{ccc}
 X_1\tilde{X}_2 \leftrightarrow \mathfrak{M}^- \leftrightarrow A_2\tilde{A}_2 \\
 X_2\tilde{X}_1 \leftrightarrow \mathfrak{M}^+ \leftrightarrow A_1\tilde{A}_1 \\
 \begin{pmatrix} \mathcal{F}[X_1\tilde{X}_1] & \mathfrak{M}^+ \\ \mathfrak{M}^- & \mathcal{F}[X_2\tilde{X}_2] \end{pmatrix} \leftrightarrow Q\tilde{Q} \leftrightarrow \begin{pmatrix} \mathfrak{M}^+ & \mathcal{F}[A_2\tilde{A}_1] \\ \mathcal{F}[A_1\tilde{A}_2] & \mathfrak{M}^- \end{pmatrix}.
 \end{array} \tag{B.8}$$

We arrive at an analogous Seiberg-like duality by observing the first and third frame of

Equation B.7:

$$\mathcal{W} = \sum_{i=1}^2 \text{Flip}[X_i \tilde{X}_i] \quad \mathcal{W} = \sum_{i \neq j}^2 \text{Flip}[A_i \tilde{A}_j] \quad (\text{B.9})$$

We can *flip* the flippers in the quiver theory shown on the LHS of Equation B.9 to obtain a Seiberg-like duality:

$$\mathcal{W} = 0 \quad \mathcal{W} = \sum_{i \neq j}^2 \text{Flip}[A_i \tilde{A}_j] + \text{Flip}[\mathfrak{M}^\pm] \quad (\text{B.10})$$

with the operator map:

$$\begin{aligned} \mathfrak{M}^+ &\leftrightarrow \mathcal{F}[A_2 \tilde{A}_1] \\ \mathfrak{M}^- &\leftrightarrow \mathcal{F}[A_1 \tilde{A}_2] \\ X \tilde{X} &\leftrightarrow \begin{pmatrix} \mathcal{F}[\mathfrak{M}^+] & A_2 \tilde{A}_2 \\ A_1 \tilde{A}_1 & \mathcal{F}[\mathfrak{M}^-] \end{pmatrix}. \end{aligned} \quad (\text{B.11})$$

We are now able to establish a fourth mirror dual frame (Equation B.12):

$$\mathcal{W} = \sum_{i \neq j}^2 \text{Flip}[A_i \tilde{A}_j] + \text{Flip}[\mathfrak{M}^\pm] \quad \mathcal{W} = 0 \quad \mathcal{W} = \sum_{i=1}^2 \text{Flip}[B_i \tilde{B}_i] \quad (\text{B.12})$$

with corresponding operator map:

$$\begin{aligned} \mathcal{F}[A_2 \tilde{A}_1] &\leftrightarrow \mathfrak{M}^+ \leftrightarrow B_2 \tilde{B}_1 \\ \mathcal{F}[A_1 \tilde{A}_2] &\leftrightarrow \mathfrak{M}^- \leftrightarrow B_1 \tilde{B}_2 \\ \begin{pmatrix} \mathcal{F}[\mathfrak{M}^+] & A_2 \tilde{A}_2 \\ A_1 \tilde{A}_1 & \mathfrak{M}^- \end{pmatrix} &\leftrightarrow Q \tilde{Q} \leftrightarrow \begin{pmatrix} \mathfrak{M}^+ & \mathcal{F}[B_1 \tilde{B}_1] \\ \mathcal{F}[B_2 \tilde{B}_2] & \mathfrak{M}^- \end{pmatrix}. \end{aligned} \quad (\text{B.13})$$

We arrive at an analogous Seiberg-like duality by observing the first and third frame of

Equation B.12:

$$(B.14)$$

$$\mathcal{W} = \sum_{i \neq j}^2 \text{Flip}[A_i \tilde{A}_j] + \text{Flip}[\mathfrak{M}^\pm] \quad \mathcal{W} = \sum_{i=1}^2 \text{Flip}[B_i \tilde{B}_i]$$

We can once again *flip* the flippers in the quiver theory shown on the RHS of Equation B.14 to obtain a Seiberg-like duality:

$$(B.15)$$

$$\mathcal{W} = \sum_{i,j} \text{Flip}[q_i \tilde{q}_j] + \text{Flip}[\mathfrak{M}^\pm] \quad \mathcal{W} = 0$$

with operator map:

$$\mathcal{F}[\mathfrak{M}^\pm] \leftrightarrow \mathfrak{M}^\pm$$

$$\begin{pmatrix} \mathcal{F}[q_2 \tilde{q}_1] & \mathcal{F}[q_1 \tilde{q}_1] \\ \mathcal{F}[q_2 \tilde{q}_2] & \mathcal{F}[q_1 \tilde{q}_2] \end{pmatrix} \leftrightarrow \begin{pmatrix} Q_1 \tilde{Q}_1 & Q_1 \tilde{Q}_2 \\ Q_2 \tilde{Q}_1 & Q_2 \tilde{Q}_2 \end{pmatrix}. \quad (B.16)$$

Thus, we have recovered the known Aharony self-duality *modulo flips* of $U(1)$ with 2 flavors [18] by only invoking basic $\mathcal{N} = 4$ Abelian mirror duality (Equation B.1). Starting from the basic $\mathcal{N} = 2$ mirror pair (Equation B.5), one can keep track of the CS and mixed BF interactions in the theory, which would yield the quiver description:

$$(B.17)$$

$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \text{Flip}[\mathfrak{M}^\pm] \quad \mathcal{W} = 0$$

Mass Deformations of Known Aharony Duals

We can obtain other pairs of Aharony duals by adding suitable real mass deformations to the dual pairs discussed above.

$U(1)_{-\frac{1}{2}}$ with $[1, 0]$ chiral fields:

The mirror pair shown in Equation B.18 can be obtained by adding a real mass deformation to the mirror pair shown in Equation B.3 and integrating out 1 chiral field:

$$\begin{array}{ccc}
\begin{array}{c} \lambda \\ \textcircled{1} \xrightarrow{Q} \boxed{1} \\ -\frac{1}{2} \end{array} & \leftrightarrow & \begin{array}{c} \boxed{1} \\ \frac{1}{2} \end{array}, X \text{ (singlet)} \\
\mathcal{W} = 0 & & \mathcal{W} = 0
\end{array} \quad (\text{B.18})$$

with the operator map:

$$\mathfrak{M}^+ \leftrightarrow X. \quad (\text{B.19})$$

Mirror symmetry maps topological symmetries to flavor symmetries - in this case, the topological symmetry of the electric CS-SQED theory ($U(1)_\lambda$) is mapped to a flavor symmetry in the dual theory, and the singlet X carries a charge +1 under it.

$U(1)_{-\frac{1}{2}}$ **with** $[1, 2]$ **chiral fields:**

Starting from the dual pair shown in Equation B.17, we can add a suitable real mass deformation and integrate out one chiral field on the RHS of the duality, which generates a CS level for the $U(1)$ gauge group. Upon mapping this deformation to the LHS, we flow to the duality shown in Equation B.20 (note that we have assigned a trial R-charge r to the fundamentals Q and anti-fundamentals $\tilde{Q}$ of the theory shown on the RHS):

$$\begin{array}{ccc}
\begin{array}{c} \boxed{1} \\ \begin{array}{c} \nearrow -\lambda - \frac{1}{2} \searrow -1 \\ \textcircled{1} \\ \nwarrow 1 \nearrow \frac{1}{2} \nwarrow -1 \nearrow 1 \\ \boxed{1} \quad \boxed{1} \\ \leftarrow 0 \rightarrow \end{array} \end{array} & \leftrightarrow & \begin{array}{c} \begin{array}{c} \nearrow -\lambda - \frac{1-r}{2} \searrow -\frac{1}{2} \\ \textcircled{1} \\ \nwarrow -1 \nearrow -\frac{1}{2} \nwarrow 1 \nearrow 1 \\ \boxed{1} \quad \boxed{1} \\ \leftarrow -\frac{\lambda}{2} - \frac{3r-1}{4} \rightarrow -\frac{\lambda}{2} - \frac{1-r}{4} \end{array} \end{array} \\
\mathcal{W} = \mathcal{W}_{\text{planar}} + \text{Flip}[\mathfrak{M}^+] & & \mathcal{W} = 0
\end{array} \quad (\text{B.20})$$

with the map of operators:

$$\begin{aligned}
\mathcal{F}[\mathfrak{M}^+] &\leftrightarrow \mathfrak{M}^- \\
\text{flippers} &\leftrightarrow Q\tilde{Q}.
\end{aligned} \quad (\text{B.21})$$

$U(1)_{-1}$ **with** $[1, 1]$ **chiral fields:**

Starting from the dual pair shown in Equation B.20, we can add another suitable real mass deformation and integrate out one antichiral field on the RHS of the duality. Upon mapping this deformation to the LHS, we flow to the duality shown in Equation B.22:

$$\begin{array}{ccc}
\begin{array}{c} \boxed{1} \\ \begin{array}{c} \nearrow -\lambda \searrow -1 \\ \textcircled{1} \\ \nwarrow 1 \nearrow -1 \\ \boxed{1} \end{array} \end{array} & \leftrightarrow & \begin{array}{c} \begin{array}{c} \nearrow -\lambda \searrow -\frac{1}{2} \\ \textcircled{1} \\ \nwarrow -1 \nearrow 1 \\ \boxed{1} \end{array} \end{array} \\
\mathcal{W} = \mathcal{W}_{\text{planar}} & & \mathcal{W} = 0
\end{array} \quad (\text{B.22})$$

with the flipper of the LHS being mapped to the gauge invariant meson $Q\tilde{Q}$ on the RHS.

B.1 An Application: Recovering the Symmetric Bifundamental Block from a Planar Abelian Quiver Gauge Theory

In Equation 2.101, it was demonstrated that N free chiral fields are mirror dual to a planar quiver gauge theory. This is schematically repeated in Figure B.1.

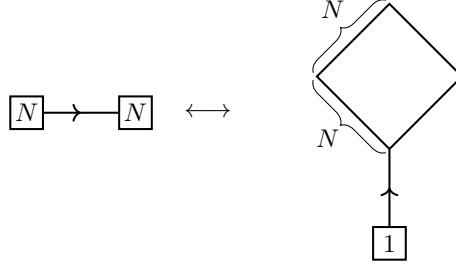


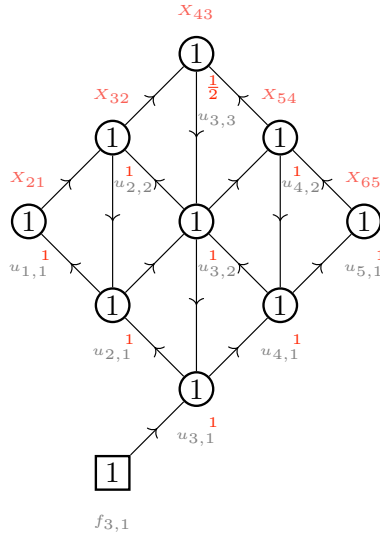
Figure B.1: The duality between the $[N, N]$ bifundamental block and a planar quiver gauge theory is shown here.

Our aim in this section is to sequentially confine the planar quivers using only the pairs of Aharony duals discussed before.

We demonstrate how to reduce the planar mirror of an $[N, N]$ bifundamental block to the planar mirror of an $[N - 1, N - 1]$ bifundamental block by following a sequence of columnwise Aharony dualizations here. Performing this sequence iteratively will confine the planar mirror.

The $[3, 3]$ block

For clarity, we fix $N = 3$. We begin with the planar mirror of the $[3, 3]$ bifundamental¹:



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

¹We follow the same conventions for the Chern–Simons, mixed Chern–Simons, and FI interactions as before, and will only mention them explicitly when they differ from these standard assignments. The FI term associated with the gauge node of fugacity $u_{3,3}$ receives an additional contribution of $-\frac{iQ}{4}$ due to mixing with the $U(1)_R$ symmetry which has been suppressed for brevity.

We now perform a sequence of local Aharony dualizations, column by column. As this is our first example, we are pedantic and provide many details. We first provide a schematic summary of the same in Figure B.2 before diving into the details.

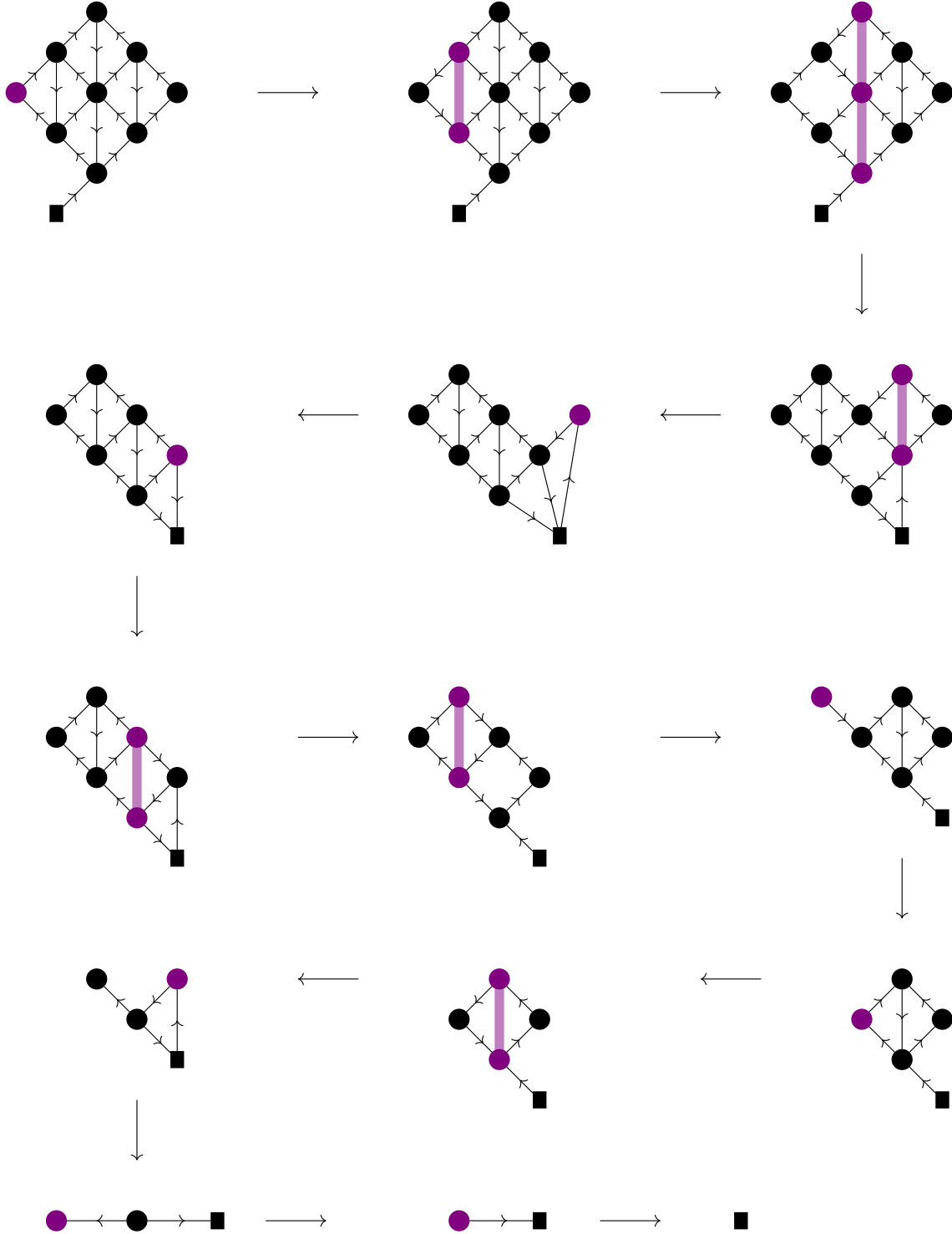
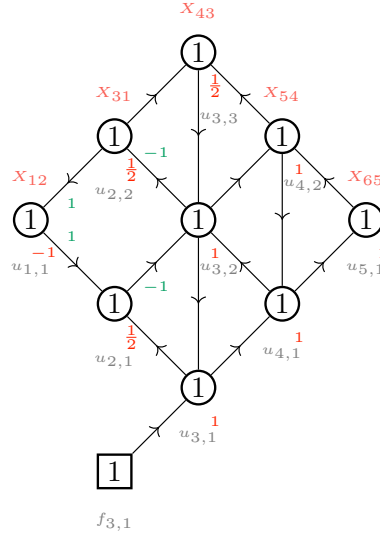


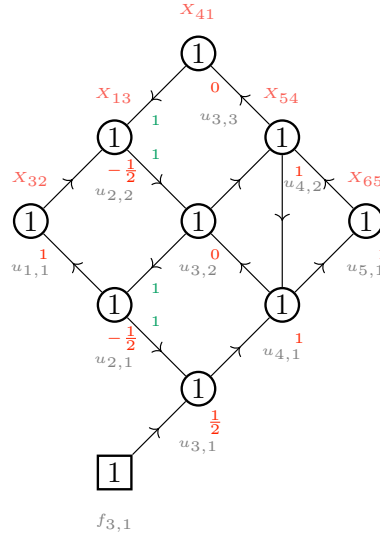
Figure B.2: Schematic summary of the dualization sequence leading to the confinement of the $[3,3]$ symmetric block. In each frame, the columns being dualized are highlighted in violet. The detailed structure of the CS, mixed CS, and FI terms, as well as the resulting singlet fields, is discussed in the main text.

1. Column 1:



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

2. Column 2^2 :

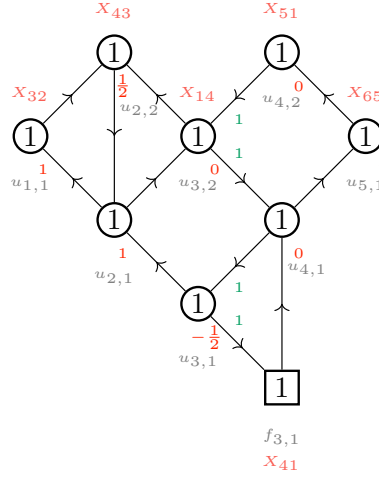


$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

3. Column 3^3 :

²As before, we suppress the mixing between the topological and $U(1)_R$ symmetries in this quiver. The gauge nodes with fugacities $u_{3,3}$ and $u_{2,2}$ each receive an additional contribution of $-\frac{iQ}{4}$, while the node $u_{2,1}$ receives a contribution of $+\frac{iQ}{4}$.

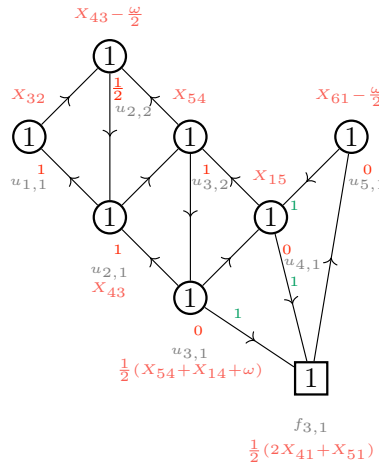
³The gauge nodes with fugacities $u_{2,2}$ and $u_{4,2}$ receive an additional contribution of $-\frac{iQ}{4}$, while the gauge node with fugacity $u_{3,1}$ receives an additional contribution of $+\frac{iQ}{4}$ to their respective FI terms.



$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopoles}$$

Notice that we make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{3,3}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_4-X_1}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

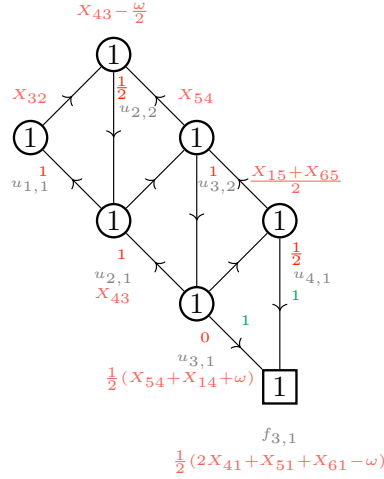
4. Column 4:



$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopoles}$$

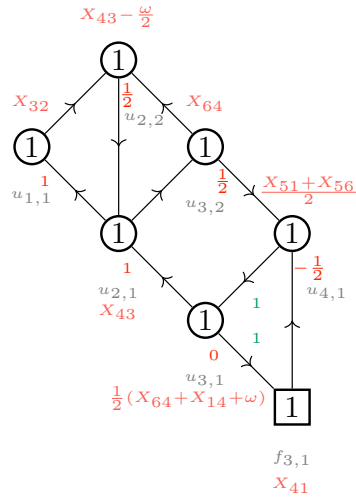
We make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{4,2}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_5-X_1}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

5. Column 5:

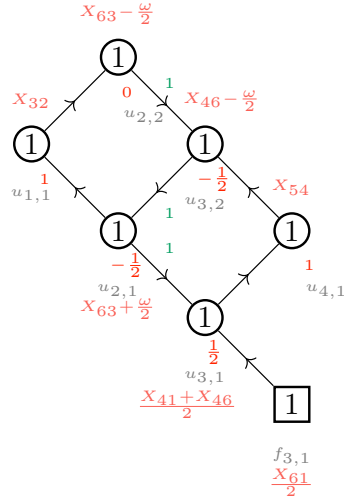


We make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{5,1}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_6-X_1}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

6. Column 4:

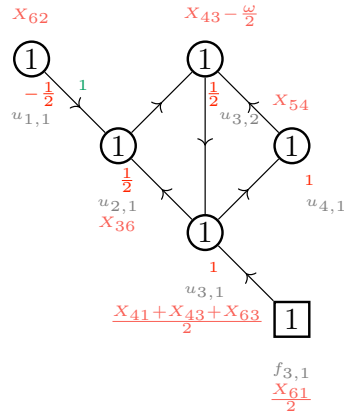


7. Column 3:



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

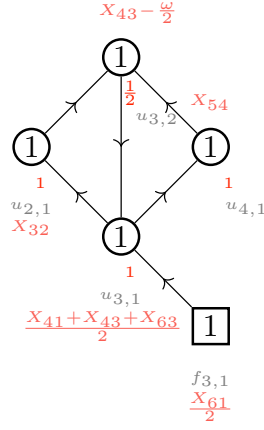
8. Column 2:



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopoles}}$$

We make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{2,2}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_6-X_3}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

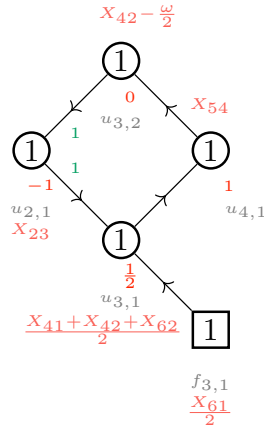
9. Column 1:



$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopoles}$$

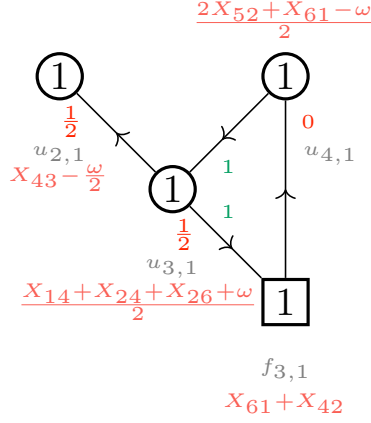
We make use of the duality shown in Equation B.18 here. Consequently, the gauge node with fugacity $u_{1,1}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_6-X_2}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry. We recognize the resulting quiver as the planar mirror of the $[2, 2]$ bifundamental block.

10. **Column 2:**



$$\mathcal{W} = \mathcal{W}_{planar} + \mathcal{W}_{monopoles}$$

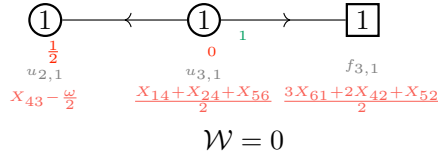
11. **Column 3:**



$$\mathcal{W} = \mathcal{W}_{\text{planar}}$$

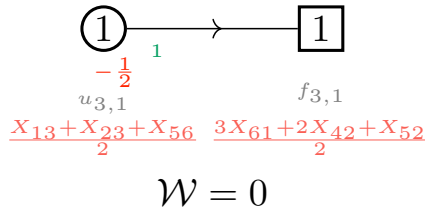
We make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{3,2}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_4-X_2}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

12. **Column 4:**



We make use of the duality shown in Equation B.3 here. Consequently, the gauge node with fugacity $u_{4,1}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_5-X_2}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

13. **Column 2:**



We make use of the duality shown in Equation B.18 here. Consequently, the gauge node with fugacity $u_{2,1}$ confines to generate a singlet. The topological symmetry of the gauge node is $U(1)_{X_4-X_3}$; consequently, the singlet generated upon dualizing and confining this node will carry a +1 charge under this symmetry.

14. We make use of the duality shown in Equation B.18 once again. The gauge node with fugacity $u_{3,1}$ confines and generates a singlet with charge +1 under $U(1)_{X_5-X_3}$.

Thus, we correctly recover all 9 components of the $[3, 3]$ bifundamental block.

Thus, we observe that the planar mirror of an $[N, N]$ bifundamental block can be system-

atically confined to yield the planar mirror of an $[N-1, N-1]$ bifundamental block. This is achieved by sequentially dualizing each column of the planar quiver—first from left to right, and then again from right to left.

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