FORWARD UNTANGLING AND APPLICATIONS TO THE UNIQUENESS PROBLEM FOR THE CONTINUITY EQUATION

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Abstract. ...

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Introduction

1. Notation and preliminaries

im_lagr_repr

1.1. **Lagrangian representations.**

Consider a vector field of the form

$$\rho(1,\boldsymbol{b}) \in \mathcal{M}(\mathbb{R}^{d+1},\mathbb{R}^{d+1}), \tag{1.1}$$

eq:vectorfield

where

$$\rho \in \mathcal{M}^+(\mathbb{R}^{d+1}), \qquad \boldsymbol{b} \in L^1(\mu;\mathbb{R}^d). \tag{1.2}$$

eq:assu_rappr_la

We assume that ρ is compactly supported and that it holds

$$\operatorname{div}(\rho(1,\boldsymbol{b})) = \mu \in \mathcal{M}(\mathbb{R}^{d+1}) \tag{1.3}$$

E_balance_transp

in the sense of distribution, i.e. $\rho(1,\boldsymbol{b})$ is a *measure-divergence vector field.* To avoid dealing with sets of ρ-negligible measure, we will assume that $\boldsymbol{b}$ is defined pointwise everywhere as Borel function.

An absolutely continuous curve $\gamma\colon I \to \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an open time interval, is a *characteristic of the vector field* $\boldsymbol{b}$ if it solves the ODE

$$\frac{d}{dt}\gamma(t) = \boldsymbol{b}(t,\gamma(t)),$$

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the equality holding $\mathcal{L}^1$-a.e. in I. As done in [BB17, Section 3.1], we will consider the metric space Υ of curves γ: more precisely, let

$$\Upsilon = \Big\{(t_1, t_2, \gamma) : t_1 < t_2,\ \gamma \in C([t_1, t_2], \mathbb{R})\Big\}$$

with the distance

$$|t_1 - t_1'| + |t_2 - t_2'| + \max\big\{|\gamma(s) - \gamma'(s)|, s \in [t_1, t_2] \cap [t_1', t_2']\big\},$$

and its subset made of characteristics

$$\Gamma = \Big\{(t_1, t_2, \gamma) \in \Upsilon : \gamma \text{ characteristic in } (t_1, t_2)\Big\}.$$

One can show that Γ is a Borel subset of Υ: indeed

$$\gamma \in \Gamma \quad \Longleftrightarrow \quad \sup_{t,s\in[t_1,t_2]\cap\mathbb{Q}} \left|\gamma(t) - \gamma(s) - \int_s^t b(\tau, \gamma(t, \tau))d\tau\right| = 0,$$

i.e. it is the 0-level set of a Borel function.

Clearly, given $t_1 < t_2$ and a function γ continuous in the closed interval $[t_1, t_2]$, it can always be extended to the real line, so that Υ can be seen as a quotient of the space $\mathbb{R}^2 \times C(\mathbb{R}, \mathbb{R}^d)$ with the quotient topology. In what follows, to shorten the notation, instead of the triplet (t_1, t_2, γ) we will write only γ, and denote its interval of definition by $[t_\gamma^-, t_\gamma^+]$. We will often consider γ as defined only in the open interval I_γ, i.e. $\gamma = \gamma\llcorner_{(t_\gamma^-, t_\gamma^+)}$: this is for convenience, since our results concern the intersection properties of families of curves in the open interval where they are characteristics.

We now recall the following important definition.

n_repr_intro **Definition 1.1** (Lagrangian representation of the vector field $\rho(1, \boldsymbol{b})$)**.** We say that a bounded, positive measure $\eta \in \mathcal{M}_b^+(\Upsilon)$ is a *Lagrangian representation of the vector field* $\rho(1, \boldsymbol{b})\,\mathcal{L}^{d+1}$ if the following conditions hold:

(1) η is concentrated on the set Γ of absolutely continuous solutions to the ODE

$$\dot\gamma(t) = \boldsymbol{b}(t, \gamma(t)), \tag{1.4}$$ eq:ode

which explicitly means for every $s, t \in I_\gamma$

$$\int_\Gamma \left|\gamma(t) - \gamma(s) - \int_s^t \mathbf{b}(\tau, \gamma(\tau))\, d\tau\right| \eta(d\gamma) = 0;$$

(2) if $(\mathrm{id}, \gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t, \gamma(t))$, then in the sense of measures

$$\rho(1, \boldsymbol{b}) = \int_\Gamma (\mathrm{id}, \gamma)_\sharp\big((1, \dot\gamma)\mathcal{L}^1\llcorner_{I_\gamma}\big)\, \eta(d\gamma);$$

(3) we can decompose the divergence μ as local superposition of Dirac masses without cancellation, i.e.

$$\mu = \int_\Gamma \Big[\delta_{t_\gamma^-, \gamma(t_\gamma^-)}(dt, dx) - \delta_{t_\gamma^+, \gamma(t_\gamma^+)}(dt, dx)\Big]\, \eta(d\gamma),$$
$$|\mu| = \int \Big[\delta_{t_\gamma^-, \gamma(t_\gamma^-)}(dt, dx) + \delta_{t_\gamma^+, \gamma(t_\gamma^+)}(dt, dx)\Big]\, \eta(d\gamma),$$

where we recall that, for every γ, the interval in which it is a characteristic is denoted by $I_\gamma = (t_\gamma^-, t_\gamma^+)$.

The existence of such a measure η is ensured by the following

Theorem 1.2 ([Smi94])**.** *Let $\rho(1, \boldsymbol{b})$ be a vector field as in (1.1), i.e. satisfying (1.2) and (1.3). Then there exists a Lagrangian representation of $\rho(1, \boldsymbol{b})$ in the sense of Definition 1.1.*

For the proof (which reduces to a reparameterization of the curves), one can adapt the proof of [BB17, Theorem 3.2]. Observe that for all γ the interval of definition is a bounded time interval (recall that we assume $\rho(1, \boldsymbol{b})$ with compact support), so that if $\mu^\pm$ is the positive/negative part of the divergence we can disintegrate η according to

$$\eta = \int_{\mathbb{R}^{d+1}} \eta_z\, \mu^-(dz) = \int_{\mathbb{R}^{d+1}} \eta_z\, \mu^+(dz), \qquad \mu^\pm = \big(t_\gamma^\pm, \gamma(t_\gamma^\pm)\big)_\sharp \eta. \tag{1.5}$$ eq:disintegr_eta

We remark finally that, by the first and second points of Definition 1.1, it follows that

$$\int_\Gamma \left[\int_{I_\gamma} |\dot\gamma| \, \mathcal{L}^1 \right] \eta(d\gamma) = \int_\Gamma \left[\int_{I_\gamma} |\boldsymbol{b}(t,\gamma(t))| \, dt \right] \eta(d\gamma) = \int_{\mathbb{R}^{d+1}} \rho |\boldsymbol{b}| \mathcal{L}^{d+1},$$

so that the total variation of η-a.e. curve is finite, and thus $\gamma(t^\pm_\gamma) \in \mathbb{R}^d$ exists.

Remark 1.3. In the case $\mu = 0$ the existence of a Lagrangian representation can also be inferred from the so called Ambrosio's Superposition Principle [Amb04].

In the paper we will use the notation

$$\mathrm{Graph}\,\gamma + B^d_r(0) = \big\{(t,x), t \in [t^-_\gamma, t^+_\gamma], x \in \gamma(t) + B^d_r(0)\big\}. \tag{1.6}$$ Equa:notation_pl

:proper_sets 1.2. **Proper sets.** Proper sets were introduced in the paper [BB17], to whom we refer the reader for a complete treatment of this class of sets. Here we limited to recall the definition and the main properties we will use in the following sections. Since the measures we consider are not absolutely continuous w.r.t. $\mathcal{L}^{d+1}$, we use the definition of [BB17, Remark 4.3], see also [BS] for the full treatment.

Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded Lipschitz function with compact support.

inner_proper **Definition 1.4** (Inner Proper Sets)**.** The open, bounded set $\Omega = \{f > h\}$ is called $\rho(1,\boldsymbol{b})$-*inner proper* if there exists a sequence $\delta_n \searrow 0$ such that the measures

$$\nu^n = \frac{1}{\delta_n} \rho(1,\boldsymbol{b}) \cdot \nabla f \llcorner_{f^{-1}(h,h+\delta_n)}$$

satisfy

$$\nu^{\delta_n} \to \nu, \qquad |\nu^{\delta_n}| \to |\nu|.$$

It is fairly easy to see that ν is the distributional trace.

roper_genera **Definition 1.5.** [Proper sets] The set $\Omega_h = \{f > h\}$ is $\rho(1,\boldsymbol{b})$-*proper* if it is inner proper, $\{-f > h\}$ is inner proper, and the two traces coincides:

$$\mathsf{Tr}\left(\rho(1,\boldsymbol{b}),\Omega\right) = \mathsf{Tr}\left(\rho(1,\boldsymbol{b}), \mathbb{R}^{d+1} \setminus \Omega\right).$$

In the following we will write *proper* instead of $\rho(1,\boldsymbol{b})$-proper when there is no ambiguity about the vector field.

Using [BB17, Lemma 4.4] or the results in [BS], we have the following proposition.

set_are_many **Proposition 1.6.** *Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is a Lipschitz function whose level sets $E_h := f^{-1}((h,+\infty))$ have compact closure. Then E_h is proper for $\mathcal{L}^1$-a.e. $h \in \mathbb{R}$. In particular*

(1) for every (t,x) the balls $\{B^{d+1}_r(t,x)\}_{r>0}$ are proper sets for $\mathcal{L}^1$-a.e. $r > 0$;
(2) for every fixed $(t,x) \in \mathbb{R}^{d+1}$ and $r, L > 0$, define the cylinder of center (t,x) and sizes r, L *as*

$$\mathrm{Cyl}^{r,L}_{t,x} := \Big\{(\tau,y) : |\tau - t| < Lr, \big|y - x - \boldsymbol{b}(t,x)(\tau - t)\big| < r\Big\}.$$

Then the cylinders $\{\mathrm{Cyl}^{r,L}_{t,x}\}_{r>0}$ (with $L > 0$ fixed) are proper sets for $\mathcal{L}^1$-a.e. $r > 0$.

Another useful property is expressed in the following proposition: under a transversality assumption of the boundaries, proper sets are closed under finite unions.

Proposition 1.7 ([BB17, Proposition 4.11])**.** *If Ω_1, Ω_2 are proper sets with*

$$\mathcal{H}^d\Big(\mathrm{Fr}\left(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega_1 \cup \partial\Omega_2\right)\Big) = 0, \tag{1.7}$$ Equa:trasnver_i

then $\Omega := \Omega_1 \cup \Omega_2$ is proper.

In [BB17] a slightly different condition on proper sets is given, because it is assumed that $\rho(1,\boldsymbol{b}) \llcorner \mathcal{L}^{d+1}$: it is required that the trace is a measure a.c. w.r.t. $\mathcal{H}^d \llcorner_{\partial\Omega}$, and that $\mathcal{H}^d$-a.e. $x \in \partial\Omega$ is a Lebesgue point of $\rho(1,\boldsymbol{b})\mathcal{L}^{d+1}$. With this definition one can prove that proper sets can be suitably perturbed in order to adapt to the special time-space structure of the vector field $\rho(1,\boldsymbol{b})$. The perturbation is made in such a way that almost all the inflow and outflow of $\rho(1,\boldsymbol{b})$ occurs on open sets which are contained in countably many time-flat hyperplanes: due to the special form of the vector field, many computations in the next sections will be simpler.

ace_control_1 **Theorem 1.8** ([BB17, Theorem 4.18]). *For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

(1) $\Omega \subset \Omega^\varepsilon \subset \Omega + B^{d+1}_\varepsilon(0)$;

(2) *if*
$$S_1^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (1,0) \text{ in a neigborhood of } (t,x)\Big\},$$
then S_1^ε is made of Lebesgue points of $\rho(1,\boldsymbol{b})$ up to a $\mathcal{H}^d$-negligible set and
$$\left| \int_{S_1^\varepsilon} \rho \mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\boldsymbol{b})\cdot \mathbf{n}]^+ \mathcal{H}^d \right| < \varepsilon;$$

(3) *if*
$$S_2^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (-1,0) \text{ in a neigborhood of } (t,x)\Big\},$$
then S_2^ε is made of Lebesgue points of $\rho(1,\boldsymbol{b})$ up to a $\mathcal{H}^d$-negligible set and
$$\left| \int_{S_2^\varepsilon} \rho \mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\boldsymbol{b})\cdot \mathbf{n}]^- \mathcal{H}^d \right| < \varepsilon.$$

As observed in [BB17, Section 7], or directly from the proof of the above theorem, we can assume that the countably many sets $\{t > \mathcal{O}(1)\}$ whose boundaries contain S_1, S_2, are proper.

In [BS] it is used a different approach, which does not need the above theorem. In this paper, however, we assume that the above theorem holds also for measure valued vector fields as follows. Write
$$\mathsf{Tr}^\pm\big(\rho(1,\boldsymbol{b}),\Omega\big)$$
as the positive/negative part of the measure $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$.

race_control **Theorem 1.9** ([BS]). *For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

l_contained_1 (1) $\Omega \subset \Omega^\varepsilon \subset \Omega + B^{d+1}_\varepsilon(0)$;

mall_trace_2 (2) *if*
$$S_1^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \Omega = \{t' > t\} \text{ in a neigborhood of } (t,x)\Big\},$$
then
$$\Big| \mathsf{Tr}^+\big(\rho(1,\boldsymbol{b}),\Omega^\varepsilon)(S_1^\varepsilon) - \mathsf{Tr}^+\big(\rho(1,\boldsymbol{b}),\Omega\big)(\partial\Omega) \Big| < \varepsilon;$$

mall_trace_3 (3) *if*
$$S_2^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \Omega = \{t' < t\} \text{ in a neigborhood of } (t,x)\Big\},$$
then
$$\Big| \mathsf{Tr}^-\big(\rho(1,\boldsymbol{b}),\Omega^\varepsilon)(S_1^\varepsilon) - \mathsf{Tr}^-\big(\rho(1,\boldsymbol{b}),\Omega\big)(\partial\Omega) \Big| < \varepsilon;$$

_proper_sets 1.2.1. *Restriction of Lagrangian representations to proper sets.* In addition to this perturbation, proper sets play an important role in connection to Lagrangian representations, as it is possible to restrict a Lagrangian representation to a proper set, in a suitable sense. Given a divergence measure vector field $\rho(1,\boldsymbol{b})$ and a proper set Ω, let $\{t_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ be a dense sequence, and label each open component of
$$\gamma^{-1}(\Omega) = \bigcup_{j\in\mathbb{N}} I^j, \qquad I^j = \big(t^{j,-}, t^{j,+}\big),$$
as follows

(1) if $t^{j,-} = t^-_\gamma$, then denote $t^{j,+} = t^{0,+}_\gamma$;
(2) if $t^{j,+} = t^+_\gamma$, then denote $t^{j,-} = t^{0,-}_\gamma$;
(3) for the remaining open intervals I^j_γ, relabel I^j as
$$I^i_\gamma = \big(t^{i,-}_\gamma, t^{i,+}_\gamma\big), \quad \text{where} \quad i = \min\big\{i' : t_{i'} \in I^j_\gamma\big\},$$
i.e. the apex i of the interval I^i_γ refers to the fact that it contains the time t_i of the dense sequence and eventually some of the t_j, $j > i$.

Let $D_0^-, D_0^+, D_i \subset \mathit{\Gamma}$ be the domains of $t^{0,-}_\gamma$, $t^{0,+}_\gamma$, $t^{i,\pm}_\gamma$ respectively. This labeling is Borel (Lemma 5.5 of [BB17]) and we can now give the following def:R_Omega **Definition 1.10.** The *restriction operators* $\mathtt{R}^{0,\pm}_\Omega$, $\mathtt{R}^i_\Omega$ and $\mathtt{R}_\Omega\gamma$ are defined respectively as

$$\mathtt{R}^{0,+}_\Omega\gamma := \gamma\llcorner_{(t^-_\gamma, t^{0,+}_\gamma)}, \qquad \mathtt{R}^{0,-}_\Omega\gamma := \gamma\llcorner_{(t^{0,-}_\gamma, t^+_\gamma)}, \qquad \mathtt{R}^i_\Omega\gamma := \gamma\llcorner_{(t^{i,-}_\gamma, t^{i,+}_\gamma)},$$
$$\mathtt{R}_\Omega\gamma := \mathtt{R}^{0,+}_\Omega\gamma \cup \mathtt{R}^{0,-}_\Omega\gamma \cup \bigcup_{i\in\mathbb{N}} \mathtt{R}^i_\Omega\gamma, \tag{1.8}$$

eq:restr_lagr

and the measures η^i_Ω are defined as

$$\eta^i_\Omega := (\mathtt{R}^i_\Omega)_\sharp\eta. \tag{1.9}$$

eq:restrict_eta_

Note that $\mathtt{R}_\Omega$ is multivalued and it is clear that if

$$\rho^i_\Omega(1,\boldsymbol{b})\,\mathcal{L}^{d+1} := \int (\mathrm{id},\gamma)_\sharp\big((1,\dot\gamma)\,\mathcal{L}^1\big)\,\eta^i_\Omega(d\gamma), \tag{1.10}$$

eq:def-rhoi

then in Ω

$$\rho(1,\boldsymbol{b}) = \sum_i \rho^i_\Omega(1,\boldsymbol{b}).$$

_trace_T_gen **Theorem 1.11** ([BB17, Theorem 6.8], [BS]). *If Ω is a proper set, the restriction operator $\mathtt{R}_\Omega$ maps a Lagrangian representation of $\rho(1,\boldsymbol{b})$ to a Lagrangian representation of $\rho(1,\boldsymbol{b})\llcorner\Omega$.*

From the definition of $\mathtt{R}_\Omega$, one can deduce the following proposition.

L_reduct_Q **Proposition 1.12** ([BB17, Proposition 6.10], [BS]). *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $N \subset \Gamma$ a Borel set. It holds*

$$\eta\big(\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_\Omega\gamma \in N\}\big) \le (\mathtt{R}^i_\Omega)_\eta(N).$$

ort_kellerer 1.3. **Optimal transport and duality.** In this section we recall some results contained in the paper [Kel84]. They have already been exploited in the setting of Lagrangian representations and we recall the main facts in this pragraph for the usefulness of the reader. Given finitely many finite measures $\mu_i \ge 0$ over Polish spaces X_i, we define the set of *admissible transference plans* $\mathsf{Adm}(\{\mu_i\}_{i\in I})$ as

$$\mathsf{Adm}(\{\mu_i\}_{i\in I}) = \big\{\pi \ge 0 : (\mathtt{p}_i)_\sharp\pi \le \mu_i\big\} \subset \mathcal{M}^+\bigg(\prod_i X_i\bigg).$$

Given a positive Borel function $h \ge 0$, consider the following duality problem:

$$\sup_{\mathsf{Adm}(\{\mu_i\})} \int h\,\pi = \inf\bigg\{\sum_i \int h_i\,\mu_i, h_i \text{ Borel}, \sum_i h_i \ge h\bigg\}. \tag{1.11}$$

eq:duality_form

We recall the following deep duality result:

lity_souslin **Theorem 1.13** ([Kel84, Theorems 2.14,2.12]). *The equality (1.11) holds if h is a Borel function, and the infimum is actually a minimum.*

Moreover, in the case of two factors X_1, X_2 and when h is a characteristic function, the infimum can be restricted to (characteristic functions of) Borel sets.

duality_sets **Proposition 1.14** ([Kel84, Proposition 3.3.]). *If $n = 2$ and $h = \mathbb{1}_B$, then the r.h.s. of (1.11) can be replaced by*

$$\inf\Big\{\mu_1(B_1) + \mu_2(B_2) : B_1, B_2 \text{ Borel such that } \mathbb{1}_{B_1} + \mathbb{1}_{B_2} \ge \mathbb{1}_B\Big\},$$

and the minimum is attained.

d_untangling 2. The local theory of forward untangling

Consider a proper set $\Omega \subset \mathbb{R}^{d+1}$, and let Ω^ε be the perturbed set constructed in Theorem 1.9. For convenience, in the first part of this section we will drop the index ε and refer to Ω^ε directly as Ω. Recall that the sets S_1, S_2 are defined in Theorem 1.9, so that essentially all inflow and outflow of $\rho(1,\boldsymbol{b})$ are occurring on open sets which are contained in finitely many time-flat hyperplanes $\{t = t_i\}$ (note: the t_i here are not the one used to define the restriction operators, but the ones corresponding to the S_1, S_2). Recall that $\mathtt{p}_t(S_1) \subset \{\{t = t_i\}$ is locally proper$\}$. Define now

$$\eta^{\mathrm{in}} := \eta\llcorner_{\{\mathrm{Graph}\,\gamma\cap S_1\neq\emptyset\}} = \int_{S_1} \eta^{\mathrm{in}}_z \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega),$$

where the last formula is the disintegration of η^{in} w.r.t. its evaluation on S_1.

We give the following

rd_untangling **Definition 2.1.** A Lagrangian representation η of $\rho(1,\boldsymbol{b})$, with $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu$, is said to be *forward untangled* if the following condition holds true: η is concentrated on a set $\Delta^{\mathrm{for}} \subset \Gamma$ made up of trajectories such that for every $\gamma, \gamma' \in \Delta^{\mathrm{for}} \times \Delta^{\mathrm{for}}$ the following implication holds:

$$\text{if there exists } t \in \big[\max\{t^-_\gamma, t^-_{\gamma'}\}, \min\{t^+_\gamma, t^+_{\gamma'}\}\big] \text{ such that } \gamma(t) = \gamma'(t) \text{ then}$$
$$\mathrm{Graph}\gamma\llcorner_{[t,\min\{t^+_\gamma, t^+_{\gamma'}\}]} \text{ coincides with } \mathrm{Graph}\,\gamma'\llcorner_{[t,\min\{t^+_\gamma, t^+_{\gamma'}\}]}.$$

This means that the trajectories can bifurcate only in the "past".

gling_notions **Remark 2.2.** Differently from the notion of untangling [BB17, Definition 8.10], where the trajectories can split in the initial time (for example when $\mu = \delta_{(t,x)}$), here this is not allowed.

2.1. **Local theory of forward untangling.** We begin by pointing out a necessary condition for a Lagrangian representation to be forward untangled. Let $\tilde{t}^-_\gamma$ be the entering time in Ω, i.e. $\gamma(\tilde{t}^-_\gamma) \in S_1$.

er_condition **Proposition 2.3.** *Let η be a forward untangled Lagrangian representation and let Ω be a perturbed proper set. Then, for every $\varpi, R > 0$ there exists $r > 0$ such that*

$$\int \frac{1}{\sigma(B^d_r(\gamma(\tilde{t}^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde{t}^-_{\gamma'}) \in \gamma(\tilde{t}^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\,\gamma'\llcorner_{[\tilde{t}^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \not\subseteq \mathrm{Graph}\,\gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \leq \varpi,$$

where

$$\sigma\big(B^d_r(\gamma(\tilde{t}^-_\gamma))\big) = \eta^{\mathrm{in}}\big(\big\{\gamma' : \gamma'(\tilde{t}^-_{\gamma'}) \in \gamma(\tilde{t}^-_\gamma) + B^d_r(0)\big\}\big) = \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(\gamma(\tilde{t}^-_\gamma) + B^d_r(0)\big).$$

See (1.6) for the notation.

Proof. The assumption that Ω is proper, and thus the inner and outer distributional traces coincide, implies that $\eta(\{\gamma(t^-_\gamma) \in \partial\Omega\}) = 0$ so that η-a.e. γ crosses $\partial\Omega$ in an inner point of $\tilde{t}^-_\gamma \in (t^-_\gamma, t^+_\gamma)$.

By the forward untangling, it follows that writing the disintegration

$$\eta^{\mathrm{in}} = \int_{S_1} \eta^{\mathrm{in}}_z \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega),$$

then for $\mathsf{Tr}^+\big(\rho(1,\boldsymbol{b}),\Omega\big)$-a.e. $z \in S_1$ there exists a curve γ_z such that

$$\eta^{\mathrm{in}}_z\big(\big\{\gamma' : \mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\gamma_z\big\}\big) = 1,$$

i.e. only the curves which are restriction of γ_z enter in Ω. The map $z \mapsto \gamma_z$ can be taken Borel, being the graph of γ_z the union of the graphs of a σ-compact subset where η^{in}_z is concentrated.

By Lusin's Theorem, for every $\delta > 0$, we can find a compact set $K_\delta \subset S_1$ with

$$\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1 \setminus K_\delta) < \delta \quad \text{and} \quad K_\delta \ni z \mapsto \gamma_z \text{ continuous w.r.t. } C^0\text{-topology.} \tag{2.1}$$ eq:lusin

By the uniform continuity on compact sets, for every $R > 0$ there exists $r_R > 0$ such that

$$z, z' \in K_\delta : \ \gamma_{z'}(\tilde{t}^-_{\gamma'}) \in \gamma_z(\tilde{t}^-_\gamma) + B^d_{r_R}(0) \Rightarrow \mathrm{Graph}\,\gamma_{z'}\llcorner_{[\tilde{t}^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \subset \mathrm{Graph}\,\gamma_z + B^d_R(0). \tag{2.2}$$ eq:unifC

Since for $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$-a.e. $z \in S_1$ it holds

$$\lim_{r \to 0} \frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) = 0$$

by definition of Lebesgue point, we can further consider a compact set $K'_\delta \subset K_\delta$ such that

$$\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(K_\delta \setminus K'_\delta) < \delta \tag{2.3}$$ Equa:choice_K_pr

the above convergence is uniform, i.e. if $z \in K'_\delta$, $r < r'$ then

$$\frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) < \delta. \tag{2.4}$$ Equa:unifo_lebes

Set now

$$r = \min\{r_R, r'\}.$$

Observe that we can write

$$\int \frac{1}{\sigma(B^d_r(\gamma(\tilde t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde t^-_{\gamma'}) \in \gamma(\tilde t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \not\subseteq \mathrm{Graph}\, \gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma)$$
$$\leq \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{\{z' : |z'-z| < r,\, \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde t^-_{\gamma_z}, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\, \gamma_z + B^d_R(0)\}} \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz), \tag{2.5}$$

Equa:inquaforwGr

because

$$\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde t^-_{\gamma'}) \in \gamma(\tilde t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \not\subseteq \mathrm{Graph}\, \gamma + B^d_R(0) \end{array} \right\}$$
$$\subset \left\{ \gamma' : \mathrm{Graph}\, \gamma' \subset \mathrm{Graph}\, \gamma_{z'}, |z'-z| < r,\, \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde t^-_{\gamma_z}, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\, \gamma_z + B^d_R(0) \right\}.$$

Now we split the integral in z of the r.h.s. of (2.5) in two terms, one on the compact set K_δ and the other in the complement. For simplicity, denote by

$$\mathcal{A}_R := \left\{ (z,z') \in S_1 \times S_1 : \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\, \gamma_z + B^d_R(0) \right\}.$$

Then we have

$$\int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{\{z' : |z'-z| < r,\, \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\, \gamma_z + B^d_R(0)\}} \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$= \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{B^d_r(z) \cap \mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$= \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$\leq \int_{(K'_\delta)^c} \Bigg\{ \underbrace{\frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(\gamma(\tilde t^-_\gamma) + B^d_r(z)\big)}_{\leq 1} \Bigg\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$\quad + \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$\overset{(2.3)}{<} 2\delta + \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz),$$

For the second integral we notice that the contribution of $z' \in B_r(z) \cap K_\delta$ is zero, in view of (2.2). Hence by Fubini Theorem

$$\int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$(K'_\delta \subset K_\delta) \quad = \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z) \cap K^c_\delta} \mathbb{1}_{\mathcal{A}_R(z)}(z') \, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$\leq \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz)$$
$$\overset{(2.4)}{<} \delta.$$

The proof is concluded by taking δ so that $3\delta \leq \varpi$. □

r_to_0untangl

Corollary 2.4. *Under the assumptions of Proposition 2.3 for every $R > 0$ it holds*

$$\lim_{r \to 0} \int \frac{1}{\sigma(B^d_r(\gamma(\tilde t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde t^-_{\gamma'}) \in \gamma(\tilde t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[\tilde t^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \not\subseteq \mathrm{Graph}\, \gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) = 0.$$

We now turn to prove the converse, which is more delicate and thus we will split the proof in several lemmata. We will denote now by η a Lagrangian representation of $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu$ in Ω (which can be taken as the restriction of a Lagrangian representation in $\mathbb{R}^{d+1}$, in view of Theorem 1.11). Here $\tilde{t}^-_\gamma = t^-_\gamma$, and we will write for shortness

$$\mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} = \mathrm{Graph}\, \gamma' \llcorner_{[\max\{t^-_{\gamma'}, t^-_\gamma\}, \min\{t^+_{\gamma'}, t^-_\gamma\}]}.$$

ward_unteng_1 **Proposition 2.5.** *Let η be a Lagrangian representation in a perturbed proper set $\Omega \subset \mathbb{R}^{d+1}$. Let $\varpi > 0$ so that for all $R > 0$ there exists $r = r(R) > 0$ such that*

$$\int_\Gamma \frac{1}{\sigma(B^d_r(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' \in \Gamma : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \leq \varpi. \tag{2.6}$$ Equa:approximat_

Then there exists a σ-continuous function

$$S_1 \ni z \mapsto \bar{\gamma}_z \in \Gamma$$

such that if

$$U = \Big\{ \gamma \in \Gamma : \mathrm{Graph}\, \gamma \subset \mathrm{Graph}\, \bar{\gamma}_{\gamma(t^-_\gamma)} \Big\},$$

then it holds

$$\eta^{\mathrm{in}}(U^c) \leq \inf_{C \geq 3/2} \left\{ 2C\varpi + \frac{\mu^-(\Omega)}{C} \right\}. \tag{2.7}$$ Equa:estim_U_eta

We begin by proving the following lemma, which shows how the piece of information contained in the hypothesis of Proposition 2.5 can be passed to the limit:

lemma:bif **Lemma 2.6.** *In the setting of Proposition 2.5, it holds*

$$\int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z \left(\left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \mathrm{Graph}\, \gamma \not\subseteq \mathrm{Graph}\, \gamma',\ \mathrm{Graph}\, \gamma' \not\subseteq \mathrm{Graph}\, \gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1,\boldsymbol{b}), \Omega)(dz) \leq \varpi. \tag{2.8}$$ eq:fine-lemma-bi

Proof. For fixed $\bar{R} \geq R$, γ we have

$$\begin{aligned} &\eta^{\mathrm{in}} \Big(\Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \Big\} \Big) \\ &\geq \eta^{\mathrm{in}} \Big(\Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar{R}}(0) \Big\} \Big). \end{aligned}$$

By keeping $\bar{R}$ fixed and sending $R \searrow 0$, we obtain a family of $\{r_n\}_{n \in \mathbb{N}}$ such that

$$\int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar{R}}(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \leq \varpi.$$

We now let $r_n \to 0$ and we make use of the following facts:

(1) the set
$$\Big\{ \gamma' : \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \Big\}$$
is open in Γ;

(2) by the properties of the disintegration, for $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$-a.e. $z \in S_1$ it holds
$$⨍_{B^d_r(z)} \eta_{z'}\, \mathsf{Tr}(\rho(1,\boldsymbol{b}), \Omega)(dz) \rightharpoonup \eta_z, \qquad \text{as measures on } \Gamma. \tag{2.9}$$ Equa:weak_conv_e

At this point one uses Fatou's Lemma and the l.s.c. of the weak convergence on open sets to obtain

$$\begin{aligned} \varpi &\geq \liminf_n \int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar{R}}(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \\ \text{(Fatou)} \quad &\geq \int \liminf_n \left\{ ⨍_{B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}}_{z'} \Big(\Big\{ \gamma' : \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar{R}}(0) \Big\} \Big) \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \eta^{\mathrm{in}}(d\gamma) \\ &\overset{(2.9)}{\geq} \int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z \Big(\Big\{ (\gamma, \gamma') : \max_{t \in [t^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \big\{ \mathrm{dist}\, (\gamma(t), \gamma'(t)) \big\} > \bar{R} \Big\} \Big) \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz). \end{aligned}$$

Finally, we send $\bar{R} \to 0$ and we use the Monotone Convergence Theorem, so that

$$\int \eta_z^{\rm in} \otimes \eta_z^{\rm in} \left(\left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \mathrm{Graph}\, \gamma \not\subseteq \mathrm{Graph}\, \gamma', \\ \mathrm{Graph}\, \gamma' \not\subseteq \mathrm{Graph}\, \gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz) \le \varpi,$$

which is what we wanted to prove. □

We now show an elementary inequality which will be very useful to conclude the argument of the proof of Proposition 2.5.

a:inequality **Lemma 2.7.** *If $D_0 \ge 3/2$ it holds*

$$1 - \alpha \le D_0 (1 - \alpha) \max \big\{ 1 - \alpha, 2(\alpha - \beta) \big\} + \frac{\beta}{D_0}, \qquad \text{for all} \quad 0 \le \beta \le \alpha \le 1.$$

We are eventually ready to prove Proposition 2.5.

Proof (of Proposition 2.5). To begin, let us define a partial order relation on the set Γ (note that we are just looking to curves contained in Ω, not defined in the whole $\mathbb{R}^{d+1}$). We consider the set

$$\mathcal{R} := \big\{ (\gamma, \gamma') \in \Gamma^2 : \mathrm{Graph}\, \gamma \subset \mathrm{Graph}\, \gamma' \big\}.$$

It is immediate to check the relation $\mathcal{R}$ is a partial order on Γ. We will write $\gamma \preccurlyeq \gamma'$ for $(\gamma, \gamma') \in \mathcal{R}$, and $\gamma \prec \gamma'$ meaning $(\gamma, \gamma') \in \mathcal{R}$ and $\gamma \neq \gamma'$. Notice that, in this language, we can rephrase the conclusion of Lemma 2.6, namely Formula (2.8), by saying that

$$\begin{aligned} \varpi &\ge \int \eta_z^{\rm in} \otimes \eta_z^{\rm in} \left(\left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \mathrm{Graph}\, \gamma \not\subseteq \mathrm{Graph}\, \gamma', \\ \mathrm{Graph}\, \gamma' \not\subseteq \mathrm{Graph}\, \gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz) \\ &= \int \eta_z^{\rm in} \otimes \eta_z^{\rm in} \Big(\Big\{ (\gamma, \gamma') : \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma),\, (\gamma, \gamma') \in \Gamma^2 \setminus \big(\mathcal{R} \cup \mathcal{R}^T \big) \Big\} \Big) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz), \end{aligned} \tag{2.10}$$ eq:varpi-relatio

where we have used the notation $\mathcal{R}^T$ to denote the set $\{(\gamma, \gamma') : (\gamma', \gamma) \in \mathcal{R}\}$.

Consider the function

$$z, \gamma \mapsto \eta_z^{\rm in} \big(\{ \gamma' : \gamma' \preccurlyeq \gamma \} \big). \tag{2.11}$$ Equa:z_gamma_eta

This function is u.s.c. in every compact set where $z \mapsto \eta_z^{\rm in}$ is weakly continuous: indeed observe that if $\gamma_n \to \gamma$ then

$$\big\{ \gamma' : \gamma' \preccurlyeq \gamma_n \big\} \quad \text{converges in Hausdorff distance to} \quad \big\{ \gamma' : \gamma' \preccurlyeq \gamma \big\},$$

so that definitely for every $\epsilon > 0$

$$\big\{ \gamma' : \gamma' \preccurlyeq \gamma_n \big\} \subset \big\{ \gamma' : \mathrm{Graph}\, \gamma' \subset \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^{d+1}_\epsilon(0) \big\}. \tag{2.12}$$ Equa:definit_co

Then using the Monotone Convergence Theorem, the u.s.c. of measures of closed sets w.r.t. weak convergence and (2.12) we get for $z_n \to z$

$$\begin{aligned} \eta_z^{\rm in} \big(\{ \gamma' : \gamma' \preccurlyeq \gamma \} \big) &= \lim_{\epsilon \searrow 0} \eta_z^{\rm in} \big(\{ \gamma' : \mathrm{Graph}\, \gamma' \subset \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^{d+1}_\epsilon(0) \} \big) \\ &\ge \lim_{\epsilon \searrow 0} \lim_n \eta_{z_n}^{\rm in} \big(\{ \gamma' : \mathrm{Graph}\, \gamma' \subset \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^{d+1}_\epsilon(0) \} \big) \\ &\ge \lim_{\epsilon \searrow 0} \lim_n \eta_{z_n}^{\rm in} \big(\{ \gamma' : \gamma' \preccurlyeq \gamma_n \} \big) \\ &\ge \lim_n \eta_{z_n}^{\rm in} \{ \gamma' : \gamma' \preccurlyeq \gamma_n \}. \end{aligned} \tag{2.13}$$ Equa:z_gamma_pr

For $z \in S_1$ let us now define

$$a_z := \sup_\gamma \eta_z^{\rm in} \big(\{ \gamma' : \gamma' \preccurlyeq \gamma \} \big).$$

Being $\gamma \mapsto \eta_z^{\rm in}(\{\gamma', \gamma' \preccurlyeq \gamma\})$ u.s.c. by (2.13), it follows that a_z is a Souslin set (or analytic) by observing that

$$\{ z : a_z \ge a \} = \mathtt{p}_z \big\{ (z, \gamma) : \eta_z^{\rm in} \big(\{ \gamma' : \gamma' \preccurlyeq \gamma \} \big) \ge a \big\},$$

i.e. a projection of a closed set [Sri98, Proposition 4.1.1]. In particular it is universally measurable.

Thus, for $z \in S_1$, for every $\varepsilon > 0$, by definition of supremum, there exists γ_z such that, having set $A_z := \big\{\gamma' : \gamma' \preccurlyeq \gamma_z\big\}$, it holds

$$\eta_z^{\mathrm{in}}\big(A_z\big) \geq a_z - \varepsilon. \tag{2.14}$$

Equa:choice_gamm

By [Sri98, Theorem 5.2.1], we can take γ_z to be $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\llcorner_{S_1}$-measurable, and hence σ-continuous by redefining γ_z on a $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$-negligible set. By prolonging the curve γ_z as

$$\gamma_z(t) \mapsto \begin{cases} \gamma_z(t) & t \in [t^-_\gamma, t^+_{\gamma_z}], \\ \gamma_z(t^+_\gamma) & t > t^+_{\gamma_z}, \end{cases}$$

and restricting it to the first exiting time, we can assume that its initial and final point belongs to S_1 and $\partial\Omega$, respectively: in particular (2.14) still holds and γ_z cannot be prolonged in Ω.

Set $B_z := \big\{\gamma' : \gamma' \prec \gamma_z\big\}$ and

$$b_z := \eta_z^{\mathrm{in}}\big(B_z\big).$$

Clearly, $b_z \leq a_z$ for $z \in S_1$; furthermore, we emphasize that B_z is the set of curves whose graph is contained in the graph of the almost-maximizer γ_z but are different from it: in view of this, these curves must have a final point inside the domain Ω, so that the following bound holds:

$$\begin{aligned} \mu^-(\Omega) &= \int_\Gamma \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta(d\gamma) \\ (\eta^{\mathrm{in}} \leq \mathtt{R}_\Omega\eta) \quad &\geq \int \bigg\{ \int_\Gamma \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta_z^{\mathrm{in}}(d\gamma)\bigg\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ \text{by above observation} \quad &\geq \int \bigg\{ \int_{B_z} \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta_z^{\mathrm{in}}(d\gamma)\bigg\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &= \int \eta_z^{\mathrm{in}}(B_z)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &= \int b_z\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz). \end{aligned} \tag{2.15}$$

Equa:divergence_

Next, we first observe that for every γ

$$\eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big) \leq a_z, \tag{2.16}$$

Equa:bound_gloca

by the very definition of a_z. Since γ_z cannot be prolonged and the implication

$$\gamma \notin A_z, \gamma \preccurlyeq \gamma' \quad \Longrightarrow \quad \gamma' \notin A_z, \tag{2.17}$$

Equa:inclus_gamm

we obtain the inclusions

$$\begin{aligned} \big\{(\gamma,\gamma') : \gamma \notin A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\} &= \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma \prec \gamma' \vee \gamma' \preccurlyeq \gamma\big\} \\ &= \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma \prec \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \\ &\underset{(2.17)}{\subset} \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \prec \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \\ &\subset \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \preccurlyeq \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \end{aligned}$$

Being the initial and final sets symmetric in γ, γ', we obtain

$$\big\{(\gamma,\gamma') : (\gamma,\gamma') \notin A_z \times A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\} \subset \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \preccurlyeq \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \tag{2.18}$$

Equa:at_least_no

Integrating w.r.t. $\eta_z^{\mathrm{in}} \times \eta_z^{\mathrm{in}}$

$$\begin{aligned} \eta_z^{\mathrm{in}} \times \eta_z^{\mathrm{in}}\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \notin A_z \times A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\}\big) &\underset{(2.18)}{\leq} 2\int_{\Gamma \setminus A_z} \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big)\eta_z^{\mathrm{in}}(d\gamma) \\ &\underset{(2.16)}{\leq} 2(1 - \eta_z^{\mathrm{in}}(A_z))a_z. \end{aligned}$$

Hence, since

$$\eta_z^{\mathrm{in}} \times \eta_z^{\mathrm{in}}(A_z \times A_z) = \eta(A_z)^2,$$

we get

$$\begin{aligned} \eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z\big(\{(\gamma,\gamma') : (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\}\big) &\leq 2(1-\eta(A_z))a_z + \eta(A_z)^2 \\ &= 2a_z - a_z^2 + \big(a_z - \eta(A_z)\big)^2. \end{aligned} \tag{2.19}$$

Equa:estima_set_

Hence

$$\begin{aligned} \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z\big(\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\}\big) &= 1 - \eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z\big(\{(\gamma,\gamma') : (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\}\big) \\ &\underset{(2.19)}{\geq} 1 - \Big(2a_z - a_z^2 + \big(a_z - \eta(A_z)\big)^2\Big) \\ &= (1-a_z)^2 - \big(a_z - \eta(A_z)\big)^2 \\ &\underset{(2.14)}{\geq} (1-a_z)^2 - \epsilon^2. \end{aligned} \tag{2.20}$$

Equa:first_estin

On the other hand, since γ_z cannot be prolonged,

$$\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\big\} \supset \big[\{\gamma_z\} \times (\Gamma \setminus A_z)\big] \cup \big[(\Gamma \setminus A_z) \times \{\gamma_z\}\big] \tag{2.21}$$

Equa:second_incl

so that

$$\begin{aligned} \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z\big(\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\}\big) &\underset{(2.21)}{\geq} 2\eta^{\mathrm{in}}_z(\{\gamma_z\})\big(1 - \eta^{\mathrm{in}}_z(A_z)\big) \\ &= 2\big(\eta^{\mathrm{in}}_z(A_z) - \eta^{\mathrm{in}}_z(B_z)\big)\big(1 - \eta^{\mathrm{in}}_z(A_z)\big) \\ &= 2\big(\eta^{\mathrm{in}}_z(A_z) - b_z\big)\big(1 - \eta^{\mathrm{in}}_z(A_z)\big) \\ &\underset{(2.14)}{\geq} 2\big(a_z - b_z - \epsilon\big)(1-a_z). \end{aligned} \tag{2.22}$$

Equa:second_esti

Hence we can continue (2.10) as

$$\begin{aligned} \varpi &\geq \int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z\big(\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\}\big)\, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ (2.20),(2.22)\quad &\geq \int_{S_1} (1-a_z)\max\{1-a_z, 2(a_z-b_z)\}\,\rho(z)\,\mathcal{H}^d(dz) - (2\epsilon+\epsilon^2)\int_{S_1} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega). \end{aligned} \tag{2.23}$$

Equa:proceed_va

On the other hand by Lemma 2.7 with $\alpha = a_z$ and $\beta = b_z$ and $D_0 \geq 3/2$ we have

$$\eta^{\mathrm{in}}_z(\Gamma \setminus A_z) \leq 1 - a_z + \epsilon \leq D_0(1-a_z)\max\big\{1-a_z, 2(a_z-b_z)\big\} + \frac{b_z}{D_0} + \epsilon,$$

so that,

$$\begin{aligned} &\int \eta^{\mathrm{in}}_z(\Gamma \setminus A_z)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &\qquad \leq D_0 \int_{S_1} \bigg[(1-a_z)\max\big\{1-a_z, a_z-b_z\big\} + \frac{b_z}{D_0}\bigg]\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) + \epsilon\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1) \\ &\qquad \underset{(2.23)}{\leq} D_0\varpi + \int \frac{b_z}{D_0}\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) + \epsilon(1+2D_0+D_0\epsilon)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1). \end{aligned}$$

Since ϵ can be chosen arbitrary small,

$$\begin{aligned} \int \eta^{\mathrm{in}}_z(\Gamma \setminus A_z)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) &\underset{(2.15)}{\leq} D_0\varpi + \frac{\mu^-(\Omega)}{D_0} + \epsilon(1+2D_0+D_0\epsilon)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1) \\ &\leq 2D_0\varpi + \frac{\mu^-(\Omega)}{D_0}, \end{aligned} \tag{2.24}$$

eq:quasi-finale

and this yields the desired conclusions: indeed, setting $U := \bigcup_{z\in S_1} A_z$, from (2.24) we have that

$$\eta^{\mathrm{in}}(U^c) \leq \inf_{C \geq 3/2}\bigg\{2C\varpi + \frac{\mu^-(\Omega)}{C}\bigg\}$$

and $\eta^{\mathrm{in}}_z \llcorner_U$ is supported on a set of curves which are restrictions of the curve γ_z by construction. □

In order to pass from uniqueness of the curves starting from the same initial point (Proposition 2.5) to forward untangling, we assume that the condition (2.6) holds for every Lagrangian representation η which is representing $\rho(1, \boldsymbol{b})\llcorner\Omega$.

untang_Omega **Proposition 2.8.** *Assume that (2.6) holds for every Lagrangian representation η. Then for every η there exists a set of forward untangled trajectories U such that*

- *$\eta^{\mathrm{in}}_z\llcorner U$ is supported on a set of curves whose graphs are a subset of a given curve $\gamma_z \in U$;*
- *it holds*

$$\eta^{\mathrm{in}}(U^c) \leq \inf_{C\geq 3/4}\bigg\{8C\varpi + \frac{\mu^-(\Omega)}{C}\bigg\}.$$

Proof. First, by Proposition 2.5 we know that for every η there exists a set U such that $\eta\llcorner U$ has the property that η^{in}_z is supported on the set of curves which are subset of a given curve γ_z. We restrict η to U, removing a set of trajectories with η-measure bounded by (2.7).

Let us define the set

$$NF = \big\{(\gamma,\gamma') : \gamma(t^-_\gamma) \neq \gamma'(t^-_{\gamma'}),\ \gamma,\gamma' \text{ cross}\big\},$$

where we say that two trajectories cross if there exists $t \in (t^-_\gamma, t^+_\gamma) \cap (t^-_{\gamma'}, t^+_{\gamma'})$ such that $\gamma(t) = \gamma'(t)$, but $(\gamma,\gamma') \notin \Delta^{\text{ for}}$, i.e. they bifurcate at a time greater than t (assuming this time to be the first time where the curves meet).

Consider the set of positive transference plans $\Pi^{\leq}(\eta^{\mathrm{in}},\eta^{\mathrm{in}})$, which are concentrated on the set NF, and suppose by contradiction that there exists a plan $\bar\pi \in \Pi^{\leq}(\eta^{\mathrm{in}},\eta^{\mathrm{in}})$ of positive measure: being NF symmetric, we can assume that also $\bar\pi$ is. Write the disintegration of $\bar\pi$ w.r.t. the marginals

$$\bar\pi = \int \bar\pi_\gamma \eta^{\mathrm{in}}(d\gamma). \tag{2.25}$$ Equa:disint_bar_

For every $(\gamma,\gamma') \in NF$, if $t_{\gamma,\gamma'}$ is the first time of intersection so that γ,γ' are certainly splitting for some $t > t_{\gamma,\gamma'}$, define

$$\tilde\gamma_{\gamma'}(t) = \begin{cases} \gamma(t) & t^-_\gamma \leq t \leq t_{\gamma,\gamma'}, \\ \gamma'(t) & t_{\gamma,\gamma'} < t \leq t^+_{\gamma'}, \end{cases} \qquad \tilde\gamma'_\gamma(t) = \begin{cases} \gamma'(t) & t^-_{\gamma'} \leq t \leq t_{\gamma,\gamma'}, \\ \gamma(t) & t_{\gamma,\gamma'} < t \leq t^+_\gamma. \end{cases}$$

This map encodes the operation of exchanging the trajectories of γ,γ' at the crossing time $t_{\gamma,\gamma'}$.

By the property $\bar\pi \in \Pi^{\leq}(\eta^{\mathrm{in}},\eta^{\mathrm{in}})$, one deduces that

$$(\mathtt{p}_\gamma)_\sharp\bar\pi \leq \eta^{\mathrm{in}}, \tag{2.26}$$ Equa:bar_pi_leq

and thus the measure

$$\tilde\eta^{\mathrm{in}} = \eta^{\mathrm{in}} - \frac{1}{2}(\mathtt{p}_\gamma)_\sharp\bar\pi + \frac{1}{2}(\tilde\gamma_{\gamma'})_\sharp\bar\pi \tag{2.27}$$ Equa:tilde_eta_

is a positive measure. Moreover, since by construction

$$\delta_{\gamma(t)} + \delta_{\gamma'(t)} = \delta_{\tilde\gamma_{\gamma'}(t)} + \delta_{\tilde\gamma'_\gamma(t)}, \tag{2.28}$$ Equa:gamma_sym_

it follows that

$$\begin{aligned} \int \delta_{\gamma(t)}\tilde\eta^{\mathrm{in}}(d\gamma) &\underset{(2.27)}{=} \int \delta_{\gamma(t)}\eta^{\mathrm{in}}(d\gamma) + \frac{1}{2}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi_\gamma(d\gamma')\eta^{\mathrm{in}}(d\gamma) \\ &\underset{(2.28)}{=} \int \delta_{\gamma(t)}\eta^{\mathrm{in}}(d\gamma) + \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma d\gamma') - \frac{1}{4}\int \big(\delta_{\tilde\gamma'_\gamma(t)} - \delta_{\gamma'(t)}\big)\bar\pi(d\gamma d\gamma') \\ \text{exchange } \gamma\to\gamma' \quad &= \int \delta_{\gamma(t)}\eta^{\mathrm{in}}(d\gamma) + \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma d\gamma') - \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma' d\gamma) \\ \bar\pi \text{ symmetric} \quad &= \int \delta_{\gamma(t)}\eta^{\mathrm{in}}(d\gamma). \end{aligned}$$

This means that $\tilde\eta^{\mathrm{in}}$ is a representation of $(t,\gamma(t))_\sharp\eta^{\mathrm{in}}$. In particular η^{in} and $\tilde\eta^{\mathrm{in}}$ have the same trace in $\partial\Omega$.

Write

$$
\begin{aligned}
\tilde{\eta}^{\mathrm{in}} &= \int_{S_1} \tilde{\eta}^{\mathrm{in}}_z \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int_{S_1} \bigg(\eta_z - \frac{1}{2} \int \|\bar{\pi}_\gamma\| \eta_z(d\gamma) + \frac{1}{2} \int (\tilde{\gamma}_{\gamma'})_\sharp \bar{\pi}_\gamma \eta_z(d\gamma) \bigg) \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz).
\end{aligned}
$$

The above formula follows from (2.25), and the fact that $\tilde{\gamma}_{\gamma'}$ has the same initial point of γ because of its definition. By $\bar{\pi} \in \Pi^{\leq}(\eta^{\mathrm{in}},\eta^{\mathrm{in}})$ one deduce that $\|\bar{\pi}_\gamma\| \leq 1$, and then

$$
\tilde{\eta}^{\mathrm{in}}_z \big(\{ \gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z \} \big) = \|\eta^{\mathrm{in}}_z\| - \frac{1}{2} \int \|\bar{\pi}_\gamma\| \eta^{\mathrm{in}}_z(d\gamma) \geq \frac{1}{2}.
$$

We have used the fact that $\mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z$ for $(\gamma,\gamma') \in NF$.

The same argument implies that if $\mathrm{Graph}\,\bar{\gamma} \not\subseteq \mathrm{Graph}\,\gamma_z$ then

$$
\begin{aligned}
\tilde{\eta}_z \big(\{ \gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \} \big) &= \eta_z \big(\{ \gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z \} \big) \\
&\quad - \frac{1}{2} \int_{\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z\}} \|\bar{\pi}_\gamma\| \eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \frac{1}{2} \int (\gamma'_\gamma)_\sharp \bar{\pi}_\gamma \big(\{ \gamma' : \mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\bar{\gamma}, \mathrm{Graph}\,\gamma' \not\subseteq \mathrm{Graph}\,\gamma_z \} \big) \eta^{\mathrm{in}}(d\gamma) \\
&= \eta_z \big(\{ \gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z \} \big) \\
&\quad - \frac{1}{2} \int_{\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z\}} \|\bar{\pi}_\gamma\| \eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \frac{1}{2} \int \bar{\pi}_\gamma \big(\{ \gamma' : \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \subset \mathrm{Graph}\,\bar{\gamma}, \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z \} \big) \eta^{\mathrm{in}}(d\gamma).
\end{aligned}
\tag{2.29}
$$

Equa:maxima_A_z_

Observing that

$$
\mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z \quad \Longrightarrow \quad \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \not\subseteq \mathrm{Graph}\,\bar{\gamma}, \tag{2.30}
$$

Equa:inclusion_g

we get for the last integral of (2.29)

$$
\begin{aligned}
&\int \bar{\pi}_\gamma \big(\{ \gamma' : \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \subset \mathrm{Graph}\,\bar{\gamma}, \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z \} \big) \eta^{\mathrm{in}}(d\gamma) \\
&\underset{(2.30)}{=} \int_{\{\gamma : \mathrm{Graph}\,\gamma \not\subseteq \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z\}} \bar{\pi}_\gamma \big(\{ \gamma' : \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \subset \mathrm{Graph}\,\bar{\gamma}, \mathrm{Graph}\,\tilde{\gamma}_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z \} \big) \eta^{\mathrm{in}}(d\gamma) \qquad (2.31) \\
&\leq \int_{\{\gamma : \mathrm{Graph}\,\gamma \not\subseteq \mathrm{Graph}\,\bar{\gamma} \cap \mathrm{Graph}\,\gamma_z\}} \|\bar{\pi}_\gamma\| \eta^{\mathrm{in}}(d\gamma).
\end{aligned}
$$

Equa:maxima_A_z_

Hence we can continue (2.29) as

$$\begin{aligned}
\tilde\eta_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\}\big) &= \eta_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta_z^{\mathrm{in}}(d\gamma) \\
&\quad + \frac{1}{2}\int \bar\pi_\gamma\big(\{\gamma' : \mathrm{Graph}\,\tilde\gamma_{\gamma'} \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subset \mathrm{Graph}\,\gamma_z\}\big)\eta^{\mathrm{in}}(d\gamma) \\
&\underset{(2.31)}{\leq} \eta_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta_z^{\mathrm{in}}(d\gamma) \\
&\quad + \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}(d\gamma) \\
&= \|\eta_z^{\mathrm{in}}\| - \frac{1}{2}\|\bar\pi_\gamma\|\eta_z^{\mathrm{in}}(d\gamma) \\
&\quad + \int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \big(\|\bar\pi_\gamma\| - 1\big)\eta^{\mathrm{in}}(d\gamma) \\
&\underset{(2.26)}{\leq} \|\eta_z^{\mathrm{in}}\| - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta_z^{\mathrm{in}}(d\gamma) \\
&= \tilde\eta^{\mathrm{in}}\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\}\big).
\end{aligned} \tag{2.32}$$

Equa:maxima_A_z

We next apply Proposition 2.5 to our case with

$$A_z = \{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\},$$

which gives by formula (2.24)

$$\|\pi\| = \int \|\bar\pi_\gamma\|\eta^{\mathrm{in}}(d\gamma) = \int \tilde\eta_z^{\mathrm{in}}\big(\Gamma \setminus \{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\}\big)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \leq 2D_0\varpi + \frac{\mu^-(\Omega)}{D_0},$$

for all $D_0 \geq 3/2$.

We now resort to Proposition 1.14 to deduce that there is a subset $M = M_1 \cup M_2 \subset U$ whose measure is at most

$$\eta^{\mathrm{in}}(M) \leq \inf_{C\geq 3/2}\left\{2D_0\varpi + \frac{\mu^-(\Omega)}{D_0}\right\}$$

and such that in $U' = U \setminus M$ the trajectories γ_z, $z \in U'$, do not cross. Adding the measure of the set $\Gamma \setminus U$ estimated in Proposition 2.5 and calling U' with U, we obtain the statement. □

3. The global theory of forward untangling

_untangl_fct

3.1. Subadditivity of untangling functional.

We now want to study how the local pieces of information contained in Propositions 2.8 can be glued in a global one. Roughly speaking, we consider here the case in which the quantity ϖ is (the mass of) a measure: we will show that a suitable functional (the *forward untangling functional*) is subadditive and this allows to compare it with a measure. We begin by giving the following

:untangl_for

Definition 3.1. Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set. The *forward untangling functional for a Lagrangian representation* η is defined as

$$\mathfrak{f}^{\mathrm{for}}(\Omega) := \inf\Big\{(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{in}}(N) : \Gamma(\Omega) \setminus N \subset \Delta^{\mathrm{for}}\Big\}. \tag{3.1}$$

eq:def_functiona

In other words, the forward untangling functional applied on a proper set Ω gives the amount of curves we have to removed (from the ones seen by $(\mathtt{R}_\Omega)_\sharp\eta^{\mathrm{in}}$) so that the remaining ones are forward untangled. We remark that $\mathfrak{f}^{\mathrm{for}}$ depends on the representation η.

We now show the following remarkable property of the forward untangling functional:

p:subadd_for **Proposition 3.2.** *The functional* $\mathfrak{f}^{\rm for}$ *defined in* (3.1) *is monotone w.r.t. inclusion and subadditive on the class of proper sets. More precisely, if* $\mathrm{U}, \mathrm{V} \subset \mathbb{R}^{d+1}$ *are proper sets whose union* $\Omega := \mathrm{U} \cup \mathrm{V}$ *is proper, then*

$$\mathfrak{f}^{\rm for}(\Omega) \leq \mathfrak{f}^{\rm for}(\mathrm{U}) + \mathfrak{f}^{\rm for}(\mathrm{V}).$$

Proof. The monotonicity follows by the elementary observation that restriction of a forward untanlged set is forward untangled. We will thus concentrate on the subadditivity.

By definition, for every $\varepsilon > 0$ there exists a set $N(\mathrm{U}) \subset \Gamma(\mathrm{U})$ such that

$$\mathfrak{f}^{\rm for}(\mathrm{U}) \geq (\mathtt{R}_{\mathrm{U}})_\sharp \eta^{\rm in}(N(\mathrm{U})) - \varepsilon$$

and

$$\Gamma(\mathrm{U}) \setminus N(\mathrm{U}) \subset \Delta^{\rm for}.$$

Let $N(\mathrm{V})$ be an analogous set for V. Set

$$N := \big\{\gamma \in \Gamma(\Omega) : \exists i \, \big(\mathtt{R}^i_{\mathrm{U}}\gamma \in N(\mathrm{U})\big)\big\} \cup \big\{\gamma \in \Gamma(\Omega) : \exists i \, \big(\mathtt{R}^i_{\mathrm{V}}\gamma \in N(\mathrm{V})\big)\big\}.$$

By Proposition 1.12

$$\begin{aligned}
\eta^{\rm in}(N) &\leq \eta^{\rm in}\big(\big\{\gamma \in \Gamma(\Omega) : \exists i \, \big(\mathtt{R}^i_{\mathrm{U}}(\gamma) \in N(\mathrm{U})\big)\big\}\big) + \eta^{\rm in}\big(\big\{\gamma \in \Gamma(\Omega) : \exists i \, \big(\mathtt{R}^i_{\mathrm{V}}(\gamma) \in N(\mathrm{V})\big)\big\}\big) \\
&\leq (\mathtt{R}_{\mathrm{U}})_\sharp \eta^{\rm in}(N(\mathrm{U})) + (\mathtt{R}_{\mathrm{V}})_\sharp \eta^{\rm in}(N(\mathrm{V})) \\
&\leq \mathfrak{f}^{\rm for}(\mathrm{U}) + \mathfrak{f}^{\rm for}(\mathrm{V}) + 2\varepsilon.
\end{aligned}$$

Being ε arbitrary and $\mathfrak{f}^{\rm for}(\Omega) \leq \eta^{\rm in}(N)$, we thus obtain that $\mathfrak{f}^{\rm for}(\Omega) \leq \mathfrak{f}^{\rm for}(\mathrm{U}) + \mathfrak{f}^{\rm for}(\mathrm{V})$ if we show that $\Gamma(\Omega) \setminus N \subset \Delta^{\rm for}$. To do this, observe that

$$\mathtt{R}_{\mathrm{U}}(\Gamma(\Omega)) \subset \Gamma(\mathrm{U}),$$

and the same for V. Hence, if $\mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \neq \emptyset$ then

(1) if

$$\mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{clos}\,\mathrm{U} \neq \emptyset,$$

and then they must coincide forward in time in $\mathrm{clos}\,\mathrm{U}$;

(2) or

$$\mathrm{Graph}\,\gamma\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{Graph}\,\gamma'\llcorner_{\mathrm{clos}\,\Omega} \cap \mathrm{clos}\,\mathrm{V} \neq \emptyset,$$

and then they must coincide forward in time in $\mathrm{clos}\,\mathrm{V}$;

Hence, if $\gamma, \gamma' \in \Gamma(U \cup V)$ and if $\gamma(\bar t) = \gamma'(\bar t)$, then $\gamma = \gamma'$ in the interval of time such that $\bar t \in \gamma^{-1}(\mathrm{clos}\,U)$ or $\bar t \in \gamma^{-1}(\mathrm{clos}\,V)$. When γ exits one of the sets and enters in the other, a simple iterative argument gives they the property of remaining together is preserved forward in time, i.e. must coincide forward in time in $\mathrm{clos}\,\mathrm{U} \cup \mathrm{clos}\,\mathrm{V} = \mathrm{clos}\,\Omega$. ☐

We are thus led to consider the following

or-untangled **Assumption 3.3.** There exist $\tau > 0$ and a non-negative measure ϖ^τ of mass τ such that for some $C \geq 1$, for all $(t,x) \in \Omega$ there exists a family of proper balls $\{B^{d+1}_r(t,x)\}_r$ such that it holds

$$\mathfrak{f}^{\rm for}(B^{d+1}_r(t,x)) \leq 8C\varpi^\tau(B^{d+1}_r(t,x)) + \frac{\mu^-(B^{d+1}_r(t,x))}{C}. \tag{3.2}$$ eq:for-estimate

For future reference let us define the measure

$$\zeta^{C,\rm for}_\tau := 8C\varpi^\tau + \frac{\mu^-}{C}.$$

By means of a standard covering argument we have the following

al-for-untan **Proposition 3.4.** *If Assumption* (3.3) *holds in a proper set* Ω *with compact closure, then*

$$\mathfrak{f}^{\rm for}(\Omega) \leq C_d \zeta^{\tau,\rm for}_C(\mathrm{clos}\,\Omega), \tag{3.3}$$ eq:zeta_measure

where C_d *is a dimensional constant.*

Proof. By Besicovitch Covering Theorem [AFP00, Theorem 2.18], for any $\varepsilon > 0$, we can cover the compact set $\operatorname{clos}\Omega$ with finitely many proper balls B_i such that (3.2) holds and

$$\sum_i \zeta_C^{\tau,\mathrm{for}}(B_i) \leq C_d \zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega) + \varepsilon.$$

Thanks to the subadditivity (and the monotonicity) of $\mathfrak{f}^{\mathrm{for}}$ we can thus write

$$\mathfrak{f}^{\mathrm{for}}(\Omega) \leq \mathfrak{f}^{\mathrm{for}}\left(\bigcup_i B_i\right) \leq \sum_i \mathfrak{f}^{\mathrm{for}}(B_i) \leq C_d \zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega) + \varepsilon$$

and sending $\varepsilon \to 0$ we obtain (3.3). □

We finally show that the validity of Assumption 3.3 is enough, thanks to the subadditivity proved in Proposition 3.2, to have that η is forward untangled.

r_untangling **Corollary 3.5.** *Suppose there exist sequences $\tau_i \searrow 0$ and $C_i \nearrow +\infty$ such that Assumption 3.3 holds for τ_i, C_i and moreover*

$$C_i \tau_i \to 0.$$

Then η is forward untangled.

Proof. It is enough to observe that under the assumptions above $\|\zeta_{C_i}^{\tau_i,\mathrm{for}}\| \to 0$. □

Using now the proof of Proposition 2.8, we conclude that if every representation η satisfies the assumptions of the above corollary, then the set of forward untangled trajectories used by every Lagrangian representation η is made of subcurves of the same family of forward untangled curves γ_z. This observation will be made precise in Theorem 4.5.

4. Uniqueness by forward untangling

In this technical section, we show how the forward untangling is related to the uniqueness problem of the continuity/transport equation.

The underlying idea is similar to the one exploited in [BB17]: considering a suitable equivalence relation on the space of curves, we want to obtain a decomposition of the space-time on which we can reduce (via the Disintegration Theorem) the PDE to one-dimensional problems. However, a technical variation occurs here in comparison with [BB17]: namely, in the case of a forward-untangled representation, the equivalence classes are not absolutely continuous curves, but they have a natural structure of tree. Thus we have to suitable adjust the reduction technique.

4.1. **Decomposition and disintegration.** Let η be a forward-untangled Lagrangian representation of $\rho(1, \boldsymbol{b})$ and let $\Delta \subset \Gamma$ be a σ-compact forward untangled set of trajectories on which η is concentrated.

:concate_def **Definition 4.1.** We say that a finite set of curves $\{\gamma_i\}_{i=0}^N \subset \Delta$ is *concatenated* if the terminal times t_i^+ are increasing and $\gamma_i(t_i^+) = \gamma_{i+1}(t_i^+)$.

For every time t we define the following partition of Δ:

$$E_x^t = \left\{ \gamma \in \Delta : \exists N \in \mathbb{N}, \{\gamma_i\}_{i=0}^N \Big(\big\{ \gamma_i \llcorner_{(-\infty,t]} \big\}_{i=0}^N \text{ concatenated } \wedge \ \gamma_N(t) = x \Big) \right\}. \tag{4.1}$$ Equa:E_t_x_def

By the forward untangling property of Δ, it is fairly easy to see that the above sets are a partition of Δ into σ-compact sets of curves which can be concatenated to a point in $\{t\} \times \mathbb{R}^d$.

Define the σ-compact subsets of $\mathbb{R}^{d+1}$

$$F_x^t = \bigcup_{\gamma \in E_x^t} \mathrm{Graph}\left(\gamma \llcorner_{(-\infty,t]}\right). \tag{4.2}$$ Equa:F_t_x_defi

t_class_tree **Lemma 4.2.** *The family of sets $\{E_x^t\}_x$ and $\{F_x^t\}_x$ are made of disjoint sets, and for every $s < t$ it holds*

$$(s,y) \in F_x^t \quad \Longrightarrow \quad E_y^s \subset E_x^t. \tag{4.3}$$ Equa:tree_struct

Moreover, for every $(s,y) \in F_x^t$, there is a unique curve connecting (s,y) to (t,x) (with $s \leq t$), and this curve is a characteristic. Finally,

$$\gamma \mapsto \{x : \gamma \in E_x^t\} \tag{4.4}$$ Equa:sigma_compa

is a map with σ-compact graph.

Proof. The second property (4.3) holds by the definition of concatenation.

If $E^t_{x'} = E^t_{x''}$, then there is a curve $\gamma \in E^t_{x'} \cap E^t_{x''}$. By forward untangling, if two curves γ', γ'' intersect $\mathrm{Graph}\,\gamma$ at t', t'', then they must coincide for $t \geq \max\{t', t''\}$. In particular, if $\{\gamma'_{i'}\}_{i'=0,\dots,N'}$ is the sequence of curves for $E^t_{x'}$ concatenating γ to $\{x'\}$, and $\{\gamma''_{i''}\}_{i''=0,\dots,N''}$ is the sequence of curves for $E^t_{x''}$ concatenating γ to $\{x''\}$, one deduces by an iterative process that

$$\mathrm{Graph}\,\gamma'_{i'} \subset \bigcup_{i''} \mathrm{Graph}\,\gamma_{i''}.$$

Hence $x' = x$.

The same reasoning shows that there is a unique curve connecting $(s, y) \in F^t_x$ to the point (t, x) with $s \leq t$, and being the finite union of characteristics with the forward untangling property, it is fairly easy to see that it is a characteristic.

The last property follows as in the analysis of [BB17, Proposition 9.1]. Indeed, the graph of the function (4.4) is the set

$$\Big\{(\gamma, x) : \exists \gamma_i, i = 0, \dots, N \Big(\gamma_i \text{ concatenated } \wedge \ \gamma_0 = \gamma, \gamma_N(t) = x\Big)\Big\},$$

which is σ-compact because

$$\Big\{(\gamma, \gamma') \in \mathit{\Delta} \times \mathit{\Delta} : \mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma' \neq \emptyset\Big\}$$

is σ-compact. □

tructur_F_t_x **Corollary 4.3.** *The set F^t_x is the union of characteristics $\tilde\gamma_{\mathfrak{b}}$, $\mathfrak{b}$ a family of indexes, each one defined in the interval $[t^-_{\mathfrak{b}}, t]$ with the forward untangling property, i.e.*

$$\tilde\gamma_{\mathfrak{b}}(s) = \tilde\gamma_{\mathfrak{b}'}(s) \quad \Longrightarrow \quad \tilde\gamma_{\mathfrak{b}}(\tau) = \tilde\gamma_{\mathfrak{b}'}(\tau) \quad \forall \tau \in [s, t].$$

Moreover $\tilde\gamma_{\mathfrak{b}}(t) = x$.

Proof. Starting from any point $(s, y) \in F^t_x$, there is a unique characteristic $\tilde\gamma_{(s,y)}$ connecting it to the final point (t, x), which is obtained by piecing together the finite set of concatenated characteristic in $\mathit{\Delta}$ connecting (s, y) to (t, x). The set of indexes $\boldsymbol{b}$ can be taken as the starting point of $\tilde\gamma$. □

We will now write the evolution of the PDE along each tree $E^{\bar t}_x$, with $\bar t$ a fixed time. For this purpose, we will restrict η to the curves which belong to some $E^{\bar t}_x$ and consider only the part of these curves contained in $(-\infty, \bar t] \times \mathbb{R}^d$. Define the map

$$r^{\bar t}(\gamma) = \gamma \llcorner_{(-\infty, \bar t]}.$$

Let

$$\eta^{\bar t} = (r^{\bar t})_\sharp (\eta \llcorner_{E^{\bar t}}), \qquad \text{with} \quad E^{\bar t} = \bigcup E^{\bar t}_x,$$

and

$$f^{\bar t} : r^{\bar t}(E^{\bar t}) \to \mathbb{R}^d \quad \text{be the quotient Borel map, } f^{\bar t}(E^{\bar t}_x) = x: \tag{4.5}$$ Equa:f_bar_t_quo

this map has σ-compact graph by (4.4). In other words, we are considering the PDE

$$\mathrm{div}\,\big(\rho(1, \boldsymbol{b}) \mathbb{1}_{(\infty, \bar t)}\big) = \mu \llcorner_{(-\infty, \bar t)} - \rho(\bar t, dx) = \mu^{\bar t},$$

and for definiteness we will consider $t \mapsto \rho(t)$ continuous from the left w.r.t. the weak convergence of measures. The existence of the measure $\rho(t)$ follows from the standard trace computation on the set $(-\infty, \bar t) \times \mathbb{R}^d$: for every test function ϕ

$$\int \phi \rho(\bar t) = \int \phi(\gamma(\bar t)) \eta(d\gamma),$$

where we consider γ defined in the semiopen interval $(t^-_\gamma, t^+_\gamma]$ (because $\rho(t)$ is continuous from the left).

Define the image measure

$$m^{\bar t} = f^{\bar t}_\sharp \eta^{\bar t}. \tag{4.6}$$ Equa:m_bar_t_de

Since

$$\rho(\bar t) = (\gamma(\bar t))_\sharp \eta,$$

it follows that $\rho(\bar t) \ll m^{\bar t}$ and then

$$\rho(\bar t, dx) = \varrho(\bar t, x) m^{\bar t}(dx).$$

By disintegration theorem

$$\eta^{\bar t} = \int \eta^{\bar t}_{\bar x} m^{\bar t}(d\bar x)$$

and hence

$$\begin{aligned}\mu^{\bar t} &= \int_\Gamma \Big[\delta_{(t^-_\gamma,\gamma(t^-_\gamma)} - \delta_{t^+_\gamma,\gamma(t^+_\gamma))}\Big]\,\eta^{\bar t}(d\gamma) \\ &= \int \bigg[\int_\Gamma \Big[\delta_{(t^-_\gamma,\gamma(t^-_\gamma)} - \delta_{t^+_\gamma,\gamma(t^+_\gamma))}\Big]\,\eta^{\bar t}_{\bar x}(d\gamma)\bigg]\, m^{\bar t}(d\bar x) =: \int \mu^{\bar t}_{\bar x} m^{\bar t}(d\bar x),\end{aligned}$$

where we have set

$$\mu^{\bar t}_{\bar x} := \int_\Gamma \Big[\delta_{(t^-_\gamma,\gamma(t^-_\gamma)} - \delta_{t^+_\gamma,\gamma(t^+_\gamma))}\Big]\,\eta^{\bar t}_{\bar x}(d\gamma). \tag{4.7}$$

Equa:disintegr_

Now we consider the projection operator (id,γ) and the measure $(\mathrm{id},\gamma)_\sharp \eta^{\bar t}_{\bar x}$. Being η a Lagrangian representation, if we define the density

$$(1,\boldsymbol{b})\rho^{\bar t}_{\bar x} = (\mathrm{id},\gamma)_\sharp\bigg(\int \big[(1,\dot\gamma)\mathcal{L}^1 \llcorner_{(t^-_\gamma,t^+_\gamma]}\big]\eta^{\bar t}_{\bar x}\bigg),$$

then by direct computation

$$\begin{aligned}\int \big[(1,\boldsymbol{b})\cdot\nabla_{t,x}\phi\big]\rho^{\bar t}_{\bar x}(t,dx)dt &= \int_\Gamma \bigg[\int_{t^-_\gamma}^{t^+_\gamma} (1,\gamma'(s))\cdot\nabla_{t,x}\phi(s,\gamma(s))\,\mathcal{L}^1(ds)\bigg]\eta^{\bar t}_{\bar x}(d\gamma) \\ &= \int_\Gamma \bigg[\int_{t^-_\gamma}^{t^+_\gamma} \frac{d}{ds}\phi(s,\gamma(s))\mathcal{L}^1(ds)\bigg]\eta^{\bar t}_{\bar x}(d\gamma) \\ &= \int_\Gamma \Big\langle \delta_{(t^-_\gamma,\gamma(t^-_\gamma)} - \delta_{t^+_\gamma,\gamma(t^+_\gamma))},\phi\Big\rangle\,\eta^{\bar t}_{\bar x}(d\gamma),\end{aligned} \tag{4.8}$$

eq:conto_diverge

hence it holds in the sense of distributions

$$\mathrm{div}(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})) = \mu^{\bar t}_{\bar x}, \qquad \rho^{\bar t}_{\bar x}(\bar t,dx) = \varrho(\bar t,\bar x)\delta_{\bar x}, \tag{4.9}$$

eq:on_tree1

for $m^{\bar t}$-a.e. $\bar x \in \mathbb{R}^d$.

The measures $\rho^{\bar t}_{\bar x}(t)$ can be obviously computed directly from the disintegration of $\rho(t)$ w.r.t. the equivalence partition $\{E^{\bar t}_{\bar x}(t)\}_{\bar x}$:

$$\rho(t,dx)\llcorner_{\cup_{\bar x}(F^{\bar t}_{\bar x})(t)} = \int \rho^{\bar t}_{\bar x}(t,dx) m^{\bar t}(d\bar x). \tag{4.10}$$

Equa:disinte_leb

In the same way, one can recover $\mu^{\bar t}_{\bar x}$ by the disintegration

$$\mu\llcorner_{\cup_{\bar x} E^{\bar t}_{\bar x}} = \int \mu^{\bar t}_{\bar x} m^{\bar t}(d\bar x). \tag{4.11}$$

Equa:disintegr_m

The balance (4.9) becomes

$$\varrho(\bar t,\bar x) = \rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)(t,x) + \mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big). \tag{4.12}$$

Equa:balance_dis

We collect these results in the following proposition.

n_along_tree **Proposition 4.4.** *If*

(1) $m^{\bar t}$ *is the measure given by* (4.6),
(2) $\rho^{\bar t}_{x}(t)$ *is the family of probability measures given by* (4.10),
(3) $\mu^{\bar t}_x$ *is the disintegration of the measure* $\mu\llcorner_{\{t\le\bar t\}}$ *on the tree* $F^{\bar t}_x$ *as in* (4.7),

then for $m^{\bar t}$*-a.e.* $\bar x$

$$\varrho(\bar t,\bar x) = \rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)(t,x) + \mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big).$$

We now prove that the partition into the sets $E^{\bar t}_{\bar x}$ is essentially unique.

ed_same_traj **Theorem 4.5.** *Assume that every Lagrangian representation of* $\rho(1,b)$ *is forward untangled. Then every* η' *is concentrated on a set of curves* $\varDelta'$ *such that if*

$$\gamma \in \varDelta', \qquad \gamma(t) \in E^{\bar t}_{\bar x}(t) \text{ for some } t^-_\gamma < t \leq \bar t, \tag{4.13}$$

Equa:assumptio_

then

$$\mathrm{Graph}\, \gamma \llcorner_{(t,\bar t]} \subset E^{\bar t}_{\bar x}. \tag{4.14}$$

Equa:subset_avr_

Proof. Let $\rho(t)$ be the left continuous density, and $\rho(t+)$ the trace of $\rho(1,\boldsymbol{b})$ in the open set $\{s > t\}$. Write the disintegration

$$\eta^{\mathrm{in}}_t = \eta \llcorner_{\{\gamma : t \in (t^-_\gamma, t^+_\gamma]\}} = \int \eta^{\mathrm{in}}_{(t,z)} \rho(t,dz)$$

By the forward untangling assumption, we have that for $\rho(t+)$-a.e. $z \in \mathbb{R}^d$ these exists a curve $\gamma_{(t,z)}(s)$ starting in (t,z) such that it holds

$$\eta^{\mathrm{in}}_{\bar t,z}\Big(\Big\{\gamma : \mathrm{Graph}\, \gamma \llcorner_{[\bar t,\infty)} \subset \mathrm{Graph}\, \gamma\Big\}\Big) = 1.$$

If we assume that every Lagrangian representation is forward untangled, one concludes that for every η' the curve $\gamma_{(t,z)}$ is the same (up to a prolongation) for $\rho(t)$-a.e. z. In particular, if $\gamma_{(t,z)}(s) \in E^{\bar t}_{\bar x}$ for some $s \in [t,\bar t]$, then the curve $\gamma_{(\bar t,z)} \llcorner_{[s,\bar t]}$ can be assumed to satisfy (4.14).

Repeating above observation for a dense sequence of times $\{t = t_i\}_{i \in \mathbb{N}}$, so that we pick every $\gamma \in \varDelta'$, we can conclude that η' is concentrated on a set of trajectories $\varDelta'$ such that if

$$\gamma \in \varDelta', \qquad \gamma(t) \in E^{\bar t}_{\bar x} \text{ for } t \leq \bar t, \tag{4.15}$$

Equa:assumptio_

then

$$\mathrm{Graph}\, \gamma \llcorner_{[t,\bar t]} \subset E^{\bar t}_{\bar x}, \tag{4.16}$$

Equa:subset_avr_

It remains to prove that the measure of trajectories which are outside $E^{\bar t}_{\bar x}$ for an initial interval of times has η'-measure 0. By a partition argument, we can assume that these trajectories $\gamma \in A$ intersect the set $\{t = \tilde t\}$ for some $\tilde t$, but the set $\{\gamma(\tilde t), \gamma \in \tilde A\}$ does not belongs to $\cup_{\bar x} E^{\bar t}(\tilde t)$. Hence it follows that the balance for η' as in (4.12) contains the additional terms $\eta'(\tilde A)$. Since the quantities in (4.12) can be computed without resorting to the Lagrangian representation, it follows that $\eta'(\tilde A) = 0$. □

en_disintegr **Remark 4.6.** One can define an equivalence relation as follows. Let $\{t_i\}_i$ be a dense countable sequence of times. For each t_i, let $F^{t_i}_x$ be the family of σ-compact sets constructed in (4.2). Define the equivalence relation E by

$$(t,x)E(t',x') \quad \Longrightarrow \quad \exists F^{t_i}_{x_i}\big((t,x),(t',x') \in F^{t_i}_{x_i}\big). \tag{4.17}$$

Equa:E_definit_

Since each F^t_x is σ-compact, it follows that E is made of σ-compact equivalence classes $F_{\mathfrak{a}}$, $\mathfrak{a} \in \mathfrak{A}$ for some index set.

However, one can make an example such that the disintegration w.r.t. E is not strongly consistent. Indeed, the problem is equivalent to ask if an equivalence relation whose graph is a countable union of graphs of equivalence relations which have a strongly consistent disintegration has a strongly consistent disintegration. This is easily seen to be false: for example let

$$E_n = \big\{(x,y) \in [0,1]^2 : x - y = \mathbb{N}2^{-n} \mod 1\big\},$$

and observe that

$$E = \big\{(x,y) : \exists k,n \in \mathbb{N}, x - y = k2^{-n} \mod 1\big\}$$

has not strongly consistent disintegration, as in the proof of [RF, Theorem 17, Chapter 2].

4.2. **Composition rule.** We assume now that every Lagrangian representation of $\rho u(1,\boldsymbol{b})$ is forward untangled for every $u \in L^\infty(\rho)$ such that

$$\mathrm{div}_{t,x}\big(\rho u(1,\boldsymbol{b})\big) = \nu \in \mathcal{M}(\mathbb{R}^{d+1}). \tag{4.18}$$

Equa:u_dive_nu

By Theorem 4.5 the trees used to decompose $\rho(1,\boldsymbol{b})$ are suitable to decompose also $\rho u(1,\boldsymbol{b})$, and thus we obtain the formula

$$\varrho(\bar t,\bar x)u(\bar t,\bar x) = \int u(t,x)\rho^{\bar t}_{\bar x}(t,dx) + \nu^{\bar t}_{\bar x}\big([t,\bar t] \times \mathbb{R}^d\big), \tag{4.19}$$

Equa:balance_di

where $\rho^{\bar t}_{\bar x}$ is given by (4.10) and $\nu^{\bar t}_{\bar x}$ is obtained by the disintegration analogous to (4.11)

$$\nu \llcorner_{\cup_{\bar x} E^{\bar t}_{\bar x}} = \int \nu^{\bar t}_{\bar x} m^{\bar t}(d\bar x). \tag{4.20}$$

Equa:disintegr_

For a given function β we now compute

$$\begin{aligned}
\varrho(\bar t,\bar x)\beta(u(\bar t,\bar x)) \underset{(4.19)}{=} {} & \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t,x)\rho^{\bar t}_{\bar x}(t,dx) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) \\
= {} & \varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg] \\
& + \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int \rho u(t)\xi^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\
& + \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) - \frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t) \\
& + \frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t).
\end{aligned}$$

In the case at t it holds $\rho^{\bar t}_{\bar x} = 0$, the above formula is right by just removing the indetermined terms.

If β is convex, by Jensens inequality

$$\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \leq \frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t), \tag{4.21}$$

Equa:jensen_comp

and then we obtain

$$\begin{aligned}
& \varrho(\bar t,\bar x)\beta(u(\bar t,\bar x)) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) \\
& \leq \varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg] \\
& \quad + \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int \rho u(t)\xi^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\
& \quad + \frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t).
\end{aligned}$$

Assume now β has at most linear growth and $\beta(0) = 0$, in particular it is uniformly Lipschitz. From the chain rule for BV function we obtain that:

(1)
$$\begin{aligned}
\varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg] \\
\leq \mathrm{Lip}(\beta)\big|\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|;
\end{aligned}$$

(2)
$$\begin{aligned}
\varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\
\leq \mathrm{Lip}(\beta)\bigg|\bigg(1 - \frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\bigg)\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg| \\
\leq \mathrm{Lip}(\beta)\|u\|_\infty\Big|\varrho(\bar t,\bar x) - \rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)\Big| \\
\overset{(4.12)}{\leq} \mathrm{Lip}(\beta)\|u\|_\infty\big|\mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|;
\end{aligned}$$

(3)
$$\frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) \leq \mathrm{Lip}(\beta)\|u\|_\infty\big|\mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|.$$

We thus obtain the following lemma.

Lemma 4.7. *For every Lipschitz convex function β it holds*

$$\varrho\beta(u)(\bar t,\bar x) \leq \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) + C_{\beta,u}\big(|\mu^{\bar t}_{\bar x}| + 2|\nu^{\bar t}_{\bar x}|\big)\big([t,\bar t)\times\mathbb{R}^d\big), \tag{4.22}$$

with $C_{\beta,u} \leq \mathrm{Lip}(\beta)\|u\|_\infty$.

4.2.1. *Construction of the composition measure.* We restrict the computation of the composition measure to the set $F^{\bar t}_{\bar x} \cap [-\infty,\bar t)$, because integrating the result w.r.t. $m^{\bar t}(d\bar x)$ gives the measure in $\cup_{\bar x} F^{\bar t}_{\bar x}$, and then repeating the computation for a dense countable sequence of times t_i we obtain the composition measure in $\mathbb{R}^d$.

A preliminary step which simplifies the computation of the composition measure is that we can write balances (4.19) and (4.22) one sets which are more general than time strips.

Assume that the set A has compact closure, $(\bar t,\bar x) \in A$ and is such that for every $(t,x) \in F^{\bar t}_{\bar x}$ the unique characteristic in $F^{\bar t}_{\bar x}$ connecting (t,x) to $(\bar t,\bar x)$ has at most one intersection with ∂A. Then, if

$$\gamma \mapsto t_{\gamma,A}$$

is the uniquen intersection time, one can define the measure

$$\rho^{\bar t}_{\bar x,A} = \big(t_\gamma,\gamma(t_{\gamma,A})\big)_\sharp \eta^{\bar t}_{\bar x}.$$

The above measure is supported on ∂A and independent of the representation, because using the intersection property of A with $F^{\bar t}_{\bar x}$, for every $(t,x) \in \partial A \cap F^{\bar t}_{\bar x}$ the tree F^t_x satisfies

$$F^t_x \setminus \{(t,x)\} \subset F^{\bar t}_{\bar x} \setminus \mathrm{clos}\,A,$$

and by writing the balances on each of these trees as in (4.8) we recover $\rho^{\bar t}_{\bar x,A}$: if $A' \subset \partial A \cap F^{\bar t}_{\bar x}$ and $\tilde t < \inf\{t : (t,x) \in A\}$, then

$$\rho^{\bar t}_{\bar x,A}(A') = \rho^{\bar t}_{\bar x}\bigg(\tilde t, \bigcup_{(t,x)\in A'} F^t_x(\tilde t)\bigg) + \mu^{\bar t}_{\bar x}\bigg(\bigcup_{(t,x)\in A'} F^t_x \cap (\tilde t,t)\times\mathbb{R}^d\bigg). \tag{4.23}$$

Using now formula (4.12), we obtain

$$\begin{aligned}
\rho^{\bar t}_{\bar x,A}(\partial A) &\underset{(4.23)}{=} \rho^{\bar t}_{\bar t}(\tilde t,\mathbb{R}^d) + \mu^{\bar t}_{\bar x}\bigg(\bigcup_{(t,x)\in\partial A} F^t_x \cap (\tilde t,t)\times\mathbb{R}^d\bigg) \\
&\underset{(4.12)}{=} \varrho(\bar t,\bar x) - \mu^{\bar t}_{\bar x}(\mathrm{clos}\,A).
\end{aligned}$$

Using the analogous formula (4.19) for the representation of $u \in L^\infty$ and the balance as above, one obtains

$$\varrho(\bar t,\bar x)u(\bar t,\bar x) = \int u\rho^{\bar t}_{\bar x,A} + \nu^{\bar t}_{\bar x}(\mathrm{clos}\,A), \tag{4.24}$$

The function $u(t,x)$ is the value computed on the top of each tree F^t_x. Using the same computations to prove Lemma 4.7, one can prove the following lemma.

Lemma 4.8. *If A is a set with compact closure such that $(\bar t,\bar x) \in A$ and each trajectory in $F^{\bar t}_{\bar x}$ has at most a single intersection with ∂A, then for every convex Lipschitz function β and $u \in L^\infty$ solving (4.18) it holds*

$$\varrho\beta(u)(\bar t,\bar x) \leq \int \beta(u(t))\rho^{\bar t}_{\bar x,A}(t) + C_{\beta,u}\big(|\mu^{\bar t}_{\bar x}| + 2|\nu^{\bar t}_{\bar x}|\big)\big(\bar A \cap [t,\bar t)\times\mathbb{R}^d\big), \tag{4.25}$$

with $C_{\beta,u} = \mathrm{Lip}(\beta)\|u\|_\infty$.

As a corollary, we can apply it to proper sets, by means of the restriction operators $\mathtt{R}^i_\Omega$.

Corollary 4.9 (Cite [BB17]). *If Ω is a $\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})$-proper set, $(\bar t,\bar x) \in \mathrm{clos}\,\Omega$ and $F^{\bar t}_{\bar x,\Omega}$ is made by the connected components of $\gamma \in E^{\bar t}_{\bar x}$ which can be concatenated to $(\bar t,\bar x)$ in Ω, then (4.25) holds.*

Proof. The only difference is that we allow $(\bar t,\bar x) \in \partial\Omega$, but since Ω is proper, by removing a set N of trajectories arbitrarily small there exists a $\delta > 0$ such that the trajectories in $F^{\bar t}_{\bar x} \setminus N$ containing $(\bar t,\bar x)$ takes values in Ω when $t \in (\bar t-\delta,\bar t)$. Thus one can apply (4.12) or (4.22). □

If we integrate w.r.t. the exiting part of the trace, $\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega)$, and observe that the entering trace $\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega)$ is given by the projection of the measure $(\mathtt{R}_\Omega)_\sharp\eta$ on the initial points $(t^-_\gamma,\gamma(t^-_\gamma))$, we obtain the following proposition.

proper_Omega **Proposition 4.10.** *If Ω is a $\rho^{\bar t}_{\bar x}(1,b)$-proper and $u\rho^{\bar t}_{\bar x}(1,b)$-proper set and (4.18) holds, then*

$$\int u\,\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) = \int u\,\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) + \nu^{\bar t}_{\bar x}(A).$$

Moreover for every Lipschitz convex function β it holds

$$\begin{aligned}\mathsf{j}^{\bar t}_{\bar x}(\Omega) := {}& \mathrm{Lip}(\beta)\|u\|_\infty\big(|\mu^{\bar t}_{\bar x}| + 2|\nu^{\bar t}_{\bar x}|\big)(\Omega)\\ &- \int \beta(u)\,\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) + \int \beta(u)\,\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) \geq 0.\end{aligned}$$

Proof. The only observation is that since $(|\mu|+|\nu|)(\partial\Omega)=0$, we do not need the left continuity as in Lemma 4.8. □

Extend j to a general Borel set E by

$$\mathsf{j}^{\bar t}_{\bar x}(E) = \inf\big\{\mathsf{j}^{\bar t}_{\bar x}(\Omega), E\subset\Omega \text{ proper}\big\}.$$

It is standard to verify that j is an outer measure, and it is additive on sets with positive distance. We then deduce from the Carathéodory criterion [AFP00, Theorem 1.49] that its restriction to the Borel sets is a positive measure:

osition_beta **Proposition 4.11.** *For every Lipschitz convex function β there exists a positive measure $\mathsf{j}^{\bar t}_{\bar x}$ such that if*

$$\mathrm{div}\,\big(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = \mu^{\bar t}_{\bar x}, \qquad \mathrm{div}\,\big(u\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = \nu^{\bar t}_{\bar x},$$

then

$$\mathrm{div}\,\big(\beta(u)\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = -\mathsf{j}^{\bar t}_{\bar x} + \mathrm{Lip}(\beta)\|u\|_\infty(|\nu|+2|\mu|). \tag{4.26}$$ Equa:j_beta_u_de

Moreover

$$\mathsf{j}^{\bar t}_{\bar x}(\mathbb{R}^{d+1}) \leq \mathrm{Lip}(\beta)\|u\|_\infty(\|\nu^{\bar t}_{\bar x}\| + 2\|\mu^{\bar t}_{\bar x}\|). \tag{4.27}$$ Equa:uniform_bou

Proof. The last estimate (2.4) follows by letting $\Omega\nearrow\mathbb{R}^d$, and recalling that $\rho(1,\boldsymbol{b})$ has compact support. □

The fact that we have the uniform bound (4.27) allows to obtain the estimate not on a single tree $F^{\bar t}_{\bar x}$ but to the solution: just consider a dense sequence of times t_i and use a standard covering argument. We thus conclude that

_compos_rule **Theorem 4.12.** *If β is a Lipschitz convex function and u solves (4.18), then there is a positive measure $\mathsf{j}=\mathsf{j}_{\beta,u}$ such that*

$$\mathrm{div}_{t,x}\,\big(\beta(u)\rho(1,\boldsymbol{b})\big) = \mathrm{Lip}(\beta)\|u\|_\infty(|\nu|+2|\mu|) - \mathsf{j}_{\beta,u},.$$

Moreover

$$\mathsf{j}_{\beta,u}(\mathbb{R}^{d+1}) \leq \mathrm{Lip}(\beta)\|u\|_\infty\big(\|\nu\|+2\|\mu\|\big).$$

In order to get a representation of the measures $\rho(1,\boldsymbol{b})$ for every function β which is the difference of two convex functions (or equivalently whose second derivative is a bounded measure), it is enough to consider the family of 1-Lipschitz functions (Kruzkhov entropies)

$$\beta_{\bar u} = |u-\bar u|,$$

and assume that $\|u\|_\infty = 1$. The formula (4.26) yields a function

$$[-1,1]\ni\bar u \ \mapsto\ \mathsf{j}_{\beta_{\bar u},u}\in\mathcal{M}^+(\mathbb{R}^{d+1})$$

which is weakly continuous by its very definition (4.26). Using the representation

$$\beta(u) = a + bu + \frac{1}{2}\int_{-1}^{1}|u-\bar u|D^2\beta(d\bar u), \qquad a,b\in\mathbb{R},$$

one obtains

ent_j_beta_u **Proposition 4.13.** *If* $\mathfrak{j}_{\beta_{\bar u},u}$ *are the composition measures for* $\beta_{\bar u}(u) = |u - \bar u|$*, then for every Lipschitz function whose second derivative is a bounded measure it holds*

$$\mathfrak{j}_{\beta,u} = a\mu + b\nu + \int_{-1}^{1} \mathfrak{j}_{\beta_{\bar u},u} D^2\beta(d\bar u). \tag{4.28}$$ Equa:compos_curv

erging_point **Remark 4.14.** The measure $\mathfrak{j}_{\beta,u}$ is related to the merging of curves due to (4.22), and if $|\beta''| \leq C$ one can verify by (4.22) that $\mathfrak{j}_{\beta,u} \leq C\mathfrak{j}_{(\cdot)^2/2,u}$.

We however are not able to take a measure which does not depends on u and such that $\mathfrak{j}_{(\cdot)^2/2,u} \leq \mathfrak{j}$: indeed, one can image a binary tree, where every branch divides in half at time 2^{-i}. Now, taking $\mu \llcorner_{t>0} = 0$, the measures ρ are

$$\rho(t \in [2^{-i-1}, 2^{-i})) = \sum_{j=1}^{2^i} 2^{-i} \delta_{x_j - tv_j - c_j}, \quad x_j, v_j, c_j \text{ some suitable constants.}$$

One can take a solution $u_{\bar i}$ of $\mathrm{div}(\rho u(1, \boldsymbol{b})) = 0$ which is 0 for $t \geq 2^{-\bar i}$ and it bifurcate to $-1, 1$ at the branching at time $2^{-\bar i}$, then remaining constant for each previous trees. It is immediate to see that

$$\mathfrak{j}_{\bar i} = -\,\mathrm{div}\left(\rho u^2(1, \boldsymbol{b})\right) = \sum_{j=1}^{2^{\bar i}} 2^{-j} \delta_{x_j - 2^{-\bar i} v_j - c_j},$$

whose mass is 1. By varying the time $\bar t$, the measure $\mathfrak{j}$ such that $\mathfrak{j}_{\bar i} \leq \mathfrak{j}$ for all i should have mass $+\infty$, hence it is not a locally bounded measure.

Ex:ac_frak_j **Example 4.15.** The measure $\mathfrak{j}_{\beta,u}$ does not seem to have any particular structure in more than one space dimension, where the flow is monotone: here we show that it can be a.c. w.r.t. the Lebesgue measure even in two space dimensions,

In [ABC13, Section 4] a Lipschitz function $H : [0,1]^2 \mapsto [0,1]$ is constructed with the following properties:

(1) for every $x_1 \in [0,1]$ the functiion $x_2 \mapsto H(x_1, x_2)$ is increasing from $H(x_1, 0) = 0$ to $H(x_1, 1) = 1$;
(2) for every $h \in [0,1]$ the level set $H^{-1}(h)$ is a curve $x_2 = f_h(x_1)$;
(3) the set N for which $\nabla H = 0$ has positive Lebesgue measure and intersect $H^{-1}(h)$ is a single point $\bar x_h$ for $\mathcal{L}^1$-a.e. h.

If ℓ is the arc length on the rectifiable set

$$H^{-1}(h) = \Big\{ x(\ell, h), \ell \in [0, L_h] \Big\}$$

let $\bar\ell_h$ be the coordinate of this intersection point on the level set $H^{-1}(h)$, $\bar x_h = x(\bar\ell_h, h)$, and define the vector field on this level set as

$$\boldsymbol{b}(x(\ell, h)) = \begin{cases} -\nabla^\perp H(x(\ell, h)) & \ell < \bar\ell_h, \\ \nabla^\perp H(x(\ell, h)) & \ell \geq \bar\ell_h. \end{cases}$$

The vector field is Borel, since the set $\{x(\ell, h)\}_{\ell, h}$ can be taken σ-compact.

Now we follow the analysis of [ABC14]. Using the disintegration

$$\mathcal{L}^2 \llcorner_{[0.1]^2} = \int \frac{\mathcal{H}^1 \llcorner_{H^{-1}(h)}}{|\nabla H|} \mathcal{L}^1(dh) + \int \delta_{\bar x_h} \mathcal{L}^1(dh),$$

the reduction of the transport equation $\mathrm{div}(\rho(1, \boldsymbol{b})) = 0$, $\rho \ll \mathcal{L}^2$, on the level set $H^{-1}(h)$ is [ABC14, Lemma 3.7]

$$\partial_t \bigg(\frac{\rho(t, \ell)}{|\nabla H(x(\ell, h))|} \bigg) - \partial_\ell \big(\mathrm{sign}(\ell - \bar\ell_h) \rho(t, \ell) \big) + \partial_t \rho(t, \bar\ell_h) \delta_{\bar\ell_h}(d\ell).$$

If the initial data is $\rho = 1$, the solution unique is

$$\rho(\ell, h) = \begin{cases} 1 & \ell \neq \bar\ell_h, \\ 1 + 2t & \ell = \bar\ell_h. \end{cases} \tag{4.29}$$ Equa:solut_H

Similarly another solution is given by

$$\rho(\ell,h)u(\ell,h) = \begin{cases} \mathrm{sign}(\ell - \bar{\ell}_h) & \ell \neq \bar{\ell}_h, \\ 0 & \ell = \bar{\ell}_h. \end{cases} \tag{4.30}$$ Equa:solut_H_u

Thus for $\beta(u) = u^2$, by comparing (4.29) with (4.30) it follows that

$$\mathrm{j}_{(\cdot)^2,u} = 2\mathcal{L}^2 \llcorner_N = 2\mathcal{L}^2 \llcorner_{\{\nabla H = 0\}}.$$

vector_field

5. Monotone vector fields

Let $A(t)$, $t \in \mathbb{R}$ be a maximal monotone operator on $\mathbb{R}^d$, i.e.

$$\forall y_1 \in A(t,x_1), y_2 \in A(t,y_2) \Big((y_1 - y_2, x_1 - x_2) \geq 0 \Big),$$

with

$$\forall R > 0 \left(\int_{B_R^{d+1}(0)} |A(t,x)| \lceil t dx < \infty \right). \tag{5.1}$$ Equa:local_contr

Since we will consider only locally defined solutions, the above integrability assumption can be clearly relaxed to some local condition and as [BG11, Section 2] we can also consider quasi-monotone operators.

Consider the ODE

$$\dot{x}(t) \in -A(t) \qquad \mathcal{L}^1\text{-a.e. } t > 0. \tag{5.2}$$ Equa:ODE_monoton

In [BG11] the following results are proved.

ess_monotone **Proposition 5.1** ([BG11, Proposition 3.3])**.** *For any initial datum $x(0) = x_0 \in \mathbb{R}^d$ there exists a unique solution $x(t)$ to (5.2) with 1-Lipschitz dependence on the initial datum.*

Theorem 5.2 ([BG11, Theorem 1.2])**.** *If $A_n(t)$ is a family of monotone operators converging in $L^1(\mathbb{R}^{d+1})$ to A, then the flow $X_n(t)$ constructed in Proposition 5.1 for the operators $A_n(t)$ converges to the corresponding flow $X(t)$ for $A(t)$.*

The flow of Proposition 5.1 is defined for every initial point and forward untangled, because of the forward uniqueness. The last result we recall is that there is a universally measurable selection

$$a(t,x) \in A(t,x)$$

such that it holds

$$\dot{X}(t,x_0) = a(t, X(t,x_0)) \quad \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}.$$

Define the set

$$\mathcal{N} = \left\{ (t,x_0) \in \mathbb{R}^{d+1} : \nexists \frac{d}{dt} X(t,x_0) \right\}$$

In [BG11, Lemma 5.3] it is proved that this set is negligible for all measures of the form $\mathcal{L}^1 \times \mu$, $\mu \in \mathcal{M}(\mathbb{R}^d)$. Hence one can define

t_monot_flow **Definition 5.3** ([BG11, Definition 5.5])**.** For the time-dependent maximal monotone operator $A(t)$, define a single-valued, everywhere defined vector field $\mathbf{b} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ by first setting

$$\mathfrak{a}(t, X(t,x_0)) = \frac{d}{dt} X(t,x_0), \qquad (t,x) \notin \mathcal{N},$$

and then extending it arbitrarily on $X(\mathcal{N})$.

Consider now

$$\rho_t + \mathrm{div}(\rho \boldsymbol{b}) = 0, \qquad \rho(t=0) = \rho_0. \tag{5.3}$$ Equa:conserv_mon

The following proposition holds.

ransport_a_mu **Proposition 5.4.** *Let $A(t)$ be a maximal monotone operator, and ρ_0 be a non-negative measure. Then there is a unique non-negative solution to the transport equation (5.3) given by the formula*

$$\rho(t) = (X(t))_\sharp \rho_0.$$

The proof is an direct consequence of the forward uniqueness of the characteristics.

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