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Benjamin-Feir instability of Stokes waves

Candidate:
Paolo Ventura

Advisors:
Prof. Massimiliano Berti
Prof. Alberto Maspero

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A Camilla
che c'è sempre stata,
e al piccolo Gioele.

Contents

Chapter 1

Introduction

In the vast realm of fluid dynamics, modulational instability –more commonly known as Benjamin-Feir instability in the context of water waves– plays a prominent role among the phenomena characterizing the wave nature of fluids.
Broadly speaking, this kind of instability concerns the modulation of an undisturbed traveling periodic wave with small boosts of much longer period, called *long-wave perturbations*, that are expected to trigger the disruption of the original waveform.
The disintegrative process of wave trains that the phenomenon exhibits is nowadays supported by a wide-spread and heterogeneous array of scientific evidence encompassing physical experiments, numerical simulations, and, more recently, rigorous analytical results.

The present thesis collects some advancements in the mathematical comprehension of the Benjamin-Feir instability in water waves. We shall describe the *linear modulational instability* of a particular class of solutions of the two-dimensional water waves system with sea depth $\mathtt{h} \in (0, +\infty]$, i.e. traveling, 2π-periodic, planar waves, known as *Stokes waves*.
Following the classic linearization procedure, we consider a small perturbation $h := h(t, x)$ of a Stokes wave of amplitude $\epsilon > 0$ and, by discarding quadratic terms $\mathcal{O}(h^2)$, we obtain the linearized water waves system along the Stokes wave

$$\partial_t h(t, x) = \mathcal{L}_\epsilon h(t, x) \,, \tag{1.0.1}$$

where $\mathcal{L}_\epsilon$ is a linear operator with 2π-periodic coefficients.
The linear modulational stability problem consists in the *spectral analysis* of the operator $\mathcal{L}_\epsilon$ regarded as acting on $2\pi N$-periodic functions, for a large natural number N.
In this context we shall describe the complete branching of the portion of the spectrum of $\mathcal{L}_\epsilon$ closest to 0, with a particular focus on the parts of spectrum outside the imaginary axis.

The existence of an eigenvalue λ with *positive real part* of $\mathcal{L}_\epsilon$ and of an associated eigenvector $e^{\mathrm{i}\,\mu x}v(x)$, for a 2π-periodic function $v := v(x) \neq 0$ and a rational value $0 < \mu \leq 1$, called *Floquet exponent*, gives rise to the following solution $h(t,x)$ of (1.0.1):

$$h(t,x) := \mathrm{Re}\big(e^{\lambda t} e^{\mathrm{i}\,\mu x} v(x)\big)\,, \qquad \partial_t h(t,x) = \mathcal{L}_\epsilon h(t,x)\,. \tag{1.0.2}$$

Since λ has positive real part, the solution $h(t,x)$ grows exponentially fast in time.

Via *Bloch-Floquet theory*, as we shall see later, we pursue the search for λ as an eigenvalue of the shifted operator $\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\,\mu x}\mathcal{L}_\epsilon e^{\mathrm{i}\,\mu x}$, which acts *only* on 2π-periodic functions.
For small values of μ and ϵ, the *Hamiltonian nature* of the operator $\mathcal{L}_{\mu,\epsilon}$, inherited from the Hamiltonicity of the water waves system, allows unstable eigenvalues, i.e. with positive real part, to bifurcate only from multiple eigenvalues of $\mathcal{L}_{0,0}$. In particular Benjamin-Feir unstable eigenvalues are expected to bifurcate from the quadruple eigenvalue 0 of $\mathcal{L}_{0,0}$ corresponding to four symmetries of the water waves system.
The core question of the thesis is the following:

- *Benjamin-Feir stable/unstable eigenvalues*: describe the global bifurcation of the four eigenvalues of the operator $\mathcal{L}_{\mu,\epsilon}$ branching off, as $\mu > 0$ and $\epsilon > 0$, from the quadruple eigenvalue of $\mathcal{L}_{0,0}$ located in the origin of the complex plane.

We provide an exhaustive answer, in full agreement with the existing numerical findings [35] and extending the analytical literature [21, 74]. Our results are divided into three chapters:

1. **Benjamin-Feir instability in infinite depth ([13], Chapter 3)**. In the idealized model of an ocean of infinite depth, four distinct eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ branch off from 0 as both parameters μ and ϵ are turned on. Two of them remain stable, i.e. purely imaginary, whereas the other two bifurcate specularly out of the imaginary axis and depict, for constant $\epsilon > 0$ as μ varies, a figure "8" that was previously conjectured by numerical simulations. The two unstable eigenvalues collide at the top of the figure "8" and then split again, this time on the imaginary axis.
   These results are presented in Theorem 1.2.2, Section 1.4 and Chapter 3.

2. **Benjamin-Feir instability in finite depth ([16], Chapter 4)**. In an ocean of finite depth $\mathtt{h} > 0$ a drastic regime shift occurs depending on whether the seabed is shallower or deeper than the *Whitham-Benjamin critical threshold* $\mathtt{h}_{\mathrm{WB}} \approx 1.363$.
   We adapt the deep-water analysis to this model and find an analytical formula for this threshold separating the deep-water unstable regime from the stable regime of

shallow water. In the latter case the 4 eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ under examination are purely imaginary, whereas, once the depth exceeds the critical threshold, the behavior of the 4 eigenvalues is in continuity with the infinite-depth case. In particular we find, for constant $\epsilon > 0$ as μ varies, a depth-dependent figure "8" depicted by two unstable eigenvalues.
These results are presented in Theorem 1.2.3, Section 1.5 and Chapter 4.

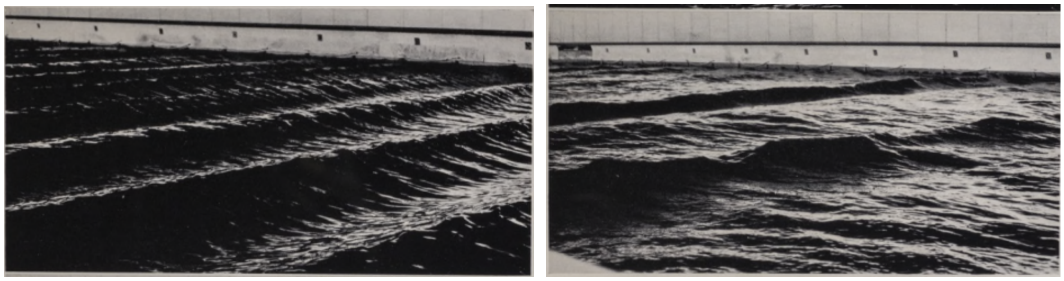
3. **Benjamin-Feir instability at the critical depth ([17], Chapter 5)**. A fine-tuned analysis of the case of critical depth shows the persistence of the pair of unstable eigenvalues, forming a degenerate figure "8", at the Whitham-Benjamin threshold. This yields an instability zone of the parameters which includes shallow depths (smaller than the critical threshold) once these are coupled with sufficiently big values of the amplitude ϵ. The result disposes of a long-standing debate within the scientific community regarding the stability/instability of Stokes waves in the transient regime between shallow and deep water.
These results are presented in Theorem 1.2.4, Section 1.6 and Chapter 5.

1.1 Historical background

Let us frame our findings regarding Benjamin-Feir instability within the historical background of the prior results obtained on the matter.

Stokes waves derive their name from Sir George Stokes, who, in his renowned article [84], computed the initial expansion of these particular traveling waves in 1847.
The existence of small amplitude Stokes waves of the water waves system (here Theorem 2.1.1) was first rigorously proved in the one-century-old works by Nekrasov [73] and Levi-Civita [67] in the case of infinite depth and Struik in the case of finite depth. Afterwards these results were extended to global branches containing extreme waves in [60, 89, 72, 32, 5, 81].

The unexpected discovery of modulational instability for Stokes waves traces its origins to the physical experiments conducted by Benjamin and Feir [10] in 1967, while they were trying to replicate Stokes waves inside large water tanks. This was the first observation in fluid dynamics of this type of instability, which is still known today as the Benjamin-Feir instability. However, we must here stress that modulational instability *per se* had already been discovered by Piliptetskii and Rustamov [80] in 1965, during experiments involving high-power lasers in organic solvents, and later received a first mathematical derivation due



Stokes wave near the wavemaker. Broken wave far from the wavemaker.

Figure 1.1: The two pictures come from [9, Figure 1] and illustrate the disintegration of the wavetrain due to modulational instability.

to Bespalov and Talanov [19] in 1966.
After the discovery that Stokes waves in deep water are unstable, many heuristic explanations were proposed, e.g. Lighthill [69] and Zakharov [97, 100]. In the light of these results it was evinced that modulational instability arises when a wave of a given periodicity is perturbed by a, however small, boost with much longer periodicity. This so-called long-wave perturbation leads to the disintegration of the wavetrain with a redistribution of the energy on a broad spectrum. More precisely the aforementioned works predict the occurrence of unstable eigenvalues (i.e. with positive real part) of the linearized equations at the Stokes wave, near the origin of the complex plane, corresponding to small Floquet exponents μ or, equivalently, to long-wave perturbations.
The same phenomenon was later predicted by Whitham [92] and Benjamin [9] for Stokes waves of wavelength $2\pi\kappa$, in finite depth $\mathtt{h}$, provided that $\kappa\mathtt{h}$ is greater than $\mathtt{h}_{\mathrm{WB}} \approx 1.363$ which we have called, in their honor, Whitham-Benjamin threshold value.
For a comprehensive historical survey on this topic, we direct the reader to the work [101].

Modulational instability has attracted significant interest from the scientific community over time. Among the numerous physical experiments (e.g. [79, 25, 26]) and numerical simulations (e.g. [44, 61, 75, 99, 87]) conducted to explore this phenomenon, special recognition is given to the work of Deconinck and Oliveras [35], who were the first to provide a comprehensive view of the $L^2(\mathbb{R})$-spectrum of the operator $\mathcal{L}_\epsilon$ near the origin of the complex plane. Their work suggests that in that region, the spectrum is distributed both on the imaginary axis –consistent with our results, as two out of the four eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ near 0 follow this pattern as μ varies, see Theorems 1.4.1-1.5.1– and out of the imaginary axis (becoming unstable) to form a figure “8”, see Figure 1.2.

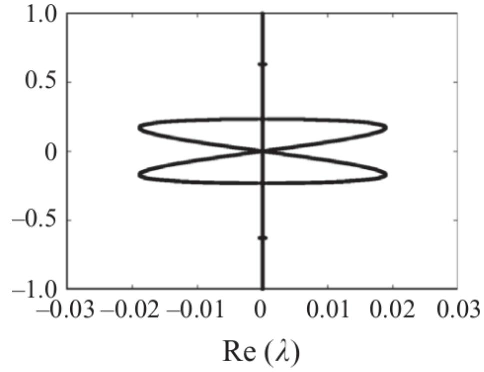


Figure 1.2: Numerical figure "8" from [35] depicted by the whole $L^2(\mathbb{R})$ spectrum of the linearized operator near the origin of the complex plane. The picture also shows other instability phenomena of lower intensity far from the origin (see [34, 48]).

Theorems 1.2.2-1.2.3 are the first analytical proofs of this long-conjectured phenomenon: the unstable eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ depict a complete figure "8" as μ varies in the interval $[-\underline{\mu}(\epsilon), \underline{\mu}(\epsilon)]$.
Our results are the first to analytically exhibit the full instability picture not only for the water wave system, but for all the fluid dynamical systems where the Benjamin-Feir phenomenon occurs, exception made for the focusing $1d$ NLS equation, for which Deconinck-Upsal [37] showed the presence of a figure "8" for elliptic solutions by exploiting the integrable structure of the equation.

Any rigorous proof of the Benjamin-Feir instability has to face the difficulty that the perturbed eigenvalues bifurcate from the *defective* eigenvalue zero. The first rigorous proof of a local branch of unstable eigenvalues close to zero for $\kappa\mathtt{h}$ larger than the Whitham-Benjamin threshold $\mathtt{h}_{\rm WB} \approx 1.363$ was obtained by Bridges-Mielke [21] in finite depth. Their method, based on a spatial dynamics and a center manifold reduction, breaks down in deep water[1]. To deal with this case Nguyen-Strauss [74] have recently developed a new approach, based on a Lyapunov-Schmidt decomposition.
Both Bridges-Mielke [21] and Nguyen-Strauss [74] reduce the spectral problem to a finite dimensional one, here a 4×4 matrix, and, in a suitable regime of parameters (μ, ϵ), prove the existence of unstable eigenvalues close to the origin with non-zero real part, namely the

[1] We quote however [50] for an analogue in infinite depth which carries most of the properties of a center manifold.

figure “x” amid the aforementioned figure “8”, see Figure 1.3.
In particular, for the case of infinite depth, we are able to prolong the local branches of

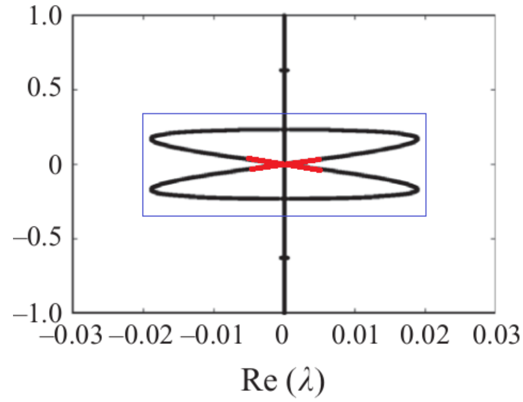


Figure 1.3: We highlighted in red the portion of figure “8” that the previous analytical results [21, 74] were able to describe, proving for the first time ever the existence of Benjamin-Feir unstable spectrum in the case of finite depth and deep water respectively. We boxed in blue the part of the spectrum of $\mathcal{L}_\epsilon$ we find with our method and that we present in the thesis.

eigenvalues discovered by Nguyen-Strauss in [74] far from the bifurcation, until they collide again on the imaginary axis and beyond.
In the case of finite depth the same holds and a precise comparison can be made between the fundamental functions that we obtain from our method and the coefficients obtained in the approach of Bridges-Mielke [21] to describe the bifurcation of unstable eigenvalues.
Thus our works on the infinite [13] and finite [16] depth cases, here Chapters 3 and 4, build upon the rigorous results [74]-[21], managing to improve the final outcome with a complete description of the spectrum near the origin of the complex plane. Moreover, as we shall see in the next section, we found a common demonstrative procedure for the two cases, albeit with the necessary adjustments due to the intrinsic differences between the two models. We are confident that this technique can be extended to many other cases of linear modulational instability.

The shallow/deep water transient. A question remained open: to determine the stability or instability of the Stokes waves at the critical Whitham-Benjamin depth $\mathtt{h}_{\rm WB}$ and analyze in detail the change of stable-vs-unstable behavior of the eigenvalues along this shallow-to-deep water transient.

Several formal responses have been provided in existing literature [55, 57, 86, 85]. The solutions for water waves within the framework of modulational instability are formally approximated by an equation that describes the wave envelope. If $\mathtt{h} < \mathtt{h}_{\mathrm{WB}}$, this equation takes the form of a defocusing cubic nonlinear Schrödinger equation (NLS), whereas if $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$, it becomes a focusing cubic NLS. This behavior aligns with the established stability/instability findings for shallow/deep water, as discussed earlier. The behavior around the critical depth $\mathtt{h} = \mathtt{h}_{\mathrm{WB}}$ corresponds to the disappearance of the cubic coefficients, necessitating the determination of a higher-order effective NLS.
In the 1970s, Johnson's formal calculations [56] hinted at a stability scenario for Stokes waves when $\mathtt{h}$ slightly exceeded $\mathtt{h}_{\mathrm{WB}}$. However, Kakutani and Michihiro [57], a few years later, derived a distinct quintic NLS equation and asserted the modulational instability of Stokes waves. This instability was further validated by Slunyaev [86], who investigated how the coefficients of the quintic effective NLS equation vary with $\mathtt{h}$.
In our work [17], here Chapter 5, we prove with mathematical rigor the occurrence of the latter scenario: Stokes waves of the pure gravity water waves equations at the critical depth are linearly unstable under long wave perturbations.

High-frequency modulational instability. Other modulationally unstable eigenvalues for the Stokes waves can be found far from the origin as showed in [34] and [48].
These unstable eigenvalues arise from the double eigenvalues in the $L^2(\mathbb{R})$-spectrum of the linearized water waves system along the Stokes wave and depict figures "0", i.e. ellipse-shaped curves called *isolas* which were described numerically in [35] and supported by formal expansions in ϵ in [34], see also [1].
Differently from the case of the quadruple eigenvalue 0, the study of the spectrum in the case of high-frequency instabilities is reduced to finding the eigenvalues of a 2x2 Hamiltonian and reversible matrix. On the other hand, this simplification is balanced by a significant computational complexity in deriving the actual instability of these finite-dimensional matrices, which can only be determined by knowing the p-th order (or higher) Taylor expansion of the matrix entries, for a positive integer parameter p that indexes, increasingly with respect to the distance from the origin, the unperturbed double eigenvalues.
We plan to adapt, in the near future, the methods presented in this thesis to the study of this problem. This could lead to a description of the full $L^2(\mathbb{R})$-spectrum of the linearized water waves system along a Stokes wave in the case of small amplitudes.

Nonliner results. In literature one can find some pioneering assaults on the nonlinear case. We bring up the outcome of Jin, Liao, and Lin [54] regarding the nonlinear modulational

instability in various fluid model equations. By assuming *a priori* that these approximate versions of water waves are linearly unstable, the three authors derive the existence of solutions $U(t,x)$ having initial datum $U(0,x)$ arbitrarily close to the traveling wave in the $H^s(\mathbb{T}_q)$ norm, but diverging from this in the $L^2(\mathbb{T}_q)$ norm.
A similar result was obtained by Chen-Su [27] concerning Stokes waves in deep water, by relaying on the modulational approximation to the water waves given by the NLS instead of working on the full system directly.
Results related to the nonlinear transversal instability of solitary traveling water waves in finite depth, which decay as they extend to infinity over the real line $\mathbb{R}$, have been established in [76]. It is important to note that in deep water, there are no solitary wave solutions, as shown in [47, 49].

Further literature on modulational instability. Modulational instability has been studied also for a variety of approximate water waves models, such as KdV, gKdV, NLS and the Whitham equation by, for instance, Whitham [93], Segur, Henderson, Carter and Hammack [83], Gallay and Haragus [42], Haragus and Kapitula [43], Bronski and Johnson [24], Johnson [55], Hur and Johnson [45], Bronski, Hur and Johnson [23], Hur and Pandey [46], Leisman, Bronski, Johnson and Marangell [66]. Also for these approximate models, numerical simulations predict a figure “8” similar to that in Figure 1.2 for the bifurcation of the unstable eigenvalues close to zero.
However, in none of these approximate models (except for the integrable NLS in [37]) the complete picture of the Benjamin-Feir instability has been rigorously proved so far.
We expect that the present approach can be adapted to describe the full bifurcation of the eigenvalues for these models too.

1.2 Benjamin-Feir unstable eigenvalues

We now state our results regarding the Benjamin-Feir unstable eigenvalues.
Let us first introduce the *Zakharov Hamiltonian formulation* of the water waves, where the water is modeled as an *Euler fluid*, in which we will take our steps.
We recall that Euler fluids form the basis of fluid dynamics theory. They are idealized fluids that are inviscid (meaning they have no internal friction or viscosity) and incompressible (meaning they have constant density). This assumption is particularly useful for studying certain aspects of fluid behavior, especially in cases where the effects of viscosity are negligible. In our context, Euler fluids provide a simplified framework for understanding

the fundamental principles governing wave propagation. By neglecting viscosity, the focus can be directed towards the interplay between nonlinearity and dispersion, which are key factors contributing to the emergence of phenomena like Benjamin-Feir instability.
However we remark that, although Euler fluids are a valuable theoretical tool, real-world fluids, including water, do have viscosity and the results obtained from the use of the model of Euler fluids should be interpreted in light of this simplification.
With this in mind, we plan to broaden this study in the future by incorporating the more comprehensive framework of viscous fluids provided by the Navier-Stokes equations.

Notation. From now on we denote the partial derivative of a function with respect to a variable with a subscript.

The water-waves system. Let us consider an ocean modeled by the cylindrical domain

$$\mathcal{D}_{\mathtt{h},\eta} := \big\{(t,x,y)\in\mathbb{R}\times\mathbb{T}\times\mathbb{R} \ : \ -\mathtt{h}<y<\eta(t,x)\big\}\,, \quad \mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}\,, \quad \mathtt{h}\in(0,+\infty]\,, \tag{1.2.1}$$

which is delimited from above by the graph $\partial\mathcal{D}_\eta = \{y=\eta(t,x)\}$ of a periodic function $\eta(t,x)$ being the water surface, and from below by the sea depth $\mathtt{h}>0$. The Euler equations for a two-dimensional, incompressible, inviscid, irrotational, fluid filling the ocean in (1.2.1), under the pure action of gravity, are

$$\begin{cases} \Phi_t + \frac{1}{2}(\Phi_x^2+\Phi_y^2) + g\eta = P & \text{at } y=\eta(x) \\ \eta_t = \Phi_y - \eta_x\,\Phi_x & \text{at } y=\eta(x) \\ \Phi_{xx}+\Phi_{yy}=0 & \text{in } \mathcal{D}_{\mathtt{h},\eta} \\ \Phi_y\to 0 & \text{as } y\to -\mathtt{h}\,, \end{cases} \tag{1.2.2}$$

where $\Phi:=\Phi(t,x,y)$ is the harmonic scalar potential of the irrotational velocity field, g is the gravity constant and $P\in\mathbb{R}$ is the Bernoulli constant, i.e. the atmospheric pressure along the free surface.
The whole dynamics of the fluid is governed by the two boundary conditions at the free surface in (1.2.2). The first one, called *dynamic boundary condition*, requires the pressure of the fluid to align with the constant atmospheric pressure P along the free surface, whereas the second one, called *kinematic boundary condition*, imposes that the fluid particles remain on the free surface during the time evolution.
The harmonic potential Φ is uniquely determined by its trace $\psi(t,x)=\Phi(t,x,\eta(t,x))$ at the free surface $y=\eta(t,x)$ as the solution of the elliptic equation

$$\Phi_{xx}+\Phi_{yy}=0 \ \text{ in } \mathcal{D}_{\mathtt{h},\eta}, \quad \Phi(t,x,\eta(t,x))=\psi(t,x)\,, \quad \Phi_y(t,x,y)\to 0 \ \text{ as } y\to-\mathtt{h}\,. \tag{1.2.3}$$

As shown by Zakharov [98] and Craig-Sulem [32], the time evolution of the fluid is completely determined by the following *Hamiltonian* system for the unknowns $(\eta(t,x),\psi(t,x))$

$$\begin{bmatrix}\eta_t\\ \psi_t\end{bmatrix}=\mathcal{J}\begin{bmatrix}\nabla_\eta\mathcal{H}\\ \nabla_\psi\mathcal{H}\end{bmatrix}\,,\qquad \mathcal{J}:=\begin{bmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{bmatrix}\,,\tag{1.2.4}$$

where ∇ denote the L^2-gradient, and the Hamiltonian

$$\mathcal{H}(\eta,\psi):=\int_{\mathbb{T}}\Big(\tfrac12\psi\,G(\eta)\psi+\tfrac12 g\eta^2-P\eta\Big)\,\mathrm{d}x$$

is the sum of the kinetic and potential energy of the fluid with $G(\eta)$ being the Dirichlet-Neumann operator of the domain $\mathcal{D}_{\mathtt{h},\eta}$, namely

$$[G(\eta;\mathtt{h})\psi](x):=[G(\eta)\psi](x):=\Phi_y(x,\eta(x))-\Phi_x(x,\eta(x))\eta_x(x)\,.\tag{1.2.5}$$

Following [32, 98], the L^2-gradient with respect to η of the kinetic energy

$$K(\eta,\psi):=\tfrac12(\psi,G(\eta)\psi)_{L^2}=\tfrac12\int_{\mathcal{D}_\eta}|\nabla\Phi|^2\,\mathrm{d}x\,,\tag{1.2.6}$$

is equal to

$$\nabla_\eta K(\eta,\psi)=-\frac12\psi_x^2+\frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi+\eta_x\psi_x\big)^2\,,\tag{1.2.7}$$

allowing us to recast system (1.2.4) as the following equivalent system

$$\eta_t=G(\eta)\psi\,,\quad \psi_t=P-g\eta-\frac{\psi_x^2}{2}+\frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi+\eta_x\psi_x\big)^2\,.\tag{1.2.8}$$

In addition to Hamiltonicity, the water waves system (1.2.8) is time-reversible with respect to the involution

$$\rho\begin{bmatrix}\eta(x)\\ \psi(x)\end{bmatrix}:=\begin{bmatrix}\eta(-x)\\ -\psi(-x)\end{bmatrix}\,,\quad \text{i.e. } \mathcal{H}\circ\rho=\mathcal{H}\,.\tag{1.2.9}$$

Stokes waves. A particular class of solutions of the water waves system (1.2.8) is formed by the Stokes waves, namely solutions of the form $\eta(t,x)=\breve{\eta}(x-ct)$ and $\psi(t,x)=\breve{\psi}(x-ct)$ for some real translational velocity c and a pair $(\breve{\eta}(x),\breve{\psi}(x))$ of 2π-periodic functions. The profile $(\breve{\eta},\breve{\psi})$ is thus an equilibrium of the following system

$$\eta_t=c\eta_x+G(\eta)\psi\,,\quad \psi_t=c\psi_x+P-g\eta-\frac{\psi_x^2}{2}+\frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi+\eta_x\psi_x\big)^2\,,\tag{1.2.10}$$

given by the water waves equations (1.2.8) in a reference frame moving at constant speed c. Existence and uniqueness of such solutions, for small amplitudes, represent a classic result of Levi-Civita [67] and Nekrasov [73] in the case of infinite depth, and Struik [88] in the case of finite depth. We now state a low-regularity version of the existence result for the Stokes of wavenumber $\kappa = 1$ with normalized constants $g = 1$ and $P = 0$ (cfr. Lemma 2.1.2), while postponing a more refined version to Theorem 2.1.1.

Theorem 1.2.1. *There exists* $\epsilon_0 := \epsilon_0(\mathtt{h}) > 0$ *and a unique family of analytic solutions*

$$(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon) \in H^1(\mathbb{T}) \times H^1(\mathbb{T}) \times \mathbb{R}$$

of the system (1.2.10)*, parameterized by* $|\epsilon| \leq \epsilon_0$*, such that* $\eta_\epsilon(x)$*,* $\psi_\epsilon(x)$*,* c_ϵ *are respectively even, odd and constant in* x *and admit the Taylor expansion in* ϵ

$$\eta_\epsilon(x) = \epsilon\cos(x) + \mathcal{O}(\epsilon^2)\,, \quad \psi_\epsilon(x) = \epsilon \mathtt{c}_\mathtt{h}^{-1}\sin(x) + \mathcal{O}(\epsilon^2)\,, \quad c_\epsilon = \mathtt{c}_\mathtt{h} + \mathcal{O}(\epsilon)\,, \tag{1.2.11}$$

with $\mathtt{c}_\mathtt{h} := \sqrt{\tanh(\mathtt{h})}$*,* $\mathtt{c}_{+\infty} = 1$ *in infinite depth, being the bifurcation value of the solution.*

We are now in a position to present the first results of this thesis.
Notation. In the following pages we denote by $\mathcal{O}(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$, $m_j, n_j \in \mathbb{N}$, analytic functions of (μ, ϵ) with values in a Banach space X which satisfy, for some $C > 0$,

$$\|\mathcal{O}(\mu^{m_j}\epsilon^{n_j})\|_X \leq C\sum_{j=1}^{p}|\mu|^{m_j}|\epsilon|^{n_j}$$

for small values of (μ, ϵ). We denote $r(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$, with or without a labeling subscript, scalar functions $\mathcal{O}(\mu^{m_1}\epsilon^{n_1}, \ldots, \mu^{m_p}\epsilon^{n_p})$ which are also *real* analytic.

Deep water. The first part of the thesis provides, for small values of the parameters ϵ and μ, the full description of the four eigenvalues near zero of the operator

$$\mathcal{L}_{\mu,\epsilon} = \begin{bmatrix} (\partial_x + \mathrm{i}\,\mu)\circ(1 + p_\epsilon(x)) & |D + \mu| \\ -(1 + a_\epsilon(x)) & (1 + p_\epsilon(x))(\partial_x + \mathrm{i}\,\mu) \end{bmatrix}, \quad p_\epsilon, a_\epsilon : \mathbb{T} \to \mathbb{R}\,, \quad p_\epsilon, a_\epsilon = \mathcal{O}(\epsilon)\,,$$

which we carry out in Section 1.3 as the Floquet shift $\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\,\mu x}\mathcal{L}_\epsilon e^{\mathrm{i}\,\mu x}$ of the linearized (and conjugated) operator $\mathcal{L}_\epsilon$ along the Stokes wave $(\eta_\epsilon, \psi_\epsilon)$ in the case of *infinite depth* $\mathtt{h} = +\infty$. We now focus on Benjamin-Feir *unstable* eigenvalues, postponing the complete result to Theorem 1.4.1.

Theorem 1.2.2. *Let* $\mathtt{h} = +\infty$*. There exist* $\epsilon_1, \mu_0 > 0$ *and an analytic function*

$$\underline{\mu} : [0, \epsilon_1) \to [0, \mu_0)\,, \qquad \textit{of the form} \quad \underline{\mu}(\epsilon) = 2\sqrt{2}\epsilon(1 + r(\epsilon))\,, \tag{1.2.12}$$

such that, for any $\epsilon \in [0,\epsilon_1)$*, the operator* $\mathcal{L}_{\mu,\epsilon}$ *has two eigenvalues* $\lambda_1^{\pm}(\mu,\epsilon)$ *given by*

$$\begin{cases} \frac{1}{2}\mathrm{i}\,\mu + \mathrm{i}\,r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{\mu}{8}\sqrt{\Delta(\mu,\epsilon)}\,, & \forall \mu \in [0,\underline{\mu}(\epsilon))\,, \\ \frac{1}{2}\mathrm{i}\,\underline{\mu}(\epsilon) + \mathrm{i}\,r(\epsilon^3)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\,\mu + \mathrm{i}\,r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\,\frac{\mu}{8}\sqrt{|\Delta(\mu,\epsilon)|}\,, & \forall \mu \in (\underline{\mu}(\epsilon),\mu_0)\,, \end{cases} \tag{1.2.13}$$

where the function $\Delta(\mu,\epsilon)$ *has the expansion*

$$\Delta(\mu,\epsilon) = 8\epsilon^2(1+r_0(\epsilon,\mu)) - \mu^2(1+r_0'(\epsilon,\mu))\,,$$

being positive when $0<\mu<\underline{\mu}(\epsilon)$ *and negative as* $\mu>\underline{\mu}(\epsilon)$*.*

As the Floquet parameter μ lies in $(0,\underline{\mu}(\epsilon))$ the eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ in (1.2.13) form the upper part of a figure "8" and have opposite non-zero real part. As μ tends to $\underline{\mu}(\epsilon)$, the two eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ collide again in the upper semiplane $\mathrm{Im}(\lambda)>0$ on the imaginary axis *far* from the origin of the complex plane. After the collision, for $\mu>\underline{\mu}(\epsilon)$, they split again, remain stable and move along the imaginary axis. For $\mu<0$ the operator $\mathcal{L}_{\mu,\epsilon}$ possesses the symmetric eigenvalues $\overline{\lambda_1^{\pm}(-\mu,\epsilon)}$ forming the lower part of the figure "8" in the semiplane $\mathrm{Im}(\lambda)<0$. Figure 1.4 portrays the whole splitting of these two eigenvalues. This figure "8" is the complete Benjamin-Feir branching of the $L^2(\mathbb{R})$-spectrum of the operator $\mathcal{L}_\epsilon$ outside of the imaginary axis predicted by Benjamin and Feir [10] and pioneered by the rigorous findings of Nguyen and Strauss [74].

Finite depth. The main result of the second part of the thesis provides, for finite values of the depth $\mathtt{h}$ and ϵ and μ small enough , the full splitting of the four eigenvalues close to zero of the operator

$$\mathcal{L}_{\mu,\epsilon}(\mathtt{h}) := \begin{bmatrix} (\partial_x+\mathrm{i}\,\mu)\circ(\mathtt{c_h}+p_\epsilon(x)) & |D+\mu|\tanh\big((\mathtt{h}+\mathtt{f}_\epsilon)|D+\mu|\big) \\ -(1+a_\epsilon(x)) & (\mathtt{c_h}+p_\epsilon(x))(\partial_x+\mathrm{i}\,\mu) \end{bmatrix},$$

which we derive in Section 1.3 as the Floquet shift $\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\,\mu x}\mathcal{L}_\epsilon e^{\mathrm{i}\,\mu x}$ of the linearized (and conjugated) operator $\mathcal{L}_\epsilon$ along the Stokes wave $(\eta_\epsilon,\psi_\epsilon)$ in the case of *finite depth* $0<\mathtt{h}<+\infty$.
As for the case of infinite depth we postpone the complete result to Theorem 1.5.1 and first focus, in Theorem 1.2.3, on the Benjamin-Feir *unstable* eigenvalues again forming, when the sea is sufficiently deep, a figure "8" in continuity with the deep-water case.

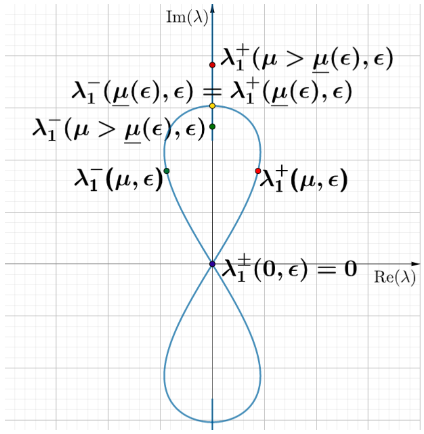


Figure 1.4: Traces of the eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ in the complex λ-plane at fixed $|\epsilon| \ll 1$ as μ varies. For $\mu \in (0, \underline{\mu}(\epsilon))$ the eigenvalues fill the portion of the 8 in $\{\mathrm{Im}(\lambda) > 0\}$ and for $\mu \in (-\underline{\mu}(\epsilon), 0)$ the symmetric portion in $\{\mathrm{Im}(\lambda) < 0\}$.

Theorem 1.2.3. (Benjamin-Feir unstable eigenvalues) *There exists a unique value* $\mathtt{h}_{WB} > 0$ *such that, for any* $\mathtt{h} > \mathtt{h}_{WB}$*, there exist* $\epsilon_1, \mu_0 > 0$ *and an analytic function* $\underline{\mu} : [0, \epsilon_1) \to [0, \mu_0)$*, of size* $\underline{\mu}(\epsilon) = r(\epsilon)$ *such that, for any* $\epsilon \in [0, \epsilon_1)$*, the operator* $\mathcal{L}_{\mu,\epsilon}$ *has two eigenvalues* $\lambda_1^{\pm}(\mu,\epsilon)$ *of the form*

$$\begin{cases} \mathrm{i}\,\frac{1}{2}C_1(\mathtt{h})\mu + \mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \frac{\mu}{8}C_2(\mathtt{h})(1 + r(\epsilon,\mu))\sqrt{\Delta_{BF}(\mathtt{h};\mu,\epsilon)}, & \forall \mu \in [0, \underline{\mu}(\epsilon)) \\ \mathrm{i}\,\frac{1}{2}C_1(\mathtt{h})\underline{\mu}(\epsilon) + \mathrm{i}\,r(\epsilon^3), & \mu = \underline{\mu}(\epsilon) \\ \mathrm{i}\,\frac{1}{2}C_1(\mathtt{h})\mu + \mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \mathrm{i}\,\frac{\mu}{8}C_2(\mathtt{h})(1 + r(\epsilon,\mu))\sqrt{|\Delta_{BF}(\mathtt{h};\mu,\epsilon)|}, & \forall \mu \in (\underline{\mu}(\epsilon), \mu_0) \end{cases} \tag{1.2.14}$$

where $C_1(\mathtt{h}), C_2(\mathtt{h}) > 0$*, for every* $\mathtt{h} > 0$*, with* $C_1(\mathtt{h}), C_2(\mathtt{h}) \to 1$ *as* $\mathtt{h} \to +\infty$*, and* $\Delta_{BF}(\mathtt{h};\mu,\epsilon)$ *is, for any* $0 < \epsilon < \epsilon_1$ *dependent on* $\mathtt{h}$*, positive for* $0 < \mu < \underline{\mu}(\epsilon)$ *and negative for* $\mu > \underline{\mu}(\epsilon)$*.*

In Section 1.5 we shall see that the value $\mathtt{h}_{\mathrm{WB}} \approx 1.363$ of Theorem 1.2.3 coincides with the critical depth found by Whitham [92] and Benjamin [9] and analytically carried out by Bridges and Mielke [21].

For $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$ as the Floquet parameter varies in $0 < \mu < \underline{\mu}(\epsilon)$, the eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ in (1.2.13) have opposite non-zero real part and form the upper part of a $\mathtt{h}$-dependent figure "8" in continuity with the case of infinite depth. The lower part of the figure "8" is depicted by the symmetric eigenvalues $\overline{\lambda_1^{\pm}(-\mu,\epsilon)}$ of the operator $\mathcal{L}_{\mu,\epsilon}$ for $\mu < 0$. This figure "8" is the whole finite-depth unstable branching of the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}_\epsilon$ predicted by Whitham

[92] and Benjamin [9] and pioneered by the rigorous findings of Bridges and Mielke [21]. In the shallow-water regime $0 < \mathtt{h} < \mathtt{h}_{\rm WB}$, for sufficiently small ϵ, the two eigenvalues in (1.2.14) remain purely imaginary.
The result, also in its complete form in Theorem 1.5.1, is not conclusive in the specific case $\mathtt{h} = \mathtt{h}_{\rm WB}$, for here, as we shall see, the first expansion of the discriminant $\Delta_{\rm BF}$ appearing in (1.2.14) degenerates and its sign, determining stability or lack thereof, has to be evinced by a higher-order expansion. This is the content of the third and last part of the thesis.

Critical threshold. The main result of the third part of the thesis proves, for the critical value of the depth $\mathtt{h} = \mathtt{h}_{\rm WB}$, the full splitting of the four eigenvalues close to zero of the operator $\mathcal{L}_{\mu,\epsilon}$ when ϵ and μ are small enough, see Theorem 1.6.1. We present here the core result that states for the first time in the analytical literature the existence of Benjamin-Feir *unstable* eigenvalues for the critical-depth case forming, as we shall detail in Theorem 1.6.2, a degenerate figure "8".

Theorem 1.2.4. (Modulational instability of the Stokes wave at $\mathtt{h} = \mathtt{h}_{WB}$) *In the critical-depth case $\mathtt{h} = \mathtt{h}_{WB}$, small amplitude Stokes waves of amplitude $\mathcal{O}(\epsilon)$ are linearly unstable subject to long wave perturbations. Actually Stokes waves are modulational unstable also at nearby depths $\mathtt{h} < \mathtt{h}_{WB}$: there is an analytic function defined for ϵ small, of the form*

$$\underline{\mathtt{h}}(\epsilon) = -c\epsilon^2 + \mathcal{O}(\epsilon^3)\,, \quad c > 0\,,$$

such that, for any $(\mathtt{h}, \epsilon)$ satisfying

$$\mathtt{h} > \mathtt{h}_{WB} + \underline{\mathtt{h}}(\epsilon)\,, \tag{1.2.15}$$

the linearized operator $\mathcal{L}_{\mu,\epsilon}$ along the Stokes wave has two eigenvalues with nontrivial real part for any Floquet exponent μ small enough, see Figure 1.5. For $\mathtt{h} = \mathtt{h}_{WB}$ the unstable eigenvalues depict a closed figure "8" as μ varies in an interval of size $[0, c_1\epsilon^2)$.

We call this figure "8" degenerate because, as we shall see in Section 1.6, it is much smaller in height and width than that of the sufficiently-deep water case $\mathtt{h} > \mathtt{h}_{\rm WB}$.
For a more rigorous statement we refer to Theorems 1.6.1 and 1.6.2 which, additionally, provide a necessary and sufficient condition for the existence of unstable eigenvalues.

1.3 Linearization along the Stokes wave

Before commenting Theorems 1.2.2, 1.2.3, 1.2.4 and entering the specific formulation of the complete results in 1.4.1, 1.5.1 and 1.6.1, let us present their common mathematical

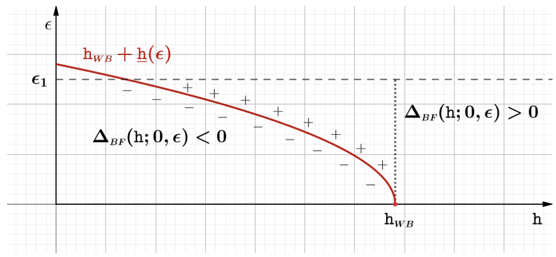


Figure 1.5: The values of $(\mathtt{h}, \epsilon)$ in $(0, \infty) \times (0, \epsilon_1)$ for which there are Benjamin-Feir unstable eigenvalues fill the zone above the red curve, where $\Delta_{\text{BF}}(\mathtt{h}; 0, \epsilon) > 0$.

framework. The content of this section is classical, and we do not dwell on it for the moment, postponing a more detailed discussion to Section 2.1 and referring the reader to the extensive literature on the subject.

The linearized water waves system along the Stokes wave of wavenumber $\kappa = 1$ is

$$\begin{bmatrix} \hat\eta_t \\ \hat\psi_t \end{bmatrix} = \underbrace{\begin{bmatrix} -G(\eta_\epsilon)B_\epsilon - \partial_x \circ (V_\epsilon - c_\epsilon) & G(\eta_\epsilon) \\ -1 + B_\epsilon(V_\epsilon - c_\epsilon)\partial_x - B_\epsilon\partial_x \circ (V_\epsilon - c_\epsilon) - B_\epsilon G(\eta_\epsilon) \circ B_\epsilon & -(V_\epsilon - c_\epsilon)\partial_x + B_\epsilon G(\eta_\epsilon) \end{bmatrix}}_{=: \mathcal{J}\mathcal{A}_\epsilon \,:\, H^1(\mathbb{T},\mathbb{R}^2) \subset L^2(\mathbb{T},\mathbb{R}^2) \xrightarrow{\text{closed}} L^2(\mathbb{T},\mathbb{R}^2)} \begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix}, \tag{1.3.1}$$

with $\mathcal{J}$ in (1.2.4) and (V_ϵ, B_ϵ) being the velocity field of the Stokes wave, i.e. the gradient of the potential Φ in (1.2.3) with $\eta = \eta_\epsilon(t,x)$, $\psi = \psi_\epsilon(t,x)$. We observe that $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon^\top$, where $\mathcal{A}^\top$ is the transposed operator with respect the scalar product of $L^2(\mathbb{T},\mathbb{R}^2)$, which shows that the real system (1.3.1) is Hamiltonian. Moreover the linear operator $\mathcal{J}\mathcal{A}_\epsilon$ is reversible, i.e. it anti-commutes with the involution ρ in (1.2.9).
The linearized system in (1.3.1) is simplified by the following conjugations.

1. *Good unknown of Alinhac.* The time-independent linear transformation

$$\begin{bmatrix} \hat\eta \\ \hat\psi \end{bmatrix} := Z_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}, \qquad Z_\epsilon = \begin{bmatrix} 1 & 0 \\ B_\epsilon & 1 \end{bmatrix}, \quad Z_\epsilon^{-1} = \begin{bmatrix} 1 & 0 \\ -B_\epsilon & 1 \end{bmatrix}, \tag{1.3.2}$$

under which the system (1.3.1) assumes the form

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \widetilde{\mathcal{L}}_\epsilon \begin{bmatrix} u \\ v \end{bmatrix}, \quad \widetilde{\mathcal{L}}_\epsilon := \begin{bmatrix} -\partial_x \circ (V_\epsilon - c_\epsilon) & G(\eta_\epsilon) \\ -1 - (V_\epsilon - c_\epsilon)(B_\epsilon)_x & -(V_\epsilon - c_\epsilon)\partial_x \end{bmatrix}, \tag{1.3.3}$$

where we regard $\widetilde{\mathcal{L}}_\epsilon$ as a closed operator $\widetilde{\mathcal{L}}_\epsilon : H^1(\mathbb{T},\mathbb{R}^2) \subset L^2(\mathbb{T},\mathbb{R}^2) \to L^2(\mathbb{T},\mathbb{R}^2)$.
The linear system (1.3.3) is still real Hamiltonian and reversible as (1.3.1), because the

transformation Z_ϵ is symplectic and reversibility preserving, i.e.

$$Z_\epsilon^\top \mathcal{J} Z_\epsilon = \mathcal{J}\,, \qquad Z_\epsilon \circ \rho = \rho \circ Z_\epsilon\,.$$

2. *Levi-Civita flattening.* A conformal change of variables to flatten the water surface. Following [7, Appendix A] one finds a diffeomorphism of $\mathbb{T}$, $x \mapsto x + \mathfrak{p}_\epsilon(x)$, with $\mathfrak{p}_\epsilon(x)$ being a 2π-periodic odd function and a constant $\mathtt{f}_\epsilon$ determined as fixed point of (see [7, (A.15)])

$$\mathfrak{p}_\epsilon = \frac{\mathcal{H}}{\tanh\big((\mathtt{h}+\mathtt{f}_\epsilon)|D|\big)}[\eta_\epsilon(x+\mathfrak{p}_\epsilon(x))]\,, \qquad \mathtt{f}_\epsilon := \frac{1}{2\pi}\int_{\mathbb{T}} \eta_\epsilon(x+\mathfrak{p}_\epsilon(x))\mathrm{d}x\,, \tag{1.3.4a}$$

which, in the case of infinite depth $\mathtt{h} = +\infty$, boils down to

$$\mathfrak{p}_\epsilon = \mathcal{H}[\eta_\epsilon(x+\mathfrak{p}_\epsilon(x))]\,. \tag{1.3.4b}$$

Here $\mathcal{H}$ is the Hilbert transform, i.e. the Fourier multiplier operator

$$\mathcal{H}(e^{\mathrm{i}\,jx}) := -\mathrm{i}\,\mathrm{sign}(j)e^{\mathrm{i}\,jx}\,, \quad \forall j \in \mathbb{Z}\setminus\{0\}\,, \quad \mathcal{H}(1) := 0\,.$$

One then writes the Dirichlet-Neumann operator as (cfr. [7, Lemma A.5])

$$G(\eta_\epsilon) = \partial_x \circ \mathfrak{P}_\epsilon^{-1} \circ \underbrace{\mathcal{H}\circ\tanh\big((\mathtt{h}+\mathtt{f}_\epsilon)|D|\big)}_{=\,\mathcal{H}\ \text{ if } \mathtt{h}=+\infty} \circ \mathfrak{P}_\epsilon\,, \quad (\mathfrak{P}_\epsilon u)(x) := u(x+\mathfrak{p}_\epsilon(x))\,. \tag{1.3.5}$$

The Levi-Civita flattening is the symplectic and reversibility-preserving mapping

$$\mathcal{P}_\epsilon := \begin{bmatrix} (1+(\mathfrak{p}_\epsilon)_x)\mathfrak{P}_\epsilon & 0 \\ 0 & \mathfrak{P}_\epsilon \end{bmatrix}\,. \tag{1.3.6}$$

Under this conjugation system (1.3.3) is recast, by (1.3.5), into the linear system (1.0.1), with $\mathcal{L}_\epsilon$ being the Hamiltonian and reversible real operator

$$\begin{aligned} \mathcal{L}_\epsilon &:= \mathcal{P}_\epsilon\,\widetilde{\mathcal{L}}_\epsilon\,\mathcal{P}_\epsilon^{-1} = \begin{bmatrix} \partial_x\circ(\mathtt{c_h}+p_\epsilon(x)) & |D|\tanh((\mathtt{h}+\mathtt{f}_\epsilon)|D|) \\ -(1+a_\epsilon(x)) & (\mathtt{c_h}+p_\epsilon(x))\partial_x \end{bmatrix} = \mathcal{J}\mathcal{B}_\epsilon \\ \mathcal{B}_\epsilon &:= \begin{bmatrix} 1+a_\epsilon(x) & -(\mathtt{c_h}+p_\epsilon(x))\partial_x \\ \partial_x\circ(\mathtt{c_h}+p_\epsilon(x)) & |D|\tanh((\mathtt{h}+\mathtt{f}_\epsilon)|D|) \end{bmatrix} = \mathcal{B}_\epsilon^*\,, \end{aligned} \tag{1.3.7a}$$

which in the case of infinite depth $\mathtt{h} = +\infty$ is

$$\mathcal{L}_\epsilon = \begin{bmatrix} \partial_x\circ(1+p_\epsilon(x)) & |D| \\ -(1+a_\epsilon(x)) & (1+p_\epsilon(x))\partial_x \end{bmatrix} = \mathcal{J}\underbrace{\begin{bmatrix} 1+a_\epsilon(x) & -(1+p_\epsilon)(x)\partial_x \\ \partial_x\circ(1+p_\epsilon(x)) & |D| \end{bmatrix}}_{=\mathcal{B}_\epsilon}\,. \tag{1.3.7b}$$

The functions $p_\epsilon := p_\epsilon(x)$ and $a_\epsilon := a_\epsilon(x)$ appearing in (1.3.7) are given by

$$\begin{aligned} \mathtt{c_h} + p_\epsilon(x) &:= \frac{c_\epsilon - V_\epsilon(x+\mathfrak{p}_\epsilon(x))}{1+(\mathfrak{p}_\epsilon)_x(x)}\,, \\ 1 + a_\epsilon(x) &:= \frac{1+(V_\epsilon(x+\mathfrak{p}_\epsilon(x))-c_\epsilon)(B_\epsilon)_x(x+\mathfrak{p}_\epsilon(x))}{1+(\mathfrak{p}_\epsilon)_x(x)}\,. \end{aligned} \tag{1.3.8}$$

We regard $\mathcal{L}_\epsilon$ as a closed operator $\mathcal{L}_\epsilon : H^1(\mathbb{T},\mathbb{R}^2) \subset L^2(\mathbb{T},\mathbb{R}^2) \to L^2(\mathbb{T},\mathbb{R}^2)$.
We are now able to reformulate the linear modulational problem into a precise statement:

- find a natural $N \geq 1$ and an *eigenvalue* $\lambda \in \sigma_{L^2(\mathbb{T}_N)}(\mathcal{L}_\epsilon)$ such that $\mathrm{Re}(\lambda) > 0$,

where $\mathbb{T}_N := \mathbb{R}/2\pi N\mathbb{Z}$. We recall that an eigenvalue is an element of the *point* spectrum of an operator. Thus λ has an associated eigenspace $\mathcal{V}_\lambda \subset L^2(\mathbb{T}_N,\mathbb{C}^2)$ and any eigenvector $w \in \mathcal{V}_\lambda$, $w \neq 0$, satisfies $\mathcal{L}_\epsilon w(x) = \lambda w(x)$.
The link between eigenvectors $w \in \mathcal{V}_\lambda$ of $\mathcal{L}_\epsilon$ and solutions $\big(\lambda,\mu,v(x)\big)$ of (1.0.2) becomes more evident in view of the *Bloch-Floquet spectral decomposition.* We have (cfr. [55])

$$\sigma_{L^2(\mathbb{T}_N)}(\mathcal{L}_\epsilon) = \bigcup_{\mu\in\Omega_N} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon})\,, \quad \sigma_{L^2(\mathbb{R})}(\mathcal{L}_\epsilon) = \bigcup_{\mu\in\Omega} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon})\,, \quad \mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\,\mu x}\mathcal{L}_\epsilon e^{\mathrm{i}\,\mu x} \tag{1.3.9}$$

where $\Omega := [-\frac{1}{2},\frac{1}{2})$ and $\Omega_N \subset \Omega\cap\mathbb{Q}$, s.t. $|\Omega_N| = N$, $N\Omega_N \subset \mathbb{Z}$.
The last identity in (1.3.9) allows us to extend the domain of the Floquet exponent μ in (1.0.2) to every value in the domain $\Omega := [-\frac{1}{2},\frac{1}{2})$. Alternatively, instead of the domain[2] Ω, one can consider any domain of the form $\Omega + m$, with $m \in \mathbb{Z}$, because of the identity

$$\mathcal{L}_{\mu+m,\epsilon} = e^{-\mathrm{i}\,mx}\mathcal{L}_{\mu,\epsilon}e^{\mathrm{i}\,mx}\,, \quad \forall m \in \mathbb{Z}\,, \tag{1.3.10}$$

which shows that the spectrum $\sigma_{L^2(\mathbb{T})}(\mathcal{L}_\mu)$ is a 1-periodic set in μ.
We remark that, since $\mathcal{L}_\epsilon$ is a real operator, $\overline{\mathcal{L}_{\epsilon,\mu}} = \mathcal{L}_{-\mu,\epsilon}$. As a first consequence the spectrum $\sigma(\mathcal{L}_{-\mu,\epsilon}) = \overline{\sigma(\mathcal{L}_{\mu,\epsilon})}$ may be studied only for $\mu > 0$. As a second consequence the operator $\mathcal{L}_{\mu,\epsilon}$ is not real anymore, apart from $\mathcal{L}_{0,0}$. To deal with complex operators we equip the space $L^2(\mathbb{T},\mathbb{C}^2)$ with the complex scalar product

$$(f,g) := \frac{1}{2\pi}\int_0^{2\pi} (f_1\overline{g_1} + f_2\overline{g_2})\,\mathrm{d}x\,, \quad \forall f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \ g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in L^2(\mathbb{T},\mathbb{C}^2)\,, \tag{1.3.11}$$

so that, from now on, $\|f\|^2 := (f,f)$. We have the following

[2] in solid state physics this is called "first zone of Brillouin".

Definition 1.3.1 (Complex Hamiltonian/Reversible operator). A complex operator $\mathcal{L} : H^1(\mathbb{T},\mathbb{C}^2) \subset L^2(\mathbb{T},\mathbb{C}^2) \overset{\text{closed}}{\longrightarrow} L^2(\mathbb{T},\mathbb{C}^2)$ is

(i) *Hamiltonian*, if it can be written as $\mathcal{L} = \mathcal{J}\mathcal{B}$, with $\mathcal{J}$ in (1.2.4) and $\mathcal{B}$ being a self-adjoint operator, namely $\mathcal{B} = \mathcal{B}^*$, where $\mathcal{B}^*$ (with domain $H^1(\mathbb{T})$) is the adjoint with respect to the complex scalar product (1.3.11) of $L^2(\mathbb{T})$;

(ii) *reversible*, if it anticommutes with the complex involution $\overline{\rho}$

$$\mathcal{L} \circ \overline{\rho} = -\overline{\rho} \circ \mathcal{L} \,, \tag{1.3.12}$$

where (cfr. (1.2.9))

$$\overline{\rho} \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \overline{\eta}(-x) \\ -\overline{\psi}(-x) \end{bmatrix} . \tag{1.3.13}$$

The Floquet shift $\mathcal{L}_{\mu,\epsilon} = e^{-\mathrm{i}\,\mu x}\mathcal{L}_\epsilon e^{\mathrm{i}\,\mu x}$ of the real operator $\mathcal{L}_\epsilon$ in (1.3.7a) is the complex *Hamiltonian* and *reversible* operator

$$\begin{aligned} \mathcal{L}_{\mu,\epsilon} &= \begin{bmatrix} (\partial_x + \mathrm{i}\,\mu) \circ (\mathtt{c_h} + p_\epsilon(x)) & |D+\mu| \tanh\big((\mathtt{h} + \mathtt{f}_\epsilon)|D+\mu|\big) \\ -(1 + a_\epsilon(x)) & (\mathtt{c_h} + p_\epsilon(x))(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(\mathtt{c_h} + p_\epsilon(x))(\partial_x + \mathrm{i}\,\mu) \\ (\partial_x + \mathrm{i}\,\mu) \circ (\mathtt{c_h} + p_\epsilon(x)) & |D+\mu| \tanh\big((\mathtt{h} + \mathtt{f}_\epsilon)|D+\mu|\big) \end{bmatrix}}_{=:\, \mathcal{B}_{\mu,\epsilon} = \mathcal{B}^*_{\mu,\epsilon}}, \end{aligned} \tag{1.3.14a}$$

in the case of infinite depth $\mathtt{h} = +\infty$

$$\begin{aligned} \mathcal{L}_{\mu,\epsilon} &= \begin{bmatrix} (\partial_x + \mathrm{i}\,\mu) \circ (1 + p_\epsilon(x)) & |D+\mu| \\ -(1 + a_\epsilon(x)) & (1 + p_\epsilon(x))(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(1 + p_\epsilon(x))(\partial_x + \mathrm{i}\,\mu) \\ (\partial_x + \mathrm{i}\,\mu) \circ (1 + p_\epsilon(x)) & |D+\mu| \end{bmatrix}}_{=\mathcal{B}_{\mu,\epsilon}} . \end{aligned} \tag{1.3.14b}$$

The derivation of $\mathcal{L}_{\mu,\epsilon}$ from $\mathcal{L}_\epsilon$ in (1.3.7) can be carried out by means of Lemma 2.1.5. We regard $\mathcal{L}_{\mu,\epsilon}$ as a closed operator $\mathcal{L}_{\mu,\epsilon} : H^1(\mathbb{T},\mathbb{C}^2) \subset L^2(\mathbb{T},\mathbb{C}^2) \to L^2(\mathbb{T},\mathbb{C}^2)$. Property (1.3.12) for $\mathcal{L}_{\mu,\epsilon}$ follows because $\mathcal{L}_\epsilon$ is a real operator which is reversible with respect to the involution ρ in (1.2.9). Equivalently, since $\mathcal{J} \circ \overline{\rho} = -\overline{\rho} \circ \mathcal{J}$, a complex Hamiltonian operator $\mathcal{L} = \mathcal{J}\mathcal{B}$ is reversible, if the self-adjoint operator $\mathcal{B}$ is *reversibility-preserving*, i.e.

$$\mathcal{B} \circ \overline{\rho} = \overline{\rho} \circ \mathcal{B} \,. \tag{1.3.15}$$

In its final form problem (1.0.2) is reformulated as follows:

• find a Floquet exponent $\mu \in \mathbb{R}$, and an eigenvalue $\lambda \in \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon})$ such that $\mathrm{Re}(\lambda) > 0$.

The complete solution of this problem, for λ near the origin of the complex plane and small parameter ϵ and μ, is given in Theorems 1.4.1, 1.5.1 and 1.6.1 deeply relies on Hamiltonicity and reversibility of the operator $\mathcal{L}_{\mu,\epsilon}$.

We aim to find the unstable eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ by a perturbation argument based on the complete knowledge of the spectrum of the Fourier multiplier

$$\mathcal{L}_{\mu,0} = \begin{bmatrix} \mathtt{c_h}(\partial_x + \mathrm{i}\,\mu) & |D+\mu| \tanh\left(\mathtt{h}|D+\mu|\right) \\ -1 & \mathtt{c_h}(\partial_x + \mathrm{i}\,\mu) \end{bmatrix}, \tag{1.3.16}$$

consisting of the purely imaginary eigenvalues $\{\lambda_k^{\pm}(\mu)\ ,\ k \in \mathbb{Z}\}$, where

$$\lambda_k^{\pm}(\mu) := \mathrm{i}\left(\mathtt{c_h}(\pm k + \mu) \mp \sqrt{|k \pm \mu| \tanh(\mathtt{h}|k \pm \mu|)}\right), \tag{1.3.17}$$

at $\mathtt{h} = +\infty$

$$\mathcal{L}_{\mu,0} = \begin{bmatrix} \partial_x + \mathrm{i}\,\mu & |D+\mu| \\ -1 & \partial_x + \mathrm{i}\,\mu \end{bmatrix}, \qquad \lambda_k^{\pm}(\mu) := \mathrm{i}\left(\pm k + \mu \mp \sqrt{|k \pm \mu|}\right). \tag{1.3.18}$$

For $\mu = 0$ we find that 0 is the quadruple eigenvalue of the real operator $\mathcal{L}_{0,0}$ given by

$$\lambda_0^+(0) = \lambda_0^-(0) = \lambda_1^+(0) = \lambda_1^-(0) = 0\,. \tag{1.3.19}$$

It is an isolated eigenvalue for $\mathcal{L}_{0,0}$, meaning that the spectrum $\sigma\left(\mathcal{L}_{0,0}\right)$ splits into two separated parts

$$\sigma\left(\mathcal{L}_{0,0}\right) = \sigma'\left(\mathcal{L}_{0,0}\right) \cup \sigma''\left(\mathcal{L}_{0,0}\right) \quad \text{where} \quad \sigma'(\mathcal{L}_{0,0}) := \{0\} \tag{1.3.20}$$

and $\sigma''(\mathcal{L}_{0,0}) := \{\lambda_k^{\sigma}(0),\ k \neq 0, 1\,, \sigma = \pm\}$.
As shown in [74, Theorem 4.1], the operator $\mathcal{L}_{0,\epsilon}$ –which has the same matrix representation as $\mathcal{L}_\epsilon$ in (1.3.7) with domain restricted to 2π-periodic functions– keeps 0 as quadruple eigenvalue.

By Kato's perturbation theory (see Lemma 2.2.1 below) for any $\mu, \epsilon \neq 0$ sufficiently small, the perturbed spectrum $\sigma\left(\mathcal{L}_{\mu,\epsilon}\right)$ admits a disjoint decomposition as

$$\sigma\left(\mathcal{L}_{\mu,\epsilon}\right) = \sigma'\left(\mathcal{L}_{\mu,\epsilon}\right) \cup \sigma''\left(\mathcal{L}_{\mu,\epsilon}\right), \qquad |\,\sigma'\left(\mathcal{L}_{\mu,\epsilon}\right)| \leq 4\,, \tag{1.3.21}$$

where $\sigma'(\mathcal{L}_{\mu,\epsilon})$ consists of 4 eigenvalues (counted with multiplicity) close to 0.
We denote by $\mathcal{V}_{\mu,\epsilon}$ the sum of all the generalized eigenspaces associated with eigenvalues in $\sigma'(\mathcal{L}_{\mu,\epsilon})$. The spectral subspace $\mathcal{V}_{\mu,\epsilon}$ has dimension 4 and is invariant by $\mathcal{L}_{\mu,\epsilon}$.
Theorems 1.4.1, 1.5.1 and 1.6.1 give analytic expressions for the four eigenvalues, produced as spectrum of a 4×4 matrix representing the operator $\mathcal{L}_{\mu,\epsilon}$ restricted to $\mathcal{V}_{\mu,\epsilon}$, in a certain region of the parameters $(\mu, \epsilon, \mathtt{h}) \in (-\mu_0, \mu_0) \times (-\epsilon_1, \epsilon_1) \times [\mathtt{h}_1, \mathtt{h}_2]$.
The figure "8" we encountered in Theorems 1.2.2, 1.2.3 and 1.2.4 is depicted by two of these four eigenvalues that, when the seabed is sufficiently deep, are unstable i.e. with opposite nonzero real part.

Our method. We prepend here some of the main common points of our method to describe the spectral branching of the operator $\mathcal{L}_{\mu,\epsilon}$, while postponing the particular description of each case after its own complete statement in Theorems 1.4.1, 1.5.1 or 1.6.1.
The first ingredient is a symplectic version of Kato's theory of similarity transformations [59, II-§4]. The Kato method is perfectly suited to study splittings of multiple isolated eigenvalues, for which regular perturbation theory might fail. It has been used in the study of infinite dimensional integrable systems [58, 63, 8, 71].
In particular we implement Kato theory for the complex operators $\mathcal{L}_{\mu,\epsilon}$ which have an Hamiltonian and reversible structure, inherited by the Hamiltonian [98, 32] and reversible [18, 7, 11] nature of the water waves equations. We show how Kato's theory can be used to prolong, in an analytic way, a symplectic and reversible basis of the generalized eigenspace of the unperturbed operator $\mathcal{L}_{0,0}$ into a (μ, ϵ)-dependent symplectic and reversible basis of the corresponding invariant subspace of $\mathcal{L}_{\mu,\epsilon}$. Thus the restriction of the canonical complex symplectic form to this subspace, is represented, in this symplectic basis, by the *constant* symplectic matrix $\mathtt{J}_4$ defined in (2.2.25), which is *independent* of (μ, ϵ). This feature simplifies considerably perturbation theory.
In this way the spectral problem is reduced to determine the eigenvalues of a 4×4 matrix, which depends analytically in μ, ϵ and it is Hamiltonian and reversible. These properties imply strong algebraic features on the matrix entries, for which we provide detailed expansions.
The second main ingredient of our method is the use of block-decoupling, in the mold of KAM theory, to simplify the 4×4 spectrum-mining process avoiding the use of the characteristic polynomial of the reduced matrix, as in the periodic Evans function approach [24, 48] or in [45, 74]. This allows to capture the whole figure "8" even in the zone where the eigenvalues fail to depend analytically on the parameters.

We use a symplectic transformation to conjugate the orginal 4×4 Hamiltonian reversible matrix obtained by the Kato method to a block-diagonal matrix whose 2×2 diagonal blocks are Hamiltonian and reversible. One of these two blocks has the eigenvalues given in (1.2.13), (1.2.14) or (1.6.7) (depending on the case), forming the figure "8" characteristic of the Benjamin-Feir instability phenomenon.

1.4 Benjamin-Feir spectrum in deep water

In the case of infinite depth our complete spectral result is the following

Theorem 1.4.1. (Complete Benjamin-Feir spectrum) *There exist $\epsilon_0, \mu_0 > 0$ and a basis $\mathtt{F}_{fin}$ of the four-dimensional vector space $\mathcal{V}_{\mu,\epsilon}$ such that, for any $0 \le \mu < \mu_0$ and $0 \le \epsilon < \epsilon_0$, the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented on $\mathtt{F}_{fin}$ by a 4×4 matrix of the form*

$$\left(\begin{array}{c|c} \mathtt{U} & 0 \\ \hline 0 & \mathtt{S} \end{array}\right), \tag{1.4.1}$$

where $\mathtt{U}$ and $\mathtt{S}$ are 2×2 matrices of the form

$$\mathtt{U} := \begin{pmatrix} \mathrm{i}\left(\frac{1}{2}\mu + r(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \\ \frac{\mu^2}{8}(1 + r_1(\epsilon, \mu)) - \epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) & \mathrm{i}\left(\frac{1}{2}\mu + r(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \end{pmatrix}, \tag{1.4.2}$$

$$\mathtt{S} := \begin{pmatrix} \mathrm{i}\,\mu(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2)) & \mu + r_{10}(\mu^2\epsilon, \mu^3) \\ -1 - r_8(\epsilon^2, \mu^2\epsilon, \mu^3) & \mathrm{i}\,\mu(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2)) \end{pmatrix}, \tag{1.4.3}$$

where in each of the two matrices the diagonal entries are identical. The eigenvalues of the matrix $\mathtt{U}$ are given by

$$\lambda_1^{\pm}(\mu, \epsilon) = \frac{1}{2}\mathrm{i}\,\mu + \mathrm{i}\,r(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2(1 + r_0(\epsilon, \mu)) - \mu^2(1 + r_0'(\epsilon, \mu))}\,. \tag{1.4.4}$$

Note that if $8\epsilon^2(1 + r_0(\epsilon, \mu)) - \mu^2(1 + r_0'(\epsilon, \mu)) > 0$, respectively < 0, the eigenvalues $\lambda_1^{\pm}(\mu, \epsilon)$ have a nontrivial real part, respectively are purely imaginary.

The eigenvalues of the matrix $\mathtt{S}$ are a pair of purely imaginary eigenvalues of the form

$$\lambda_0^{\pm}(\mu, \epsilon) = \mp\mathrm{i}\,\sqrt{\mu}(1 + r'(\epsilon^2, \mu\epsilon, \mu^2)) + \mathrm{i}\,\mu(1 + r_9(\epsilon^2, \mu\epsilon, \mu^2))\,. \tag{1.4.5}$$

For $\epsilon = 0$ the eigenvalues $\lambda_1^{\pm}(\mu, 0), \lambda_0^{\pm}(\mu, 0)$ coincide with those in (1.3.18).

Let us comment on the result.

1. Validity region. Theorem 1.4.1 gives analytic expressions (1.4.4)-(1.4.5) for both stable and unstable eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ near 0 that holds in the region of the parameters $[0,\mu_0)\times[0,\epsilon_0)$. The analytic curve $\underline{\mu}(\epsilon)=2\sqrt{2}\epsilon(1+r(\epsilon))$ in (1.2.12), with tangent line at $\epsilon=0$ given by $\mu=2\sqrt{2}\epsilon$, divides the validity region in an "unstable" part where two eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ with non-trivial real part arise, and a "stable" part where all the four eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ are purely imaginary, see Figure 1.6.

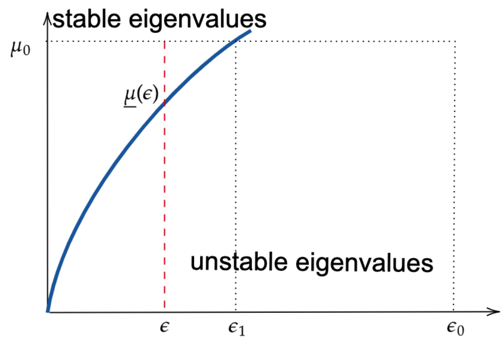


Figure 1.6: The blue line is the analytic curve defined implicitly by $8\epsilon^2(1+r_0(\epsilon,\mu))-\mu^2(1+r_0'(\epsilon,\mu))=0$. For values of μ below this curve, the two eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ have opposite real part. For μ above the curve, $\lambda_1^{\pm}(\mu,\epsilon)$ are purely imaginary.

2. Regularity break. The eigenvalues (1.4.4), that coincide with those in (1.2.13), are *not analytic* in (μ,ϵ) close to the value $(\underline{\mu}(\epsilon),\epsilon)$ where $\lambda_1^{\pm}(\mu,\epsilon)$ collide at the top of the figure "8" far from 0, but only Hölder continuous. Any attempt to describe the whole figure "8" by supposing *a priori* the eigenvalues to be analytic in (μ,ϵ) is thus doomed to fail. This might explain why, in previous approaches, the validity region of the parameters is way narrower than the one presented here and able to render only the figure "x" amid the figure "8", see Figure 1.3. On the other hand, the 2×2 matrix $\mathtt{U}$, whose spectrum is given by (1.4.4), *is analytic* with respect to the parameters (μ,ϵ).

3. Stability interval. For larger values of the Floquet parameter μ, the *perturbative principle of Hamiltonian spectra* that we described in Section 2.1 guarantees that the eigenvalues will remain on the imaginary axis until the Floquet exponent μ reaches values close to the next "collision" between two other eigenvalues of $\mathcal{L}_{\mu,0}$ in (1.3.18). For water waves in infinite depth this value is close to $\mu=1/4$, with eigenvalues close to $\mathrm{i}\,3/4$.

We conclude this section with a detailed description of our approach.

Ideas and scheme of proof. We first write the operator $\mathcal{L}_{\mu,\epsilon} = \mathrm{i}\,\mu + \mathscr{L}_{\mu,\epsilon}$ as in (2.2.1) and we aim to construct a basis of $\mathcal{V}_{\mu,\epsilon}$ to represent $\mathscr{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}$ as a convenient 4×4 matrix. The unperturbed operator $\mathscr{L}_{0,0}|_{\mathcal{V}_{0,0}}$ possesses 0 as isolated eigenvalue with algebraic multiplicity 4 and generalized kernel $\mathcal{V}_{0,0}$ spanned by the vectors $\{f_1^{\pm}, f_0^{\pm}\}$ in (2.1.15b), (2.1.16).

Exploiting Kato's theory of similarity transformations for separated eigenvalues we prolong the unperturbed symplectic basis $\{f_1^{\pm}, f_0^{\pm}\}$ of $\mathcal{V}_{0,0}$ into a symplectic basis of $\mathcal{V}_{\mu,\epsilon}$ (cfr. Definition 2.2.6), depending analytically on μ, ϵ. In Lemma 2.2.1 we construct the transformation operator $U_{\mu,\epsilon}$, see (2.2.11), which is invertible and analytic in μ, ϵ, and maps isomorphically $\mathcal{V}_{0,0}$ into $\mathcal{V}_{\mu,\epsilon}$. Furthermore, since $\mathscr{L}_{\mu,\epsilon}$ is Hamiltonian and reversible, we prove in Lemma 2.2.2 that the operator $U_{\mu,\epsilon}$ is symplectic and reversibility preserving. This implies that the vectors $f_k^{\sigma}(\mu,\epsilon) := U_{\mu,\epsilon} f_k^{\sigma}$, $k = 0, 1$, $\sigma = \pm$, form a symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$, according to Definition 2.2.6.

This construction has the following interpretation in the setting of complex symplectic structures, cfr. [6, 38]. The complex symplectic form (2.2.20) restricted to the symplectic subspace $\mathcal{V}_{\mu,\epsilon}$ is represented, in the (μ,ϵ)-dependent symplectic basis $f_k^{\sigma}(\mu,\epsilon)$, by the *constant* antisymmetric matrix $\mathtt{J}_4$ defined in (2.2.25), for *any* value of (μ,ϵ). In this sense $U_{\mu,\epsilon}$ is acting as a "Darboux transformation". Consequently, the Hamiltonian and reversible operator $\mathscr{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}$ is represented, in the symplectic basis $f_k^{\sigma}(\mu,\epsilon)$, by a 4×4 matrix of the form $\mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ with $\mathtt{B}_{\mu,\epsilon}$ selfadjoint, see Lemma 2.2.10. This property simplifies considerably the perturbation theory of the spectrum (we refer to [78] for a discussion, in a different context, of the difficulties raised by parameter-dependent symplectic forms).

We then modify the basis $\{f_k^{\sigma}(\mu,\epsilon)\}$ to construct a new symplectic and reversible basis $\{g_k^{\sigma}(\mu,\epsilon)\}$ of $\mathcal{V}_{\mu,\epsilon}$, still depending analytically on μ, ϵ, with the additional property that $g_1^{-}(0,\epsilon)$ has zero space average; this property plays a crucial role in the expansion obtained in Lemma 3.2.5, necessary to exhibit the Benjamin-Feir instability phenomenon, see Remark 3.2.6. By construction, the eigenvalues of the 4×4 matrix $\mathtt{L}_{\mu,\epsilon}$, representing the action of the operator $\mathscr{L}_{\mu,\epsilon}$ on the basis $\{g_k^{\sigma}(\mu,\epsilon)\}$, coincide with the portion of the spectrum $\sigma'(\mathscr{L}_{\mu,\epsilon})$ close to zero, defined in (1.3.21). In Proposition 3.2.2 we prove that the 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}$ has the form

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix}, \tag{1.4.6}$$

where $\mathtt{J}_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $E = E^*$, $G = G^*$ and F are 2×2 matrices having the expansions (3.2.5)-(3.2.7). To compute these expansions –from which the Benjamin-Feir instability will emerge– we use two ingredients. First we Taylor expand $(\mu,\epsilon)\mapsto U_{\mu,\epsilon}$ in Lemma 3.1.3. The Taylor expansion of $U_{\mu,\epsilon}$ is not a symplectic operator, but this is no longer important to compute the expansions (3.2.5)-(3.2.7) of the matrix $\mathtt{L}_{\mu,\epsilon}$. We used the fact that $U_{\mu,\epsilon}$ is symplectic to prove the Hamiltonian structure (1.4.6) of $\mathtt{L}_{\mu,\epsilon}$. The second ingredient is a careful analysis of $\mathtt{L}_{0,\epsilon}$ and $\partial_\mu \mathtt{L}_{\mu,\epsilon}|_{\mu=0}$. In particular we prove that the $(2,2)$-entry of the matrix E in (3.2.5) does not have any term $\mathcal{O}(\epsilon^m)$ nor $\mathcal{O}(\mu\epsilon^m)$ *for any* $m\in\mathbb{N}$. These terms would be dangerous because they might change the sign of the entry $(2,2)$ of the matrix E in (3.2.5) which instead is always negative. This is crucial to prove the Benjamin-Feir instability, as we explain below. We show the absence of terms $\mathcal{O}(\epsilon^m)$, $m\in\mathbb{N}$, fully exploiting the structural information (2.1.17) concerning the four dimensional generalized Kernel of the operator $\mathcal{L}_{0,\epsilon}$ for any $\epsilon>0$, see Lemma 3.2.4. The absence of terms $\mathcal{O}(\mu\epsilon^m)$, $m\in\mathbb{N}$, is due to the properties of the basis $\{g_k^\sigma(\mu,\epsilon)\}$ (see Remark 3.2.6) and it is the motivation for modifying the original basis $\{f_k^\sigma(\mu,\epsilon)\}$.

Thanks to this analysis, the 2×2 matrix

$$\mathtt{J}_2E=\begin{pmatrix} -\mathrm{i}\left(\frac{\mu}{2}+r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\frac{\mu^2}{8}(1+r_5(\epsilon,\mu)) \\ -\epsilon^2(1+r_1'(\epsilon,\mu\epsilon^2))+\frac{\mu^2}{8}(1+r_1''(\epsilon,\mu)) & -\mathrm{i}\left(\frac{\mu}{2}+r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right)\end{pmatrix} \tag{1.4.7}$$

possesses two eigenvalues with non-zero real part –we say that it exhibits the Benjamin-Feir phenomenon– as long as the two off-diagonal elements have the same sign, which happens for $0<\mu<\overline{\mu}(\epsilon)$ with $\overline{\mu}(\epsilon)\sim 2\sqrt{2}\epsilon$. On the other hand the 2×2 matrix $\mathtt{J}_2G$ has purely imaginary eigenvalues for $\mu>0$ of order $\mathcal{O}(\sqrt{\mu})$. In order to prove that the complete 4×4 matrix $\mathtt{L}_{\mu,\epsilon}$ in (1.4.6) possesses Benjamin-Feir unstable eigenvalues as well, our aim is to eliminate the coupling term $\mathtt{J}_2F$. This is done in Section 3.3 by a block diagonalization procedure, inspired by KAM theory. This is a singular perturbation problem because the spectrum of the matrices $\mathtt{J}_2E$ and $\mathtt{J}_2G$ tends to 0 as $\mu\to 0$. We construct a symplectic and reversibility preserving block-diagonalization transformation in three steps:

1. *First step of block-diagonalization (Section 3.3.1).* Note that the spectral gap between the 2 block matrices $\mathtt{J}_2E$ and $\mathtt{J}_2G$ is of order $\mathcal{O}(\sqrt{\mu})$, whereas the entry F_{11} of the matrix F has size $\mathcal{O}(\epsilon^3)$. In Section 3.3.1 we perform a symplectic and reversibility-preserving change of coordinates removing F_{11} and conjugating $\mathtt{L}_{\mu,\epsilon}$ to a new Hamiltonian and reversible matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$ whose block-off-diagonal matrix $\mathtt{J}_2F^{(1)}$ has size $\mathcal{O}(\mu\epsilon,\mu^3)$ and $\mathtt{J}_2E^{(1)}$ has the same form (1.4.7), and therefore possesses Benjamin-Feir unstable eigenvalues as well. This transformation is inspired by the Jordan normal form of $\mathtt{L}_{0,\epsilon}$.

2. *Second step of block-diagonalization (Section 3.3.2).* We next perform a step of

block-diagonalization to decrease further the size of the off-diagonal blocks: by applying a procedure inspired by KAM theory we obtain (at least) a $\mathcal{O}(\mu^2)$ factor in each entries of $F^{(2)}$ in (3.3.14) (by contrast note the presence of $\mathcal{O}(\mu\epsilon)$ entries in $F^{(1)}$). To achieve this, we construct a linear change of variables that conjugates the matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$ to the new Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$ in (3.3.13), where the new off-diagonal matrix $\mathtt{J}_2 F^{(2)}$ is much smaller than $\mathtt{J}_2 F^{(1)}$. The delicate point, for which we perform Step 2 separately from Step 3, is to estimate the new block-diagonal matrices after the conjugation, and prove that $\mathtt{J}_2 E^{(2)}$ has still the form (1.4.7) – thus possessing Benjamin-Feir unstable eigenvalues. Let us elaborate on this. In order to reduce the size of $\mathtt{J}_2 F^{(1)}$, we conjugate $\mathtt{L}^{(1)}_{\mu,\epsilon}$ by the symplectic matrix $\exp(S^{(1)})$, where $S^{(1)}$ is a Hamiltonian matrix with the same form of $\mathtt{J}_2 F^{(1)}$, see (3.3.12). The transformed matrix $\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S^{(1)})\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S^{(1)})$ has the Lie expansion[3]

$$\begin{aligned}
\mathtt{L}^{(2)}_{\mu,\epsilon} = & \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \\
& + \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & 0 \end{pmatrix} + \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \right] \\
& + \frac{1}{2}\left[S^{(1)}, \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \right]\right] + \left[S^{(1)}, \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & 0 \end{pmatrix} \right] + \text{h.o.t.}
\end{aligned} \tag{1.4.8}$$

The first line in the right hand side of (1.4.8) is the original block-diagonal matrix, the second line of (1.4.8) is a purely off-diagonal matrix and the third line is the sum of two block-diagonal matrices and "h.o.t." collects terms of much smaller size. We determine $S^{(1)}$ in such a way that the second line of (1.4.8) vanishes (this equation would be referred to as the "homological equation" in the context of KAM theory). In this way the remaining off-diagonal matrices (appearing in the h.o.t. remainder) are much smaller in size. We then compute the block-diagonal corrections in the third line of (1.4.8) and show that the new block-diagonal matrix $\mathtt{J}_2 E^{(2)}$ has again the form (1.4.7) (clearly with different remainders, but of the same order) and thus displays Benjamin-Feir instability. This last step is delicate because $S^{(1)} = \mathcal{O}(\epsilon, \mu^2)$ and $\mathtt{J}_2 F^{(1)} = \mathcal{O}(\mu\epsilon, \mu^3)$ and so the matrix in the third line of (1.4.8) could a priori have elements of size $\mathcal{O}(\mu\epsilon^2)$. Adding a term of size $\mathcal{O}(\mu\epsilon^2)$ to the (1,2)-entry of the matrix $\mathtt{J}_2 E^{(1)}$, which has the form $-\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu))$ as in (1.4.7), could make it positive. In such a case the eigenvalues of $\mathtt{J}_2 E^{(2)}$ would be purely imaginary, and the Benjamin-Feir instability would disappear. Actually, estimating individually each components, we show that no contribution of size $\mathcal{O}(\mu\epsilon^2)$ appears in the (1,2)-entry.

One further comment is needed. We solve the required homological equation without

[3] recall that $\exp(S)L\exp(-S) = \sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}^n_S(L)$, where $\mathrm{ad}^0_S(L) := L$, $\mathrm{ad}^n_S(L) = [S, \mathrm{ad}^{n-1}_S(L)]$ for $n \geq 1$.

diagonalizing $\mathtt{J}_2 E^{(1)}$ and $\mathtt{J}_2 G^{(1)}$ (as done typically in KAM theory). Note that diagonalization is not even possible at $\mu \sim 2\sqrt{2}\epsilon$ where $\mathtt{J}_2 E^{(1)}$ becomes a Jordan block (here its eigenvalues fail to be analytic). We use a direct linear algebra argument that enables to preserve the analyticity in μ, ϵ of the transformed 4×4 matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$.

3. *Complete block-diagonalization (Section 3.3.3).* As a last step in Lemma 3.3.8 we perform, by means of a standard implicit function theorem, a symplectic and reversibility preserving transformation that block-diagonalize $\mathtt{L}^{(2)}_{\mu,\epsilon}$ completely. The invertibility properties and estimates required to apply the implicit function theorem rely on the solution of the homological equation obtained in Step 2. The off-diagonal matrix $\mathtt{J}_2 F^{(2)}$ is small enough to directly prove that the block-diagonal matrix $\mathtt{J}_2 E^{(3)}$ has the same form of $\mathtt{J}_2 E^{(2)}$, thus possesses Benjamin-Feir unstable eigenvalues (without distinguishing the entries as we do in Step 2).

In conclusion, the original matrix $\mathtt{L}_{\mu,\epsilon}$ in (1.4.6) has been conjugated to the Hamiltonian and reversible matrix (1.4.1). This proves Theorem 1.4.1 and Theorem 1.2.2.

1.5 Benjamin-Feir spectrum in finite depth

Before stating the main result let us introduce the "Whitham-Benjamin" function

$$\mathtt{e}_{\mathrm{WB}} := \mathtt{e}_{\mathrm{WB}}(\mathtt{h}) := \frac{1}{\mathtt{c_h}}\Big[\frac{9\mathtt{c}_\mathtt{h}^8 - 10\mathtt{c}_\mathtt{h}^4 + 9}{8\mathtt{c}_\mathtt{h}^6} - \frac{1}{\mathtt{h} - \frac{1}{4}\mathtt{e}_{12}^2}\Big(1 + \frac{1-\mathtt{c}_\mathtt{h}^4}{2} + \frac{3}{4}\frac{(1-\mathtt{c}_\mathtt{h}^4)^2}{\mathtt{c}_\mathtt{h}^2}\mathtt{h}\Big)\Big], \tag{1.5.1}$$

where $\mathtt{c_h} = \sqrt{\tanh(\mathtt{h})}$ is the bifurcation value of the Stokes wave (cfr. Theorem 1.2.1), and

$$\mathtt{e}_{12} := \mathtt{e}_{12}(\mathtt{h}) := \mathtt{c_h} + \mathtt{c}_\mathtt{h}^{-1}(1-\mathtt{c}_\mathtt{h}^4)\mathtt{h} > 0\,, \quad \forall \mathtt{h} > 0\,. \tag{1.5.2}$$

The function $\mathtt{e}_{\mathrm{WB}}(\mathtt{h})$ is well defined for any $\mathtt{h} > 0$ because the denominator $\mathtt{h} - \frac{1}{4}\mathtt{e}_{12}^2 > 0$ in (1.5.1) is positive for any $\mathtt{h} > 0$, see Lemma 4.3.7. The function (1.5.1) coincides, up to a non zero factor, with the celebrated function obtained by Whitham [92], Benjamin [9] and Bridges-Mielke [21] which has as *unique root* the threshold depth between the shallow and the deep-water regimes.

The value $\mathtt{h}_{\mathrm{WB}} > 0$ in Theorem 1.2.3, that we call *Whitham-Benjamin critical depth*, is thus the zero of the Whitham-Benjamin function $\mathtt{e}_{\mathrm{WB}}(\mathtt{h})$, numerically approximated by

$$\mathtt{h}_{\mathrm{WB}} \approx 1.363\,. \tag{1.5.3}$$

The Whitham-Benjamin function $\mathtt{e}_{\mathrm{WB}}(\mathtt{h})$ is negative for $0 < \mathtt{h} < \mathtt{h}_{\mathrm{WB}}$, positive for $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$ and asymptotically 1^- as $\mathtt{h} \to +\infty$, see Figure 1.7.

Before stating of Theorem 1.2.3, let us also define the positive coefficient

$$\mathtt{e}_{22} := \mathtt{e}_{22}(\mathtt{h}) := \frac{(1-\mathtt{c}_\mathtt{h}^4)(1+3\mathtt{c}_\mathtt{h}^4)\mathtt{h}^2 + 2\mathtt{c}_\mathtt{h}^2(\mathtt{c}_\mathtt{h}^4-1)\mathtt{h} + \mathtt{c}_\mathtt{h}^4}{\mathtt{c}_\mathtt{h}^3} > 0\,, \quad \forall \mathtt{h} > 0\,. \tag{1.5.4}$$

We observe that the functions $\mathtt{e}_{12}(\mathtt{h}) > \mathtt{c}_\mathtt{h}$ and $\mathtt{e}_{22}(\mathtt{h}) > 0$ are positive for any $\mathtt{h} > 0$, tend to 0 as $\mathtt{h} \to 0^+$ and to 1 as $\mathtt{h} \to +\infty$, see Lemma 4.2.6.

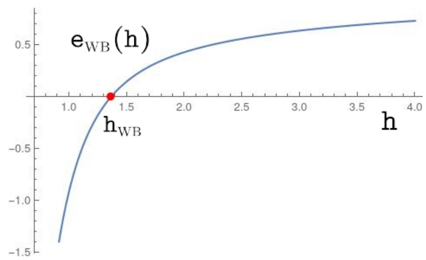


Figure 1.7: Plot of the Whitham-Benjamin function $\mathtt{e}_{\rm WB}(\mathtt{h})$. The red dot shows its unique root $\mathtt{h}_{\rm WB} = 1.363\ldots$. which is the celebrated "shallow/sufficiently deep" water threshold predicted independently by Whitham (cfr.[92] p.49) and Benjamin (cfr.[9] p.68), and recovered in the rigorous proof of Bridges-Mielke [21, p. 183].

In the case of finite depth our complete spectral result is the following

Theorem 1.5.1. (Complete Benjamin-Feir spectrum) *There exist $\epsilon_0, \mu_0 > 0$, uniformly for the depth $\mathtt{h}$ in any compact set of $(0, +\infty)$, and a basis $\mathtt{F}_{fin}$ of the four-dimensional vector space $\mathcal{V}_{\mu,\epsilon}$ such that, for any $0 < \mu < \mu_0$ and $0 \le \epsilon < \epsilon_0$, the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented on the basis $\mathtt{F}_{fin}$ by a 4×4 matrix of the form*

$$\left(\begin{array}{c|c} \mathtt{U} & 0 \\ \hline 0 & \mathtt{S} \end{array}\right), \tag{1.5.5}$$

where $\mathtt{U}$ and $\mathtt{S}$ are 2×2 matrices, with identical diagonal entries each, of the form

$$\mathtt{U} = \begin{pmatrix} \mathrm{i}\left((\mathtt{c}_\mathtt{h} - \tfrac{1}{2}\mathtt{e}_{12})\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\mathtt{e}_{22}\tfrac{\mu}{8}(1 + r_5(\epsilon,\mu)) \\ -\mu\epsilon^2 \mathtt{e}_{WB} + r_1'(\mu\epsilon^3, \mu^2\epsilon^2) + \mathtt{e}_{22}\tfrac{\mu^3}{8}(1 + r_1''(\epsilon,\mu)) & \mathrm{i}\left((\mathtt{c}_\mathtt{h} - \tfrac{1}{2}\mathtt{e}_{12})\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \end{pmatrix},$$

$$\mathtt{S} = \begin{pmatrix} \mathrm{i}\,\mathtt{c}_\mathtt{h}\mu + \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \\ -\mu + r_8(\mu\epsilon^2, \mu^3\epsilon) & \mathrm{i}\,\mathtt{c}_\mathtt{h}\mu + \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) \end{pmatrix}, \tag{1.5.6}$$

where $\mathtt{e}_{WB}$, $\mathtt{e}_{12}, \mathtt{e}_{22}$ are defined in (1.5.1), (1.5.2), (1.5.4). *The eigenvalues of $\mathtt{U}$ have the form*

$$\lambda_1^\pm(\mu,\epsilon) = \mathrm{i}\,\frac{1}{2}\check{\mathtt{c}}_\mathtt{h}\mu + \mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \pm \tfrac{1}{8}\mu\sqrt{\mathtt{e}_{22}(\mathtt{h})(1 + r_5(\epsilon,\mu))}\sqrt{\Delta_{BF}(\mathtt{h};\mu,\epsilon)}\,, \tag{1.5.7}$$

where $\breve{\mathtt{c}}_{\mathtt{h}} := 2\mathtt{c}_{\mathtt{h}} - \mathtt{e}_{12}(\mathtt{h})$ *and* $\Delta_{BF}(\mathtt{h};\mu,\epsilon)$ *is the Benjamin-Feir discriminant function*

$$\Delta_{BF}(\mathtt{h};\mu,\epsilon) := 8\mathtt{e}_{WB}(\mathtt{h})\epsilon^2 - 8r_1'(\epsilon^3,\mu\epsilon^2) - \mathtt{e}_{22}(\mathtt{h})\mu^2(1+r_1''(\epsilon,\mu)) \,. \tag{1.5.8}$$

As $\mathtt{e}_{22}(\mathtt{h}) > 0$*, they have non-zero real part if and only if* $\Delta_{BF}(\mathtt{h};\mu,\epsilon) > 0$*.*

The eigenvalues of the matrix $\mathtt{S}$ *are a pair of purely imaginary eigenvalues of the form*

$$\lambda_0^{\pm}(\mu,\epsilon) = \mathrm{i}\,\mathtt{c}_{\mathtt{h}}\mu(1+r_9(\epsilon^2,\mu\epsilon)) \mp \mathrm{i}\sqrt{\mu\tanh(\mathtt{h}\mu)}(1+r(\epsilon)) \,. \tag{1.5.9}$$

For $\epsilon = 0$ *the eigenvalues* $\lambda_1^{\pm}(\mu,0), \lambda_0^{\pm}(\mu,0)$ *coincide with those in* (1.3.17)*.*

Theorem 1.2.3 descends from Theorem 1.5.1 with $\mathtt{h}_{\mathrm{WB}}$ being the Whitham-Benjamin threshold in (1.5.3),

$$\underline{\mu}(\epsilon) = \mathtt{e}_{\mathtt{h}}\epsilon(1+r(\epsilon)) \,, \quad \mathtt{e}_{\mathtt{h}} := \sqrt{\frac{8\mathtt{e}_{\mathrm{WB}}(\mathtt{h})}{\mathtt{e}_{22}(\mathtt{h})}} \,, \tag{1.5.10}$$

$C_1(\mathtt{h}) := \breve{\mathtt{c}}_{\mathtt{h}}$ and $C_2(\mathtt{h}) := \sqrt{\mathtt{e}_{22}(\mathtt{h})}$. Figure 1.8 portrays the whole splitting of these two eigenvalues.

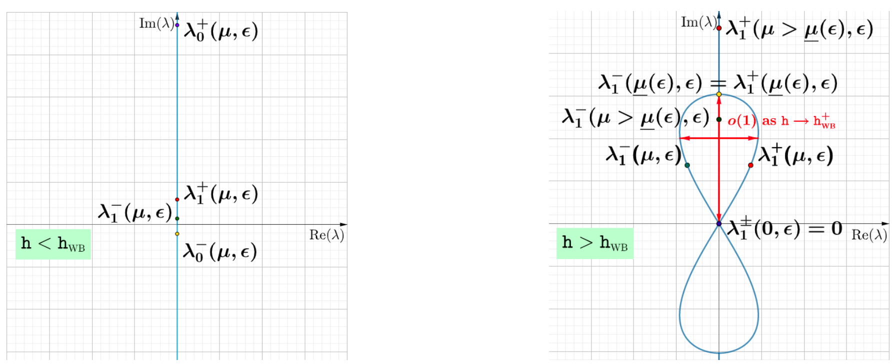


Figure 1.8: The picture on the left shows, in the "shallow" water regime $\mathtt{h} < \mathtt{h}_{\mathrm{WB}}$, the eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ and $\lambda_0^{\pm}(\mu,\epsilon)$ which are purely imaginary. The picture on the right shows, in the "sufficiently deep" water regime $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$, a first approximation of the eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ in the complex λ-plane at fixed $|\epsilon| \ll 1$ as μ varies. This figure "8 " depends on $\mathtt{h}$ and shrinks to 0 as $\mathtt{h} \to \mathtt{h}_{\mathrm{WB}}^+$, see Figure 1.9. As $\mathtt{h} \to +\infty$ the spectrum resembles the one in deep water.

Remark 1.5.2. At $\epsilon = 0$, the eigenvalues in (1.5.7) have the Taylor expansion

$$\lambda_1^{\pm}(\mu,0) = \mathrm{i}\,(\mathtt{c}_{\mathtt{h}} - \frac{1}{2}\mathtt{e}_{12}(\mathtt{h}))\mu \pm \mathrm{i}\,\frac{\mathtt{e}_{22}(\mathtt{h})}{8}\mu^2 + \mathcal{O}(\mu^3) \,,$$

which coincides with the one of $\lambda_1^{\pm}(\mu)$ in (1.3.17), in view of the coefficients $\mathtt{e}_{12}(\mathtt{h})$ and $\mathtt{e}_{22}(\mathtt{h})$ defined in (1.5.2), (1.5.4).

Let us comment Thoerem 1.5.1.

1. The exact figure "8". The figure "8" depicted by the two eigenvalues in (1.5.7) is well approximated by the curves

$$\mu \mapsto \Big(\pm \frac{\mu}{8}\sqrt{\mathtt{e}_{22}}\sqrt{8\mathtt{e}_{\mathrm{WB}}\epsilon^2 - \mathtt{e}_{22}\mu^2},\ \tfrac{1}{2}\breve{\mathtt{c}}_{\mathtt{h}}\mu \Big)\,, \tag{1.5.11}$$

see Figure 1.8, which also portrays the Benjamin-Feir spectrum in the stable case of shallow water. A better approximation of the figure "8" can be obtained by expanding the analytic remainders r, r_1, r_1'', r_2 which are explicitly computable. Actually, we tackle part of this task in the last part of the thesis to analyze the critical-depth case. However we remark that expansion (1.5.7)-(1.5.8) is sufficient to provide an *exact* description of the figure "8" as union of two convex regions (i.e. the upper and lower halves) encircled by a closed curve having 0 as double point and tangents at the top and bottom of the figure "8" parallel to the real axis. The figure "8" is symmetric with respect to the real axis due to the reality of the operator $\mathcal{L}_\epsilon$, and symmetric with respect to the imaginary axis as a consequence of its Hamiltonian nature.

2. Relation with Bridges-Mielke [21]. To facilitate a precise comparison between the outcomes of the paper of Bridges and Mielke [21] and Theorem 1.5.1, we elaborate on the connection of the functions with their respective coefficients. The expression for the Whitham-Benjamin function, denoted as $\mathtt{e}_{\mathrm{WB}}$, in equation (4.2.5), takes the form $\mathtt{e}_{\mathrm{WB}} = (\mathtt{c}_{\mathtt{h}}\mathtt{h})^{-1}\nu(F)$, where $\nu(F)$ is explicitly defined in [21, formula (6.17)], and $F = \mathtt{c}_{\mathtt{h}}\mathtt{h}^{-\frac{1}{2}}$ represents the Froude number, as indicated in [21, formula (3.4)]. Additionally, the value of the term $\mathtt{e}_{12}$ within equation (1.5.2) corresponds to $\mathtt{e}_{12} = 2c_g$, where, by [21, formula (3.8)], $c_g = \frac{1}{2}\mathtt{c}_{\mathtt{h}}(1 + F^{-2}\mathrm{sech}^2(\mathtt{h}))$ defines the group velocity. Lastly, the term $\mathtt{e}_{22}(\mathtt{h})$ is proportional to $\dot{c}_g$, where $\dot{c}_g$ signifies the derivative of the group velocity as defined in [21, formula (6.15)]. Notably, for gravity waves, this derivative is consistently negative across all depths.

3. Behavior near the Whitham-Benjamin depth $\mathtt{h}_{\mathrm{WB}}$. As $\mathtt{h} \to \mathtt{h}_{\mathrm{WB}}^{+}$ the first approximation in (1.5.11) of the figure "8" of Benjamin-Feir unstable eigenvalues collapses to zero, see Figure 1.9. In particular

$$\max_{\mu\in[0,\underline{\mu}(\epsilon)]} \mathrm{Re}\,\lambda_1^{+}(\mu,\epsilon) = \mathrm{Re}\,\lambda_1^{+}(\mu_{\max},\epsilon) = \frac{1}{2}\mathtt{e}_{\mathrm{WB}}(\mathtt{h})\epsilon^2 + r(\epsilon^3) \tag{1.5.12}$$

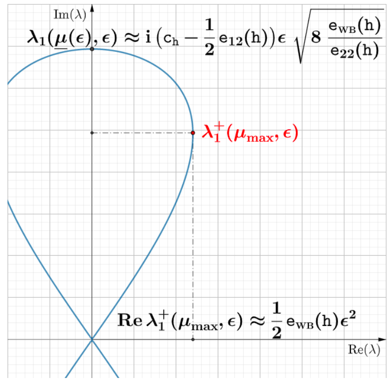


Figure 1.9: in a first approximation The Benjamin-Feir eigenvalue $\lambda_1^+(\mu_{\max}, \epsilon)$ in (1.5.12) has maximal real part $\frac{1}{2}\mathtt{e}_{\mathrm{WB}}(\mathtt{h})\epsilon^2$ which shrinks to zero as $\mathtt{h} \to \mathtt{h}_{\mathrm{WB}}^+$ making the whole figure "8" collapse.

degenerates to $r(\epsilon^3)$ as $\mathtt{h} \to \mathtt{h}_{\mathrm{WB}}^+$.

4. Shallow water regime. In Theorem 1.5.1 we prove that the four eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ close to zero remain purely imaginary for ϵ sufficiently small in the shallow-water regime $0 < \mathtt{h} < \mathtt{h}_{\mathrm{WB}}$. The expansion of the eigenvalues in Theorem 1.5.1 becomes singular when $\mathtt{h} \to 0^+$, coherently with the expansion of the Stokes waves in (A.3.1).

5. Unstable Floquet exponents and amplitudes (μ, ϵ). In Theorem 1.5.1 we actually prove that the expansion (1.2.14) of the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ holds for any value of (μ, ϵ) in a larger rectangle $[0, \mu_0) \times [0, \epsilon_0)$, and there exist Benjamin-Feir unstable eigenvalues if and only if the analytic function $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)$ in (1.5.8) is positive. The zero set of $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)$ is an analytic variety which, for $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$, is, restricted to the rectangle $[0, \mu_0) \times [0, \epsilon_1)$, the graph of the analytic function $\underline{\mu}(\epsilon) = \mathtt{e}_{\mathtt{h}}\epsilon(1 + r(\epsilon))$ in (1.5.10). This function is tangent at $\epsilon = 0$ to the straight line $\mu = \mathtt{e}_{\mathtt{h}}\epsilon$, and divides $[0, \mu_0) \times [0, \epsilon_1)$ in the region where $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon) > 0$ –i.e. where the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ have non-trivial real part– from the "stable" one where all the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ are purely imaginary, see Figure 1.10. In the region $[0, \mu_0) \times [\epsilon_1, \epsilon_0)$ the higher order polynomial approximations of $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)$ (which are computable) will determine the sign of $\Delta_{\mathrm{BF}}(\mathtt{h}; \mu, \epsilon)$.

6. Deep water limit. In the present form, the expansions of the eigenvalues in 1.2.3 and 1.5.1 may not admit a limit as $\mathtt{h} \to +\infty$. Indeed the remainders in the expansions of the eigenvalues are uniform only on compact sets of depths $\mathtt{h} \in (0, +\infty)$. Mathematically, the difference between the two cases becomes evident by analyzing the asymptotic behavior of

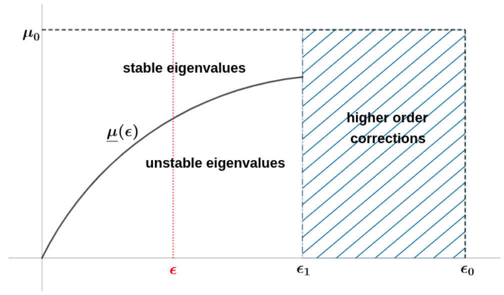


Figure 1.10: The solid curve portrays the graph of the real analytic function $\underline{\mu}(\epsilon)$ in (1.5.10) as $\mathtt{h} > \mathtt{h}_{\mathrm{WB}}$. For values of μ below this curve, the two eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ have non zero real part. For μ above the curve, $\lambda_1^{\pm}(\mu,\epsilon)$ are purely imaginary. In the region $[\epsilon_1,\epsilon_0)\times[0,\mu_0)$ the eigenvalues are real/purely imaginary depending on the higher order corrections given by Theorem 1.5.1, which determine the sign of $\Delta_{\mathrm{BF}}(\mathtt{h};\mu,\epsilon)$.

$\tanh(\mathtt{h}\mu)$ or similar quantities. In the idealized deep water case $\mathtt{h}=+\infty$, the term $\tanh(\mathtt{h}\mu)$ is replaced with 1, *no matter how small* the Floquet exponent μ is. On the contrary, for constant finite depth $\mathtt{h}>0$ we have $\tanh(\mathtt{h}\mu)=O(\mu\mathtt{h})$ as $\mu\to 0$. Additional intermediate scaling regimes $\mathtt{h}\mu\sim 1$, $\mathtt{h}\mu\ll 1$, $\mathtt{h}\mu\gg 1$ are possible.
The passage from the finite to the infinite-depth model is intrinsically delicate. It is well-known (e.g. see [30]) that intermediate long-wave regimes of the water-waves equations formally lead to different physically-relevant limit equations as Boussinesq, KdV, NLS, Benjamin-Ono, etc...

7. Behavior at the Whitham-Benjamin threshold $\mathtt{h}_{\mathrm{WB}}$. The analysis of Theorem 1.5.1 is not conclusive for the critical depth $\mathtt{h}=\mathtt{h}_{\mathrm{WB}}$, since, by recalling that $\mathtt{e}_{\mathrm{WB}}(\mathtt{h}_{\mathrm{WB}})=0$, the Benjamin-Feir discriminant function (1.5.8) reduces to

$$\Delta_{\mathrm{BF}}(\mathtt{h}_{\mathrm{WB}};\mu,\epsilon)=r(\epsilon^3)+r(\mu\epsilon^2)-\mathtt{e}_{22}(\mathtt{h}_{\mathrm{WB}})\mu^2(1+r_1''(\epsilon,\mu))\,. \tag{1.5.13}$$

Thus the quadratic expansion in (1.5.13) is not sufficient anymore to determine the sign of $\Delta_{\mathrm{BF}}(\mathtt{h}_{\mathrm{WB}};\mu,\epsilon)$, which could be either positive or negative depending on the sign of the term $r(\epsilon^3)$, provided μ is small enough. The analysis of this case will be the content of the last part of the thesis.

We conclude this section describing in detail our approach.

Ideas and scheme of proof. The first step is to exploit as in the case of infinite-depth Kato's theory to prolong the unperturbed symplectic basis $\{f_1^{\pm},f_0^{\pm}\}$ of $\mathcal{V}_{0,0}$ in (2.1.15a)-

(2.1.16) into a symplectic basis $\{f_k^\sigma(\mu,\epsilon), k=0,1,\sigma=\pm\}$ of the spectral subspace $\mathcal{V}_{\mu,\epsilon}$ associated with $\sigma'(\mathcal{L}_{\mu,\epsilon})$ in (1.3.21), depending analytically on μ,ϵ. Its expansion in μ,ϵ is provided in Lemma 4.1.2. This procedure reduces our spectral problem to determine the eigenvalues of the 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}$ (Lemma 2.2.10), representing the action of the operator $\mathcal{L}_{\mu,\epsilon} - \mathrm{i}\,\mathtt{c_h}\mu$ on $\{f_k^\sigma(\mu,\epsilon)\}$. In Proposition 4.2.1 we prove that

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix} \qquad \text{where} \quad \mathtt{J}_4 = \begin{pmatrix} \mathtt{J}_2 & 0 \\ 0 & \mathtt{J}_2 \end{pmatrix}, \quad \mathtt{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.5.14}$$

and the 2×2 matrices E, G, F have the expansions (4.2.2)-(4.2.4). In finite depth this computation is much more involved than in deep water, as we need to track the exact dependence of the matrix entries with respect to $\mathtt{h}$. In particular the matrix E is

$$E = \begin{pmatrix} \mathtt{e}_{11}\epsilon^2(1+r_1'(\epsilon,\mu\epsilon)) - \mathtt{e}_{22}\frac{\mu^2}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu^2}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix} \tag{1.5.15}$$

where the coefficients $\mathtt{e}_{11}$ and $\mathtt{e}_{22}$, defined in (4.2.5) and (1.5.4), are strictly positive *for any* value of $\mathtt{h}>0$. Thus the submatrix $\mathtt{J}_2 E$ has a pair of eigenvalues with nonzero real part, for any value of $\mathtt{h}>0$, provided $0<\mu<\overline{\mu}(\epsilon)\sim\epsilon$. On the other hand, it has to come out that the complete 4×4 matrix $\mathtt{L}_{\mu,\epsilon}$ possesses unstable eigenvalues *if and only if* the depth exceeds the celebrated Whitham-Benjamin threshold $\mathtt{h}_{\mathrm{WB}}\sim 1.363\ldots$. Indeed the correct eigenvalues of $\mathtt{L}_{\mu,\epsilon}$ are not a small perturbation of those of $\begin{pmatrix} \mathtt{J}_2 E & 0 \\ 0 & \mathtt{J}_2 G \end{pmatrix}$ and will emerge only after one non-perturbative step of block diagonalization. This was not the case in the infinitely deep water case, where the corresponding submatrix $\mathtt{J}_2 E$ was sufficient to obtain the Benjamin-Feir eigenvalues, and we only had to check their stability under perturbation.

Remark 1.5.3. We emphasize that (1.5.15) is not a simple Taylor expansion in μ,ϵ: note that the $(2,2)$-entry in (1.5.15) does not have any term $\mathcal{O}(\epsilon^m)$ nor $\mathcal{O}(\mu\epsilon^m)$ *for any* $m\in\mathbb{N}$. These terms could change the sign of the entry $(2,2)$ which instead, in (1.5.15), is always negative (recall that $\mathtt{e}_{22}(\mathtt{h})>0$). We prove the absence of terms ϵ^m exploiting the structural information (2.1.17) concerning the four dimensional generalized Kernel of the operator $\mathcal{L}_{0,\epsilon}$ for any $\epsilon>0$, see Lemma 4.2.2. We also note that the 2×2 matrices $\mathtt{J}_2 E$ and $\mathtt{J}_2 G$ in (1.5.14) have both eigenvalues of size $\mathcal{O}(\mu)$. As already mentioned, this is a crucial difference with the deep water case, where the eigenvalues of $\mathtt{J}_2 G$ are $\mathcal{O}(\sqrt{\mu})$.

In order to determine the spectrum of the matrix $\mathtt{L}_{\mu,\epsilon}$ in (1.5.14), we perform a block diagonalization of $\mathtt{L}_{\mu,\epsilon}$ to eliminate the coupling term $\mathtt{J}_2 F$ (which has size ϵ, see (4.2.4)).

We proceed, in Section 4.3, in three steps:

1. *Symplectic rescaling.* We first perform a symplectic rescaling which is singular at $\mu = 0$, see Lemma 4.3.1, obtaining the matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$. The effects are twofold: (i) the diagonal elements of

$$E^{(1)} = \begin{pmatrix} \mathtt{e}_{11}\mu\epsilon^2(1+r_1'(\epsilon,\mu\epsilon)) - \mathtt{e}_{22}\frac{\mu^3}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix} \tag{1.5.16}$$

have size $\mathcal{O}(\mu)$, as well as those of $G^{(1)}$, and (ii) the matrix $F^{(1)}$ has the smaller size $\mathcal{O}(\mu\epsilon)$.

2. *Non-perturbative step of block-diagonalization (Section 4.3.1).* Inspired by KAM theory, we perform one step of block decoupling to decrease further the size of the off-diagonal blocks. This step *modifies* the matrix $\mathtt{J}_2 E^{(1)}$ in a substantial way, by a term $\mathcal{O}(\mu\epsilon^2)$. Let us explain better this step. In order to reduce the size of $\mathtt{J}_2 F^{(1)}$, we conjugate $\mathtt{L}^{(1)}_{\mu,\epsilon}$ by the symplectic matrix $\exp(S^{(1)})$, where $S^{(1)}$ is a Hamiltonian matrix with the same form of $\mathtt{J}_2 F^{(1)}$, see (4.3.9). The transformed matrix $\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S^{(1)})\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S^{(1)})$ has the Lie expansion[4]

$$\begin{aligned} \mathtt{L}^{(2)}_{\mu,\epsilon} &= \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & 0 \end{pmatrix} + \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \right] \\ &+ \frac{1}{2}\left[S^{(1)}, \left[S^{(1)}, \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix} \right]\right] + \left[S^{(1)}, \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & 0 \end{pmatrix} \right] + \text{h.o.t.} \end{aligned} \tag{1.5.17}$$

The first line in the right hand side of (1.5.17) is the previous block-diagonal matrix, the second line of (1.5.17) is a purely off-diagonal matrix and the third line is the sum of two block-diagonal matrices and “h.o.t.” collects terms of much smaller size. $S^{(1)}$ is determined in such a way that the second line of (1.5.17) vanishes, and therefore the remaining off-diagonal matrices (appearing in the h.o.t. remainder) are smaller in size. Unlike the infinitely deep water case, the block-diagonal corrections in the third line of (1.5.17) are *not* perturbative, modifying substantially the block-diagonal part. More precisely we obtain that $\mathtt{L}^{(2)}_{\mu,\epsilon}$ has the form (4.3.10) with

$$E^{(2)} := \begin{pmatrix} \mu\epsilon^2\mathtt{e}_{\mathrm{WB}} + r_1'(\mu\epsilon^3,\mu^2\epsilon^2) - \mathtt{e}_{22}\frac{\mu^3}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix}.$$

Note the appearance of the Whitham-Benjamin function $\mathtt{e}_{\mathrm{WB}}(\mathtt{h})$ in the (1,1)-entry of $E^{(2)}$, which changes sign at the critical depth $\mathtt{h}_{\mathrm{WB}}$, see Figure 1.7, unlike the coefficient

[4] recall that $\exp(S)L\exp(-S) = \sum_{n\geq 0}\frac{1}{n!}\mathrm{ad}^n_S(L)$, where $\mathrm{ad}^0_S(L) := L$, $\mathrm{ad}^n_S(L) = [S, \mathrm{ad}^{n-1}_S(L)]$ for $n \geq 1$.

$\mathtt{e}_{11}(\mathtt{h}) > 0$ in (1.5.16). If $\mathtt{e}_{\rm WB}(\mathtt{h}) > 0$ and ϵ and μ are sufficiently small, the matrix $\mathtt{J}_2 E^{(2)}$ has eigenvalues with non-zero real part (recall that $\mathtt{e}_{22}(\mathtt{h}) > 0$ for any $\mathtt{h}$). On the contrary, if $\mathtt{e}_{\rm WB}(\mathtt{h}) < 0$, then the eigenvalues of $\mathtt{J}_2 E^{(2)}$ lay on the imaginary axis.

3. *Complete block-diagonalization (Section 4.3.2).* In Lemma 4.3.9 we completely block-diagonalize $\mathtt{L}_{\mu,\epsilon}^{(2)}$ by means of a standard implicit function theorem, finally proving that $\mathtt{L}_{\mu,\epsilon}$ is conjugated to the matrix (1.5.5).

1.6 Benjamin-Feir spectrum near the critical depth $\mathtt{h}_{\rm WB}$

As described in Section 1.5 the stability analysis of the Stokes waves in the case of finite depth is completely determined by the sign of the Benjamin-Feir discriminant function $\Delta_{\rm BF}(\mathtt{h};\mu,\epsilon)$ in (1.5.8). In particular, for any $\mathtt{h} > \mathtt{h}_{\rm WB}$ it results $\mathtt{e}_{\rm WB}(\mathtt{h}) > 0$ and therefore for μ and ϵ small enough $\Delta_{\rm BF}(\mathtt{h};\mu,\epsilon) > 0$ proving the existence of unstable eigenvalues. On the contrary for $\mathtt{h} < \mathtt{h}_{\rm WB}$ one has $\mathtt{e}_{\rm WB}(\mathtt{h}) < 0$ and consequently $\Delta_{\rm BF}(\mathtt{h};\mu,\epsilon) < 0$ implying the stability of the two eigenvalues.

In the case of critical depth $\mathtt{h} = \mathtt{h}_{\rm WB}$ our complete spectral result is the following

Theorem 1.6.1 (**Complete Benjamin-Feir spectrum).** *There exist $\epsilon_0, \mu_0 > 0$, uniformly for the depth $\mathtt{h}$ in any compact set of $(0,+\infty)$, and a basis $\mathtt{F}_{fin}$ of the four-dimensional vector space $\mathcal{V}_{\mu,\epsilon}$ such that, for any $0 < \mu < \mu_0$ and $0 \le \epsilon < \epsilon_0$, the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented on the basis $\mathtt{F}_{fin}$ by a 4×4 matrix of the form*

$$\begin{pmatrix} \mathtt{U} & 0 \\ 0 & \mathtt{S} \end{pmatrix}, \tag{1.6.1}$$

where $\mathtt{U}$ and $\mathtt{S}$ are 2×2 matrices, with identical purely imaginary diagonal entries each, of the form

$$\mathtt{U} = \begin{pmatrix} \mathrm{i}\big((\mathtt{c}_{\mathtt{h}} - \tfrac12 \mathtt{e}_{12})\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\big) & -\mathtt{e}_{22}\tfrac{\mu}{8}(1+r_5(\epsilon,\mu)) \\ -\tfrac{\mu}{8}\Delta_{BF}(\mathtt{h};\mu,\epsilon) & \mathrm{i}\big((\mathtt{c}_{\mathtt{h}} - \tfrac12 \mathtt{e}_{12})\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\big) \end{pmatrix}, \tag{1.6.2}$$

$$\mathtt{S} = \begin{pmatrix} \mathrm{i}\,\mathtt{c}_{\mathtt{h}}\mu + \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \\ -\mu + r_8(\mu\epsilon^2,\mu^3\epsilon) & \mathrm{i}\,\mathtt{c}_{\mathtt{h}}\mu + \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) \end{pmatrix}. \tag{1.6.3}$$

The Benjamin-Feir discriminant function $\Delta_{BF}(\mathtt{h};\mu,\epsilon)$ in (1.6.2) has the form

$$\Delta_{BF}(\mathtt{h};\mu,\epsilon) := 8\mathtt{e}_{WB}(\mathtt{h})\epsilon^2 + 8\eta_{WB}(\mathtt{h})\epsilon^4 + r_1(\epsilon^5,\mu\epsilon^3) - \mathtt{e}_{22}(\mathtt{h})\mu^2(1 + r_1''(\epsilon,\mu)) \tag{1.6.4}$$

where $\mathtt{e}_{\mathrm{WB}}(\mathtt{h})$ *is the Whitham-Benjamin function in* (1.5.1), $\mathtt{e}_{22}(\mathtt{h}) > 0$ *is in* (1.5.4), *and the value of* $\eta_{\mathrm{WB}}(\mathtt{h})$ *is explicitly computable (cfr. [17, (1.20)]). In particular*

$$\eta_{WB}(\mathtt{h}_{WB}) \approx 5.65555 > 0\,, \tag{1.6.5}$$

is strictly positive.

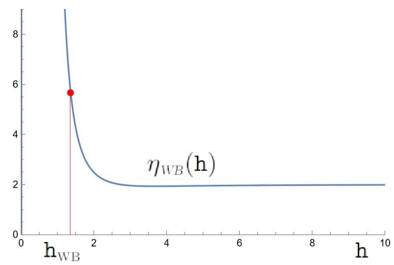


Figure 1.11: The plot of the function $\eta_{\mathrm{WB}}(\mathtt{h})$ looks positive for every depth.

By Theorem 1.6.1 and (1.6.5) we deduce Theorem 1.2.4 since eigenvalues with nonzero real part appear whenever the Benjamin-Feir discriminant $\Delta_{\mathrm{BF}}(\mathtt{h};\mu,\epsilon) > 0$.

In the following corollary of Theorem 1.6.1 we describe the unstable eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ at the critical depth $\mathtt{h} = \mathtt{h}_{\mathrm{WB}}$. Analogous results hold for *any* pair $(\mathtt{h},\epsilon)$ satisfying (1.2.15).

Theorem 1.6.2. (Benjamin-Feir unstable eigenvalues at $\mathtt{h} = \mathtt{h}_{WB}$.) *There exist* $\epsilon_1, \mu_0 > 0$ *and an analytic function* $\underline{\mu}(\cdot) : [0,\epsilon_1) \to [0,\mu_0)$ *of the form*

$$\underline{\mu}(\epsilon) = \underline{c}\,\epsilon^2(1 + r(\epsilon))\,, \quad \underline{c} := \sqrt{\frac{8\eta_{WB}(\mathtt{h}_{WB})}{\mathtt{e}_{22}(\mathtt{h}_{WB})}}\,, \tag{1.6.6}$$

such that, for any $\epsilon \in [0,\epsilon_1)$, *the operator* $\mathcal{L}_{\mu,\epsilon}$ *has two eigenvalues* $\lambda_1^{\pm}(\mu,\epsilon)$

$$\begin{cases} \mathrm{i}\,\frac{1}{2}\check{\mathtt{c}}_{\mathtt{h}}\mu + \mathrm{i}\,r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{1}{8}\mu\sqrt{\mathtt{e}_{22}(\mathtt{h}_{WB})}(1 + r(\epsilon,\mu))\sqrt{\Delta_{BF}(\mathtt{h}_{WB};\mu,\epsilon)} & 0 < \mu < \underline{\mu}(\epsilon) \\ \mathrm{i}\,\frac{1}{2}\check{\mathtt{c}}_{\mathtt{h}}\mu + \mathrm{i}\,r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\,\frac{1}{8}\mu\sqrt{\mathtt{e}_{22}(\mathtt{h}_{WB})}(1 + r(\epsilon,\mu))\sqrt{|\Delta_{BF}(\mathtt{h}_{WB};\mu,\epsilon)|} & \underline{\mu}(\epsilon) \le \mu < \mu_0\,, \end{cases} \tag{1.6.7}$$

with $\check{\mathtt{c}}_{\mathtt{h}} := 2\mathtt{c}_{\mathtt{h}} - \mathtt{e}_{12}(\mathtt{h}) > 0$ *and*

$$\Delta_{BF}(\mathtt{h}_{WB};\mu,\epsilon) = 8\eta_{WB}(\mathtt{h}_{WB})\epsilon^4 + r_1(\epsilon^5,\mu\epsilon^3) - \mathtt{e}_{22}(\mathtt{h}_{WB})\mu^2(1 + r_1''(\epsilon,\mu))\,.$$

A first approximation of the degenerate figure “8” obtained by discarding remainders in (1.6.7) is given in Figure 1.12. We now prove Theorem 1.6.2 relying on Theorem 1.6.1.

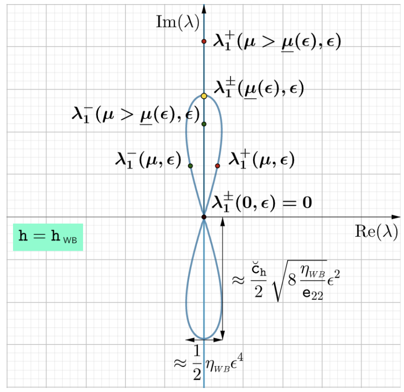


Figure 1.12: The degenerate figure "8" at the critical depth. It has a smaller height and width than the deep-water one in Figure 1.8.

Proof of Theorem 1.6.2. Since $\Delta_{\text{BF}}(\mathtt{h}_{\text{WB}}; 0, \epsilon) = 8\eta_{\text{WB}}(\mathtt{h}_{\text{WB}})\epsilon^4(1 + r(\epsilon))$, it results that $\Delta_{\text{BF}}(\mathtt{h}_{\text{WB}}; \mu, \epsilon) > 0$, for any $\mu \in (0, \underline{\mu}(\epsilon))$ as in (1.6.6) and ϵ small enough. The unstable eigenvalues $\lambda_1^\pm(\mu, \epsilon)$ in (1.6.6) are those of the matrix $\mathtt{U}$ in (1.6.2). In order to determine the value $\mu = \underline{\mu}(\epsilon)$ such that $\lambda_1^\pm(\mu, \epsilon)$ touches the imaginary axis far from the origin, we set $\mu = c\epsilon^2$ so that $\Delta_{\text{BF}}(\mathtt{h}_{\text{WB}}; \mu, \epsilon) = 0$ if and only if

$$0 = \epsilon^{-4}\Delta_{\text{BF}}(\mathtt{h}_{\text{WB}}; c\epsilon^2, \epsilon) = 8\eta_{\text{WB}}(\mathtt{h}_{\text{WB}})(1 + r(\epsilon)) + r_1(c\epsilon) - \mathtt{e}_{22}(\mathtt{h}_{\text{WB}})c^2(1 + r_1(\epsilon)) .$$

This equation is solved by an analytic function $\epsilon \mapsto c_\epsilon = \underline{c}(1 + r(\epsilon))$ with $\underline{c}$ defined in (1.6.6). □

We conclude this section describing the main steps of the proof and the organization of this part of the thesis.

Ideas and scheme of proof. We recall from Section 1.5 that, through the results in Chapter 2.2, the finite-depth instability matter was reduced to the problem of determining the eigenvalues of the 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}\mathtt{B}_{\mu,\epsilon}$ in (2.2.25). In Section 5.1 we provide the Taylor expansion of the matrix $\mathtt{B}_{\mu,\epsilon}$ in (5.1.2) at an order of accuracy higher than in Proposition 4.2.1. In particular in Proposition 5.1.1 we compute the coefficients of the Taylor expansion up to order 4 in the matrix entries (5.1.5a)-(5.1.5c) which enter in the constant $\eta_{\text{WB}}(\mathtt{h})$ (cfr. (5.2.9)) appearing in the Benjamin-Feir discriminant function (1.6.4). This explicit computation requires the knowledge of the Taylor expansions of the Kato spectral projections $P_{\mu,\epsilon}$ up to cubic order, that we provide in Section 5.1.2, relying on complex analysis. In order to perform effective computations we observe several analytical cancellations in Sections 5.1.3 and 5.1.4, which reduce considerably the number of explicit scalar products to compute. The proof of Proposition 5.1.1 requires ultimately the knowledge of the Taylor expansion up to order four of the auxiliary functions (2.1.7)-(1.3.4a) appearing in the Alinhac and Levi-Civita transformations. and of the functions $a_\epsilon(x), p_\epsilon(x)$ in the operator $\mathcal{L}_{\mu,\epsilon}$ in (1.3.14a), which are derived in Appendix A.4. Such expansions are derived from the one for the Stokes waves that we prove in Appendix A.3. Finally in Section 5.2 we implement more steps of the block-diagonalization procedure of the finite-depth case which provides the block-diagonal matrix (1.6.2) and we analytically compute the expansion of the Benjamin-Feir discriminant function $\Delta_{\rm BF}(\mathtt{h};\mu,\epsilon)$, in particular of the constant $\eta_{\rm WB}(\mathtt{h})$ in (5.2.9) and its positive value at $\mathtt{h}=\mathtt{h}_{\rm WB}$.

We point out that the constant $\eta_{\rm WB}$ in (5.2.9) is *analytically* computed in terms of the coefficients (5.2.4) which, in turn, are expressed in terms of the coefficients ϕ_{21}, ϕ_{22}, γ_{12}, η_{12}, γ_{11}, ϕ_{11}, γ_{22}, ϕ_{12}, $\mathtt{f}_{11}$, and ultimately $\eta_2^{[0]},\dots,\eta_4^{[4]},\psi_2^{[2]},\dots,\psi_4^{[4]}$, c_2, c_4 of the Stokes wave provided in Appendix A.3. Then we used the software Mathematica® to compute how the coefficients of the Stokes wave, of the functions $a_\epsilon(x)$, $p_\epsilon(x)$ in (1.3.8), and $\eta_{\rm WB}(\mathtt{h})$ in (5.2.9) depend on $\mathtt{h}$, starting from their algebraic formulas. The Mathematica code employed can be found at `https://git-scm.sissa.it/amaspero/benjamin-feir-instability`.

Chapter 2

Spectral reduction

In this chapter we extensively describe our method to reduce the problem of studying the $L^2(\mathbb{R})$-spectrum of $\mathcal{L}_\epsilon$ in (1.3.7) to the study of a 4×4 matrix.

2.1 Properties of $\mathcal{L}_\epsilon$ and $\mathcal{L}_{\mu,\epsilon}$

In this section we derive some useful properties of the linearized operator $\mathcal{L}_\epsilon$ and of its Floquet shift $\mathcal{L}_{\mu,\epsilon}$ on which we will rely in the sequel. The content of this section will sometimes revisit that of section 1.3, shedding light on some details we just outlined during the introduction.

Stokes waves. We reformulate the existence and uniqueness Theorem 1.2.1 in the following setting of spaces of periodic analytic functions

$$H^{\sigma,s} := H^{\sigma,s}(\mathbb{T}) := \Big\{ u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}\,k\cdot x} \ : \ \ \|u\|^2_{H^{\sigma,s}} := \sum_{k\in\mathbb{Z}} e^{2\sigma|k|}\langle k\rangle^{2s}\,|u_k|^2 < \infty \Big\} . \quad (2.1.1)$$

If $\sigma = 0$ the space $H^{0,s}$ is the usual Sobolev space H^s. If $\sigma > 0$, a periodic function $u(x)$ belongs to $H^{\sigma,s}(\mathbb{T})$, if and only if it admits an analytic extension in the strip $|y| < \sigma$ and the traces at the boundaries $u(\cdot + \mathrm{i}\,y)$, $|y| = \sigma$, belong to the Sobolev space $H^s := H^s(\mathbb{T})$ (cfr. [15, Appendix B.1]). Moreover we denote with $H^{\sigma,s}_{\mathtt{ev}}$, respectively $H^{\sigma,s}_{\mathtt{odd}}$, the subspace of $H^{\sigma,s}$ collecting even, respectively odd, functions.

The vector field associated to system (1.2.10) in the above spatial setting is

$$F : H^{\sigma,s+1}(\mathbb{T}) \times H^{\sigma,s+1}(\mathbb{T}) \times \mathbb{R} \times (0,+\infty] \times \mathbb{R}_{>0} \times \mathbb{R} \longrightarrow H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \quad (2.1.2)$$

$$(\eta,\psi,c\,;\,\mathtt{h},g,P) \mapsto \Big(c\eta_x + G(\eta;\mathtt{h})\psi\,,\ c\psi_x + P - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta;\mathtt{h})\psi + \eta_x\psi_x\big)^2 \Big) .$$

A Stokes wave is a *zero* of F. Let $\mathrm{B}(r) := \{x \in \mathbb{R} : |x| < r\}$ be the real ball of radius r centered in 0, we have the following

Theorem 2.1.1. (Stokes waves) *For any $\sigma \geq 0$, $s > 5/2$, wavenumber $\kappa \in \mathbb{N}$, depth $\mathtt{h} \in (0, +\infty]$, positive gravity constant $g > 0$ and atmospheric pressure $P \in \mathbb{R}$ there exists $\epsilon_0 := \epsilon_0(\sigma, s, \kappa, \mathtt{h}, g, P) > 0$ and a unique family of solutions*

$$(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon) \in H^{\sigma,s}_{\mathtt{ev}}(\mathbb{T}) \times H^{\sigma,s}_{\mathtt{odd}}(\mathbb{T}) \times \mathbb{R}$$

of the system (1.2.10)*, parameterized by $|\epsilon| \leq \epsilon_0$, such that*

1. *the map $\epsilon \mapsto (\eta_\epsilon, \psi_\epsilon, c_\epsilon)$, $\mathrm{B}(\epsilon_0) \to H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$ is analytic;*
2. *the solutions $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$ have the expansion*

$$\begin{aligned}
&(\eta_\epsilon(x), \psi_\epsilon(x)) = \epsilon \Big(\frac{P}{g} + \sqrt{\kappa} \cos(\kappa x), \sqrt{\frac{g}{\tanh(\mathtt{h}\kappa)}} \sin(\kappa x) \Big) + O(\epsilon^2) \, , \\
&c_\epsilon \to \mathtt{c}_{\mathtt{h},\kappa,g} := \sqrt{\frac{g \tanh(\mathtt{h}\kappa)}{\kappa}} \quad as \quad \epsilon \to 0 \, ;
\end{aligned} \tag{2.1.3}$$

3. *the solutions $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$ depend analytically on the parameters $\mathtt{h}, g > 0$, $P \in \mathbb{R}$.*

Nowadays the existence of traveling solutions of (1.2.10) is derived through the analytic Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue, see e.g. [20]. We remark that C^1 traveling waves are actually real analytic, see Lewy [68] and Nicholls-Reitich [77]. We give an explicit proof of Theorem 2.1.1 in the case of infinite depth in Appendix A.2.

In the sequel we shall always consider the Stokes wave given by Theorem 2.1.1 of wavenumber $\kappa = 1$, with gravity constant $g = 1$ and external pressure $P = 0$. The following result ensures that such choice implies no loss of generality.

Lemma 2.1.2 (Symmetries of the Stokes waves). *Let $\big(\eta(x), \psi(x), c, \mathtt{h}, 1, 0\big)$ be a zero of F, i.e. equilibrium solutions of* (1.2.10) *with ocean depth $\mathtt{h} > 0$, gravity constant $g = 1$ and Bernoulli constant $P = 0$. Then:*

1. *for any $\Phi_0 \in \mathbb{R}$ one has $F\big(\eta(x), \psi(x) + \Phi_0, c\, ; \, \mathtt{h}, 1, 0\big) = 0$;*
2. *for any $\theta \in \mathbb{R}$ one has $F\big(\eta(x+\theta), \psi(x+\theta), c\, ; \, \mathtt{h}, 1, 0\big) = 0$;*
3. *for any $P \neq 0$ one has $F\big(\eta(x) - P, \psi(x), c\, ; \, \mathtt{h} + P, 1, P\big) = 0$;*

4. *for any $g>0$ one has* $F(\eta(x), \sqrt{g}\psi(x), \sqrt{g}\, c\,; \mathtt{h}, g, 0) = 0$.

Lemma 2.1.2 follows from the following symmetries of the Dirichlet-Neumann operator.

Lemma 2.1.3. *The Dirichlet-Neumann operator in* (1.2.5) *enjoys the following symmetries:*

$$\begin{aligned}
&1.\ G(\eta^\vee)[\psi^\vee] = (G(\eta)[\psi])^\vee, \text{ where } f^\vee(x) := f(-x);\\
&2.\ G(\eta)[\psi+\Phi_0] = G(\eta)[\psi] \text{ for any real } \Phi_0;\\
&3.\ G(\mathtt{h},\eta{+}P)[\psi] = G(\mathtt{h}{+}P,\eta)[\psi] \text{ and } G(+\infty,\eta{+}P)[\psi] = G(+\infty,\eta)[\psi]\\
&\quad \text{for any real } P;\\
&4.\ \tau_\theta G(\eta)\psi = G(\tau_\theta\eta)[\tau_\theta\psi] \text{ for any real } \theta, \text{ where } \tau_\theta u(x) := u(x+\theta).
\end{aligned} \tag{2.1.4}$$

In particular by symmetry (2.1.4)-1 we obtain the time-reversibility property (1.2.9). In Proposition A.3.1 we prove the fourth-order Taylor expansion in (A.3.2) of the Stokes wave of wavenumber $\kappa = 1$, with $g = 1$ and $P = 0$.

Linearization along a Stokes wave. By linearizing the traveling water wave system in (1.2.10) along the family $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$ of Stokes waves given by Theorem 2.1.1 with wavenumber $\kappa = 1$, $g = 1$ and $P = 0$ we obtain the linear system

$$(\hat\eta_t\,,\ \hat\psi_t) = \mathrm{d}_{(\eta,\psi)} F(\eta_\epsilon, \psi_\epsilon, c_\epsilon; \mathtt{h}, 1, 0)[\hat\eta\,,\ \hat\psi]\,, \tag{2.1.5}$$

with F in (2.1.2). The explicit matrix form of (2.1.5) given in (1.3.1) requires deriving the Dirichlet-Neumann operator $G(\eta)$ in (1.2.5) with respect to an increment in η. For this purpose, we employ the so-called *shape derivative* introduced in [64], which is given by

$$\mathrm{d}_\eta G(\eta)[\hat\eta, \psi] = -G(\eta)[B(\eta,\psi)\hat\eta] - (V(\eta,\psi)\hat\eta)_x\,, \tag{2.1.6}$$

where

$$V(\eta,\psi) := -B(\eta,\psi)\eta_x + \psi_x\,, \quad B(\eta,\psi) := \frac{G(\eta)\psi + \psi_x\eta_x}{1+\eta_x^2}\,. \tag{2.1.7}$$

Once evaluated at a solution (η, ψ) of the water wave system (1.2.8), the functions (V, B) are the horizontal and vertical components of the velocity field (Φ_x, Φ_y) at the free surface. Moreover they inherit regularity and parity from (η, ψ), hence along a Stokes wave

$$\epsilon \mapsto (V_\epsilon, B_\epsilon) := \big(V(\eta_\epsilon,\psi_\epsilon), B(\eta_\epsilon,\psi_\epsilon)\big)$$

is an analytic mapping from $\mathrm{B}(\epsilon_0)$ to $H^{\sigma,s-1}_{\mathtt{ev}}(\mathbb{T}) \times H^{\sigma,s-1}_{\mathtt{odd}}(\mathbb{T})$. The propagation of regularity extends to the Levi-Civita flattening in (1.3.6), where the solution $\epsilon \mapsto \big(\mathfrak{p}_\epsilon(x), \mathtt{f}_\epsilon\big)$ of (1.3.4a) is an analytic mapping from $\mathrm{B}(\epsilon_0)$ to $H^{\sigma,s}_{\mathtt{odd}}(\mathbb{T}) \times \mathbb{R}$. Finally, the functions $a_\epsilon(x)$ and $p_\epsilon(x)$ in (1.3.8) inherit regularity and parity from V_ϵ, B_ϵ and $\mathfrak{p}_\epsilon$, i.e. the mapping $\epsilon \mapsto (p_\epsilon, a_\epsilon)$ is analytic from $\mathrm{B}(\epsilon_0)$ to $H^{\sigma,s-1}_{\mathtt{ev}}(\mathbb{T}) \times H^{\sigma,s-2}_{\mathtt{ev}}(\mathbb{T})$. We have the following

Proposition 2.1.4. *The functions $p_\epsilon(x)$ and $a_\epsilon(x)$ in* (1.3.8) *and the ϵ-dependent constant $\mathtt{f}_\epsilon$ in* (1.3.4a) *admit the Taylor expansion*

$$\begin{aligned} p_\epsilon(x) &= p_1^{[1]}\epsilon\cos(x) + \epsilon^2\big(p_2^{[0]} + p_2^{[2]}\cos(2x)\big) + \epsilon^3\big(p_3^{[1]}\cos(x) + p_3^{[3]}\cos(3x)\big) \\ &\qquad + \epsilon^4\big(p_4^{[0]} + p_4^{[2]}\cos(2x) + p_4^{[4]}\cos(4x)\big) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{2.1.8a}$$

$$\begin{aligned} a_\epsilon(x) &= a_1^{[1]}\epsilon\cos(x) + \epsilon^2\big(a_2^{[0]} + a_2^{[2]}\cos(2x)\big) + \epsilon^3\big(a_3^{[1]}\cos(x) + a_3^{[3]}\cos(3x)\big) \\ &\qquad + \epsilon^4\big(a_4^{[0]} + a_4^{[2]}\cos(2x) + a_4^{[4]}\cos(4x)\big) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{2.1.8b}$$

$$\mathtt{f}_\epsilon = \epsilon^2\mathtt{f}_2 + \epsilon^4\mathtt{f}_4 + \mathcal{O}(\epsilon^5)\,, \qquad \mathtt{c_h} = \sqrt{\tanh(\mathtt{h})}\,, \tag{2.1.8c}$$

with the Taylor coefficients given in (A.4.22)-(A.4.23) *and* (A.4.8) *for arbitrary depth* $\mathtt{h} > 0$.

Let us report here the second-order coefficients of the expansion (2.1.8) for the infinite depth case $\mathtt{h} = +\infty$ ($\mathtt{c_h} = 1$):

$$p_1^{[1]} = a_1^{[1]} = p_2^{[2]} = a_2^{[2]} = -2\,, \quad p_2^{[0]} = \frac{3}{2}\,, \quad a_2^{[0]} = 2\,, \tag{2.1.9}$$

which will be fundamental for the proof of instability in the infinite-depth case.

Let us now detail one of the main tools for the study the modulations of a periodic wave caused by long-wave perturbations: Bloch-Floquet theory.

Bloch-Floquet decomposition. The space $L^2(\mathbb{T}_N)$, $\mathbb{T}_N := \mathbb{R}/2\pi N\mathbb{Z}$ admits the splitting

$$L^2(\mathbb{T}_N) = \bigoplus_{\mu\in\Omega_N} e^{\mathrm{i}\,\mu x}L^2(\mathbb{T})\,, \quad \Omega_N := \begin{cases} \{-\frac12 + \frac{1}{2N}, \dots, \frac12 - \frac{1}{2N}\}\,, & \text{if } N \text{ is odd}\,, \\ \{-\frac12, \dots, \frac12 - \frac1N\}\,, & \text{if } N \text{ is even}\,, \end{cases} \tag{2.1.10}$$

where in fact any translation of the set Ω_N by an integer provides an identical splitting. The decomposition follows by dividing the Fourier expansion of $u \in L^2(\mathbb{T}_N)$ as follows

$$u(x) = \sum_{j\in\mathbb{Z}} \breve{u}_j e^{\mathrm{i}\,\frac{j}{N}x} = \sum_{\mu\in\Omega_N}\sum_{k\in\mathbb{Z}} \breve{u}_{kN+\mu N} e^{\mathrm{i}\,\frac{(k+\mu)N}{N}x} = \sum_{\mu\in\Omega_N} e^{\mathrm{i}\,\mu x} \underbrace{\sum_{k\in\mathbb{Z}} \breve{u}_{(k+\mu)N} e^{\mathrm{i}\,kx}}_{=:\,\breve{u}(x;\mu)\,\in\, L^2(\mathbb{T})}\,,$$

where the 2π-periodic *Bloch waves* $\breve{u}(x;\mu)$ are uniquely determined.

Let $\mathcal{L}$ be an operator on $L^2(\mathbb{T}_N)$. The Bloch-Floquet expansion of its action on $u \in L^2(\mathbb{T}_N)$ gives

$$\mathcal{L}u(x) = \sum_{\mu\in\Omega_N} \mathcal{L}e^{\mathrm{i}\,\mu x}\breve{u}(x,\mu) = \sum_{\mu\in\Omega_N} e^{\mathrm{i}\,\mu x}\mathcal{L}_\mu\breve{u}(x,\mu)\,, \quad \mathcal{L}_\mu := e^{-\mathrm{i}\,\mu x}\mathcal{L}e^{\mathrm{i}\,\mu x}\,. \tag{2.1.11}$$

The last sum in (2.1.11) is the Bloch-Floquet decomposition of $\mathcal{L}u(x)$ provided $\mathcal{L}_\mu$ preserves 2π-periodic functions. This is the case for pseudodifferential operators with 2π-periodic coefficients (such as $\mathcal{L}_\epsilon$ in (1.3.7)), as shown in the following extension of [74, Lemma 3.5].

Lemma 2.1.5. *Let $\ell(x,\xi)$ be a 2π-periodic in x symbol of a pseudodifferential operator $\mathcal{L} := \ell(x,D) : H^{s_1}(\mathbb{T}) \to H^{s_2}(\mathbb{T})$. Then*

$$\mathcal{L}_\mu = e^{-\mathrm{i}\,\mu x}\ell(x,D)e^{\mathrm{i}\,\mu x} = \ell(x,D+\mu) : H^{s_1}(\mathbb{T}) \to H^{s_2}(\mathbb{T}) , \quad \forall \mu \in \mathbb{R} . \tag{2.1.12}$$

Proof. Let $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}\,kx}$ be a 2π-periodic function. Then $e^{\mathrm{i}\,\mu x}u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}\,(k+\mu)x}$ and

$$\begin{aligned} e^{-\mathrm{i}\,\mu x}\mathcal{L}[e^{\mathrm{i}\,\mu x}u(x)] &= \sum_{k\in\mathbb{Z}} u_k\, \ell(x,k+\ell)e^{\mathrm{i}\,(k+\mu)x} \\ &= \sum_{k\in\mathbb{Z}} u_k\, \ell(x,k+\mu)e^{\mathrm{i}\,kx} = \ell(x,D+\mu)u(x) . \end{aligned} \tag{2.1.13}$$

Clearly $\ell(x,\xi+\mu)$ satisfies the same decay estimates of $\ell(x,\xi)$ and since $\ell(x,\xi)$ is 2π-periodic in x then $\ell(x,D+\mu)u(x)$ in (2.1.13) is. □

As shown in (1.3.9), the spectrum of an operator $\mathcal{L}$ on $L^2(\mathbb{T}_N)$ preserving Floquet fibers (e.g. satisfying the conditions of Lemma 2.1.5) can be decomposed similarly to (2.1.10). Remarkably, the decomposition in (1.3.9) also extends to the whole spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})$, as discussed in [55] and related sources.

The semiperturbed spectra. The unstable eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ are obtained, in our method, by a perturbation argument based on the complete knowledge of the spectrum of $\mathcal{L}_{0,0}$.

As we shall see in detail in Section 2.2, this is possible for two main reasons. Let $\Sigma \subset \mathbb{C}$ be the interior of a compact portion of the complex plane without parts of the spectrum of $\mathcal{L}_{0,0}$ lying on its boundary $\partial\Sigma$. Then for small values of $\epsilon > 0$ and $\mu > 0$ we have that:

- isolated eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ inside Σ evolve continuously with respect to the parameters;
- the number of the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ inside Σ, counted with multiplicity, is constant.

Moreover, the Hamiltonicity of these operators gives the following spectral symmetry.

Lemma 2.1.6. *Let $\mathcal{L}$ be a complex Hamiltonian operator, as in Definition 1.3.1,*

$$\mathcal{L} = \mathcal{J}\mathcal{B} : H^1(\mathbb{T},\mathbb{C}^2) \subset L^2(\mathbb{T},\mathbb{C}^2) \overset{\text{closed}}{\longrightarrow} L^2(\mathbb{T},\mathbb{C}^2) .$$

Then

$$\lambda \in \sigma_{L^2}(\mathcal{L}) \Longleftrightarrow -\overline{\lambda} \in \sigma_{L^2}(\mathcal{L}) . \tag{2.1.14}$$

Proof. We prove the contrapositive. Let $z \in \mathbb{C}$ in the resolvent set of $\mathcal{L}$. Then the operator

$$\mathcal{L} - z = \mathcal{J}(\mathcal{B} - \mathcal{J}^{-1}z)\mathcal{J}^*\mathcal{J}^{-*} = \mathcal{J}[\mathcal{J}(\mathcal{B} - \mathcal{J}^{-*}\overline{z})]^*\mathcal{J}^{-*} \overset{\mathcal{J}^*=-\mathcal{J}}{=} \mathcal{J}(\mathcal{L}^* + z)\mathcal{J}^{-*}$$

is invertible. In particular $-z$ is in the resolvent set of $\mathcal{L}^*$ or, equivalently, $-\overline{z}$ is in the resolvent set of $\mathcal{L}$, which implies (2.1.14). □

By Lemma 2.1.6 we deduce the following perturbative principle for Hamiltonian spectra: *no unstable eigenvalue of $\mathcal{L}_{\mu,\epsilon}$ originates from a simple, isolated, purely-imaginary eigenvalue of $\mathcal{L}_{0,0}$* by perturbation in ϵ or μ. Indeed any contradiction of this statement would violate either the continuity of the spectrum with respect to the parameters or the local constancy of the number of eigenvalues.
The principle restricts the search for eigenvalues with positive real part of $\mathcal{L}_{\mu,\epsilon}$ in proximity of multiple eigenvalues of the unperturbed operator $\mathcal{L}_{0,0}$. According to Lemma 2.1.6, any of these unstable eigenvalues consistently appears alongside (at least) one twin eigenvalue symmetric with respect to the imaginary axis.

The perturbative procedure to obtain the spectrum of $\mathcal{L}_{\mu,\epsilon}$ from that of $\mathcal{L}_{0,0}$ is significantly enhanced when considering the exact information about the spectra of the semiperturbed operators $\mathcal{L}_{\mu,0}$ and $\mathcal{L}_{0,\epsilon}$, which we will now describe.

Lemma 2.1.7. *The spectrum of the Fourier multiplier $\mathcal{L}_{\mu,0}$ in* (1.3.16) *is the double family of eigenvalues in* (1.3.17) *and, for the case of infinite depth, in* (1.3.18).

Proof. For every $k \in \mathbb{Z}$, the action of the operator $\mathcal{L}_{\mu,0}$ restricted to the invariant subspace

$$\operatorname{span} \mathtt{F}_k\,, \quad \mathtt{F}_k := \left\{ \begin{bmatrix}1\\0\end{bmatrix} e^{\mathrm{i}\,kx}, \begin{bmatrix}0\\1\end{bmatrix} e^{\mathrm{i}\,kx} \right\}$$

is represented on the basis $\mathtt{F}_k$ by the matrix

$$\begin{pmatrix} \mathrm{i}\,\mathtt{c_h}(k+\mu) & |k+\mu|\,\tanh\left(\mathtt{h}|k+\mu|\right) \\ -1 & \mathrm{i}\,\mathtt{c_h}(k+\mu) \end{pmatrix},$$

whose spectrum is $\{\lambda_k^+(\mu), \lambda_k^-(\mu)\}$. □

As already stressed in the introduction, Benjamin-Feir instability arises by perturbation from the quadruple eigenvalue 0 of $\mathcal{L}_{0,0}$. The Kernel of $\mathcal{L}_{0,0}$ is three-dimensional, with a real basis given by

$$f_1^+ := \begin{bmatrix} \mathtt{c_h}^{1/2}\cos(x) \\ \mathtt{c_h}^{-1/2}\sin(x) \end{bmatrix}, \quad f_1^- := \begin{bmatrix} -\mathtt{c_h}^{1/2}\sin(x) \\ \mathtt{c_h}^{-1/2}\cos(x) \end{bmatrix}, \qquad f_0^- := \begin{bmatrix}0\\1\end{bmatrix}, \tag{2.1.15a}$$

being in the infinite-depth $\mathtt{h} = +\infty$ case

$$f_1^+ := \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}, \quad f_1^- := \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}, \qquad f_0^- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.1.15b}$$

Thus 0 is *defective* as eigenvalue of $\mathcal{L}_{0,0}$. To complete (2.1.15) to a real basis of the whole generalized eigenspace we add the generalized eigenvector

$$f_0^+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathcal{L}_{0,0} f_0^+ = -f_0^- . \tag{2.1.16}$$

We now adapt the proof of [74, Theorem 4.1] to show that the operator $\mathcal{L}_{0,\epsilon}$ keeps 0 as quadruple eigenvalue with a particular Jordan structure.

Lemma 2.1.8. *The operator $\mathcal{L}_{0,\epsilon}$ has a four-dimensional generalized Kernel $\mathcal{V}_{0,\epsilon}$ such that*

$$\Big(\mathcal{L}_{0,\epsilon}|_{\mathcal{V}_{0,\epsilon}}\Big)^2 = 0 . \tag{2.1.17}$$

Proof. We show the above structure, that is clearly invariant by conjugations, for the generalized kernel of the operator $\mathrm{d}_{(\eta,\psi)} F\big(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon\, ;\, \mathtt{h}, 1, 0\big)$ in (2.1.5)-(1.3.1).
Let $\mathrm{S}_\epsilon := \big(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon\, ;\, \mathtt{h}, 1, 0\big)$, by the symmetries of Lemma 2.1.2 we have that

$$V_1 := [0, 1] \qquad \text{and} \qquad V_2 := [(\eta_\epsilon(x))_x, (\psi_\epsilon)_x]$$

are kernel vectors since

$$0 = \partial_{\Phi_0}\big|_{\Phi_0=0} F\big(\eta_\epsilon(x), \psi_\epsilon(x) + \Phi_0, c_\epsilon\, ;\, \mathtt{h}, 1, 0\big) = \mathrm{d}_{(\eta,\psi)} F(\mathrm{S}_\epsilon) V_1 , \tag{2.1.18a}$$

and

$$0 = \partial_\theta\big|_{\theta=0} F\big(\eta_\epsilon(x+\theta), \psi_\epsilon(x+\theta), c_\epsilon\, ;\, \mathtt{h}, 1, 0\big) = \mathrm{d}_{(\eta,\psi)} F(\mathrm{S}_\epsilon) V_2 . \tag{2.1.18b}$$

Similarly, by denoting with a dot the derivative with respect to ϵ,

$$0 = \partial_\epsilon F\big(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon\, ;\, \mathtt{h}, 1, 0\big) = \mathrm{d}_{(\eta,\psi)} F(\mathrm{S}_\epsilon)[\dot\eta_\epsilon, \dot\psi_\epsilon] + \partial_c F(\mathrm{S}_\epsilon) \dot c_\epsilon , \tag{2.1.18c}$$

where, by (2.1.2),

$$\partial_c F(\mathrm{S}_\epsilon) = \big[(\eta_\epsilon)_x, (\psi_\epsilon)_x\big] = V_2 . \tag{2.1.19}$$

Thus $V_3 := [\dot\eta_\epsilon, \dot\psi_\epsilon]$ is a generalized kernel vector, since by (2.1.18c) and (2.1.19)

$$\mathrm{d}_{(\eta,\psi)} F(\mathrm{S}_\epsilon) V_3 = -\dot c_\epsilon V_2 . \tag{2.1.20}$$

This shows that $\mathcal{V}_{0,\epsilon}$ is at least three-dimensional. By Lemma 2.1.14 its dimension is exactly four. Indeed $\dim \mathcal{V}_{0,0} = 4$, given that $\mathcal{V}_{0,0}$ is spanned by the vectors in (2.1.15)-(2.1.16), and a hypothetical simple eigenvalue emerging from 0 as $\epsilon \neq 0$ cannot move away from the imaginary axis due to Hamiltonicity, nor can it move along the imaginary axis due to the reality of $\mathcal{L}_{0,\epsilon}$, thus it must remain in 0.
To prove (2.1.17), we note that the operator $\mathcal{L}_{0,\epsilon}$ is represented on any real symplectic basis (cfr. Definition 2.2.6) by a 4×4 real Hamiltonian matrix (cfr. Definition 2.2.12) $\mathtt{L}_{0,\epsilon}$ on $\mathcal{V}_{0,\epsilon}$. By Hamiltonicity and reality the rank of the matrix $\mathtt{L}_{0,\epsilon}^2$ is even, namely there exists a non-negative integer $n \in \mathbb{N}_0$ such that

$$0 \leq \dim \mathtt{Rg}\, \mathtt{L}_{0,\epsilon}^2 = 2n\,, \quad \dim \mathtt{Ker}\, \mathtt{L}_{0,\epsilon}^2 \geq 3\,, \quad \dim \mathtt{Rg}\, \mathtt{L}_{0,\epsilon}^2 + \dim \mathtt{Ker}\, \mathtt{L}_{0,\epsilon}^2 = 4\,,$$

implying that the rank of $\mathtt{L}_{0,\epsilon}^2$ vanishes, namely $\mathtt{L}_{0,\epsilon}^2$ is the zero matrix. □

We are now in a position to introduce our symplectic version of Kato perturbation theory to follow the branching of the quadruple eigenvalue 0 of $\mathcal{L}_{0,0}$ as both the parameters ϵ and μ are turned on.

2.2 Symplectic Kato theory

In this section we apply Kato's similarity transformation theory [59, I-§4-6, II-§4] to study the splitting of the eigenvalues of $\mathcal{L}_{\mu,\epsilon}$ in (1.3.14) near 0 for small values of μ and ϵ. First of all it is convenient to decompose the operator $\mathcal{L}_{\mu,\epsilon}$ in (1.3.14) as

$$\mathcal{L}_{\mu,\epsilon} = \mathrm{i}\, \mathtt{c_h} \mu + \mathscr{L}_{\mu,\epsilon}\,, \qquad \mu > 0\,, \tag{2.2.1}$$

where $\mathscr{L}_{\mu,\epsilon}$ is the operator

$$\mathscr{L}_{\mu,\epsilon} := \begin{bmatrix} \partial_x \circ (\mathtt{c_h} + p_\epsilon(x)) + \mathrm{i}\, \mu\, p_\epsilon(x) & |D+\mu|\, \tanh\big((\mathtt{h} + \mathtt{f}_\epsilon)|D+\mu|\big) \\ -(1 + a_\epsilon(x)) & (\mathtt{c_h} + p_\epsilon(x))\partial_x + \mathrm{i}\, \mu\, p_\epsilon(x) \end{bmatrix}, \tag{2.2.2a}$$

whereas in the case of infinite depth $\mathtt{h} = +\infty$ one has

$$\mathscr{L}_{\mu,\epsilon} := \begin{bmatrix} \partial_x \circ (1 + p_\epsilon(x)) + \mathrm{i}\, \mu\, p_\epsilon(x) & |D| + \mu(\mathrm{sgn}(D) + \Pi_0) \\ -(1 + a_\epsilon(x)) & (1 + p_\epsilon(x))\partial_x + \mathrm{i}\, \mu\, p_\epsilon(x) \end{bmatrix}, \tag{2.2.2b}$$

since $\mathtt{c}_{+\infty} = 1$ and

$$|D+\mu| = |D| + \mu\big(\mathrm{sgn}(D) + \Pi_0\big)\,, \qquad \Pi_0 f(x) := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\mathrm{d}x\,, \tag{2.2.3}$$

$$\operatorname{sgn}(D)\sum_{k\geq 0} a_k\cos(kx)+\sum_{k\geq 1} b_k\sin(kx) := \mathrm{i}\sum_{k\geq 1} a_k\sin(kx)-\mathrm{i}\sum_{k\geq 1} b_k\cos(kx)\,. \tag{2.2.4}$$

The operator $\mathscr{L}_{\mu,\epsilon}$ has the form $\mathscr{L}_{\mu,\epsilon}=\mathcal{J}\,\mathcal{B}_{\mu,\epsilon}$, where

$$\mathcal{B}_{\mu,\epsilon} := \begin{bmatrix} 1+a_\epsilon(x) & -(\mathtt{c_h}+p_\epsilon(x))\partial_x-\mathrm{i}\,\mu\,p_\epsilon(x) \\ \partial_x\circ(\mathtt{c_h}+p_\epsilon(x))+\mathrm{i}\,\mu\,p_\epsilon(x) & |D+\mu|\,\tanh\big((\mathtt{h}+\mathtt{f}_\epsilon)|D+\mu|\big)\end{bmatrix}, \tag{2.2.5a}$$

which, in the case of infinite depth $\mathtt{h}=+\infty$, becomes

$$\mathcal{B}_{\mu,\epsilon} := \begin{bmatrix} 1+a_\epsilon(x) & -((1+p_\epsilon(x))\partial_x-\mathrm{i}\,\mu\,p_\epsilon(x)) \\ \partial_x\circ(1+p_\epsilon(x))+\mathrm{i}\,\mu\,p_\epsilon(x) & |D|+\mu(\operatorname{sgn}(D)+\Pi_0)\end{bmatrix}. \tag{2.2.5b}$$

The operator $\mathcal{B}_{\mu,\epsilon}$ is selfadjoint, and reversibility-preserving, i.e. fulfills (1.3.15). Thus $\mathscr{L}_{\mu,\epsilon}$ is *Hamiltonian* and *reversible*, namely it satisfies, by (1.3.12),

$$\mathscr{L}_{\mu,\epsilon}\circ\overline{\rho}=-\overline{\rho}\circ\mathscr{L}_{\mu,\epsilon}\,,\qquad \overline{\rho}\text{ defined in (1.3.13)}\,. \tag{2.2.6}$$

We also observe that $\mathcal{B}_{0,\epsilon}$ is a real operator.

The effect on the spectrum of $\mathscr{L}_{\mu,\epsilon}$ of the shift in (2.2.1) is but a translation along the imaginary axis of the quantity $\mathrm{i}\,\mathtt{c_h}\mu$, namely

$$\sigma(\mathcal{L}_{\mu,\epsilon})=\mathrm{i}\,\mathtt{c_h}\mu+\sigma(\mathscr{L}_{\mu,\epsilon})\,,\qquad \Big(\sigma(\mathcal{L}_{\mu,\epsilon})=\mathrm{i}\,\mu+\sigma(\mathscr{L}_{\mu,\epsilon})\ \text{ at }\mathtt{h}=+\infty\Big).$$

In the sequel we focus on the analysis of the spectrum of $\mathscr{L}_{\mu,\epsilon}$.
We also note that $\mathscr{L}_{0,\epsilon}=\mathcal{L}_{0,\epsilon}$ for any $\epsilon\geq 0$. In particular $\mathscr{L}_{0,0}$ has zero as isolated eigenvalue with algebraic multiplicity 4, geometric multiplicity 3 and generalized kernel spanned by the vectors $\{f_1^+,f_1^-,f_0^+,f_0^-\}$ in (2.1.15)-(2.1.16). Furthermore its spectrum is separated as in (1.3.20). In general $\mathscr{L}_{0,\epsilon}$ has zero as isolated eigenvalue and fulfills (2.1.17). We also remark that, in view of (2.2.3), the operator $\mathscr{L}_{\mu,\epsilon}$, in the case of infinite depth $\mathtt{h}=+\infty$, is linear in μ. We remind that $\mathscr{L}_{\mu,\epsilon}:Y\subset X\to X$ has domain $Y:=H^1(\mathbb{T}):=H^1(\mathbb{T},\mathbb{C}^2)$ and range $X:=L^2(\mathbb{T}):=L^2(\mathbb{T},\mathbb{C}^2)$.

In the next lemma we construct the transformation operators which map isomorphically the unperturbed spectral subspace into the perturbed ones.

Lemma 2.2.1. *Let Γ be a closed, counterclockwise-oriented curve around 0 in the complex plane separating $\sigma'(\mathscr{L}_{0,0})=\{0\}$ and the other part of the spectrum $\sigma''(\mathscr{L}_{0,0})$ in (1.3.20). There exist $\epsilon_0,\mu_0>0$ such that for any $(\mu,\epsilon)\in B(\mu_0)\times B(\epsilon_0)$ the following statements hold:*

1. *The curve* Γ *belongs to the resolvent set of the operator* $\mathscr{L}_{\mu,\epsilon} : Y \subset X \to X$ *in* (2.2.2).

2. *The operators*

$$P_{\mu,\epsilon} := -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} \mathrm{d}\lambda : X \to Y \tag{2.2.7}$$

are well defined projections commuting with $\mathscr{L}_{\mu,\epsilon}$, *i.e.*

$$P^2_{\mu,\epsilon} = P_{\mu,\epsilon} , \quad P_{\mu,\epsilon}\mathscr{L}_{\mu,\epsilon} = \mathscr{L}_{\mu,\epsilon}P_{\mu,\epsilon} . \tag{2.2.8}$$

The map $(\mu, \epsilon) \mapsto P_{\mu,\epsilon}$ *is analytic from* $B(\mu_0) \times B(\epsilon_0)$ *to* $\mathcal{L}(X, Y)$.

3. *The domain* Y *of the operator* $\mathscr{L}_{\mu,\epsilon}$ *decomposes as the direct sum*

$$Y = \mathcal{V}_{\mu,\epsilon} \oplus Ker(P_{\mu,\epsilon}) , \quad \mathcal{V}_{\mu,\epsilon} := Rg(P_{\mu,\epsilon}) = Ker(\mathrm{Id} - P_{\mu,\epsilon}) , \tag{2.2.9}$$

of the closed subspaces $\mathcal{V}_{\mu,\epsilon}$, $Ker(P_{\mu,\epsilon})$ *of* Y, *which are invariant under* $\mathscr{L}_{\mu,\epsilon}$,

$$\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon} , \qquad \mathscr{L}_{\mu,\epsilon} : Ker(P_{\mu,\epsilon}) \to Ker(P_{\mu,\epsilon}) .$$

Moreover

$$\begin{aligned} &\sigma(\mathscr{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ inside } \Gamma\} = \sigma(\mathscr{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}) = \sigma'(\mathscr{L}_{\mu,\epsilon}),\\ &\sigma(\mathscr{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ outside } \Gamma\} = \sigma(\mathscr{L}_{\mu,\epsilon}|_{Ker(P_{\mu,\epsilon})}) = \sigma''(\mathscr{L}_{\mu,\epsilon}) \ , \end{aligned} \tag{2.2.10}$$

proving the "semicontinuity property" (1.3.21) *of separated parts of the spectrum.*

4. *The projections* $P_{\mu,\epsilon}$ *are similar one to each other: the transformation operators*

$$U_{\mu,\epsilon} := \big(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\big)^{-1/2} \big[P_{\mu,\epsilon}P_{0,0} + (\mathrm{Id} - P_{\mu,\epsilon})(\mathrm{Id} - P_{0,0})\big] , \tag{2.2.11}$$

where $(\mathrm{Id} - R)^{-\frac{1}{2}}$ *is defined, for any operator* R *satisfying* $\|R\|_{\mathcal{L}(Y)} < 1$, *by the power series*

$$(\mathrm{Id} - R)^{-\frac{1}{2}} := \sum_{k=0}^{\infty} \binom{-1/2}{k} (-R)^k = \mathrm{Id} + \frac{1}{2}R + \frac{3}{8}R^2 + \mathcal{O}(R^3) , \tag{2.2.12}$$

are bounded and invertible in Y *and in* X, *with inverse*

$$U^{-1}_{\mu,\epsilon} = \big[P_{0,0}P_{\mu,\epsilon} + (\mathrm{Id} - P_{0,0})(\mathrm{Id} - P_{\mu,\epsilon})\big]\big(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\big)^{-1/2} , \tag{2.2.13}$$

and

$$U_{\mu,\epsilon}P_{0,0}U^{-1}_{\mu,\epsilon} = P_{\mu,\epsilon} , \qquad U^{-1}_{\mu,\epsilon}P_{\mu,\epsilon}U_{\mu,\epsilon} = P_{0,0} . \tag{2.2.14}$$

The map $(\mu, \epsilon) \mapsto U_{\mu,\epsilon}$ *is analytic from* $B(\mu_0) \times B(\epsilon_0)$ *to* $\mathcal{L}(Y)$.

5. *The subspaces* $\mathcal{V}_{\mu,\epsilon} = Rg(P_{\mu,\epsilon})$ *are isomorphic one to each other:* $\mathcal{V}_{\mu,\epsilon} = U_{\mu,\epsilon}\mathcal{V}_{0,0}$. *In particular* $\dim \mathcal{V}_{\mu,\epsilon} = \dim \mathcal{V}_{0,0} = 4$, *for any* $(\mu,\epsilon) \in B(\mu_0) \times B(\epsilon_0)$.

We prove Lemma 2.2.1 in the case of infinite depth and redirect to [16, Lemma 3.1] for the very few changes to adopt in the case of finite depth.

Proof. 1. For any $\lambda \in \mathbb{C}$ we decompose $\mathscr{L}_{\mu,\epsilon} - \lambda = \mathscr{L}_{0,0} - \lambda + \mathcal{R}_{\mu,\epsilon}$ where $\mathscr{L}_{0,0} = \begin{bmatrix} \partial_x & |D| \\ -1 & \partial_x \end{bmatrix}$ and

$$\mathcal{R}_{\mu,\epsilon} := \mathscr{L}_{\mu,\epsilon} - \mathscr{L}_{0,0} = \begin{bmatrix} (\partial_x + \mathrm{i}\,\mu)p_\epsilon(x) & \mu g(D) \\ -a_\epsilon(x) & p_\epsilon(x)(\partial_x + \mathrm{i}\,\mu) \end{bmatrix} : Y \to X\,, \tag{2.2.15}$$

having used also (2.2.3) and setting $g(D) := \mathrm{sgn}(D) + \Pi_0$. For any $\lambda \in \Gamma$, the operator $\mathscr{L}_{0,0} - \lambda$ is invertible and its inverse is the Fourier multiplier matrix operator

$$(\mathscr{L}_{0,0} - \lambda)^{-1} = \mathrm{Op}\left(\frac{1}{(\mathrm{i}\,k - \lambda)^2 + |k|} \begin{bmatrix} \mathrm{i}\,k - \lambda & -|k| \\ 1 & \mathrm{i}\,k - \lambda \end{bmatrix} \right) : X \to Y\,.$$

Hence, for $|\epsilon| < \epsilon_0$ and $|\mu| < \mu_0$ small enough, uniformly on the compact set Γ, the operator $(\mathscr{L}_{0,0} - \lambda)^{-1}\mathcal{R}_{\mu,\epsilon} : Y \to Y$ is bounded, with small operatorial norm. Then $\mathscr{L}_{\mu,\epsilon} - \lambda$ is invertible by Neumann series and

$$(\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} = \big(\mathrm{Id} + (\mathscr{L}_{0,0} - \lambda)^{-1}\mathcal{R}_{\mu,\epsilon}\big)^{-1}(\mathscr{L}_{0,0} - \lambda)^{-1} : X \to Y\,. \tag{2.2.16}$$

This proves that Γ belongs to the resolvent set of $\mathscr{L}_{\mu,\epsilon}$.
2. By the previous point the operator $P_{\mu,\epsilon}$ is well defined and bounded $X \to Y$. It clearly commutes with $\mathscr{L}_{\mu,\epsilon}$. The projection property $P^2_{\mu,\epsilon} = P_{\mu,\epsilon}$ is a classical result based on complex integration, see [59], and we omit it. The map $(\mu,\epsilon) \to (\mathscr{L}_{0,0} - \lambda)^{-1}\mathcal{R}_{\mu,\epsilon} \in \mathcal{L}(Y)$ is analytic. Since the map $T \mapsto (\mathrm{Id} + T)^{-1}$ is analytic in $\mathcal{L}(Y)$ (for $\|T\|_{\mathcal{L}(Y)} < 1$) the operators $(\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1}$ in (2.2.16) and $P_{\mu,\epsilon}$ in $\mathcal{L}(X,Y)$ are analytic as well with respect to (μ,ϵ).
3. The decomposition (2.2.9) is a consequence of $P_{\mu,\epsilon}$ being a continuous projection in $\mathcal{L}(Y)$. The invariance of the subspaces follows since $P_{\mu,\epsilon}$ and $\mathscr{L}_{\mu,\epsilon}$ commute. To prove (2.2.10) define for an arbitrary $\lambda_0 \notin \Gamma$ the operator

$$R_{\mu,\epsilon}(\lambda_0) := -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{1}{\lambda - \lambda_0} (\mathcal{L}_{\mu,\epsilon} - \lambda)^{-1}\, \mathrm{d}\lambda \ :\ X \to Y\ .$$

If λ_0 is outside Γ, one has $R_{\mu,\epsilon}(\lambda_0)(\mathscr{L}_{\mu,\epsilon} - \lambda_0) = (\mathscr{L}_{\mu,\epsilon} - \lambda_0)R_{\mu,\epsilon}(\lambda_0) = P_{\mu,\epsilon}$ and thus $\lambda_0 \notin \sigma(\mathscr{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}})$. For λ_0 inside Γ, $R_{\mu,\epsilon}(\lambda_0)(\mathscr{L}_{\mu,\epsilon} - \lambda_0) = (\mathscr{L}_{\mu,\epsilon} - \lambda_0)R_{\mu,\epsilon}(\lambda_0) = P_{\mu,\epsilon} - \mathrm{Id}$

and thus $\lambda_0 \notin \sigma(\mathscr{L}_{\mu,\epsilon}|_{Ker(P_{\mu,\epsilon})})$. Then (2.2.10) follows.
4. By (2.2.7), the resolvent identity $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$ and (2.2.15), we write

$$P_{\mu,\epsilon} - P_{0,0} = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} \mathcal{R}_{\mu,\epsilon} (\mathscr{L}_{0,0} - \lambda)^{-1} \mathrm{d}\lambda \,.$$

Then $\|P_{\mu,\epsilon} - P_{0,0}\|_{\mathcal{L}(Y)} < 1$ for $|\epsilon| < \epsilon_0$, $|\mu| < \mu_0$ small enough and the operators $U_{\mu,\epsilon}$ in (2.2.11) are well defined in $\mathcal{L}(Y)$ (actually $U_{\mu,\epsilon}$ are also in $\mathcal{L}(X)$). The invertibility of $U_{\mu,\epsilon}$ and formula (2.2.14) are proved in [59], Chapter I, Section 4.6, for the pairs of projections $Q = P_{\mu,\epsilon}$ and $P = P_{0,0}$. The analyticity of $(\mu,\epsilon) \mapsto U_{\mu,\epsilon} \in \mathcal{L}(Y)$ follows by the analyticity $(\mu,\epsilon) \mapsto P_{\mu,\epsilon} \in \mathcal{L}(Y)$ and of the map $T \mapsto (\mathrm{Id} - T)^{-\frac{1}{2}}$ in $\mathcal{L}(Y)$ for $\|T\|_{\mathcal{L}(Y)} < 1$.
5. It follows from the conjugation formula (2.2.14). □

The Hamiltonian and reversible nature of the operator $\mathscr{L}_{\mu,\epsilon}$, see (2.2.5) and (2.2.6), imply additional algebraic properties for spectral projections $P_{\mu,\epsilon}$ and the transformation operators $U_{\mu,\epsilon}$.

Lemma 2.2.2. *For any $(\mu,\epsilon) \in B(\mu_0) \times B(\epsilon_0)$, the following holds true:*

(i) The projections $P_{\mu,\epsilon}$ defined in (2.2.7) *are (complex) skew-Hamiltonian, namely $\mathfrak{J} P_{\mu,\epsilon}$ are skew-Hermitian*

$$\mathfrak{J} P_{\mu,\epsilon} = P_{\mu,\epsilon}^* \mathfrak{J} \,, \tag{2.2.17}$$

and reversibility preserving, i.e. $\overline{\rho} P_{\mu,\epsilon} = P_{\mu,\epsilon} \overline{\rho}$.

(ii) The transformation operators $U_{\mu,\epsilon}$ in (2.2.11) *are symplectic, namely*

$$U_{\mu,\epsilon}^* \mathfrak{J} U_{\mu,\epsilon} = \mathfrak{J} \,,$$

and reversibility preserving.

(iii) $P_{0,\epsilon}$ and $U_{0,\epsilon}$ are real operators, i.e. $\overline{P_{0,\epsilon}} = P_{0,\epsilon}$ and $\overline{U_{0,\epsilon}} = U_{0,\epsilon}$.

Remark 2.2.3. The term (complex) skew-Hamiltonian is used in [39, Section 6] for matrices.

Proof. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a counter-clockwise oriented parametrization of Γ.
(i) Since $\mathscr{L}_{\mu,\epsilon}$ is Hamiltonian, it results $\mathscr{L}_{\mu,\epsilon} \mathfrak{J} = -\mathfrak{J} \mathscr{L}_{\mu,\epsilon}^*$ on Y. Then, for any scalar λ in the resolvent set of $\mathscr{L}_{\mu,\epsilon}$, the number $-\lambda$ belongs to the resolvent of $\mathscr{L}_{\mu,\epsilon}^*$ and

$$\mathfrak{J}(\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} = -(\mathscr{L}_{\mu,\epsilon}^* + \lambda)^{-1} \mathfrak{J} \,. \tag{2.2.18}$$

Taking the adjoint of (2.2.7), we have

$$P^*_{\mu,\epsilon} = \frac{1}{2\pi \mathrm{i}} \int_0^1 \left(\mathcal{L}^*_{\mu,\epsilon} - \overline{\gamma}(t)\right)^{-1} \dot{\overline{\gamma}}(t)\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left(\mathcal{L}^*_{\mu,\epsilon} + \lambda\right)^{-1} \mathrm{d}\lambda\,, \tag{2.2.19}$$

because the path $-\overline{\gamma}(t)$ winds around the origin clockwise. We conclude that

$$\mathcal{J} P_{\mu,\epsilon} \overset{(2.2.7)}{=} -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \mathcal{J}\left(\mathscr{L}_{\mu,\epsilon} - \lambda\right)^{-1} \mathrm{d}\lambda \overset{(2.2.18)}{=} \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left(\mathscr{L}^*_{\mu,\epsilon} + \lambda\right)^{-1} \mathcal{J} \mathrm{d}\lambda \overset{(2.2.19)}{=} P^*_{\mu,\epsilon}\mathcal{J}\,.$$

Let us now prove that $P_{\mu,\epsilon}$ is reversibility preserving. By (2.2.6) one has $(\mathscr{L}_{\mu,\epsilon} - \lambda)\overline{\rho} = \overline{\rho}(-\mathscr{L}_{\mu,\epsilon} - \overline{\lambda})$ and, for any scalar λ in the resolvent set of $\mathscr{L}_{\mu,\epsilon}$, we have $\overline{\rho}(\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1} = -(\mathscr{L}_{\mu,\epsilon} + \overline{\lambda})^{-1}\overline{\rho}$, using also that $(\overline{\rho})^{-1} = \overline{\rho}$. Thus, recalling (2.2.7) and (1.3.13), we have

$$\overline{\rho} P_{\mu,\epsilon} = \frac{1}{2\pi \mathrm{i}} \int_0^1 -\left(\mathscr{L}_{\mu,\epsilon} + \overline{\gamma}(t)\right)^{-1} \dot{\overline{\gamma}}(t)\mathrm{d}t\,\overline{\rho} = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathscr{L}_{\mu,\epsilon} - \lambda)^{-1}\mathrm{d}\lambda\,\overline{\rho} = P_{\mu,\epsilon}\overline{\rho}\,,$$

because the path $-\overline{\gamma}(t)$ winds around the origin clockwise.

(ii) If an operator A is skew-Hamiltonian then A^k, $k \in \mathbb{N}$, is skew-Hamiltonian as well. As a consequence, being the projections $P_{\mu,\epsilon}$, $P_{0,0}$ and their difference skew-Hamiltonian, the operator $\left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2}$ defined as in (2.2.12) is skew Hamiltonian as well. Hence, by (2.2.11) we get

$$\mathcal{J} U_{\mu,\epsilon} = \left[\left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2}\right]^* \left[P_{0,0}P_{\mu,\epsilon} + (\mathrm{Id} - P_{0,0})(\mathrm{Id} - P_{\mu,\epsilon})\right]^* \mathcal{J} \overset{(2.2.13)}{=} U^{-*}_{\mu,\epsilon}\mathcal{J}$$

and therefore $U^*_{\mu,\epsilon}\mathcal{J}U_{\mu,\epsilon} = \mathcal{J}$. Finally the operator $U_{\mu,\epsilon}$ defined in (2.2.11) is reversibility-preserving just as $\overline{\rho}$ commutes with $P_{\mu,\epsilon}$ and $P_{0,0}$.

(iii) By (2.2.7) and since $\mathscr{L}_{0,\epsilon}$ is a real operator, we have

$$\overline{P_{0,\epsilon}} = \frac{1}{2\pi \mathrm{i}} \int_0^1 \left(\mathscr{L}_{0,\epsilon} - \overline{\gamma}(t)\right)^{-1} \dot{\overline{\gamma}}(t)\mathrm{d}t = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \left(\mathscr{L}_{0,\epsilon} - \lambda\right)^{-1} \mathrm{d}\lambda = P_{0,\epsilon}$$

because the path $\overline{\gamma}(t)$ winds around the origin clockwise, proving that the operator $P_{0,\epsilon}$ is real. Then the operator $U_{0,\epsilon}$ defined in (2.2.11) is real as well. □

By the previous lemma, the linear involution $\overline{\rho}$ commutes with the spectral projections $P_{\mu,\epsilon}$ and then $\overline{\rho}$ leaves invariant the subspaces $\mathcal{V}_{\mu,\epsilon} = \mathrm{Rg}(P_{\mu,\epsilon})$.

Let us discuss the implications of the previous lemma in the setting of complex symplectic structures, presented for example in [6, 38]. The infinite dimensional complex space $L^2(\mathbb{T}, \mathbb{C}^2)$, with scalar product (1.3.11), is equipped with the *complex symplectic form*

$$\mathcal{W}_c : L^2(\mathbb{T}, \mathbb{C}^2) \times L^2(\mathbb{T}, \mathbb{C}^2) \to \mathbb{C}\,, \quad \mathcal{W}_c(f, g) := (\mathcal{J} f, g)\,, \tag{2.2.20}$$

which is sesquilinear, skew-Hermitian and non-degenerate, cfr. Definition 1 in [38]. The skew-Hamiltonian property (2.2.17) of the projection $P_{\mu,\epsilon}$ implies the following lemma.

Lemma 2.2.4. *For any (μ,ϵ), the linear subspace $\mathcal{V}_{\mu,\epsilon} = Rg(P_{\mu,\epsilon})$ is a complex symplectic subspace of $L^2(\mathbb{T},\mathbb{C}^2)$, namely the symplectic form $\mathcal{W}_c$ in (2.2.20), restricted to $\mathcal{V}_{\mu,\epsilon}$, is non-degenerate.*

Proof. Let $\tilde f \in \mathcal{V}_{\mu,\epsilon}$, thus $\tilde f = P_{\mu,\epsilon}\tilde f$. Suppose that $\mathcal{W}_c(\tilde f,\tilde g) = 0$ for any $\tilde g = P_{\mu,\epsilon} g \in \mathcal{V}_{\mu,\epsilon}$, $g \in L^2(\mathbb{T},\mathbb{C}^2)$. Thus

$$0 = \mathcal{W}_c(\tilde f,\tilde g) = (\mathcal{J}\tilde f, P_{\mu,\epsilon} g) = (P^*_{\mu,\epsilon}\mathcal{J}\tilde f, g) \overset{(2.2.17)}{=} (\mathcal{J}P_{\mu,\epsilon}\tilde f, g) = (\mathcal{J}\tilde f, g)\,.$$

We deduce that $\mathcal{J}\tilde f = 0$ and then $\tilde f = 0$. □

Remark 2.2.5. In view of Lemma 2.2.2-(ii) the transformation operator $U_{\mu,\epsilon}$ is symplectic, namely preserves the symplectic form (2.2.20), i.e. $\mathcal{W}_c(U_{\mu,\epsilon} f, U_{\mu,\epsilon} g) = \mathcal{W}_c(f,g)$, for any $f,g \in L^2(\mathbb{T},\mathbb{C}^2)$.

Symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$. It is convenient to represent the Hamiltonian and reversible operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ in a basis which is symplectic and reversible, according to the following definition.

Definition 2.2.6. (Symplectic reversible basis) A $\mathcal{V}_{\mu,\epsilon}$ basis $\mathtt{F} := \{\mathtt{f}_1^+,\, \mathtt{f}_1^-,\, \mathtt{f}_0^+,\, \mathtt{f}_0^-\}$ is

- *symplectic* if, for any $k, k' = 0, 1$,

$$\begin{aligned} &\left(\mathcal{J}\mathtt{f}_k^-\,,\, \mathtt{f}_k^+\right) = 1\,, \quad (\mathcal{J}\mathtt{f}_k^\sigma, \mathtt{f}_k^\sigma) = 0\,,\ \forall \sigma = \pm\,; \\ &\text{if } k \neq k' \text{ then } (\mathcal{J}\mathtt{f}_k^\sigma, \mathtt{f}_{k'}^{\sigma'}) = 0\,,\ \forall \sigma,\sigma' = \pm\ . \end{aligned} \tag{2.2.21}$$

- *reversible* if

$$\begin{aligned} &\overline{\rho}\mathtt{f}_1^+ = \mathtt{f}_1^+, \quad \overline{\rho}\mathtt{f}_1^- = -\mathtt{f}_1^-, \quad \overline{\rho}\mathtt{f}_0^+ = \mathtt{f}_0^+, \quad \overline{\rho}\mathtt{f}_0^- = -\mathtt{f}_0^-, \\ &\text{i.e. } \overline{\rho}\mathtt{f}_k^\sigma = \sigma\mathtt{f}_k^\sigma\,,\ \forall \sigma = \pm, k = 0, 1\,. \end{aligned} \tag{2.2.22}$$

Remark 2.2.7. By Remark 2.2.5, the operator $U_{\mu,\epsilon}$ maps a symplectic basis in a symplectic basis.

In the next lemma we outline a property of a reversible basis. We use the following notation along the thesis: we denote by $even(x)$ a real 2π-periodic function which is even in x, and by $odd(x)$ a real 2π-periodic function which is odd in x.

Lemma 2.2.8. *The real and imaginary parts of the elements of a reversible basis* $\mathtt{F} = \{\mathtt{f}_k^\pm\}$, $k = 0, 1$, *enjoy the following parity properties*

$$\mathtt{f}_k^+(x) = \begin{bmatrix} even(x) + \mathrm{i}\, odd(x) \\ odd(x) + \mathrm{i}\, even(x) \end{bmatrix}, \quad \mathtt{f}_k^-(x) = \begin{bmatrix} odd(x) + \mathrm{i}\, even(x) \\ even(x) + \mathrm{i}\, odd(x) \end{bmatrix}. \tag{2.2.23}$$

Proof. By the definition of the involution $\overline{\rho}$ in (1.3.13), we get

$$\mathtt{f}_k^+(x) = \begin{bmatrix} a(x) + \mathrm{i}\, b(x) \\ c(x) + \mathrm{i}\, d(x) \end{bmatrix} = \overline{\rho}\mathtt{f}_k^+(x) = \begin{bmatrix} a(-x) - \mathrm{i}\, b(-x) \\ -c(-x) + \mathrm{i}\, d(-x) \end{bmatrix} \Longrightarrow a, d \text{ even}, \ b, c \text{ odd}.$$

The properties of $\mathtt{f}_k^-$ follow similarly. □

We now expand a vector of $\mathcal{V}_{\mu,\epsilon}$ along a symplectic basis.

Lemma 2.2.9. *Let* $\mathtt{F} = \{\mathtt{f}_1^+, \mathtt{f}_1^-, \mathtt{f}_0^+, \mathtt{f}_0^-\}$ *be a symplectic basis of* $\mathcal{V}_{\mu,\epsilon}$. *Then any* $\mathtt{f}$ *in* $\mathcal{V}_{\mu,\epsilon}$ *has the expansion*

$$\mathtt{f} = -\left(\mathcal{J}\mathtt{f}\,,\, \mathtt{f}_1^-\right)\mathtt{f}_1^+ + \left(\mathcal{J}\mathtt{f}\,,\, \mathtt{f}_1^+\right)\mathtt{f}_1^- - \left(\mathcal{J}\mathtt{f}\,,\, \mathtt{f}_0^-\right)\mathtt{f}_0^+ + \left(\mathcal{J}\mathtt{f}\,,\, \mathtt{f}_0^+\right)\mathtt{f}_0^-. \tag{2.2.24}$$

Proof. We decompose $\mathtt{f} = \alpha_1^+\mathtt{f}_1^+ + \alpha_1^-\mathtt{f}_1^- + \alpha_0^+\mathtt{f}_0^+ + \alpha_0^-\mathtt{f}_0^-$ for suitable coefficients $\alpha_k^\sigma \in \mathbb{C}$. By applying $\mathcal{J}$, taking the L^2 scalar products with the vectors $\{\mathtt{f}_k^\sigma\}_{\sigma=\pm,k=0,1}$, using (2.2.21) and noting that $\left(\mathcal{J}\mathtt{f}_k^+\,,\, \mathtt{f}_k^-\right) = -1$, we get the expression of the coefficients α_k^σ as in (2.2.24). □

We now represent $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ with respect to a symplectic and reversible basis.

Lemma 2.2.10. *The* 4×4 *matrix that represents the Hamiltonian and reversible operator* $\mathscr{L}_{\mu,\epsilon} = \mathcal{J}\mathcal{B}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ *with respect to a symplectic and reversible basis* $\mathtt{F} = \{\mathtt{f}_1^+, \mathtt{f}_1^-, \mathtt{f}_0^+, \mathtt{f}_0^-\}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *is*

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4\mathtt{B}_{\mu,\epsilon}, \quad \mathtt{J}_4 := \left(\begin{array}{c|c} \mathtt{J}_2 & 0 \\ \hline 0 & \mathtt{J}_2 \end{array}\right), \quad \mathtt{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \textit{where} \quad \mathtt{B}_{\mu,\epsilon} = \mathtt{B}_{\mu,\epsilon}^* \tag{2.2.25}$$

is the self-adjoint matrix

$$\mathtt{B}_{\mu,\epsilon} = \begin{pmatrix} \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^+\,,\mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^-\,,\mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^+\,,\mathtt{f}_1^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^-\,,\mathtt{f}_1^+\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^+\,,\mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^-\,,\mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^+\,,\mathtt{f}_1^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^-\,,\mathtt{f}_1^-\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^+\,,\mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^-\,,\mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^+\,,\mathtt{f}_0^+\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^-\,,\mathtt{f}_0^+\right) \\ \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^+\,,\mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_1^-\,,\mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^+\,,\mathtt{f}_0^-\right) & \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_0^-\,,\mathtt{f}_0^-\right) \end{pmatrix}. \tag{2.2.26}$$

The entries of the matrix $\mathtt{B}_{\mu,\epsilon}$ *are alternatively real or purely imaginary: for any* $\sigma = \pm$, $k = 0, 1$,

$$\left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_k^\sigma\,,\, \mathtt{f}_{k'}^\sigma\right) \ \textit{is real}, \qquad \left(\mathcal{B}_{\mu,\epsilon}\mathtt{f}_k^\sigma\,,\, \mathtt{f}_{k'}^{-\sigma}\right) \ \textit{is purely imaginary}. \tag{2.2.27}$$

Proof. Lemma 2.2.9 implies that

$$\mathscr{L}_{\mu,\epsilon}\mathtt{f}^{\sigma}_{k} = -\sum_{k'=0,1,\sigma'=\pm} \sigma' \big(\mathcal{J}\mathscr{L}_{\mu,\epsilon}\mathtt{f}^{\sigma}_{k}, \mathtt{f}^{-\sigma'}_{k'}\big)\mathtt{f}^{\sigma'}_{k'} = \sum_{k'=0,1,\sigma'=\pm} \sigma' \big(\mathcal{B}_{\mu,\epsilon}\mathtt{f}^{\sigma}_{k}, \mathtt{f}^{-\sigma'}_{k'}\big)\mathtt{f}^{\sigma'}_{k'} \,.$$

Then the matrix representing the operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ with respect to the basis $\mathtt{F}$ is given by $\mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ with $\mathtt{B}_{\mu,\epsilon}$ in (2.2.26). The matrix $\mathtt{B}_{\mu,\epsilon}$ is selfadjoint because $\mathcal{B}_{\mu,\epsilon}$ is a selfadjoint operator. We now prove (2.2.27). By recalling (1.3.13) and (1.3.11) it results

$$(f\,,\,g) = \overline{(\overline{\rho} f\,,\,\overline{\rho} g)}\,. \tag{2.2.28}$$

Then, by (2.2.28), since $\mathcal{B}_{\mu,\epsilon}$ is reversibility-preserving and (2.2.22), we get

$$\big(\mathcal{B}_{\mu,\epsilon}\,\mathtt{f}^{\sigma}_{k}, \mathtt{f}^{\sigma'}_{k'}\big) = \overline{\big(\overline{\rho}\mathcal{B}_{\mu,\epsilon}\,\mathtt{f}^{\sigma}_{k}\,,\,\overline{\rho}\mathtt{f}^{\sigma'}_{k'}\big)} = \overline{\big(\mathcal{B}_{\mu,\epsilon}\overline{\rho}\,\mathtt{f}^{\sigma}_{k}\,,\,\overline{\rho}\mathtt{f}^{\sigma'}_{k'}\big)} = \sigma\sigma'\,\overline{\big(\mathcal{B}_{\mu,\epsilon}\,\mathtt{f}^{\sigma}_{k}\,,\,\mathtt{f}^{\sigma'}_{k'}\big)}\,,$$

which proves (2.2.27). □

Remark 2.2.11. The complex symplectic form $\mathcal{W}_c$ in (2.2.20) restricted to the symplectic subspace $\mathcal{V}_{\mu,\epsilon}$ is represented, in *any* symplectic basis (cfr. (2.2.21)), by the matrix $\mathtt{J}_4$ in (2.2.25), acting in $\mathbb{C}^4$ with the standard complex scalar product.

Hamiltonian and reversible matrices. In the sequel we frequently deal with matrices of the form obtained in Lemma 2.2.10. We shall use the following terminology.

Definition 2.2.12. A $2n \times 2n$, $n = 1, 2$, matrix of the form $\mathtt{L} = \mathtt{J}_{2n}\mathtt{B}$ is

1. *Hamiltonian* if $\mathtt{B}$ is a self-adjoint matrix, i.e. $\mathtt{B} = \mathtt{B}^*$;
2. *Reversible* if $\mathtt{B}$ is reversibility-preserving, i.e. $\rho_{2n} \circ \mathtt{B} = \mathtt{B} \circ \rho_{2n}$, where

$$\rho_4 := \begin{pmatrix} \rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, \qquad \rho_2 := \begin{pmatrix} \mathfrak{c} & 0 \\ 0 & -\mathfrak{c} \end{pmatrix}, \tag{2.2.29}$$

and $\mathfrak{c} : z \mapsto \overline{z}$ is the conjugation of the complex plane. Equivalently, $\rho_{2n} \circ \mathtt{L} = -\mathtt{L} \circ \rho_{2n}$.

In the sequel we shall mainly deal with 4×4 Hamiltonian and reversible matrices. The transformations preserving the Hamiltonian structure are called *symplectic*, and satisfy

$$Y^*\mathtt{J}_4 Y = \mathtt{J}_4\,. \tag{2.2.30}$$

If Y is symplectic then Y^* and Y^{-1} are symplectic as well. A Hamiltonian matrix $\mathtt{L} = \mathtt{J}_4\mathtt{B}$, with $\mathtt{B} = \mathtt{B}^*$, is conjugated through Y in the new Hamiltonian matrix

$$\mathtt{L}_1 = Y^{-1}\mathtt{L}Y = Y^{-1}\mathtt{J}_4 Y^{-*} Y^*\mathtt{B}Y = \mathtt{J}_4\mathtt{B}_1 \quad \text{where} \quad \mathtt{B}_1 := Y^*\mathtt{B}Y = \mathtt{B}_1^*\,. \tag{2.2.31}$$

Note that the matrix ρ_4 in (2.2.29) represents the action of the involution $\overline{\rho} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ defined in (1.3.13) in a reversible basis (cfr. (2.2.22)). A 4×4 matrix $\mathtt{B} = (\mathtt{B}_{ij})_{i,j=1,\dots,4}$ is reversibility-preserving if and only if its entries are alternatively real and purely imaginary, namely $\mathtt{B}_{ij}$ is real when $i+j$ is even and purely imaginary otherwise, as in (2.2.27). A 4×4 complex matrix $\mathtt{L} = (\mathtt{L}_{ij})_{i,j=1,\dots,4}$ is reversible if and only if $\mathtt{L}_{ij}$ is purely imaginary when $i+j$ is even and real otherwise.

In the sequel we shall use that the flow of a Hamiltonian reversibility-preserving matrix is symplectic and reversibility-preserving.

Lemma 2.2.13. *Let Σ be a self-adjoint and reversible matrix, then $\exp(\tau \mathtt{J}_4 \Sigma)$, $\tau \in \mathbb{R}$, is a reversibility-preserving symplectic matrix.*

Proof. The flow $\varphi(\tau) := \exp(\tau \mathtt{J}_4 \Sigma)$ solves $\frac{d}{d\tau}\varphi(\tau) := \mathtt{J}_4 \Sigma \varphi(\tau)$, with $\varphi(0) = \mathrm{Id}$. Then $\psi(\tau) := \varphi(\tau)^* \mathtt{J}_4 \varphi(\tau) - \mathtt{J}_4$ satisfies $\psi(0) = 0$ and $\frac{d}{d\tau}\psi(\tau) = \varphi(\tau)^* \mathtt{J}_4^* \mathtt{J}_4 \varphi(\tau) + \varphi(\tau)^* \mathtt{J}_4 \mathtt{J}_4 \varphi(\tau) = 0$. Then $\psi(\tau) = 0$ for any τ and $\varphi(\tau)$ is symplectic. The matrix $\exp(\tau \mathtt{J}_4 \Sigma) = \sum_{n \geq 0} \frac{1}{n!}(\tau \mathtt{J}_4 \Sigma)^n$ is reversibility-preserving since each $(\mathtt{J}_4 \Sigma)^n$, $n \geq 0$, is reversibility-preserving. $\square$

Chapter 3

Benjamin-Feir instability in deep water

In this chapter we prove the full description of the Benjamin-Feir instability phenomenon in the case of infinite depth given in Theorem 1.4.1 and its "corollary" Theorem 1.2.2.

3.1 Expansion of the Kato basis

In this section we use the transformation operators $U_{\mu,\epsilon}$ obtained in Chapter 2.2 to construct a symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$ and, in Proposition 3.2.2, we compute the 4×4 Hamiltonian and reversible matrix representing $\mathscr{L}_{\mu,\epsilon} \colon \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ on such basis.

First basis of $\mathcal{V}_{\mu,\epsilon}$. In view of Lemma 2.2.1, the first basis of $\mathcal{V}_{\mu,\epsilon}$ that we consider is

$$\begin{aligned} &\mathcal{F} := \{ f_1^+(\mu,\epsilon),\ f_1^-(\mu,\epsilon),\ f_0^+(\mu,\epsilon),\ f_0^-(\mu,\epsilon) \}\,, \\ &f_k^\sigma(\mu,\epsilon) := U_{\mu,\epsilon} f_k^\sigma\,,\ \sigma = \pm\,,\ k = 0,1\,, \end{aligned} \tag{3.1.1}$$

obtained applying the transformation operators $U_{\mu,\epsilon}$ in (2.2.11) to the vectors

$$f_1^+ = \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}, \quad f_1^- = \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}, \quad f_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.1.2}$$

which form a basis of $\mathcal{V}_{0,0} = \mathrm{Rg}(P_{0,0})$, cfr. (2.1.15b)-(2.1.16). Note that the real valued vectors $\{f_1^\pm, f_0^\pm\}$ are orthonormal with respect to the scalar product (1.3.11), and satisfy

$$\mathcal{J} f_1^+ = -f_1^-\,, \qquad \mathcal{J} f_1^- = f_1^+, \qquad \mathcal{J} f_0^+ = -f_0^-\,, \qquad \mathcal{J} f_0^- = f_0^+\,, \tag{3.1.3}$$

thus forming a symplectic and reversible basis for $\mathcal{V}_{0,0}$, according to Definition 2.2.6.

In view of Remarks 2.2.5 and 2.2.7, the symplectic operators $U_{\mu,\epsilon}$ transform, for any (μ, ϵ) small, the symplectic basis (3.1.2) of $\mathcal{V}_{0,0}$, into the symplectic basis (3.1.1):

Lemma 3.1.1. *The basis* $\mathcal{F}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *defined in* (3.1.1)*, is symplectic and reversible, i.e. satisfies* (2.2.21) *and* (2.2.22)*. Each map* $(\mu, \epsilon) \mapsto f_k^\sigma(\mu, \epsilon)$ *is analytic from* $B(\mu_0) \times B(\epsilon_0)$ *to* $H^1(\mathbb{T})$*.*

Proof. Since by Lemma 2.2.2-(*ii*) the maps $U_{\mu,\epsilon}$ are symplectic and reversibility-preserving the transformed vectors $f_1^+(\mu, \epsilon), \ldots, f_0^-(\mu, \epsilon)$ are symplectic orthogonals and reversible as well as the unperturbed ones $f_1^+, \ldots, f_0^-$. The analyticity of $f_k^\sigma(\mu, \epsilon)$ follows from the analyticity property of $U_{\mu,\epsilon}$ proved in Lemma 2.2.1. □

In the next lemma we provide a suitable expansion of the vectors $f_k^\sigma(\mu, \epsilon)$ in (μ, ϵ). We denote by $even_0(x)$ a real, even, 2π-periodic function with zero space average. In what follows $\mathcal{O}(\mu^m \epsilon^n) \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix}$ denotes an analytic map in (μ, ϵ) ranging in $H^1(\mathbb{T}, \mathbb{C}^2)$, whose first component is $even(x)$ and the second one $odd(x)$; similar meaning for $\mathcal{O}(\mu^m \epsilon^n) \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix}$, and similar notation.

Lemma 3.1.2. (Expansion of the basis $\mathcal{F}$) *For small values of* (μ, ϵ) *the basis* $\mathcal{F}$ *in* (3.1.1) *has the following expansion*

$$
\begin{aligned}
f_1^+(\mu, \epsilon) &= \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix} + \mathrm{i}\,\frac{\mu}{4} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} + \epsilon \begin{bmatrix} 2\cos(2x) \\ \sin(2x) \end{bmatrix} \\
&\quad + \mathcal{O}(\mu^2) \begin{bmatrix} even_0(x) + \mathrm{i}\, odd(x) \\ odd(x) + \mathrm{i}\, even_0(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} + \mathrm{i}\,\mu\epsilon \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2)\,,
\end{aligned} \tag{3.1.4}
$$

$$
\begin{aligned}
f_1^-(\mu, \epsilon) &= \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix} + \mathrm{i}\,\frac{\mu}{4} \begin{bmatrix} \cos(x) \\ -\sin(x) \end{bmatrix} + \epsilon \begin{bmatrix} -2\sin(2x) \\ \cos(2x) \end{bmatrix} \\
&\quad + \mathcal{O}(\mu^2) \begin{bmatrix} odd(x) + \mathrm{i}\, even_0(x) \\ even_0(x) + \mathrm{i}\, odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix} + \mathrm{i}\,\mu\epsilon \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2)\,,
\end{aligned} \tag{3.1.5}
$$

$$
f_0^+(\mu, \epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \begin{bmatrix} \cos(x) \\ -\sin(x) \end{bmatrix} + \mathcal{O}(\epsilon^2) \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} + \mathrm{i}\,\mu\epsilon \begin{bmatrix} odd(x) \\ even_0(x) \end{bmatrix} + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2)\,, \tag{3.1.6}
$$

$$
f_0^-(\mu, \epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu\epsilon \left(\begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} + \mathrm{i} \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} \right) + \mathcal{O}(\mu^2\epsilon, \mu\epsilon^2)\,, \tag{3.1.7}
$$

where the remainders $\mathcal{O}()$ are vectors in $H^1(\mathbb{T})$. For $\mu = 0$ the basis $\{f_k^{\pm}(0,\epsilon), k = 0,1\}$ is real and

$$f_1^+(0,\epsilon) = \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix},\ f_1^-(0,\epsilon) = \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix},\ f_0^+(0,\epsilon) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix},\ f_0^-(0,\epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.1.8}$$

The rest of the section is devoted to the proof of Lemma 3.1.2.
We first Taylor-expand the transformation operators $U_{\mu,\epsilon}$ defined in (2.2.11). We denote ∂_ϵ with a prime and ∂_μ with a dot.

Lemma 3.1.3. *The first jets of $U_{\mu,\epsilon}P_{0,0}$ are*

$$U_{0,0}P_{0,0} = P_{0,0}\,, \quad U'_{0,0}P_{0,0} = P'_{0,0}P_{0,0}\,, \quad \dot{U}_{0,0}P_{0,0} = \dot{P}_{0,0}P_{0,0}\,, \tag{3.1.9}$$

$$\dot{U}'_{0,0}P_{0,0} = (\dot{P}'_{0,0} - \frac{1}{2}P_{0,0}\dot{P}'_{0,0})P_{0,0}\,, \tag{3.1.10}$$

where

$$P'_{0,0} = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,, \tag{3.1.11}$$

$$\dot{P}_{0,0} = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,, \tag{3.1.12}$$

and

$$\dot{P}'_{0,0} = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda \tag{3.1.13a}$$

$$-\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda \tag{3.1.13b}$$

$$+\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,. \tag{3.1.13c}$$

The operators $\mathscr{L}'_{0,0}$ and $\dot{\mathscr{L}}_{0,0}$ are

$$\mathscr{L}'_{0,0} = \begin{bmatrix} \partial_x \circ p_1(x) & 0 \\ -a_1(x) & p_1(x)\circ\partial_x \end{bmatrix}, \quad \dot{\mathscr{L}}_{0,0} = \begin{bmatrix} 0 & \mathrm{sgn}(D) + \Pi_0 \\ 0 & 0 \end{bmatrix}, \tag{3.1.14}$$

with $a_1(x) = p_1(x) = -2\cos(x)$, cfr. (2.1.8)-(2.1.9). The operator $\dot{\mathscr{L}}'_{0,0}$ is

$$\dot{\mathscr{L}}'_{0,0} = \begin{bmatrix} \mathrm{i}\,p_1(x) & 0 \\ 0 & \mathrm{i}\,p_1(x) \end{bmatrix}. \tag{3.1.15}$$

Proof. By (2.2.11) and (2.2.12) one has the Taylor expansion in $\mathcal{L}(Y)$

$$U_{\mu,\epsilon}P_{0,0} = P_{\mu,\epsilon}P_{0,0} + \frac{1}{2}(P_{\mu,\epsilon} - P_{0,0})^2 P_{\mu,\epsilon}P_{0,0} + \mathcal{O}(P_{\mu,\epsilon} - P_{0,0})^4 \, ,$$

where $\mathcal{O}(P_{\mu,\epsilon} - P_{0,0})^4 = \mathcal{O}(\epsilon^4, \epsilon^3\mu, \epsilon^2\mu^2, \epsilon\mu^3, \mu^4) \in \mathcal{L}(Y)$. Consequently one derives (3.1.9), (3.1.10), using also the identity $\dot{P}_{0,0}P'_{0,0}P_{0,0} + P'_{0,0}\dot{P}_{0,0}P_{0,0} = -P_{0,0}\dot{P}'_{0,0}P_{0,0}$, which follows differentiating $P^2_{\mu,\epsilon} = P_{\mu,\epsilon}$. Differentiating (2.2.7) one gets (3.1.11)–(3.1.13c). Formulas (3.1.14)-(3.1.15) follow by (2.2.2b). □

By the previous lemma we have the Taylor expansion

$$f^\sigma_k(\mu,\epsilon) = f^\sigma_k + \epsilon P'_{0,0}f^\sigma_k + \mu\dot{P}_{0,0}f^\sigma_k + \mu\epsilon(\dot{P}'_{0,0} - \frac{1}{2}P_{0,0}\dot{P}'_{0,0})f^\sigma_k + \mathcal{O}(\mu^2,\epsilon^2)\, . \tag{3.1.16}$$

In order to compute the vectors $P'_{0,0}f^\sigma_k$ and $\dot{P}_{0,0}f^\sigma_k$ using (3.1.11) and (3.1.12), it is useful to know the action of $(\mathscr{L}_{0,0} - \lambda)^{-1}$ on the vectors

$$f^+_k := \begin{bmatrix}\cos(kx)\\ \sin(kx)\end{bmatrix}, \quad f^-_k := \begin{bmatrix}-\sin(kx)\\ \cos(kx)\end{bmatrix}, \quad f^+_{-k} := \begin{bmatrix}\cos(kx)\\ -\sin(kx)\end{bmatrix}, \quad f^-_{-k} := \begin{bmatrix}\sin(kx)\\ \cos(kx)\end{bmatrix}, \quad k \in \mathbb{N}\, . \tag{3.1.17}$$

Lemma 3.1.4. *The space $H^1(\mathbb{T})$ decomposes as $H^1(\mathbb{T}) = \mathcal{V}_{0,0} \oplus \mathcal{U} \oplus \mathcal{W}_{H^1}$, with $\mathcal{W}_{H^1} := \overline{\bigoplus_{k=2}^{\infty} \mathcal{W}_k}^{H^1}$, where the subspaces $\mathcal{V}_{0,0}, \mathcal{U}$ and $\mathcal{W}_k$, defined below, are invariant under $\mathscr{L}_{0,0}$ and the following properties hold:*

(i) $\mathcal{V}_{0,0} = \mathit{span}\{f^+_1, f^-_1, f^+_0, f^-_0\}$ *is the generalized kernel of* $\mathscr{L}_{0,0}$. *For any* $\lambda \neq 0$ *the operator* $\mathscr{L}_{0,0} - \lambda : \mathcal{V}_{0,0} \to \mathcal{V}_{0,0}$ *is invertible and*

$$(\mathscr{L}_{0,0} - \lambda)^{-1}f^+_1 = -\frac{1}{\lambda}f^+_1\, , \quad (\mathscr{L}_{0,0} - \lambda)^{-1}f^-_1 = -\frac{1}{\lambda}f^-_1\, , \quad (\mathscr{L}_{0,0} - \lambda)^{-1}f^-_0 = -\frac{1}{\lambda}f^-_0\, , \tag{3.1.18}$$

$$(\mathscr{L}_{0,0} - \lambda)^{-1}f^+_0 = -\frac{1}{\lambda}f^+_0 + \frac{1}{\lambda^2}f^-_0\, . \tag{3.1.19}$$

(ii) $\mathcal{U} := \mathit{span}\Big\{f^+_{-1}, f^-_{-1}\Big\}$. *For any* $\lambda \neq \pm 2\mathrm{i}$ *the operator* $\mathscr{L}_{0,0} - \lambda : \mathcal{U} \to \mathcal{U}$ *is invertible and*

$$(\mathscr{L}_{0,0}-\lambda)^{-1}f^+_{-1} = \frac{1}{\lambda^2+4}\Big(-\lambda f^+_{-1} + 2f^-_{-1}\Big)\, , \quad (\mathscr{L}_{0,0}-\lambda)^{-1}f^-_{-1} = \frac{1}{\lambda^2+4}\Big(-2f^+_{-1} - \lambda f^-_{-1}\Big)\, . \tag{3.1.20}$$

(iii) *Each subspace* $\mathcal{W}_k := \mathrm{span}\Big\{f_k^+,\ f_k^-, f_{-k}^+,\ f_{-k}^-\Big\}$ *is invariant under* $\mathscr{L}_{0,0}$. *Let* $\mathcal{W}_{L^2} := \overline{\bigoplus_{k=2}^{\infty} \mathcal{W}_k}^{L^2}$. *For any* $|\lambda| < \frac{1}{2}$, *the operator* $\mathscr{L}_{0,0} - \lambda : \mathcal{W}_{H^1} \to \mathcal{W}_{L^2}$ *is invertible and, for any* $f \in \mathcal{W}_{L^2}$,

$$(\mathscr{L}_{0,0} - \lambda)^{-1} f = (\partial_x^2 + |D|)^{-1} \begin{bmatrix} \partial_x & -|D| \\ 1 & \partial_x \end{bmatrix} f + \lambda \varphi_f(\lambda, x)\,, \tag{3.1.21}$$

for some analytic function $\lambda \mapsto \varphi_f(\lambda, \cdot) \in H^1(\mathbb{T}, \mathbb{C}^2)$.

Proof. By inspection the spaces $\mathcal{V}_{0,0}$, $\mathcal{U}$ and $\mathcal{W}_k$ are invariant under $\mathscr{L}_{0,0}$ and, by Fourier series, they decompose $H^1(\mathbb{T}, \mathbb{C}^2)$.
(i) Formulas (3.1.18)-(3.1.19) follow using that f_1^+, f_1^-, f_0^- are in the kernel of $\mathscr{L}_{0,0}$, and $\mathscr{L}_{0,0} f_0^+ = -f_0^-$.
(ii) Formula (3.1.20) follows using that $\mathscr{L}_{0,0} f_{-1}^+ = -2 f_{-1}^-$ and $\mathscr{L}_{0,0} f_{-1}^- = 2 f_{-1}^+$.
(iii) Let $\mathcal{W} := \mathcal{W}_{H^1}$. The operator $(\mathscr{L}_{0,0} - \lambda \mathrm{Id})\big|_{\mathcal{W}}$ is invertible for any $\lambda \notin \{\pm \mathrm{i}\sqrt{|k|} \pm \mathrm{i}\, k, k \geq 2, k \in \mathbb{N}\}$ and $(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} = (\partial_x^2 + |D|)^{-1} \begin{bmatrix} \partial_x & -|D| \\ 1 & \partial_x \end{bmatrix}_{|\mathcal{W}}$. In particular, by Neumann series, for any λ such that $|\lambda| \|(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\|_{\mathcal{L}(\mathcal{W}_{L^2}, H^1(\mathbb{T}))} < 1$, e.g. for any $|\lambda| < 1/2$,

$$(\mathscr{L}_{0,0}|_{\mathcal{W}} - \lambda)^{-1} = (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} \big(\mathrm{Id} - \lambda (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\big)^{-1} = (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} \sum_{k \geq 0} ((\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} \lambda)^k\,.$$

Formula (3.1.21) follows with $\varphi_f(\lambda, x) := (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} \sum_{k \geq 1} \lambda^{k-1} [(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}]^k f$. □

We shall also use the following formulas, obtained by (3.1.14) and (3.1.2):

$$\begin{aligned} &\mathscr{L}'_{0,0} f_1^+ = 2 \begin{bmatrix} \sin(2x) \\ 0 \end{bmatrix}, \quad \mathscr{L}'_{0,0} f_1^- = 2 \begin{bmatrix} \cos(2x) \\ 0 \end{bmatrix}, \quad \mathscr{L}'_{0,0} f_0^+ = 2 \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}, \quad \mathscr{L}'_{0,0} f_0^- = 0\,, \\ &\dot{\mathscr{L}}_{0,0} f_1^+ = -\mathrm{i} \begin{bmatrix} \cos(x) \\ 0 \end{bmatrix}, \quad \dot{\mathscr{L}}_{0,0} f_1^- = \mathrm{i} \begin{bmatrix} \sin(x) \\ 0 \end{bmatrix}, \quad \dot{\mathscr{L}}_{0,0} f_0^+ = 0, \quad \dot{\mathscr{L}}_{0,0} f_0^- = f_0^+\,. \end{aligned} \tag{3.1.22}$$

We finally compute $P'_{0,0} f_k^\sigma$ and $\dot{P}_{0,0} f_k^\sigma$.

Lemma 3.1.5. *One has*

$$\begin{aligned} &P'_{0,0} f_1^+ = \begin{bmatrix} 2\cos(2x) \\ \sin(2x) \end{bmatrix}, \quad P'_{0,0} f_1^- = \begin{bmatrix} -2\sin(2x) \\ \cos(2x) \end{bmatrix}, \quad P'_{0,0} f_0^+ = f_{-1}^+\,, \quad P'_{0,0} f_0^- = 0\,, \\ &\dot{P}_{0,0} f_1^+ = \frac{\mathrm{i}}{4} f_{-1}^-\,, \quad \dot{P}_{0,0} f_1^- = \frac{\mathrm{i}}{4} f_{-1}^+\,, \quad \dot{P}_{0,0} f_0^+ = 0\,, \quad \dot{P}_{0,0} f_0^- = 0\,. \end{aligned} \tag{3.1.23}$$

Proof. We first compute $P'_{0,0}f_1^+$. By (3.1.11), (3.1.18) and (3.1.22) we deduce

$$P'_{0,0}f_1^+ = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix}2\sin(2x)\\0\end{bmatrix}\mathrm{d}\lambda\,.$$

We note that $\begin{bmatrix}2\sin(2x)\\0\end{bmatrix}$ belongs to $\mathcal{W}$, being equal to $f_{-2}^- - f_2^-$ (recall (3.1.17)). By (3.1.21) there is an analytic function $\lambda \mapsto \varphi(\lambda,\cdot) \in H^1(\mathbb{T},\mathbb{C}^2)$ so that

$$P'_{0,0}f_1^+ = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{1}{\lambda}\Big(\begin{bmatrix}-2\cos(2x)\\-\sin(2x)\end{bmatrix} + \lambda\varphi(\lambda)\Big)\mathrm{d}\lambda = \begin{bmatrix}2\cos(2x)\\ \sin(2x)\end{bmatrix},$$

using the residue Theorem. Similarly one computes $P'_{0,0}f_1^-$. By (3.1.11), (3.1.18) and (3.1.22), one has $P'_{0,0}f_0^- = 0$. Next we compute $P'_{0,0}f_0^+$. By (3.1.11), (3.1.18), (3.1.19) and (3.1.22) we get

$$P'_{0,0}f_0^+ = -\frac{2}{2\pi\mathrm{i}}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}f_{-1}^-\mathrm{d}\lambda \overset{(3.1.20)}{=} -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\Big(-\frac{4}{\lambda(\lambda^2+4)}f_{-1}^+ - \frac{2}{\lambda^2+4}f_{-1}^-\Big)\mathrm{d}\lambda = f_{-1}^+\,,$$

where in the last step we used the residue theorem. We compute now $\dot P_{0,0}f_1^+$. First we have $\dot P_{0,0}f_1^+ = \frac{\mathrm{i}}{2\pi\mathrm{i}}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix}\cos(x)\\0\end{bmatrix}\mathrm{d}\lambda$ and then, writing $\begin{bmatrix}\cos(x)\\0\end{bmatrix} = \frac{1}{2}(f_1^+ + f_{-1}^+)$ and using (3.1.20), we conclude

$$\dot P_{0,0}f_1^+ = \frac{\mathrm{i}}{2}\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\Big(-\frac{1}{\lambda^2}f_1^+ - \frac{1}{\lambda^2+4}f_{-1}^+ + \frac{2}{\lambda(\lambda^2+4)}f_{-1}^-\Big)\mathrm{d}\lambda = \frac{\mathrm{i}}{4}f_{-1}^-$$

using again the residue theorem. The computations of $\dot P_{0,0}f_1^-$, $\dot P_{0,0}f_0^+$, $\dot P_{0,0}f_0^-$ are analogous. □

So far we have obtained the linear terms of the expansions (3.1.4), (3.1.5), (3.1.6), (3.1.7). We now provide further information about the expansion of the basis at $\mu = 0$.

Lemma 3.1.6. *The basis* $\{f_k^\sigma(0,\epsilon),\ k=0,1,\sigma=\pm\}$ *is real. For any* ϵ *it results* $f_0^-(0,\epsilon) \equiv f_0^-$. *The property* (3.1.8) *holds.*

Proof. The reality of the basis $f_k^\sigma(0,\epsilon)$ is a consequence of Lemma 2.2.2-(iii). Since by (2.2.2b) $\mathscr{L}_{0,\epsilon}f_0^- = 0$ for any ϵ, we deduce $(\mathscr{L}_{0,\epsilon}-\lambda)^{-1}f_0^- = -\frac{1}{\lambda}f_0^-$ and then, using also the residue theorem,

$$P_{0,\epsilon}f_0^- = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma(\mathscr{L}_{0,\epsilon}-\lambda)^{-1}f_0^-\mathrm{d}\lambda = f_0^-\,.$$

In particular $P_{0,\epsilon}f_0^- = P_{0,0}f_0^-$, for any ϵ and we get, by (2.2.11), $f_0^-(0,\epsilon) = U_{0,\epsilon}f_0^- = f_0^-$, for any ϵ.

Let us prove property (3.1.8). In view of (2.2.23) and since the basis is real, we know that $f_k^+(0,\epsilon) = \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix}$, $f_k^-(0,\epsilon) = \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix}$, for any $k = 0, 1$. By Lemma 3.1.1 the basis $\{f_k^\sigma(0,\epsilon)\}$ is symplectic (cfr. (2.2.21)) and, since $\mathcal{J} f_0^-(0,\epsilon) = \mathcal{J} f_0^- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for any ϵ, we get

$$0 = \left(\mathcal{J} f_0^-(0,\epsilon)\,,\, f_1^+(0,\epsilon)\right) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_1^+(0,\epsilon)\right), \quad 1 = (\mathcal{J} f_0^-(0,\epsilon), f_0^+(0,\epsilon)) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_0^+(0,\epsilon)\right).$$

Thus the first component of both $f_1^+(0,\epsilon)$ and $f_0^+(0,\epsilon) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has zero average, proving (3.1.8). □

We now provide further information about the expansion of the basis at $\epsilon = 0$.

Lemma 3.1.7. *For any small μ, we have $f_0^+(\mu,0) \equiv f_0^+$ and $f_0^-(\mu,0) \equiv f_0^-$. Moreover the vectors $f_1^+(\mu,0)$ and $f_1^-(\mu,0)$ have both components with zero space average.*

Proof. The operator $\mathscr{L}_{\mu,0} = \begin{bmatrix} \partial_x & |D+\mu| \\ -1 & \partial_x \end{bmatrix}$ leaves invariant the subspace $\mathcal{Z} := \mathrm{span}\{f_0^+,\, f_0^-\}$ since $\mathscr{L}_{\mu,0}f_0^+ = -f_0^-$ and $\mathscr{L}_{\mu,0}f_0^- = \mu f_0^+$. The operator $\mathscr{L}_{\mu,0}|_{\mathcal{Z}}$ has the two eigenvalues $\pm \mathrm{i}\sqrt{\mu}$, which, for small μ, lie inside the loop Γ around 0 in (2.2.7). Then, by (2.2.10), we have $\mathcal{Z} \subseteq \mathcal{V}_{\mu,0} = \mathrm{Rg}(P_{\mu,0})$ and

$$P_{\mu,0}f_0^\pm = f_0^\pm, \quad f_0^\pm(\mu,0) = U_{\mu,0}f_0^\pm = f_0^\pm, \text{ for any } \mu \text{ small}.$$

The basis $\{f_k^\sigma(\mu,0)\}$ is symplectic. Then, since $\mathcal{J} f_0^+ = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathcal{J} f_0^- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have

$$0 = \left(\mathcal{J} f_0^+(\mu,0)\,,\, f_1^\sigma(\mu,0)\right) = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}, f_1^\sigma(\mu,0)\right), \ \ 0 = \left(\mathcal{J} f_0^-(\mu,0), f_1^\sigma(\mu,0)\right) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, f_1^\sigma(\mu,0)\right),$$

namely both the components of $f_1^\pm(\mu,0)$ have zero average. □

We finally consider the $\mu\epsilon$ term in the expansion (3.1.16) of the vectors $f_k^\sigma(\mu,\epsilon)$, $k = 0, 1$, $\sigma = \pm$.

Lemma 3.1.8. *The derivatives* $(\partial_\mu\partial_\epsilon f_k^\sigma)(0,0) = \left(\dot{P}'_{0,0} - \frac{1}{2}P_{0,0}\dot{P}'_{0,0}\right) f_k^\sigma$ *satisfy*

$$
\begin{aligned}
&(\partial_\mu\partial_\epsilon f_1^+)(0,0) = \mathrm{i}\begin{bmatrix} odd(x) \\ even(x)\end{bmatrix}, \qquad (\partial_\mu\partial_\epsilon f_1^-)(0,0) - = \mathrm{i}\begin{bmatrix} even(x) \\ odd(x)\end{bmatrix}, \\
&(\partial_\mu\partial_\epsilon f_0^+)(0,0) = \mathrm{i}\begin{bmatrix} odd(x) \\ even_0(x)\end{bmatrix}, \qquad (\partial_\mu\partial_\epsilon f_0^-)(0,0) = \frac{1}{2}\begin{bmatrix} \sin(x) \\ \cos(x)\end{bmatrix} + \mathrm{i}\begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix}.
\end{aligned}
\tag{3.1.24}
$$

Proof. We decompose the Fourier multiplier operator $\dot{\mathscr{L}}_{0,0}$ in (3.1.14) as

$$
\dot{\mathscr{L}}_{0,0} = \dot{\mathscr{L}}_{0,0}^{(I)} + \dot{\mathscr{L}}_{0,0}^{(II)}, \qquad \dot{\mathscr{L}}_{0,0}^{(I)} := \begin{bmatrix} 0 & \operatorname{sgn}(D) \\ 0 & 0 \end{bmatrix}, \qquad \dot{\mathscr{L}}_{0,0}^{(II)} := \begin{bmatrix} 0 & \Pi_0 \\ 0 & 0 \end{bmatrix},
$$

and, accordingly, we write $\dot{P}'_{0,0} = (3.1.13\text{a})^{(I)} + (3.1.13\text{a})^{(II)} + (3.1.13\text{b})^{(I)} + (3.1.13\text{b})^{(II)} +$ (3.1.13c) defining

$$
(3.1.13\text{a})^{(I)} := -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}^{(I)}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda, \tag{3.1.25}
$$

$$
(3.1.13\text{a})^{(II)} := -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}^{(II)}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda, \tag{3.1.26}
$$

$$
(3.1.13\text{b})^{(I)} := -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}^{(I)}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda, \tag{3.1.27}
$$

$$
(3.1.13\text{b})^{(II)} := -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}^{(II)}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda. \tag{3.1.28}
$$

Note that the operators $(3.1.13\text{a})^{(I)}$, $(3.1.13\text{b})^{(I)}$ and (3.1.13c) are purely imaginary because $\dot{\mathscr{L}}_{0,0}^{(I)}$ is purely imaginary, $\mathscr{L}'_{0,0}$ in (3.1.14) is real and $\dot{\mathscr{L}}'_{0,0}$ in (3.1.15) is purely imaginary (argue as in Lemma 2.2.2-(*iii*)). Then, applied to the real vectors f_k^σ, $k=0,1$, $\sigma=\pm$, give purely imaginary vectors.

We first compute $(\partial_\mu\partial_\epsilon f_1^+)(0,0)$. Using (3.1.18) and (3.1.22) we get

$$
(3.1.13\text{a})^{(II)} f_1^+ = \frac{2}{2\pi\mathrm{i}}\oint_\Gamma \frac{1}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}^{(II)}(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix}\sin(2x) \\ 0\end{bmatrix}\mathrm{d}\lambda = 0
$$

because, by Lemma 3.1.4, $(\mathscr{L}_{0,0}-\lambda)^{-1}\begin{bmatrix}\sin(2x) \\ 0\end{bmatrix} \in \mathcal{W}$ and therefore it is a vector with zero average, so in the kernel of $\dot{\mathscr{L}}_{0,0}^{(II)}$. In addition $(3.1.13\text{b})^{(II)} f_1^+ = 0$ since $\dot{\mathscr{L}}_{0,0}^{(II)}(\mathscr{L}_{0,0}-\lambda)^{-1} f_1^+ = 0$. All together $\dot{P}'_{0,0} f_1^+$ is a purely imaginary vector. Since $P_{0,0}$ is a real operator, also $(\dot{P}'_{0,0} - \frac{1}{2}P_{0,0}\dot{P}'_{0,0})f_1^+$ is purely imaginary, and Lemma 2.2.8 implies that $(\partial_\mu\partial_\epsilon f_1^+)(0,0)$

has the claimed structure in (3.1.24). In the same way one proves the structure for $(\partial_\mu \partial_\epsilon f_1^-)(0,0)$.

Next we prove that $(\partial_\mu \partial_\epsilon f_0^+)(0,0)$, in addition to being purely imaginary, has zero average. We have, by (3.1.19) and (3.1.22)

$$(3.1.13\text{a})^{(I)} f_0^+ := \frac{2}{2\pi \mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \dot{\mathscr{L}}_{0,0}^{(I)} (\mathscr{L}_{0,0} - \lambda)^{-1} \frac{1}{\lambda} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \mathrm{d}\lambda$$

and since the operators $(\mathscr{L}_{0,0} - \lambda)^{-1}$ and $\dot{\mathscr{L}}_{0,0}^{(I)}$ are both Fourier multipliers, hence they preserve the absence of average of the vectors, then $(3.1.13\text{a})^{(I)} f_0^+$ has zero average. In addition $(3.1.13\text{a})^{(II)} f_0^+ = 0$ as $\dot{\mathscr{L}}_{0,0}^{(II)} (\mathscr{L}_{0,0} - \lambda)^{-1} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} = 0$. Next $(3.1.13\text{b})^{(I)} f_0^+ = 0$ since $\dot{\mathscr{L}}_{0,0}^{(I)} f_0^\pm = 0$. Using also that $\dot{\mathscr{L}}_{0,0}^{(II)} f_0^+ = 0$ and $\dot{\mathscr{L}}_{0,0}^{(II)} f_0^- = f_0^+$,

$$\begin{aligned} (3.1.13\text{b})^{(II)} f_0^+ &\overset{(3.1.19)}{=} -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \mathscr{L}_{0,0}' (\mathscr{L}_{0,0} - \lambda)^{-1} \frac{1}{\lambda^2} f_0^+ \mathrm{d}\lambda \\ &\overset{(3.1.19),(3.1.22)}{=} \frac{2}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda^3} (\mathscr{L}_{0,0} - \lambda)^{-1} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix} \mathrm{d}\lambda = 0 \end{aligned}$$

using (3.1.20) and the residue theorem. Finally, by (3.1.19) and (3.1.15) where $p_1(x) = -2\cos(x)$,

$$(3.1.13\text{c}) f_0^+ = -\frac{\mathrm{i}\, 2}{2\pi \mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \Big(-\frac{1}{\lambda} \begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} + \frac{1}{\lambda^2} \begin{bmatrix} 0 \\ \cos(x) \end{bmatrix} \Big) \mathrm{d}\lambda$$

is a vector with zero average. We conclude that $\dot{P}_{0,0}' f_0^+$ is an imaginary vector with zero average, as well as $(\partial_\mu \partial_\epsilon f_0^+)(0,0)$ since $P_{0,0}$ sends zero average functions in zero average functions. Finally, by Lemma 2.2.8, $(\partial_\mu \partial_\epsilon f_0^+)(0,0)$ has the claimed structure in (3.1.24).

We finally consider $(\partial_\mu \partial_\epsilon f_0^-)(0,0)$. By (3.1.18) and $\mathscr{L}_{0,0}' f_0^- = 0$ (cfr. (3.1.22)), it results

$$(3.1.13\text{a})^{(M)} f_0^- = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{(\mathscr{L}_{0,0} - \lambda)^{-1}}{\lambda} \dot{\mathscr{L}}_{0,0}^{(M)} (\mathscr{L}_{0,0} - \lambda)^{-1} \mathscr{L}_{0,0}' f_0^- \mathrm{d}\lambda = 0\,, \quad M = I, II\ .$$

Next by (3.1.18) and $\dot{\mathscr{L}}_{0,0}^{(I)} f_0^- = 0$ we get $(3.1.13\text{b})^{(I)} f_0^- = 0$. Then, since $\dot{\mathscr{L}}_{0,0}^{(II)} f_0^- = f_0^+$,

$$\begin{aligned} (3.1.13\text{b})^{(II)} f_0^- &\overset{(3.1.18)-(3.1.19)}{=} \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{(\mathscr{L}_{0,0} - \lambda)^{-1}}{\lambda} \mathscr{L}_{0,0}' \Big(-\frac{1}{\lambda} f_0^+ + \frac{1}{\lambda^2} f_0^- \Big) \mathrm{d}\lambda \\ &\overset{(3.1.22),(3.1.20)}{=} -\frac{2}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda^2} \frac{1}{\lambda^2 + 4} (-2 f_{-1}^+ - \lambda f_{-1}^-) \mathrm{d}\lambda = \frac{1}{2} f_{-1}^- = \frac{1}{2} \begin{bmatrix} \sin(x) \\ \cos(x) \end{bmatrix}\,, \end{aligned}$$

which is the only real term of $(\partial_\mu\partial_\epsilon f_0^-)(0,0)$ in (3.1.24). Finally by (3.1.18) and (3.1.15)

$$(3.1.13c) f_0^- = \frac{2\mathrm{i}}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\frac{1}{\lambda}\begin{bmatrix}0\\ \cos(x)\end{bmatrix}\mathrm{d}\lambda = -\frac{\mathrm{i}}{2}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix}$$

by (3.1.18), (3.1.20) and the residue theorem. In conclusion $\dot P'_{0,0} f_0^- = \frac{1}{2}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix} - \frac{\mathrm{i}}{2}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix} \in \mathcal{U}$ and, since $P_{0,0}|_{\mathcal{U}} = 0$, we find that $\left(\dot P'_{0,0} - \frac{1}{2}P_{0,0}\dot P'_{0,0}\right) f_0^- = \frac{1}{2}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix} - \frac{\mathrm{i}}{2}\begin{bmatrix}\cos(x)\\ \sin(x)\end{bmatrix}$. □

This completes the proof of Lemma 3.1.2.

3.2 Matrix representation of $\mathscr{L}_{\mu,\epsilon}$ on $\mathcal{V}_{\mu,\epsilon}$

Before representing the operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ we slightly modify the basis $\mathcal{F}$ in (3.1.1) into another symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$ with an additional property. Note that the second component of the vector $f_1^-(0,\epsilon)$ is an even function whose space average is not necessarily zero, cfr. (3.1.8). Thus we introduce the new symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$

$$\mathcal{G} := \{g_1^+(\mu,\epsilon),\ g_1^-(\mu,\epsilon),\ g_0^+(\mu,\epsilon),\ g_0^-(\mu,\epsilon)\},$$

defined by

$$\begin{aligned} g_1^+(\mu,\epsilon) &:= f_1^+(\mu,\epsilon), \qquad & g_1^-(\mu,\epsilon) &:= f_1^-(\mu,\epsilon) - n(\mu,\epsilon) f_0^-(\mu,\epsilon),\\ g_0^+(\mu,\epsilon) &:= f_0^+(\mu,\epsilon) + n(\mu,\epsilon) f_1^+(\mu,\epsilon), \qquad & g_0^-(\mu,\epsilon) &:= f_0^-(\mu,\epsilon), \end{aligned} \tag{3.2.1}$$

with

$$n(\mu,\epsilon) := \frac{\left(f_1^-(\mu,\epsilon)\,,\, f_0^-(\mu,\epsilon)\right)}{\|f_0^-(\mu,\epsilon)\|^2}. \tag{3.2.2}$$

Note that $n(\mu,\epsilon)$ is real, because, in view of (2.2.28) and Lemma 3.1.1,

$$n(\mu,\epsilon) := \frac{\overline{\left(\overline{\rho} f_1^-(\mu,\epsilon)\,,\, \overline{\rho} f_0^-(\mu,\epsilon)\right)}}{\|f_0^-(\mu,\epsilon)\|^2} = \frac{\overline{\left(f_1^-(\mu,\epsilon)\,,\, f_0^-(\mu,\epsilon)\right)}}{\|f_0^-(\mu,\epsilon)\|^2} = \overline{n(\mu,\epsilon)}. \tag{3.2.3}$$

This new basis has the property that $g_1^-(0,\epsilon)$ has zero average, see (3.2.13). We shall exploit this feature crucially in Lemma 3.2.5, see remark 3.2.6.

Lemma 3.2.1. *The basis $\mathcal{G}$ in* (3.2.1) *is symplectic and reversible, i.e. it satisfies* (2.2.21) *and* (2.2.22). *Each map $(\mu,\epsilon) \mapsto g_k^\sigma(\mu,\epsilon)$ is analytic as a map $B(\mu_0)\times B(\epsilon_0) \to H^1(\mathbb{T},\mathbb{C}^2)$.*

Proof. The vectors $g_k^{\pm}(\mu,\epsilon)$, $k=0,1$ satisfy (2.2.21) and (2.2.22) because $f_k^{\pm}(\mu,\epsilon)$, $k=0,1$ satisfy the same properties as well, and $n(\mu,\epsilon)$ is real. The analyticity of $g_k^{\sigma}(\mu,\epsilon)$ follows from the corresponding property of the basis $\mathcal{F}$. □

We now state the main result of this section.

Proposition 3.2.2. *The matrix that represents the Hamiltonian and reversible operator $\mathscr{L}_{\mu,\epsilon}:\mathcal{V}_{\mu,\epsilon}\to\mathcal{V}_{\mu,\epsilon}$ in the symplectic and reversible basis $\mathcal{G}$ of $\mathcal{V}_{\mu,\epsilon}$ defined in* (3.2.1)*, is a Hamiltonian matrix $\mathtt{L}_{\mu,\epsilon}=\mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$, where $\mathtt{B}_{\mu,\epsilon}$ is a self-adjoint and reversibility preserving (i.e. satisfying* (2.2.27)*) 4×4 matrix of the form*

$$\mathtt{B}_{\mu,\epsilon}=\begin{pmatrix} E & F \\ F^* & G\end{pmatrix}, \qquad E=E^*\,, \quad G=G^*\,, \tag{3.2.4}$$

where E,F,G are the 2×2 matrices

$$E:=\begin{pmatrix} \epsilon^2(1+r_1'(\epsilon,\mu\epsilon^2))-\frac{\mu^2}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac12\mu+r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac12\mu+r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\frac{\mu^2}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix} \tag{3.2.5}$$

$$G:=\begin{pmatrix} 1+r_8(\epsilon^3,\mu^2\epsilon,\mu\epsilon^2,\mu^3) & -\mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \\ \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon,\mu^3) & \mu+r_{10}(\mu^2\epsilon,\mu^3) \end{pmatrix} \tag{3.2.6}$$

$$F=\begin{pmatrix} r_3(\epsilon^3,\mu\epsilon^2,\mu^2\epsilon,\mu^3) & \mathrm{i}\,r_4(\mu\epsilon,\mu^3) \\ \mathrm{i}\,r_6(\mu\epsilon,\mu^3) & r_7(\mu^2\epsilon,\mu^3) \end{pmatrix}. \tag{3.2.7}$$

The rest of this section is devoted to the proof of Proposition 3.2.2. The first step is to provide the following expansion in (μ,ϵ) of the basis $\mathcal{G}$.

Lemma 3.2.3. (Expansion of the basis $\mathcal{G}$) *For small values of (μ,ϵ), the basis $\mathcal{G}$ defined in* (3.2.1) *has the following expansion*

$$g_1^+(\mu,\epsilon)=\begin{bmatrix}\cos(x)\\ \sin(x)\end{bmatrix}+\mathrm{i}\,\frac{\mu}{4}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix}+\epsilon\begin{bmatrix}2\cos(2x)\\ \sin(2x)\end{bmatrix} \tag{3.2.8}$$
$$+\mathcal{O}(\mu^2)\begin{bmatrix} even_0(x)+\mathrm{i}\,odd(x)\\ odd(x)+\mathrm{i}\,even_0(x)\end{bmatrix}+\mathcal{O}(\epsilon^2)\begin{bmatrix}even_0(x)\\ odd(x)\end{bmatrix}+\mathrm{i}\,\mu\epsilon\begin{bmatrix}odd(x)\\ even(x)\end{bmatrix}+\mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,,$$

$$g_1^-(\mu,\epsilon)=\begin{bmatrix}-\sin(x)\\ \cos(x)\end{bmatrix}+\mathrm{i}\,\frac{\mu}{4}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix}+\epsilon\begin{bmatrix}-2\sin(2x)\\ \cos(2x)\end{bmatrix} \tag{3.2.9}$$
$$+\mathcal{O}(\mu^2)\begin{bmatrix} odd(x)+\mathrm{i}\,even_0(x)\\ even_0(x)+\mathrm{i}\,odd(x)\end{bmatrix}+\mathcal{O}(\epsilon^2)\begin{bmatrix}odd(x)\\ even_0(x)\end{bmatrix}+\mathrm{i}\,\mu\epsilon\begin{bmatrix}even(x)\\ odd(x)\end{bmatrix}+\mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,,$$

$$g_0^+(\mu,\epsilon) = \begin{bmatrix}1\\0\end{bmatrix} + \epsilon\begin{bmatrix}\cos(x)\\-\sin(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\begin{bmatrix}even_0(x)\\odd(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix}odd(x)\\even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2) \tag{3.2.10}$$

$$g_0^-(\mu,\epsilon) = \begin{bmatrix}0\\1\end{bmatrix} + \mu\epsilon\left(\begin{bmatrix}\sin(x)\\\cos(x)\end{bmatrix} + \mathrm{i}\begin{bmatrix}even_0(x)\\odd(x)\end{bmatrix}\right) + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,. \tag{3.2.11}$$

In particular, at $\mu = 0$, the basis $\{g_k^\sigma(0,\epsilon), \sigma = \pm, k = 0,1\}$ is real,

$$\begin{aligned} g_1^+(0,\epsilon) &= \begin{bmatrix}even_0(x)\\odd(x)\end{bmatrix}, \qquad g_1^-(0,\epsilon) = \begin{bmatrix}odd(x)\\even_0(x)\end{bmatrix}, \\ g_0^+(0,\epsilon) &= \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}even_0(x)\\odd(x)\end{bmatrix}, \qquad g_0^-(0,\epsilon) = \begin{bmatrix}0\\1\end{bmatrix}, \end{aligned} \tag{3.2.12}$$

and, for any ϵ,

$$\int_{\mathbb{T}} g_1^-(0,\epsilon)\,\mathrm{d}x = 0\,. \tag{3.2.13}$$

Proof. First note that, by (3.1.8), $f_0^-(0,\epsilon) = \begin{bmatrix}0\\1\end{bmatrix}$, and thus $g_1^-(0,\epsilon)$ in (3.2.1) reduces to

$$g_1^-(0,\epsilon) = f_1^-(0,\epsilon) - \Big(f_1^-(0,\epsilon), \begin{bmatrix}0\\1\end{bmatrix}\Big)\begin{bmatrix}0\\1\end{bmatrix},$$

which satisfies (3.2.13), recalling also that the first component of $f_1^-(0,\epsilon)$ is odd. In order to prove (3.2.8)-(3.2.11) we note that $n(\mu,\epsilon)$ in (3.2.2) is real by (3.2.3), and satisfies, by (3.1.5), (3.1.7),

$$n(\mu,\epsilon) = \frac{1}{1 + r(\mu^2\epsilon,\mu\epsilon^2)}\Big[r(\epsilon^2) + \mu\epsilon\Big(\begin{bmatrix}-\sin(x)\\\cos(x)\end{bmatrix}, \begin{bmatrix}\sin(x)\\\cos(x)\end{bmatrix}\Big) + r(\mu^2\epsilon,\mu\epsilon^2)\Big] = r(\epsilon^2,\mu^2\epsilon,\mu\epsilon^2)\,.$$

Hence, in view of (3.1.4)-(3.1.7), the vectors $g_k^\sigma(\mu,\epsilon)$ satisfy the expansion (3.2.8)-(3.2.11). Finally at $\mu = 0$ the vectors $g_k^\pm(0,\epsilon)$, $k = 0,1$, are real being real linear combinations of real vectors. □

We start now the proof of Proposition 3.2.2. It is useful to decompose $\mathcal{B}_{\mu,\epsilon}$ in (2.2.5b) as

$$\mathcal{B}_{\mu,\epsilon} = \mathcal{B}_\epsilon + \mathcal{B}^\flat + \mathcal{B}^\sharp\,,$$

where $\mathcal{B}_\epsilon$, $\mathcal{B}^\flat$, $\mathcal{B}^\sharp$ are the self-adjoint and reversibility preserving operators

$$\mathcal{B}_\epsilon := \mathcal{B}_{0,\epsilon} := \begin{bmatrix} 1 + a_\epsilon(x) & -(1 + p_\epsilon(x))\partial_x \\ \partial_x \circ (1 + p_\epsilon(x)) & |D| \end{bmatrix}, \tag{3.2.14}$$

$$\mathcal{B}^\flat := \mu \begin{bmatrix} 0 & 0 \\ 0 & g(D) \end{bmatrix}, \qquad g(D) = \mathrm{sgn}(D) + \Pi_0 \,, \tag{3.2.15}$$

$$\mathcal{B}^\sharp := \mu \begin{bmatrix} 0 & -\mathrm{i}\, p_\epsilon \\ \mathrm{i}\, p_\epsilon & 0 \end{bmatrix}. \tag{3.2.16}$$

Note that the operators $\mathcal{B}^\flat$, $\mathcal{B}^\sharp$ are linear in μ. In order to prove (3.2.4)-(3.2.7) we exploit the representation Lemma 2.2.10 and compute perturbatively the 4×4 matrices, associated, as in (2.2.26), to the self-adjoint and reversibility preserving operators $\mathcal{B}_\epsilon$, $\mathcal{B}^\flat$ and $\mathcal{B}^\sharp$, in the basis $\mathcal{G}$.

Lemma 3.2.4. (Expansion of $\mathtt{B}_\epsilon$) *The self-adjoint and reversibility preserving matrix $\mathtt{B}_\epsilon := \mathtt{B}_\epsilon(\mu)$ associated, as in* (2.2.26)*, with the self-adjoint and reversibility preserving operator $\mathcal{B}_\epsilon$, defined in* (3.2.14)*, with respect to the basis $\mathcal{G}$ of $\mathcal{V}_{\mu,\epsilon}$ in* (3.2.1)*, expands as*

$$\mathtt{B}_\epsilon = \left(\begin{array}{cc|cc} \epsilon^2 + \frac{\mu^2}{8} + r_1(\epsilon^3, \mu\epsilon^4) & \mathrm{i}\, r_2(\mu\epsilon^3) & r_3(\epsilon^3, \mu\epsilon^2) & \mathrm{i}\, r_4(\mu\epsilon^3) \\ -\mathrm{i}\, r_2(\mu\epsilon^3) & \frac{\mu^2}{8} & \mathrm{i}\, r_6(\mu\epsilon) & 0 \\ \hline r_3(\epsilon^3, \mu\epsilon^2) & -\mathrm{i}\, r_6(\mu\epsilon) & 1 + r_8(\epsilon^3, \mu\epsilon^2) & \mathrm{i}\, r_9(\mu\epsilon^2) \\ -\mathrm{i}\, r_4(\mu\epsilon^3) & 0 & -\mathrm{i}\, r_9(\mu\epsilon^2) & 0 \end{array}\right) + \mathcal{O}(\mu^2\epsilon, \mu^3)\,. \tag{3.2.17}$$

Proof. We expand the matrix $\mathtt{B}_\epsilon(\mu)$ as

$$\mathtt{B}_\epsilon(\mu) = \mathtt{B}_\epsilon(0) + \mu(\partial_\mu \mathtt{B}_\epsilon)(0) + \frac{\mu^2}{2}(\partial_\mu^2 \mathtt{B}_0)(0) + \mathcal{O}(\mu^2\epsilon, \mu^3)\,. \tag{3.2.18}$$

The matrix $\mathtt{B}_\epsilon(0)$. The main result of this long paragraph is to prove that the matrix $\mathtt{B}_\epsilon(0)$ has the expansion (3.2.22). The matrix $\mathtt{B}_\epsilon(0)$ is real, because the operator $\mathcal{B}_\epsilon$ is real and the basis $\{g_k^\pm(0,\epsilon)\}_{k=0,1}$ is real. Consequently, by (2.2.27), its matrix elements $(\mathtt{B}_\epsilon(0))_{i,j}$ are real whenever $i+j$ is even and vanish for $i+j$ odd. In addition $g_0^-(0,\epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by (3.2.12), and, by (3.2.14), we get $\mathcal{B}_\epsilon g_0^-(0,\epsilon) = 0$, for any ϵ. We deduce that the self-adjoint matrix $\mathtt{B}_\epsilon(0)$ has the form

$$\mathtt{B}_\epsilon(0) = \Big(\mathcal{B}_\epsilon\, g_k^\sigma(0,\epsilon),\, g_{k'}^{\sigma'}(0,\epsilon)\Big)_{k,k'=0,1,\sigma,\sigma'=\pm} = \left(\begin{array}{cc|cc} \mathtt{a} & 0 & \alpha & 0 \\ 0 & \mathtt{b} & 0 & 0 \\ \hline \alpha & 0 & \mathtt{c} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{3.2.19}$$

with $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$, α real numbers depending on ϵ. We claim that $\mathtt{b} = 0$ for any ϵ. As a first step we prove that

$$\text{either } \ \mathtt{b} = 0\,, \qquad \text{or } \ \mathtt{b} \neq 0 \ \text{ and } \ \mathtt{a} = 0 = \alpha\,. \tag{3.2.20}$$

By Lemma 2.1.8 the operator $\mathscr{L}_{0,\epsilon} \equiv \mathcal{L}_{0,\epsilon}$ (cfr. (2.2.1)) satisfies $\mathscr{L}^2_{0,\epsilon} = 0$ on $\mathcal{V}_{0,\epsilon}$ being exactly the generalized Kernel of $\mathcal{L}_{0,\epsilon}$, by Lemma 2.2.1. Thus the matrix

$$\mathtt{L}_\epsilon(0) := \mathtt{J}_4 \mathtt{B}_\epsilon(0) = \left(\begin{array}{cc|cc} 0 & \mathtt{b} & 0 & 0 \\ -\mathtt{a} & 0 & -\alpha & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\alpha & 0 & -\mathtt{c} & 0 \end{array}\right), \tag{3.2.21}$$

which represents $\mathscr{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$, satisfies $\mathtt{L}^2_\epsilon(0) = 0$, namely

$$\mathtt{L}^2_\epsilon(0) = \left(\begin{array}{cc|cc} -\mathtt{ab} & 0 & -\alpha\mathtt{b} & 0 \\ 0 & -\mathtt{ab} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -\alpha\mathtt{b} & 0 & 0 \end{array}\right) = 0\,.$$

This implies (3.2.20). We now prove that the matrix $\mathtt{B}_\epsilon(0)$ defined in (3.2.19) expands as

$$\mathtt{B}_\epsilon(0) = \left(\begin{array}{cc|cc} \mathtt{a} & 0 & \alpha & 0 \\ 0 & \mathtt{b} & 0 & 0 \\ \hline \alpha & 0 & \mathtt{c} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|cc} \epsilon^2 + r(\epsilon^3) & 0 & r(\epsilon^3) & 0 \\ 0 & 0 & 0 & 0 \\ \hline r(\epsilon^3) & 0 & 1 + r(\epsilon^3) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{3.2.22}$$

We expand the operator $\mathcal{B}_\epsilon$ in (3.2.14) as

$$\mathcal{B}_\epsilon = \mathcal{B}_0 + \epsilon\mathcal{B}_1 + \epsilon^2\mathcal{B}_2 + \mathcal{O}(\epsilon^3),\ \mathcal{B}_0 := \begin{bmatrix} 1 & -\partial_x \\ \partial_x & |D| \end{bmatrix},\ \mathcal{B}_j := \begin{bmatrix} a_j(x) & -p_j(x)\partial_x \\ \partial_x \circ p_j(x) & 0 \end{bmatrix},\ j = 1, 2\,, \tag{3.2.23}$$

where the remainder term $\mathcal{O}(\epsilon^3) \in \mathcal{L}(Y, X)$ and, by (2.1.8)-(2.1.9),

$$a_1(x) = p_1(x) = -2\cos(x)\,, \quad a_2(x) = 2 - 2\cos(2x)\,, \quad p_2(x) = \frac{3}{2} - 2\cos(2x)\,. \tag{3.2.24}$$

• *Expansion of* $\mathtt{a} = \epsilon^2 + r(\epsilon^3)$. By (3.2.8) we split the real function $g_1^+(0,\epsilon)$ as

$$g_1^+(0,\epsilon) = f_1^+ + \epsilon g_{1_1}^+ + \epsilon^2 g_{1_2}^+ + \mathcal{O}(\epsilon^3),\quad f_1^+ = \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix},\ g_{1_1}^+ := \begin{bmatrix} 2\cos(2x) \\ \sin(2x) \end{bmatrix},\ g_{1_2}^+ := \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix}, \tag{3.2.25}$$

where both $g_{1_2}^+$ and $\mathcal{O}(\epsilon^3)$ are vectors in $H^1(\mathbb{T})$. Since $\mathcal{B}_0 f_1^+ = \mathcal{J}^{-1}\mathscr{L}_{0,0} f_1^+ = 0$, and both $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators, it results

$$\begin{aligned}\mathtt{a} &= \left(\mathcal{B}_\epsilon g_1^+(0,\epsilon)\,,\, g_1^+(0,\epsilon)\right)\\ &= \epsilon\left(\mathcal{B}_1 f_1^+\,,\, f_1^+\right) + \epsilon^2\left[\left(\mathcal{B}_2 f_1^+\,,\, f_1^+\right) + 2\left(\mathcal{B}_1 f_1^+\,,\, g_{1_1}^+\right) + \left(\mathcal{B}_0 g_{1_1}^+\,,\, g_{1_1}^+\right)\right] + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{3.2.26}$$

By (3.2.23) one has

$$\mathcal{B}_1 f_1^+ = \begin{bmatrix} 0 \\ 2\sin(2x)\end{bmatrix}, \quad \mathcal{B}_2 f_1^+ = \begin{bmatrix} \frac12\cos(x) \\ 3\sin(3x) - \frac12\sin(x)\end{bmatrix}, \quad \mathcal{B}_0 g_{1_1}^+ = \begin{bmatrix} 0 \\ -2\sin(2x)\end{bmatrix} = -\mathcal{B}_1 f_1^+. \tag{3.2.27}$$

Then the ϵ^2-term of $\mathtt{a}$ is $\left(\mathcal{B}_2 f_1^+\,,\, f_1^+\right) + \left(\mathcal{B}_1 f_1^+\,,\, g_{1_1}^+\right)$ and, by (3.2.26), (3.2.27), (3.2.25), a direct computation gives $\mathtt{a} = \epsilon^2 + r(\epsilon^3)$ as stated in (3.2.22).

In particular, for $\epsilon \neq 0$ sufficiently small, one has $\mathtt{a} \neq 0$ and the second alternative in (3.2.20) is ruled out, implying $\mathtt{b} = 0$.

• *Expansion of* $\mathtt{c} = 1 + r(\epsilon^3)$. By (3.2.10) we split the real-valued function $g_0^+(0,\epsilon)$ as

$$g_0^+(0,\epsilon) = f_0^+ + \epsilon g_{0_1}^+ + \epsilon^2 g_{0_2}^+ + \mathcal{O}(\epsilon^3)\,, \quad f_0^+ = \begin{bmatrix}1\\0\end{bmatrix}, \ g_{0_1}^+ := \begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix}, \ g_{0_2}^+ := \begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix}. \tag{3.2.28}$$

Since, by (2.1.15b) and (3.2.23), $\mathcal{B}_0 f_0^+ = f_0^+$, and both $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators,

$$\begin{aligned}\mathtt{c} &= \left(\mathcal{B}_\epsilon g_0^+(0,\epsilon)\,,\, g_0^+(0,\epsilon)\right)\\ &= 1 + \epsilon\left(\mathcal{B}_1 f_0^+\,,\, f_0^+\right) + \epsilon^2\left[\left(\mathcal{B}_2 f_0^+\,,\, f_0^+\right) + 2\left(\mathcal{B}_1 f_0^+\,,\, g_{0_1}^+\right) + \left(\mathcal{B}_0 g_{0_1}^+\,,\, g_{0_1}^+\right)\right] + r(\epsilon^3)\,,\end{aligned} \tag{3.2.29}$$

where we also used $\|f_0^+\| = 1$ and $(f_0^+, g_{0_1}^+) = (f_0^+, g_{0_2}^+) = 0$. By (3.2.23), (3.2.24) one has

$$\mathcal{B}_1 f_0^+ = 2\begin{bmatrix} -\cos(x) \\ \sin(x)\end{bmatrix}, \quad \mathcal{B}_2 f_0^+ = \begin{bmatrix} 2 - 2\cos(2x) \\ 4\sin(2x)\end{bmatrix}, \quad \mathcal{B}_0 g_{0_1}^+ = 2\begin{bmatrix} \cos(x) \\ -\sin(x)\end{bmatrix} = -\mathcal{B}_1 f_0^+. \tag{3.2.30}$$

Then the ϵ^2-term of $\mathtt{c}$ is $\left(\mathcal{B}_2 f_0^+\,,\, f_0^+\right) + \left(\mathcal{B}_1 f_0^+\,,\, g_{0_1}^+\right)$ and, by (3.2.28)-(3.2.30), we conclude that $\mathtt{c} = 1 + r(\epsilon^3)$ as stated in (3.2.22).

• *Expansion of* $\alpha = \mathcal{O}(\epsilon^3)$. By (3.2.25), (3.2.28) and since $\mathcal{B}_0, \mathcal{B}_1$ are self-adjoint and real we have

$$\alpha = \left(\mathcal{B}_\epsilon g_1^+(0,\epsilon)\,,\, g_0^+(0,\epsilon)\right) = \left(\mathcal{B}_0 f_1^+\,,\, f_0^+\right) + \epsilon\left[\left(\mathcal{B}_1 f_1^+\,,\, f_0^+\right) + \left(\mathcal{B}_0 f_1^+\,,\, g_{0_1}^+\right) + \left(\mathcal{B}_0 g_{1_1}^+\,,\, f_0^+\right)\right] +$$

$$\epsilon^2\big[\left(\mathcal{B}_2 f_1^+\,,\, f_0^+\right)+\left(\mathcal{B}_1 f_1^+\,,\, g_{0_1}^+\right)+\left(\mathcal{B}_1 f_0^+\,,\, g_{1_1}^+\right)+\left(\mathcal{B}_0 g_{1_2}^+\,,\, f_0^+\right)+\left(\mathcal{B}_0 g_{1_1}^+\,,\, g_{0_1}^+\right)+\left(\mathcal{B}_0 f_1^+\,,\, g_{0_2}^+\right)\big]+r(\epsilon^3)\,.$$

Recalling that $\mathcal{B}_0 f_1^+ = 0$ and $\mathcal{B}_0 f_0^+ = f_0^+$, we arrive at

$$\begin{aligned}\alpha = \epsilon\,&\Big[\left(\mathcal{B}_1 f_1^+\,,\, f_0^+\right)+\left(g_{1_1}^+\,,\, f_0^+\right)\Big]\\ &+\epsilon^2\big[\left(\mathcal{B}_2 f_1^+\,,\, f_0^+\right)+\left(\mathcal{B}_1 f_1^+\,,\, g_{0_1}^+\right)+\left(\mathcal{B}_1 f_0^+\,,\, g_{1_1}^+\right)+\left(g_{1_2}^+\,,\, f_0^+\right)+\left(\mathcal{B}_0 g_{1_1}^+\,,\, g_{0_1}^+\right)\big]+r(\epsilon^3)=r(\epsilon^3)\,,\end{aligned}$$

using that, by (3.2.25), (3.2.27), (3.2.28) (3.2.30), all the scalar products in the formula vanish.

We have proved the expansion (3.2.22).

Linear terms in μ. We now compute the terms of $\mathtt{B}_\epsilon(\mu)$ that are linear in μ. It results

$$\partial_\mu \mathtt{B}_\epsilon(0) = X + X^* \qquad \text{where} \qquad X := \big(\mathcal{B}_\epsilon g_k^\sigma(0,\epsilon), (\partial_\mu g_{k'}^{\sigma'})(0,\epsilon)\big)_{k,k'=0,1,\sigma,\sigma'=\pm}\,. \tag{3.2.31}$$

We now prove that

$$X = \left(\begin{array}{cc|cc} \mathcal{O}(\epsilon^4) & 0 & \mathcal{O}(\epsilon^2) & 0\\ \mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon) & 0\\ \hline \mathcal{O}(\epsilon^4) & 0 & \mathcal{O}(\epsilon^2) & 0\\ \mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0\end{array}\right). \tag{3.2.32}$$

The matrix $\mathtt{L}_\epsilon(0)$ in (3.2.21) where $\mathtt{b}=0$, represents the action of the operator $\mathcal{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$ in the basis $\{g_k^\sigma(0,\epsilon)\}$ and then we deduce that $\mathcal{L}_{0,\epsilon} g_1^-(0,\epsilon)=0$, $\mathcal{L}_{0,\epsilon} g_0^-(0,\epsilon)=0$. Thus also $\mathcal{B}_\epsilon g_1^-(0,\epsilon)=0$, $\mathcal{B}_\epsilon g_0^-(0,\epsilon)=0$, for every ϵ, and the second and the fourth column of the matrix X in (3.2.32) are zero. In order to compute the other two columns we use the expansion of the derivatives, where denoting with a dot the derivative w.r.t. μ,

$$\dot g_1^+(0,\epsilon) = \frac{\mathrm{i}}{4}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix} + \mathrm{i}\,\epsilon\begin{bmatrix} odd(x)\\ even(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\,, \qquad \dot g_0^+(0,\epsilon) = \mathrm{i}\,\epsilon\begin{bmatrix} odd(x)\\ even_0(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.2.33}$$

$$\dot g_1^-(0,\epsilon) = \frac{\mathrm{i}}{4}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix} + \mathrm{i}\,\epsilon\begin{bmatrix} even(x)\\ odd(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\,, \quad \dot g_0^-(0,\epsilon) = \epsilon\Big(\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix} + \mathrm{i}\begin{bmatrix} even_0(x)\\ odd(x)\end{bmatrix}\Big) + \mathcal{O}(\epsilon^2)$$

that follow by (3.2.8)-(3.2.11). In view of (3.1.3), (3.2.8)-(3.2.11), (3.2.21) and since $\mathcal{B}_\epsilon g_k^\sigma(0,\epsilon) = -\mathcal{J}\mathcal{L}_\epsilon g_k^\sigma(0,\epsilon)$, we have

$$\mathcal{B}_\epsilon g_1^+(0,\epsilon) = (\epsilon^2 + r(\epsilon^3))\,\mathcal{J} g_1^-(0,\epsilon) + r(\epsilon^3)\,\mathcal{J} f_0^- = \epsilon^2\begin{bmatrix}\cos(x)\\ \sin(x)\end{bmatrix} + r(\epsilon^3)\Big(\begin{bmatrix}1\\ 0\end{bmatrix} + \begin{bmatrix} even_0(x)\\ odd(x)\end{bmatrix}\Big)\,,$$

$$\mathcal{B}_\epsilon g_0^+(0,\epsilon) = r(\epsilon^3)\mathcal{J} g_1^-(0,\epsilon) + (1+r(\epsilon^3))\mathcal{J} f_0^- = \begin{bmatrix}1\\0\end{bmatrix} + r(\epsilon^3)\Big(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}even_0(x)\\ odd(x)\end{bmatrix}\Big) . \quad (3.2.34)$$

The other two columns of the matrix X in (3.2.31) have the expansion (3.2.32), by (3.2.33) and (3.2.34).

Quadratic terms in μ. By denoting with a double dot the double derivative w.r.t. μ, we have

$$\partial_\mu^2 \mathtt{B}_0(0) = \Big(\mathcal{B}_0 f_k^\sigma\,,\, \ddot g_{k'}^{\sigma'}(0,0)\Big) + \Big(\ddot g_k^\sigma(0,0)\,,\, \mathcal{B}_0 f_k^{\sigma'}\Big) + 2\,\Big(\mathcal{B}_0 \dot g_k^\sigma(0,0)\,,\, \dot g_{k'}^{\sigma'}(0,0)\Big) =: Y + Y^* + 2Z\,. \quad (3.2.35)$$

We claim that $Y=0$. Indeed, its first, second and fourth column are zero, since $\mathcal{B}_0 f_k^\sigma = 0$ for $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$. The third column is also zero by noting that $\mathcal{B}_0 f_0^+ = f_0^+$ and

$$\ddot g_1^+(0,0) = \begin{bmatrix}even_0(x) + \mathrm{i}\, odd(x)\\ odd(x) + \mathrm{i}\, even_0(x)\end{bmatrix},\ \ \ddot g_1^-(0,0) = \begin{bmatrix}odd(x) + \mathrm{i}\, even_0(x)\\ even_0(x) + \mathrm{i}\, odd(x)\end{bmatrix},\ \ \ddot g_0^+(0,0) = \ddot g_0^-(0,0) = 0\,.$$

We claim that

$$Z = \Big(\mathcal{B}_0 \dot g_k^\sigma(0,0)\,,\, \dot g_{k'}^{\sigma'}(0,0)\Big)_{\substack{k,k'=0,1,\\ \sigma,\sigma'=\pm}} = \left(\begin{array}{cc|cc} \frac18 & 0 & 0 & 0\\ 0 & \frac18 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right) . \quad (3.2.36)$$

Indeed, by (3.2.33), we have $\dot g_0^+(0,0) = \dot g_0^-(0,0) = 0$. Therefore the last two columns of Z, and by self-adjointness the last two rows, are zero. By (3.2.33), $\dot g_1^+(0,0) = \frac{\mathrm{i}}{4}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix}$ and $\dot g_1^-(0,0) = \frac{\mathrm{i}}{4}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix}$, so that $\mathcal{B}_0 \dot g_1^+(0,0) = \frac{\mathrm{i}}{2}\begin{bmatrix}\sin(x)\\ \cos(x)\end{bmatrix}$ and $\mathcal{B}_0 \dot g_1^-(0,0) = \frac{\mathrm{i}}{2}\begin{bmatrix}\cos(x)\\ -\sin(x)\end{bmatrix}$, and we obtain the matrix (3.2.36) computing the scalar products.

In conclusion (3.2.18), (3.2.31), (3.2.32), (3.2.35), the fact that $Y=0$ and (3.2.36) imply (3.2.17), using also the selfadjointness of $\mathtt{B}_\epsilon$ and (2.2.27). $\square$

We now consider $\mathcal{B}^\flat$.

Lemma 3.2.5. (Expansion of $\mathtt{B}^\flat$) *The self-adjoint and reversibility-preserving matrix* $\mathtt{B}^\flat$ *associated, as in* (2.2.26), *to the self-adjoint and reversibility-preserving operator* $\mathcal{B}^\flat$, *defined in* (3.2.15)*, with respect to the basis* $\mathcal{G}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *in* (3.2.1)*, admits the expansion*

$$\mathtt{B}^\flat = \left(\begin{array}{cc|cc} -\frac{\mu^2}{4} & \mathrm{i}\,(\frac{\mu}{2}+r_2(\mu\epsilon^2)) & 0 & 0 \\ -\mathrm{i}\,(\frac{\mu}{2}+r_2(\mu\epsilon^2)) & -\frac{\mu^2}{4} & \mathrm{i}\,r_6(\mu\epsilon) & 0 \\ \hline 0 & -\mathrm{i}\,r_6(\mu\epsilon) & 0 & 0 \\ 0 & 0 & 0 & \mu \end{array}\right) + \mathcal{O}(\mu^2\epsilon,\mu^3)\,. \tag{3.2.37}$$

Proof. We have to compute the expansion of the matrix entries $(\mathcal{B}^\flat g_k^\sigma(\mu,\epsilon), g_{k'}^{\sigma'}(\mu,\epsilon))$. The operator $\mathcal{B}^\flat$ in (3.2.15) is linear in μ and by (3.2.8), (3.2.9), (3.2.13) and the identities $\mathrm{sgn}(D)\sin(kx) = -\mathrm{i}\,\cos(kx)$ and $\mathrm{sgn}(D)\cos(kx) = \mathrm{i}\,\sin(kx)$ for any $k\in\mathbb{N}$, we have

$$\mathcal{B}^\flat g_1^+(\mu,\epsilon) = -\mathrm{i}\,\mu\begin{bmatrix}0\\ \cos(x)\end{bmatrix} - \frac{\mu^2}{4}\begin{bmatrix}0\\ \sin(x)\end{bmatrix} - \mathrm{i}\,\mu\epsilon\begin{bmatrix}0\\ \cos(2x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)\,,$$

$$\mathcal{B}^\flat g_1^-(\mu,\epsilon) = \mathrm{i}\,\mu\begin{bmatrix}0\\ \sin(x)\end{bmatrix} - \frac{\mu^2}{4}\begin{bmatrix}0\\ \cos(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix}0\\ \sin(2x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ odd(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)\,.$$

Note that $\mu\begin{bmatrix}0 & 0\\ 0 & \Pi_0\end{bmatrix} g_1^-(\mu,\epsilon) = \mathcal{O}(\mu^3\epsilon,\mu^2\epsilon^2)$ thanks to the property (3.2.13) of the basis $\mathcal{G}$. In addition, by (3.2.10)-(3.2.11), we get that

$$\mathcal{B}^\flat g_0^+(\mu,\epsilon) = \mathrm{i}\,\mu\epsilon\begin{bmatrix}0\\ \cos(x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon)\,,\quad \mathcal{B}^\flat g_0^-(\mu,\epsilon) = \begin{bmatrix}0\\ \mu\end{bmatrix} + \mathcal{O}(\mu^2\epsilon)\,.$$

Taking the scalar products of the above expansions of $\mathcal{B}^\flat g_k^\sigma(\mu,\epsilon)$ with the functions $g_{k'}^{\sigma'}(\mu,\epsilon)$ expanded as in (3.2.8)-(3.2.11) we deduce (3.2.37). □

Remark 3.2.6. The $(2,2)$ entry in the matrix $\mathtt{B}^\flat$ in (3.2.37) has no terms $\mathcal{O}(\mu\epsilon^k)$, thanks to property (3.2.13). This property is fundamental in order to verify that the $(2,2)$ entry of the matrix E in (3.2.5) starts with $-\frac{\mu^2}{8}$ and therefore it is negative for μ small. Such property does not hold for the first basis $\mathcal{F}$ defined in (3.1.1), and this motivates the use of the second basis $\mathcal{G}$.

Finally we consider $\mathcal{B}^\sharp$.

Lemma 3.2.7. (Expansion of $\mathtt{B}^\sharp$) *The self-adjoint and reversibility-preserving matrix* $\mathtt{B}^\sharp$ *associated, as in* (2.2.26)*, to the self-adjoint and reversibility-preserving operators* $\mathcal{B}^\sharp$*,* *defined in* (3.2.16)*, with respect to the basis* $\mathcal{G}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *in* (3.2.1)*, admits the expansion*

$$\mathtt{B}^\sharp = \left(\begin{array}{cc|cc} 0 & \mathrm{i}\, r_2(\mu\epsilon^2) & 0 & \mathrm{i}\, r_4(\mu\epsilon) \\ -\mathrm{i}\, r_2(\mu\epsilon^2) & 0 & -\mathrm{i}\, r_6(\mu\epsilon) & 0 \\ \hline 0 & \mathrm{i}\, r_6(\mu\epsilon) & 0 & -\mathrm{i}\, r_9(\mu\epsilon^2) \\ -\mathrm{i}\, r_4(\mu\epsilon) & 0 & \mathrm{i}\, r_9(\mu\epsilon^2) & 0 \end{array}\right) + \mathcal{O}(\mu^2\epsilon)\,. \tag{3.2.38}$$

Proof. Since $\mathcal{B}^\sharp = -\mathrm{i}\,\mu p_\epsilon \mathcal{J}$ and $p_\epsilon = \mathcal{O}(\epsilon)$ by (2.1.8)-(2.1.9), we have the expansion

$$\big(\mathcal{B}^\sharp g_k^\sigma(\mu,\epsilon), g_{k'}^{\sigma'}(\mu,\epsilon)\big) = \big(\mathcal{B}^\sharp g_k^\sigma(0,\epsilon), g_{k'}^{\sigma'}(0,\epsilon)\big) + \mathcal{O}(\mu^2\epsilon)\,. \tag{3.2.39}$$

We claim that the matrix entries $(\mathcal{B}^\sharp g_k^\sigma(0,\epsilon), g_{k'}^\sigma(0,\epsilon))$, $k,k'=0,1$ are zero. Indeed they are real by (2.2.27), and also purely imaginary, since the operator $\mathcal{B}^\sharp$ is purely imaginary[1] and the basis $\{g_k^\pm(0,\epsilon)\}_{k=0,1}$ is real. Hence $\mathtt{B}^\sharp$ has the form

$$\mathtt{B}^\sharp = \left(\begin{array}{cc|cc} 0 & \mathrm{i}\,\beta & 0 & \mathrm{i}\,\delta \\ -\mathrm{i}\,\beta & 0 & -\mathrm{i}\,\gamma & 0 \\ \hline 0 & \mathrm{i}\,\gamma & 0 & \mathrm{i}\,\eta \\ -\mathrm{i}\,\delta & 0 & -\mathrm{i}\,\eta & 0 \end{array}\right) + \mathcal{O}(\mu^2\epsilon) \quad \text{where} \quad \begin{cases} \Big(\mathcal{B}^\sharp g_1^-(0,\epsilon)\,,\, g_1^+(0,\epsilon)\Big) =: \mathrm{i}\,\beta\,, \\ \Big(\mathcal{B}^\sharp g_1^-(0,\epsilon)\,,\, g_0^+(0,\epsilon)\Big) =: \mathrm{i}\,\gamma\,, \\ \Big(\mathcal{B}^\sharp g_0^-(0,\epsilon)\,,\, g_1^+(0,\epsilon)\Big) =: \mathrm{i}\,\delta\,, \\ \Big(\mathcal{B}^\sharp g_0^-(0,\epsilon)\,,\, g_0^+(0,\epsilon)\Big) =: \mathrm{i}\,\eta\,, \end{cases} \tag{3.2.40}$$

and α, β, γ, δ are real numbers. As $\mathcal{B}^\sharp = \mathcal{O}(\mu\epsilon)$ in $\mathcal{L}(Y)$, we get immediately that $\gamma = r(\mu\epsilon)$ and $\delta = r(\mu\epsilon)$. Next we compute the expansion of β and η. We split the operator $\mathcal{B}^\sharp$ in (3.2.16) as

$$\mathcal{B}^\sharp = \mathrm{i}\,\mu\epsilon\mathcal{B}_1^\sharp + \mathcal{O}(\mu\epsilon^2)\,, \qquad \mathcal{B}_1^\sharp := -p_1(x)\mathcal{J}\,, \tag{3.2.41}$$

with $p_1(x)$ in (3.2.24) and $\mathcal{O}(\mu\epsilon^2) \in \mathcal{L}(Y)$. By (3.2.41) and the expansion (3.2.8)-(3.2.11), $g_1^+(0,\epsilon) = f_1^+ + \mathcal{O}(\epsilon)$, $g_1^-(0,\epsilon) = f_1^- + \mathcal{O}(\epsilon)$, $g_0^+(0,\epsilon) = f_0^+ + \mathcal{O}(\epsilon)$, $g_0^-(0,\epsilon) = \begin{bmatrix}0\\1\end{bmatrix}$ we obtain

$$\beta = \mu\epsilon\left(\mathcal{B}_1^\sharp f_1^-\,,\, f_1^+\right) + r(\mu\epsilon^2)\,, \quad \eta = \mu\epsilon\left(\mathcal{B}_1^\sharp f_0^-\,,\, f_0^+\right) + r(\mu\epsilon^2)\,.$$

Computing $\mathcal{B}_1^\sharp f_1^- = \begin{bmatrix}1+\cos(2x)\\ \sin(2x)\end{bmatrix}$, $\mathcal{B}_1^\sharp f_0^- = \begin{bmatrix}2\cos(x)\\ 0\end{bmatrix}$ and the various scalar products with the vectors f_k^σ in (3.1.2), we get $\beta = r(\mu\epsilon^2)$, $\eta = r(\mu\epsilon^2)$. Using also (3.2.39) and (3.2.40), one gets (3.2.38). □

Lemmata 3.2.4, 3.2.5 and 3.2.7 imply Proposition 3.2.2.

[1] An operator $\mathcal{A}$ is *purely imaginary* if $\overline{\mathcal{A}} = -\mathcal{A}$. A purely imaginary operator sends real functions into purely imaginary ones.

3.3 Block decoupling

The 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}$ obtained in Proposition 3.2.2, has the form

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2 E & \mathtt{J}_2 F \\ \mathtt{J}_2 F^* & \mathtt{J}_2 G \end{pmatrix}, \tag{3.3.1}$$

where E, G, F are the 2×2 matrices in (3.2.5)-(3.2.7). In particular $\mathtt{J}_2 E$ has the form

$$\mathtt{J}_2 E = \begin{pmatrix} -\mathrm{i}\left(\frac{\mu}{2} + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \\ -\epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) + \frac{\mu^2}{8}(1 + r_1''(\epsilon, \mu)) & -\mathrm{i}\left(\frac{\mu}{2} + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \end{pmatrix} \tag{3.3.2}$$

and therefore possesses two eigenvalues with non-zero real part ("Benjamin-Feir" eigenvalues), as long as its two off-diagonal entries have the same sign, see the discussion below (1.4.7). In order to prove that also the full 4×4 matrix $\mathtt{L}_{\mu,\epsilon}$ in (3.3.1) possesses Benjamin-Feir unstable eigenvalues, we aim to eliminate the coupling term $\mathtt{J}_2 F$ by a change of variables. More precisely in this section we conjugate the matrix $\mathtt{L}_{\mu,\epsilon}$ in (3.3.1) to the Hamiltonian and reversible *block-diagonal* matrix $\mathtt{L}^{(3)}_{\mu,\epsilon}$ in (3.3.35),

$$\mathtt{L}^{(3)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix},$$

where $\mathtt{J}_2 E^{(3)}$ is a 2×2 matrix with the same form as (3.3.2) (clearly with different remainders, but of the same order). The spectrum of the 4×4 matrix $\mathtt{L}^{(3)}_{\mu,\epsilon}$, which coincides with that of $\mathtt{L}_{\mu,\epsilon}$, contains the Benjamin-Feir unstable eigenvalues of the 2×2 matrix $\mathtt{J}_2 E^{(3)}$ (it turns out that the two eigenvalues of $\mathtt{J}_2 G^{(3)}$ are purely imaginary). This will prove Theorem 1.4.1.

The block-diagonalization of $\mathtt{L}_{\mu,\epsilon}$ is achieved in three steps, in Lemma 3.3.1, Lemma 3.3.2, and finally Lemma 3.3.8. Motivations and goals of each step were described at the end of Section 1.4.

3.3.1 First step of block decoupling

We write the matrices E, F, G in (3.2.4) as

$$E = \begin{pmatrix} E_{11} & \mathrm{i}\, E_{12} \\ -\mathrm{i}\, E_{12} & E_{22} \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & \mathrm{i}\, F_{12} \\ \mathrm{i}\, F_{21} & F_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & \mathrm{i}\, G_{12} \\ -\mathrm{i}\, G_{12} & G_{22} \end{pmatrix} \tag{3.3.3}$$

where the real numbers E_{ij}, F_{ij}, G_{ij}, $i, j = 1, 2$, have the expansion given in (3.2.5)-(3.2.7).

Lemma 3.3.1. *Conjugating the Hamiltonian and reversible matrix* $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}$ *obtained in Proposition 3.2.2 through the symplectic and reversibility-preserving* 4×4*-matrix*

$$Y = \mathrm{Id}_4 + m \begin{pmatrix} 0 & -P \\ Q & 0 \end{pmatrix} \quad \text{with} \quad Q := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ P := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ m := m(\mu,\epsilon) := -\frac{F_{11}(\mu,\epsilon)}{G_{11}(\mu,\epsilon)}, \tag{3.3.4}$$

where $m = r(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3)$ *is a real number, we obtain the Hamiltonian and reversible matrix*

$$\mathtt{L}^{(1)}_{\mu,\epsilon} := Y^{-1} \mathtt{L}_{\mu,\epsilon} Y = \mathtt{J}_4 \mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(1)} & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & \mathtt{J}_2 G^{(1)} \end{pmatrix} \tag{3.3.5}$$

where $\mathtt{B}^{(1)}_{\mu,\epsilon}$ *is a self-adjoint and reversibility-preserving* 4×4 *matrix*

$$\mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix}, \quad E^{(1)} = [E^{(1)}]^*, \ G^{(1)} = [G^{(1)}]^*, \tag{3.3.6}$$

where the 2×2 *matrices* $E^{(1)}$, $G^{(1)}$ *have the same expansion* (3.2.5)-(3.2.6) *of* E, G *and*

$$F^{(1)} = \begin{pmatrix} 0 & \mathrm{i}\, r_4(\mu\epsilon, \mu^3) \\ \mathrm{i}\, r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \end{pmatrix}. \tag{3.3.7}$$

Note that the entry $F^{(1)}_{11}$ *is* 0*, the other entries of* $F^{(1)}$ *have the same size as for* F *in* (3.2.7)*.*

Proof. The matrix Y is symplectic, i.e. (2.2.30) holds, and since m is real, it is reversibility preserving, i.e. satisfies (2.2.27). By (2.2.31),

$$\mathtt{B}^{(1)}_{\mu,\epsilon} = Y^* \mathtt{B}_{\mu,\epsilon} Y = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix}, \tag{3.3.8}$$

where, by (3.3.4) and (3.3.3), the self-adjoint matrices $E^{(1)}, G^{(1)}$ are

$$\begin{aligned} E^{(1)} &:= E + m(QF^* + FQ) + m^2 QGQ = E + \begin{pmatrix} 2mF_{11} + m^2 G_{11} & -\mathrm{i}\, m F_{21} \\ \mathrm{i}\, m F_{21} & 0 \end{pmatrix}, \\ G^{(1)} &:= G - m(PF + F^*P) + m^2 PEP = G + \begin{pmatrix} 0 & \mathrm{i}\, m F_{21} \\ -\mathrm{i}\, m F_{21} & -2mF_{22} + m^2 E_{22} \end{pmatrix}. \end{aligned} \tag{3.3.9}$$

Similarly, the off-diagonal 2×2 matrix $F^{(1)}$ is

$$F^{(1)} := F + m(QG - EP) - m^2 QF^*P = \begin{pmatrix} 0 & \mathrm{i}\,(F_{12} + mG_{12} - mE_{12} + m^2F_{21}) \\ \mathrm{i}\,F_{21} & F_{22} - mE_{22} \end{pmatrix}, \tag{3.3.10}$$

where we have used that the first entry of this matrix is $F_{11} + mG_{11} = 0$, by the definition of m in (3.3.4). By (3.3.8)-(3.3.10) and (3.2.5)-(3.2.7) we deduce the expansion of $\mathtt{B}^{(1)}_{\mu,\epsilon}$ in (3.3.7), (3.3.6) and consequently that of (3.3.5). □

3.3.2 Second step of block decoupling

We now perform a further step of block decoupling, obtaining the new Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$ in (3.3.13) where the 2×2 matrix $\mathtt{J}_2 E^{(2)}$ has still the Benjamin-Feir unstable eigenvalues and the size of the new coupling matrix $\mathtt{J}_2 F^{(2)}$ is much smaller than $\mathtt{J}_2 F^{(1)}$. In particular note that the entries of $F^{(2)}$ in (3.3.14) have size $\mathcal{O}(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7)$ whereas those of $F^{(1)}$ in (3.3.7) are $\mathcal{O}(\mu\epsilon^3, \mu^3)$.

Lemma 3.3.2. (Step of block decoupling) *There exists a 2×2 reversibility-preserving matrix X, analytic in (μ,ϵ), of the form*

$$X = \begin{pmatrix} x_{11} & \mathrm{i}\,x_{12} \\ \mathrm{i}\,x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} r_{11}(\mu^2, \mu\epsilon) & \mathrm{i}\,r_{12}(\mu^3, \mu\epsilon) \\ \mathrm{i}\,r_{21}(\epsilon, \mu^2) & r_{22}(\mu^3, \mu\epsilon) \end{pmatrix}, \quad x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{R}, \tag{3.3.11}$$

such that, by conjugating the Hamiltonian and reversible matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$, defined in (3.3.5), with the symplectic and reversibility-preserving 4×4 matrix

$$\exp\left(S^{(1)}\right), \quad \textit{where} \quad S^{(1)} := \mathtt{J}_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}, \quad \Sigma := \mathtt{J}_2 X, \tag{3.3.12}$$

we get the Hamiltonian and reversible matrix

$$\mathtt{L}^{(2)}_{\mu,\epsilon} := \exp\left(S^{(1)}\right)\mathtt{L}^{(1)}_{\mu,\epsilon}\exp\left(-S^{(1)}\right) = \mathtt{J}_4\mathtt{B}^{(2)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2E^{(2)} & \mathtt{J}_2F^{(2)} \\ \mathtt{J}_2[F^{(2)}]^* & \mathtt{J}_2G^{(2)} \end{pmatrix}, \tag{3.3.13}$$

where the 2×2 self-adjoint and reversibility-preserving matrices $E^{(2)}$, $G^{(2)}$ have the same expansion of $E^{(1)}$, $G^{(1)}$, namely of E, G, given in (3.2.5)-(3.2.6), and

$$F^{(2)} = \begin{pmatrix} F^{(2)}_{11} & \mathrm{i}\,F^{(2)}_{12} \\ \mathrm{i}\,F^{(2)}_{21} & F^{(2)}_{22} \end{pmatrix} = \begin{pmatrix} r_3(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7) & \mathrm{i}\,r_4(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) \\ \mathrm{i}\,r_6(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) & r_7(\mu^3\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \end{pmatrix}. \tag{3.3.14}$$

Remark 3.3.3. The new matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$ in (3.3.13) is still analytic in (μ,ϵ), as was $\mathtt{L}^{(1)}_{\mu,\epsilon}$. This is not obvious a priori, since the spectrum of the matrices $\mathtt{J}_2E^{(1)}$ and $\mathtt{J}_2G^{(1)}$ is shrinking to zero as $(\mu,\epsilon)\to 0$. As we shall see in Lemma 3.3.5, analyticity is proved by a careful computation which ensures that no singularities are introduced in the new matrix during the block decoupling. This would not have been possible if we had not performed the first step of block decoupling in Section 3.3.1.

The rest of the section is devoted to the proof of Lemma 3.3.2. We denote for simplicity $S = S^{(1)}$.

The matrix $\exp(S)$ is symplectic and reversibility preserving because the matrix S in (3.3.12) is Hamiltonian and reversibility preserving, cfr. Lemma 2.2.13. Note that S is reversibility preserving since X has the form (3.3.11).

We now expand in Lie series the Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S)\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S)$.

We split $\mathtt{L}^{(1)}_{\mu,\epsilon}$ into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}^{(1)}_{\mu,\epsilon} = D^{(1)} + R^{(1)}\,,$$
$$D^{(1)} := \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2E^{(1)} & 0 \\ 0 & \mathtt{J}_2G^{(1)} \end{pmatrix}, \quad R^{(1)} := \begin{pmatrix} 0 & \mathtt{J}_2F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & 0 \end{pmatrix}. \tag{3.3.15}$$

In order to construct a transformation which eliminates the main part of the off-diagonal part $R^{(1)}$, we conjugate $\mathtt{L}^{(1)}_{\mu,\epsilon}$ by a symplectic matrix $\exp(S)$ generated as the flow of a Hamiltonian matrix S with the same form of $R^{(1)}$. By a Lie expansion we obtain

$$\begin{aligned}
&\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S)\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S) \\
&= D^{(1)} + \Big[S\,,\,D^{(1)}\Big] + \frac{1}{2}[S,[S,D^{(1)}]] + R^{(1)} + [S,R^{(1)}] \qquad (3.3.16)\\
&+ \frac{1}{2}\int_0^1 (1-\tau)^2\exp(\tau S)\mathrm{ad}_S^3(D^{(1)})\exp(-\tau S)\,\mathrm{d}\tau + \int_0^1 (1-\tau)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\,\mathrm{d}\tau
\end{aligned}$$

where $\mathrm{ad}_A(B) := [A,B] := AB - BA$ denotes the commutator between linear operators A, B.

We look for a 4×4 matrix S as in (3.3.12) which solves the homological equation

$$R^{(1)} + \Big[S\,,\,D^{(1)}\Big] = 0$$

which, recalling (3.3.15), amounts to eliminate the off-diagonal part

$$\begin{pmatrix} 0 & \mathtt{J}_2F^{(1)} + \mathtt{J}_2\Sigma D_0 - D_1\mathtt{J}_2\Sigma \\ \mathtt{J}_2[F^{(1)}]^* + \mathtt{J}_2\Sigma^* D_1 - D_0\mathtt{J}_2\Sigma^* & 0 \end{pmatrix} = 0\,. \tag{3.3.17}$$

Note that the equation $\mathtt{J}_2 F^{(1)} + \mathtt{J}_2\Sigma D_0 - D_1 \mathtt{J}_2 \Sigma = 0$ implies also $\mathtt{J}_2 [F^{(1)}]^* + \mathtt{J}_2\Sigma^* D_1 - D_0 \mathtt{J}_2 \Sigma^* = 0$ and viceversa. Thus, writing $\Sigma = \mathtt{J}_2 X$, namely $X = -\mathtt{J}_2\Sigma$, the equation (3.3.17) is equivalent to solve the "Sylvester" equation

$$D_1 X - X D_0 = -\mathtt{J}_2 F^{(1)} \,. \tag{3.3.18}$$

Recalling (3.3.15), (3.3.11) and (3.3.3), it amounts to solve the 4×4 real linear system

$$\underbrace{\begin{pmatrix} G_{12}^{(1)} - E_{12}^{(1)} & G_{11}^{(1)} & E_{22}^{(1)} & 0 \\ G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} & 0 & -E_{22}^{(1)} \\ E_{11}^{(1)} & 0 & G_{12}^{(1)} - E_{12}^{(1)} & -G_{11}^{(1)} \\ 0 & -E_{11}^{(1)} & -G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} \end{pmatrix}}_{=:\mathcal{A}} \underbrace{\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}}_{=:\vec{x}} = \underbrace{\begin{pmatrix} -F_{21} \\ F_{22} \\ -F_{11} \\ F_{12} \end{pmatrix}}_{=:\vec{f}} . \tag{3.3.19}$$

Recall that, by (3.3.7), $F_{11} = 0$.

We solve this system using the following result, verified by a direct calculus.

Lemma 3.3.4. *The determinant of the matrix*

$$A := \begin{pmatrix} a & b & c & 0 \\ d & a & 0 & -c \\ e & 0 & a & -b \\ 0 & -e & -d & a \end{pmatrix} \tag{3.3.20}$$

where a, b, c, d, e are real numbers, is

$$\det A = a^4 - 2a^2(bd + ce) + (bd - ce)^2 \,. \tag{3.3.21}$$

If $\det A \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a\left(a^2 - bd - ce\right) & b\left(-a^2 + bd - ce\right) & -c\left(a^2 + bd - ce\right) & -2abc \\ d\left(-a^2 + bd - ce\right) & a\left(a^2 - bd - ce\right) & 2acd & -c\left(-a^2 - bd + ce\right) \\ -e\left(a^2 + bd - ce\right) & 2abe & a\left(a^2 - bd - ce\right) & b\left(a^2 - bd + ce\right) \\ -2ade & -e\left(-a^2 - bd + ce\right) & d\left(a^2 - bd + ce\right) & a\left(a^2 - bd - ce\right) \end{pmatrix} . \tag{3.3.22}$$

As the Sylvester matrix $\mathcal{A}$ in (3.3.19) has the form (3.3.20) with (cfr. (3.2.5), (3.2.6))

$$\begin{gathered} a = G_{12}^{(1)} - E_{12}^{(1)} = -\frac{\mu}{2}(1 + r(\epsilon^2, \mu\epsilon, \mu^2)) \,, \quad b = G_{11}^{(1)} = 1 + r(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3) \,, \\ c = E_{22}^{(1)} = -\frac{\mu^2}{8}(1 + r(\epsilon, \mu)) \,, \quad d = G_{22}^{(1)} = \mu(1 + r(\mu\epsilon, \mu^2)) \,, \quad e = E_{11}^{(1)} = r(\epsilon^2, \mu^2) \,, \end{gathered} \tag{3.3.23}$$

we use (3.3.21) to compute

$$\det \mathcal{A} = \mu^2(1 + r(\mu, \epsilon^3)) . \tag{3.3.24}$$

Moreover, by (3.3.22), we have

$$\mathcal{A}^{-1} = \frac{1}{\mu} \begin{pmatrix} \frac{\mu}{2}(1+r(\epsilon,\mu)) & 1+r(\epsilon,\mu) & \frac{\mu^2}{8}(1+r(\epsilon,\mu)) & -\frac{\mu^2}{8}(1+r(\epsilon,\mu)) \\ \mu(1+r(\epsilon,\mu)) & \frac{\mu}{2}(1+r(\epsilon,\mu)) & \frac{\mu^3}{8}(1+r(\epsilon,\mu)) & -\frac{\mu^2}{8}(1+r(\epsilon,\mu)) \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & \frac{\mu}{2}(1+r(\epsilon,\mu)) & -1+r(\epsilon,\mu) \\ \mu r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & -\mu(1+r(\epsilon,\mu)) & \frac{\mu}{2}(1+r(\epsilon,\mu)) \end{pmatrix} . \tag{3.3.25}$$

Therefore, for any $\mu \neq 0$, there exists a unique solution $\vec{x} = \mathcal{A}^{-1}\vec{f}$ of the linear system (3.3.19), namely a unique matrix X which solves the Sylvester equation (3.3.18).

Lemma 3.3.5. *The matrix solution X of the Sylvester equation* (3.3.18) *is analytic in (μ, ϵ) and admits an expansion as in* (3.3.11).

Proof. The expansion (3.3.11) of the coefficients $x_{ij} = [\mathcal{A}^{-1}\vec{f}]_{ij}$ follows, for any $\mu \neq 0$ small, by (3.3.25) and the expansions of F_{ij} in (3.3.7). In particular each x_{ij} admits an analytic extension at $\mu = 0$ and the resulting matrix X still solves (3.3.18) at $\mu = 0$ (note that, for $\mu = 0$, one has $F^{(1)} = 0$ and the Sylvester equation does not have a unique solution). □

Since the matrix S solves the homological equation $\left[S\, ,\, D^{(1)}\right] + R^{(1)} = 0$ we deduce by (3.3.16) that

$$\mathtt{L}^{(2)}_{\mu,\epsilon} = D^{(1)} + \frac{1}{2}\left[S\, ,\, R^{(1)}\right] + \frac{1}{2}\int_0^1 (1-\tau^2)\, \exp(\tau S)\, \mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S) \mathrm{d}\tau . \tag{3.3.26}$$

The matrix $\frac{1}{2}\left[S\, ,\, R^{(1)}\right]$ is, by (3.3.12), (3.3.15), the block-diagonal Hamiltonian and reversible matrix

$$\frac{1}{2}\left[S\, ,\, R^{(1)}\right] = \begin{pmatrix} \frac{1}{2}\mathtt{J}_2(\Sigma \mathtt{J}_2 [F^{(1)}]^* - F^{(1)}\mathtt{J}_2\Sigma^*) & 0 \\ 0 & \frac{1}{2}\mathtt{J}_2(\Sigma^* \mathtt{J}_2 F^{(1)} - [F^{(1)}]^*\mathtt{J}_2\Sigma) \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2\tilde{E} & 0 \\ 0 & \mathtt{J}_2\tilde{G} \end{pmatrix} , \tag{3.3.27}$$

where, since $\Sigma = \mathtt{J}_2 X$,

$$\tilde{E} := \mathrm{Sym}\big(\mathtt{J}_2 X \mathtt{J}_2 [F^{(1)}]^*\big) , \qquad \tilde{G} := \mathrm{Sym}\big(X^* F^{(1)}\big) , \tag{3.3.28}$$

denoting $\mathrm{Sym}(A) := \frac{1}{2}(A + A^*)$.

Lemma 3.3.6. *The self-adjoint and reversibility-preserving matrices $\tilde{E}, \tilde{G}$ in* (3.3.28) *have the form*

$$\tilde{E} = \begin{pmatrix} r_1(\mu\epsilon^2, \mu^3\epsilon, \mu^5) & \mathrm{i}\, r_2(\mu^2\epsilon^2, \mu^3\epsilon, \mu^5) \\ -\mathrm{i}\, r_2(\mu^2\epsilon^2, \mu^3\epsilon, \mu^5) & r_5(\mu^2\epsilon^2, \mu^4\epsilon, \mu^5) \end{pmatrix} , \tilde{G} = \begin{pmatrix} r_8(\mu\epsilon^2, \mu^3\epsilon, \mu^5) & \mathrm{i}\, r_9(\mu^3\epsilon, \mu^2\epsilon^2, \mu^5) \\ \mathrm{i}\, r_9(\mu^3\epsilon, \mu^2\epsilon^2, \mu^5) & r_{10}(\mu^4\epsilon, \mu^2\epsilon^2, \mu^6) \end{pmatrix} . \tag{3.3.29}$$

Proof. For simplicity set $F = F^{(1)}$. By (3.3.11), (3.3.7) and since $F_{11} = 0$ (cfr. (3.3.7)), one has

$$\mathtt{J}_2 X \mathtt{J}_2 F^* = \begin{pmatrix} x_{21}F_{12} & \mathrm{i}\,(x_{22}F_{21} + x_{21}F_{22}) \\ \mathrm{i}\,x_{11}F_{12} & x_{12}F_{21} - x_{11}F_{22} \end{pmatrix} = \begin{pmatrix} r(\mu\epsilon^2, \mu^3\epsilon, \mu^5) & \mathrm{i}\,r(\mu^2\epsilon^2, \mu^3\epsilon, \mu^5) \\ \mathrm{i}\,r(\mu^2\epsilon^2, \mu^3\epsilon, \mu^5) & r(\mu^2\epsilon^2, \mu^4\epsilon, \mu^5) \end{pmatrix}$$

and, adding its symmetric (cfr. (3.3.28)), the expansion of $\tilde{E}$ in (3.3.29) follows. For $\tilde{G}$ one has

$$X^* F = \begin{pmatrix} x_{21}F_{21} & \mathrm{i}\,(x_{11}F_{12} - x_{21}F_{22}) \\ \mathrm{i}\,x_{22}F_{21} & x_{22}F_{22} + x_{12}F_{12} \end{pmatrix} = \begin{pmatrix} r(\mu\epsilon^2, \mu^3\epsilon, \mu^5) & \mathrm{i}\,r(\mu^3\epsilon, \mu^2\epsilon^2, \mu^5) \\ \mathrm{i}\,r(\mu^4\epsilon, \mu^2\epsilon^2, \mu^6) & r(\mu^4\epsilon, \mu^2\epsilon^2, \mu^6) \end{pmatrix}$$

and the expansion of $\tilde{G}$ in (3.3.29) follows by symmetrizing. □

We now show that the last term in (3.3.26) is very small.

Lemma 3.3.7. *The 4×4 Hamiltonian and reversibility matrix*

$$\frac{1}{2}\int_0^1 (1-\tau^2)\, \exp(\tau S)\, \mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S)\, \mathrm{d}\tau = \begin{pmatrix} \mathtt{J}_2\widehat{E} & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2[F^{(2)}]^* & \mathtt{J}_2\widehat{G} \end{pmatrix} \tag{3.3.30}$$

where the 2×2 self-adjoint and reversible matrices $\widehat{E} = \begin{pmatrix} \widehat{E}_{11} & \mathrm{i}\,\widehat{E}_{12} \\ -\mathrm{i}\,\widehat{E}_{12} & \widehat{E}_{22} \end{pmatrix}$, $\widehat{G} = \begin{pmatrix} \widehat{G}_{11} & \mathrm{i}\,\widehat{G}_{12} \\ -\mathrm{i}\,\widehat{G}_{12} & \widehat{G}_{22} \end{pmatrix}$ *have entries*

$$\widehat{E}_{ij}\ , \widehat{G}_{ij} = \mu^2 r(\epsilon^3, \mu\epsilon^2, \mu^3\epsilon, \mu^5)\,, \quad i,j = 1,2\,, \tag{3.3.31}$$

and the 2×2 reversible matrix $F^{(2)}$ admits an expansion as in (3.3.14).

Proof. Since S and $R^{(1)}$ are Hamiltonian and reversibility-preserving then $\mathrm{ad}_S R^{(1)} = [S, R^{(1)}]$ is Hamiltonian and reversibility-preserving as well. Thus each $\exp(\tau S)\, \mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S)$ is Hamiltonian and reversibility-preserving, and formula (3.3.30) holds. In order to estimate its entries we first compute $\mathrm{ad}_S^2(R^{(1)})$. Using the form of S in (3.3.12) and $[S, R^{(1)}]$ in (3.3.27) one gets

$$\mathrm{ad}_S^2(R^{(1)}) = \begin{pmatrix} 0 & \mathtt{J}_2\tilde{F} \\ \mathtt{J}_2\tilde{F}^* & 0 \end{pmatrix} \qquad \text{where} \qquad \tilde{F} := 2\left(\Sigma \mathtt{J}_2 \tilde{G} - \tilde{E}\mathtt{J}_2\Sigma\right) \tag{3.3.32}$$

and $\tilde{E}$, $\tilde{G}$ are defined in (3.3.28). In order to estimate $\tilde{F}$, we write $\tilde{G} = \begin{pmatrix} \tilde{G}_{11} & \mathrm{i}\,\tilde{G}_{12} \\ -\mathrm{i}\,\tilde{G}_{12} & \tilde{G}_{22} \end{pmatrix}$, $\tilde{E} = \begin{pmatrix} \tilde{E}_{11} & \mathrm{i}\,\tilde{E}_{12} \\ -\mathrm{i}\,\tilde{E}_{12} & \tilde{E}_{22} \end{pmatrix}$ and, by (3.3.29), (3.3.11) and $\Sigma = \mathtt{J}_2 X$, we obtain

$$\Sigma \mathtt{J}_2 \tilde{G} = \begin{pmatrix} x_{21}\tilde{G}_{12} - x_{22}\tilde{G}_{11} & \mathrm{i}\,(x_{21}\tilde{G}_{22} - x_{22}\tilde{G}_{12}) \\ \mathrm{i}\,(x_{11}\tilde{G}_{12} + x_{12}\tilde{G}_{11}) & -x_{11}\tilde{G}_{22} - x_{12}\tilde{G}_{12} \end{pmatrix} = \begin{pmatrix} r(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7) & \mathrm{i}\,r(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) \\ \mathrm{i}\,r(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) & r(\mu^3\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \end{pmatrix},$$

$$\tilde{E}\,\mathtt{J}_2\Sigma = \begin{pmatrix} \tilde{E}_{12}x_{21} - \tilde{E}_{11}x_{11} & -\mathrm{i}\,(\tilde{E}_{11}x_{12} + \tilde{E}_{12}x_{22}) \\ \mathrm{i}\,(\tilde{E}_{12}x_{11} - \tilde{E}_{22}x_{21}) & -\tilde{E}_{12}x_{12} - \tilde{E}_{22}x_{22} \end{pmatrix} = \begin{pmatrix} r(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7) & \mathrm{i}\,r(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \\ \mathrm{i}\,r(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) & r(\mu^3\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \end{pmatrix}.$$

Thus the matrix $\tilde{F}$ in (3.3.32) has an expansion as in (3.3.14). Then, for any $\tau \in [0,1]$, the matrix $\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S) = \mathrm{ad}_S^2(R^{(1)})(1 + \mathcal{O}(\mu,\epsilon))$. In particular the matrix $F^{(2)}$ in (3.3.30) has the same expansion of $\tilde{F}$, whereas the matrices $\widehat{E}$, $\widehat{G}$ have entries at least as in (3.3.31). □

Proof of Lemma 3.3.2. It follows by Lemmata 3.3.6 and 3.3.7. The matrix $E^{(2)} := E^{(1)} + \tilde{E} + \widehat{E}$ has the same expansion of $E^{(1)}$ in (3.2.5). The same holds for $G^{(2)}$. □

3.3.3 Complete block decoupling and proof of the main results

We now block-diagonalize the 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}^{(2)}$ in (3.3.13). First we split it into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}_{\mu,\epsilon}^{(2)} = D^{(2)} + R^{(2)}\,,$$
$$D^{(2)} := \begin{pmatrix} D_1^{(2)} & 0 \\ 0 & D_0^{(2)} \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2 E^{(2)} & 0 \\ 0 & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \quad R^{(2)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & 0 \end{pmatrix}. \tag{3.3.33}$$

Lemma 3.3.8. *There exist a 4×4 reversibility-preserving Hamiltonian matrix $S^{(2)} := S^{(2)}(\mu,\epsilon)$ of the form* (3.3.12)*, analytic in (μ,ϵ), of size $\mathcal{O}(\epsilon^3, \mu\epsilon^2, \mu^3\epsilon, \mu^5)$, and a 4×4 block-diagonal reversible Hamiltonian matrix $P := P(\mu,\epsilon)$, analytic in (μ,ϵ), of size $\mu^2\mathcal{O}(\epsilon^4, \mu^4\epsilon^3, \mu^6\epsilon^2, \mu^8\epsilon, \mu^{10})$, such that*

$$\mathtt{L}_{\mu,\epsilon}^{(3)} := \exp(\mu S^{(2)})\,\mathtt{L}_{\mu,\epsilon}^{(2)}\,\exp(-\mu S^{(2)}) = D^{(2)} + P\,. \tag{3.3.34}$$

In particular

$$\mathtt{L}_{\mu,\epsilon}^{(3)} = \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix} \tag{3.3.35}$$

where $E^{(3)}$ and $G^{(3)}$ are selfadjoint and reversibility-preserving matrices of the form (3.2.5)-(3.2.6)*.*

Proof. We set for brevity $S = S^{(2)}$. The equation (3.3.34) is equivalent to the system

$$\begin{cases} \Pi_D\big(e^{\mu S}(D^{(2)} + R^{(2)})e^{-\mu S}\big) - D^{(2)} = P \\ \Pi_\varnothing\big(e^{\mu S}(D^{(2)} + R^{(2)})e^{-\mu S}\big) = 0\,, \end{cases} \tag{3.3.36}$$

where Π_D is the projection onto the block-diagonal matrices and $\Pi_\varnothing$ onto the block-off-diagonal ones. The second equation in (3.3.36) is equivalent, by a Lie expansion, and since $[S, R^{(2)}]$ is block-diagonal, to

$$R^{(2)} + \mu \left[S\,,\, D^{(2)}\right] + \mu^2 \underbrace{\Pi_\varnothing \int_0^1 (1-\tau) e^{\mu\tau S} \mathrm{ad}_S^2 (D^{(2)} + R^{(2)}) e^{-\mu\tau S} \mathrm{d}\tau}_{=:\mathcal{R}(S)} = 0\,. \tag{3.3.37}$$

The “nonlinear homological equation" (3.3.37), i.e. $[S, D^{(2)}] = -\frac{1}{\mu} R^{(2)} - \mu\mathcal{R}(S)$, is equivalent to solve the 4×4 real linear system

$$\mathcal{A}\vec{x} = \vec{f}(\mu,\epsilon,\vec{x})\,, \quad \vec{f}(\mu,\epsilon,\vec{x}) = \mu\vec{v}(\mu,\epsilon) + \mu^2 \vec{g}(\mu,\epsilon,\vec{x}) \tag{3.3.38}$$

associated, as in (3.3.19), to (3.3.37). The vector $\mu\vec{v}(\mu,\epsilon)$ is associated with $-\frac{1}{\mu}R^{(2)}$ with $R^{(2)}$ in (3.3.33). The vector $\mu^2\vec{g}(\mu,\epsilon,\vec{x})$ is associated with the matrix $-\mu\mathcal{R}(S)$, which is a Hamiltonian and reversible block-off-diagonal matrix (i.e of the form (3.3.15)), of size $\mathcal{R}(S) = \mathcal{O}(\mu)$ since $\Pi_\varnothing \mathrm{ad}_S^2(D^{(2)}) = 0$. The function $\vec{g}(\mu,\epsilon,\vec{x})$ is quadratic in $\vec{x}$. In view of (3.3.14) one has

$$\mu^2 \vec{v}(\mu,\epsilon) := (-F_{21}^{(2)}, F_{22}^{(2)}, -F_{11}^{(2)}, F_{12}^{(2)})^\top, \quad F_{ij}^{(2)} = \mu^2 r(\epsilon^3, \mu\epsilon^2, \mu^3\epsilon, \mu^5)\,. \tag{3.3.39}$$

System (3.3.38) is equivalent to $\vec{x} = \mathcal{A}^{-1}\vec{f}(\mu,\epsilon,\vec{x})$ and, writing $\mathcal{A}^{-1} = \frac{1}{\mu}\mathcal{B}(\mu,\epsilon)$ (cfr. (3.3.25)), to

$$\vec{x} = \mathcal{B}(\mu,\epsilon)\vec{v}(\mu,\epsilon) + \mu\mathcal{B}(\mu,\epsilon)\vec{g}(\mu,\epsilon,\vec{x})\,.$$

By the implicit function theorem this equation admits a unique small solution $\vec{x} = \vec{x}(\mu,\epsilon)$, analytic in (μ,ϵ), with size $\mathcal{O}(\epsilon^3, \mu\epsilon^2, \mu^3\epsilon, \mu^5)$ as $\vec{v}$ in (3.3.39). The claimed estimate of P follows by the the first equation of (3.3.36) and the estimate for S and of $R^{(2)}$ obtained by (3.3.14). □

Proof of Theorems 1.4.1. By Lemma 3.3.8 and recalling (2.2.1) the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented by the 4×4 Hamiltonian and reversible matrix

$$\mathrm{i}\,\mu + \exp(\mu S^{(2)}) \mathtt{L}_{\mu,\epsilon}^{(2)} \exp(-\mu S^{(2)}) = \mathrm{i}\,\mu + \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix} =: \begin{pmatrix} \mathtt{U} & 0 \\ 0 & \mathtt{S} \end{pmatrix}\,,$$

where the matrices $E^{(3)}$ and $G^{(3)}$ expand as in (3.2.5)-(3.2.6). Consequently the matrices $\mathtt{U}$ and $\mathtt{S}$ have an expansion as in (1.4.2), (1.4.3). Theorem 1.4.1 is proved. The unstable eigenvalues in Theorem 1.2.2 arise from the block $\mathtt{U}$. Its bottom-left entry vanishes for $\frac{\mu^2}{8}(1 + r_1'(\mu,\epsilon)) = \epsilon^2(1 + r_1''(\mu,\epsilon))$, which, by taking square roots, amounts to solve $\mu = 2\sqrt{2}\epsilon(1 + r(\mu,\epsilon))$. By the implicit function theorem, it admits a unique analytic solution $\underline{\mu}(\epsilon) = 2\sqrt{2}\epsilon(1 + r(\epsilon))$. The proof of Theorem 1.2.2 is complete.

Chapter 4

Benjamin-Feir instability in finite depth

In this chapter we prove the full description of the Benjamin-Feir instability phenomenon in the case of finite depth given in Theorem 1.5.1 and its "corollary" Theorem 1.2.3.

4.1 Expansion of the Kato basis

Using the transformation operators $U_{\mu,\epsilon}$ in (2.2.11), we construct the basis of $\mathcal{V}_{\mu,\epsilon}$

$$\mathcal{F} := \{ f_1^+(\mu,\epsilon),\ f_1^-(\mu,\epsilon),\ f_0^+(\mu,\epsilon),\ f_0^-(\mu,\epsilon) \}\,, \quad f_k^\sigma(\mu,\epsilon) := U_{\mu,\epsilon} f_k^\sigma\,,\ \sigma = \pm\,,\ k = 0,1\,, \tag{4.1.1}$$

where

$$f_1^+ = \begin{bmatrix} \mathtt{c_h}^{1/2}\cos(x) \\ \mathtt{c_h}^{-1/2}\sin(x) \end{bmatrix}, \quad f_1^- = \begin{bmatrix} -\mathtt{c_h}^{1/2}\sin(x) \\ \mathtt{c_h}^{-1/2}\cos(x) \end{bmatrix}, \quad f_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{4.1.2}$$

form a basis of $\mathcal{V}_{0,0} = \mathrm{Rg}(P_{0,0})$, cfr. (2.1.15a)-(2.1.16). Note that the real valued vectors $\{f_1^\pm, f_0^\pm\}$ form a symplectic and reversible basis for $\mathcal{V}_{0,0}$, according to Definition 2.2.6. Then, by Lemma 2.2.2 and 2.2.1 we deduce that (cfr. Lemma 4.1 in [13]):

Lemma 4.1.1. *The basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ defined in* (4.1.1)*, is symplectic and reversible, i.e. satisfies* (2.2.21) *and* (2.2.22)*. Each map $(\mu,\epsilon) \mapsto f_k^\sigma(\mu,\epsilon)$ is analytic as a map $B(\mu_0) \times B(\epsilon_0) \to H^1(\mathbb{T})$.*

In the next lemma we expand the vectors $f_k^\sigma(\mu,\epsilon)$ in (μ,ϵ). We denote by $even_0(x)$ a real, even, 2π-periodic function with zero space average. In the sequel $\mathcal{O}(\mu^m\epsilon^n)\begin{bmatrix} even(x) \\ odd(x) \end{bmatrix}$

denotes an analytic map in (μ,ϵ) with values in $H^1(\mathbb{T},\mathbb{C}^2)$, whose first component is $even(x)$ and the second one $odd(x)$; similar meaning for $\mathcal{O}(\mu^m\epsilon^n)\begin{bmatrix} odd(x) \\ even(x)\end{bmatrix}$, etc...

Lemma 4.1.2. (Expansion of the basis $\mathcal{F}$) *For small values of (μ,ϵ) the basis $\mathcal{F}$ in* (4.1.1) *has the expansion*

$$f_1^+(\mu,\epsilon) = \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12}\cos(x) \\ \mathtt{c}_{\mathtt{h}}^{-\frac12}\sin(x)\end{bmatrix} + \mathrm{i}\,\frac{\mu}{4}\gamma_{\mathtt{h}}\begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12}\sin(x) \\ \mathtt{c}_{\mathtt{h}}^{-\frac12}\cos(x)\end{bmatrix} + \epsilon\begin{bmatrix}\alpha_{\mathtt{h}}\cos(2x) \\ \beta_{\mathtt{h}}\sin(2x)\end{bmatrix} \tag{4.1.3}$$

$$+\,\mathcal{O}(\mu^2)\begin{bmatrix} even_0(x) + \mathrm{i}\,odd(x) \\ odd(x) + \mathrm{i}\,even_0(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix} odd(x) \\ even(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,,$$

$$f_1^-(\mu,\epsilon) = \begin{bmatrix} -\mathtt{c}_{\mathtt{h}}^{\frac12}\sin(x) \\ \mathtt{c}_{\mathtt{h}}^{-\frac12}\cos(x)\end{bmatrix} + \mathrm{i}\,\frac{\mu}{4}\gamma_{\mathtt{h}}\begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12}\cos(x) \\ -\mathtt{c}_{\mathtt{h}}^{-\frac12}\sin(x)\end{bmatrix} + \epsilon\begin{bmatrix}-\alpha_{\mathtt{h}}\sin(2x) \\ \beta_{\mathtt{h}}\cos(2x)\end{bmatrix} \tag{4.1.4}$$

$$+\,\mathcal{O}(\mu^2)\begin{bmatrix} odd(x) + \mathrm{i}\,even_0(x) \\ even_0(x) + \mathrm{i}\,odd(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\begin{bmatrix} odd(x) \\ even(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix} even(x) \\ odd(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,,$$

$$f_0^+(\mu,\epsilon) = \begin{bmatrix} 1 \\ 0\end{bmatrix} + \epsilon\delta_{\mathtt{h}}\begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12}\cos(x) \\ -\mathtt{c}_{\mathtt{h}}^{-\frac12}\sin(x)\end{bmatrix} + \mathcal{O}(\epsilon^2)\begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix} odd(x) \\ even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,, \tag{4.1.5}$$

$$f_0^-(\mu,\epsilon) = \begin{bmatrix} 0 \\ 1\end{bmatrix} + \mathrm{i}\,\mu\epsilon\begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu\epsilon^2)\,, \tag{4.1.6}$$

where the remainders $\mathcal{O}()$ are vectors in $H^1(\mathbb{T})$ and

$$\alpha_{\mathtt{h}} := \frac12\mathtt{c}_{\mathtt{h}}^{-\frac{11}{2}}(3+\mathtt{c}_{\mathtt{h}}^4)\,,\quad \beta_{\mathtt{h}} := \frac14\mathtt{c}_{\mathtt{h}}^{-\frac{13}{2}}(1+\mathtt{c}_{\mathtt{h}}^4)(3-\mathtt{c}_{\mathtt{h}}^4)\,,\quad \gamma_{\mathtt{h}} := 1+\frac{\mathtt{h}(1-\mathtt{c}_{\mathtt{h}}^4)}{\mathtt{c}_{\mathtt{h}}^2}\,,\quad \delta_{\mathtt{h}} := \frac{3+\mathtt{c}_{\mathtt{h}}^4}{4\mathtt{c}_{\mathtt{h}}^{\frac52}}\,. \tag{4.1.7}$$

For $\mu=0$ the basis $\{f_k^\pm(0,\epsilon), k=0,1\}$ is real and

$$f_1^+(0,\epsilon) = \begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix},\ f_1^-(0,\epsilon) = \begin{bmatrix} odd(x) \\ even(x)\end{bmatrix},\ f_0^+(0,\epsilon) = \begin{bmatrix} 1 \\ 0\end{bmatrix} + \begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix},\ f_0^-(0,\epsilon) = \begin{bmatrix} 0 \\ 1\end{bmatrix}. \tag{4.1.8}$$

The rest of the section is devoted to the proof of Lemma 4.1.2.
We first Taylor-expand the transformation operators $U_{\mu,\epsilon}$ defined in (2.2.11). We denote ∂_ϵ with a prime and ∂_μ with a dot.

Lemma 4.1.3. *The first jets of* $U_{\mu,\epsilon}P_{0,0}$ *are*

$$U_{0,0}P_{0,0} = P_{0,0}\,, \quad U'_{0,0}P_{0,0} = P'_{0,0}P_{0,0}\,, \quad \dot U_{0,0}P_{0,0} = \dot P_{0,0}P_{0,0}\,, \tag{4.1.9}$$

$$\dot U'_{0,0}P_{0,0} = (\dot P'_{0,0} - \tfrac{1}{2}P_{0,0}\dot P'_{0,0})P_{0,0}\,, \tag{4.1.10}$$

where

$$P'_{0,0} = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,, \tag{4.1.11}$$

$$\dot P_{0,0} = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,, \tag{4.1.12}$$

and

$$\dot P'_{0,0} = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda \tag{4.1.13a}$$

$$-\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\mathscr{L}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda \tag{4.1.13b}$$

$$+\frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}'_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda\,. \tag{4.1.13c}$$

The operators $\mathscr{L}'_{0,0}$ *and* $\dot{\mathscr{L}}_{0,0}$ *are*

$$\mathscr{L}'_{0,0} = \begin{bmatrix} \partial_x \circ p_1(x) & 0 \\ -a_1(x) & p_1(x)\circ\partial_x \end{bmatrix}, \qquad \dot{\mathscr{L}}_{0,0} = \begin{bmatrix} 0 & \operatorname{sgn}(D)m(D) \\ 0 & 0 \end{bmatrix}, \tag{4.1.14}$$

where $\operatorname{sgn}(D)$ *is defined in* (2.2.4) *and* $m(D)$ *is the real, even operator*

$$m(D) := \tanh(\mathtt{h}|D|) + \mathtt{h}|D|(1-\tanh^2(\mathtt{h}|D|)) \tag{4.1.15}$$

and $a_1(x)$ *and* $p_1(x)$ *are given in Proposition 2.1.4.*

The operator $\dot{\mathscr{L}}'_{0,0}$ *is*

$$\dot{\mathscr{L}}'_{0,0} = \begin{bmatrix} \mathrm{i}\,p_1(x) & 0 \\ 0 & \mathrm{i}\,p_1(x) \end{bmatrix}. \tag{4.1.16}$$

Proof. By (2.2.11) and (2.2.12) one has the Taylor expansion in $\mathcal{L}(Y)$

$$U_{\mu,\epsilon}P_{0,0} = P_{\mu,\epsilon}P_{0,0} + \frac{1}{2}(P_{\mu,\epsilon}-P_{0,0})^2P_{\mu,\epsilon}P_{0,0} + \mathcal{O}(P_{\mu,\epsilon}-P_{0,0})^4\,,$$

where $\mathcal{O}(P_{\mu,\epsilon}-P_{0,0})^4 = \mathcal{O}(\epsilon^4,\epsilon^3\mu,\epsilon^2\mu^2,\epsilon\mu^3,\mu^4) \in \mathcal{L}(Y)$. Consequently one derives (4.1.9), (4.1.10), using also the identity $\dot P_{0,0}P'_{0,0}P_{0,0} + P'_{0,0}\dot P_{0,0}P_{0,0} = -P_{0,0}\dot P'_{0,0}P_{0,0}$, which follows differentiating $P^2_{\mu,\epsilon} = P_{\mu,\epsilon}$. Differentiating (2.2.7) one gets (4.1.11)-(4.1.13c). Formulas (4.1.14)-(4.1.16) follow by (2.2.5a) using also that the Fourier multiplier $\Pi_0\big(\tanh(\mathtt{h}|D|) + \mathtt{h}|D|(1-\tanh^2(\mathtt{h}|D|))\big) = 0$. □

By the previous lemma we have the Taylor expansion

$$f_k^\sigma(\mu,\epsilon) = f_k^\sigma + \epsilon P'_{0,0} f_k^\sigma + \mu \dot{P}_{0,0} f_k^\sigma + \mu\epsilon(\dot{P}'_{0,0} - \frac{1}{2} P_{0,0}\dot{P}'_{0,0}) f_k^\sigma + \mathcal{O}(\mu^2,\epsilon^2)\,. \tag{4.1.17}$$

In order to compute the vectors $P'_{0,0} f_k^\sigma$ and $\dot{P}_{0,0} f_k^\sigma$ using (4.1.11) and (4.1.12), it is useful to know the action of $(\mathscr{L}_{0,0} - \lambda)^{-1}$ on the vectors

$$\begin{aligned} f_k^+ &:= \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\cos(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\sin(kx) \end{bmatrix}, \quad f_k^- := \begin{bmatrix} -\mathtt{c}_\mathtt{h}^{1/2}\sin(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\cos(kx) \end{bmatrix}, \\ f_{-k}^+ &:= \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\cos(kx) \\ -\mathtt{c}_\mathtt{h}^{-1/2}\sin(kx) \end{bmatrix}, \quad f_{-k}^- := \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\sin(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\cos(kx) \end{bmatrix}, \quad k \in \mathbb{N}\,. \end{aligned} \tag{4.1.18}$$

Lemma 4.1.4. *The space $H^1(\mathbb{T})$ decomposes as $H^1(\mathbb{T}) = \mathcal{V}_{0,0} \oplus \mathcal{U} \oplus \mathcal{W}_{H^1}$, with $\mathcal{W}_{H^1} = \overline{\bigoplus_{k=2}^\infty \mathcal{W}_k}^{H^1}$ where the subspaces $\mathcal{V}_{0,0}, \mathcal{U}$ and $\mathcal{W}_k$, defined below, are invariant under $\mathscr{L}_{0,0}$ and the following properties hold:*

(i) $\mathcal{V}_{0,0} = span\{f_1^+, f_1^-, f_0^+, f_0^-\}$ is the generalized kernel of $\mathscr{L}_{0,0}$. For any $\lambda \neq 0$ the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{V}_{0,0} \to \mathcal{V}_{0,0}$ is invertible and

$$(\mathscr{L}_{0,0} - \lambda)^{-1} f_1^+ = -\frac{1}{\lambda} f_1^+\,, \quad (\mathscr{L}_{0,0} - \lambda)^{-1} f_1^- = -\frac{1}{\lambda} f_1^-\,, \quad (\mathscr{L}_{0,0} - \lambda)^{-1} f_0^- = -\frac{1}{\lambda} f_0^-\,, \tag{4.1.19}$$

$$(\mathscr{L}_{0,0} - \lambda)^{-1} f_0^+ = -\frac{1}{\lambda} f_0^+ + \frac{1}{\lambda^2} f_0^-\,. \tag{4.1.20}$$

(ii) $\mathcal{U} := span\Big\{f_{-1}^+, f_{-1}^-\Big\}$. For any $\lambda \neq \pm 2\mathrm{i}$ the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{U} \to \mathcal{U}$ is invertible and

$$\begin{aligned} (\mathscr{L}_{0,0} - \lambda)^{-1} f_{-1}^+ &= \frac{1}{\lambda^2 + 4\mathtt{c}_\mathtt{h}^2}\Big(-\lambda f_{-1}^+ + 2\mathtt{c}_\mathtt{h} f_{-1}^-\Big)\,, \\ (\mathscr{L}_{0,0} - \lambda)^{-1} f_{-1}^- &= \frac{1}{\lambda^2 + 4\mathtt{c}_\mathtt{h}^2}\Big(-2\mathtt{c}_\mathtt{h} f_{-1}^+ - \lambda f_{-1}^-\Big)\,. \end{aligned} \tag{4.1.21}$$

(iii) Each subspace $\mathcal{W}_k := span\Big\{f_k^+,\ f_k^-, f_{-k}^+,\ f_{-k}^-\Big\}$ is invariant under $\mathscr{L}_{0,0}$. Let $\mathcal{W}_{L^2} = \overline{\bigoplus_{k=2}^\infty \mathcal{W}_k}^{L^2}$. For any $|\lambda| < \delta(\mathtt{h})$ small enough, the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{W}_{H^1} \to \mathcal{W}_{L^2}$ is invertible and for any $f \in \mathcal{W}_{L^2}$

$$(\mathscr{L}_{0,0} - \lambda)^{-1} f = \big(\mathtt{c}_\mathtt{h}^2 \partial_x^2 + |D|\tanh(\mathtt{h}|D|)\big)^{-1} \begin{bmatrix} \mathtt{c}_\mathtt{h}\partial_x & -|D|\tanh(\mathtt{h}|D|) \\ 1 & \mathtt{c}_\mathtt{h}\partial_x \end{bmatrix} f + \lambda \varphi_f(\lambda, x)\,, \tag{4.1.22}$$

for some analytic function $\lambda \mapsto \varphi_f(\lambda,\cdot) \in H^1(\mathbb{T},\mathbb{C}^2)$.

Proof. By inspection the spaces $\mathcal{V}_{0,0}$, $\mathcal{U}$ and $\mathcal{W}_k$ are invariant under $\mathscr{L}_{0,0}$ and, by Fourier series, they decompose $H^1(\mathbb{T},\mathbb{C}^2)$. Formulas (4.1.19)-(4.1.20) follow using that f_1^+, f_1^-, f_0^- are in the kernel of $\mathscr{L}_{0,0}$, and $\mathscr{L}_{0,0}f_0^+ = -f_0^-$. Formula (4.1.21) follows using that $\mathscr{L}_{0,0}f_{-1}^+ = -2\mathtt{c_h}f_{-1}^-$ and $\mathscr{L}_{0,0}f_{-1}^- = 2\mathtt{c_h}f_{-1}^+$. Let us prove item (iii). Let $\mathcal{W} := \mathcal{W}_{H^1}$. The operator $(\mathscr{L}_{0,0} - \lambda\mathrm{Id})\big|_{\mathcal{W}}$ is invertible for any $\lambda \notin \{\pm\mathrm{i}\sqrt{|k|\tanh(\mathtt{h}|k|)} \pm \mathrm{i}\,k\mathtt{c_h}, k\geq 2, k\in\mathbb{N}\}$ and

$$(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1} = \Big(\mathtt{c_h}^2\partial_x^2 + |D|\tanh(\mathtt{h}|D|)\Big)^{-1}\begin{bmatrix}\mathtt{c_h}\partial_x & -|D|\tanh(\mathtt{h}|D|))\\ 1 & \mathtt{c_h}\partial_x\end{bmatrix}_{|\mathcal{W}}.$$

By Neumann series, for any λ such that $|\lambda|\|(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\|_{\mathcal{L}(\mathcal{W},H^1(\mathbb{T}))} < 1$ we have

$$(\mathscr{L}_{0,0}|_{\mathcal{W}} - \lambda)^{-1} = (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}(\mathrm{Id} - \lambda(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1})^{-1} = (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\sum_{k\geq 0}((\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\lambda)^k.$$

Formula (4.1.22) follows with $\varphi_f(\lambda,x) := (\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}\sum_{k\geq1}\lambda^{k-1}[(\mathscr{L}_{0,0}|_{\mathcal{W}})^{-1}]^k f$. □

We shall also use the following formulas obtained by (4.1.14), (4.1.15) and (4.1.2):

$$\begin{aligned}
&\mathscr{L}'_{0,0}f_1^+ = \begin{bmatrix} 2\mathtt{c_h}^{-1/2}\sin(2x) \\ \frac12\mathtt{c_h}^{5/2}(1-\mathtt{c_h}^{-4})(1+\cos(2x))\end{bmatrix}, \qquad \mathscr{L}'_{0,0}f_1^- = \begin{bmatrix} 2\,\mathtt{c_h}^{-1/2}\cos(2x) \\ -\frac12\mathtt{c_h}^{5/2}(1-\mathtt{c_h}^{-4})\sin(2x)\end{bmatrix},\\
&\mathscr{L}'_{0,0}f_0^+ = \begin{bmatrix} 2\mathtt{c_h}^{-1}\sin(x) \\ \left(\mathtt{c_h}^2+\mathtt{c_h}^{-2}\right)\cos(x)\end{bmatrix}, \qquad \mathscr{L}'_{0,0}f_0^- = 0,\\
&\dot{\mathscr{L}}_{0,0}f_1^+ = -\mathrm{i}\,b(\mathtt{h})\begin{bmatrix}\cos(x)\\0\end{bmatrix}, \qquad \dot{\mathscr{L}}_{0,0}f_1^- = \mathrm{i}\,b(\mathtt{h})\begin{bmatrix}\sin(x)\\0\end{bmatrix}, \quad b(\mathtt{h}) := \mathtt{c_h}^{-1/2}(\mathtt{c_h}^2 + \mathtt{h}(1-\mathtt{c_h}^4)),\\
&\dot{\mathscr{L}}_{0,0}f_0^+ = 0, \qquad \dot{\mathscr{L}}_{0,0}f_0^- = 0.
\end{aligned} \tag{4.1.23}$$

Remark. In deep water we have $\dot{\mathscr{L}}_{0,0}f_0^- = f_0^+$ (cfr. formula (A.14) in [13]). In finite depth instead $\dot{\mathscr{L}}_{0,0}f_0^- = 0$ because the Fourier multiplier $\mathrm{sgn}(D)m(D)$ in (4.1.15) vanishes on the constants.

We finally compute $P'_{0,0}f_k^\sigma$ and $\dot P_{0,0}f_k^\sigma$.

Lemma 4.1.5. *One has*

$$\begin{aligned}
&P'_{0,0}f_1^+ = \begin{bmatrix}\frac12\mathtt{c_h}^{-\frac{11}{2}}(3+\mathtt{c_h}^4)\cos(2x)\\ \frac14\mathtt{c_h}^{-\frac{13}{2}}(1+\mathtt{c_h}^4)(3-\mathtt{c_h}^4)\sin(2x)\end{bmatrix}, \quad P'_{0,0}f_1^- = \begin{bmatrix}-\frac12\mathtt{c_h}^{-\frac{11}{2}}(3+\mathtt{c_h}^4)\sin(2x)\\ \frac14\mathtt{c_h}^{-\frac{13}{2}}(1+\mathtt{c_h}^4)(3-\mathtt{c_h}^4)\cos(2x)\end{bmatrix},\\
&P'_{0,0}f_0^+ = \tfrac14\mathtt{c_h}^{-\frac52}(3+\mathtt{c_h}^4)f_{-1}^+, \quad P'_{0,0}f_0^- = 0, \quad \dot P_{0,0}f_0^+ = 0, \quad \dot P_{0,0}f_0^- = 0,\\
&\dot P_{0,0}f_1^+ = \frac{\mathrm{i}}{4}(1+\mathtt{c_h}^{-2}\mathtt{h}(1-\mathtt{c_h}^4))f_{-1}^-, \quad \dot P_{0,0}f_1^- = \frac{\mathrm{i}}{4}(1+\mathtt{c_h}^{-2}\mathtt{h}(1-\mathtt{c_h}^4))f_{-1}^+.
\end{aligned} \tag{4.1.24}$$

Proof. We first compute $P'_{0,0} f_1^+$. By (4.1.11), (4.1.19) and (4.1.23) we deduce

$$P'_{0,0} f_1^+ = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda} (\mathscr{L}_{0,0} - \lambda)^{-1} \begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1/2} \sin(2x) \\ \frac{1}{2}\mathtt{c}_\mathtt{h}^{5/2}(1-\mathtt{c}_\mathtt{h}^{-4})(1+\cos(2x)) \end{bmatrix} \mathrm{d}\lambda \,.$$

We note that $\begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1/2} \sin(2x) \\ \frac{1}{2}\mathtt{c}_\mathtt{h}^{5/2}(1-\mathtt{c}_\mathtt{h}^{-4})(1+\cos(2x)) \end{bmatrix} = \frac{1}{2}\mathtt{c}_\mathtt{h}^{5/2}(1-\mathtt{c}_\mathtt{h}^{-4}) f_0^- + \mathcal{W}$. Therefore by (4.1.19) and (4.1.22) there is an analytic function $\lambda \mapsto \varphi(\lambda, \cdot) \in H^1(\mathbb{T}, \mathbb{C}^2)$ so that

$$P'_{0,0} f_1^+ = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda} \Big(-\frac{\mathtt{c}_\mathtt{h}^{5/2}(1-\mathtt{c}_\mathtt{h}^{-4})}{2\lambda} f_0^- - \frac{1+\mathtt{c}_\mathtt{h}^4}{4\mathtt{c}_\mathtt{h}^6} \begin{bmatrix} 2\mathtt{c}_\mathtt{h} \frac{\mathtt{c}_\mathtt{h}^{-\frac{1}{2}}(3+\mathtt{c}_\mathtt{h}^4)}{1+\mathtt{c}_\mathtt{h}^4} \cos(2x) \\ \mathtt{c}_\mathtt{h}^{-\frac{1}{2}}(3-\mathtt{c}_\mathtt{h}^4)\sin(2x) \end{bmatrix} + \lambda\varphi(\lambda) \Big) \mathrm{d}\lambda \,,$$

where we exploited the identity $\tanh(2\mathtt{h}) = \frac{2\mathtt{c}_\mathtt{h}^2}{1+\mathtt{c}_\mathtt{h}^4}$ in applying (4.1.22). Thus, by means of residue Theorem we obtain the first identity in (4.1.24). Similarly one computes $P'_{0,0} f_1^-$. By (4.1.11), (4.1.19) and (4.1.23), one has $P'_{0,0} f_0^- = 0$. Next we compute $P'_{0,0} f_0^+$. By (4.1.11), (4.1.19), (4.1.20) and (4.1.23) we get

$$P'_{0,0} f_0^+ = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda} (\mathscr{L}_{0,0} - \lambda)^{-1} \begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1}\sin(x) \\ (\mathtt{c}_\mathtt{h}^2 + \mathtt{c}_\mathtt{h}^{-2})\cos(x) \end{bmatrix} \mathrm{d}\lambda \,.$$

Next we decompose $\begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1}\sin(x) \\ (\mathtt{c}_\mathtt{h}^2 + \mathtt{c}_\mathtt{h}^{-2})\cos(x) \end{bmatrix} = \underbrace{\frac{1}{2}\mathtt{c}_\mathtt{h}^{-\frac{3}{2}}(\mathtt{c}_\mathtt{h}^4+3)}_{=:\alpha} f_{-1}^- + \underbrace{\frac{1}{2}\mathtt{c}_\mathtt{h}^{-\frac{3}{2}}(\mathtt{c}_\mathtt{h}^4-1)}_{=:\beta} f_1^-$. By (4.1.23) and (4.1.21) we get

$$P'_{0,0} f_0^+ = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \Big(-\frac{2\alpha \mathtt{c}_\mathtt{h}}{\lambda(\lambda^2+4\mathtt{c}_\mathtt{h}^2)} f_{-1}^+ - \frac{\alpha}{\lambda^2+4\mathtt{c}_\mathtt{h}^2} f_{-1}^- + \frac{\beta}{\lambda^2} f_1^- \Big) \mathrm{d}\lambda = \frac{\alpha}{2\mathtt{c}_\mathtt{h}} f_{-1}^+ \,,$$

where in the last step we used the residue theorem. We compute now $\dot{P}_{0,0} f_1^+$. First we have $\dot{P}_{0,0} f_1^+ = \frac{\mathrm{i}}{2\pi\mathrm{i}} b(\mathtt{h}) \oint_\Gamma \frac{1}{\lambda} (\mathscr{L}_{0,0} - \lambda)^{-1} \begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} \mathrm{d}\lambda$, where $b(\mathtt{h})$ is in (4.1.23), and then, writing $\begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} = \frac{1}{2}\mathtt{c}_\mathtt{h}^{-\frac{1}{2}}(f_1^+ + f_{-1}^+)$ and using (4.1.21), we conclude using again the residue theorem $\dot{P}_{0,0} f_1^+ = \frac{\mathrm{i}}{4}(1 + \mathtt{h}(1-\mathtt{c}_\mathtt{h}^4)\mathtt{c}_\mathtt{h}^{-2}) f_{-1}^-$. The computation of $\dot{P}_{0,0} f_1^-$ is analogous. Finally, in view of (4.1.23), we have

$$\dot{P}_{0,0} f_0^+ = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0} (\frac{1}{\lambda^2} f_0^- - \frac{1}{\lambda} f_0^+) \mathrm{d}\lambda = 0 \,,$$
$$\dot{P}_{0,0} f_0^- = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{1}{\lambda} (\mathcal{L}_{0,0} - \lambda)^{-1} \dot{\mathcal{L}}_{0,0} f_0^- \mathrm{d}\lambda = 0 \,.$$

In conclusion all the formulas in (4.1.24) are proved. □

So far we have obtained the linear terms of the expansions (4.1.3), (4.1.4), (4.1.5), (4.1.6). We now provide further information about the expansion of the basis at $\mu = 0$. The proof of the next lemma follows as Lemma 3.1.6.

Lemma 4.1.6. *The basis* $\{f_k^\sigma(0,\epsilon),\ k = 0,1, \sigma = \pm\}$ *is real. For any* ϵ $f_0^-(0,\epsilon) \equiv f_0^-$. *The property* (4.1.8) *holds.*

We now provide further information about the expansion of the basis at $\epsilon = 0$. The following lemma follows as Lemma 3.1.7. The key observation is that the operator $\mathscr{L}_{\mu,0}|_{\mathcal{Z}}$, where $\mathcal{Z}$ is the invariant subspace $\mathcal{Z} := \text{span}\{f_0^+,\ f_0^-\}$, has the two eigenvalues $\pm\mathrm{i}\sqrt{\mu\tanh(\mathtt{h}\mu)}$, which, for small μ, lie inside the loop Γ around 0 in (2.2.7).

Lemma 4.1.7. *For any small* μ, *we have* $f_0^+(\mu,0) \equiv f_0^+$ *and* $f_0^-(\mu,0) \equiv f_0^-$. *Moreover the vectors* $f_1^+(\mu,0)$ *and* $f_1^-(\mu,0)$ *have both components with zero space average.*

We finally consider the $\mu\epsilon$ term in the expansion (4.1.17).

Lemma 4.1.8. *The derivatives* $(\partial_\mu\partial_\epsilon f_k^\sigma)(0,0) = \left(\dot P'_{0,0} - \frac{1}{2}P_{0,0}\dot P'_{0,0}\right) f_k^\sigma$ *satisfy*

$$\begin{aligned}
(\partial_\mu\partial_\epsilon f_1^+)(0,0) &= \mathrm{i}\begin{bmatrix} odd(x) \\ even(x)\end{bmatrix}, \qquad (\partial_\mu\partial_\epsilon f_1^-)(0,0)- = \mathrm{i}\begin{bmatrix} even(x) \\ odd(x)\end{bmatrix}, \\
(\partial_\mu\partial_\epsilon f_0^+)(0,0) &= \mathrm{i}\begin{bmatrix} odd(x) \\ even_0(x)\end{bmatrix}, \qquad (\partial_\mu\partial_\epsilon f_0^-)(0,0) = \mathrm{i}\begin{bmatrix} even_0(x) \\ odd(x)\end{bmatrix}.
\end{aligned} \tag{4.1.25}$$

Proof. We prove that $\dot P'_{0,0} =$ (4.1.13a) + (4.1.13b) + (4.1.13c) is purely imaginary. This follows since the operators in (4.1.13a), (4.1.13b) and (4.1.13c) are purely imaginary because $\dot{\mathscr{L}}_{0,0}$ is purely imaginary, $\mathscr{L}'_{0,0}$ in (4.1.14) is real and $\dot{\mathscr{L}}'_{0,0}$ in (4.1.16) is purely imaginary (argue as in Lemma 2.2.2-(*iii*)). Then, applied to the real vectors f_k^σ, $k = 0,1$, $\sigma = \pm$, give purely imaginary vectors.

The property (2.2.23) implies that $(\partial_\mu\partial_\epsilon f_k^\sigma)(0,0)$ have the claimed parity structure in (4.1.25). We shall now prove that $(\partial_\mu\partial_\epsilon f_0^\pm)(0,0)$ have zero average. We have, by (4.1.20) and (4.1.23)

$$(4.1.13\text{a})f_0^+ := \frac{1}{2\pi\mathrm{i}}\oint_\Gamma (\mathscr{L}_{0,0}-\lambda)^{-1}\dot{\mathscr{L}}_{0,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\frac{1}{\lambda}\begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1}\sin(x) \\ \left(\mathtt{c}_\mathtt{h}^2 + \mathtt{c}_\mathtt{h}^{-2}\right)\cos(x)\end{bmatrix}\mathrm{d}\lambda$$

and since the operators $(\mathscr{L}_{0,0}-\lambda)^{-1}$ and $\dot{\mathscr{L}}_{0,0}$ are both Fourier multipliers, hence they preserve the absence of average of the vectors, then $(4.1.13\text{a})f_0^+$ has zero average. Next

$(4.1.13\text{b})f_0^+ = 0$ since $\dot{\mathscr{L}}_{0,0} f_0^\pm = 0$, cfr. (2.2.4). Finally, by (4.1.20) and (4.1.16) where $p_1(x) = p_1^{[1]}\cos(x)$,

$$(4.1.13\text{c})f_0^+ = \frac{\mathrm{i}\,p_1^{[1]}}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1}\Big(-\frac{1}{\lambda}\begin{bmatrix}\cos(x)\\ 0\end{bmatrix} + \frac{1}{\lambda^2}\begin{bmatrix}0\\ \cos(x)\end{bmatrix}\Big)\,\mathrm{d}\lambda$$

is a vector with zero average. We conclude that $\dot{P}'_{0,0} f_0^+$ is an imaginary vector with zero average, as well as $(\partial_\mu\partial_\epsilon f_0^+)(0,0)$ since $P_{0,0}$ sends zero average functions in zero average functions. Finally, by (2.2.23), $(\partial_\mu\partial_\epsilon f_0^+)(0,0)$ has the claimed structure in (4.1.25).

We finally consider $(\partial_\mu\partial_\epsilon f_0^-)(0,0)$. By (4.1.19) and $\mathscr{L}'_{0,0} f_0^- = 0$ (cfr. (4.1.23)), it results

$$(4.1.13\text{a})f_0^- = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{(\mathscr{L}_{0,0} - \lambda)^{-1}}{\lambda} \dot{\mathscr{L}}_{0,0} (\mathscr{L}_{0,0} - \lambda)^{-1} \mathscr{L}'_{0,0} f_0^- \,\mathrm{d}\lambda = 0\,.$$

Next by (4.1.19) and $\dot{\mathscr{L}}_{0,0} f_0^- = 0$ we get $(4.1.13\text{b})f_0^- = 0$. Finally by (4.1.19) and (4.1.16)

$$(4.1.13\text{c})f_0^- = -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{0,0} - \lambda)^{-1} \frac{1}{\lambda}\begin{bmatrix}0\\ \mathrm{i}\,p_1^{[1]}\cos(x)\end{bmatrix}\mathrm{d}\lambda$$

has zero average since $(\mathscr{L}_{0,0} - \lambda)^{-1}$ is a Fourier multiplier (and thus preserves average absence). □

This completes the proof of Lemma 4.1.2.

4.2 Matrix representation of $\mathscr{L}_{\mu,\epsilon}$ on $\mathcal{V}_{\mu,\epsilon}$

The main result of this section is the following

Proposition 4.2.1. *The matrix that represents the Hamiltonian and reversible operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ in the symplectic and reversible basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ defined in* (4.1.1)*, is a Hamiltonian matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}$, where $\mathtt{B}_{\mu,\epsilon}$ is a self-adjoint and reversibility preserving (i.e. satisfying* (2.2.27)*) 4×4 matrix of the form*

$$\mathtt{B}_{\mu,\epsilon} = \begin{pmatrix} E & F \\ F^* & G \end{pmatrix}, \qquad E = E^*\,, \quad G = G^*\,, \tag{4.2.1}$$

where E, F, G are the 2×2 matrices

$$E := \begin{pmatrix} \mathtt{e}_{11}\epsilon^2(1 + r_1'(\epsilon,\mu\epsilon)) - \mathtt{e}_{22}\frac{\mu^2}{8}(1 + r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu^2}{8}(1 + r_5(\epsilon,\mu)) \end{pmatrix} \tag{4.2.2}$$

$$G := \begin{pmatrix} 1 + r_8(\epsilon^2, \mu^2\epsilon) & -\mathrm{i}\, r_9(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\, r_9(\mu\epsilon^2, \mu^2\epsilon) & \mu \tanh(\mathtt{h}\mu) + r_{10}(\mu^2\epsilon) \end{pmatrix} \tag{4.2.3}$$

$$F := \begin{pmatrix} \mathtt{f}_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon) & \mathrm{i}\,\mu\epsilon \mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + \mathrm{i}\, r_4(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\, r_6(\mu\epsilon) & r_7(\mu^2\epsilon) \end{pmatrix}, \tag{4.2.4}$$

with $\mathtt{e}_{12}$ *and* $\mathtt{e}_{22}$ *given in* (1.5.2) *and* (1.5.4) *respectively, and*

$$\mathtt{e}_{11} := \frac{9\mathtt{c}_{\mathtt{h}}^8 - 10\mathtt{c}_{\mathtt{h}}^4 + 9}{8\mathtt{c}_{\mathtt{h}}^7} = \frac{9(1-\mathtt{c}_{\mathtt{h}}^4)^2 + 8\mathtt{c}_{\mathtt{h}}^4}{8\mathtt{c}_{\mathtt{h}}^7} > 0\,, \qquad \mathtt{f}_{11} := \tfrac{1}{2}\mathtt{c}_{\mathtt{h}}^{-\frac{3}{2}}(1-\mathtt{c}_{\mathtt{h}}^4)\,. \tag{4.2.5}$$

The rest of this section is devoted to the proof of Proposition 4.2.1.

We decompose $\mathcal{B}_{\mu,\epsilon}$ in (2.2.5a) as

$$\mathcal{B}_{\mu,\epsilon} = \mathcal{B}_\epsilon + \mathcal{B}^\flat + \mathcal{B}^\sharp\,,$$

where $\mathcal{B}_\epsilon$, $\mathcal{B}^\flat$, $\mathcal{B}^\sharp$ are the self-adjoint and reversibility preserving operators

$$\mathcal{B}_\epsilon := \mathcal{B}_{0,\epsilon} := \begin{bmatrix} 1 + a_\epsilon(x) & -(\mathtt{c}_{\mathtt{h}} + p_\epsilon(x))\partial_x \\ \partial_x \circ (\mathtt{c}_{\mathtt{h}} + p_\epsilon(x)) & |D|\tanh((\mathtt{h} + \mathtt{f}_\epsilon)|D|) \end{bmatrix}, \tag{4.2.6}$$

$$\mathcal{B}^\flat := \begin{bmatrix} 0 & 0 \\ 0 & |D+\mu|\tanh((\mathtt{h}+\mathtt{f}_\epsilon)|D+\mu|) - |D|\tanh((\mathtt{h}+\mathtt{f}_\epsilon)|D|) \end{bmatrix}, \tag{4.2.7}$$

$$\mathcal{B}^\sharp := \mu \begin{bmatrix} 0 & -\mathrm{i}\, p_\epsilon \\ \mathrm{i}\, p_\epsilon & 0 \end{bmatrix}, \tag{4.2.8}$$

where the operator $\mathcal{B}^\flat$ is analytic in μ.

In order to prove (4.2.1)-(4.2.4) we exploit the representation Lemma 2.2.10 and compute perturbatively the 4×4 matrices, associated, as in (2.2.26), to the self-adjoint and reversibility preserving operators $\mathcal{B}_\epsilon$, $\mathcal{B}^\flat$ and $\mathcal{B}^\sharp$, in the basis $\mathcal{F}$.

Lemma 4.2.2. (Expansion of $\mathtt{B}_\epsilon$) *The self-adjoint and reversibility preserving matrix* $\mathtt{B}_\epsilon := \mathtt{B}_\epsilon(\mu)$ *associated, as in* (2.2.26), *with the self-adjoint and reversibility preserving operator* $\mathcal{B}_\epsilon$ *defined in* (4.2.6), *with respect to the basis* $\mathcal{F}$ *of* $\mathcal{V}_{\mu,\epsilon}$ *in* (4.1.1), *expands as*

$$\mathtt{B}_\epsilon = \left(\begin{array}{cc|cc} \mathtt{e}_{11}\epsilon^2 + \zeta_{\mathtt{h}}\mu^2 + r_1(\epsilon^3, \mu\epsilon^3) & \mathrm{i}\, r_2(\mu\epsilon^2) & \mathtt{f}_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2) & \mathrm{i}\, r_4(\mu\epsilon^3) \\ -\mathrm{i}\, r_2(\mu\epsilon^2) & \zeta_{\mathtt{h}}\mu^2 & \mathrm{i}\, r_6(\mu\epsilon) & 0 \\ \hline \mathtt{f}_{11}\epsilon + r_3(\epsilon^3, \mu\epsilon^2) & -\mathrm{i}\, r_6(\mu\epsilon) & 1 + r_8(\epsilon^2, \mu\epsilon^2) & \mathrm{i}\, r_9(\mu\epsilon^2) \\ -\mathrm{i}\, r_4(\mu\epsilon^3) & 0 & -\mathrm{i}\, r_9(\mu\epsilon^2) & 0 \end{array}\right) + \mathcal{O}(\mu^2\epsilon, \mu^3)\,, \tag{4.2.9}$$

where $\mathtt{e}_{11}$, $\mathtt{f}_{11}$ *are defined respectively in* (4.2.5)*, and*

$$\zeta_{\mathtt{h}} := \tfrac{1}{8}\mathtt{c}_{\mathtt{h}}\gamma_{\mathtt{h}}^2 \,. \tag{4.2.10}$$

Proof. We expand the matrix $\mathtt{B}_\epsilon(\mu)$ as

$$\mathtt{B}_\epsilon(\mu) = \mathtt{B}_\epsilon(0) + \mu(\partial_\mu \mathtt{B}_\epsilon)(0) + \frac{\mu^2}{2}(\partial_\mu^2 \mathtt{B}_0)(0) + \mathcal{O}(\mu^2\epsilon, \mu^3)\,. \tag{4.2.11}$$

The matrix $\mathtt{B}_\epsilon(0)$. The main result of this long paragraph is to prove that the matrix $\mathtt{B}_\epsilon(0)$ has the expansion (4.2.15). The matrix $\mathtt{B}_\epsilon(0)$ is real, because the operator $\mathcal{B}_\epsilon$ is real and the basis $\{f_k^{\pm}(0,\epsilon)\}_{k=0,1}$ is real. Consequently, by (2.2.27), its matrix elements $(\mathtt{B}_\epsilon(0))_{i,j}$ are real whenever $i+j$ is even and vanish for $i+j$ odd. In addition $f_0^-(0,\epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by (4.1.8), and, by (4.2.6), we get $\mathcal{B}_\epsilon f_0^-(0,\epsilon) = 0$, for any ϵ. We deduce that the self-adjoint matrix $\mathtt{B}_\epsilon(0)$ has the form

$$\mathtt{B}_\epsilon(0) = \Big(\mathcal{B}_\epsilon\, f_k^\sigma(0,\epsilon),\, f_{k'}^{\sigma'}(0,\epsilon)\Big)_{k,k'=0,1,\sigma,\sigma'=\pm} = \left(\begin{array}{cc|cc} E_{11}(0,\epsilon) & 0 & F_{11}(0,\epsilon) & 0 \\ 0 & E_{22}(0,\epsilon) & 0 & 0 \\ \hline F_{11}(0,\epsilon) & 0 & G_{11}(0,\epsilon) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{4.2.12}$$

where $E_{11}(0,\epsilon)$, $E_{22}(0,\epsilon)$, $G_{11}(0,\epsilon)$, $F_{11}(0,\epsilon)$ are real. We claim that $E_{22}(0,\epsilon) = 0$ for any ϵ. As a first step, as in the infinite-depth case, we prove that

$$\text{either} \quad E_{22}(0,\epsilon) \equiv 0\,, \qquad \text{or} \quad E_{11}(0,\epsilon) \equiv 0 \equiv F_{11}(0,\epsilon)\,. \tag{4.2.13}$$

Indeed by Lemma (2.1.8) we have $\mathscr{L}_{0,\epsilon}^2 = 0$ on $\mathcal{V}_{0,\epsilon}$. Thus the matrix

$$\mathtt{L}_\epsilon(0) := \mathtt{J}_4\mathtt{B}_\epsilon(0) = \left(\begin{array}{cc|cc} 0 & E_{22}(0,\epsilon) & 0 & 0 \\ -E_{11}(0,\epsilon) & 0 & -F_{11}(0,\epsilon) & 0 \\ \hline 0 & 0 & 0 & 0 \\ -F_{11}(0,\epsilon) & 0 & -G_{11}(0,\epsilon) & 0 \end{array}\right), \tag{4.2.14}$$

which represents $\mathscr{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$, satisfies $\mathtt{L}_\epsilon^2(0) = 0$, namely

$$\mathtt{L}_\epsilon^2(0) = -\left(\begin{array}{cc|cc} (E_{11}E_{22})(0,\epsilon) & 0 & (F_{11}E_{22})(0,\epsilon) & 0 \\ 0 & (E_{11}E_{22})(0,\epsilon) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & (F_{11}E_{22})(0,\epsilon) & 0 & 0 \end{array}\right) = 0$$

which implies (4.2.13). We now prove that the matrix $\mathtt{B}_\epsilon(0)$ defined in (4.2.12) expands as

$$\mathtt{B}_\epsilon(0)=\left(\begin{array}{cc|cc}\mathtt{e}_{11}\epsilon^2+r(\epsilon^3) & 0 & \mathtt{f}_{11}\epsilon+r(\epsilon^3) & 0\\ 0 & 0 & 0 & 0\\ \hline \mathtt{f}_{11}\epsilon+r(\epsilon^3) & 0 & 1+r(\epsilon^2) & 0\\ 0 & 0 & 0 & 0\end{array}\right) \tag{4.2.15}$$

where $\mathtt{e}_{11}$ and $\mathtt{f}_{11}$ are in (4.2.21) and (4.2.24). We expand the operator $\mathcal{B}_\epsilon$ in (4.2.6) as

$$\begin{aligned}&\mathcal{B}_\epsilon=\mathcal{B}_0+\epsilon\mathcal{B}_1+\epsilon^2\mathcal{B}_2+\mathcal{O}(\epsilon^3),\quad \mathcal{B}_0:=\begin{bmatrix}1 & -\mathtt{c_h}\partial_x\\ \mathtt{c_h}\partial_x & |D|\tanh(\mathtt{h}|D|)\end{bmatrix},\\ &\mathcal{B}_1:=\begin{bmatrix}a_1(x) & -p_1(x)\partial_x\\ \partial_x\circ p_1(x) & 0\end{bmatrix},\quad \mathcal{B}_2:=\begin{bmatrix}a_2(x) & -p_2(x)\partial_x\\ \partial_x\circ p_2(x) & -\mathtt{f}_2\partial_x^2(1-\tanh^2(\mathtt{h}|D|))\end{bmatrix},\end{aligned} \tag{4.2.16}$$

where the remainder term $\mathcal{O}(\epsilon^3)\in\mathcal{L}(Y,X)$, the functions a_1, p_1, a_2, p_2 are given in (2.1.4) and, in view of (A.4.8), $\mathtt{f}_2:=\frac14\mathtt{c_h}^{-2}(\mathtt{c_h}^4-3)$.

• *Expansion of* $E_{11}(0,\epsilon)=\mathtt{e}_{11}\epsilon^2+r(\epsilon^3)$. By (4.1.3) we split the real function $f_1^+(0,\epsilon)$ as

$$\begin{aligned}&f_1^+(0,\epsilon)=f_1^++\epsilon f_{1_1}^++\epsilon^2 f_{1_2}^++\mathcal{O}(\epsilon^3),\\ &f_1^+=\begin{bmatrix}\mathtt{c_h}^{\frac12}\cos(x)\\ \mathtt{c_h}^{-\frac12}\sin(x)\end{bmatrix},\ f_{1_1}^+:=\begin{bmatrix}\alpha_{\mathtt h}\cos(2x)\\ \beta_{\mathtt h}\sin(2x)\end{bmatrix},\ f_{1_2}^+:=\begin{bmatrix}even_0(x)\\ odd(x)\end{bmatrix},\end{aligned} \tag{4.2.17}$$

where both $f_{1_2}^+$ and $\mathcal{O}(\epsilon^3)$ are vectors in $H^1(\mathbb{T})$. Since $\mathcal{B}_0 f_1^+=\mathcal{J}^{-1}\mathscr{L}_{0,0}f_1^+=0$, and both $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators, it results

$$\begin{aligned}E_{11}(0,\epsilon)&=\left(\mathcal{B}_\epsilon f_1^+(0,\epsilon)\,,\,f_1^+(0,\epsilon)\right)\\ &=\epsilon\left(\mathcal{B}_1f_1^+\,,\,f_1^+\right)+\epsilon^2\left[\left(\mathcal{B}_2f_1^+\,,\,f_1^+\right)+2\left(\mathcal{B}_1f_1^+\,,\,f_{1_1}^+\right)+\left(\mathcal{B}_0f_{1_1}^+\,,\,f_{1_1}^+\right)\right]+\mathcal{O}(\epsilon^3).\end{aligned} \tag{4.2.18}$$

By (4.2.16) one has

$$\mathcal{B}_1f_1^+=\begin{bmatrix}A_1(1+\cos(2x))\\ B_1\sin(2x)\end{bmatrix},\quad \mathcal{B}_2f_1^+=\begin{bmatrix}A_2\cos(x)+A_3\cos(3x)\\ B_2\sin(x)+B_3\sin(3x)\end{bmatrix},\quad \mathcal{B}_0f_{1_1}^+=\begin{bmatrix}A_4\cos(2x)\\ B_4\sin(2x)\end{bmatrix}, \tag{4.2.19}$$

with

$$\begin{aligned}&A_1:=\tfrac12(a_1^{[1]}\mathtt{c_h}^{\frac12}-p_1^{[1]}\mathtt{c_h}^{-\frac12}),\qquad B_1:=-p_1^{[1]}\mathtt{c_h}^{\frac12},\\ &A_2:=\mathtt{c_h}^{\frac12}a_2^{[0]}-\mathtt{c_h}^{-\frac12}p_2^{[0]}+\tfrac12\mathtt{c_h}^{\frac12}a_2^{[2]}-\tfrac12\mathtt{c_h}^{-\frac12}p_2^{[2]},\qquad A_4:=\alpha_{\mathtt h}-2\beta_{\mathtt h}\mathtt{c_h},\\ &B_2:=-\mathtt{c_h}^{\frac12}p_2^{[0]}-\tfrac12\mathtt{c_h}^{\frac12}p_2^{[2]}+\mathtt{c_h}^{-\frac12}\mathtt{f}_2(1-\mathtt{c_h}^4),\qquad B_4:=-2\alpha_{\mathtt h}\mathtt{c_h}+\frac{4\mathtt{c_h}^2}{1+\mathtt{c_h}^4}\beta_{\mathtt h}.\end{aligned} \tag{4.2.20}$$

By (4.2.19) and (4.2.17), we deduce

$$E_{11}(0,\epsilon)=\mathtt{e}_{11}\epsilon^2+r(\epsilon^3)\,,\quad \mathtt{e}_{11}:=\frac{1}{2}\big(A_2\mathtt{c}_{\mathtt{h}}^{\frac12}+B_2\mathtt{c}_{\mathtt{h}}^{-\frac12}+2\alpha_{\mathtt{h}}A_1+2B_1\beta_{\mathtt{h}}+\alpha_{\mathtt{h}}A_4+\beta_{\mathtt{h}}B_4\big)\,. \tag{4.2.21}$$

By (4.2.21), (4.2.20), (4.1.7) and (A.4.22)-(A.4.23) we obtain (4.2.5). Since $\mathtt{e}_{11}>0$ the second alternative in (4.2.13) is ruled out, implying $E_{22}(0,\epsilon)\equiv 0$.

• *Expansion of* $G_{11}(0,\epsilon)=1+r(\epsilon^2)$. By (4.1.5) we split the real-valued function $f_0^+(0,\epsilon)$ as

$$f_0^+(0,\epsilon)=f_0^+ +\epsilon f_{0_1}^+ +\epsilon^2 f_{0_2}^+ +\mathcal{O}(\epsilon^3)\,,\ \ f_0^+=\begin{bmatrix}1\\0\end{bmatrix}\,,\ f_{0_1}^+:=\delta_{\mathtt{h}}\begin{bmatrix}\mathtt{c}_{\mathtt{h}}^{\frac12}\cos(x)\\-\mathtt{c}_{\mathtt{h}}^{-\frac12}\sin(x)\end{bmatrix}\,,\ f_{0_2}^+:=\begin{bmatrix}even_0(x)\\odd(x)\end{bmatrix}\,. \tag{4.2.22}$$

Since, by (2.1.15a) and (4.2.16), $\mathcal{B}_0 f_0^+ = f_0^+$, using that $\mathcal{B}_0$, $\mathcal{B}_1$ are self-adjoint real operators, and $\|f_0^+\|=1$, $(f_0^+,f_{0_1}^+)$, we have $G_{11}(0,\epsilon)=\Big(\mathcal{B}_\epsilon f_0^+(0,\epsilon)\,,\,f_0^+(0,\epsilon)\Big)=1+\epsilon\Big(\mathcal{B}_1 f_0^+\,,\,f_0^+\Big)+r(\epsilon^2)$. By (4.2.16) and (A.4.22)-(A.4.23) one has

$$\mathcal{B}_1 f_0^+=\begin{bmatrix}a_1^{[1]}\cos(x)\\-p_1^{[1]}\sin(x)\end{bmatrix} \tag{4.2.23}$$

and, by (4.2.22), we deduce $G_{11}(0,\epsilon)=1+r(\epsilon^2)$.

• *Expansion of* $F_{11}(0,\epsilon)=\mathtt{f}_{11}\epsilon+r(\epsilon^3)$. By (4.2.16), (4.2.17), (4.2.22), using that $\mathcal{B}_0,\mathcal{B}_1$ are self-adjoint and real, and $\mathcal{B}_0 f_1^+=0$, $\mathcal{B}_0 f_0^+=f_0^+$, we obtain

$$\begin{aligned}F_{11}(0,\epsilon)&=\epsilon\Big[\Big(\mathcal{B}_1 f_1^+\,,\,f_0^+\Big)+\Big(f_{1_1}^+\,,\,f_0^+\Big)\Big]\\&\quad+\epsilon^2\Big[\Big(\mathcal{B}_2 f_1^+\,,\,f_0^+\Big)+\Big(\mathcal{B}_1 f_1^+\,,\,f_{0_1}^+\Big)+\Big(\mathcal{B}_1 f_0^+\,,\,f_{1_1}^+\Big)+\Big(f_{1_2}^+\,,\,f_0^+\Big)+\Big(\mathcal{B}_0 f_{1_1}^+\,,\,f_{0_1}^+\Big)\Big]+r(\epsilon^3)\,.\end{aligned}$$

By (4.2.17), (4.2.19), (4.2.20), (4.2.22), (4.2.23), all these scalar products vanish but the first one, and then

$$F_{11}(0,\epsilon)=\mathtt{f}_{11}\epsilon+r(\epsilon^3)\,,\quad \mathtt{f}_{11}:=A_1=\tfrac12\big(a_1^{[1]}\mathtt{c}_{\mathtt{h}}^{\frac12}-p_1^{[1]}\mathtt{c}_{\mathtt{h}}^{-\frac12}\big)\,, \tag{4.2.24}$$

which, by substituting the expressions of $a_1^{[1]}$, $p_1^{[1]}$ in Proposition 2.1.4, gives the expression in (4.2.5).

The expansion (4.2.15) in proved.

Linear terms in μ. We now compute the terms of $\mathtt{B}_\epsilon(\mu)$ that are linear in μ. It results

$$\partial_\mu \mathtt{B}_\epsilon(0)=X+X^*\qquad\text{where}\qquad X:=\big(\mathcal{B}_\epsilon f_k^\sigma(0,\epsilon),(\partial_\mu f_{k'}^{\sigma'})(0,\epsilon)\big)_{k,k'=0,1,\sigma,\sigma'=\pm}\,. \tag{4.2.25}$$

We now prove that

$$X = \left(\begin{array}{cc|cc} \mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0 \\ \mathcal{O}(\epsilon^2) & 0 & \mathcal{O}(\epsilon) & 0 \\ \hline \mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0 \\ \mathcal{O}(\epsilon^3) & 0 & \mathcal{O}(\epsilon^2) & 0 \end{array}\right). \tag{4.2.26}$$

The matrix $\mathtt{L}_\epsilon(0)$ in (4.2.14) where $E_{22}(0,\epsilon) = 0$, represents the action of the operator $\mathscr{L}_{0,\epsilon} : \mathcal{V}_{0,\epsilon} \to \mathcal{V}_{0,\epsilon}$ in the basis $\{f_k^\sigma(0,\epsilon)\}$ and then we deduce that $\mathscr{L}_{0,\epsilon} f_1^-(0,\epsilon) = 0$, $\mathscr{L}_{0,\epsilon} f_0^-(0,\epsilon) = 0$. Thus also $\mathcal{B}_\epsilon f_1^-(0,\epsilon) = 0$, $\mathcal{B}_\epsilon f_0^-(0,\epsilon) = 0$, and the second and the fourth column of the matrix X in (4.2.26) are zero. To compute the other two columns we use the expansion of the derivatives. In view of (4.1.3)-(4.1.6) and by denoting with a dot the derivative w.r.t. μ, one has

$$\begin{aligned} \dot f_1^+(0,\epsilon) &= \frac{\mathrm{i}}{4}\gamma_{\mathtt{h}} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12} \sin(x) \\ \mathtt{c}_{\mathtt{h}}^{-\frac12} \cos(x) \end{bmatrix} + \mathrm{i}\,\epsilon \begin{bmatrix} odd(x) \\ even(x) \end{bmatrix} + \mathcal{O}(\epsilon^2), \quad \dot f_0^+(0,\epsilon) = \mathrm{i}\,\epsilon \begin{bmatrix} odd(x) \\ even_0(x) \end{bmatrix} + \mathcal{O}(\epsilon^2), \\ \dot f_1^-(0,\epsilon) &= \frac{\mathrm{i}}{4}\gamma_{\mathtt{h}} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{\frac12} \cos(x) \\ -\mathtt{c}_{\mathtt{h}}^{-\frac12} \sin(x) \end{bmatrix} + \mathrm{i}\,\epsilon \begin{bmatrix} even(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2), \quad \dot f_0^-(0,\epsilon) = \mathrm{i}\,\epsilon \begin{bmatrix} even_0(x) \\ odd(x) \end{bmatrix} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.2.27}$$

In view of (1.2.4), (4.1.3)-(4.1.6), (4.2.14), (4.1.8), (4.2.21),(4.2.24), and since $\mathcal{B}_\epsilon f_k^\sigma(0,\epsilon) = -\mathcal{J}\mathscr{L}_\epsilon f_k^\sigma(0,\epsilon)$, we have

$$\begin{aligned} \mathcal{B}_\epsilon f_1^+(0,\epsilon) &= E_{11}(0,\epsilon)\,\mathcal{J} f_1^-(0,\epsilon) + F_{11}(0,\epsilon)\,\mathcal{J} f_0^- = \epsilon \begin{bmatrix} \mathtt{f}_{11} \\ 0 \end{bmatrix} + \epsilon^2 \mathtt{e}_{11} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{-\frac12} \cos(x) \\ \mathtt{c}_{\mathtt{h}}^{\frac12} \sin(x) \end{bmatrix} + \mathcal{O}(\epsilon^3), \\ \mathcal{B}_\epsilon f_0^+(0,\epsilon) &= F_{11}(0,\epsilon)\,\mathcal{J} f_1^-(0,\epsilon) + G_{11}(0,\epsilon)\,\mathcal{J} f_0^- = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \epsilon \mathtt{f}_{11} \begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{-\frac12} \cos(2x) \\ \mathtt{c}_{\mathtt{h}}^{\frac12} \sin(2x) \end{bmatrix} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.2.28}$$

We deduce (4.2.26) by (4.2.27) and (4.2.28).

Quadratic terms in μ. By denoting with a double dot the double derivative w.r.t. μ, we have

$$\partial_\mu^2 \mathtt{B}_0(0) = \Big(\mathcal{B}_0 f_k^\sigma\,,\, \ddot f_{k'}^{\sigma'}(0,0)\Big) + \Big(\ddot f_k^\sigma(0,0)\,,\, \mathcal{B}_0 f_{k'}^{\sigma'}\Big) + 2\,\Big(\mathcal{B}_0 \dot f_k^\sigma(0,0)\,,\, \dot f_{k'}^{\sigma'}(0,0)\Big) =: Y + Y^* + 2Z\,. \tag{4.2.29}$$

We claim that $Y = 0$. Indeed, its first, second and fourth column are zero, since $\mathcal{B}_0 f_k^\sigma = 0$

for $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$. The third column is also zero by noting that $\mathcal{B}_0 f_0^+ = f_0^+$ and

$$\ddot{f}_1^+(0,0) = \begin{bmatrix} even_0(x) + \mathrm{i}\, odd(x) \\ odd(x) + \mathrm{i}\, even_0(x) \end{bmatrix}, \quad \ddot{f}_1^-(0,0) = \begin{bmatrix} odd(x) + \mathrm{i}\, even_0(x) \\ even_0(x) + \mathrm{i}\, odd(x) \end{bmatrix}, \quad \ddot{f}_0^+(0,0) = \ddot{f}_0^-(0,0) = 0\,.$$

We claim that

$$Z = \Big(\mathcal{B}_0 \dot{f}_k^\sigma(0,0)\,,\, \dot{f}_{k'}^{\sigma'}(0,0)\Big)_{\substack{k,k'=0,1,\\ \sigma,\sigma'=\pm}} = \left(\begin{array}{cc|cc} \zeta_{\mathtt{h}} & 0 & 0 & 0 \\ 0 & \zeta_{\mathtt{h}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{4.2.30}$$

with $\zeta_{\mathtt{h}}$ as in (4.2.10). Indeed, by (4.2.27), we have $\dot{f}_0^+(0,0) = \dot{f}_0^-(0,0) = 0$. Therefore the last two columns of Z, and by self-adjointness the last two rows, are zero. By (4.2.16), (4.2.27) we obtain the matrix (4.2.30) with

$$\zeta_{\mathtt{h}} := \Big(\mathcal{B}_0 \dot{f}_1^+(0,0)\,,\, \dot{f}_1^+(0,0)\Big) = \Big(\mathcal{B}_0 \dot{f}_1^-(0,0)\,,\, \dot{f}_1^-(0,0)\Big) = \tfrac{1}{8}\mathtt{c}_{\mathtt{h}}\gamma_{\mathtt{h}}^2\,.$$

In conclusion (4.2.11), (4.2.25), (4.2.26), (4.2.29), the fact that $Y = 0$ and (4.2.30) imply (4.2.9), using also the selfadjointness of $\mathtt{B}_\epsilon$ and (2.2.27). □

We now consider $\mathcal{B}^\flat$.

Lemma 4.2.3. (Expansion of $\mathtt{B}^\flat$) *The self-adjoint and reversibility-preserving matrix $\mathtt{B}^\flat$ associated, as in* (2.2.26)*, to the self-adjoint and reversibility-preserving operator $\mathcal{B}^\flat$, defined in* (4.2.7)*, with respect to the basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ in* (4.1.1)*, admits the expansion*

$$\mathtt{B}^\flat = \left(\begin{array}{cc|cc} -\frac{\mu^2}{4}\boldsymbol{b}_{\mathtt{h}} & \mathrm{i}\,(\frac{\mu}{2}\mathtt{e}_{12} + r_2(\mu\epsilon^2)) & 0 & 0 \\ -\mathrm{i}\,(\frac{\mu}{2}\mathtt{e}_{12} + r_2(\mu\epsilon^2)) & -\frac{\mu^2}{4}\boldsymbol{b}_{\mathtt{h}} & \mathrm{i}\, r_6(\mu\epsilon) & 0 \\ \hline 0 & -\mathrm{i}\, r_6(\mu\epsilon) & 0 & 0 \\ 0 & 0 & 0 & \mu\tanh(\mathtt{h}\mu) \end{array}\right) + \mathcal{O}(\mu^2\epsilon, \mu^3) \tag{4.2.31}$$

where $\mathtt{e}_{12}$ is defined in (1.5.2) *and*

$$\boldsymbol{b}_{\mathtt{h}} := \gamma_{\mathtt{h}}\mathtt{c}_{\mathtt{h}} + \mathtt{c}_{\mathtt{h}}^{-1}\mathtt{h}(1-\mathtt{c}_{\mathtt{h}}^4)(\gamma_{\mathtt{h}} - 2(1-\mathtt{c}_{\mathtt{h}}^2\mathtt{h}))\,. \tag{4.2.32}$$

Proof. We have to compute the expansion of the matrix entries $(\mathcal{B}^\flat f_k^\sigma(\mu,\epsilon), f_{k'}^{\sigma'}(\mu,\epsilon))$. First, by (4.1.6), (4.2.7) and since $\mathtt{f}_\epsilon = O(\epsilon^2)$ (cfr. (2.1.8c)) we have

$$\mathcal{B}^\flat f_0^-(\mu,\epsilon) = \begin{bmatrix} 0 \\ \mu\tanh(\mathtt{h}\mu) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{O}(\mu^2\epsilon) \end{bmatrix}.$$

Hence, by (4.1.3)-(4.1.6), the entries of the last column (and row) of $\mathtt{B}^\flat$ are

$$(\mathcal{B}^\flat f_0^-(\mu,\epsilon), f_1^+(\mu,\epsilon)) = \mathcal{O}(\mu^2\epsilon)\,, \quad (\mathcal{B}^\flat f_0^-(\mu,\epsilon), f_1^-(\mu,\epsilon)) = \mu\tanh(\mathtt{h}\mu)\mathcal{O}(\epsilon^2) + \mathcal{O}(\mu^2\epsilon^2) = \mathcal{O}(\mu^2\epsilon^2)$$
$$(\mathcal{B}^\flat f_0^-(\mu,\epsilon), f_0^+(\mu,\epsilon)) = \mathcal{O}(\mu^2\epsilon,\mu^3)\,, \quad (\mathcal{B}^\flat f_0^-(\mu,\epsilon), f_0^-(\mu,\epsilon)) = \mu\tanh(\mathtt{h}\mu) + \mathcal{O}(\mu^2\epsilon)\,,$$

in agreement with (4.2.31).

In order to compute the other matrix entries we expand $\mathcal{B}^\flat$ in (4.2.7) at $\mu = 0$, obtaining

$$\mathcal{B}^\flat = \mu\mathcal{B}_1^\flat(0) + \mu\mathcal{R}^\flat(\epsilon) + \mu^2\mathcal{B}_2^\flat + \mathcal{O}(\mu^2\epsilon,\mu^3)\,, \quad \text{where}$$
$$\mathcal{B}_1^\flat(0) := \Big[\mathtt{h}D(1-\tanh^2(\mathtt{h}|D|)) + \operatorname{sgn}(D)\tanh(\mathtt{h}|D|)\Big]\Pi_{\mathbb{I}}\,, \quad \Pi_{\mathbb{I}} := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id}\end{bmatrix}, \tag{4.2.33}$$
$$\mathcal{R}^\flat(\epsilon) := \mathcal{O}(\epsilon^2)\Pi_{\mathbb{I}}\,, \qquad \mathcal{B}_2^\flat := \Big[\mathtt{h}(1-\tanh^2(\mathtt{h}|D|))(1-\mathtt{h}\tanh(\mathtt{h}|D|)|D|)\Big]\Pi_{\mathbb{I}}\,.$$

We note that

$$\mu(\mathcal{R}^\flat(\epsilon)f_k^\sigma(\mu,\epsilon), f_{k'}^{\sigma'}(\mu,\epsilon)) = \mu(\mathcal{R}^\flat f_k^\sigma(0,\epsilon), f_{k'}^{\sigma'}(0,\epsilon)) + \mathcal{O}(\mu^2\epsilon^2) = \begin{cases}\mathcal{O}(\mu^2\epsilon^2) & \text{if } \sigma = \sigma'\,,\\ \mathcal{O}(\mu\epsilon^2) & \text{if } \sigma \neq \sigma'\,.\end{cases} \tag{4.2.34}$$

Indeed, if $\sigma = \sigma'$, $(\mathcal{R}^\flat f_k^\sigma(0,\epsilon), f_{k'}^{\sigma'}(0,\epsilon))$ is real by (2.2.27), but purely imaginary too, since the operator $\mathcal{R}^\flat$ is purely imaginary (as $\mathcal{B}^\flat$ is) and the basis $\{f_k^\pm(0,\epsilon)\}_{k=0,1}$ is real. The terms (4.2.34) contribute to $r_2(\mu\epsilon^2)$ and $r_6(\epsilon\mu)$ in (4.2.31).

Next we compute the other scalar products. By (4.1.3), (4.2.33), and the identities $\operatorname{sgn}(D)\sin(kx) = -\mathrm{i}\cos(kx)$ and $\operatorname{sgn}(D)\cos(kx) = \mathrm{i}\sin(kx)$ for any $k\in\mathbb{N}$, we have

$$\mu\mathcal{B}_1^\flat(0)f_1^+(\mu,\epsilon) = -\mathrm{i}\,\mu\flat_1\begin{bmatrix}0\\ \cos(x)\end{bmatrix} - \frac{\mu^2}{4}\gamma_{\mathtt{h}}\flat_1\begin{bmatrix}0\\ \sin(x)\end{bmatrix} - \mathrm{i}\,\mu\epsilon\flat_2\begin{bmatrix}0\\ \cos(2x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)$$

where

$$\flat_1 := \mathtt{c}_{\mathtt{h}}^{-\frac12}(\mathtt{c}_{\mathtt{h}}^2 + (1-\mathtt{c}_{\mathtt{h}}^4)\mathtt{h})$$
$$\flat_2 := \beta_{\mathtt{h}}\Big(\tanh(2\mathtt{h}) + 2\mathtt{h}(1-\tanh^2(2\mathtt{h}))\Big) = \beta_{\mathtt{h}}\Big(\frac{2\mathtt{c}_{\mathtt{h}}^2}{1+\mathtt{c}_{\mathtt{h}}^4} + 2\mathtt{h}(1-\frac{4\mathtt{c}_{\mathtt{h}}^4}{(1+\mathtt{c}_{\mathtt{h}}^4)^2})\Big)\,. \tag{4.2.35}$$

Similarly $\mu^2\mathcal{B}_2^\flat f_1^+(\mu,\epsilon) = \mu^2\flat_3\begin{bmatrix}0\\ \sin(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)$, where

$$\flat_3 := \mathtt{h}(1-\tanh^2(\mathtt{h}))(1-\tanh(\mathtt{h})\mathtt{h})\mathtt{c}_{\mathtt{h}}^{-\frac12} = \mathtt{h}(1-\mathtt{c}_{\mathtt{h}}^4)(1-\mathtt{c}_{\mathtt{h}}^2\mathtt{h})\mathtt{c}_{\mathtt{h}}^{-\frac12}\,. \tag{4.2.36}$$

Analogously, using (4.1.4),

$$\mu\mathcal{B}_1^\flat(0)f_1^-(\mu,\epsilon) = \mathrm{i}\,\mu\flat_1\begin{bmatrix}0\\ \sin(x)\end{bmatrix} - \frac{\mu^2}{4}\gamma_{\mathtt{h}}\flat_1\begin{bmatrix}0\\ \cos(x)\end{bmatrix} + \mathrm{i}\,\mu\epsilon\flat_3\begin{bmatrix}0\\ \sin(2x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ odd(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)\,,$$

and $\mu^2\mathcal{B}_2^\flat f_1^-(\mu,\epsilon) = \mu^2\flat_3\begin{bmatrix}0\\ \cos(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon,\mu^3)$, with $\flat_j$, $j=1,2,3$, defined in (4.2.35) and (4.2.36). In addition, by (4.1.5)-(4.1.6), we get that

$$\mu\mathcal{B}_1^\flat(0)f_0^+(\mu,\epsilon) = \mathrm{i}\,\mu\epsilon\delta_{\mathtt{h}}\flat_1\begin{bmatrix}0\\ \cos(x)\end{bmatrix} + \mathrm{i}\,\mathcal{O}(\mu\epsilon^2)\begin{bmatrix}0\\ even_0(x)\end{bmatrix} + \mathcal{O}(\mu^2\epsilon)\,,\ \ \mu^2\mathcal{B}_2^\flat f_0^+(\mu,\epsilon) = \begin{bmatrix}0\\ \mathcal{O}(\mu^2\epsilon)\end{bmatrix}$$

with $\flat_1$ in (4.2.35). By taking the scalar products of the above expansions of $\mathcal{B}^\flat f_k^\sigma(\mu,\epsilon)$ with the functions $f_{k'}^{\sigma'}(\mu,\epsilon)$ expanded as in (4.1.3)-(4.1.6) we obtain that (recall that the scalar product is conjugate-linear in the second component)

$$\begin{aligned}
&\big(\mu\mathcal{B}_1^\flat(0)f_1^+(\mu,\epsilon), f_1^+(\mu,\epsilon)\big)\,,\ \big(\mu\mathcal{B}_1^\flat(0)f_1^-(\mu,\epsilon), f_1^-(\mu,\epsilon)\big) = -\frac{\mu^2}{4}\gamma_{\mathtt{h}}\flat_1\mathtt{c}_{\mathtt{h}}^{-\frac12} + \mathcal{O}(\mu^2\epsilon,\mu^3)\\
&\big(\mu^2\mathcal{B}_2^\flat f_1^+(\mu,\epsilon), f_1^+(\mu,\epsilon)\big)\,,\ \big(\mu^2\mathcal{B}_2^\flat f_1^-(\mu,\epsilon), f_1^-(\mu,\epsilon)\big) = \frac{\mu^2}{2}\flat_3\mathtt{c}_{\mathtt{h}}^{-\frac12} + \mathcal{O}(\mu^2\epsilon,\mu^3)
\end{aligned}$$

and, recalling (4.2.33), (4.2.35), (4.2.36), we deduce the expansion of the entries $(1,1)$ and $(2,2)$ of the matrix $\mathtt{B}^\flat$ in (4.2.31) with $\mathbf{b}_{\mathtt{h}} = \mathtt{c}_{\mathtt{h}}^{-\frac12}(\gamma_{\mathtt{h}}\flat_1 - 2\flat_3)$ in (4.2.32). Moreover

$$\big(\mu\mathcal{B}_1^\flat(0)f_1^-(\mu,\epsilon), f_1^+(\mu,\epsilon)\big) = \mathrm{i}\,\frac{\mu}{2}\mathtt{e}_{12} + \mathcal{O}(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\,,\ \ \big(\mu^2\mathcal{B}_2^\flat f_1^-(\mu,\epsilon), f_1^+(\mu,\epsilon)\big) = \mathcal{O}(\mu^3,\mu^2\epsilon)\,,$$

where $\mathtt{e}_{12} := \flat_1\mathtt{c}_{\mathtt{h}}^{-\frac12}$ is equal to (1.5.2). Finally we obtain

$$\begin{aligned}
\big(\mu(\mathcal{B}_1^\flat(0)+\mu\mathcal{B}_2^\flat)f_1^-(\mu,\epsilon), f_0^+(\mu,\epsilon)\big) &= \mathcal{O}(\mu\epsilon,\mu^3)\\
\big(\mu(\mathcal{B}_1^\flat(0)+\mu\mathcal{B}_2^\flat)f_1^+(\mu,\epsilon), f_0^+(\mu,\epsilon)\big) &= \mathcal{O}(\mu^3,\mu^2\epsilon)\,,\\
\big(\mu(\mathcal{B}_1^\flat(0)+\mu\mathcal{B}_2^\flat)f_0^+(\mu,\epsilon), f_0^+(\mu,\epsilon)\big) &= \mathcal{O}(\mu^2\epsilon^2)\,.
\end{aligned}$$

The expansion (4.2.31) is proved. □

Finally we consider $\mathcal{B}^\sharp$.

Lemma 4.2.4. (Expansion of $\mathtt{B}^\sharp$) *The self-adjoint and reversibility-preserving matrix $\mathtt{B}^\sharp$ associated, as in* (2.2.26)*, to the self-adjoint and reversibility-preserving operators $\mathcal{B}^\sharp$, defined in* (4.2.8)*, with respect to the basis $\mathcal{F}$ of $\mathcal{V}_{\mu,\epsilon}$ in* (4.1.1)*, admits the expansion*

$$\mathtt{B}^\sharp = \left(\begin{array}{cc|cc}
0 & \mathrm{i}\,r_2(\mu\epsilon^2) & 0 & \mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac12} + \mathrm{i}\,r_4(\mu\epsilon^2)\\
-\mathrm{i}\,r_2(\mu\epsilon^2) & 0 & -\mathrm{i}\,r_6(\mu\epsilon) & 0\\
\hline
0 & \mathrm{i}\,r_6(\mu\epsilon) & 0 & -\mathrm{i}\,r_9(\mu\epsilon^2)\\
-\mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac12} - \mathrm{i}\,r_4(\mu\epsilon^2) & 0 & \mathrm{i}\,r_9(\mu\epsilon^2) & 0
\end{array}\right) + \mathcal{O}(\mu^2\epsilon)\,. \tag{4.2.37}$$

Proof. Since $\mathcal{B}^\sharp = -\mathrm{i}\,\mu p_\epsilon \mathcal{J}$ and $p_\epsilon = \mathcal{O}(\epsilon)$ by (2.1.8a), we have the expansion

$$(\mathcal{B}^\sharp f_k^\sigma(\mu,\epsilon), f_{k'}^{\sigma'}(\mu,\epsilon)) = (\mathcal{B}^\sharp f_k^\sigma(0,\epsilon), f_{k'}^{\sigma'}(0,\epsilon)) + \mathcal{O}(\mu^2\epsilon)\,. \tag{4.2.38}$$

The matrix entries $(\mathcal{B}^\sharp f_k^\sigma(0,\epsilon), f_{k'}^\sigma(0,\epsilon))$, $k,k' = 0,1$, $\sigma = \{\pm\}$ are zero, because they are simultaneously real by (2.2.27), and purely imaginary, being the operator $\mathcal{B}^\sharp$ purely imaginary and the basis $\{f_k^\pm(0,\epsilon)\}_{k=0,1}$ real. Hence $\mathtt{B}^\sharp$ has the form

$$\mathtt{B}^\sharp = \left(\begin{array}{cc|cc} 0 & \mathrm{i}\,\beta & 0 & \mathrm{i}\,\delta \\ -\mathrm{i}\,\beta & 0 & -\mathrm{i}\,\gamma & 0 \\ \hline 0 & \mathrm{i}\,\gamma & 0 & \mathrm{i}\,\eta \\ -\mathrm{i}\,\delta & 0 & -\mathrm{i}\,\eta & 0 \end{array}\right) + \mathcal{O}(\mu^2\epsilon) \quad \text{where} \quad \begin{cases} \Big(\mathcal{B}^\sharp f_1^-(0,\epsilon)\,,\, f_1^+(0,\epsilon)\Big) =: \mathrm{i}\,\beta\,, \\ \Big(\mathcal{B}^\sharp f_1^-(0,\epsilon)\,,\, f_0^+(0,\epsilon)\Big) =: \mathrm{i}\,\gamma\,, \\ \Big(\mathcal{B}^\sharp f_0^-(0,\epsilon)\,,\, f_1^+(0,\epsilon)\Big) =: \mathrm{i}\,\delta\,, \\ \Big(\mathcal{B}^\sharp f_0^-(0,\epsilon)\,,\, f_0^+(0,\epsilon)\Big) =: \mathrm{i}\,\eta\,, \end{cases} \tag{4.2.39}$$

and α, β, γ, δ are real numbers. As $\mathcal{B}^\sharp = \mathcal{O}(\mu\epsilon)$ in $\mathcal{L}(Y)$, we deduce that $\gamma = r(\mu\epsilon)$. Let us compute the expansion of β, δ and η. By (A.4.22) and (1.2.4) we write the operator $\mathcal{B}^\sharp$ in (4.2.8) as

$$\mathcal{B}^\sharp = \mathrm{i}\,\mu\epsilon\mathcal{B}_1^\sharp + \mathcal{O}(\mu\epsilon^2)\,, \quad \mathcal{B}_1^\sharp := 2\mathtt{c}_\mathtt{h}^{-1}\cos(x)\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}, \tag{4.2.40}$$

with $\mathcal{O}(\mu\epsilon^2) \in \mathcal{L}(Y)$. In view of (4.1.3)-(4.1.6), $f_1^\pm(0,\epsilon) = f_1^\pm + \mathcal{O}(\epsilon)$, $f_0^+(0,\epsilon) = f_0^+ + \mathcal{O}(\epsilon)$, $f_0^-(0,\epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where f_k^σ are in (4.1.2). By (4.2.40) we have $\mathcal{B}_1^\sharp f_1^- = \begin{bmatrix} \mathtt{c}_\mathtt{h}^{-\frac{3}{2}}(1+\cos(2x)) \\ \mathtt{c}_\mathtt{h}^{-\frac{1}{2}}\sin(2x) \end{bmatrix}$, $\mathcal{B}_1^\sharp f_0^- = \begin{bmatrix} 2\mathtt{c}_\mathtt{h}^{-1}\cos(x) \\ 0 \end{bmatrix}$ and then

$$\begin{aligned} \beta &= \mu\epsilon\left(\mathcal{B}_1^\sharp f_1^-\,,\, f_1^+\right) + r(\mu\epsilon^2) = r(\mu\epsilon^2)\,, \\ \delta &= \mu\epsilon\left(\mathcal{B}_1^\sharp f_0^-\,,\, f_1^+\right) + r(\mu\epsilon^2) = \mu\epsilon\mathtt{c}_\mathtt{h}^{-\frac{1}{2}} + r(\mu\epsilon^2)\,, \\ \eta &= \mu\epsilon\left(\mathcal{B}_1^\sharp f_0^-\,,\, f_0^+\right) + r(\mu\epsilon^2) = r(\mu\epsilon^2)\,. \end{aligned}$$

This proves (4.2.37). □

Lemmata 4.2.2, 4.2.3, 4.2.4 imply (4.2.1) where the matrix E has the form (4.2.2) and

$$\mathtt{e}_{22} := 2(\mathbf{b}_\mathtt{h} - 4\zeta_\mathtt{h}) = 2\gamma_\mathtt{h}\mathtt{c}_\mathtt{h} + 2\mathtt{c}_\mathtt{h}^{-1}\mathtt{h}(1-\mathtt{c}_\mathtt{h}^4)(\gamma_\mathtt{h} - 2(1-\mathtt{c}_\mathtt{h}^2\mathtt{h})) - \mathtt{c}_\mathtt{h}\gamma_\mathtt{h}^2\,,$$

with $\mathbf{b}_\mathtt{h}$ in (4.2.32) and $\zeta_\mathtt{h}$ in (4.2.10). The term $\mathtt{e}_{22}$ has the expansion in (1.5.4). Moreover

$$G := G(\mu,\epsilon) = \begin{pmatrix} 1 + r_8(\epsilon^2,\mu^2\epsilon,\mu^3) & -\mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \\ \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon,\mu^3) & \mu\tanh(\mathtt{h}\mu) + r_{10}(\mu^2\epsilon,\mu^3) \end{pmatrix} \tag{4.2.41}$$

$$F := F(\mu,\epsilon) = \begin{pmatrix} \mathtt{f}_{11}\epsilon + r_3(\epsilon^3,\mu\epsilon^2,\mu^2\epsilon,\mu^3) & \mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + \mathrm{i}\,r_4(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \\ \mathrm{i}\,r_6(\mu\epsilon,\mu^3) & r_7(\mu^2\epsilon,\mu^3) \end{pmatrix} . \qquad (4.2.42)$$

In order to deduce the expansion (4.2.3)-(4.2.4) of the matrices F, G we exploit further information for

$$\mathscr{L}_{\mu,0} := \mathcal{J}\mathcal{B}_{\mu,0}\,, \quad \mathcal{B}_{\mu,0} := \begin{bmatrix} 1 & -\mathtt{c}_{\mathtt{h}}\partial_x \\ \mathtt{c}_{\mathtt{h}}\partial_x & |D+\mu|\,\tanh\big(\mathtt{h}|D+\mu|\big) \end{bmatrix} . \qquad (4.2.43)$$

We have

Lemma 4.2.5. *At $\epsilon = 0$ the matrices are $F(\mu,0) = 0$ and $G(\mu,0) = \begin{pmatrix} 1 & 0 \\ 0 & \mu\tanh(\mathtt{h}\mu) \end{pmatrix}$.*

Proof. By Lemma 4.1.7 and (4.2.43) we have $\mathcal{B}_{\mu,0} f_0^+(\mu,0) = f_0^+$ and $\mathcal{B}_{\mu,0} f_0^-(\mu,0) = \mu\tanh(\mathtt{h}\mu) f_0^-$, for any μ. Then the lemma follows recalling (2.2.26) and the fact that $f_1^+(\mu,0)$ and $f_1^-(\mu,0)$ have zero space average by Lemma 4.1.7. □

In view of Lemma 4.2.5 we deduce that the matrices (4.2.41) and (4.2.42) have the form (4.2.3) and (4.2.4). This completes the proof of Proposition 4.2.1.

We now show that the constant $\mathtt{e}_{22}$ in (1.5.4) is positive for any depth $\mathtt{h} > 0$.

Lemma 4.2.6. *For any $\mathtt{h} > 0$ the term $\mathtt{e}_{22}$ in (1.5.4) is positive, $\mathtt{e}_{22} \to 0$ as $\mathtt{h} \to 0^+$ and $\mathtt{e}_{22} \to 1$ as $\mathtt{h} \to +\infty$. As a consequence for any $\mathtt{h}_0 > 0$ the term $\mathtt{e}_{22}$ is bounded from below uniformly in $\mathtt{h} > \mathtt{h}_0$.*

Proof. The quantity $z := \mathtt{c}_{\mathtt{h}}^2 = \tanh(\mathtt{h})$ is in $(0,1)$ for any $\mathtt{h} > 0$. Then the quadratic polynomial $(0,+\infty) \ni \mathtt{h} \mapsto (1-z^2)(1+3z^2)\mathtt{h}^2 + 2z(z^2-1)\mathtt{h} + z^2$ is positive because its discriminant $-4z^4(1-z^2)$ is negative as $0 < z^2 < 1$. The limits for $\mathtt{h} \to 0^+$ and $\mathtt{h} \to +\infty$ follow by inspection. □

4.3 Block decoupling and emergence of the Whitham-Benjamin function

In this section we block-decouple the 4×4 Hamiltonian matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ obtained in Proposition 4.2.1.

We first perform a singular symplectic and reversibility-preserving change of coordinates.

Lemma 4.3.1. (Singular symplectic rescaling) *The conjugation of the Hamiltonian and reversible matrix* $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ *obtained in Proposition 4.2.1 through the symplectic and reversibility-preserving* 4×4*-matrix*

$$Y := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \quad \textit{with} \quad Q := \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix}, \quad \mu > 0\,, \tag{4.3.1}$$

yields the Hamiltonian and reversible matrix

$$\mathtt{L}^{(1)}_{\mu,\epsilon} := Y^{-1}\mathtt{L}_{\mu,\epsilon}Y = \mathtt{J}_4\mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2E^{(1)} & \mathtt{J}_2F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & \mathtt{J}_2G^{(1)} \end{pmatrix} \tag{4.3.2}$$

where $\mathtt{B}^{(1)}_{\mu,\epsilon}$ *is a self-adjoint and reversibility-preserving* 4×4 *matrix*

$$\mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix}, \quad E^{(1)} = [E^{(1)}]^*\,,\ G^{(1)} = [G^{(1)}]^*\,, \tag{4.3.3}$$

where the 2×2 *reversibility-preserving matrices* $E^{(1)}$, $G^{(1)}$ *and* $F^{(1)}$ *extend analytically at* $\mu = 0$ *with the following expansion*

$$E^{(1)} = \begin{pmatrix} \mathtt{e}_{11}\mu\epsilon^2(1+r_1'(\epsilon,\mu\epsilon)) - \mathtt{e}_{22}\frac{\mu^3}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix}, \tag{4.3.4}$$

$$G^{(1)} = \begin{pmatrix} \mu + r_8(\mu\epsilon^2,\mu^3\epsilon) & -\mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) \\ \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \end{pmatrix}, \tag{4.3.5}$$

$$F^{(1)} = \begin{pmatrix} \mathtt{f}_{11}\mu\epsilon + r_3(\mu\epsilon^3,\mu^2\epsilon^2,\mu^3\epsilon) & \mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + \mathrm{i}\,r_4(\mu\epsilon^2,\mu^2\epsilon) \\ \mathrm{i}\,r_6(\mu\epsilon) & r_7(\mu\epsilon) \end{pmatrix} \tag{4.3.6}$$

where $\mathtt{e}_{11}, \mathtt{e}_{12}, \mathtt{e}_{22}, \mathtt{f}_{11}$ *are defined in* (4.2.5), (1.5.2), (1.5.4).

Remark 4.3.2. The matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$, a priori defined only for $\mu \neq 0$, extends analytically to the zero matrix at $\mu = 0$. For $\mu \neq 0$ the spectrum of $\mathtt{L}^{(1)}_{\mu,\epsilon}$ coincides with the spectrum of $\mathtt{L}_{\mu,\epsilon}$.

Proof. The matrix Y is symplectic, i.e. (2.2.30) holds, and since μ is real, it is reversibility preserving, i.e. satisfies (2.2.27). By (2.2.31),

$$\mathtt{B}^{(1)}_{\mu,\epsilon} = Y^*\mathtt{B}_{\mu,\epsilon}Y = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)} \end{pmatrix},$$

with, Q being self-adjoint, $E^{(1)} = QEQ = [E^{(1)}]^*$, $G^{(1)} = QGQ = [G^{(1)}]^*$ and $F^{(1)} = QFQ$. In view of (4.2.2)-(4.2.4), we obtain (4.3.4)-(4.3.6). ☐

4.3.1 Non-perturbative step of block decoupling

We first verify that the quantity $D_{\mathtt{h}} := \mathtt{h} - \frac{1}{4}\mathtt{e}_{12}^2$ is nonzero for any $\mathtt{h} > 0$. In view of the comment 3 after Theorem 1.2.3, we have that $D_{\mathtt{h}} = \mathtt{h} - c_g^2$. The non-degeneracy property $D_{\mathtt{h}} \neq 0$ corresponds to that in Bridges-Mielke [21, p.183] and [92, p.409].

Lemma 4.3.3. *For any* $\mathtt{h} > 0$

$$\mathtt{D}_{\mathtt{h}} := \mathtt{h} - \tfrac{1}{4}\mathtt{e}_{12}^2 > 0\,, \quad \textit{and} \quad \lim_{\mathtt{h}\to 0^+} \mathtt{D}_{\mathtt{h}} = 0\,. \tag{4.3.7}$$

Proof. We write $\mathtt{D}_{\mathtt{h}} = (\sqrt{\mathtt{h}} + \frac{1}{2}\mathtt{e}_{12})(\sqrt{\mathtt{h}} - \frac{1}{2}\mathtt{e}_{12})$ whose first factor is positive for $\mathtt{h} > 0$. We claim that also the second factor is positive. In view of (1.5.2) it is equal to $\frac{1}{2}\mathtt{c}_{\mathtt{h}}^{-1} f(\mathtt{h})$ with

$$f(\mathtt{h}) := \big(\sqrt{\mathtt{h}}\tanh(\mathtt{h}) - \sqrt{\mathtt{h}} + \sqrt{\tanh(\mathtt{h})}\big)\big(\sqrt{\mathtt{h}}\tanh(\mathtt{h}) + \sqrt{\mathtt{h}} - \sqrt{\tanh(\mathtt{h})}\big) =: q(\mathtt{h})p(\mathtt{h})\,.$$

The function $p(\mathtt{h})$ is positive since $\mathtt{h} > \tanh(\mathtt{h})$ for any $\mathtt{h} > 0$. We claim that also the function $q(\mathtt{h})$ is positive. Indeed its derivative

$$q'(\mathtt{h}) = \frac{1-\tanh(\mathtt{h})}{2\sqrt{\mathtt{h}}\sqrt{\tanh(\mathtt{h})}}\Big(- \sqrt{\tanh(\mathtt{h})} + \sqrt{\mathtt{h}} + \sqrt{\mathtt{h}}\tanh(\mathtt{h})\Big) + \sqrt{\mathtt{h}}(1-\tanh^2(\mathtt{h})) > 0$$

for any $\mathtt{h} > 0$. Since $q(0) = 0$ we deduce that $q(\mathtt{h}) > 0$ for any $\mathtt{h} > 0$. This proves the lemma. □

We now state the main result of this section.

Lemma 4.3.4. (Step of block decoupling) *There exists a* 2×2 *reversibility-preserving matrix* X*, analytic in* (μ,ϵ)*, of the form*

$$\begin{aligned} X &:= \begin{pmatrix} x_{11} & \mathrm{i}\,x_{12} \\ \mathrm{i}\,x_{21} & x_{22} \end{pmatrix} \qquad \textit{with} \quad x_{ij} \in \mathbb{R}\,,\ i,j = 1,2\,, \\ &= \begin{pmatrix} r_{11}(\epsilon) & \mathrm{i}\,r_{12}(\epsilon) \\ -\mathrm{i}\,\frac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}(\mathtt{e}_{12}\mathtt{f}_{11} + 2\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}})\epsilon + \mathrm{i}\,r_{21}(\epsilon^2,\mu\epsilon) & \frac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}(\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\mathtt{e}_{12} + 2\mathtt{h}\mathtt{f}_{11})\epsilon + r_{22}(\epsilon^2,\mu\epsilon) \end{pmatrix}, \end{aligned} \tag{4.3.8}$$

where $\mathtt{e}_{12}$, $\mathtt{f}_{11}$ *are defined in* (1.5.2), (4.2.5) *and* $\mathtt{D}_{\mathtt{h}}$ *is the positive constant in* (4.3.7)*, such that the following holds true. By conjugating the Hamiltonian and reversible matrix* $\mathtt{L}_{\mu,\epsilon}^{(1)}$*, defined in* (4.3.2)*, with the symplectic and reversibility-preserving* 4×4 *matrix*

$$\exp\big(S^{(1)}\big)\,, \quad \textit{where} \quad S^{(1)} := \mathtt{J}_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}, \quad \Sigma := \mathtt{J}_2 X\,, \tag{4.3.9}$$

we get the Hamiltonian and reversible matrix

$$\mathtt{L}^{(2)}_{\mu,\epsilon} := \exp\Big(S^{(1)}\Big)\,\mathtt{L}^{(1)}_{\mu,\epsilon}\exp\Big(-S^{(1)}\Big) = \mathtt{J}_4\mathtt{B}^{(2)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(2)} & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2[F^{(2)}]^* & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \tag{4.3.10}$$

where the reversibility-preserving 2×2 *self-adjoint matrix* $[E^{(2)}]^* = E^{(2)}$ *has the form*

$$E^{(2)} = \begin{pmatrix} \mu\epsilon^2\mathtt{e}_{WB} + r_1'(\mu\epsilon^3,\mu^2\epsilon^2) - \mathtt{e}_{22}\frac{\mu^3}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac12\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac12\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu}{8}(1+r_5(\epsilon,\mu)) \end{pmatrix}, \tag{4.3.11}$$

where

$$\mathtt{e}_{WB} = \mathtt{e}_{11} - \mathtt{D}_{\mathtt{h}}^{-1}(\mathtt{c}_{\mathtt{h}}^{-1} + \mathtt{h}\mathtt{f}_{11}^2 + \mathtt{e}_{12}\mathtt{f}_{11}\mathtt{c}_{\mathtt{h}}^{-\frac12}) \tag{4.3.12}$$

(with constants $\mathtt{e}_{11}$*,* $\mathtt{D}_{\mathtt{h}}$*,* $\mathtt{f}_{11}$*,* $\mathtt{e}_{12}$*, defined in* (4.2.5)*,* (4.3.7)*,* (1.5.2)*), is the Whitham-Benjamin function defined in* (1.5.1)*, the reversibility-preserving* 2×2 *self-adjoint matrix* $[G^{(2)}]^* = G^{(2)}$ *has the form*

$$G^{(2)} = \begin{pmatrix} \mu + r_8(\mu\epsilon^2,\mu^3\epsilon) & -\mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) \\ \mathrm{i}\,r_9(\mu\epsilon^2,\mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \end{pmatrix}, \tag{4.3.13}$$

and

$$F^{(2)} = \begin{pmatrix} r_3(\mu\epsilon^3) & \mathrm{i}\,r_4(\mu\epsilon^3) \\ \mathrm{i}\,r_6(\mu\epsilon^3) & r_7(\mu\epsilon^3) \end{pmatrix}. \tag{4.3.14}$$

The rest of the section is devoted to the proof of Lemma 4.3.4. For simplicity let $S = S^{(1)}$.

The matrix $\exp(S)$ is symplectic and reversibility-preserving because the matrix S in (4.3.9) is Hamiltonian and reversibility-preserving, by Lemma 2.2.8. Note that S is reversibility preserving since X has the form (4.3.8).

We now expand in Lie series the Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S)\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S)$.

We split $\mathtt{L}^{(1)}_{\mu,\epsilon}$ into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}^{(1)}_{\mu,\epsilon} = D^{(1)} + R^{(1)}, \tag{4.3.15}$$

$$D^{(1)} := \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} := \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix}, \quad R^{(1)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & 0 \end{pmatrix},$$

and we perform the Lie expansion

$$\mathtt{L}^{(2)}_{\mu,\epsilon} = \exp(S)\mathtt{L}^{(1)}_{\mu,\epsilon}\exp(-S) = D^{(1)} + \Big[S\,,\,D^{(1)}\Big] + \frac12[S,[S,D^{(1)}]] + R^{(1)} + [S,R^{(1)}] \tag{4.3.16}$$

$$+\frac{1}{2}\int_0^1 (1-\tau)^2 \exp(\tau S)\mathrm{ad}_S^3(D^{(1)})\exp(-\tau S)\,\mathrm{d}\tau + \int_0^1 (1-\tau)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\,\mathrm{d}\tau\,.$$

We look for a 4×4 matrix S as in (4.3.9) (which is Hamiltonian, reversibility-preserving and off-diagonal as the term $R^{(1)}$ we wish to eliminate) that solves the homological equation $R^{(1)} + \left[S\,,\,D^{(1)}\right] = 0$, which, recalling (4.3.15), reads

$$\begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} + \mathtt{J}_2\Sigma D_0 - D_1\mathtt{J}_2\Sigma \\ \mathtt{J}_2[F^{(1)}]^* + \mathtt{J}_2\Sigma^* D_1 - D_0\mathtt{J}_2\Sigma^* & 0 \end{pmatrix} = 0\,. \tag{4.3.17}$$

Note that the equation $\mathtt{J}_2 F^{(1)} + \mathtt{J}_2\Sigma D_0 - D_1\mathtt{J}_2\Sigma = 0$ implies also $\mathtt{J}_2[F^{(1)}]^* + \mathtt{J}_2\Sigma^* D_1 - D_0\mathtt{J}_2\Sigma^* = 0$ and viceversa. Thus, writing $\Sigma = \mathtt{J}_2 X$, namely $X = -\mathtt{J}_2\Sigma$, the equation (4.3.17) amounts to solve the "Sylvester" equation

$$D_1 X - X D_0 = -\mathtt{J}_2 F^{(1)}\,. \tag{4.3.18}$$

We write the matrices $E^{(1)}, F^{(1)}, G^{(1)}$ in (4.3.2) as

$$E^{(1)} = \begin{pmatrix} E_{11}^{(1)} & \mathrm{i}\,E_{12}^{(1)} \\ -\mathrm{i}\,E_{12}^{(1)} & E_{22}^{(1)} \end{pmatrix}\,,\quad F^{(1)} = \begin{pmatrix} F_{11}^{(1)} & \mathrm{i}\,F_{12}^{(1)} \\ \mathrm{i}\,F_{21}^{(1)} & F_{22}^{(1)} \end{pmatrix}\,,\quad G^{(1)} = \begin{pmatrix} G_{11}^{(1)} & \mathrm{i}\,G_{12}^{(1)} \\ -\mathrm{i}\,G_{12}^{(1)} & G_{22}^{(1)} \end{pmatrix} \tag{4.3.19}$$

where the real numbers $E_{ij}^{(1)}, F_{ij}^{(1)}, G_{ij}^{(1)}$, $i,j=1,2$, have the expansion in (4.3.4), (4.3.5), (4.3.6). Thus, by (4.3.15), (4.3.8) and (4.3.19), the equation (4.3.18) amounts to solve the 4×4 real linear system

$$\underbrace{\begin{pmatrix} G_{12}^{(1)} - E_{12}^{(1)} & G_{11}^{(1)} & E_{22}^{(1)} & 0 \\ G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} & 0 & -E_{22}^{(1)} \\ E_{11}^{(1)} & 0 & G_{12}^{(1)} - E_{12}^{(1)} & -G_{11}^{(1)} \\ 0 & -E_{11}^{(1)} & -G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} \end{pmatrix}}_{=:\mathcal{A}} \underbrace{\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}}_{=:\vec{x}} = \underbrace{\begin{pmatrix} -F_{21}^{(1)} \\ F_{22}^{(1)} \\ -F_{11}^{(1)} \\ F_{12}^{(1)} \end{pmatrix}}_{=:\vec{f}}\,. \tag{4.3.20}$$

We solve this system using the following result, verified by a direct calculation.

Lemma 4.3.5. *The determinant of the matrix*

$$A := \begin{pmatrix} a & b & c & 0 \\ d & a & 0 & -c \\ e & 0 & a & -b \\ 0 & -e & -d & a \end{pmatrix} \tag{4.3.21}$$

where a, b, c, d, e are real numbers, is

$$\det A = a^4 - 2a^2(bd + ce) + (bd - ce)^2 = (bd - a^2)^2 - 2ce\big(a^2 + bd - \frac{1}{2}ce\big)\,. \qquad (4.3.22)$$

If $\det A \neq 0$ *then* A *is invertible and*

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a\left(a^2 - bd - ce\right) & b\left(-a^2 + bd - ce\right) & -c\left(a^2 + bd - ce\right) & -2abc \\ d\left(-a^2 + bd - ce\right) & a\left(a^2 - bd - ce\right) & 2acd & -c\left(-a^2 - bd + ce\right) \\ -e\left(a^2 + bd - ce\right) & 2abe & a\left(a^2 - bd - ce\right) & b\left(a^2 - bd + ce\right) \\ -2ade & -e\left(-a^2 - bd + ce\right) & d\left(a^2 - bd + ce\right) & a\left(a^2 - bd - ce\right) \end{pmatrix}. \qquad (4.3.23)$$

The Sylvester matrix $\mathcal{A}$ in (4.3.20) has the form (4.3.21) where, by (4.3.4)-(4.3.6) and since $\tanh(\mathtt{h}\mu) = \mathtt{h}\mu + r(\mu^3)$,

$$\begin{aligned} &a = G_{12}^{(1)} - E_{12}^{(1)} = -\mathtt{e}_{12}\frac{\mu}{2}(1 + r(\epsilon^2, \mu\epsilon, \mu^2))\,,\ b = G_{11}^{(1)} = \mu + r_8(\mu\epsilon^2, \mu^3\epsilon)\,, \qquad (4.3.24) \\ &c = E_{22}^{(1)} = -\mathtt{e}_{22}\frac{\mu}{8}(1 + r_5(\epsilon, \mu))\,,\ d = G_{22}^{(1)} = \mu\mathtt{h} + r(\mu\epsilon, \mu^3)\,,\ e = E_{11}^{(1)} = r(\mu\epsilon^2, \mu^3)\,, \end{aligned}$$

where $\mathtt{e}_{12}$ and $\mathtt{e}_{22}$, defined respectively in (1.5.2), (1.5.4), are positive for any $\mathtt{h} > 0$.

By (4.3.22), the determinant of the matrix $\mathcal{A}$ is

$$\det \mathcal{A} = (bd - a^2)^2 + r(\mu^4\epsilon^2, \mu^6) = \mu^4 \mathtt{D}_{\mathtt{h}}^2(1 + r(\epsilon, \mu^2)) \qquad (4.3.25)$$

where $\mathtt{D}_{\mathtt{h}}$ is defined in (4.3.7). By (4.3.23), (4.3.24), (4.3.25) and, since $\mathtt{D}_{\mathtt{h}} = \mathtt{h} - \frac{1}{4}\mathtt{e}_{12}^2$, we obtain

$$\mathcal{A}^{-1} = (1 + r(\epsilon, \mu))\frac{1}{\mu\mathtt{D}_{\mathtt{h}}^2}\begin{pmatrix} \frac{1}{2}\mathtt{e}_{12}\mathtt{D}_{\mathtt{h}} & \mathtt{D}_{\mathtt{h}} & \frac{1}{32}\mathtt{e}_{22}(\mathtt{e}_{12}^2 + 4\mathtt{h}) & -\frac{1}{8}\mathtt{e}_{12}\,\mathtt{e}_{22} \\ \mathtt{h}\mathtt{D}_{\mathtt{h}} & \frac{1}{2}\mathtt{e}_{12}\mathtt{D}_{\mathtt{h}} & \frac{1}{8}\mathtt{e}_{12}\mathtt{e}_{22}\mathtt{h} & -\frac{1}{32}\mathtt{e}_{22}\,(\mathtt{e}_{12}^2 + 4\mathtt{h}) \\ r(\epsilon^2, \mu^2) & r(\epsilon^2, \mu^2) & \frac{1}{2}\mathtt{e}_{12}\mathtt{D}_{\mathtt{h}} & -\mathtt{D}_{\mathtt{h}} \\ r(\epsilon^2, \mu^2) & r(\epsilon^2, \mu^2) & -\mathtt{h}\mathtt{D}_{\mathtt{h}} & \frac{1}{2}\mathtt{e}_{12}\mathtt{D}_{\mathtt{h}} \end{pmatrix}. \qquad (4.3.26)$$

Therefore, for any $\mu \neq 0$, there exists a unique solution $\vec{x} = \mathcal{A}^{-1}\vec{f}$ of the linear system (4.3.20), namely a unique matrix X which solves the Sylvester equation (4.3.18).

Lemma 4.3.6. *The matrix solution X of the Sylvester equation* (4.3.18) *is analytic in* (μ, ϵ)*, and admits an expansion as in* (4.3.8).

Proof. By (4.3.20), (4.3.26), (4.3.19), (4.3.6) we obtain, for any $\mu \neq 0$

$$\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \frac{1}{\mathtt{D}_\mathtt{h}^2} \begin{pmatrix} \frac12 \mathtt{e}_{12}\mathtt{D}_\mathtt{h} & \mathtt{D}_\mathtt{h} & \frac{1}{32}\mathtt{e}_{22}(\mathtt{e}_{12}^2+4\mathtt{h}) & -\frac18 \mathtt{e}_{12}\,\mathtt{e}_{22} \\ \mathtt{h}\mathtt{D}_\mathtt{h} & \frac12 \mathtt{e}_{12}\mathtt{D}_\mathtt{h} & \frac18 \mathtt{e}_{12}\mathtt{e}_{22}\mathtt{h} & -\frac{1}{32}\mathtt{e}_{22}\,(\mathtt{e}_{12}^2+4\mathtt{h}) \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & \frac12 \mathtt{e}_{12}\mathtt{D}_\mathtt{h} & -\mathtt{D}_\mathtt{h} \\ r(\epsilon^2,\mu^2) & r(\epsilon^2,\mu^2) & -\mathtt{h}\mathtt{D}_\mathtt{h} & \frac12 \mathtt{e}_{12}\mathtt{D}_\mathtt{h} \end{pmatrix} \begin{pmatrix} r(\epsilon) \\ r(\epsilon) \\ -\mathtt{f}_{11}\epsilon + r(\epsilon^3,\mu\epsilon^2,\mu^2\epsilon) \\ \mathtt{c}_\mathtt{h}^{-\frac12}\epsilon + r(\epsilon^2,\mu\epsilon) \end{pmatrix} (1+r(\epsilon,\mu)),$$

which proves (4.3.8). In particular each x_{ij} admits an analytic extension at $\mu = 0$. Note that, for $\mu = 0$, one has $E^{(2)} = G^{(2)} = F^{(2)} = 0$ and the Sylvester equation reduces to tautology. □

Since the matrix S solves the homological equation $\left[S\,,\,D^{(1)}\right] + R^{(1)} = 0$, identity (4.3.16) simplifies to

$$\mathtt{L}_{\mu,\epsilon}^{(2)} = D^{(1)} + \frac12\left[S\,,\,R^{(1)}\right] + \frac12\int_0^1 (1-\tau^2)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\mathrm{d}\tau\,. \tag{4.3.27}$$

The matrix $\frac12\left[S\,,\,R^{(1)}\right]$ is, by (4.3.9), (4.3.15), the block-diagonal Hamiltonian and reversible matrix

$$\begin{aligned} &\frac12\left[S\,,\,R^{(1)}\right] \\ &= \begin{pmatrix} \frac12 \mathtt{J}_2(\Sigma \mathtt{J}_2 [F^{(1)}]^* - F^{(1)}\mathtt{J}_2\Sigma^*) & 0 \\ 0 & \frac12 \mathtt{J}_2(\Sigma^*\mathtt{J}_2 F^{(1)} - [F^{(1)}]^*\mathtt{J}_2\Sigma) \end{pmatrix} = \begin{pmatrix} \mathtt{J}_2\tilde{E} & 0 \\ 0 & \mathtt{J}_2\tilde{G} \end{pmatrix}, \end{aligned} \tag{4.3.28}$$

where, since $\Sigma = \mathtt{J}_2 X$,

$$\tilde{E} := \mathbf{Sym}\big(\mathtt{J}_2 X \mathtt{J}_2 [F^{(1)}]^*\big)\,, \qquad \tilde{G} := \mathbf{Sym}\big(X^* F^{(1)}\big)\,, \tag{4.3.29}$$

denoting $\mathbf{Sym}(A) := \frac12(A + A^*)$.

Lemma 4.3.7. *The self-adjoint and reversibility-preserving matrices $\tilde{E}$, $\tilde{G}$ in* (4.3.29) *have the form*

$$\begin{aligned} &\tilde{E} = \begin{pmatrix} \tilde{\mathtt{e}}_{11}\mu\epsilon^2 + \tilde{r}_1(\mu\epsilon^3,\mu^2\epsilon^2) & \mathrm{i}\,\tilde{r}_2(\mu\epsilon^2) \\ -\mathrm{i}\,\tilde{r}_2(\mu\epsilon^2) & \tilde{r}_5(\mu\epsilon^2) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} \tilde{r}_8(\mu\epsilon^2) & \mathrm{i}\,\tilde{r}_9(\mu\epsilon^2) \\ -\mathrm{i}\,\tilde{r}_9(\mu\epsilon^2) & \tilde{r}_{10}(\mu\epsilon^2) \end{pmatrix}, \\ &\tilde{\mathtt{e}}_{11} := -\mathtt{D}_\mathtt{h}^{-1}\big(\mathtt{c}_\mathtt{h}^{-1} + \mathtt{h}\mathtt{f}_{11}^2 + \mathtt{e}_{12}\mathtt{f}_{11}\mathtt{c}_\mathtt{h}^{-\frac12}\big)\,. \end{aligned} \tag{4.3.30}$$

Proof. For simplicity we set $F = F^{(1)}$. By (4.3.8), (4.3.6), one has

$$\mathtt{J}_2 X \mathtt{J}_2 F^* = \begin{pmatrix} x_{21}F_{12} - x_{22}F_{11} & \mathrm{i}\,(x_{21}F_{22} + x_{22}F_{21}) \\ \mathrm{i}\,(x_{11}F_{12} + x_{12}F_{11}) & -x_{11}F_{22} + x_{12}F_{21} \end{pmatrix} = \begin{pmatrix} \tilde{\mathtt{e}}_{11}\mu\epsilon^2 + r(\mu\epsilon^3,\mu^2\epsilon^2) & \mathrm{i}\,r(\mu\epsilon^2) \\ \mathrm{i}\,r(\mu\epsilon^2) & r(\mu\epsilon^2) \end{pmatrix}$$

with $\tilde{\mathtt{e}}_{11}$ defined in (4.3.30). The expansion of $\tilde{E}$ in (4.3.30) follows in view of (4.3.29). Since $X = \mathcal{O}(\epsilon)$ by (4.3.8) and $F = O(\mu\epsilon)$ by (4.3.6) we deduce that $X^*F = \mathcal{O}(\mu\epsilon^2)$ and the expansion of $\tilde{G}$ in (4.3.30) follows. □

Note that the term $\tilde{\mathtt{e}}_{11}\mu\epsilon^2$ in the matrix $\tilde{E}$ in (4.3.29)-(4.3.30), has the same order of the $(1,1)$-entry of $E^{(1)}$ in (4.3.4), thus will contribute to the Whitham-Benjamin function $\mathtt{e}_{\rm WB}$ in the $(1,1)$-entry of $E^{(2)}$ in (4.3.11). Finally we show that the last term in (4.3.27) is small.

Lemma 4.3.8. *The 4×4 Hamiltonian and reversibility matrix*

$$\frac{1}{2}\int_0^1 (1-\tau^2)\, \exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S)\,\mathrm{d}\tau = \begin{pmatrix} \mathtt{J}_2\widehat{E} & \mathtt{J}_2F^{(2)} \\ \mathtt{J}_2[F^{(2)}]^* & \mathtt{J}_2\widehat{G} \end{pmatrix} \tag{4.3.31}$$

where the 2×2 self-adjoint and reversible matrices $\widehat{E}$, $\widehat{G}$ have entries

$$\widehat{E}_{ij}\ ,\widehat{G}_{ij} = r(\mu\epsilon^3)\,, \quad i,j=1,2\,, \tag{4.3.32}$$

and the 2×2 reversible matrix $F^{(2)}$ admits an expansion as in (4.3.14).

Proof. Since S and $R^{(1)}$ are Hamiltonian and reversibility-preserving then $\mathrm{ad}_S R^{(1)} = [S, R^{(1)}]$ is Hamiltonian and reversibility-preserving as well. Thus each $\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S)$ is Hamiltonian and reversibility-preserving, and formula (4.3.31) holds. In order to estimate its entries we first compute $\mathrm{ad}_S^2(R^{(1)})$. Using the form of S in (4.3.9) and $[S, R^{(1)}]$ in (4.3.28) one gets

$$\mathrm{ad}_S^2(R^{(1)}) = \begin{pmatrix} 0 & \mathtt{J}_2\tilde{F} \\ \mathtt{J}_2\tilde{F}^* & 0 \end{pmatrix} \qquad \text{where} \qquad \tilde{F} := 2\left(\Sigma\mathtt{J}_2\tilde{G} - \tilde{E}\mathtt{J}_2\Sigma\right) \tag{4.3.33}$$

and $\tilde{E}$, $\tilde{G}$ are defined in (4.3.29). Since $\tilde{E}, \tilde{G} = \mathcal{O}(\mu\epsilon^2)$ by (4.3.30), and $\Sigma = \mathtt{J}_2 X = \mathcal{O}(\epsilon)$ by (4.3.8), we deduce that $\tilde{F} = \mathcal{O}(\mu\epsilon^3)$. Then, for any $\tau\in[0,1]$, the matrix $\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\, \exp(-\tau S) = \mathrm{ad}_S^2(R^{(1)})(1+\mathcal{O}(\mu,\epsilon))$. In particular the matrix $F^{(2)}$ in (4.3.31) has the same expansion of $\tilde{F}$, namely $F^{(2)} = \mathcal{O}(\mu\epsilon^3)$, and the matrices $\widehat{E}$, $\widehat{G}$ have entries as in (4.3.32). □

Proof of Lemma 4.3.4. It follows by (4.3.27)-(4.3.28), (4.3.15) and Lemmata 4.3.7 and 4.3.8. The matrix $E^{(2)} := E^{(1)} + \tilde{E} + \widehat{E}$ has the expansion in (4.3.11), with $\mathtt{e}_{\rm WB} = \mathtt{e}_{11} + \tilde{\mathtt{e}}_{11}$ as in (4.3.12). Similarly $G^{(2)} := G^{(1)} + \tilde{G} + \widehat{G}$ has the expansion in (4.3.13). □

4.3.2 Complete block decoupling and proof of the main results

We now block-diagonalize the 4×4 Hamiltonian and reversible matrix $\mathtt{L}^{(2)}_{\mu,\epsilon}$ in (4.3.10). First we split it into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}^{(2)}_{\mu,\epsilon} = D^{(2)} + R^{(2)} ,$$
$$D^{(2)} := \begin{pmatrix} \mathtt{J}_2 E^{(2)} & 0 \\ 0 & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \quad R^{(2)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & 0 \end{pmatrix}. \tag{4.3.34}$$

Lemma 4.3.9. *There exist a 4×4 reversibility-preserving Hamiltonian matrix $S^{(2)} := S^{(2)}(\mu, \epsilon)$ of the form (4.3.9), analytic in (μ, ϵ), of size $\mathcal{O}(\epsilon^3)$, and a 4×4 block-diagonal reversible Hamiltonian matrix $P := P(\mu, \epsilon)$, analytic in (μ, ϵ), of size $\mathcal{O}(\mu\epsilon^6)$ such that*

$$\exp(S^{(2)})(D^{(2)} + R^{(2)}) \exp(-S^{(2)}) = D^{(2)} + P . \tag{4.3.35}$$

Proof. We set for brevity $S = S^{(2)}$. The equation (4.3.35) is equivalent to the system

$$\begin{cases} \Pi_D (e^S (D^{(2)} + R^{(2)}) e^{-S}) - D^{(2)} = P \\ \Pi_\varnothing (e^S (D^{(2)} + R^{(2)}) e^{-S}) = 0 , \end{cases} \tag{4.3.36}$$

where Π_D is the projection onto the block-diagonal matrices and $\Pi_\varnothing$ onto the block-off-diagonal ones. The second equation in (4.3.36) is equivalent, by a Lie expansion, and since $[S, R^{(2)}]$ is block-diagonal, to

$$R^{(2)} + \Big[S \, , \, D^{(2)} \Big] + \underbrace{\Pi_\varnothing \int_0^1 (1-\tau) e^{\tau S} \mathrm{ad}_S^2 (D^{(2)} + R^{(2)}) e^{-\tau S} \mathrm{d}\tau}_{=:\mathcal{R}(S)} = 0 . \tag{4.3.37}$$

The “nonlinear homological equation" (4.3.37),

$$[S, D^{(2)}] = -R^{(2)} - \mathcal{R}(S) , \tag{4.3.38}$$

is equivalent to solving the 4×4 real linear system

$$\mathcal{A} \vec{x} = \vec{f}(\mu, \epsilon, \vec{x}) , \quad \vec{f}(\mu, \epsilon, \vec{x}) = \mu \vec{v}(\mu, \epsilon) + \mu \vec{g}(\mu, \epsilon, \vec{x}) \tag{4.3.39}$$

associated, as in (4.3.20), to (4.3.38). The vector $\mu\vec{v}(\mu, \epsilon)$ is associated with $-R^{(2)}$ where $R^{(2)}$ is in (4.3.34). The vector $\mu\vec{g}(\mu, \epsilon, \vec{x})$ is associated with the matrix $-\mathcal{R}(S)$, which is a Hamiltonian and reversible block-off-diagonal matrix (i.e of the form (4.3.15)). The factor μ

is present in $D^{(2)}$ and $R^{(2)}$, see (4.3.11), (4.3.13), (4.3.14) and the analytic function $\vec{g}(\mu,\epsilon,\vec{x})$ is quadratic in $\vec{x}$ (for the presence of ad_S^2 in $\mathcal{R}(S)$). In view of (4.3.14) one has

$$\mu\vec{v}(\mu,\epsilon) := (-F_{21}^{(2)}, F_{22}^{(2)}, -F_{11}^{(2)}, F_{12}^{(2)})^\top, \quad F_{ij}^{(2)} = r(\mu\epsilon^3)\,. \tag{4.3.40}$$

System (4.3.39) is equivalent to $\vec{x} = \mathcal{A}^{-1}\vec{f}(\mu,\epsilon,\vec{x})$ and, writing $\mathcal{A}^{-1} = \frac{1}{\mu}\mathcal{B}(\mu,\epsilon)$ (cfr. (4.3.26)), to

$$\vec{x} = \mathcal{B}(\mu,\epsilon)\vec{v}(\mu,\epsilon) + \mathcal{B}(\mu,\epsilon)\vec{g}(\mu,\epsilon,\vec{x})\,.$$

By the implicit function theorem this equation admits a unique small solution $\vec{x} = \vec{x}(\mu,\epsilon)$, analytic in (μ,ϵ), with size $\mathcal{O}(\epsilon^3)$ as $\vec{v}$ in (4.3.40). Then the first equation of (4.3.36) gives $P = [S, R^{(2)}] + \Pi_D \int_0^1 (1-\tau) e^{\tau S}\mathrm{ad}_S^2(D^{(2)} + R^{(2)})e^{-\tau S}\mathrm{d}\tau$, and its estimate follows from those of S and $R^{(2)}$ (see (4.3.14)). □

Proof of Theorems 1.5.1 and 1.2.3. By Lemma 4.3.9 and recalling (2.2.1) the operator $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ is represented by the 4×4 Hamiltonian and reversible matrix

$$\mathrm{i}\,\mathtt{c_h}\mu + \exp(S^{(2)})\mathtt{L}_{\mu,\epsilon}^{(2)}\exp(-S^{(2)}) = \mathrm{i}\,\mathtt{c_h}\mu + \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix} =: \begin{pmatrix} \mathtt{U} & 0 \\ 0 & \mathtt{S} \end{pmatrix},$$

where the matrices $E^{(3)}$ and $G^{(3)}$ expand as in (4.3.11), (4.3.13). Consequently the matrices $\mathtt{U}$ and $\mathtt{S}$ expand as in (1.5.6). Theorem 1.5.1 is proved. Theorem 1.2.3 is a straightforward corollary. The function $\underline{\mu}(\epsilon)$ in (1.5.10) is defined as the implicit solution of the function $\Delta_{\mathrm{BF}}(\mathtt{h};\mu,\epsilon)$ in (1.5.8) for ϵ small enough, depending on $\mathtt{h}$.

Chapter 5

Benjamin-Feir instability at the Whitham-Benjamin threshold

In this chapter we prove the full description of the Benjamin-Feir instability phenomenon at the critical depth $\mathtt{h}_{\text{WB}}$ given in Theorem 1.6.1.

5.1 Expansion of $\mathtt{B}_{\mu,\epsilon}$

In this section we provide the Taylor expansion of the matrix $\mathtt{B}_{\mu,\epsilon}$ in (4.2.1), i.e. (5.1.2), at an order of accuracy higher than in Proposition 4.2.1. In particular we compute the quadratic terms $\gamma_{11}\epsilon^2$, $\phi_{21}\mu\epsilon$, the cubic ones $\eta_{12}\mu\epsilon^2$, $\gamma_{12}\mu\epsilon^2$,$\phi_{11}\epsilon^3$, $\phi_{22}\mu^2\epsilon$, and the quartic terms $\eta_{11}\epsilon^4$, $\gamma_{22}\mu^2\epsilon^2$, $\phi_{12}\mu\epsilon^3$ in the matrices (5.1.5a)-(5.1.5c) below. These are the coefficients which enter in the constant η_{WB} (cfr. (5.2.9)) of the Benjamin-Feir discriminant function (1.6.4).

We recall from Proposition 4.2.1 that the operator $\mathscr{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ in (2.2.5) defined for $(\mu,\epsilon) \in B_{\mu_0}(0) \times B_{\epsilon_0}(0)$ is represented on the basis $\mathcal{F}$ in (4.1.1) by the 4×4 Hamiltonian and reversible matrix

$$\mathtt{L}_{\mu,\epsilon} = \mathtt{J}\mathtt{B}_{\mu,\epsilon} \quad \text{where} \quad \mathtt{J} := \mathtt{J}_4 := \begin{pmatrix} \mathtt{J}_2 & 0 \\ 0 & \mathtt{J}_2 \end{pmatrix}, \quad \mathtt{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.1.1}$$

with the 4×4 matrix $\mathtt{B}_{\mu,\epsilon}$ decomposing as

$$\mathtt{B}_{\mu,\epsilon} = \begin{pmatrix} E(\mu,\epsilon) & F(\mu,\epsilon) \\ F^*(\mu,\epsilon) & G(\mu,\epsilon) \end{pmatrix} \tag{5.1.2}$$

where E, G are the 2×2 self-adjoint matrices

$$E(\mu,\epsilon) := \begin{pmatrix} E_{11}(\mu,\epsilon) & \mathrm{i}\, E_{12}(\mu,\epsilon) \\ -\mathrm{i}\, E_{12}(\mu,\epsilon) & E_{22}(\mu,\epsilon) \end{pmatrix} := \begin{pmatrix} (\mathfrak{B}_{\mu,\epsilon} f_1^+, f_1^+) & (\mathfrak{B}_{\mu,\epsilon} f_1^-, f_1^+) \\ (\mathfrak{B}_{\mu,\epsilon} f_1^+, f_1^-) & (\mathfrak{B}_{\mu,\epsilon} f_1^-, f_1^-) \end{pmatrix}, \tag{5.1.3a}$$

$$G(\mu,\epsilon) := \begin{pmatrix} G_{11}(\mu,\epsilon) & \mathrm{i}\, G_{12}(\mu,\epsilon) \\ -\mathrm{i}\, G_{12}(\mu,\epsilon) & G_{22}(\mu,\epsilon) \end{pmatrix} := \begin{pmatrix} (\mathfrak{B}_{\mu,\epsilon} f_0^+, f_0^+) & (\mathfrak{B}_{\mu,\epsilon} f_0^-, f_0^+) \\ (\mathfrak{B}_{\mu,\epsilon} f_0^+, f_0^-) & (\mathfrak{B}_{\mu,\epsilon} f_0^-, f_0^-) \end{pmatrix}, \tag{5.1.3b}$$

and

$$F(\mu,\epsilon) := \begin{pmatrix} F_{11}(\mu,\epsilon) & \mathrm{i}\, F_{12}(\mu,\epsilon) \\ \mathrm{i}\, F_{21}(\mu,\epsilon) & F_{22}(\mu,\epsilon) \end{pmatrix} := \begin{pmatrix} (\mathfrak{B}_{\mu,\epsilon} f_0^+, f_1^+) & (\mathfrak{B}_{\mu,\epsilon} f_0^-, f_1^+) \\ (\mathfrak{B}_{\mu,\epsilon} f_0^+, f_1^-) & (\mathfrak{B}_{\mu,\epsilon} f_0^-, f_1^-) \end{pmatrix}. \tag{5.1.3c}$$

Here, in view of Lemmata 2.2.10, 2.2.2 and 2.2.1, we have introduced the operator

$$\mathfrak{B}_{\mu,\epsilon} := P_{0,0}^* \, U_{\mu,\epsilon}^* \, \mathcal{B}_{\mu,\epsilon} \, U_{\mu,\epsilon} \, P_{0,0} \,. \tag{5.1.4}$$

The main result of this section is the following proposition.

Proposition 5.1.1. *The 2×2 matrices $E := E(\mu,\epsilon)$, $F := F(\mu,\epsilon)$, $G := G(\mu,\epsilon)$ defined in* (5.1.3) *admit the expansion*

$$E = \begin{pmatrix} \mathtt{e}_{11}\epsilon^2(1+r_1'(\epsilon^3,\mu\epsilon)) + \eta_{11}\epsilon^4 - \mathtt{e}_{22}\frac{\mu^2}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac12 \mathtt{e}_{12}\mu + \eta_{12}\mu\epsilon^2 + r_2(\mu\epsilon^3,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac12 \mathtt{e}_{12}\mu + \eta_{12}\mu\epsilon^2 + r_2(\mu\epsilon^3,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu^2}{8}(1+r_5(\epsilon^2,\mu)) \end{pmatrix} \tag{5.1.5a}$$

$$G = \begin{pmatrix} 1+\gamma_{11}\epsilon^2 + r_8(\epsilon^3,\mu\epsilon^2,\mu^2\epsilon) & -\mathrm{i}\,\gamma_{12}\mu\epsilon^2 - \mathrm{i}\, r_9(\mu\epsilon^3,\mu^2\epsilon) \\ \mathrm{i}\,\gamma_{12}\mu\epsilon^2 + \mathrm{i}\, r_9(\mu\epsilon^3,\mu^2\epsilon) & \mu\tanh(\mathtt{h}\mu) + \gamma_{22}\mu^2\epsilon^2 + r_{10}(\mu^2\epsilon^3,\mu^3\epsilon) \end{pmatrix} \tag{5.1.5b}$$

$$F = \begin{pmatrix} \mathtt{f}_{11}\epsilon + \phi_{11}\epsilon^3 + r_3(\epsilon^4,\mu\epsilon^2,\mu^2\epsilon) & \mathrm{i}\,\mu\epsilon\mathtt{c}_\mathtt{h}^{-\frac12} + \mathrm{i}\,\phi_{12}\mu\epsilon^3 + \mathrm{i}\, r_4(\mu\epsilon^4,\mu^2\epsilon^2,\mu^3\epsilon) \\ \mathrm{i}\,\phi_{21}\mu\epsilon + \mathrm{i}\, r_6(\mu\epsilon^3,\mu^2\epsilon) & \phi_{22}\mu^2\epsilon + r_7(\mu^2\epsilon^3,\mu^3\epsilon) \end{pmatrix}, \tag{5.1.5c}$$

where the coefficients

$$\mathtt{e}_{11} := \frac{9\mathtt{c}_\mathtt{h}^8 - 10\mathtt{c}_\mathtt{h}^4 + 9}{8\mathtt{c}_\mathtt{h}^7} = \frac{9(1-\mathtt{c}_\mathtt{h}^4)^2 + 8\mathtt{c}_\mathtt{h}^4}{8\mathtt{c}_\mathtt{h}^7} > 0\,, \qquad \mathtt{f}_{11} := \tfrac12 \mathtt{c}_\mathtt{h}^{-\frac32}(1-\mathtt{c}_\mathtt{h}^4)\,, \tag{5.1.6a}$$

$$\mathtt{e}_{12} := \mathtt{c}_\mathtt{h} + \mathtt{c}_\mathtt{h}^{-1}(1-\mathtt{c}_\mathtt{h}^4)\mathtt{h} > 0\,, \tag{5.1.6b}$$

$$\mathtt{e}_{22} := \frac{(1-\mathtt{c}_\mathtt{h}^4)(1+3\mathtt{c}_\mathtt{h}^4)\mathtt{h}^2 + 2\mathtt{c}_\mathtt{h}^2(\mathtt{c}_\mathtt{h}^4-1)\mathtt{h} + \mathtt{c}_\mathtt{h}^4}{\mathtt{c}_\mathtt{h}^3} > 0\,, \tag{5.1.6c}$$

were computed in Proposition 4.2.1, whereas

$$\eta_{11} := \frac{1}{256\mathtt{c}_\mathtt{h}^{19}(\mathtt{c}_\mathtt{h}^2+1)}\big(-36\mathtt{c}_\mathtt{h}^{26} - 108\mathtt{c}_\mathtt{h}^{24} - 261\mathtt{c}_\mathtt{h}^{22} - 73\mathtt{c}_\mathtt{h}^{20} + 1429\mathtt{c}_\mathtt{h}^{18} + 1237\mathtt{c}_\mathtt{h}^{16} \tag{5.1.6d}$$

$$- 3666\mathtt{c}_{\mathtt{h}}^{14} - 3450\mathtt{c}_{\mathtt{h}}^{12} + 3774\mathtt{c}_{\mathtt{h}}^{10} + 3654\mathtt{c}_{\mathtt{h}}^{8} - 873\mathtt{c}_{\mathtt{h}}^{6} - 765\mathtt{c}_{\mathtt{h}}^{4} + 81\mathtt{c}_{\mathtt{h}}^{2} + 81\big)\,,$$

$$\eta_{12} := \frac{\mathtt{c}_{\mathtt{h}}^2(3\mathtt{c}_{\mathtt{h}}^{12} - 8\mathtt{c}_{\mathtt{h}}^8 + 3\mathtt{c}_{\mathtt{h}}^4 + 18) - (\mathtt{c}_{\mathtt{h}}^{16} - 2\mathtt{c}_{\mathtt{h}}^{12} + 12\mathtt{c}_{\mathtt{h}}^8 - 38\mathtt{c}_{\mathtt{h}}^4 + 27)\mathtt{h}}{16\mathtt{c}_{\mathtt{h}}^9}\,, \tag{5.1.6e}$$

$$\gamma_{11} := \frac{-\mathtt{c}_{\mathtt{h}}^8 + 6\mathtt{c}_{\mathtt{h}}^4 - 5}{8\mathtt{c}_{\mathtt{h}}^4}\,, \quad \gamma_{12} := \frac{2\mathtt{c}_{\mathtt{h}}^{12} - \mathtt{c}_{\mathtt{h}}^8 - 9}{16\mathtt{c}_{\mathtt{h}}^7}\,, \quad \gamma_{22} := \frac{\mathtt{c}_{\mathtt{h}}^4 - 5}{4\mathtt{c}_{\mathtt{h}}^2}\,, \tag{5.1.6f}$$

$$\phi_{11} := \frac{10\mathtt{c}_{\mathtt{h}}^{20} + 4\mathtt{c}_{\mathtt{h}}^{18} - 7\mathtt{c}_{\mathtt{h}}^{16} - 6\mathtt{c}_{\mathtt{h}}^{14} - 99\mathtt{c}_{\mathtt{h}}^{12} + 257\mathtt{c}_{\mathtt{h}}^8 - 6\mathtt{c}_{\mathtt{h}}^6 - 171\mathtt{c}_{\mathtt{h}}^4 + 18}{64\mathtt{c}_{\mathtt{h}}^{27/2}}\,, \tag{5.1.6g}$$

$$\phi_{12} := \frac{2\mathtt{c}_{\mathtt{h}}^{18} - 2\mathtt{c}_{\mathtt{h}}^{16} - 33\mathtt{c}_{\mathtt{h}}^{14} - 27\mathtt{c}_{\mathtt{h}}^{12} + 34\mathtt{c}_{\mathtt{h}}^{10} + 34\mathtt{c}_{\mathtt{h}}^8 - 33\mathtt{c}_{\mathtt{h}}^6 - 27\mathtt{c}_{\mathtt{h}}^4 + 18\mathtt{c}_{\mathtt{h}}^2 + 18}{32\mathtt{c}_{\mathtt{h}}^{25/2}(\mathtt{c}_{\mathtt{h}}^2 + 1)} \tag{5.1.6h}$$

$$\phi_{21} := \frac{\mathtt{c}_{\mathtt{h}}^2(\mathtt{c}_{\mathtt{h}}^4 - 5) - (\mathtt{c}_{\mathtt{h}}^8 + 2\mathtt{c}_{\mathtt{h}}^4 - 3)\mathtt{h}}{8\mathtt{c}_{\mathtt{h}}^{7/2}}\,, \quad \phi_{22} := \frac{-\mathtt{c}_{\mathtt{h}}^4\mathtt{h} + \mathtt{c}_{\mathtt{h}}^2 + \mathtt{h}}{4\mathtt{c}_{\mathtt{h}}^{5/2}}\,. \tag{5.1.6i}$$

The rest of the section is devoted to the proof of this proposition.

In Proposition 4.2.1 we showed that the matrices E, G, F in (5.1.5a), (5.1.5b), (5.1.5c) admit the following expansions

$$E(\mu,\epsilon) = \begin{pmatrix} \mathtt{e}_{11}\epsilon^2 - \mathtt{e}_{22}\frac{\mu^2}{8} & \mathrm{i}\,\frac{1}{2}\mathtt{e}_{12}\mu \\ -\mathrm{i}\,\frac{1}{2}\mathtt{e}_{12}\mu & -\mathtt{e}_{22}\frac{\mu^2}{8} \end{pmatrix} + \underbrace{\begin{pmatrix} r_1(\epsilon^3, \mu^2\epsilon, \mu^3) & \mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ -\mathrm{i}\,r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3) & r_5(\mu^2\epsilon, \mu^3) \end{pmatrix}}_{=:\mathcal{E}(\mu,\epsilon)}, \tag{5.1.7a}$$

$$G(\mu,\epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \mu\tanh(\mathtt{h}\mu) \end{pmatrix} + \underbrace{\begin{pmatrix} r_8(\epsilon^2, \mu^2\epsilon) & -\mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\,r_9(\mu\epsilon^2, \mu^2\epsilon) & r_{10}(\mu^2\epsilon)\,, \end{pmatrix}}_{=:\Gamma(\mu,\epsilon)} \tag{5.1.7b}$$

$$F(\mu,\epsilon) := \begin{pmatrix} \mathtt{f}_{11}\epsilon & \mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon) & \mathrm{i}\,r_4(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\,r_6(\mu\epsilon) & r_7(\mu^2\epsilon) \end{pmatrix}}_{=:\Phi(\mu,\epsilon)}\,. \tag{5.1.7c}$$

In order to get the expansion of $\mathcal{E}(\mu,\epsilon)$, $\Gamma(\mu,\epsilon)$ and $\Phi(\mu,\epsilon)$ in Proposition 5.1.1 we first expand the operators $\mathcal{B}_{\mu,\epsilon}$ in (2.2.5a) (Section 5.1.1), the projection $P_{\mu,\epsilon}$ in (2.2.7) (Section 5.1.2) and the operator $\mathfrak{B}_{\mu,\epsilon}$ in (5.1.4) (Section 5.1.3). Finally we prove Proposition 5.1.1 in Section 5.1.4.

Notation. For an operator $A = A(\mu,\epsilon)$ we denote its Taylor coefficients as

$$A_{i,j} := \frac{1}{i!j!}(\partial_\mu^i \partial_\epsilon^j A)(0,0)\,, \qquad A_k := A_k(\mu,\epsilon) := \sum_{\substack{i+j=k \\ i,j\geq 0}} A_{i,j}\mu^i\epsilon^j\,. \tag{5.1.8}$$

Moreover we shall occasionally split $A_{i,j} = A_{i,j}^{[\mathtt{ev}]} + A_{i,j}^{[\mathtt{odd}]}$, where $A_{i,j}^{[\mathtt{ev}]}$ is the part of the operator $A_{i,j}$ having only even harmonics, whereas $A_{i,j}^{[\mathtt{odd}]}$ is the part having only odd ones.

5.1.1 Expansion of $\mathcal{B}_{\mu,\epsilon}$

In the sequel $\mathcal{O}_5$ means an operator which maps $H^1(\mathbb{T},\mathbb{C}^2)$ into $L^2(\mathbb{T},\mathbb{C}^2)$-functions with size ϵ^5, $\mu\epsilon^4$, $\mu^2\epsilon^3$, $\mu^3\epsilon^2$, $\mu^4\epsilon$ or μ^5.

Lemma 5.1.2. *The operator $\mathcal{B}_{\mu,\epsilon}$ in* (2.2.5a) *has the Taylor expansion*

$$\mathcal{B}_{\mu,\epsilon} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{O}_5\,, \tag{5.1.9}$$

where

$$\mathcal{B}_0 = \begin{bmatrix} 1 & -\mathtt{c_h}\partial_x \\ \mathtt{c_h}\partial_x & |D|\tanh(\mathtt{h}|D|) \end{bmatrix}, \tag{5.1.10a}$$

$$\mathcal{B}_1 = \epsilon \begin{bmatrix} a_1(x) & -p_1(x)\partial_x \\ \partial_x \circ p_1(x) & 0 \end{bmatrix} + \mu\ell_{1,0}(|D|)\Pi_{\mathtt{s}}\,, \tag{5.1.10b}$$

$$\mathcal{B}_2 = \epsilon^2 \begin{bmatrix} a_2(x) & -p_2(x)\partial_x \\ \partial_x \circ p_2(x) & \ell_{0,2}(|D|) \end{bmatrix} - \mathrm{i}\,\mu\epsilon p_1(x)\mathcal{J} + \mu^2\ell_{2,0}(|D|)\Pi_{\mathtt{ev}}\,, \tag{5.1.10c}$$

$$\mathcal{B}_3 = \epsilon^3 \begin{bmatrix} a_3(x) & -p_3(x)\partial_x \\ \partial_x \circ p_3(x) & 0 \end{bmatrix} - \mathrm{i}\,\mu\epsilon^2 p_2(x)\mathcal{J} + \mu^3\ell_{3,0}(|D|)\Pi_{\mathtt{s}}, \tag{5.1.10d}$$

$$\mathcal{B}_4 = \epsilon^4 \begin{bmatrix} a_4(x) & -p_4(x)\partial_x \\ \partial_x \circ p_4(x) & \ell_{0,4}(|D|) \end{bmatrix} - \mathrm{i}\,\mu\epsilon^3 p_3(x)\mathcal{J} + \mu^2\epsilon^2\ell_{2,2}(|D|)\Pi_{\mathtt{ev}} + \mu^4\ell_{4,0}(|D|)\Pi_{\mathtt{ev}}\,, \tag{5.1.10e}$$

and $p_i(x)$ and $a_i(x)$, $i = 1, \dots, 4$, are computed in (A.4.22a)-(A.4.23a)*, $\mathcal{J}$ is the symplectic matrix in* (1.2.4)*,*

$$\Pi_{\mathtt{s}} := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{sgn}(D) \end{bmatrix}, \qquad \Pi_{\mathtt{ev}} := \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Id} \end{bmatrix}, \tag{5.1.11}$$

and

$$\ell_{1,0}(|D|) = \tanh(\mathtt{h}|D|) + \mathtt{h}|D|\big(1 - \tanh^2(\mathtt{h}|D|)\big)\,, \tag{5.1.12a}$$

$$\ell_{2,0}(|D|) = \mathtt{h}\big(1 - \tanh^2(\mathtt{h}|D|)\big)\big(1 - \mathtt{h}|D|\tanh(\mathtt{h}|D|)\big)\,, \tag{5.1.12b}$$

$$\ell_{0,2}(|D|) = \mathtt{f}_2|D|^2\big(1 - \tanh^2(\mathtt{h}|D|)\big)\,, \tag{5.1.12c}$$

$$\ell_{2,2}(|D|) = \mathtt{f}_2\big(1 - \tanh^2(\mathtt{h}|D|)\big)\left(-\mathtt{h}^2|D|^2 + 3\mathtt{h}^2|D|^2\tanh^2(\mathtt{h}|D|) - 4\mathtt{h}|D|\tanh(\mathtt{h}|D|) + 1\right), \tag{5.1.12d}$$

$$\ell_{0,4}(|D|) = \mathtt{f}_4|D|^2(1-\tanh^2(\mathtt{h}|D|)) - \mathtt{f}_2^2|D|^3\tanh(\mathtt{h}|D|)(1-\tanh^2(\mathtt{h}|D|)), \tag{5.1.12e}$$

with $\mathtt{f}_2$ *and* $\mathtt{f}_4$ *in* (A.4.8).

Proof. By Taylor-expanding (2.2.5a). □

We observe that, using the notation introduced in (5.1.8), we have

$$\mathcal{B}_{i,j}^{[\mathtt{ev}]} = \begin{cases} \mathcal{B}_{i,j} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}, \end{cases} \qquad \mathcal{B}_{i,j}^{[\mathtt{odd}]} = \begin{cases} 0 & \text{if } j \text{ is even}, \\ \mathcal{B}_{i,j} & \text{if } j \text{ is odd}. \end{cases} \tag{5.1.13}$$

5.1.2 Expansion of the projection $P_{\mu,\epsilon}$

The projections $P_{\mu,\epsilon}$ in (2.2.7) admit the expansion

$$P_{\mu,\epsilon} = P_0 + P_1 + P_2 + P_3 + \mathcal{O}_4, \tag{5.1.14}$$

where

$$\begin{aligned} &P_0 := P_{0,0}, \quad P_1 := \mathcal{P}[\mathcal{B}_1] \\ &P_2 := \mathcal{P}[\mathcal{B}_2] + \mathcal{P}[\mathcal{B}_1,\mathcal{B}_1], \\ &P_3 := \mathcal{P}[\mathcal{B}_3] + \mathcal{P}[\mathcal{B}_2,\mathcal{B}_1] + \mathcal{P}[\mathcal{B}_1,\mathcal{B}_2] + \mathcal{P}[\mathcal{B}_1,\mathcal{B}_1,\mathcal{B}_1], \end{aligned} \tag{5.1.15}$$

and

$$\begin{aligned} &\mathcal{P}[A_1] := \frac{1}{2\pi \mathrm{i}}\oint_\Gamma (\mathcal{L}_{0,0}-\lambda)^{-1}\mathcal{J}A_1(\mathcal{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda, \quad \text{and for } k\geq 2 \\ &\mathcal{P}[A_1,\dots,A_k] := \frac{(-1)^{k+1}}{2\pi\mathrm{i}}\oint_\Gamma (\mathcal{L}_{0,0}-\lambda)^{-1}\mathcal{J}A_1(\mathcal{L}_{0,0}-\lambda)^{-1}\dots\mathcal{J}A_k(\mathcal{L}_{0,0}-\lambda)^{-1}\mathrm{d}\lambda. \end{aligned} \tag{5.1.16}$$

In virtue of (5.1.8), (5.1.15)-(5.1.16) and (5.1.13) we obtain

$$P_{i,j}^{[\mathtt{ev}]} = \begin{cases} P_{i,j} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}, \end{cases} \qquad P_{i,j}^{[\mathtt{odd}]} = \begin{cases} 0 & \text{if } j \text{ is even}, \\ P_{i,j} & \text{if } j \text{ is odd}. \end{cases} \tag{5.1.17}$$

Action of $P_{\ell,j}$ on the unperturbed vectors. We now collect how the operators $P_{\ell,j}$ act on the vectors $f_1^+, f_1^-, f_0^+, f_0^-$ in (4.1.2). We denote

$$f_{-1}^+ := \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\cos(x) \\ -\mathtt{c}_\mathtt{h}^{-1/2}\sin(x) \end{bmatrix}, \qquad f_{-1}^- := \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\sin(x) \\ \mathtt{c}_\mathtt{h}^{-1/2}\cos(x) \end{bmatrix}. \tag{5.1.18}$$

We first consider the first order jets $P_{0,1}$ and $P_{1,0}$ of P_1.

Lemma 5.1.3. (First order jets) *The action of the jets $P_{0,1}$ and $P_{1,0}$ of P_1 in* (5.1.15) *on the basis in* (4.1.2) *is*

$$\begin{aligned} &P_{0,1}f_1^+ = \begin{bmatrix} \mathfrak{a}_{0,1}\cos(2x) \\ \mathfrak{b}_{0,1}\sin(2x) \end{bmatrix}, \quad P_{0,1}f_1^- = \begin{bmatrix} -\mathfrak{a}_{0,1}\sin(2x) \\ \mathfrak{b}_{0,1}\cos(2x) \end{bmatrix}, \\ &P_{0,1}f_0^+ = \mathfrak{u}_{0,1}f_{-1}^+, \quad P_{0,1}f_0^- = 0, \quad P_{1,0}f_0^+ = 0, \quad P_{1,0}f_0^- = 0, \\ &P_{1,0}f_1^+ = \mathrm{i}\,\mathfrak{u}_{1,0}f_{-1}^-, \quad P_{1,0}f_1^- = \mathrm{i}\,\mathfrak{u}_{1,0}f_{-1}^+, \end{aligned} \tag{5.1.19}$$

where

$$\begin{aligned} &\mathfrak{a}_{0,1} := \frac{1}{2}\mathtt{c}_\mathtt{h}^{-\frac{11}{2}}(3+\mathtt{c}_\mathtt{h}^4), \quad \mathfrak{b}_{0,1} := \frac{1}{4}\mathtt{c}_\mathtt{h}^{-\frac{13}{2}}(1+\mathtt{c}_\mathtt{h}^4)(3-\mathtt{c}_\mathtt{h}^4), \\ &\mathfrak{u}_{0,1} := \tfrac{1}{4}\mathtt{c}_\mathtt{h}^{-\frac{5}{2}}(3+\mathtt{c}_\mathtt{h}^4), \quad \mathfrak{u}_{1,0} := \tfrac{1}{4}\big(1+\mathtt{c}_\mathtt{h}^{-2}\mathtt{h}(1-\mathtt{c}_\mathtt{h}^4)\big). \end{aligned} \tag{5.1.20}$$

Proof. See (4.1.24). □

Lemma 5.1.4. (Second order jets). *The action of the jet $P_{0,2}$ of P_2 in* (5.1.15) *on the basis in* (4.1.2) *is given by*

$$P_{0,2}f_1^+ = \mathfrak{n}_{0,2}f_1^+ + \mathfrak{u}_{0,2}^+f_{-1}^+ + \begin{bmatrix} \mathfrak{a}_{0,2}\cos(3x) \\ \mathfrak{b}_{0,2}\sin(3x) \end{bmatrix}, \quad P_0P_{0,2}f_0^+ = 0, \tag{5.1.21a}$$

$$P_{0,2}f_1^- = \mathfrak{n}_{0,2}f_1^- + \mathfrak{u}_{0,2}^-f_{-1}^- + \begin{bmatrix} \widetilde{\mathfrak{a}}_{0,2}\sin(3x) \\ \widetilde{\mathfrak{b}}_{0,2}\cos(3x) \end{bmatrix}, \quad P_{0,2}f_0^- = 0, \tag{5.1.21b}$$

where $\widetilde{\mathfrak{a}}_{0,2}, \widetilde{\mathfrak{b}}_{0,2} \in \mathbb{R}$ and

$$\begin{aligned} &\mathfrak{n}_{0,2} := \frac{\mathtt{c}_\mathtt{h}^{12}+\mathtt{c}_\mathtt{h}^8-9\mathtt{c}_\mathtt{h}^4-9}{8\mathtt{c}_\mathtt{h}^{12}}, \quad \mathfrak{u}_{0,2}^+ := \frac{-2\mathtt{c}_\mathtt{h}^{12}-7\mathtt{c}_\mathtt{h}^8+8\mathtt{c}_\mathtt{h}^4+9}{32\mathtt{c}_\mathtt{h}^8}, \quad \mathfrak{u}_{0,2}^- := \frac{2\mathtt{c}_\mathtt{h}^{12}-11\mathtt{c}_\mathtt{h}^8+20\mathtt{c}_\mathtt{h}^4-3}{32\mathtt{c}_\mathtt{h}^8}, \\ &\mathfrak{a}_{0,2} := \frac{3(\mathtt{c}_\mathtt{h}^{12}+17\mathtt{c}_\mathtt{h}^8+51\mathtt{c}_\mathtt{h}^4+27)}{64\mathtt{c}_\mathtt{h}^{23/2}}, \quad \mathfrak{b}_{0,2} := \frac{3(3\mathtt{c}_\mathtt{h}^{12}-5\mathtt{c}_\mathtt{h}^8+25\mathtt{c}_\mathtt{h}^4+9)}{64\mathtt{c}_\mathtt{h}^{25/2}}. \end{aligned} \tag{5.1.21c}$$

The action of the jet $P_{2,0}$ on the vector f_0^- in (4.1.2) *is*

$$P_{2,0}f_0^- = 0. \tag{5.1.21d}$$

The action of the jet $P_{1,1}$ on the basis in (4.1.2) *is*

$$\begin{aligned} &P_{1,1}f_1^+ = \mathrm{i}\,\widetilde{\mathfrak{m}}_{1,1}f_0^- + \mathrm{i}\begin{bmatrix} \widetilde{\mathfrak{a}}_{1,1}\sin(2x) \\ \widetilde{\mathfrak{b}}_{1,1}\cos(2x) \end{bmatrix}, \qquad P_{1,1}f_1^- = \mathrm{i}\begin{bmatrix} \mathfrak{a}_{1,1}\cos(2x) \\ \mathfrak{b}_{1,1}\sin(2x) \end{bmatrix}, \\ &P_{1,1}f_0^- = -\frac{\mathrm{i}}{2}\mathtt{c}_\mathtt{h}^{-3/2}f_{-1}^+, \qquad P_{1,1}f_0^+ = \mathrm{i}\,\widetilde{\mathfrak{n}}_{1,1}f_1^- + \mathrm{i}\,\widetilde{\mathfrak{u}}_{1,1}f_{-1}^-, \end{aligned} \tag{5.1.21e}$$

where $\widetilde{\mathfrak{m}}_{1,1}, \widetilde{\mathfrak{a}}_{1,1}, \widetilde{\mathfrak{b}}_{1,1}, \widetilde{\mathfrak{n}}_{1,1}, \widetilde{\mathfrak{u}}_{1,1} \in \mathbb{R}$ *and*

$$\begin{aligned} \mathfrak{a}_{1,1} &:= -\frac{3(\mathtt{c}_\mathtt{h}^8 - 6\mathtt{c}_\mathtt{h}^4 + 5)\mathtt{h} - 3\mathtt{c}_\mathtt{h}^2(\mathtt{c}_\mathtt{h}^4 + 3)}{8\mathtt{c}_\mathtt{h}^{15/2}}, \\ \mathfrak{b}_{1,1} &:= \frac{(\mathtt{c}_\mathtt{h}^8 + 8\mathtt{c}_\mathtt{h}^4 - 9)\mathtt{h} + 3(\mathtt{c}_\mathtt{h}^6 + \mathtt{c}_\mathtt{h}^2)}{8\mathtt{c}_\mathtt{h}^{17/2}}. \end{aligned} \tag{5.1.22}$$

(Third order jets). *The jets* $P_{0,3}$, $P_{1,2}$ *and* $P_{2,1}$ *of* P_3 *in* (5.1.15) *act on* f_1^+, f_0^- *as*

$$\begin{aligned} P_{0,3} f_1^+ &= \begin{bmatrix} \mathfrak{a}_{0,3}\cos(2x) \\ \mathfrak{b}_{0,3}\sin(2x) \end{bmatrix} + \begin{bmatrix} \widetilde{\mathfrak{a}}_{0,3}\cos(4x) \\ \widetilde{\mathfrak{b}}_{0,3}\sin(4x) \end{bmatrix}, \quad P_{0,3} f_0^- = 0, \\ P_{1,2} f_0^- &= \mathrm{i}\begin{bmatrix} \mathfrak{a}_{1,2}\cos(2x) \\ \mathfrak{b}_{1,2}\sin(2x) \end{bmatrix}, \quad P_{2,1} f_0^- = \widetilde{\mathfrak{n}}_{2,1} f_1^- + \widetilde{\mathfrak{u}}_{2,1} f_{-1}^-, \end{aligned} \tag{5.1.23}$$

where $\widetilde{\mathfrak{a}}_{0,3}, \widetilde{\mathfrak{b}}_{0,3}, \widetilde{\mathfrak{n}}_{2,1}, \widetilde{\mathfrak{u}}_{2,1} \in \mathbb{R}$ *and*

$$\begin{aligned} \mathfrak{a}_{0,3} &:= \frac{1}{64\mathtt{c}_\mathtt{h}^{35/2}(\mathtt{c}_\mathtt{h}^2+1)}\Big(6\mathtt{c}_\mathtt{h}^{22} + 2\mathtt{c}_\mathtt{h}^{20} + 27\mathtt{c}_\mathtt{h}^{18} + 21\mathtt{c}_\mathtt{h}^{16} - 379\mathtt{c}_\mathtt{h}^{14} \\ &\quad - 361\mathtt{c}_\mathtt{h}^{12} + 575\mathtt{c}_\mathtt{h}^{10} + 581\mathtt{c}_\mathtt{h}^8 - 243\mathtt{c}_\mathtt{h}^6 - 225\mathtt{c}_\mathtt{h}^4 - 162\mathtt{c}_\mathtt{h}^2 - 162\Big), \\ \mathfrak{b}_{0,3} &:= \frac{1}{128\mathtt{c}_\mathtt{h}^{37/2}(\mathtt{c}_\mathtt{h}^2+1)}\Big(6\mathtt{c}_\mathtt{h}^{26} + 10\mathtt{c}_\mathtt{h}^{24} + 35\mathtt{c}_\mathtt{h}^{22} + 21\mathtt{c}_\mathtt{h}^{20} - 146\mathtt{c}_\mathtt{h}^{18} - 146\mathtt{c}_\mathtt{h}^{16} \\ &\quad - 46\mathtt{c}_\mathtt{h}^{14} - 34\mathtt{c}_\mathtt{h}^{12} + 470\mathtt{c}_\mathtt{h}^{10} + 482\mathtt{c}_\mathtt{h}^8 - 333\mathtt{c}_\mathtt{h}^6 - 315\mathtt{c}_\mathtt{h}^4 - 162\mathtt{c}_\mathtt{h}^2 - 162\Big), \\ \mathfrak{a}_{1,2} &= -\frac{\mathtt{c}_\mathtt{h}^4+3}{4\mathtt{c}_\mathtt{h}^7}, \quad \mathfrak{b}_{1,2} := \frac{\mathtt{c}_\mathtt{h}^4+1}{4\mathtt{c}_\mathtt{h}^4}. \end{aligned} \tag{5.1.24}$$

The rest of the section is devoted to the proof of Lemma 5.1.4.

We denote, for any $k \in \mathbb{N}$,

$$\begin{aligned} f_k^+ &:= \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\cos(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\sin(kx) \end{bmatrix}, \quad f_k^- := \begin{bmatrix} -\mathtt{c}_\mathtt{h}^{1/2}\sin(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\cos(kx) \end{bmatrix}, \\ f_{-k}^+ &:= \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\cos(kx) \\ -\mathtt{c}_\mathtt{h}^{-1/2}\sin(kx) \end{bmatrix}, \quad f_{-k}^- := \begin{bmatrix} \mathtt{c}_\mathtt{h}^{1/2}\sin(kx) \\ \mathtt{c}_\mathtt{h}^{-1/2}\cos(kx) \end{bmatrix}, \end{aligned} \tag{5.1.25}$$

and we define for any $k \in \mathbb{Z}$ the spaces

$$\mathcal{W}_k := \mathrm{span}\Big\{f_k^+,\ f_k^-, f_{-k}^+,\ f_{-k}^-\Big\}, \quad \mathcal{W}_k^\sigma := \mathrm{span}_\mathbb{R}\{f_k^\sigma,\ f_{-k}^\sigma\}, \quad \sigma = \pm. \tag{5.1.26}$$

We have the following

Lemma 5.1.5. *The jets of the operator $\mathcal{B}_{\mu,\epsilon}$ in (5.1.10) act on the spaces in (5.1.26) as follows*[1]

$$\mathcal{B}_{\ell,j}\mathcal{W}_k^\sigma = \underbrace{\mathrm{i}^\ell \mathcal{W}_{k-j}^{(-1)^\ell\sigma} +_{\mathbb{R}} \mathrm{i}^\ell \mathcal{W}_{k-j+2}^{(-1)^\ell\sigma} +_{\mathbb{R}} \cdots +_{\mathbb{R}} \mathrm{i}^\ell \mathcal{W}_{k+j}^{(-1)^\ell\sigma}}_{j+1 \text{ terms}}, \quad f_0^- \notin \mathcal{B}_{0,j}\mathcal{W}_k^-, \tag{5.1.27}$$

with $\ell, j = 0, \ldots, 4$, while the operator $\mathcal{J}$ in (1.2.4) acts as $\mathcal{J}\mathcal{W}_k^\pm = \mathcal{W}_k^\mp$.

Proof. The first formula in (5.1.27) follows by (5.1.10)-(5.1.11). Let us prove the second statement by contradiction supposing that there exists $g \in \mathcal{W}_k^-$ such that $\mathcal{B}_{0,j}g = f_0^-$. Then $1 = (f_0^-, f_0^-) = (\mathcal{B}_{0,j}g, f_0^-) = (g, \mathcal{B}_{0,j}f_0^-) = 0$, by (5.1.10), which is a contradiction. $\square$

We now give an extended version of Lemma 4.1.4.

Lemma 5.1.6. *The space $H^1(\mathbb{T})$ decomposes as $H^1(\mathbb{T}) = \mathcal{V}_{0,0} \oplus \mathcal{U} \oplus \mathcal{W}_{H^1}$, with $\mathcal{W}_{H^1} = \overline{\bigoplus_{k\geq 2} \mathcal{W}_k}^{H^1}$ where the subspaces $\mathcal{V}_{0,0}, \mathcal{U}$ and $\mathcal{W}_k$, defined below, are invariant under $\mathscr{L}_{0,0}$ and the following properties hold:*

(i) $\mathcal{V}_{0,0} = \mathit{span}\{f_1^+, f_1^-, f_0^+, f_0^-\}$ *is the generalized kernel of $\mathscr{L}_{0,0}$. For any $\lambda \neq 0$ the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{V}_{0,0} \to \mathcal{V}_{0,0}$ is invertible and*

$$(\mathscr{L}_{0,0}-\lambda)^{-1} f_1^+ = -\frac{1}{\lambda} f_1^+\,,\ (\mathscr{L}_{0,0}-\lambda)^{-1} f_1^- = -\frac{1}{\lambda} f_1^-,\ (\mathscr{L}_{0,0}-\lambda)^{-1} f_0^- = -\frac{1}{\lambda} f_0^-\,, \tag{5.1.28a}$$

$$(\mathscr{L}_{0,0}-\lambda)^{-1} f_0^+ = -\frac{1}{\lambda} f_0^+ + \frac{1}{\lambda^2} f_0^-\,. \tag{5.1.28b}$$

(ii) $\mathcal{U} := \mathit{span}\left\{f_{-1}^+, f_{-1}^-\right\}$*. For any $\lambda \neq \pm 2\mathrm{i}$ the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{U} \to \mathcal{U}$ is invertible and*

$$\begin{aligned}(\mathscr{L}_{0,0}-\lambda)^{-1} f_{-1}^+ &= \frac{1}{\lambda^2+4\mathtt{c}_\mathtt{h}^2}\left(-\lambda f_{-1}^+ + 2\mathtt{c}_\mathtt{h} f_{-1}^-\right),\\ (\mathscr{L}_{0,0}-\lambda)^{-1} f_{-1}^- &= \frac{1}{\lambda^2+4\mathtt{c}_\mathtt{h}^2}\left(-2\mathtt{c}_\mathtt{h} f_{-1}^+ - \lambda f_{-1}^-\right).\end{aligned} \tag{5.1.28c}$$

(iii) Each subspace $\mathcal{W}_k$ in (5.1.26) is invariant under $\mathscr{L}_{0,0}$. For any $|\lambda| < \delta(\mathtt{h})$ small enough and any natural $k \geq 2$, the operator $\mathscr{L}_{0,0} - \lambda : \mathcal{W}_k \to \mathcal{W}_k$ is invertible and for any $f \in \mathcal{W}_k$ and any natural number N

$$(\mathscr{L}_{0,0}-\lambda)^{-1} f = \mathscr{L}_{0,0}^{\dashv} f + \lambda (\mathscr{L}_{0,0}^{\dashv})^2 f + \cdots + \lambda^{N-1} (\mathscr{L}_{0,0}^{\dashv})^N f + \lambda^N \varphi_{f,N}(\lambda, x)\,, \tag{5.1.28d}$$

[1] the sum is direct if $j \leq k$, otherwise some spaces may overlap.

for some analytic function $\lambda \mapsto \varphi_{f,N}(\lambda,\cdot) \in \mathcal{W}_k$, *where* $\mathscr{L}_{0,0}^{\dashv} : \mathcal{W}_k \to \mathcal{W}_k$ *is*

$$\mathscr{L}_{0,0}^{\dashv} := \big(\mathtt{c}_\mathtt{h}^2\partial_x^2 + |D|\tanh(\mathtt{h}|D|)\big)^{-1}\begin{bmatrix}\mathtt{c}_\mathtt{h}\partial_x & -|D|\tanh(\mathtt{h}|D|)\\ 1 & \mathtt{c}_\mathtt{h}\partial_x\end{bmatrix}, \quad \mathscr{L}_{0,0}^{\dashv}\mathcal{W}_k^{\pm} = \mathcal{W}_k^{\mp}. \tag{5.1.29}$$

Remark 5.1.7. We will use in the sequel the following decomposition formula

$$\begin{aligned}\begin{bmatrix}\mathtt{a}\cos(x)\\ \mathtt{b}\sin(x)\end{bmatrix} &= \frac{1}{2}(\mathtt{a}\mathtt{c}_\mathtt{h}^{-\frac12} + \mathtt{b}\mathtt{c}_\mathtt{h}^{\frac12})f_1^+ + \frac12(\mathtt{a}\mathtt{c}_\mathtt{h}^{-\frac12} - \mathtt{b}\mathtt{c}_\mathtt{h}^{\frac12})f_{-1}^+,\\ \begin{bmatrix}\mathtt{a}\sin(x)\\ \mathtt{b}\cos(x)\end{bmatrix} &= \frac{1}{2}(\mathtt{b}\mathtt{c}_\mathtt{h}^{\frac12} - \mathtt{a}\mathtt{c}_\mathtt{h}^{-\frac12})f_1^- + \frac12(\mathtt{b}\mathtt{c}_\mathtt{h}^{\frac12} + \mathtt{a}\mathtt{c}_\mathtt{h}^{-\frac12})f_{-1}^-,\end{aligned} \qquad \forall \mathtt{a},\mathtt{b}\in\mathbb{C}. \tag{5.1.30}$$

Notation. We denote by $\mathcal{O}(\lambda)$ an analytic function having a zero of order 1 at $\lambda = 0$ and $\mathcal{O}_Z(\lambda^m)$ an analytic function with valued in a subspace Z having a zero of order m at $\lambda = 0$. We denote with $\mathcal{O}(\lambda^{-1} : \lambda)$ any function having a Laurent series at $\lambda = 0$ of the form $\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\lambda^j$. We denote by $f_{\mathcal{W}_k}$ a function in $\mathcal{W}_k$.

If $h(\lambda) = h_0 + \mathcal{O}(\lambda^{-1} : \lambda)$, $h_0 \in \mathbb{C}$, then, by the residue theorem,

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{h(\lambda)}{\lambda}\mathrm{d}\lambda = h_0. \tag{5.1.31}$$

We prepend to the proof of Lemma 5.1.4 a list of results given by straight-forward computations.

Lemma 5.1.8 (Action of $(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}$ on $\mathcal{V}_{0,0}$, $\mathcal{U}$ and $\mathcal{W}_2$)**.** *One has*

$$\begin{aligned}
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ &= \frac{\zeta_1^+}{\lambda}f_0^- + \mathrm{A}_2^+ + \lambda\mathrm{B}_2^- + \lambda^2 f_{\mathcal{W}_2^+} + \lambda^3 f_{\mathcal{W}_2^-} + \mathcal{O}_{\mathcal{W}_2}(\lambda^4),\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^- &= \mathrm{A}_2^- + \lambda\mathrm{B}_2^+ + \lambda^2 f_{\mathcal{W}_2^-} + \lambda^3 f_{\mathcal{W}_2^+} + \mathcal{O}_{\mathcal{W}_2}(\lambda^4),\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_0^+ &= \frac{\zeta_0^+}{\lambda}f_1^- + \frac{\alpha_0^+}{\lambda^2+4\mathtt{c}_\mathtt{h}^2}f_{-1}^+ + \lambda\frac{\beta_0^+}{\lambda^2+4\mathtt{c}_\mathtt{h}^2}f_{-1}^-,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_0^- &= 0,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^+ &= \frac{\zeta_{-1}^+}{\lambda}f_0^- + \mathrm{A}_{-2}^+ + \lambda f_{\mathcal{W}_2^-} + \mathcal{O}_{\mathcal{W}_2}(\lambda^2),\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^- &= f_{\mathcal{W}_2^-} + \mathcal{O}_{\mathcal{W}_2}(\lambda),
\end{aligned} \tag{5.1.32}$$

where ζ_1^+, ζ_0^+, $\alpha_0^+, \beta_0^+, \zeta_{-1}^+$ *are real numbers, and*

$$\mathrm{A}_2^+(x) := \begin{bmatrix}\dfrac{-a_1^{[1]}\mathtt{c}_\mathtt{h} + (2+\mathtt{c}_\mathtt{h}^4)p_1^{[1]}}{2\mathtt{c}_\mathtt{h}^{9/2}}\cos(2x)\\ -\dfrac{(\mathtt{c}_\mathtt{h}^4+1)(a_1^{[1]}\mathtt{c}_\mathtt{h} - 2p_1^{[1]})}{4\mathtt{c}_\mathtt{h}^{11/2}}\sin(2x)\end{bmatrix}, \ \mathrm{B}_2^-(x) := \begin{bmatrix}\dfrac{(\mathtt{c}_\mathtt{h}^4+1)\big((\mathtt{c}_\mathtt{h}^4+4)p_1^{[1]} - 2a_1^{[1]}\mathtt{c}_\mathtt{h}\big)}{4\mathtt{c}_\mathtt{h}^{19/2}}\sin(2x)\\ \dfrac{(\mathtt{c}_\mathtt{h}^4+1)\big(a_1^{[1]}\mathtt{c}_\mathtt{h}(\mathtt{c}_\mathtt{h}^4+2) - (3\mathtt{c}_\mathtt{h}^4+4)p_1^{[1]}\big)}{8\mathtt{c}_\mathtt{h}^{21/2}}\cos(2x)\end{bmatrix},$$

$$\mathrm{A}_2^-(x) := \begin{bmatrix} \frac{\left(a_1^{[1]}\mathtt{c_h} - (\mathtt{c_h}^4+2)p_1^{[1]}\right)}{2\mathtt{c_h}^{9/2}}\sin(2x) \\ -\frac{(\mathtt{c_h}^4+1)(a_1^{[1]}\mathtt{c_h} - 2p_1^{[1]})}{4\mathtt{c_h}^{11/2}}\cos(2x) \end{bmatrix}, \quad \mathrm{B}_2^+(x) := \begin{bmatrix} \frac{(\mathtt{c_h}^4+1)\left((\mathtt{c_h}^4+4)p_1^{[1]} - 2a_1^{[1]}\mathtt{c_h}\right)}{4\mathtt{c_h}^{19/2}}\cos(2x) \\ -\frac{(\mathtt{c_h}^4+1)\left(a_1^{[1]}\mathtt{c_h}(\mathtt{c_h}^4+2) - (3\mathtt{c_h}^4+4)p_1^{[1]}\right)}{8\mathtt{c_h}^{21/2}}\sin(2x) \end{bmatrix},$$

$$\mathrm{A}_{-2}^+(x) := \begin{bmatrix} \frac{(\mathtt{c_h}^3 p_1^{[1]} - a_1^{[1]})}{2\mathtt{c_h}^{7/2}}\cos(2x) \\ -\frac{a_1^{[1]}(\mathtt{c_h}^4+1)}{4\mathtt{c_h}^{9/2}}\sin(2x) \end{bmatrix}. \tag{5.1.33}$$

Moreover

$$\begin{aligned}
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathrm{A}_2^+ &= \frac{\zeta_2^+}{\lambda}f_1^- + \frac{\alpha_2^+}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + \mathrm{A}_3^+ + \frac{\lambda\beta_2^+}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + \lambda f_{\mathcal{W}_3^-} + \mathcal{O}_{\mathcal{W}_3}(\lambda^2)\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathrm{B}_2^- &= \frac{\zeta_3^+}{\lambda}f_1^+ + \frac{\alpha_3^+}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + f_{\mathcal{W}_3^-} + \frac{\lambda\beta_3^+}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + \mathcal{O}_{\mathcal{W}_3}(\lambda)\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathrm{A}_2^- &= \frac{\zeta_2^-}{\lambda}f_1^+ + \frac{\alpha_2^-}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + f_{\mathcal{W}_3^-} + \frac{\lambda\beta_2^-}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + \lambda f_{\mathcal{W}_3^+} + \mathcal{O}_{\mathcal{W}_3}(\lambda^2)\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathrm{B}_2^+ &= \frac{\zeta_3^-}{\lambda}f_1^- + \frac{\alpha_3^-}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + f_{\mathcal{W}_3^+} + \frac{\lambda\beta_3^-}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + \mathcal{O}_{\mathcal{W}_3}(\lambda)\,,
\end{aligned} \tag{5.1.34}$$

where ζ_2^-, $\beta_2^\pm$, $\alpha_3^\pm$ and $\beta_3^\pm$ are real numbers and

$$\begin{aligned}
\zeta_2^+ &:= -\frac{(a_1^{[1]})^2\mathtt{c_h}^2 - 2a_1^{[1]}(\mathtt{c_h}^4+2)\mathtt{c_h}p_1^{[1]} + (3\mathtt{c_h}^4+4)(p_1^{[1]})^2}{8\mathtt{c_h}^5}\,, \qquad (5.1.35)\\
\alpha_2^+ &:= -\alpha_2^- := -\frac{(a_1^{[1]})^2\mathtt{c_h} - 2a_1^{[1]}(\mathtt{c_h}^4+1)p_1^{[1]} + \mathtt{c_h}^3(p_1^{[1]})^2}{4\mathtt{c_h}^3}\,,\\
\zeta_3^+ &:= \zeta_3^- := \frac{(\mathtt{c_h}^4+1)(a_1^{[1]}\mathtt{c_h} - 2p_1^{[1]})\left((\mathtt{c_h}^4+2)p_1^{[1]} - a_1^{[1]}\mathtt{c_h}\right)}{8\mathtt{c_h}^{10}}\,,\\
\mathrm{A}_3^+(x) &:= \begin{bmatrix} \frac{(a_1^{[1]})^2(\mathtt{c_h}^4+3)\mathtt{c_h}^2 - 2\mathtt{c_h}p_1^{[1]}a_1^{[1]}(\mathtt{c_h}^8+9\mathtt{c_h}^4+6) + (11\mathtt{c_h}^8+29\mathtt{c_h}^4+12)(p_1^{[1]})^2}{32\mathtt{c_h}^{19/2}}\cos(3x) \\ \frac{(3\mathtt{c_h}^4+1)\left((a_1^{[1]})^2\mathtt{c_h}^2 - 2\mathtt{c_h}p_1^{[1]}a_1^{[1]}(\mathtt{c_h}^4+2) + (3\mathtt{c_h}^4+4)(p_1^{[1]})^2\right)}{32\mathtt{c_h}^{21/2}}\sin(3x) \end{bmatrix}.
\end{aligned}$$

Proof. Use the operator $\mathcal{B}_{0,1}$ in (5.1.10b), Lemma 5.1.6 and that

$$\frac{1}{2\tanh(2\mathtt{h}) - 4\mathtt{c_h}^2} = -\frac{1+\mathtt{c_h}^4}{4\mathtt{c_h}^6}\,, \quad \frac{1}{3\tanh(3\mathtt{h}) - 9\mathtt{c_h}^2} = -\frac{1+3\mathtt{c_h}^4}{24\mathtt{c_h}^6}\,,$$

which comes from the classical identity $\tanh(a+b) = \frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)}$. ☐

Lemma 5.1.9 (Action of $(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}$ on $\mathcal{V}_{0,0}$ and $\mathcal{U}$)**.** *One has*

$$
\begin{aligned}
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_1^+ = -\mathrm{i}\,\frac{\mu_{\mathtt{h}}}{\lambda}f_1^+ + \mathrm{i}\,\frac{2\mathtt{c}_{\mathtt{h}}\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^- - \mathrm{i}\,\frac{\lambda\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^+\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_1^- = -\mathrm{i}\,\frac{\mu_{\mathtt{h}}}{\lambda}f_1^- + \mathrm{i}\,\frac{2\mathtt{c}_{\mathtt{h}}\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^+ + \mathrm{i}\,\lambda\frac{\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^-\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_0^\pm = 0\,, \qquad (5.1.36)\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_{-1}^+ = \mathrm{i}\,\frac{\mu_{\mathtt{h}}}{\lambda}f_1^+ - \mathrm{i}\,\frac{2\mathtt{c}_{\mathtt{h}}\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^- + \mathrm{i}\,\frac{\lambda\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^+\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_{-1}^- = -\mathrm{i}\,\frac{\mu_{\mathtt{h}}}{\lambda}f_1^- + \mathrm{i}\,\frac{2\mathtt{c}_{\mathtt{h}}\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^+ + \mathrm{i}\,\lambda\frac{\mu_{\mathtt{h}}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^-\,,
\end{aligned}
$$

with

$$\mu_{\mathtt{h}} := \frac{(\mathtt{c}_{\mathtt{h}}^4-1)\mathtt{h}-\mathtt{c}_{\mathtt{h}}^2}{2\mathtt{c}_{\mathtt{h}}}\,. \qquad (5.1.37)$$

Proof. We apply the operator $\mathcal{B}_{1,0}=\ell_{1,0}(|D|)\Pi_{\mathtt{s}}$ in (5.1.10b) to the vectors in $\mathcal{V}_{0,0}$ and use (5.1.30) and (5.1.28a)-(5.1.28c). □

Lemma 5.1.10 (Action of $(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}$ on $\mathcal{V}_{0,0}$)**.** *One has*

$$
\begin{aligned}
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_1^+ = \frac{\tau_1^+}{\lambda}f_1^- + \frac{\ell_1^+}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^+ + \mathrm{L}_3^+ + \frac{\lambda m_1^+}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^- + O_{\mathcal{W}_3}(\lambda)\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_1^- = \frac{\tau_1^-}{\lambda}f_1^+ + \frac{\ell_1^-}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}f_{-1}^- + f_{\mathcal{W}_3^-} + \mathcal{O}_{\mathcal{U}\oplus\mathcal{W}_3}(\lambda)\,, \qquad (5.1.38)\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_0^+ = \frac{\tau_0^+}{\lambda}f_0^- + f_{\mathcal{W}_2^+} + \mathcal{O}_{\mathcal{W}_2}(\lambda)\,, \quad (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_0^- = 0\,,
\end{aligned}
$$

where m_1^+, τ_1^-, τ_0^+, *are real numbers, and*

$$
\begin{aligned}
&\tau_1^+ := \frac{1}{4\mathtt{c}_{\mathtt{h}}}\big(2a_2^{[0]}\mathtt{c}_{\mathtt{h}}^2 + a_2^{[2]}\mathtt{c}_{\mathtt{h}}^2 + 2\mathtt{f}_2(1-\mathtt{c}_{\mathtt{h}}^4) - 4\mathtt{c}_{\mathtt{h}}p_2^{[0]} - 2\mathtt{c}_{\mathtt{h}}p_2^{[2]}\big)\,,\\
&\ell_1^+ := \frac{1}{2}\big(\mathtt{c}_{\mathtt{h}}^2(2a_2^{[0]}+a_2^{[2]}) - 2\mathtt{f}_2(1-\mathtt{c}_{\mathtt{h}}^4)\big)\,, \quad \ell_1^- := \frac{1}{2}\big(\mathtt{c}_{\mathtt{h}}^2(-2a_2^{[0]}+a_2^{[2]}) + 2\mathtt{f}_2(1-\mathtt{c}_{\mathtt{h}}^4)\big)\,,\\
&\mathrm{L}_3^+(x) := \begin{bmatrix} -\dfrac{\big(a_2^{[2]}\mathtt{c}_{\mathtt{h}}(\mathtt{c}_{\mathtt{h}}^4+3) - 2(5\mathtt{c}_{\mathtt{h}}^4+3)p_2^{[2]}\big)}{16\mathtt{c}_{\mathtt{h}}^{9/2}}\cos(3x) \\ -\dfrac{(3\mathtt{c}_{\mathtt{h}}^4+1)(a_2^{[2]}\mathtt{c}_{\mathtt{h}}-2p_2^{[2]})}{16\mathtt{c}_{\mathtt{h}}^{11/2}}\sin(3x)\end{bmatrix}\,. \qquad (5.1.39)
\end{aligned}
$$

Proof. We apply the operators $\mathcal{B}_{0,2}$ in (5.1.10c) and $\mathcal{J}$ in (1.2.4) to the vectors in $\mathcal{V}_{0,0}$. Then we use (5.1.30) and Lemma 5.1.6 to obtain (5.1.38)-(5.1.39). □

Lemma 5.1.11 (Action of $(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}$ on $\mathcal{V}_{0,0}$ and $\mathcal{U}$)**.** *One has*

$$\begin{aligned}
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_1^+ &= -\frac{\mathrm{i}\,\mathtt{c_h}^{-\frac12}}{\lambda^2}f_0^- + \frac{\mathrm{i}\,\mathtt{c_h}^{-\frac12}}{\lambda}f_0^+ + \mathrm{i}\,f_{\mathcal{W}_2^-} + \mathcal{O}_{\mathcal{W}}(\lambda)\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_1^- &= \frac{\mathrm{i}\,\mathtt{c_h}^{-\frac32}}{\lambda}f_0^- + \mathrm{i}\,\mathrm{Q}_2^+ + \lambda\mathrm{i}\,f_{\mathcal{W}_2^-} + \mathcal{O}_{\mathcal{W}_2}(\lambda^2)\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_0^+ &= \frac{\mathrm{i}\,\mathtt{c_h}^{-\frac32}}{\lambda}f_1^+ - \frac{2\mathrm{i}\,\mathtt{c_h}^{-\frac12}}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + \lambda\frac{\mathrm{i}\,\mathtt{c_h}^{-\frac32}}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+\,,\\
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_0^- &= \frac{\mathrm{i}\,\mathtt{c_h}^{-\frac12}}{\lambda}f_1^- + \frac{2\mathrm{i}\,\mathtt{c_h}^{\frac12}}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + \lambda\frac{\mathrm{i}\,\mathtt{c_h}^{-\frac12}}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^-\,,
\end{aligned} \tag{5.1.40}$$

where

$$\mathrm{Q}_2^+(x) := \begin{bmatrix} \dfrac{(\mathtt{c_h}^4+3)p_1^{[1]}}{4\mathtt{c_h}^{9/2}}\cos(2x) \\ \dfrac{3(\mathtt{c_h}^4+1)p_1^{[1]}}{8\mathtt{c_h}^{11/2}}\sin(2x) \end{bmatrix}. \tag{5.1.41}$$

Proof. We have $\mathcal{B}_{1,1} = -\mathrm{i}\,p_1(x)\mathcal{J}$ by (5.1.10c), with $p_1(x) = p_1^{[1]}\cos(x)$ in (A.4.22a) and $p_1^{[1]} = -2\mathtt{c_h}^{-1}$. Use also Lemma 5.1.6. ☐

Lemma 5.1.12 (Action of $(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{2,0}$ on f_0^-)**.** *One has*

$$(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{2,0}f_0^- = \frac{\mathtt{h}}{\lambda^2}f_0^- - \frac{\mathtt{h}}{\lambda}f_0^+\,. \tag{5.1.42}$$

Proof. We apply $\mathcal{B}_{2,0} = \ell_{2,0}(|D|)\Pi_{\mathtt{ev}}$ in (5.1.10c) and (5.1.28b). ☐

Lemma 5.1.13 (Action of $((\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1})^2$ on $\mathcal{V}_{0,0}$)**.** *One has*

$$\begin{aligned}
[(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^2 f_1^+ &= \frac{\zeta_2^+}{\lambda}f_1^- + \frac{\alpha_2^+}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^+ + \zeta_3^+ f_1^+ + \mathrm{A}_2^+ \\
&\quad + \lambda(f_{\mathcal{W}_1^-} + f_{\mathcal{W}_3^-}) + \lambda^2 f_{\mathcal{W}_1^+} + \mathcal{O}_{\mathcal{W}_3}(\lambda^2) + \mathcal{O}(\lambda^3)\,,\\
[(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^2 f_1^- &= \frac{\zeta_2^-}{\lambda}f_1^+ + \frac{\alpha_2^-}{\lambda^2+4\mathtt{c_h}^2}f_{-1}^- + \zeta_3^- f_1^- + f_{\mathcal{W}_3^-} + \lambda(f_{\mathcal{W}_1^+} + f_{\mathcal{W}_3^+}) \\
&\quad + \lambda^2 f_{\mathcal{W}_1^-} + \mathcal{O}_{\mathcal{W}_3}(\lambda^2) + \mathcal{O}(\lambda^3)\,,\\
[(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^2 f_0^+ &= \zeta_0^+\mathrm{B}_2^+ + \frac{\alpha_0^+}{4\mathtt{c_h}^2}\mathrm{A}_{-2}^+ + \mathcal{O}(\lambda^{-1}:\lambda)\,, \quad [(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^2 f_0^- = 0\,.
\end{aligned} \tag{5.1.43}$$

Proof. Apply twice Lemma 5.1.8 and use that

$$\begin{aligned}
&\lambda^2 (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1} f_{\mathcal{W}_2^{\pm}} = \lambda\alpha_5^{\pm} f_1^{\mp} + \lambda^2 f_{\mathcal{W}_1^{\pm}} + \mathcal{O}_{\mathcal{W}_3}(\lambda^2) + \mathcal{O}(\lambda^3)\,,\\
&\lambda^3 (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\ f_{\mathcal{W}_2^{\mp}} = \lambda^2\alpha_6^{\pm} f_1^{\pm} + \mathcal{O}(\lambda^3)\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}_{\mathcal{W}_2}(\lambda^4) = \mathcal{O}(\lambda^3)
\end{aligned}$$

where $\alpha_5^{\pm}$, $\alpha_6^{\pm}$ are real numbers. □

We further list a series of identities to exploit later.

By applying first Lemma 5.1.9 and then Lemma 5.1.8 we get

$$\begin{aligned}
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0} f_1^{+} = -\mathrm{i}\,\mu_{\mathtt{h}}\Big[\frac{\zeta_1^{+}}{\lambda^2} f_0^{-} + \frac{1}{\lambda}\mathrm{A}_2^{+} + f_{\mathcal{W}_2^{-}} + \alpha_7 f_0^{-} + \mathcal{O}_{\mathcal{W}_2}(\lambda) + \mathcal{O}_{\mathcal{W}_0}(\lambda^2)\Big]\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0} f_1^{-} = -\mathrm{i}\,\mu_{\mathtt{h}}\Big[\frac{1}{\lambda}\Big(\mathrm{A}_2^{-} - \frac{2\mathtt{c}_{\mathtt{h}}\zeta_{-1}^{+}}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2} f_0^{-}\Big) + \mathrm{J}_2^{+} + \lambda f_{\mathcal{W}_2^{-}} + \mathcal{O}_{\mathcal{W}_2}(\lambda^2)\Big]\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0} f_0^{\pm} = 0\,, \qquad (5.1.44)
\end{aligned}$$

where α_7 is a real number and

$$\mathrm{J}_2^{+}(x) := \mathrm{B}_2^{+}(x) - \frac{1}{2\mathtt{c}_{\mathtt{h}}}\mathrm{A}_{-2}^{+}(x) \overset{(5.1.33)}{=} \begin{bmatrix} \dfrac{(5\mathtt{c}_{\mathtt{h}}^4+4)p_1^{[1]} - a_1^{[1]}\mathtt{c}_{\mathtt{h}}(\mathtt{c}_{\mathtt{h}}^4+2)}{4\mathtt{c}_{\mathtt{h}}^{19/2}}\cos(2x) \\ \dfrac{(\mathtt{c}_{\mathtt{h}}^4+1)\big((3\mathtt{c}_{\mathtt{h}}^4+4)p_1^{[1]} - 2a_1^{[1]}\mathtt{c}_{\mathtt{h}}\big)}{8\mathtt{c}_{\mathtt{h}}^{21/2}}\sin(2x) \end{bmatrix}. \qquad (5.1.45)$$

By applying first Lemma 5.1.8 and then Lemma 5.1.9, and since $\mathcal{J}\mathcal{B}_{1,0}\mathcal{W}_k \subseteq \mathcal{W}_k$, we get

$$\begin{aligned}
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1} f_1^{+} = \mathrm{i}\, f_{\mathcal{W}_2^{-}} + \mathcal{O}_{\mathcal{W}_2}(\lambda)\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1} f_1^{-} = \mathrm{i}\,\mathrm{S}_2^{+} + \mathrm{i}\,\lambda f_{\mathcal{W}_2^{-}} + \mathcal{O}_{\mathcal{W}_2}(\lambda^2)\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1} f_0^{+} = \mathrm{i}\, f_{\mathcal{W}_1^{-}} + \mathcal{O}(\lambda^{-1}:\lambda)\,,\\
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1} f_0^{-} = 0\,,
\end{aligned} \qquad (5.1.46)$$

where, using also (5.1.29), (5.1.10b)–(5.1.12a), (5.1.33)

$$\mathrm{i}\,\mathrm{S}_2^{+}(x) := (\mathscr{L}_{0,0})^{-1}\mathcal{J}\mathcal{B}_{1,0}\mathrm{A}_2^{-} = \mathrm{i}\begin{bmatrix} \dfrac{(\mathtt{c}_{\mathtt{h}}^8\mathtt{h} + \mathtt{c}_{\mathtt{h}}^6 - 2\mathtt{c}_{\mathtt{h}}^4\mathtt{h} + \mathtt{c}_{\mathtt{h}}^2 + \mathtt{h})(a_1^{[1]}\mathtt{c}_{\mathtt{h}} - 2p_1^{[1]})}{4\mathtt{c}_{\mathtt{h}}^{21/2}}\cos(2x) \\ \dfrac{(\mathtt{c}_{\mathtt{h}}^8\mathtt{h} + \mathtt{c}_{\mathtt{h}}^6 - 2\mathtt{c}_{\mathtt{h}}^4\mathtt{h} + \mathtt{c}_{\mathtt{h}}^2 + \mathtt{h})(a_1^{[1]}\mathtt{c}_{\mathtt{h}} - 2p_1^{[1]})}{8\mathtt{c}_{\mathtt{h}}^{23/2}}\sin(2x) \end{bmatrix}. \qquad (5.1.47)$$

We are now in position to prove Lemmata 5.1.4.

Second-order jets. The proof is divided in three parts, one for each group of formulas in (5.1.21).

Computation of $P_{0,2}f_j^\sigma$. Since $P_{0,\epsilon}f_0^- = f_0^-$ (cfr. Lemma 4.1.6) we have

$$P_{0,2}f_0^- = 0\,. \tag{5.1.48}$$

On the other hand, for $f_j^\sigma \in \{f_1^+, f_1^-, f_0^+\}$, in view of (5.1.15) we have, by (5.1.28a)-(5.1.28b) and since $\mathcal{B}_{0,1}f_0^- = \mathcal{B}_{0,2}f_0^- = 0$,

$$\begin{aligned} P_{0,2}f_j^\sigma &= -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,2}f_j^\sigma\,\mathrm{d}\lambda \\ &\quad+\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_j^\sigma\,\mathrm{d}\lambda =: \mathrm{I}_j^\sigma + \mathrm{II}_j^\sigma\,. \end{aligned} \tag{5.1.49}$$

In case $f_j^\sigma = f_0^+$ one readily sees, in view of (5.1.38) for I_0^+ and (5.1.43) for II_0^+, that $P_{0,2}f_0^+ \in \mathcal{W}_2^+$ which implies the second statement in (5.1.21a). We now compute the remaining four terms.

First by Lemma 5.1.10 and the residue theorem

$$\mathrm{I}_1^+ = -\frac{\ell_1^+}{4\mathtt{c}_\mathtt{h}^2}f_{-1}^+ - \mathrm{L}_3^+\,, \qquad \mathrm{I}_1^- = -\frac{\ell_1^-}{4\mathtt{c}_\mathtt{h}^2}f_{-1}^- + f_{\mathcal{W}_3^-}\,. \tag{5.1.50}$$

Then by (5.1.43) and the residue theorem

$$\mathrm{II}_1^+ = \frac{\alpha_2^+}{4\mathtt{c}_\mathtt{h}^2}f_{-1}^+ + \zeta_3^+ f_1^+ + \mathrm{A}_3^+\,, \qquad \mathrm{II}_1^- = \frac{\alpha_2^-}{4\mathtt{c}_\mathtt{h}^2}f_{-1}^- + \zeta_3^- f_1^- + f_{\mathcal{W}_3^-}\,. \tag{5.1.51}$$

In conclusion we have formulae (5.1.21a), (5.1.21b) with

$$P_{0,2}f_1^+ = \underbrace{\frac{\alpha_2^+ - \ell_1^+}{4\mathtt{c}_\mathtt{h}^2}}_{\mathfrak{u}_{0,2}^+} f_{-1}^+ + \underbrace{\zeta_3^+}_{\mathfrak{n}_{0,2}} f_1^+ \underbrace{-\mathrm{L}_3^+ + \mathrm{A}_3^+}_{\begin{bmatrix}\mathfrak{a}_{0,2}\cos(3x)\\ \mathfrak{b}_{0,2}\sin(3x)\end{bmatrix}}, \quad P_{0,2}f_1^- = \underbrace{\frac{\alpha_2^- - \ell_1^-}{4\mathtt{c}_\mathtt{h}^2}}_{\mathfrak{u}_{0,2}^-} f_{-1}^- + \underbrace{\zeta_3^-}_{\mathfrak{n}_{0,2}} f_1^- + f_{\mathcal{W}_3^-}\,,$$

and we obtain the explicit expression (5.1.21c) of the coefficients given by

$$\mathfrak{n}_{0,2} := \zeta_3^+\,, \quad \mathfrak{u}_{0,2}^+ := \frac{\alpha_2^+ - \ell_1^+}{4\mathtt{c}_\mathtt{h}^2}\,, \quad \mathfrak{u}_{0,2}^- := \frac{\alpha_2^- - \ell_1^-}{4\mathtt{c}_\mathtt{h}^2}\,, \quad \begin{bmatrix}\mathfrak{a}_{0,2}\cos(3x)\\ \mathfrak{b}_{0,2}\sin(3x)\end{bmatrix} := \mathrm{A}_3^+(x) - \mathrm{L}_3^+(x)\,,$$

with ζ_3^+, $\alpha_2^\pm$, A_3^+ in (5.1.35) and $\ell_1^\pm$, L_3^+ in (5.1.39).

Computation of $P_{2,0}f_0^-$. Since $P_{\mu,0}f_0^- = f_0^-$ (cfr. Lemma 4.1.7) we have

$$P_{2,0}f_0^- = 0\,. \tag{5.1.52}$$

Computation of $P_{1,1}f_j^\sigma$. In case $f_j^\sigma \in \{f_1^+, f_1^-, f_0^-\}$ by (5.1.15), (5.1.16) and (5.1.28a), we have

$$P_{1,1}f_j^\sigma = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,1}f_j^\sigma \mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,0}f_j^\sigma \mathrm{d}\lambda$$
$$+\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_j^\sigma \mathrm{d}\lambda =: \mathrm{III}_j^\sigma + \mathrm{IV}_j^\sigma + \mathrm{V}_j^\sigma\,, \tag{5.1.53a}$$

whereas, by (5.1.28b), (5.1.36), (5.1.32)

$$P_{1,1}f_0^+ = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,1}f_0^+ \mathrm{d}\lambda + \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda^2}\mathcal{J}\mathcal{B}_{1,1}f_0^- \mathrm{d}\lambda$$
$$+\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_0^+ \mathrm{d}\lambda =: \mathrm{III}_0^+ + \mathrm{IV}_0^+ + \mathrm{V}_0^+\,. \tag{5.1.53b}$$

When $f_j^\sigma = f_1^+$ one readily sees, in view of (5.1.40) for III_1^+, (5.1.44) for IV_1^+ and (5.1.46) for V_1^+, that $P_{1,1}f_1^+ \in \mathrm{i}\,\mathcal{W}_0^- \oplus_{\mathbb{R}} \mathrm{i}\,\mathcal{W}_2^-$ as stated in (5.1.21e). Similarly, when $f_j^\sigma = f_0^+$ one has, in view of (5.1.40) for III_0^+ and IV_0^+ and (5.1.46) for V_0^+, that $P_{1,1}f_0^+ \in \mathrm{i}\,\mathcal{W}_1^-$ as stated in (5.1.21e).

We now compute the remaining terms. By Lemma 5.1.11 and the residue theorem

$$\mathrm{III}_1^- = -\mathrm{i}\,\mathrm{Q}_2^+\,,\quad \mathrm{III}_0^- = -\frac{\mathrm{i}}{2\mathtt{c}_\mathtt{h}^{3/2}}f_{-1}^+\,. \tag{5.1.54}$$

By (5.1.44) we have

$$\mathrm{IV}_1^- = -\mathrm{i}\,\mu_\mathtt{h}\mathrm{J}_2^+,\quad \mathrm{IV}_0^- = 0\,. \tag{5.1.55}$$

By (5.1.46) we have

$$\mathrm{V}_1^- = \mathrm{i}\,\mathrm{S}_2^+\,,\quad \mathrm{V}_0^- = 0\,. \tag{5.1.56}$$

In conclusion we have formulae (5.1.21e) with

$$P_{1,1}f_1^- = \underbrace{-\mathrm{i}\,\mathrm{Q}_2^+ + \mathrm{i}\,\mathrm{S}_2^+ - \mathrm{i}\,\mu_\mathtt{h}\mathrm{J}_2^+}_{\mathrm{i}\begin{bmatrix}\mathfrak{a}_{1,1}\cos(2x)\\ \mathfrak{b}_{1,1}\sin(2x)\end{bmatrix}}\,,\quad P_{1,1}f_0^- = -\frac{\mathrm{i}}{2\mathtt{c}_\mathtt{h}^{3/2}}f_{-1}^+\,, \tag{5.1.57}$$

and we obtain the explicit expression (5.1.22) of the coefficients given by

$$\begin{bmatrix}\mathfrak{a}_{1,1}\cos(2x)\\ \mathfrak{b}_{1,1}\sin(2x)\end{bmatrix} := -\mathrm{Q}_2^+ + \mathrm{S}_2^+ - \mu_\mathtt{h}\mathrm{J}_2^+$$

with S_2^+ in (5.1.47), $\mu_{\mathtt{h}}$ in (5.1.37), Q_2^+ in (5.1.41) and J_2^+ in (5.1.45).
This concludes the proof of Lemma 5.1.4. □

Third-order jets.

Computation of $P_{0,3}f_j^\sigma$. Similarly to (5.1.48) we have $P_{0,3}f_0^- = 0$ (as stated in (5.1.23)). Let us now compute $P_{0,3}f_1^+$. By (5.1.15), (5.1.16) and (5.1.28a)

$$
\begin{aligned}
P_{0,3}f_1^+ = &-\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,3}f_1^+ \mathrm{d}\lambda \\
&+\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ \mathrm{d}\lambda \\
&+\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_1^+ \mathrm{d}\lambda \\
&-\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ \mathrm{d}\lambda \\
&=: \mathrm{VI}+\mathrm{VII}+\mathrm{VIII}+\mathrm{IX}\,.
\end{aligned}
\tag{5.1.58}
$$

We now compute these four terms.

VI) By (5.1.10d) we have $\mathcal{B}_{0,3} = \begin{bmatrix} a_3(x) & -p_3(x)\partial_x \\ \partial_x \circ p_3(x) & 0 \end{bmatrix}$ with $p_3(x), a_3(x)$ in (A.4.22b)-(A.4.23b). Then $\mathcal{B}_{0,3}f_0^- = 0$ whereas $\mathcal{J}\mathcal{B}_{0,3}f_1^+ = \alpha_{10}f_0^- + \mathrm{W}_2^- + f_{\mathcal{W}_4^-}$ where $\alpha_{10} \in \mathbb{R}$ and

$$
\mathrm{W}_2^-(x) := \begin{bmatrix} -\mathtt{c}_{\mathtt{h}}^{\frac12}(p_3^{[1]}+p_3^{[3]})\sin(2x) \\ \dfrac{-\mathtt{c}_{\mathtt{h}}(a_3^{[1]}+a_3^{[3]})+(p_3^{[1]}+p_3^{[3]})}{2\mathtt{c}_{\mathtt{h}}^{\frac12}}\cos(2x) \end{bmatrix}. \tag{5.1.59}
$$

Hence by (5.1.28d) we get

$$
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,3}f_1^+ = \mathscr{L}_{0,0}^{\dashv}(\mathrm{W}_2^- + f_{\mathcal{W}_4^-}) + \mathcal{O}(\lambda^{-1}:\lambda)\,,
$$

and, by (5.1.31), the term VI in (5.1.58) is

$$
\mathrm{VI} = -\mathscr{L}_{0,0}^{\dashv}\mathrm{W}_2^- + \widetilde{f}_{\mathcal{W}_4^+}\,, \tag{5.1.60}
$$

with

$$
\mathscr{L}_{0,0}^{\dashv}\mathrm{W}_2^-(x) = -\begin{bmatrix}\kappa_1\cos(2x) \\ \varsigma_1\sin(2x)\end{bmatrix}, \quad
\begin{aligned}
\kappa_1 &:= \frac{\mathtt{c}_{\mathtt{h}}(a_3^{[1]}+a_3^{[3]}) - (\mathtt{c}_{\mathtt{h}}^4+2)(p_3^{[1]}+p_3^{[3]})}{2\mathtt{c}_{\mathtt{h}}^{9/2}}\,, \\
\varsigma_1 &:= \frac{(\mathtt{c}_{\mathtt{h}}^4+1)(\mathtt{c}_{\mathtt{h}}(a_3^{[1]}+a_3^{[3]}) - 2(p_3^{[1]}+p_3^{[3]}))}{4\mathtt{c}_{\mathtt{h}}^{11/2}}\,.
\end{aligned}
\tag{5.1.61}
$$

VII) By Lemma 5.1.8 and since $\mathcal{B}_{0,2}f_0^- = 0$, we get

$$(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ = (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}\mathrm{A}_2^+ + \lambda(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_{\mathcal{W}_2^-} + \lambda^2(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_{\mathcal{W}_2^+} + \mathcal{O}(\lambda)\,. \tag{5.1.62}$$

Applying the operators $\mathcal{B}_{0,2}$ in (5.1.10c) and $\mathcal{J}$ in (1.2.4) to the vector A_2^+ in (5.1.33) one obtains

$$\mathcal{J}\mathcal{B}_{0,2}\mathrm{A}_2^+ = \mathrm{X}_2^- + f_{\mathcal{W}_0^-} + f_{\mathcal{W}_4^-}\,, \tag{5.1.63a}$$

where

$$\mathrm{X}_2^-(x) := \begin{bmatrix} -\dfrac{(\mathtt{c_h}^4-1)^2\mathtt{f}_2(a_1^{[1]}\mathtt{c_h}-2p_1^{[1]}) + \mathtt{c_h}(\mathtt{c_h}^4+1)p_2^{[0]}((\mathtt{c_h}^4+2)p_1^{[1]}-a_1^{[1]}\mathtt{c_h})}{\mathtt{c_h}^{11/2}(\mathtt{c_h}^4+1)}\sin(2x) \\ -\dfrac{a_2^{[0]}\mathtt{c_h}((\mathtt{c_h}^4+2)p_1^{[1]}-a_1^{[1]}\mathtt{c_h}) + (\mathtt{c_h}^4+1)p_2^{[0]}(a_1^{[1]}\mathtt{c_h}-2p_1^{[1]})}{2\mathtt{c_h}^{11/2}}\cos(2x)\end{bmatrix}. \tag{5.1.63b}$$

On the other hand, in view of (5.1.27) and Lemma 5.1.6, one has

$$\lambda(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_{\mathcal{W}_2^-} = \mathcal{O}(\lambda)\,,\quad \lambda^2(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_{\mathcal{W}_2^+} = \mathcal{O}(\lambda)\,. \tag{5.1.63c}$$

Then Lemmata 5.1.10, 5.1.6, (5.1.62) and (5.1.63) give

$$(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ = \mathscr{L}_{0,0}^{\dashv}(\mathrm{X}_2^- + f_{\mathcal{W}_4^-}) + \mathcal{O}(\lambda^{-1}:\lambda)\,,$$

and, by (5.1.31), the term VII in (5.1.58) is

$$\mathrm{VII} = \mathscr{L}_{0,0}^{\dashv}\mathrm{X}_2^- + \widetilde{f}_{\mathcal{W}_4^+}\,, \tag{5.1.64}$$

with, by (5.1.29) and (5.1.63b),

$$\mathscr{L}_{0,0}^{\dashv}\mathrm{X}_2^-(x) = \begin{bmatrix}\kappa_2\cos(2x)\\ \varsigma_2\sin(2x)\end{bmatrix} \tag{5.1.65}$$

with

$$\kappa_2 := \frac{1}{2\mathtt{c_h}^{21/2}}\Big[a_1^{[1]}\mathtt{c_h}\big(a_2^{[0]}\mathtt{c_h}^2 + (\mathtt{c_h}^4-1)^2\mathtt{f}_2 - 2(\mathtt{c_h}^4+1)\mathtt{c_h}p_2^{[0]}\big) + p_1^{[1]}\big(-a_2^{[0]}(\mathtt{c_h}^4+2)\mathtt{c_h}^2 - 2(\mathtt{c_h}^4-1)^2\mathtt{f}_2 + (\mathtt{c_h}^8+5\mathtt{c_h}^4+4)\mathtt{c_h}p_2^{[0]}\big)\Big]\,,$$

$$\varsigma_2 := \frac{1}{4\mathtt{c_h}^{23/2}}\Big[a_1^{[1]}\mathtt{c_h}\big(a_2^{[0]}\mathtt{c_h}^2(\mathtt{c_h}^4+1) + (\mathtt{c_h}^4-1)^2\mathtt{f}_2 - \mathtt{c_h}(\mathtt{c_h}^8+3\mathtt{c_h}^4+2)p_2^{[0]}\big) + p_1^{[1]}\big(-a_2^{[0]}(\mathtt{c_h}^8+3\mathtt{c_h}^4+2)\mathtt{c_h}^2 - 2(\mathtt{c_h}^4-1)^2\mathtt{f}_2 + (3\mathtt{c_h}^8+7\mathtt{c_h}^4+4)\mathtt{c_h}p_2^{[0]}\big)\Big]\,.$$

VIII) By Lemma 5.1.10, (5.1.39) and Lemma 5.1.6 we get

$$
\begin{aligned}
&(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_1^+ = \\
&\quad \frac{\tau_1^+}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^- + \frac{\ell_1^+}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^+ + \mathscr{L}_{0,0}^{\dashv}\mathcal{J}\mathcal{B}_{0,1}\mathrm{L}_3^+ \\
&\quad + \frac{\lambda m_1^+}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^- + (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}_{\mathcal{W}_3}(\lambda) + \mathcal{O}(\lambda)\,.
\end{aligned}
\tag{5.1.66}
$$

Applying the operators $\mathcal{B}_{0,1}$ in (5.1.10b) and $\mathcal{J}$ in (1.2.4) to the vector L_3^+ in (5.1.39) one obtains

$$
\mathcal{J}\mathcal{B}_{0,1}\mathrm{L}_3^+ = \mathrm{Y}_2^- + f_{\mathcal{W}_4^-}\,,
\tag{5.1.67a}
$$

where

$$
\mathrm{Y}_2^-(x) := \begin{bmatrix} \dfrac{p_1^{[1]}\big(a_2^{[2]}\mathtt{c}_{\mathtt{h}}(3+\mathtt{c}_{\mathtt{h}}^4)-2(3+5\mathtt{c}_{\mathtt{h}}^4)p_2^{[2]}\big)}{16\mathtt{c}_{\mathtt{h}}^{9/2}}\sin(2x) \\ -\dfrac{3(1+3\mathtt{c}_{\mathtt{h}}^4)p_1^{[1]}(a_2^{[2]}\mathtt{c}_{\mathtt{h}}-2p_2^{[2]})+a_1^{[1]}\mathtt{c}_{\mathtt{h}}\big(-a_2^{[2]}\mathtt{c}_{\mathtt{h}}(3+\mathtt{c}_{\mathtt{h}}^4)+2(3+5\mathtt{c}_{\mathtt{h}}^4)p_2^{[2]}\big)}{32\mathtt{c}_{\mathtt{h}}^{11/2}}\cos(2x) \end{bmatrix}.
\tag{5.1.67b}
$$

In view of (5.1.27) and Lemma 5.1.6, one obtains by inspection

$$
\frac{\lambda m_1^+}{\lambda^2+4\mathtt{c}_{\mathtt{h}}^2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^- = \mathcal{O}(\lambda)\,, \quad (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}_{\mathcal{W}_3}(\lambda) = \mathcal{O}(\lambda)\,.
\tag{5.1.67c}
$$

Then (5.1.66), Lemmata 5.1.8, 5.1.6 and (5.1.67) give

$$
(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,2}f_1^+ = \tau_1^+\mathrm{B}_2^+ + \frac{\ell_1^+}{4\mathtt{c}_{\mathtt{h}}^2}\mathrm{A}_{-2}^+ + \mathscr{L}_{0,0}^{\dashv}(\mathrm{Y}_2^- + f_{\mathcal{W}_4^-}) + \mathcal{O}(\lambda^{-1}:\lambda)\,,
$$

and, by (5.1.31), the term VIII in (5.1.58) is

$$
\mathrm{VIII} = \tau_1^+\mathrm{B}_2^+ + \frac{\ell_1^+}{4\mathtt{c}_{\mathtt{h}}^2}\mathrm{A}_{-2}^+ + \mathscr{L}_{0,0}^{\dashv}\mathrm{Y}_2^- + \widetilde{f}_{\mathcal{W}_4^+}\,,
\tag{5.1.68}
$$

where

$$
\mathscr{L}_{0,0}^{\dashv}\mathrm{Y}_2^-(x) = \begin{bmatrix} \kappa_3\cos(2x) \\ \varsigma_3\sin(2x) \end{bmatrix}.
\tag{5.1.69}
$$

with

$$
\kappa_3 := \frac{a_1^{[1]}\mathtt{c}_{\mathtt{h}}\big(a_2^{[2]}\mathtt{c}_{\mathtt{h}}(\mathtt{c}_{\mathtt{h}}^4+3)-2(5\mathtt{c}_{\mathtt{h}}^4+3)p_2^{[2]}\big)-a_2^{[2]}\mathtt{c}_{\mathtt{h}}(\mathtt{c}_{\mathtt{h}}^8+13\mathtt{c}_{\mathtt{h}}^4+6)p_1^{[1]}+2(\mathtt{c}_{\mathtt{h}}^4+3)(5\mathtt{c}_{\mathtt{h}}^4+2)p_1^{[1]}p_2^{[2]}}{32\mathtt{c}_{\mathtt{h}}^{19/2}}\,,
$$

$$\varsigma_3 := \frac{(\mathtt{c_h}^4+1)\big[a_1^{[1]}\mathtt{c_h}\big(a_2^{[2]}\mathtt{c_h}(\mathtt{c_h}^4+3)-2(5\mathtt{c_h}^4+3)p_2^{[2]}\big)-2a_2^{[2]}\mathtt{c_h}(5\mathtt{c_h}^4+3)p_1^{[1]}+4(7\mathtt{c_h}^4+3)p_1^{[1]}p_2^{[2]}\big]}{64\mathtt{c_h}^{21/2}}\,.$$

IX) By (5.1.43) we get

$$[(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^3 f_1^+ = \frac{\zeta_2^+}{\lambda}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^- \tag{5.1.70a}$$

$$\begin{aligned}&+\frac{\alpha_2^+}{\lambda^2+4\mathtt{c_h}^2}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{-1}^+ + \zeta_3^+(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_1^+ + (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathrm{A}_3^+\\ &+\lambda(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\left(f_{\mathcal{W}_1^-}+f_{\mathcal{W}_3^-}\right)+\lambda^2(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{\mathcal{W}_1^+}\\ &+(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}_{\mathcal{W}_3}(\lambda^2)+(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}(\lambda^3)\,.\end{aligned} \tag{5.1.70b}$$

In view of (5.1.27) and Lemma 5.1.6 the terms in the two lines in (5.1.70b) are

$$\lambda(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\left(f_{\mathcal{W}_1^-}+f_{\mathcal{W}_3^-}\right)=\mathcal{O}(\lambda)\,,\quad \lambda^2(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}f_{\mathcal{W}_1^+}=\mathcal{O}(\lambda)\,,$$
$$(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}_{\mathcal{W}_3}(\lambda^2)=\mathcal{O}(\lambda^2)\,,\quad (\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}\mathcal{O}(\lambda^3)=\mathcal{O}(\lambda)\,.$$

The remaining terms in (5.1.70), again by Lemma 5.1.8, are

$$[(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}]^3 f_1^+ = \zeta_2^+\mathrm{B}_2^+ + \frac{\alpha_2^+}{4\mathtt{c_h}^2}\mathrm{A}_{-2}^+ + \zeta_3^+\mathrm{A}_2^+ + \mathscr{L}_{0,0}^{\dashv}\mathcal{J}\mathcal{B}_{0,1}\mathrm{A}_3^+ + \mathcal{O}(\lambda^{-1}:\lambda)\,. \tag{5.1.71}$$

By applying the operators $\mathcal{B}_{0,1}$ in (5.1.10b) and $\mathcal{J}$ in (1.2.4) to the vector A_3^+ in (5.1.35) one gets

$$\mathcal{J}\mathcal{B}_{0,1}\mathrm{A}_3^+ = \mathrm{Z}_2^- + f_{\mathcal{W}_4^-} \tag{5.1.72}$$

where

$$\mathrm{Z}_2^-(x) := -\begin{bmatrix} \dfrac{p_1^{[1]}\Big((a_1^{[1]})^2\big(\mathtt{c_h}^4+3\big)\mathtt{c_h}^2-2a_1^{[1]}p_1^{[1]}\big(\mathtt{c_h}^8+9\mathtt{c_h}^4+6\big)\mathtt{c_h}+(p_1^{[1]})^2\big(11\mathtt{c_h}^8+29\mathtt{c_h}^4+12\big)\Big)}{32\mathtt{c_h}^{19/2}}\sin(2x) \\ \dfrac{(a_1^{[1]}\mathtt{c_h}-p_1^{[1]})\Big((a_1^{[1]})^2\big(\mathtt{c_h}^4+3\big)\mathtt{c_h}^2-2a_1^{[1]}p_1^{[1]}\big(\mathtt{c_h}^8+13\mathtt{c_h}^4+6\big)\mathtt{c_h}+3(p_1^{[1]})^2\big(9\mathtt{c_h}^8+15\mathtt{c_h}^4+4\big)\Big)}{64\mathtt{c_h}^{21/2}}\cos(2x)\end{bmatrix}. \tag{5.1.73}$$

Finally, by (5.1.71) and (5.1.72), we get that the term IX in (5.1.58) is

$$\mathrm{IX} = -\zeta_2^+\mathrm{B}_2^+ - \frac{\alpha_2^+}{4\mathtt{c_h}^2}\mathrm{A}_{-2}^+ - \zeta_3^+\mathrm{A}_2^+ - \mathscr{L}_{0,0}^{\dashv}\left(\mathrm{Z}_2^- + f_{\mathcal{W}_4^-}\right)\,, \tag{5.1.74}$$

where

$$\mathscr{L}_{0,0}^{\dashv}\mathrm{Z}_2^- = \begin{bmatrix} \kappa_4 \cos(2x) \\ \\ \varsigma_4 \sin(2x) \end{bmatrix} \tag{5.1.75}$$

where

$$\kappa_4 := \frac{1}{64\mathtt{c}_{\mathtt{h}}^{29/2}}\big(-(a_1^{[1]})^3(\mathtt{c}_{\mathtt{h}}^4+3)\mathtt{c}_{\mathtt{h}}^3 + (a_1^{[1]})^2 p_1^{[1]}(3\mathtt{c}_{\mathtt{h}}^8+31\mathtt{c}_{\mathtt{h}}^4+18)\mathtt{c}_{\mathtt{h}}^2$$
$$- a_1^{[1]}(p_1^{[1]})^2(2\mathtt{c}_{\mathtt{h}}^{12}+49\mathtt{c}_{\mathtt{h}}^8+101\mathtt{c}_{\mathtt{h}}^4+36)\mathtt{c}_{\mathtt{h}} + (p_1^{[1]})^3(11\mathtt{c}_{\mathtt{h}}^{12}+67\mathtt{c}_{\mathtt{h}}^8+86\mathtt{c}_{\mathtt{h}}^4+24)\big)$$
$$\varsigma_4 := \frac{(\mathtt{c}_{\mathtt{h}}^4+1)}{128\mathtt{c}_{\mathtt{h}}^{31/2}}\big(-(a_1^{[1]})^3(\mathtt{c}_{\mathtt{h}}^4+3)\mathtt{c}_{\mathtt{h}}^3 + 2(a_1^{[1]})^2 p_1^{[1]}(\mathtt{c}_{\mathtt{h}}^8+14\mathtt{c}_{\mathtt{h}}^4+9)\mathtt{c}_{\mathtt{h}}^2$$
$$- a_1^{[1]}(p_1^{[1]})^2(31\mathtt{c}_{\mathtt{h}}^8+89\mathtt{c}_{\mathtt{h}}^4+36)\mathtt{c}_{\mathtt{h}} + 2(p_1^{[1]})^3(19\mathtt{c}_{\mathtt{h}}^8+37\mathtt{c}_{\mathtt{h}}^4+12)\big)$$

In conclusion, by (5.1.58), (5.1.60), (5.1.64), (5.1.68) and (5.1.74) we deduce that

$$P_{0,3}f_1^+ = \mathscr{L}_{0,0}^{\dashv}\Big(-\mathrm{W}_2^- + \mathrm{X}_2^- + \mathrm{Y}_2^- - \mathrm{Z}_2^-\Big) + (\tau_1^+ - \zeta_2^+)\mathrm{B}_2^+ + \frac{\ell_1^+ - \alpha_2^+}{4\mathtt{c}_{\mathtt{h}}^2}\mathrm{A}_{-2}^+ - \zeta_3^+\mathrm{A}_2^+ + \widetilde{f}_{\mathrm{W}_4^+}$$

which proves the expansion of $P_{0,3}f_1^+$ in (5.1.23) with

$$\begin{bmatrix} \mathfrak{a}_{0,3}\cos(2x) \\ \mathfrak{b}_{0,3}\sin(2x) \end{bmatrix} := \begin{bmatrix} (\kappa_1+\kappa_2+\kappa_3-\kappa_4)\cos(2x) \\ (\varsigma_1+\varsigma_2+\varsigma_3-\varsigma_4)\sin(2x) \end{bmatrix}$$
$$+ (\tau_1^+ - \zeta_2^+)\mathrm{B}_2^+(x) + \frac{\ell_1^+ - \alpha_2^+}{4\mathtt{c}_{\mathtt{h}}^2}\mathrm{A}_{-2}^+(x) - \zeta_3^+\mathrm{A}_2^+(x)\,,$$

with κ_i, ς_i, $i = 1, \dots, 4$ in (5.1.61), (5.1.65), (5.1.69), (5.1.75), B_2^+, A_{-2}^+ A_2^+ in (5.1.33), ζ_2^+, α_2^+, ζ_3^+ in (5.1.35) and ℓ_1^+, τ_1^+ in (5.1.39), resulting in the coefficients $\mathfrak{a}_{0,3}$ and $\mathfrak{b}_{0,3}$ in (5.1.24).

Computation of $P_{1,2}f_0^-$. By (5.1.15), (5.1.16) and the fact that $\mathcal{B}_{1,0}f_0^- = \mathcal{B}_{0,1}f_0^- = \mathcal{B}_{0,2}f_0^- = 0$, the term $P_{1,2}f_0^-$ reduces to

$$P_{1,2}f_0^- = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,2}f_0^-\,\mathrm{d}\lambda$$
$$+ \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_0^-\,\mathrm{d}\lambda =: \mathrm{X} + \mathrm{XI}\,. \tag{5.1.76}$$

We now compute the two terms.

X) By (5.1.103) we have $\mathcal{J}\mathcal{B}_{1,2}f_0^- = \mathrm{i}\,a_3 f_0^- + \mathrm{i}\,\mathrm{W}_{-2}^-$ with $\mathrm{W}_{-2}^-(x) := p_2^{[2]}\begin{bmatrix} 0 \\ \cos(2x) \end{bmatrix}$ and, by (5.1.28),

$$\mathrm{X} \overset{(5.1.31)}{=} -\mathrm{i}\,\mathscr{L}_{0,0}^{\dashv}\mathrm{W}_{-2}^-\,, \quad \mathscr{L}_{0,0}^{\dashv}\mathrm{W}_{-2}^-(x) \overset{(5.1.29)}{=} p_2^{[2]}\begin{bmatrix} \mathtt{c}_{\mathtt{h}}^{-4}\cos(2x) \\ \frac{1+\mathtt{c}_{\mathtt{h}}^4}{2\mathtt{c}_{\mathtt{h}}^5}\sin(2x) \end{bmatrix}. \tag{5.1.77}$$

XI) By Lemmata 5.1.11, 5.1.8 one has

$$(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_0^- = \mathrm{i}\,\mathtt{c}_\mathtt{h}^{-\frac12}\mathrm{B}_2^+ + \mathrm{i}\,\frac{1}{2\mathtt{c}_\mathtt{h}^{3/2}}\mathrm{A}_{-2}^+ + \mathcal{O}(\lambda^{-1}:\lambda)\,,$$

and therefore

$$\mathrm{XI} = \mathrm{i}\,\mathtt{c}_\mathtt{h}^{-\frac12}\mathrm{B}_2^+ + \mathrm{i}\,\frac{1}{2\mathtt{c}_\mathtt{h}^{3/2}}\mathrm{A}_{-2}^+\,. \tag{5.1.78}$$

In conclusion, by (5.1.76), (5.1.77) and (5.1.78)

$$P_{1,2}f_0^- = -\mathrm{i}\,(\mathscr{L}_{0,0}^{\dashv})\mathrm{W}_{-2}^- + \mathrm{i}\,\mathtt{c}_\mathtt{h}^{-\frac12}\mathrm{B}_2^+ + \mathrm{i}\,\frac{1}{2\mathtt{c}_\mathtt{h}^{3/2}}\mathrm{A}_{-2}^+\,, \tag{5.1.79}$$

which, in view of (5.1.77), proves the expansion of $P_{1,2}f_0^-$ in (5.1.23) with

$$\begin{bmatrix}\mathfrak{a}_{1,2}\cos(2x)\\ \mathfrak{b}_{1,2}\sin(2x)\end{bmatrix} := \begin{bmatrix}-p_2^{[2]}\mathtt{c}_\mathtt{h}^{-4}\cos(2x)\\ -p_2^{[2]}\frac{1+\mathtt{c}_\mathtt{h}^4}{2\mathtt{c}_\mathtt{h}^5}\sin(2x)\end{bmatrix} + \mathtt{c}_\mathtt{h}^{-\frac12}\mathrm{B}_2^+(x) + \frac{1}{2\mathtt{c}_\mathtt{h}^{3/2}}\mathrm{A}_{-2}^+(x)\,,$$

with B_2^+ and A_{-2}^+ in (5.1.33), resulting in the coefficients $\mathfrak{a}_{1,2}$ and $\mathfrak{b}_{1,2}$ given in (5.1.24).

Computation of $P_{2,1}f_0^-$. By (5.1.15) and the fact that $\mathcal{B}_{1,0}f_0^- = \mathcal{B}_{0,1}f_0^-$ and $\mathcal{B}_{2,1}=0$ the term $P_{2,1}f_0^-$ reduces to

$$\begin{aligned} P_{2,1}f_0^- &= \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{0,1}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{2,0}f_0^-\mathrm{d}\lambda\\ &\quad+\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{(\mathscr{L}_{0,0}-\lambda)^{-1}}{\lambda}\mathcal{J}\mathcal{B}_{1,0}(\mathscr{L}_{0,0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{1,1}f_0^-\mathrm{d}\lambda\,. \end{aligned}$$

By repeated use of (5.1.27) and Lemma 5.1.6 one finds that $P_{2,1}f_0^- \in \mathcal{W}_1^-$ as stated in (5.1.23).

5.1.3 Expansion of $\mathfrak{B}_{\mu,\epsilon}$

In this section we provide the expansion of the operator $\mathfrak{B}_{\mu,\epsilon}$ defined in (5.1.4). We introduce the notation $\mathbf{Sym}[A] := \frac12 A + \frac12 A^*$.

Lemma 5.1.14 (Expansion of $\mathfrak{B}_{\mu,\epsilon}$). *The operator $\mathfrak{B}_{\mu,\epsilon}$ in* (5.1.4) *has the Taylor expansion*

$$\mathfrak{B}_{\mu,\epsilon} = \sum_{j=0}^{4}\mathfrak{B}_j + \mathcal{O}_5 \tag{5.1.80}$$

where

$$\mathfrak{B}_0 := P_0^*\mathcal{B}_0 P_0, \quad \mathfrak{B}_1 := P_0^*\mathcal{B}_1 P_0, \quad \mathfrak{B}_2 := P_0^*\mathbf{Sym}[\mathcal{B}_2 + \mathcal{B}_1 P_1]P_0\,, \tag{5.1.81a}$$

$$\mathfrak{B}_3 := P_0^*\mathbf{Sym}[\mathcal{B}_3 + \mathcal{B}_2 P_1 + \mathcal{B}_1(\mathrm{Id}-P_0)P_2]P_0\,, \tag{5.1.81b}$$

$$\mathfrak{B}_4 := P_0^*\mathbf{Sym}[\mathcal{B}_4 + \mathcal{B}_3 P_1 + \mathcal{B}_2(\mathrm{Id}-P_0)P_2 + \mathcal{B}_1(\mathrm{Id}-P_0)P_3 - \mathcal{B}_1 P_1 P_0 P_2 + \mathfrak{N} P_0 P_2]P_0\,, \tag{5.1.81c}$$

the operators $P_0,\ldots,P_3$ *are defined in* (5.1.15) *and*

$$\mathfrak{N} := \frac{1}{4}(P_2^*\mathcal{B}_0 - \mathcal{B}_0 P_2) = -\mathfrak{N}^*\,. \tag{5.1.82}$$

It results

$$(\mathfrak{N} f_k^\sigma, f_{k'}^{\sigma'}) = 0\,, \quad \forall f_k^\sigma, f_{k'}^{\sigma'} \in \{f_1^+, f_1^-, f_0^-\}\,. \tag{5.1.83}$$

Proof. In order to expand $\mathfrak{B}_{\mu,\epsilon}$ in (5.1.4) we first expand $U_{\mu,\epsilon}P_0$. In view of (2.2.11) we have, introducing the analytic function $g(x) := (1-x)^{-\frac{1}{2}}$ for $|x|<1$,

$$U_{\mu,\epsilon}P_0 = g((P_{\mu,\epsilon}-P_0)^2)\,P_{\mu,\epsilon}\,P_0 = P_{\mu,\epsilon}\,g((P_{\mu,\epsilon}-P_0)^2)\,P_0\ , \tag{5.1.84}$$

using that $(P_{\mu,\epsilon}-P_0)^2$ commutes with $P_{\mu,\epsilon}$, and so does $g((P_{\mu,\epsilon}-P_0)^2)$. The Taylor expansion $g(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \mathcal{O}(x^3)$ implies that

$$g((P_{\mu,\epsilon}-P_0)^2) = \mathrm{Id} + \underbrace{\frac{1}{2}(P_{\mu,\epsilon}-P_0)^2}_{=:g_2} + \underbrace{\frac{3}{8}(P_{\mu,\epsilon}-P_0)^4}_{=:g_4} + \mathcal{O}_6\,, \tag{5.1.85}$$

where $\mathcal{O}_6 = \mathcal{O}((P_{\mu,\epsilon}-P_0)^6) \in \mathcal{L}(Y)$.

Furthermore, since $P_{\mu,\epsilon}\mathscr{L}_{\mu,\epsilon} = \mathscr{L}_{\mu,\epsilon}P_{\mu,\epsilon}$ (see Lemma 2.2.1- item 2), applying $\mathcal{J}$ to both sides and using (2.2.2), yields

$$P_{\mu,\epsilon}^*\mathcal{B}_{\mu,\epsilon} = \mathcal{B}_{\mu,\epsilon}P_{\mu,\epsilon} \qquad \text{where} \qquad P_{\mu,\epsilon}^2 = P_{\mu,\epsilon}\,. \tag{5.1.86}$$

Therefore the operator $\mathfrak{B}_{\mu,\epsilon}$ in (5.1.4) has the expansion

$$\begin{aligned}
\mathfrak{B}_{\mu,\epsilon} &\overset{(5.1.84)}{=} P_0^*\,g((P_{\mu,\epsilon}-P_0)^2)^*\,P_{\mu,\epsilon}^*\,\mathcal{B}_{\mu,\epsilon}\,P_{\mu,\epsilon}\,g((P_{\mu,\epsilon}-P_0)^2)\,P_0 \\
&\overset{(5.1.86)}{=} P_0^*\,g((P_{\mu,\epsilon}-P_0)^2)^*\,\mathcal{B}_{\mu,\epsilon}\,P_{\mu,\epsilon}\,g((P_{\mu,\epsilon}-P_0)^2)\,P_0 \\
&\overset{(5.1.85)}{=} P_0^*\,(\mathrm{Id}+g_2^*+g_4^*+\mathcal{O}_6)\,\mathcal{B}_{\mu,\epsilon}\,P_{\mu,\epsilon}\,(\mathrm{Id}+g_2+g_4+\mathcal{O}_6)\,P_0 \\
&= \mathbf{Sym}[P_0^*(\mathcal{B}_{\mu,\epsilon}P_{\mu,\epsilon} + 2\mathcal{B}_{\mu,\epsilon}P_{\mu,\epsilon}g_2 + g_2^*\mathcal{B}_{\mu,\epsilon}P_{\mu,\epsilon}g_2 + 2\mathcal{B}_{\mu,\epsilon}P_{\mu,\epsilon}g_4)P_0]\,+\mathcal{O}_6
\end{aligned} \tag{5.1.87}$$

using (5.1.86) and that $g_2 = \mathcal{O}_2$ and $g_4 = \mathcal{O}_4$.

A further analysis of the term (5.1.87) relies on the following lemma.

Lemma 5.1.15. *Let Π_0^+ be the orthogonal projection on f_0^+ in (4.1.2) and $\Pi^\angle := \mathrm{Id} - \Pi_0^+$. One has*

$$\mathcal{B}_0\,P_0 = P_0^*\,\mathcal{B}_0 = \Pi_0^+\,, \qquad \mathcal{B}_0\,P_1 + \mathcal{B}_1\,P_0 = P_1^*\,\mathcal{B}_0 + P_0^*\,\mathcal{B}_1\,, \tag{5.1.88}$$

$$P_0\,P_1\,P_0 = 0\,, \qquad P_0\,P_2\,P_0 = -P_1^2\,P_0 = -P_0\,P_1^2\,, \tag{5.1.89}$$

$$(P_{\mu,\epsilon} - P_0)^2 P_0 = P_0(\mathrm{Id} - P_{\mu,\epsilon})\,P_0\,, \qquad (P_{\mu,\epsilon} - P_0)^4 P_0 = P_0\big((\mathrm{Id} - P_{\mu,\epsilon})P_0\big)^2\,, \tag{5.1.90}$$

$$\mathcal{J}P_j = P_j^*\mathcal{J}\,, \quad \forall j\in\mathbb{N}_0\,, \qquad P_0^*\mathcal{B}_0P_j\Pi^\angle P_0 = \Pi_0^+P_j\Pi^\angle P_0\,. \tag{5.1.91}$$

Proof. We deduce that $\mathcal{B}_0\,P_0 = \Pi_0^+$ because $\mathcal{B}_0 f_1^+ = \mathcal{B}_0 f_1^- = \mathcal{B}_0 f_0^- = 0$, $\mathcal{B}_0 f_0^+ = f_0^+$ and the first identity in (5.1.88) follows also since $P_0^*\mathcal{B}_0 = \mathcal{B}_0 P_0$ by (5.1.86). The second one follows by expanding the identity in (5.1.86) at order 1, using the expansions of $P_{\mu,\epsilon}$ and $B_{\mu,\epsilon}$ in (5.1.14) and (5.1.9). The identities in (5.1.89) follow by expanding the identity $P_{\mu,\epsilon}^2 = P_{\mu,\epsilon}$ at order 1 and 2, getting $P_1P_0 + P_0P_1 = P_1$ and $P_2P_0 + P_1^2 + P_0P_2 = P_2$, and applying P_0 to the right and the left of the identities above. The first identity in (5.1.90) is verified using that $P_{\mu,\epsilon}^2 = P_{\mu,\epsilon}$ and the second one follows by applying the first one twice. Finally the first identity in (5.1.91) follows by expanding the identity $\mathcal{J}P_{\mu,\epsilon} = P_{\mu,\epsilon}^*\mathcal{J}$ in (2.2.2) into homogeneous orders. The last identity in (5.1.91) descends from the first of (5.1.88), since for any $g \in L^2(\mathbb{T},\mathbb{C}^2)$ and $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$ one has

$$\big(P_0^*\mathcal{B}_0P_jf_k^\sigma, g\big) = \big(P_jf_k^\sigma, \mathcal{B}_0P_0g\big) = \big(P_jf_k^\sigma, \Pi_0^+g\big) = \big(\Pi_0^+P_jf_k^\sigma, g\big)\,.$$

This concludes the proof of the lemma. □

By (5.1.90) and (5.1.89) the Taylor expansions of g_2P_0 and g_4P_0 in (5.1.85) are

$$g_2P_0 = \frac{1}{2}P_0(\mathrm{Id} - P_{\mu,\epsilon})P_0 = -\frac{1}{2}P_0\,P_2\,P_0 - \frac{1}{2}P_0\,P_3\,P_0 - \frac{1}{2}P_0\,P_4\,P_0 + \mathcal{O}_5\,, \tag{5.1.92a}$$

$$g_4P_0 = \frac{3}{8}P_0(\mathrm{Id} - P_{\mu,\epsilon})P_0(\mathrm{Id} - P_{\mu,\epsilon})P_0 = \frac{3}{8}P_0\,P_2\,P_0\,P_2\,P_0 + \mathcal{O}_5\,. \tag{5.1.92b}$$

We now Taylor expand the operators in (5.1.87) and collect the terms of the same order.
Expression of $\mathfrak{B}_0$: The term of order 0 in (5.1.87) is simply $\mathfrak{B}_0 = P_0^*\mathcal{B}_0P_0$.
Expression of $\mathfrak{B}_1$: The term of order 1 is

$$\mathfrak{B}_1 = \frac{1}{2}P_0^*\big(\mathcal{B}_0P_1 + \mathcal{B}_1P_0 + P_1^*\mathcal{B}_0 + P_0^*\mathcal{B}_1\big)P_0 = P_0^*\,\mathcal{B}_1\,P_0$$

using (5.1.89) and (5.1.88) and that $\mathcal{B}_0, \mathcal{B}_1$ are self-adjoint.
Expression of $\mathfrak{B}_2$: We compute the terms of order 2 in (5.1.87). By (5.1.92a) we get

$$\mathfrak{B}_2 = \mathbf{Sym}[P_0^*\big(\mathcal{B}_2P_0 + \mathcal{B}_1P_1 + \mathcal{B}_0P_2 - \mathcal{B}_0P_0P_2\big)P_0]\,. \tag{5.1.93}$$

Moreover

$$P_0^* (\mathcal{B}_0 P_2 - \mathcal{B}_0 P_0 P_2) P_0 = P_0^* \mathcal{B}_0 (\mathrm{Id} - P_0) P_2 P_0 \stackrel{(5.1.88)}{=} \mathcal{B}_0 P_0 (\mathrm{Id} - P_0) P_2 P_0 = 0 \ ,$$

and from (5.1.93) descends the expression of $\mathfrak{B}_2$ in (5.1.81a).

<u>Expression of $\mathfrak{B}_3$:</u> We compute the terms of order 3 in (5.1.87). By (5.1.92a), identity (5.1.81b) follows from

$$\begin{aligned} \mathfrak{B}_3 &= \mathbf{Sym}[P_0^* (\mathcal{B}_3 + \mathcal{B}_2 P_1 + \mathcal{B}_1 P_2 + \mathcal{B}_0 P_3 - (\mathcal{B}_0 P_1 + \mathcal{B}_1 P_0) P_0 \, P_2 \, P_0 - \mathcal{B}_0 \, P_0 \, P_3 \, P_0) P_0] \\ &= \mathbf{Sym}[P_0^* (\mathcal{B}_3 + \mathcal{B}_2 P_1 + \mathcal{B}_1 P_2 - \mathcal{B}_1 P_0 P_2) P_0], \end{aligned}$$

where we used $P_0^* \mathcal{B}_0 P_3 = P_0^* \mathcal{B}_0 P_0 P_3$ and $P_0^* \mathcal{B}_0 P_1 P_0 P_2 P_0 = 0$ by (5.1.88) and (5.1.89).

<u>Expression of $\mathfrak{B}_4$:</u> At the fourth order we get, in view of (5.1.92a) and (5.1.92b),

$$\begin{aligned} \mathfrak{B}_4 = \mathbf{Sym}\Big[P_0^* \Big(& \mathcal{B}_0 P_4 + \mathcal{B}_1 P_3 + \mathcal{B}_2 P_2 + \mathcal{B}_3 P_1 + \mathcal{B}_4 - \mathcal{B}_0 P_0 P_4 P_0 - (\mathcal{B}_0 P_1 + \mathcal{B}_1 P_0) P_0 P_3 P_0 \\ & - (\mathcal{B}_2 P_0 + \mathcal{B}_1 P_1 + \mathcal{B}_0 P_2) P_0 P_2 P_0 + \frac{3}{4} \mathcal{B}_0 P_0 P_2 P_0 P_2 P_0 + \frac{1}{4} P_2^* P_0^* \mathcal{B}_0 P_0 P_2 P_0 \Big) P_0\Big] \\ = \mathbf{Sym}\Big[P_0^* \Big(& \mathcal{B}_1 (\mathrm{Id} - P_0) P_3 + \mathcal{B}_2 (\mathrm{Id} - P_0) P_2 + \mathcal{B}_3 P_1 + \mathcal{B}_4 - \mathcal{B}_1 P_1 P_0 P_2 P_0 \\ & - \frac{1}{4} \mathcal{B}_0 (P_0 P_2 P_0)^2 + \frac{1}{4} P_2^* \mathcal{B}_0 P_0 P_2 P_0 \Big) P_0\Big] \ , \end{aligned} \tag{5.1.94}$$

where to pass from the first to the second line we used $P_0^* \mathcal{B}_0 P_4 = P_0^* \mathcal{B}_0 P_0 P_4$ (by (5.1.88)) and $P_0^* \mathcal{B}_0 P_1 P_0 P_3 P_0 = 0$ (by (5.1.88) and (5.1.89)). We sum up the last two terms in (5.1.94) into $\mathbf{Sym}[P_0^* \mathfrak{N} P_0 P_2 P_0]$ where $\mathfrak{N}$ is in (5.1.82). We observe that, in view of (5.1.88)-(5.1.91), we have, for any $f_k^\sigma, f_{k'}^{\sigma'} \in \{f_1^+, f_1^-, f_0^-\}$, that (5.1.83) holds. Thus we obtain (5.1.81c). In conclusion, we have proved formula (5.1.80). □

Action of the jets of $\mathfrak{B}_{\mu,\epsilon}$ on the kernel vectors. We now collect how the operators $\mathfrak{B}_{i,j}$ (cfr. (5.1.8)) acts on the vectors f_1^+, f_1^-, f_0^-.

Lemma 5.1.16. *The first jets of the operator $\mathfrak{B}_{\mu,\epsilon}$ in* (5.1.4) *act, for $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$, as*

$$\mathfrak{B}_{0,2} f_k^\sigma = P_0^* (\mathcal{B}_{0,2} + \mathcal{B}_{0,1} P_{0,1}) f_k^\sigma \ , \quad \mathfrak{B}_{2,0} f_k^\sigma = P_0^* (\mathcal{B}_{2,0} + \mathcal{B}_{1,0} P_{1,0}) f_k^\sigma \ , \tag{5.1.95a}$$

$$\mathfrak{B}_{1,1} f_k^\sigma = P_0^* (\mathcal{B}_{1,1} + \mathcal{B}_{1,0} P_{0,1} + \mathcal{B}_{0,1} P_{1,0} + \frac{1}{2} \Pi_0^+ P_{1,1}) f_k^\sigma \ , \tag{5.1.95b}$$

$$\begin{aligned} \mathfrak{B}_{0,3} f_k^\sigma &= P_0^* (\mathcal{B}_{0,3} + \mathcal{B}_{0,2} P_{0,1} + \mathcal{B}_{0,1} P_{0,2} - \mathbf{Sym}[\mathcal{B}_{0,1} P_0 P_{0,2}]) f_k^\sigma \ , \\ \mathfrak{B}_{3,0} f_k^\sigma &= P_0^* (\mathcal{B}_{3,0} + \mathcal{B}_{2,0} P_{1,0} + \mathcal{B}_{1,0} P_{2,0} - \mathbf{Sym}[\mathcal{B}_{1,0} P_0 P_{2,0}]) f_k^\sigma \ , \end{aligned} \tag{5.1.95c}$$

$$
\begin{aligned}
\mathfrak{B}_{1,2} f_k^\sigma &= P_0^* \big(\mathcal{B}_{1,2} + \mathcal{B}_{1,1} P_{0,1} + \mathcal{B}_{0,2} P_{1,0} + \mathcal{B}_{1,0} P_{0,2} + \mathcal{B}_{0,1} P_{1,1} + \frac{1}{2} \Pi_0^+ P_{1,2} \\
&\qquad - \mathbf{Sym}[\mathcal{B}_{1,0} P_0 P_{0,2} + \mathcal{B}_{0,1} P_0 P_{1,1}] \big) f_k^\sigma \,, && (5.1.95d)\\
\mathfrak{B}_{2,1} f_k^\sigma &= P_0^* \big(\mathcal{B}_{2,1} + \mathcal{B}_{1,1} P_{1,0} + \mathcal{B}_{2,0} P_{0,1} + \mathcal{B}_{0,1} P_{2,0} + \mathcal{B}_{1,0} P_{1,1} + \frac{1}{2} \Pi_0^+ P_{2,1} \\
&\qquad - \mathbf{Sym}[\mathcal{B}_{0,1} P_0 P_{2,0} + \mathcal{B}_{1,0} P_0 P_{1,1}] \big) f_k^\sigma \,, && (5.1.95e)\\
\mathfrak{B}_{0,4} f_k^\sigma &= P_0^* \big(\mathcal{B}_{0,4} + \mathcal{B}_{0,3} P_{0,1} + \mathcal{B}_{0,2} P_{0,2} + \mathcal{B}_{0,1} P_{0,3} \\
&\qquad - \mathbf{Sym}[\mathcal{B}_{0,2} P_0 P_{0,2} + \mathcal{B}_{0,1} P_0 P_{0,3} + \mathcal{B}_{0,1} P_{0,1} P_0 P_{0,2} - \mathfrak{N}_{0,2} P_0 P_{0,2}] \big) f_k^\sigma \,, && (5.1.95f)\\
\mathfrak{B}_{2,2} f_k^\sigma &= P_0^* \big(\mathcal{B}_{2,2} + \mathcal{B}_{1,2} P_{1,0} + \mathcal{B}_{2,1} P_{0,1} + \mathcal{B}_{0,2} P_{2,0} + \mathcal{B}_{1,1} P_{1,1} + \mathcal{B}_{2,0} P_{0,2} + \mathcal{B}_{0,1} P_{2,1} + \mathcal{B}_{1,0} P_{1,2} \\
&\quad + \frac{1}{2} \Pi_0^+ P_{2,2} - \mathbf{Sym}[\mathcal{B}_{0,2} P_0 P_{2,0} + \mathcal{B}_{1,1} P_0 P_{1,1} + \mathcal{B}_{2,0} P_0 P_{0,2} + \mathcal{B}_{0,1} P_0 P_{2,1} + \mathcal{B}_{1,0} P_0 P_{1,2} \\
&\qquad + \mathcal{B}_{0,1} P_{0,1} P_0 P_{2,0} + \mathcal{B}_{1,0} P_{0,1} P_0 P_{1,1} + \mathcal{B}_{0,1} P_{1,0} P_0 P_{1,1} + \mathcal{B}_{1,0} P_{1,0} P_0 P_{0,2} \\
&\qquad - \mathfrak{N}_{2,0} P_0 P_{0,2} - \mathfrak{N}_{0,2} P_0 P_{2,0} - \mathfrak{N}_{1,1} P_0 P_{1,1}] \big) f_k^\sigma \,, && (5.1.95g)\\
\mathfrak{B}_{1,3} f_k^\sigma &= P_0^* \big(\mathcal{B}_{1,3} + \mathcal{B}_{0,3} P_{1,0} + \mathcal{B}_{1,2} P_{0,1} + \mathcal{B}_{0,2} P_{1,1} + \mathcal{B}_{1,1} P_{0,2} + \mathcal{B}_{1,0} P_{0,3} + \mathcal{B}_{0,1} P_{1,2} \\
&\quad + \frac{1}{2} \Pi_0^+ P_{1,3} - \mathbf{Sym}[\mathcal{B}_{0,2} P_0 P_{1,1} + \mathcal{B}_{1,1} P_0 P_{0,2} + \mathcal{B}_{1,0} P_0 P_{0,3} + \mathcal{B}_{0,1} P_0 P_{1,2} && (5.1.95h)\\
&\quad + \mathcal{B}_{1,0} P_{0,1} P_0 P_{0,2} + \mathcal{B}_{0,1} P_{1,0} P_0 P_{0,2} + \mathcal{B}_{0,1} P_{0,1} P_0 P_{1,1} - \mathfrak{N}_{1,1} P_0 P_{0,2} - \mathfrak{N}_{0,2} P_0 P_{1,1}] \big) f_k^\sigma \,,
\end{aligned}
$$

with $\mathcal{B}_j$, $j = 0, \dots, 4$, *in* (5.1.10) *and* P_j, $j = 0, \dots, 3$, *in* (5.1.15).

The proof of (5.1.95) relies on formulas (5.1.81a)–(5.1.81c) and Lemmata 5.1.17, 5.1.18 below.

Lemma 5.1.17. *Let* $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$. *For any* $j \in \mathbb{N}$ *we have*

$$
P_0^* \mathbf{Sym}[\mathcal{B}_j + \mathcal{B}_{j-1} P_1 + \dots + \mathcal{B}_1 P_{j-1}] P_0 f_k^\sigma = P_0^* \big(\mathcal{B}_j + \mathcal{B}_{j-1} P_1 + \dots + \mathcal{B}_1 P_{j-1} + \tfrac{1}{2} \Pi_0^+ P_j \big) P_0 f_k^\sigma \,, \tag{5.1.96}
$$

where Π_0^+ *is the orthogonal projection on* f_0^+.

Proof. By identity (5.1.86) the operator $\mathcal{B}_{\mu,\epsilon} P_{\mu,\epsilon}$ is, like $\mathcal{B}_{\mu,\epsilon}$, self-adjoint, hence its j-th jet fulfills

$$
\mathbf{Sym}[\mathcal{B}_j P_0 + \dots + \mathcal{B}_1 P_{j-1}] = \mathcal{B}_j P_0 + \dots + \mathcal{B}_1 P_{j-1} + \mathcal{B}_0 P_j - \mathbf{Sym}[\mathcal{B}_0 P_j] \,. \tag{5.1.97}
$$

We claim that, for $f_k^\sigma \in \{f_1^+, f_1^-, f_0^-\}$ we have

$$
P_0^* (\mathcal{B}_0 P_j - \mathbf{Sym}[\mathcal{B}_0 P_j]) P_0 f_k^\sigma = \frac{1}{2} P_0^* \Pi_0^+ P_j P_0 f_k^\sigma \,, \tag{5.1.98}
$$

which, together with (5.1.97), proves (5.1.96). Claim (5.1.98) follows, by observing that f_k^σ fulfills $\mathcal{B}_0 f_k^\sigma = 0$ and $\Pi^\angle P_0 f_k^\sigma = f_k^\sigma$ (cfr. Lemma 5.1.15), then

$$P_0^* \mathbf{Sym}[\mathcal{B}_0 P_j] P_0 f_k^\sigma = \frac{1}{2} P_0^* \mathcal{B}_0 P_j f_k^\sigma + \frac{1}{2} P_0^* P_j^* \mathcal{B}_0 f_k^\sigma = \frac{1}{2} P_0^* \mathcal{B}_0 P_j \Pi^\angle P_0 f_k^\sigma \overset{(5.1.91)}{=} \frac{1}{2} \Pi_0^+ P_j f_k^\sigma \,. \tag{5.1.99}$$

Using again that $P_0^* \mathcal{B}_0 P_j f_k^\sigma = \Pi_0^+ P_j f_k^\sigma$ we obtain (5.1.98). □

Lemma 5.1.18. *For any $f \in \{f_1^+, f_1^-, f_0^-\}$ and $j \in \mathbb{N}$ we have $\Pi_0^+ P_{0,j} f = \Pi_0^+ P_{j,0} f = 0$.*

Proof. We have that $\Pi_0^+ P_{0,j} f = 0$ if and only if $(P_{0,j} f, f_0^+) = 0$. By (4.1.8) we have that $P_{0,\epsilon} f_0^- = f_0^-$ for any ϵ and we have the chain of identities

$$(P_{0,\epsilon} f, f_0^+) = -(\mathcal{J} P_{0,\epsilon} f, \mathcal{J} f_0^+) \overset{(2.2.2), \mathcal{J} f_0^+ = -f_0^-}{=} (P_{0,\epsilon}^* \mathcal{J} f, f_0^-) = (\mathcal{J} f, P_{0,\epsilon} f_0^-) = (\mathcal{J} f, f_0^-) = 0$$

for any $f \in \{f_1^+, f_1^-, f_0^-\}$, deducing, in particular, that $(P_{0,j} f, f_0^+) = 0$. The proof that $\Pi_0^+ P_{j,0} f = 0$ is obtained similarly, exploiting that $P_{\mu,0} f_0^- = f_0^-$ as proved in Lemma 4.1.7. □

In virtue of (5.1.8), (5.1.13) and (5.1.17) and in view of (5.1.81)-(5.1.82) we obtain

$$\mathfrak{B}_{i,j}^{[\mathtt{ev}]} = \begin{cases} \mathfrak{B}_{i,j} & \text{if } j \text{ is even}\,, \\ 0 & \text{if } j \text{ is odd}\,, \end{cases} \qquad \mathfrak{B}_{i,j}^{[\mathtt{odd}]} = \begin{cases} 0 & \text{if } j \text{ is even}\,, \\ \mathfrak{B}_{i,j} & \text{if } j \text{ is odd}\,. \end{cases} \tag{5.1.100}$$

5.1.4 Proof of Proposition 5.1.1

Proposition 5.1.1 is a direct consequence of the next proposition.

Proposition 5.1.19. *The 2×2 matrices $\mathcal{E} := \mathcal{E}(\mu, \epsilon)$, $\Gamma := \Gamma(\mu, \epsilon)$, $\Phi := \Phi(\mu, \epsilon)$ in* (5.1.7a)-(5.1.7c) *admit the expansions*

$$\mathcal{E} := \begin{pmatrix} \widetilde{\eta}_{11} \epsilon^3 + \eta_{11} \epsilon^4 + r_1(\epsilon^5, \mu\epsilon^3, \mu^2\epsilon, \mu^3) & \mathrm{i}\,\left(\eta_{12} \mu\epsilon^2 + r_2(\mu\epsilon^3, \mu^2\epsilon, \mu^3)\right) \\ -\mathrm{i}\,\left(\eta_{12}\mu\epsilon^2 + r_2(\mu\epsilon^3, \mu^2\epsilon, \mu^3)\right) & \widetilde{\eta}_{22} \mu^2 \epsilon + r_5(\mu^2\epsilon^2, \mu^3) \end{pmatrix}, \tag{5.1.101a}$$

$$\Gamma := \begin{pmatrix} \gamma_{11}\epsilon^2 + r_8(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon) & -\mathrm{i}\,\gamma_{12}\mu\epsilon^2 - \mathrm{i}\, r_9(\mu\epsilon^3, \mu^2\epsilon) \\ \mathrm{i}\,\gamma_{12}\mu\epsilon^2 + \mathrm{i}\, r_9(\mu\epsilon^3, \mu^2\epsilon) & \widetilde{\gamma}_{22}\mu^2\epsilon + \gamma_{22}\mu^2\epsilon^2 + r_{10}(\mu^2\epsilon^3, \mu^3\epsilon) \end{pmatrix}, \tag{5.1.101b}$$

$$\Phi := \begin{pmatrix} \phi_{11}\epsilon^3 + r_3(\epsilon^4, \mu\epsilon^2, \mu^2\epsilon) & \mathrm{i}\,\widetilde{\phi}_{12}\mu\epsilon^2 + \mathrm{i}\,\phi_{12}\mu\epsilon^3 + \mathrm{i}\,\widetilde{\psi}_{12}\mu^2\epsilon + \mathrm{i}\, r_4(\mu\epsilon^4, \mu^2\epsilon^2, \mu^3\epsilon) \\ \mathrm{i}\,\phi_{21}\mu\epsilon + \mathrm{i}\,\widetilde{\phi}_{21}\mu\epsilon^2 + \mathrm{i}\, r_6(\mu\epsilon^3, \mu^2\epsilon) & \phi_{22}\mu^2\epsilon + \widetilde{\phi}_{22}\mu^2\epsilon^2 + r_7(\mu^2\epsilon^3, \mu^3\epsilon) \end{pmatrix}, \tag{5.1.101c}$$

where

$$\widetilde{\eta}_{11} := (\mathfrak{B}_{0,3} f_1^+, f_1^+) = 0\,, \qquad \widetilde{\eta}_{22} := (\mathfrak{B}_{2,1} f_1^-, f_1^-) = 0\,, \qquad \widetilde{\gamma}_{22} := (\mathfrak{B}_{2,1} f_0^-, f_0^-) = 0\,,$$
$$\mathrm{i}\,\widetilde{\phi}_{12} := (\mathfrak{B}_{1,2} f_0^-, f_1^+) = 0\,, \quad \mathrm{i}\,\widetilde{\phi}_{21} := (\mathfrak{B}_{1,2} f_0^+, f_1^-) = -(\mathfrak{B}_{1,2} f_1^-, f_0^+) = 0\,,$$
$$\widetilde{\phi}_{22} := (\mathfrak{B}_{2,2} f_0^-, f_1^-) = 0\,, \qquad \mathrm{i}\,\widetilde{\psi}_{12} := (\mathfrak{B}_{2,1} f_0^-, f_1^+) = 0\,, \tag{5.1.102a}$$

whereas the coefficients

$$\begin{aligned} &\eta_{11} := (\mathfrak{B}_{0,4} f_1^+, f_1^+)\,, \qquad \mathrm{i}\,\eta_{12} := (\mathfrak{B}_{1,2} f_1^-, f_1^+)\,, \\ &\gamma_{11} := (\mathfrak{B}_{0,2} f_0^+, f_0^+)\,, \qquad \mathrm{i}\,\gamma_{12} := (\mathfrak{B}_{1,2} f_0^-, f_0^+)\,, \qquad \gamma_{22} := (\mathfrak{B}_{2,2} f_0^-, f_0^-)\,, \\ &\phi_{11} := (\mathfrak{B}_{0,3} f_0^+, f_1^+) = (\mathfrak{B}_{0,3} f_1^+, f_0^+)\,, \quad \mathrm{i}\,\phi_{12} := (\mathfrak{B}_{1,3} f_0^-, f_1^+)\,, \\ &\mathrm{i}\,\phi_{21} := (\mathfrak{B}_{1,1} f_0^+, f_1^-) = -(\mathfrak{B}_{1,1} f_1^-, f_0^+)\,, \quad \phi_{22} = (\mathfrak{B}_{2,1} f_0^-, f_1^-)\,, \end{aligned} \tag{5.1.102b}$$

are given in (5.1.6d)-(5.1.6i).

The rest of the section is devoted to the proof of Proposition 5.1.19.

Lemma 5.1.20. *The coefficients* $\widetilde{\eta}_{11}, \widetilde{\eta}_{22}, \widetilde{\gamma}_{22}, \widetilde{\phi}_{12}, \widetilde{\phi}_{21}, \widetilde{\phi}_{22}$ *in* (5.1.102a) *vanish.*

Proof. The first six coefficients in (5.1.102a) are (use also the self-adjointness of the jets of $\mathfrak{B}_{\mu,\epsilon}$)

$$(\mathfrak{B}_{0,3}^{[\mathtt{ev}]} f_1^+, f_1^+)\,,\ (\mathfrak{B}_{2,1}^{[\mathtt{ev}]} f_1^-, f_1^-)\,,\ (\mathfrak{B}_{2,1}^{[\mathtt{ev}]} f_0^-, f_0^-)\,,\ (\mathfrak{B}_{1,2}^{[\mathtt{odd}]} f_0^-, f_1^+)\,,\ (f_0^+, \mathfrak{B}_{1,2}^{[\mathtt{odd}]} f_1^-)\,,\ (\mathfrak{B}_{2,2}^{[\mathtt{odd}]} f_0^-, f_1^-)\,,$$

which are zero because, by (5.1.100), the operators $\mathfrak{B}_{0,3}^{[\mathtt{ev}]} \mathfrak{B}_{1,2}^{[\mathtt{odd}]} = \mathfrak{B}_{2,1}^{[\mathtt{ev}]} = \mathfrak{B}_{2,2}^{[\mathtt{odd}]} = 0$. □

For the computation of the other coefficients we use the following lemma.

Lemma 5.1.21. *We have*

$$\mathcal{B}_{0,1} f_1^+ = \begin{bmatrix} \frac12 (a_1^{[1]} \mathtt{c}_\mathtt{h}^{\frac12} - p_1^{[1]} \mathtt{c}_\mathtt{h}^{-\frac12}) \cos(2x) \\ -p_1^{[1]} \mathtt{c}_\mathtt{h}^{\frac12} \sin(2x) \end{bmatrix} + \mathfrak{h}^{[0]}(x)\,, \tag{5.1.103}$$

$$\mathcal{B}_{1,0} f_1^+ = \begin{bmatrix} 0 \\ -\mathrm{i}\, \mathtt{c}_\mathtt{h}^{-\frac12} (\mathtt{c}_\mathtt{h}^2 + \mathtt{h}(1-\mathtt{c}_\mathtt{h}^4)) \cos(x) \end{bmatrix}, \quad \mathcal{B}_{1,1} f_1^+ = \frac{\mathrm{i}\, p_1^{[1]}}{2} \begin{bmatrix} -\mathtt{c}_\mathtt{h}^{-\frac12} \sin(2x) \\ \mathtt{c}_\mathtt{h}^{\frac12} \cos(2x) \end{bmatrix} + \mathfrak{h}^{[0]}(x)\,,$$

$$\mathcal{B}_{0,2} f_1^+ = \begin{bmatrix} ((a_2^{[0]} + \frac12 a_2^{[2]}) \mathtt{c}_\mathtt{h}^{\frac12} - (p_2^{[0]} + \frac12 p_2^{[2]}) \mathtt{c}_\mathtt{h}^{-\frac12}) \cos(x) \\ (\mathtt{f}_2 (1-\mathtt{c}_\mathtt{h}^4) \mathtt{c}_\mathtt{h}^{-\frac12} - (p_2^{[0]} + \frac12 p_2^{[2]}) \mathtt{c}_\mathtt{h}^{\frac12}) \sin(x) \end{bmatrix} + \begin{bmatrix} \frac12 (a_2^{[2]} \mathtt{c}_\mathtt{h}^{\frac12} - p_2^{[2]} \mathtt{c}_\mathtt{h}^{-\frac12}) \cos(3x) \\ -\frac32 p_2^{[2]} \mathtt{c}_\mathtt{h}^{\frac12} \sin(3x) \end{bmatrix},$$

$$\mathcal{B}_{0,3} f_1^+ = \begin{bmatrix} \frac12 (a_3^{[1]} \mathtt{c}_\mathtt{h}^{\frac12} - p_3^{[1]} \mathtt{c}_\mathtt{h}^{-\frac12}) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac12 (a_3^{[1]} \mathtt{c}_\mathtt{h}^{\frac12} + a_3^{[3]} \mathtt{c}_\mathtt{h}^{\frac12} - p_3^{[1]} \mathtt{c}_\mathtt{h}^{-\frac12} - p_3^{[3]} \mathtt{c}_\mathtt{h}^{-\frac12}) \cos(2x) \\ -(p_3^{[1]} + p_3^{[3]}) \mathtt{c}_\mathtt{h}^{\frac12} \sin(2x) \end{bmatrix} + \mathfrak{h}^{[4]}(x)\,,$$

$$\mathcal{B}_{0,4}f_1^+ = \begin{bmatrix} (\mathtt{c}_\mathtt{h}^{\frac12}(a_4^{[0]}+\frac12 a_4^{[2]}) - \mathtt{c}_\mathtt{h}^{-\frac12}(p_4^{[0]}+\frac12 p_4^{[2]}))\cos(x) \\ (\mathtt{c}_\mathtt{h}^{-\frac12}(1-\mathtt{c}_\mathtt{h}^4)(\mathtt{f}_4-\mathtt{f}_2^2\mathtt{c}_\mathtt{h}^2) - \mathtt{c}_\mathtt{h}^{\frac12}(p_4^{[0]}+\frac12 p_4^{[2]})\sin(x) \end{bmatrix} + \mathfrak{h}^{[3,5]}(x)\,,$$

$$\mathcal{B}_{1,0}f_1^- = \begin{bmatrix} 0 \\ \mathrm{i}\,\mathtt{c}_\mathtt{h}^{-\frac12}(\mathtt{c}_\mathtt{h}^2+\mathtt{h}(1-\mathtt{c}_\mathtt{h}^4))\sin(x) \end{bmatrix}, \quad \mathcal{B}_{1,1}f_1^- = -\frac{\mathrm{i}\,p_1^{[1]}}{2}\begin{bmatrix} \mathtt{c}_\mathtt{h}^{-\frac12} \\ 0 \end{bmatrix} + \mathfrak{h}^{[2]}(x)\,,$$

$$\mathcal{B}_{1,2}f_1^- = -\mathrm{i}\begin{bmatrix} \mathtt{c}_\mathtt{h}^{-\frac12}(p_2^{[0]}+\frac12 p_2^{[2]})\cos(x) \\ \mathtt{c}_\mathtt{h}^{\frac12}(p_2^{[0]}-\frac12 p_2^{[2]})\sin(x) \end{bmatrix} + \mathfrak{h}^{[3]}(x)\,,$$

$$\mathcal{B}_{0,1}f_0^+ = \begin{bmatrix} a_1^{[1]}\cos(x) \\ -p_1^{[1]}\sin(x) \end{bmatrix}, \quad \mathcal{B}_{0,2}f_0^+ = \begin{bmatrix} a_2^{[0]} \\ 0 \end{bmatrix} + \begin{bmatrix} a_2^{[2]}\cos(2x) \\ -2p_2^{[2]}\sin(2x) \end{bmatrix},$$

$$\mathcal{B}_{1,1}f_0^- = -\mathrm{i}\,p_1^{[1]}\begin{bmatrix} \cos(x) \\ 0 \end{bmatrix}, \quad \mathcal{B}_{1,2}f_0^- = -\mathrm{i}\,p_2^{[0]}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \mathrm{i}\,p_2^{[2]}\begin{bmatrix} \cos(2x) \\ 0 \end{bmatrix},$$

$$\mathcal{B}_{1,3}f_0^- = -\mathrm{i}\,p_3^{[1]}\begin{bmatrix} \cos(x) \\ 0 \end{bmatrix} + \mathfrak{h}^{[3]}(x)\,, \quad \mathcal{B}_{2,2}f_0^- = \begin{bmatrix} 0 \\ \mathtt{f}_2 \end{bmatrix},$$

with $p_j^{[i]}$ *and* $a_j^{[i]}$*,* $j=1,\dots,4$*,* $i=0,\dots,j$*, in* (A.4.22)*,* (A.4.23) *and* $\mathtt{f}_2$ *in* (A.4.8) *and where* $\mathfrak{h}^{[\kappa_1,\dots,\kappa_\ell]}(x)$ *denotes a function supported on Fourier modes* $\kappa_1,\dots,\kappa_\ell\in\mathbb{N}_0$*.*

Proof. By (5.1.10)-(5.1.11) and (4.1.2). □

We now compute the remaining coefficients in (5.1.102).

Computation of γ_{11}. In view of (5.1.102b) and (5.1.81a) we have

$$\gamma_{11} = \underbrace{(\mathcal{B}_{0,2}f_0^+, f_0^+)}_{a_2^{[0]} \text{ by (5.1.10c)}} + \underbrace{\frac12(\mathcal{B}_{0,1}P_{0,1}f_0^+, f_0^+) + \frac12(\mathcal{B}_{0,1}f_0^+, P_{0,1}f_0^+)}_{\mathfrak{u}_{0,1}\left(f_{-1}^+,\mathcal{B}_{0,1}f_0^+\right) \text{ by (5.1.19)}}\,. \tag{5.1.104}$$

By (5.1.103) and (5.1.18) it results that (5.1.104) is equal to

$$\gamma_{11} = a_2^{[0]} + \frac12\mathfrak{u}_{0,1}(a_1^{[1]}\mathtt{c}_\mathtt{h}^{\frac12} + p_1^{[1]}\mathtt{c}_\mathtt{h}^{-\frac12})\,,$$

which in view of (A.4.22), (5.1.20) gives the term in (5.1.6f).

Computation of ϕ_{21}. In view of (5.1.102b) and (5.1.95b) we have

$$\begin{aligned} \mathrm{i}\,\phi_{21} &= -(\mathfrak{B}_{1,1}f_1^-, f_0^+) \\ &= -(\mathcal{B}_{1,1}f_1^-, f_0^+) - \underbrace{(\mathcal{B}_{0,1}P_{1,0}f_1^-, f_0^+)}_{\mathrm{i}\,\mathfrak{u}_{1,0}\left(f_{-1}^+,\mathcal{B}_{0,1}f_0^+\right) \text{ by (5.1.19)}} - \underbrace{(\mathcal{B}_{1,0}P_{0,1}f_1^-, f_0^+)}_{0 \text{ by (5.1.19)}} - \underbrace{\frac12(\Pi_0^+P_{1,1}f_1^-, f_0^+)}_{0 \text{ by (5.1.21e)}}\,. \end{aligned}$$

By (5.1.103), (4.1.2) and (5.1.18) it results

$$\phi_{21} = \frac{1}{2}\big(\mathtt{c}_\mathtt{h}^{-\frac12} p_1^{[1]} - \mathfrak{u}_{1,0}\mathtt{c}_\mathtt{h}^{\frac12} a_1^{[1]} - \mathfrak{u}_{1,0}\mathtt{c}_\mathtt{h}^{-\frac12} p_1^{[1]}\big)$$

which in view of (A.4.22), (5.1.20) gives the term in (5.1.6i).

Computation of η_{12}. In view of (5.1.102b) and (5.1.95d), (5.1.19), Lemmata 5.1.4 and 5.1.4 we have

$$\begin{aligned}
\mathrm{i}\,\eta_{12} = {}& (\mathcal{B}_{1,2}f_1^-, f_1^+) + \underbrace{(\mathcal{B}_{1,1}P_{0,1}f_1^-, f_1^+)}_{\big(\begin{bmatrix}-\mathfrak{a}_{0,1}\sin(2x)\\ \mathfrak{b}_{0,1}\cos(2x)\end{bmatrix},\mathcal{B}_{1,1}f_1^+\big)\text{ by (5.1.19)}} + \underbrace{(\mathcal{B}_{0,2}P_{1,0}f_1^-, f_1^+)}_{\mathrm{i}\,\mathfrak{u}_{1,0}\left(f_{-1}^+,\mathcal{B}_{0,2}f_1^+\right)\text{ by (5.1.19)}} \\
&+ \underbrace{(\mathcal{B}_{1,0}P_{0,2}f_1^-, f_1^+)}_{\underline{\mathfrak{n}_{0,2}(\mathcal{B}_{1,0}f_1^-,f_1^+)}+\mathfrak{u}_{0,2}^-(f_{-1}^-,\mathcal{B}_{1,0}f_1^+)\text{ by (5.1.21b)}} + \underbrace{(\mathcal{B}_{0,1}P_{1,1}f_1^-, f_1^+)}_{\mathrm{i}\big(\begin{bmatrix}\mathfrak{a}_{1,1}\cos(2x)\\ \mathfrak{b}_{1,1}\sin(2x)\end{bmatrix},\mathcal{B}_{0,1}f_1^+\big)\text{ by (5.1.21e)}} \\
&+ \underbrace{\frac12(\Pi_0^+P_{1,2}f_1^-, f_1^+)}_{=0\text{ since }\Pi_0^+f_1^+=0}\ \underbrace{-\frac12(\mathcal{B}_{1,0}P_0P_{0,2}f_1^-, f_1^+)}_{\underline{-\frac12\mathfrak{n}_{0,2}(\mathcal{B}_{1,0}f_1^-,f_1^+)}\text{ by (5.1.21b)}}\ \underbrace{-\frac12(\mathcal{B}_{0,1}P_0P_{1,1}f_1^-, f_1^+)}_{=0\text{ by (5.1.21e)}} \\
&\underbrace{-\frac12(\mathcal{B}_{1,0}f_1^-, P_0P_{0,2}f_1^+)}_{\underline{-\frac12\mathfrak{n}_{0,2}(\mathcal{B}_{1,0}f_1^-,f_1^+)}\text{ by (5.1.21a)}}\ \underbrace{-\frac12(\mathcal{B}_{0,1}f_1^-, P_0P_{1,1}f_1^+)}_{\overset{(5.1.21e)}{=}+\mathrm{i}\,\frac12\widetilde{\mathfrak{m}}_{1,1}(f_1^-,\mathcal{B}_{0,1}f_0^-)=0}\ ,
\end{aligned}$$

where the three underlined terms cancel out. Hence, by (5.1.103), (4.1.2) and (5.1.18),

$$\begin{aligned}
\eta_{12} = {}& -p_2^{[0]} - \frac14 p_1^{[1]}(\mathfrak{b}_{0,1}\mathtt{c}_\mathtt{h}^{\frac12} + \mathfrak{a}_{0,1}\mathtt{c}_\mathtt{h}^{-\frac12}) + \mathfrak{u}_{1,0}\big(\frac12\mathtt{c}_\mathtt{h}a_2^{[0]} + \frac14\mathtt{c}_\mathtt{h}a_2^{[2]} - \frac12\mathtt{c}_\mathtt{h}^{-1}\mathtt{f}_2(1-\mathtt{c}_\mathtt{h}^4)\big) \\
&+ \frac12\mathfrak{u}_{0,2}^-(\mathtt{c}_\mathtt{h}^2 + \mathtt{h}(1-\mathtt{c}_\mathtt{h}^4))\mathtt{c}_\mathtt{h}^{-1} + \frac14\mathtt{c}_\mathtt{h}^{\frac12}a_1^{[1]}\mathfrak{a}_{1,1} - \frac12\mathtt{c}_\mathtt{h}^{\frac12}p_1^{[1]}\mathfrak{b}_{1,1} - \frac14\mathtt{c}_\mathtt{h}^{-\frac12}p_1^{[1]}\mathfrak{a}_{1,1}
\end{aligned}$$

which in view of (A.4.22), (5.1.20) and (5.1.21c) gives the term in (5.1.6e).

Computation of γ_{12}. By (5.1.102b) and (5.1.95d), (5.1.19), Lemma 5.1.4 and 5.1.4 and since $\mathcal{B}_{0,1}f_0^- = 0$ and $\mathcal{B}_{1,0}f_0^- = 0$ we have

$$\begin{aligned}
\mathrm{i}\,\gamma_{12} = {}& (\mathcal{B}_{1,2}f_0^-, f_0^+)\underbrace{+(\mathcal{B}_{1,1}P_{0,1}f_0^-, f_0^+)}_{=0}\underbrace{+(\mathcal{B}_{0,2}P_{1,0}f_0^-, f_0^+)}_{=0} \\
&\underbrace{+(\mathcal{B}_{1,0}P_{0,2}f_0^-, f_0^+)}_{=0}\underbrace{+(\mathcal{B}_{0,1}P_{1,1}f_0^-, f_0^+)}_{-\mathrm{i}\,\frac12\mathtt{c}_\mathtt{h}^{-3/2}(f_{-1}^+,\mathcal{B}_{0,1}f_0^+)}\underbrace{+\frac12(\Pi_0^+P_{1,2}f_0^-, f_0^+)}_{=0}
\end{aligned}$$

$$\underbrace{-\frac{1}{2}(\mathcal{B}_{1,0}P_0P_{0,2}f_0^-, f_0^+)}_{=0}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}P_0P_{1,1}f_0^-, f_0^+)}_{=0}$$
$$\underbrace{-\frac{1}{2}(\mathcal{B}_{1,0}f_0^-, P_0P_{0,2}f_0^+)}_{=0}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}f_0^-, P_0P_{1,1}f_0^+)}_{=0}.$$

So, by (5.1.103), (4.1.2) and (5.1.18),

$$\gamma_{12} = -p_2^{[0]} - \frac{1}{4}\mathtt{c_h}^{-3/2}(a_1^{[1]}\mathtt{c_h}^{\frac{1}{2}} + p_1^{[1]}\mathtt{c_h}^{-\frac{1}{2}})$$

which in view of (A.4.22) gives the term (5.1.6f).

Computation of ϕ_{11}. By (5.1.102b) and (5.1.95c), (5.1.19), Lemma 5.1.4 and 5.1.4 we have

$$\phi_{11} = (\mathcal{B}_{0,3}f_1^+, f_0^+) + \underbrace{(\mathcal{B}_{0,2}P_{0,1}f_1^+, f_0^+)}_{\left(\begin{bmatrix}\mathfrak{a}_{0,1}\cos(2x)\\ \mathfrak{b}_{0,1}\sin(2x)\end{bmatrix}, \mathcal{B}_{0,2}f_0^+\right)} + \underbrace{(\mathcal{B}_{0,1}P_{0,2}f_1^+, f_0^+)}_{\mathfrak{n}_{0,2}(f_1^+, \mathcal{B}_{0,1}f_0^+) + \mathfrak{u}_{0,2}^+(f_{-1}^+, \mathcal{B}_{0,1}f_0^+)}$$
$$\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}P_0P_{0,2}f_1^+, f_0^+)}_{-\frac{1}{2}\mathfrak{n}_{0,2}(f_1^+, \mathcal{B}_{0,1}f_0^+)}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}f_1^+, P_0P_{0,2}f_0^+)}_{\overset{(5.1.21a)}{=}0}.$$

Thus, by (5.1.103), (4.1.2) and (5.1.18),

$$\begin{aligned}\phi_{11} = \frac{1}{2}a_3^{[1]}\mathtt{c_h}^{\frac{1}{2}} - \frac{1}{2}p_3^{[1]}\mathtt{c_h}^{-\frac{1}{2}} + \frac{1}{2}\mathfrak{a}_{0,1}a_2^{[2]} - \mathfrak{b}_{0,1}p_2^{[2]}\\ + \frac{1}{4}\mathfrak{n}_{0,2}(a_1^{[1]}\mathtt{c_h}^{\frac{1}{2}} - p_1^{[1]}\mathtt{c_h}^{-\frac{1}{2}}) + \frac{1}{2}\mathfrak{u}_{0,2}^+(a_1^{[1]}\mathtt{c_h}^{\frac{1}{2}} + p_1^{[1]}\mathtt{c_h}^{-\frac{1}{2}})\end{aligned}$$

which in view of (A.4.22), (5.1.20), (5.1.21c), gives the term (5.1.6g).

Computation of ϕ_{22}. By (5.1.102b) and (5.1.95e), (5.1.19), Lemma 5.1.4 and 5.1.4 and since $\mathcal{B}_{2,1} = 0$ and $\mathcal{B}_{0,1}f_0^- = 0$, $\mathcal{B}_{1,0}f_0^- = 0$ we have

$$\phi_{22} = \underbrace{(\mathcal{B}_{2,1}f_0^-, f_1^-)}_{0} + \underbrace{(\mathcal{B}_{1,1}P_{1,0}f_0^-, f_1^-)}_{0}\underbrace{+(\mathcal{B}_{2,0}P_{0,1}f_0^-, f_1^-)}_{0}$$
$$\underbrace{+(\mathcal{B}_{0,1}P_{2,0}f_0^-, f_1^-)}_{0}\underbrace{+(\mathcal{B}_{1,0}P_{1,1}f_0^-, f_1^-)}_{-\frac{\mathrm{i}}{2}\mathtt{c_h}^{-3/2}\left(f_{-1}^+, \mathcal{B}_{1,0}f_1^-\right)} + \underbrace{\frac{1}{2}(\Pi_0^+P_{2,1}f_0^-, f_1^-)}_{\frac{1}{2}\left(\tilde{\mathfrak{n}}_{2,1}\Pi_0^+f_1^- + \tilde{\mathfrak{u}}_{2,1}\Pi_0^+f_{-1}^-, f_1^-\right)=0}$$
$$\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}P_0P_{2,0}f_0^-, f_1^-)}_{=0} - \underbrace{\frac{1}{2}(\mathcal{B}_{1,0}P_0P_{1,1}f_0^-, f_1^-)}_{=0}$$

$$-\underbrace{\frac{1}{2}(\mathcal{B}_{0,1}f_0^-, P_0P_{2,0}f_1^-)}_{=0}-\underbrace{\frac{1}{2}(\mathcal{B}_{1,0}f_0^-, P_0P_{1,1}f_1^-)}_{=0}.$$

So, by (5.1.103) and (5.1.18),

$$\phi_{22}=\frac{1}{4}\mathtt{c_h}^{-\frac{5}{2}}(\mathtt{c_h}^2+\mathtt{h}(1-\mathtt{c_h}^4))$$

which is the term (5.1.6i).

Computation of $\widetilde{\psi}_{12}$. By (5.1.102a), (5.1.95e) and since $\mathcal{B}_{2,1}=0$, $P_{1,0}f_0^-=0$, $P_{0,1}f_0^-=0$ by (5.1.19), Lemmata 5.1.4 and 5.1.4 this term is given by

$$\begin{aligned}\mathrm{i}\,\widetilde{\psi}_{12}=&\underbrace{(\mathcal{B}_{2,1}f_0^-, f_1^+)}_{=0}+\underbrace{(\mathcal{B}_{1,1}P_{1,0}f_0^-, f_1^+)}_{=0}+\underbrace{(\mathcal{B}_{2,0}P_{0,1}f_0^-, f_1^+)}_{=0}\\&+\underbrace{(\mathcal{B}_{0,1}P_{2,0}f_0^-, f_1^+)}_{=0\ \text{by}\,(5.1.21\text{d})}+\underbrace{(\mathcal{B}_{1,0}P_{1,1}f_0^-, f_1^+)}_{-\frac{\mathrm{i}}{2}\mathtt{c_h}^{-\frac{3}{2}}\left(f_{-1}^+,\mathcal{B}_{1,0}f_1^+\right)\ \text{by}(5.1.21\text{e})}+\underbrace{\frac{1}{2}(\Pi_0^+P_{2,1}f_0^-, f_1^+)}_{=0\ \text{as}\ \Pi_0^+f_1^+=0}\\&-\underbrace{\frac{1}{2}(\mathcal{B}_{0,1}P_0P_{2,0}f_0^-, f_1^+)}_{=0\ \text{by}\,(5.1.21\text{d})}-\frac{1}{2}\underbrace{(\mathcal{B}_{1,0}P_0P_{1,1}f_0^-, f_1^+)}_{=0\ \text{by}\,(5.1.21\text{e})}\\&-\frac{1}{2}\underbrace{(\mathcal{B}_{0,1}f_0^-, P_0P_{2,0}f_1^+)}_{=0\ \text{since}\ \mathcal{B}_{0,1}f_0^-=0}-\frac{1}{2}\underbrace{(\mathcal{B}_{1,0}f_0^-, P_0P_{1,1}f_1^+)}_{=0\ \text{since}\ \mathcal{B}_{1,0}f_0^-=0}\end{aligned}$$

and finally, by (5.1.18) and (5.1.103),

$$\mathrm{i}\,\widetilde{\psi}_{12}=-\tfrac{\mathrm{i}}{2}\mathtt{c_h}^{-3/2}(f_{-1}^+,\mathcal{B}_{1,0}f_1^+)=0\,.$$

Computation of η_{11}. By (5.1.102b), (5.1.95f), (5.1.19), Lemma 5.1.4 and Lemma 5.1.4 and (5.1.83), we have

$$\begin{aligned}\eta_{11}=&(\mathcal{B}_{0,4}f_1^+, f_1^+)+\underbrace{(\mathcal{B}_{0,3}P_{0,1}f_1^+, f_1^+)}_{\left(\begin{bmatrix}\mathfrak{a}_{0,1}\cos(2x)\\ \mathfrak{b}_{0,1}\sin(2x)\end{bmatrix},\mathcal{B}_{0,3}f_1^+\right)}+\underbrace{(\mathcal{B}_{0,2}P_{0,2}f_1^+, f_1^+)}_{\left(\underline{\mathfrak{n}_{0,2}f_1^+}+\mathfrak{u}_{0,2}^+f_{-1}^++\begin{bmatrix}\mathfrak{a}_{0,2}\cos(3x)\\ \mathfrak{b}_{0,2}\sin(3x)\end{bmatrix},\mathcal{B}_{0,2}f_1^+\right)}\\&+\underbrace{(\mathcal{B}_{0,1}P_{0,3}f_1^+, f_1^+)}_{\left(\begin{bmatrix}\mathfrak{a}_{0,3}\cos(2x)\\ \mathfrak{b}_{0,3}\sin(2x)\end{bmatrix},\mathcal{B}_{0,1}f_1^+\right)}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,2}P_0P_{0,2}f_1^+, f_1^+)}_{\underline{-\frac{1}{2}\mathfrak{n}_{0,2}\left(\mathcal{B}_{0,2}f_1^+,f_1^+\right)}}\\&\underbrace{-\frac{1}{2}(\mathcal{B}_{0,2}f_1^+, P_0P_{0,2}f_1^+)}_{\underline{-\frac{1}{2}\mathfrak{n}_{0,2}\left(\mathcal{B}_{0,2}f_1^+,f_1^+\right)}}-\underbrace{\frac{1}{2}(\mathcal{B}_{0,1}P_0P_{0,3}f_1^+, f_1^+)}_{=0}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\underbrace{(\mathcal{B}_{0,1}f_1^+,P_0P_{0,3}f_1^+)}_{=0}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}P_{0,1}P_0P_{0,2}f_1^+,f_1^+)}_{-\frac{1}{2}\mathfrak{n}_{0,2}\left(\begin{bmatrix}\mathfrak{a}_{0,1}\cos(2x)\\ \mathfrak{b}_{0,1}\sin(2x)\end{bmatrix},\mathcal{B}_{0,1}f_1^+\right)}\underbrace{-\frac{1}{2}(\mathcal{B}_{0,1}f_1^+,P_{0,1}P_0P_{0,2}f_1^+)}_{-\frac{1}{2}\mathfrak{n}_{0,2}\left(\begin{bmatrix}\mathfrak{a}_{0,1}\cos(2x)\\ \mathfrak{b}_{0,1}\sin(2x)\end{bmatrix},\mathcal{B}_{0,1}f_1^+\right)}\\
&+\frac{1}{2}\underbrace{(\mathfrak{N}_{0,2}P_0P_{0,2}f_1^+,f_1^+)}_{=0}+\underbrace{\frac{1}{2}(f_1^+,\mathfrak{N}_{0,2}P_0P_{0,2}f_1^+)}_{=0},
\end{aligned}
$$

where the three underlined terms cancel out. Thus, in view of (5.1.103), (4.1.2) and (5.1.18), we get

$$
\begin{aligned}
\eta_{11}=&\ \mathtt{c_h}\frac{a_4^{[0]}}{2}+\mathtt{c_h}\frac{a_4^{[2]}}{4}-p_4^{[0]}-\frac{p_4^{[2]}}{2}+\frac{1}{2\mathtt{c_h}}(1-\mathtt{c_h}^4)(\mathtt{f}_4-\mathtt{f}_2^2\mathtt{c_h}^2)\\
&+\frac{1}{4}(\mathtt{c_h}^{\frac{1}{2}}(a_3^{[1]}+a_3^{[3]})-\mathtt{c_h}^{-\frac{1}{2}}(p_3^{[1]}+p_3^{[3]}))\mathfrak{a}_{0,1}-\frac{1}{2}\mathtt{c_h}^{\frac{1}{2}}(p_3^{[3]}+p_3^{[1]})\mathfrak{b}_{0,1}\\
&+\frac{1}{4}(a_2^{[2]}\mathtt{c_h}^{\frac{1}{2}}-p_2^{[2]}\mathtt{c_h}^{-\frac{1}{2}})\mathfrak{a}_{0,2}-\frac{3}{4}\mathtt{c_h}^{\frac{1}{2}}p_2^{[2]}\mathfrak{b}_{0,2}\\
&+\frac{1}{2}\mathfrak{u}_{0,2}^+(\mathtt{c_h}a_2^{[0]}+\frac{1}{2}\mathtt{c_h}a_2^{[2]}-\mathtt{c_h}^{-1}\mathtt{f}_2(1-\mathtt{c_h}^4))\\
&+\frac{1}{4}\mathfrak{a}_{0,3}(a_1^{[1]}\mathtt{c_h}^{\frac{1}{2}}-p_1^{[1]}\mathtt{c_h}^{-\frac{1}{2}})-\frac{1}{2}\mathfrak{b}_{0,3}\mathtt{c_h}^{\frac{1}{2}}p_1^{[1]}\\
&-\frac{1}{4}\mathfrak{n}_{0,2}\mathfrak{a}_{0,1}(a_1^{[1]}\mathtt{c_h}^{\frac{1}{2}}-p_1^{[1]}\mathtt{c_h}^{-\frac{1}{2}})+\frac{1}{2}\mathfrak{n}_{0,2}\mathfrak{b}_{0,1}\mathtt{c_h}^{\frac{1}{2}}p_1^{[1]}
\end{aligned}
\tag{5.1.105}
$$

which in view of (A.4.22), (A.4.23c), (5.1.20), (5.1.21c), (5.1.24) gives (5.1.6d).

Computation of γ_{22}. By (5.1.102b), (5.1.95g), where we exploit that $(\mathbf{Sym}[A]f,f)=\mathfrak{Re}(Af,f)$, (5.1.19), Lemma 5.1.4, Lemma 5.1.4 and since $\mathcal{B}_{0,1}f_0^-=0$ and $\mathcal{B}_{1,0}f_0^-=0$ we have

$$
\begin{aligned}
\gamma_{22}=&\underbrace{(\mathcal{B}_{2,2}f_0^-,f_0^-)}_{\mathtt{f}_2}+\underbrace{(\mathcal{B}_{1,2}P_{1,0}f_0^-,f_0^-)}_{=0}+\underbrace{(\mathcal{B}_{2,1}P_{0,1}f_0^-,f_0^-)}_{=0}+\underbrace{(\mathcal{B}_{0,2}P_{2,0}f_0^-,f_0^-)}_{=0}\\
&+\underbrace{(\mathcal{B}_{1,1}P_{1,1}f_0^-,f_0^-)}_{-\frac{\mathrm{i}}{2}\mathtt{c_h}^{-3/2}(f_{-1}^+,\mathcal{B}_{1,1}f_0^-)}+\underbrace{(\mathcal{B}_{2,0}P_{0,2}f_0^-,f_0^-)}_{=0}+\underbrace{(\mathcal{B}_{0,1}P_{2,1}f_0^-,f_0^-)}_{(P_{2,1}f_0^-,\mathcal{B}_{0,1}f_0^-)=0}+\underbrace{(\mathcal{B}_{1,0}P_{1,2}f_0^-,f_0^-)}_{=0}+\frac{1}{2}\underbrace{(\Pi_0^+P_{2,2}f_0^-,f_0^-)}_{=0}\\
&\underbrace{-\mathfrak{Re}(\mathcal{B}_{0,2}P_0P_{2,0}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{1,1}P_0P_{1,1}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{2,0}P_0P_{0,2}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{0,1}P_0P_{2,1}f_0^-,f_0^-)}_{(P_0P_{2,1}f_0^-,\mathcal{B}_{0,1}f_0^-)=0}\\
&\underbrace{-\mathfrak{Re}(\mathcal{B}_{1,0}P_0P_{1,2}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{0,1}P_{0,1}P_0P_{2,0}f_0^-,f_0^-)}_{=0}\\
&\underbrace{-\mathfrak{Re}(\mathcal{B}_{1,0}P_{0,1}P_0P_{1,1}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{0,1}P_{1,0}P_0P_{1,1}f_0^-,f_0^-)}_{=0}\underbrace{-\mathfrak{Re}(\mathcal{B}_{1,0}P_{1,0}P_0P_{0,2}f_0^-,f_0^-)}_{=0}
\end{aligned}
$$

$$\underbrace{+\mathfrak{Re}\big(\mathfrak{N}_{2,0}P_0P_{0,2}f_0^-,f_0^-\big)}_{=0}\underbrace{+\mathfrak{Re}\big(\mathfrak{N}_{0,2}P_0P_{2,0}f_0^-,f_0^-\big)}_{=0}\underbrace{+\mathfrak{Re}\big(\mathfrak{N}_{1,1}P_0P_{1,1}f_0^-,f_0^-\big)}_{=0}.$$

Then by (5.1.10e), (5.1.11), (5.1.12d), (5.1.18) and (5.1.103) we get

$$\gamma_{22}=\mathtt{f}_2+\frac{p_1^{[1]}}{4\mathtt{c_h}}$$

which, in view of (A.4.8), (A.4.22c) gives (5.1.6f).

Computation of ϕ_{12}. By (5.1.102b), (5.1.95h), (5.1.19), Lemma 5.1.4, Lemma 5.1.4, (5.1.83) and since $\mathcal{B}_{0,2}f_0^-=0$, $\mathcal{B}_{1,0}f_0^-=0$ and $\mathcal{B}_{0,1}f_0^-=0$ we have

$$\begin{aligned}
\mathrm{i}\,\phi_{12}=&\ \big(\mathcal{B}_{1,3}f_0^-,f_1^+\big)+\underbrace{\big(\mathcal{B}_{0,3}P_{1,0}f_0^-,f_1^+\big)}_{=0}+\underbrace{\big(\mathcal{B}_{1,2}P_{0,1}f_0^-,f_1^+\big)}_{=0}\\
&+\underbrace{\big(\mathcal{B}_{0,2}P_{1,1}f_0^-,f_1^+\big)}_{-\frac{\mathrm{i}}{2}\mathtt{c_h}^{-3/2}\left(f_{-1}^+,\mathcal{B}_{0,2}f_1^+\right)}+\underbrace{\big(\mathcal{B}_{1,1}P_{0,2}f_0^-,f_1^+\big)}_{=0}+\underbrace{\big(\mathcal{B}_{1,0}P_{0,3}f_0^-,f_1^+\big)}_{=0}+\underbrace{\big(\mathcal{B}_{0,1}P_{1,2}f_0^-,f_1^+\big)}_{\mathrm{i}\left(\begin{bmatrix}\mathfrak{a}_{1,2}\cos(2x)\\ \mathfrak{b}_{1,2}\sin(2x)\end{bmatrix},\mathcal{B}_{0,1}f_1^+\right)}\\
&+\underbrace{\frac12\big(\Pi_0^+P_{1,3}f_0^-,f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,2}P_0P_{1,1}f_0^-,f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,2}f_0^-,P_0P_{1,1}f_1^+\big)}_{=0}\\
&-\frac12\underbrace{\big(\mathcal{B}_{1,1}P_0P_{0,2}f_0^-,f_1^+\big)}_{=0}\underbrace{-\frac12\big(\mathcal{B}_{1,1}f_0^-,P_0P_{0,2}f_1^+\big)}_{-\frac12\mathfrak{n}_{0,2}\left(\mathcal{B}_{1,1}f_0^-,f_1^+\right)}-\frac12\underbrace{\big(\mathcal{B}_{1,0}P_0P_{0,3}f_0^-,f_1^+\big)}_{=0}\\
&-\frac12\underbrace{\big(\mathcal{B}_{1,0}f_0^-,P_0P_{0,3}f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,1}P_0P_{1,2}f_0^-,f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,1}f_0^-,P_0P_{1,2}f_1^+\big)}_{=0}\\
&-\frac12\underbrace{\big(\mathcal{B}_{1,0}P_{0,1}P_0P_{0,2}f_0^-,f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{1,0}f_0^-,P_{0,1}P_0P_{0,2}f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,1}P_{1,0}P_0P_{0,2}f_0^-,f_1^+\big)}_{=0}\\
&-\frac12\underbrace{\big(\mathcal{B}_{0,1}f_0^-,P_{1,0}P_0P_{0,2}f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,1}P_{0,1}P_0P_{1,1}f_0^-,f_1^+\big)}_{=0}-\frac12\underbrace{\big(\mathcal{B}_{0,1}f_0^-,P_{0,1}P_0P_{1,1}f_1^+\big)}_{=0}\\
&+\underbrace{\frac12\big(\mathfrak{N}_{1,1}P_0P_{0,2}f_0^-,f_1^+\big)}_{=0}+\underbrace{\frac12\big(f_0^-,\mathfrak{N}_{1,1}P_0P_{0,2}f_1^+\big)}_{=0}\\
&+\underbrace{\frac12\big(\mathfrak{N}_{0,2}P_0P_{1,1}f_0^-,f_1^+\big)}_{=0}+\underbrace{\frac12\big(f_0^-,\mathfrak{N}_{0,2}P_0P_{1,1}f_1^+\big)}_{-\frac{\mathrm{i}}{2}\widetilde{\mathfrak{m}}_{1,1}\left(f_0^-,\mathfrak{N}_{0,2}f_0^-\right)\overset{(5.1.83)}{=}0}.
\end{aligned}$$

Hence by (5.1.103), (4.1.2), (5.1.18), we have

$$\begin{aligned}\phi_{12} = &-\frac{1}{2}p_3^{[1]}\mathtt{c}_{\mathtt{h}}^{1/2} - \frac{1}{4}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}(a_2^{[0]} + \frac{1}{2}a_2^{[2]}) + \frac{1}{4}\mathtt{c}_{\mathtt{h}}^{-\frac{5}{2}}\mathtt{f}_2(1-\mathtt{c}_{\mathtt{h}}^4) \\ &+ \frac{1}{4}\mathfrak{a}_{1,2}(a_1^{[1]}\mathtt{c}_{\mathtt{h}}^{\frac{1}{2}} - p_1^{[1]}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}) - \frac{1}{2}\mathfrak{b}_{1,2}\mathtt{c}_{\mathtt{h}}^{\frac{1}{2}}p_1^{[1]} + \frac{1}{4}\mathfrak{n}_{0,2}p_1^{[1]}\mathtt{c}_{\mathtt{h}}^{\frac{1}{2}}\,,\end{aligned}$$

which, in view of (A.4.22), (5.1.24) gives the term (5.1.6h).

5.2 Block decoupling and proof of Theorem 1.6.1

In this section we prove Theorem 1.6.1 by block-decoupling the 4×4 Hamiltonian matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ in (5.1.1) obtained in Proposition 5.1.1, expanding the computations of the finite-depth case at a higher degree of accuracy.

We first perform the singular symplectic and reversibility-preserving change of coordinates in Lemma 4.3.1.

Lemma 5.2.1. (Singular symplectic rescaling) *The conjugation of the Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon} = \mathtt{J}_4\mathtt{B}_{\mu,\epsilon}$ in* (5.1.1) *obtained in Proposition 5.1.1 through the symplectic and reversibility-preserving 4×4-matrix*

$$Y := \begin{pmatrix} Q & 0 \\ 0 & Q\end{pmatrix} \quad with \quad Q := \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}}\end{pmatrix}\,, \quad \mu > 0\,,$$

yields the Hamiltonian and reversible matrix

$$\mathtt{L}^{(1)}_{\mu,\epsilon} := Y^{-1}\mathtt{L}_{\mu,\epsilon}Y = \mathtt{J}_4\mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2E^{(1)} & \mathtt{J}_2F^{(1)} \\ \mathtt{J}_2[F^{(1)}]^* & \mathtt{J}_2G^{(1)}\end{pmatrix} \tag{5.2.1}$$

where $\mathtt{B}^{(1)}_{\mu,\epsilon}$ is a self-adjoint and reversibility-preserving 4×4 matrix

$$\mathtt{B}^{(1)}_{\mu,\epsilon} = \begin{pmatrix} E^{(1)} & F^{(1)} \\ [F^{(1)}]^* & G^{(1)}\end{pmatrix}\,, \quad E^{(1)} = [E^{(1)}]^*\,,\ G^{(1)} = [G^{(1)}]^*\,,$$

where the 2×2 reversibility-preserving matrices $E^{(1)} := E^{(1)}(\mu,\epsilon)$, $G^{(1)} := G^{(1)}(\mu,\epsilon)$ and $F^{(1)} := F^{(1)}(\mu,\epsilon)$ extend analytically at $\mu = 0$ with the expansion

$$E^{(1)} = \begin{pmatrix} \mathtt{e}_{11}\mu\epsilon^2(1+r_1'(\epsilon^3,\mu\epsilon)) + \eta_{11}\mu\epsilon^4 - \mathtt{e}_{22}\frac{\mu^3}{8}(1+r_1''(\epsilon,\mu)) & \mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + \eta_{12}\mu\epsilon^2 + r_2(\mu\epsilon^3,\mu^2\epsilon,\mu^3)\right) \\ -\mathrm{i}\left(\frac{1}{2}\mathtt{e}_{12}\mu + \eta_{12}\mu\epsilon^2 + r_2(\mu\epsilon^3,\mu^2\epsilon,\mu^3)\right) & -\mathtt{e}_{22}\frac{\mu}{8}(1+r_5(\epsilon^2,\mu))\end{pmatrix} \tag{5.2.2a}$$

$$G^{(1)} := \begin{pmatrix} \mu + \gamma_{11}\mu\epsilon^2 + r_8(\mu\epsilon^3, \mu^2\epsilon^2, \mu^3\epsilon) & -\mathrm{i}\,\gamma_{12}\mu\epsilon^2 - \mathrm{i}\,r_9(\mu\epsilon^3, \mu^2\epsilon) \\ \mathrm{i}\,\gamma_{12}\mu\epsilon^2 + \mathrm{i}\,r_9(\mu\epsilon^3, \mu^2\epsilon) & \tanh(\mathtt{h}\mu) + \gamma_{22}\mu\epsilon^2 + r_{10}(\mu\epsilon^3, \mu^2\epsilon) \end{pmatrix} \tag{5.2.2b}$$

$$F^{(1)} := \begin{pmatrix} \mathtt{f}_{11}\mu\epsilon + \phi_{11}\mu\epsilon^3 + r_3(\mu\epsilon^4, \mu^2\epsilon^2, \mu^3\epsilon) & \mathrm{i}\,\mu\epsilon\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + \mathrm{i}\,\phi_{12}\mu\epsilon^3 + \mathrm{i}\,r_4(\mu\epsilon^4, \mu^2\epsilon^2, \mu^3\epsilon) \\ \mathrm{i}\,\phi_{21}\mu\epsilon + \mathrm{i}\,r_6(\mu\epsilon^3, \mu^2\epsilon) & \phi_{22}\mu\epsilon + r_7(\mu\epsilon^3, \mu^2\epsilon) \end{pmatrix} \tag{5.2.2c}$$

where the coefficients appearing in the entries are the same of (5.1.5).

Note that the matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$, initially defined only for $\mu \neq 0$, extends analytically to the zero matrix at $\mu = 0$. For $\mu \neq 0$ the spectrum of $\mathtt{L}^{(1)}_{\mu,\epsilon}$ coincides with the spectrum of $\mathtt{L}_{\mu,\epsilon}$.

Non-perturbative step of block decoupling. The following lemma computes the first order Taylor expansions (5.2.4) of the matrix entries in (5.2.3) and then the expansion (5.2.8) at a higher degree of accuracy with respect to Lemma 4.3.4.

Lemma 5.2.2. (Step of block decoupling) *There exists a* 2×2 *reversibility-preserving matrix* X*, analytic in* (μ, ϵ)*, of the form*

$$X := \begin{pmatrix} x_{11} & \mathrm{i}\,x_{12} \\ \mathrm{i}\,x_{21} & x_{22} \end{pmatrix}, \qquad x_{ij} \in \mathbb{R},\ i, j = 1, 2, \tag{5.2.3}$$

with

$$x_{11} = x_{11}^{(1)}\epsilon + r(\epsilon^3, \mu\epsilon), \qquad x_{12} = x_{12}^{(1)}\epsilon + r(\epsilon^3, \mu\epsilon) \tag{5.2.4a}$$
$$x_{21} = x_{21}^{(1)}\epsilon + x_{21}^{(3)}\epsilon^3 + r(\epsilon^4, \mu\epsilon^2, \mu^2\epsilon), \qquad x_{22} = x_{22}^{(1)}\epsilon + x_{22}^{(3)}\epsilon^3 + r(\epsilon^4, \mu\epsilon^2, \mu^2\epsilon),$$

where

$$x_{21}^{(1)} := -\tfrac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}\big(\mathtt{e}_{12}\mathtt{f}_{11} + 2\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\big), \quad x_{22}^{(1)} := \tfrac{1}{2}\mathtt{D}_{\mathtt{h}}^{-1}\big(\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\mathtt{e}_{12} + 2\mathtt{h}\mathtt{f}_{11}\big), \tag{5.2.4b}$$

and

$$\begin{aligned} x_{11}^{(1)} &:= \mathtt{D}_{\mathtt{h}}^{-1}\big(\tfrac{1}{16}\mathtt{e}_{12}\mathtt{e}_{22}x_{21}^{(1)} - \tfrac{1}{2}\mathtt{e}_{12}\phi_{21} + \phi_{22} - \tfrac{1}{8}\mathtt{e}_{22}x_{22}^{(1)}\big), \\ x_{12}^{(1)} &:= \mathtt{D}_{\mathtt{h}}^{-1}\big(\tfrac{1}{8}\mathtt{h}\mathtt{e}_{22}x_{21}^{(1)} - \mathtt{h}\phi_{21} + \tfrac{1}{2}\mathtt{e}_{12}\phi_{22} - \tfrac{1}{16}\mathtt{e}_{12}\mathtt{e}_{22}x_{22}^{(1)}\big), \\ x_{21}^{(3)} &:= \mathtt{D}_{\mathtt{h}}^{-1}\big(-\tfrac{1}{2}\mathtt{e}_{11}\mathtt{e}_{12}x_{11}^{(1)} + \tfrac{1}{2}(\gamma_{12} + \eta_{12})\mathtt{e}_{12}x_{21}^{(1)} + \tfrac{1}{2}\mathtt{e}_{12}\gamma_{11}x_{22}^{(1)} \\ &\qquad - \tfrac{1}{2}\phi_{11}\mathtt{e}_{12} - \mathtt{e}_{11}x_{12}^{(1)} - \gamma_{22}x_{21}^{(1)} - (\gamma_{12} + \eta_{12})x_{22}^{(1)} - \phi_{12}\big), \\ x_{22}^{(3)} &:= \mathtt{D}_{\mathtt{h}}^{-1}\big(\mathtt{h}\mathtt{e}_{11}x_{11}^{(1)} - \mathtt{h}(\gamma_{12} + \eta_{12})x_{21}^{(1)} - \mathtt{h}\gamma_{11}x_{22}^{(1)} + \mathtt{h}\phi_{11} \\ &\qquad + \tfrac{1}{2}\mathtt{e}_{11}\mathtt{e}_{12}x_{12}^{(1)} + \tfrac{1}{2}\mathtt{e}_{12}\gamma_{22}x_{21}^{(1)} + \tfrac{1}{2}\mathtt{e}_{12}(\gamma_{12} + \eta_{12})x_{22}^{(1)} + \tfrac{1}{2}\mathtt{e}_{12}\phi_{12}\big), \end{aligned} \tag{5.2.4c}$$

with $\mathtt{e}_{12}$*,* $\mathtt{e}_{22}$*,* $\mathtt{e}_{11}$*,* $\phi_{21}, \phi_{22}, \gamma_{12}, \eta_{12}, \gamma_{11}, \phi_{11}, \gamma_{22}, \phi_{12}, \mathtt{f}_{11}$ *computed in* (5.1.6) *and, we report here formula* (4.3.7)*,*

$$\mathtt{D}_{\mathtt{h}} := \mathtt{h} - \frac{1}{4}\mathtt{e}_{12}^2 > 0\,, \quad \forall \mathtt{h} > 0\,, \tag{5.2.5}$$

such that the following holds true. By conjugating the Hamiltonian and reversible matrix $\mathtt{L}^{(1)}_{\mu,\epsilon}$*, defined in* (5.2.1)*, with the symplectic and reversibility-preserving* 4×4 *matrix*

$$\exp\big(S^{(1)}\big)\,, \qquad \text{where} \qquad S^{(1)} := \mathtt{J}_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}, \qquad \Sigma := \mathtt{J}_2 X\,, \tag{5.2.6}$$

we get the Hamiltonian and reversible matrix

$$\mathtt{L}^{(2)}_{\mu,\epsilon} := \exp\big(S^{(1)}\big)\mathtt{L}^{(1)}_{\mu,\epsilon} \exp\big(-S^{(1)}\big) = \mathtt{J}_4 \mathtt{B}^{(2)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(2)} & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \tag{5.2.7}$$

where the reversibility-preserving 2×2 *self-adjoint matrix* $E^{(2)}$ *has the form*

$$E^{(2)} = \begin{pmatrix} \mathtt{e}_{WB}\mu\epsilon^2 + \eta_{WB}\mu\epsilon^4 + r_1'(\mu\epsilon^5, \mu^2\epsilon^3) - \mathtt{e}_{22}\frac{\mu^3}{8}(1 + r_1''(\epsilon,\mu)) & \mathrm{i}\,\big(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) \\ -\mathrm{i}\,\big(\frac{1}{2}\mathtt{e}_{12}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\big) & -\mathtt{e}_{22}\frac{\mu}{8}(1 + r_5(\epsilon,\mu)) \end{pmatrix}, \tag{5.2.8}$$

where $\mathtt{e}_{WB}$ *is the Whitham-Benjamin function in* (1.5.1) *and*

$$\begin{aligned} \eta_{WB} = {}& \eta_{11} + x_{21}^{(1)}\phi_{12} + x_{21}^{(3)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} - x_{22}^{(1)}\phi_{11} - x_{22}^{(3)}\mathtt{f}_{11} + \frac{3}{2}(x_{21}^{(1)})^2 x_{22}^{(1)}\phi_{22} + (x_{21}^{(1)})^2 x_{12}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} \\ & - \frac{3}{2}x_{21}^{(1)}x_{12}^{(1)}x_{22}^{(1)}\mathtt{f}_{11} + \frac{3}{2}(x_{22}^{(1)})^2 x_{21}^{(1)}\phi_{21} - \frac{3}{2}x_{22}^{(1)}x_{11}^{(1)}x_{21}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + (x_{22}^{(1)})^2 x_{11}^{(1)}\mathtt{f}_{11} \end{aligned} \tag{5.2.9}$$

with $x_{11}^{(1)}$*,* $x_{12}^{(1)}$*,* $x_{22}^{(1)}$*,* $x_{21}^{(1)}$*,* $x_{21}^{(3)}$*,* $x_{22}^{(3)}$ *in* (5.2.4) *and the remaining coefficients in* (5.1.6)*, whereas the reversibility-preserving* 2×2 *self-adjoint matrix* $G^{(2)}$ *has the form*

$$G^{(2)} = \begin{pmatrix} \mu + r_8(\mu\epsilon^2, \mu^3\epsilon) & -\mathrm{i}\, r_9(\mu\epsilon^2, \mu^2\epsilon) \\ \mathrm{i}\, r_9(\mu\epsilon^2, \mu^2\epsilon) & \tanh(\mathtt{h}\mu) + r_{10}(\mu\epsilon) \end{pmatrix}, \tag{5.2.10}$$

and finally

$$F^{(2)} = \begin{pmatrix} r_3(\mu\epsilon^3) & \mathrm{i}\, r_4(\mu\epsilon^3) \\ \mathrm{i}\, r_6(\mu\epsilon^3) & r_7(\mu\epsilon^3) \end{pmatrix}. \tag{5.2.11}$$

The rest of the section is devoted to the proof of Lemma 5.2.2. In Lemma 4.3.4 we proved the existence of a matrix X as in (5.2.3) such that we obtain (5.2.7) with matrices $G^{(2)}, F^{(2)}$ as in (5.2.10)-(5.2.11) and a 2×2-self adjoint and reversibility preserving matrix

$E^{(2)}$ whose first entry has the form $[E^{(2)}]_{11} = \mathtt{e}_{\rm WB}\mu\epsilon^2 + r_1(\mu\epsilon^3, \mu^2\epsilon^2)$. The main result of Lemma 5.2.2 is that the first entry $[E^{(2)}]_{11}$ has the better expansion

$$[E^{(2)}]_{11} = \mathtt{e}_{\rm WB}\mu\epsilon^2 + r_1(\mu\epsilon^3, \mu^2\epsilon^2) = \mathtt{e}_{\rm WB}\mu\epsilon^2 + \eta_{\rm WB}\mu\epsilon^4 + r_1'(\mu\epsilon^5, \mu^2\epsilon^3)$$

with $\eta_{\rm WB}$ computed in (5.2.9), which is relevant to determine the stability/instablity of the Stokes wave at the critical depth. Clearly we could compute explicitly also other Taylor coefficients of the matrix entries of $E^{(2)}, G^{(2)}, F^{(2)}$, but it is not needed.

The coefficients $x_{21}^{(1)}$ and $x_{22}^{(1)}$ in (5.2.4b) were already computed in Lemma 4.3.4.

We now expand in Lie series the Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}^{(2)} = \exp(S)\mathtt{L}_{\mu,\epsilon}^{(1)}\exp(-S)$ where for simplicity we set $S := S^{(1)}$. We split $\mathtt{L}_{\mu,\epsilon}^{(1)}$ into its 2×2-diagonal and off-diagonal Hamiltonian and reversible matrices

$$\mathtt{L}_{\mu,\epsilon}^{(1)} = D^{(1)} + R^{(1)}\,, \tag{5.2.12}$$

$$D^{(1)} := \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} := \begin{pmatrix} \mathtt{J}_2 E^{(1)} & 0 \\ 0 & \mathtt{J}_2 G^{(1)} \end{pmatrix}, \quad R^{(1)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} \\ \mathtt{J}_2 [F^{(1)}]^* & 0 \end{pmatrix},$$

and we perform the Lie expansion

$$\mathtt{L}_{\mu,\epsilon}^{(2)} = \exp(S)\mathtt{L}_{\mu,\epsilon}^{(1)}\exp(-S) = D^{(1)} + [S, D^{(1)}] + \frac{1}{2}[S,[S,D^{(1)}]] + R^{(1)} + [S, R^{(1)}] \tag{5.2.13}$$
$$+ \frac{1}{2}\int_0^1 (1-\tau)^2 \exp(\tau S)\mathrm{ad}_S^3(D^{(1)})\exp(-\tau S)\,\mathrm{d}\tau + \int_0^1 (1-\tau)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\,\mathrm{d}\tau$$

where $\mathrm{ad}_A(B) := [A,B] := AB - BA$ denotes the commutator between the linear operators A, B.

We look for a 4×4 matrix S as in (5.2.6) that solves the homological equation $R^{(1)} + \left[S\,,\, D^{(1)}\right] = 0$, which, recalling (5.2.12), reads

$$\begin{pmatrix} 0 & \mathtt{J}_2 F^{(1)} + \mathtt{J}_2\Sigma D_0 - D_1\mathtt{J}_2\Sigma \\ \mathtt{J}_2 \left[F^{(1)}\right]^* + \mathtt{J}_2\Sigma^* D_1 - D_0\mathtt{J}_2\Sigma^* & 0 \end{pmatrix} = 0\,. \tag{5.2.14}$$

Writing $\Sigma = \mathtt{J}_2 X$, namely $X = -\mathtt{J}_2\Sigma$, the equation (5.2.14) amounts to solve the "Sylvester" equation

$$D_1 X - X D_0 = -\mathtt{J}_2 F^{(1)}\,. \tag{5.2.15}$$

We write the matrices $E^{(1)}, F^{(1)}, G^{(1)}$ in (5.2.1) as

$$E^{(1)} = \begin{pmatrix} E_{11}^{(1)} & \mathrm{i}\,E_{12}^{(1)} \\ -\mathrm{i}\,E_{12}^{(1)} & E_{22}^{(1)} \end{pmatrix}, \quad F^{(1)} = \begin{pmatrix} F_{11}^{(1)} & \mathrm{i}\,F_{12}^{(1)} \\ \mathrm{i}\,F_{21}^{(1)} & F_{22}^{(1)} \end{pmatrix}, \quad G^{(1)} = \begin{pmatrix} G_{11}^{(1)} & \mathrm{i}\,G_{12}^{(1)} \\ -\mathrm{i}\,G_{12}^{(1)} & G_{22}^{(1)} \end{pmatrix} \tag{5.2.16}$$

where the real numbers $E_{ij}^{(1)}, F_{ij}^{(1)}, G_{ij}^{(1)}$, $i, j = 1, 2$, have the expansion in (5.2.2a), (5.2.2b), (5.2.2c). Thus, by (5.2.12), (5.2.3) and (5.2.16), the equation (5.2.15) amounts to solve the 4×4 real linear system

$$\underbrace{\begin{pmatrix} G_{12}^{(1)} - E_{12}^{(1)} & G_{11}^{(1)} & E_{22}^{(1)} & 0 \\ G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} & 0 & -E_{22}^{(1)} \\ E_{11}^{(1)} & 0 & G_{12}^{(1)} - E_{12}^{(1)} & -G_{11}^{(1)} \\ 0 & -E_{11}^{(1)} & -G_{22}^{(1)} & G_{12}^{(1)} - E_{12}^{(1)} \end{pmatrix}}_{=:\mathcal{A}} \underbrace{\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}}_{=:\vec{x}} = \underbrace{\begin{pmatrix} -F_{21}^{(1)} \\ F_{22}^{(1)} \\ -F_{11}^{(1)} \\ F_{12}^{(1)} \end{pmatrix}}_{=:\vec{f}} . \tag{5.2.17}$$

As in the finite-depth case system (5.2.17) admits a unique solution. We now prove that it has the form (5.2.4).

Lemma 5.2.3. *The vector $\vec{x} = (x_{11}, x_{12}, x_{21}, x_{22})$ with entries in* (5.2.4) *solves* (5.2.17).

Proof. Since $\tanh(\mathtt{h}\mu) = \mathtt{h}\mu + r(\mu^3)$, we have

$$\begin{aligned} &G_{12}^{(1)} - E_{12}^{(1)} = -\mathtt{e}_{12}\frac{\mu}{2} - (\gamma_{12} + \eta_{12})\mu\epsilon^2 + r(\mu\epsilon^3, \mu^2\epsilon, \mu^3)\,, \\ &G_{11}^{(1)} = \mu + \gamma_{11}\mu\epsilon^2 + r_8(\mu\epsilon^3, \mu^2\epsilon^2, \mu^3\epsilon)\,, \quad E_{22}^{(1)} = -\mathtt{e}_{22}\frac{\mu}{8}(1 + r_5(\epsilon^2, \mu))\,, \\ &G_{22}^{(1)} = \mu\mathtt{h} + \gamma_{22}\mu\epsilon^2 + r(\mu\epsilon^3, \mu^2\epsilon, \mu^3)\,, \quad E_{11}^{(1)} = \mathtt{e}_{11}\mu\epsilon^2 + r(\mu\epsilon^4, \mu^2\epsilon^3, \mu^3)\,, \end{aligned} \tag{5.2.18}$$

with coefficients $\mathtt{e}_{12}$, γ_{12}, η_{12}, γ_{11}, $\mathtt{e}_{22}$, γ_{22} and $\mathtt{e}_{11}$ computed in (5.1.6). We exploit that the terms $x_{21}^{(1)}$ and $x_{22}^{(2)}$ have been already computed in Lemma 4.3.4, in order to get $x_{11}^{(1)}$ and $x_{12}^{(2)}$ in (5.2.4) as solutions of the system

$$\begin{pmatrix} -\frac{1}{2}\mathtt{e}_{12} & 1 \\ \mathtt{h} & -\frac{1}{2}\mathtt{e}_{12} \end{pmatrix} \begin{pmatrix} x_{11}^{(1)} \\ x_{12}^{(1)} \end{pmatrix} = \begin{pmatrix} \frac{1}{8}\mathtt{e}_{22}x_{21}^{(1)} - \phi_{21} \\ \phi_{22} - \frac{1}{8}\mathtt{e}_{22}x_{22}^{(1)} \end{pmatrix}, \quad \det \begin{pmatrix} -\frac{1}{2}\mathtt{e}_{12} & 1 \\ \mathtt{h} & -\frac{1}{2}\mathtt{e}_{12} \end{pmatrix} \overset{(5.2.5)}{=} -\mathtt{D}_{\mathtt{h}} < 0\,, \tag{5.2.19}$$

given, using also (5.2.2c), by the first two lines in (5.2.17) at order $\mu\epsilon$. Similarly, $x_{21}^{(3)}$ and $x_{22}^{(3)}$ in (5.2.4) solve the system

$$\begin{pmatrix} -\frac{1}{2}\mathtt{e}_{12} & -1 \\ -\mathtt{h} & -\frac{1}{2}\mathtt{e}_{12} \end{pmatrix} \begin{pmatrix} x_{21}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} -\mathtt{e}_{11}x_{11}^{(1)} + (\gamma_{12} + \eta_{12})x_{21}^{(1)} + \gamma_{11}x_{22}^{(1)} - \phi_{11} \\ \mathtt{e}_{11}x_{12}^{(1)} + \gamma_{22}x_{21}^{(1)} + (\gamma_{12} + \eta_{12})x_{22}^{(1)} + \phi_{12} \end{pmatrix}, \tag{5.2.20}$$

which comes, also by (5.2.2c), from the last two lines of (5.2.17) at order $\mu\epsilon^3$. The solutions of (5.2.19)-(5.2.20) are given in (5.2.4c). $\square$

We now prove the expansion (5.2.8). Since the matrix S solves the homological equation $\left[S\,,\,D^{(1)}\right]+R^{(1)}=0$, identity (5.2.13) simplifies to

$$\mathtt{L}^{(2)}_{\mu,\epsilon}=D^{(1)}+\frac{1}{2}[S,R^{(1)}]+\frac{1}{2}\int_0^1(1-\tau^2)\,\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\mathrm{d}\tau\,. \tag{5.2.21}$$

By plugging the Lie expansion

$$\begin{aligned}&\exp(\tau S)\,\mathrm{ad}_S^2(R^{(1)})\,\exp(-\tau S)\\&\quad=\mathrm{ad}_S^2(R^{(1)})+\tau\mathrm{ad}_S^3(R^{(1)})+\tau^2\int_0^1(1-\tau')\exp(\tau'\tau S)\,\mathrm{ad}_S^4(R^{(1)})\,\exp(-\tau'\tau S)\,\mathrm{d}\tau'\end{aligned}$$

into (5.2.21) we get

$$\mathtt{L}^{(2)}_{\mu,\epsilon}=D^{(1)}+\frac{1}{2}[S,R^{(1)}]+\frac{1}{3}\mathrm{ad}_S^2(R^{(1)})+\frac{1}{8}\mathrm{ad}_S^3(R^{(1)}) \tag{5.2.22a}$$

$$+\frac{1}{2}\int_0^1(1-\tau^2)\tau^2\int_0^1(1-\tau')\exp(\tau\tau' S)\mathrm{ad}_S^4(R^{(1)})\exp(-\tau\tau' S)\mathrm{d}\tau'\mathrm{d}\tau\,. \tag{5.2.22b}$$

Next we compute the commutators in the expansion (5.2.22a).

Lemma 5.2.4. *One has*

$$\frac{1}{2}[S,R^{(1)}]=\begin{pmatrix}\mathtt{J}_2\tilde E_1 & 0\\ 0 & \mathtt{J}_2\tilde G_1\end{pmatrix} \tag{5.2.23}$$

where $\tilde E_1$, $\tilde G_1$ are self-adjoint and reversibility-preserving matrices of the form

$$\begin{aligned}\tilde E_1&=\begin{pmatrix}\tilde{\mathtt{e}}_{11}\mu\epsilon^2+\tilde\eta_{11}^{(a)}\mu\epsilon^4+\tilde r_1(\mu\epsilon^5,\mu^2\epsilon^3,\mu^3\epsilon^2) & \mathrm{i}\left(\tilde{\mathtt{e}}_{12}\mu\epsilon^2+\tilde r_2(\mu\epsilon^4,\mu^2\epsilon^2)\right)\\ -\mathrm{i}\left(\tilde{\mathtt{e}}_{12}\mu\epsilon^2+\tilde r_2(\mu\epsilon^4,\mu^2\epsilon^2)\right) & \tilde r_5(\mu\epsilon^2)\end{pmatrix},\\ \tilde G_1&=\begin{pmatrix}\tilde{\mathtt{g}}_{11}\mu\epsilon^2+\tilde r_8(\mu\epsilon^4,\mu^2\epsilon^2) & \mathrm{i}\left(\tilde{\mathtt{g}}_{12}\mu\epsilon^2+\tilde r_9(\mu\epsilon^4,\mu^2\epsilon^2)\right)\\ -\mathrm{i}\left(\tilde{\mathtt{g}}_{12}\mu\epsilon^2+\tilde r_9(\mu\epsilon^4,\mu^2\epsilon^2)\right) & \tilde{\mathtt{g}}_{22}\mu\epsilon^2+\tilde r_{10}(\mu\epsilon^4,\mu^2\epsilon^2)\end{pmatrix},\end{aligned} \tag{5.2.24}$$

where

$$\begin{aligned}&\tilde{\mathtt{e}}_{11}:=x_{21}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac12}-x_{22}^{(1)}\mathtt{f}_{11}\,,\quad\tilde\eta_{11}^{(a)}:=x_{21}^{(1)}\phi_{12}+x_{21}^{(3)}\mathtt{c}_{\mathtt{h}}^{-\frac12}-x_{22}^{(1)}\phi_{11}-x_{22}^{(3)}\mathtt{f}_{11}\,,\\&\tilde{\mathtt{e}}_{12}:=-\tilde{\mathtt{g}}_{12}:=\frac{1}{2}\big(x_{21}^{(1)}\phi_{22}+x_{22}^{(1)}\phi_{21}-x_{11}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac12}-x_{12}^{(1)}\mathtt{f}_{11}\big)\,,\\&\tilde{\mathtt{g}}_{11}:=x_{11}^{(1)}\mathtt{f}_{11}+x_{21}^{(1)}\phi_{21}\,,\quad\tilde{\mathtt{g}}_{22}:=x_{22}^{(1)}\phi_{22}+x_{12}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac12}\,.\end{aligned} \tag{5.2.25}$$

Proof. By (5.2.6), (5.2.12), and since $\Sigma=\mathtt{J}_2X$, we have

$$\frac{1}{2}\left[S\,,\,R^{(1)}\right]=\begin{pmatrix}\mathtt{J}_2\tilde E_1 & 0\\ 0 & \mathtt{J}_2\tilde G_1\end{pmatrix},\quad\tilde E_1:=\mathbf{Sym}\big[\mathtt{J}_2X\mathtt{J}_2[F^{(1)}]^*\big]\,,\quad\tilde G_1:=\mathbf{Sym}\big[X^*F^{(1)}\big]\,, \tag{5.2.26}$$

where $\mathbf{Sym}[A] := \frac{1}{2}(A + A^*)$, see (4.3.28)-(4.3.29). By (5.2.3), (5.2.16), setting $F = F^{(1)}$, we have

$$\mathtt{J}_2 X \mathtt{J}_2 F^* = \begin{pmatrix} x_{21}F_{12} - x_{22}F_{11} & \mathrm{i}\,(x_{21}F_{22} + x_{22}F_{21}) \\ \mathrm{i}\,(x_{11}F_{12} + x_{12}F_{11}) & -x_{11}F_{22} + x_{12}F_{21} \end{pmatrix}, \tag{5.2.27}$$

$$X^* F = \begin{pmatrix} x_{11}F_{11} + x_{21}F_{21} & \mathrm{i}\,(x_{11}F_{12} - x_{21}F_{22}) \\ \mathrm{i}\,(x_{22}F_{21} - x_{12}F_{11}) & x_{22}F_{22} + x_{12}F_{12} \end{pmatrix}, \tag{5.2.28}$$

and the expansions in (5.2.24) with the coefficients given in (5.2.25) follow by (5.2.27), (5.2.28), (5.2.4) and (5.2.2c). □

Lemma 5.2.5. *One has*

$$\frac{1}{3}\mathrm{ad}_S^2(R^{(1)}) = \begin{pmatrix} 0 & \mathtt{J}_2\tilde{F} \\ \mathtt{J}_2\tilde{F}^* & 0 \end{pmatrix}, \tag{5.2.29}$$

where $\tilde{F}$ is a reversibility-preserving matrix of the form

$$\tilde{F} = \begin{pmatrix} \tilde{\mathtt{f}}_{11}\mu\epsilon^3 + \tilde{r}_3(\mu\epsilon^5, \mu^2\epsilon^3) & \mathrm{i}\,\tilde{\mathtt{f}}_{12}\mu\epsilon^3 + \mathrm{i}\,\tilde{r}_4(\mu\epsilon^5, \mu^2\epsilon^3) \\ \mathrm{i}\,\tilde{r}_6(\mu\epsilon^3) & \tilde{r}_7(\mu\epsilon^3) \end{pmatrix}, \tag{5.2.30}$$

with

$$\begin{aligned} \tilde{\mathtt{f}}_{11} &:= \frac{4}{3}x_{21}^{(1)}x_{11}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} - \frac{4}{3}x_{22}^{(1)}x_{11}^{(1)}\mathtt{f}_{11} - \frac{4}{3}x_{22}^{(1)}x_{21}^{(1)}\phi_{21} + \frac{2}{3}x_{21}^{(1)}x_{12}^{(1)}\mathtt{f}_{11} - \frac{2}{3}(x_{21}^{(1)})^2\phi_{22}\,, \\ \tilde{\mathtt{f}}_{12} &:= \frac{4}{3}x_{21}^{(1)}x_{22}^{(1)}\phi_{22} + \frac{4}{3}x_{12}^{(1)}x_{21}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} - \frac{4}{3}x_{12}^{(1)}x_{22}^{(1)}\mathtt{f}_{11} + \frac{2}{3}(x_{22}^{(1)})^2\phi_{21} - \frac{2}{3}x_{11}^{(1)}x_{22}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}}\,. \end{aligned} \tag{5.2.31}$$

Proof. Using the form of S in (5.2.6) and $[S, R^{(1)}]$ in (5.2.23) we deduce (5.2.29) with

$$\tilde{F} := \frac{2}{3}\big(\mathtt{J}_2 X \mathtt{J}_2 \tilde{G}_1 + \tilde{E}_1 X\big) \tag{5.2.32}$$

where $\tilde{E}_1$ and $\tilde{G}_1$ are the matrices in (5.2.24). Writing $\tilde{E}_1 = \begin{pmatrix} [\tilde{E}_1]_{11} & \mathrm{i}\,[\tilde{E}_1]_{12} \\ -\mathrm{i}\,[\tilde{E}_1]_{12} & [\tilde{E}_1]_{22} \end{pmatrix}$, $\tilde{G}_1 = \begin{pmatrix} [\tilde{G}_1]_{11} & \mathrm{i}\,[\tilde{G}_1]_{12} \\ -\mathrm{i}\,[\tilde{G}_1]_{12} & [\tilde{G}_1]_{22} \end{pmatrix}$ we have, in view of (5.2.3),

$$\begin{aligned} \mathtt{J}_2 X \mathtt{J}_2 \tilde{G}_1 &= \begin{pmatrix} x_{21}[\tilde{G}_1]_{12} - x_{22}[\tilde{G}_1]_{11} & \mathrm{i}\,(x_{21}[\tilde{G}_1]_{22} - x_{22}[\tilde{G}_1]_{12}) \\ \mathrm{i}\,(x_{11}[\tilde{G}_1]_{12} + x_{12}[\tilde{G}_1]_{11}) & -x_{11}[\tilde{G}_1]_{22} - x_{12}[\tilde{G}_1]_{12} \end{pmatrix}, \\ \tilde{E}_1 X &= \begin{pmatrix} x_{11}[\tilde{E}_1]_{11} - x_{21}[\tilde{E}_1]_{12} & \mathrm{i}\,(x_{12}[\tilde{E}_1]_{11} + x_{22}[\tilde{E}_1]_{12}) \\ \mathrm{i}\,(x_{21}[\tilde{E}_1]_{22} - x_{11}[\tilde{E}_1]_{12}) & x_{12}[\tilde{E}_1]_{12} + x_{22}[\tilde{E}_1]_{22} \end{pmatrix}. \end{aligned} \tag{5.2.33}$$

By (5.2.32), (5.2.33), (5.2.4) and (5.2.24) we deduce that the matrix $\tilde{F}$ has the expansion (5.2.30) with

$$\tilde{\mathtt{f}}_{11} = \frac{2}{3}(x_{21}^{(1)}\tilde{\mathtt{g}}_{12} - x_{22}^{(1)}\tilde{\mathtt{g}}_{11} + x_{11}^{(1)}\tilde{\mathtt{e}}_{11} - x_{21}^{(1)}\tilde{\mathtt{e}}_{12}) ,$$
$$\tilde{\mathtt{f}}_{12} = \frac{2}{3}(x_{21}^{(1)}\tilde{\mathtt{g}}_{22} - x_{22}^{(1)}\tilde{\mathtt{g}}_{12} + x_{12}^{(1)}\tilde{\mathtt{e}}_{11} + x_{22}^{(1)}\tilde{\mathtt{e}}_{12}) ,$$

which, by (5.2.25), gives (5.2.31). □

Lemma 5.2.6. *One has*

$$\frac{1}{8}\mathrm{ad}_S^3(R^{(1)}) = \begin{pmatrix} \mathtt{J}_2\tilde{E}_3 & 0 \\ 0 & \mathtt{J}_2\tilde{G}_3 \end{pmatrix} , \tag{5.2.34}$$

where the self-adjoint and reversibility-preserving matrices $\tilde{E}_3, \tilde{G}_3$ *in* (5.2.34) *have entries of size* $\mathcal{O}(\mu\epsilon^4)$. *In particular the first entry of the matrix* $\tilde{E}_3$ *has the expansion*

$$[\tilde{E}_3]_{11} = \tilde{\eta}_{11}^{(b)}\mu\epsilon^4 + r(\mu\epsilon^5, \mu^2\epsilon^4) \tag{5.2.35}$$

with

$$\begin{aligned}\tilde{\eta}_{11}^{(b)} := {}& \frac{3}{2}(x_{21}^{(1)})^2 x_{22}^{(1)}\phi_{22} + (x_{21}^{(1)})^2 x_{12}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} - \frac{3}{2}x_{21}^{(1)}x_{12}^{(1)}x_{22}^{(1)}\mathtt{f}_{11} \\ &+ \frac{3}{2}(x_{22}^{(1)})^2 x_{21}^{(1)}\phi_{21} - \frac{3}{2}x_{22}^{(1)}x_{11}^{(1)}x_{21}^{(1)}\mathtt{c}_{\mathtt{h}}^{-\frac{1}{2}} + (x_{22}^{(1)})^2 x_{11}^{(1)}\mathtt{f}_{11} .\end{aligned} \tag{5.2.36}$$

Proof. Since $\frac{1}{8}\mathrm{ad}_S^3(R^{(1)}) = \frac{3}{8}[S, \frac{1}{3}\mathrm{ad}_S^2(R^{(1)})]$ and using (5.2.29), the identity (5.2.34) holds with

$$\tilde{E}_3 := \frac{3}{4}\mathbf{Sym}\big[\mathtt{J}_2 X \mathtt{J}_2[\tilde{F}]^*\big] , \qquad \tilde{G}_3 := \frac{3}{4}\mathbf{Sym}\big[X^*\tilde{F}\big] . \tag{5.2.37}$$

Since, by (5.2.4) the matrix X in (5.2.3) has entries of size $\mathcal{O}(\epsilon)$ and the matrix $\tilde{F}$ in (5.2.30) has entries of size $\mathcal{O}(\mu\epsilon^3)$ we deduce that the matrices $\tilde{E}_3$, $\tilde{G}_3$ in (5.2.37) have entries of size $\mathcal{O}(\mu\epsilon^4)$. By (5.2.37) and denoting $\tilde{F} = \begin{pmatrix} \tilde{F}_{11} & \mathrm{i}\,\tilde{F}_{12} \\ \mathrm{i}\,\tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix}$, we deduce, similarly to (5.2.27), that $[\tilde{E}_3]_{11} = \frac{3}{4}(x_{21}\tilde{F}_{12} - x_{22}\tilde{F}_{11})$ which, by (5.2.4) and (5.2.30), gives (5.2.35) with $\tilde{\eta}_{11}^{(b)} = \frac{3}{4}(x_{21}^{(1)}\tilde{\mathtt{f}}_{12} - x_{22}^{(1)}\tilde{\mathtt{f}}_{11})$ which by (5.2.31) gives (5.2.36). □

Finally we show that the term in (5.2.22b) is small.

Lemma 5.2.7. *The* 4×4 *Hamiltonian and reversible matrix* $\begin{pmatrix} \mathtt{J}_2\widehat{E} & \mathtt{J}_2\widehat{F} \\ \mathtt{J}_2[\widehat{F}]^* & \mathtt{J}_2\widehat{G} \end{pmatrix}$ *given by* (5.2.22b) *has the* 2×2 *self-adjoint and reversibility-preserving blocks* $\widehat{E}$, $\widehat{G}$ *and the* 2×2 *reversibility-preserving block* $\widehat{F}$ *all with entries of size* $\mathcal{O}(\mu\epsilon^5)$.

Proof. By the Hamiltonian and reversibility properties of S and $R^{(1)}$ the matrix $\mathrm{ad}_S^4(R^{(1)})$ is Hamiltonian and reversible and the same holds, for any $\tau, \tau' \in [0,1]$, for

$$\exp(\tau\tau' S)\,\mathrm{ad}_S^4(R^{(1)})\,\exp(-\tau\tau' S) = [S, \mathrm{ad}_S^3(R^{(1)})](1 + \mathcal{O}(\mu,\epsilon)) \,. \tag{5.2.38}$$

The claimed estimate on the entries of the matrix given by (5.2.22b) follows by (5.2.38) and because S in (5.2.6) has entries of size $\mathcal{O}(\epsilon)$ and $\mathrm{ad}_S^3(R^{(1)})$ in (5.2.34) has entries of size $\mathcal{O}(\mu\epsilon^4)$. □

Proof of Lemma 5.2.2. It follows by (5.2.22), (5.2.12) and Lemmata 5.2.4, 5.2.5, 5.2.6 and 5.2.7. The matrix $E^{(2)} := E^{(1)} + \tilde{E}_1 + \tilde{E}_3 + \widehat{E}$ has the expansion in (5.2.8), with

$$\mathtt{e}_{\rm WB} := \mathtt{e}_{11} + \tilde{\mathtt{e}}_{11} = \mathtt{e}_{11} - \mathtt{D}_{\mathtt{h}}^{-1}\big(\mathtt{c}_{\mathtt{h}}^{-1} + \mathtt{h}\mathtt{f}_{11}^2 + \mathtt{e}_{12}\mathtt{f}_{11}\mathtt{c}_{\mathtt{h}}^{-\frac12}\big)\,, \qquad \eta_{\rm WB} := \eta_{11} + \tilde{\eta}_{11}^{(a)} + \tilde{\eta}_{11}^{(b)}\,,$$

as in (5.2.9). Furthermore $G^{(2)} := G^{(1)} + \tilde{G}_1 + \tilde{G}_3 + \widehat{G}$ has the expansion in (5.2.10) and $F^{(2)} := \tilde{F} + \widehat{F}$ has the expansion in (5.2.11). □

Complete block decoupling and proof of the main result. Finally Theorem 1.6.1 is proved, as Theorem 1.5.1 in the case of finite-depth, by block-diagonalizing the 4×4 Hamiltonian and reversible matrix $\mathtt{L}_{\mu,\epsilon}^{(2)}$ in (5.2.7),

$$\mathtt{L}_{\mu,\epsilon}^{(2)} = D^{(2)} + R^{(2)}\,, \quad D^{(2)} := \begin{pmatrix} \mathtt{J}_2 E^{(2)} & 0 \\ 0 & \mathtt{J}_2 G^{(2)} \end{pmatrix}, \quad R^{(2)} := \begin{pmatrix} 0 & \mathtt{J}_2 F^{(2)} \\ \mathtt{J}_2 [F^{(2)}]^* & 0 \end{pmatrix}. \tag{5.2.39}$$

The next lemma reports the content of Lemma 4.3.9.

Lemma 5.2.8. *There exist a 4×4 reversibility-preserving Hamiltonian matrix $S^{(2)} := S^{(2)}(\mu,\epsilon)$ of the form* (5.2.6)*, analytic in (μ,ϵ), of size $\mathcal{O}(\epsilon^3)$, and a 4×4 block-diagonal reversible Hamiltonian matrix $P := P(\mu,\epsilon)$, analytic in (μ,ϵ), of size $\mathcal{O}(\mu\epsilon^6)$ such that*

$$\exp(S^{(2)})(D^{(2)} + R^{(2)})\exp(-S^{(2)}) = D^{(2)} + P\,. \tag{5.2.40}$$

By (5.2.40), (5.2.8)-(5.2.10) and the fact that P has size $\mathcal{O}(\mu\epsilon^6)$ we deduce Theorem 1.6.1: there exists a symplectic and reversibility-preserving linear map that conjugates the matrix $\mathrm{i}\,\mathtt{c}_{\mathtt{h}}\mu + \mathtt{L}_{\mu,\epsilon}$ (which represents $\mathcal{L}_{\mu,\epsilon}$) with $\mathtt{L}_{\mu,\epsilon}$ in (5.1.1) into the Hamiltonian and reversible matrix (1.6.1) with $\mathtt{U}$ in (1.6.2) and $\mathtt{S}$ in (1.6.3). The function $\Delta_{\rm BF}(\mathtt{h};\mu,\epsilon)$ expands as in (1.6.4).

Appendix A

Stokes waves

We devote the whole appendix to the Stokes waves. We present four results:

1. in Section A.1 we prove as in [15, Section 2], for the infinite-depth case (we direct to [77] for the finite-depth case), that the Dirichlet-Neumann operator is analytic with respect to the sea surface η in the framework given by the analytic domains in (2.1.1);

2. in Section A.2 we prove as in [15, Section 3], for the infinite-depth case, the existence of the Stokes waves as an analytic 1-parameter family of equilibrium solutions of (1.2.10) in the analytic spaces introduced in (2.1.1);

3. in Section A.3 we expand, as done in [16, Appendix B]-[17, Appendix A.1], the Stokes waves up to the fourth order in Taylor;

4. finally, in Section A.4, following [17, Appendix A.2] we Taylor-expand at the fourth order the coefficients of the linear operator $\mathcal{L}_\epsilon$ in (1.3.7), obtained from the linearized traveling water waves system (1.3.1) by conjugating with the Alinhac and Levi-Civita transformations.

In each section we shall present the rigorous statements outlined above and their proof.

A.1 Analytic properties of the Dirichlet-Neumann operator

In this section we prove an analyticity result (Theorem A.1.1) for the Dirichlet-Neumann map $\eta \mapsto G(\eta)\psi$ in (1.2.5) in the case of infinite depth. The novelty of this Theorem with respect to the previous literature, e.g. [28, 31, 29, 95, 96, 64, 65], is that we consider η and

ψ within the same analytic space $H^{\sigma,s}$, instead of considering η more regular than ψ.
Let us consider the cylindrical domain

$$\mathcal{D}_\eta^{(d)} := \{(x,y) \in \mathbb{T}^d \times \mathbb{R} \ : \ y < \eta(x)\}, \quad d \geq 1, \tag{A.1.1}$$

which generalizes the one defined in (1.2.1) to arbitrary horizontal dimension and suppresses the time variable.
We regard the Dirichlet-Neumann operator, which in dimension d takes the form

$$[G(\eta)\psi](x) := (\partial_y \Phi)(x, \eta(x)) - \nabla\eta(x) \cdot (\nabla\Phi)(x, \eta(x)), \tag{A.1.2}$$

as acting between spaces of periodic analytic functions defined in (A.1.3) below. We suppose that the functions η and ψ belong to the spaces of periodic functions

$$H^{\sigma,s} := H^{\sigma,s}(\mathbb{T}^d) := \Big\{ u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{\mathrm{i}\, k\cdot x} \ : \ \|u\|^2_{H^{\sigma,s}} := \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k\rangle^{2s} |u_k|^2 < \infty \Big\} \tag{A.1.3}$$

where, for any $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we set

$$|k|_1 := |k_1| + \cdots + |k_d|, \quad \langle k\rangle := \max(1, |k|), \quad |k| := \Big(\sum_{j=1}^d k_j^2 \Big)^{1/2}.$$

Clearly, if the dimension $d = 1$ then $|k| = |k|_1$.

If $\sigma = 0$ the space $H^{0,s}$ is the usual Sobolev space H^s. If $\sigma > 0$, a periodic function $u(x)$ belongs to $H^{\sigma,s}(\mathbb{T}^d)$, if and only if it admits an analytic extension in the strip $|y|_\infty := \max\{|y_1|, \ldots, |y_d|\} < \sigma$ and the traces at the boundaries $u(\cdot + \mathrm{i}\, y)$, $|y|_\infty = \sigma$, belong to the Sobolev space $H^s := H^s(\mathbb{T}^d)$. This characterization is proved in [15, Appendix B.1], together with the property that the spaces $H^{\sigma,s}$ form, for $s > d/2$, an algebra with respect to the product of functions and satisfy tame estimates.

The main result of this section is the following theorem.

Let $B^{\sigma,s}(r)$ denote the open ball in $H^{\sigma,s}$ of center 0 and radius $r > 0$.

Theorem A.1.1. (Dirichlet-Neumann operator) *Let $\sigma \geq 0$ and s, s_0 such that $s + \frac{1}{2}, s_0 \in \mathbb{N}$, and $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. Then there exists $\epsilon_0 := \epsilon_0(s) > 0$ such that the Dirichlet-Neumann operator mapping*

$$\eta \mapsto G(\eta), \quad H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, H^{\sigma,s-1}),$$

is analytic and fulfills the tame estimate

$$\|G(\eta)\psi\|_{H^{\sigma,s-1}} \leq C(s)\big(\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}} \|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}}\big). \tag{A.1.4}$$

We remark that, in Theorem A.1.1, the functions η, ψ have the same analytic regularity. As we are about to see, the proof of such result relies on a regularizing flattening method (following [64, 2]) together with a perturbative argument in suitable functional spaces.

The rest of the section is devoted to the proof of Theorem A.1.1.

The first step is to straighten the free surface.

Regularizing diffeomorphism. Following [64, 2] we apply the regularizing change of variables

$$x = x' , \qquad y = \rho(x', y') , \quad \rho(x', y') := y' + e^{y'|D|}\eta(x') , \tag{A.1.5}$$

where $e^{y|D|}$ is the Fourier multiplier

$$\left(e^{y|D|}g\right)(x) := \sum_{k\in\mathbb{Z}^d} g_k \, e^{y|k|} \, e^{\mathrm{i}\, k\cdot x} , \quad \forall \; g(x) = \sum_{k\in\mathbb{Z}^d} g_k \, e^{\mathrm{i}\, k\cdot x} .$$

Note that

$$\rho(x', 0) = \eta(x') , \qquad \lim_{y'\to-\infty} \rho(x', y') - y' = \eta_0 ,$$

and, since

$$\partial_{y'}\rho(x', y') = 1 + e^{y|'D|}|D|\eta ,$$

if $\sup_{y'<0} \|e^{y'|D|}|D|\eta\|_{L^\infty(\mathbb{T}^d)} < 1$ the change of coordinates (A.1.5) is a diffeomorphism between the domain $\mathcal{D}_\eta = \{(x, y) : \; y \leq \eta(x)\}$ and the flat half-cylinder $\{(x', y') : \; y' \leq 0\} = \mathbb{T}^d \times \mathbb{R}_{\leq 0}$ where $\mathbb{R}_{\leq 0} := (-\infty, 0]$. By the change of variables (A.1.5) the derivatives ∂_y and ∇_x become respectively

$$\Lambda_1 = \frac{1}{\partial_{y'}\rho}\partial_{y'} , \qquad \Lambda_2 = \nabla_{x'} - \frac{\nabla_{x'}\rho}{\partial_{y'}\rho}\partial_{y'} ,$$

and the transformed harmonic function

$$\varphi(x', y') := \Phi(x', y' + \rho(x', y'))$$

solves the elliptic problem

$$\begin{cases} (\Lambda_1^2 + \Lambda_2^2)\varphi = 0 \\ \varphi(x, 0) = \psi(x) \\ \partial_y\varphi(x, y) \to 0 \qquad \text{as } y \to -\infty . \end{cases} \tag{A.1.6}$$

By means of the chain rule, system (A.1.6) is rewritten (cfr. [2]) as the perturbed elliptic problem (we rename the variables x', y' as x, y)

$$\begin{cases} \Delta_{x,y}\varphi = F(\eta)[\varphi] \\ \varphi(x,0) = \psi(x) \\ \partial_y \varphi(x,y) \to 0 \quad \text{as } y \to -\infty\,, \end{cases} \tag{A.1.7}$$

where

$$F(\eta)[\varphi] := \Big(\alpha(\eta)\partial_y^2 + \beta(\eta)\Delta + \gamma(\eta)\cdot\nabla\partial_y + \delta(\eta)\partial_y\Big)\,\varphi \tag{A.1.8}$$

with, since $\nabla\rho(x,y) = e^{y|D|}\nabla\eta$ and $\partial_y\rho(x,y) = 1 + e^{y|D|}|D|\eta$,

$$\begin{aligned} \alpha(\eta) &:= 1 - \frac{1+|\nabla\rho|^2}{\partial_y\rho} = \frac{e^{y|D|}|D|\eta - |e^{y|D|}\nabla\eta|^2}{1+e^{y|D|}|D|\eta}\,, \\ \beta(\eta) &:= 1 - \partial_y\rho = -e^{y|D|}|D|\eta\,, \\ \gamma(\eta) &:= 2\nabla\rho = 2e^{y|D|}\nabla\eta\,, \\ \delta(\eta) &:= \frac{1}{\partial_y\rho}\Big(-2\nabla\rho\cdot\nabla\partial_y\rho + \partial_y\rho\Delta\rho + \frac{1+|\nabla\rho|^2}{\partial_y\rho}\partial_y^2\rho\Big)\,. \end{aligned} \tag{A.1.9}$$

In the new variables (A.1.5), the Dirichlet-Neumann operator defined in (A.1.2) becomes

$$[G(\eta)\psi](\cdot) = -\nabla\eta\cdot\nabla\varphi(\cdot,0) + \frac{1+|\nabla\eta|^2(\cdot)}{1+(|D|\eta)(\cdot)}\,(\partial_y\varphi)(\cdot,0)\,. \tag{A.1.10}$$

Function spaces. In order to state our main existence result for the solutions of (A.1.7), we introduce some function spaces. Given $s \in \mathbb{N}_0$, $\sigma, a \geq 0$, we define

$$\mathcal{H}^{\sigma,s,a} := \Big\{ u(x,y) = \sum_{k\in\mathbb{Z}^d} u_k(y)e^{\mathrm{i}\,k\cdot x} \,:\, \mathbb{T}^d \times (-\infty, 0] \to \mathbb{C} \;:\; \|u\|_{\sigma,s,a} < \infty \Big\} \tag{A.1.11}$$

endowed with the norm

$$\begin{aligned} \|u\|_{\sigma,s,a}^2 &:= \sum_{j=0}^{s} \|\partial_y^j u\|^2_{L^{2,a}(\mathbb{R}_{\le 0}, H^{\sigma,s-j})} \\ &= \sum_{j=0}^{s} \int_{-\infty}^{0} \|\partial_y^j u(\cdot,y)\|^2_{H^{\sigma,s-j}}\, e^{-2ay}\mathrm{d}y \\ &= \sum_{j=0}^{s} \int_{-\infty}^{0} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1}\,\langle k\rangle^{2(s-j)}\,|\partial_y^j u_k(y)|^2 e^{2a|y|}\mathrm{d}y \end{aligned} \tag{A.1.12}$$

$$= \sum_{j=0}^{s} \sum_{k\in\mathbb{Z}^d} e^{2\sigma|k|_1} \langle k\rangle^{2(s-j)} \|\partial_y^j u_k\|^2_{L^{2,a}} \tag{A.1.13}$$

where, given a Hilbert space X, we have used the notation

$$\|u\|^2_{L^{2,a}(\mathbb{R}_{\le 0},X)} := \int_{-\infty}^{0} \|u(y)\|^2_X e^{-2ay}\,\mathrm{d}y = \int_{-\infty}^{0} \|u(y)\|^2_X e^{2a|y|}\,\mathrm{d}y\,. \tag{A.1.14}$$

Remark A.1.2. For $\sigma = a = 0$, the space $H^{0,s,0}$ coincides with the Sobolev space $H^s(\mathbb{T}^d \times \mathbb{R}_{\le 0})$ of L^2 functions $u : \mathbb{T}^d \times \mathbb{R}_{\le 0} \to \mathbb{C}$ possessing weak derivatives $\partial^\alpha u$ in L^2, for any multiindex $\alpha \in \mathbb{N}^{d+1}$ with modulus $|\alpha| \le s$, with equivalent norm $\|u\|_s^2 = \sum_{\alpha\in\mathbb{N}^{d+1},|\alpha|\le s} \|\partial^\alpha u\|^2_{L^2}$.

We point out that, for any $s \in \mathbb{N}$,

$$\|u\|^2_{\sigma,s,a} = \|u\|^2_{L^{2,a}(\mathbb{R}_{\le 0},H^{\sigma,s})} + \|\partial_y u\|^2_{\sigma,s-1,a}\,,$$

and, by (A.1.12) and $\|\partial_{x_i} v\|_{H^{\sigma,s-1}} \le \|v\|_{H^{\sigma,s}}$, we directly get the following simple lemma.

Lemma A.1.3. *Let $s \in \mathbb{N}$, $\sigma \ge 0$, $a \ge 0$. The linear maps*

$$\partial_{x_i} : \mathcal{H}^{\sigma,s,a} \mapsto \mathcal{H}^{\sigma,s-1,a}\,,\ \forall i = 1,\ldots,d, \quad \partial_y : \mathcal{H}^{\sigma,s,a} \mapsto \mathcal{H}^{\sigma,s-1,a}\,,$$

are continuous.

We also denote

$$\mathbb{C} \oplus \mathcal{H}^{\sigma,s,a} := \left\{c + u(x,y)\,,\ c \in \mathbb{C}\,,\ u \in \mathcal{H}^{\sigma,s,a}\right\}, \quad \Pi : \mathbb{C} \oplus \mathcal{H}^{\sigma,s,a} \to \mathcal{H}^{\sigma,s,a},\ \Pi[c+u] = u\,, \tag{A.1.15}$$

and, with a small abuse of notation, given a function $g \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s,a}$, we denote its norm by $\|g\|_{\sigma,s,a} := \|\Pi g\|_{\sigma,s,a} + |g - \Pi g|$. The function spaces $\mathcal{H}^{\sigma,s,a}$ and $\mathbb{C} \oplus \mathcal{H}^{\sigma,s,a}$ are modeled to mimic the decay of the harmonic function $\underline{\varphi}$ in (A.1.30) as $y \to -\infty$, cfr. Lemma A.1.6.

We now list a series of properties of the spaces $\mathcal{H}^{\sigma,s,a}$ used in the sequel. Their proofs can be found in [15, Appendix B.2].

Lemma A.1.4 (Trace). *Let $\sigma \ge 0$, $s \in \mathbb{R}$. Then one has*

$$\|u\|_{C^0(\mathbb{R}_{\le 0},H^{\sigma,s})} \le \|u\|_{L^2(\mathbb{R}_{\le 0},H^{\sigma,s+\frac{1}{2}})} + \|\partial_y u\|_{L^2(\mathbb{R}_{\le 0},H^{\sigma,s-\frac{1}{2}})}. \tag{A.1.16}$$

In particular, the trace operator

$$\Gamma(u) := u(\cdot,0) := u|_{y=0} \tag{A.1.17}$$

is, for any $s \in \mathbb{N}_0$*,* $a \geq 0$*, a linear bounded map between* $\mathcal{H}^{\sigma,s+1,a} \to H^{\sigma,s+\frac{1}{2}}$*, satisfying*

$$\|\Gamma(u)\|_{H^{\sigma,s+\frac{1}{2}}} \leq \|u\|_{\sigma,s+1,0} \leq \|u\|_{\sigma,s+1,a} \,. \tag{A.1.18}$$

If $s > \frac{d+1}{2}$, the space $\mathcal{H}^{\sigma,s,a}$ is an algebra with respect to the product of functions and the following tame estimates hold.

Proposition A.1.5 (Tame). *Let* $\sigma, a \geq 0$ *and* $s \geq s_0 > \frac{d+1}{2}$*,* $s, s_0 \in \mathbb{N}$*. Then there exist positive constants* $C_s \geq 1$ *(non-decreasing in* s*) such that, for any* $u \in \mathcal{H}^{\sigma,s,0}$ *and* $v \in \mathcal{H}^{\sigma,s,a}$*,*

$$\|uv\|_{\sigma,s,a} \leq C_s\big(\|u\|_{\sigma,s,0}\,\|v\|_{\sigma,s_0,a} + \|u\|_{\sigma,s_0,0}\,\|v\|_{\sigma,s,a}\big)\,. \tag{A.1.19}$$

In particular one has

$$\|u^j\|_{\sigma,s,a} \leq \big(2C_s\|u\|_{\sigma,s_0,a}\big)^{j-1}\|u\|_{\sigma,s,a}\,, \quad \forall j \geq 1\,. \tag{A.1.20}$$

The next lemma proves the continuity of the harmonic function $e^{y|D|}g$, which solves the Dirichlet-Neumann elliptic problem (A.1.30), with respect to the Dirichlet datum g at $y = 0$.

Lemma A.1.6 (Harmonic propagator). *Let* $\sigma \geq 0$ *and* $s + \frac{1}{2} \in \mathbb{N}$*. Then, for any* $g \in H^{\sigma,s}$*, the function*

$$(e^{y|D|}g)(x) := \sum_{k\in\mathbb{Z}^d} g_k\, e^{y|k|}\, e^{\mathrm{i}\,k\cdot x}$$

belongs to $\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$*,* $a \in (0,1)$*, and the linear map*

$$H^{\sigma,s} \to \mathcal{H}^{\sigma,s+\frac{1}{2},a}\,, \quad g \mapsto \Pi[e^{y|D|}g] = e^{y|D|}g - g_0\,,$$

is continuous.

We now come back to Theorem A.1.1. The key result of its proof is the following proposition regarding the solution of the elliptic problem (A.1.7).

The parameter $a \in (0,1)$ plays a technical role in studying the decay as $y \to -\infty$ of the solution of the elliptic problem (A.1.6). *In the sequel we fix* $a = \frac{1}{2}$*.*

Proposition A.1.7. *Let* $\sigma \geq 0$ *and* s*,* s_0 *such that* $s + \frac{1}{2}$*,* $s_0 \in \mathbb{N}$ *and* $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$*. Then there exist* $\epsilon_0 := \epsilon_0(s) > 0$ *and, for any* $\eta \in H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ *and* $\psi \in H^{\sigma,s}$*, a unique solution* $\varphi \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$ *of the elliptic problem* (A.1.7)*, satisfying*

$$\|\Pi\varphi\|_{\sigma,s+\frac{1}{2},a} \leq C(s)\big(\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}}\big)\,. \tag{A.1.21}$$

Moreover $\varphi = \Psi(\eta)[\psi]$*, where* Ψ *is an analytic map* $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})$*, and* $\Psi(0)\psi = e^{y|D|}\psi$*.*

Postponing the proof of this proposition, we first use it to deduce Theorem A.1.1.

Proof of Theorem A.1.1. By Proposition A.1.7, for any $\eta \in H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ and $\psi \in H^{\sigma,s}$, there exists a unique solution $\varphi \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}$ of (A.1.7). The Dirichlet-Neumann operator is computed in (A.1.10). Since $\varphi(x,0) = \psi(x)$, using the trace operator $\Gamma(u) = u(\cdot,0)$ in (A.1.17), and recalling the definition of Π in (A.1.15), we rewrite (A.1.10) as

$$\begin{aligned} G(\eta)\psi &= -\nabla\eta\cdot\nabla\psi + \frac{1+|\nabla\eta|^2}{1+(|D|\eta)}\,\Gamma[\partial_y\varphi] \\ &= \underbrace{-\nabla\eta\cdot\nabla\psi}_{=:G_1(\eta)\psi} + \underbrace{\Gamma[\partial_y\Pi\varphi]}_{=:G_2(\eta)\psi} + \underbrace{\frac{|\nabla\eta|^2-(|D|\eta)}{1+(|D|\eta)}\,\Gamma[\partial_y\Pi\varphi]}_{=:G_3(\eta)\psi}\,. \end{aligned} \tag{A.1.22}$$

We prove that each map

$$G_i : H^{\sigma,s}\cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(H^{\sigma,s}, H^{\sigma,s-1})\,, \quad i=1,2,3\,, \quad \text{is analytic}\,, \tag{A.1.23}$$

and fulfills the tame estimate (A.1.4). Regarding $G_1(\eta)\psi$, it suffices to note that it is linear in η and by the following tame estimate (cfr. [15, Lemma B.2])

$$\|fg\|_{H^{\sigma,s}} \le C_{s,s_0}\big(\|f\|_{H^{\sigma,s}}\|g\|_{H^{\sigma,s_0}} + \|f\|_{H^{\sigma,s_0}}\|g\|_{H^{\sigma,s}}\big)\,, \tag{A.1.24}$$

one has

$$\|\nabla\eta\cdot\nabla\psi\|_{H^{\sigma,s-1}} \lesssim_s \|\eta\|_{H^{\sigma,s_0+\frac{1}{2}}}\|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{1}{2}}}\,. \tag{A.1.25}$$

Next we consider $G_2(\eta)\psi = \Gamma[\partial_y\Pi\Psi(\eta)\psi]$. By Lemma A.1.3 and A.1.4, the map $\varphi \mapsto \Gamma[\partial_y\Pi\varphi] \in \mathcal{L}(\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac{1}{2},a}, H^{\sigma,s-1})$ which, together with the analyticity of $\eta \mapsto \Psi(\eta)$ stated in Proposition A.1.7, implies the analyticity of $\eta \mapsto G_2(\eta)$ as in (A.1.23). Moreover by (A.1.18), Lemma A.1.3 and (A.1.21), we have

$$\|\Gamma[\partial_y\Pi\varphi]\|_{H^{\sigma,s-1}} \le \|\partial_y\Pi\varphi\|_{\sigma,s-\frac{1}{2},0} \lesssim_s \|\psi\|_{H^{\sigma,s}} + \|\eta\|_{H^{\sigma,s}}\|\psi\|_{H^{\sigma,s_0+\frac{3}{2}}}\,. \tag{A.1.26}$$

Finally consider $G_3(\eta)\psi = f(\eta)G_2(\eta)\psi$, where $f(\eta)$ is the multiplication operator by the function

$$f(\eta) = \frac{|\nabla\eta|^2-(|D|\eta)}{1+(|D|\eta)} = (|\nabla\eta|^2-(|D|\eta))\sum_{j=0}^{\infty}(-|D|\eta)^j\,. \tag{A.1.27}$$

By [15, Lemma B.2] we have that $\|(|D|\eta)^j\|_{H^{\sigma,s-1}} \le (C(s)\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}})^j\|\eta\|_{H^{\sigma,s}}$ for any $j\in\mathbb{N}$, and therefore $f(\eta)$ in (A.1.27) is bounded, on the domain $H^{\sigma,s}\cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, by

$$\|f(\eta)\|_{H^{\sigma,s-1}} \lesssim_s \|\eta\|_{H^{\sigma,s}}\ . \tag{A.1.28}$$

Moreover $f(\eta)$ in (A.1.27) is a series of analytic functions uniformly convergent on the sets $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, $\forall R > 0$. Thus, by Weierstrass theorem, $\eta \mapsto f(\eta)$ is analytic on $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$, and, by the arbitrariness of R, on the whole open set $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to H^{\sigma,s-1}$. We conclude that also $G_3(\eta)$ is analytic as stated in (A.1.23). Finally, (A.1.28) and (A.1.26) imply that $G_3(\eta)$ satisfies the tame estimate (A.1.4). $\square$

Remark A.1.8. It follows from the proof that $G(0)\psi = G_2(0)\psi = \Gamma[\partial_y \Pi\Psi(0)\psi]$, which, together with $\Psi(0)\psi = e^{y|D|}\psi$, recovers the identity $G(0)\psi = |D|\psi$.

The final paragraph is devoted to the proof of Proposition A.1.7.

Proof of Proposition A.1.7: the perturbative argument. We look for a solution φ of (A.1.7) of the form

$$\varphi(x,y) = \underline{\varphi}(x,y) + u(x,y) \tag{A.1.29}$$

where $\underline{\varphi}$ is the harmonic solution of

$$\begin{cases} \Delta_{x,y}\underline{\varphi} = 0 \\ \underline{\varphi}(x,0) = \psi(x) \\ \partial_y\underline{\varphi}(x,y) \to 0 \qquad \text{as } y \to -\infty\,, \end{cases} \qquad \text{i.e.} \qquad \underline{\varphi}(x,y) := e^{y|D|}\psi(x)\,, \tag{A.1.30}$$

whereas u solves the elliptic problem

$$\begin{cases} \Delta_{x,y}u = F(\eta)[\phi + u]\,, \\ u(x,0) = 0 \\ \partial_y u(x,y) \to 0 \qquad \text{as } y \to -\infty\,, \end{cases} \tag{A.1.31}$$

with $\phi := \underline{\varphi}$. The harmonic function $\underline{\varphi} = e^{y|D|}\psi$ ia estimated by Lemma A.1.6.
The solution of system (A.1.31) is given by the following lemma.

Lemma A.1.9. *Let $\sigma \geq 0$ and s, s_0 such that $s + \frac{1}{2}$, $s_0 \in \mathbb{N}$ and $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. Then there exist $\epsilon_0 := \epsilon_0(s) > 0$ and a unique analytic map*

$$\eta \mapsto U(\eta)\,, \quad U : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \longrightarrow \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}\big)\,,$$

such that $u = U(\eta)[\phi] = U(\eta)[\Pi\phi]$, with Π in (A.1.15)*, solves* (A.1.31)*, satisfying*

$$\|\Pi U(\eta)[\phi]\|_{\sigma,s+\frac{1}{2},a} \leq C(s)\big(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}}\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \|\eta\|_{H^{\sigma,s}}\|\Pi\phi\|_{\sigma,s_0+2,a}\big)\,. \tag{A.1.32}$$

The proof of Lemma A.1.9 relies on Lemmata A.1.10 and A.1.11 below.

Given a function $g(x,y)$ defined in $\mathbb{T}^d \times (-\infty, 0)$, we first consider the linear elliptic problem

$$\begin{cases} \Delta_{x,y} u = g \\ u(x,0) = 0 \\ \partial_y u(x,y) \to 0 \quad \text{ as } y \to -\infty \,. \end{cases} \tag{A.1.33}$$

The following key lemma is proved in [15, Appendix C].

Lemma A.1.10 (Elliptic regularity). *Fix $\sigma \geq 0$, $s \in \mathbb{N}_0$ and $a \in (0,1)$. For any $g \in \mathcal{H}^{\sigma,s,a}$, the elliptic problem* (A.1.33) *has a unique solution $u := L(g) \in \mathbb{C} \oplus \mathcal{H}^{\sigma,s+2,a}$. The linear map*

$$L : \mathcal{H}^{\sigma,s,a} \to \mathbb{C} \oplus \mathcal{H}^{\sigma,s+2,a} \,, \quad g \mapsto L(g) \,,$$

is continuous, i.e. there exists $C_a > 0$ such that $\|Lg\|_{\sigma,s+2,a} \leq C_a \|g\|_{\sigma,s,a}$.

Thanks to Lemma A.1.10, we recast the nonlinear elliptic problem (A.1.31) into the equation

$$(\mathrm{Id} - L \circ F(\eta))[u] = L \circ F(\eta)[\phi] \,. \tag{A.1.34}$$

Note that the linear operator $\mathrm{Id} - L \circ F(\eta)$ depends non-linearly on η and that, recalling (A.1.8), $F(\eta)[\phi] = F(\eta)[\Pi\phi]$ depends only on the component $\Pi\phi \in \mathcal{H}^{\sigma,s,a}$ of ϕ defined in (A.1.15), for the presence of the derivatives $\partial_y, \partial_{yy}, \nabla\partial_y$. In the next lemma we study the regularity of the nonlinear map $\eta \mapsto F(\eta)$.

Lemma A.1.11. *Let $\sigma \geq 0$, $s + \frac{1}{2}, s_0 \in \mathbb{N}$ with $s - \frac{3}{2} \geq s_0 > \frac{d+1}{2}$. There exists $\epsilon_0 := \epsilon_0(s) > 0$ such that the nonlinear map*

$$\begin{aligned} F : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) &\to \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}, \mathcal{H}^{\sigma,s-\frac{3}{2},a}\big) \,, \\ \eta \qquad &\mapsto \qquad \{\, \phi \mapsto F(\eta)[\phi] \,\} \,, \end{aligned}$$

defined in (A.1.8) *is analytic and satisfies the tame estimate*

$$\|F(\eta)[\phi]\|_{\sigma,s-\frac{3}{2},a} \leq C(s) \big(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \big) \,. \tag{A.1.35}$$

Proof. We write $F(\eta)[\phi]$ in (A.1.8) as

$$F(\eta)[\phi] = \mathcal{F}_1[\alpha(\eta), \phi] + \mathcal{F}_2[\beta(\eta), \phi] + \sum_{j=1}^{d} \mathcal{F}_{3j}[\gamma_j(\eta), \phi] + \mathcal{F}_4[\delta(\eta), \phi]$$

with bilinear maps

$$\mathcal{F}_1[g,\phi] := g\partial_y^2\phi\,, \quad \mathcal{F}_2[g,\phi] := g\Delta\phi\,, \quad \mathcal{F}_{3j}[g,\phi] := g\partial_{x_j}\partial_y\phi\,, \quad \mathcal{F}_4[g,\phi] := g\partial_y\phi\,.$$

In view of (A.1.19), Lemma A.1.3 and (A.1.15), each of these maps is bounded $\mathcal{H}^{\sigma,s-\frac{3}{2},a} \times (\mathbb{C}\oplus\mathcal{H}^{\sigma,s+\frac{1}{2},a}) \to \mathcal{H}^{\sigma,s-\frac{3}{2},a}$ and any $\mathcal{F}\in\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_{3j},\mathcal{F}_4\}$ fulfills the tame estimates

$$\|\mathcal{F}[g,\phi]\|_{\sigma,s-\frac{3}{2},a} \lesssim_s \|g\|_{\sigma,s-\frac{3}{2},a}\|\Pi\phi\|_{\sigma,s_0+2,a} + \|g\|_{\sigma,s_0,a}\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a}\,. \tag{A.1.36}$$

We claim that the maps

$$\begin{aligned} &H^{\sigma,s}\cap B^{\sigma,s_0+\frac{1}{2}}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{1}{2},a}\,, \quad \eta\mapsto\alpha(\eta)\,,\beta(\eta)\,,\gamma_j(\eta)\,, \quad j=1,\dots,d\,,\\ &H^{\sigma,s}\cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{3}{2},a}\,, \quad \eta\mapsto\delta(\eta)\,, \end{aligned} \tag{A.1.37}$$

are analytic and, for any $s\geq s_0+\frac{3}{2}$, $j=1,\dots,d$,

$$\|\alpha(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\beta(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\gamma_j(\eta)\|_{\sigma,s-\frac{1}{2},a},\ \|\delta(\eta)\|_{\sigma,s-\frac{3}{2},a} \leq C(s)\|\eta\|_{H^{\sigma,s}}\,. \tag{A.1.38}$$

It is clear that these properties, together with (A.1.36), imply the Lemma.

Let us consider first $\alpha(\eta)$, defined in (A.1.9), which we rewrite as

$$\alpha(\eta) = \Big(1-\frac{1}{\partial_y\rho(\eta)}\Big) + \Big(1-\frac{1}{\partial_y\rho(\eta)}\Big)|\nabla\rho(\eta)|^2 - |\nabla\rho(\eta)|^2\,.$$

We first prove that $\eta\mapsto 1-\frac{1}{\partial_y\rho(\eta)}$ is analytic as a map $H^{\sigma,s}\cap B^{\sigma,s_0}(\epsilon_0)\to\mathcal{H}^{\sigma,s-\frac{1}{2},a}$. We first note that Lemma A.1.6 implies

$$\|\partial_y\rho(\eta)-1\|_{\sigma,s-\frac{1}{2},a} = \|e^{y|D|}|D|\eta\|_{\sigma,s-\frac{1}{2},a} \lesssim_s \|\eta\|_{H^{\sigma,s}}\,. \tag{A.1.39}$$

Then by (A.1.20) and (A.1.39) the series

$$1-\frac{1}{\partial_y\rho(\eta)} = -\sum_{j\geq1}(1-\partial_y\rho(\eta))^j \tag{A.1.40}$$

is bounded by

$$\Big\|\frac{1}{\partial_y\rho(\eta)}-1\Big\|_{\sigma,s-\frac{1}{2},a} \leq \|\partial_y\rho(\eta)-1\|_{\sigma,s-\frac{1}{2},a}\sum_{j\geq1}\big(2C_s\|\partial_y\rho(\eta)-1\|_{\sigma,s_0,a}\big)^{j-1} \leq C(s)\|\eta\|_{H^{\sigma,s}} \tag{A.1.41}$$

provided $\|\eta\|_{H^{\sigma,s_0+\frac{1}{2}}} < \epsilon_0(s)$ is small enough. The series (A.1.40) of analytic functions in uniformly convergent in $\mathcal{H}^{\sigma,s-\frac{1}{2},a}$ on the domain $\eta\in B^{\sigma,s}(R)\cap B^{\sigma,s_0}(\epsilon_0)$, $\forall R>0$, thus it defines an analytic map on $H^{\sigma,s} \cap B^{\sigma,s_0}(\epsilon_0)$. Moreover the linear map $\eta \mapsto \nabla\rho(\eta) = e^{y|D|}\nabla\eta$ is, by Lemma A.1.6, bounded between $H^{\sigma,s} \to \mathcal{H}^{\sigma,s-\frac{1}{2},a}$. Therefore $\alpha(\eta)$ is the product of analytic functions $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{1}{2}}(\epsilon_0) \to \mathcal{H}^{\sigma,s-\frac{1}{2},a}$, and using the tame estimate (A.1.19) we get (A.1.38).

The analyticity and the estimates of the functions $\eta \mapsto \beta(\eta), \gamma_j(\eta)$, $j = 1, \ldots, d$ stated in (A.1.37) follow similarly. Finally consider $\delta(\eta)$ in (A.1.9). The biggest loss of derivatives follows from the linear maps $\eta \mapsto \Delta\rho(\eta)$, $\partial_y^2\rho(\eta)$, $\nabla\partial_y\rho(\eta)$ which, by Lemmata A.1.3 and A.1.6, are bounded between $H^{\sigma,s} \to \mathcal{H}^{\sigma,s-\frac{3}{2},a}$. Moreover $\delta(\eta)$ satisfies the estimate $\|\delta(\eta)\|_{H^{\sigma,s-\frac{3}{2},a}} \leq C(s, \|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}}) \|\eta\|_{H^{\sigma,s}}$ for any $s - \frac{3}{2} \geq s_0$. □

Proof of Lemma A.1.9. For any $s \geq s_0 + \frac{3}{2}$ such that $s + \frac{1}{2} \in \mathbb{N}$, by Lemmata A.1.10 and A.1.11, the map

$$\eta \mapsto P(\eta) := L \circ F(\eta)\,, \quad H^{\sigma,s} \cap B^{\sigma,s_0+\frac{1}{2}}(\epsilon_0) \to \mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})\,,$$

is analytic and, for positive constants $C(s) \geq C'(s_0) > 0$, in view of (A.1.15),

$$\begin{aligned} \|P(\eta)[\phi]\|_{\sigma,s+\frac{1}{2},a} &\leq C(s) \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}\right) \\ \|P(\eta)[\phi]\|_{\sigma,s_0+2,a} &\leq C'(s_0) \|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s_0+2,a}\,. \end{aligned} \tag{A.1.42}$$

We claim that, for any $\eta \in H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ and $\epsilon_0(s) > 0$ small enough, the operator $\mathrm{Id} - P(\eta)$ is invertible in $\mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})$ and the inverse map

$$\eta \mapsto (\mathrm{Id} - P(\eta))^{-1} = \sum_{j=0}^{\infty} P(\eta)^j[\phi]\,, \quad H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \to \mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})\,, \tag{A.1.43}$$

is analytic. As each $\eta \mapsto P(\eta)^j$ is analytic $H^{\sigma,s} \cap B^{\sigma,s_0+\frac{1}{2}}(\epsilon_0) \to \mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})$, the claim follows by proving that the series (A.1.43) converges uniformly in $\mathcal{L}(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a})$ for $\eta \in B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ for any $R > 0$. By (A.1.42) we have, for any $j \in \mathbb{N}$,

$$\|P(\eta)^j[\phi]\|_{\sigma,s_0+2,a} \leq \left(C'(s_0)\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}}\right)^j \|\Pi\phi\|_{\sigma,s_0+2,a}\,, \tag{A.1.44}$$

and, by induction, we prove that

$$\|P(\eta)^j[\phi]\|_{\sigma,s+\frac{1}{2},a} \leq C(s)^j \|\eta\|^{j-1}_{H^{\sigma,s_0+\frac{3}{2}}} \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + j\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}\right). \tag{A.1.45}$$

Indeed, for $j = 1$ this is (A.1.42). Then assuming that (A.1.45) holds for j, we get

$$\|P(\eta)^{j+1}[\phi]\|_{\sigma,s+\frac{1}{2},a} \overset{\text{(A.1.42)}}{\leq} C(s) \big(\underbrace{\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|P(\eta)^j[\phi]\|_{\sigma,s+\frac{1}{2},a}}_{=:A} + \underbrace{\|\eta\|_{H^{\sigma,s}} \|P(\eta)^j[\phi]\|_{\sigma,s_0+2,a}}_{=:B} \big).$$

By the inductive hypothesis the first term is bounded by

$$A \le C(s)^j \|\eta\|^j_{H^{\sigma,s_0+\frac{3}{2}}} \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + j\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}\right),$$

whereas, by (A.1.44),

$$B \le (C'(s_0)\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}})^j \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a},$$

and we deduce, as $C'(s_0) \le C(s)$, that

$$\|P(\eta)^{j+1}[\phi]\|_{\sigma,s+\frac{1}{2},a} \le C(s)^{j+1} \|\eta\|^j_{H^{\sigma,s_0+\frac{3}{2}}} \left(\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + (j+1)\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}\right),$$

which proves (A.1.45) at the step $j+1$.

By (A.1.45), the series in (A.1.43) is bounded by

$$\begin{aligned}
\|(\mathrm{Id} - P(\eta))^{-1}[\phi]\|_{\sigma,s+\frac{1}{2},a} &\le \sum_{j\ge 0} \|P(\eta)^j[\phi]\|_{\sigma,s+\frac{1}{2},a} \\
&\le \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + \sum_{j\ge 1} C(s)^j \|\eta\|^{j-1}_{H^{\sigma,s_0+\frac{3}{2}}} \|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} \|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} \\
&\qquad + \sum_{j\ge 1} C(s)^j \|\eta\|^{j-1}_{H^{\sigma,s_0+\frac{3}{2}}} \, j \, \|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a} \\
&\le 2\|\Pi\phi\|_{\sigma,s+\frac{1}{2},a} + C\|\eta\|_{H^{\sigma,s}} \|\Pi\phi\|_{\sigma,s_0+2,a}
\end{aligned} \tag{A.1.46}$$

provided $\|\eta\|_{H^{\sigma,s_0+\frac{3}{2}}} < \epsilon_0(s)$ is sufficiently small. In particular this shows the claim on the uniform convergence of the series on $B^{\sigma,s}(R) \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0)$ for any $R > 0$.

The analytic map

$$U : H^{\sigma,s} \cap B^{\sigma,s_0+\frac{3}{2}}(\epsilon_0) \longrightarrow \mathcal{L}\big(\mathbb{C} \oplus \mathcal{H}^{\sigma,s+\frac{1}{2},a}\big), \quad U(\eta)[\phi] := (\mathrm{Id} - L \circ F(\eta))^{-1}[L \circ F(\eta)[\phi]],$$

defines the unique solution $u = U(\eta)[\phi]$ of (A.1.34) and, consequently, of system (A.1.31). By (A.1.46) and (A.1.42) we deduce (A.1.32). This proves Lemma A.1.9. □

Proof of Proposition A.1.7. It follows with $\varphi = \Psi(\eta)[\psi] = e^{y|D|}\psi + U(\eta)[e^{y|D|}\psi]$, see (A.1.29), (A.1.30) and Lemma A.1.9. □

A.2 Bifurcation of the Stokes waves

In this section we prove Theorem 2.1.1 in the case of infinite depth, namely the following

Theorem A.2.1. (Stokes waves) *For any* $\sigma \geq 0$, $s > 5/2$ *and* $k \in \mathbb{N}$, *there exists* $\epsilon_0 := \epsilon_0(\sigma, s, k) > 0$ *and a unique family of solutions*

$$(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon) \in H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$$

of the system (1.2.10), *parameterized by* $|\epsilon| \leq \epsilon_0$, *such that*

1. *the map* $\epsilon \mapsto (\eta_\epsilon, \psi_\epsilon, c_\epsilon)$, $B(\epsilon_0) \to H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}(\mathbb{T}) \times \mathbb{R}$ *is analytic;*

2. $\eta_\epsilon(x)$ *is even,* $\eta_\epsilon(x)$ *has zero average,* $\psi_\epsilon(x)$ *is odd;*

3. *the solutions* $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$ *have the expansion*

$$(\eta_\epsilon(x), \psi_\epsilon(x)) = \epsilon(\sqrt{k}\cos(kx), \sqrt{g}\sin(kx)) + O(\epsilon^2)\,, \quad c_\epsilon \to \sqrt{\frac{g}{k}} \quad \text{as} \quad \epsilon \to 0\,. \tag{A.2.1}$$

We remark that a proof of Theorem 2.1.1 for arbitrary depth $\mathtt{h} > 0$ can be given by mimicking the following proof for the deep-water case. On the other hand, in the case of finite depth and adding surface tension, Theorem 2.1.1 is proved by Nicholls-Reitich [77].

With the aid of the analyticity result of Theorem A.1.1, we can adapt the classical bifurcation procedure to obtain a solution $(\eta_\epsilon(x), \psi_\epsilon(x))$ with $\eta_\epsilon(x)$ and $\psi_\epsilon(x)$ in the same analytic space $H^{\sigma,s}$. This is fundamental to deduce the exact regularity, through the functions $p_\epsilon(x)$ and $a_\epsilon(x)$ in (1.3.8), of the operator $\mathcal{L}_{\mu,\epsilon}$ in (1.3.14). This essentially ensures, thanks to the algebra property of the spaces $H^{\sigma,s}$, that every formula presented so far is analytic with respect to ϵ.

The rest of the section is devoted to the proof of Theorem A.2.1. It is based on the application of the analytic Crandall-Rabinowitz Theorem A.2.2 below. For the proof we refer e.g to [20], and Theorem 4.1 in Chap. 5 of [4] for its smooth version.

Theorem A.2.2 (Crandall-Rabinowitz bifurcation Theorem)**.** *Let* X, Y *be Banach spaces and* $U \subset X$ *be an open neighbourhood of 0. Let* $F : U \times \mathbb{R} \to Y$, $F(u, c)$, *be an analytic map satisfying* $F(0, c) = 0$ *for any* $c \in \mathbb{R}$. *Let* c^* *be such that* $L := \mathrm{d}_u F(0, c^*) \in \mathcal{L}(X, Y)$ *is not invertible and*

1. *Ker*$(L) = \mathrm{span}\langle u^* \rangle$, $u^* \in X$, *is* 1*-dimensional;*

2. *the range* $R :=$ *Rng*(L) *is closed and* codim $R = 1$*;*

3. *(transversality)* $\partial_c \mathrm{d}_u F(0, c^*)[u^*] \notin R\,.$

Then there exist $\epsilon_* > 0$ *and an analytic function*

$$(-\epsilon_*, \epsilon_*) \to U \times \mathbb{R}, \quad \epsilon \mapsto (u_\epsilon, c_\epsilon), \qquad u_\epsilon = \epsilon u^* + O(\epsilon^2), \quad c_\epsilon = c^* + O(\epsilon),$$

such that $F(u_\epsilon, c_\epsilon) = 0$ *for any* $|\epsilon| < \epsilon_*$.

Theorem A.2.1 is proved by applying Theorem A.2.2 to the nonlinear operator

$$\begin{aligned} F :\ & (H^{\sigma,s}_{\mathtt{ev}_0} \cap B^{\sigma,s_0}(\epsilon_0)) \times H^{\sigma,s}_{\mathtt{odd}} \times \mathbb{R} \longrightarrow H^{\sigma,s-1}_{\mathtt{odd}} \times H^{\sigma,s-1}_{\mathtt{ev}_0}, \ \sigma \geq 0, \ s > 5/2, \\ & F(\eta, \psi, c) := \Big(c\eta_x + G(\eta)\psi \ , \ c\psi_x - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)} (G(\eta)\psi + \eta_x \psi_x)^2 \Big) \end{aligned} \tag{A.2.2}$$

where $H^{\sigma,s}_{\mathtt{ev}_0}$, respectively $H^{\sigma,s}_{\mathtt{odd}}$, denote the space of even, respectively odd, and average-free real valued functions in $H^{\sigma,s}$ defined in (2.1.1), and $\epsilon_0 := \epsilon_0(\sigma, s, s_0) > 0$ is provided by Theorem A.1.1. Note that a real function $(\eta, \psi) \in H^{\sigma,s}_{\mathtt{ev}_0} \times H^{\sigma,s}_{\mathtt{odd}}$ admits a Fourier series expansion

$$\begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} = \sum_{k \geq 1} \begin{bmatrix} \eta_k \cos(kx) \\ \psi_k \sin(kx) \end{bmatrix} \quad \text{with norm} \quad \|(\eta, \psi)\|^2_{H^{\sigma,s}} \simeq \sum_{k \geq 1} e^{2\sigma|k|} \langle k \rangle^{2s} (\eta_k^2 + \psi_k^2). \tag{A.2.3}$$

The fact that the nonlinear operator F in (A.2.2) maps a pair of functions (η, ψ) which are odd/even in x into a pair of functions which are even/odd in x is verified thanks to the reversibility property (1.2.9). Moreover, the second component of F has zero average thanks to the following lemma.

Lemma A.2.3. *Let* $G(\eta)$ *be the Dirichlet-Neumann operator defined in* (1.2.5)*. Then*

$$\int_{\mathbb{T}} -\frac{1}{2}\psi_x^2 + \frac{1}{2(1+\eta_x^2)} (G(\eta)\psi + \eta_x \psi_x)^2 \, \mathrm{d}x = 0. \tag{A.2.4}$$

Proof. By (4), the kinetic energy $K(\eta, \psi) = \frac{1}{2}(\psi, G(\eta)\psi)_{L^2}$ in (1.2.6) satisfies $K(\eta+m, \psi) = K(\eta, \psi)$ for any $m \in \mathbb{R}$. Thus

$$0 = \frac{\mathrm{d}}{\mathrm{d}m} K(\eta + m, \psi) = \mathrm{d}_\eta K(\eta, \psi)[1] = (\nabla_\eta K(\eta, \psi), 1)_{L^2} = \int_{\mathbb{T}} \nabla_\eta K(\eta, \psi) \, \mathrm{d}x.$$

In view of (1.2.7), the identity (A.2.4) is proved. □

We now start verifying the assumptions of the Crandall-Rabinowitz Theorem A.2.2. First, by Theorem A.1.1, the nonlinear operator F defined in (A.2.2) is *analytic*. Moreover, by inspection,

$$F(0, 0, c) = 0, \quad \forall c \in \mathbb{R}.$$

The possible bifurcation values of non-trivial solutions of $F(\eta,\psi,c)=0$ are those speeds c such that the linearized operator

$$\mathrm{d}_{(\eta,\psi)}F(0,0,c):\ H^{\sigma,s}_{\mathtt{ev}_0}\times H^{\sigma,s}_{\mathtt{odd}}\to H^{\sigma,s-1}_{\mathtt{odd}}\times H^{\sigma,s-1}_{\mathtt{ev}_0}\,,\qquad \begin{bmatrix}\hat\eta\\ \hat\psi\end{bmatrix}\mapsto \begin{bmatrix} c\partial_x & |D|\\ -g & c\partial_x\end{bmatrix}\begin{bmatrix}\hat\eta\\ \hat\psi\end{bmatrix}\,, \tag{A.2.5}$$

has a nontrivial kernel. In the next lemma we characterize such values.

Lemma A.2.4. (Bifurcation speeds) *The kernel of* $\mathrm{d}_{(\eta,\psi)}F(0,0,c)$ *in* (A.2.5) *is nontrivial if and only if*

$$c=\pm\sqrt{\frac{g}{k}}\qquad \textit{for some } k\in\mathbb{N}\,. \tag{A.2.6}$$

For any $k\in\mathbb{N}$*, the Kernel of* $L:=\mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$*, where we set* $c_k^*:=\sqrt{\frac{g}{k}}$*, is one dimensional and*

$$\mathit{Ker}(L)=\langle u^*\rangle\quad \textit{with}\quad u^*:=\begin{bmatrix}\sqrt{k}\cos(kx)\\ \sqrt{g}\sin(kx)\end{bmatrix}\,. \tag{A.2.7}$$

Proof. By the Fourier expansion (A.2.3), it results that the kernel of $\mathrm{d}_{(\eta,\psi)}F(0,0,c)$ is nontrivial if and only if at least one of the matrices $\begin{bmatrix}-ck & k\\ -g & ck\end{bmatrix}$, $k\in\mathbb{N}$, has zero determinant. This is verified provided $c^2k=g$ for some $k\in\mathbb{N}$, i.e. (A.2.6) holds. In addition, a vector $\begin{bmatrix}\eta(x)\\ \psi(x)\end{bmatrix}=\sum_{j\geq 1}\begin{bmatrix}\eta_j\cos(jx)\\ \psi_j\sin(jx)\end{bmatrix}$ belongs to the Kernel of $\mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$ if and only if

$$\begin{bmatrix}-c_k^*j & j\\ -g & c_k^*j\end{bmatrix}\begin{bmatrix}\eta_j\\ \psi_j\end{bmatrix}=0\,,\quad \forall j\geq 1\,. \tag{A.2.8}$$

If $j\neq k$ then

$$\det\begin{bmatrix}-c_k^*j & j\\ -g & c_k^*j\end{bmatrix}=-(c_k^*)^2j^2+gj=j^2\big((c_j^*)^2-(c_k^*)^2\big)\neq 0\,, \tag{A.2.9}$$

since the map $k\mapsto (c_k^*)^2=g/k$ is injective on $\mathbb{N}$. Hence $\eta_j=\psi_j=0$ for any $j\neq k$. On the other hand, if $j=k$ then (A.2.8) is solved provided $\sqrt{g}\eta_k=\sqrt{k}\psi_k$, proving (A.2.7). $\square$

We apply Theorem A.2.2 with $c_k^*:=\sqrt{\frac{g}{k}}$. By Lemma A.2.4 assumption 1 holds. The next lemma verifies the assumptions 2)-3).

Lemma A.2.5. *The range* $R := RngL$*,* $L = \mathrm{d}_{(\eta,\psi)}F(0,0,c_k^*)$*, is*

$$R = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in H^{\sigma,s-1}_{\mathtt{odd}}(\mathbb{T}) \times H^{\sigma,s-1}_{\mathtt{ev}_0}(\mathbb{T}) : \quad \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} f_k \sin(kx) \\ c_k^* f_k \cos(kx) \end{bmatrix} + \sum_{j\geq 1, j\neq k} \begin{bmatrix} f_j \sin(jx) \\ g_j \cos(jx) \end{bmatrix} \right\} . \tag{A.2.10}$$

In particular R *is closed and* $\operatorname{codim} R = 1$*.*

The vector $(\partial_c \mathrm{d}_{(\eta,\psi)}F)(0,0,c_k^*) \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix}$ *does not belong to* R*.*

Proof. A vector $\begin{bmatrix} f \\ g \end{bmatrix} \in H^{\sigma,s-1}_{\mathtt{odd}}(\mathbb{T}) \times H^{\sigma,s-1}_{\mathtt{ev}_0}(\mathbb{T})$ belongs to R if and only if there is $\begin{bmatrix} \eta \\ \psi \end{bmatrix} \in H^{\sigma,s}_{\mathtt{ev}_0} \times H^{\sigma,s}_{\mathtt{odd}}$ such that, recalling (A.2.5) and (A.2.3),

$$\begin{bmatrix} -c_k^* j & j \\ -g & c_k^* j \end{bmatrix} \begin{bmatrix} \eta_j \\ \psi_j \end{bmatrix} = \begin{bmatrix} f_j \\ g_j \end{bmatrix} \quad \forall j \geq 1 \quad \text{where} \quad \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \sum_{j\geq 1} \begin{bmatrix} f_j \sin(jx) \\ g_j \cos(jx) \end{bmatrix} . \tag{A.2.11}$$

For any $j \neq k$, by (A.2.9), system (A.2.11) has the unique solution

$$\eta_j = \frac{1}{g}\frac{\sqrt{k}}{k-j}(\sqrt{g} f_j - \sqrt{k} g_j) , \quad \psi_j = \frac{1}{j\sqrt{g}}\frac{\sqrt{k}}{k-j}(\sqrt{kg} f_j - j g_j) . \tag{A.2.12}$$

If $j = k$, the system (A.2.11) is solvable if and only if

$$\sqrt{g} f_k = \sqrt{k} g_k \tag{A.2.13}$$

and a solution is $\eta_k = -\frac{1}{\sqrt{kg}} f_k$, $\psi_k = 0$. By (A.2.12) we deduce that $|\eta_j|, |\psi_j| \leq \frac{C_k}{j}(|f_j| + |g_j|)$, for any $j \in \mathbb{N} \setminus \{k\}$, implying that $(\eta,\psi) \in H^{\sigma,s}$, actually

$$\|\eta\|_{H^{\sigma,s}}, \ \|\psi\|_{H^{\sigma,s}} \leq C_k (\|f\|_{H^{\sigma,s-1}} + \|g\|_{H^{\sigma,s-1}}) .$$

In conclusion, the range R of L has the form (A.2.10), by (A.2.13) and $c_k^* = \sqrt{g/k}$.

Finally differentiating (A.2.5) one computes

$$(\partial_c \mathrm{d}_{(\eta,\psi)}F)(0,0,c_k^*) \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix} = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix} \begin{bmatrix} \sqrt{k}\cos(kx) \\ \sqrt{g}\sin(kx) \end{bmatrix} = \begin{bmatrix} -k^{\frac{3}{2}}\sin(kx) \\ k\sqrt{g}\cos(kx) \end{bmatrix}$$

which does not belong to the range R in (A.2.10). □

All the assumptions of the Crandall-Rabinowitz Theorem are verified.

This concludes the proof of Theorem A.2.1.

A.3 Expansion of the Stokes waves

In this part of the appendix we prove the following

Proposition A.3.1 (Expansion of Stokes waves). *The Stokes waves $\eta_\epsilon(x)$, $\psi_\epsilon(x)$ and the speed c_ϵ in Theorem 2.1.1 admit the expansion*

$$\begin{aligned}\eta_\epsilon(x) &= \epsilon\cos(x) + \epsilon^2(\eta_2^{[0]} + \eta_2^{[2]}\cos(2x)) + \epsilon^3(\eta_3^{[1]}\cos(x) + \eta_3^{[3]}\cos(3x)) \\ &\quad + \epsilon^4(\eta_4^{[0]} + \eta_4^{[2]}\cos(2x) + \eta_4^{[4]}\cos(4x)) + \mathcal{O}(\epsilon^5)\,,\end{aligned} \tag{A.3.1a}$$

$$\begin{aligned}\psi_\epsilon(x) &= \epsilon \mathtt{c_h}^{-1}\sin(x) + \epsilon^2\psi_2^{[2]}\sin(2x) + \epsilon^3(\psi_3^{[1]}\sin(x) + \psi_3^{[3]}\sin(3x)) \\ &\quad + \epsilon^4(\psi_4^{[2]}\sin(2x) + \psi_4^{[4]}\sin(4x)) + \mathcal{O}(\epsilon^5)\,,\end{aligned} \tag{A.3.1b}$$

$$c_\epsilon = \mathtt{c_h} + \epsilon^2 c_2 + \epsilon^4 c_4 + \mathcal{O}(\epsilon^5)\,, \qquad \mathtt{c_h} := \mathtt{c}_{\mathtt{h},1,1} = \sqrt{\tanh(\mathtt{h})}\,, \tag{A.3.1c}$$

with the Taylor coefficients, for arbitrary depth $\mathtt{h} > 0$, given by

$$\eta_2^{[0]} := \frac{\mathtt{c_h}^4 - 1}{4\mathtt{c_h}^2}\,, \qquad \eta_2^{[2]} := \frac{3 - \mathtt{c_h}^4}{4\mathtt{c_h}^6}\,, \qquad \psi_2^{[2]} := \frac{3 + \mathtt{c_h}^8}{8\mathtt{c_h}^7}\,, \tag{A.3.2a}$$

$$c_2 := \frac{9 - 10\mathtt{c_h}^4 + 9\mathtt{c_h}^8}{16\mathtt{c_h}^7} + \frac{(1 - \mathtt{c_h}^4)}{2\mathtt{c_h}}\eta_2^{[0]} = \frac{-2\mathtt{c_h}^{12} + 13\mathtt{c_h}^8 - 12\mathtt{c_h}^4 + 9}{16\mathtt{c_h}^7}\,, \tag{A.3.2b}$$

$$\begin{aligned}\eta_3^{[1]} &:= \frac{-2\mathtt{c_h}^{12} + 3\mathtt{c_h}^8 + 3}{16\mathtt{c_h}^8(1 + \mathtt{c_h}^2)}\,, \qquad \eta_3^{[3]} := \frac{-3\mathtt{c_h}^{12} + 9\mathtt{c_h}^8 - 9\mathtt{c_h}^4 + 27}{64\mathtt{c_h}^{12}}\,, \\ \psi_3^{[1]} &:= \frac{2\mathtt{c_h}^{12} - 3\mathtt{c_h}^8 - 3}{16\mathtt{c_h}^7(1 + \mathtt{c_h}^2)}\,, \qquad \psi_3^{[3]} := \frac{-9\mathtt{c_h}^{12} + 19\mathtt{c_h}^8 + 5\mathtt{c_h}^4 + 9}{64\mathtt{c_h}^{13}}\,,\end{aligned} \tag{A.3.2c}$$

$$\begin{aligned}\eta_4^{[0]} &:= \frac{-4\mathtt{c_h}^{20} - 4\mathtt{c_h}^{18} + 17\mathtt{c_h}^{16} + 6\mathtt{c_h}^{14} - 48\mathtt{c_h}^8 + 6\mathtt{c_h}^6 + 36\mathtt{c_h}^4 - 9}{64\mathtt{c_h}^{14}}\,, \\ \eta_4^{[2]} &:= \frac{1}{384\mathtt{c_h}^{18}(\mathtt{c_h}^2 + 1)}\Big(- 24\mathtt{c_h}^{22} + 285\mathtt{c_h}^{18} + 177\mathtt{c_h}^{16} - 862\mathtt{c_h}^{14} - 754\mathtt{c_h}^{12} \\ &\quad + 1116\mathtt{c_h}^{10} + 1080\mathtt{c_h}^8 - 162\mathtt{c_h}^6 - 54\mathtt{c_h}^4 - 81\mathtt{c_h}^2 - 81\Big)\,, \\ \eta_4^{[4]} &:= \frac{21\mathtt{c_h}^{20} + \mathtt{c_h}^{16} - 262\mathtt{c_h}^{12} + 522\mathtt{c_h}^8 + 81\mathtt{c_h}^4 + 405}{384\mathtt{c_h}^{18}(\mathtt{c_h}^4 + 5)}\,, \\ \psi_4^{[2]} &:= \frac{1}{768\mathtt{c_h}^{19}(\mathtt{c_h}^2 + 1)}\Big(- 12\mathtt{c_h}^{26} - 36\mathtt{c_h}^{24} + 57\mathtt{c_h}^{22} + 93\mathtt{c_h}^{20} + 51\mathtt{c_h}^{18} - 21\mathtt{c_h}^{16}\,, \\ &\quad - 646\mathtt{c_h}^{14} - 502\mathtt{c_h}^{12} + 1098\mathtt{c_h}^{10} + 1098\mathtt{c_h}^8 - 243\mathtt{c_h}^6 - 135\mathtt{c_h}^4 - 81\mathtt{c_h}^2 - 81\Big)\,, \\ \psi_4^{[4]} &:= \frac{-21\mathtt{c_h}^{24} + 60\mathtt{c_h}^{20} + 343\mathtt{c_h}^{16} - 1648\mathtt{c_h}^{12} + 3177\mathtt{c_h}^8 + 756\mathtt{c_h}^4 + 405}{1536\mathtt{c_h}^{19}(\mathtt{c_h}^4 + 5)}\,,\end{aligned} \tag{A.3.2d}$$

$$c_4 = \frac{1}{1024\mathtt{c_h}^{19}(\mathtt{c_h}^2 + 1)}\Big(56\mathtt{c_h}^{30} + 88\mathtt{c_h}^{28} - 272\mathtt{c_h}^{26} - 528\mathtt{c_h}^{24} - 7\mathtt{c_h}^{22} + 497\mathtt{c_h}^{20} \tag{A.3.2e}$$

$$+1917\mathtt{c}_{\mathtt{h}}^{18}+1437\mathtt{c}_{\mathtt{h}}^{16}-4566\mathtt{c}_{\mathtt{h}}^{14}-4038\mathtt{c}_{\mathtt{h}}^{12}+4194\mathtt{c}_{\mathtt{h}}^{10}+3906\mathtt{c}_{\mathtt{h}}^{8}-891\mathtt{c}_{\mathtt{h}}^{6}-675\mathtt{c}_{\mathtt{h}}^{4}+81\mathtt{c}_{\mathtt{h}}^{2}+81\Big)\,.$$

The rest of the section is devoted to the proof of Proposition A.3.1.

By Theorem 2.1.1 the Stokes waves admit the Taylor expansion

$$\begin{aligned}
\eta_\epsilon(x) &= \epsilon\eta_1(x)+\epsilon^2\eta_2(x)+\epsilon^3\eta_3(x)++\epsilon^4\eta_4(x)+\mathcal{O}(\epsilon^5)\,,\\
\psi_\epsilon(x) &= \epsilon\psi_1(x)+\epsilon^2\psi_2(x)+\epsilon^3\psi_3(x)+\epsilon^4\psi_4(x)+\mathcal{O}(\epsilon^5)\,,\\
c_\epsilon &= \mathtt{c}_{\mathtt{h}}+\epsilon c_1+\epsilon^2 c_2+\epsilon^3 c_3+\epsilon^4 c_4+\mathcal{O}(\epsilon^5)\,,
\end{aligned} \tag{A.3.3}$$

where $\eta_1=\cos(x)$, $\psi_1=\mathtt{c}_{\mathtt{h}}^{-1}\sin(x)$ and η_i is $even(x)$ and ψ_i is $odd(x)$ for $i=2,\ldots,4$.

We solve order by order in ϵ the equations (1.2.10), that we rewrite as

$$\begin{cases} -c\,\psi_x+\eta+\dfrac{\psi_x^2}{2}-\dfrac{\eta_x^2}{2(1+\eta_x^2)}(c-\psi_x)^2=0\\ c\,\eta_x+G(\eta)\psi=0\,,\end{cases} \tag{A.3.4}$$

having substituted $G(\eta)\psi$ with $-c\,\eta_x$ in the first equation.

We Taylor expand the Dirichlet-Neumann operator $G(\eta)$ as

$$G(\eta)=G_0+G_1(\eta)+G_2(\eta)+G_3(\eta)+\mathcal{O}(\eta^4)$$

where, by [29, formulae (39)-(40)],

$$\begin{aligned}
&G_0 := D\tanh(\mathtt{h}D)=|D|\tanh(\mathtt{h}|D|)\,,\\
&G_1(\eta) := -\partial_x\eta\partial_x-G_0\eta G_0\,,\\
&G_2(\eta) := -\frac12 G_0\partial_x\eta^2\partial_x+\frac12\partial_x^2\eta^2G_0-G_0\eta G_1(\eta)\,,\\
&G_3(\eta) := \frac16\partial_x^3\eta^3\partial_x+\frac16 G_0\partial_x^2\eta^3G_0-G_0\eta G_2(\eta)+\frac12\partial_x^2\eta^2G_1(\eta)\\
&G_4(\eta) := \frac1{24}G_0\partial_x^3\eta^4\partial_x-\frac1{24}\partial_x^4\eta^4G_0+\frac12\partial_x^2\eta^2G_2(\eta)+\frac16 G_0\partial_x^2\eta^3G_1(\eta)-G_0\eta G_3(\eta)\,.
\end{aligned} \tag{A.3.5}$$

Remark A.3.2. In order to check that (A.3.5) coincides with [29, formulae (39)-(40)] use the identity $D^2=-\partial_x^2$. We point out that (A.3.5) coincides with [32, formulae (2.13)-(2.14)] and the recursion formulae of [94, p. 24].

By linearizing system (A.3.4) at $(\eta,\psi,c)=(0,0,\mathtt{c}_{\mathtt{h}})$ we get the linear system

$$\mathcal{B}_0\begin{bmatrix}\widehat\eta\\ \widehat\psi\end{bmatrix}=0\,,\qquad \mathcal{B}_0:=\begin{bmatrix}1 & -\mathtt{c}_{\mathtt{h}}\partial_x\\ \mathtt{c}_{\mathtt{h}}\partial_x & G_0\end{bmatrix}\,, \tag{A.3.6}$$

with $\mathcal{B}_0=\mathcal{B}_0^*$ with respect to the scalar product of $L^2(\mathbb{T},\mathbb{R}^2)$.

To apply the bifurcation procedure we study kernel and range of $\mathcal{B}_0$ in the following

Lemma A.3.3. *The kernel of the operator* $\mathcal{B}_0$ *in* (A.3.6) *is*

$$K := Ker\, \mathcal{B}_0 = span\Big\{ \begin{bmatrix} \cos(x) \\ \mathtt{c}_{\mathtt{h}}^{-1} \sin(x) \end{bmatrix} \Big\} \tag{A.3.7a}$$

and its range $R := Rn\, \mathcal{B}_0 = K^{\perp_{L^2}}$ *is given by* $R = R_0 \oplus R_1 \oplus R_\varnothing$*, where*

$$\begin{aligned} &R_0 := span\Big\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Big\}, \quad R_1 := span\Big\{ \begin{bmatrix} -\cos(x) \\ \mathtt{c}_{\mathtt{h}} \sin(x) \end{bmatrix} \Big\}, \\ &R_\varnothing := \overline{\bigoplus_{k=2}^{\infty} R_k}\,, \quad R_k := span\Big\{ \begin{bmatrix} \cos(kx) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sin(kx) \end{bmatrix} \Big\}. \end{aligned} \tag{A.3.7b}$$

Consequently there exists a unique self-adjoint bounded linear operator $\mathcal{B}_0^{-1} : R \to R$

$$\mathcal{B}_0^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathcal{B}_0^{-1} \begin{bmatrix} -\cos(x) \\ \mathtt{c}_{\mathtt{h}} \sin(x) \end{bmatrix} = \frac{1}{1+\mathtt{c}_{\mathtt{h}}^2} \begin{bmatrix} -\cos(x) \\ \mathtt{c}_{\mathtt{h}} \sin(x) \end{bmatrix}, \tag{A.3.8}$$

$$\mathcal{B}_0^{-1} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \big(|D| \tanh(\mathtt{h}|D|) + \mathtt{c}_{\mathtt{h}}^2 \partial_x^2\big)^{-1} \begin{bmatrix} |D|\tanh(\mathtt{h}|D|) & \mathtt{c}_{\mathtt{h}} \partial_x \\ -\mathtt{c}_{\mathtt{h}} \partial_x & 1 \end{bmatrix} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \ \forall \begin{bmatrix} f \\ g \end{bmatrix} \in R_\varnothing\,,$$

such that $\mathcal{B}_0 \mathcal{B}_0^{-1} = \mathcal{B}_0^{-1} \mathcal{B}_0\big|_R = \mathrm{Id}_R$.

Second order in ϵ**.** By plugging the expansion (A.3.3) in system (A.3.4) and discarding cubic terms we find the linear system

$$\mathcal{B}_0 \begin{bmatrix} \eta_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} c_1 (\psi_1)_x - \frac12 (\psi_1)_x^2 + \frac12 (G_0 \psi_1)^2 \\ -c_1 (\eta_1)_x - G_1(\eta_1)\psi_1 \end{bmatrix}, \tag{A.3.9}$$

where $\mathcal{B}_0$ is the self-adjoint operator in (A.3.6). System (A.3.9) admits a solution if and only if its right-hand term is orthogonal to the Kernel of $\mathcal{B}_0$ in (A.3.7a), namely

$$\Big(\begin{bmatrix} c_1 (\psi_1)_x - \frac12 (\psi_1)_x^2 + \frac12 (G_0 \psi_1)^2 \\ -c_1 (\eta_1)_x - G_1(\eta_1)\psi_1 \end{bmatrix}, \begin{bmatrix} \cos(x) \\ \mathtt{c}_{\mathtt{h}}^{-1} \sin(x) \end{bmatrix} \Big) = 0\,. \tag{A.3.10}$$

In view of the explicit first-order expansion in (A.3.3) and the identity $\tanh(2\mathtt{h}) = \dfrac{2\mathtt{c}_{\mathtt{h}}^2}{1+\mathtt{c}_{\mathtt{h}}^4}$, it results

$$G_0 \psi_1 = \mathtt{c}_{\mathtt{h}} \sin(x)\,, \quad G_1(\eta_1)\psi_1 = \frac{1-\mathtt{c}_{\mathtt{h}}^4}{\mathtt{c}_{\mathtt{h}}(1+\mathtt{c}_{\mathtt{h}}^4)} \sin(2x)\,. \tag{A.3.11}$$

so that (A.3.10) implies $c_1 = 0$, in agreement with (A.3.1c). Equation (A.3.9) reduces to

$$\begin{bmatrix} 1 & -\mathtt{c_h}\partial_x \\ \mathtt{c_h}\partial_x & G_0 \end{bmatrix} \begin{bmatrix} \eta_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}(\mathtt{c_h}^{-2} - \mathtt{c_h}^2) - \frac{1}{4}(\mathtt{c_h}^{-2} + \mathtt{c_h}^2)\cos(2x) \\ -\frac{1-\mathtt{c_h}^4}{\mathtt{c_h}(1+\mathtt{c_h}^4)}\sin(2x) \end{bmatrix}. \tag{A.3.12}$$

Setting $\eta_2 = \eta_2^{[0]} + \eta_2^{[2]}\cos(2x)$ and $\psi_2 = \psi_2^{[2]}\sin(2x)$, system (A.3.12) amounts to

$$\begin{cases} \eta_2^{[0]} + (\eta_2^{[2]} - 2\mathtt{c_h}\psi_2^{[2]})\cos(2x) = -\frac{1}{4}\left(\mathtt{c_h}^{-2} - \mathtt{c_h}^2\right) - \frac{1}{4}\left(\mathtt{c_h}^{-2} + \mathtt{c_h}^2\right)\cos(2x) \\ (-2\mathtt{c_h}\eta_2^{[2]} + 2\psi_2^{[2]}\tanh(2\mathtt{h}))\sin(2x) = -\frac{1-\mathtt{c_h}^4}{\mathtt{c_h}(1+\mathtt{c_h}^4)}\sin(2x)\,, \end{cases}$$

which is solved by the coefficients $\eta_2^{[0]}$, $\eta_2^{[2]}$, $\psi_2^{[2]}$ given in (A.3.2a).

Third order in ϵ. By plugging the expansion (A.3.3) in system (A.3.4) and discarding quartic terms we find the following linear system

$$\mathcal{B}_0\begin{bmatrix} \eta_3 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} c_2(\psi_1)_x - (\psi_1)_x(\psi_2)_x - (\eta_1)_x^2(\psi_1)_x\mathtt{c_h} + (\eta_1)_x(\eta_2)_x\mathtt{c_h}^2 \\ -c_2(\eta_1)_x - G_1(\eta_1)\psi_2 - G_1(\eta_2)\psi_1 - G_2(\eta_1)\psi_1 \end{bmatrix} =: \begin{bmatrix} f_3 \\ g_3 \end{bmatrix}. \tag{A.3.13}$$

In view of (A.3.1) we have

$$\begin{aligned} (\psi_1)_x(\psi_2)_x &= \frac{\psi_2^{[2]}}{\mathtt{c_h}}(\cos(x) + \cos(3x))\,, \\ (\eta_1)_x^2(\psi_1)_x\mathtt{c_h} &= \frac{1}{4}(\cos(x) - \cos(3x))\,, \end{aligned} \qquad \mathtt{c_h}^2(\eta_1)_x(\eta_2)_x = \mathtt{c_h}^2\eta_2^{[2]}(\cos(x) - \cos(3x))\,. \tag{A.3.14}$$

By means of (A.3.5) and since

$$\tanh(\mathtt{h}) = \mathtt{c_h}^2\,, \quad \tanh(2\mathtt{h}) = \frac{2\mathtt{c_h}^2}{1+\mathtt{c_h}^4}\,, \quad \tanh(3\mathtt{h}) = \frac{3\mathtt{c_h}^2 + \mathtt{c_h}^6}{1+3\mathtt{c_h}^4}\,, \tag{A.3.15}$$

whereby

$$G_0\psi_2 = \frac{4\mathtt{c_h}^2}{1+\mathtt{c_h}^4}\psi_2^{[2]}\sin(2x)\,, \tag{A.3.16}$$

we have, in view of (A.3.11) too,

$$\begin{aligned} G_1(\eta_1)\psi_2 &= \psi_2^{[2]}\frac{1-\mathtt{c_h}^4}{1+\mathtt{c_h}^4}\sin(x) + 3\psi_2^{[2]}\frac{1-2\mathtt{c_h}^4+\mathtt{c_h}^8}{1+4\mathtt{c_h}^4+3\mathtt{c_h}^8}\sin(3x)\,, \\ G_2(\eta_1)\psi_1 &= \frac{\mathtt{c_h}}{4}\frac{3\mathtt{c_h}^4-1}{1+\mathtt{c_h}^4}\sin(x) - \frac{3}{4}\mathtt{c_h}\frac{\mathtt{c_h}^8-4\mathtt{c_h}^4+3}{1+4\mathtt{c_h}^4+3\mathtt{c_h}^8}\sin(3x)\,, \\ G_1(\eta_2)\psi_1 &= \frac{1}{\mathtt{c_h}}(\eta_2^{[0]}(1-\mathtt{c_h}^4) + \frac{1}{2}\eta_2^{[2]}(1+\mathtt{c_h}^4))\sin(x) + \frac{3}{2\mathtt{c_h}}\eta_2^{[2]}\frac{1-\mathtt{c_h}^8}{1+3\mathtt{c_h}^4}\sin(3x)\,. \end{aligned} \tag{A.3.17}$$

System (A.3.13) has a solution if and only if the right hand side is orthogonal to the Kernel of $\mathcal{B}_0$ given in (A.3.7a), namely

$$
\begin{aligned}
0 &= \Big(\begin{bmatrix} f_3(x) \\ g_3(x) \end{bmatrix} , \begin{bmatrix} \cos(x) \\ \mathtt{c_h}^{-1} \sin(x) \end{bmatrix} \Big) \\
&= \Big(\begin{bmatrix} c_2(\psi_1)_x - (\psi_1)_x(\psi_2)_x - (\eta_1)_x^2(\psi_1)_x \mathtt{c_h} + (\eta_1)_x(\eta_2)_x \mathtt{c_h}^2 \\ -c_2(\eta_1)_x - G_1(\eta_1)\psi_2 - G_1(\eta_2)\psi_1 - G_2(\eta_1)\psi_1 \end{bmatrix} , \begin{bmatrix} \cos(x) \\ \mathtt{c_h}^{-1} \sin(x) \end{bmatrix} \Big) .
\end{aligned} \tag{A.3.18}
$$

The kernel equation (A.3.18) determines uniquely c_2 to be, using also (A.3.14) and (A.3.17), the term in (A.3.2b). Consequently the right-hand side of system (A.3.13) is explicitly given by

$$
\begin{bmatrix} f_3(x) \\ g_3(x) \end{bmatrix} = \begin{bmatrix} f_3^{[1]} \cos(x) + f_3^{[3]} \cos(3x) \\ g_3^{[1]} \sin(x) + g_3^{[3]} \sin(3x) \end{bmatrix} , \tag{A.3.19}
$$

with

$$
\begin{aligned}
f_3^{[1]} &:= \frac{-2\mathtt{c_h}^{12} + 3\mathtt{c_h}^8 + 3}{16\mathtt{c_h}^8} , \quad f_3^{[3]} := \frac{3\mathtt{c_h}^8 - 6\mathtt{c_h}^4 - 3}{8\mathtt{c_h}^8} , \\
g_3^{[1]} &:= \frac{2\mathtt{c_h}^{12} - 3\mathtt{c_h}^8 - 3}{16\mathtt{c_h}^7} , \quad g_3^{[3]} := \frac{-6\mathtt{c_h}^8 + 15\mathtt{c_h}^4 - 9}{4\mathtt{c_h}^7(1 + 3\mathtt{c_h}^4)} .
\end{aligned} \tag{A.3.20}
$$

The choice of c_2 in (A.3.2b) as solution of the kernel equation (A.3.18) ensures the existence of a solution

$$
\begin{bmatrix} \eta_3(x) \\ \psi_3(x) \end{bmatrix} := \begin{bmatrix} \eta_3^{[1]} \cos(x) + \eta_3^{[3]} \cos(3x) \\ \psi_3^{[1]} \sin(x) + \psi_3^{[3]} \sin(3x) \end{bmatrix} := \mathcal{B}_0^{-1} \begin{bmatrix} f_3(x) \\ g_3(x) \end{bmatrix} .
$$

By Lemma A.3.3 and (A.3.15) we have

$$
\begin{aligned}
\begin{bmatrix} \eta_3^{[3]} \cos(3x) \\ \psi_3^{[3]} \sin(3x) \end{bmatrix} = \mathcal{B}_0^{-1} \begin{bmatrix} \beta \cos(3x) \\ \delta \sin(3x) \end{bmatrix} &= -\frac{1 + 3\mathtt{c_h}^4}{24\mathtt{c_h}^6} \begin{bmatrix} |D| \tanh(\mathtt{h}|D|) & \mathtt{c_h}\partial_x \\ -\mathtt{c_h}\partial_x & 1 \end{bmatrix} \begin{bmatrix} f_3^{[3]} \cos(3x) \\ g_3^{[3]} \sin(3x) \end{bmatrix} \\
&= -\frac{1 + 3\mathtt{c_h}^4}{24\mathtt{c_h}^6} \begin{bmatrix} 3\big(\frac{3\mathtt{c_h}^2 + \mathtt{c_h}^6}{1 + 3\mathtt{c_h}^4} f_3^{[3]} + \mathtt{c_h} g_3^{[3]}\big) \cos(3x) \\ (3\mathtt{c_h} f_3^{[3]} + g_3^{[3]}) \sin(3x) \end{bmatrix} ,
\end{aligned} \tag{A.3.21}
$$

and $\begin{bmatrix} \eta_3^{[1]} \cos(x) \\ \psi_3^{[1]} \sin(x) \end{bmatrix} = \mathcal{B}_0^{-1} \begin{bmatrix} f_3^{[1]} \cos(x) \\ g_3^{[1]} \sin(x) \end{bmatrix} = \frac{1}{1+\mathtt{c_h}^2} \begin{bmatrix} f_3^{[1]} \cos(x) \\ g_3^{[1]} \sin(x) \end{bmatrix}$. The coefficients in (A.3.2c) follow.

Fourth order in ϵ. By plugging the expansion (A.3.3) in system (A.3.4) and discarding quintic terms we find the linear system

$$
\mathcal{B}_0 \begin{bmatrix} \eta_4 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} f_4 \\ g_4 \end{bmatrix} , \tag{A.3.22}
$$

with

$$f_4 := c_3(\psi_1)_x + c_2(\psi_2)_x - (\psi_1)_x(\psi_3)_x - \frac{1}{2}(\psi_2)_x^2 + \mathtt{c_h}(\eta_1)_x^2(c_2 - (\psi_2)_x) \qquad \text{(A.3.23a)}$$
$$+ (\eta_1)_x(\eta_3)_x\mathtt{c_h}^2 + \frac{1}{2}(\eta_2)_x^2\mathtt{c_h}^2 + \frac{1}{2}(\eta_1)_x^2(\psi_1)_x^2 - \frac{1}{2}\mathtt{c_h}^2(\eta_1)_x^4 - 2\mathtt{c_h}(\eta_1)_x(\eta_2)_x(\psi_1)_x\,,$$
$$g_4 := -c_3(\eta_1)_x - c_2(\eta_2)_x - G_1(\eta_1)\psi_3 - G_1(\eta_2)\psi_2 - G_1(\eta_3)\psi_1 \qquad \text{(A.3.23b)}$$
$$- G_2'(\eta_1)[\eta_2]\psi_1 - G_2(\eta_1)\psi_2 - G_3(\eta_1)\psi_1,$$

where, in view of (A.3.5),

$$G_2'(\eta)[\hat\eta] := -G_0\partial_x\eta\hat\eta\partial_x + \partial_x^2\eta\hat\eta G_0 - G_0\hat\eta G_1(\eta) - G_0\eta G_1(\hat\eta)\,. \qquad \text{(A.3.24)}$$

Let us inspect the terms in (A.3.23a). In view of (A.3.1) we have

$$c_3(\psi_1)_x = \mathtt{c_h}^{-1}c_3\cos(x)\,, \quad c_3(\eta_1)_x = -c_3\sin(x)\,, \quad c_2(\psi_2)_x = 2c_2\psi_2^{[2]}\cos(2x)\,, \qquad \text{(A.3.25)}$$
$$c_2(\eta_2)_x = -2c_2\eta_2^{[2]}\sin(2x)\,, \quad \frac{1}{2}(\psi_2)_x^2 = (\psi_2^{[2]})^2 + (\psi_2^{[2]})^2\cos(4x)\,,$$
$$(\psi_1)_x(\psi_3)_x = \frac{1}{2}\mathtt{c_h}^{-1}\psi_3^{[1]} + \frac{1}{2}\mathtt{c_h}^{-1}(\psi_3^{[1]} + 3\psi_3^{[3]})\cos(2x) + \frac{3}{2}\mathtt{c_h}^{-1}\psi_3^{[3]}\cos(4x)\,,$$
$$\mathtt{c_h}(\eta_1)_x^2(\psi_2)_x = -\frac{1}{2}\mathtt{c_h}\psi_2^{[2]} + \mathtt{c_h}\psi_2^{[2]}\cos(2x) - \frac{1}{2}\mathtt{c_h}\psi_2^{[2]}\cos(4x)\,,$$
$$\frac{1}{2}\mathtt{c_h}^2(\eta_2)_x^2 = \mathtt{c_h}^2(\eta_2^{[2]})^2 - \mathtt{c_h}^2(\eta_2^{[2]})^2\cos(4x)\,, \quad \mathtt{c_h}c_2(\eta_1)_x^2 = \frac{1}{2}\mathtt{c_h}c_2(1-\cos(2x))\,,$$
$$\mathtt{c_h}^2(\eta_1)_x(\eta_3)_x = \frac{1}{2}\mathtt{c_h}^2\eta_3^{[1]} + \frac{1}{2}\mathtt{c_h}^2(-\eta_3^{[1]} + 3\eta_3^{[3]})\cos(2x) - \frac{3}{2}\mathtt{c_h}^2\eta_3^{[3]}\cos(4x)\,,$$
$$\frac{1}{2}(\eta_1)_x^2(\psi_1)_x^2 = \frac{\mathtt{c_h}^{-2}}{16}(1-\cos(4x))\,, \quad \frac{1}{2}\mathtt{c_h}^2(\eta_1)_x^4 = \frac{\mathtt{c_h}^2}{16}(3 - 4\cos(2x) + \cos(4x))\,,$$
$$2\mathtt{c_h}(\eta_1)_x(\eta_2)_x(\psi_1)_x = \eta_2^{[2]} - \eta_2^{[2]}\cos(4x)\,.$$

Let us inspect the terms in (A.3.23b). In view of (A.3.5), (A.3.3), (A.3.15), whereby

$$G_0\psi_3 = \mathtt{c_h}^2\psi_3^{[1]}\sin(x) + 3\mathtt{c_h}^2\frac{3+\mathtt{c_h}^4}{1+3\mathtt{c_h}^4}\psi_3^{[3]}\sin(3x)\,, \qquad \text{(A.3.26)}$$

and since

$$\tanh(4\mathtt{h}) = \frac{4\mathtt{c_h}^2 + 4\mathtt{c_h}^6}{1 + 6\mathtt{c_h}^4 + \mathtt{c_h}^8}\,, \qquad \text{(A.3.27)}$$

we have

$$G_1(\eta_1)\psi_3 = \Big(\frac{1-\mathtt{c_h}^4}{1+\mathtt{c_h}^4}\psi_3^{[1]} + \frac{3(1-\mathtt{c_h}^4)^2\psi_3^{[3]}}{(1+\mathtt{c_h}^4)(1+3\mathtt{c_h}^4)}\Big)\sin(2x) + 6\frac{(1-\mathtt{c_h}^4)^3\psi_3^{[3]}\sin(4x)}{(1+3\mathtt{c_h}^4)(1+6\mathtt{c_h}^4+\mathtt{c_h}^8)}\,,$$

$$G_1(\eta_2)\psi_2 = 4\frac{(1-\mathtt{c}_{\mathtt{h}}^4)^2}{(1+\mathtt{c}_{\mathtt{h}}^4)^2}\psi_2^{[2]}\eta_2^{[0]}\sin(2x) + 4\frac{(1-\mathtt{c}_{\mathtt{h}}^4)^2\psi_2^{[2]}\eta_2^{[2]}}{1+6\mathtt{c}_{\mathtt{h}}^4+\mathtt{c}_{\mathtt{h}}^8}\sin(4x)\,, \tag{A.3.28}$$

$$G_1(\eta_3)\psi_1 = \Big(\frac{1-\mathtt{c}_{\mathtt{h}}^4}{\mathtt{c}_{\mathtt{h}}(1+\mathtt{c}_{\mathtt{h}}^4)}\eta_3^{[1]} + \frac{1+3\mathtt{c}_{\mathtt{h}}^4}{\mathtt{c}_{\mathtt{h}}(1+\mathtt{c}_{\mathtt{h}}^4)}\eta_3^{[3]}\Big)\sin(2x) + 2\frac{(1-\mathtt{c}_{\mathtt{h}}^4)(1+3\mathtt{c}_{\mathtt{h}}^4)}{\mathtt{c}_{\mathtt{h}}(1+6\mathtt{c}_{\mathtt{h}}^4+\mathtt{c}_{\mathtt{h}}^8)}\eta_3^{[3]}\sin(4x)\,.$$

In view of (A.3.1), (A.3.24), (A.3.15), (A.3.11), (A.3.17) and (A.3.27), we have

$$G_2'(\eta_1)[\eta_2]\psi_1 = \Big(\frac{8\mathtt{c}_{\mathtt{h}}^5\eta_2^{[2]}}{(1+3\mathtt{c}_{\mathtt{h}}^4)(1+\mathtt{c}_{\mathtt{h}}^4)} - \frac{4\mathtt{c}_{\mathtt{h}}(1-\mathtt{c}_{\mathtt{h}}^4)}{(1+\mathtt{c}_{\mathtt{h}}^4)^2}\eta_2^{[0]}\Big)\sin(2x) - \frac{8\mathtt{c}_{\mathtt{h}}(1-\mathtt{c}_{\mathtt{h}}^8)\eta_2^{[2]}}{(1+3\mathtt{c}_{\mathtt{h}}^4)(1+6\mathtt{c}_{\mathtt{h}}^4+\mathtt{c}_{\mathtt{h}}^8)}\sin(4x)\,. \tag{A.3.29}$$

By (A.3.1), (A.3.5), (A.3.17), (A.3.15) and (A.3.27) we have

$$G_2(\eta_1)\psi_2 = -\frac{8\mathtt{c}_{\mathtt{h}}^2(1-\mathtt{c}_{\mathtt{h}}^4)\psi_2^{[2]}}{(1+\mathtt{c}_{\mathtt{h}}^4)^2(1+3\mathtt{c}_{\mathtt{h}}^4)}\sin(2x) - \frac{16\mathtt{c}_{\mathtt{h}}^2(1-\mathtt{c}_{\mathtt{h}}^4)^2\psi_2^{[2]}}{(1+\mathtt{c}_{\mathtt{h}}^4)(1+3\mathtt{c}_{\mathtt{h}}^4)(1+6\mathtt{c}_{\mathtt{h}}^4+\mathtt{c}_{\mathtt{h}}^8)}\sin(4x)\,. \tag{A.3.30}$$

Finally, by (A.3.1), (A.3.5), (A.3.17), (A.3.11), (A.3.15) and (A.3.27), we have

$$G_3(\eta_1)\psi_1 = \frac{-1+14\mathtt{c}_{\mathtt{h}}^4-9\mathtt{c}_{\mathtt{h}}^8}{3\mathtt{c}_{\mathtt{h}}(1+\mathtt{c}_{\mathtt{h}}^4)^2(1+3\mathtt{c}_{\mathtt{h}}^4)}\sin(2x) + 2\frac{-1+15\mathtt{c}_{\mathtt{h}}^4-23\mathtt{c}_{\mathtt{h}}^8+9\mathtt{c}_{\mathtt{h}}^{12}}{3\mathtt{c}_{\mathtt{h}}(1+\mathtt{c}_{\mathtt{h}}^4)(1+3\mathtt{c}_{\mathtt{h}}^4)(1+6\mathtt{c}_{\mathtt{h}}^4+\mathtt{c}_{\mathtt{h}}^8)}\sin(4x)\,. \tag{A.3.31}$$

By (A.3.23), (A.3.25), (A.3.28), (A.3.29), (A.3.30), (A.3.31) and (A.3.2a)-(A.3.2c) system (A.3.22) reads as

$$\begin{bmatrix} f_4(x) \\ g_4(x) \end{bmatrix} = \begin{bmatrix} f_4^{[0]} + \mathtt{c}_{\mathtt{h}}^{-1}c_3\cos(x) + f_4^{[2]}\cos(2x) + f_4^{[4]}\cos(4x) \\ c_3\sin(x) + g_4^{[2]}\sin(2x) + g_4^{[4]}\sin(4x) \end{bmatrix}\,, \tag{A.3.32}$$

with

$$\begin{aligned} f_4^{[0]} &= \frac{-4\mathtt{c}_{\mathtt{h}}^{20}-4\mathtt{c}_{\mathtt{h}}^{18}+17\mathtt{c}_{\mathtt{h}}^{16}+6\mathtt{c}_{\mathtt{h}}^{14}-48\mathtt{c}_{\mathtt{h}}^8+6\mathtt{c}_{\mathtt{h}}^6+36\mathtt{c}_{\mathtt{h}}^4-9}{64\mathtt{c}_{\mathtt{h}}^{14}}\,, \quad f_4^{[4]} = \frac{7\mathtt{c}_{\mathtt{h}}^{16}-48\mathtt{c}_{\mathtt{h}}^{12}+126\mathtt{c}_{\mathtt{h}}^8-168\mathtt{c}_{\mathtt{h}}^4-45}{128\mathtt{c}_{\mathtt{h}}^{14}}\,, \\ f_4^{[2]} &= \frac{4\mathtt{c}_{\mathtt{h}}^{22}+12\mathtt{c}_{\mathtt{h}}^{20}-27\mathtt{c}_{\mathtt{h}}^{18}-31\mathtt{c}_{\mathtt{h}}^{16}+78\mathtt{c}_{\mathtt{h}}^{14}+66\mathtt{c}_{\mathtt{h}}^{12}-72\mathtt{c}_{\mathtt{h}}^{10}-84\mathtt{c}_{\mathtt{h}}^8+6\mathtt{c}_{\mathtt{h}}^6-6\mathtt{c}_{\mathtt{h}}^4+27\mathtt{c}_{\mathtt{h}}^2+27}{128\mathtt{c}_{\mathtt{h}}^{14}(1+\mathtt{c}_{\mathtt{h}}^2)}\,, \\ g_4^{[2]} &= \frac{3\mathtt{c}_{\mathtt{h}}^{16}-12\mathtt{c}_{\mathtt{h}}^{14}-39\mathtt{c}_{\mathtt{h}}^{12}+18\mathtt{c}_{\mathtt{h}}^{10}+139\mathtt{c}_{\mathtt{h}}^8-225\mathtt{c}_{\mathtt{h}}^4+18\mathtt{c}_{\mathtt{h}}^2+18}{48\mathtt{c}_{\mathtt{h}}^9(1+\mathtt{c}_{\mathtt{h}}^4)}\,, \\ g_4^{[4]} &= \frac{-21\mathtt{c}_{\mathtt{h}}^{20}+61\mathtt{c}_{\mathtt{h}}^{16}+14\mathtt{c}_{\mathtt{h}}^{12}-198\mathtt{c}_{\mathtt{h}}^8+279\mathtt{c}_{\mathtt{h}}^4-135}{48\mathtt{c}_{\mathtt{h}}^{13}(\mathtt{c}_{\mathtt{h}}^8+6\mathtt{c}_{\mathtt{h}}^4+1)}\,. \end{aligned} \tag{A.3.33}$$

As a consequence of (A.3.32) we have

$$\Big(\begin{bmatrix} f_4(x) \\ g_4(x) \end{bmatrix}, \begin{bmatrix} \cos(x) \\ \mathtt{c}_{\mathtt{h}}^{-1}\sin(x) \end{bmatrix} \Big) = \mathtt{c}_{\mathtt{h}}^{-1}c_3\,.$$

By Lemma A.3.3, with $c_3 = 0$ as stated in (A.3.1c), one ensures that the system (A.3.22) is solved by

$$\begin{bmatrix}\eta_4(x)\\ \psi_4(x)\end{bmatrix} = \mathcal{B}_0^{-1}\begin{bmatrix}f_4(x)\\ g_4(x)\end{bmatrix}, \tag{A.3.34}$$

where, in view of (A.3.8) and (A.3.32),

$$\begin{bmatrix}\eta_4(x)\\ \\ \psi_4(x)\end{bmatrix} = \begin{bmatrix}\eta_4^{[0]} + \eta_4^{[2]}\cos(2x) + \eta_4^{[4]}\cos(4x)\\ \\ \psi_4^{[2]}\sin(2x) + \psi_4^{[4]}\sin(4x)\end{bmatrix}, \tag{A.3.35}$$

with, in view of (A.3.15) and (A.3.27) too,

$$\eta_4^{[0]} := f_4^{[0]}, \quad \begin{bmatrix}\eta_4^{[2]}\\ \\ \psi_4^{[2]}\end{bmatrix} = -\frac{1+\mathtt{c}_\mathtt{h}^4}{4\mathtt{c}_\mathtt{h}^6}\begin{bmatrix}\frac{4\mathtt{c}_\mathtt{h}^2}{1+\mathtt{c}_\mathtt{h}^4} & 2\mathtt{c}_\mathtt{h}\\ 2\mathtt{c}_\mathtt{h} & 1\end{bmatrix}\begin{bmatrix}f_4^{[2]}\\ \\ g_4^{[2]}\end{bmatrix} = -\frac{1+\mathtt{c}_\mathtt{h}^4}{4\mathtt{c}_\mathtt{h}^6}\begin{bmatrix}\frac{4\mathtt{c}_\mathtt{h}^2}{1+\mathtt{c}_\mathtt{h}^4}f_4^{[2]} + 2\mathtt{c}_\mathtt{h}g_4^{[2]}\\ 2\mathtt{c}_\mathtt{h}f_4^{[2]} + g_4^{[2]}\end{bmatrix}, \tag{A.3.36}$$

$$\begin{bmatrix}\eta_4^{[4]}\\ \\ \psi_4^{[4]}\end{bmatrix} = -\frac{1+6\mathtt{c}_\mathtt{h}^4+\mathtt{c}_\mathtt{h}^8}{16\mathtt{c}_\mathtt{h}^6(5+\mathtt{c}_\mathtt{h}^4)}\begin{bmatrix}16\mathtt{c}_\mathtt{h}^2\frac{1+\mathtt{c}_\mathtt{h}^4}{1+6\mathtt{c}_\mathtt{h}^4+\mathtt{c}_\mathtt{h}^8} & 4\mathtt{c}_\mathtt{h}\\ 4\mathtt{c}_\mathtt{h} & 1\end{bmatrix}\begin{bmatrix}f_4^{[4]}\\ \\ g_4^{[4]}\end{bmatrix} = -\frac{1+6\mathtt{c}_\mathtt{h}^4+\mathtt{c}_\mathtt{h}^8}{16\mathtt{c}_\mathtt{h}^6(5+\mathtt{c}_\mathtt{h}^4)}\begin{bmatrix}16\mathtt{c}_\mathtt{h}^2\frac{1+\mathtt{c}_\mathtt{h}^4}{1+6\mathtt{c}_\mathtt{h}^4+\mathtt{c}_\mathtt{h}^8}f_4^{[4]} + 4\mathtt{c}_\mathtt{h}g_4^{[4]}\\ 4\mathtt{c}_\mathtt{h}f_4^{[4]} + g_4^{[4]}\end{bmatrix}.$$

By (A.3.35) we conclude the proof of (A.3.1a)-(A.3.1b) and, in view of (A.3.36) and (A.3.33), of (A.3.2d). To conclude the proof of (A.3.2) we compute the term c_4.

Fifth order in ϵ. By plugging the expansion (A.3.3) in system (A.3.4) and discarding sextic terms we find the linear system

$$\mathcal{B}_0\begin{bmatrix}\eta_5\\ \psi_5\end{bmatrix} = \begin{bmatrix}\mathtt{f}_5\\ \mathtt{g}_5\end{bmatrix} + c_4\begin{bmatrix}(\psi_1)_x\\ -(\eta_1)_x\end{bmatrix}, \tag{A.3.37}$$

with

$$\begin{aligned}\mathtt{f}_5 := {}& c_2(\psi_3)_x - (\psi_1)_x(\psi_4)_x - (\psi_2)_x(\psi_3)_x + (\eta_1)_x^2\big(-\mathtt{c}_\mathtt{h}(\psi_3)_x - c_2(\psi_1)_x + (\psi_1)_x(\psi_2)_x\big)\\ &+ (\eta_1)_x(\eta_2)_x\big((\psi_1)_x^2 + 2\mathtt{c}_\mathtt{h}c_2 - 2\mathtt{c}_\mathtt{h}(\psi_2)_x\big) - \mathtt{c}_\mathtt{h}(\psi_1)_x\big((\eta_2)_x^2 - (\eta_1)_x^4 + 2(\eta_1)_x(\eta_3)_x\big)\\ &+ \mathtt{c}_\mathtt{h}^2\big((\eta_1)_x(\eta_4)_x + (\eta_2)_x(\eta_3)_x - 2(\eta_1)_x^3(\eta_2)_x\big), \end{aligned}\tag{A.3.38a}$$

$$\begin{aligned}\mathtt{g}_5 := {}& -c_2(\eta_3)_x - G_1(\eta_1)\psi_4 - G_1(\eta_2)\psi_3 - G_1(\eta_3)\psi_2 - G_1(\eta_4)\psi_1 - G_2(\eta_1)\psi_3\\ &- G_2(\eta_2)\psi_1 - G_2'(\eta_1)[\eta_2,\psi_2] - G_2'(\eta_1)[\eta_3,\psi_1] - G_3(\eta_1)\psi_2 - G_3'(\eta_1)[\eta_2,\psi_1] - G_4(\eta_1)\psi_1, \end{aligned}\tag{A.3.38b}$$

where G_2' is in (A.3.24) and, by (A.3.5),

$$G_3'(\eta)[\hat\eta] = \frac12\partial_x^3\eta^2\hat\eta\partial_x + \frac12 G_0\partial_x^2\eta^2\hat\eta G_0 - G_0\hat\eta G_2(\eta) - G_0\eta G_2'(\eta)[\hat\eta] + \partial_x^2\eta\hat\eta G_1(\eta) + \frac12\partial_x^2\eta^2 G_1(\hat\eta). \tag{A.3.39}$$

The term c_4 is obtained by imposing the expression on the right-hand side of (A.3.37) to be orthogonal to the kernel of the operator $\mathcal{B}_0$ in (A.3.7a), obtaining, in view of (1.3.11),

$$c_4 = -\frac{(\mathtt{f}_5, \eta_1) + (\mathtt{g}_5, \psi_1)}{((\psi_1)_x, \eta_1) - ((\eta_1)_x, \psi_1)} = -\mathtt{c_h}\Big((\mathtt{f}_5, \eta_1) + (\mathtt{g}_5, \psi_1)\Big)\,. \tag{A.3.40}$$

By (A.3.38) and (A.3.1) we find that c_4 in (A.3.40) has the explicit expression in (A.3.2e).

Remark A.3.4. Expansion (A.3.1)-(A.3.2) coincides with that in [41, formulae (12)-(14)], provided one rescales properly their amplitude $\varepsilon_{\rm Fen} = \epsilon + f(\mathtt{h})\epsilon^3$ with a suitable $f(\mathtt{h})$, translates their bottom to $d := \mathtt{h} + \epsilon^2\eta_2^{[0]} + \epsilon^4\eta_4^{[0]} + \mathcal{O}(\epsilon^5)$ (in [41] the water surface η has zero average) and removes from the velocity potential a shear term $-\overline{u}x$ (which corresponds to a Galilean reference frame).

A.4 Fourth-order expansion of the operator $\mathcal{L}_{\mu,\epsilon}$

In this section we compute the fourth order expansion of the functions $a_\epsilon(x)$ and $p_\epsilon(x)$ and of the constant $\mathtt{f}_\epsilon$ in (2.1.8).
We begin from the Taylor expansion of the Levi-Civita flattening.

Lemma A.4.1. (Fourth order expansion of $\mathfrak{p}(x)$ and $\mathtt{f}_\epsilon$) *The function* $\mathfrak{p}(x)$ *in* (1.3.4a) *admits the following Taylor expansion*

$$\begin{aligned}\mathfrak{p}(x) = \epsilon\mathtt{c_h}^{-2}\sin(x) + \epsilon^2\mathfrak{p}_2^{[2]}\sin(2x) + \epsilon^3\big(\mathfrak{p}_3^{[1]}\sin(x) + \mathfrak{p}_3^{[3]}\sin(3x)\big)\\ + \epsilon^4\big(\mathfrak{p}_4^{[2]}\sin(2x) + \mathfrak{p}_4^{[4]}\sin(4x)\big) + \mathcal{O}(\epsilon^5)\end{aligned} \tag{A.4.1}$$

with coefficients

$$\mathfrak{p}_2^{[2]} := \frac{3 + 4\mathtt{c_h}^4 + \mathtt{c_h}^8}{8\mathtt{c_h}^8}\,, \tag{A.4.2}$$

$$\mathfrak{p}_3^{[1]} := \frac{4\mathtt{c_h}^{14} + 2\mathtt{c_h}^{12} - 17\mathtt{c_h}^{10} - 14\mathtt{c_h}^8 + 10\mathtt{c_h}^6 + 10\mathtt{c_h}^4 - 15\mathtt{c_h}^2 - 12}{16\mathtt{c_h}^{10}(\mathtt{c_h}^2+1)}\,, \tag{A.4.3}$$

$$\mathfrak{p}_3^{[3]} := \frac{9 + 41\mathtt{c_h}^4 + 43\mathtt{c_h}^8 + 3\mathtt{c_h}^{12}}{64\mathtt{c_h}^{14}}\,, \tag{A.4.4}$$

$$\begin{aligned}\mathfrak{p}_4^{[2]} := -\frac{1}{256\mathtt{c_h}^{20}(\mathtt{c_h}^2+1)}\big(8\mathtt{c_h}^{24} - 57\mathtt{c_h}^{22} - 37\mathtt{c_h}^{20} + 199\mathtt{c_h}^{18} + 175\mathtt{c_h}^{16} + 238\mathtt{c_h}^{14}\\ + 190\mathtt{c_h}^{12} - 130\mathtt{c_h}^{10} - 178\mathtt{c_h}^8 + 171\mathtt{c_h}^6 + 135\mathtt{c_h}^4 + 27\mathtt{c_h}^2 + 27\big)\,,\end{aligned} \tag{A.4.5}$$

$$\mathfrak{p}_4^{[4]} := \frac{\mathtt{c_h}^{24} + 44\mathtt{c_h}^{20} + 557\mathtt{c_h}^{16} + 2528\mathtt{c_h}^{12} + 3595\mathtt{c_h}^8 + 1332\mathtt{c_h}^4 + 135}{512\mathtt{c_h}^{20}(\mathtt{c_h}^4+5)}\,. \tag{A.4.6}$$

The real constant $\mathtt{f}_\epsilon$ *in* (1.3.7a) *has the Taylor expansion*

$$\mathtt{f}_\epsilon = \epsilon^2 \mathtt{f}_2 + \epsilon^4 \mathtt{f}_4 + \mathcal{O}(\epsilon^5) \tag{A.4.7}$$

with coefficients

$$\begin{aligned} \mathtt{f}_2 &:= \frac{\mathtt{c}_\mathtt{h}^4 - 3}{4\mathtt{c}_\mathtt{h}^2}\,, \\ \mathtt{f}_4 &:= \frac{1}{64\mathtt{c}_\mathtt{h}^{14}(\mathtt{c}_\mathtt{h}^2+1)}\big(-4\mathtt{c}_\mathtt{h}^{22} - 8\mathtt{c}_\mathtt{h}^{20} + 5\mathtt{c}_\mathtt{h}^{18} + 23\mathtt{c}_\mathtt{h}^{16} + 40\mathtt{c}_\mathtt{h}^{14} + 22\mathtt{c}_\mathtt{h}^{12} - 78\mathtt{c}_\mathtt{h}^{10} \\ &\qquad - 72\mathtt{c}_\mathtt{h}^{8} + 72\mathtt{c}_\mathtt{h}^{6} + 54\mathtt{c}_\mathtt{h}^{4} - 27\mathtt{c}_\mathtt{h}^{2} - 27\big)\,. \end{aligned} \tag{A.4.8}$$

Proof. We expand

$$\begin{aligned} \mathfrak{p}(x) &= \epsilon\mathfrak{p}_1(x) + \epsilon^2\mathfrak{p}_2(x) + \epsilon^3\mathfrak{p}_3(x) + \epsilon^4\mathfrak{p}_4(x) + \mathcal{O}(\epsilon^5)\,, \\ \mathtt{f}_\epsilon &= \epsilon^2\mathtt{f}_2 + \epsilon^3\mathtt{f}_3 + \epsilon^4\mathtt{f}_4 + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{A.4.9}$$

Let us denote derivatives w.r.t x with a prime $'$. We first note that the constant $\mathtt{f}_\epsilon = \mathcal{O}(\epsilon^2)$ because $\eta_1(x) = \cos(x)$ has zero average. Then, by (1.3.4a),

$$\mathfrak{p}(x) = \frac{\mathcal{H}}{\tanh(\mathtt{h}|D|)}\big[\epsilon\eta_1 + \epsilon^2(\eta_2 + \eta_1'\mathfrak{p}_1) + \mathcal{O}(\epsilon^3)\big]\,,$$

and, using that

$$\mathcal{H}\cos(kx) = \sin(kx)\,, \quad \forall k \in \mathbb{N}\,,$$

we get

$$\mathfrak{p}_1(x) = \frac{\mathcal{H}}{\tanh(\mathtt{h}|D|)}\cos(x) = \mathtt{c}_\mathtt{h}^{-2}\sin(x)\,, \tag{A.4.10}$$

$$\mathfrak{p}_2(x) = \frac{\mathcal{H}}{\tanh(\mathtt{h}|D|)}(\eta_1'\mathfrak{p}_1 + \eta_2) = \frac{(1+\mathtt{c}_\mathtt{h}^4)(\mathtt{c}_\mathtt{h}^4+3)}{8\mathtt{c}_\mathtt{h}^8}\sin(2x)\,. \tag{A.4.11}$$

As a consequence, by (1.3.4a),

$$\mathtt{f}_\epsilon = \frac{\epsilon^2}{2\pi}\int_\mathbb{T}(\eta_2 + \eta_1'\mathfrak{p}_1)\mathrm{d}x + \mathcal{O}(\epsilon^3) = \epsilon^2(\eta_2^{[0]} - \tfrac12\mathtt{c}_\mathtt{h}^{-2}) + \mathcal{O}(\epsilon^3) = \epsilon^2\frac{\mathtt{c}_\mathtt{h}^4 - 3}{4\mathtt{c}_\mathtt{h}^2} + \mathcal{O}(\epsilon^3)\,. \tag{A.4.12}$$

By (1.3.4a), (A.3.1) and (A.4.9), we get

$$\mathtt{f}_3 = \frac{1}{2\pi}\int_\mathbb{T}\big(\eta_3(x) + \eta_2'(x)\mathfrak{p}_1(x) + \eta_1'(x)\mathfrak{p}_2(x) + \frac12\eta_1''(x)\mathfrak{p}_1^2(x)\big)\mathrm{d}x = 0$$

as stated in the expansion (A.4.7). In view of (1.3.4a) and (A.4.12) we have

$$\begin{aligned}\mathfrak{p}_3(x) = -\mathrm{i}\,\frac{\mathrm{sgn}(D)}{\tanh(\mathtt{h}|D|)}\Big(\eta_3(x) + \eta_2'(x)\mathfrak{p}_1(x) + \eta_1'(x)\mathfrak{p}_2(x) + \frac{1}{2}\eta_1''(x)\mathfrak{p}_1^2(x)\Big)\\ + \mathtt{f}_2\frac{\partial_x}{\tanh^2(\mathtt{h}|D|)}(1-\tanh^2(\mathtt{h}|D|))\eta_1(x)\,.\end{aligned} \tag{A.4.13}$$

In view of (A.4.13), (A.3.1) and (A.4.1)-(A.4.2), we have

$$\begin{aligned}\mathfrak{p}_3(x) = \frac{16\eta_3^{[1]}\mathtt{c}_\mathtt{h}^8 - 16\eta_2^{[2]}\mathtt{c}_\mathtt{h}^6 - 3 - 6\mathtt{c}_\mathtt{h}^4 - \mathtt{c}_\mathtt{h}^8 - 16\mathtt{f}_2\mathtt{c}_\mathtt{h}^6 + 16\mathtt{f}_2\mathtt{c}_\mathtt{h}^{10}}{16\mathtt{c}_\mathtt{h}^{10}}\sin(x)\\ + \frac{(1+3\mathtt{c}_\mathtt{h}^4)(8\eta_3^{[3]}\mathtt{c}_\mathtt{h}^4 + 8\eta_2^{[2]}\mathtt{c}_\mathtt{h}^2 + 4\mathtt{c}_\mathtt{h}^4\mathfrak{p}_2^{[2]} + 1)}{8\mathtt{c}_\mathtt{h}^6(3+\mathtt{c}_\mathtt{h}^4)}\sin(3x)\,,\end{aligned} \tag{A.4.14}$$

which, by (A.3.2a)-(A.3.2c), is (A.4.4). By (1.3.4a), (A.3.1), (A.4.2) and (A.4.4), we have

$$\begin{aligned}\mathtt{f}_4 &= \frac{1}{2\pi}\int_{\mathbb{T}}\big(\eta_4(x) + \eta_3'(x)\mathfrak{p}_1(x) + \eta_2'(x)\mathfrak{p}_2(x) + \eta_1'(x)\mathfrak{p}_3(x)\\ &\quad + \frac{1}{2}\eta_2''(x)\mathfrak{p}_1^2(x) + \eta_1''(x)\mathfrak{p}_2(x)\mathfrak{p}_1(x) + \frac{1}{6}\eta_1'''(x)\mathfrak{p}_1^3(x)\big)\mathrm{d}x\\ &= \eta_4^{[0]} - \frac{\eta_3^{[1]}}{2\mathtt{c}_\mathtt{h}^2} - \mathfrak{p}_2^{[2]}\eta_2^{[2]} - \frac{1}{2}\mathfrak{p}_3^{[1]} + \frac{\eta_2^{[2]}}{2\mathtt{c}_\mathtt{h}^4} - \frac{\mathfrak{p}_2^{[2]}}{4\mathtt{c}_\mathtt{h}^2} + \frac{1}{16\mathtt{c}_\mathtt{h}^6}\,,\end{aligned} \tag{A.4.15}$$

which gives the fourth-order coefficient in (A.4.8). Finally, by (1.3.4a) and (A.4.8),

$$\begin{aligned}\mathfrak{p}_4(x) &= -\mathrm{i}\,\frac{\mathrm{sgn}(D)}{\tanh(\mathtt{h}|D|)}\big(\eta_4(x) + \eta_3'(x)\mathfrak{p}_1(x) + \eta_2'(x)\mathfrak{p}_2(x) + \eta_1'(x)\mathfrak{p}_3(x)\\ &\quad + \frac{1}{2}\eta_2''(x)\mathfrak{p}_1^2(x) + \eta_1''(x)\mathfrak{p}_2(x)\mathfrak{p}_1(x) + \frac{1}{6}\eta_1'''(x)\mathfrak{p}_1^3(x)\big)\\ &\quad + \mathtt{f}_2\frac{\partial_x}{\tanh^2(\mathtt{h}|D|)}(1-\tanh^2(\mathtt{h}|D|))\big(\eta_2(x) + \eta_1'(x)\mathfrak{p}_1(x)\big)\\ &= \frac{1+\mathtt{c}_\mathtt{h}^4}{2\mathtt{c}_\mathtt{h}^2}\big(\frac{1}{2}\mathfrak{p}_3^{[1]} - \frac{1}{2}\mathfrak{p}_3^{[3]} - \frac{1}{12\mathtt{c}_\mathtt{h}^6} - \frac{\eta_2^{[2]}}{\mathtt{c}_\mathtt{h}^4} + \frac{\eta_3^{[1]}}{2\mathtt{c}_\mathtt{h}^2} - \frac{3\eta_3^{[3]}}{2\mathtt{c}_\mathtt{h}^2} + \eta_4^{[2]}\big)\sin(2x)\\ &\quad + \frac{1+6\mathtt{c}_\mathtt{h}^4+\mathtt{c}_\mathtt{h}^8}{4\mathtt{c}_\mathtt{h}^2(1+\mathtt{c}_\mathtt{h}^4)}\big(\frac{\mathfrak{p}_2^{[2]}}{4\mathtt{c}_\mathtt{h}^2} + \frac{3\eta_3^{[3]}}{2\mathtt{c}_\mathtt{h}^2} + \mathfrak{p}_2^{[2]}\eta_2^{[2]} + \frac{1}{2}\mathfrak{p}_3^{[3]} + \frac{1}{48\mathtt{c}_\mathtt{h}^6} + \frac{\eta_2^{[2]}}{2\mathtt{c}_\mathtt{h}^4} + \eta_4^{[4]}\big)\sin(4x)\\ &\quad - \mathtt{f}_2\frac{(1-\mathtt{c}_\mathtt{h}^4)^2(1+2\mathtt{c}_\mathtt{h}^2\eta_2^{[2]})}{4\mathtt{c}_\mathtt{h}^6}\sin(2x)\,,\end{aligned}$$

which, by (A.3.2a)-(A.3.2c), gives (A.4.1) with the coefficients in (A.4.5)-(A.4.6). □

We now give the fourth-order Taylor expansion of the velocity field $\big(V(x), B(x)\big)$ of the Stokes waves in (2.1.7).

Lemma A.4.2 (Expansion of $B(x)$ and $V(x)$)**.** *The functions $B(x)$ and $V(x)$ in* (2.1.7) *admit the following Taylor expansion*

$$\begin{aligned} B(x) &= \epsilon B_1(x) + \epsilon^2 B_2(x) + \epsilon^3 B_3(x) + \epsilon^4 B_4(x) + \mathcal{O}(\epsilon^5)\,, \\ V(x) &= \epsilon V_1(x) + \epsilon^2 V_2(x) + \epsilon^3 V_3(x) + \epsilon^4 V_4(x) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{A.4.16}$$

where

$$\begin{aligned} &B_1(x) = \mathtt{c_h} \sin(x)\,, \quad B_2(x) = B_2^{[2]} \sin(2x)\,, \quad B_2^{[2]} := \frac{3 - 2\mathtt{c_h}^4}{2\mathtt{c_h}^5}\,, \\ &V_1(x) = \mathtt{c_h}^{-1} \cos(x)\,, \quad V_2(x) = \frac{\mathtt{c_h}}{2} + V_2^{[2]} \cos(2x)\,, \quad V_2^{[2]} := \frac{3 - \mathtt{c_h}^8}{4\mathtt{c_h}^7}\,, \end{aligned} \tag{A.4.17}$$

and

$$\begin{aligned} &B_3(x) = B_3^{[1]} \sin(x) + B_3^{[3]} \sin(3x)\,, \quad B_4(x) = B_4^{[2]} \sin(2x) + B_4^{[4]} \sin(4x)\,, \\ &V_3(x) = V_3^{[1]} \cos(x) + V_3^{[3]} \cos(3x)\,, \quad V_4(x) = V_4^{[0]} + V_4^{[2]} \cos(2x) + V_4^{[4]} \cos(4x)\,, \end{aligned} \tag{A.4.18}$$

with

$$\begin{aligned} &B_3^{[1]} := \frac{6 + 3\mathtt{c_h}^2 - 8\mathtt{c_h}^4 - 8\mathtt{c_h}^6 + 6\mathtt{c_h}^8 + 3\mathtt{c_h}^{10} - 4\mathtt{c_h}^{12} - 2\mathtt{c_h}^{14}}{16\mathtt{c_h}^7(1 + \mathtt{c_h}^2)}\,, \quad B_3^{[3]} := \frac{81 - 99\mathtt{c_h}^4 + 43\mathtt{c_h}^8 - \mathtt{c_h}^{12}}{64\mathtt{c_h}^{11}}\,, \\ &V_3^{[1]} := \frac{2\mathtt{c_h}^{12} - 15\mathtt{c_h}^8 - 12\mathtt{c_h}^6 + 24\mathtt{c_h}^4 + 24\mathtt{c_h}^2 - 3}{16\mathtt{c_h}^7(1 + \mathtt{c_h}^2)}\,, \quad V_3^{[3]} := \frac{21\mathtt{c_h}^{12} - 39\mathtt{c_h}^8 + 15\mathtt{c_h}^4 + 27}{64\mathtt{c_h}^{13}}\,, \end{aligned} \tag{A.4.19a}$$

and

$$\begin{aligned} V_4^{[0]} &:= \frac{-2\mathtt{c_h}^{18} - 6\mathtt{c_h}^{16} + 3\mathtt{c_h}^{14} + 9\mathtt{c_h}^{12} - 33\mathtt{c_h}^6 - 27\mathtt{c_h}^4 + 36\mathtt{c_h}^2 + 36}{32\mathtt{c_h}^{11}(\mathtt{c_h}^2 + 1)}\,, \\ B_4^{[2]} &:= \frac{1}{192\mathtt{c_h}^{17}(\mathtt{c_h}^2 + 1)}\big(- 24\mathtt{c_h}^{22} + 24\mathtt{c_h}^{20} + 354\mathtt{c_h}^{18} + 210\mathtt{c_h}^{16} \\ &\quad - 943\mathtt{c_h}^{14} - 835\mathtt{c_h}^{12} + 927\mathtt{c_h}^{10} + 855\mathtt{c_h}^8 - 81\mathtt{c_h}^6 + 27\mathtt{c_h}^4 - 81\mathtt{c_h}^2 - 81\big)\,, \\ V_4^{[2]} &:= \frac{1}{384\mathtt{c_h}^{19}(\mathtt{c_h}^2 + 1)}\big(12\mathtt{c_h}^{26} + 36\mathtt{c_h}^{24} - 9\mathtt{c_h}^{22} - 45\mathtt{c_h}^{20} + 357\mathtt{c_h}^{18} + 285\mathtt{c_h}^{16} \\ &\quad - 1060\mathtt{c_h}^{14} - 988\mathtt{c_h}^{12} + 1584\mathtt{c_h}^{10} + 1584\mathtt{c_h}^8 - 243\mathtt{c_h}^6 - 135\mathtt{c_h}^4 - 81\mathtt{c_h}^2 - 81\big)\,, \\ B_4^{[4]} &:= \frac{6\mathtt{c_h}^{20} - 47\mathtt{c_h}^{16} - 100\mathtt{c_h}^{12} + 522\mathtt{c_h}^8 - 594\mathtt{c_h}^4 + 405}{96\mathtt{c_h}^{17}(\mathtt{c_h}^4 + 5)}\,, \\ V_4^{[4]} &:= \frac{9\mathtt{c_h}^{24} - 96\mathtt{c_h}^{20} - 377\mathtt{c_h}^{16} + 1484\mathtt{c_h}^{12} - 1413\mathtt{c_h}^8 + 756\mathtt{c_h}^4 + 405}{384\mathtt{c_h}^{19}(\mathtt{c_h}^4 + 5)}\,. \end{aligned} \tag{A.4.19b}$$

Proof. A direct computation shows that (A.4.16) and (A.4.17) hold.
On the other hand, in view of (2.1.7) and (A.3.1), (A.3.2), the third order terms are

$$\begin{aligned} B_3(x) &= -\mathtt{c_h}\eta_3'(x) + \mathtt{c_h}(\eta_1'(x))^3 + \psi_1'(x)\eta_2'(x) + (\psi_2'(x) - c_2)\eta_1'(x) \\ &= \big(\mathtt{c_h}\eta_3^{[1]} - \frac{3}{4}\mathtt{c_h} - \frac{1}{\mathtt{c_h}}\eta_2^{[2]} + \psi_2^{[2]} + c_2\big)\sin(x) + \big(3\mathtt{c_h}\eta_3^{[3]} + \frac{1}{4}\mathtt{c_h} - \frac{1}{\mathtt{c_h}}\eta_2^{[2]} - \psi_2^{[2]}\big)\sin(3x) \end{aligned} \tag{A.4.20a}$$

and

$$\begin{aligned} V_3(x) &= \psi_3'(x) - B_1(x)\eta_2'(x) - B_2(x)\eta_1'(x) \\ &= \big(\psi_3^{[1]} + \mathtt{c_h}\eta_2^{[2]} + \frac{1}{2}B_2^{[2]}\big)\cos(x) + \big(3\psi_3^{[3]} - \mathtt{c_h}\eta_2^{[2]} - \frac{1}{2}B_2^{[2]}\big)\cos(3x)\,. \end{aligned} \tag{A.4.20b}$$

The fourth order terms are given by

$$\begin{aligned} B_4(x) &= \psi_3'(x)\eta_1'(x) + (\psi_2'(x) - c_2 + 3\mathtt{c_h}(\eta_1'(x))^2)\eta_2'(x) + \psi_1'(x)\eta_3'(x) - \psi_1'(x)(\eta_1'(x))^3 - \mathtt{c_h}\eta_4'(x)\,, \\ &= \big(\frac{3}{2}\psi_3^{[3]} - \frac{1}{2}\psi_3^{[1]} + 2c_2\eta_2^{[2]} - 3\mathtt{c_h}\eta_2^{[2]} - \frac{1}{2\mathtt{c_h}}\eta_3^{[1]} - \frac{3}{2\mathtt{c_h}}\eta_3^{[3]} + \frac{1}{4\mathtt{c_h}} + 2\mathtt{c_h}\eta_4^{[2]}\big)\sin(2x) \\ &\quad \big(-\frac{3}{2}\psi_3^{[3]} - 2\eta_2^{[2]}\psi_2^{[2]} + \frac{3}{2}\mathtt{c_h}\eta_2^{[2]} - \frac{3}{2\mathtt{c_h}}\eta_3^{[3]} - \frac{1}{8\mathtt{c_h}} + 4\mathtt{c_h}\eta_4^{[4]}\big)\sin(4x)\,, \\ V_4(x) &= \psi_4'(x) - B_1(x)\eta_3'(x) - B_2(x)\eta_2'(x) - B_3(x)\eta_1'(x) \\ &= \frac{1}{2}\mathtt{c_h}\eta_3^{[1]} + B_2^{[2]}\eta_2^{[2]} + \frac{1}{2}B_3^{[1]} + \big(4\psi_4^{[4]} - \frac{3}{2}\mathtt{c_h}\eta_3^{[3]} - B_2^{[2]}\eta_2^{[2]} - \frac{1}{2}B_3^{[3]}\big)\cos(4x) \\ &\quad + \big(2\psi_4^{[2]} - \frac{1}{2}\mathtt{c_h}\eta_3^{[1]} + \frac{3}{2}\mathtt{c_h}\eta_3^{[3]} - \frac{1}{2}B_3^{[1]} + \frac{1}{2}B_3^{[3]}\big)\cos(2x)\,. \end{aligned} \tag{A.4.20c}$$

From (A.4.20) we obtain (A.4.18) with the coefficients in (A.4.19). □

Remark A.4.3. The expansion of the functions V and B in (A.4.16) coincide respectively with the horizontal and vertical derivative of the velocity potential Φ in [41, formula (12)] after the rescaling procedure outlined in Remark A.3.4.

We now provide the fourth order expansion of the functions $p_\epsilon(x)$ and $a_\epsilon(x)$ in (1.3.8).

Proposition A.4.4. *The functions $p_\epsilon(x)$ and $a_\epsilon(x)$ in* (1.3.8) *have a Taylor expansion*

$$\begin{aligned} p_\epsilon(x) &= \epsilon p_1(x) + \epsilon^2 p_2(x) + \epsilon^3 p_3(x) + \epsilon^4 p_4(x) + \mathcal{O}(\epsilon^5)\,, \\ a_\epsilon(x) &= \epsilon a_1(x) + \epsilon^2 a_2(x) + \epsilon^3 a_3(x) + \epsilon^4 a_4(x) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{A.4.21}$$

with

$$p_1(x) = p_1^{[1]}\cos(x)\,, \quad p_1^{[1]} := -2\mathtt{c_h}^{-1}\,, \tag{A.4.22a}$$

$$p_2(x) = p_2^{[0]} + p_2^{[2]}\cos(2x)\,, \quad p_2^{[0]} := \frac{9+12\mathsf{c}_\mathsf{h}^4+5\mathsf{c}_\mathsf{h}^8-2\mathsf{c}_\mathsf{h}^{12}}{16\mathsf{c}_\mathsf{h}^7}\,, \quad p_2^{[2]} := -\frac{3+\mathsf{c}_\mathsf{h}^4}{2\mathsf{c}_\mathsf{h}^7}\,,$$

$$p_3(x) = p_3^{[1]}\cos(x) + p_3^{[3]}\cos(3x)\,, \quad p_3^{[3]} := -\frac{\mathsf{c}_\mathsf{h}^{12}+17\mathsf{c}_\mathsf{h}^8+51\mathsf{c}_\mathsf{h}^4+27}{32\mathsf{c}_\mathsf{h}^{13}}\,, \tag{A.4.22b}$$

$$p_3^{[1]} := \frac{-2\mathsf{c}_\mathsf{h}^{14}+14\mathsf{c}_\mathsf{h}^{10}+11\mathsf{c}_\mathsf{h}^8-10\mathsf{c}_\mathsf{h}^6-10\mathsf{c}_\mathsf{h}^4+24\mathsf{c}_\mathsf{h}^2+21}{8\mathsf{c}_\mathsf{h}^9(\mathsf{c}_\mathsf{h}^2+1)}\,,$$

$$p_4(x) = p_4^{[0]} + p_4^{[2]}\cos(2x) + p_4^{[4]}\cos(4x)\,, \tag{A.4.22c}$$

$$p_4^{[0]} := \frac{1}{1024\mathsf{c}_\mathsf{h}^{19}(\mathsf{c}_\mathsf{h}^2+1)}\big(56\mathsf{c}_\mathsf{h}^{30}+88\mathsf{c}_\mathsf{h}^{28}-208\mathsf{c}_\mathsf{h}^{26}-336\mathsf{c}_\mathsf{h}^{24}+441\mathsf{c}_\mathsf{h}^{22}+369\mathsf{c}_\mathsf{h}^{20}-995\mathsf{c}_\mathsf{h}^{18}$$
$$-899\mathsf{c}_\mathsf{h}^{16}-630\mathsf{c}_\mathsf{h}^{14}-294\mathsf{c}_\mathsf{h}^{12}+1026\mathsf{c}_\mathsf{h}^{10}+1314\mathsf{c}_\mathsf{h}^8-27\mathsf{c}_\mathsf{h}^6+189\mathsf{c}_\mathsf{h}^4+81\mathsf{c}_\mathsf{h}^2+81\big)\,,$$

$$p_4^{[2]} := \frac{1}{64\mathsf{c}_\mathsf{h}^{19}(\mathsf{c}_\mathsf{h}^2+1)}\big(-12\mathsf{c}_\mathsf{h}^{22}-4\mathsf{c}_\mathsf{h}^{20}-19\mathsf{c}_\mathsf{h}^{18}-7\mathsf{c}_\mathsf{h}^{16}+350\mathsf{c}_\mathsf{h}^{14}$$
$$+314\mathsf{c}_\mathsf{h}^{12}-256\mathsf{c}_\mathsf{h}^{10}-268\mathsf{c}_\mathsf{h}^8+198\mathsf{c}_\mathsf{h}^6+162\mathsf{c}_\mathsf{h}^4+27\mathsf{c}_\mathsf{h}^2+27\big)\,,$$

$$p_4^{[4]} := \frac{-\mathsf{c}_\mathsf{h}^{20}-39\mathsf{c}_\mathsf{h}^{16}-366\mathsf{c}_\mathsf{h}^{12}-850\mathsf{c}_\mathsf{h}^8-657\mathsf{c}_\mathsf{h}^4-135}{64\mathsf{c}_\mathsf{h}^{19}(\mathsf{c}_\mathsf{h}^4+5)}\,,$$

and

$$a_1(x) = a_1^{[1]}\cos(x)\,, \quad a_1^{[1]} := -(\mathsf{c}_\mathsf{h}^2+\mathsf{c}_\mathsf{h}^{-2})\,,$$
$$a_2(x) = a_2^{[0]} + a_2^{[2]}\cos(2x)\,, \quad a_2^{[0]} := \frac{3}{2}+\frac{1}{2\mathsf{c}_\mathsf{h}^4}\,, \quad a_2^{[2]} := \frac{9\mathsf{c}_\mathsf{h}^8-14\mathsf{c}_\mathsf{h}^4-3}{4\mathsf{c}_\mathsf{h}^8}\,, \tag{A.4.23a}$$

$$a_3(x) = a_3^{[1]}\cos(x) + a_3^{[3]}\cos(3x)\,, \quad a_3^{[3]} := \frac{-\mathsf{c}_\mathsf{h}^{16}-98\mathsf{c}_\mathsf{h}^{12}+252\mathsf{c}_\mathsf{h}^8-318\mathsf{c}_\mathsf{h}^4-27}{64\mathsf{c}_\mathsf{h}^{14}}\,,$$

$$a_3^{[1]} := \frac{4\mathsf{c}_\mathsf{h}^{18}+6\mathsf{c}_\mathsf{h}^{16}-11\mathsf{c}_\mathsf{h}^{14}-12\mathsf{c}_\mathsf{h}^{12}-45\mathsf{c}_\mathsf{h}^{10}-48\mathsf{c}_\mathsf{h}^8+93\mathsf{c}_\mathsf{h}^6+90\mathsf{c}_\mathsf{h}^4+27\mathsf{c}_\mathsf{h}^2+24}{16\mathsf{c}_\mathsf{h}^{10}(\mathsf{c}_\mathsf{h}^2+1)}\,,$$

(A.4.23b)

$$a_4(x) = a_4^{[0]} + a_4^{[2]}\cos(2x) + a_4^{[4]}\cos(4x)\,, \tag{A.4.23c}$$

$$a_4^{[0]} := \frac{-12\mathsf{c}_\mathsf{h}^{20}-31\mathsf{c}_\mathsf{h}^{18}-17\mathsf{c}_\mathsf{h}^{16}+40\mathsf{c}_\mathsf{h}^{14}+46\mathsf{c}_\mathsf{h}^{12}-150\mathsf{c}_\mathsf{h}^{10}-132\mathsf{c}_\mathsf{h}^8+84\mathsf{c}_\mathsf{h}^6+90\mathsf{c}_\mathsf{h}^4+9\mathsf{c}_\mathsf{h}^2+9}{32\mathsf{c}_\mathsf{h}^{16}(\mathsf{c}_\mathsf{h}^2+1)}\,,$$

$$a_4^{[2]} := \frac{1}{128\mathsf{c}_\mathsf{h}^{20}(\mathsf{c}_\mathsf{h}^2+1)}\big(-72\mathsf{c}_\mathsf{h}^{24}-431\mathsf{c}_\mathsf{h}^{22}-211\mathsf{c}_\mathsf{h}^{20}+1767\mathsf{c}_\mathsf{h}^{18}$$
$$+1623\mathsf{c}_\mathsf{h}^{16}-2142\mathsf{c}_\mathsf{h}^{14}-2070\mathsf{c}_\mathsf{h}^{12}+1022\mathsf{c}_\mathsf{h}^{10}+854\mathsf{c}_\mathsf{h}^8+333\mathsf{c}_\mathsf{h}^6+297\mathsf{c}_\mathsf{h}^4+27\mathsf{c}_\mathsf{h}^2+27\big)\,,$$

$$a_4^{[4]} := \frac{9\mathsf{c}_\mathsf{h}^{24}+238\mathsf{c}_\mathsf{h}^{20}-233\mathsf{c}_\mathsf{h}^{16}-1676\mathsf{c}_\mathsf{h}^{12}+743\mathsf{c}_\mathsf{h}^8-3042\mathsf{c}_\mathsf{h}^4-135}{128\mathsf{c}_\mathsf{h}^{20}(\mathsf{c}_\mathsf{h}^4+5)}\,.$$

Proof. The first two jets $p_1(x), p_2(x)$ in (A.4.22a) and $a_1(x), a_2(x)$ in (A.4.23a) of the expansion (A.4.21) come by direct computation. Let us explicitly show the third order terms. In view of (1.3.8), (A.4.16) and (A.4.1), we have

$$\begin{aligned} p_3(x) &= \mathtt{c_h}\big(- \mathfrak{p}_3'(x) + 2\mathfrak{p}_1'(x)\mathfrak{p}_2'(x) - (\mathfrak{p}_1'(x))^3\big) + V_1(x)\big(\mathfrak{p}_2'(x) - (\mathfrak{p}_1'(x))^2\big) \\ &\quad - \big(c_2 - V_2(x) - V_1'(x)\mathfrak{p}_1(x)\big)\mathfrak{p}_1'(x) \\ &\quad - V_3(x) - V_2'(x)\mathfrak{p}_1(x) - V_1'(x)\mathfrak{p}_2(x) - \frac{1}{2}V_1''(x)\mathfrak{p}_1^2(x) \\ &= \Big(- \mathtt{c_h}\mathfrak{p}_3^{[1]} + \frac{7}{2\mathtt{c_h}}\mathfrak{p}_2^{[2]} - \frac{13}{8}\mathtt{c_h}^{-5} - \frac{c_2}{\mathtt{c_h}^2} + \frac{1}{2\mathtt{c_h}} + \frac{3V_2^{[2]}}{2\mathtt{c_h}^2} - V_3^{[1]}\Big)\cos(x) \\ &\quad + \Big(- 3\mathtt{c_h}\mathfrak{p}_3^{[3]} + \frac{5}{2\mathtt{c_h}}\mathfrak{p}_2^{[2]} - \frac{3}{8\mathtt{c_h}^5} - \frac{V_2^{[2]}}{2\mathtt{c_h}^2} - V_3^{[3]}\Big)\cos(3x)\,, \end{aligned} \tag{A.4.24a}$$

$$\begin{aligned} a_3(x) &= -\mathfrak{p}_3'(x) + 2\mathfrak{p}_1'(x)\mathfrak{p}_2'(x) - (\mathfrak{p}_1'(x))^3 \\ &\quad - \mathtt{c_h}\big(B_3'(x) + B_2''(x)\mathfrak{p}_1(x) + B_1''(x)\mathfrak{p}_2(x) + \frac{1}{2}B_1'''(x)\mathfrak{p}_1^2(x)\big) \\ &\quad - p_1(x)\big(B_2'(x) + B_1''(x)\mathfrak{p}_1(x)\big) - p_2(x)B_1'(x)\,, \\ &= \Big(- \mathfrak{p}_3^{[1]} + 2\mathtt{c_h}^{-2}\mathfrak{p}_2^{[2]} - \frac{3}{4}\mathtt{c_h}^{-6} - \mathtt{c_h}B_3^{[1]} + 2\mathtt{c_h}^{-1}B_2^{[2]} + \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_2^{[2]} \\ &\quad + \frac{1}{8}\mathtt{c_h}^{-2} - p_1^{[1]}B_2^{[2]} + \frac{1}{4}\mathtt{c_h}^{-1}p_1^{[1]} - \mathtt{c_h}p_2^{[0]} - \frac{1}{2}\mathtt{c_h}p_2^{[2]}\Big)\cos(x) \\ &\quad + \Big(- 3\mathfrak{p}_3^{[3]} + 2\mathtt{c_h}^{-2}\mathfrak{p}_2^{[2]} - \frac{1}{4}\mathtt{c_h}^{-6} - 3\mathtt{c_h}B_3^{[3]} - 2\mathtt{c_h}^{-1}B_2^{[2]} \\ &\quad - \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_2^{[2]} - \frac{1}{8}\mathtt{c_h}^{-2} - p_1^{[1]}B_2^{[2]} - \frac{1}{4}\mathtt{c_h}^{-1}p_1^{[1]} - \frac{1}{2}\mathtt{c_h}p_2^{[2]}\Big)\cos(3x)\,, \end{aligned} \tag{A.4.24b}$$

and

$$\begin{aligned} p_4(x) &= c_4 - V_4(x) - V_3'(x)\mathfrak{p}_1(x) - V_2'(x)\mathfrak{p}_2(x) - V_1'(x)\mathfrak{p}_3(x) \\ &\quad - \frac{1}{2}V_2''(x)\mathfrak{p}_1^2(x) - V_1''(x)\mathfrak{p}_2(x)\mathfrak{p}_1(x) - \frac{1}{6}V_1'''(x)\mathfrak{p}_1^3(x) \\ &\quad + \mathfrak{p}_1'(x)\big(V_3(x) + V_2'(x)\mathfrak{p}_1(x) + V_1'(x)\mathfrak{p}_2(x) + \frac{1}{2}V_1''(x)\mathfrak{p}_1^2(x)\big) \\ &\quad + \big((\mathfrak{p}_1'(x))^2 - \mathfrak{p}_2'(x)\big)\big(c_2 - V_2(x) - V_1'(x)\mathfrak{p}_1(x)\big) \\ &\quad + V_1(x)\big(\mathfrak{p}_3'(x) - 2\mathfrak{p}_2'(x)\mathfrak{p}_1'(x) + (\mathfrak{p}_1'(x))^3\big) \\ &\quad + \mathtt{c_h}\big(- \mathfrak{p}_4'(x) + 2\mathfrak{p}_3'(x)\mathfrak{p}_1'(x) + (\mathfrak{p}_2'(x))^2 - 3\mathfrak{p}_2'(x)(\mathfrak{p}_1'(x))^2 + (\mathfrak{p}_1'(x))^4\big)\,, \\ &= \frac{3}{4}\mathtt{c_h}^{-7} + \frac{1}{2}c_2\mathtt{c_h}^{-4} - \frac{5}{4}\mathtt{c_h}^{-4}V_2^{[2]} - 2\mathtt{c_h}^{-3}\mathfrak{p}_2^{[2]} - \frac{1}{4}\mathtt{c_h}^{-3} + \mathtt{c_h}^{-2}V_3^{[1]} + 2\mathtt{c_h}(\mathfrak{p}_2^{[2]})^2 \\ &\quad + 2\mathtt{c_h}^{-1}\mathfrak{p}_3^{[1]} + 2\mathfrak{p}_2^{[2]}V_2^{[2]} + c_4 - V_4^{[0]} + \Big(\frac{13}{12}\mathtt{c_h}^{-7} + \frac{1}{2}\mathtt{c_h}^{-4}c_2 + \frac{1}{2}\mathtt{c_h}^{-4}V_2^{[2]} - 6\mathtt{c_h}^{-3}\mathfrak{p}_2^{[2]} \end{aligned} \tag{A.4.25a}$$

$$- \frac{1}{4}\mathtt{c_h}^{-3} + 2\mathtt{c_h}^{-2}V_3^{[3]} + \mathtt{c_h}\mathfrak{p}_2^{[2]} + \mathtt{c_h}^{-1}\mathfrak{p}_3^{[1]} + 5\mathtt{c_h}^{-1}\mathfrak{p}_3^{[3]} - 2\mathtt{c_h}\mathfrak{p}_4^{[2]} - 2\mathfrak{p}_2^{[2]}c_2 - V_4^{[2]}\big)\cos(2x)$$
$$+ \big(\frac{1}{6}\mathtt{c_h}^{-7} - \frac{1}{4}\mathtt{c_h}^{-4}V_2^{[2]} - 2\mathtt{c_h}^{-3}\mathfrak{p}_2^{[2]} - \mathtt{c_h}^{-2}V_3^{[3]} + 2\mathtt{c_h}(\mathfrak{p}_2^{[2]})^2 + 4\mathtt{c_h}^{-1}\mathfrak{p}_3^{[3]} - 4\mathtt{c_h}\mathfrak{p}_4^{[4]} - V_4^{[4]}\big)\cos(4x)$$

$$\begin{aligned} a_4(x) = & -\mathfrak{p}_4'(x) + (\mathfrak{p}_2'(x))^2 + 2\mathfrak{p}_1'(x)\mathfrak{p}_3'(x) - 3(\mathfrak{p}_1'(x))^2\mathfrak{p}_2'(x) + (\mathfrak{p}_1'(x))^4 \\ & - \mathtt{c_h}\big(B_4'(x) + B_3''(x)\mathfrak{p}_1(x) + B_2''(x)\mathfrak{p}_2(x) + B_1''(x)\mathfrak{p}_3(x) \\ & + \frac{1}{2}B_2'''(x)\mathfrak{p}_1^2(x) + B_1'''(x)\mathfrak{p}_1(x)\mathfrak{p}_2(x) + \frac{1}{6}B_1^{IV}(x)\mathfrak{p}_1^3(x)\big) \\ & - p_1(x)\big(B_3'(x) + B_2''(x)\mathfrak{p}_1(x) + B_1''(x)\mathfrak{p}_2(x) + \frac{1}{2}B_1'''(x)\mathfrak{p}_1^2(x)\big) \\ & - p_2(x)\big(B_2'(x) + B_1''(x)\mathfrak{p}_1(x)\big) - p_3(x)B_1'(x) \\ = & \frac{3}{8}\mathtt{c_h}^{-8} - \frac{1}{16}\mathtt{c_h}^{-4} - \frac{3}{2}\mathtt{c_h}^{-4}\mathfrak{p}_2^{[2]} + \frac{1}{16}\mathtt{c_h}^{-3}p_1^{[1]} - \mathtt{c_h}^{-3}B_2^{[2]} + \mathtt{c_h}^{-2}B_2^{[2]}p_1^{[1]} + \mathtt{c_h}^{-2}\mathfrak{p}_3^{[1]} \\ & + \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_3^{[1]} + 2\mathtt{c_h}\mathfrak{p}_2^{[2]}B_2^{[2]} + \frac{1}{4}\mathtt{c_h}\mathfrak{p}_2^{[2]}p_1^{[1]} - \frac{1}{2}\mathtt{c_h}p_3^{[1]} + \frac{1}{2}\mathtt{c_h}^{-1}B_3^{[1]} + \frac{1}{2}\mathtt{c_h}^{-1}p_2^{[0]} - \frac{1}{4}\mathtt{c_h}^{-1}p_2^{[2]} \\ & + 2(\mathfrak{p}_2^{[2]})^2 + \frac{1}{4}\mathfrak{p}_2^{[2]} - B_2^{[2]}p_2^{[2]} - \frac{1}{2}B_3^{[1]}p_1^{[1]} + \big(\frac{1}{2}\mathtt{c_h}^{-8} - 3\mathtt{c_h}^{-4}\mathfrak{p}_2^{[2]} + \frac{1}{12}\mathtt{c_h}^{-4} + 2\mathtt{c_h}^{-3}B_2^{[2]} \\ & - \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_3^{[1]} + \mathtt{c_h}^{-2}\mathfrak{p}_3^{[1]} + \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_3^{[3]} + 3\mathtt{c_h}^{-2}\mathfrak{p}_3^{[3]} - \frac{1}{2}\mathtt{c_h}^{-1}B_3^{[1]} + \frac{9}{2}\mathtt{c_h}^{-1}B_3^{[3]} - 2\mathtt{c_h}B_4^{[2]} - \frac{1}{2}\mathtt{c_h}^{-1}p_2^{[0]} \\ & + \frac{1}{2}\mathtt{c_h}^{-1}p_2^{[2]} - \frac{1}{2}\mathtt{c_h}p_3^{[1]} - \frac{1}{2}\mathtt{c_h}p_3^{[3]} - 2B_2^{[2]}p_2^{[0]} - \frac{1}{2}B_3^{[1]}p_1^{[1]} - \frac{3}{2}B_3^{[3]}p_1^{[1]} - 2\mathfrak{p}_4^{[2]}\big)\cos(2x) \\ & + \big(\frac{1}{8}\mathtt{c_h}^{-8} - \frac{3}{2}\mathtt{c_h}^{-4}\mathfrak{p}_2^{[2]} - \frac{1}{48}\mathtt{c_h}^{-4} - \mathtt{c_h}^{-3}B_2^{[2]} - \frac{1}{16}\mathtt{c_h}^{-3}p_1^{[1]} - \mathtt{c_h}^{-2}B_2^{[2]}p_1^{[1]} - \frac{1}{2}\mathtt{c_h}^2\mathfrak{p}_3^{[3]} \\ & + 3\mathtt{c_h}^{-2}\mathfrak{p}_3^{[3]} - 2\mathtt{c_h}\mathfrak{p}_2^{[2]}B_2^{[2]} - \frac{1}{4}\mathtt{c_h}\mathfrak{p}_2^{[2]}p_1^{[1]} - \frac{9}{2}\mathtt{c_h}^{-1}B_3^{[3]} - 4\mathtt{c_h}B_4^{[4]} - \frac{1}{4}\mathtt{c_h}^{-1}p_2^{[2]} \\ & - \frac{1}{2}\mathtt{c_h}p_3^{[3]} + 2(\mathfrak{p}_2^{[2]})^2 - \frac{1}{4}\mathfrak{p}_2^{[2]} - B_2^{[2]}p_2^{[2]} - \frac{3}{2}B_3^{[3]}p_1^{[1]} - 4\mathfrak{p}_4^{[4]}\big)\cos(4x)\,. \end{aligned} \tag{A.4.25b}$$

The expansions of $p_3(x)$, $a_3(x)$, $p_4(x)$ and $a_4(x)$ in (A.4.22a)-(A.4.23a) descend from (A.4.24a), (A.4.24b), (A.4.25a) and (A.4.25b) respectively, in view of (A.4.6), (A.4.4) (A.4.2), (A.4.17), (A.3.2b), (A.4.20b), (A.4.20a) and (A.4.20c)-(A.4.19b). □

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