



Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Fourier transform of BV functions and applications

Thomas Beretti^a, Luca Gennaioli^{b,*}^a *International School for Advanced Studies (SISSA), Italy*^b *University of Warwick, Department of Mathematics, United Kingdom*

ARTICLE INFO

Article history:

Received 5 October 2025

Accepted 10 September 2026

Available online 29 September 2026

Communicated by Guido De Philippis

MSC:

primary 42B37

secondary 11K38, 26B30, 28A75

Keywords:

Heat kernel

BV functions

Fourier transform

Discrepancy

ABSTRACT

This paper investigates the relation between the Fourier transform of BV (bounded variation) functions and their jump sets. We introduce the notion of L^2 -jump product and obtain a weighted Plancherel identity for BV functions. As a corollary, we get a characterization of sets of finite perimeter in terms of their Fourier transform. Moreover, we sharpen a result of Herz on the set-theoretic derivative of the Fourier transform of characteristic functions of sets. Last, we obtain sharp bounds on the quadratic discrepancy of BV functions, and as a consequence, we generalize the classic estimates of Beck and Montgomery.

© 2026 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

Contents

1. Introduction	2
2. Fourier asymptotics and BV functions	7
3. Fourier transform of the derivative of sets	20
4. Discrepancy of BV functions	22
5. Appendix	28
Funding	29
Data availability	30
References	30

* Corresponding author.

E-mail addresses: thomas.beretti@sissa.it (T. Beretti), luca.gennaioli@warwick.ac.uk (L. Gennaioli).

1. Introduction

This work presents a novel bridge between harmonic analysis and geometric measure theory. The Fourier transform has proved to be an effective tool when studying differential equations. As is well known, it allows linear differential problems to be turned into algebraic ones. Moreover, since Sobolev norms are equivalent to weighted L^2 -norms of the Fourier transform, information on the latter can be exploited for studying the regularity of solutions of PDEs. Another example of such duality is the relation between the Fourier restriction problem and the decay of solutions of some wave problems, such as the Schrödinger equation or the Klein-Gordon equation, and we refer the reader to the seminal paper of Strichartz [28].

In Section 2, we show that, for functions of bounded variation, the problem of estimating the short-time behaviour of the (relative) heat content is dual to determining the limit behaviour of averages of the Fourier transform. Since the former retrieves several quantities of geometric interest and is strictly related to perimeters, the latter will naturally inherit such a property. In general, the problem of estimating the Fourier transform has a long history, and we refer the reader to the classic work of Hlawka [20], Herz [18], and to the study in [11].

In what follows, d stands for a positive integer, and for a positive value R , we write B_R to denote the ball of radius R centered at the origin. The Fourier transform of a function u is either denoted by $\mathcal{F}\{u\}$ or $\hat{u}$. Last, $\mathcal{H}^d$ stands for the d -dimensional Hausdorff measure.

We turn to the main result of Section 2. Before proceeding with its statement, we introduce a special geometric quantity. In particular, the reader may find relevant geometric definitions and notation on BV functions at the start of the same section.

Definition 1.1 (L^2 -jump product). Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then we define the L^2 -jump product of u and v as

$$\mathcal{J}(u, v) = \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} \, d\mathcal{H}^{d-1},$$

with the convention that $\mathcal{J}(u) = \mathcal{J}(u, u)$.

We proceed with stating what we interpret as a weighted Plancherel identity.

Theorem 1.2. Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then it holds

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \overline{\hat{v}}(\xi) \, d\xi = \mathcal{J}(u, v). \quad (1.1)$$

In particular, if $u = v$, we get

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) \, d\xi = \mathcal{J}(u). \quad (1.2)$$

We make a few comments on the latter theorem. By classic properties of the Fourier transform, we can rewrite (1.2) as

$$\lim_{R \rightarrow +\infty} \frac{1}{2R} \int_{B_R} |\mathcal{F}\{Du\}|^2(\xi) \, d\xi = \mathcal{J}(u), \quad (1.3)$$

where Du stands for the distributional gradient of u . As a well-known property, the distributional gradient of a BV function decomposes in an absolutely continuous part, a Cantor part, and a jump part. In (1.3), the jump part alone plays a role in the limit, and we give the following heuristic interpretation. The Fourier transform tells how influential a certain oscillation is, and when looking at increasingly high frequencies, one expects to detect faster oscillations. Since the oscillation frequency is “infinity” at the jump set of a function, then the jump part of the gradient is the only one caught up in the limit.

We also mention that, by considering $d = 1$ in (1.3), we recover Wiener’s lemma for compactly supported measures with vanishing total mass. Indeed, every such measure μ is the gradient of a (bounded and compactly supported) function of bounded variation, whose jumps occur exactly at the atoms of μ .

Last, it is worth comparing Theorem 1.2 with the results in [29], where analogous estimates for real-valued measures are obtained under different hypotheses. Namely, for a non-negative integer k , the author considers measures $\mathcal{H}_{|E}^k$ that are the restriction of the k -dimensional Hausdorff measure to a k -regular set E , and that are locally uniformly k -dimensional, that is, there exists a positive constant c such that

$$\mathcal{H}_{|E}^k(B_r(x)) \leq cr^k \quad \forall x \in \mathbb{R}^d \quad \forall r \in (0, 1].$$

We remark that a k -regular set E , as defined in [16], is a set whose upper and lower k -density with respect to $\mathcal{H}^k$ are jointly equal to 1, and by the theorem of Marstrand [24], it follows that the condition of k being a non-negative integer is necessary. Moreover, the techniques employed by the author, such as maximal operators and covering theorems, differ from ours in nature. We address his question of removing the regularity assumptions in the case of measures that are also distributional gradients, and this is done by employing tools from the theory of BV functions, exploiting the decomposition of the distributional gradient and the good behaviour of its jump part. For completeness, we also mention the preceding study in [4] concerning measures supported on $\mathcal{C}^1$ submanifolds of $\mathbb{R}^d$, where the authors obtain similar estimates.

We also mention that in [6, Theorem 1.1], the authors generalized (1.2) to a more general family of symbols.

Integrating by parts (1.2), we immediately deduce this first corollary to Theorem 1.2.

Corollary 1.3. *Let $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then*

$$\lim_{R \rightarrow +\infty} 2\pi^2 R \int_{B_R^c} |\hat{u}|^2(\xi) \, d\xi = \mathcal{J}(u). \quad (1.4)$$

The latter equality is significant for at least two reasons, the first being that it quantifies the error committed by cutting off the higher frequencies in Plancherel identity. Namely, we obtain

$$\int_{\mathbb{R}^d} |u|^2(x) \, dx = \int_{B_R} |\hat{u}|^2(\xi) \, d\xi + \frac{\mathcal{J}(u)}{2\pi^2 R} + o(R^{-1}) \quad \text{as } R \rightarrow +\infty.$$

Moreover, by letting $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter, and by setting $u = \mathbb{1}_\Omega$ in equation (1.4), we get

$$\lim_{R \rightarrow +\infty} 2\pi^2 R \int_{B_R^c} |\widehat{\mathbb{1}_\Omega}|^2(\xi) \, d\xi = \text{Per}(\Omega),$$

which extends the estimates in [25, Ch. 6] to any dimension and for any set of finite perimeter.

Dealing with characteristic functions of sets, we further obtain this second corollary to Theorem 1.2. Again, we refer to the preliminary part of Section 2 for the notion of reduced boundary.

Corollary 1.4. *Let Ω_1 and Ω_2 be sets of finite measure and perimeter of $\mathbb{R}^d$, then it holds*

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 \widehat{\mathbb{1}_{\Omega_1}}(\xi) \overline{\widehat{\mathbb{1}_{\Omega_2}}}(\xi) \, d\xi = \int_{\partial^* \Omega_1 \cap \partial^* \Omega_2} \nu_{\partial^* \Omega_1} \cdot \nu_{\partial^* \Omega_2} \, d\mathcal{H}^{d-1}. \quad (1.5)$$

In particular, if $\Omega_1 = \Omega_2 = \Omega$, we get

$$\lim_{R \rightarrow +\infty} \frac{2\pi^2}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}_\Omega}|^2(\xi) \, d\xi = \text{Per}(\Omega). \quad (1.6)$$

Moreover, if for a set $\Omega \subset \mathbb{R}^d$ of finite measure it holds

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}_\Omega}|^2(\xi) \, d\xi < +\infty, \quad (1.7)$$

then Ω is of finite perimeter. On the other hand, there exists $\Omega \subset [0, 1]^d$ such that $\text{Per}(\Omega) = +\infty$ and

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}_\Omega}|^2(\xi) \, d\xi = 0. \quad (1.8)$$

The latter criterion thus characterizes sets of finite perimeter in terms of their Fourier transform, without involving distributional calculus and derivatives, while (1.8) shows that the lim sup in (1.7) cannot be replaced by a lim inf. Additionally, (1.6) improves on the estimates in [18, Thm. 2], where an upper bound for convex sets is obtained. For characterizations of sets of finite perimeter through the finiteness of different quadratic functionals, namely variances of linear statistics of determinantal point processes, we refer the reader to [22].

In Section 3, again regarding sets of finite perimeter, we address another question of geometric measure theory and harmonic analysis. Namely, we characterize the set-theoretic derivative of the Fourier transform of sets of finite perimeter. Here, $\mathcal{SM}(\Omega)$ stands for the outer Minkowski content of the set Ω , and we refer the reader to Definition 3.1. The following result improves on [18, Thm. 1], in which the author only considered convex sets.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter, then*

$$\lim_{h \rightarrow 0^+} \frac{\widehat{\mathbb{1}_{\Omega+hB_1}}(\xi) - \widehat{\mathbb{1}_\Omega}(\xi)}{h} = \mathcal{F}\{|D\mathbb{1}_\Omega|\}(\xi) \quad \forall \xi \in \mathbb{R}^d \quad (1.9)$$

if and only if $\text{Per}(\Omega) = \mathcal{SM}(\Omega)$.

The hypotheses under which we obtain (1.9) are the weakest possible, and as an example, any set with Lipschitz boundary satisfies our hypotheses, as the reader may verify in [1]. In addition, our proof appears more straightforward than the one in [18, Thm. 1] since it relies on classic functional analysis techniques and geometric measure theory.

In Section 4, we show how the asymptotic relation in (1.2) is key for improving on a major result in the theory of irregularities of distribution, also known as discrepancy theory. In what follows, we assume the dimension d to be greater than or equal to 2.

For a set $\Omega \subset \mathbb{T}^d$ and for a configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$ (that is, a collection of N not necessarily distinct points counted with multiplicity), the discrepancy of $\mathcal{P}_N$ with respect to Ω refers to the quantity

$$\mathcal{D}(\Omega; \mathcal{P}_N) = \sum_{p \in \mathcal{P}_N} \mathbb{1}_\Omega(p) - N|\Omega|.$$

The field had a pioneering advancement due to a seminal paper of Roth (see [27]), where a new geometric point of view was introduced, leading to a broader employment of harmonic analysis. A later significant development was due to the work of Beck [8] concerning discrepancy over the similarity group, and such a problem turned out to be

strictly related to estimating Fourier transforms. Before proceeding with our contribution, we need to introduce adequate notation.

Consider a real function $u \in L^1(\mathbb{R}^d)$, and let $\tau \in \mathbb{R}^d$ be a translation factor, let $\delta > 0$ be a dilation factor, and let $\rho: \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a rotation; in particular, we identify $\text{SO}(d)$ with the set of all rotations. We define the transformation

$$[\tau, \delta, \rho]u(x) = u(\rho(x - \tau)/\delta).$$

Moreover, we consider the periodization functional $\mathfrak{P}: L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{T}^d)$ defined in the sense that

$$\mathfrak{P}\{u\}(x) = \sum_{n \in \mathbb{Z}^d} u(x + n).$$

Then, for a function $u \in L^1(\mathbb{R}^d)$ and for a configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$, we extend the previous notion of discrepancy by defining

$$\mathcal{D}(u; \mathcal{P}_N) = \sum_{p \in \mathcal{P}_N} \mathfrak{P}\{u\}(p) - N\hat{u}(\mathbf{0}). \quad (1.10)$$

In this setting, we define the quadratic discrepancy over the similarity group as

$$\mathcal{D}_2(u; \mathcal{P}_N) = \int_{\text{SO}(d)} \int_0^1 \int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 \, d\tau \, d\delta \, d\rho, \quad (1.11)$$

where $d\tau$ is the normalized Lebesgue measure on $\mathbb{T}^d$, $d\delta$ is the Lebesgue measure on $[0, 1]$, and $d\rho$ is the Haar measure on $\text{SO}(d)$ normalized so that its total mass equals $\mathcal{H}^{d-1}(\mathbb{S}^{d-1})$.

It is time to state the sharp bounds we obtain, starting from the upper one.

Theorem 1.6. *There exists a positive dimensional constant C_d such that for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it holds*

$$\limsup_{N \rightarrow +\infty} N^{(1-d)/d} \inf_{\mathcal{P}_N \subset \mathbb{T}^d} \mathcal{D}_2(u; \mathcal{P}_N) \leq C_d \mathcal{J}(u).$$

As the following result shows, the latter estimate is the best achievable with respect to the order of N and the dependence on u .

Theorem 1.7. *There exists a positive dimensional constant C_d such that for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with compact support, it holds*

$$\liminf_{N \rightarrow +\infty} N^{(1-d)/d} \inf_{\mathcal{P}_N \subset \mathbb{T}^d} \mathcal{D}_2(u; \mathcal{P}_N) \geq C_d \mathcal{J}(u).$$

In [8], the latter lower bound is obtained for characteristic functions of convex compact sets (with non-empty interior) of $\mathbb{R}^d$, and in the same setting, the study in [7] proved the order $N^{(1-d)/d}$ to be sharp, although a generic set-dependent constant replaces the dependence on the perimeter. In an independent work, Montgomery [25, Ch. 6] showed that the same lower bound holds for every compact set (with non-empty interior) of $\mathbb{R}^2$ with piecewise- $\mathcal{C}^1$ boundary, hence trading convexity for a requirement of regularity. In particular, his work employed Fourier series and a refinement of an argument of Cassels [14] for estimating sums of complex exponentials, and we briefly treat the latter in the last section.

Last, in the appendix, we adapt an argument of [18] and obtain a functional inequality that retrieves a non-sharp version of the celebrated isoperimetric inequality (when specialized to sets).

Acknowledgments. We thank Luca Brandolini, Camillo Brena, Michele Caselli, Leonardo Colzani, Davide Donati, Giacomo Gigante, Antonio Pedro Ramos, Giorgio Stefani, and Giancarlo Travaglini for the valuable feedback and discussions. We also thank the anonymous referee for the suggestions, which improved the manuscript.

2. Fourier asymptotics and BV functions

We shall introduce the main tools from the theory of PDEs, harmonic analysis and geometric measure theory. The interested reader can consult [15] and [21] for a better account on PDEs and harmonic analysis. For what concerns geometric measure theory, we refer the reader to [3] and [23].

First, for a function $u \in L^1(\mathbb{R}^d)$, we define its Fourier transform as

$$\mathcal{F}\{u\}(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^d} u(x) e^{-2\pi i \xi \cdot x} dx,$$

with clear extension to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. This choice of normalization implies the following version of Plancherel identity: for $u, v \in L^2(\mathbb{R}^d)$, it holds

$$\int_{\mathbb{R}^d} u(x) v(x) dx = \int_{\mathbb{R}^d} \hat{u}(\xi) \bar{\hat{v}}(\xi) d\xi,$$

with $\bar{\hat{v}}$ being the complex conjugate of $\hat{v}$.

Now, consider a vector-valued measure $\mu = (\mu_1, \dots, \mu_k)$ of $\mathbb{R}^d$. We define the total variation of μ as the measure $|\mu|$ which, to any Borel set $E \subset \mathbb{R}^d$, assigns the value

$$|\mu|(E) = \sup \left\{ \sum_n \|\mu(E_n)\|_{\mathbb{R}^k} : (E_n)_n \text{ is a Borel partition of } E \right\}.$$

We say that a vector-valued measure μ is finite if

$$|\mu|(\mathbb{R}^d) < +\infty,$$

and we shall denote the vector space of finite measures of $\mathbb{R}^d$ with values in $\mathbb{R}^k$ with $\mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$, with the convention that $\mathcal{M}(\mathbb{R}^d) = \mathcal{M}(\mathbb{R}^d; \mathbb{R}^1)$. Equipping such spaces with the norm

$$\|\mu\|_{\text{TV}} = |\mu|(\mathbb{R}^d)$$

makes them Banach spaces. We also remark that it makes sense to compute the Fourier transform of a measure, and indeed, the Fourier transform

$$\mathcal{F} : \mathcal{M}(\mathbb{R}^d) \rightarrow \mathcal{C}_b(\mathbb{R}^d)$$

is a linear and continuous operator with norm equal to 1, and the same holds component-wise if we consider the operator

$$\mathcal{F} : \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k) \rightarrow \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^k).$$

Now, we proceed with defining the class of functions of bounded variation.

Definition 2.1 (BV functions). We say that $u \in L^1(\mathbb{R}^d)$ is a function of bounded variation if $Du \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$, where Du is the distributional gradient of u . We denote the space of such functions with $\text{BV}(\mathbb{R}^d)$.

For a Borel set E and a Borel measure μ , we shall write $\mu|_E$ to denote the Borel measure such that $\mu|_E(A) = \mu(E \cap A)$ for any Borel set A . Now, we shall recall the decomposition of the gradient of a BV function as in [3, Sec. 3.7]. Indeed, if $u \in \text{BV}(\mathbb{R}^d)$, then Du can be written as

$$Du = \nabla u \mathcal{L}^d + (u^\vee - u^\wedge) \nu_{S_u} \mathcal{H}_{|S_u}^{d-1} + D^c u. \quad (2.1)$$

In the previous decomposition, ∇u is the density of the absolutely continuous part with respect to $\mathcal{L}^d$, and $D^c u$ is the Cantor part of the distributional gradient. The second term, often denoted with $D^j u$, is the jump part of the gradient, and we briefly explain the notation used for it. Here, S_u denotes the approximate discontinuity set of u , while ν_{S_u} stands for its measure-theoretic normal. Namely, we have that $x \in S_u^c$ if there exists $a \in \mathbb{R}$ such that

$$\lim_{\rho \rightarrow 0^+} \int_{B_\rho(x)} |u(y) - a| \, dy = 0, \quad (2.2)$$

and on the other hand, for $x \in S_u$, we have

$$u^\vee(x) = \inf \{t \in [-\infty, +\infty] : \lim_{\rho \rightarrow 0^+} \rho^{-d} |\{u > t\} \cap B_\rho(x)| = 0\},$$

$$u^\wedge(x) = \sup \{t \in [-\infty, +\infty] : \lim_{\rho \rightarrow 0^+} \rho^{-d} |\{u < t\} \cap B_\rho(x)| = 0\}.$$

We shall use the fact that there exists a $\mathcal{H}^{d-1}$ -rectifiable Borel set $J_u \subseteq S_u$, referred to as the jump set of u , such that $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$ and that allows us to rewrite (2.1) as

$$Du = \nabla u \mathcal{L}^d + (u^+ - u^-) \nu_{J_u} \mathcal{H}^{d-1}_{|J_u} + D^c u. \quad (2.3)$$

Here, u^+ and u^- are the one-sided approximate limits of u , defined in such a way that

$$\lim_{\rho \rightarrow 0^+} \int_{B_\rho^+(x)} |u(y) - u^+(x)| dy = \lim_{\rho \rightarrow 0^+} \int_{B_\rho^-(x)} |u(y) - u^-(x)| dy = 0,$$

where, denoting by $\langle \cdot, \cdot \rangle$ the inner product of $\mathbb{R}^d$, we have

$$B_\rho^+(x) = \{y \in B_\rho(x) : \langle y - x, \nu_{J_u} \rangle > 0\},$$

$$B_\rho^-(x) = \{y \in B_\rho(x) : \langle y - x, \nu_{J_u} \rangle < 0\}.$$

We proceed with the definition of sets of finite perimeter.

Definition 2.2 (*Finite perimeter set*). For a Borel set $\Omega \subset \mathbb{R}^d$, we define the perimeter of Ω as

$$\text{Per}(\Omega) = |D\mathbf{1}_\Omega|(\mathbb{R}^d),$$

and we say that Ω is of finite perimeter if $\mathbf{1}_\Omega \in \text{BV}(\mathbb{R}^d)$.

Due to the isoperimetric inequality, if Ω is of finite perimeter, then either Ω or Ω^c is of finite measure. In the sequel, we shall assume that our sets of finite perimeter come with finite measure.

For a set of finite perimeter, it is possible to define the notion of reduced boundary, which identifies the more “regular” points of the boundary. In particular, this coincides with the jump set of the characteristic function of the set itself.

Definition 2.3 (*Reduced boundary*). For a set of finite perimeter $\Omega \subset \mathbb{R}^d$, we define its reduced boundary $\partial^* \Omega$ as the Borel set of all the points $x \in \text{supp}|D\mathbf{1}_\Omega|$ such that the limit

$$\nu_\Omega(x) = \lim_{r \rightarrow 0^+} \frac{D\mathbf{1}_\Omega(B_r(x))}{|D\mathbf{1}_\Omega|(B_r(x))}$$

exists and has norm equal to 1.

Remark 2.4. Observe that $\partial^*\Omega$ is always a subset of the topological boundary, and it may happen that the two do not coincide. Trivially, the reduced boundary of a square consists of its topological boundary except for the 4 points at the corners.

We are now in the position to recall part of the celebrated structural result for sets of finite perimeter, namely De Giorgi theorem (see [3, Thm. 3.59]).

Theorem 2.5 (*De Giorgi*). *Let $\Omega \subset \mathbb{R}^d$ be a set of finite perimeter, then $\partial^*\Omega$ is countably $\mathcal{H}^{d-1}$ -rectifiable and $|D\mathbf{1}_\Omega| = \mathcal{H}^{d-1}_{|\partial^*\Omega|}$.*

Remark 2.6. We do not delve into the definition of rectifiability in the above sense, and again, we refer the reader to [3] or [23]. However, one can think of it as a measure-theoretic notion of being $(d-1)$ -dimensional.

Now, following the notation in [26], for a function $u \in L^1$, we denote by $T(t)u$ the unique solution to the differential problem

$$\begin{cases} \partial_t v = \Delta_x v & \text{in } \mathbb{R}^d \times (0, +\infty) \\ v(x, 0) = u & \text{on } \mathbb{R}^d \end{cases}, \quad (2.4)$$

evaluated at time $t > 0$. Moreover, we recall the explicit representation via convolution

$$(T(t)u)(x) = \int_{\mathbb{R}^d} G_d(x-y, t)u(y) \, dy,$$

where G_d stands for the classic Gaussian heat kernel

$$G_d(z, t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|z|^2}{4t}}. \quad (2.5)$$

Now, we recall two results on the (relative) heat content. First, we state [26, Thm. 4.3].

Theorem 2.7. *Let $u, v \in \text{BV}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then it holds*

$$\lim_{t \rightarrow 0^+} \frac{\sqrt{\pi}}{\sqrt{t}} \langle u - T(t)u, v \rangle = \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} \, d\mathcal{H}^{d-1}. \quad (2.6)$$

Remark 2.8. For our purposes, given $u, v \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we consider the map

$$H(t) = \int_{\mathbb{R}^d} (T(t^2)u)(x)v(x) \, dx. \quad (2.7)$$

By the latter result, we can only deduce that H is (right) differentiable at $t = 0$. In order to prove Theorem 1.2, we actually need

$$\lim_{t \rightarrow 0^+} H'(t) \quad \text{to exist,}$$

and in this case, the limit would clearly coincide with the right-hand term of (2.6). We address such a matter with Proposition 2.10.

Secondly, we state [26, Thm. 3.4]. The following is a sufficient condition for a set to be of finite perimeter in terms of its (relative) heat content, and we use it in the proof of Corollary 1.4.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ be such that either $|\Omega| < +\infty$ or $|\Omega^c| < +\infty$. If*

$$\liminf_{t \rightarrow 0^+} \frac{1}{t} \int_{\Omega^c} T(t^2) \mathbb{1}_\Omega(x) \, dx < +\infty,$$

then Ω has finite perimeter.

We proceed with the proof of the main technical result of the section.

Proposition 2.10. *Let $u, v \in \text{BV}(\mathbb{R}^d)$, with $u \in L^\infty(\mathbb{R}^d)$ or $v \in L^\infty(\mathbb{R}^d)$, then*

$$\lim_{t \rightarrow 0^+} H'(t) = -\frac{\mathcal{J}(u, v)}{\sqrt{\pi}}. \quad (2.8)$$

Proof. Without loss of generality, let us assume $u \in L^\infty(\mathbb{R}^d)$. Moreover, let H be defined as in (2.7). Clearly $H \in \mathcal{C}^1((0, +\infty))$, and by (2.4), for every $t \in (0, +\infty)$, we are allowed to write

$$H'(t) = 2t \int_{\mathbb{R}^d} \Delta_x(T(t^2)u)(x)v(x) \, dx = -2t \int_{\mathbb{R}^d} \nabla_x(T(t^2)u)(x) \, dDv(x). \quad (2.9)$$

We shall now compute the term $\nabla_x(T(t^2)u)(x)$ explicitly. Namely, we have

$$\begin{aligned} \nabla_x(T(t^2)u)(x) &= \int_{\mathbb{R}^d} \nabla_x G_d(x-y, t^2) u(y) \, dy \\ &= -\frac{1}{2t^2} \int_{\mathbb{R}^d} (x-y) \frac{e^{-\frac{|x-y|^2}{4t^2}}}{(4\pi t^2)^{d/2}} u(y) \, dy \\ &= -\frac{1}{(4\pi)^{d/2} 2t} \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x-tz) \, dz. \end{aligned}$$

Therefore, (2.9) becomes

$$H'(t) = \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x - tz) \, dz \, dDv(x). \quad (2.10)$$

Now, let us consider

$$g(x, t) = \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} u(x - tz) \, dz, \quad (2.11)$$

and split the integral in (2.10) as

$$H'(t) = I_1(t) + I_2(t),$$

with

$$I_1(t) := \frac{1}{(4\pi)^{d/2}} \int_{J_u} g(x, t) \, dDv(x),$$

$$I_2(t) := \frac{1}{(4\pi)^{d/2}} \int_{S_u^c} g(x, t) \, dDv(x).$$

Here, as in (2.1), S_u and J_u stand respectively for the approximate discontinuity set and jump set of u (as in (2.3)). In particular, we are allowed to write the latter decomposition since $\mathcal{H}^{d-1}(S_u \setminus J_u) = 0$. Moreover, observe that $g(\cdot, t)$ is a continuous function such that

$$\sup_{x \in \mathbb{R}^d} |g(x, t)| \leq \|u\|_{L^\infty} \int_{\mathbb{R}^d} |z| e^{-\frac{|z|^2}{4}} \, dz = \frac{2^{d+1} \pi^{d/2}}{\Gamma(d/2)} \Gamma((d+1)/2) \|u\|_{L^\infty}. \quad (2.12)$$

Claim: $\lim_{t \rightarrow 0^+} I_2(t) = 0$.

Let us prove the claim. Observe that, since the map $z \mapsto z e^{-|z|^2/4}$ has vanishing integral, for every $x \in S_u^c$ we are allowed to write

$$g(x, t) = \int_{\mathbb{R}^d} z e^{-\frac{|z|^2}{4}} (u(x - tz) - \tilde{u}(x)) \, dz,$$

where $\tilde{u}(x)$ stands for the approximate limit of u at x . Now, fix $R > 0$ and estimate g as

$$\begin{aligned} |g(x, t)| &\leq \int_{B_R} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz \\ &\quad + \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz \\ &\leq \int_{B_R} |z| e^{-\frac{|z|^2}{4}} |u(x - tz) - \tilde{u}(x)| \, dz + 2\|u\|_{L^\infty} \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} \, dz. \end{aligned}$$

We shall now take the $\limsup$ as $t \rightarrow 0^+$ of the previous expression, so that we get

$$\limsup_{t \rightarrow 0^+} |g(x, t)| \leq 2\|u\|_{L^\infty} \int_{B_R^c} |z| e^{-\frac{|z|^2}{4}} dz,$$

since x is a point of approximate continuity. Finally, by taking the $\limsup$ as $R \rightarrow +\infty$, we obtain that

$$\lim_{t \rightarrow 0^+} g(x, t) = 0 \quad \forall x \in S_u^c.$$

Using the fact that $|Dv|(\mathbb{R}^d) < +\infty$ and the bound in (2.12), we obtain the claim by a dominated convergence argument.

We are thus left to study what happens in the set J_u . First, we have that J_u is countably $\mathcal{H}^{d-1}$ -rectifiable, hence we can apply [3, Prop. 3.92] to deduce that $|D^c v|(J_u) = 0$. Moreover, the set J_u has Lebesgue measure zero, and therefore, it only remains to deal with

$$I_1(t) = \frac{1}{(4\pi)^{d/2}} \int_{J_u \cap J_v} g(x, t) \cdot \nu_{J_v}(v^+ - v^-) d\mathcal{H}^{d-1}(x). \quad (2.13)$$

Now, define

$$f(x, t) := \langle g(x, t), \nu_{J_v}(x) \rangle,$$

and consider

$$\begin{aligned} H_{\nu_{J_u}(x)}^+ &= \{z \in \mathbb{R}^d : \langle z, \nu_{J_u}(x) \rangle \geq 0\}, \\ H_{\nu_{J_u}(x)}^- &= \{z \in \mathbb{R}^d : \langle z, \nu_{J_u}(x) \rangle \leq 0\}. \end{aligned}$$

By exploiting the symmetries of the integrand in (2.13), we get

$$\begin{aligned} f(x, t) &= \int_{H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^-(x)) dz \\ &\quad + u^-(x) \int_{H_{\nu_{J_u}}^+} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} dz \\ &\quad + \int_{H_{\nu_{J_u}}^-} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^+(x)) dz \\ &\quad + u^+(x) \int_{H_{\nu_{J_u}}^-} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} dz \end{aligned}$$

$$=: A + B + C + D.$$

Claim: $\lim_{t \rightarrow 0^+} A = \lim_{t \rightarrow 0^+} C = 0$ for every $x \in J_u \cap J_v$.

First, let us consider the term A ; in particular, the term C can be treated analogously. We have

$$\begin{aligned} A &= \int_{B_R \cap H_{\nu}^+ J_u} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^-(x)) \, dz \\ &+ \int_{B_R^c \cap H_{\nu}^+ J_u} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^-(x)) \, dz. \end{aligned}$$

By the definition of $u^-(x)$, the first integral on the right-hand side goes to zero as $t \rightarrow 0^+$. For the second integral, as in (2.12), we have that for every $t \in (0, +\infty)$ it holds

$$\left| \int_{B_R^c \cap H_{\nu}^+ J_u} \langle z, \nu_{J_v}(x) \rangle e^{-\frac{|z|^2}{4}} (u(x - tz) - u^-(x)) \, dz \right| \leq 2\|u\|_{L^\infty} \int_{B_R^c} |z| e^{-|z|^2/4} \, dz,$$

and therefore, it converges to 0 as $R \rightarrow +\infty$. This proves the claim.

Now observe that, since for $\mathcal{H}^{d-1}$ -a.e. $x \in J_u \cap J_v$ it holds

$$\nu_{J_v}(x) = \nu_{J_u}(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle,$$

then we get

$$\begin{aligned} B &= \frac{(4\pi)^{d/2}}{\sqrt{\pi}} u^-(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle, \\ D &= -\frac{(4\pi)^{d/2}}{\sqrt{\pi}} u^+(x) \langle \nu_{J_u}(x), \nu_{J_v}(x) \rangle. \end{aligned}$$

A straightforward application of dominated convergence finally gives

$$\lim_{t \rightarrow 0^+} I_1(t) = -\frac{1}{\sqrt{\pi}} \int_{J_u \cap J_v} (u^+ - u^-)(v^+ - v^-) \nu_{J_u} \cdot \nu_{J_v} \, d\mathcal{H}^{d-1},$$

which is the thesis. $\square$

Before proceeding with the proof of Theorem 1.2, we recall the integral Abelian-Tauberian theorem. The reader may find related proofs in the appendix of [5].

Theorem 2.11 (*Abelian-Tauberian*). *Let ν be a σ -finite positive Borel measure and let $\gamma \in [0, +\infty)$, then it holds*

$$\lim_{t \rightarrow 0^+} t^\gamma \int_0^{+\infty} e^{-t\lambda} d\nu(\lambda) = C \quad \Longleftrightarrow \quad \lim_{a \rightarrow +\infty} \frac{\nu([0, a])}{a^\gamma} = \frac{C}{\Gamma(\gamma + 1)}.$$

In the proof of Corollary 1.4, we use a special one-sided version of the latter theorem, in which the existence of the limit is not required. Again, we refer to the appendix of [5].

Theorem 2.12. *Let ν be a σ -finite positive Borel measure and let $\gamma \in [0, +\infty)$. If*

$$\limsup_{a \rightarrow \infty} \frac{\nu([0, a])}{a^\gamma} \leq C < +\infty$$

for some $C \geq 0$, then

$$\limsup_{t \rightarrow 0^+} t^\gamma \int_0^\infty e^{-t\lambda} d\nu(\lambda) \leq C\Gamma(\gamma + 1).$$

Finally, we have gathered all the tools to prove Theorem 1.2.

Proof of Theorem 1.2. First, we shall prove (1.2). Let u be as in the assumption, and similarly to (2.7), consider the quantity

$$H(t) = \int_{\mathbb{R}^d} (T(t^2)u)(x)u(x) dx.$$

By Plancherel identity and by the explicit expression of the heat kernel in (2.5), we get

$$H(t) = \int_{\mathbb{R}^d} \mathcal{F}\{T(t^2)u\}(\xi) \overline{\mathcal{F}\{u\}}(\xi) d\xi = \int_{\mathbb{R}^d} e^{-4\pi^2|\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi.$$

Observe that $t \mapsto H(t)$ is continuous in $[0, 1]$, and by Plancherel identity and dominated convergence, it follows that $H(0) = \|u\|_{L^2}^2$. Again, by dominated convergence, we obtain

$$H'(t) = -8\pi^2 t \int_{\mathbb{R}^d} |\xi|^2 e^{-4\pi^2|\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi,$$

and in particular, H' is continuous on $(0, 1]$. Therefore, thanks to Proposition 2.10, we get

$$t \int_{\mathbb{R}^d} |\xi|^2 e^{-4\pi^2|\xi|^2 t^2} |\hat{u}|^2(\xi) d\xi \rightarrow \frac{\mathcal{J}(u)}{8\pi^{\frac{5}{2}}}.$$

Now, let us convert to polar coordinates and obtain

$$t \int_0^{+\infty} e^{-4\pi^2 r^2 t^2} r^{d+1} g(r) \, dr \rightarrow \frac{\mathcal{J}(u)}{8\pi^{\frac{5}{2}}},$$

where we have defined

$$g(r) := \int_{\mathbb{S}^{d-1}} |\hat{u}|^2(rv) \, d\sigma(v).$$

Let us write $\delta := 4\pi^2 t^2$, and then, by the change of variable $r^2 = s$, we get

$$\int_0^{+\infty} e^{-\delta s} s^{\frac{d}{2}} g(\sqrt{s}) \, ds \sim \frac{\mathcal{J}(u)}{2\pi^{\frac{3}{2}}\sqrt{\delta}} \quad \text{as } \delta \rightarrow 0^+, \quad (2.14)$$

where we use the symbol “ $\sim$ ” to denote the asymptotic behaviour. Now, consider

$$\Phi(s) := \int_0^s z^{\frac{d}{2}} g(\sqrt{z}) \, dz = 2 \int_0^{\sqrt{s}} y^{d+1} g(y) \, dy = 2 \int_{B_{\sqrt{s}}} |\xi|^2 |\hat{u}|^2(\xi) \, d\xi.$$

With the previous definition in mind, we can write (2.14) as

$$\int_0^{+\infty} e^{-\delta s} \, d\Phi(s) \sim \frac{\mathcal{J}(u)}{2\pi^{\frac{3}{2}}\sqrt{\delta}} \quad \text{as } \delta \rightarrow 0^+.$$

Finally, we apply Theorem 2.11 and obtain

$$\Phi(R) \sim \sqrt{R} \frac{\mathcal{J}(u)}{\pi^2} \quad \text{as } R \rightarrow +\infty,$$

which (by replacing R with R^2) means that

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) \, d\xi = \frac{\mathcal{J}(u)}{2\pi^2}. \quad (2.15)$$

In order to prove (1.1), we shall proceed by polarization. Again, let u and v satisfy our hypotheses. We exploit the fact that

$$\Re\{\hat{u}(\xi)\bar{\hat{v}}(\xi)\} = \frac{|\widehat{(u+v)}(\xi)|^2}{2} - \frac{|\hat{u}(\xi)|^2}{2} - \frac{|\hat{v}(\xi)|^2}{2}.$$

Indeed, by the latter equation and by (1.2), we obtain

$$\begin{aligned}
\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} |\xi|^2 \hat{u}(\xi) \bar{\hat{v}}(\xi) \, d\xi &= \frac{1}{4\pi^2} \left(\int_{J_{u+v}} ((u+v)^+ - (u+v)^-)^2 \, d\mathcal{H}^{d-1} \right. \\
&\quad - \int_{J_u} (u^+ - u^-)^2 \, d\mathcal{H}^{d-1} \\
&\quad \left. - \int_{J_v} (v^+ - v^-)^2 \, d\mathcal{H}^{d-1} \right). \tag{2.16}
\end{aligned}$$

Now, we write

$$\begin{aligned}
J_{u+v} &\subseteq (J_u \setminus J_v) \cup (J_v \setminus J_u) \cup (J_u \cap J_v), \\
J_u &= (J_u \setminus J_v) \cup (J_u \cap J_v), \\
J_v &= (J_v \setminus J_u) \cup (J_u \cap J_v),
\end{aligned}$$

and we split the integrals at the right-hand side of (2.16) accordingly. In particular, we observe that if $x \in J_u \cap J_v$, then it holds

$$(u+v)^{\pm}(x) = u^{\pm}(x) + v^{\pm}(x).$$

Here, we have chosen the orientations of the normals so that $\nu_{J_u} = \nu_{J_v}$ holds $\mathcal{H}^{d-1}$ -almost everywhere on $J_u \cap J_v$. Finally, a simple rearrangement of the terms gives (1.1). $\square$

Now, we proceed with the proof of the first corollary to Theorem 1.2.

Proof of Corollary 1.3. Before performing the limit in (1.2), multiply both sides by R^{-3} and then integrate with respect to R from $L > 0$ to $+\infty$, so as to get

$$\int_L^\infty R^{-3} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) \, d\xi \, dR \sim \frac{\mathcal{J}(u)}{2\pi^2 L} \quad \text{as } L \rightarrow +\infty.$$

Now, we integrate by parts (justified by (1.2)) the left-hand side above and obtain

$$\frac{1}{2L^2} \int_{B_L} |\xi|^2 |\hat{u}|^2(\xi) \, d\xi + \frac{1}{2} \int_{B_L^\xi} |\hat{u}|^2(\xi) \, d\xi \sim \frac{\mathcal{J}(u)}{2\pi^2 L} \quad \text{as } L \rightarrow +\infty.$$

Finally, by applying (1.2) a second time, we end up with

$$\int_{B_L^\xi} |\hat{u}|^2(\xi) \, d\xi \sim \frac{\mathcal{J}(u)}{2\pi^2 L} \quad \text{as } L \rightarrow +\infty,$$

which is indeed (1.4). $\square$

We proceed with the proof of the second and last corollary, which contains the characterization of sets of finite perimeter discussed in the introduction.

Proof of Corollary 1.4. The relations (1.5) and (1.6) easily follow from Theorem 1.2 and the definition of reduced boundary. Hence, let us suppose that (1.7) holds. By Theorem 2.12, we have

$$\limsup_{\delta \rightarrow 0^+} \sqrt{\delta} \int_0^{+\infty} e^{-\delta s} d\Phi(s) < +\infty,$$

with (we shall neglect the constants)

$$\Phi(s) = \int_0^s z^{\frac{d}{2}} g(\sqrt{z}) dz \quad \text{and} \quad g(r) = \int_{\mathbb{S}^{d-1}} |\widehat{\mathbf{1}}_\Omega|^2(rv) d\sigma(v).$$

By a change of variables, we obtain

$$\limsup_{t \rightarrow 0^+} t \int_{\mathbb{R}^d} e^{-|\xi|^2 t^2} |\xi|^2 |\widehat{\mathbf{1}}_\Omega|^2(\xi) d\xi < +\infty.$$

By Plancherel identity, the latter inequality implies that

$$\limsup_{t \rightarrow 0^+} |H'(t)| < +\infty,$$

where H' is computed as in the proof of Theorem 1.2 with $u = v = \mathbf{1}_\Omega$. Hence, by the mean value theorem and by the monotonicity properties of H , it follows that

$$0 \leq \limsup_{t \rightarrow 0^+} \frac{H(0) - H(t)}{t} < +\infty. \quad (2.17)$$

Exploiting the definition of H , we turn (2.17) into

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \int_{\Omega^c} (T(t^2) \mathbf{1}_\Omega)(x) dx < +\infty,$$

and we can infer that Ω is of finite perimeter by Theorem 2.9.

As for (1.8), we shall use a result from [19]. Let $E \subset [0, 1]^d$ be the bounded set provided by [19, Thm. 2.1 and Sec. 2], then $\text{Per}(E) = +\infty$ and, by considering the rescaled standard Gaussian $\gamma_\varepsilon(x) = G_d(x, \varepsilon^2/2)$, it holds

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{1}{|\log \varepsilon|} \int_{\mathbb{R}^d} |\xi| |\mathcal{F}\{\gamma_\varepsilon * \mathbf{1}_E\}|^2(\xi) d\xi = 0. \quad (2.18)$$

Assume by contradiction that (1.8) fails for E , namely that there exist $\delta, R_0 > 0$ such that for all $L \geq R_0$ it holds

$$\frac{1}{L} \int_{B_L} |\xi|^2 |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi \geq \delta.$$

Dividing both sides by L and integrating between R_0 and R gives

$$\int_{R_0}^R \frac{1}{L^2} \int_0^L r^{d+1} g_E(r) \, dr \, dL \geq \delta \log(R/R_0), \quad \text{with} \quad g_E(r) = \int_{\mathbb{S}^{d-1}} |\widehat{\mathbb{1}}_E|^2(rv) \, d\sigma(v).$$

Integrating by parts then gives

$$\begin{aligned} & -\frac{1}{R} \int_{B_R} |\xi|^2 |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi + \frac{1}{R_0} \int_{B_{R_0}} |\xi|^2 |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi + \int_{B_R \setminus B_{R_0}} |\xi| |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi \\ & \geq \delta \log(R/R_0). \end{aligned}$$

The first term on the left-hand side is non-positive and the second does not depend on R . Hence, dividing by $\log R$ and letting $R \rightarrow +\infty$, we obtain

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \int_{B_R} |\xi| |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi \geq \delta.$$

Now, as in the proof of Theorem 1.2, we have $\mathcal{F}\{\gamma_{1/R}\}(\xi) = \mathcal{F}\{G_d(\cdot, 1/(2R^2))\}(\xi) = e^{-2\pi^2|\xi|^2/R^2}$, and hence $|\mathcal{F}\{\gamma_{1/R}\}|^2(\xi) = e^{-4\pi^2|\xi|^2/R^2} \geq e^{-4\pi^2}$ for every $\xi \in B_R$. Therefore, we get

$$\begin{aligned} \int_{\mathbb{R}^d} |\xi| |\mathcal{F}\{\gamma_{1/R} * \mathbb{1}_E\}|^2(\xi) \, d\xi &= \int_{\mathbb{R}^d} |\xi| e^{-4\pi^2|\xi|^2/R^2} |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi \\ &\geq e^{-4\pi^2} \int_{B_R} |\xi| |\widehat{\mathbb{1}}_E|^2(\xi) \, d\xi, \end{aligned}$$

and thus

$$\liminf_{R \rightarrow +\infty} \frac{1}{\log R} \int_{\mathbb{R}^d} |\xi| |\mathcal{F}\{\gamma_{1/R} * \mathbb{1}_E\}|^2(\xi) \, d\xi \geq e^{-4\pi^2} \delta > 0,$$

which contradicts (2.18) with $\varepsilon = 1/R$. $\square$

3. Fourier transform of the derivative of sets

We shall write Ω_r to denote the set

$$\Omega_r = \{x \in \mathbb{R}^d : \text{dis}(\Omega, x) \leq r\},$$

which coincides with the Minkowski sum $\Omega + B_r$. We have the following definition.

Definition 3.1 (*outer Minkowski content*). We say that a Borel set $\Omega \subseteq \mathbb{R}^d$ has a finite outer Minkowski content if the limit

$$\mathcal{SM}(\Omega) = \lim_{r \rightarrow 0^+} \frac{|\Omega_r \setminus \Omega|}{r}$$

exists and is finite.

Moreover, we give the following notion of convergence for sequences of measures.

Definition 3.2. We say that a sequence of measures $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ weakly converges to a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, if for every $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$ it holds

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} \varphi \, d\mu_n = \int_{\mathbb{R}^d} \varphi \, d\mu.$$

The following result is essential to obtain Theorem 1.5, and it is a refinement of [1, Prop. 2].

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure and perimeter. Then $\mathcal{SM}(\Omega) = \text{Per}(\Omega)$ if and only if the measures*

$$\mu_h := \frac{\mathbb{1}_{\Omega+hB_1} - \mathbb{1}_\Omega}{h} \tag{3.1}$$

weakly converge to $|D\mathbb{1}_\Omega|$.

Proof. First, let us assume that

$$\mathcal{SM}(\Omega) = \text{Per}(\Omega) < +\infty.$$

By Portmanteau theorem (see [12, Thm. 8.2.3]), and since the assumption implies that

$$\lim_{h \rightarrow 0^+} \mu_h(\mathbb{R}^d) = \text{Per}(\Omega),$$

then it is enough to show that

$$\liminf_{h \rightarrow 0^+} \mu_h(U) \geq |D\mathbf{1}_\Omega|(U) \quad \text{for every open set } U.$$

Therefore, consider an open set U and define the distance function

$$\text{dis}_\Omega(x) := \inf\{|x - y| : y \in \Omega\}.$$

It is clear that such a function is Lipschitz continuous, and as in [2, p. 11], we have that

$$|\nabla \text{dis}_\Omega| = 1 \quad \mathcal{L}^d\text{-almost everywhere in } \Omega^c.$$

Moreover, since $|\Omega_r \setminus \Omega| \geq |\overline{\Omega} \setminus \Omega|$ for every $r > 0$, the assumption $\mathcal{SM}(\Omega) < +\infty$ forces $|\overline{\Omega} \setminus \Omega| = 0$; in particular, $\{\text{dis}_\Omega \leq h\} \setminus \Omega$ coincides with $\{0 < \text{dis}_\Omega \leq h\}$ up to a Lebesgue-negligible set. Given the previous observations, we apply Coarea formula to obtain

$$\begin{aligned} \mu_h(U) &= \frac{1}{h} \int_{\{0 < \text{dis}_\Omega \leq h\}} \mathbf{1}_U |\nabla \text{dis}_\Omega| \, d\mathcal{H}^d = \frac{1}{h} \int_0^h \text{Per}(\{\text{dis}_\Omega > t\}; U) \, dt \\ &= \int_0^1 \text{Per}(\{\text{dis}_\Omega > hr\}; U) \, dr. \end{aligned}$$

Since the sets $\{\text{dis}_\Omega > hr\}$ converge locally in measure to $\{\text{dis}_\Omega > 0\}$ as $h \rightarrow 0^+$, and the latter set coincides with Ω^c up to a Lebesgue-negligible set, then, by the lower semicontinuity of the perimeter (see [3, Prop. 3.38]) and by Fatou lemma, we get

$$\liminf_{h \rightarrow 0^+} \mu_h(U) \geq \text{Per}(\Omega^c; U) = \text{Per}(\Omega; U) = |D\mathbf{1}_\Omega|(U).$$

This proves the first part of the result.

Conversely, let μ_h be weakly convergent to $|D\mathbf{1}_\Omega|$. Then this implies the boundedness of the sequence, and moreover, we have that

$$\mu_h(\mathbb{R}^d) \rightarrow |D\mathbf{1}_\Omega|(\mathbb{R}^d) = \text{Per}(\Omega),$$

which gives the thesis. $\square$

In light of the latter proposition, we give a short proof of Theorem 1.5.

Proof of Theorem 1.5. Notice that the function

$$x \mapsto e^{-2\pi i \xi \cdot x}$$

is bounded and continuous for all $\xi \in \mathbb{R}^d$. If the measures μ_h (defined as in (3.1)) converge in duality with $\mathcal{C}_b(\mathbb{R}^d)$ to $|D\mathbf{1}_\Omega|$, then we have

$$\lim_{h \rightarrow 0^+} \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d\mu_h(x) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} d|D\mathbb{1}_\Omega|(x),$$

which is equivalent to (1.9). The converse implication is trivial since one can test the pointwise convergence of the Fourier transforms at $\xi = 0$, easily implying the thesis. $\square$

4. Discrepancy of BV functions

We now turn our attention to the quadratic discrepancy and prove Theorem 1.6 and Theorem 1.7. Let us start by exploiting the convolutional structure of (1.10). Namely, if we let μ_L be the Lebesgue measure on $\mathbb{T}^d$, and for $p \in \mathbb{T}^d$ we consider $\mu_D(p)$ to be the Dirac delta centered at p , then by setting

$$\tilde{\mu} = \sum_{p \in \mathcal{P}_N} \mu_D(-p) - N\mu_L,$$

for a function $u \in L^1(\mathbb{R}^d)$, we may rewrite (1.10) as

$$\mathcal{D}(u; \mathcal{P}_N) = \int_{\mathbb{T}^d} \mathfrak{P}\{u\}(x) d\tilde{\mu}(-x) = (\mathfrak{P}\{u\} * \tilde{\mu})(\mathbf{0}).$$

Now, for $f \in L^1(\mathbb{T}^d)$ or $f \in \mathcal{M}(\mathbb{T}^d)$, let $\tilde{\mathcal{F}}\{f\}: \mathbb{Z}^d \rightarrow \mathbb{C}$ be the function of the Fourier coefficients of f . Then, it is not difficult to see that for $u \in L^1(\mathbb{R}^d)$, and for every $n \in \mathbb{Z}^d$, it holds

$$\tilde{\mathcal{F}} \circ \mathfrak{P}\{u\}(n) = \mathcal{F}\{u\}(n).$$

Moreover, by classic properties of the Fourier transform, it holds

$$\mathcal{F}\{[\mathbf{0}, \delta, \rho]u\} = \delta^d[\mathbf{0}, \delta^{-1}, \rho]\mathcal{F}\{u\}. \quad (4.1)$$

Therefore, by applying Plancherel identity on $\mathbb{T}^d$, we get

$$\begin{aligned} \int_{\mathbb{T}^d} |\mathcal{D}([\tau, \delta, \rho]u; \mathcal{P}_N)|^2 d\tau &= \int_{\mathbb{T}^d} |(\mathfrak{P}\{[\mathbf{0}, \delta, \rho]u\} * \tilde{\mu})|^2(\tau) d\tau \\ &= \sum_{n \in \mathbb{Z}^d} |\tilde{\mathcal{F}}\{\tilde{\mu}\}(n)|^2 |\tilde{\mathcal{F}} \circ \mathfrak{P}\{[\mathbf{0}, \delta, \rho]u\}(n)|^2 \\ &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 |\mathcal{F}\{[\mathbf{0}, \delta, \rho]u\}(n)|^2, \end{aligned}$$

where, for the sake of notation, we have set $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Finally, by (4.1) and the symmetry of $\text{SO}(d)$, we may rewrite (1.11) as

$$\mathcal{D}_2(u; \mathcal{P}_N) = \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 \int_{\text{SO}(d)} \int_0^1 \delta^{2d} |[\mathbf{0}, \delta^{-1}, \rho] \mathcal{F}\{u\}(n)|^2 d\delta d\rho,$$

and further, by a change of variable, we get

$$\mathcal{D}_2(u; \mathcal{P}_N) = \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi. \quad (4.2)$$

The rest of the section relies on the following asymptotic relation resulting from integrating by parts in (1.2). Namely, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi \sim \frac{|n|^d \mathcal{J}(u)}{2d\pi^2} \quad \text{as } |n| \rightarrow \infty. \quad (4.3)$$

Hence, let us start by proving the upper bound in Theorem 1.6.

Proof of Theorem 1.6. We assume that $\mathcal{J}(u) \neq 0$; in the degenerate case $\mathcal{J}(u) = 0$, a similar argument applies. We write $c_i = c_i(d)$ with $i \in \mathbb{N}$ to denote generic dimensional constants throughout the proof. By (4.3), there exists $n_0 \in \mathbb{N}$ such that for every n with $|n| \geq n_0$, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) d\xi \leq \frac{|n|^d \mathcal{J}(u)}{d\pi^2}.$$

First, we prove the upper bound for all positive integers N such that $N^{1/d} \geq n_0$ and $N^{1/d} \in \mathbb{N}$. Consider the configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$ to be the periodization of

$$\frac{[0, N^{1/d})^d \cap \mathbb{Z}^d}{N^{1/d}} \subset [0, 1)^d. \quad (4.4)$$

Hence, it is not difficult to see that

$$\sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} = \begin{cases} N & \text{if } n \in N^{1/d} \mathbb{Z}^d \\ 0 & \text{else} \end{cases},$$

and therefore, by (4.2) and the assumptions on N , we get

$$\begin{aligned}
\mathcal{D}_2(u; \mathcal{P}_N) &= \sum_{m \in \mathbb{Z}_*^d} N^2 |N^{1/d} m|^{-2d-1} \int_{|\xi| \leq N^{1/d} |m|} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\
&\leq \frac{1}{d\pi^2} \mathcal{J}(u) N^2 \sum_{m \in \mathbb{Z}_*^d} |N^{1/d} m|^{-d-1} \\
&\leq c_1 \mathcal{J}(u) N^{(d-1)/d}.
\end{aligned}$$

Now, we prove the upper bound for every positive integer N . As k goes through the positive integers, we consider the recursive decomposition of N defined as

$$N_k = \max \left\{ n \in \mathbb{N} : n^{1/d} \in \mathbb{N}, n \leq N - \sum_{j=1}^{k-1} N_j \right\},$$

where improper sums are zeros. By the latter definition, we get that

$$N - \sum_{j=1}^{k-1} N_j \leq (N_k^{1/d} + 1)^d \leq N_k + c_2 N_k^{(d-1)/d},$$

and therefore, for every integer $k \geq 1$, it holds

$$N_{k+1} \leq N - \sum_{j=1}^k N_j \leq c_2 N_k^{(d-1)/d}. \quad (4.5)$$

Hence, since by definition $N_1 \leq N$, then we recursively get

$$N_{k+1}^{1/d} \leq c_2^{1/d} N_k^{(d-1)/d^2} \leq c_2^{(k-1)/d} N^{(d-1)^{k-1}/d^k}. \quad (4.6)$$

Now, set $K = K(d)$ to be the smallest integer such that

$$\frac{(d-1)^K}{d^K} \leq \frac{d-1}{4d},$$

and in particular, notice that by (4.6), we have

$$N_{K+1}^{(d-1)/d} \leq c_3 N^{(d-1)/(4d)}. \quad (4.7)$$

Now, consider the remainder

$$N_0 = N - \sum_{j=1}^{K+1} N_j,$$

and by (4.5) and (4.7), it is immediate to see that

$$N_0 \leq c_2 c_3 N^{(d-1)/(4d)}. \quad (4.8)$$

Last, consider the constant depending on u defined as

$$M(u) = \sup_{|n| \geq 1} |n|^{-d} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi,$$

and in particular, notice that this quantity is indeed finite. Finally, for $1 \leq j \leq K+1$, we assign a uniform lattice $\mathcal{P}_{N_j}$ to every N_j as in (4.4), and by proceeding as in the first part of the proof, we get the estimate

$$\mathcal{D}_2(u; \mathcal{P}_{N_j}) \leq \min \left(c_1 \mathcal{J}(u) N_j^{(d-1)/d}, c_4 M(u) n_0^{d-1} \right), \quad (4.9)$$

where the minimum involved depends on whether or not it holds $N_j^{1/d} \geq n_0$. On the other hand, for every configuration of N_0 points $\mathcal{P}_{N_0} \subset \mathbb{T}^d$, by the inequality in (4.8), we get

$$\begin{aligned} \mathcal{D}_2(u; \mathcal{P}_{N_0}) &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_{N_0}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\ &\leq c_5 M(u) N^{(d-1)/(2d)}. \end{aligned} \quad (4.10)$$

Therefore, by (4.9) and (4.10), we get

$$\begin{aligned} \mathcal{D}_2(u; \mathcal{P}_N) &= \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{j=0}^{K+1} \sum_{p \in \mathcal{P}_{N_j}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\ &\leq (K+2) \sum_{j=0}^{K+1} \sum_{n \in \mathbb{Z}_*^d} \left| \sum_{p \in \mathcal{P}_{N_j}} e^{2\pi i p \cdot n} \right|^2 |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \\ &\leq (K+2) M(u) \left(c_4 (K+1) n_0^{d-1} + c_5 N^{(d-1)/(2d)} \right) + \\ &\quad + c_1 (K+2) \mathcal{J}(u) \sum_{j=1}^{K+1} N_j^{(d-1)/d} \\ &\leq c_5 M(u) N^{(d-1)/(2d)} + C_d \mathcal{J}(u) N^{(d-1)/d}, \end{aligned}$$

so that, dividing by $N^{(d-1)/d}$ and by taking the lim sup as $N \rightarrow +\infty$, the conclusion follows. $\square$

Before proceeding with the proof of the lower bound, we need a technical lemma. We briefly present a result of Cassels [14] and Montgomery [25]. The following proof is

inspired by Siegel's analytic proof of Minkowski's convex body theorem. In particular, variations of such an argument proved suitable for tackling various questions of quadratic discrepancy, and we refer to [13] and [9] for some recent developments. Moreover, we mention that an analogous argument holds on manifolds, as the interested reader may verify in [10] and [17].

Lemma 4.1 (*Cassels-Montgomery*). *For every origin-symmetric convex body $\Omega \subset \mathbb{R}^d$, and for every finite configuration of points $\{p_j\}_{j=1}^N \subset \mathbb{T}^d$, it holds*

$$\sum_{m \in \Omega \cap \mathbb{Z}_*^d} \left| \sum_{j=1}^N e^{2\pi i m \cdot p_j} \right|^2 \geq 2^{-d} |\Omega| N - N^2.$$

Proof. Consider the auxiliary sets $A_\Omega(x) \subset \mathbb{Z}^d$ defined by

$$A_\Omega(x) = (x + \Omega/2) \cap \mathbb{Z}^d.$$

Notice that

$$\int_{\mathbb{T}^d} \#A_\Omega(x) \, dx = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} \mathbf{1}_{\Omega/2}(n - x) \, dx = \int_{\mathbb{R}^d} \mathbf{1}_{\Omega/2}(x) \, dx = 2^{-d} |\Omega|,$$

and therefore, we find a point

$$\tilde{x} \in [0, 1)^d \quad \text{such that} \quad \#A_\Omega(\tilde{x}) \geq 2^{-d} |\Omega|.$$

Now, consider the non-negative trigonometric polynomial

$$T(y) = \frac{1}{\#A_\Omega(\tilde{x})} \left| \sum_{n \in A_\Omega(\tilde{x})} e^{2\pi i n \cdot y} \right|^2 = \frac{1}{\#A_\Omega(\tilde{x})} \sum_{n, m \in A_\Omega(\tilde{x})} e^{2\pi i (n-m) \cdot y},$$

and notice that the function of its Fourier coefficients $\widehat{T}: \mathbb{Z}^d \rightarrow \mathbb{R}$ is non-negative as well and, since $n, m \in A_\Omega(\tilde{x})$ imply $(n-m) \in \Omega$, then its support is contained in Ω . Further, observe that we have

$$T(0) = \#A_\Omega(\tilde{x}) \geq 2^{-d} |\Omega|,$$

moreover, it is not difficult to notice that for all $n \in \mathbb{Z}^d$, it holds

$$0 \leq \widehat{T}(n) \leq \widehat{T}(\mathbf{0}) = \int_{\mathbb{T}^d} T(x) \, dx = 1.$$

Therefore, it follows that

$$\begin{aligned}
\sum_{n \in \Omega \cap \mathbb{Z}^d} \left| \sum_{j=1}^N e^{2\pi i n \cdot p_j} \right|^2 &\geq \sum_{n \in \Omega \cap \mathbb{Z}^d} \widehat{T}(n) \left| \sum_{j=1}^N e^{2\pi i n \cdot p_j} \right|^2 \\
&= \sum_{j=1}^N \sum_{\ell=1}^N \sum_{n \in \Omega \cap \mathbb{Z}^d} \widehat{T}(n) e^{2\pi i n \cdot (p_j - p_\ell)} \\
&= \sum_{j=1}^N \sum_{\ell=1}^N T(p_j - p_\ell) \geq N 2^{-d} |\Omega|,
\end{aligned}$$

and the conclusion easily follows. $\square$

Finally, we use the previous lemma to prove the lower bound in Theorem 1.7.

Proof of Theorem 1.7. We write $c_i = c_i(d)$ with $i \in \mathbb{N}$ to denote generic dimensional constants throughout the proof. By (4.3), there exists $n_0 \in \mathbb{N}$ such that for every n with $|n| \geq n_0$, it holds

$$\int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \geq \frac{|n|^d \mathcal{J}(u)}{4d\pi^2}.$$

Hence, consider

$$m(u) = \min_{1 \leq |n| \leq n_0} |n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi.$$

Since (by hypothesis) u is of compact support, we have that $\hat{u}$ is of exponential type, and therefore $\hat{u} \equiv 0$ in B_1 would imply $u \equiv 0$. Hence, we may assume that $m(u) > 0$. We prove the lower bound for all the positive integers N such that

$$\frac{\mathcal{J}(u)}{4d\pi^2} N^{-(d+1)/d} \leq m(u),$$

and for every configuration of N points $\mathcal{P}_N \subset \mathbb{T}^d$. Indeed, within this assumption, for all $n \in B_{N^{1/d}}$ it holds

$$|n|^{-2d-1} \int_{B_{|n|}} |\xi|^{d+1} |\hat{u}|^2(\xi) \, d\xi \geq \frac{\mathcal{J}(u)}{4d\pi^2} N^{-(d+1)/d}.$$

Therefore, by applying the latter inequality in (4.2), we get that for a dimensional constant $\kappa \geq 1$ to be chosen later, it holds

$$\mathcal{D}_2(u; \mathcal{P}_N) \geq \sum_{n \in \mathbb{Z}_*^d \cap B_{(\kappa N)^{1/d}}} \left| \sum_{p \in \mathcal{P}_N} e^{2\pi i p \cdot n} \right|^2 \frac{\mathcal{J}(u)}{4d\pi^2} (\kappa N)^{-(d+1)/d}.$$

Finally, we can apply Lemma 4.1 with $\Omega = B_{(\kappa N)^{1/d}}$ and, denoting $\omega_d = |B_1|$, obtain

$$\mathcal{D}_2(u; \mathcal{P}_N) \geq c_1 \mathcal{J}(u) (\kappa N)^{-(d+1)/d} (2^{-d} \omega_d \kappa N^2 - N^2),$$

and by setting $\kappa = 2^{d+1} \omega_d^{-1}$, it follows that

$$\mathcal{D}_2(u; \mathcal{P}_N) \geq c_2 \mathcal{J}(u) N^{(d-1)/d}.$$

Since $m(u) > 0$, the latter holds for every N large enough, and the thesis follows with $C_d = c_2$. $\square$

5. Appendix

For simplicity, in what follows we denote $\|\cdot\|_{L^p}$ with $\|\cdot\|_p$.

Lemma 5.1. *There exists a positive dimensional constant C_d such that, for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and for every $v \in \mathbb{S}^{d-1}$, it holds*

$$\frac{1}{R} \int_{B_R} |\langle \xi, v \rangle|^2 |\hat{u}|^2(\xi) d\xi \leq C_d \|u\|_\infty |D_v u|(\mathbb{R}^d) \quad \forall R > 0. \quad (5.1)$$

In particular, by integrating over $\mathbb{S}^{d-1}$, we get

$$\frac{1}{R} \int_{B_R} |\xi|^2 |\hat{u}|^2(\xi) d\xi \leq C_d \|u\|_\infty |Du|(\mathbb{R}^d) \quad \forall R > 0. \quad (5.2)$$

Proof. First, we fix a vector $v \in \mathbb{S}^{d-1}$ and a $h > 0$. By standard properties of the Fourier transform and by Plancherel identity, we have

$$\int_{\mathbb{R}^d} \frac{|u(x + hv) - u(x)|^2}{h} dx = \frac{2}{h} \int_{\mathbb{R}^d} (1 - \cos(2\pi h \langle \xi, v \rangle)) |\hat{u}|^2(\xi) d\xi.$$

The left-hand side is bounded from above by $2\|u\|_\infty |D_v u|(\mathbb{R}^d)$, where $|D_v u|$ stands for the directional total variation. Now, we exploit the fact that if $\xi \in B_{1/(2\pi h)}$, then there exists a positive absolute constant c such that

$$ch^2 |\langle \xi, v \rangle|^2 \leq 2(1 - \cos(2\pi h \langle \xi, v \rangle)).$$

Therefore, we get

$$h \int_{B_{1/h}} |\langle \xi, v \rangle|^2 |\hat{u}|^2(\xi) d\xi \leq C_d \|u\|_\infty |D_v u|(\mathbb{R}^d),$$

which is (5.1). Finally, by setting $R = 1/h$ and by integrating over $\mathbb{S}^{d-1}$, we obtain (5.2). $\square$

Remark 5.2. By integrating by parts, we can reformulate (5.2) as

$$\int_{B_R^c} |\hat{u}|^2(\xi) d\xi \leq C_d \frac{\|u\|_\infty |Du|(\mathbb{R}^d)}{R}.$$

Last, we proceed with showing an interesting inequality. In particular, we obtain a non-sharp isoperimetric inequality by substituting $u = \mathbb{1}_\Omega$ in the following proposition.

Proposition 5.3. *There exists a positive dimensional constant C_d such that, for every $u \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it holds*

$$\|u\|_2^2 \leq C_d \|u\|_1^{\frac{2}{d+1}} \|u\|_\infty^{\frac{d}{d+1}} \left(|Du|(\mathbb{R}^d) \right)^{\frac{d}{d+1}}.$$

Proof. First, we use Plancherel identity to write

$$\int_{\mathbb{R}^d} |u|^2(x) dx = \int_{B_R} |\hat{u}|^2(\xi) d\xi + \int_{B_R^c} |\hat{u}|^2(\xi) d\xi.$$

By the fact that $\|\mathcal{F}\|_{L^1 \rightarrow L^\infty} \leq 1$, and by (5.2), we obtain

$$\|u\|_2^2 \leq \omega_d R^d \|u\|_1^2 + C_d \frac{\|u\|_\infty |Du|(\mathbb{R}^d)}{R}.$$

By optimizing over $R > 0$, we get the optimal value

$$R = \left(\frac{C_d \|u\|_\infty |Du|(\mathbb{R}^d)}{d \omega_d \|u\|_1^2} \right)^{\frac{1}{d+1}}.$$

Therefore, it follows that

$$\|u\|_2^2 \leq C_d \|u\|_1^{\frac{2}{d+1}} \|u\|_\infty^{\frac{d}{d+1}} \left(|Du|(\mathbb{R}^d) \right)^{\frac{d}{d+1}},$$

and this is the thesis. $\square$

Funding

L.G. is supported by UK Research and Innovation (UKRI) under the Horizon Europe funding guarantee [grant number EP/Z000297/1].

Data availability

No data was used for the research described in the article.

References

- [1] Luigi Ambrosio, Andrea Colesanti, Elena Villa, Outer Minkowski content for some classes of closed sets, *Math. Ann.* 342 (4) (2008) 727–748.
- [2] L. Ambrosio, N. Dancer, *Calculus of Variations and Partial Differential Equations*, Springer-Verlag, Berlin, 2000, Topics on geometrical evolution problems and degree theory, Papers from the Summer School held in Pisa, September 1996.
- [3] Luigi Ambrosio, Nicola Fusco, Diego Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
- [4] S. Agmon, L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics, *J. Anal. Math.* 30 (1976) 1–38.
- [5] Luigi Ambrosio, Shouhei Honda, David Tewodrose, Short-time behavior of the heat kernel and Weyl’s law on $RCD^*(K, N)$ spaces, *Ann. Glob. Anal. Geom.* 53 (1) (2018) 97–119.
- [6] J. Antezana, M. Levi, J. Marzo, J. Ortega-Cerdà, *Functions of Bounded Variation and Point Processes*, 2026.
- [7] J. Beck, W.W.L. Chen, Note on irregularities of distribution. II, *Proc. Lond. Math. Soc.* (3) 61 (2) (1990) 251–272.
- [8] József Beck, Irregularities of distribution. I, *Acta Math.* 159 (1–2) (1987) 1–49.
- [9] Thomas Beretti, *The Fourier Transform of Planar Convex Bodies and Discrepancy over Intervals of Rotations*, 2025.
- [10] Luca Brandolini, Bianca Gariboldi, Giacomo Gigante, On a sharp lemma of Cassels and Montgomery on manifolds, *Math. Ann.* 379 (3–4) (2021) 1807–1834.
- [11] Joaquim Bruna, Alexander Nagel, Stephen Wainger, Convex hypersurfaces and Fourier transforms, *Ann. Math.* (2) 127 (2) (1988) 333–365.
- [12] V.I. Bogachev, *Measure Theory*. Vol. I, II, Springer-Verlag, Berlin, 2007.
- [13] Luca Brandolini, Giancarlo Travaglini, Irregularities of distribution and geometry of planar convex sets, *Adv. Math.* 396 (2022) 108162.
- [14] J.W.S. Cassels, On the sums of powers of complex numbers, *Acta Math. Acad. Sci. Hung.* 7 (1956) 283–289.
- [15] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.
- [16] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge Tracts in Mathematics, vol. 85, Cambridge University Press, Cambridge, 1986.
- [17] Bianca Gariboldi, Giacomo Gigante, Almost positive kernels on compact Riemannian manifolds, *Math. Z.* 302 (2) (2022) 783–801.
- [18] C.S. Herz, Fourier transforms related to convex sets, *Ann. Math.* (2) 75 (1962) 81–92.
- [19] Felipe Hernández, Some properties of a Hilbertian norm for perimeter, *Pure Appl. Funct. Anal.* 4 (3) (2019) 559–572.
- [20] Edmund Hlawka, Über Integrale auf konvexen Körpern. I, *Monatshefte Math.* 54 (1950) 1–36.
- [21] Lars Hörmander, *Linear Partial Differential Operators*, Grundlehren der mathematischen Wissenschaften, vol. 116, Springer-Verlag/Academic Press, Inc., Publishers, Berlin-Göttingen-Heidelberg/New York, 1963.
- [22] Matteo Levi, Jordi Marzo, Joaquim Ortega-Cerdà, Linear statistics of determinantal point processes and norm representations, *Int. Math. Res. Not.* (19) (2024) 12869–12903.
- [23] Francesco Maggi, *Sets of Finite Perimeter and Geometric Variational Problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012, An introduction to geometric measure theory.
- [24] John M. Marstrand, The (φ, s) regular subsets of n -space, *Trans. Am. Math. Soc.* 113 (1964) 369–392.
- [25] Hugh L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics, vol. 84, Conference Board of

the Mathematical Sciences/by the American Mathematical Society, Washington, DC/Providence, RI, 1994.

- [26] Michele Miranda Jr., Diego Pallara, Fabio Paronetto, Marc Preunkert, Short-time heat flow and functions of bounded variation in $\mathbb{R}^N$, *Ann. Fac. Sci. Toulouse Math.* (6) 16 (1) (2007) 125–145.
- [27] K.F. Roth, On irregularities of distribution, *Mathematika* 1 (1954) 73–79.
- [28] Robert S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math. J.* 44 (3) (1977) 705–714.
- [29] Robert S. Strichartz, Fourier asymptotics of fractal measures, *J. Funct. Anal.* 89 (1) (1990) 154–187.