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Invariants of almost complex and almost symplectic manifolds

Candidate:
Lorenzo Sillari

Advisor:
Prof. Adriano Tomassini

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Abstract

We study invariants of almost complex, almost symplectic and almost Hermitian manifolds.

First, we use the Nijenhuis tensor to measure how far an almost complex structure is from being integrable. We prove that the space of maximally non-integrable structures is open and dense or empty in the space of almost complex structures. Then, we provide a technique that allows to produce almost complex structures whose Nijenhuis tensor has arbitrary prescribed rank on parallelizable manifolds, and we use the rank to classify invariant structures on 6-dimensional nilmanifolds.

From a cohomological point of view, we give definitions of Bott–Chern and Aeppli cohomologies of almost complex manifolds, based on the operators d and d^c, and of their almost symplectic counterparts, based on the operators d and d^Λ. These cohomologies generalize the usual notions of Bott–Chern and Aeppli cohomologies and of symplectic cohomologies of Tseng and Yau. We also explain the importance of the operators δ and $\bar{\delta}$, a suitable generalization of the complex operators ∂ and $\bar{\partial}$. In the non-integrable setting, these cohomologies do not admit a natural bigrading. However, they naturally have a $\mathbb{Z}_2$-splitting induced by the parity of forms.

Finally, we deal with spaces of harmonic forms on compact almost Hermitian manifolds. We describe a series of Laplacians that generalize the complex Bott–Chern and Aeppli Laplacians and the symplectic Laplacians of Tseng and Yau. We formulate a general version of Kodaira–Spencer's problem on the metric-independence of the dimensions of the kernels of such Laplacians. It turns out that these invariants are especially well-behaved on 4-manifolds. If the 4-manifold admits an almost Kähler metric, we find a series of metric-independent invariants, solving the generalized Kodaira–Spencer's problem. Motivated by the complex case, we show that the invariants we defined have a strong link with topological

invariants. On almost Kähler 4-manifolds, they essentially reduce to topological numbers and to the almost complex invariants $h^1_{d+d^c}$ and h^-_J. Motivated by a conjecture of Li and Zhang on the generic vanishing of h^-_J, we conjecture that $h^1_{d+d^c}$ generically vanishes. We are able to confirm our conjecture on high-dimensional manifolds.

We complement the theoretical results with a large number of examples on locally homogeneous manifolds of dimension 4 and 6, where we explicitly compute the rank of the Nijenhuis tensor, the almost complex cohomologies and the spaces of harmonic forms.

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Contents

Introduction

Almost complex structures appeared for the first time in the work of Ehresmann [36] and the role they play in the understanding of complex and symplectic structures is well-known to experts. In this thesis we are especially interested in almost complex structures as a generalization of complex structures, but one should not forget their importance in symplectic geometry and the theory of pseudo-holomorphic curves, see [44], [65] and [66].

On the complex side, existence of almost complex structures is the main known obstruction to the existence of complex structures. However, understanding if an almost complex manifold admits a complex structure is an extremely difficult problem. A direct way to show that a manifold M is complex is to take an almost complex structure J on it and to check if its Nijenhuis tensor vanishes at every point. Such a J is called *integrable*, and Newlander–Nirenberg's theorem guarantees that it corresponds to a complex structure on M [74]. If $\dim_{\mathbb{R}} M = 2$, every almost complex structure is integrable by dimensional reasons. Since we are mostly interested in non-integrable structures, we will not discuss again the 2-dimensional case in this thesis. If $\dim_{\mathbb{R}} M = 4$, there are examples of almost complex 4-manifolds that do not admit complex structures. The first examples are due to Van de Ven [105], Yau [112] and Brotherton [18]. The problem of whether or not an almost complex 4-manifold admits a complex structure is now well-understood in terms of Enriques–Kodaira classification of complex surfaces, or, more recently, in terms of almost complex invariants [30]. If $\dim_{\mathbb{R}} M \geq 6$, there is no known example of an almost complex manifold without complex structures. Finding such an example or showing that every almost complex $2m$-manifold with $2m \geq 6$ admits a complex structure is today an open problem [111].

When dealing with complex and symplectic manifolds, we have several invariants of cohomological or analytical nature at our disposal. The ones relevant for this thesis are the Nijenhuis tensor, the classical Dolbeault, Bott–Chern and Aeppli cohomologies, the symplectic cohomologies of Tseng and Yau, see [100] and [101], and, after fixing a suitable Riemannian metric, the spaces of harmonic forms of complex and symplectic Laplacians. A general definition of these invariants valid for almost complex or almost symplectic manifolds could provide a way to study the properties of those structures and to compare them to integrable complex structures or to symplectic structures, giving further insight on the problem of finding integrable structures.

The search for invariants of almost complex structures that generalize complex invariants is a very natural problem that can be dated back to Kodaira and Spencer [47]. In spite of that, significant progresses in the field are all very recent and concentrated after 2018, when [29] appeared in its first version on the arXiv. The rise in interest is mostly due to the works of Cirici and Wilson [28] and [29], focused on the introduction of Dolbeault cohomology of almost complex manifolds and the study of harmonic forms on almost Kähler manifolds, and to those of Holt and Zhang on the original Kodaira–Spencer's problem, see [52] and [53]. We give later in this introduction a review of the literature on the topic.

The goal of this thesis is to define new almost complex and almost Hermitian invariants that generalize the cohomologies of complex manifolds and the spaces of harmonic forms of Hermitian manifolds. We analyze the relations between the invariants we introduce and other known invariants of almost complex manifolds, and we study how they are related to integrability of almost complex structures. Ultimately, we are able to show that certain invariants are a sufficient tool to distinguish between different almost complex structures and we underline a link with the topology of the underlying manifold that becomes especially significant on 4-manifolds. The definition of many of the invariants we introduce naturally extends to almost symplectic manifolds. We will take some detour from almost complex structures to discuss that in details. The invariants studied in this thesis are mostly of three kinds: the rank of the Nijenhuis tensor, cohomological invariants and spaces of harmonic forms.

The rank of the Nijenhuis tensor

Let (M, J) be an almost complex manifold and let

$$N_J \colon TM \otimes TM \longrightarrow TM$$

be its Nijenhuis tensor. The complex rank of the distribution $\operatorname{Im} N_J \subseteq TM$ provides an invariant of the almost complex structure

$$\operatorname{rk} N_J \colon M \to \mathbb{N}$$

that can be thought as a measure of non-integrability for J. On the one hand, we find complex structures, for which $\operatorname{rk} N_J$ identically vanishes. On the other hand, we have *maximally non-integrable* structures, for which $\operatorname{rk} N_J$ is maximal at every point. This natural class of structures, present in the literature also under the name of *totally non-integrable structures* in [11] and [69], is one of the main sources of examples of non-integrable structures. Among the others, we wish to recall [23], [25], [58], [72] and [107], where the rank of the Nijenhuis tensor is explicitly studied and related to properties of almost complex structures. To motivate the abundance of maximally non-integrable structures in the literature, one has to appeal to their flexibility: a generic local perturbation of an almost complex structure is maximally non-integrable and every Nijenhuis tensor is locally induced by an almost complex structure, see [58] and [69]. When passing to global statements, two aspects must be addressed: existence and density.

Existence of maximally non-integrable structures follows from the work of Coelho, Placini and Stelzig [32], where a h-principle for almost complex structures is established. As a consequence, if $\dim_{\mathbb{R}} M \geq 10$, maximally non-integrable structures always exist, while on compact manifolds of lower dimension there are obstructions to their existence.

Density of maximally non-integrable structures is the first original result of this thesis. Denote by $\mathcal{J}$ the space of almost complex structures on M.

Theorem A. *Let M be a compact almost complex manifold and let $\mathcal{C}$ be a path-connected component of $\mathcal{J}$. Let $\mathcal{C}_k$ be the subspace of $\mathcal{C}$ of almost complex structures whose Nijenhuis tensor has rank at least k at every point of M. Then $\mathcal{C}_k$ is either open and dense or empty in $\mathcal{C}$.*

Density of structures has to be intended in the C^∞-topology induced on $\mathcal{J}$ by thinking of its elements as smooth sections of the twistor bundle.

Corollary B. *Let M be a compact almost complex manifold. Then the space of maximally non-integrable almost complex structures on M is either open and dense or empty in each path-connected component of $\mathcal{J}$.*

The main tools used in proving Theorem A are local estimates for the rank of N_J along small deformations of almost complex structures and existence of real analytic sections of real analytic vector bundles.

A preferred class of manifolds on which to study the rank of N_J is that of parallelizable manifolds, in particular solvmanifolds. On parallelizable manifolds, we can describe small deformations of almost complex structures in terms of complex valued functions and controlling the rank of N_J amounts to finding solutions to a system of PDEs. While the set-up is general, we can solve the PDEs on specific examples to produce a large number of explicit complex and almost complex structures whose N_J has arbitrary constant rank on solvmanifolds of dimension 4 and 6. Note that these structures are not necessarily invariant. Indeed, even though every solvmanifold admits maximally non-integrable structures, there are many examples where every invariant structure is not maximally non-integrable. Motivated by this, we focus on 6-dimensional nilmanifolds and study the possible values of the rank of invariant structures, providing a complete classification. Note that in dimension 6 the possible values for $\operatorname{rk} N_J$ go from 0 to 3, and that the case of vanishing rank was already treated by Salamon [81].

Theorem C. *Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold and let $\mathfrak{g}$ be the Lie algebra of G. Then*

(i) M admits an invariant almost complex structure of rank 3 if and only if $\mathfrak{g}$ is isomorphic to one of

$$\begin{array}{ll}
(0,0,12,13,14+23,34-25), & (0,0,12,13,14,34-25),\\
(0,0,12,13,14+23,24+15), & (0,0,12,13,14,23+15),\\
(0,0,12,13,23,14), & (0,0,12,13,23,14-25),\\
(0,0,12,13,23,14+25), & (0,0,0,12,14-23,15+34),\\
(0,0,0,12,14,15+23), & (0,0,0,12,14,15+23+24),\\
(0,0,0,12,14,15+24), & (0,0,0,12,13,14+35),\\
(0,0,0,12,23,14+35), & (0,0,0,12,23,14-35),\\
(0,0,0,12,14,24), & (0,0,0,12,13-24,14+23),\\
(0,0,0,12,14,13-24), & (0,0,0,12,13+14,24),\\
(0,0,0,12,13,14+23), & (0,0,0,12,13,24),\\
(0,0,0,12,13,23); &
\end{array}$$

*(ii) M does **not** admit an invariant almost complex structure of rank 2 if and only if $\mathfrak{g}$ is isomorphic to one of*

$$\begin{array}{ll}
(0,0,0,12,13,23), & (0,0,0,0,0,12+34),\\
(0,0,0,0,0,12), & (0,0,0,0,0,0);
\end{array}$$

*(iii) M does **not** admit an invariant almost complex structure of rank 1 if and only if $\mathfrak{g}$ is isomorphic to one of*

$$(0,0,12,13,14+23,34-25), \quad (0,0,0,0,0,0).$$

The classification for 6-dimensional nilmanifolds and the techniques used in the proof show that there is a connection between algebraic properties of the Lie algebras and the rank of N_J. Under suitable assumptions, we can establish a similar constraint for $\operatorname{rk} N_J$ in terms of the topology of the underlying manifold valid in any dimension.

Theorem D. *Let $M = \Gamma\backslash G$ be a solvmanifold of completely solvable type and let J be an invariant almost complex structure on M. Then we have that*

$$\operatorname{rk} N_J \leq \dim_{\mathbb{R}} M - b_1(M).$$

We will see later that the topology determines almost entirely the invariants of arbitrary structures (not necessarily invariant) on compact almost complex 4-manifolds.

Bott–Chern and Aeppli cohomologies

On complex manifolds, not necessarily compact, Bott–Chern and Aeppli cohomologies are well-understood and widely studied invariants of the complex structure. They have been introduced by Bott and Chern [17] and by Aeppli [1], respectively, see also the works of Bigolin [14] and [15]. Unlike Dolbeault cohomology, they take into account the simultaneous action of ∂ and $\bar\partial$ and, if the manifold is compact, they are finite-dimensional and their complex dimensions can be used to characterize deep cohomological properties of the complex structure, like the $\partial\bar\partial$-lemma [6].

The first generalization of complex cohomologies to almost complex manifolds was given by Cirici and Wilson [29], who introduced an almost complex version of Dolbeault cohomology. Later, Coelho, Placini and Stelzig [32] gave a definition of Bott–Chern and Aeppli cohomologies of almost complex manifolds based on the use of the operators ∂ and $\bar\partial$. In this thesis, we give a different definition of Bott-Chern and Aeppli cohomologies that is based on the use of the operators d and d^c. If (M, J) is an almost complex manifold, we define its Bott–Chern and Aeppli cohomologies as the usual Bott–Chern and Aeppli cohomologies of the operators d and d^c computed on a suitable subcomplex $B^\bullet$ of the complex of forms and on the associate quotient complex $C^\bullet$, respectively. Our construction of cohomologies is inspired by the definition of Coelho, Placini and Stelzig, while the choice of the operators d and d^c is closer to the original definition of Bott and Chern and resembles the choice made by Tseng and Yau in their definition of symplectic cohomologies. While in the complex case this change of perspective yields the same cohomologies, the resulting almost complex cohomologies of the operators d and d^c are different from those built using ∂ and $\bar\partial$. On the same subcomplex $B^\bullet$, one can naturally define also the Dolbeault, Bott–Chern and Aeppli cohomologies of the operators $\delta := \partial + \bar\mu$ and $\bar\delta := \bar\partial + \mu$. In analogy with the complex case, Bott–Chern and Aeppli cohomologies defined using d and d^c or δ and $\bar\delta$ coincide. Thus δ and $\bar\delta$ appear to be appropriate generalizations of ∂ and $\bar\partial$, at least in the context of Bott–Chern and Aeppli cohomologies.

One last feature of using the operators d and d^c is that they naturally induce a $\mathbb{Z}_2$-splitting of differential forms into an even and an odd part. Thus, we define even and odd versions of de Rham and Bott–Chern cohomologies. Denote by $H^k_{d+d^c}$ the almost complex Bott–Chern cohomology, by $(H^k_{d+d^c})^{even}$ and $(H^k_{d+d^c})^{odd}$ its even and odd versions, respectively, and by H^k_{even} and H^k_{odd} the even and odd versions of de Rham cohomology, respectively. We see that the $\mathbb{Z}_2$-splitting of forms passes to Bott–Chern cohomologies.

Theorem E. *There is a natural map*

$$\begin{aligned} H^k_{d+d^c} &\longrightarrow H^k_{even} + H^k_{odd}, \\ [\alpha]_{d+d^c} &\longmapsto [\alpha^{even}]_d + [\alpha^{odd}]_d \end{aligned}$$

that induces a $\mathbb{Z}_2$-graded decomposition of Bott–Chern cohomology

$$H^k_{d+d^c} = (H^k_{d+d^c})^{even} \oplus (H^k_{d+d^c})^{odd}.$$

Existence of a $\mathbb{Z}_2$-splitting in the context of symplectic cohomologies was already observed by Tseng and Yau in Section 5 of [100].

On the almost symplectic side, we note that the differential admits a decomposition $d = d_0 + d_1 + \dots$ induced by the Lefschetz bigrading of forms. The operator d_0 is cohomological and the decomposition of d allows to define a spectral sequence from the cohomology $H^{\bullet,\bullet}_{d_0}$ to de Rham cohomology. Remarkably, the vanishing of the operator d_0 encodes interesting geometric properties.

Theorem F. *Let (M, ω) be a compact almost symplectic $2m$-manifold. If $2m = 4$, then $d_0 = 0$. If $2m \geq 6$, then $d_0 = 0$ if and only if ω is locally conformally symplectic.*

With the same idea we used for Bott–Chern and Aeppli cohomologies of almost complex manifolds, we define an almost symplectic version of the symplectic cohomologies of Tseng and Yau obtained as the cohomologies of the operators d and d^Λ computed on a suitable subcomplex and the associated quotient complex.

Spaces of harmonic forms

Classical Hodge theory for compact Hermitian manifolds establishes an isomorphism between Dolbeault, Bott–Chern and Aeppli cohomologies and certain spaces of harmonic forms obtained as the kernel of suitable self-adjoint elliptic operators, called the Dolbeault, Bott–Chern and Aeppli Laplacians, respectively. By the general theory of self-adjoint elliptic operators, the spaces of harmonic forms are finite-dimensional vector spaces on compact manifolds. Thanks to the isomorphism with the cohomologies, this implies that the dimensions of the spaces of harmonic forms, which in principle depend on the choice of Hermitian metric, are actually metric-independent.

As already observed by Kodaira, the Dolbeault Laplacian is an elliptic operator even on almost Hermitian manifolds. Hence, the dimension of its kernel computed on (p, q)-forms is an invariant $h^{p,q}_{\bar\partial}$ of the almost Hermitian structure. Motivated by the isomorphism valid in the complex case, Kodaira and Spencer posed the following problem, see Problem 20 in [47].

Kodaira–Spencer's problem. Is $h^{p,q}_{\bar\partial}$ independent of the choice of the [almost] Hermitian structure? If not, give some other definition of the $h^{p,q}_{\bar\partial}$ which depends only on the almost complex structure and which generalizes the $h^{p,q}_{\bar\partial}$ of a complex manifold.

The first (negative) answer was given by Holt and Zhang almost 70 years later [53]. They exhibited an example of a compact almost complex 4-manifold for which the value of $h^{0,1}_{\bar\partial}$ varies for different choices of metric. A metric-independent definition of the numbers $h^{p,q}_{\bar\partial}$ valid on arbitrary almost complex manifolds has not yet appeared in the literature, while in dimension 4 a solution was proposed by Cirici and Wilson [30]. Nevertheless, several authors carried out an intense work in the study of the spaces of harmonic forms associated to various Laplacians in the hope of finding a solution to Kodaira–Spencer's problem. The main differences among the works on the spaces of harmonic forms lie in which differential operators are used to build the Laplacians and in the choice of considering *Hodge-type* Laplacians of the second order or *Bott–Chern-type* Laplacians of the fourth order. The possible choices for the operators are $\bar\partial$, $\bar\delta$ or d for the Hodge-type Laplacian and the pairs of operators $(\partial, \bar\partial)$, $(\delta, \bar\delta)$ or (d, d^c) for the Bott–Chern-type Laplacians. We give an exhaustive review of the literature on the topic.

- The approach using ∂ and $\bar\partial$ has been studied by Cattaneo, Tardini and Tomassini [26], Holt [48], Holt and Piovani [49], Holt and Zhang [52, 53], Piovani and Tomassini [79], Tardini and Tomassini [94, 95], for what concerns the Hodge-type Laplacian $\Delta_{\bar\partial}$ and by Holt [48], Holt and Piovani [49], Piovani and Tardini [77], Piovani and Tomassini [78], for what concerns the Bott–Chern-type Laplacian $\Delta_{\partial+\bar\partial}$. The related invariants are the numbers $h^{p,q}_{\bar\partial}$ and $h^{p,q}_{\partial+\bar\partial}$.

- The approach using δ and $\bar\delta$ has been introduced and studied by Tardini and Tomassini [96]. They considered the Hodge-type Laplacian $\Delta_{\bar\delta}$ and the Bott–Chern-type Laplacian $\Delta_{\delta+\bar\delta}$. The dimensions of the related spaces of harmonic forms are $h^k_{\bar\delta}$ and $h^k_{\delta+\bar\delta}$.

- The approach using d and d^c was introduced by Cirici and Wilson for the Hodge-type Laplacian Δ_d on almost Kähler manifolds [31] and was furhter studied by Holt, Piovani and Tomassini on almost complex manifolds [51].

In this thesis, we introduce for the first time the Bott–Chern-type Laplacian Δ_{d+d^c}. In this case, the invariants are $h_d^{p,q}$ and $h^k_{d+d^c}$.

The focus of the research is in studying the properties of the spaces of harmonic forms and the dependence on the metric of their dimensions. One of the main goals of this thesis is to address the problem in full generality. We formulate the following problem, that appears as a natural generalization of Kodaira–Spencer's problem.

Generalized Kodaira–Spencer's problem.
Let (M, J) be a compact almost complex manifold. Fix an almost Hermitian structure and consider the associated numbers $h^k_{\bar\partial}$, $h^k_{\partial+\bar\partial}$, $h^k_{\bar\delta}$, $h^k_{\delta+\bar\delta}$, $h_d^{p,q}$ and $h^k_{d+d^c}$.

- Are these numbers independent of the choice of almost Hermitian structure?
- Are they independent of the choice of almost Kähler structure?

To address the problem, we summarize and improve the known results. We also introduce new spaces of harmonic forms built using the operators d and d^c. The main result we obtain on the metric-independence is valid for almost Kähler 4-manifolds.

Theorem G. *Let (M, J) be a compact almost complex 4-manifold admitting a J-compatible almost Kähler metric. Then the numbers $h^k_{\bar\delta}$, $h_d^{p,q}$, $h^k_{\delta+\bar\delta}$ and $h^k_{d+d^c}$ do not depend on the choice of J-compatible almost Kähler metric.*

If we think of the description of complex and Kähler surfaces given by the Enriques–Kodaira classification, it is not surprising that assuming low-dimension or existence of an almost Kähler metric yields a better behaviour in terms of metric-independence. Theorem G provides a full answer to the generalized Kodaira–Spencer's problem valid for almost Kähler 4-manifolds. In its proof it is essential to understand the actions of the Hodge $*$ operator and the symplectic $*_s$ operator, and the interplay between those actions and the bidegree of forms induced by the almost complex structure. Along the way, we establish several results that are interesting on their own, since they hold on almost complex manifolds of arbitrary dimension or that do not admit almost Kähler metrics.

Once established that certain invariants are metric-independent, the next step is to use them to distinguish between different almost complex structures. The other natural step is to determine precisely how they are related to each other, and the two problems are often intertwined. As an example, on compact complex surfaces, the dimensions of Dolbeault, Bott–Chern and Aeppli cohomology groups not only are metric-independent, but they are completely determined by the topology of the underlying manifold, see [10], and all complex structures on a given smooth manifold have the same cohomological invariants.

This is very different from the situation in the almost complex case, where we have genuine almost complex invariants. Actually, we are able to determine the precise dependence on topological constants of the almost Kähler invariants considered in Theorem G.

Theorem H. *Let (M, J) be a compact almost complex 4-manifold admitting a J-compatible almost Kähler metric. Then, for every choice of J-compatible almost Kähler metric, the invariants $h^k_{\bar{\delta}}$, $h^{p,q}_d$, $h^k_{\delta+\bar{\delta}}$ and $h^k_{d+d^c}$ are completely determined by:*

- *the oriented topology of the underlying manifold (more precisely, by the numbers b_1 and b^-);*
- *the almost complex invariant $h^1_{d+d^c}$;*
- *the almost complex invariant h^-_J.*

Furthermore, the invariants $h^1_{d+d^c}$ and h^-_J do not completely determine each other.

We point out that the results of Theorems G and H are partially valid even without the almost Kähler assumption: the numbers $h^k_{d+d^c}$, for $k \neq 3$, are almost complex invariants on arbitrary almost Hermitian 4-manifolds and they depend only on the topology, on $h^1_{d+d^c}$ and on h^-_J. It is not known if the number $h^3_{d+d^c}$ depends on the choice of metric in the non-almost Kähler case.

The utility of the numbers $h^1_{d+d^c}$ and h^-_J, the only degrees of freedom for almost Kähler invariants, is much wider. We show that they can be used to distinguish between different almost complex structures. In particular, the number $h^1_{d+d^c}$ is able to do so even when other invariants like $\operatorname{rk} N_J$, h^-_J, or symplectic invariants, fail. Motivated by this and the fact that $h^1_{d+d^c}$ is completely known for compact complex surfaces, we formulate the following conjecture.

Conjecture. Let M be a compact almost complex $2m$-manifold. Then, the number $h^1_{d+d^c}$ vanishes for a generic almost complex structure.

As a consequence of Theorem A, we are able to confirm the conjecture for manifolds that admit maximally non-integrable structures, in particular for manifolds of dimension at least 10 or for homogeneous spaces.

Aside from the main results described above, the thesis contains a careful description of the relations occurring among different spaces of harmonic forms and a discussion of how they are related with the cohomologies. We also establish several lemmas that allow to simplify or to avoid direct computations of the spaces of harmonic forms. This is applied to a large number of examples, explicitly showing how to compute the rank of N_J and the spaces of harmonic forms on:

- every 6-dimensional nilmanifold;

- every 4-dimensional solvmanifold admitting a complex structure and certain 6-dimensional solvmanifolds;
- compact complex surfaces obtained as a quotient of non-solvable Lie groups, like the Hopf surface.

Outline of the thesis

This thesis collects the results of the published works [87], [88], [89], [90] and of the preprint [85], together with material that appears here for the first time. It is structured as follows.

Chapter 1 presents the fundamentals of almost Hermitian geometry, describing the action of compatible triples on the space of forms and the integrability conditions for almost complex and almost symplectic structures. It also contains some well-known result on multicomplexes, on homogeneous manifolds and on small deformations of almost complex structures. Minor original results are present in Sections 1.2, 1.3 and 1.4.

Chapter 2 contains a thorough study of the rank of the Nijenhuis tensor. Sections 2.1 and 2.2 focus on theoretical aspects of almost complex structures, recalling the important results of [32] and proving Theorem A using real analytic curves of almost complex structures. In Sections 2.3 and 2.4, we underline algebraic and topological constraints on the rank of N_J. There, we prove the classification of invariant structures on 6-dimensional nilmanifolds of Theorem C. We also provide an explicit method to produce structures of prescribed rank on parallelizable manifolds and we apply it to obtain a large number of examples.

Chapter 3 is entirely devoted to almost complex and almost symplectic cohomologies. On almost complex manifolds, we define the Bott–Chern and Aeppli cohomologies of the operators d and d^c and several cohomologies of the operators δ and $\bar{\delta}$. We study their main properties and how they are related to each other. On almost symplectic manifolds, we define the almost symplectic cohomologies of the operators d and d^Λ, and we study the spectral sequence going from the cohomologies of the operators d_j to the de Rham cohomology. Finally, we underline how, in the non-integrable setting, the de Rham and Bott–Chern cohomologies naturally admit a $\mathbb{Z}_2$-splitting into even and odd forms.

Chapter 4 deals with spaces of harmonic forms on compact almost Hermitian manifolds. After introducing these spaces in full generality, we move to the 4-dimensional case to study their dependence on the choice of metric. Here is where we prove Theorem G, that solves the generalized Kodaira–Spencer's problem on almost Kähler 4-manifolds. In the rest of the chapter, we describe the relations among different spaces of harmonic forms and their inclusion into the cohomologies of Chapter 3. We conclude the chapter with explicit computations of cohomologies and of harmonic forms on the Kodaira–Thurston manifold.

Chapter 5 consists of two parts. The first part, Sections 5.1 and 5.2, takes a detour from almost complex structures and focuses on integrable structures. We precisely determine Bott–Chern numbers of compact complex surfaces in terms of topological constants and we compute the dimensions of the spaces of harmonic forms of invariant Hermitian structures on complex surfaces diffeomorphic to solvmanifolds. In the second part, Sections 5.3 and 5.4, we determine the dependence on the topology of the almost Kähler invariants computed in Chapter 4, proving Theorem H. We especially focus on the invariant $h^1_{d+d^c}$ on arbitrary almost complex manifolds and we formulate a conjecture on its vanishing, proving it on high-dimensional manifolds.

CHAPTER 1

Background material

This chapter collects background material that will be used consistently throughout the thesis. Definitions and results that play a more specific role will be recalled outside of this chapter, right before they are needed or whenever they are relevant for the discussion. Any classical textbook on complex geometry will be a good reference for the basic results. We recommend [10], [43], [54], [55] and [71]. More recent results, which do not yet appear in books, are accompanied by a reference to the relevant article. Sections 1.2, 1.3 and 1.4 contain minor original results.

1.1 Multicomplexes and spectral sequences

The definitions we give in this section can be stated in more generality for arbitrary abelian groups. Since we always deal with the de Rham algebra of a smooth manifold, we give every definition taking into account also the algebra product that, on forms, is induced by the wedge product.

A *differential* $(\mathbb{Z}, \mathbb{Z})$*-graded multicomplex* $(A^{\bullet,\bullet}, d)$, sometimes simply *bigraded multicomplex* or *multicomplex*, is a $(\mathbb{Z}, \mathbb{Z})$-graded algebra $\{A^{p,q}\}_{p,q\in\mathbb{Z}}$ endowed with a linear differential d satisfying the equation $d^2 = 0$ and the graded Leibniz rule

$$d(a \cdot b) = da \cdot b + (-1)^{p+q}\, a \cdot db,$$

where $a \in A^{p,q}$, $b \in A^{r,s}$ and $\cdot$ denotes the algebra product. In terms of the bigrading, it admits a decomposition

$$d = \sum_{j=0}^{j_{max}} d_j \tag{1.1.1}$$

with

$$d_j \colon A^{p,q} \longrightarrow A^{p+j,q-j+1}, \quad j = 0, \ldots, j_{max}.$$

If p and q run over a finite number of indices we say that the multicomplex is *bounded*. A *double complex*, or *bicomplex*, is a multicomplex with only two differentials. A *subcomplex* $(B^{\bullet,\bullet}, d)$ of $(A^{\bullet,\bullet}, d)$ is a bigraded subalgebra $B^{\bullet,\bullet} \subseteq A^{\bullet,\bullet}$ preserved by the action of the differential d. The associated *quotient complex* $(C^{\bullet,\bullet}, d)$ is obtained by taking $C^{p,q} := A^{p,q}/B^{p,q}$ endowed with the differential induced by d.

The bigrading on $A^{\bullet,\bullet}$ naturally induces a grading taking $A^k := \bigoplus_{p+q=k} A^{p,q}$. The (graded) complex $A^\bullet$ inherits a *filtration* induced by the bigrading, i.e., a sequence of subcomplexes

$$A^k = F^0 A^k \supseteq F^1 A^k \supseteq F^2 A^k \supseteq \ldots \supseteq 0$$

given by

$$F^p A^k := \bigoplus_{j \geq p} A^{j,k-j}$$

and compatible with the action of the differential d in the sense that

$$d(F^p A^k) \subseteq F^p A^{k+1}.$$

The *bigraded cohomology* of the multicomplex is the quotient

$$H^{p,q}(A^{\bullet,\bullet}, d) := \frac{F^p(\ker d \cap A^q)}{F^p(\operatorname{Im} d \cap A^q)}.$$

A *bigraded spectral sequence* $(E_*^{\bullet,\bullet}, d_*)$ is a sequence of multicomplexes $(E_r^{\bullet,\bullet}, d_r)$ indexed by an integer $r \geq 0$, each one called *page* of the spectral sequence, endowed with a differential

$$d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad d_r^2|_{E_r^{p,q}} = 0,$$

and such that each page is isomorphic to the cohomology of the previous one

$$E_{r+1}^{\bullet,\bullet} \cong H^{\bullet,\bullet}(E_r^{\bullet,\bullet}, d_r).$$

We say that the spectral sequence *degenerates at page* r_0 if for all $r \geq r_0$ the differential d_r is trivial. In such a case there is an isomorphism

$$(E_r^{\bullet,\bullet}, d_r = 0) \cong (E_{r_0}^{\bullet,\bullet}, d_{r_0}) \quad \text{for all } r \geq r_0.$$

If a spectral sequence degenerates, we set $E_\infty^{\bullet,\bullet} := E_{r_0}^{\bullet,\bullet}$, we say that the spectral sequence *converges* to $E_\infty^{\bullet,\bullet}$ and we write $E_0^{p,q} \Rightarrow E_\infty^{p,q}$. The following is a classical result on spectral sequences, see unnumbered Proposition on page 440 in [43].

Proposition 1.1.1. *Every filtered complex $(F^\bullet A^\bullet, d)$ has a natural associated spectral sequence obtained setting*

$$E_0^{p,q} := \frac{F^p A^{p+q}}{F^{p+1} A^{p+q}}$$

and requiring that $E_{r+1}^{\bullet,\bullet} \cong H^{\bullet,\bullet}(E_r^{\bullet,\bullet}, d_r)$, where each differential d_r is induced by d.

As a direct consequence, every multicomplex admits an associated spectral sequence. Moreover, the differentials d_r of the spectral sequence can be explicitly described, up to isomorphism, in terms of the original decomposition (1.1.1), providing a useful framework for explicit computations, see [33] for the case of double complexes and [62] for the general case of multicomplexes.

Finally, we remark that every spectral sequence arising from a bounded multicomplex always degenerates in a finite number of steps, see Section 5.3 in [19].

1.2 Almost Hermitian manifolds

In this section we describe the space of forms of almost Hermitian manifolds and the main differential operators studied in this thesis.

Let M be a smooth manifold. Suppose that M has even real dimension $2m$. Denote by TM its tangent bundle and by T^*M its cotangent bundle. The bundles of real k-forms and of complex k-forms are the bundles

$$\Lambda^k_{\mathbb{R}} := \bigwedge^k T^*M \quad \text{and} \quad \Lambda^k := \bigwedge^k T^*M^{\mathbb{C}},$$

respectively, where $T^*M^{\mathbb{C}}$ denotes the complexified cotangent bundle. The corresponding spaces of smooth sections are denoted by $A^k_{\mathbb{R}}$ and A^k, and they are called spaces of real and complex k-forms, respectively. We endow M with two geometric structures:

- an *almost complex structure* on M is an endomorphism of its tangent bundle $J \in \operatorname{End}(TM)$ such that $J^2 = -\operatorname{Id}_{TM}$;
- an *almost symplectic structure* on M is a real non-degenerate 2-form $\omega \in A^2_{\mathbb{R}}$.

It is not restrictive to also endow M with a Riemannian metric g. We say that the triple (J, ω, g) is a *compatible triple* if for every X and Y vector fields on M we have that

$$g(X, Y) = \omega(X, JY). \tag{1.2.1}$$

A compatible triple is also called an *almost Hermitian structure*. Given any two elements among J, ω and g, the remaining one is uniquely determined by the compatibility condition (1.2.1). For instance, if J and ω are assigned, we say that

- J is *ω-tamed* if $\omega(X, JX) > 0$ for every non-zero vector field X;
- ω is *J-invariant* if $\omega(JX, JY) = \omega(X, Y)$ for all vector fields X and Y.

If both conditions are satisfied, then $\omega(X, JY)$ is symmetric and positive, determining uniquely a Riemannian metric.

We adopt the following terminology: the term *almost Hermitian manifold* could refer to a manifold admitting an almost Hermitian structure, without choosing a specific structure, or to a quadruple (M, J, ω, g). Similarly, the term *almost complex manifold* refers either to a smooth manifold admitting at least one almost complex structure or to a pair (M, J). The same holds for *almost symplectic manifolds*.

Since there are no constraints for the existence of a Riemannian metric, the spaces of almost complex manifolds, of almost symplectic manifolds and of almost Hermitian manifolds coincide with each other. We will change the name depending on which structure we are dealing with.

Action of compatible triples on the space of forms

The existence of a compatible triple induces three operators on A^k: the Hodge $*$ operator, the operator J and the Lefschetz operator L.

The Hodge $*$ operator is induced by the Riemannian metric g. For α and $\beta \in A^k$, it is defined by the relation

$$\alpha \wedge \overline{*\beta} = g(\alpha, \beta) \operatorname{Vol}_g,$$

where Vol_g is the volume form of the metric. Its action sends A^k into A^{2m-k} and vice versa. If M is compact, the metric induces an L^2-Hermitian product on $A^{p,q}$ defined by

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \overline{*\beta}. \tag{1.2.2}$$

Let $P\colon A^k \to A^{k+l}$ be any operator of (possibly negative) degree l. The *formal adjoint of* P is the unique operator $P^*\colon A^k \to A^{k-l}$ such that the equality

$$\langle P\alpha, \beta \rangle = \langle \alpha, P^*\beta \rangle$$

holds for all $\alpha \in A^k$ and $\beta \in A^{k+l}$. If M is not compact, the Hermitian product is defined on the space of forms with compact support or on the space of L^2-forms, see [50] for a recent approach to L^2-forms related to the theory we present in Chapter 4.

The almost complex structure J acts on k-forms by duality. For $\alpha \in A^k$ and $X_1, \dots, X_k$ vector fields, the duality action is

$$J^*\alpha(X_1, \dots, X_k) \coloneqq \alpha(JX_1, \dots, JX_k).$$

We will omit the superscript $*$ and denote it simply by J. Then J is an endomorphism of k-forms satisfying $J^2 = (-1)^k \operatorname{Id}$. On 1-forms, we have that $J^2 = -\operatorname{Id}$, thus $T^*M^{\mathbb{C}}$ decomposes into $(\pm i)$-eigenbundles denoted by $T^*M^{1,0}$ and $T^*M^{0,1}$, respectively. The bundle of complex (p,q)-forms is the bundle

$$\Lambda^{p,q} \coloneqq \bigwedge^p T^*M^{1,0} \otimes \bigwedge^q T^*M^{0,1}.$$

The space of its smooth sections $A^{p,q}$ is the *space of complex (p,q)-forms* and complex conjugation defines an isomorphism $A^{p,q} \xrightarrow{\sim} A^{q,p}$. The space of k-form decomposes as the direct sum of bigraded spaces as

$$A^k = \bigoplus_{p+q=k} A^{p,q}$$

and there is a natural projection $\pi^{p,q} \colon A^\bullet \to A^{p,q}$. Then J acts on $\alpha \in A^{p,q}$ as $J\alpha = (-1)^q\, i^{p+q}\alpha$ and it preserves the bigrading. The space of forms endowed with the wedge product and the bigrading induced by J inherits the structure of *bigraded algebra* $(A^{\bullet,\bullet}, \wedge)$ with

$$\wedge \colon A^{p,q} \times A^{r,s} \longrightarrow A^{p+r,q+s}.$$

The almost symplectic structure ω acts on k-forms by the Lefschetz operator and its dual operator. The *Lefschetz operator* L is the operator

$$\begin{aligned} L \colon A^k &\longrightarrow A^{k+2}, \\ \alpha &\longmapsto \omega \wedge \alpha. \end{aligned}$$

The *dual Lefschetz operator* $\Lambda \colon A^k \to A^{k-2}$ is defined as contraction by ω. A k-form α, with $k \le m$, is called *primitive* if $\Lambda\alpha = 0$, or, equivalently, if $L^{m-k+1}\alpha = 0$. Denote by $\mathcal{P}^k$ the space of primitive k-forms. Every function and 1-form is primitive and, by definition, there are no primitive k-forms for $k > m$.

It is convenient to have two other (equivalent) descriptions of the dual Lefschetz operator. The first one is a local description. Fix local coordinates $\{x_j\}_{j=1}^{2m}$ and let

$$\omega = \frac{1}{2}\sum_{j,k} \omega_{jk}\, dx^{jk}$$

be a local expression for ω. The dual Lefschetz operator is given by

$$\Lambda := \frac{1}{2}\sum_{j,k} \omega^{jk}\iota_{\partial_j}\iota_{\partial_k},$$

where $\partial_j := \partial/\partial x_j$ and the matrix (ω^{jk}) is the inverse of (ω_{jk}).

The second description is given in terms of formal adjoints. Extend the action of ω from vector fields to differential forms by duality. Then for α and $\beta \in A^k$ the *symplectic adjoint* P^{*_s} of an operator P is defined by the identity

$$\omega(P\alpha, \beta) = \omega(\alpha, P^{*_s}\beta).$$

The dual Lefschetz operator Λ is the symplectic adjoint of L. More explicitly, consider the *symplectic* $*_s$ *operator* defined by the relation

$$\alpha \wedge \overline{*_s\,\beta} = \omega(\alpha, \beta)\,\frac{\omega^m}{m!}.$$

This is a symplectic version of the Hodge $*$ operator defined for a Riemannian metric and it sends A^k into A^{2m-k}. The dual Lefschetz operator is

$$\Lambda := *_s\, L\, *_s\,.$$

Since ω is non-degenerate, for each $k = 1, \dots, m$, powers of the Lefschetz operator give an isomorphism

$$L^k\colon A^{m-k} \xrightarrow{\ \sim\ } A^{m+k}, \tag{1.2.3}$$

called the *Lefschetz isomorphism.* The inverse of the map L^k is Λ^k. It is important to observe that in general L and Λ do not commute and that $\Lambda^k L^k = \mathrm{Id}$ only on $(m-k)$-forms, while $L^k\Lambda^k = \mathrm{Id}$ only on $(m+k)$-forms. On forms of arbitrary degree, the action of (powers of) L and Λ is more complicated and we need to take into account the role played by their commutator, the operator $H := [\Lambda, L]$.

The action of powers of L on primitive forms allows to define a *symplectic bidegree.* Consider the spaces

$$\mathcal{L}^{r,s} := \{\omega^r \wedge P^s : P^s \in \mathcal{P}^s\} \subseteq A^{2r+s}.$$

The triple (L, Λ, H) defines a representation of $\mathfrak{sl}(2,\mathbb{C})$ acting on $A^\bullet$ that induces a *Lefschetz decomposition* of the space A^k into bigraded components, see Théorème 3 on page 26 in [108], namely

$$A^k = \bigoplus_{j \geq \max\{k-m,0\}} \mathcal{L}^{j,k-2j}.$$

There is a natural projection $\pi^{r,s}\colon A^\bullet \to \mathcal{L}^{r,s}$. We refer to the bigrading $\mathcal{L}^{\bullet,\bullet}$ as the *Lefschetz bigrading* or *symplectic bigrading*. The Lefschetz isomorphism (1.2.3) is compatible with the bigrading and it gives isomorphisms

$$L^{m-2r-s}\colon \mathcal{L}^{r,s} \xrightarrow{\sim} \mathcal{L}^{m-r-s,s}.$$

In contrast to what happens for the bigrading induced by an almost complex structure, there is no algebra structure induced by the wedge product on $\mathcal{L}^{\bullet,\bullet}$ since the wedge product of primitive forms is not necessarily a primitive form. Nevertheless, we still have that

$$\mathcal{L}^{r,s} \wedge \mathcal{L}^{t,u} \subseteq \bigoplus_{j \geq r+t} \mathcal{L}^{j,s+u-2j}. \tag{1.2.4}$$

The compatibility condition (1.2.1) allows to express $*$ as

$$* = *_s J = J *_s, \tag{1.2.5}$$

while Théorème 2 on page 23 in [108] gives the explicit expression for the action of the Hodge $*$ on each summand of the Lefschetz decomposition

$$*L^r P^k = (-1)^{\frac{k(k+1)}{2}} \frac{r!}{(m-k-r)!} L^{m-k-r} J P^k, \tag{1.2.6}$$

with $P^s \in \mathcal{P}^s$. In particular, by comparing (1.2.5) and (1.2.6), we see that $*_s$ acts on forms of bidegree (p,q) as a multiple of the Lefschetz operator.

Bigraded decomposition of the differential

First, we consider the almost complex point of view and the action of d on the space of (p,q)-forms. An operator is said to be *bigraded of bidegree* (r,s) if it sends (p,q)-forms into $(p+r,q+s)$-forms. The first observation is that d is **not** a bigraded operator. Nevertheless, we have some control on its action: while a priori we only know that $d(A^{p,q}) \subseteq A^{p+q+1}$, it turns out that

$$d(A^{p,q}) \subseteq A^{p+2,q-1} \oplus A^{p+1,q} \oplus A^{p,q+1} \oplus A^{p-1,q+2}.$$

Projecting on the various components of its image, we decompose d as the sum of four operators, $d = \mu + \partial + \bar\partial + \bar\mu$, where

$$\begin{aligned} \mu &:= \pi^{p+2,q-1} \circ d|_{A^{p,q}}, & \partial &:= \pi^{p+1,q} \circ d|_{A^{p,q}}, \\ \bar\partial &:= \pi^{p,q+1} \circ d|_{A^{p,q}}, & \bar\mu &:= \pi^{p-1,q+2} \circ d|_{A^{p,q}}. \end{aligned}$$

The operators ∂ and $\bar\partial$ are conjugate to each other, in the sense that $\bar\partial\alpha = \overline{\partial\bar\alpha}$, and they have bidegree $(1,0)$ and $(0,1)$, respectively. The same is true for μ and $\bar\mu$, that have bidegree $(2,-1)$ and $(-1,2)$, respectively. Separating by bidegree the terms of the equation $d^2 = (\mu + \partial + \bar\partial + \bar\mu)^2 = 0$, we obtain the relations

$$\begin{cases} \bar\mu^2 = 0, \\ \bar\mu\bar\partial + \bar\partial\bar\mu = 0, \\ \bar\partial^2 + \bar\mu\partial + \partial\bar\mu = 0, \\ \partial\bar\partial + \bar\partial\partial + \mu\bar\mu + \bar\mu\mu = 0 \end{cases} \tag{1.2.7}$$

and the conjugate equations, that imply that $(A^{\bullet,\bullet}, \mu, \partial, \bar\partial, \bar\mu)$ is a multicomplex endowed with four differential, in the sense of Section 1.1.

In addition to μ, ∂, $\bar\partial$ and $\bar\mu$, we introduce several operators obtained as linear combinations of them that will play an essential role in the rest of the thesis. First, we consider the operator $d^c := J^{-1}dJ$. Taking into account the explicit action of J on (p,q)-forms, we can write d^c in terms of μ, ∂, $\bar\partial$ and $\bar\mu$ as

$$d^c = i(\mu - \partial + \bar\partial - \bar\mu).$$

From the definition, it follows immediately that $(d^c)^2 = 0$, while a short computation yields

$$dd^c + d^c d = 4i(\bar\partial^2 - \partial^2).$$

The operator δ and its conjugate $\bar\delta$ are defined by the equations

$$\delta := \frac{1}{2}(d + id^c) = \partial + \bar\mu \quad \text{and} \quad \bar\delta := \frac{1}{2}(d - id^c) = \bar\partial + \mu. \tag{1.2.8}$$

This gives the decompositions

$$d = \delta + \bar\delta \quad \text{and} \quad d^c = i(\bar\delta - \delta). \tag{1.2.9}$$

By (1.2.7), (1.2.8) and (1.2.9), we have that

$$\delta^2 = -\bar\delta^2 = \frac{i}{4}(dd^c + d^c d) = \partial^2 - \bar\partial^2 \quad \text{and} \quad \delta\bar\delta = -\frac{i}{4}(dd^c - d^c d). \tag{1.2.10}$$

In particular, we observe that for the pairs of operators (d, d^c) and $(\delta, \bar\delta)$ we have

$$\begin{cases} d^2 = 0, \\ (d^c)^2 = 0, \\ dd^c + d^c d \neq 0, \end{cases} \quad \text{and} \quad \begin{cases} \delta^2 \neq 0, \\ \bar\delta^2 \neq 0, \\ \delta\bar\delta + \bar\delta\delta = 0, \end{cases} \tag{1.2.11}$$

while the operators ∂ and $\bar\partial$ neither square to zero nor anti-commute.

Now, we consider the almost symplectic point of view and the action of d on the space $\mathcal{L}^{r,s}$. As it happens for the almost complex bigrading, the differential does not preserve the symplectic bigrading. We see that, even though we have some control on the action of d, the situation is more complicated than in the almost complex case, since it involves the differential of ω. The Lefschetz decomposition of $d\omega$ is

$$d\omega = H + \omega \wedge \theta, \tag{1.2.12}$$

where $H \in A^3_{\mathbb{R}}$ is real primitive 3-form and $\theta \in A^1_{\mathbb{R}}$ is a real (primitive by degree reasons) 1-form. The form θ is usually called the *Lee form* of ω.

Proposition 1.2.1. *Let (M, ω) be an almost symplectic manifold. Then*

$$d(\mathcal{L}^{r,s}) \subseteq \bigoplus_{j\geq 0} \mathcal{L}^{r-1+j,s+3-2j}.$$

Proof. Let $\omega^r \wedge P^s \in \mathcal{L}^{r,s}$. Then $P^s \in \mathcal{L}^{0,s} = \mathcal{P}^s$ and $dP^s \in \bigoplus_{j\geq 0} \mathcal{L}^{j,2-sj}$. By (1.2.4) and (1.2.12), we have that

$$d(\mathcal{L}^{r,s}) \subseteq (\mathcal{L}^{0,3} \oplus \mathcal{L}^{1,1}) \wedge (\bigoplus_{j\geq 0} \mathcal{L}^{j,2-sj}) \subseteq \bigoplus_{j\geq 0} \mathcal{L}^{r-1+j,s+3-2j}. \qquad \square$$

Projecting on the components of the image of $d|_{\mathcal{L}^{r,s}}$, we get the decomposition

$$d = \sum_{j\geq 0}^{j_{max}} d_j, \tag{1.2.13}$$

with $d_j := \pi^{r-1+j,s+3-2j} \circ d|_{\mathcal{L}^{r,s}}$ and j_{max} to be determined. Observe that the operator d_j has symplectic bidegree $(j-1, 3-2j)$. For a fixed bidegree (r, s), we can be more precise on the vanishing of certain differentials.

Lemma 1.2.2. *Consider the action of d on $\mathcal{L}^{r,s}$ and decompose the differential according to* (1.2.13)*. Then*

(i) if $r + s = m$, then $d_1 = 0$;

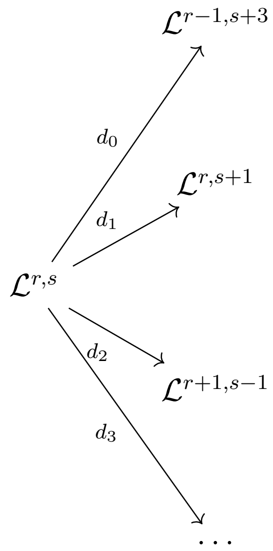
(ii) if $r + s \in \{m-1, m\}$ or if $r = 0$, then $d_0 = 0$;

(iii) $d_j = 0$ for all $j > \frac{s+3}{2}$.

Proof. We prove only claim (*iii*). Claims (*i*) and (*ii*) have a similar, simpler, proof. The proof starts by arranging the bigraded spaces of forms in the *Lefschetz* *pyramid*

$$\begin{array}{ccccccccc}
 & & & & \mathcal{L}^{0,m} & & & & \\
 & & & \mathcal{L}^{0,m-1} & & \mathcal{L}^{1,m-1} & & & \\
 & & \mathcal{L}^{0,m-2} & & \mathcal{L}^{1,m-2} & & \mathcal{L}^{2,m-2} & & \\
 & \iddots & & & \vdots & & & \ddots & \\
\mathcal{L}^{0,0} & & \cdots & & \cdots & & \cdots & & \mathcal{L}^{m,0}
\end{array}$$

see also [101]. The action of the differential on $\mathcal{L}^{r,s}$ corresponds to the arrows



and an arrow with target outside of the pyramid corresponds to a vanishing differential. The maximum value of j for which $d_j \neq 0$ depends on the number of terms in the Lefschetz decomposition

$$A^{2r+s+1} = \ldots \oplus \mathcal{L}^{r+1,s-1} \oplus \mathcal{L}^{r+2,s-3} \oplus \ldots$$

The last term in the decomposition is $\mathcal{L}^{r+s/2,1}$ if s is even, or $\mathcal{L}^{r+(s+1)/2,0}$ if s is odd, giving the upper bound $j \leq \lfloor \frac{s+3}{2} \rfloor$. $\square$

In particular, since primitive forms have at most bidegree m, we have that $j_{max} = \lfloor \frac{m+3}{2} \rfloor$ in (1.2.13).

Separating bidegrees in the equation $d^2 = 0$, we get the relations

$$\sum_{j=0}^{k} d_j d_{k-j} = 0 \quad \text{for each } k \geq 0, \tag{1.2.14}$$

that are analogous to the relations (1.2.7) obtained for almost complex operators.

We need to define one more differential, which has a role similar to that played by d^c on almost complex manifolds: the *symplectic co-differential* d^Λ introduced in [21], see also [13], [64], [99], [100], [101] and [110] for further relevant developments. The operator d^Λ is defined as the symplectic adjoint of d. On k-forms, it has the explicit expression

$$d^\Lambda = (-1)^{k+1} *_s d *_s .$$

The operators d and d^Λ both square to zero, but in general they do not anti-commute. By the action of d on $\mathcal{L}^{r,s}$, we determine that d^Λ acts on $\mathcal{L}^{r,s}$ as

$$d^\Lambda(\mathcal{L}^{r,s}) \subseteq \bigoplus_{j\geq 0} \mathcal{L}^{r-2+j,s+3-2j}.$$

By taking projections on the components of the image, we get the decomposition

$$d^\Lambda = \sum_{j\geq 0} d^\Lambda_j \quad \text{with} \quad d^\Lambda_j := \pi^{r-2+j,s+3-2j} \circ d^\Lambda|_{\mathcal{L}^{r,s}}.$$

Each projection d^Λ_j is the symplectic adjoint of d_j as given in (1.2.13), namely, we have that

$$d^\Lambda_j = (-1)^{k+1} *_s d_j *_s$$

on k-forms. Finally, we observe that, once a compatible triple (J, ω, g) is fixed, the almost complex differential d^c and the symplectic co-differential d^Λ completely determine each other by the relation

$$(d^c)^* = d^\Lambda. \tag{1.2.15}$$

1.3 Integrable structures

In this section we discuss *integrability conditions* of almost complex and almost symplectic structures that, if satisfied, allow to develop rich cohomological theories.

Let (M, J) be an almost complex manifold. The *Nijenhuis tensor* of J is the $(2, 1)$-tensor defined by

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \tag{1.3.1}$$

for all vector fields X and Y. If $N_J = 0$, we say that J is *integrable*. A deep theorem due to Newlander and Nirenberg establishes that an almost complex structure induces a structure of complex manifold on M (the manifold admits local complex coordinates with holomorphic change of coordinates) if and only if its Nijenhuis tensor vanishes [74]. A *complex structure* is an integrable almost complex structure and a *complex manifold* is a pair (M, J) with J integrable.

Similarly, let (M, ω) be an almost symplectic manifold. If $d\omega = 0$, we say that the almost symplectic structure is *integrable* and we call it a *symplectic structure*. A *symplectic manifold* is a pair (M, ω) with $d\omega = 0$.

When we fix a compatible Riemannian metric, the terminology is more specific. An almost Hermitian manifold (M, J, ω, g) is said to be *almost Kähler* if $d\omega = 0$, *Hermitian* if $N_J = 0$, or *Kähler* if both $d\omega = 0$ and $N_J = 0$.

1.3.1 Complex manifolds

Let (M, J), with $N_J = 0$, be a complex manifold. The condition $N_J = 0$ holds if and only if $\bar{\mu} = 0$, since the operators μ and $\bar{\mu}$ computed on 1-forms can be identified with the dual of the complexified Nijenhuis tensor via the relation

$$\mu + \bar{\mu} = -\frac{1}{4}(N_J^{\mathbb{C}})^*.$$

More in general, integrability of J is equivalent to the vanishing of any of the operators

$$N_J, \quad \bar{\mu}, \quad \bar{\partial}^2, \quad \bar{\delta}^2, \quad dd^c + d^c d,$$

or of their complex conjugate. Under the integrability assumption, the operators δ and $\bar{\delta}$ coincide with ∂ and $\bar{\partial}$ and the equations (1.2.9) simplify to

$$d = \partial + \bar{\partial} \quad \text{and} \quad d^c = i(\bar{\partial} - \partial). \tag{1.3.2}$$

The relations (1.2.7) and (1.2.11) reduce to

$$\begin{cases} \partial^2 = 0, \\ \bar{\partial}^2 = 0, \\ \partial\bar{\partial} + \bar{\partial}\partial = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} d^2 = 0, \\ (d^c)^2 = 0, \\ dd^c + d^c d = 0. \end{cases}$$

These are precisely the conditions for which $(A^{\bullet,\bullet}, \partial, \bar{\partial})$ and $(A^{\bullet,\bullet}, d, d^c)$ are double complexes as defined in Section 1.1, hence it is natural to study the cohomologies of the operators involved. Taking separately the operators d and d^c, we obtain the cohomologies

$$H_d^k := \frac{\ker d \cap A^k}{\operatorname{Im} d \cap A^k} \quad \text{and} \quad H_{d^c}^k := \frac{\ker d^c \cap A^k}{\operatorname{Im} d^c \cap A^k},$$

which are called *complex de Rham cohomology* and d^c*-cohomology*, respectively. Starting with the operators ∂ and $\bar{\partial}$, we obtain the cohomologies

$$H_\partial^{p,q} := \frac{\ker \partial \cap A^{p,q}}{\operatorname{Im} \partial \cap A^{p,q}} \quad \text{and} \quad H_{\bar{\partial}}^{p,q} := \frac{\ker \bar{\partial} \cap A^{p,q}}{\operatorname{Im} \bar{\partial} \cap A^{p,q}},$$

which are called *∂-cohomology* and *$\bar\partial$-cohomology*, respectively. It is not hard to verify that the action of J gives the isomorphism $H^k_d \cong H^k_{d^c}$, while complex conjugation give the isomorphism $H^{p,q}_\partial \cong H^{q,p}_{\bar\partial}$. We will mostly focus on the de Rham cohomology and, for historical reasons, on the $\bar\partial$-cohomology, which is also known as *Dolbeault cohomology*.

There are two other natural cohomologies associated to a double complex that take into account the simultaneous action of the differentials: the *Bott–Chern cohomology* [17] and the *Aeppli cohomology* [1]. The Bott–Chern cohomology of the operators d and d^c is

$$H^{p,q}_{BC} := \frac{\ker d \cap \ker d^c \cap A^{p,q}}{\operatorname{Im} dd^c \cap A^{p,q}},$$

while their Aeppli cohomology is

$$H^{p,q}_{A} := \frac{\ker dd^c \cap A^{p,q}}{(\operatorname{Im} d + \operatorname{Im} d^c) \cap A^{p,q}}.$$

By (1.3.2), if we replace d and d^c by ∂ and $\bar\partial$ in the definition of the Bott–Chern and Aeppli cohomologies, the cohomology groups do not change.

There are natural inclusions of Bott–Chern cohomology into Dolbeault and de Rham cohomologies and natural arrows from Dolbeault and de Rham cohomologies into Aeppli cohomology that are all induced by the identity on the representatives. The link between Dolbeault cohomology and complex de Rham cohomology is more subtle and follows from the existence of a spectral sequence arising from the double complex, as detailed in the following example.

Example 1.3.1 (Frölicher spectral sequence). The double complex $(A^{\bullet,\bullet}, \partial, \bar\partial)$ admits a natural spectral sequence induced by the filtration of its rows, indexed by p, or of its columns, indexed by q, as done in Section 1.1. Choosing the row filtration corresponds to choosing the cohomology of the $\bar\partial$ operator. Explicitly, the row filtration is

$$F^p A^{p+q} := \bigoplus_{j\geq p} A^{j,k-j}.$$

By Proposition 1.1.1, the associated spectral sequence at page 0 is

$$E_0^{p,q} = \frac{F^p A^{p+q}}{F^{p+1}A^{p+q}} = \frac{A^{p,q} \oplus A^{p+1,q-1} \oplus A^{p+2,q-2} \oplus \dots}{A^{p+1,q-1} \oplus A^{p+2,q-2} \oplus \dots} \cong A^{p,q}$$

and the differential induced by d is $d_0 = \bar\partial$, since we are taking the quotient modulo

$$A^{p+1,q} \oplus A^{p+2,q-1} \oplus \dots$$

The first page of the spectral sequence is the cohomology of $(E_0^{p,q}, d_0)$ and it is isomorphic to the Dolbeault cohomology, giving $E_1^{p,q} \cong H_{\bar\partial}^{p,q}$. The differential induced on $E_1^{\bullet,\bullet}$ is $d_1 = \partial$ acting on Dolbeault cohomology classes. Since the double complex of bigraded forms is bounded, the spectral sequence converges in at most m steps to $E_\infty^{p,q}$, a subspace of the cohomology of the differential $d = \partial + \bar\partial$, that is, of the de Rham cohomology group H_d^{p+q}. This induces a bigrading the on de Rham cohomology in terms of a decomposition

$$H_d^k \cong \bigoplus_{p+q=k} E_\infty^{p,q}.$$

The spectral sequence we built is known as *Frölicher* (or *Hodge–de Rham*) *spectral sequence* [39], and is denoted by $H_{\bar\partial}^{p,q} \Rightarrow H_d^{p+q}$. ■

The relations among the cohomologies described above and the Frölicher spectral sequence allow to arrange them in the following diagram

$$
\begin{array}{ccccccc}
 & & & H_{BC}^{p,q} & & & \\
 & \swarrow & \swarrow & & \searrow & \searrow & \\
H_{\bar\partial}^{p,q} & \Longrightarrow & H_d^{p+q} & \xleftrightarrow[J]{\sim} & H_{d^c}^{p+q} & \Longleftarrow & H_{\partial}^{p,q} \\
 & \searrow & \searrow & & \swarrow & \swarrow & \\
 & & & H_A^{p,q} & & &
\end{array} \tag{1.3.3}
$$

Spaces of harmonic forms

Suppose that M is **compact**. Fix a J-compatible Riemannian metric and consider its Hodge $*$ operator. The formal adjoints, see Section 1.2, of the operators d, d^c, ∂ and $\bar\partial$ have the explicit expressions

$$d^* = - * d *, \quad (d^c)^* = -J^{-1} * d * J, \quad \partial^* = - * \bar\partial * \quad \text{and} \quad \bar\partial^* = - * \partial *,$$

respectively. The Laplacians of the operators d and $\bar\partial$ are

$$\Delta_d := dd^* + d^*d \quad \text{and} \quad \Delta_{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial,$$

respectively. The Laplacians of the operators d^c and ∂ are obtained from Δ_d and $\Delta_{\bar\partial}$ by the action of J and of complex conjugation, respectively. The corresponding spaces of harmonic forms are

$$
\begin{aligned}
\mathcal{H}_d^k &:= A^k \cap \ker \Delta_d = \{\alpha \in A^k : d\alpha = 0 \text{ and } d^*\alpha = 0\},\\
\mathcal{H}_{d^c}^k &:= A^k \cap \ker \Delta_{d^c} = \{\alpha \in A^k : d^c\alpha = 0 \text{ and } (d^c)^*\alpha = 0\},\\
\mathcal{H}_{\partial}^{p,q} &:= A^{p,q} \cap \ker \Delta_{\partial} = \{\alpha \in A^{p,q} : \partial\alpha = 0 \text{ and } \partial^*\alpha = 0\} \quad \text{and}\\
\mathcal{H}_{\bar\partial}^{p,q} &:= A^{p,q} \cap \ker \Delta_{\bar\partial} = \{\alpha \in A^{p,q} : \bar\partial\alpha = 0 \text{ and } \bar\partial^*\alpha = 0\}.
\end{aligned}
$$

The following classical result in operator theory is fundamental in the study of harmonic forms and Hodge theory.

Lemma 1.3.2. *Every self-adjoint elliptic operator on a compact manifold has finite dimensional kernel and its kernel and image are in orthogonal direct sum.*

All of the Laplacians that we described are second-order self-adjoint elliptic operators, so that the spaces $\mathcal{H}^k_d$, $\mathcal{H}^k_{d^c}$, $\mathcal{H}^{p,q}_{\partial}$ and $\mathcal{H}^{p,q}_{\bar\partial}$ are finite-dimensional vector spaces over $\mathbb{C}$. Classical Hodge theory establishes isomorphisms between the cohomologies and the spaces of harmonic forms

$$H^k_d \cong \mathcal{H}^k_d, \quad H^k_{d^c} \cong \mathcal{H}^k_{d^c}, \quad H^{p,q}_{\partial} \cong \mathcal{H}^{p,q}_{\partial} \quad \text{and} \quad H^{p,q}_{\bar\partial} \cong \mathcal{H}^{p,q}_{\bar\partial},$$

so that the dimensions of the cohomology groups, hence of the spaces of harmonic forms, provide a series of invariants. The numbers

$$b_k := \dim_{\mathbb{C}} H^k_d = \dim_{\mathbb{C}} H^k_{d^c}$$

are called *Betti numbers*, and they depend only on the homeomorphism type of the underlying manifold. The numbers

$$h^{p,q}_{\bar\partial} := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial} = \dim_{\mathbb{C}} H^{q,p}_{\partial}.$$

are called *Hodge numbers*, and they depend on the complex structure J, but not on the choice of J-compatible Riemannian metric.

The theory develops very similarly for Bott–Chern and Aeppli cohomologies. One can consider the *Bott–Chern Laplacian*

$$\Delta_{BC} = \partial\bar\partial(\partial\bar\partial)^* + (\partial\bar\partial)^*\partial\bar\partial + \partial^*\bar\partial(\partial^*\bar\partial)^* + (\partial^*\bar\partial)^*\partial^*\bar\partial + \partial^*\partial + \bar\partial^*\bar\partial \qquad (1.3.4)$$

and the *Aeppli Laplacian*

$$\Delta_A = \partial\bar\partial(\partial\bar\partial)^* + (\partial\bar\partial)^*\partial\bar\partial + \partial^*\bar\partial(\partial^*\bar\partial)^* + (\partial^*\bar\partial)^*\partial^*\bar\partial + \partial\partial^* + \bar\partial\bar\partial^*. \qquad (1.3.5)$$

These are fourth-order self-adjoint elliptic operators [82]. Their kernels

$$\mathcal{H}^{p,q}_{BC} := A^{p,q} \cap \ker \Delta_{BC} \quad \text{and} \quad \mathcal{H}^{p,q}_{A} := A^{p,q} \cap \ker \Delta_{A}$$

are the spaces of *Bott–Chern harmonic forms* and of *Aeppli harmonic forms*, respectively. Their explicit expressions are

$$\mathcal{H}^{p,q}_{BC} = \{\alpha \in A^{p,q} : \partial\alpha = 0,\ \bar\partial\alpha = 0 \text{ and } (\partial\bar\partial)^*\alpha = 0\}$$

and

$$\mathcal{H}^{p,q}_{A} = \{\alpha \in A^{p,q} : \partial\bar\partial\alpha = 0,\ \partial^*\alpha = 0 \text{ and } \bar\partial^*\alpha = 0\}.$$

Then, there are isomorphisms

$$H^{p,q}_{BC} \cong \mathcal{H}^{p,q}_{BC} \quad \text{and} \quad H^{p,q}_{A} \cong \mathcal{H}^{p,q}_{A}.$$

Since the Hodge $*$ induces an isomorphism $\mathcal{H}^{p,q}_{BC} \cong \mathcal{H}^{m-q,m-p}_{A}$, it is not restrictive to focus only on Bott–Chern harmonic forms. By complex conjugation, we also have $\mathcal{H}^{p,q}_{BC} \cong \mathcal{H}^{q,p}_{BC}$. The complex dimensions of the Bott–Chern cohomology groups

$$h^{p,q}_{BC} := \dim_{\mathbb{C}} H^{p,q}_{BC} = \dim_{\mathbb{C}} H^{m-p,m-q}_{A}$$

are the *Bott–Chern numbers* of the complex structure.

Remark 1.3.3. *It is relevant to observe that in the literature there are several non-equivalent definitions of Bott–Chern and Aeppli Laplacians. Nevertheless, all their spaces of harmonic forms are isomorphic to each other and any property valid for our choice of Laplacians holds in general. For a detailed discussion we refer to [50] and to the original references [56], [82] and [106]. In Chapter 4, we adopt another possible definition of Bott–Chern and Aeppli Laplacians based on the use of* d *and* d^c. *On complex manifolds, it is equivalent to* (1.3.4) *and* (1.3.5).

1.3.2 Symplectic manifolds

This section summarizes the theory of the symplectic cohomologies introduced by Tseng and Yau [100] and further studied in [99], [101] and [102].

Let (M, ω), with $d\omega = 0$, be a symplectic manifold. We study the action of the differential d on bigraded forms. There are two natural double complexes associated to $\mathcal{L}^{\bullet,\bullet}$, namely $(\mathcal{L}^{\bullet,\bullet}, d, d^\Lambda)$ and $(\mathcal{L}^{\bullet,\bullet}, \partial_+, \partial_-)$, built in analogy with $(A^{\bullet,\bullet}, d, d^c)$ and $(A^{\bullet,\bullet}, \partial, \bar{\partial})$, respectively. For the first one, we note that if $d\omega = 0$, then d^Λ is the commutator

$$d^\Lambda = [d, \Lambda],$$

and the operators d and d^Λ satisfy the equations of a double complex

$$d^2 = 0, \quad (d^\Lambda)^2 = 0 \quad \text{and} \quad dd^\Lambda + d^\Lambda d = 0.$$

As we already observed in Section 1.2, in general $dd^\Lambda + d^\Lambda d \neq 0$. It is not known if the converse implication is also true. We formulate this as a question.

Question. Let (M, ω) be an almost symplectic manifold such that we have $dd^\Lambda + d^\Lambda d = 0$ on every form. Is ω necessarily d-closed?

On compact manifolds, we provide an answer when $2m = 4$.

Lemma 1.3.4. *Let* (M, ω) *be a compact almost symplectic* $2m$*-manifold such that* d *and* d^Λ *anti-commute. Then* $d\omega^{m-1} = 0$. *In particular, if* $2m = 4$, *then* $d\omega = 0$.

Proof. By computing $dd^\Lambda + d^\Lambda d$ on an arbitrary function $f \in C^\infty(M)$, we see that

$$0 = (dd^\Lambda + d^\Lambda d)f = d^\Lambda df = *_s\, d *_s df = -\frac{1}{(m-1)!} *_s d(\omega^{m-1} \wedge df) =$$
$$= -\frac{1}{(m-2)!} *_s d\omega^{m-1} \wedge df,$$

where we used (1.2.6). Since $*_s$ is an isomorphism, we have that

$$d\omega^{m-1} \wedge df = 0 \tag{1.3.6}$$

for every $f \in C^\infty(M)$. Fix $x \in M$ and let $\{x_j\}_{j=1}^{2m}$ be coordinate functions in a neighborhood U of x. In local coordinates, we can write

$$d\omega^{m-1} = \sum_{j=1}^{2m} \omega_j\, dx^{1\ldots\hat{j}\ldots 2m},$$

where $\hat{j}$ denotes missing indices. If we choose f as a smooth extension of x_j from U to M, equation (1.3.6) implies that $\omega_j = 0$ for all $j = 1, \ldots, 2m$. Therefore we have

$$d\omega^{m-1} = 0$$

on U and, since x is arbitrary, on M. In dimension $2m = 4$, this immediately implies $d\omega = 0$. □

Remark 1.3.5. *Metrics whose fundamental form satisfies $d\omega^{m-1} = 0$ are known in the literature as* (almost) balanced metrics *[67], or* semi-Kähler *[42]. In Lemma 1.3.4, we have proved that the semi-Kähler condition is equivalent to asking that $d\omega$ is primitive, or that d and d^Λ anti-commute* ***on functions****.*

To define the double complex $(\mathcal{L}^{\bullet,\bullet}, \partial_+, \partial_-)$, consider the action of d on $\mathcal{L}^{r,s}$. Under the assumption $d\omega = 0$, the decomposition (1.2.13) simplifies to

$$d(\mathcal{L}^{r,s}) \subseteq \mathcal{L}^{r,s+1} \oplus \mathcal{L}^{r+1,s-1}$$

and induces a decomposition of d as the sum $d = \partial_+ + L\partial_-$, where

$$\partial_+ := \pi^{r,s+1} \circ d|_{\mathcal{L}^{r,s}} \quad \text{and} \quad \partial_- := \Lambda \circ \pi^{r+1,s-1} \circ d|_{\mathcal{L}^{r,s}}.$$

Separating by bidegree the terms of the equation $(\partial_+ + L\partial_-)^2 = 0$ and taking into account that ∂_+ and ∂_- commute with the Lefschetz operator, we see that they satisfy the equations of a double complex

$$\partial_+^2 = 0, \quad \partial_-^2 = 0 \quad \text{and} \quad \partial_+\partial_- + \partial_-\partial_+ = 0,$$

when acting on primitive k-forms with $k < m$. In terms of the d_j appearing in (1.2.13), we have

$$\partial_+ = d_1 \quad \text{and} \quad \partial_- = \Lambda d_2.$$

Any operator among d, d^Λ, ∂_+ and ∂_- commutes with L, so that it is enough to define their cohomologies on primitive forms. The full cohomologies are recovered by the action of the Lefschetz operator. The cohomologies of the double complex $(\mathcal{L}^{\bullet,\bullet}, \partial_+, \partial_-)$ are

$$PH^k_{\partial_+} := \frac{\ker \partial_+ \cap \mathcal{P}^k}{\operatorname{Im} \partial_+ \cap \mathcal{P}^k} \quad \text{and} \quad PH^k_{\partial_-} := \frac{\ker \partial_- \cap \mathcal{P}^k}{\operatorname{Im} \partial_- \cap \mathcal{P}^k},$$

together with their Bott–Chern and Aeppli counterparts

$$PH^k_{\partial_+ + \partial_-} := \frac{\ker \partial_+ \cap \ker \partial_- \cap \mathcal{P}^k}{\operatorname{Im} \partial_+\partial_- \cap \mathcal{P}^k} \quad \text{and} \quad PH^k_{\partial_+\partial_-} := \frac{\ker \partial_+\partial_- \cap \mathcal{P}^k}{\operatorname{Im}(\partial_+ + \partial_-) \cap \mathcal{P}^k}.$$

In a similar fashion, we define the cohomologies of the double complex $(\mathcal{L}^{\bullet,\bullet}, d, d^\Lambda)$. Clearly, we obtain the de Rham cohomology and the d^Λ-*cohomology*

$$H^k_{d^\Lambda} := \frac{\ker d^\Lambda \cap A^k}{\operatorname{Im} d^\Lambda \cap A^k},$$

that are isomorphic to each other, see diagram (1.3.7). Then we have the symplectic cohomologies

$$H^k_{d+d^\Lambda} := \frac{\ker d \cap \ker d^\Lambda \cap A^k}{\operatorname{Im} dd^\Lambda \cap A^k} \quad \text{and} \quad H^k_{dd^\Lambda} := \frac{\ker dd^\Lambda \cap A^k}{\operatorname{Im}(d + d^\Lambda) \cap A^k}.$$

They also admit a primitive version which contains the same information as the full cohomology. We refer collectively to all of them as *symplectic cohomologies.* In full analogy with the theory of complex cohomologies, there are equalities

$$\frac{\ker \partial_+ \cap \ker \partial_- \cap A^k}{\operatorname{Im} \partial_+\partial_- \cap A^k} = \frac{\ker d \cap \ker d^\Lambda \cap A^k}{\operatorname{Im} dd^\Lambda \cap A^k},$$

and

$$\frac{\ker \partial_+\partial_- \cap A^k}{\operatorname{Im}(\partial_+ + \partial_-) \cap A^k} = \frac{\ker dd^\Lambda \cap A^k}{\operatorname{Im}(d + d^\Lambda) \cap A^k}.$$

We can arrange the cohomologies in a diagram

$$\begin{array}{ccccccc} & & & H^k_{d+d^\Lambda} & & & \\ & \swarrow & \swarrow & & \searrow & \searrow & \\ H^k_{\partial_+} & & H^k_d & \overset{\sim}{\longleftrightarrow} & H^k_{d^\Lambda} & & H^k_{\partial_-} \\ & \searrow & \searrow & & \swarrow & \swarrow & \\ & & & H^k_{dd^\Lambda} & & & \end{array} \tag{1.3.7}$$

where the arrows are defined by the natural inclusion. The isomorphism between H^k_d and $H^k_{d^\Lambda}$ is given by any choice of almost complex structure compatible with the symplectic form ω. Using the general theory of multicomplexes, one can define a spectral sequences that relates $H^k_{\partial_+}$ to de Rham cohomology, see Section 3.4.

Spaces of harmonic forms

Suppose that M is **compact**. Let g be an ω-compatible Riemannian metric on M, i.e., a metric such that

$$\omega(\cdot, J\cdot) = g(\cdot,\cdot)$$

for some almost complex structure J (we do not require $N_J = 0$). Denote by d^* and $(d^\Lambda)^*$ the formal adjoints of d and d^Λ. The Hodge-type Laplacians of d and d^Λ are

$$\Delta_d := dd^* + d^*d \quad \text{and} \quad \Delta_{d^\Lambda} := d^\Lambda(d^\Lambda)^* + (d^\Lambda)^* d^\Lambda,$$

respectively. Similarly, one defines the Hodge-type Laplacians of ∂_+ and ∂_- by

$$\Delta_{\partial_+} := \partial_+\partial_+^* + \partial_+^*\partial_+ \quad \text{and} \quad \Delta_{\partial_-} := \partial_-\partial_-^* + \partial_-^*\partial_-,$$

respectively, and their Bott–Chern-type and Aeppli-type Laplacians by

$$\Delta_{\partial_+ + \partial_-} := \partial_+\partial_-(\partial_+\partial_-)^* + (\partial_+\partial_-)^*\partial_+\partial_- + \partial_+^*\partial_-(\partial_+^*\partial_-)^* + (\partial_+^*\partial_-)^*\partial_+^*\partial_- + \partial_+^*\partial_+ + \partial_-^*\partial_-$$

and

$$\Delta_{\partial_+\partial_-} := \partial_+\partial_-(\partial_+\partial_-)^* + (\partial_+\partial_-)^*\partial_+\partial_- + \partial_+^*\partial_-(\partial_+^*\partial_-)^* + (\partial_+^*\partial_-)^*\partial_+^*\partial_- + \partial_+\partial_+^* + \partial_-\partial_-^*,$$

respectively. The corresponding Laplacians for d and d^Λ are

$$\Delta_{d+d^\Lambda} := dd^\Lambda(dd^\Lambda)^* + (dd^\Lambda)^* dd^\Lambda + d^*d^\Lambda(d^*d^\Lambda)^* + (d^*d^\Lambda)^* d^*d^\Lambda + d^*d + (d^\Lambda)^* d^\Lambda$$

and

$$\Delta_{dd^\Lambda} := dd^\Lambda(dd^\Lambda)^* + (dd^\Lambda)^* dd^\Lambda + d(d^\Lambda)^*(d(d^\Lambda)^*)^* + (d(d^\Lambda)^*)^* d(d^\Lambda)^* + dd^* + d^\Lambda(d^\Lambda)^*.$$

Hodge theory for symplectic manifolds develops in the same way as Hodge theory for complex manifolds: all the symplectic Laplacians are elliptic and self-adjoint. Thanks to the compactness of M, they have finite dimensional kernels and the kernels are isomorphic to the corresponding cohomologies. Even the consequences of Hodge theory are the same, i.e., the symplectic cohomology groups are finite-dimensional on compact manifolds and the dimensions of the spaces of harmonic forms, which in principle depend on the metric, are actually metric-independent, providing the symplectic invariants

$$h^k_{\partial_+}, \quad h^k_{\partial_-}, \quad h^k_{\partial_+ + \partial_-} = h^k_{d+d^\Lambda} \quad \text{and} \quad h^k_{\partial_+\partial_-} = h^k_{dd^\Lambda}.$$

Remark 1.3.6. *We recall that the original definition of symplectic cohomologies and symplectic harmonic forms was given for real forms. Since the symplectic Laplacians are real operators, computing them on complex forms yields the same theory.*

1.4 Small deformations of almost complex structures

In this section we describe almost complex structures in terms of the twistor bundle and we study their small deformations. We prove that every continuous curve of almost complex structures can be approximated by a real analytic curve of almost complex structures. For more details on the twistor space we refer to [9] and [76], while for small deformations we refer to [12] and [54].

Let M be a **compact** almost complex $2m$-manifold. We interpret almost complex structures as sections of a suitable bundle, called the *twistor bundle*. Let $x \in M$ and let T_xM be the tangent space at x. The twistor bundle of M is the fiber bundle $\mathrm{Tw} \to M$ whose fiber is

$$\mathrm{Tw}_x \coloneqq \{J_x \in \mathrm{End}(T_xM) : J_x^2 = -\mathrm{Id}_{T_xM}\},$$

the space of linear complex structures on T_xM. A smooth section of the twistor bundle corresponds to a local choice of complex structure depending smoothly on x, hence to a global almost complex structure J on M. While the twistor bundle is defined for arbitrary even dimensional smooth manifolds, it will admit a global section if and only if M admits an almost complex structure.

Denote by $\mathcal{J}$ the space of almost complex structures on M, or, equivalently, the space of smooth sections of $\mathrm{Tw} \to M$. Since $\mathcal{J}$ is the space of sections of a smooth vector bundle, it naturally admits the structure of a Fréchet manifold induced by a family of seminorms. In particular, we endow $\mathcal{J}$ with the C^k-topology, for $k = 0, \dots, \infty$, induced by such seminorms as follows. Fix a Riemannian metric on M. Denote by $\|\cdot\|$ the induced norm on $\mathrm{End}(TM)$ and by ∇ the induced connection on TM extended to $\mathrm{End}(TM)$. Two points J_0 and $J_1 \in \mathcal{J}$ are ε-close in the C^k-topology if

$$\sup_{j \le k} \max_{x \in M} \|\nabla^j (J_0 - J_1)|_x\| < \varepsilon.$$

In the case $k = \infty$ we take the supremum over $k \in \mathbb{N}$. For a general discussion on topologies on spaces of sections we refer to [57] and [68].

Let $J_0 \in \mathcal{J}$ and let

$$T_{J_0}\mathcal{J} = \{L \in \mathrm{End}(TM) : LJ_0 + J_0L = 0\}$$

be the tangent space to $\mathcal{J}$ at J_0. Set $I = [0, 1] \subset \mathbb{R}$ and denote by $\Delta_\varepsilon \subset \mathbb{C}$ the (open) disk of radius ε centered at the origin.

Definition 1.4.1. A *(real) curve of almost complex structures* is a continuous map from I to $\mathcal{J}$. We write it as a family of almost complex structures J_t depending continuously on the parameter $t \in I$. If J_t depends smoothly on t, we say that J_t is a *smooth curve of almost complex structures.*

Definition 1.4.2. Fix $J_0 \in \mathcal{J}$. A *small deformation* of J_0 is a family of almost complex structures parametrized by a complex parameter $s \in \Delta_\varepsilon$ of the form

$$J_s = (I + L_s)J_0(I + L_s)^{-1}, \quad s \in \Delta_\varepsilon, \tag{1.4.1}$$

where $L_s \in T_{J_0}\mathcal{J}$ and $L_s = sL + o(s)$. If $L_s = sL$, we say that J_s is a *first order deformation* of J_0.

A small deformation of J_0 naturally defines a curve of almost complex structures passing through J_0 parametrized by $s \in \Delta_\varepsilon$. Taking the restriction of s to the real axis, we obtain a real curve of almost complex structures parametrized by $t \in (-\varepsilon, \varepsilon)$ and, up to reparametrization, a curve defined in a neighborhood of I. If $J_1 \in \mathcal{J}$ is an almost complex structure close enough to J_0 in the C^0-topology, then J_1 can be written as a small deformation of J_0.

A different way of thinking of an almost complex structure J is in terms of a splitting of the complexified tangent bundle

$$TM^{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1},$$

where $TM^{1,0}$ and $TM^{0,1}$ are the $(\pm i)$-eigenbundles of J extended to $TM^{\mathbb{C}}$, respectively. Every almost complex structure determines uniquely such a splitting. Conversely, every splitting

$$TM^{\mathbb{C}} = L \oplus \bar{L}$$

determines uniquely J as the endomorphism of $TM^{\mathbb{C}}$ whose $(\pm i)$-eigenbundles are L and $\bar{L}$, respectively. This fact will be used consistently in the rest of this section and in Chapter 2, as well as in explicit examples.

A curve of almost complex structures J_t defines a family of splittings

$$TM^{\mathbb{C}} = TM_t^{1,0} \oplus TM_t^{0,1}, \quad t \in I.$$

If t is small enough and J_t is induced by a small deformation of J_0, we can view $TM_t^{1,0}$ as a graph over $TM_0^{1,0}$ obtained as

$$TM_t^{1,0} = (\mathrm{Id} + \Psi(t))\, TM_0^{1,0},$$

where $\Psi(t) = \sum_{j=0}^{\infty} \psi_j\, t^j$ is a power series with coefficients

$$\psi_j \in T^*M_0^{0,1} \otimes TM_0^{1,0}$$

and $\psi_0 = 0$. With respect to the description of J_t as a small deformation of J_0, we have that

$$\Psi(t) = \frac{1}{2}(L_t - iJ_0L_t),$$

while L_t can be recovered from J_0 and J_t via the formula

$$L_t = (I - J_0J_t)^{-1}(I + J_0J_t), \tag{1.4.2}$$

see Remark 3.1 in [12]. Conversely, a power series $\Psi(t)$ with $\Psi(0) = 0$ does not necessarily give rise to a small deformation of almost complex structures and one has to deal with convergence issues. When we talk about small deformations of almost complex structures we always mean that the power series in t is convergent in a neighborhood of 0.

Among curves of almost complex structures, a special role is played by those that are locally described by a formal deformation of almost complex structures that is actually convergent.

Definition 1.4.3. We say that a curve of almost complex structures J_t is *real analytic* if for each $t_0 \in I$ there exists $\delta > 0$ such that

$$J_t = (I + L_{t-t_0})J_{t_0}(I + L_{t-t_0})^{-1} \quad \text{for } t \in (t_0 - \delta, t_0 + \delta),$$

and if the power series described by L_{t-t_0} is convergent in a neighborhood of t_0 with positive radius of convergence. It will be clear from the proof of Lemma 1.4.6 that one can think of real analytic curves as $C^{\omega,\infty}$-sections of a suitable fiber bundle in the sense of [57].

To deal with real analytic curves it is convenient to endow the twistor bundle with an analytic structure.

Definition 1.4.4. An *analytic structure* (or *C^ω-structure*) on M is the datum of a maximal atlas whose transition functions are analytic. M endowed with a C^ω-structure is an *analytic manifold.* If M and N are analytic manifolds, an *analytic function* (or *C^ω-function*) $f\colon M \to N$ is a function that is analytic after composing with local charts on M and N.

Definition 1.4.5. An *analytic fiber bundle* (or *C^ω-fiber bundle*) over M is a fiber bundle $\pi\colon E \to M$ where E is an analytic manifold and π is a surjective analytic map. An *analytic section* (or *C^ω-section*) of a C^ω-fiber bundle is a section that is also a C^ω-function.

Every smooth manifold admits an analytic structure compatible with its smooth structure. This guarantees that the total space of the twistor bundle admits an analytic structure. Nevertheless, endowing $\mathrm{Tw} \to M$ with a structure of analytic bundle requires some additional care.

First, we need an approximation result for analytic sections. By Steenrod's approximation theorem, see Section 6.7 in [91], every continuous section σ_0 of a smooth fiber bundle $B \to M$ can be approximated by a smooth section σ_∞, once a metric is fixed on B. Furthermore, the section σ_∞ can be taken in such a way that it coincides with σ_0 on a closed subset of B on which σ_0 is already smooth. The problem of approximating a smooth section σ_∞ with a C^ω-section on a C^ω-fiber bundle was left open by Steenrod, but a solution was found later by Shiga, see [83] or Proposition 2 in [84]. The approximation is arbitrarily good in the C^∞-topology of the space of sections and can be performed outside of a closed subset on which σ_∞ is already analytic. We refer to Section 30.12 in [57] for a modern treatment of the approximation problem.

Consider $\pi : \mathrm{Tw} \to M$ as a smooth vector bundle. Fix a C^ω-structure on M compatible with the underlying smooth structure. Endow the total space Tw with a C^ω-structure induced by those on M and on Tw_x. Then π is a smooth map between analytic manifolds and can be approximated by an analytic map π^ω. Since the C^ω-structure on Tw is induced by the one on M, the map π^ω is still a bundle projection and the C^ω-fiber bundle structure is well-defined.

The second result we need is a deep theorem due to Grauert [41] and Morrey [70]. They show that the choice of analytic structure we made on M is essentially unique: two C^ω-structures compatible with the same smooth structure on M are C^ω-diffeomorphic. In particular, this guarantees that every smooth invariant we compute is independent of the initial choice of analytic structure on M, thus it is independent of the choice of analytic structure on Tw.

Lemma 1.4.6. *Let M be a compact almost complex manifold and let J_0 and J_1 be two almost complex structures on M. Suppose that there exists a continuous curve of almost complex structures J_t, $t \in I$, between J_0 and J_1. Then J_t can be approximated by a real analytic curve of almost complex structures.*

Proof. For an arbitrary smooth manifold N, continuous curves from I to N can be seen as continuous sections of the product fiber bundle $I \times N \to I$. Hence, we can think of J_t as a section of the fiber bundle $I \times \mathrm{Tw} \to I \times M$ that depends continuously on $t \in I$ and smoothly on $x \in M$. Endow Tw with a C^ω-fiber bundle structure as described above. Then J_t is a continuous section of a C^ω-fiber bundle. By Proposition 2 in [84], we approximate it by a C^ω-section $\tilde{J}_t$. By [83], up to a small perturbation with fixed endpoints of J_t in a neighborhood of $t = 0$ and $t = 1$, the approximation can be performed keeping the endpoints fixed since $\{0\} \times M$ and $\{1\} \times M$ are closed C^ω-submanifolds of $I \times M$. To prove that $\tilde{J}_t$ is a real analytic curve of almost complex structures in the sense of Definition 1.4.3, fix

$t_0 \in I$ and let $t \in I$ be close enough to t_0. As in (1.4.2), consider the operator

$$\tilde{L}_t \coloneqq (I - \tilde{J}_{t_0}\tilde{J}_t)^{-1}(I + \tilde{J}_{t_0}\tilde{J}_t).$$

Then $\tilde{L}_t$ defines a small deformation $\tilde{J}_t^{def}$ of $\tilde{J}_{t_0}$. We need to prove that $\tilde{J}_t^{def}$ coincides with the curve $\tilde{J}_t$ in a neighborhood of t_0. Since $\tilde{J}_t$ is analytic in t, so is $\tilde{L}_t$. The same is true for $\tilde{J}_t^{def}$, that is completely determined by $\tilde{L}_t$ in a small neighborhood of t_0. Hence $\tilde{J}_t$ and $\tilde{J}_t^{def}$ are both analytic in t. Taking derivatives of (1.4.1) and (1.4.2), it is immediate to check that their derivatives at $t = t_0$ coincide, so that $\tilde{J}_t = \tilde{J}_t^{def}$. □

1.5 Invariant structures on quotients of Lie groups

Let G be a connected and simply connected Lie group of dimension $2m$, let $\mathfrak{g}$ be its Lie algebra and let $\mathfrak{g}^*$ be its dual Lie algebra. Fix a trivialization of TG by left-invariant vector fields so that

$$TG \cong C^\infty(G)\langle e_1, \ldots, e_{2m}\rangle, \quad e_j \in \mathfrak{g}.$$

Let $e^1, \ldots, e^{2m}$ be a basis of $\mathfrak{g}^*$ dual to $e_1, \ldots, e_{2m}$. Then $\mathfrak{g}$ and $\mathfrak{g}^*$ are the span

$$\mathfrak{g} = \mathbb{R}\langle e_1, \ldots, e_{2m}\rangle \quad \text{and} \quad \mathfrak{g}^* = \mathbb{R}\langle e^1, \ldots, e^{2m}\rangle.$$

Consider the graded algebra

$$A^k_{\mathfrak{g}} \coloneqq \bigwedge^k \mathfrak{g}^*$$

endowed with the differential defined as the dual of the Lie bracket: for $X, Y \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$, we set

$$da(X,Y) \coloneqq -a([X,Y])$$

and extend it as a graded derivation to $A^\bullet_{\mathfrak{g}}$. The differential graded algebra $(A^\bullet_{\mathfrak{g}}, d)$ is the *Chevalley–Eilenberg complex of* $\mathfrak{g}$ [27]. The cohomology of the Chevalley–Eilenberg complex is the *Lie algebra cohomology of* $\mathfrak{g}$

$$H^k(\mathfrak{g}) \coloneqq \frac{\ker d \cap A^k_{\mathfrak{g}}}{\operatorname{Im} d \cap A^k_{\mathfrak{g}}}.$$

The Lie algebra structure of $\mathfrak{g}$ is completely described in terms of the differentials $de^j \in \bigwedge^2 \mathfrak{g}^*$. For brevity, we denote $e^k \wedge e^l$ by e^{kl} and we write

$$\mathfrak{g} = (c^1_{kl}kl, \ldots, c^{2m}_{kl}kl),$$

where we are summing over repeated indices and we mean that

$$de^j = \sum_{k,l} c^j_{kl} e^{kl}.$$

As an example, the Lie algebra of the complex 3-dimensional Heisenberg group is isomorphic to the 6-dimensional real Lie algebra with structure equations

$$de^5 = e^{13} - e^{24}, \quad de^6 = e^{14} + e^{23} \quad \text{and} \quad de^j = 0 \quad \text{for } j = 1, 2, 3, 4.$$

It will be simply denoted by

$$\mathfrak{g} = (0, 0, 0, 0, 13 - 24, 14 + 23).$$

Endow $\mathfrak{g}$ with a compatible triple $(J, \omega, \langle \cdot, \cdot \rangle)$, where $J \in \mathrm{End}(\mathfrak{g})$ is such that $J^2 = -\mathrm{Id}$, ω is a non-degenerate element of $A^2_{\mathfrak{g}}$ and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{g}$. In addition, we ask that they satisfy the compatibility condition

$$\langle X, Y \rangle = \omega(X, JY)$$

for all X and $Y \in \mathfrak{g}$. If $\mathfrak{g}$ is endowed with a compatible triple, the Chevalley–Eilenberg complex admits a bigrading defined in terms of J or ω and the differential admits decompositions analogous to those presented in Section 1.2. If $N_J = 0$ or $d\omega = 0$, then one can compute the complex or symplectic cohomologies of $\mathfrak{g}$ as the corresponding cohomologies of the Chevalley–Eilenberg complex.

Observe that Lie groups are parallelizable, thus G will always admit an almost complex structure. We focus on a special class of structures that has a stronger connection with the Lie algebra of G. A *left-invariant almost complex structure* on G is an almost complex structure that is invariant under left translation. Left-invariant almost complex structures descend naturally to $\mathfrak{g}$. Conversely, every endomorphism $J \in \mathrm{End}(\mathfrak{g})$ satisfying $J^2 = -\mathrm{Id}$ induces a left-invariant structure on G by $C^\infty(G)$-linearity. Similarly, a left-invariant non-degenerate 2-form corresponds to a non-degenerate element of $\bigwedge^2 \mathfrak{g}^*$ and a left-invariant metric on G corresponds to an inner product on $\mathfrak{g}$.

Since in most cases G will not be compact, to deal with compact manifolds we assume that G admits a co-compact discrete subgroup Γ. Consider the left-action of elements in Γ and the quotient $M \coloneqq \Gamma \backslash G$. Then M is a compact smooth $2m$-manifold whose tangent space coincides with that of G and is isomorphic to $\mathfrak{g}$. In particular, M is parallelizable and admits an almost complex structure.

The *derived series* $\{\mathfrak{g}^{(k)}\}_{k \geq 1}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^{(1)} \coloneqq [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad \mathfrak{g}^{(k+1)} \coloneqq [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \quad \text{for } k \geq 1.$$

If $\mathfrak{g}^{(k)} = \{0\}$ for some k, we say that $\mathfrak{g}$ is *solvable*. A solvable Lie algebra is called *completely solvable* if $\operatorname{ad} X$ has real eigenvalues for all $X \in \mathfrak{g}$. The *descending central series* $\{\mathfrak{g}^k\}_{k\geq 1}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad \mathfrak{g}^{k+1} := [\mathfrak{g}, \mathfrak{g}^k] \quad \text{for } k \geq 1.$$

If $\mathfrak{g}^k = \{0\}$ for some k, we say that $\mathfrak{g}$ is *nilpotent*. In particular, every nilpotent Lie algebra is completely solvable and every completely solvable Lie algebra is solvable. We say that G is solvable, completely solvable or nilpotent if its Lie algebra is solvable, completely solvable or nilpotent, respectively.

If $\Gamma \subseteq G$ is a co-compact discrete subgroup and G is solvable or nilpotent we call $M = \Gamma\backslash G$ *solvmanifold* or *nilmanifold*, respectively. If G is completely solvable, we say that M is a solvmanifold *of completely solvable type*. An almost complex structure on M is said to be *invariant* if it is induced on M by an almost complex structure on $\mathfrak{g}$. An almost complex structure on G is said to be Γ-invariant if it is invariant under left translation by elements of Γ. In terms of the basis of $\mathfrak{g}$, an almost complex structure J sends e_j to a linear combination of $e_1, \dots, e_{2m}$ with coefficients in

- $\mathbb{R}$, if J is an invariant structure on M or a left-invariant structure on G;
- $C^\infty(M)$, if J is an arbitrary almost complex structure on M or a Γ-invariant structure on G;
- $C^\infty(G)$, if J is an arbitrary almost complex structure on M.

Note that smooth functions on M correspond to smooth functions on G that are also Γ-periodic.

For invariant structures on M, one should consider two kind of cohomologies. The first one is the genuine cohomology of the structure computed on the smooth manifold. The second one is the *invariant cohomology*, i.e., the cohomology of the corresponding structure on $\mathfrak{g}$ computed on the Chevalley–Eilenberg complex. Similarly, for a left-invariant structure on a Lie group we can consider its usual cohomology or its *left-invariant cohomology*. In general the two cohomologies are different. However, under certain assumption we know that the de Rham cohomology of M can be computed using invariant forms, see [46] and [75].

Theorem 1.5.1. *The de Rham cohomology of nilmanifolds and of solvmanifolds of completely solvable type is isomorphic to the invariant de Rham cohomology.*

In general, understanding if invariant and non-invariant cohomologies coincide is a difficult task and, in the case of Dolbeault cohomology, it is the object of open conjectures, see, for instance, [80]. In Chapter 5, we will compare invariant and non-invariant harmonic forms.

CHAPTER 2

Rank of the Nijenhuis tensor

In this chapter we deal with the Nijenhuis tensor of almost complex manifolds. We study the local behaviour of its rank along small deformations of the almost complex structure and we prove that maximally non-integrable structures are generic: they form a set that is either open and dense or empty in each path-connected component of the space of almost complex structures. Then we focus on parallelizable manifolds. We describe a method that allows to produce almost complex structures whose Nijenhuis tensor has arbitrary prescribed rank. On homogeneous spaces, we show that, under certain assumptions, the rank of the Nijenhuis tensor admits an upper bound in terms of topological invariants. On 6-dimensional nilmanifolds, we provide a classification of invariant almost complex structures according to the rank of their Nijenhuis tensor.

2.1 The rank as a measure of non-integrability

Let (M, J) be an almost complex $2m$-manifold and let N_J be the Nijenhuis tensor of J, see (1.3.1). Its image defines a distribution $\mathcal{V} \subseteq TM$ given by $\mathcal{V}_x := \mathrm{Im}(N_J|_x)$ at every $x \in M$. The Nijenhuis tensor has symmetries induced by the action of J on vector fields. We have that

$$N_J(JX, Y) = N_J(X, JY) = -JN_J(X, Y). \tag{2.1.1}$$

Hence, the distribution $\mathcal{V}$ is invariant under the action of J and it has even real rank at every point. The *rank of the Nijenhuis tensor* is the map

$$\begin{aligned} \operatorname{rk} N_J \colon M &\longrightarrow \mathbb{N}, \\ x &\longmapsto \frac{1}{2}\dim_{\mathbb{R}} \mathcal{V}_x = \dim_{\mathbb{C}} \mathcal{V}_x \end{aligned}$$

and in general it is not constant on M. An important observation is that the rank of N_J corresponds to the complex dimension of the image of the operator $\bar{\mu}$ computed on $(1,0)$-forms

$$\operatorname{rk} N_J|_x = \dim_{\mathbb{C}} \bar{\mu}_x(A^{1,0}_x).$$

For brevity, we will sometime refer to the rank of N_J as the rank of J. If $2m \geq 6$, its possible values vary between 0 and m. If $2m = 4$, the only possible values are 0 and 1. If $\operatorname{rk} N_J|_x = 0$ for all $x \in M$, then $N_J = 0$ and J is integrable. One can think of the rank as measure of how far is J from being integrable and it is natural to ask which kind of structures are the farthest possible from the integrable ones.

Definition 2.1.1. An almost complex structure J is said to be *maximally non-integrable* if for all $x \in M$ we have that

$$\operatorname{rk} N_J|_x = \begin{cases} m & \text{if } 2m \geq 6, \\ 1 & \text{if } 2m = 4. \end{cases}$$

Integrable and maximally non-integrable structures are the extremal cases among structures of constant rank. An intermediate but still relevant situation is that of everywhere non-integrable structures.

Definition 2.1.2. An almost complex structure J is said to be *everywhere non-integrable* if $N_J|_x \neq 0$ for all $x \in M$.

In dimension $2m = 4$, the notions of maximally non-integrable and everywhere non-integrable structure coincide. As the following example shows, it is not hard, on certain manifolds, to find structures of constant rank for every choice of rank. In Section 2.3, we develop a general framework that allows to produce structures of arbitrary constant rank on parallelizable manifolds and that provides a large number of examples.

Example 2.1.3 (Almost complex structures of constant rank on tori).
Let T be a $2m$-dimensional torus and let J be an invariant almost complex structure on T. Then J corresponds to an almost complex structure on the Lie algebra of $\mathbb{R}^{2m}$, the universal cover of T. Since $\mathfrak{g} := \operatorname{Lie}(\mathbb{R}^{2m})$ is abelian, we have

that $N_J(X,Y) = 0$ for all X and $Y \in \mathfrak{g}$, and every invariant structure on T must be integrable. Nevertheless, T admits non-invariant almost complex structures of arbitrary constant rank k whose construction, which we now recall, is performed in Theorem 14 and Remark 15 in [32].

Think of T as a quotient of $\mathbb{C}^m$ by the standard lattice $(\mathbb{Z}[2\pi i])^m$. This induces complex coordinates on T. Denote by $\{dz^j\}_{j=1}^m$ the corresponding global frame of $(1,0)$-forms and consider the forms $\{\omega^j\}_{j=1}^m$ defined by

$$\omega^j := dz^j + f_j d\bar{z}^j, \quad j = 1, \dots m,$$

where $f_j \in C^\infty(T)$. As long as the constraint $|f_j|^2 \neq 1$ is satisfied for all j, the forms ω^j define an almost complex structure on T by requiring that they span the $(1,0)$-tangent bundle $T^{1,0}M$. The choice of functions

$$f_j = \begin{cases} A + e^{i\,\Re(z_j+1)} & j = 1, \dots k-1, \\ A + e^{i\,\Re(z_1)} & j = k, \\ 0 & j = k+1, \dots, m, \end{cases}$$

where $A \in \mathbb{C}$ is a constant such that $|A| \geq 2$, gives an almost complex structure of constant rank k on T. The only case not covered by the above argument is $k = 2$ in dimension $2m = 6$, that can be obtained with a slight modification of the ω^j. We refer to [32] for the details. ■

Maximally non-integrable structures not only are the less integrable almost complex structure. They also appear in the literature as a preferred source of examples when dealing with almost complex structures, see Introduction. This is essentially due to the fact that such structures are the *generic* almost complex structure, while integrable structures appear as the less generic kind of almost complex structures. Heuristically speaking, given a linear morphism of vector spaces $L \in \mathrm{Hom}(V,W)$, with $\dim V = \dim W$, “having image of maximal dimension” is a genericity condition for L: it can be perturbed to an invertible operator since invertible matrices form a dense subset of $\mathrm{Hom}(V,W)$. However, the generalization of this fact to almost complex structures is not straightforward. This is the topic of Section 2.2, where we show that maximally non-integrable structures form an open and dense set in each path-connected component of the space of almost complex structures, as soon as they exist, see Theorem 2.2.4 and Corollary 2.2.5.

Actually, maximally non-integrable structures might not exist on arbitrary manifolds. The following result is essentially due to Armstrong [8], if $2m = 4$, and to Bryant [20], if $2m = 6$. It gives necessary and sufficient conditions for existence of maximally non-integrable structures on compact manifolds of low dimension.

Theorem 2.1.4. *Let M be a compact $2m$-manifold and let χ and σ be the Euler characteristic and the signature of M, respectively. Then M admits a maximally non-integrable almost complex structure if and only if*

- $(1 - b_1 + b^+)$ *is even and* $5\chi + 6\sigma = 0$, *if* $2m = 4$;
- *it admits a spin structure, if* $2m = 6$.

In dimension 8 there are necessary conditions [32]. In dimension at least 10, maximally non-integrable structures always exist on arbitrary (not necessarily compact) manifolds. The result on existence of maximally non-integrable structures on high-dimensional manifolds is a consequence of a more general result on the flexibility (genericity) of almost complex structures: an h-principle [32]. In order to deal with flexibility of structures, it is convenient to introduce the spaces of structures whose rank is bounded from below. For every k, consider the set

$$\mathcal{J}_k \coloneqq \{J \in \operatorname{End}(TM) : J^2 = -\operatorname{Id} \text{ and } \operatorname{rk} N_J|_x \geq k \text{ for all } x \in M\}.$$

The set $\mathcal{J}_0$ is the space of all almost complex structures on M. We denote it simply by $\mathcal{J}$, see Section 1.4. The space $\mathcal{J}_1$ is the space of everywhere non-integrable structures, while $\mathcal{J}_m$ is the space of maximally non-integrable structures (if $2m \geq 6$). For a fixed $J \in \mathcal{J}$, let $\mathcal{N}_J$ be the set of $(2,1)$-tensors satisfying (2.1.1). The space of *formal Nijenhuis tensors* associated to $\mathcal{J}_k$ is the space

$$\mathcal{N}_k \coloneqq \{(J, N) : J \in \mathcal{J},\ N \in \mathcal{N}_J \text{ and } \operatorname{rk} N|_x \geq k \text{ for all } x \in M\}.$$

At least locally, every $(2,1)$-tensor satisfying (2.1.1) is the Nijenhuis tensor of some (local) almost complex structure, see Theorem 5 in [58] or Proposition 1.1 in [69]. As it happens for almost complex structures, a simple local description of the Nijenhuis tensor does not translate to a simple global one.

Theorem 2.1.5 (h-principle, Theorem A in [32])**.** *Let M be an almost complex manifold. For every integer k the map sending J to its Nijenhuis tensor induces a homotopy equivalence*

$$\mathcal{J}_k \longrightarrow \mathcal{N}_k.$$

By transversality, it is possible to establish existence of special structures on high dimensional manifolds.

Corollary 2.1.6 (Corollary A.1 in [32])**.** *Every almost complex structure on an almost complex $2m$-manifold is homotopic to an everywhere non-integrable one if $2m \geq 6$ and to a maximally non-integrable if $2m \geq 10$.*

In particular, maximally non-integrable structures always exist on manifolds of dimension at least 10, while everywhere non-integrable structures always exist on manifolds of dimension at least 6.

2.2 Density of maximally non-integrable structures

Let (M, J) be a compact almost complex manifold. The compactness assumption guarantees that the local description of small deformations of almost complex structures given by (1.4.1) holds, cf. page 257 in [54]. Let J_t, $|t| < \varepsilon$, be a small deformation of $J = J_0$, see Section 1.4, and let N_{J_t} be the Nijenhuis tensor of J_t. Along small deformations of almost complex structures we are able to control the rank of the Nijenhuis tensor.

Lemma 2.2.1. *Let M be a compact almost complex $2m$-manifold. Let J_0 be an almost complex structure on M and let*

$$J_t = (I + L_t)J_0(I + L_t)^{-1}, \quad t \in (-\varepsilon, \varepsilon),$$

be a small deformation of J_0. Then for every $a \in (0, \varepsilon)$ and for every $x \in M$, we have that

$$\operatorname{rk} N_{J_t}|_x \geq \max\{\operatorname{rk} N_{J_0}|_x, \operatorname{rk} N_{J_a}|_x\}$$

for all $t \in [-a, a]$ except a finite number.

Proof. Fix $x \in M$. Let U be a small neighborhood of x and let $\{\omega_t^j\}_{j=1}^m$ be a local co-frame of $(1, 0)$-forms with respect to J_t. The operator $\bar{\mu}_t$ computed on the co-frame can be written as

$$\bar{\mu}_t\, \omega_t^j = \sum_{k<l} G_{kl}^j(t)\, \omega_t^{\bar{k}\bar{l}},$$

where $G_{kl}^j \in C^\infty(M \times (-\varepsilon, \varepsilon))$ is a power series in the t variable that converges on $(-\varepsilon, \varepsilon)$. Fix $a \in (0, \varepsilon)$ and set

$$k_x \coloneqq \max\{\operatorname{rk} N_{J_0}|_x, \operatorname{rk} N_{J_a}|_x\}.$$

Let G be the matrix $(G_{kl}^j(t))$ and consider all the determinants of the $k_x \times k_x$ minors of G. They are power series in t that converge on $[-a, a]$, therefore each of them either has a finite number of zeros or identically vanishes. Since $k_x = \max\{\operatorname{rk} N_{J_0}|_x, \operatorname{rk} N_{J_a}|_x\}$, at least one determinant does not identically vanish along the curve J_t, showing that $\operatorname{rk} N_{J_t}|_x \geq k_x$ for all $t \in [-a, a]$ except a finite number. $\square$

Let $\mathcal{C}$ be a path-connected component of the space of almost complex structures. For every $k = 0, \dots, m$, consider the set

$$\mathcal{C}_k \coloneqq \mathcal{C} \cap \mathcal{J}_k = \{J \in \mathcal{C} : \operatorname{rk} N_J|_x \geq k \text{ for all } x \in M\}.$$

To extend the results of Lemma 2.2.1 from local to global, we use a compactness argument and the existence of real analytic curves, see Definition 1.4.3.

Proposition 2.2.2. *Let M be a compact almost complex $2m$-manifold. Let J_0 and J_1 be two almost complex structures on M. Suppose that there exists a real analytic curve of almost complex structures J_t joining J_0 and J_1. If $J_t \in \mathcal{C}_k$ for some $t \in I$, then $J_t \in \mathcal{C}_k$ for all $t \in I$ except a finite number.*

Proof. Without loss of generality, we can assume that J_t is defined on $(-\varepsilon, 1+\varepsilon)$. Since J_t is a real analytic curve, for each $t^* \in I$ there exists a compact neighborhood I^* of t^* such that J_t is a small deformation of J_{t^*} for all $t \in I^*$. Cover I with such neighborhoods and, by compactness of I, extract a finite subcover $\{t_j, I_j\}_{j=0}^{r}$ with $0 = t_0 < t_1 < \ldots < t_r = 1$ and $I_j \cap I_{j+1} \neq \varnothing$, for $j = 0, \ldots, r-1$. On each I_j, the power series describing the small deformation of J_{t_j} is convergent.

Since M is compact, we can cover it with a finite number of neighborhoods $\{x_s, U_s\}_{s=1}^{q}$ such that $T_{x_s}U_s \cong U_s \times \mathbb{R}^{2m}$. By assumption, there exist $\tilde{j} \in \{0, \ldots, r\}$ and $\tilde{t} \in I_{\tilde{j}}$ such that $J_{\tilde{t}} \in \mathcal{C}_k$. By Lemma 2.2.1 applied locally on U_s to the curve J_t, $t \in I_{\tilde{j}}$, we have that $\operatorname{rk} N_{J_t}|_x \geq k$ for all $t \in I_{\tilde{j}}$ except a finite number. The result of Lemma 2.2.1 applies to all $x \in U_s$ for the same values of t. Two consecutive intervals have non-empty overlap and we can repeat the argument on each I_j. We conclude that $\operatorname{rk} N_{J_t}|_x \geq k$ for all $x \in U_s$ and for all $t \in I$ except a finite number. Finally, since we covered M with a finite number of open sets, we have that $J_t \in \mathcal{C}_k$ for all $t \in I$ except a finite number. □

The proposition can be reformulated as follows:

Corollary 2.2.3. *Let M be a compact almost complex manifold and let J_t, $t \in I$, be a real analytic curve of almost complex structures on M. Then for every $x \in M$ we have that*

$$\operatorname{rk} N_{J_t}|_x = \max_{t \in I}\{\operatorname{rk} N_{J_t}|_x\}$$

for all $t \in I$ except a finite number.

Proof. This is a direct consequence of Proposition 2.2.2 and an elementary fact: level sets of a continuous real function f defined on an interval I and satisfying the following property

$$\left\{\begin{array}{c}\text{for all } x \in I \text{ there exists } \varepsilon_x \text{ such that for all } y \in I_\varepsilon := (x-\varepsilon, x+\varepsilon) \\ \text{we have } f(y) = \max_{I_\varepsilon} f \text{ except for a finite number of } y\end{array}\right\}$$

are discrete, except for the level set $f^{-1}(\max f)$. □

We are ready to state and prove the main result on the density of $\mathcal{C}_k$.

Theorem 2.2.4. *Let M be a compact almost complex manifold and let $\mathcal{C}$ be a path-connected component of $\mathcal{J}$. Let $\mathcal{C}_k$ be the subspace of $\mathcal{C}$ of almost complex structures of rank at least k at every point of M. Then $\mathcal{C}_k$ is either open and dense or empty in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a path-connected component of $\mathcal{J}$. Suppose that $\mathcal{C}_k \subseteq \mathcal{C}$ is not empty. Let $J_1 \in \mathcal{C}_k$ and let $J_0 \in \mathcal{C}$ be any almost complex structure. Let J_t be a continuous curve of almost complex structures joining J_0 and J_1. Up to small perturbations, we can assume that J_t is a real analytic curve by Lemma 1.4.6. By Proposition 2.2.2, and since $J_1 \in \mathcal{C}_k$, we have that $J_t \in \mathcal{C}_k$ for all $t \in I$ except a finite number. In particular, every neighborhood of J_0 contains almost complex structures in $\mathcal{C}_k$. Openness of $\mathcal{C}_k$ follows by lower semi-continuity of the rank of N_{J_t} with respect to t, and by the fact that if $J_0 \in \mathcal{C}_k$, then almost complex structures close to J_0 in the C^0-topology are small deformations of J_0. □

As a direct consequence, we deduce that maximally non-integrable structures are generic almost complex structures.

Corollary 2.2.5. *Let M be a compact almost complex manifold. Then the space of maximally non-integrable almost complex structures on M is either open and dense or empty in each path-connected component of $\mathcal{J}$.*

By Theorem 2.2.4 and the existence of structures on high-dimensional manifolds, we get the following approximation theorem.

Theorem 2.2.6. *Let M be a compact almost complex $2m$-manifold. Then every almost complex structure on M can be approximated arbitrarily well in the C^∞-topology by*

- *an everywhere non-integrable structure if $2m \geq 6$;*
- *a maximally non-integrable structure if $2m \geq 10$.*

Proof. By Corollary 2.1.6, existence of everywhere or maximally non-integrable structures is guaranteed on each path-connected component in dimension $2m \geq 6$ or $2m \geq 10$, respectively. Approximation in the C^0-topology is a direct consequence of the existence of these structures and of Theorem 2.2.4. To prove approximation in the C^∞-topology, it is enough to observe that the perturbed curve of almost complex structures $\tilde{J}_t$ obtained in Lemma 1.4.6 is analytic in t. □

Example 2.2.7 (**Invariant structures on solvmanifolds.**). By Corollary A.4 in [32], every invariant almost complex structure J_0 on a solvmanifold is homotopic to a maximally non-integrable one. By Theorem 2.2.6, one can find maximally non-integrable structures in each neighborhood of J_0. We show with an explicit example that this is the case for structure of arbitrary constant rank.

Proposition 2.2.8. *Let M be any complex parallelizable 3-dimensional solvmanifold. Then there exists an integrable structure J_0 on M for which we can find curves of almost complex structures J_s^k, $k \in \{0, 1, 2, 3\}$, such that* (*i*) $J_0^k = J_0$;

(*ii*) J_s^k *has constant rank* k *for all* $s \neq 0$, $|s| < \epsilon$.

In particular, each neighborhood of J_0 *contains almost complex structures whose Nijenhuis tensor has arbitrary constant rank.*

Proof. We give a description of complex parallelizable 3-dimensional solvmanifolds in Section 2.3.3. We show here the computations for maximally non-integrable structures on the Nakamura manifold $\mathcal{N}$. Consider the curve of almost complex structures on $\mathcal{N}$ defined by the co-frame of $(1,0)$-forms

$$\omega_s^1 := \phi^1 + s\phi^{\bar{2}} + s\phi^{\bar{3}}, \quad \omega_s^2 := \phi^2 + \frac{s}{2}\phi^{\bar{2}} \quad \text{and} \quad \omega_s^3 := \phi^3 + \frac{s}{2}\phi^{\bar{3}},$$

obtained deforming the standard complex structure on $\mathcal{N}$ in the direction of the maximally non-integrable almost complex structure (2.3.11). The same computations performed in Section 2.3.3 show that

$$\bar{\mu}_s\omega_s^1 = -\frac{4s}{4-|s|^2}\,\omega^{\bar{1}\bar{2}} + \frac{4s}{4-|s|^2}\,\omega^{\bar{1}\bar{3}} + \frac{32|s|^2}{4-|s|^2}\,\omega^{\bar{2}\bar{3}},$$

$$\bar{\mu}_s\omega_s^2 = \frac{2s}{4-|s|^2}\,\omega^{\bar{1}\bar{2}} + \frac{4s^3+8|s|^2}{(4-|s|^2)^2}\,\omega^{\bar{2}\bar{3}} \quad \text{and}$$

$$\bar{\mu}_s\omega_s^3 = \frac{2s}{4-|s|^2}\,\omega^{\bar{1}\bar{3}} - \frac{4s^3+8|s|^2}{(4-|s|^2)^2}\,\omega^{\bar{2}\bar{3}},$$

giving a maximally non integrable structure for all $s \neq 0$ such that $|s| \neq 2$. Computations for the other cases proceed in the same way, replacing (2.3.11) with (2.3.12) or (2.3.13). □

With a slight modification of the example above, one can produce structures of constant rank k_0 that admit a neighborhood containing structures of arbitrary constant rank k_1 for each $k_1 \geq k_0$. This is done replacing $\mathcal{N}$ and the structures considered in Proposition 2.2.8 with 6-dimensional nilmanifolds endowed with the structures provided in Table 2.1. ■

2.3 Structures of prescribed rank on parallelizable manifolds

We illustrate a general method to build almost complex structures of prescribed, possibly constant, rank on parallelizable $2m$-manifolds. Starting from an arbitrary almost complex structure J_0, we build an almost complex structure J_1 depending on J_0 and m^2 smooth functions. Computing $\operatorname{rk} N_{J_1}$ amounts to finding solutions of a system of first order PDEs involving the smooth functions parametrizing J_1. Despite the lack of a general solution, on specific manifold we are able to solve the system and to produce the desired structures.

2.3.1 Outline of the general procedure

Let M be a parallelizable $2m$-manifold. Fix a frame of vector fields $\{E_j\}_{j=1}^{2m}$ giving a parallelism of TM and let J_0 be an almost complex structure on M. The choice of J_0 determines a co-frame of $(1,0)$-forms $\{\phi^j\}_{j=1}^m$, and vice versa. Assume that the differentials $d\phi^j$ are known. Consider on M the almost complex structure J_1 defined by the co-frame of $(1,0)$-forms

$$\omega^j := \phi^j + f_k^j\, \phi^{\bar{k}}, \quad j = 1, \dots m,$$

where we are summing over repeated indices and the f_k^j are complex-valued smooth functions on M. Let Φ be the matrix

$$\Phi := (f_k^j) \in M_{m\times m}(C^\infty(M))$$

and let P be the matrix

$$P := \begin{bmatrix} \mathrm{Id}_m & \Phi \\ \bar{\Phi} & \mathrm{Id}_m \end{bmatrix}. \tag{2.3.1}$$

The forms ω^j are independent as long as

$$D := \det(P) \in C^\infty(M)$$

is a never-vanishing function on M. In such a case, J_1 is a well-defined almost complex structure. Since we described J_1 in terms of differential forms, instead of computing $\operatorname{rk} N_{J_1}$, it is convenient to compute the rank of the map $\bar{\mu}_1$ associated to J_1 on $(1,0)$-forms. The differential of ω^j is

$$d\omega^j = d\phi^j + f_k^j\, d\phi^{\bar{k}} + df_k^j \wedge \phi^{\bar{k}}.$$

We have to express it in function of $\{\omega^j, \omega^{\bar{j}}\}_{j=1}^m$ and then take its $(0,2)$-bidegree part *with respect to the bigrading induced by* J_1.

Let $\{\psi_j, \psi_{\bar{j}}\}_{j=1}^m$ be a frame of vector fields dual to $\{\omega^j, \omega^{\bar{j}}\}_{j=1}^m$. In such a frame, we can write for any $h \in C^\infty(M)$ the projection on bidegree $(0,2)$ as

$$(dh \wedge \phi^{\bar{k}})^{0,2} = F_{lp}^k(h)\, \omega^{\bar{l}\bar{p}},$$

where F^k_{lp} are suitable $(0,1)$-vector fields in $C^\infty(M)\langle\psi_{\bar{j}}\rangle_{j=1}^m$. By assumption, we are given an explicit expression for $d\phi^j$ and $d\phi^{\bar{j}}$ that can be written in terms of smooth functions on M and of the 2-forms ϕ^{jk}, $\phi^{j\bar{k}}$ and $\phi^{\bar{j}\bar{k}}$. By inverting the matrix P, one can compute the co-frame $\{\phi^j, \phi^{\bar{j}}\}_{j=1}^m$ in function of the co-frame $\{\omega^j, \omega^{\bar{j}}\}_{j=1}^m$. Taking the projection on bidegree $(0,2)$, we obtain an explicit expression for $\bar{\mu}_1\omega^j$ in terms of the functions f^j_k, of their first order derivatives $F^k_{lp}(f^j_k)$ and of a basis of $(0,2)$-forms $\{\omega^{\bar{j}\bar{k}}\}$. The rank of $\bar{\mu}_1$ can be prescribed imposing conditions on the f^j_k and the $F^k_{lp}(f^j_k)$. Finding functions satisfying such constraints provides a structure with the desired rank.

In practice, the approach we described is strongly limited by the difficulty of the computations involved, both on the side of the linear algebra, and on that of solving the final system of PDEs. In our applications, we will put ourselves in the best case scenario, making assumptions based on the following remarks:

1. the almost complex structure J_1 depends on the initial choice of J_0. The standard choice

$$J_0 E_k = E_{m+k} \quad \text{and} \qquad J_0 E_{m+k} = -E_k, \quad k = 1, \dots, m,$$

   allows to immediately find the starting co-frame of $(1,0)$-forms and their differentials;

2. increasing the dimension of M drastically increases the difficulty of the computations. We will focus on manifolds of dimension 4 and 6;

3. by choosing a manifold for which smooth functions can be explicitly written, or at least put in a manageable form, the final system of PDEs can be solved more easily.

2.3.2 Dimension 4: the Kodaira–Thurston manifold

We use the Kodaira–Thurston manifold as a toy model for the computations of the rank of $\bar{\mu}$. The 4-dimensional examples are less significant since the only possible values of the rank are 0 and 1. Nevertheless, they allow to greatly simplify computations and to clearly illustrate the ideas involved.

We begin by briefly recalling the construction of the Kodaira–Thurston manifold. Consider the 3-dimensional real Heisenberg group

$$\mathbb{H}_3 := \left\{ \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Kodaira–Thurston manifold $\mathcal{KT}$ is the 4-dimensional nilmanifold defined by

$$\mathcal{KT} := \mathbb{H}_3/(\mathbb{H}_3 \cap \mathrm{SL}(3,\mathbb{Z})) \times S^1.$$

Denote by t the coordinate on S^1. A basis of invariant vector fields on $\mathcal{KT}$ is

$$\{e_1 = \partial_t,\, e_2 = \partial_x,\, e_3 = \partial_y + x\,\partial_z,\, e_4 = \partial_z\}$$

and the dual basis of invariant 1-forms is

$$\{e^1 = dt,\, e^2 = dx,\, e^3 = dy,\, e^4 = dz - x\,dy\}.$$

The only non-vanishing Lie bracket of vector fields is $[e_2, e_3] = e_4$ and the only non-vanishing differential is $de^4 = -e^{23}$. It is well-known that $\mathcal{KT}$ admits both complex and symplectic structures, but it has no Kähler structure [98]. It also admits non-integrable almost complex structures.

We will perform the construction described in Section 2.3.1 twice: first we start from a complex structure and produce maximally non-integrable structures. Then we start from a maximally non-integrable structure and produce integrable structures.

From complex to almost complex

Let J_0 be the complex structure on $\mathcal{KT}$ given by

$$J_0\,e_1 := -e_4 \quad \text{and} \quad J_0\,e_2 := e_3.$$

A corresponding basis of $(1,0)$-forms for J_0 is given by

$$\phi^1 := dx + i\,dy \quad \text{and} \quad \phi^2 := dz - x\,dy + i\,dt,$$

with differentials

$$d\phi^1 = 0 \quad \text{and} \quad d\phi^2 = -\frac{i}{2}\,\phi^{1\bar{1}}.$$

Consider the co-frame of $(1,0)$-forms given by

$$\omega^1 := \phi^1 + e\,\phi^{\bar{1}} + f\,\phi^{\bar{2}} \quad \text{and} \quad \omega^2 := \phi^2 + g\,\phi^{\bar{1}} + h\,\phi^{\bar{2}},$$

where e, f, g and $h \in C^\infty(\mathcal{KT})$ are complex-valued smooth functions on $\mathcal{KT}$. Requiring that ω^1 and ω^2 have bidegree $(1,0)$ determines an almost complex structure J_1 on $\mathcal{KT}$ as long as

$$D := \det(P) = 1 - \bar{f}g - f\bar{g} + |f|^2|g|^2 - |e|^2 - |h|^2 + |e|^2|h|^2 - fg\bar{e}\bar{h} - \bar{f}\bar{g}eh \in C^\infty(\mathcal{KT})$$

never vanishes on $\mathcal{KT}$, where P is as in (2.3.1). We characterize integrability of J_1 in terms of conditions on the functions e, f, g, h and their derivatives. Direct computations show that

$$P^{-1} = \frac{1}{D}\begin{bmatrix} 1-\bar{f}g-|h|^2 & \bar{f}e+f\bar{h} & e|h|^2-e-fg\bar{h} & g|f|^2-f-\bar{f}eh \\ \bar{e}g+\bar{g}h & 1-f\bar{g}-|e|^2 & f|g|^2-g-\bar{g}eh & h|e|^2-h-fg\bar{e} \\ \bar{e}|h|^2-\bar{e}-\bar{f}\bar{g}h & \bar{g}|f|^2-\bar{f}-f\bar{e}\bar{h} & 1-f\bar{g}-|h|^2 & f\bar{e}+\bar{f}h \\ \bar{f}|g|^2-\bar{g}-g\bar{e}\bar{h} & \bar{h}|e|^2-\bar{h}-\bar{f}\bar{g}e & e\bar{g}+g\bar{h} & 1-\bar{f}g-|e|^2 \end{bmatrix}$$

Hence, the expressions of the ϕ^j in function of the ω^j are

$$\phi^1 = \tfrac{1}{D}\left[(1-\bar{f}g-|h|^2)\,\omega^1 + (\bar{f}e+f\bar{h})\,\omega^2 + (e|h|^2-e-fg\bar{h})\,\omega^{\bar{1}} + (g|f|^2-f-\bar{f}eh)\,\omega^{\bar{2}}\right]$$

and

$$\phi^2 = \tfrac{1}{D}\left[(\bar{e}g+\bar{g}h)\,\omega^1 + (1-f\bar{g}-|e|^2)\,\omega^2 + (f|g|^2-g-\bar{g}eh)\,\omega^{\bar{1}} + (h|e|^2-h-fg\bar{e})\,\omega^{\bar{2}}\right].$$

The differentials of the ω^j are

$$\begin{aligned} d\omega^1 &= f\, d\phi^{\bar{2}} + de\wedge\phi^{\bar{1}} + df\wedge\phi^{\bar{2}} \quad \text{and} \\ d\omega^2 &= (1+h)\, d\phi^{\bar{2}} + dg\wedge\phi^{\bar{1}} + dh\wedge\phi^{\bar{2}}. \end{aligned} \tag{2.3.2}$$

Since

$$\phi^{1\bar{1}} = \frac{1}{D}\left[-\bar{f}\,\omega^{12} - (1-|h|^2)\,\omega^{1\bar{1}} + \bar{f}h\,\omega^{1\bar{2}} - f\bar{h}\,\omega^{\bar{1}2} - |f|^2\,\omega^{2\bar{2}} + f\,\omega^{\bar{1}\bar{2}}\right],$$

we have that

$$(d\phi^{\bar{2}})^{0,2} = -\frac{i}{2}\,(\phi^{1\bar{1}})^{0,2} = -\frac{i}{2}\frac{f}{D}\omega^{\bar{1}\bar{2}}. \tag{2.3.3}$$

Denote by $\{\xi_j,\, \xi_{\bar{j}}\}$ the frame dual to $\{\phi^j,\, \phi^{\bar{j}}\}$ and by $\{\psi_j,\, \psi_{\bar{j}}\}$ the one dual to $\{\omega^j,\, \omega^{\bar{j}}\}$. By duality, the frame $\{\psi_j, \psi_{\bar{j}}\}$ depends on $\{\xi_j, \xi_{\bar{j}}\}$ via $(P^{-1})^t$. The $(0,2)$-bidegree part of the 2-forms $dh\wedge\phi^{\bar{j}}$ is

$$(dh\wedge\phi^{\bar{j}})^{0,2} = F^j(h)\,\omega^{\bar{1}\bar{2}}, \tag{2.3.4}$$

where

$$\begin{aligned} F^1(h) &:= \frac{1}{D}\left[(f\bar{e}+\bar{f}h)\,\psi_{\bar{1}}(h) - (1-f\bar{g}-|h|^2)\,\psi_{\bar{2}}(h)\right] \quad \text{and} \\ F^2(h) &:= \frac{1}{D}\left[(1-\bar{f}g-|e|^2)\,\psi_{\bar{1}}(h) - (e\bar{g}+\bar{g}h)\,\psi_{\bar{2}}(h)\right]. \end{aligned} \tag{2.3.5}$$

Finally, we can compute $\bar{\mu}_1\omega^j$ taking the $(0,2)$-bidegree part of (2.3.2). By (2.3.3) and (2.3.4), we obtain

$$\bar{\mu}_1\omega^1 = \left(-\frac{i}{2D}f^2 + F^1(e) + F^2(f)\right)\omega^{\bar{1}\bar{2}} \quad \text{and}$$
$$\bar{\mu}_1\omega^2 = \left(-\frac{i}{2D}f(1+h) + F^1(g) + F^2(h)\right)\omega^{\bar{1}\bar{2}}.$$

The system of PDEs

$$\begin{cases} -\frac{i}{2D}f^2 + F^1(e) + F^2(f) = 0, \\ -\frac{i}{2D}f(1+h) + F^1(g) + F^2(h) = 0, \end{cases}$$

allows to control integrability of J_1. First, we look for constant solutions. They correspond to invariant almost complex structures on $\mathcal{KT}$. The system reduces to

$$\begin{cases} -\frac{i}{2}f^2 = 0, \\ -\frac{i}{2}f(1+h) = 0. \end{cases}$$

We can conclude that J_1 is integrable if and only if $f = 0$. In any other case, its rank is equal to 1. This is true as long as D does not vanish, i.e., as long as $|e| \neq 1$ and $|h| \neq 1$.

We now aim at finding maximally non-integrable almost complex structures that are not invariant, looking for non-constant functions such that at least one between $\bar{\mu}_1\omega^1$ and $\bar{\mu}_1\omega^2$ never vanishes. To simplify the computations, we take e and h to be identically zero. We must find f and $g \in C^\infty(\mathcal{KT})$ such that at every point either

$$-\frac{i}{2D}f^2 + F^2(f) \neq 0 \quad \text{or} \quad -\frac{i}{2D}f + F^1(g) \neq 0.$$

The terms involved have the explicit expressions

$$\psi_1 = \frac{1}{D}\left[(1-\bar{f}g)\,\xi_1 + \bar{g}(\bar{f}g-1)\,\xi_{\bar{2}}\right] \quad \text{and}$$
$$\psi_2 = \frac{1}{D}\left[(1-f\bar{g})\,\xi_2 + \bar{f}(f\bar{g}-1)\,\xi_{\bar{1}}\right],$$

where $D = (1-f\bar{g})(1-\bar{f}g)$ must be never-vanishing. By substituting

$$F^1(g) = -\frac{1}{D}(1-f\bar{g})\,\psi_{\bar{2}}(g) = -\frac{1}{D}(\xi_{\bar{2}} - f\xi_1)(g) \quad \text{and}$$
$$F^2(f) = \frac{1}{D}(1-\bar{f}g)\,\psi_{\bar{1}}(f) = \frac{1}{D}(\xi_{\bar{1}} - g\xi_2)(f)$$

in the PDEs, we have that it must be

$$-\frac{i}{2D}f^2 + \frac{1}{D}(\xi_{\bar{1}} - g\xi_2)(f) \neq 0 \quad \text{or} \quad -\frac{i}{2D}f - \frac{1}{D}(\xi_{\bar{2}} - f\xi_1)(g) \neq 0.$$

Proposition 2.3.1. *For every never vanishing $f \in C^\infty(\mathcal{KT})$, there exists a maximally non-integrable almost complex structure J_f on $\mathcal{KT}$.*

Proof. We impose $g = 0$, so that the equations reduce to

$$-\frac{i}{2D} f^2 + \frac{1}{D}\xi_{\bar{1}}(f) \neq 0 \quad \text{or} \quad -\frac{i}{2D} f \neq 0.$$

Any never vanishing f provides a maximally non-integrable almost complex structure obtained taking $\omega^1 = \phi^1 + f\,\phi^{\bar{2}}$ and $\omega^2 = \phi^2$. Note that if $e = g = h = 0$, then $D = 1$, and the resulting structure is well defined. □

From almost complex to complex

Let J_0 be the almost complex structure of constant rank 1 given by

$$J_0\, e_1 \coloneqq -e_2 \quad \text{and} \quad J_0\, e_3 \coloneqq -e_4. \tag{2.3.6}$$

A basis of $(1,0)$-forms is

$$\phi^1 \coloneqq dx + i\,dt \quad \text{and} \quad \phi^2 \coloneqq dz - x\,dy + i\,dy,$$

and the differentials are

$$d\phi^1 = 0 \quad \text{and} \quad d\phi^2 = \frac{i}{4}\,(\phi^{12} - \phi^{1\bar{2}} + \phi^{\bar{1}2} - \phi^{\bar{1}\bar{2}}).$$

Proceeding in the same way as in the previous paragraph, we build a structure J_1 depending on functions e, f, g and $h \in C^\infty(\mathcal{KT})$. The 2-forms ϕ^{12} and $\phi^{1\bar{2}}$ expressed in the co-frame $\{\omega^j, \omega^{\bar{j}}\}$ are

$$\phi^{12} = \frac{1}{D}\left[\omega^{12} - g\,\omega^{1\bar{1}} - h\,\omega^{1\bar{2}} - e\,\omega^{\bar{1}2} + f\,\omega^{2\bar{2}} + (eh - fg)\,\omega^{\bar{1}\bar{2}}\right] \quad \text{and}$$

$$\phi^{1\bar{2}} = \frac{1}{D}\left[-\bar{h}\,\omega^{12} + g\bar{h}\,\omega^{1\bar{1}} + (1 - \bar{f}g)\,\omega^{1\bar{2}} + e\bar{h}\,\omega^{\bar{1}2} + \bar{f}e\,\omega^{2\bar{2}} - e\,\omega^{\bar{1}\bar{2}}\right],$$

and the only non-zero differential is

$$\begin{aligned} d\phi^2 = \frac{i}{4D}\Big[&(1 + \bar{h} - \bar{e} - \bar{e}\bar{h} + \bar{f}\bar{g})\,\omega^{12} + (-g\bar{h} - \bar{g}h - g - \bar{g})\,\omega^{1\bar{1}} \\ &+(-1 + \bar{f}g + \bar{e}h - h + \bar{e})\,\omega^{1\bar{2}} + (1 - f\bar{g} - e\bar{h} - e + \bar{h})\,\omega^{\bar{1}2} \\ &+(-\bar{f}e - f\bar{e} + f + \bar{f})\,\omega^{2\bar{2}} + (-1 + e - h + eh - fg)\,\omega^{\bar{1}\bar{2}}\Big]. \end{aligned}$$

The corresponding operator $\bar{\mu}_1$ computed on $(1,0)$-forms is given by

$$\bar{\mu}_1\omega^1 = \left(-\frac{i}{4D}(1 - e + h - eh + fg)f + F^1(e) + F^2(f)\right)\omega^{\bar{1}\bar{2}} \quad \text{and}$$

$$\bar{\mu}_1\omega^2 = \left(-\frac{i}{4D}(1-e+h-eh+fg)(1+h)+F^1(g)+F^2(h)\right)\omega^{\bar{1}\bar{2}},$$

where the F^j are as in (2.3.5). In order to find an integrable structure we must solve the following system of PDEs

$$\begin{cases} -\frac{i}{4}(1-e+h-eh+fg)f+(f\bar{e}+\bar{f}h)\,\psi_{\bar{1}}(e)-(1-f\bar{g}-|h|^2)\,\psi_{\bar{2}}(e) \\ +(1-\bar{f}g-|e|^2)\,\psi_{\bar{1}}(f)-(\bar{g}e+g\bar{h})\,\psi_{\bar{2}}(f)=0, \\ -\frac{i}{4}(1-e+h-eh+fg)(1+h)+(f\bar{e}+\bar{f}h)\,\psi_{\bar{1}}(g)-(1-f\bar{g}-|h|^2)\,\psi_{\bar{2}}(g) \\ +(1-\bar{f}g-|e|^2)\,\psi_{\bar{1}}(h)-(\bar{g}e+g\bar{h})\,\psi_{\bar{2}}(h)=0, \end{cases} \tag{2.3.7}$$

where the operators $\psi_{\bar{j}}$ have the expressions

$$\psi_{\bar{1}} = \frac{1}{D}\Big[(\bar{e}|h|^2-\bar{e}-\bar{f}\bar{g}h)\,\xi_1+(\bar{g}|f|^2-\bar{f}-f\bar{e}\bar{h})\,\xi_2 \\ +(1-f\bar{g}-|h|^2)\,\xi_{\bar{1}}+(f\bar{e}+\bar{f}h)\,\xi_{\bar{2}}\Big] \quad \text{and}$$

$$\psi_{\bar{2}} = \frac{1}{D}\Big[(\bar{f}|g|^2-\bar{g}-g\bar{e}\bar{h})\,\xi_1+(\bar{h}|e|^2-\bar{h}-\bar{f}\bar{g}e)\,\xi_2 \\ +(e\bar{g}+g\bar{h})\,\xi_{\bar{1}}+(1-\bar{f}g-|e|^2)\,\xi_{\bar{2}}\Big].$$

Invariant complex structures, i.e., constant solutions, are obtained solving

$$\begin{cases} -\frac{i}{4}(1-e+h-eh+fg)f=0, \\ -\frac{i}{4}(1-e+h-eh+fg)(1+h)=0. \end{cases}$$

The solutions $\{f=0,\, h=-1\}$ or $\{f=0,\, e=1\}$ lead to a vanishing D, thus J_1 is an invariant complex structure if and only if $1-e+h-eh+fg=0$ and D does not vanish. This can be easily achieved by setting $f=e-1\neq 0$ and $g=h+1\neq 0$.

Regarding non-constant solutions, finding an expression for the general solutions of (2.3.7) is a difficult task. We rather provide a family of explicit solutions.

Proposition 2.3.2. *For every never vanishing $f\in C^\infty(\mathcal{KT})$ depending only on the x variable, there exists an integrable almost complex structure J_f on $\mathcal{KT}$.*

Proof. Recall that the frame $\{\xi_1,\xi_2\}$ of $(1,0)$-vector fields can be written in terms of real vector fields as

$$\xi_1=\frac{1}{2}(\partial_x-i\partial_t) \quad \text{and} \quad \xi_2=\frac{1}{2}(\partial_z-i(\partial_y+x\partial_z)).$$

Let $f \in C^\infty(\mathcal{KT})$ be a never vanishing function that depends only on the x variable, e.g., $f(x) = A + \cos(2\pi x)$, where A is a real constant such that $A > 1$ or $A < -1$. It is not hard to check that the choice

$$e = f + 1, \quad f, \quad g = f \quad \text{and} \quad h = f - 1,$$

gives a solution of (2.3.7), and that the corresponding J_1 is a well-defined complex structure on $\mathcal{KT}$. □

We conclude pointing out that the technique employed in this section allows to build new complex structures on the Kodaira–Thurston manifold. The same idea could be used to produce new complex structures on other manifolds as soon as one can show that the associated PDE admits at least one solution.

2.3.3 Dimension 6: holomorphically parallelizable complex solvmanifolds

We explicitly compute almost complex structures of arbitrary constant rank on complex parallelizable solvmanifolds of complex dimension 3. There are three such manifolds [73], namely the 6-dimensional torus, the *Iwasawa manifold* and the holomorphically parallelizable *Nakamura manifold*. By Corollary A.4 in [32], they all admit a maximally non-integrable almost complex structure. However, only the Nakamura manifold admits an invariant one.

2.3.3.1 Torus

For the case of the torus, we refer to Theorem 14 in [32] or to Example 2.1.3.

2.3.3.2 Iwasawa Manifold

For the Iwasawa manifold, we only show the results of the computations, since they follows closely those performed in Section 2.3.2 for the Kodaira–Thurston manifold, with the added difficulty of working in dimension 6. We find the possible rank for invariant structures and fill the gaps exhibiting explicit non-invariant structures.

Let $\mathbb{H}_3^{\mathbb{C}}$ be the complex Heisenberg group

$$\mathbb{H}_3^{\mathbb{C}} := \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}$$

whit the group operation induced by matrix multiplication. The Iwasawa manifold is the quotient

$$\mathcal{I} := \mathbb{H}_3^{\mathbb{C}} / (\mathbb{H}_3^{\mathbb{C}} \cap \mathrm{SL}(3, \mathbb{Z}[i])).$$

The complex structure inherited from $\mathbb{C}$ induces on $\mathcal{I}$ a basis of $(1,0)$-forms $\{\phi^j\}_{j=1}^3$ whose differentials are

$$d\phi^1 = 0, \quad d\phi^2 = 0 \quad \text{and} \quad d\phi^3 = -\phi^{12}.$$

This corresponds to a complex structure on the Lie algebra

$$(0,0,0,0,13-24,14+23).$$

Consider the 1-forms

$$\begin{aligned}\omega^1 &:= \phi^1 + e\,\phi^{\bar{1}} + f\,\phi^{\bar{2}} + g\,\phi^{\bar{3}},\\ \omega^2 &:= \phi^2 + p\,\phi^{\bar{1}} + q\,\phi^{\bar{2}} + r\,\phi^{\bar{3}},\\ \omega^3 &:= \phi^3 + s\,\phi^{\bar{1}} + t\,\phi^{\bar{2}} + u\,\phi^{\bar{3}},\end{aligned}$$

where e,f,g,p,q,r,s,t and $u \in C^\infty(\mathcal{I})$. Let P be as in (2.3.1). Requiring the ω^j to have bidegree $(1,0)$ defines an almost complex structure J_1 on $\mathcal{I}$ as long as $D := \det(P) \in C^\infty(\mathcal{I})$ never vanishes.

We first focus on invariant structures, that correspond to taking the functions $e,\dots,t$ and u to be constant. Following Section 2.3.1, we are able to build invariant structures of rank 1 and 2, see also Table 2.1. An invariant structure of rank 2 is given by

$$\omega^1 := \phi^1 + \phi^{\bar{3}}, \quad \omega^2 := \phi^2 + 2\phi^{\bar{2}} \quad \text{and} \quad \omega^3 := \phi^3.$$

An invariant structure of rank 1 is given by

$$\omega^1 := \phi^1, \quad \omega^2 := \phi^2 + \phi^{\bar{3}} \quad \text{and} \quad \omega^3 := \phi^3.$$

The following proposition on structures of rank 3 follows either from Corollary 2.4.3 or from Theorem 2.4.15.

Proposition 2.3.3. *The Iwasawa manifold does not admit invariant maximally non-integrable almost complex structures.*

To find maximally non-integrable structures, let e,f,g,q,r and u be complex smooth functions on $\mathcal{I}$, and consider the following $(1,0)$-forms:

$$\begin{aligned}\omega^1 &:= \phi^1 + e\,\phi^{\bar{1}} + f\,\phi^{\bar{2}} + g\,\phi^{\bar{3}},\\ \omega^2 &:= \phi^2 + q\,\phi^{\bar{2}} + r\,\phi^{\bar{3}},\\ \omega^3 &:= \phi^3 + u\,\phi^{\bar{3}}.\end{aligned}$$

They define an almost complex structure J_1 as long as

$$D = (1-|e|^2)\,(1-|q|^2)\,(1-|u|^2) \neq 0.$$

Following again Section 2.3.1, for any $h \in C^\infty(\mathcal{I})$ we write

$$(dh \wedge \phi^{\bar{j}})^{0,2} = \sum_{\substack{k,\, l=1 \\ k<l}}^{3} F^{\bar{j}}_{\bar{k}\bar{l}}(h)\, \omega^{\bar{k}\bar{l}}.$$

The explicit expressions for the $F^{\bar{j}}_{\bar{k}\bar{l}}$ in terms of the frame $\{\xi_j, \xi_{\bar{j}}\}$ dual to $\{\phi^j, \phi^{\bar{j}}\}$ are

$$\begin{aligned}
F^{\bar{1}}_{\bar{1}\bar{2}} &= \frac{1}{D}[(1-|u|^2)\, f\, \xi_1 + (1-|u|^2)\, q\, \xi_2 - (1-|u|^2)\, \xi_{\bar{2}}],\\
F^{\bar{1}}_{\bar{1}\bar{3}} &= \frac{1}{D}[(g - g|q|^2 + fr\bar{q} + fu\bar{r})\, \xi_1 + (r + qu\bar{r})\, \xi_2 - u\, \xi_3 - (r\bar{q} + u\bar{r})\, \xi_{\bar{2}} + \xi_{\bar{3}}],\\
F^{\bar{1}}_{\bar{2}\bar{3}} &= \frac{1}{D}[(-r|f|^2 + gq\bar{f} - fu\bar{g})\, \xi_1 + (-gq\bar{e} + fr\bar{e} - qu\bar{g})\, \xi_2 + u(f\bar{e} + q\bar{f})\, \xi_3\\
&\qquad + (g\bar{e} + r\bar{f} + u\bar{g})\, \xi_{\bar{2}} - (f\bar{e} + q\bar{f})\, \xi_{\bar{3}}],\\
F^{\bar{2}}_{\bar{1}\bar{2}} &= \frac{1}{D}[-(1-|u|^2)\, e\, \xi_1 + (1-|u|^2)\, \xi_{\bar{1}}],\\
F^{\bar{2}}_{\bar{1}\bar{3}} &= \frac{1}{D}[-e(r\bar{q} + u\bar{r})\, \xi_1 + (r\bar{q} + u\bar{r})\, \xi_{\bar{1}}],\\
F^{\bar{2}}_{\bar{2}\bar{3}} &= \frac{1}{D}[(g + er\bar{f} + eu\bar{g})\, \xi_1 + (1-|e|^2)\, r\, \xi_2 + (1-|e|^2)\, \xi_3\\
&\qquad - (g\bar{e} + r\bar{f} + u\bar{g})\, \xi_{\bar{1}} - (1-|e|^2)\, \xi_{\bar{3}}],\\
F^{\bar{3}}_{\bar{1}\bar{2}} &= 0,\\
F^{\bar{3}}_{\bar{1}\bar{3}} &= \frac{1}{D}[e\, \xi_1 - \xi_{\bar{1}}],\\
F^{\bar{3}}_{\bar{2}\bar{3}} &= \frac{1}{D}[-(f + eq\bar{f})\, \xi_1 - (1-|e|^2)\, q\, \xi_2 + (f\bar{e} + q\bar{f})\, \xi_{\bar{1}} + (1-|e|^2)\, \xi_{\bar{2}}].
\end{aligned}$$

This allows to write $\bar{\mu}_1$ on $(1,0)$-forms as

$$\begin{aligned}
\bar{\mu}_1\omega^1 &= \Big(-g\,(1-|u|^2) + F^{\bar{1}}_{\bar{1}\bar{2}}(e) + F^{\bar{2}}_{\bar{1}\bar{2}}(f)\Big)\, \omega^{\bar{1}\bar{2}}\\
&\quad + \Big(-g\,(r\bar{q} + u\bar{r}) + F^{\bar{1}}_{\bar{1}\bar{3}}(e) + F^{\bar{2}}_{\bar{1}\bar{3}}(f) + F^{\bar{3}}_{\bar{1}\bar{3}}(g)\Big)\, \omega^{\bar{1}\bar{3}}\\
&\quad + \Big(g\,(g\bar{e} + r\bar{f} + u\bar{g}) + F^{\bar{1}}_{\bar{2}\bar{3}}(e) + F^{\bar{2}}_{\bar{2}\bar{3}}(f) + F^{\bar{3}}_{\bar{2}\bar{3}}(g)\Big)\, \omega^{\bar{2}\bar{3}},\\
\bar{\mu}_1\omega^2 &= \Big(-r\,(1-|u|^2) + F^{\bar{2}}_{\bar{1}\bar{2}}(q)\Big)\, \omega^{\bar{1}\bar{2}}\\
&\quad + \Big(-r\,(r\bar{q} + u\bar{r}) + F^{\bar{2}}_{\bar{1}\bar{3}}(q) + F^{\bar{3}}_{\bar{1}\bar{3}}(r)\Big)\, \omega^{\bar{1}\bar{3}}\\
&\quad + \Big(r\,(g\bar{e} + r\bar{f} + u\bar{g}) + F^{\bar{2}}_{\bar{2}\bar{3}}(q) + F^{\bar{3}}_{\bar{2}\bar{3}}(r)\Big)\, \omega^{\bar{2}\bar{3}},
\end{aligned}$$

$$\begin{aligned}\bar{\mu}_1\omega^3 = &\left(-(eq+u)\,(1-|u|^2)\right)\omega^{\bar{1}\bar{2}}\\ &+\Big(-(er+ru\bar{q}+u\,(eq+u)\,\bar{r})+F^{\bar{3}}_{\bar{1}\bar{3}}(u)\Big)\,\omega^{\bar{1}\bar{3}}\\ &+\Big((gq-fr+gu\bar{e}+ru\bar{f}+u\,(eq+u)\,\bar{g})+F^{\bar{3}}_{\bar{2}\bar{3}}(u)\Big)\,\omega^{\bar{2}\bar{3}}.\end{aligned}$$

To further simplify the computations, we impose $e = u = 0$. Taking the determinant of the 3×3 matrix defined by the coefficients of $\bar{\mu}_1\omega^j$ with respect to the basis $\{\omega^{\bar{k}\bar{l}}\}$, we see that J_1 is maximally non-integrable if and only if the function

$$G := (g\,q - f\,r)\left[g\,\xi_{\bar{1}}(r) - r\,\xi_{\bar{1}}(g) + \frac{1}{D}(\xi_{\bar{1}}(g)\,\xi_{\bar{1}}(q) - \xi_{\bar{1}}(f)\,\xi_{\bar{1}}(r))\right]$$

never vanishes on $\mathcal{I}$. In terms of the z_j coordinates, we have that $\xi_{\bar{1}} = \frac{\partial}{\partial z_{\bar{1}}}$. Denote by x_1 the real part of z_1. The following choice of functions leads to a non-vanishing G at every point:

$$\begin{aligned}&g(x_1) = \cos(2\pi x_1), \quad r(x_1) = \sin(2\pi x_1),\\ &q(x_1) = \frac{1}{2}\cos(2\pi x_1) \quad \text{and} \quad f(x_1) = \frac{1}{2}\sin(2\pi x_1).\end{aligned} \tag{2.3.8}$$

This provides a non-invariant maximally non-integrable almost complex structure on $\mathcal{I}$. Furthermore, it is immediate to check, using the explicit expression for $\bar{\mu}_1$, that the functions

$$f = 0, \quad q = 0, \quad g = h \quad \text{and} \quad r = h, \tag{2.3.9}$$

where $h \in C^\infty(\mathcal{I})$ is any never-vanishing function, give a non-invariant structure of constant rank 1 on $\mathcal{I}$, while the choice

$$f = 0, \quad q = 0, \quad g = \sin(2\pi x_1) \quad \text{and} \quad r = \cos(2\pi x_1), \tag{2.3.10}$$

gives a non-invariant structure of constant rank 2. Using the prototype structures we just defined, we build families of non-invariant almost complex structures of prescribed constant rank

Proposition 2.3.4. *The Iwasawa manifold admits the following families of almost complex structures of constant rank:*

- *a family J^3_h of maximally non-integrable structures parametrized by never-vanishing $h \in C^\infty(\mathcal{I})$ such that $\xi_{\bar{1}}(h) = 0$ and $|h| \leq 1$;*
- *a family J^2_h of structures of constant rank 2 parametrized by never-vanishing $h \in C^\infty(\mathcal{I})$ such that $\xi_{\bar{1}}(h) = 0$;*

- *a family* J_h^1 *of structures of constant rank* 1 *parametrized by never-vanishing* $h \in C^\infty(\mathcal{I})$.

Proof. Let $h \in C^\infty(\mathcal{I})$ be a never-vanishing function and let J_h be the almost complex structure defined by the $(1,0)$-forms

$$\begin{aligned}\omega^1 &:= \phi^1 + hf\,\phi^{\bar{2}} + hg\,\phi^{\bar{3}},\\ \omega^2 &:= \phi^2 + hq\,\phi^{\bar{2}} + hr\,\phi^{\bar{3}},\\ \omega^3 &:= \phi^3,\end{aligned}$$

where f, g, q and $r \in C^\infty(\mathcal{I})$. The family of structures J_h^1 is obtained choosing the functions f, g, q and r as in (2.3.9), since in this case the rank of J_h is 1. Further assuming that $\xi_{\bar{1}}(h) = 0$, the choice of f, g, q and r as in (2.3.10), provides a family of structures of constant rank 2. Finally, if we also assume $|h| \leq 1$, and take f, g, q and r as in (2.3.8), then J_h is a well-defined maximally non-integrable almost complex structure. □

2.3.3.3 Nakamura manifold

The Nakamura manifold admits invariant structures of any constant rank. Since the computations are substantially the same as in Section 2.3.3.2, we omit the details.

Let G be the Lie group $\mathbb{C} \ltimes_\psi \mathbb{C}^2$, with coordinates z_1, z_2, z_3, where

$$\psi(z_1) := \begin{bmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{bmatrix}.$$

The Nakamura manifold is the quotient

$$\mathcal{N} := \Gamma \backslash G,$$

where $\Gamma \subseteq G$ is a suitable lattice [73]. A basis of holomorphic $(1,0)$-forms that trivializes the complexified tangent bundle is given by

$$\phi^1 := dz^1, \quad \phi^2 := e^{-z_1} dz^2 \quad \text{and} \quad \phi^3 := e^{z_1} dz^3,$$

and the differentials are

$$d\phi^1 = 0, \quad d\phi^2 = -\phi^{12} \quad \text{and} \quad d\phi^3 = \phi^{13}.$$

The Nakamura manifold admits invariant structures of all possible ranks. We give an explicit example for each of them. A structure of rank 3 is defined by the $(1,0)$-forms

$$\omega^1 := \phi^1 + \phi^{\bar{2}} + \phi^{\bar{3}}, \quad \omega^2 := \phi^2 + \frac{1}{2}\phi^{\bar{2}} \quad \text{and} \quad \omega^3 := \phi^3 + \frac{1}{2}\phi^{\bar{3}}. \tag{2.3.11}$$

A structure of rank 2 is given by

$$\omega^1 := \phi^1, \quad \omega^2 := \phi^2 + \phi^{\bar{3}} \quad \text{and} \quad \omega^3 := \phi^3 + 2\phi^{\bar{2}}, \tag{2.3.12}$$

while a structure of rank 1 is given by

$$\omega^1 := \phi^1, \quad \omega^2 := \phi^2 + \phi^{\bar{3}} \quad \text{and} \quad \omega^3 := \phi^3. \tag{2.3.13}$$

2.4 Invariant almost complex structures on compact quotients of Lie groups

In this section we compute the rank of the Nijhenuis tensor of almost complex structures on 6-dimensional nilpotent real Lie algebras. As a consequence, for each 6-dimensional nilmanifold we determine whether or not it admits an invariant almost complex structure whose Nijenhuis tensor has a given rank. If such a structure exists, we provide an explicit choice of complex parameters that allows to build it starting from an assigned almost complex structure. We also deduce a topological upper bound for the rank of N_J on solvmanifolds of completely solvable type.

2.4.1 Classification of invariant structures on 6-dimensional nilpotent real Lie algebras

We determine the possible values of the rank of the Nijenhuis tensor of almost complex structures on 6-dimensional nilpotent real Lie algebras. There are 34 isomorphism classes of 6-dimensional nilpotent Lie algebras, see [63] or [81]. Adopting the notation of Section 1.5, we give a list of them in the first column of Table 2.1. The main result is the following classification theorem.

Theorem 2.4.1. *A* 6*-dimensional nilpotent real Lie algebra* $\mathfrak{g}$ *admits an almost complex structure whose Nijenhuis tensor has rank* 3 *if and only if it is isomorphic* *to one of*

$$
\begin{array}{ll}
(0,0,12,13,14+23,34-25), & (0,0,12,13,14,34-25),\\
(0,0,12,13,14+23,24+15), & (0,0,12,13,14,23+15),\\
(0,0,12,13,23,14), & (0,0,12,13,23,14-25),\\
(0,0,12,13,23,14+25), & (0,0,0,12,14-23,15+34),\\
(0,0,0,12,14,15+23), & (0,0,0,12,14,15+23+24),\\
(0,0,0,12,14,15+24), & (0,0,0,12,13,14+35),\\
(0,0,0,12,23,14+35), & (0,0,0,12,23,14-35),\\
(0,0,0,12,14,24), & (0,0,0,12,13-24,14+23),\\
(0,0,0,12,14,13-24), & (0,0,0,12,13+14,24),\\
(0,0,0,12,13,14+23), & (0,0,0,12,13,24),\\
(0,0,0,12,13,23). &
\end{array}
$$

A 6*-dimensional nilpotent real Lie algebra* $\mathfrak{g}$ *does* ***not*** *admit an almost complex structure whose Nijenhuis tensor has rank* 2 *if and only if it is isomorphic to one of*

$$
\begin{array}{ll}
(0,0,0,12,13,23), & (0,0,0,0,0,12+34),\\
(0,0,0,0,0,12), & (0,0,0,0,0,0).
\end{array}
$$

A 6*-dimensional nilpotent real Lie algebra* $\mathfrak{g}$ *does* ***not*** *admit an almost complex structure whose Nijenhuis tensor has rank* 1 *if and only if it is isomorphic to one of*

$$
(0,0,12,13,14+23,34-25), \quad (0,0,0,0,0,0).
$$

The proof of the theorem is a collection of smaller results, each one dealing with a different value of the rank. We proceed to determine the rank of the almost complex structures existing on each of them. For the rest of the section we will directly work with almost complex structures defined on the elements of $\mathfrak{g}^*$, adopting the corresponding notation.

Structures of rank 3

Lie algebras for which $A^1_{\mathbb{R}} \cap \ker d$ is high-dimensional *never* admit maximally non-integrable almost complex structures. This is a direct consequence of the following lemma, that holds for arbitrary Lie algebras in any dimension.

Lemma 2.4.2. *Let* $\mathfrak{g}$ *be a* $2m$*-dimensional Lie algebra and let* k *be the real dimension of* $A^1_{\mathbb{R}} \cap \ker d$*. Then for any almost complex structure* J *on* $\mathfrak{g}$ *we have that*

$$
\operatorname{rk} N_J \leq 2m - k.
$$

Proof. Let J be an almost complex structure on $\mathfrak{g}$. Then

$$\begin{aligned}\operatorname{rk} N_J &= \dim_{\mathbb{C}}(A^{0,2} \cap \operatorname{Im} \bar{\mu}) \leq \dim_{\mathbb{C}}(A^2 \cap \operatorname{Im} d) \\ &= \dim_{\mathbb{C}} \mathbb{C}\langle de^1, \ldots, de^{2m}\rangle = 2m - k.\end{aligned}$$ □

It follows immediately that several Lie algebras do not admit maximally non-integrable almost complex structures.

Corollary 2.4.3. *Let* $\mathfrak{g}$ *be any* 6*-dimensional nilpotent Lie algebra with* $b_1 \geq 4$. *Then* $\mathfrak{g}$ *does not admit maximally non-integrable almost complex structures.*

There are three 6-dimensional nilpotent Lie algebras without maximally non-integrable structures that are not covered by Corollary 2.4.3. Also in this case, non-existence follows from a general result.

Proposition 2.4.4. *Let* $\mathfrak{g}$ *be a* $2m$*-dimensional Lie algebra such that*

$$A^2_{\mathbb{R}} \cap \operatorname{Im} d \subseteq e^1 \wedge A^1_{\mathbb{R}}.$$

If $2m \geq 6$, *then* $\mathfrak{g}$ *does not admit maximally non-integrable almost complex structures.*

Proof. Let J be an almost complex structure on $\mathfrak{g}$. Consider the $(1,0)$-form

$$\omega^1 := e^1 + iJe^1$$

and complete it to a basis of $(1,0)$-forms $\{\omega^1, \ldots, \omega^m\}$. By the assumption on $A^2_{\mathbb{R}} \cap \operatorname{Im} d$, we have that

$$A^2 \cap \operatorname{Im} d = \mathbb{C}\langle de^1, \ldots de^6\rangle \subseteq e^1 \wedge A^1.$$

Taking the projection on bidegree $(0,2)$, we conclude that

$$\begin{aligned}\operatorname{rk} N_J &= \dim_{\mathbb{C}}(A^{0,2} \cap \operatorname{Im} \bar{\mu}) \\ &\leq \dim_{\mathbb{C}} \mathbb{C}\langle (e^1)^{0,1}\rangle \wedge (A^1)^{0,1} \\ &= \dim_{\mathbb{C}}(\omega^{\bar{1}} \wedge A^{0,1}) = m - 1,\end{aligned}$$

where $(\cdot)^{0,1}$ denotes the projection on bidegree $(0,1)$. If $2m \geq 6$, then $\operatorname{rk} N_J$ cannot be maximal. □

Corollary 2.4.5. *None of the following Lie algebras admits a maximally non-integrable almost complex structure:*

$$(0,0,0,12,13,14), \quad (0,0,0,12,14,15), \quad (0,0,12,13,14,15). \tag{2.4.1}$$

The following existence result completes the classification of Lie algebras admitting a maximally non-integrable structure.

Proposition 2.4.6. *Any* 6*-dimensional nilpotent Lie algebra with* $b_1 \leq 3$ *and not listed in* (2.4.1) *admits a maximally non-integrable almost complex structure.*

Proof. We prove existence of a maximally non-integrable almost complex structure by explicitly building it. Let $\mathfrak{g} = \mathbb{R}\langle e_1, \ldots, e_6\rangle$ be a nilpotent Lie algebra. Consider the almost complex structure J_0 defined by the co-frame of $(1,0)$-forms

$$\phi^1 := e^1 + ie^2, \quad \phi^2 := e^3 + ie^4 \quad \text{and} \quad \phi^3 := e^5 + ie^6.$$

Let J_1 be the almost complex structure defined by the $(1,0)$-forms

$$\begin{aligned} \omega^1 &:= \phi^1 + e\,\phi^{\bar 1} + f\,\phi^{\bar 2} + g\,\phi^{\bar 3},\\ \omega^2 &:= \phi^2 + p\,\phi^{\bar 1} + q\,\phi^{\bar 2} + r\,\phi^{\bar 3},\\ \omega^3 &:= \phi^3 + s\,\phi^{\bar 1} + t\,\phi^{\bar 2} + u\,\phi^{\bar 3}, \end{aligned}$$

with e, f, g, p, q, r, s, t and u complex parameters satisfying the condition $\det(P) \neq 0$ and P defined as in (2.3.1). For the choice of parameters described in the second column of Table 2.1, J_1 is a maximally non-integrable almost complex structure on the corresponding Lie algebra. □

Structures of rank 2

We begin with an elementary fact, of which we give a proof for completeness.

Lemma 2.4.7. *Let* $\mathfrak{g}^* = \mathbb{R}\langle e^1, \ldots, e^{2m}\rangle$ *be the dual of a Lie algebra* $\mathfrak{g}$*. Fix an almost complex structure* J *on* $\mathfrak{g}^*$*. Suppose that for some indices* $j, k \in \{1, \ldots, 2m\}$ *we have that* $(e^{jk})^{0,2} = 0$*. Then* $(e^k)^{0,1}$ *and* $(e^j)^{0,1}$ *are proportional to each other.*

Proof. Fix a basis of $(0,1)$-forms $\{\omega^{\bar j}\}_{j=1}^m$. In terms of the basis, we can write

$$\begin{aligned} (e^j)^{0,1} &= A_1\,\omega^{\bar 1} + \ldots + A_m\,\omega^{\bar m} \quad \text{and}\\ (e^k)^{0,1} &= B_1\,\omega^{\bar 1} + \ldots + B_m\,\omega^{\bar m}, \end{aligned}$$

with A_j and $B_j \in \mathbb{C}$ for $j = 1, \ldots, m$. The condition $(e^{jk})^{0,2} = 0$ implies that the matrix

$$\begin{bmatrix} A_1 & \cdots & A_m \\ B_1 & \cdots & B_m \end{bmatrix}$$

has rank 1. Since $(e^j)^{0,1}$ and $(e^k)^{0,1}$ cannot be the zero form because they are the projection on bidegree $(0,1)$ of a real form, the only possibility is that $(e^j)^{0,1}$ and $(e^k)^{0,1}$ are proportional to each other. □

We are ready to classify Lie algebras that admit a structure of rank 2.

Proposition 2.4.8. *Every 6-dimensional nilpotent Lie algebra different from*

$$\begin{aligned}&(0,0,0,12,13,23), \quad (0,0,0,0,0,12+34),\\ &(0,0,0,0,0,12) \quad \textit{or} \quad (0,0,0,0,0,0)\end{aligned} \tag{2.4.2}$$

admits an almost complex structure of rank 2*.*

Proof. By Lemma 2.4.2, any algebra among $(0,0,0,0,0,12+34)$, $(0,0,0,0,0,12)$ and $(0,0,0,0,0,0)$ admits only structures of at most rank 1. We directly prove that there are no structures of rank 2 on the Lie algebra $\mathfrak{g} = (0,0,0,12,13,23)$.

Let J be an almost complex structure on $\mathfrak{g}$. We study the $(0,2)$-bidegree part of the forms e^{12}, e^{13} and e^{23}. First, suppose that $(e^{jk})^{0,2} = 0$ for some $jk \in \{12, 13, 23\}$. Due to the symmetries in the indices $\{1,2,3\}$ and $\{4,5,6\}$ of the Lie algebra, we can assume that $(e^{12})^{0,2} = 0$. By Lemma 2.4.7, we have that $(e^1)^{0,1}$ and $(e^2)^{0,1}$ are proportional to the same $(0,1)$-form α. The forms

$$\phi^j := e^j + iJe^j, \quad j = 1, \dots, 6,$$

are a set of generators of $A^{1,0}$. If $(e^1)^{0,1}$ and $(e^2)^{0,1}$ are proportional to α, then we have $\bar{\mu}\phi^j \in \mathbb{C}\langle \alpha \wedge (e^3)^{0,1} \rangle$, that implies $\operatorname{rk} N_J \leq 1$.

Suppose now that $(e^{jk})^{0,2} \neq 0$ for all $jk \in \{12, 13, 23\}$. We further split the argument into three cases.

Case 1: two among $(e^{12})^{0,2}$, $(e^{13})^{0,2}$ and $(e^{23})^{0,2}$ are multiple of each other. Again, for the symmetries of the Lie algebra we can assume that

$$(e^1)^{0,1} \wedge (e^2)^{0,1} = A(e^1)^{0,1} \wedge (e^3)^{0,1},$$

with $A \in \mathbb{C} \setminus \{0\}$. This implies that

$$(e^2)^{0,1} = A(e^3)^{0,1} + \gamma, \tag{2.4.3}$$

where γ is a $(0,1)$-form in the kernel of the map $(e^1)^{0,1} \wedge \bullet$. Consider the form

$$\omega^{\bar{1}} := e^1 - iJe^1$$

and complete it to a basis of $(0,1)$-forms $\{\omega^{\bar{1}}, \omega^{\bar{2}}, \omega^{\bar{3}}\}$, so that

$$\gamma = C_1\omega^{\bar{1}} + C_2\omega^{\bar{2}} + C_3\omega^{\bar{3}}, \quad \text{with } C_j \in \mathbb{C}. \tag{2.4.4}$$

The condition $(e^1)^{0,1} \wedge \gamma = 0$ gives $C_2 = C_3 = 0$. Thus γ is a multiple of $\omega^{\bar{1}}$, hence of $(e^1)^{0,1}$, since $\omega^{\bar{1}} = 2(e^1)^{0,1}$. By (2.4.3) and (2.4.4), we have that

$$(e^2)^{0,1} \wedge (e^3)^{0,1} = (A(e^3)^{0,1} + \gamma) \wedge (e^3)^{0,1} = 2C_1(e^{13})^{0,2}.$$

In particular, this gives that $(e^{23})^{0,2}$ is a multiple of $(e^{13})^{0,2}$. Since all of the $(e^{jk})^{0,2}$, for $jk \in \{12, 13, 23\}$, are multiple of each other, N_J has at most rank 1.

Case 2: one among $(e^{12})^{0,2}$, $(e^{13})^{0,2}$ and $(e^{23})^{0,2}$ is a linear combination of the remaining two. By symmetry, we can assume that

$$(e^{12})^{0,2} = A(e^{13})^{0,2} + B(e^{23})^{0,2} = (A(e^1)^{0,1} + B(e^2)^{0,1}) \wedge (e^3)^{0,1},$$

where A and $B \in \mathbb{C}$ are both non-zero, or else we would reduce to Case 1. Since $(e^{12})^{0,2} \neq 0$, consider the forms

$$\omega^{\bar{j}} := e^j - iJe^j, \quad j = 1, 2,$$

and complete them to a basis of $(0,1)$-forms $\{\omega^{\bar{1}}, \omega^{\bar{2}}, \omega^{\bar{3}}\}$. Proceeding as in the previous case, it is straightforward to see that

$$(e^3)^{0,1} = C(e^1)^{0,1} + D(e^2)^{0,1},$$

which implies that the forms $(e^{13})^{0,2}$ and $(e^{23})^{0,2}$ are both multiple of $(e^{12})^{0,2}$. The conclusion that $\operatorname{rk} N_J \leq 1$ follows as in the first case.

Case 3: all of the forms $(e^{12})^{0,2}$, $(e^{13})^{0,2}$ and $(e^{23})^{0,2}$ are independent over $\mathbb{C}$. We prove that $\bar{\mu}$ has necessarily rank 3. Consider the $(1,0)$-forms

$$\phi^j := e^j + iJe^j, \quad j = 1, 2, 3.$$

The projections on bidegree $(0,2)$ of the forms e^{jk} are independent over $\mathbb{C}$ by assumption. They can be expressed in terms of the ϕ^j as

$$(e^{12})^{0,2} = \frac{1}{4}\phi^{\bar{1}\bar{2}}, \quad (e^{13})^{0,2} = \frac{1}{4}\phi^{\bar{1}\bar{3}} \quad \text{and} \quad (e^{23})^{0,2} = \frac{1}{4}\phi^{\bar{2}\bar{3}}.$$

This implies that also the ϕ^j, for $j = 1, 2, 3$, are independent, and so they are a basis of $(1,0)$-forms. In terms of the real basis e^j, we can write J in block form as

$$J = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

From the identity $J^2 = -\operatorname{Id}$, we obtain the relation

$$A^2 + BC = -\operatorname{Id}. \tag{2.4.5}$$

Computing the rank of $\bar{\mu}$ amounts to compute the rank of the matrix B, since

$$\bar{\mu}\phi^j = -\frac{i}{4}(B_{j1}\phi^{\bar{1}\bar{2}} + B_{j2}\phi^{\bar{1}\bar{3}} + B_{j3}\phi^{\bar{2}\bar{3}}), \quad j = 1, 2, 3.$$

The complex 1-forms $\{\phi^j, \phi^{\bar{j}}\}_{j=1}^3$ can be written in terms of the e^j as

$$(\phi^1, \phi^2, \phi^3, \phi^{\bar{1}}, \phi^{\bar{2}}, \phi^{\bar{3}})^t = Q(e^1, e^2, e^3, e^4, e^5, e^6)^t,$$

where Q is the block matrix

$$Q = \begin{bmatrix} \mathrm{Id} - iA & -iB \\ \\ \mathrm{Id} + iA & iB \end{bmatrix}.$$

Since $\{\phi^j, \phi^{\bar{j}}\}_{j=1}^3$ is a basis of 1-forms, Q must be invertible and the matrix

$$Q^*Q = \begin{bmatrix} 2(\mathrm{Id} - iA)(\mathrm{Id} + iA) & i[(I - iA)B - (I + iA)B] \\ -i[B(I + iA) - B(I - iA)] & B^2 \end{bmatrix}$$

is positive definite. In particular, its principal minor

$$(\mathrm{Id} + iA)(\mathrm{Id} - iA) = \mathrm{Id} + A^2$$

is also positive definite, and by (2.4.5) it is equal to BC. This forces B, and thus $\bar{\mu}$, to have rank 3, concluding the proof of the fact that $(0, 0, 0, 12, 13, 23)$ has no almost complex structure whose Nijenhuis tensor has precisely rank 2.

Finally, we prove that each 6-dimensional nilpotent Lie algebras not listed in (2.4.2) admits an almost complex structure of rank 2 by explicitly building it, as we did in the proof of Proposition 2.4.6. The explicit choice of constants providing the desired structure can be found in the third column of Table 2.1. □

Structures of rank 1

The only 6-dimensional nilpotent Lie algebras without almost complex structures of rank 1 are

$$(0, 0, 0, 0, 0, 0) \quad \text{and} \quad (0, 0, 12, 13, 14 + 23, 34 - 25). \qquad (2.4.6)$$

For the former, this is again an immediate consequence of Lemma 2.4.2, while for the latter it is the content of the following proposition.

Proposition 2.4.9. *Any almost complex structure on the Lie algebra*

$$(0, 0, 12, 13, 14 + 23, 34 - 25)$$

has at least rank 2.

The proof of Proposition 2.4.9 proceeds assuming the existence of a structure of rank 1 on $\mathfrak{g} = (0, 0, 12, 13, 14 + 23, 34 - 25)$ and splitting the argument into several main cases, according to whether or not the projection on bidegree $(0, 2)$ of d-exact 2-forms is zero. In each of the cases, we reach an absurd by contradicting the following well-known lemma on the linear algebra of almost complex structures (or a direct consequence of it), of which we give a proof for the sake of completeness.

Lemma 2.4.10. *Let V be a $2m$-dimensional real vector space and let $\{e_j\}_{j=1}^{2m}$ be a basis of V. Fix an almost complex structure J on V and consider the projection $\pi^{0,1}\colon V^{\mathbb{C}} \to V^{0,1}$. Then the space*

$$S := \mathbb{C}\langle \pi^{0,1}(e_j), \pi^{0,1}(e_k), \pi^{0,1}(e_l)\rangle,$$

with $j \neq k$, $k \neq l$ and $j \neq l$ has at least complex dimension 2.

Proof. Summing over repeated indices, we can write $Je_j = J_j^k e_k$, with $J_j^k \in \mathbb{R}$. Since e_j is a real vector, its $(0, 1)$-bidegree part cannot be zero and is given by

$$\pi^{0,1}(e_j) = \frac{1}{2}(e_j + iJe_j).$$

Thus S is at least 1-dimensional. Assume by contradiction that S has precisely dimension 1. Then the vectors $(e_k + iJe_k)$ and $(e_l + iJe_l)$ are proportional, up to non-vanishing complex constants, to $(e_j + iJe_j)$, that is,

$$(e_j + iJe_j) = A(e_k + iJe_k) = B(e_l + iJe_l), \qquad \text{with } A \text{ and } B \in \mathbb{C} \setminus \{0\} .$$

Writing explicitly the vectors in terms of the element of the basis, we obtain a system of equations with coefficients A, B and J_j^k. We are interested in the part involving only e_j and e_k:

$$\begin{cases} 1 + iJ_j^j = iAJ_j^k = iBJ_j^l, \\ \\ iJ_k^j = A(1 + iJ_k^k) = iBJ_k^l. \end{cases}$$

From the first equation we have that $J_j^k \neq 0$ and $J_j^l \neq 0$, obtaining the value for the constants

$$A = -i\frac{(1 + iJ_j^j)}{J_j^k} \quad \text{and} \quad B = -i\frac{(1 + iJ_j^j)}{J_j^l}.$$

Substituting in the second equation, we are left with

$$iJ_k^j = -i\frac{(1 + iJ_j^j)(1 + iJ_k^k)}{J_j^k} = \frac{J_k^l}{J_j^l}(1 + iJ_j^j).$$

From the equality $iJ^j_k = J^l_k(1 + iJ^j_j)/J^l_j$, we deduce that both J^l_k and J^j_k must vanish. The remaining equation $(1 + iJ^j_j)(1 + iJ^k_k) = 0$ leads to the contradiction $(J^j_j)^2 = -1$, concluding the proof. □

We also need the following lemma.

Lemma 2.4.11. *Let $\mathfrak{g}^* = \mathbb{R}\langle e^1, \dots, e^{2m}\rangle$ be the dual of a Lie algebra $\mathfrak{g}$. Fix an almost complex structure J on $\mathfrak{g}^*$. Suppose that for some $j, k \in \{1, \dots, 2m\}$, with $j < k$, the form $(e^k)^{0,1}$ is proportional to $(e^j)^{0,1}$. Then Je^j and $Je^k \in \mathbb{R}\langle e^j, e^k\rangle$.*

Proof. With the same notation of the proof of Lemma 2.4.10, we have, passing to the dual, that

$$(e^j)^{0,1} = -\frac{i}{2}(J^j_p + i\delta^j_p)e^p.$$

The forms $(e^j)^{0,1}$ and $(e^k)^{0,1}$ are multiple of each other if and only if the matrix

$$\begin{bmatrix} J^j_1 & \cdots & J^j_j + i & \cdots & J^j_k & \cdots & J^j_{2m} \\ J^k_1 & \cdots & J^k_j & \cdots & J^k_k + i & \cdots & J^k_{2m} \end{bmatrix}$$

has rank 1. Imposing that the determinant of each of its 2×2 minors vanishes and separating real and imaginary part, we see that J must satisfy the relations

$$\begin{cases} J^k_k = -J^j_j, \\ (J^j_j)^2 + J^j_k J^k_j = -1, \\ J^j_p = J^k_p = 0, & \text{if } p \notin \{j, k\}. \end{cases} \tag{2.4.7}$$

By (2.4.7), we conclude that $Je^j = J^j_j e^j + J^j_k e^k$ and $Je^k = J^k_j e^j - J^j_j e^k$. □

We can now prove that any almost complex structure on $\mathfrak{g}$ has at least rank 2.

Proof of Proposition 2.4.9. By [81], there are no complex structures on $\mathfrak{g}$. Let J be an almost complex structure on $\mathfrak{g}$ and assume by contradiction that it has precisely rank 1. The $(1,0)$-forms $\{\phi^j := e^j + iJe^j\}_{j=1}^6$ are a set of generators of $A^{1,0}$, so we can compute the rank of $\bar{\mu}$ focusing only on the ϕ^j. We have that

$$\bar{\mu}\phi^j = i(J^j_k - i\delta^j_k)(de^k)^{0,2}.$$

The rank of $\bar{\mu}$ is the rank of a suitable submatrix obtained removing from $J - i\,\mathrm{Id}_6$ the columns corresponding to the indices for which $(de^k)^{0,2} = 0$, and taking linear combinations of such columns if $(de^j)^{0,2}$ is a non-zero multiple of $(de^k)^{0,2}$. Since $de^1 = de^2 = 0$, we work with a 6×4 matrix and we identify for each case which among the $(de^k)^{0,2}$ vanish and which are proportional to each other.

Case A: $(e^{12})^{0,2} = (e^{13})^{0,2} = 0$. By Lemma 2.4.7, the assumption $(e^{12})^{0,2} = 0$ implies that $(e^1)^{0,1}$ and $(e^2)^{0,1}$ are proportional. Similarly, we have that $(e^1)^{0,1}$ is proportional to $(e^3)^{0,1}$, contradicting Lemma 2.4.10.

Case B: $(e^{12})^{0,2} = 0$ and $(e^{13})^{0,2} \neq 0$. Again by Lemma 2.4.7, we have that $(e^2)^{0,1} = A(e^1)^{0,1}$ for some $A \in \mathbb{C} \setminus \{0\}$. Consider the 2-form

$$\alpha \coloneqq (e^{14+23})^{0,2} = (e^1)^{0,1} \wedge ((e^4)^{0,1} + A(e^3)^{0,1}).$$

If α is not a multiple of $(e^{13})^{0,2}$, then $(e^4)^{0,1}$ is independent of $(e^1)^{0,1}$ and $(e^3)^{0,1}$, giving a basis of $(1,0)$-forms $\{\phi^1, \phi^3, \phi^4\}$. In terms of such a basis, we have that

$$(e^{13})^{0,2} = \frac{1}{4}\phi^{\bar{1}\bar{3}}, \quad (e^{14+23})^{0,2} = \frac{1}{4}(\phi^{\bar{1}\bar{4}} + A\phi^{\bar{1}\bar{3}})$$

and

$$(e^{34-25})^{0,2} = \frac{1}{4}\phi^{\bar{3}\bar{4}} + \phi^{\bar{1}} \wedge \gamma,$$

for some $(0,1)$-form γ. These $(0,2)$-forms are independent, thus the rank of $\bar{\mu}$ is determined by the corresponding columns of $J - i\,\mathrm{Id}_6$, i.e., by the rank of the 6×3 matrix $U \coloneqq (J^j_k + i\delta^j_k)$, with $j = 1, \ldots, 6$, and $k = 4, 5, 6$. Since $\bar{\mu}$ has rank 1, so does U, and we can apply repeatedly Lemma 2.4.11 to its columns to deduce the condition $J^3_3 = -i$, reaching an absurd.
The proof of Case B is concluded if we prove that α cannot be a multiple of $(e^{13})^{0,2}$. Assume by contradiction that $(e^{14+23})^{0,2}$ is a multiple of $(e^{13})^{0,2}$. Then necessarily $(e^4)^{0,1}$ is a linear combination

$$(e^4)^{0,1} = B(e^1)^{0,1} + C(e^3)^{0,1}.$$

If $B \neq 0$, we can redefine the elements of the basis e^j setting

$$\hat{e}^4 \coloneqq e^4 - Ce^3,$$

so that $(\hat{e}^4)^{0,1} = B(e^1)^{0,1} = B/A\,(e^2)^{0,1}$. This contradicts Lemma 2.4.10.
If $B = 0$, then $(e^4)^{0,1} = C(e^3)^{0,1}$, giving a simple expression for the projection on bidegree $(0,2)$ of de^6:

$$\beta \coloneqq (e^{34-25})^{0,2} = -A(e^1)^{0,1} \wedge (e^5)^{0,1}.$$

If β is proportional to $(e^{13})^{0,2}$, then

$$(e^5)^{0,1} = D(e^1)^{0,1} + E(e^3)^{0,1}.$$

As above, if $D = 0$, we immediately get a contradiction to Lemma 2.4.10. The same follows when $D \neq 0$, by redefining $\hat{e}^5 \coloneqq e^5 - Ee^3$.

If β is not proportional to $(e^{13})^{0,2}$, then the matrix that determines the rank of $\bar{\mu}$ is

$$\begin{bmatrix} 0 & 0 & J_4^3 & J_4^4 + i & J_4^5 & J_4^6 \\ 0 & 0 & J_6^3 & J_6^4 & J_6^5 & J_6^6 + i \end{bmatrix}^t.$$

By Lemma 2.4.11 applied to $(e^3)^{0,1}$ and $(e^4)^{0,1}$, it must be $J_6^3 = J_6^4 = 0$. Imposing the condition $\operatorname{rk}\bar{\mu} = 1$, we easily obtain the contradiction $J_3^3 = -i$, proving that α is not a multiple of $(e^{13})^{0,2}$, and thus proving our claim in Case B.

Case C: $(\mathbf{e^{12}})^{\mathbf{0,2}} \neq \mathbf{0}$. The proof is similar to that of case B, with slightly longer computations. □

An existence result completes the classification.

Proposition 2.4.12. *Every 6-dimensional nilpotent Lie algebra different from (2.4.6) admits an almost complex structure of rank 1.*

Proof. The existence of a structure of rank 1 is proved as in Proposition 2.4.6. The constants providing the desired structure are presented in the fourth column of Table 2.1. □

Complex structures

The classification of 6-dimensional nilpotent Lie algebras admitting a complex structure was carried out by Salamon [81]. For the sake of completeness, in the last column of Table 2.1 we give explicit constants that allow to obtain examples of complex structures following the idea of the proof of Proposition 2.4.6. The only Lie algebras on which a complex structure cannot be obtained in this way are $(0, 0, 0, 12, 23, 14 - 35)$ and $(0, 0, 0, 0, 12, 14 + 25)$. In these two cases, it is immediate to check that the co-frame of $(1, 0)$-forms

$$\phi^1 := e^1 + ie^3, \quad \phi^2 := e^4 + ie^5, \quad \phi^3 := -e^2 + ie^6 \tag{2.4.8}$$

defines a complex structure on $(0, 0, 0, 12, 23, 14 - 35)$, while the co-frame

$$\phi^1 := e^1 + ie^2, \quad \phi^2 := e^4 + ie^5, \quad \phi^3 := e^3 + ie^6 \tag{2.4.9}$$

defines a complex structure on $(0, 0, 0, 0, 12, 14 + 25)$.

Remark 2.4.13. *By Theorem 2.5 in [81], complex structures on 6-dimensional nilpotent Lie algebras have two types of canonical basis. Type (I) has the form*

$$\omega^1 = e^1 - ie^2, \quad \omega^1 = e^3 - ie^4, \quad \omega^1 = e^5 - ie^6,$$

while type (II) has the form

$$\omega^1 = e^1 - ie^2, \quad \omega^1 = e^4 - ie^5, \quad \omega^1 = e^3 - ie^6.$$

We point out that when we give examples of explicit structures, we are deforming a (possibly non-integrable) structure with a basis of type (I). *The Lie algebras on which a complex structure cannot be obtained in this way are precisely those that admit only complex structures of type* (II) *[81].*

2.4.2 Consequences on locally homogeneous manifolds

The classification by rank of almost complex structures on 6-dimensional nilpotent Lie algebras allows to establish which 6-dimensional nilmanifolds admit an invariant almost complex structure of a certain rank.

Theorem 2.4.14. *Let* $M = \Gamma\backslash G$ *be a* 6*-dimensional nilmanifold and let* $\mathfrak{g}$ *be the Lie algebra of* G*. Then* M *admits an invariant almost complex structure of rank* k *if and only if* $\mathfrak{g}$ *admits an almost complex structure of rank* k*, according to Theorem 2.4.1.*

Proof. Fix an almost complex structure J on M. There is a bijection between invariant almost complex structures on M and almost complex structures on $\mathfrak{g}$, see Section 1.5, and the rank of N_J on M is equal to the rank of the almost complex structure induced by J on $\mathfrak{g}$. This, together with Theorem 2.4.1, proves our claim. □

More in general, we can establish a topological upper bound for the rank of invariant almost complex structures on solvmanifolds of completely solvable type of arbitrary dimension.

Theorem 2.4.15. *Let* $M = \Gamma\backslash G$ *be a solvmanifold of completely solvable type. Let* J *be an invariant almost complex structure on* M*. Then we have that*

$$\operatorname{rk} N_J \leq \dim_{\mathbb{R}} M - b_1(M).$$

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Given any invariant almost complex structure J on M, there is a corresponding almost complex structure $\tilde{J}$ on $\mathfrak{g}$, and $\operatorname{rk} N_J = \operatorname{rk} N_{\tilde{J}}$. Let $b_1(\mathfrak{g})$ be the real dimension of the first Lie algebra cohomology group of $\mathfrak{g}$

$$H^1(\mathfrak{g}) = (\ker d \cap A^1)/(\operatorname{Im} d \cap A^1) = (\ker d \cap A^1),$$

see also Section 1.5. By Lemma 2.4.2, we have that

$$\operatorname{rk} N_J = \operatorname{rk} N_{\tilde{J}} \leq 2m - \dim_{\mathbb{R}}(A^1_{\mathbb{R}} \cap \ker d) = 2m - b_1(\mathfrak{g}).$$

By Hattori's theorem [46], see also Section 1.5, there is an isomorphism

$$H^k(\mathfrak{g}) \cong H^k_d$$

with the de Rham cohomology of M, so that $b_1(\mathfrak{g}) = b_1(M)$. □

2.4.3 Table of the possible ranks

This section contains the table summarizing the possible ranks of almost complex structures on 6-dimensional nilpotent Lie algebras. The first column lists the 34 isomorphism type of Lie algebras. The remaining columns list whether or not a structure of prescribed rank exists on each of them. When such a structure exists, the choice of parameters

$$\begin{bmatrix} e & f & g \\ p & q & r \\ s & t & u \end{bmatrix}$$

provided in the table, allows to obtain it starting from a fixed almost complex structure, see proof of Proposition 2.4.6. The non-existence of structures is proved in Section 2.4.1. By the word *generic*, we mean that a generic almost complex structure will have the corresponding rank.

Remark 2.4.16. *The computations for the rank presented in Table 2.1 have been checked using* Wolfram Mathematica, Version 13.1*.*

Table 2.1: Classification of invariant almost complex structures by rank on 6-dimensional nilpotent Lie algebras.

Lie Algebra	Rank 3	Rank 2	Rank 1	Rank 0
$(0,0,12,13,14+23,34-25)$	Generic $\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$	No	No
$(0,0,12,13,14,34-25)$	Generic $\begin{bmatrix} 0&0&1\\0&0&0\\0&2&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&-\frac{1}{2}&-\frac{1}{4}+\frac{i}{2}\\0&0&-\frac{1}{2}\\0&0&0 \end{bmatrix}$	No
$(0,0,12,13,14,15)$	No	Generic $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&1&1\\0&-4&1 \end{bmatrix}$	No
$(0,0,12,13,14+23,24+15)$	Generic $\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&1&1\\0&-2+2\sqrt{2}&1 \end{bmatrix}$	No
$(0,0,12,13,14,23+15)$	Generic $\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$	Yes $\begin{bmatrix} 0&0&0\\0&0&2\\0&2&1 \end{bmatrix}$	No

$(0,0,12,13,23,14)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,12,13,23,14-25)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&2\\0&2&1\end{bmatrix}$	No
$(0,0,12,13,23,14+25)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&1&-1\\0&2&1\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&1&-1\\0&2&i\end{bmatrix}$
$(0,0,0,12,14-23,15+34)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,14,15+23)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,14,15+23+24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,14,15+24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,14,15)$	No	Generic $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,13,14+35)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,23,14+35)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,12,23,14-35)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes See (2.4.8)
$(0,0,0,12,14,24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&1&1\\0&2&1\\0&0&2\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&2\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$(0,0,0,12,13-24,14+23)$	Generic $\begin{bmatrix}0&0&1\\0&2&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&\frac{1}{2}\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
$(0,0,0,12,14,13-24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&3\end{bmatrix}$
$(0,0,0,12,13+14,24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&\frac{-1+2i}{5}\end{bmatrix}$
$(0,0,0,12,13,14+23)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&-\frac{1}{3}\end{bmatrix}$
$(0,0,0,12,13,24)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&2i&0\\0&0&-3i\end{bmatrix}$
$(0,0,0,12,13,14)$	No	Generic $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&2\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
$(0,0,0,12,13,23)$	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	No	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&2\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
$(0,0,0,0,12,15+34)$	No	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,0,12,15)$	No	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	No
$(0,0,0,0,12,14+25)$	No	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes See (2.4.9)
$(0,0,0,0,13-24,14+23)$	No	Generic $\begin{bmatrix}0&0&1\\0&2&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
$(0,0,0,0,12,14+23)$	No	Generic $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	Yes $\begin{bmatrix}0&-i&0\\0&0&0\\0&0&0\end{bmatrix}$

$(0,0,0,0,12,34)$	No	Generic $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$(0,0,0,0,12,13)$	No	Generic $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & -1/6 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
$(0,0,0,0,0,12+34)$	No	No	Generic $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$(0,0,0,0,0,12)$	No	No	Generic $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	Yes $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$(0,0,0,0,0,0)$	No	No	No	Generic

CHAPTER 3

Cohomologies of non-integrable structures

In this chapter we review the existing cohomologies of almost complex manifolds and we introduce three new cohomologies: the Bott–Chern and Aeppli cohomologies of the operators d and d^c; the cohomologies of the operators δ and $\bar{\delta}$ and the associated Bott–Chern and Aeppli cohomologies; the J-even and J-odd cohomologies induced by the parity of the action of J on forms. We also explain how to extend our definitions to almost symplectic manifolds.

3.1 A short review of the existing cohomologies

In this section we briefly recall the definitions of several relevant cohomologies of almost complex manifolds, namely the J-invariant and J-anti-invariant cohomologies of Draghici, Li and Zhang, the Dolbeault cohomology of Cirici and Wilson, and the Bott–Chern and Aeppli cohomologies of Coelho, Placini and Stelzig.

In addition, it is worth mentioning that there are two cohomologies that do not play a role in this thesis but are related to our studies. They are the *transverse Dolbeault cohomology* introduced by Cahen, Gutt and Gutt [24], and the *refined Dolbeault cohomology* introduced by Lin [61]. We invite the reader to consult the original references for the details.

The J-invariant and J-anti-invariant cohomologies

The J-invariant and J-anti-invariant cohomologies have been introduced and initially studied by Draghici, Li and Zhang in [34], [35] and [60], with the purpose of understanding the tamed and compatible cones of almost complex manifolds.

Let (M, J) be an almost complex manifold. Then J acts on k-forms as $J^2 = (-1)^k \operatorname{Id}$, see Section 1.2. In particular, on 2-forms J is an involution and Λ^2 decomposes into its (± 1)-eigenspaces

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-.$$

Smooth sections of Λ^+ and Λ^- are called *J-invariant* and *J-anti-invariant* forms, respectively, and they are denoted by

$$A^+ = \{\alpha \in A^2 : J\alpha = \alpha\} \quad \text{and} \quad A^- = \{\alpha \in A^2 : J\alpha = -\alpha\}.$$

The bidegree of α completely determines its J-invariance and we have

$$A^+ = A^{1,1} \quad \text{and} \quad A^- = A^{2,0} \oplus A^{0,2}.$$

The *J-invariant* and *J-anti-invariant* cohomologies of J are the subgroups of de Rham cohomology given by

$$H_J^+ := \{[\alpha] \in H_d^2 : \alpha \in A^+\} \quad \text{and} \quad H_J^- := \{[\alpha] \in H_d^2 : \alpha \in A^-\},$$

respectively. On compact manifolds, they are finite-dimensional since they are subgroups of de Rham cohomology. We denote by h_J^+ and h_J^- their respective dimensions, that are an invariant of the almost complex structure. In dimension 4, the numbers $h_J^\pm$ also occur as the dimension of the spaces of harmonic forms

$$\mathcal{H}_J^{1,1} := A^{1,1} \cap \ker \Delta_d \quad \text{and} \quad \mathcal{H}_J^{(2,0)(0,2)} := (A^{2,0} \oplus A^{0,2}) \cap \Delta_d, \tag{3.1.1}$$

and we have

$$h_J^+ = \dim_{\mathbb{C}} \mathcal{H}_J^{1,1} \quad \text{and} \quad h_J^- = \dim_{\mathbb{C}} \mathcal{H}_J^{(2,0)(0,2)}.$$

For this reason, it is common to find the notation

$$H_J^{1,1} = H_J^+ \quad \text{and} \quad H_J^{(2,0)(0,2)} = H_J^-.$$

The Dolbeault, Bott–Chern and Aeppli cohomologies of almost complex manifolds

The first generalization of Dolbeault cohomology to almost complex manifolds was introduced by Cirici and Wilson [29].

Let (M, J) be an almost complex manifold. Since $\bar\mu^2 = 0$, one can consider the *$\bar\mu$-cohomology*

$$H_{\bar\mu}^{p,q} := \frac{\ker \bar\mu \cap A^{p,q}}{\operatorname{Im} \bar\mu \cap A^{p,q}}.$$

Then $\bar\partial$ induces a well-defined map $\bar\partial\colon H^{p,q}_{\bar\mu} \to H^{p,q+1}_{\bar\mu}$ that satisfies $\bar\partial^2 = 0$. The *Dolbeault cohomology of* (M, J) is the cohomology

$$H^{p,q}_{Dol} := \frac{\ker\bar\partial \cap H^{p,q}_{\bar\mu}}{\operatorname{Im}\bar\partial \cap H^{p,q}_{\bar\mu}}.$$

If J is integrable, then $H^{p,q}_{Dol}$ coincides with the usual Dolbeault cohomology. In general $H^{p,q}_{Dol}$ is infinite-dimensional, even on compact manifolds [32]. Furthermore, finite-dimensionality of the Dolbeault cohomology completely characterizes integrability of J on almost complex 4-manifolds [30]. For a comparison of Dolbeault cohomology of almost complex manifolds and J-invariant cohomology, we refer to [86].

A notion of Bott–Chern and Aeppli cohomology related to the Dolbeault cohomology was introduced by Coelho, Placini and Stelzig [32]. Consider the subcomplex of $A^{\bullet,\bullet}$ given by

$$B^{\bullet,\bullet} := A^{\bullet,\bullet} \cap \ker\mu \cap \ker\bar\mu \cap \ker\partial^2 \cap \ker\bar\partial^2$$

and the quotient complex

$$C^{\bullet,\bullet} := A^{\bullet,\bullet}/(\operatorname{Im}\mu + \operatorname{Im}\bar\mu + \operatorname{Im}\partial^2 + \operatorname{Im}\bar\partial^2).$$

Then the *Bott–Chern cohomology of* (M, J) is the cohomology

$$H^{p,q}_{BC} := \frac{\ker\partial \cap \ker\bar\partial \cap B^{p,q}}{\operatorname{Im}\partial\bar\partial \cap B^{p,q}}$$

and the *Aeppli cohomology of* (M, J) is the cohomology

$$H^{p,q}_{A} := \frac{\ker\partial\bar\partial \cap C^{p,q}}{(\operatorname{Im}\partial + \operatorname{Im}\bar\partial) \cap C^{p,q}}.$$

If J is integrable, these cohomologies coincide with the usual Bott–Chern and Aeppli cohomologies. Together with the Dolbeault cohomology, they fit into a diagram

$$\begin{array}{ccccccc} & & & H^{p,q}_{BC} & & & \\ & \swarrow & \swarrow & & \searrow & \searrow & \\ H^{p,q}_{Dol} & \Longrightarrow & H^{p+q}_{d} & \xleftrightarrow[J]{\sim} & H^{p+q}_{d^c} & \Longleftarrow & \overline{H^{p,q}_{Dol}} \\ & \searrow & \searrow & & \swarrow & \swarrow & \\ & & & H^{p,q}_{A} & & & \end{array}$$

where $H^{p,q}_{Dol} \Rightarrow H^{p+q}_{d}$ is a version of the Frölicher spectral sequence of Example 1.3.1 valid for multicomplexes, see [29]. For an almost symplectic version of the spectral sequence, see Section 3.4.

3.2 Cohomologies of the operators d and d^c

Let (M, J) be an almost complex manifold with non-integrable J. Let d^c be the conjugate of d by the action of J. Then d and d^c satisfy the equations $d^2 = 0$ and $(d^c)^2 = 0$, see Section 1.2. Since J is not integrable, we have $dd^c + d^c d \neq 0$. Thus, $(A^\bullet, d, d^c)$ is not a double complex and its Bott–Chern and Aeppli cohomologies are not well-defined.

We introduce a subcomplex and a quotient complex of $A^\bullet$, see Section 1.1, on which d and d^c anti-commute. A similar construction has been performed also in [32] to give a different definition of Bott–Chern and Aeppli cohomologies of almost complex manifolds, see Section 3.1.

Definition 3.2.1. Consider the subcomplex $B^\bullet \subseteq A^\bullet$ given by

$$B^\bullet := A^\bullet \cap \ker(dd^c + d^c d)$$

and the quotient complex of $A^\bullet$ given by

$$C^\bullet := \frac{A^\bullet}{\operatorname{Im}(dd^c + d^c d)}.$$

Proposition 3.2.2. *The complexes $(B^\bullet, d, d^c)$ and $(C^\bullet, d, d^c)$ are $\mathbb{Z}$-graded double complexes.*

Proof. If $\alpha \in B^k$, then $d\alpha \in B^{k+1}$. Indeed, since d and d^c anti-commute on $B^\bullet$ and $d^2 = 0$, we have that

$$(dd^c + d^c d)d\alpha = dd^c d\alpha = -d(dd^c)\alpha = 0.$$

Similarly, if $\alpha \in B^k$, then $d^c\alpha \in B^{k+1}$. Finally, we have that $dd^c + d^c d = 0$ on $B^\bullet$ by definition. If α and $\beta \in B^\bullet$, then

$$(dd^c + d^c d)(\alpha \wedge \beta) = (dd^c + d^c d)\alpha \wedge \beta + \alpha \wedge (dd^c + d^c d)\beta = 0,$$

so that $\alpha \wedge \beta \in B^\bullet$. The proof for $(C^\bullet, d, d^c)$ is similar. □

We are ready to define Bott–Chern and Aeppli cohomologies for almost complex manifolds.

Definition 3.2.3. Let (M, J) be an almost complex manifold. The *Bott–Chern cohomology of* (M, J) is the Bott–Chern cohomology of $(B^\bullet, d, d^c)$, i.e.,

$$H^k_{d+d^c} := \frac{\ker(d\colon B^k \to B^{k+1}) \cap \ker(d^c\colon B^k \to B^{k+1})}{\operatorname{Im}(dd^c\colon B^{k-2} \to B^k)}.$$

The *Aeppli cohomology of* (M, J) is the Aeppli cohomology of $(C^\bullet, d, d^c)$, i.e.,

$$H^k_{dd^c} := \frac{\ker(dd^c\colon C^k \to C^{k+2})}{\operatorname{Im}(d\colon C^{k-1} \to C^k) + \operatorname{Im}(d^c\colon C^{k-1} \to C^k)}.$$

In the complex case our definition of Bott–Chern and Aeppli cohomologies coincides with the usual one: if J is integrable then $dd^c + d^c d = 0$ on all forms, so that $B^\bullet = A^\bullet$ and $C^\bullet = A^\bullet$. We now describe the basic properties of Bott–Chern and Aeppli cohomologies.

Proposition 3.2.4. *Aeppli cohomology has the structure of module over Bott–Chern cohomology. In particular, there is a well-defined pairing*

$$\begin{aligned} H^k_{d+d^c} \times H^\ell_{dd^c} &\longrightarrow H^{k+\ell}_{dd^c}, \\ ([\alpha]_{d+d^c}, [\gamma]_{dd^c}) &\longmapsto [\alpha \wedge \gamma]_{dd^c}. \end{aligned}$$

Proof. Let $[\alpha]_{d+d^c} \in H^k_{d+d^c}$. Then, we can write

$$[\alpha]_{d+d^c} = \alpha + dd^c\beta,$$

where $d\alpha = 0$, $d^c\alpha = 0$ and $\beta \in B^{k-2}$. Pick any $[\gamma]_{dd^c} \in H^\ell_{dd^c}$, so that we have

$$[\gamma]_{dd^c} = \gamma + d\eta + d^c\theta$$

up to $(dd^c + d^c d)$-exact forms, where $dd^c\gamma = [0]_{C^\bullet}$. To prove the proposition it is enough to check that $[\alpha \wedge \gamma]_{dd^c}$ is a well-defined cohomology class. We have that $dd^c(\alpha \wedge \gamma) = [0]_{C^\bullet}$ since α is d-closed and d^c-closed, and γ is dd^c-closed. Moreover, the cohomology class does not depend on the choice of representative since

$$(\alpha + dd^c\beta) \wedge (\gamma + d\eta + d^c\theta) = \alpha \wedge \gamma + \alpha \wedge d\eta + \alpha \wedge d^c\theta + dd^c\beta \wedge \gamma + dd^c\beta \wedge d\eta + dd^c\beta \wedge d^c\theta.$$

When we pass to cohomology, the second, third, fifth and sixth terms on the right hand side vanish because they are d-exact or d^c-exact. The fourth term is both d and d^c-exact, since when computing Aeppli cohomology, we are considering classes in $C^\bullet$. □

Proposition 3.2.5. *There is a diagram of cohomologies*

$$\begin{array}{ccccc} & & H^k_{d+d^c} & & \\ & \swarrow & & \searrow & \\ H^k_d & & \xleftrightarrow[J]{\sim} & & H^k_{d^c} \\ & \searrow & & \swarrow & \\ & & H^k_{dd^c} & & \end{array}$$

that generalizes diagram (1.3.3) *to the case of almost complex manifolds.*

Proof. Let $[\alpha]_{d+d^c} \in H^k_{d+d^c}$. Then, we can write $[\alpha]_{d+d^c} = \alpha + dd^c\beta$. On the one side, we have that $d\alpha = 0$ and that $dd^c\beta$ is d-exact. Thus, the form α defines a de Rham cohomology class. On the other side, we also have $d^c\alpha = 0$ and $dd^c\beta = -d^c d\beta$, so it also defines a d^c-cohomology class. Hence, the morphisms from Bott–Chern cohomology to the de Rham and d^c-cohomologies are given by the identity on the representatives. Let now $[\gamma]_d \in H^k_d$. Then, we have that

$$dd^c\gamma = (dd^c + d^c d)\gamma,$$

which is the zero class in $C^\bullet$, so that γ defines an Aeppli cohomology class. Changing representative in the de Rham cohomology class modifies γ by a d-exact form, preserving the corresponding Aeppli cohomology class. The same proof holds for a d^c-cohomology class. Thus, the morphisms from the de Rham and d^c-cohomologies to Aeppli cohomology are given by the identity on the representatives composed with the projection on $C^\bullet$. □

The morphisms going from the Bott–Chern cohomology to the de Rham and d^c-cohomologies are not injective nor surjective. The same holds for those going from the de Rham and d^c-cohomologies to the Aeppli cohomology. This is true even if J is integrable, see, for example, [2].

The next proposition establishes that Bott–Chern and Aeppli cohomologies are preserved under pseudo-holomorphic maps and are almost complex invariants. A map between almost complex manifolds $f\colon (M, J) \to (M', J')$ is *pseudo-holomorphic* if

$$df \circ J = J' \circ df. \tag{3.2.1}$$

Theorem 3.2.6. *Let (M, J) and (M', J') be two almost complex manifolds. Let $f\colon M \to M'$ be a pseudo-holomorphic map. Then f induces a morphism of differential $\mathbb{Z}$-graded algebras*

$$f^*\colon H^\bullet_{d+d^c}(M', J') \longrightarrow H^\bullet_{d+d^c}(M, J),$$

and a morphism of differential $\mathbb{Z}$-graded modules over Bott–Chern cohomology

$$f^*\colon H^\bullet_{dd^c}(M', J') \longrightarrow H^\bullet_{dd^c}(M, J).$$

If in addition f is a diffeomorphism, then f^* is an isomorphism.

Proof. The pullback f^* commutes with d. By (3.2.1), it also commutes with d^c. In particular, it sends the double complexes $(B^\bullet, d, d^c)$ and $(C^\bullet, d, d^c)$ defined for (M', J') into the analogous objects defined for (M, J), and it defines a morphism on the cohomologies. If f is also a diffeomorphism, then $M = M'$ and df is an isomorphism. Thus $J = df^{-1} \circ J' \circ df$ and f^* is an isomorphism with inverse $(f^{-1})^*$. □

In principle, one could be tempted to define a bigraded version of Bott–Chern and Aeppli cohomologies for d and d^c. However, if J is not integrable, the operator dd^c does not preserve the bigrading and giving such a definition would require to impose further conditions on the subcomplex $B^\bullet$ that would make the construction less natural. Nevertheless, if $(p,q) \in \{(1,0),(0,1),(2,0),(0,2)\}$, one can still consider the bigraded spaces

$$H^{p,q}_{d+d^c} := \ker d \cap A^{p,q},$$

since d-closed (p,q)-forms coincide with d^c-closed (p,q)-forms, while for $(p,q) = (1,1)$ one can set

$$H^{1,1}_{d+d^c} := \frac{\ker d \cap A^{1,1}}{\operatorname{Im}(dd^c\colon \ker(dd^c + d^c d) \cap C^\infty(M) \to \ker(dd^c + d^c d) \cap A^{1,1})}.$$

Using the bigraded cohomology groups we just defined, we can prove a bigraded splitting in the case of 1-forms.

Proposition 3.2.7. *Let (M,J) be an almost complex manifold. Then*

$$H^1_{d+d^c} = H^{1,0}_{d+d^c} \oplus H^{0,1}_{d+d^c}.$$

Proof. A Bott–Chern cohomology class in $H^1_{d+d^c}$ is given by a 1-form α that is d-closed and d^c-closed. Furthermore, there are no dd^c-exact 1-forms for degree reasons. Split α according to the bidegree as

$$\alpha = \alpha^{1,0} + \alpha^{0,1}.$$

Since α is both d-closed and d^c-closed, the forms $\alpha^{1,0}$ and $\alpha^{0,1}$ are both d-closed and d^c-closed since they have pure bidegree. Hence, they define two Bott–Chern cohomology classes in $H^{1,0}_{d+d^c}$ and $H^{1,0}_{d+d^c}$, respectively. The converse inclusion is immediate. □

We point out that a bigraded splitting does not hold outside of 1-forms unless J is integrable. In the integrable case, the cohomology groups we computed on k-forms naturally split into bigraded components that coincide with the usual Bott–Chern and Aeppli cohomology. We give here a cohomological proof of this fact. For a similar statement valid for spaces of harmonic forms on compact Hermitian manifolds, see Corollary 4.1.3.

Lemma 3.2.8. *Let (M,J) be a complex manifold. Then*

$$H^k_{d+d^c} = \bigoplus_{p+q=k} H^{p,q}_{BC} \quad \text{and} \quad H^k_{dd^c} = \bigoplus_{p+q=k} H^{p,q}_{A}.$$

Proof. We prove the claim for Bott–Chern cohomology. The claim for Aeppli cohomology is proved similarly. The inclusion $\bigoplus_{p+q=k} H^{p,q}_{BC} \subseteq H^k_{d+d^c}$ is immediate. For the opposite inclusion, let $[\alpha]_{BC} \in H^k_{d+d^c}$ and let

$$\alpha = \sum_{p+q=k} \alpha^{p,q}$$

be the bidegree decomposition of α. Since α is both d-closed and d^c-closed, we have that the forms

$$\alpha^{even} := \sum_{p \text{ even}} \alpha^{p,q} \quad \text{and} \quad \alpha^{odd} := \sum_{p \text{ odd}} \alpha^{p,q}$$

are both d-closed. Consider the equation

$$0 = d\alpha^{ev} = \bar{\partial}\alpha^{0,k} + \partial\alpha^{0,k} + \bar{\partial}\alpha^{2,k-2} + \dots$$

and separate the terms by bidegree. Since the operators ∂ and $\bar{\partial}$ have bidegree $(1,0)$ and $(0,1)$, respectively, and since two summands of α^{even} differ in bidegree p by at least 2, all the terms $\partial\alpha^{p,q}$ and $\bar{\partial}\alpha^{p,q}$, for p even, have different bidegree. Hence, for all (p,q) with p even, it must be $\partial\alpha^{p,q} = \bar{\partial}\alpha^{p,q} = 0$. With a similar reasoning applied to α^{odd}, we have that $d\alpha^{p,q} = 0$ for all (p,q), so that $\alpha^{p,q} \in A^{p,q} \cap \ker d \cap \ker d^c$. This shows that each $\alpha^{p,q}$ defines a cohomology class in $H^{p,q}_{BC}$. Choosing another representative in $[\alpha]_{BC}$ we add a term $dd^c\beta$. Since in the complex case dd^c has bidegree $(1,1)$, the projection of $dd^c\beta$ on each of its components of bidegree (p,q) is still dd^c-exact and gives a well-defined cohomology class $[\alpha^{p,q}]_{BC} \in H^{p,q}_{BC}$. □

3.3 Cohomologies of the operators δ and $\bar{\delta}$

By (1.2.10), on an arbitrary almost complex manifold, the operators δ and $\bar{\delta}$ anti-commute but do not square to zero, and their cohomology is not well-defined. Nevertheless, again by (1.2.10), the space of forms where $dd^c + d^c d = 0$ coincides with the space of forms where $\bar{\delta}^2 = 0$. Thus, the subcomplex $B^\bullet$ seems to be a natural space on which to define the cohomologies of δ and $\bar{\delta}$.

Definition 3.3.1. The *δ-cohomology of* (M,J) is the cohomology

$$H^k_\delta := \frac{\ker(\delta\colon B^k \longrightarrow B^{k+1})}{\operatorname{Im}(\delta\colon B^{k-1} \longrightarrow B^k)}.$$

The *$\bar{\delta}$-cohomology of* (M,J) is the cohomology

$$H^k_{\bar{\delta}} := \frac{\ker(\bar{\delta}\colon B^k \longrightarrow B^{k+1})}{\operatorname{Im}(\bar{\delta}\colon B^{k-1} \longrightarrow B^k)}.$$

In the same fashion as Definition 3.2.3, we define the Bott–Chern and Aeppli cohomologies of the operators δ and $\bar\delta$.

Definition 3.3.2. The $(\delta+\bar\delta)$-*cohomology* of (M,J) is

$$H^k_{\delta+\bar\delta} := \frac{\ker(\delta\colon B^k\to B^{k+1})\cap\ker(\bar\delta\colon B^k\to B^{k+1})}{\operatorname{Im}(\delta\bar\delta\colon B^{k-2}\to B^k)}.$$

The $\delta\bar\delta$-*cohomology of* (M,J) is

$$H^k_{dd^c} := \frac{\ker(\delta\bar\delta\colon C^k\to C^{k+2})}{\operatorname{Im}(\delta\colon C^{k-1}\to C^k)+\operatorname{Im}(\bar\delta\colon C^{k-1}\to C^k)}.$$

In the integrable case, Bott–Chern and Aeppli cohomologies built using the operators d and d^c or ∂ and $\bar\partial$ coincide. This is still true in the non-integrable case for Bott–Chern and Aeppli cohomologies defined using the operators d and d^c or δ and $\bar\delta$.

Proposition 3.3.3. *Let (M,J) be an almost complex manifold. Then*

$$H^\bullet_{d+d^c} = H^\bullet_{\delta+\bar\delta} \quad \text{and} \quad H^\bullet_{dd^c} = H^\bullet_{\delta\bar\delta}.$$

Proof. By (1.2.8) and (1.2.9), we have that

$$\ker d\cap\ker d^c = \ker\delta\cap\ker\bar\delta.$$

By (1.2.10), there is an equality

$$\operatorname{Im}(dd^c\colon B^{k-2}\longrightarrow B^k) = \operatorname{Im}(\delta\bar\delta\colon B^{k-2}\longrightarrow B^k),$$

so that we conclude that

$$H^k_{d+d^c} = H^k_{\delta+\bar\delta}.$$

The equality between Aeppli cohomology and $\delta\bar\delta$-cohomology follows with a similar reasoning, since we are computing the cohomologies on forms in $C^\bullet$. □

3.4 Cohomologies of the operators d and d^Λ

In this section we describe almost symplectic cohomologies built using the operators d and d^Λ and the decomposition of the differential (1.2.13) induced by the symplectic bigrading of forms.

Let (M,ω) be an almost symplectic manifold. Let $\mathcal{L}^{r,s}$ be the space of forms of Lefschetz bidegree (r,s) and let $d = d_0 + d_1 + \dots$ be the decomposition of the differential induced by the Lefschetz bigrading, with $d_j\colon \mathcal{L}^{r,s}\to\mathcal{L}^{r-1+j,s+3-2j}$, see Section 1.2. This provides us with a multicomplex $(\mathcal{L}^{\bullet,\bullet}, d = d_0 + \ldots)$. The associated total complex is the de Rham complex $(A^\bullet, d)$, where

$$A^k = \bigoplus_{2r+s=k} \mathcal{L}^{r,s}$$

is given by the Lefschetz decomposition. The natural filtration to put on the multicomplex is the *Lefschetz filtration* [99]

$$F^p A^k = \bigoplus_{r \leq p} \mathcal{L}^{r,k-2r}.$$

In particular, the space $F^0 A^k$ coincides with the space of primitive k-forms. Following Section 1.1, every filtered complex admits a natural spectral sequence.

Theorem 3.4.1. *The multicomplex $(\mathcal{L}^{r,s}, d = d_0 + \ldots)$ admits a natural spectral sequence $(E_q^{r,s}, d^q)$ converging to de Rham cohomology $E_q^{r,s} \Rightarrow H_d^{2r+s}$ whose first page is isomorphic to the cohomology of the d_0 operator.*

In the symplectic case, the first non-trivial page of the spectral sequence is isomorphic to the cohomology of the ∂_+ operator and we have convergence $H_{\partial_+}^{r,s} \Rightarrow H_d^{2r+s}$.

Proof. The multicomplex $(\mathcal{L}^{r,s}, d = d_0 + \ldots)$ admits the Lefschetz filtration and it is bounded by Lemma 1.2.2. Thus, by Proposition 1.1.1, it admits an associated spectral sequence that degenerates in a finite number of steps. Furthermore, the spectral sequence converges to the cohomology of the total complex $(A^\bullet, d)$, which is the de Rham cohomology, inducing a bigrading on H_d^{2r+s} that generalizes the symplectic bigrading induced by the Lefschetz decomposition, see [100]. By (1.2.14), we have that $d_0^2 = 0$ and its cohomology is well-defined. By [62], we can write down explicitly the pages of the spectral sequence in terms of the differentials. The first page is

$$E_1^{r,s} \cong \frac{\{\omega^r \wedge \beta : d_0(\omega^r \wedge \beta) = 0 \text{ and } \beta \in \mathcal{P}^s\}}{\{d_0(\omega^{r+1} \wedge \gamma) : \gamma \in \mathcal{P}^{s-3}\}},$$

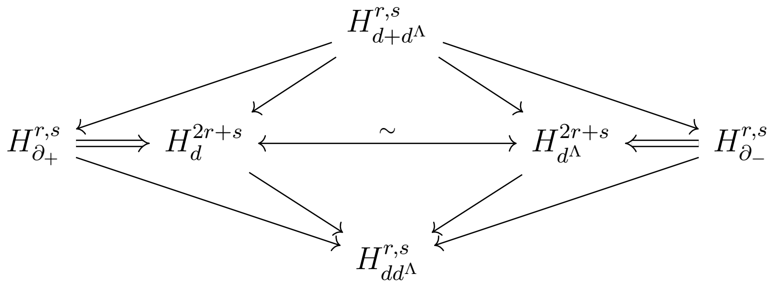
which is isomorphic to the cohomology of the d_0 operator. The second page is

$$E_2^{r,s} \cong \frac{\{\omega^r \wedge \beta : d_0(\omega^r \wedge \beta_1) = 0 \text{ and } d_1(\omega^r \wedge \beta_1) + d_0(\omega^{r+1} \wedge \beta_2) = 0\}}{\{d_0(\omega^{r+1} \wedge \gamma_1) + d_1(\omega^r \wedge \gamma_2) : d_0(\omega^r \wedge \gamma_2) = 0\}},$$

with $\beta_1 \in \mathcal{P}^s$, $\beta_2 \in \mathcal{P}^{s-2}$, $\gamma_1 \in \mathcal{P}^{s-3}$ and $\gamma_2 \in \mathcal{P}^{s-1}$.

In the symplectic case, all the differentials vanish except for d_1 and d_2, which coincide with ∂_+ and ∂_-, respectively. We have that $E_1^{r,s} \cong \mathcal{L}^{r,s}$ and $E_2^{r,s} \cong H_{\partial_+}^{r,s} \cong \omega^r \wedge PH_{\partial_+}^s$, where $PH_{\partial_+}^s$ is the primitive symplectic cohomology as defined in Section 1.3.2. $\square$

We focus for a moment on the symplectic case and on the double complex $(\mathcal{L}^{r,s}, \partial_+, \partial_-)$. The spectral sequence we defined allows to write a more precise version of diagram (1.3.7)



By general facts on spectral sequences of double complexes, the maps considered in the above diagram are isomorphisms if and only if the dd^Λ-lemma (or, equivalently, the $\partial_+\partial_-$-lemma) holds, see [4] for more details. In particular, if the dd^Λ-lemma holds, the spectral sequence degenerates at the first page and the de Rham cohomology splits under the action of the Lefschetz operator as

$$H^k_d = \bigoplus_{j\geq 0} \omega^j \wedge PH^{k-2j}_d.$$

In such a case, the symplectic manifold satisfies the Hard Lefschetz condition.

Going back to almost symplectic manifolds, we point out an interesting geometric interpretation of the operator d_0. Recall that an almost symplectic structure ω is called *locally conformally symplectic*, see [103] and [104], if

$$d\omega = \omega \wedge \theta, \quad \text{with } d\theta = 0.$$

Theorem 3.4.2. *Let (M, ω) be a compact almost symplectic $2m$-manifold. If $2m = 4$, then $d_0 = 0$. If $2m \geq 6$, then $d_0 = 0$ if and only if ω is locally conformally symplectic.*

Proof. On almost symplectic manifold, the Lefschetz decomposition of $d\omega$ is

$$d\omega = H + \omega \wedge \theta.$$

We show that d_0 identically vanishes if and only if $H = 0$. If $d_0 = 0$, in particular we have

$$0 = d_0\omega = \pi^{0,3}(d\omega) = H.$$

Conversely, suppose that H identically vanishes and let $\omega^r \wedge \beta \in \mathcal{L}^{r,s}$. Then, we have

$$d_0(\omega^r \wedge \beta) = \pi^{r-1,s+3} \circ d(\omega^r \wedge \beta) = \pi^{r-1,s+3}(\omega^r \wedge d\beta + r\omega^r \wedge \theta \wedge \beta) = 0,$$

proving our claim. On 4-manifolds, the form H identically vanishes since there are no primitive 3-forms, so that $d_0 = 0$ always. If $2m \geq 6$, the vanishing of H is equivalent to having $d\omega = \omega \wedge \theta$. From the equation

$$0 = d^2\omega = d(\omega \wedge \theta) = \omega \wedge d\theta,$$

we see that $d\theta = 0$ and that ω is locally conformally symplectic. □

We are ready to define almost symplectic versions of the symplectic cohomologies of Tseng and Yau using the same idea of Bott–Chern and Aeppli cohomologies of almost complex manifolds.

Definition 3.4.3. Consider the subcomplex $B^\bullet \subseteq A^\bullet$ given by

$$B^\bullet := A^\bullet \cap \ker(dd^\Lambda + d^\Lambda d)$$

and the quotient complex of $A^\bullet$ given by

$$C^\bullet := \frac{A^\bullet}{\operatorname{Im}(dd^\Lambda + d^\Lambda d)}.$$

The complexes $(B^\bullet, d, d^\Lambda)$ and $(C^\bullet, d, d^\Lambda)$ are $\mathbb{Z}$-graded double complexes.

Definition 3.4.4. Let (M, ω) be an almost symplectic manifold. The *almost symplectic cohomologies of* (M, ω) are

$$H^k_{d+d^\Lambda} := \frac{\ker(d\colon B^k \to B^{k+1}) \cap \ker(d^\Lambda\colon B^k \to B^{k-1})}{\operatorname{Im}(dd^\Lambda\colon B^k \to B^k)}$$

and

$$H^k_{dd^\Lambda} := \frac{\ker(dd^\Lambda\colon C^k \to C^k)}{\operatorname{Im}(d\colon C^{k-1} \to C^k) + \operatorname{Im}(d^\Lambda\colon C^{k+1} \to C^k)}.$$

If ω is symplectic then d and d^Λ anti-commute and our definition coincides with the one given by Tseng and Yau. The almost symplectic cohomologies are a well-defined invariant of the almost symplectic structure. The proof follows closely that of Theorem 3.2.6, therefore it is omitted. With a slight abuse of terminology, we say that a map between almost symplectic manifolds $f\colon (M, \omega) \to (M', \omega')$ is a *symplectomorphism* if

$$f^*\omega' = \omega.$$

Theorem 3.4.5. *Let (M, ω) and (M', ω') be two almost symplectic manifolds. Let $f\colon M \to M'$ be a symplectomorphism. Then f induces a morphism of differential $\mathbb{Z}$-graded algebras*

$$f^*\colon H^\bullet_{d+d^\Lambda}(M', \omega') \longrightarrow H^\bullet_{d+d^\Lambda}(M, \omega),$$

and a morphism of differential $\mathbb{Z}$-graded modules over $H^\bullet_{d+d^\Lambda}$

$$f^*\colon H^\bullet_{dd^\Lambda}(M', \omega') \longrightarrow H^\bullet_{dd^\Lambda}(M, \omega).$$

If in addition f is a diffeomorphism, then f^* is an isomorphism.

3.5 Cohomologies of even and odd forms

In this section we describe several cohomologies induced by the action of J on k-forms. In a natural way they admit a $\mathbb{Z}_2$-grading that induces a $\mathbb{Z}_2$-splitting of Bott–Chern cohomology. A similar splitting appears naturally in the context of the symplectic cohomologies, see Section 5 in [100].

Recall that J^2 acts on k-forms as $(-1)^k\,\mathrm{Id}$, see Section 1.2. This naturally induces a splitting of Λ^k into the $(\pm i)$-eigenbundles of J if k is odd, or into its (± 1)-eigenbundles if k is even. In terms of the bigrading of forms, it is easy to determine the eigenbundles. For example, if $k \equiv 0 \mod 4$ the $(+1)$-eigenbundle of J is given by $\bigoplus_{p \text{ even}} \Lambda^{p,k-p}$. This motivates the definition of the following spaces of forms. We set

$$A^k_{even} \coloneqq \bigoplus_{\substack{p+q=k \\ p \text{ even}}} A^{p,q} \quad \text{and} \quad A^k_{odd} \coloneqq \bigoplus_{\substack{p+q=k \\ p \text{ odd}}} A^{p,q}. \tag{3.5.1}$$

Definition 3.5.1. If $\alpha \in A^k_{even}$, we say that α is an *even k-form*. Similarly, if $\alpha \in A^k_{odd}$, we say that α is an *odd k-form*.

This provides a direct sum decomposition

$$A^k = A^k_{even} \oplus A^k_{odd}, \tag{3.5.2}$$

that allows to split $\alpha \in A^k$ into an *even part* and an *odd part*

$$\alpha = \alpha^{even} + \alpha^{odd},$$

given by the projections on the subspaces defined in (3.5.1). The same splitting can be given for forms in B^k, see Definition 3.5.6. It is immediate from the definition to check that the wedge product of forms of the same parity is even, while the wedge product of forms of opposite parity is odd. Thus the decomposition (3.5.2) endows $A^\bullet$ with a $\mathbb{Z}_2$-graded algebra structure. With respect to the $\mathbb{Z}_2$-grading on A^k, we have that complex conjugation reverses the parity of forms when k is odd, while it preserves the parity when k is even.

Definition 3.5.2. Let $P\colon A^\bullet \to A^\bullet$ be a differential operator. We say that P is *even* if it preserves the parity of forms, that is, if

$$P(A^\bullet_{even}) \subseteq A^\bullet_{even} \quad \text{and} \quad P(A^\bullet_{odd}) \subseteq A^\bullet_{odd}.$$

We say that P is *odd* if it reverses the parity of forms, that is, if

$$P(A^\bullet_{even}) \subseteq A^\bullet_{odd} \quad \text{and} \quad P(A^\bullet_{odd}) \subseteq A^\bullet_{even}.$$

The differential d is neither even nor odd, but it admits a splitting into an even and an odd part that correspond to the operators $\bar{\delta}$ and δ, respectively. Indeed, when computing d on even forms, δ is the projection of d onto odd forms, while $\bar{\delta}$ is the projection onto even forms. The parities of most of the operators we study can be determined without effort and are collected in the next proposition.

Proposition 3.5.3. *We have the following properties:*

(i) *the composition of operators of the same parity is even;*

(ii) *the composition of operators of opposite parity is odd;*

(iii) *the operators δ and $\bar{\delta}$ are odd and even, respectively;*

(iv) *the operator $4i\delta\bar{\delta} = dd^c - d^c d$ is odd;*

(v) *the operator $-4i\delta^2 = dd^c + d^c d$ is even;*

(vi) *the operator dd^c restricted to $B^\bullet$ is odd.*

Proof. We have that (i) and (ii) follow immediately from the definition of even and odd operators, that (iii) follows from the explicit expression of δ and $\bar{\delta}$ in terms of μ, ∂, $\bar{\partial}$ and $\bar{\mu}$, and that (iv) and (v) are a consequence of ($i - iii$). For (vi), let $\alpha \in B^\bullet$. Then $(dd^c + d^c d)\alpha = -4i\delta^2\alpha = 4i\bar{\delta}^2\alpha = 0$. In particular, by (1.2.9), we also have that

$$dd^c\alpha = i(\delta + \bar{\delta})(\bar{\delta} - \delta)\alpha = i(\bar{\delta}^2 + 2\delta\bar{\delta} - \delta^2)\alpha = 2i\delta\bar{\delta}\alpha.$$

Thus, when restricted to $B^\bullet$, the operator dd^c coincides with $\delta\bar{\delta}$ up to a constant and it is odd. □

Using even and odd forms, one can define the associated subgroups of the de Rham cohomology.

Definition 3.5.4. Let (M, J) be an almost complex manifold. The *J-even cohomology* of (M, J) is

$$H^k_{even}(M, J) := \{[\alpha] \in H^k_{dR} : \alpha \in A^k_{even}\}.$$

The *J-odd cohomology* of (M, J) is

$$H^k_{odd}(M, J) := \{[\alpha] \in H^k_{dR} : \alpha \in A^k_{odd}\}.$$

The J-even and J-odd cohomologies are a special case of the cohomology H_J^S, see [5], obtained by taking $S = \{(p,q) : p \text{ is even}\}$ and $S = \{(p,q) : p \text{ is odd}\}$, respectively. They also generalize the notions of cohomologies introduced by Draghici, Li and Zhang, see Section 3.1. Indeed, if $k = 2$, we have that

$$H^2_{even} = H_J^- \quad \text{and} \quad H^2_{odd} = H_J^+.$$

Apart from generalizing well-known cohomologies, the J-even and J-odd cohomologies arise naturally in the context of Bott–Chern cohomology of almost complex manifolds.

Theorem 3.5.5. *There is a natural map*

$$\begin{aligned} H^k_{d+d^c} &\longrightarrow H^k_{even} + H^k_{odd}, \\ [\alpha]_{d+d^c} &\longmapsto [\alpha^{even}]_d + [\alpha^{odd}]_d. \end{aligned} \tag{3.5.3}$$

Proof. The theorem is a consequence of the fact that a k-form α is d-closed and d^c-closed if and only if its even and odd parts are both d-closed. Decompose $\alpha = \alpha^{even} + \alpha^{odd}$. The almost complex structure J acts on even and odd forms as multiplication by a constant, and the two constants differ by a sign. More precisely, we have that

$$J\alpha^{even} = (-i)^k \alpha^{even} \quad \text{and} \quad J\alpha^{odd} = -(-i)^k \alpha^{odd},$$

so that

$$d\alpha = 0 \Leftrightarrow d\alpha^{even} + d\alpha^{odd} = 0 \quad \text{and} \quad d^c\alpha = 0 \Leftrightarrow d\alpha^{even} - d\alpha^{odd} = 0.$$

In particular, the form α defines a Bott–Chern cohomology class if and only if α^{even} and α^{odd} define two de Rham cohomology classes. This provides the map (3.5.3). □

We recall that, if $k = 2$ and (M, J) is a compact almost complex 4-manifold, then de Rham cohomology splits as the direct sum of J-invariant and J-anti-invariant cohomologies, see Theorem 2.3 in [35]. Since de Rham cohomology classes are defined up to d-exact forms and d is neither an even nor an odd operator, we have that $H^k_{even} \cap H^k_{odd} \neq \{0\}$. In particular, in general there is no splitting

$$H^k_d = H^k_{even} \oplus H^k_{odd}.$$

However, we can define cohomologies in the context of Bott–Chern cohomology that take into account the parity of the forms involved.

Definition 3.5.6. Consider the subcomplexes

$$B^\bullet_{even} \coloneqq B^\bullet \cap A^\bullet_{even} \quad \text{and} \quad B^\bullet_{odd} \coloneqq B^\bullet \cap A^\bullet_{odd}.$$

Let $(H^k_{d+d^c})^{even}$ be the space of Bott–Chern cohomology classes computed on even forms

$$(H^k_{d+d^c})^{even} \coloneqq \frac{\ker d \cap \ker d^c \cap A^k_{even}}{\operatorname{Im}(dd^c \colon B^{k-2}_{odd} \to B^k_{even})},$$

and let $(H^k_{d+d^c})^{odd}$ be the space of Bott–Chern cohomology classes computed on odd forms

$$(H^k_{d+d^c})^{odd} \coloneqq \frac{\ker d \cap \ker d^c \cap A^k_{odd}}{\operatorname{Im}(dd^c \colon B^{k-2}_{even} \to B^k_{odd})}.$$

Using these cohomologies, we can prove the desired splitting.

Corollary 3.5.7. *The map* (3.5.3) *induces a* $\mathbb{Z}_2$*-graded decomposition of Bott–Chern cohomology*

$$H^k_{d+d^c} = (H^k_{d+d^c})^{even} \oplus (H^k_{d+d^c})^{odd}.$$

Proof. Let $[\alpha]_{d+d^c} \in H^k_{d+d^c}$ be a Bott–Chern cohomology class. Writing the representatives according to the parity of the forms, we have that

$$[\alpha]_{d+d^c} = \alpha^{even} + \alpha^{odd} + (dd^c\beta)^{even} + (dd^c\beta)^{odd},$$

where $\beta \in B^{k-2}$. By (vi) of Proposition 3.5.3, we know that dd^c is an odd operator on $B^\bullet$. Furthermore, the space B^{k-2} splits into its even and an odd part, so that

$$(dd^c\beta)^{even} = dd^c(\beta^{odd}) \quad \text{and} \quad (dd^c\beta)^{odd} = dd^c(\beta^{even}),$$

proving that the even and odd parts of a dd^c-exact form are both dd^c-exact. Finally, since $dd^c + d^cd$ is even, the even and odd part of a form in $B^\bullet$ are still in $B^\bullet$. This implies that the even and odd part of any representative provide a well-defined splitting of Bott–Chern cohomology. □

The bigraded splitting of Proposition 3.2.7 follows as an easy consequence of Corollary 3.5.7 in degree 1.

CHAPTER 4

Harmonic forms on almost Hermitian manifolds

In this chapter we introduce several Laplacians on compact almost Hermitian manifolds that generalize the Bott–Chern and the Aeppli Laplacians of complex manifolds and the symplectic Laplacians of symplectic manifolds. We extensively study the associated spaces of harmonic forms, determining their basic properties, their symmetries and the relations occurring among spaces of harmonic forms of different Laplacians. We are mainly interested in solving a problem posed by Kodaira and Spencer: find a generalization of Hodge numbers and Bott–Chern numbers from complex to almost complex manifolds. In our context, the problem reduces to determine whether or not the dimensions of the space of harmonic forms we introduced depend on the choice of J-compatible Riemannian metric. While a general result on metric-independence at the moment seems out of reach, if the manifold has dimension 4 or the almost Hermitian structure is almost Kähler, the situation drastically improves and we are able to provide a solution to the Kodaira–Spencer's problem. Noteworthy, all the spaces of harmonic forms we introduce injects into the cohomologies defined in Chapter 3.

4.1 Almost Hermitian Laplacians

Let (M, g) be a compact Riemannian manifold. Let P and Q be first-order differential operators on $A^\bullet$ of degree ± 1. Denote by P^* and Q^* their formal adjoints. The *Hodge-type Laplacian of P* is the operator of the second order

$$\Delta_P \coloneqq PP^* + P^*P.$$

The *Bott–Chern-type Laplacian of P and Q* is the operator of the fourth order

$$\Delta_{P+Q} \coloneqq PQQ^*P^* + Q^*P^*PQ + P^*QQ^*P + Q^*PP^*Q + P^*P + Q^*Q,$$

while the *Aeppli-type Laplacian of P and Q* is the operator of the fourth order

$$\Delta_{PQ} \coloneqq PQQ^*P^* + Q^*P^*PQ + P^*QQ^*P + Q^*PP^*Q + PP^* + QQ^*.$$

Observe that the Bott–Chern-type and the Aeppli-type Laplacians differ from each other only for a term of the second order and that their fourth-order part are **not** symmetric in P and Q. The *space of P-harmonic k-forms* is the space

$$\mathcal{H}^k_P \coloneqq A^k \cap \ker \Delta_P.$$

Similarly, the *spaces of $(P+Q)$-harmonic k-forms* and *of PQ-harmonic k-forms* are

$$\mathcal{H}^k_{P+Q} \coloneqq A^k \cap \ker \Delta_{P+Q} \quad \text{and} \quad \mathcal{H}^k_{PQ} \coloneqq A^k \cap \ker \Delta_{PQ},$$

respectively. If the space of forms admits a bigrading, it is natural to define a bigraded version of the spaces of harmonic forms obtained as the kernel of the Laplacians restricted to bigraded forms. The corresponding spaces of harmonic (p,q)-forms are denoted by $\mathcal{H}^{p,q}_P$, $\mathcal{H}^{p,q}_{P+Q}$ and $\mathcal{H}^{p,q}_{PQ}$, respectively. Using the L^2-inner product of forms defined in (1.2.2), we have an explicit description of the spaces of harmonic forms:

$$\begin{aligned}
\mathcal{H}^k_P &= \{\alpha \in A^k : P\alpha = 0 \text{ and } P^*\alpha = 0\},\\
\mathcal{H}^k_{P+Q} &= \{\alpha \in A^k : P\alpha = 0,\ Q\alpha = 0 \text{ and } (PQ)^*\alpha = 0\},\\
\mathcal{H}^k_{PQ} &= \{\alpha \in A^k : PQ\alpha = 0,\ P^*\alpha = 0 \text{ and } Q^*\alpha = 0\}.
\end{aligned}$$

Let (M, J, ω, g) be a compact almost Hermitian $2m$-manifold. We make explicit choices for P and Q as differential operators induced by the almost Hermitian structure. We first focus on the almost complex point of view. Consider the pairs of almost complex operators

$$(P,Q) \in \{(d, d^c), (\partial, \bar\partial), (\delta, \bar\delta)\}.$$

Then we have the Hodge-type Laplacians

$$\Delta_d, \quad \Delta_{d^c}, \quad \Delta_\partial, \quad \Delta_{\bar\partial}, \quad \Delta_\delta \quad \text{and} \quad \Delta_{\bar\delta}.$$

The corresponding spaces of graded harmonic forms are

$$\mathcal{H}^k_d, \quad \mathcal{H}^k_{d^c}, \quad \mathcal{H}^k_\partial, \quad \mathcal{H}^k_{\bar\partial}, \quad \mathcal{H}^k_\delta \quad \text{and} \quad \mathcal{H}^k_{\bar\delta}.$$

Since the action of J induces a bigrading on forms, we also have bigraded versions

$$\mathcal{H}_d^{p,q}, \quad \mathcal{H}_{d^c}^{p,q}, \quad \mathcal{H}_\partial^{p,q}, \quad \mathcal{H}_{\bar\partial}^{p,q}, \quad \mathcal{H}_\delta^{p,q} \quad \text{and} \quad \mathcal{H}_{\bar\delta}^{p,q}.$$

Similarly, one can consider the Bott–Chern-type Laplacians

$$\Delta_{d+d^c}, \quad \Delta_{d^c+d}, \quad \Delta_{\partial+\bar\partial}, \quad \Delta_{\bar\partial+\partial}, \quad \Delta_{\delta+\bar\delta}, \quad \text{and} \quad \Delta_{\bar\delta+\delta},$$

and the Aeppli-type Laplacians

$$\Delta_{dd^c}, \quad \Delta_{d^cd}, \quad \Delta_{\partial\bar\partial}, \quad \Delta_{\bar\partial\partial}, \quad \Delta_{\delta\bar\delta}, \quad \text{and} \quad \Delta_{\bar\delta\delta},$$

together with the associated spaces of graded or bigraded harmonic forms. To reduce the number of spaces we have to deal with, we observe the following elementary symmetries of the Laplacians under the action of J, of the Hodge $*$ and of complex conjugation. The almost complex structure induces the symmetries

$$\Delta_d J = J\Delta_{d^c}, \quad \Delta_{d+d^c} J = J\Delta_{d^c+d} \quad \text{and} \quad \Delta_{dd^c} J = J\Delta_{d^cd},$$

that give isomorphisms of graded harmonic forms

$$\mathcal{H}_d^k \cong \mathcal{H}_{d^c}^k, \quad \mathcal{H}_{d+d^c}^k \cong \mathcal{H}_{d^c+d}^k \quad \text{and} \quad \mathcal{H}_{dd^c}^k \cong \mathcal{H}_{d^cd}^k, \tag{4.1.1}$$

respectively, together with similar isomorphisms for bigraded harmonic forms. In the same way, complex conjugation gives the isomorphisms

$$\mathcal{H}_\partial^k \cong \mathcal{H}_{\bar\partial}^k, \quad \mathcal{H}_\delta^k \cong \mathcal{H}_{\bar\delta}^k, \quad \mathcal{H}_{\partial+\bar\partial}^k \cong \mathcal{H}_{\bar\partial+\partial}^k \quad \text{and} \quad \mathcal{H}_{\delta+\bar\delta}^k \cong \mathcal{H}_{\bar\delta+\delta}^k. \tag{4.1.2}$$

The Hodge $*$ operator gives a series of isomorphisms between forms of different degree. Up to the isomorphisms (4.1.1) and (4.1.2), they reduce to

$$\mathcal{H}_d^k \cong \mathcal{H}_d^{2m-k}, \quad \mathcal{H}_{\bar\partial}^k \cong \mathcal{H}_{\bar\partial}^{2m-k} \quad \text{and} \quad \mathcal{H}_{\bar\delta}^k \cong \mathcal{H}_{\bar\delta}^{2m-k}$$

for harmonic forms coming from the Hodge-type Laplacians, and to

$$\mathcal{H}_{d+d^c}^k \cong \mathcal{H}_{d^cd}^{2m-k}, \quad \mathcal{H}_{\partial+\bar\partial}^k \cong \mathcal{H}_{\bar\partial\partial}^{2m-k} \quad \text{and} \quad \mathcal{H}_{\delta+\bar\delta}^k \cong \mathcal{H}_{\bar\delta\delta}^{2m-k}$$

for harmonic forms coming from the Bott–Chern-type and the Aeppli-type Laplacians. In particular, up to isomorphism, we can restrict ourselves to the study of the spaces of graded harmonic forms

$$\mathcal{H}_d^k, \quad \mathcal{H}_{\bar\partial}^k, \quad \mathcal{H}_{\bar\delta}^k, \quad \mathcal{H}_{d+d^c}^k, \quad \mathcal{H}_{\partial+\bar\partial}^k, \quad \mathcal{H}_{\delta+\bar\delta}^k,$$

and to their bigraded version. For the Hermitian case, a detailed study of the symmetries induced by the action of compatible triples on the space of forms can be found in [109].

From the almost symplectic point of view, we consider the pair $(P, Q) = (d, d^\Lambda)$. The associated Laplacians are

$$\Delta_d, \quad \Delta_{d^\Lambda}, \quad \Delta_{d+d^\Lambda}, \quad \Delta_{d^\Lambda+d}, \quad \Delta_{dd^\Lambda} \quad \text{and} \quad \Delta_{d^\Lambda d},$$

while the corresponding spaces of graded harmonic forms are

$$\mathcal{H}_d, \quad \mathcal{H}_{d^\Lambda}, \quad \mathcal{H}_{d+d^\Lambda}, \quad \mathcal{H}_{d^\Lambda+d}, \quad \mathcal{H}_{dd^\Lambda} \quad \text{and} \quad \mathcal{H}_{d^\Lambda d}.$$

In this case, computing bigraded harmonic forms means to consider the restriction of the Laplacians to the spaces $\mathcal{L}^{r,s}$. We are particularly interested in the *spaces of primitive harmonic s-forms*, that is, the kernels of the Laplacians computed on $\mathcal{L}^{0,s}$, which we denote by $\mathcal{PH}^s_d$, $\mathcal{PH}^s_{d+d^\Lambda}$ and $\mathcal{PH}^s_{dd^\Lambda}$. The almost symplectic Laplacians satisfy the symmetries induced by the symplectic $*_s$ operator

$$*_s\,\Delta_{d+d^\Lambda} = \Delta_{d^\Lambda+d}\, *_s \quad \text{and} \quad *_s\,\Delta_{dd^\Lambda} = \Delta_{d^\Lambda d}\, *_s,$$

that give isomorphisms

$$\mathcal{H}^k_{d+d^\Lambda} \cong \mathcal{H}^{2m-k}_{d^\Lambda+d} \quad \text{and} \quad \mathcal{H}^k_{dd^\Lambda} \cong \mathcal{H}^{2m-k}_{d^\Lambda d}.$$

Similarly, the symmetries induced by J are

$$J\Delta_d = -\Delta_{d^\Lambda} J, \quad J\Delta_{d+d^\Lambda} = \Delta_{dd^\Lambda} J \quad \text{and} \quad J\Delta_{d^\Lambda+d} = \Delta_{d^\Lambda d} J,$$

that give isomorphisms

$$\mathcal{H}^k_d \cong \mathcal{H}^k_{d^\Lambda}, \quad \mathcal{H}^k_{d+d^\Lambda} \cong \mathcal{H}^k_{dd^\Lambda} \quad \text{and} \quad \mathcal{H}^k_{d^\Lambda+d} \cong \mathcal{H}^k_{d^\Lambda d},$$

respectively. Up to isomorphism, we are left with the spaces

$$\mathcal{H}^k_d, \quad \mathcal{H}^k_{d+d^\Lambda}, \quad \mathcal{PH}^k_d \quad \text{and} \quad \mathcal{PH}^k_{d+d^\Lambda}.$$

An essential property in developing Hodge theory on complex or symplectic manifolds is the ellipticity of the Laplacians. It turns out that ellipticity of the operators we consider is independent of the integrability of J or of ω.

Theorem 4.1.1. *Let η be any symbol among d, $\bar\partial$, $\bar\delta$, $d+d^c$, $\partial+\bar\partial$, $\delta+\bar\delta$ and $d+d^\Lambda$. Then the Laplacian Δ_η is a self-adjoint elliptic operator and there is decomposition of* ***graded*** *k-forms*

$$A^k = \mathcal{H}^k_\eta \overset{\perp}{\oplus} \operatorname{Im}\Delta_\eta. \tag{4.1.3}$$

Furthermore, the space $\mathcal{H}^k_\eta$ is finite-dimensional.

Proof. The fact that Δ_η is self-adjoint follows from the definition. Ellipticity of Δ_d is established by classical Hodge theory of Riemannian manifolds. Ellipticity of the remaining Laplacians follows by showing that they coincide, up to lower-order terms, with an elliptic operator. We write down the proof for $\eta = d + d^c$. The remaining cases follow similarly. Denote by $\cong$ the equality up to terms of order at most one. Then we have that:

- $dd^c + d^c d$ has order one, so that $dd^c \cong -d^c d$;
- $d(d^c)^* + (d^c)^* d$ has order one by the Kähler identities for almost Hermitian manifolds, see [28] or [37], so that $d(d^c)^* \cong -(d^c)^* d$.

We can conclude that

$$\begin{aligned} \Delta_{d+d^c} &\cong dd^c(dd^c)^* + (dd^c)^* dd^c + d^* d^c (d^* d^c)^* + (d^* d^c)^* d^* d^c \cong \\ &\cong dd^* d^c (d^c)^* + d^* d (d^c)^* d^c + d^* dd^c (d^c)^* + dd^* (d^c)^* d^c = \\ &= \Delta_d \Delta_{d^c} \cong (\Delta_d)^2, \end{aligned}$$

which is elliptic. The proofs involving the operator d^Λ follow similarly taking into account that, by (1.2.15), on almost Hermitian manifolds we have $d^\Lambda = (d^c)^*$. The orthogonal direct sum decomposition between image and kernel of the Laplacians and the finite-dimensionality of the kernel follow from the theory of self-adjoint elliptic operators on compact manifolds, see Lemma 1.3.2. □

Since the spaces of harmonic (p, q)-forms, with $p + q = k$, are subspaces of the spaces of harmonic k-forms, they are finite-dimensional. The complex dimensions of the spaces of harmonic forms are denoted by

$$h^k_\eta := \dim_{\mathbb{C}} \mathcal{H}^k_\eta \quad \text{and} \quad h^{p,q}_\eta := \dim_{\mathbb{C}} \mathcal{H}^{p,q}_\eta,$$

and they depend on the initial choice of almost Hermitian structure. If needed, we specify the almost Hermitian structure from which the dimensions depend by writing $h^k_\eta(J, \omega, g)$ and $h^{p,q}_\eta(J, \omega, g)$.

In general, a decomposition similar to (4.1.3) does not hold for bigraded forms unless Δ_η preserves the bigrading.

Proposition 4.1.2. *Let (M, J, ω, g) be a compact almost Hermitian manifold. Then there is an inclusion*

$$\bigoplus_{p+q=k} \mathcal{H}^{p,q}_\eta \subseteq \mathcal{H}^k_\eta \tag{4.1.4}$$

that induces an inequality

$$\sum_{p+q=k} h^{p,q}_\eta \leq h^k_\eta. \tag{4.1.5}$$

If Δ_η preserves the bigrading, then the equality holds in (4.1.4) *and* (4.1.5).

Proof. The inclusion (4.1.4) is immediate since η-harmonic (p,q)-forms, with $p+q=k$, are in particular η-harmonic k-forms. Suppose that Δ_η preserves the bigrading of forms. Let $\alpha \in \mathcal{H}^k_\eta$ and let

$$\alpha = \sum_{p+q=k} \alpha^{p,q}$$

be its bidegree decomposition. Then the bidegree decomposition of $\Delta_\eta\alpha$ is

$$0 = \Delta_\eta\alpha = \Delta_\eta(\sum_{p+q=k} \alpha^{p,q}) = \sum_{p+q=k} \Delta_\eta\alpha^{p,q}.$$

Since Δ_η preserves the bigrading, each summand $\Delta_\eta\alpha^{p,q}$ must vanish separately, showing the direct sum decomposition. The claims on the dimensions of the spaces of harmonic forms follow easily. □

Motivated by Proposition 4.1.2, we ask the following question:

Question: does the equality in (4.1.4) or (4.1.5) imply that the Laplacian Δ_η preserves the bigrading?

There are several instances of Laplacians that preserve the bigrading: the Laplacians Δ_{d+d^c} and Δ_{dd^c} on complex manifolds and the Laplacians $\Delta_{\bar\partial}$ and $\Delta_{\partial+\bar\partial}$ on almost complex manifolds. This immediately gives a bigraded decomposition of the spaces of harmonic forms.

Corollary 4.1.3. *Let (M,J) be a compact complex manifold. Then, for every choice of J-compatible metric, we have*

$$\mathcal{H}^k_{d+d^c} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{BC} \quad \text{and} \quad \mathcal{H}^k_{dd^c} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{A}.$$

In particular, $h^k_{d+d^c} = \sum_{p+q=k} h^{p,q}_{BC}$ *and* $h^k_{dd^c} = \sum_{p+q=k} h^{p,q}_{A}$.

Corollary 4.1.4. *Let (M,J) be a compact almost complex manifold. Then, for every choice of J-compatible metric, we have*

$$\mathcal{H}^k_{\bar\partial} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\bar\partial} \quad \text{and} \quad \mathcal{H}^k_{\partial+\bar\partial} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\partial+\bar\partial}.$$

In particular, $h^k_{\bar\partial} = \sum_{p+q=k} h^{p,q}_{\bar\partial}$ *and* $h^k_{\partial+\bar\partial} = \sum_{p+q=k} h^{p,q}_{\partial+\bar\partial}$.

Corollary 4.1.3, together with Hodge theory, provides a different proof of Lemma 3.2.8 valid on compact complex manifolds.

Next, we show that the numbers h^k_η are preserved under morphisms of almost Hermitian manifolds, so that they define almost Hermitian invariants.

Theorem 4.1.5. *Let (M, J, ω, g) and (M', J', ω', g') be two compact almost Hermitian manifolds of the same dimension and let $f\colon M \to M'$ be a surjective smooth map preserving the compatible triple, i.e., satisfying*

$$df \circ J = J' \circ df, \quad f^*\omega' = \omega \quad \text{and} \quad f^*g' = g. \tag{4.1.6}$$

Then we have that

$$h^k_\eta(J', \omega', g') \leq h^k_\eta(J, \omega, g).$$

If in addition f is a diffeomorphism, then

$$h^k_\eta(J', \omega', g') = h^k_\eta(J, \omega, g).$$

If any two conditions among those in (4.1.6) hold, then also the remaining one is satisfied.

Proof. If α is η-harmonic with respect to the almost Hermitian structure (J', ω', g'), then $f^*\alpha$ is η-harmonic with respect to (J, ω, g) by (4.1.6). Furthermore, the differential df is injective because $f^*g' = g$. Since M and M' have he same dimension, it is also surjective at every point of M. We show that the pull-back f^* is injective. Let $\alpha \in A^k(M')$ and assume that $f^*\alpha|_x = 0$ for all $x \in M$. Fix $x' \in M'$. Let $v'_1, \ldots, v'_k \in T_{x'}M'$. Since the maps f and df are surjective, then there exist $v_1, \ldots, v_k \in T_xM$ such that $df|_x v_j = v'_j$, where $j = 1, \ldots, k$, and $x \in f^{-1}(x')$. Therefore, we have that

$$\alpha|_{x'}(v'_1, \ldots, v'_k) = \alpha|_{x'}(df|_x v_1, \ldots, df|_x v_k) = f^*\alpha|_x(v_1, \ldots, v_k) = 0.$$

For any $x' \in M'$, this gives $\alpha|_{x'} = 0$ and consequently $\alpha = 0$, showing the inequality $h^k_\eta(J', \omega', g') \leq h^k_\eta(J, \omega, g)$. If in addition f is a diffeomorphism, applying the above argument to

$$f^{-1} : M' \to M$$

we obtain that $h^k_P(J, g) \leq h^k_{P'}(J', g')$, proving the equality. □

The last step of classical Hodge theory is to show that the dimensions of the spaces of harmonic forms are independent of the choice of Riemannian metric. For instance, consider the space $\mathcal{H}^k_d$ consisting of d-harmonic forms. Its dimension is independent of any choice and is actually a topological invariant, the k-th Betti number of the manifold M. Similarly, integrability of J guarantees that there are isomorphisms between Dolbeault, Bott–Chern and Aeppli cohomologies and the corresponding spaces of harmonic forms on Hermitian manifolds. The same is true for the symplectic cohomologies on almost Kähler manifolds. In view of that, we formulate the following problem.

Generalized Kodaira–Spencer's problem.
Let (M, J) be a compact almost complex manifold. Fix a J-compatible Riemannian metric and let η be any symbol among d, $\bar\partial$, $\bar\delta$, $d + d^c$, $\partial + \bar\partial$ and $\delta + \bar\delta$. Determine if the numbers h^k_η and $h^{p,q}_\eta$ depend on the choice of Riemannian metric.

The first simple observation is that, for every choice of η, the lowest and top degree numbers are always equal to 1 since harmonic functions are necessarily constant and since harmonic top forms are constant multiples of the volume form.

4.2 Dependence on the choice of metric

In this section we study the dependence on the metric of the numbers h^k_η on almost Hermitian 4-manifolds. Thanks to the restriction on the dimension, we are able to show that several of them do not depend on the choice of metric. The situation significantly improves if we assume that the metric is almost Kähler, where we find a solution to the generalized Kodaira–Spencer's problem, see Theorems 4.2.3 and 4.2.7.

The section is structured as follows: we begin each paragraph with the description of the known metric-dependence (or independence) of the invariants. This is summarized in several diagrams where we adopt the notation

- ☺ : almost complex invariant
- ! : almost Kähler, possibly almost complex invariant
- 😐 : almost Kähler, not almost complex invariant
- ☹ : metric-dependent

The rest of each paragraph contains the proof of the main statement. The proof will be based on results spread across several papers, that will be properly referenced, and several original results that we prove on the go.

Dependence on the metric of the numbers $h^k_{\bar\delta}$, $h^{p,q}_{\bar\partial}$, $h^k_{\partial+\bar\partial}$ and $h^{p,q}_{\partial+\bar\partial}$
For these numbers the situation is almost entirely known.

Theorem 4.2.1. *Dependence on the metric of the numbers $h^{p,q}_{\bar\partial}$ is described by the diagram*

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}_{\bar\partial} & & h^{0,1}_{\bar\partial} & \\ h^{2,0}_{\bar\partial} & & h^{1,1}_{\bar\partial} & & h^{0,2}_{\bar\partial} \\ & h^{2,1}_{\bar\partial} & & h^{1,2}_{\bar\partial} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & \text{☺} & & \\ & \text{☺} & & \text{☹} & \\ \text{☺} & & \text{😐} & & \text{☺} \\ & \text{☹} & & \text{☺} & \\ & & \text{☺} & & \end{array} \tag{4.2.1}$$

Dependence on the metric of the numbers $h^{p,q}_{\partial+\bar\partial}$ is described by the diagram

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}_{\partial+\bar\partial} & & h^{0,1}_{\partial+\bar\partial} & \\ h^{2,0}_{\partial+\bar\partial} & & h^{1,1}_{\partial+\bar\partial} & & h^{0,2}_{\partial+\bar\partial} \\ & h^{2,1}_{\partial+\bar\partial} & & h^{1,2}_{\partial+\bar\partial} & \\ & & 1 & & \end{array} \qquad (4.2.2)$$

Dependence on the metric of the numbers $h^k_{\bar\partial}$ and $h^k_{\partial+\bar\partial}$ is described by the table

k	0	1	2	3	4
$h^k_{\bar\partial}$	☺	☹	😐	☹	☺
$h^k_{\partial+\bar\partial}$	☺	☺	☺	☹	☺

Recall that the isomorphism induced by the Hodge $*$ gives an equality

$$h^{p,q}_{\bar\partial} = h^{m-p,m-q}_{\bar\partial}. \qquad (4.2.3)$$

In general, the numbers $h^{p,0}_{\bar\partial}$ do not depend on the choice of metric by bidegree reasons. Holt and Zhang showed that there exists an almost complex 4-manifold such that $h^{0,1}_{\bar\partial}$ varies for different choices of almost Kähler metric, see [52] and [53]. By (4.2.3), we know the situation also for $h^{2,0}_{\bar\partial}$, $h^{2,1}_{\bar\partial}$ and $h^{1,2}_{\bar\partial}$. The remaining number $h^{1,1}_{\bar\partial}$ is independent of the choice of almost Kähler metric [53] but it depends on the choice of almost Hermitian metric [94]. This proves the situation described in diagram (4.2.1). By Lemma 4.1 and Theorem 4.2 in [48], the numbers $h^{p,q}_{\partial+\bar\partial}$ are metric-independent for (p,q) any among $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$ and $(1,1)$. For $(p,q)=(2,1)$ and $(p,q)=(1,2)$, we have the following result.

Lemma 4.2.2. *Let (M,J) be a compact almost complex 4-manifold admitting a J-compatible almost Kähler metric. Then for every choice of J-compatible almost Kähler metric we have that*

$$\mathcal{H}^{2,1}_{\partial+\bar\partial} \cong \mathcal{H}^{1,0}_{\bar\partial} \quad \textit{and} \quad \mathcal{H}^{1,2}_{\partial+\bar\partial} \cong \mathcal{H}^{0,1}_{\bar\partial}.$$

Proof. The proofs of the two isomorphisms are similar. We prove only the first one, which is provided by the Hodge $*$ operator. Let $\alpha^{1,0} \in \mathcal{H}^{1,0}_{\bar\partial}$. Then $\bar\partial\alpha = 0$. After taking the Hodge $*$, we obtain that $\partial * \alpha^{1,0} = 0$ by bidegree reasons, that $\bar\partial * \alpha^{1,0}$ is proportional to $\bar\partial(\omega \wedge \alpha^{1,0}) = \omega \wedge \bar\partial\alpha^{1,0} = 0$ and that $\partial\bar\partial * (*\alpha^{1,0}) = 0$. This proves the inclusion

$$*(\mathcal{H}^{1,0}_{\bar\partial}) \subseteq \mathcal{H}^{2,1}_{\partial+\bar\partial}.$$

For the opposite inclusion, we need to use the almost Kähler identities [31]. Since the Hodge $*$ is an isomorphism between $(2,1)$-forms and $(1,0)$-forms, any form in

$\mathcal{H}^{2,1}_{\partial+\bar\partial}$ can be written as $\omega\wedge\alpha^{1,0}$ for some $\alpha^{1,0}\in A^{1,0}$. Moreover, since $\bar\partial(\omega\wedge\alpha^{1,0})=0$, the form $\bar\partial\alpha^{1,0}$ is primitive, and since $\partial\bar\partial*(\omega\wedge\alpha^{1,0})=0$, we have $\partial\bar\partial\alpha^{1,0}=0$. By the almost Kähler identities, we have that

$$i\bar\partial^*\bar\partial\alpha^{1,0}=[\Lambda,\partial]\bar\partial\alpha^{1,0}=\Lambda\partial\bar\partial\alpha^{1,0}-\partial\Lambda\bar\partial\alpha^{1,0}=0,$$

where in the last equality we used the fact that $\bar\partial\alpha^{1,0}$ is primitive and $\partial\bar\partial\alpha^{1,0}=0$. This shows that $\bar\partial\alpha^{1,0}=0$ and proves the opposite inclusion. □

By diagram (4.2.1) and Lemma 4.2.2 we conclude that $h^{1,2}_{\bar\partial}$ depends on the choice of almost Kähler metric, while $h^{2,1}_{\bar\partial}$ does not. This gives the situation described in diagram (4.2.2). Finally, by Corollary 4.1.4, it is easy to determine dependence on the metric of the dimensions of the spaces of graded harmonic forms.

Dependence on the metric of the numbers $h^k_{\bar\delta}$, $h^{p,q}_{\bar\delta}$, $h^k_{\delta+\bar\delta}$ and $h^{p,q}_{\delta+\bar\delta}$ The Laplacians associated to the operators δ and $\bar\delta$ do not preserve the bigrading. Hence, there is a substantial difference in considering graded or bigraded invariants. Furthermore, the spaces of $\bar\delta$-harmonic and $(\delta+\bar\delta)$-harmonic forms involve in a non-trivial way the action of μ and $\bar\mu$, so that they differ from the spaces of $\bar\partial$-harmonic and $(\partial+\bar\partial)$-harmonic forms.

Theorem 4.2.3. *Dependence on the metric of the numbers $h^{p,q}_{\bar\delta}$ is described by the diagram*

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}_{\bar\delta} & & h^{0,1}_{\bar\delta} & \\ h^{2,0}_{\bar\delta} & & h^{1,1}_{\bar\delta} & & h^{0,2}_{\bar\delta} \\ & h^{2,1}_{\bar\delta} & & h^{1,2}_{\bar\delta} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & ☺ & & \\ & ☺ & & ! & \\ ☺ & & ☺ & & ☺ \\ & ! & & ☺ & \\ & & ☺ & & \end{array} \tag{4.2.4}$$

Dependence on the metric of the numbers $h^{p,q}_{\delta+\bar\delta}$ is described by the diagram

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0}_{\delta+\bar\delta} & & h^{0,1}_{\delta+\bar\delta} & \\ h^{2,0}_{\delta+\bar\delta} & & h^{1,1}_{\delta+\bar\delta} & & h^{0,2}_{\delta+\bar\delta} \\ & h^{2,1}_{\delta+\bar\delta} & & h^{1,2}_{\delta+\bar\delta} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & ☺ & & \\ & ☺ & & ☺ & \\ ☺ & & ☺ & & ☺ \\ & ! & & ! & \\ & & ☺ & & \end{array} \tag{4.2.5}$$

Dependence on the metric of the numbers $h^k_{\bar\delta}$ and $h^k_{\delta+\bar\delta}$ is described by the table

k	0	1	2	3	4
$h^k_{\bar\delta}$	☺	!	☺	!	☺
$h^k_{\delta+\bar\delta}$	☺	☺	☺	!	☺

In degree 1, the spaces $\mathcal{H}^{1,0}_{\delta+\bar\delta}$, $\mathcal{H}^{0,1}_{\delta+\bar\delta}$ and $\mathcal{H}^{1}_{\delta+\bar\delta}$ are independent of the choice of metric since the equation $\delta\bar\delta * \alpha = 0$ is trivially satisfied for bidegree reasons. A short computation shows the equalities of spaces

$$\mathcal{H}^{1,0}_{\bar\delta} = \mathcal{H}^{1,0}_{\bar\partial} \quad \text{and} \quad \mathcal{H}^{0,1}_{\bar\delta} = \mathcal{H}^{0,1}_{\bar\partial} \cap \ker\mu,$$

that give metric-independence for $h^{1,0}_{\bar\delta}$. We also observe that there are bigraded decompositions

$$\mathcal{H}^{1}_{\bar\delta} = \mathcal{H}^{1,0}_{\bar\delta} \oplus \mathcal{H}^{0,1}_{\bar\delta} \quad \text{and} \quad \mathcal{H}^{3}_{\bar\delta} = \mathcal{H}^{2,1}_{\bar\delta} \oplus \mathcal{H}^{1,2}_{\bar\delta}. \tag{4.2.6}$$

Let $\mathcal{H}^{(2,0)(0,2)}_{J}$ be the space of harmonic forms defined in (3.1.1). In degree 2, we prove the following decompositions of harmonic forms.

Theorem 4.2.4. *Let (M,J) be a compact almost complex 4-manifold. Fix a J-compatible almost Hermitian metric on M. Then:*

- *there is a decomposition of $\bar\delta$-harmonic 2-forms*

$$\mathcal{H}^{2}_{\bar\delta} = \mathcal{H}^{1,1}_{\bar\partial} \oplus \mathcal{H}^{(2,0)(0,2)}_{J};$$

- *there is a decomposition of $(\delta+\bar\delta)$-harmonic 2-forms*

$$\mathcal{H}^{2}_{\delta+\bar\delta} = \mathcal{H}^{1,1}_{\partial+\bar\partial} \oplus \mathcal{H}^{(2,0)(0,2)}_{J}.$$

Proof. It is immediate to verify that the inclusions

$$\mathcal{H}^{1,1}_{\bar\partial} \oplus \mathcal{H}^{(2,0)(0,2)}_{J} \subseteq \mathcal{H}^{2}_{\bar\delta} \quad \text{and} \quad \mathcal{H}^{1,1}_{\partial+\bar\partial} \oplus \mathcal{H}^{(2,0)(0,2)}_{J} \subseteq \mathcal{H}^{2}_{\delta+\bar\delta}$$

hold. For the opposite inclusions, we first prove that $\mathcal{H}^{2}_{\bar\delta} \subseteq \mathcal{H}^{1,1}_{\bar\partial} \oplus \mathcal{H}^{(2,0)(0,2)}_{J}$. Let $\alpha \in \mathcal{H}^{2}_{\bar\delta}$, so that $\bar\delta\alpha = 0$ and $\delta * \alpha = 0$. Writing α as the sum of bigraded forms $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$ and imposing that α is $\bar\delta$-harmonic, we have that

$$\begin{aligned} 0 = \bar\delta\alpha &= (\bar\partial + \mu)(\alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}) = \\ &= \bar\partial\alpha^{2,0} + \bar\partial\alpha^{1,1} + \bar\partial\alpha^{0,2} + \mu\alpha^{2,0} + \mu\alpha^{1,1} + \mu\alpha^{0,2} = \\ &= \bar\partial\alpha^{2,0} + \mu\alpha^{0,2} + \bar\partial\alpha^{1,1}, \end{aligned}$$

since several terms vanish by bidegree reasons. Separating the bidegrees, we obtain that

$$\bar\partial\alpha^{1,1} = 0 \quad \text{and} \quad \bar\partial\alpha^{2,0} + \mu\alpha^{0,2} = 0. \tag{4.2.7}$$

Similarly, since $(2,0)$ and $(0,2)$-forms are self-dual, we see that

$$\begin{aligned}0 = \delta * \alpha &= (\partial + \bar{\mu}) * (\alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}) = \\ &= (\partial + \bar{\mu})(\alpha^{2,0} + *\alpha^{1,1} + \alpha^{0,2}) = \\ &= \partial\alpha^{0,2} + \bar{\mu}\alpha^{2,0} + \partial * \alpha^{1,1},\end{aligned}$$

from which we get the equations

$$\partial * \alpha^{1,1} = 0 \quad \text{and} \quad \partial\alpha^{0,2} + \bar{\mu}\alpha^{2,0} = 0. \tag{4.2.8}$$

Combining (4.2.7) and (4.2.8), we immediately deduce that $\alpha^{1,1} \in \mathcal{H}^{1,1}_{\bar{\partial}}$ and that $\alpha^{2,0} + \alpha^{0,2}$ is d-closed and self-dual, thus d-harmonic, proving the first part of the theorem. For the second part, we prove the inclusion $\mathcal{H}^2_{\delta+\bar{\delta}} \subseteq \mathcal{H}^{1,1}_{\partial+\bar{\partial}} \oplus \mathcal{H}^{(2,0)(0,2)}_J$. Let $\alpha \in \mathcal{H}^2_{\delta+\bar{\delta}}$. Then $\delta\alpha = 0$, $\bar{\delta}\alpha = 0$ and $\delta\bar{\delta} * \alpha = 0$. Writing α as the sum of forms of pure bidegree, following the same computations of the first part of the proof of this theorem and imposing the conditions $\delta\alpha = 0$ and $\bar{\delta}\alpha = 0$, we get that

$$\begin{aligned}\partial\alpha^{1,1} = 0, &\quad (\partial + \bar{\mu})(\alpha^{2,0} + \alpha^{0,2}) = 0, \\ \bar{\partial}\alpha^{1,1} = 0, &\quad (\bar{\partial} + \mu)(\alpha^{2,0} + \alpha^{0,2}) = 0.\end{aligned} \tag{4.2.9}$$

Starting from the equation $\delta\bar{\delta} * \alpha = 0$, we observe that in general

$$\delta\bar{\delta} = (\partial + \bar{\mu})(\bar{\partial} + \mu) = \partial\bar{\partial} + \partial\mu + \bar{\mu}\bar{\partial} + \bar{\mu}\mu, \tag{4.2.10}$$

while on 4-manifolds equation (4.2.10) simplifies to $\delta\bar{\delta} = \partial\bar{\partial} + \bar{\mu}\mu$ and the operator $\delta\bar{\delta}$ has bidegree $(1,1)$. Thus, by bidegree, we have that

$$0 = \delta\bar{\delta} * \alpha = (\partial\bar{\partial} + \bar{\mu}\mu) * \alpha^{1,1} = \partial\bar{\partial} * \alpha^{1,1}. \tag{4.2.11}$$

Finally, from (4.2.9) and (4.2.11), we conclude that $\alpha^{1,1} \in \mathcal{H}^{1,1}_{\partial+\bar{\partial}}$ and $\alpha^{2,0} + \alpha^{0,2} \in \mathcal{H}^{(2,0)(0,2)}_J$. This completes the proof of the theorem. □

The dimension h^-_J of the space $\mathcal{H}^{(2,0)(0,2)}_J$ depends only on the almost complex structure, see Section 3.1. From the decompositions of Theorem 4.2.4 and from the fact that μ and $\bar{\mu}$ vanish on $(1,1)$-forms on 4-manifolds, it follows that $h^{1,1}_{\bar{\delta}}$ and $h^{1,1}_{\delta+\bar{\delta}}$ depend on the metric as $h^{1,1}_{\bar{\partial}}$ and $h^{1,1}_{\partial+\bar{\partial}}$, respectively. In particular, $h^{1,1}_{\bar{\delta}}$ is an almost Kähler invariant, while $h^{1,1}_{\delta+\bar{\delta}}$ is an almost complex invariant. In bidegree $(2,0)$, we have the following lemma.

Lemma 4.2.5. *Let (M,J) be a compact almost complex 4-manifold. For any choice of J-compatible metric, we have that*

$$\mathcal{H}^{2,0}_{\bar{\delta}} = \mathcal{H}^{2,0}_d.$$

Furthermore, the dimension $h^{2,0}_{\bar{\delta}}$ is independent of the choice of metric.

Proof. Let $\alpha \in A^{2,0}$. Since $(2,0)$-forms are self-dual, metric-independence follows by observing that

$$\mathcal{H}_d^{2,0} = A^{2,0} \cap \ker d.$$

To prove the equality of the spaces of harmonic forms, let $\alpha \in \mathcal{H}_{\bar\delta}^{2,0}$. Then $\bar\delta\alpha = 0$ and $\bar\mu * \alpha = 0$. Since α is self-dual, we also have that $\bar\mu\alpha = 0$ and, by bidegree, $\partial\alpha = 0$. In particular, $d\alpha = d * \alpha = 0$. □

Note that by Lemma 5.6 in [31], if J is not integrable then $h_d^{2,0} = h_{\bar\delta}^{2,0} = 0$. By the isomorphisms

$$\mathcal{H}_{\bar\delta}^{2,0} \cong \mathcal{H}_{\bar\delta}^{0,2} \quad \text{and} \quad \mathcal{H}_d^{2,0} \cong \mathcal{H}_d^{0,2}$$

induced by the Hodge $*$ and complex conjugation, we get metric-independence of the numbers $h_{\bar\delta}^{2,0}$, $h_{\bar\delta}^{0,2}$, $h_d^{2,0}$ and $h_d^{0,2}$. This leaves only the numbers $h_{\bar\delta}^{0,1} = h_{\bar\delta}^{2,1}$, $h_{\delta+\bar\delta}^{2,1} = h_{\delta+\bar\delta}^{1,2}$ and $h_{\delta+\bar\delta}^{3}$. Furthermore, since the action of δ and $\bar\delta$ induces a decomposition of 3-forms into bigraded forms and since $\delta\bar\delta$ has bidegree $(1,1)$ on 4-manifolds, we also have $h_{\delta+\bar\delta}^{1,2} = \frac{1}{2}h_{\delta+\bar\delta}^{3}$. Under the almost Kähler assumption, we are able to prove their metric-independence.

Theorem 4.2.6. *Let (M,J) be a compact almost complex 4-manifold admitting a J-compatible almost Kähler metric. Then the numbers $h_{\bar\delta}^{k}$, $h_{\bar\delta}^{p,q}$, $h_{\delta+\bar\delta}^{k}$ and $h_{\delta+\bar\delta}^{p,q}$ do not depend on the choice of J-compatible almost Kähler metric.*

Proof. By Proposition 6.10 in [96], on almost Kähler manifolds we have that $\mathcal{H}_{\bar\delta}^{k} = \mathcal{H}_{\delta+\bar\delta}^{k}$. Therefore, we just need to prove the theorem for $h_{\bar\delta}^{k}$ or $h_{\delta+\bar\delta}^{k}$. As already observed, the number $h_{\delta+\bar\delta}^{1}$ is independent of the choice of almost Hermitian metric. By Theorem 4.2.4, also $h_{\delta+\bar\delta}^{2}$ is metric-independent. Finally, we have that

$$h_{\delta+\bar\delta}^{1} = h_{\bar\delta}^{1} = h_{\bar\delta}^{3} = h_{\delta+\bar\delta}^{3},$$

proving metric-independence of $h_{\bar\delta}^{k} = h_{\delta+\bar\delta}^{k}$. For bigraded invariants, the only number of which we do not know metric-independence is $h_{\bar\delta}^{0,1}$. Let $\alpha \in \mathcal{H}_{\bar\delta}^{0,1}$. Since

$$\mathcal{H}_{\bar\delta}^{0,1} = \mathcal{H}_{\bar\partial}^{0,1} \cap \ker\mu,$$

the only non-trivial differential is $\partial\alpha = \delta\alpha$. We show that actually $\delta\alpha = 0$. This gives the equality $h_{\bar\delta}^{0,1} = h_d^{0,1}$, hence the metric-independence by Corollary 5.9 in [31]. The norm of $\delta\alpha$ is

$$\|\delta\alpha\|^2 = \int_M \delta\alpha \wedge \overline{*\delta\alpha} = \int_M (\delta+\bar\delta)\alpha \wedge *(\delta+\bar\delta)\bar\alpha = \int_M d\alpha \wedge *d\bar\alpha.$$

The form $d\bar\alpha$ is a primitive $(1,1)$-form, hence anti-self-dual, since $d\bar\alpha = \bar\partial\bar\alpha$ and

$$\omega \wedge \bar\partial\bar\alpha = \bar\partial(\omega\wedge\bar\alpha) = \bar\partial * \bar\alpha = \overline{\partial * \alpha} = 0$$

by the vanishing of the differentials on α. By Stokes's theorem, we conclude that

$$\|\delta\alpha\|^2 = \int_M d\alpha \wedge *d\bar{\alpha} = \int_M d\alpha \wedge *d\bar{\alpha} = 0. \qquad \square$$

This proves the situation described in the diagrams (4.2.4) and (4.2.5). The dependence on the metric of the spaces of graded harmonic forms follows by the direct sum decompositions of equation (4.2.6) and of Theorem 4.2.4.

Dependence on the metric of the numbers $h_d^{p,q}$, $h_{d+d^c}^k$ and $h_{d+d^c}^{p,q}$ The numbers $h_d^{p,q}$ and $h_{d+d^c}^{p,q}$ are symmetric in p and q, while $h_d^{p,q}$ satisfy the additional symmetry $h_d^{p,q} = h_d^{m-p,m-q}$.

Theorem 4.2.7. *Dependence on the metric of the numbers $h_d^{p,q}$ is described by the diagram*

$$\begin{array}{ccccc} & & 1 & & \\ & h_d^{1,0} & & h_d^{0,1} & \\ h_d^{2,0} & & h_d^{1,1} & & h_d^{0,2} \\ & h_d^{2,1} & & h_d^{1,2} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & \text{☺} & & \\ & \text{😐} & & \text{😐} & \\ \text{☺} & & \text{😐} & & \text{☺} \\ & \text{😐} & & \text{😐} & \\ & & \text{☺} & & \end{array} \tag{4.2.12}$$

Dependence on the metric of the numbers $h_{d+d^c}^{p,q}$ is described by the diagram

$$\begin{array}{ccccc} & & 1 & & \\ & h_{d+d^c}^{1,0} & & h_{d+d^c}^{0,1} & \\ h_{d+d^c}^{2,0} & & h_{d+d^c}^{1,1} & & h_{d+d^c}^{0,2} \\ & h_{d+d^c}^{2,1} & & h_{d+d^c}^{1,2} & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & \text{☺} & & \\ & \text{☺} & & \text{☺} & \\ \text{☺} & & \text{☺} & & \text{☺} \\ & ! & & ! & \\ & & \text{☺} & & \end{array} \tag{4.2.13}$$

Dependence on the metric of the numbers $h_{d+d^c}^k$ is described by the table

k	0	1	2	3	4
$h_{d+d^c}^k$	☺	☺	☺	!	☺

In degree 1, the numbers $h_{d+d^c}^{1,0}$ and $h_{d+d^c}^{0,1}$ do not depend on the choice of metric for degree reasons and the action of d and d^c induces a bigraded decomposition of the space $\mathcal{H}_{d+d^c}^1$ into bigraded forms, that gives $h_{d+d^c}^{1,0} = h_{d+d^c}^{1,0} = \frac{1}{2} h_{d+d^c}^1$. By Theorem 4.1 in [51], the numbers $h_d^{1,0}$ and $h_d^{0,1}$ depend on the choice of metric and by Corollary 5.9 in [31], they do not depend on the choice of almost Kähler metric.

In degree 2, the numbers $h_d^{2,0}$ and $h_d^{0,2}$ do not depend on the choice of metric by Lemma 4.2.5. Let $\alpha \in \mathcal{H}_d^{2,0}$. Then $d\alpha = 0$. Since α is anti-self-dual, we have that

$$(dd^c)^*\alpha = *d^c d * \alpha = *d^c d\alpha = 0.$$

Thus, there are equalities of spaces

$$\mathcal{H}^{2,0}_{d+d^c} = A^{2,0} \cap \ker d = \mathcal{H}^{2,0}_d,$$

which imply metric-independence for the numbers $h^{2,0}_{d+d^c}$ and $h^{0,2}_{d+d^c}$. To show that also $h^2_{d+d^c}$ is metric-independent, we establish a decomposition for $\mathcal{H}^2_{d+d^c}$. Consider the decomposition of ω with respect to the Hodge Laplacian

$$\omega = h(\omega) + d\eta + d^*\mu, \tag{4.2.14}$$

where $h(\omega)$ is d-harmonic, $\eta \in A^1$ and $\mu \in A^3$. Define the 2-form γ_0 as

$$\gamma_0 := -d * \mu - d^*\mu. \tag{4.2.15}$$

Note that we have

$$*\gamma_0 = - * d * \mu + *^2 d * \mu = d^*\mu + d * \mu = -\gamma_0, \tag{4.2.16}$$

hence γ_0 is anti-self-dual. Since anti-self-dual forms have necessarily bidegree $(1,1)$, we also have that

$$J\gamma_0 = \gamma_0. \tag{4.2.17}$$

Theorem 4.2.8. *Let $(M, J, \tilde{\omega}, \tilde{g})$ be a compact almost Hermitian 4-manifold and let (ω, g) be a Gauduchon metric in the same conformal class of $(\tilde{\omega}, \tilde{g})$. Then*

$$\mathcal{H}^2_{d+d^c} = \mathbb{C}\langle \omega + \gamma_0 \rangle \oplus \mathcal{H}^-_g \oplus \mathcal{H}^{(2,0)(0,2)}_J.$$

In particular, $h^2_{d+d^c} = b^- + 1 + h^-_J$ and it is metric independent.

Proof. Let g be a metric in the same conformal class of $\tilde{g}$. If $\alpha \in A^2$, then $*_g\alpha = *_{\tilde{g}}\alpha$. As a consequence, the space $\mathcal{H}^2_{d+d^c}$ is invariant under conformal changes of metric. In each conformal class of metric there always exists a *Gauduchon metric* [40], i.e., a metric for which $dd^c\omega = d^c d\omega = 0$. Therefore, we assume that g is a Gauduchon metric in the same conformal class of $\tilde{g}$.

We first prove the inclusion $\mathbb{C}\langle \omega + \gamma_0 \rangle \oplus \mathcal{H}^-_g \oplus \mathcal{H}^{(2,0)(0,2)}_J \subseteq \mathcal{H}^2_{d+d^c}$. The form $\omega + \gamma_0$ is d-closed since

$$d(\omega + \gamma_0) = d(h(\omega) + d\eta - d * \mu) = 0,$$

it is d^c-closed by (4.2.17) and it is $(dd^c)^*$-closed by (4.2.16) and because the metric is Gauduchon. Forms in $\mathcal{H}^-_g \oplus \mathcal{H}^{(2,0)(0,2)}_J$ are necessarily $(d + d^c)$-harmonic since they are d-harmonic and have bidegree either $(1,1)$ or $(2,0) + (0,2)$.
We now prove the opposite inclusion. Let $\alpha \in \mathcal{H}^2_{d+d^c}$. Write α using the Lefschetz and bidegree decomposition as

$$\alpha = f\omega + \gamma^{1,1} + \gamma^{(2,0)(0,2)},$$

with $\gamma^{1,1}$ and $\gamma^{(2,0)(0,2)}$ primitive forms and $f \in C^\infty(M)$. Since $d\alpha = 0$ and $d^c\alpha = 0$, we have that

$$d(f\omega + \gamma^{1,1}) = 0 \tag{4.2.18}$$

and

$$d\gamma^{(2,0)(0,2)} = 0. \tag{4.2.19}$$

The form $\gamma^{(2,0)(0,2)}$ is d-closed and primitive, thus d-harmonic, which implies that $\gamma^{(2,0)(0,2)} \in \mathcal{H}_J^{(2,0)(0,2)}$. On the other side, we have that

$$\begin{aligned} 0 = (dd^c)^*\alpha &= - * d^c d * (f\omega + \gamma^{1,1} + \gamma^{(2,0)(0,2)}) = \\ &= - * d^c d(f\omega - \gamma^{1,1} + \gamma^{(2,0)(0,2)}) = \\ &= -2 * d^c d(f\omega), \end{aligned}$$

where in the last equality we used (4.2.18) and (4.2.19). Since the metric is Gauduchon, we have that

$$0 = d^c d(f\omega) = d^c df \wedge \omega - df \wedge d^c\omega + d^c f \wedge d\omega.$$

Consider the real operator $P \colon C^\infty(M) \to C^\infty(M)$ given by

$$P(f) := *(d^c df \wedge \omega - df \wedge d^c\omega + d^c f \wedge d\omega).$$

By the same argument of the proof of Theorem 4.3 in [78], P is strongly elliptic, thus f must be constant. Consider the form

$$\beta^{1,1} := \gamma^{1,1} - f\gamma_0.$$

The form $\beta^{1,1}$ is anti-self-dual by (4.2.16), and it is also d-closed by (4.2.14), (4.2.15) and (4.2.18). Therefore, it is d-harmonic and $\beta^{1,1} \in \mathcal{H}_g^-$. The claim follows writing α as

$$\alpha = f(\omega + \gamma_0) + \beta^{1,1} + \gamma^{(2,0)(0,2)},$$

with f constant, $\beta^{1,1} \in \mathcal{H}_g^-$ and $\gamma^{(2,0)(0,2)} \in \mathcal{H}_J^{(2,0)(0,2)}$. □

The space $\mathcal{H}^{1,1}_{d+d^c}$ coincides with the $(1,1)$-bidegree part of the decomposition of Theorem 4.2.8, hence $h^{1,1}_{d+d^c} = b^- + 1$ and it is metric-independent. For the space $\mathcal{H}^{1,1}_d$, by Theorem 5.7 in [31] or by [53], the dimension $h^{1,1}_d$ is independent of the choice of almost Kähler metric since it coincides with $h^{1,1}_{\bar\partial}$. However, in general it depends on the conformal class of metric. This is the content of Theorem 3.1 in [48], of which we give now an alternative proof.

Theorem 4.2.9. *Let (M, J) be a compact almost complex 4-manifold. Let g be a J-compatible metric. Then*

$$h^{1,1}_d = \begin{cases} b^- + 1 & \text{if } g \text{ is conformally almost Kähler,} \\ b^- & \text{otherwise.} \end{cases}$$

Proof. The spaces $\mathcal{H}_d^{1,1}$ and $\mathcal{H}_{d+d^c}^{1,1}$ are invariant under conformal changes of metric. Hence, we can suppose that g is a Gauduchon metric. Let $\alpha \in \mathcal{H}_d^{1,1}$. From the equations $d\alpha = 0$ and $d * \alpha = 0$, we have that $d^c d * \alpha = 0$, which shows the inclusion $\mathcal{H}_d^{1,1} \subseteq \mathcal{H}_{d+d^c}^{1,1}$. By the decomposition of Theorem 4.2.8, we have the chain of inclusions

$$\mathcal{H}_d^{1,1} \subseteq \mathcal{H}_{d+d^c}^{1,1} = \mathcal{H}_g^- \oplus \mathbb{C}\langle \omega + \gamma_0 \rangle.$$

Suppose that $h_d^{1,1} = b^- + 1$. Then there is an equality of vector spaces $\mathcal{H}_d^{1,1} = \mathcal{H}_{d+d^c}^{1,1}$ by dimensional reasons. In particular, the form $\omega + \gamma_0$ is d-closed and d^*-closed. By (4.2.16), we have

$$d(\omega + \gamma_0) = 0 \quad \text{and} \quad d(\omega - \gamma_0) = 0,$$

which implies that $d\omega = 0$ and that the Gauduchon metric is actually almost Kähler. Conversely, if g is conformally equivalent to an almost Kähler metric, then the almost Kähler metric and the Gauduchon metric in the conformal class of g coincide by uniqueness of the Gauduchon metric [40]. In this case, the form $\omega + \gamma_0$ is d-harmonic and $\mathcal{H}_{d+d^c}^{1,1} = \mathcal{H}_d^{1,1}$ since $\mathcal{H}_g^-$ is trivially contained in $\mathcal{H}_d^{1,1}$. This gives the equality $h_d^{1,1} = h_{d+d^c}^{1,1} = b^- + 1$. □

This proves the situation described in diagram (4.2.12). In degree 3, there is no general result for the numbers $h_{d+d^c}^{2,1}$, $h_{d+d^c}^{1,2}$ and $h_{d+d^c}^{3}$. Nevertheless, if the metric is almost Kähler they are all metric-independent. The result for $h_{d+d^c}^{3}$ is a consequence of Corollary 4.3.4, that gives the equality $h_{d+d^c}^{3} = b_1$. In bidegrees $(2,1)$ and $(1,2)$, it is a consequence of the following lemma.

Lemma 4.2.10. *Let (M, J) be a compact almost complex 4-manifold. For any choice of J-compatible almost Kähler metric there is an isomorphism*

$$\mathcal{H}_{d+d^c}^{2,1} \cong \mathcal{H}_{\bar\partial}^{1,0}.$$

In particular, the numbers $h_{d+d^c}^{2,1}$ and $h_{d+d^c}^{1,2}$ do not depend on the choice of almost Kähler metric.

Proof. The isomorphism between $\mathcal{H}_{d+d^c}^{2,1}$ and $\mathcal{H}_{\bar\partial}^{1,0}$ is given by the Hodge $*$ operator. Let $\alpha^{1,0} \in \mathcal{H}_{\bar\partial}^{1,0}$. By the proof of Corollary 5.9 in [31], we have that $d\alpha^{1,0} = 0$. Up to a constant, the Hodge $*$ acts as wedge product by ω. Thus we have

$$d * \alpha^{1,0} = d(\omega \wedge \alpha^{1,0}) = \omega \wedge d\alpha^{1,0} = 0.$$

Clearly, we have that $d^c * \alpha^{1,0} = 0$, while

$$d^c d * (*\alpha^{1,0}) = d^c d\alpha^{1,0} = 0.$$

This proves the inclusion $*\mathcal{H}^{1,0}_{\bar\partial} \subseteq \mathcal{H}^{2,1}_{d+d^c}$. For the opposite inclusion, let $\alpha^{2,1} \in \mathcal{H}^{2,1}_{d+d^c}$. The action of the Hodge $*$ send $\alpha^{2,1}$ to $\alpha^{1,0}$, with $\alpha^{2,1} = \omega \wedge \alpha^{1,0}$. From the equation

$$0 = d\alpha^{2,1} = d(\omega \wedge \alpha^{1,0}) = \omega \wedge \bar\partial\alpha^{1,0},$$

we see that $\bar\partial\alpha^{1,0}$ is primitive, hence it is anti-self-dual. From the equation $d^c d * \alpha^{2,1} = 0$, we see that

$$0 = dJd\alpha^{1,0} = 2(\partial\bar\partial + \bar\partial^2)\alpha^{1,0}.$$

Taking only the term in bidegree $(2,1)$ and using anti-self-duality of $\bar\partial\alpha^{1,0}$, we conclude that

$$0 = *\partial\bar\partial\alpha^{1,0} = - * \partial * \bar\partial\alpha^{1,0} = \bar\partial^*\bar\partial\alpha^{1,0},$$

so that $\bar\partial\alpha^{1,0} = 0$ and $\alpha^{1,0} \in \mathcal{H}^{1,0}_{\bar\partial}$. □

With this we have shown the situation described in diagram (4.2.13). The claim for $h^k_{d+d^c}$ follows easily.

We conclude this paragraph with an alternative proof of the fact that, on almost Kähler 4-manifolds, there is an equality $\mathcal{H}^{1,0}_{\bar\partial} = \mathcal{H}^{1,0}_d$. The original proof is due to Cirici and Wilson [31], see also Lemma 4.1 in [61].

Lemma 4.2.11. *Let (M, J) be a compact almost complex 4-manifold endowed with an almost Kähler metric. Then*

$$A^{1,0} \cap \ker\bar\partial \subseteq A^{1,0} \cap \ker d \cap \ker d^*.$$

Proof. Note that a $(1,0)$-form α is d-harmonic if and only if $\bar\partial\alpha = \partial\alpha = \bar\mu\alpha = \partial^*\alpha = 0$. Indeed, the remaining equations are automatically satisfied by bidegree reasons. Suppose that $\bar\partial\alpha = 0$. We have to prove that $\partial\alpha = \bar\mu\alpha = \partial^*\alpha = 0$. Observe that $\partial\alpha$ has bidegree $(2,0)$, while $\bar\mu\alpha$ has bidegree $(0,2)$. Hence $\delta\alpha = 0$ if and only if $\partial\alpha = 0$ and $\bar\mu\alpha = 0$. We have that

$$\|\delta\alpha\|^2 = \int_M \delta\alpha \wedge \overline{*\delta\alpha} = \int_M (\delta + \bar\delta)\alpha \wedge *(\delta + \bar\delta)\bar\alpha = \int_M d\alpha \wedge *d\bar\alpha,$$

where in the second equality we used the fact that $\bar\delta\alpha = \bar\partial\alpha = 0$. Since $d\bar\alpha = \bar\partial\bar\alpha + \mu\bar\alpha$ has bidegree $(2,0)+(0,2)$ for any choice of almost Hermitian metric, $d\bar\alpha$ is necessarily a self-dual form, and we have

$$\|\delta\alpha\|^2 = \int_M d\alpha \wedge *d\bar\alpha = \int_M d\alpha \wedge d\bar\alpha = \int_M d(\alpha \wedge d\bar\alpha) = 0,$$

by Stokes' theorem, showing that $\partial\alpha = \bar\mu\alpha = 0$. For the last equation, we have that

$$\partial^*\alpha = - * \bar\partial * \alpha = i * \bar\partial(\omega \wedge \alpha),$$

where ω is the fundamental form of the almost Kähler metric. Finally, we compute that

$$\bar{\partial}(\omega \wedge \alpha) = \bar{\partial}\omega \wedge \alpha + \omega \wedge \bar{\partial}\alpha = 0,$$

completing the proof of the lemma. Note that this last equation is the only instance in the proof where we use that the metric is almost Kähler. □

Dependence on the metric of the numbers $h^k_{d+d^\Lambda}$

On almost Hermitian 4-manifolds, the problem of metric-independence for the numbers $h^k_{d+d^\Lambda}$ reduces to computing the spaces of $(d+d^\Lambda)$-harmonic primitive forms. Clearly, we have $h^0_{d+d^\Lambda} = h^4_{d+d^\Lambda} = 1$. Every 1-form is primitive, and we have the following proposition valid in arbitrary dimension.

Proposition 4.2.12. *Let (M, J, ω, g) be a compact almost Hermitian $2m$-manifold. Then there is an inclusion $\mathcal{H}^1_{d+d^\Lambda} \subseteq \mathcal{H}^1_d$. If $d\omega = 0$, then $\mathcal{H}^1_{d+d^\Lambda} = \mathcal{H}^1_d$.*

Proof. Let $\alpha \in \mathcal{H}^1_{d+d^\Lambda}$. Then α is d-closed and $(dd^\Lambda)^*$-closed. From the equation $(dd^\Lambda)^*\alpha = 0$, we have that

$$0 = dJ * d * \alpha = -dJd^*\alpha = -dd^*\alpha,$$

since J acts trivially on functions. Therefore, we have that $d^*\alpha = 0$ and that α is d-harmonic. If $d\omega = 0$, the opposite inclusion also holds. Indeed, let $\alpha \in \mathcal{H}^1_d$. Then we immediately have $d\alpha = 0$ and $(dd^\Lambda)^*\alpha = d^c d^*\alpha = 0$. Moreover, since $d\omega = 0$, we also have $d^\Lambda\alpha = (d\Lambda - \Lambda d)\alpha = 0$ since α is d-closed and has degree 1. □

Remark 4.2.13. *The equality $h^1_{d+d^\Lambda} = b_1$ on symplectic manifolds is a well-known fact, see Lemma 2.7 in [13].*

In degree 2, there is a decomposition between primitive forms and multiples of the fundamental form ω.

Theorem 4.2.14. *Let (M, J, ω, g) be a compact almost Hermitian 4-manifold and let $\mathcal{PH}^2_{d+d^\Lambda}$ be the space of $(d+d^\Lambda)$-harmonic primitive 2-forms. Then*

$$\mathcal{H}^2_{d+d^\Lambda} = \begin{cases} \mathbb{C}\langle\omega\rangle \oplus \mathcal{PH}^2_{d+d^\Lambda} & \text{if } d\omega = 0, \\ \mathcal{PH}^2_{d+d^\Lambda} & \text{if } d\omega \neq 0. \end{cases}$$

Proof. The inclusions $\mathbb{C}\langle\omega\rangle \oplus \mathcal{PH}^2_{d+d^\Lambda} \subseteq \mathcal{H}^2_{d+d^\Lambda}$ and $\mathcal{PH}^2_{d+d^\Lambda} \subseteq \mathcal{H}^2_{d+d^\Lambda}$ are immediate. For the opposite inclusions, let $\alpha \in \mathcal{H}^2_{d+d^\Lambda}$ and let

$$\alpha = f\omega + \gamma^{1,1} + \gamma^{(2,0)(0,2)}$$

be its Lefschetz and bidegree decomposition. Since α is d-closed and d^Λ-closed, we have that

$$0 = dJ * \alpha = d(f\omega - \gamma^{1,1} - \gamma^{(2,0)(0,2)}) = d(2f\omega - \alpha) = 2d(f\omega). \qquad (4.2.20)$$

Thus, we have $dd^c(f\omega) = d^c d(f\omega) = 0$ and, with the same argument of the proof of Theorem 4.2.8, we deduce that f is constant. Since $f\omega$ is d-closed, d^Λ closed and self-dual, it is also $(d + d^\Lambda)$-harmonic. In particular, the form $\gamma^{1,1} + \gamma^{(2,0)(0,2)}$ is primitive and $(d + d^\Lambda)$-harmonic. If $d\omega = 0$, we obtain that

$$\mathcal{H}^2_{d+d^\Lambda} = \mathbb{C}\langle\omega\rangle \oplus \mathcal{P}\mathcal{H}^2_{d+d^\Lambda},$$

proving the first part of the theorem. For the second part, by (4.2.20) we have that

$$0 = d(f\omega) = f d\omega,$$

with f constant. In particular, if $d\omega \neq 0$, then $f = 0$. □

4.3 Relations among spaces of harmonic forms

In this section we prove general inclusions or equalities among the spaces of harmonic forms introduced in Section 4.1 that will be used in Section 4.5 and Chapter 5 for explicit computations. Sometimes we require additional assumptions on the integrability of the involved structures.

We begin with a series of equalities whose proof follows easily from the explicit description of the spaces of harmonic forms.

Proposition 4.3.1. *Let (M, J, ω, g) be a compact almost Hermitian $2m$-manifold. Then there are equalities of spaces*

$$\mathcal{H}^k_d \cap \mathcal{H}^k_{d^c} = \mathcal{H}^k_\delta \cap \mathcal{H}^k_{\bar\delta}, \quad \mathcal{H}^k_{d+d^c} \cap \mathcal{H}^k_{\delta+\bar\delta} = \mathcal{H}^k_{d^c+d} \cap \mathcal{H}^k_{\delta+\bar\delta}$$

and

$$\mathcal{H}^k_{dd^c} \cap \mathcal{H}^k_{\delta\bar\delta} = \mathcal{H}^k_{d^c d} \cap \mathcal{H}^k_{\delta\bar\delta}.$$

We now establish a series of inclusions of spaces of harmonic forms.

Theorem 4.3.2. *Let (M, J, ω, g) be a compact almost Kähler $2m$-manifold. Then there is an injection*

$$\mathcal{H}^k_{d+d^c} \hookrightarrow \mathcal{H}^k_{d+d^\Lambda}.$$

In particular, we have that $h^k_{d+d^c} \leq h^k_{d+d^\Lambda}$.

Proof. Let $\alpha \in \mathcal{H}^k_{d+d^c}$. Since α is d-closed, d^c-closed and $(dd^c)^*$-closed, we have that

$$0 = (d^c d)^*(J\alpha) = d^* d^\Lambda(J\alpha) = d^*(d\Lambda - \Lambda d)(J\alpha) = d^* d(\Lambda J\alpha),$$

which implies $d\Lambda(J\alpha) = d^\Lambda(J\alpha) = 0$. Furthermore, we also have $d(J\alpha) = 0$. In particular, the form $J\alpha$ is both d-closed and d^Λ-closed, and it defines a symplectic cohomology class $[J\alpha]_{d+d^\Lambda} \in H^k_{d+d^\Lambda}$. Taking the $(d + d^\Lambda)$-harmonic representative of $[J\alpha]_{d+d^\Lambda}$, we have a well-defined map $\mathcal{H}^k_{d+d^c} \to \mathcal{H}^k_{d+d^\Lambda}$. The map is injective because if $J\alpha = d^\Lambda d\beta$ for some $\beta \in A^k$, then

$$0 = -Jd\alpha = d^c(J\alpha) = (d^\Lambda)^* d^\Lambda d\beta,$$

giving $J\alpha = d^\Lambda d\beta = 0$. □

The opposite inclusion in general does not hold. For instance, one can endow the Kodaira–Thurston manifold with an almost Kähler structure such that $h^1_{d+d^c} = 2$ and $h^1_{d+d^\Lambda} = b_1 = 3$, see Proposition 4.5.1. Nevertheless, we can prove the opposite inclusion on $(2m-1)$-forms.

Theorem 4.3.3. *Let (M, J, g, ω) be a compact almost Hermitian $2m$-manifold. Then we have*

$$\mathcal{H}^{2m-1}_{d^\Lambda+d} \subseteq \mathcal{H}^{2m-1}_{d^c+d}.$$

If in addition $d\omega = 0$, then $\mathcal{H}^{2m-1}_{d+d^\Lambda} = \mathcal{H}^{2m-1}_{d^\Lambda+d} = \mathcal{H}^{2m-1}_{d+d^c}$.

Proof. Let $\alpha \in \mathcal{H}^{2m-1}_{d^\Lambda+d}$. Since α is $(d^\Lambda d)^*$-closed, we have that

$$0 = d^* J^{-1} dJ\alpha = d^* dJ\alpha,$$

where we used the fact that J acts trivially on top-forms. Hence, we conclude that $d^c\alpha = 0$. By the equation $d^\Lambda\alpha = 0$, we have that $dJ * \alpha = 0$. Thus $J\alpha$ is $(d + d^c)$-harmonic, since

$$(dd^c)^* J\alpha = d^\Lambda d^* J\alpha = 0,$$

so that α is $(d^c + d)$-harmonic. This proves the inclusion $\mathcal{H}^{2m-1}_{d^\Lambda+d} \subseteq \mathcal{H}^{2m-1}_{d^c+d}$. If $d\omega = 0$, by Theorem 4.3.2 we have that

$$\mathcal{H}^{2m-1}_{d^\Lambda+d} \subseteq \mathcal{H}^{2m-1}_{d^c+d} \cong \mathcal{H}^{2m-1}_{d+d^c} \hookrightarrow \mathcal{H}^{2m-1}_{d+d^\Lambda} = \mathcal{H}^{2m-1}_{d^\Lambda+d},$$

giving the equality of the spaces and concluding the proof. □

Corollary 4.3.4. *Let (M, J, ω, g) be a compact almost Hermitian $2m$-manifold. Then we have $h^1_{d+d^\Lambda} \leq h^{2m-1}_{d+d^c}$. If $d\omega = 0$, then we have $h^{2m-1}_{d+d^c} = b_1$.*

Proposition 4.3.5. *Let (M, J, g, ω) be a compact almost Kähler 4-manifold. Then we have*

$$\mathcal{H}^2_{d+d^c} \subseteq \mathcal{H}^2_{d+d^\Lambda}.$$

The inclusion can be strict.

Proof. Let $\alpha \in \mathcal{H}^2_{d+d^c}$. Using the decomposition of Theorem 4.2.8, we can write

$$\alpha = c\,\omega + \gamma^{1,1} + \alpha^{2,0} + \alpha^{0,2},$$

with c constant, $\gamma^{1,1}$ primitive and anti-self-dual and $\alpha^{2,0} + \alpha^{0,2}$ primitive and self-dual. Then we have

$$d^\Lambda \alpha = (d\Lambda - \Lambda d)\alpha = dc = 0$$

and

$$dd^\Lambda * \alpha = dd^\Lambda(c\omega - \gamma^{1,1} + \alpha^{2,0} + \alpha^{0,2}) = dd^\Lambda(\alpha - 2\gamma^{1,1}) = -2dd^\Lambda\gamma^{1,1} = 0,$$

so that $\alpha \in \mathcal{H}^2_{d+d^\Lambda}$. For the second part of the proposition, we give an explicit example. On the Kodaira–Thurston manifold endowed with the almost Kähler structure of Section 4.5, we have that $h^1_{d+d^c} = 2$ and $h^2_{d+d^c} = 4$, while $h^1_{d+d^\Lambda} = 3$ and $h^2_{d+d^\Lambda} = 5$, see Example 3.4 in [100]. □

Theorem 4.3.6. *Let (M, J, ω, g) be a compact almost Kähler $2m$-manifold. Suppose that we have $\mathcal{H}^k_{d^c} \subseteq \mathcal{H}^k_{d+d^c}$ for some k. Then $\mathcal{H}^k_{d+d^c} = \mathcal{H}^k_d$ and $h^k_{d+d^c} = b_k$.*

Proof. Let $\alpha \in \mathcal{H}^k_{d+d^c}$ and let

$$\alpha = h_{d^c}(\alpha) + d^c\eta + (d^c)^*\gamma$$

be its Hodge decomposition with respect to Δ_{d^c}. Since α is d^c-closed, we have that $(d^c)^*\gamma = 0$. The form

$$d^c\eta = \alpha - h_{d^c}(\alpha)$$

is $(d + d^c)$-harmonic because it is the difference of two harmonic forms by the assumption on $\mathcal{H}^k_{d^c}$. From the equation $(dd^c)^*\alpha = 0$, we deduce that

$$(d^c)^*d^*d^c\eta = (d^c)^*d^*(\alpha - h_{d^c}(\alpha)) = 0.$$

By (1.2.15) and $d\omega = 0$, we have that

$$0 = (d^c)^*d^*d^c\eta = d^\Lambda d^*(d^\Lambda)^*\eta = -d^\Lambda(d^\Lambda)^*d^*\eta,$$

which implies $\langle d^\Lambda(d^\Lambda)^*d^*\eta, d^*\eta\rangle = 0$. Therefore $(d^\Lambda)^*d^*\eta = -d^*(d^\Lambda)^*\eta = 0$. Finally, we have that $d\alpha = 0$ since $\alpha \in \mathcal{H}^k_{d+d^\Lambda}$ and that $d^*\alpha = d^*(d^\Lambda)^*\eta = 0$. This gives $\alpha \in \mathcal{H}^k_d$ and $\mathcal{H}^k_{d+d^c} \subseteq \mathcal{H}^k_d$. To conclude, observe that

$$\mathcal{H}^k_{d+d^c} \subseteq \mathcal{H}^k_d \cong \mathcal{H}^k_{d^c} \subseteq \mathcal{H}^k_{d+d^c},$$

which gives the equality of the spaces. □

We also have a Hermitian counterpart of Theorem 4.3.6 that allows to explicitly compute the numbers $h^k_{d+d^\Lambda}$ when J is integrable.

Theorem 4.3.7. *Let (M, J, ω, g) be a compact Hermitian $2m$-manifold. Suppose that we have $\mathcal{H}^k_{d^\Lambda} \subseteq \mathcal{H}^k_{d+d^\Lambda}$ for some k. Then $\mathcal{H}^k_{d+d^\Lambda} = \mathcal{H}^k_d$ and $h^k_{d+d^\Lambda} = b_k$.*

Proof. The proof follows closely that of Theorem 4.3.6, replacing d^c by $(d^c)^*$. Let $\alpha \in \mathcal{H}^k_{d+d^\Lambda}$ and let

$$\alpha = h_{d^\Lambda}(\alpha) + d^\Lambda\eta + (d^\Lambda)^*\gamma$$

be its Hodge decomposition with respect to Δ_{d^Λ}. Since α is d^Λ-closed, we have $(d^\Lambda)^*\gamma = 0$. From the equation $(dd^\Lambda)^*\alpha = 0$, we deduce that

$$d^c d^* d^\Lambda\eta = d^c d^*(\alpha - h_{d^\Lambda}(\alpha)) = 0.$$

By (1.2.15) and integrability of J, we have

$$0 = d^c d^* d^\Lambda\eta = d^c d^*(d^c)^*\eta = -d^c(d^c)^* d^*\eta,$$

which implies $(d^c)^* d^*\eta = -d^*(d^c)^*\eta = 0$. Since $d\alpha = 0$, $d^*\alpha = d^*(d^c)^*\eta = 0$ and

$$\mathcal{H}^k_{d+d^\Lambda} \subseteq \mathcal{H}^k_d \cong \mathcal{H}^k_{d^\Lambda} \subseteq \mathcal{H}^k_{d+d^\Lambda},$$

the theorem is proved. □

We conclude this section with a decomposition for $\mathcal{H}^3_{\delta+\bar\delta}$ valid on almost Hermitian 4-manifolds.

Lemma 4.3.8. *Let (M, J) be a compact almost complex 4-manifold. Then for every choice of J-compatible Hermitian metric we have that*

$$\mathcal{H}^3_{\delta+\bar\delta} = \mathcal{H}^{2,1}_{\partial+\bar\partial} \oplus \overline{\mathcal{H}^{2,1}_{\partial+\bar\partial}}.$$

Proof. Let $\alpha \in \mathcal{H}^{2,1}_{\partial+\bar\partial}$. Then $\bar\partial\alpha = 0$ and $\partial\bar\partial * \alpha = 0$. Observe that

$$\delta\alpha = (\partial + \bar\mu)\alpha = 0$$

by bidegree reasons and that

$$\bar\delta\alpha = (\bar\partial + \mu)\alpha = \bar\partial\alpha = 0$$

by bidegree reasons and $\bar\partial\alpha = 0$. By (4.2.11), we have that

$$\delta\bar\delta * \alpha = (\partial\bar\partial + \bar\mu\mu) * \alpha = \partial\bar\partial * \alpha = 0$$

since $\mu\alpha = 0$. This shows the inclusion $\mathcal{H}^{2,1}_{\partial+\bar\partial} \subseteq \mathcal{H}^3_{\delta+\bar\delta}$. Noting that the equations $\delta\alpha = 0$, $\bar\delta\alpha = 0$, $\delta\bar\delta * \alpha = 0$ are symmetric by complex conjugation, we also have $\overline{\mathcal{H}^{2,1}_{\partial+\bar\partial}} \subseteq \mathcal{H}^3_{\delta+\bar\delta}$. For the opposite inclusion

$$\mathcal{H}^3_{\delta+\bar\delta} \subseteq \mathcal{H}^{2,1}_{\partial+\bar\partial} \oplus \overline{\mathcal{H}^{2,1}_{\partial+\bar\partial}},$$

let $\alpha \in \mathcal{H}^3_{\delta+\bar\delta}$. Write α as the sum of bigraded forms $\alpha = \alpha^{2,1} + \alpha^{1,2}$. By bidegree reasons and the equation $\delta\alpha = 0$, we have

$$0 = \delta\alpha = (\partial + \bar\mu)(\alpha^{2,1} + \alpha^{1,2}) = \partial\alpha^{1,2}.$$

Similarly, from the equation $\bar\delta\alpha = 0$, we deduce that $\bar\partial\alpha^{2,1} = 0$. Finally, from the equation $\delta\bar\delta * \alpha = 0$, we get

$$0 = \delta\bar\delta * \alpha = (\partial\bar\partial + \bar\mu\mu) * (\alpha^{2,1} + \alpha^{1,2}).$$

Since $\delta\bar\delta$ has bidegree $(1,1)$, we can separate the bidegrees to get two equations

$$\begin{cases} \partial\bar\partial + \bar\mu\mu * \alpha^{2,1} = 0, \\ \partial\bar\partial + \bar\mu\mu * \alpha^{1,2} = 0. \end{cases}$$

Observing that $\bar\mu\mu * \alpha^{2,1} = 0$ (for bidegree reasons), that $\mu\bar\mu * \alpha^{1,2} = 0$ (bidegree reasons) and that $\partial\bar\partial + \bar\mu\mu = -\bar\partial\partial - \mu\bar\mu$, all of our equations reduce to

$$\begin{cases} \bar\partial\alpha^{2,1} = 0, \\ \partial\bar\partial * \alpha^{2,1} = 0, \end{cases} \qquad \begin{cases} \partial\alpha^{1,2} = 0, \\ \bar\partial\partial * \alpha^{1,2} = 0, \end{cases}$$

proving that $\alpha^{2,1} \in \mathcal{H}^{2,1}_{\partial+\bar\partial}$ and $\overline{\alpha^{1,2}} \in \mathcal{H}^{2,1}_{\partial+\bar\partial}$, and thus our lemma. □

4.4 Relations with the cohomologies

A natural question to ask is whether or not the Bott–Chern and Aeppli cohomologies introduced in Section 3.2 are isomorphic to some space of harmonic forms. In general, this is not the case since Proposition 4.5.2 shows that $H^2_{d+d^c}$ might be infinite-dimensional even on compact manifolds. However, we have an equality between Bott–Chern cohomology and $(d + d^c)$-harmonic forms in degrees $k = 0$ and $k = 1$, and an inclusion in other degrees. A similar result holds for Aeppli cohomology, providing an isomorphism $H^1_{d+d^c} \cong H^{2m-1}_{dd^c}$.

Proposition 4.4.1. *Let (M, J) be a compact almost complex manifold. Then for any choice of J-compatible metric we have*

$$H^0_{d+d^c} = \mathcal{H}^0_{d+d^c} = \mathbb{C} \quad \textit{and} \quad H^1_{d+d^c} = \mathcal{H}^1_{d+d^c}.$$

In particular, the cohomology group $H^1_{d+d^c}$ *is finite-dimensional. Furthermore, there is an inclusion*

$$\mathcal{H}^k_{d+d^c} \cup \mathcal{H}^k_{d^c+d} \hookrightarrow H^k_{d+d^c}.$$

Proof. The fact that $H^0_{d+d^c} = \mathcal{H}^0_{d+d^c} = \mathbb{C}$ and $H^1_{d+d^c} = \mathcal{H}^1_{d+d^c}$ follows immediately from the explicit expression of Bott–Chern cohomology and of $(d+d^c)$-harmonic forms. Let now $\alpha \in \mathcal{H}^k_{d+d^c}$. Then $d\alpha = 0$ and $d^c\alpha = 0$, so that the identity defines a map from $\mathcal{H}^k_{d+d^c}$ to $H^k_{d+d^c}$. Assume that α defines the zero class in Bott–Chern cohomology, that is $\alpha = dd^c\beta$, with $\beta \in B^\bullet$. Since α is harmonic, we have that

$$0 = (dd^c)^*\alpha = (dd^c)^* dd^c\beta,$$

and $0 = \langle (dd^c)^* dd^c\beta, \beta\rangle = \|dd^c\beta\|^2$. This implies that $\alpha = dd^c\beta = 0$, proving injectivity. The same argument applies to (d^c+d)-harmonic forms. □

A similar statement is true for Aeppli cohomology.

Proposition 4.4.2. *Let (M, J) be a compact almost complex manifold. Then for any choice of J-compatible metric we have*

$$H^0_{dd^c} = \mathcal{H}^0_{dd^c} = \mathbb{C}, \quad H^{2m}_{dd^c} \cong \mathcal{H}^{2m}_{dd^c} \cong \mathbb{C} \quad \text{and} \quad H^{2m-1}_{dd^c} \cong \mathcal{H}^{2m-1}_{dd^c}.$$

In particular, the cohomology group $H^{2m-1}_{dd^c}$ *is finite-dimensional. Furthermore, there is an inclusion*

$$\mathcal{H}^k_{dd^c} \cup \mathcal{H}^k_{d^cd} \hookrightarrow H^k_{dd^c}.$$

Proof. We first prove the injectivity of

$$\mathcal{H}^k_{dd^c} \hookrightarrow H^k_{dd^c}.$$

Injectivity for $\mathcal{H}^k_{d^cd} \hookrightarrow H^k_{dd^c}$ follows with a similar proof. Let $\alpha \in \mathcal{H}^k_{dd^c}$. Then $dd^c\alpha = 0$, so that the projection on the quotient complex C^k defines a map from $\mathcal{H}^k_{dd^c}$ to $H^k_{dd^c}$. Assume that α defines the zero class in Aeppli cohomology, that is $\alpha = d\beta + d^c\gamma$. Since α is harmonic, we have that

$$0 = d^*\alpha = d^*d\beta + d^*d^c\gamma \quad \text{and} \quad 0 = (d^c)^*\alpha = (d^c)^*d\beta + (d^c)^*d^c\gamma.$$

As a consequence we obtain that

$$\|d\beta\|^2 + \langle d\beta, d^c\gamma\rangle = 0 \quad \text{and} \quad \|d^c\gamma\|^2 + \langle d\beta, d^c\gamma\rangle = 0.$$

In particular, we have that $\|d\beta + d^c\gamma\|^2 = 0$, proving injectivity.
We now prove the isomorphisms in degree $k \in \{0, 2m-1, 2m\}$. The statement

for functions follows immediately from the definition. Let $\alpha \in A^{2m}$. Using the harmonic decomposition for Δ_{dd^c} of Theorem 4.1.1, we have that

$$\begin{aligned}\alpha &= c\,Vol + \Delta_{dd^c}\beta = \\ &= c\,Vol + dd^c(dd^c)^*\beta + d(d^c)^*d^cd^*\beta \\ &+ d^cd^*d(d^c)^*\beta + dd^*\beta + d^c(d^c)^*\beta = \\ &= c\,Vol + d\gamma + d^c\eta,\end{aligned}$$

where γ and η are $(2m-1)$-forms and c is a constant. Note that a similar decomposition holds also for $(2m-1)$-forms since $(dd^c)^*dd^c\beta = 0$ if $\beta \in A^{2m-1}$. Passing to Aeppli cohomology, we obtain the existence of harmonic representatives. Uniqueness follows from the injectivity of $\mathcal{H}^k_{dd^c} \hookrightarrow H^k_{dd^c}$. □

Using the isomorphism $\mathcal{H}^1_{d+d^c} \cong \mathcal{H}^{2m-1}_{d^cd}$ induced by the Hodge $*$ operator, we deduce an isomorphism for Bott–Chern and Aeppli cohomologies.

Corollary 4.4.3. *Let (M, J) be a compact almost complex $2m$-manifold. Then there is an isomorphism*

$$H^1_{d+d^c} \cong H^{2m-1}_{dd^c}.$$

4.5 Cohomologies and harmonic forms on the Kodaira–Thurston manifold

In this section we compute the Bott–Chern and Aeppli cohomologies and the spaces of harmonic forms for an almost Kähler structure on the Kodaira–Thurston manifold.

Let $\mathcal{KT}$ be the Kodaira–Thurston manifold as defined in Section 2.3.2. Consider the symplectic form

$$\omega_0 := e^{12} + e^{34},$$

the ω_0-compatible almost complex structure J_0 given in (2.3.6) and the associated metric g_0. Then $(\mathcal{KT}, J_0)$ admits a co-frame of $(1,0)$-forms $\{\phi^1, \phi^2\}$ with differentials

$$d\phi^1 = 0 \quad \text{and} \quad d\phi^2 = \frac{i}{4}\,(\phi^{12} - \phi^{1\bar{2}} + \phi^{\bar{1}2} - \phi^{\bar{1}\bar{2}}).$$

Denote by $\{\xi_1, \xi_2\}$ the dual frame of $(1,0)$-vector fields. We first compute the spaces of $(d + d^c)$-harmonic forms.

Proposition 4.5.1. *The spaces of* $(d+d^c)$*-harmonic forms on* $\mathcal{KT}$ *endowed with the almost Kähler structure* (J_0, ω_0, g_0) *are*

$$\begin{aligned}
\mathcal{H}^0_{d+d^c} &= \mathbb{C},\\
\mathcal{H}^1_{d+d^c} &= \mathbb{C}\langle \phi^1,\, \phi^{\bar 1}\rangle,\\
\mathcal{H}^2_{d+d^c} &= \mathbb{C}\langle \phi^{12}+\phi^{\bar 1\bar 2},\, \phi^{1\bar 2}+\phi^{\bar 1 2},\, \phi^{1\bar 1},\, \phi^{2\bar 2}\rangle,\\
\mathcal{H}^3_{d+d^c} &= \mathbb{C}\langle \phi^{12\bar 2},\, \phi^{2\bar 1\bar 2},\, \phi^{12\bar 1}+\phi^{1\bar 1\bar 2}\rangle,\\
\mathcal{H}^4_{d+d^c} &= \mathbb{C}\langle \phi^{12\bar 1\bar 2}\rangle.
\end{aligned}$$

Proof. The claim for $k \in \{0,4\}$ is immediate. For $k=1$, we need to establish which 1-forms are both d-closed and d^c-closed. This is equivalent to finding d-closed $(1,0)$-forms. Let $\alpha \in A^{1,0}$. Then

$$\alpha = f\phi^1 + g\phi^2,$$

with f and $g \in C^\infty(M)$. Writing explicitly the equation $d\alpha = 0$ and separating the bidegree of the forms, we deduce that f must be constant and $g=0$, hence we have the equality $\mathcal{H}^1_{d+d^c} = \mathbb{C}\langle \phi^1, \phi^{\bar 1}\rangle$.

We now compute invariant $(d+d^c)$-harmonic 2-forms and we show that there is no non-invariant $(d+d^c)$-harmonic 2-form. Let $\alpha \in A^2$. Assume that α is invariant. In terms of a basis of invariant 2-forms, we have

$$\alpha = a\phi^{12} + e\phi^{1\bar 1} + f\phi^{1\bar 2} + g\phi^{\bar 1 2} + h\phi^{2\bar 2} + b\phi^{\bar 1\bar 2},$$

with a, b, e, f, g and $h \in \mathbb{C}$. By computing separately the differential on the even and the odd part of α, we see that the constants must satisfy $a=b$ and $f=g$. Moreover, the condition $(dd^c)^*\alpha = 0$ is satisfied since all invariant 3-forms are d-closed. This gives the inclusion

$$\mathbb{C}\langle \phi^{12}+\phi^{\bar 1\bar 2}, \phi^{1\bar 2}+\phi^{\bar 1 2}, \phi^{1\bar 1}, \phi^{2\bar 2}\rangle \subseteq \mathcal{H}^2_{d+d^c}.$$

Moreover, we also have that

$$b_2(\mathcal{KT}) = 4 = \dim_{\mathbb{C}}(\mathbb{C}\langle \phi^{12}+\phi^{\bar 1\bar 2}, \phi^{1\bar 2}+\phi^{\bar 1 2}, \phi^{1\bar 1}, \phi^{2\bar 2}\rangle) \le h^2_{d+d^c} \le b_2(\mathcal{KT}),$$

by Theorem 4.2.8. This implies the equality of the spaces and that all $(d+d^c)$-harmonic 2-forms are invariant. Finally, we know the space of $(d+d^c)$-harmonic 3-forms thanks to Theorem 4.3.3 and the computations of Section 3.4 in [100] rewritten in terms of complex forms. □

We now compute the almost complex Bott–Chern cohomology group $H^2_{d+d^c}$ of $(\mathcal{KT}, J_0)$ and we show that it is infinite-dimensional. To compute the Bott–Chern cohomology of 2-forms, let

$$\alpha = a\phi^{12} + e\phi^{1\bar 1} + f\phi^{1\bar 2} + g\phi^{\bar 1 2} + h\phi^{2\bar 2} + b\phi^{\bar 1\bar 2},$$

with a, b, e, f, g and $h \in C^\infty(M)$, be a 2-form. Direct computations show that α is d-closed and d^c-closed if and only if there exist functions a, b, e, f, g and h satisfying the system

$$\begin{cases} \xi_{\bar{2}}(a) = \xi_2(b) = 0, \\ \frac{i}{4}(a-b) + \xi_{\bar{1}}(a) = 0, \\ \frac{i}{4}(a-b) + \xi_1(b) = 0, \\ \xi_1(h) = \xi_2(f), \\ \xi_{\bar{1}}(h) = -\xi_{\bar{2}}(g), \\ \frac{i}{4}(f-g) - \xi_2(e) - \xi_1(g) = 0 \\ \frac{i}{4}(f-g) + \xi_{\bar{2}}(e) - \xi_{\bar{1}}(f) = 0. \end{cases} \qquad (*)$$

Separating the bidegree, we introduce the spaces

$$\mathcal{H} := \{a\phi^{12} + b\phi^{\bar{1}\bar{2}} : (*) \text{ holds } \}$$

and

$$\mathcal{I} := \{e\phi^{1\bar{1}} + f\phi^{1\bar{2}} + g\phi^{\bar{1}2} + h\phi^{2\bar{2}} : (*) \text{ holds } \}$$

Thus, we can describe the Bott–Chern cohomology.

Proposition 4.5.2. *The second Bott–Chern cohomology group of $\mathcal{KT}$ endowed with the almost complex structure* (2.3.6) *is*

$$H^2_{d+d^c} \cong \mathcal{H} \oplus \frac{\mathcal{I}}{\mathcal{M}},$$

where

$$\mathcal{M} = \{\xi_1\xi_{\bar{1}}(\theta)\,\phi^{1\bar{1}} : \theta \in C^\infty(M) \text{ and } \xi_2(\theta) = \xi_{\bar{2}}(\theta) = 0\}.$$

In particular, the space $H^2_{d+d^c}$ is infinite-dimensional.

Observe that the splitting of $H^2_{d+d^c}$ as the sum of the space $\mathcal{H}$ and the quotient $\mathcal{I}/\mathcal{M}$ corresponds to the splitting of Bott–Chern cohomology into even and odd part of Corollary 3.5.7.

Proof. The space of d-closed and d^c-closed 2-forms is given by

$$\mathcal{H} \oplus \mathcal{I}.$$

To compute Bott–Chern cohomology, we have to quotient by $dd^c\theta$, where $\theta \in C^\infty(M)$ is such that $(dd^c + d^c d)\theta = 0$. By (1.2.10), the function θ must be ∂^2-closed and $\bar{\partial}^2$-closed. From the first condition, we have that

$$0 = \partial^2\theta = (\xi_1\xi_2(\theta) - \xi_2\xi_1(\theta) + \frac{i}{4}\xi_2(\theta))\phi^{12} = -\frac{i}{4}\xi_{\bar{2}}(\theta)\phi^{12},$$

where the last equality follows from the commutator relation

$$[\xi_1, \xi_2] = -\frac{i}{4}(\xi_2 + \xi_{\bar{2}}).$$

This implies that $\xi_{\bar{2}}(\theta) = 0$. Similarly, we have that $\xi_2(\theta) = 0$. Finally, since $(dd^c + d^c d)\theta = 0$, we can compute

$$dd^c\theta = 2i\xi_1\xi_{\bar{1}}(\theta)\,\phi^{1\bar{1}},$$

showing that we have to quotient by $\mathcal{M}$. To show that $H^2_{d+d^c}$ contains an infinite-dimensional subspace, consider

$$\mathcal{S} \coloneqq \{h\phi^{2\bar{2}} : \xi_1(h) = 0 \text{ and } \xi_{\bar{1}}(h) = 0\}.$$

Clearly $\mathcal{S} \subset H^2_{d+d^c}$ and it is infinite-dimensional because it strictly contains the family of functions $\{cos(2\pi ny)\}_{n\in\mathbb{N}}$. □

CHAPTER 5

Almost complex and topological invariants

In this chapter we discuss the relations between the almost complex invariants defined in Chapter 4 and the topological invariants of the underlying manifold. The classical theory of compact complex surfaces shows that their Hodge and Bott–Chern numbers depend only on topological invariants. Motivated by this, we find an explicit expression for Bott–Chern numbers in terms of topological constants. The study of the dependence on the topology for almost Kähler invariants on 4-manifolds inspires a conjecture on the generic vanishing of $h^1_{d+d^c}$, which we are able to prove in high dimension.

5.1 Bott–Chern diamond of compact complex surfaces

The goal of this section is to compute Bott–Chern numbers of compact complex surfaces and to show that they depend only on the topology of the underlying manifold. More precisely, they do not depend on the choice of complex structure, but only on the numbers b_1, b^+ and b^-. This is a result that was already implicitly contained in the work of Teleman [97], see also [92].

Let (M, J) be a compact complex surface without boundary. We are interested in the following invariants, see also Section 1.3.1:

- the *Betti numbers* b_k, for $k = 0, \dots, 4$. Since M is an oriented closed manifold, its Betti numbers reduce to

k	0	1	2	3	4
b_k	1	b_1	b_2	b_1	1

- the *Hodge numbers* $h^{p,q}_{\bar\partial}$, for $p,q = 0,1,2$, that can be arranged in the so called *Hodge diamond*

$$\begin{array}{ccccc} & & h^{0,0}_{\bar\partial} & & \\ & h^{1,0}_{\bar\partial} & & h^{0,1}_{\bar\partial} & \\ h^{2,0}_{\bar\partial} & & h^{1,1}_{\bar\partial} & & h^{0,2}_{\bar\partial} \\ & h^{2,1}_{\bar\partial} & & h^{1,2}_{\bar\partial} & \\ & & h^{2,2}_{\bar\partial} & & \end{array}$$

- the *Bott–Chern and Aeppli numbers* $h^{p,q}_{BC}$ and $h^{p,q}_{A}$, for $p,q = 0,1,2$. By duality between Bott–Chern and Aeppli cohomologies, we have that $h^{p,q}_{BC} = h^{m-p,m-q}_{A}$ for all p and q. Thus, knowing Bott–Chern numbers completely determines Aeppli numbers. We arrange Bott–Chern numbers in the *Bott–Chern diamond*

$$\begin{array}{ccccc} & & h^{0,0}_{BC} & & \\ & h^{1,0}_{BC} & & h^{0,1}_{BC} & \\ h^{2,0}_{BC} & & h^{1,1}_{BC} & & h^{0,2}_{BC} \\ & h^{2,1}_{BC} & & h^{1,2}_{BC} & \\ & & h^{2,2}_{BC} & & \end{array}$$

It is a well-known fact that, while a priori Hodge numbers depend on the choice of complex structure, for compact complex surfaces they actually depend only on the first Betti number b_1 and on the positive and negative self-intersection numbers b^+ and b^-, with $b^+ + b^- = b_2$. For the sake of completeness, we give here a precise statement, whose proof follows from Theorems 2.7 and 2.14 of Chapter 4 in [10].

Theorem 5.1.1. *Let (M,J) be a compact complex surface. If b_1 is even, then the Hodge diamond of (M,J) is*

$$\begin{array}{ccccc} & & 1 & & \\ & \frac{b_1}{2} & & \frac{b_1}{2} & \\ \frac{b^+-1}{2} & & b^- + 1 & & \frac{b^+-1}{2} \\ & \frac{b_1}{2} & & \frac{b_1}{2} & \\ & & 1 & & \end{array}$$

If b_1 is odd, then the Hodge diamond of (M, J) is

$$\begin{array}{ccccc} & & 1 & & \\ & \frac{b_1-1}{2} & & \frac{b_1+1}{2} & \\ \frac{b^+}{2} & & b^- & & \frac{b^+}{2} \\ & \frac{b_1+1}{2} & & \frac{b_1-1}{2} & \\ & & 1 & & \end{array}$$

Corollary 5.1.2. *Hodge numbers of compact complex surfaces depend only on the topology of the underlying manifold.*

We state and prove a similar result valid for Bott–Chern numbers.

Theorem 5.1.3. *Let (M, J) be a compact complex surface. If b_1 is even, then the Bott–Chern diamond of (M, J) is*

$$\begin{array}{ccccc} & & 1 & & \\ & \frac{b_1}{2} & & \frac{b_1}{2} & \\ \frac{b^+-1}{2} & & b^- + 1 & & \frac{b^+-1}{2} \\ & \frac{b_1}{2} & & \frac{b_1}{2} & \\ & & 1 & & \end{array}$$

If b_1 is odd, then the Bott–Chern diamond of (M, J) is

$$\begin{array}{ccccc} & & 1 & & \\ & \frac{b_1-1}{2} & & \frac{b_1-1}{2} & \\ \frac{b^+}{2} & & b^- + 1 & & \frac{b^+}{2} \\ & \frac{b_1+1}{2} & & \frac{b_1+1}{2} & \\ & & 1 & & \end{array}$$

Proof. As we observed for general spaces of harmonic forms, we have $h^{0,0}_{BC} = h^{2,2}_{BC} = 1$ for every compact complex surface. By [97], see also [3] and [7], on compact complex surfaces we have that

$$h^{1,0}_{BC} + h^{0,1}_{BC} + h^{2,1}_{BC} + h^{1,2}_{BC} = 2b_1 \tag{5.1.1}$$

and that

$$h^{2,0}_{BC} + h^{1,1}_{BC} + h^{0,2}_{BC} = \begin{cases} b_2 & \text{if } b_1 \text{ is even,} \\ b_2 + 1 & \text{if } b_1 \text{ is odd.} \end{cases} \tag{5.1.2}$$

Since Bott–Chern numbers are symmetric in p and q, we can simplify (5.1.1) and (5.1.2) to get

$$h^{1,0}_{BC} + h^{2,1}_{BC} = b_1 \tag{5.1.3}$$

and

$$2h^{2,0}_{BC} + h^{1,1}_{BC} = \begin{cases} b_2 & \text{if } b_1 \text{ is even,} \\ b_2 + 1 & \text{if } b_1 \text{ is odd.} \end{cases} \tag{5.1.4}$$

If b_1 is even, then (M, J) admits a Kähler metric, see [22] or [59]. Since the numbers $h^{p,q}_{BC}$ are independent of the choice of metric, it is enough to compute them for a Kähler metric. On Kähler manifolds, there is an isomorphism $H^{p,q}_{BC} \cong H^{p,q}_{\bar\partial}$. Therefore the Bott–Chern diamond of compact complex surfaces with b_1 even coincides with their Hodge diamond given in Theorem 5.1.1.

Suppose now that b_1 is odd. For any choice of metric g, we have $\mathcal{H}^{1,0}_{BC} = \mathcal{H}^{1,0}_{\bar\partial}$. Indeed, writing explicitly the spaces of harmonic forms, we have that

$$\mathcal{H}^{1,0}_{BC} = A^{1,0} \cap \ker\partial \cap \ker\bar\partial \quad \text{and} \quad \mathcal{H}^{1,0}_{\bar\partial} = A^{1,0} \cap \ker\bar\partial.$$

By Lemma 2.1 of Chapter 4 in [10], every holomorphic form on a compact complex surface is d-closed, thus $A^{1,0} \cap \ker\bar\partial = A^{1,0} \cap \ker\partial \cap \ker\bar\partial$, giving the equality of the two spaces, and allowing to deduce that

$$h^{1,0}_{BC} = h^{0,1}_{BC} = h^{1,0}_{\bar\partial} = \frac{b_1 - 1}{2}.$$

By (5.1.3), we also obtain

$$h^{2,1}_{BC} = h^{1,2}_{BC} = \frac{b_1 + 1}{2}.$$

The number $h^{1,1}_{BC}$ can be computed either using Lemma 2.3 in [97], using Theorem 4.2.8 applied to an integrable J together with Lemma 3.2.8, or applying Theorem 4.2 in [48], since in the complex case d and d^c or ∂ and $\bar\partial$ are interchangeable in the definition of Bott–Chern cohomology. It turns out that, for Bott–Chern numbers, we have

$$h^{1,1}_{BC} = b^- + 1. \tag{5.1.5}$$

Finally, by (5.1.4) and (5.1.5), we have

$$h^{2,0}_{BC} = h^{0,2}_{BC} = \frac{b^+}{2},$$

concluding the proof. □

Note that one has $h^{1,1}_{BC} = b^- + 1$ independently of the parity of b_1, in contrast to what happens for $h^{1,1}_{\bar\partial}$.

Corollary 5.1.4. *Bott–Chern and Aeppli numbers of compact complex surfaces depend only on the topology of the underlying manifold.*

5.2 The numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ of invariant compatible triples

In this section we determine the numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ of compact complex surfaces that are diffeomorphic to solvmanifolds endowed with an invariant compatible triple. Along the way, we establish results valid more in general for compact quotients of Lie groups by a lattice. For a short description of compact quotients of Lie groups, we refer to Section 1.5.

The main result of this section is the following:

Theorem 5.2.1. *Let M be a 4-dimensional solvmanifold endowed with an invariant compatible triple (J, ω, g) with integrable J. Then the numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ are independent of the choice of invariant compatible triple.*

The proof of Theorem 5.2.1 follows from several lemmas that are valid in a slightly more general setting and that will be useful later for explicit computations, see Example 5.2.5. As a consequence of the proof, we will also compute the numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ of compact complex surfaces diffeomorphic to solvmanifolds. The resulting numbers are summarized in Tables 5.1 and 5.2.

We begin by taking care of 1-forms.

Lemma 5.2.2. *Let $M = \Gamma\backslash G$ be a compact quotient of a Lie group by a lattice endowed with an invariant almost symplectic structure ω. Then $h^1_{d+d^\Lambda} = b_1$ and it is metric-independent.*

Proof. Fix a compatible metric g. The inequality $h^1_{d+d^\Lambda} \leq b_1$ holds for arbitrary almost Hermitian manifolds. For the opposite inequality, let $\alpha \in \mathcal{H}^1_d$ be a d-harmonic 1-form. Then $d\alpha = 0$ and $(dd^\Lambda)^*\alpha = d^c d^*\alpha = 0$ for any choice of compatible metric. Since ω is invariant, the 3-form $*_s\alpha$ is invariant, thus it is d-closed and we have $d^\Lambda\alpha = 0$. □

The following lemma allows us to deal with the space $\mathcal{H}^3_{d+d^\Lambda}$.

Lemma 5.2.3. *Let (M, J, ω, g) be a compact Hermitian $2m$-manifold. Suppose that for every $\gamma \in \mathcal{H}^1_{dd^c}$ we have that*

$$(d^c)^*d^*d\gamma = 0.$$

Then, we have $\mathcal{H}^{2m-1}_{d+d^\Lambda} = \mathcal{H}^{2m-1}_{d+d^c} \cap \ker\Delta_{d+d^\Lambda}$ and $h^{2m-1}_{d+d^\Lambda} \leq h^{2m-1}_{d+d^c}$.

Proof. By the isomorphism $\mathcal{H}^{2m-1}_{d+d^\Lambda} \cong \mathcal{H}^1_{d^\Lambda d}$ induced by $*_s$, it is enough to compute $d^\Lambda d$-harmonic 1-forms. Let $\{\gamma_1, \ldots, \gamma_t\}$ be a basis of $\mathcal{H}^1_{d+d^c}$, let $\alpha \in \mathcal{H}^1_{d^\Lambda d}$ and let

$$\alpha = \sum_{j=1}^{t} A_j\gamma_j + (dd^c)^*\beta + df + d^cg$$

be the harmonic decomposition of α with respect to Δ_{dd^c}, with $A_j \in \mathbb{C}$, f and $g \in C^\infty(M)$ and $\beta \in A^3$. Since $(d^\Lambda)^*\alpha = d^c\alpha = 0$ and $\gamma_j \in \mathcal{H}^1_{dd^c}$, we have

$$0 = dd^c\alpha = dd^c(dd^c)^*\beta,$$

obtaining $(dd^c)^*\beta = 0$. On the other side, we have that

$$0 = d^\Lambda d\alpha = \sum_{j=1}^{t} A_j (d^c)^* d\gamma_j + (d^c)^* dd^c g.$$

Taking the inner product with dg and using the assumption $(d^c)^* d^* d\gamma_j = 0$, we can write

$$0 = \sum_{j=1}^{t} A_j \langle (d^c)^* d^* d\gamma_j, g\rangle + \langle dd^c g, d^c dg\rangle = -\|dd^c g\|^2,$$

which implies that g is constant. Finally, from

$$0 = d^*\alpha = d^* df$$

we deduce that f is constant, that $\alpha = \sum_j A_j\gamma_j \in \mathcal{H}^1_{dd^c}$ and that $\mathcal{H}^1_{d^\Lambda d}$ can be computed as $\mathcal{H}^1_{dd^c} \cap \ker \Delta_{d^\Lambda d}$. □

Proposition 5.2.4. *Let $M = \Gamma\backslash G$ be a 4-dimensional compact quotient of a Lie group by a lattice and let (J, ω, g) be an invariant compatible triple on M with J integrable. Then we have*

$$\mathcal{H}^3_{d+d^\Lambda} = \mathcal{H}^{2,1}_d \cup \mathcal{H}^{1,2}_d.$$

Proof. Since the compatible triple is invariant, any 1-form γ automatically satisfies the equation $(d^c)^* d^* d\gamma = 0$. By Lemma 5.2.3, we have that

$$\mathcal{H}^3_{d+d^\Lambda} \cong \mathcal{H}^1_{d^\Lambda d} = \mathcal{H}^1_{dd^c} \cap \ker \Delta_{d^\Lambda d} = A^1 \cap \ker \Delta_{dd^c} \cap \ker \Delta_{d^\Lambda d}.$$

Let $\gamma \in A^1 \cap \ker \Delta_{dd^c} \cap \ker \Delta_{d^\Lambda d}$. Since $\Delta_{dd^c}\gamma = 0$ and $\Delta_{d^\Lambda d}\gamma = 0$, the form γ must satisfy the equations

$$d^c\gamma = 0, \quad d^*\gamma = 0, \quad (d^c)^*\gamma = 0, \quad d^c d\gamma = 0 \quad \text{and} \quad d^\Lambda d\gamma = 0.$$

We show that $d\gamma = 0$. Let

$$d\gamma = f\omega + \beta^{1,1} + \beta^{(2,0)(0,2)} \tag{5.2.1}$$

be the Lefschetz and bidegree decomposition of $d\gamma$, with $\beta^{1,1}$ and $\beta^{(2,0)(0,2)}$ primitive forms and $f \in C^\infty(M)$. From the equations

$$\begin{aligned} 0 &= d^2\gamma = d(f\omega + \beta^{1,1} + \beta^{(2,0)(0,2)}), \\ 0 &= d^c d\gamma = -Jd(f\omega + \beta^{1,1} - \beta^{(2,0)(0,2)}) \quad \text{and} \\ 0 &= d^\Lambda d\gamma = -*_s\, d(f\omega - \beta^{1,1} - \beta^{(2,0)(0,2)}), \end{aligned}$$

we get $d(f\omega) = 0$, $d\beta^{1,1} = 0$ and $d\beta^{(2,0)(0,2)} = 0$. Since each of the terms in (5.2.1) is d-closed, we have that

$$d^* d\gamma = - * d(f\omega - \beta^{1,1} + \beta^{(2,0)(0,2)}) = 0,$$

which implies that $d\gamma = 0$ and that γ is both d-harmonic and d^c-harmonic. As a consequence, the $(1,0)$-bidegree part and $(0,1)$-bidegree part of γ are both d-harmonic. This shows that $\mathcal{H}^1_{d^\Lambda d} = \mathcal{H}^{1,0}_d \cup \mathcal{H}^{0,1}_d$ and, after applying the isomorphism given by the Hodge $*$, it concludes the proof of the proposition. □

We are ready for the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. Let M be a 4-dimensional solvmanifold admitting a complex structure. By [45], M is one of the following:

(A) a complex torus;

(B) a hyperelliptic surface;

(C) an Inoue surface of type $\mathcal{S}_M$;

(D) a primary Kodaira surface;

(E) a secondary Kodaira surface;

(F) an Inoue surface of type $\mathcal{S}^{\pm}$.

By Theorem 5.1.3, the numbers $h^k_{d+d^c}$ depend only on b_1, b^+ and b^- and not on the choice of compatible triple (not necessarily invariant).

All invariant structures on the torus and the hyperelliptic surface (cases (A) and (B)) are Kähler structures, therefore the numbers $h^k_{d+d^\Lambda}$ coincide with the Betti numbers and they are independent of the choice of compatible triple.

Cases (C), (E) and (F) can be treated simultaneously since they have the same Betti numbers, namely $b_1 = 1$ and $b_2 = 0$. By Theorem 5.1.3, we have that $h^1_{d+d^c} = b_1 - 1 = 0$, that $h^2_{d+d^c} = b_2 + 1 = 1$ and that $h^3_{d+d^c} = b_1 + 1 = 2$. By Lemma 5.2.2, we have $h^1_{d+d^\Lambda} = b_1 = 1$. By Theorem 4.3.7 and the fact that there are no d-harmonic 2-forms since $b_2 = 0$, we have that $h^2_{d+d^\Lambda} = b_2 = 0$. By Proposition 5.2.4, we have that $\mathcal{H}^3_{d+d^\Lambda} = \mathcal{H}^{2,1}_d \cup \mathcal{H}^{1,2}_d$. In particular, since $\mathcal{H}^{2,1}_d \cong \mathcal{H}^{1,2}_d$ via complex conjugation, the number $h^3_{d+d^\Lambda}$ must be even. Moreover, the intersection of $\mathcal{H}^{2,1}_d$ and $\mathcal{H}^{1,2}_d$ is trivial and both spaces inject into $\mathcal{H}^3_d$. Thus $h^3_{d+d^\Lambda}$ is an even number and satisfies $h^3_{d+d^\Lambda} \leq b_1 = 1$. This shows that $h^3_{d+d^\Lambda} = 0$.

Case (D) has to be treated separately, since the Betti numbers in this case are $b_1 = 3$ and $b_2 = 4$. By Theorem 5.1.3, we immediately have $h^1_{d+d^c} = 2$, $h^2_{d+d^c} = 5$ and $h^1_{d+d^c} = 4$. By Lemma 5.2.2, we have that $h^1_{d+d^\Lambda} = b_1 = 3$. By Proposition

5.2.4, with the same reasoning as in cases (C), (E) and (F), we know that $h^3_{d+d^\Lambda}$ is even and $h^3_{d+d^\Lambda} \le b_1 = 3$, so that either $h^3_{d+d^\Lambda} = 0$ or $h^3_{d+d^\Lambda} = 2$. We show that $h^3_{d+d^\Lambda} = 2$ by showing that there is at least one $(d + d^\Lambda)$-harmonic 3-form. Fix an invariant compatible triple (J, ω, g). By Theorem 3.1 in [81], since primary Kodaira surfaces are nilmanifolds $\Gamma\backslash G$ and the complex structure is invariant, we can find, up to a linear transformation of the Lie algebra of G, a basis of invariant $(1, 0)$-forms $\{\phi^1, \phi^2\}$ such that $d\phi^1 = 0$ and $d\phi^2 = \phi^{1\bar{1}}$. Since g is an invariant metric, the $(2, 1)$-form $*\phi^1$ is invariant and $d(*\phi^1) = 0$. Moreover, we also have $d^*(*\phi^1) = - * d\phi^1 = 0$. This implies that $*\phi^1 \in \mathcal{H}^{2,1}_d$ and that $h^3_{d+d^\Lambda} = 2$.

To conclude the proof, we show that $\mathcal{H}^2_d = \mathcal{H}^2_{d^c} \subseteq \mathcal{H}^2_{d+d^\Lambda}$, which implies that $h^2_{d+d^\Lambda} = b_2 = 4$ by Theorem 4.3.7. Consider the decomposition

$$\mathcal{H}^2_d = \mathcal{H}^+_g \oplus \mathcal{H}^-_g$$

between self-dual and anti-self-dual harmonic forms. Forms in $\mathcal{H}^-_g$ are anti-self-dual, thus they have bidegree $(1, 1)$ and they are primitive. Moreover, they are also d-closed. If $\alpha \in \mathcal{H}^-_g$, then $d\alpha = 0$, $d^\Lambda\alpha = - *_s d *_s \alpha = *_s d\alpha = 0$ and $(dd^\Lambda)^*\alpha = d^c d^*\alpha = d^c * d\alpha = 0$. This gives the inclusion $\mathcal{H}^-_g \subseteq \mathcal{H}^2_{d+d^\Lambda}$. For the inclusion $\mathcal{H}^+_g \subseteq \mathcal{H}^2_{d+d^\Lambda}$, we observe that on primary Kodaira surfaces we have $b^+ = 2$ and, as long as the compatible triple is invariant, we also have $\mathcal{H}^+_g = \mathbb{C}\langle \phi^{12}, \phi^{\bar{1}\bar{2}}\rangle$, where $\{\phi^1, \phi^2\}$ is the preferred basis of [81] we considered above in the proof, up to normalization. Indeed, we have that $d\phi^{12} = 0$ and $*\phi^{12}$ is an invariant $(2, 0)$-form since the metric is invariant. Finally, also $*_s\phi^{12}$ is an invariant $(2, 0)$-form since ω is invariant, and one has that $d\phi^{12} = 0$, $d^\Lambda\phi^{12} = - *_s d *_s \phi^{12} = 0$ and $(dd^\Lambda)^*\phi^{12} = d^c d^*\phi^{12} = 0$, proving that $\mathcal{H}^+_g \subseteq \mathcal{H}^2_{d+d^\Lambda}$ and that $h^2_{d+d^\Lambda} = 4$. □

The results proved in this section and in Section 5.1 can be used to compute the numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ in the case of complex surfaces not necessarily diffeomorphic to solvmanifolds, as we illustrate in the example below.

Example 5.2.5 (Hopf surface). Let $M = S^1 \times S^3$ be the Hopf surface. There exists a parallelism for T^*M given by $\{e^1, e^2, e^3, e^4\}$, with differentials

$$de^1 = 0, \quad de^2 = -e^{34}, \quad de^3 = e^{24} \quad \text{and} \quad de^4 = -e^{23}.$$

Note that M is the quotient of a non-solvable Lie group by a lattice.

Proposition 5.2.6. *Let (M, J, ω, g) be the Hopf surface endowed with an invariant compatible triple with J integrable. Then the numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ are*

k	0	1	2	3	4
$h^k_{d+d^c}$	1	0	1	2	1
$h^k_{d+d^\Lambda}$	1	1	0	0	1

Proof. By Theorem 5.1.3, we have that $h^1_{d+d^c} = 0$, $h^2_{d+d^c} = 1$ and $h^3_{d+d^c} = 2$. To compute $h^k_{d+d^\Lambda}$, we resort to several different arguments. Lemma 5.2.2 gives that $h^1_{d+d^\Lambda} = b_1 = 1$. Since there is no d-harmonic 2-form, Theorem 4.3.7 tells us that $h^2_{d+d^\Lambda} = 0$. Finally, by Proposition 5.2.4 and the same reasoning used in the proof of Theorem 5.2.1, we conclude that $h^3_{d+d^\Lambda} = 0$. □

The fact that almost complex and almost symplectic invariants are determined by the topology of the underlying manifold is not surprising, especially on solvmanifolds, see Theorem 2.4.15. ■

Table 5.1: The numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ of the complex torus, the hyperelliptic surface and the Inoue surface $\mathcal{S}_M$.

	(A) Complex torus		(B) Hyperelliptic surface		(C) Inoue surface $\mathcal{S}_M$	
k	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$
0	1	1	1	1	1	1
1	4	4	2	2	0	1
2	6	6	2	2	1	0
3	4	4	2	2	2	0
4	1	1	1	1	1	1

Table 5.2: The numbers $h^k_{d+d^c}$ and $h^k_{d+d^\Lambda}$ of the primary and secondary Kodaira surface and the Inoue surface $\mathcal{S}^\pm$.

	(D) Primary Kodaira surface		(E) Secondary Kodaira surface		(F) Inoue surface $\mathcal{S}^\pm$	
k	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$	$h^k_{d+d^c}$	$h^k_{d+d^\Lambda}$
0	1	1	1	1	1	1
1	2	3	0	1	0	1
2	5	4	1	0	1	0
3	4	2	2	0	2	0
4	1	1	1	1	1	1

5.3 The number $h^1_{d+d^c}$ as an almost complex invariant

In this section we show that the numbers $h^1_{d+d^c}$ and h^-_J are the only almost complex invariants of almost Kähler 4-manifolds that do not depend on the topology of the underlying manifold. We also explain how to use $h^1_{d+d^c}$ to distinguish between different almost complex structures.

In Section 4.2, we saw that on compact almost complex 4-manifolds, the numbers $h^k_{\bar\delta}$, $h^{p,q}_d$, $h^k_{\delta+\bar\delta}$ and $h^k_{d+d^c}$ are almost Käler invariants. We now determine their precise dependence from the topology of the underlying manifold and other almost complex invariants.

Theorem 5.3.1. *Let (M, J) be a compact almost complex 4-manifold admitting a J-compatible almost Kähler metric. Then for every choice of J-compatible almost Kähler metric, the invariants $h^k_{\bar\delta}$, $h^{p,q}_d$, $h^k_{\delta+\bar\delta}$ and $h^k_{d+d^c}$ are completely determined by:*

- *the oriented topology of the underlying manifold (more precisely, by the numbers b_1 and b^-);*
- *the almost complex invariant $h^1_{d+d^c}$;*
- *the almost complex invariant h^-_J.*

Furthermore, the invariants $h^1_{d+d^c}$ and h^-_J do not completely determine each other.

In the proof of the theorem, we explicitly compute the numbers $h^k_{\bar\delta}$, $h^{p,q}_d$, $h^k_{\delta+\bar\delta}$ and $h^k_{d+d^c}$ in terms of b_1, b^-, $h^1_{d+d^c}$ and h^-_J, see Table 5.3.

Proof. Fix an arbitrary J-compatible almost Kähler metric. The cases $k = 0$ and $k = 4$ are easy to deal with. For the remaining values of k, note that by Proposition 6.10 in [96], we have that $h^k_{\bar\delta} = h^k_{\delta+\bar\delta}$. By Theorem 4.2.4, we have that $h^2_{\delta+\bar\delta} = b^- + 1 + h^-_J$. By the isomorphism given by the Hodge $*$, we also have $h^1_{\delta+\bar\delta} = h^3_{\delta+\bar\delta}$. Hence, the only degrees of freedom for $h^k_{\bar\delta}$ and $h^k_{\delta+\bar\delta}$ are $h^1_{\delta+\bar\delta}$ and h^-_J.

By Theorem 4.2.8, we have that $h^2_{d+d^c} = b^- + 1 + h^-_J$ and, by Corollary 4.3.4, we obtain $h^3_{d+d^c} = b_1$. Again, the only degrees of freedom for $h^k_{d+d^c}$ are $h^1_{d+d^c}$ and h^-_J. Furthermore, we observe that since $d = \delta + \bar\delta$ and $d^c = i(\bar\delta - \delta)$, one immediately deduces that $h^1_{\delta+\bar\delta} = h^1_{d+d^c}$.

For the numbers $h^{p,q}_d$, by Lemma 5.6 in [31], we have that $h^{2,0}_d = h^{0,2}_d = 0$. By Theorem 4.2.9, we have that $h^{1,1}_d = b^- + 1$. By the symmetries induced by the Hodge $*$ and by the results of [31], we finally obtain $h^{1,0}_d = h^{0,1}_d = h^{2,1}_d = h^{1,2}_d = h^{1,0}_{\bar\partial}$.

To conclude the proof of the first part of the theorem, we prove that on compact almost Kähler 4-manifolds $h^1_{d+d^c} = 2h^{1,0}_d$. By definition, one has that

$$\mathcal{H}^1_{d+d^c} = A^1 \cap \ker d \cap \ker d^c = (A^{1,0} \cap \ker d) \oplus (A^{0,1} \cap \ker d),$$

hence $h^1_{d+d^c} = 2\dim_{\mathbb{C}}(A^{1,0} \cap \ker d)$. Again by definition, we also have that $h^{1,0}_d = \dim_{\mathbb{C}}(A^{1,0} \cap \ker d \cap \ker d^*)$. Finally, note that

$$A^{1,0} \cap \ker d \cap \ker d^* \subseteq A^{1,0} \cap \ker d \subseteq A^{1,0} \cap \ker \bar{\partial} = A^{1,0} \cap \ker d \cap \ker d^*,$$

where the last equality follows from Lemma 4.2.11. This implies the equality of spaces

$$\mathcal{H}^{1,0}_d = \mathcal{H}^{1,0}_{\bar{\partial}} = A^{1,0} \cap \ker d$$

and shows that $h^1_{d+d^c} = 2h^{1,0}_d$. The second part of the theorem follows from the fact that there exists a symplectic 4-manifold (M, ω) and a curve of almost complex structures J_t, with $t \in (-\epsilon, \epsilon)$, such that:

- ω is an almost Kähler metric for each J_t;
- $h^1_{d+d^c}(J_t)$ varies for different values of t;
- $h^-_{J_t} = 0$ for all $t \in (-\epsilon, \epsilon)$.

The symplectic 4-manifold and the curve of almost complex structures that we have to consider are those given in Example 5.3.2 below, where we also prove the first two claims we made on J_t. To prove that $h^-_{J_t} = 0$ for all $t \in (-\epsilon, \epsilon)$, note that, for this specific example, we have $b^+ = b^- = 1$ and that, by Corollary 3.4 in [35], if $b^+ = 1$ then $h^-_J = 0$ for all tamed almost complex structures. □

Table 5.3: The numbers $h^k_{\bar{\delta}}$, $h^{p,q}_d$, $h^k_{\delta+\bar{\delta}}$ and $h^k_{d+d^c}$ of compact almost Kähler 4-manifolds.

k	1		2			3	
(p,q)	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	$(2,1)$	$(1,2)$
$h^k_{\bar{\delta}}$	$h^1_{d+d^c}$		$b^- + 1 + h^-_J$			$h^1_{d+d^c}$	
$h^k_{\delta+\bar{\delta}}$	$h^1_{d+d^c}$		$b^- + 1 + h^-_J$			$h^1_{d+d^c}$	
$h^{p,q}_d$	$\frac{1}{2}h^1_{d+d^c}$	$\frac{1}{2}h^1_{d+d^c}$	0	$b^- + 1$	0	$\frac{1}{2}h^1_{d+d^c}$	$\frac{1}{2}h^1_{d+d^c}$
$h^k_{d+d^c}$	$h^1_{d+d^c}$		$b^- + 1 + h^-_J$			b_1	

Example 5.3.2 (The number $h^1_{d+d^c}$ distinguishes between almost Kähler structures). Let Sol(3) be the only 3-dimensional solvable, non-nilpotent, Lie group. Let M be the 4-manifold obtained as a quotient of $\mathrm{Sol}(3) \times \mathbb{R}$ by a co-compact lattice. A co-frame of left-invariant forms on M is given by $\{e^1, e^2, e^3, e^4\}$ with structure equations

$$de^3 = -e^{13} \quad \text{and} \quad de^4 = e^{14}.$$

The manifold M admits no complex structures, see [16], but it admits a symplectic structure

$$\omega := e^{12} + e^{34}.$$

For $t \in \mathbb{R}$ small enough, let J_t be the family of ω-compatible almost Kähler structures defined in Section 6.1 in [38]. Explicitly, a basis of $(1,0)$-forms is given by

$$\phi^1_t = e^1 + i\left(\frac{1+t^2}{1-t^2}e^2 - \frac{2t}{1-t^2}e^4\right) \quad \text{and}$$
$$\phi^2_t = e^3 + i\left(\frac{2t}{1-t^2}e^2 + \frac{1+t^2}{1-t^2}e^4\right).$$

Direct computations show that real forms are expressed in terms of complex forms as

$$e^1 = \frac{1}{2}(\phi^1_t + \phi^{\bar{1}}_t), \quad e^3 = \frac{1}{2}(\phi^2_t + \phi^{\bar{2}}_t)$$

and

$$e^4 = -\frac{i}{2}\,\frac{(1-t^2)(1+t^2)}{1+6t^2+t^4}\left(\phi^2_t - \phi^{\bar{2}}_t - \frac{2t}{1+t^2}(\phi^1_t - \phi^{\bar{1}}_t)\right).$$

Consequently, the differentials of the $(1,0)$-co-frame are

$$d\phi^1_t = -\frac{1}{2}\,\frac{t(1+t^2)}{1+6t^2+t^4}\left(\phi^{12}_t - \phi^{1\bar{2}}_t + \phi^{\bar{1}2}_t - \phi^{\bar{1}\bar{2}}_t + \frac{4t}{1+t^2}\phi^{1\bar{1}}_t\right)$$

and

$$d\phi^2_t = -\tfrac{1}{4}(\phi^{12}_t + \phi^{1\bar{2}}_t + \phi^{\bar{1}2}_t + \phi^{\bar{1}\bar{2}}_t) + \tfrac{1}{4}\,\tfrac{(1+t^2)^2}{1+6t^2+t^4}\left(\phi^{12}_t - \phi^{1\bar{2}}_t + \phi^{\bar{1}2}_t - \phi^{\bar{1}\bar{2}}_t + \tfrac{4t}{1+t^2}\phi^{1\bar{1}}_t\right).$$

Recall that $h^1_{d+d^c}(J_t) = 2\dim_{\mathbb{C}}(\ker d \cap A^{1,0}_t)$ and let

$$\alpha = f\phi^1_t + g\phi^2_t$$

be a d-closed $(1,0)$-form. In particular, we have that

$$0 = \bar{\mu}\alpha = \left(\frac{1}{2}\,\frac{t(1+t^2)}{1+6t^2+t^4}f - \frac{1}{4}\,g\Big(1 + \frac{(1+t^2)^2}{1+6t^2+t^4}\Big)\right)\phi^{\bar{1}\bar{2}}_t,$$

which implies that

$$g = c_t f, \quad \text{with } c_t = \frac{t(1+t^2)}{1+4t^2+t^4}. \tag{5.3.1}$$

Taking the coefficients of $d\alpha$ corresponding to $\phi_t^{\bar{1}2}$ and $\phi_t^{1\bar{1}}$, we get the following system of equations

$$\begin{cases} \xi_{\bar{1}}(g) - \frac{1}{2}\frac{t(1+t^2)}{1+6t^2+t^4} f - \frac{1}{4} g \left(1 - \frac{(1+t^2)^2}{1+6t^2+t^4}\right) = 0, \\ -\xi_{\bar{1}}(f) - \frac{2t^2}{1+6t^2+t^4} f + \frac{t(1+t^2)}{1+6t^2+t^4} g = 0. \end{cases} \tag{5.3.2}$$

Combining (5.3.1) and (5.3.2), we deduce that:

- for $t \neq 0$ small enough, it must be $f = g = 0$. Thus $\alpha = 0$ and $h^1_{d+d^c}(J_t) = 0$;
- for $t = 0$, we have $g = 0$. From the equation $d\alpha = 0$, it is not hard to see that f must be constant, that $\alpha = f\phi_0^1$ and that $h^1_{d+d^c}(J_t) = 2$. ■

As a consequence, we have that $h^1_{d+d^c}$ can be used to distinguish between almost complex structures compatible with the same symplectic form.

Proposition 5.3.3. *There exists a compact symplectic 4-manifold (M, ω) admitting two ω-compatible almost Kähler structures J_1 and J_2 such that*

$$h^1_{d+d^c}(J_1) \neq h^1_{d+d^c}(J_2).$$

In particular, $h^1_{d+d^c}$ distinguishes between almost complex structures compatible with the same symplectic form.

The utility of $h^1_{d+d^c}$ in distinguishing between almost complex structure is not limited to low-dimensional examples.

Proposition 5.3.4. *There exists a smooth 6-manifold admitting two almost complex structures J_1 and J_2 such that*

$$h^1_{d+d^c}(J_1) \neq h^1_{d+d^c}(J_2).$$

In particular, $h^1_{d+d^c}$ allows to distinguish between almost complex structures. Moreover, one can take J_1 and J_2 in such a way that

$$\operatorname{rk} N_{J_1|_x} = \operatorname{rk} N_{J_2|_x}$$

at every point x and $h^1_{d+d^c}(J_1) \neq h^1_{d+d^c}(J_2)$. Thus $h^1_{d+d^c}$ allows to distinguish between almost complex structures whose Nijenhuis tensors have the same rank.

Proof. Let $\mathcal{I}$ be the Iwasawa manifold endowed with the standard complex structure determined by the $(1,0)$-co-frame

$$\phi^1 := dz^1, \quad \phi^2 := dz^2 \quad \text{and} \quad \phi^2 := dz^3 - z_1 dz^2,$$

with differentials

$$d\phi^1 = 0, \quad d\phi^2 = 0 \quad \text{and} \quad d\phi^3 = -\phi^{12}.$$

Let J_1 be the almost complex structure defined by the $(1,0)$-forms

$$\omega^1 := \phi^1, \quad \omega^2 := \phi^2 + \phi^{\bar{3}} \quad \text{and} \quad \omega^3 := \phi^3,$$

with differentials

$$d\omega^1 = 0, \quad d\omega^2 = -\omega^{\bar{1}\bar{2}} + \omega^{\bar{1}3} \quad \text{and} \quad d\omega^3 = -\omega^{12} + \omega^{1\bar{3}}.$$

Then $\operatorname{rk} N_{J_1} = 1$ at every point. We show that

$$h^1_{d+d^c}(J_1) = 2\dim_{\mathbb{C}}(\ker d \cap A^{1,0}) = 2.$$

Let $\alpha = f\omega^1 + g\omega^2 + h\omega^3 \in \ker d \cap A^{1,0}$, with f, g and $h \in C^\infty(M)$. Then

$$0 = d\alpha = df \wedge \omega^1 + dg \wedge \omega^2 + dh \wedge \omega^3 - g\omega^{\bar{1}\bar{2}} + g\omega^{\bar{1}3} - h\omega^{12} + h\omega^{1\bar{3}}.$$

Taking the coefficient of $\omega^{\bar{1}\bar{2}}$, we see that $g = 0$, while taking the coefficients of $\omega^{\bar{j}3}$, for $j = 1,2,3$, we obtain $\bar{\partial} h = 0$, so that h is constant. This leaves us with the equation

$$df \wedge \omega^1 - h\omega^{12} + h\omega^{1\bar{3}} = 0. \tag{5.3.3}$$

Let $\{\psi_j\}_{j=1}^3$ be the dual frame to $\{\omega^j\}_{j=1}^3$. Explicitly, we can write

$$\psi_1 = \frac{\partial}{\partial z_1}, \quad \psi_2 = \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3} \quad \text{and} \quad \psi_3 = \frac{\partial}{\partial z_3} - \frac{\partial}{\partial z_{\bar{2}}} - \bar{z}_1 \frac{\partial}{\partial z_{\bar{3}}}.$$

In particular, it is immediate to check that $[\psi_j, \psi_{\bar{j}}] = 0$ for $j = 1,2,3$. By (5.3.3), we have that $\psi_{\bar{1}}(f) = \psi_{\bar{2}}(f) = 0$ and $\psi_{\bar{3}}(f) = h$, which implies that

$$(\psi_1\psi_{\bar{1}} + \psi_2\psi_{\bar{2}} + \psi_3\psi_{\bar{3}})f = \psi_3(h) = 0,$$

since h is constant. The operator $(\psi_1\psi_{\bar{1}} + \psi_2\psi_{\bar{2}} + \psi_3\psi_{\bar{3}})$ is real and elliptic, thus f is constant and $h = 0$, proving that $h^1_{d+d^c}(J_1) = 2$.
Let J_2 be the almost complex structure defined by the $(1,0)$-forms

$$\omega^1 := \phi^1, \quad \omega^2 := \phi^2 \quad \text{and} \quad \omega^3 := \phi^3 + \frac{1}{2}\phi^{\bar{3}}.$$

We still have $\operatorname{rk} N_{J_2} = 1$. In the same way as we did in the first part of the proof, one sees that $h^1_{d+d^c}(J_2) = 4$, proving the claim. □

Remark 5.3.5. *By taking products of M with a complex torus, we produce examples in higher dimension. More examples where $h^1_{d+d^c}$ distinguishes between almost complex structures can be found directly on the torus T^{2m}. By Example 2.1.3, there exists almost complex structures J_k on T^{2m}, with $k \in \{0, \dots, m\}$ and $2m \geq 6$, such that $\operatorname{rk} N_{J_k} = k$. A straightforward computation shows that,* for those specific structures, *we have $h^1_{d+d^c} = 2(m-k)$.*

Proposition 5.3.4 shows that the number $h^1_{d+d^c}$ distinguishes between almost complex structures whose Nijenhuis tensors have both rank 1. However, it cannot distinguish between maximally non-integrable almost complex structures on manifolds of dimension at least 6.

Proposition 5.3.6. *Let (M, J) be an almost complex $2m$-manifold. If $2m \geq 6$ and there exists $x \in M$ such that $\operatorname{rk} N_J|_x$ is maximal, then $h^1_{d+d^c} = 0$.*

Proof. By assumption, the map $\bar{\mu}_x \colon A^{1,0}_x \to A^{0,2}_x$ is injective. Therefore, there are no global d-closed $(1,0)$-forms. Indeed, assume by contradiction that $\alpha \in A^{1,0}$ is d-closed and non-zero. Then in a neighborhood of $x \in M$ we would have $\bar{\mu}\alpha = 0$, contradicting the assumption that $\bar{\mu}_x$ is injective. □

A similar statement for 4-manifolds cannot hold. For instance, the almost complex structure on the Kodaira–Thurston manifold we consider in Section 4.5 is maximally non-integrable and it has $h^1_{d+d^c} = 2$. Another obstruction for the possible values of $h^1_{d+d^c}$ is given by the first Betti number.

Lemma 5.3.7. *Let (M, J) be an almost complex $2m$-manifold. Then*

$$h^1_{d+d^c} \leq b_1.$$

Proof. We prove that $\mathcal{H}^1_{d+d^c}$ injects into de Rham cohomology. From the definition, it follows that

$$\mathcal{H}^1_{d+d^c} = (A^{1,0} \cap \ker d) \cup (A^{0,1} \cap \ker d).$$

Let $\alpha \in \mathcal{H}^1_{d+d^c}$. Then $\alpha = \alpha^{1,0} + \alpha^{0,1}$, with $\alpha^{1,0}$ and $\alpha^{0,1}$ both d-closed, so that α defines a de Rham class. To prove injectivity, suppose that

$$\alpha^{1,0} + \alpha^{0,1} = df$$

for some $f \in C^\infty(M)$. By bidegree, we have that $\alpha^{1,0} = \partial f$ and $\alpha^{0,1} = \bar{\partial} f$. Since $d\alpha^{1,0} = 0$, we have that $\partial^2 f + \bar{\partial}\partial f + \bar{\mu}\partial f = 0$. In particular, by bidegree reasons we have $\bar{\partial}\partial f = 0$, which implies that f is constant. This gives $\alpha^{1,0} = \partial f = 0$ and $\alpha^{0,1} = \bar{\partial} f = 0$. □

Corollary 5.3.8. *Let (M, J) be an almost complex $2m$-manifold such that $b_1 \in \{0, 1\}$. Then $h^1_{d+d^c} = 0$.*

We invite the reader to compare the results of Lemma 5.3.7 and Corollary 5.3.8 with Corollary 4.6 in [31] and Lemma 4.2 in [51].

5.4 A conjecture on $h^1_{d+d^c}$

In their study of J-anti-invariant cohomology, Draghici, Li and Zhang conjectured that the number h^-_J generically vanishes on almost complex 4-manifolds, that is, that the space of almost complex structures admits an open and dense subset on which $h^-_J = 0$, see Conjecture 2.4 in [34]. The conjecture was proved in [34] in the case $b^+ = 1$, and in [93] in the general case. The number h^-_J, as well as many invariants we consider in this thesis, admits an interpretation as the dimension of the kernel of a suitable elliptic operator. By classical results in operator theory, it is upper semi-continuous with respect to small deformation of the metric and its value can only decrease. Hence, it seems natural to ask if a vanishing result similar to that for h^-_J holds for other almost complex invariants.

In Theorem 5.3.1, we established that the only almost complex invariants of almost Kähler 4-manifolds are the numbers $h^1_{d+d^c}$ and h^-_J. Since we already know the generic vanishing of h^-_J, we focus on the number $h^1_{d+d^c}$. We formulate the following conjecture.

Conjecture. Let M be a compact almost complex $2m$-manifold and let $\mathcal{J}$ be the space of almost complex structures on M. Then $h^1_{d+d^c}$ generically vanishes on $\mathcal{J}$.

Thanks to the approximation theorem for maximally non-integrable structures, we are able to confirm the conjecture on manifolds of dimension at least 10.

Theorem 5.4.1. *Let M be a compact almost complex manifold of dimension $2m \geq 10$. Then for a generic almost complex structure we have $h^1_{d+d^c} = 0$.*

Proof. By Theorem 2.2.6, the space of maximally non-integrable structures is open and dense in the space of almost complex structures. By Proposition 5.3.6, if J is maximally non-integrable and $2m \geq 6$ we have that $h^1_{d+d^c}(J) = 0$. □

More in general, on every almost complex manifold of dimension $2m \geq 6$, one has that $h^1_{d+d^c} = 0$ generically on each path connected component of $\mathcal{J}$ on which there exists at least one maximally non-integrable structure. By Corollary 5.3.8, the conjecture also holds on almost complex manifolds with $b_1 = 0$ or $b_1 = 1$ of every dimension.

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