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Soliton gas for the Nonlinear Schrödinger equation

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MA PER SEGUIR VIRTUTE E CANOSCENZA
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Ai miei Nonni, Peppe e Lina,
ed alle mie Zie, Pia e Gina.
Senza di voi, io non sarei qui.

Abstract

In this thesis, we define and study the properties of solutions of a gas of solitons for the Focusing Nonlinear Schrödinger equation. A gas of solitons is an initial data with an infinite number of solitons that is defined via a suitable limit. A N soliton solution is characterised, via the scattering problem, by 2N spectral points, $\{z_j, \overline{z}_j\}_{j=1}^N$ and the corresponding norming constants $\{c_j\}_{j=1}^N$, with z_i and c_j complex numbers. We formulate the inverse spectral problem for the focusing nonlinear Schrödinger equation via a suitable Riemann-Hilbert problem that is amenable to the limit $N \to \infty$. Assuming that the spectral points $\{z_j\}_{j=1}^N$ fill uniformly some domain $\mathscr{D}$ in the upper half plane, we show that in the limit $N \to \infty$ the inverse problem of a soliton gas can be described both by a Riemann-Hilbert problem and by a $\overline{\partial}$-problem. For particular choices of the domains $\mathscr{D}$, we prove that there is a shielding effect and the soliton gas behaves, for (x, t) in compact sets, as a finite multi-soliton solution. In particular when $\mathscr{D}$ is a disk, the soliton gas is a one-soliton solution.

Next we study the case when $\mathscr{D}$ is an ellipse with foci located on the imaginary axis. We prove that the corresponding soliton gas initial data has a step-like oscillatory behaviour decaying exponentially at $+\infty$ while for $x \to -\infty$ the oscillations are described by the Jacobi elliptic function. The long-time asymptotic behaviour of such initial data depends on the ratio of the amplitude of the foci. If such ratio is above a certain threshold then the solution has oscillations described by genus one and three theta-functions in different sectors of the (x, t)-plane otherwise the oscillations are described by genus one, two and three theta functions. The asymptotic solution remains exponentially decreasing in the right-most sector of the (x, t) plane.

Finally, inspired by the soliton gas solution we develop an extension of the Its-Izergin-Korepin-Slavnov theory of *integrable operators* acting on a domain of the complex plane. We show that the resolvent of an integrable operator $\mathcal{K}$ acting on a domain of the complex plane is obtained from the solution of a $\overline{\partial}$-problem. When the problem depends on auxiliary parameters, the related Malgrange one form is closed and coincide with the logarithmic derivative of the Hilbert-Carleman determinant of the operator $\mathcal{K}$. If the $\overline{\partial}$-problem is related to the inverse spectral problem of the soliton gas, then the Hilbert-Carleman determinant is a τ-function of the Kadomtsev-Petviashvili (KP) or Nonlinear Schrödinger hierarchies.

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Contents

Introduction

Overview

In the contemporary realm of Integrable Systems theory, Lax's definition of integrability plays a crucial role, and it is centred around the existence of a pair of linear operators L and A called *Lax's pair.* This definition offers a significant avenue for solving and determining constants of motion within integrable systems characterised by an infinite number of degrees of freedom, namely integrable nonlinear evolution partial differential equations (PDEs). Specifically, an evolution partial differential equation $u_t = K(u, u_x, u_{xx}, \dots)$, where K is in general a rational function of the dependent variable $u = u(x,t)$ and its spatial derivatives, is integrable if there exists a pair of linear operators $L(u), A(u)$, such that :

$$\frac{\mathrm{d}L(u)}{\mathrm{d}t} = [A(u), L(u)] \Leftrightarrow u_t = K(u). \tag{0.0.1}$$

From the above formulation it follows that the eigenvalues of $L(u)$ are the constants of motions of the PDE.

This procedure was first discovered for the Korteweg de Vries (KdV) equation

$$u_t - 6uu_x - u_{xxx} = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}^+, \;\; u(x,t) \in \mathbb{R}, \tag{0.0.2}$$

by Gardner, Green, Kruskal and Miura [47] (1967) that were able to "linearize" the KdV equation, with initial data vanishing at infinity, via the two linear operators,

$$L(u) = -\partial_x^2 + u,$$
$$A(u) = 4\partial_x^3 - 6u\partial_x - 3u_x,$$

that now bare the name of Lax operators. Indeed in 1968 P. Lax [70] made the crucial observation that this procedure could be applied to other equations. Subsequently, Zakharov and Shabat [93] (1971) were able to find the Lax pair for the cubic nonlinear Schödinger (NLS) equation and later Ablowitz, Kaup, Newell, and Segur, [2] (1974) extended the procedure to several physically important nonlinear wave systems now known as the AKNS class, that includes the KdV and the NLS equations.

The existence of a Lax pair permits to solve the PDEs by linearizing it in the scattering coordinates for the linear operator L of the Lax pair. The reconstruction of the solution is called Inverse Scattering Transform (IST).

The logic of the procedure is as follows. Let $u_t = K(u, u_x, \dots)$ be a nonlinear evolutionary PDE in one space dimension that can be linearized via the Lax pair

$$L(u)\, f(z,x,t) = zf(z,x,t) \tag{0.0.3a}$$
$$f_t(z,x,t) = A(u)\, f(z,x,t). \tag{0.0.3b}$$

Then the construction of the solution $u(x,t)$ from the initial data $u(x,0)$ consists of the following steps:

1. At time $t = 0$ we solve the scattering problem (0.0.3a) and we find the *scattering data* $\Sigma(z)$;
2. from the evolution equation (0.0.3b), we determine how the scattering data evolves in time $\Sigma(z,t)$;
3. we reconstruct the solution $u(x,t)$ of the PDEs from the scattering data $\Sigma(z,t)$ by solving an *Inverse Scattering Problem*.

This method facilitated the discovery of distinctive category of solutions for these PDEs known as *solitons solutions*. Solitons are localised waves travelling at constant velocity and interacting elastically. These solutions are distinguished by an initial datum that decays at infinity exponential fast and a pure discrete spectrum for the operator L. Another category of solutions comprises quasi-periodic solutions, expressed through *Theta functions*.

For a review about the Inverse Scattering Theory and solitons, we refer to [3, 37, 45].

Another pivotal element in integrable system is the concept of τ-function introduced in the context of isomonodromic and isospectral deformations. It had been introduced in the 80s by the Kyoto school, with the aim [66] to construct a generalisation of the theta functions appearing, as particular solutions of some nonlinear integrable equations. In the theory of isomonodromic deformations, tau functions are constructed starting from a certain differential 1-form ω defined on the space of the deformation parameters [66]. Under the hypothesis that the parameters are of isomonodromic type, the form ω is closed and the tau function τ is defined (locally and up to a multiplicative constant) by the formula

$$\delta \ln \tau := \omega, \tag{0.0.4}$$

where δ denotes the total differential with respect to the parameters.

On the side of isospectral deformations, Sato [80] defined the tau function starting from his interpretation of the KP hierarchy in terms of the geometry of Grassmannian manifolds. To each solution of the KP hierarchy, one can associate a point W in an infinite dimensional Grassmannian, and the related tau function is the formal series

$$\tau_W := \sum_{\mathsf{Y} \in \mathbb{Y}} s_{\mathsf{Y}} W_{\mathsf{Y}}, \tag{0.0.5}$$

where $\mathbb{Y}$ is the set of partitions, $\{s_{\mathsf{Y}\in\mathbb{Y}}\}$ are the Schur polynomials and $\{W_{\mathsf{Y}\in\mathbb{Y}}\}$ is the set of the Plücker coordinates of W. In [81], Segal and Wilson provided an analytic version of Sato's theory, where formal series are replaced by L^2 functions, and rewrote the tau function as the (analytically well-defined) Fredholm determinant of a certain projection operator. Around the same years Hirota [60] show that the KP τ-functions satisfies a differential equation that is now known as *"Hirota bilinear relation"*. To give a precise historical account, the first example of τ-function in the context of isospectral deformations appeared in 1976 in the work by Dyson [40] who expressed the solution of the KdV equation as a Fredholm determinant of an operator built from the scattering data. On the isomonodromic side, the first example of τ-function is due to Widom [90] in 1974 in the context of Toepltiz operators.

The equivalence between the two definitions of τ functions has been understood only recently [59], [26].

The concept of τ-function has found applications in other branches of mathematics, including random matrix theory [5,58], combinatorics [75,76], enumerative geometry [6, 68].

This thesis comprises two distinct yet interconnected parts. In the first part, we study the Inverse Scattering Transform for non-standard initial data. Specifically, we consider initial data arising form the limiting case when the number N of solitons tends to infinity, also call *soliton gas solution*. These solutions are of particular significance as they enable the study of a broader class of initial data for the Integrable PDEs considered. In the second part, taking inspiration from this class of data, we develop the theory of integrable operators acting on domains of the complex plane and derive their τ function as a Fredholm determinant.

In our work we study the focusing Nonlinear Schrödinger equation (NLS)

$$i\psi_t + \frac{\psi_{xx}}{2} + |\psi|\psi = 0, \tag{0.0.6}$$

where $\psi = \psi(x,t)$ is a complex function of the real variable x and the positive variable t. The spectrum of the Lax operator for the N soliton solution is characterised by a discrete spectrum $\{z_j, \overline{z}_j\}_{j=1}^N$ in the complex plane and N complex numbers $\{c_j\}_1^N$, called norming constants. We consider the case where the points $\{z_j\}_1^N$ and their complex conjugates, as $N \to \infty$, accumulate uniformly in a simply connected domain $\mathscr{D}$ and $\mathscr{D}^*$ respectively. Additionally, the norming constants scale as N^{-1} in the form $c_j := \frac{\mathcal{A}}{N}\beta(z_j, \overline{z}_j)$, where $\mathcal{A}$ is the area of $\mathscr{D}$ and $\beta(z, \overline{z})$ is a smooth and bounded function in $\mathscr{D}$. We reformulate the inverse problem of the N soliton solution as a Riemann-Hilbert problem (RHP) and we can define a limiting Riemann-Hilbert problem as $N \to \infty$. This limiting RHP can also be reformulated as a $\overline{\partial}$-problem. The solution $\psi(x,t)$ of the NLS equation is obtained from the solution of the $\overline{\partial}$-problem. We study the properties of the solution of the soliton gas for specific choices of the domain $\mathscr{D}$ and the interpolating function β.

In the second part of the thesis our focus shifts to the study of the theory of integrable operator acting on domains on the complex plain. Specifically, in the case of integrable operators acting on curves Σ, the work of Its, Izergin, Korepin and Slavnon [62, 63] established a connection between such operators and a specific Riemann-Hilbert problem. Their findings revealed that both the resolvent and the logarithmic derivative of the Fredholm determinant of the operator are expressed in terms of the solution to the Riemann-Hilbert problem. In particular, if the Riemann-Hilbert problem for the resolvent operator is solvable, then the Fredholm determinant does not vanish and it corresponds to the τ-function of the related integrable system.

We extend this theory to encompass integrable operator acting on domains. We demonstrate that the resolvent and the Hilbert-Carleman determinant (i.e. a generalization of the Fredholm determinant) of the operator can be expressed in terms of solutions of a $\overline{\partial}$-problem. Furthermore, we establish that, in some instances, the Hilbert-Carleman determinant actually generates a solution of the KP hierarchy. As an illustrative example, we present the $\overline{\partial}$-problem introduced in part one. Through a straightforward generalization, we prove that the entire *NLS hierarchy* can be recover form the same $\overline{\partial}$-problem.

The structure of the thesis is the following:

In Chapter 1, we introduce the Inverse Scattering Problem for the focusing NLS equation (0.0.6). Building upon the results of Girotti et al. [53], we derive the $\overline{\partial}$-problem for the soliton gas of (0.0.6). Our exploration reveals that this problem can be explicitly solved for specific choices of $\mathscr{D}$ and $\beta(z, \overline{z})$, resulting in a solution $\psi(x, t)$ defined for (x, t) in compact sets. This kind of phenomenon, called *soliton shielding*, is substantiated by both analytical solutions and numerical evidence. The infinity soliton limit is reinforced by proving the convergence of the *tau*-function τ_N, for the N soliton solution, to a Fredholm determinant of an integrable operator $\mathcal{K}$ (Theorem 1.3.1). This Theorem, together with the results achieved in Chapter 5, demonstrate the existence and uniqueness of a solution for the $\overline{\partial}$-problem for (1.1.1). The result of this chapter are taken from our paper *"Soliton shielding of the focusing nonlinear Schrödinger equation"* Physics Review Letter (2023) [13], made in collaboration with M. Bertola and T. Grava.

In Chapter 2, we turn our attention to the scenario where $\mathscr{D}$ takes the form of an elliptic domain, and $\beta(z, \overline{z})$ is analytic. We establish a connection between this problem and an Inverse scattering problem featuring asymptotic step-like initial data. Specifically, as $x \to +\infty$, $\psi_0(x)$ undergoes exponentially decay, while at $x \to -\infty$, it exhibits behavior akin to a *genus one wave* (Theorem 2.2.5).

In Chapter 3 and 4, we explore the long-time asymptotic of $\psi(x, t)$ for $t > 0$ using the Nonlinear steepest Descent theory developed by Deift and Zhou [35]. Through this investigation, we unveil that by manipulating certain parameters, the solution $\psi(x, t)$ exhibits different behaviours in distinct sectors of the (x, t) plane (Theorem 3.0.1 and 4.0.1).

In Chapter 5 we extend the IIKS theory for Integrable operator $\mathcal{K}$ acting on a Hilbert space $L^2(\mathscr{D}, \mathrm{d}^2 w) \otimes \mathbb{C}^n$. We establish a connection between the existence of the resolvent of $\mathcal{K}$ and the existence of a solution to a $\overline{\partial}$-problem (Theorem 5.1.3). Additionally, we prove that the *Hilbert-Carleman determinant*, a generalization of the *Fredholm determinant* for Hilbert-Schmidt operators, not only is a τ-function but also that it solves the KP equation (Theorem 5.3.1). As an example, we consider the $\overline{\partial}$-problem for the NLS equation introduced in Chapter 1 and we extend it to the NLS hierarchy. The result of this chapter are taken from our paper *"Integrable operators, $\overline{\partial}$-problems, KP and NLS hierarchy"*, arXiv:2307.13119v2 [12] written in collaboration with M. Bertola and T. Grava.

We will now describe our result in more detail.

Soliton gas solution

The concept of soliton gas was initially introduced by Zakharov [92] in the context of KdV equation. Specifically, Zakharov's focus was on the interaction of a single trial soliton with a background of infinitely many solitons, and widely spaced, a condition referred to as a *"rarefied"* soliton gas. The soliton density at time t and position x is $\rho(k,x,t)$, where k is the spectral parameter and the condition of having a rarefied gas is

$$\int_{-\infty}^{\infty} \rho(k,x,t) dx \ll 1.$$

Zakharov found out that the velocity of the soliton with spectral parameter k changes according to the following law:

$$v(k,x,t) = 4k^2 + \int_0^{+\infty} \log\left|\frac{k_1+k}{k_1-k}\right| (k^2-k_1^2)\rho(k_1,x,t)\mathrm{d}k_1 \tag{0.0.7}$$

where the density $\rho(k,x,t)$ satisfies the conservation law

$$\partial_t \rho(k,x,t) + \partial_x(v(k,x,t)\rho(k,x,t)) = 0. \tag{0.0.8}$$

Further research by El e. al [42–44] generalize the kinetic equation (0.0.7) to the case of a dense gases for the KdV equation. The main idea was to model the KdV soliton gas as the thermodynamic solitonic limit of a finite gap solution. In other words, for a given N gap solution of KdV, they approached the limiting scenario by shrinking the bands exponentially fast in N while the gap are close at a speed of order $1/N$. This procedure led to the derivation of the following kinetic equation:

$$v(k,x,t) = 4k^2 + \frac{1}{k}\int_0^{+\infty} \log\left|\frac{k+k_1}{k-k_1}\right| \rho(k_1,x,t)[v(k,x,t)-v(k_1,x,t)]\mathrm{d}k_1. \tag{0.0.9}$$

Later on, thanks to the works of Tobvis, El, Roberti, Congy et al. [28, 29, 41, 88] it was possible to expand this theory to encompass the focusing NLS and to explore additional scenarios, such as the *"breather gas"*. In recent years, research groups in England

and France have been investigating this phenomenon numerically and experimentally. These studies have unveil connections with other fields of mathematics and physics. We summarize some of the most important results:

- In [4] and [50], it has been found that random waves in nonlinear conservative media, dubbed *integrable turbulence*, exhibit properties of a dense bound-state soliton gas.
- In [83] and [51], it is shown numerically that the soliton gas theory could be instrumental for the development of the statistical description of the rogue wave formation.
- In [19], it is established a relation between the kinetic equation of a soliton gas and the *generalize hydrodynamic* (GHD) of integrable many-body systems, both quantum and classical.
- In [77] and [87], soliton gas. is generated in a water-tank from random (and non-random) initial data.

For a complete and comprehensive review we refer to [86].

Along this research line, Girotti et al. [53] constructed asymptotic solutions for the KdV equation, providing a description of a particular class of soliton gas states. They started by investigating the inverse problem for the N-soliton solution of the KdV equation and then send $N \to +\infty$, assuming that the spectral points $\{i\kappa_j\}_{j=0}^N \in i\mathbb{R}_+$ and their complex conjugate are uniformly distributed in finite lines (or "bands") $(i\alpha_1, i\alpha_2) \cup (-i\alpha_2, -i\alpha_1)$. For $\lambda \in (\alpha_1, \alpha_2) \cup (-\alpha_2, -\alpha_1)$, they derived a RHP with high oscillatory jumps on the real "bands"

$$X(\lambda,x,t)_+ = X(\lambda,x,t)_- \begin{cases} \begin{bmatrix} 1 & 0 \\ -ir(\lambda)\mathrm{e}^{8\lambda t\left(\lambda^2 - \frac{x}{4t}\right)} & 1 \end{bmatrix} & \text{for } \lambda \in (\alpha_1, \alpha_2), \\ \begin{bmatrix} 1 & ir(\lambda)\mathrm{e}^{-8\lambda t\left(\lambda^2 - \frac{x}{4t}\right)} \\ 0 & 1 \end{bmatrix} & \text{for } \lambda \in (-\alpha_2, -\alpha_1), \end{cases} \tag{0.0.10}$$

with boundary conditions

$$X(\lambda,x,t) \sim \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}(\lambda^{-1}) \text{ as } \lambda \to \infty \tag{0.0.11}$$

$$X(-\lambda,x,t) = X(\lambda,x,t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{0.0.12}$$

They investigated the long time asymptotic of this new RHP using the Nonlinear Steepest Descend technique. Their findings revealed that the solution $q(x,t)$ has a step like behaviour. Further for large times the solution behaves as a dispersive shock wave:

$$q(x,t) = \begin{cases} \alpha_2^2 - \alpha_1^2 - 2\alpha_2^2 \mathrm{dn}^2(\alpha(x - 2(\alpha_2^2 + \alpha_1^2)t + \phi) + K(m)\,|m) + \mathcal{O}(t^{-1}) \text{ as } \dfrac{x}{4t} < \eta_{crit} \\ \mathcal{O}(\mathrm{e}^{-Ct}) \text{ with } C > 0, \text{ as } \dfrac{x}{4t} > \alpha_2^2, \end{cases} \tag{0.0.13}$$

where $m = \alpha_1^2/\alpha_2^2$, $\phi \in \mathbb{R}$ and the critical value η_{crit} is obtained by a Whitham modulation equation. In the interval $\eta_{crit} < \frac{x}{4t} < \alpha_2^2$ there is a modular region which connects the two different behaviours. This new perspective on soliton gas was inspired by the concept of the *primitive potential*, introduced by Dyachenko, D. Zakharov and V. Zakharov in [39].

A connection with the kinetic theory of soliton gases was established in [54], where the same ideas were applied to the mKdV equation. The authors found a method to recover the density of state (DOS) of the soliton gas and the eq (0.0.9) from quantities defined in the Nonlinear Steepest Descend theory.

Our objective is to extend this methodology also for focusing NLS equation (0.0.6). Since the nonlinear spectral points $\{z_k, \overline{z_k}\}_{k=1}^N$ can be distributed in all the complex plane except the real line, we specifically consider the scenario where, as $N \to +\infty$, the points z_k uniformly fill a simply connected domains $\mathscr{D}$ in the upper half complex plane and consequently $\overline{z_k}$ fill uniformly the domain $\overline{\mathscr{D}}$. As demonstrated in Chapter 1, the inverse problem of the N-soliton solution for NLS can be reformulated both as a Riemann-Hilbert problem or a $\overline{\partial}$-problem

$$\begin{aligned}&\partial_{\overline{z}}\Gamma(z) = \Gamma(z)\begin{bmatrix} 0 & -\beta^*(z,\overline{z})\mathrm{e}^{-2i(z^2t+zx)}\chi_{\overline{\mathscr{D}}} \\ \beta(z,\overline{z})\mathrm{e}^{2i(z^2t+zx)}\chi_{\mathscr{D}} & 0\end{bmatrix}, \\ &\Gamma(z) \sim \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty,\end{aligned} \tag{0.0.14}$$

where $\beta^*(z,\overline{z}) = \overline{\beta(\overline{z},z)}$.

The existence and uniqueness of the solution is demonstrated by combining three results: the Theorem 1.3.1, the Theorem 5.1.3 and the Lemma 5.1.2.

For specific cases, the problem (0.0.14) can be solved exactly. Initially, we consider a special class of domains $\mathscr{D}$ known as quadrature domains. We establish that in compact sets of $(x,t) \in \mathbb{R}^2$ the solution $\psi(x,t)$ of focusing NLS is an n-soliton solution, where n corresponds to the number of points of the quadrature domain $\mathscr{D}$. We call this phenomenon *"soliton shielding"*.

Additionally, we explore the case where $\mathscr{D}$ takes the form of an ellipse. For simplicity, we impose that the foci E_1, E_2 of the ellipse lie in the imaginary axis. In this specific scenario, as detailed in Chapter 2, the analytic part of the solution of the $\overline{\partial}$-problem (0.0.14) satisfies a RHP

$$\begin{aligned}&\Gamma(z)_+ = \Gamma(z)_-\begin{bmatrix} 1 & -\overline{r(\overline{z})}\mathrm{e}^{-2i(zx+z^2t)}\chi_{[\overline{E_2},\overline{E_1}]} \\ r(z)\mathrm{e}^{2i(zx+z^2t)}\chi_{[E_1,E_2]} & 1\end{bmatrix} \\ &\Gamma(z) \sim \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty,\end{aligned} \tag{0.0.15}$$

with $r(z)$ an analytic function, bounded in the segment $[E_1, E_2]$. This RHP is similar to the one studied by de Monvel, Lennels and Shepelsky in [24, 31, 74] for the IST of NLS with step-like initial data.

We investigate the long time asymptotic of this RHP (0.0.15), revealing that the behaviour of $\psi(x,t)$ adopts a step-like oscillatory behaviour akin the scenario observed in the KdV equation. The main distinction arises in the middle region, where the behaviour change, resulting in an increase of genus of the wave. In particular, there is a critical value m_c of the parameter $m = \frac{\mathrm{Im}\,(E_1)^2}{\mathrm{Im}\,(E_2)^2}$, such that two distinct cases emerges:

- for $m > m_c$, only one intermediate region exists (see Figure 1), wherein the wave transitions from genus one to genus three before decreasing exponentially fast. This case will be examined and proved in chapter 3 (Theorem 3.0.1);
- for $m < m_c$, three intermediate regions emerges (see Figure 2). The first one is still of genus one, albeit the topology of the RHP is different from the previews one, denoted as (genus $1)_s$. The second region features a solution of genus 2 and the final region exhibits a solution of genus 3. This case will be treated and proven in chapter 4.

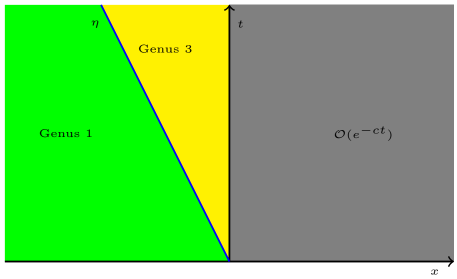


Figure 1: The (x,t) plane in the case where $m > m_c$.

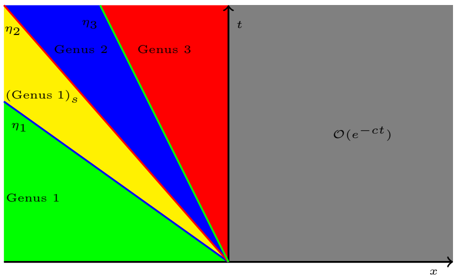


Figure 2: The (x,t) plane in the case where $m < m_c$.

τ–function and IIKS Theory

A fundamental concept in the theory of solvable integrable systems is the τ–function, (see [59] for a comprehensive historical perspective). In many instances, these τ–functions coincide with the Fredholm determinant of an integral operator.

A notable class of integral operators are the so called *integrable operators*: the theory of such operators has its roots in the work of Jimbo et al. [65] that ultimately led to the construction by Its, Izergin, Korepin and Slavnov [62] of a Riemann-Hilbert problem to express the kernel of their resolvent operators. Initially motivated by the theory of quantum integrable models, the theory of integrable operators has found applications in various fields of mathematics such as random matrices and integrable partial differential equations: for example the *gap probabilities* in determinantal random point processes (and more generally the generating function of occupation numbers) are expressible as a Fredholm determinant [84], [32] and this is at the core of the celebrated Tracy-Widom distribution for fluctuations of the largest eigenvalue of a random matrix in a Gaussian Unitary Ensemble [89]. An integrable operator is an integral operator acting on $L^2(\Sigma, |dw|) \otimes \mathbb{C}^n$ of the form

$$\mathcal{K}[v](z) = \int_\Sigma \hat{\mathcal{K}}(z,w)v(w)dw, \quad z \in \Sigma,$$

where Σ is some oriented contour in the complex plane and the kernel $\hat{\mathcal{K}}(z,w) \in Mat(n \times n, \mathbb{C})$ has a special form

$$\hat{\mathcal{K}}(z,w) := \frac{p^T(z)\,q(w)}{z-w}, \quad p(z), q(z) \in Mat(r \times n, \mathbb{C}), \tag{0.0.16}$$

where p and q are rectangular $r \times n$ matrices and for the time being we only assume that p and q are smooth along the connected components of Σ. The condition for K to be non-singular requires

$$p^T(z)\,q(z) \equiv 0.$$

In the most relevant applications, the operators of the form (0.0.16) are trace class operators.

An important observation in [62] is that the resolvent operator $\mathcal{R} := (\mathrm{Id} - \mathcal{K})^{-1} - \mathrm{Id}$, where Id is the identity operator, is of the same class, namely

$$\mathcal{R}[v](z) = \int_\Sigma \hat{\mathcal{R}}(z,w)v(w)dw, \quad z \in \Sigma,$$

where the resolvent kernel has also the form of an integrable operator:

$$\hat{\mathcal{R}}(z,w) := \frac{P^T(z)\,Q(w)}{z-w}, \quad P(z), Q(z) \in Mat(r \times n, \mathbb{C}). \tag{0.0.17}$$

Here $P^T(z) = (\mathrm{Id} - \mathcal{K})^{-1}p^T$ and $Q = q(\mathrm{Id} - \mathcal{K})^{-1}$, where $(\mathrm{Id} - \mathcal{K})^{-1}$ in the first relation is acting to the right while in the second relation its action is to the left. Another crucial observation of [62] (see also the introduction of [61]) is that the determination of $\mathcal{R}$ is equivalent to the solution of an associated RHP for a $r \times r$ matrix $\Gamma(z)$ analytic in $\mathbb{C}\backslash\Sigma$ that satisfies the boundary value relation (sometimes referred to as "jump relation")

$$\begin{aligned} &\Gamma_+(z) = \Gamma_-(z)M(z), \ \ z \in \Sigma, \quad M(z) = \mathbf{1} + 2\pi i p(z) q^T(z) \\ &\Gamma(z) \to \mathbf{1}, \quad \text{as } |z| \to \infty. \end{aligned} \tag{0.0.18}$$

Here $\Gamma_\pm(z)$ denote the boundary values of the matrix $\Gamma(z)$ as z approaches from the left and right of the oriented contour Σ and $\mathbf{1}$ is the identity matrix in $\mathrm{Mat}(r \times r, \mathbb{C})$. The matrices P and Q that define the resolvent kernel (0.0.17) are related to the solution Γ of the RHP (0.0.18) by the relation

$$P(z) = \Gamma(z)p(z), \quad Q(z) = (\Gamma(z)^T)^{-1}q(z). \tag{0.0.19}$$

This formulation implies that the existence of the solution of the RHP (0.0.18) is equivalent to the existence of the resolvent $\mathcal{R}$. This connection between the Fredholm determinant and the RHP has been exploited in several contexts where the kernel depends on large parameters and the study of the asymptotic behaviour of the Fredholm determinant is obtained via the Deift–Zhou nonlinear steepest descent method of the corresponding RHP [35]. This analysis has been successfully implemented for $n = 1$ and $r = 2$ in a large class of kernels originating in random matrices, orthogonal polynomials, probability and partial differential equations see for example (see e.g. [21, 33, 35, 59, 63]).

An enlargement of the class of integrable operators (0.0.16) was studied by Bertola and Cafasso [11] who considered Hankel composition operators that have been reduced to integrable operators in Fourier space.

Recently Bothner [22] and Krajenbrink [69] enlarged the class of Hankel composition operators that can be studied via Riemann-Hilbert problems. Applications are obtained in [23], [25].

In chapter 5, we enlarge the class of integrable operators by considering operators $\mathcal{K}$ acting on $L^2(\mathscr{D}, \mathrm{d}^2 w) \otimes \mathbb{C}^n$ where $\mathscr{D}$ is a bounded domain of the complex plane with a matrix kernel $\hat{\mathcal{K}}(z, \overline{z}, w, \overline{w}) \in Mat(n \times n, \mathbb{C})$, namely

$$\mathcal{K}[v](z, \overline{z}) = \iint_{\mathscr{D}} \hat{\mathcal{K}}(z, \overline{z}, w, \overline{w}) v(w) \frac{\mathrm{d}\overline{w} \wedge \mathrm{d}w}{2i}, \quad z, \overline{z} \in \mathscr{D}, \tag{0.0.20}$$

$$\hat{\mathcal{K}}(z, \overline{z}, w, \overline{w}) := \frac{p^T(z, \overline{z})\, q(w, \overline{w})}{z - w},$$

$$p^T(z, \overline{z})\, q(z, \overline{z}) \equiv 0 \equiv (\partial_{\overline{z}} p(z, \overline{z}))^T q(z, \overline{z}), \quad p, q \in \mathcal{C}^\infty(\mathscr{D}, Mat(r \times n, \mathbb{C})). \tag{0.0.21}$$

The kernel $\hat{\mathcal{K}}(z, \overline{z}, w, \overline{w})$ and the corresponding integral operator $\mathcal{K}$ is a Hilbert–Schmidt operator with a well–defined and continuous value on the diagonal in $\mathscr{D} \times \mathscr{D}$.

Our results are the following.

- In Section 5.1 we show that the resolvent of the integral operator $\mathrm{Id} - \mathcal{K}$ is obtained through the solution of a $\overline{\partial}$-Problem (instead of a Riemann-Hilbert problem) for a matrix–valued function Γ:

$$\begin{aligned} &\partial_{\overline{z}} \Gamma(z, \overline{z}) = \Gamma(z, \overline{z}) M(z, \overline{z}); \qquad \Gamma(z, \overline{z}) \underset{z \to \infty}{\to} \mathbf{1}, \\ &M(z, \overline{z}) = \pi p(z, \overline{z}) q^T(z, \overline{z}) \chi_{\mathscr{D}}(z), \end{aligned} \tag{0.0.22}$$

where $\chi_{\mathscr{D}}(z)$ is the characteristic function of the domain $\mathscr{D}$. Note that the matrix $M(z,\overline{z})$ is nilpotent because of (0.0.21). Furthermore we show, in analogy with integrable operators defined on contours, that the kernel of the resolvent is

$$\hat{\mathcal{R}}(z,\overline{z},w,\overline{w}) = \frac{P(z,\overline{z})^T\, Q(w,\overline{w})}{z-w}, \quad P(z,\overline{z}) = \Gamma(z,\overline{z})p(z,\overline{z}), \quad Q(z,\overline{z}) = \Gamma^{-1}(z,\overline{z})q(z,\overline{z})$$

where Γ solves the $\overline{\partial}$-problem (0.0.22).

- In Section 5.2 we consider the regularized determinant (Hilbert-Carleman determinant, equation 7.8 [55]) of the operator $\mathcal{K}$. This is defined as the Fredholm determinant of the trace class operator $\mathcal{T}_{\mathcal{K}} := \mathrm{Id} - (\mathrm{Id} - \mathcal{K})e^{\mathcal{K}}$, namely

$$\det\nolimits_2(\mathrm{Id} - \mathcal{K}) := \det(\mathrm{Id} - \mathcal{T}_{\mathcal{K}}) = \det((\mathrm{Id} - \mathcal{K})e^{\mathcal{K}}). \tag{0.0.23}$$

Assuming the operator $\mathcal{K}$ depends on some parameters $\boldsymbol{t} = (t_1,\ldots,t_j,\ldots)$, we show that the logarithmic total derivative of the Hilbert-Carleman determinant is a one closed form:

$$\delta \log \det\nolimits_2(\mathrm{Id} - \mathcal{K}) = \omega, \qquad \delta = \sum_{j=1}^{\infty} \partial_{t_j} \mathrm{d}t_j \tag{0.0.24}$$

$$\omega := -\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta M(z)\right) \frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}. \tag{0.0.25}$$

Since $\delta\omega = 0$, in analogy with the literature on Riemann–Hilbert problems on contours [9, 10] we call ω the Malgrange one form of the $\overline{\partial}$-Problem. Since ω is closed is locally exact and therefore there is a function, called the τ function of the $\overline{\partial}$-Problem defined by

$$\delta \log \tau = \omega\,.$$

Therefore we have that

$$\tau = \det\nolimits_2(\mathrm{Id} - \mathcal{K}), \tag{0.0.26}$$

up to a constant. We also show (Subsection 5.2.1) that under the less restrictive assumption that $M(z,\overline{z},\boldsymbol{t})$ is traceless but not nilpotent, we can associate to the $\overline{\partial}$-problem

$$\partial_{\overline{z}}\Gamma(z,\overline{z},\boldsymbol{t}) = \Gamma(z,\overline{z})M(z,\overline{z},\boldsymbol{t}); \qquad \Gamma(z,\overline{z},\boldsymbol{t}) \underset{z\to\infty}{\to} \mathbf{1}, \tag{0.0.27}$$

a Malgrange one-form ω as in the formula (0.0.25). This enables us to define the τ-function of the $\overline{\partial}$-problem by the relation

$$\delta \log \tau = \omega. \tag{0.0.28}$$

We remark that the time deformation is completely general as long as the solution of the the $\overline{\partial}$-problem (0.0.27) exists.

- Finally in Section 5.3 we use the results of the previous section by considering the $\overline{\partial}$-problem (0.0.27) where M is a 2×2 matrix of the form

$$M(z,\bar{z},\boldsymbol{t}) = \mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} M_0(z,\bar{z}) \mathrm{e}^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j$ and $M_0(z,\bar{z})$ is a traceless matrix compactly supported on $\mathscr{D}$; we show that the corresponding τ-function (0.0.28) of the $\overline{\partial}$-problem (0.0.27) is a Kadomtsev-Petviashvili (KP) τ-function, namely it satisfies Hirota bilinear relations for the KP hierarchy (see e.g. [59]).

We then specialize the matrix M of the $\overline{\partial}$-problem in (0.0.27) to the nilpotent and traceless form

$$M(z,\overline{z};x,t) = \pi \mathrm{e}^{-i(zx+z^2t)\sigma_3} p(z,\overline{z}) q^T(z,\overline{z}) \mathrm{e}^{i(zx+z^2t)\sigma_3}, \quad x\in\mathbb{R},\ \ t\geq 0$$

with

$$p(z,\overline{z}) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} -\sqrt{\beta^*(z,\overline{z})}\chi_{\mathscr{D}^*}(z) \\ \sqrt{\beta(z,\overline{z})}\chi_{\mathscr{D}}(z) \end{bmatrix}, \tag{0.0.29}$$

$$q(z,\overline{z}) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \sqrt{\beta(z,\overline{z})}\chi_{\mathscr{D}}(z) \\ \sqrt{\beta^*(z,\overline{z})}\chi_{\mathscr{D}^*}(z) \end{bmatrix} \tag{0.0.30}$$

where $\beta^*(z,\overline{z}) = \overline{\beta(\overline{z},z)}$ is a smooth function and $\chi_{\mathscr{D}}$, $\chi_{\overline{\mathscr{D}}}$ are respectively the characteristic functions of a simply connected domain $\mathscr{D}\subset\mathbb{C}^+$ and its conjugate $\overline{\mathscr{D}}$. Here $\mathbb{C}^+$ is the upper half space. We show that the τ-function of the $\overline{\partial}$-problem (0.0.27) is the τ-function for the focusing Nonlinear Schrödinger (NLS) equation and coincides with Hilbert-Carleman determinant of the operator $\mathcal{K}$ with integrable kernel $\hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) = \frac{p^T(z,\overline{z})\,q(w,\overline{w})}{z-w}$, namely

$$\partial_x^2 \log\tau(x,t) = \partial_x^2 \log\det{}_2(\mathrm{Id}\,-\mathcal{K}) = |\psi(x,t)|^2,$$

where the complex function $\psi(x,t)$ solves the focusing Nonlinear Schrödinger equation (NLS) (0.0.6).

Chapter 1

The Soliton gas for NLS: the $\overline{\partial}$-problem

In this chapter we introduce the problem of the soliton gas solution for the focusing NLS equation (0.0.6). Following the ideas of Girotti et al. [53], we choose to address the problem through Inverse Scattering Transform, introduced for NLS by Zakharov and Shabat in [94]. The main difference with respect to the KdV and mKdV equations is that the nonlinear discrete spectrum $\{z_j\}_{j=1}^N$ lies in all the complex upper half plane. We consider the N soliton solutions with point spectrum $\{z_j\}_{j=1}^N$ that are sample from a constant density function of a bounded domain $\mathscr{D}$ in the upper half complex plane and, as N grows to infinity, these points will fill the domain uniformly.

This chapter is organized as follows:

- In section 1.1, we recover the theory of IST for the NLS equation by describing the various steps and stating the most important results.

- In section 1.2, we rewrite the Inverse Scattering Problem for the N-soliton solution as a Riemann-Hilbert problem and then we perform the limit $N \to +\infty$. Such limit can also be interpreted as a $\overline{\partial}$-problem (0.0.14) on the domain $\mathscr{D}$.

- In section 1.4, we consider some particular choices of the domain $\mathscr{D}$ for which the $\overline{\partial}$-problem (0.0.14) can be explicitly solved. In these cases the infinity soliton limit is reduced to a solution with a finite number of solitons. We call this effect '*soliton shielding*".

1.1 The Inverse Scattering Transform for the NLS equation

The Inverse scattering Transform (IST) for the Nonlinear Schrödinger equation (NLS)

$$i\psi_t + \frac{\psi_{xx}}{2} \pm |\psi|^2\psi = 0, \tag{1.1.1}$$

where the sign $\pm$ separate the focusing (+) case from the defocusing (−) case, was introduced by Zakharov and Shabat in 1980 [94]. Respect to the IST that the was developed by Miura et al. in [72] for KdV equation, the Lax pair is given by 2×2 matrices

$$\partial_x \vec{w} = U\vec{w} := \begin{bmatrix} -iz & \psi \\ \mp\overline{\psi} & iz \end{bmatrix} \vec{w} \tag{1.1.2}$$

and

$$\partial_t \vec{w} = V\vec{w} := \begin{bmatrix} -iz^2 \pm \frac{i}{2}|\psi|^2 & z\psi + \frac{i}{2}\psi_x \\ \mp z\overline{\psi} \pm \frac{i}{2}\overline{\psi_x} & iz \mp \frac{i}{2}|\psi|^2 \end{bmatrix} \vec{w}. \tag{1.1.3}$$

The compatibility conditions between equations (1.1.2) and (1.1.3), which coincide with the zero-curvature condition of the matrices $U(z, x, t)$ and $V(z, x, t)$, is equivalent to the NLS equation (1.1.1)

$$i\psi_t + \frac{\psi_{xx}}{2} \pm |\psi|^2\psi = 0 \quad \Leftrightarrow \quad \partial_t U - \partial_x V + [U, V] \equiv 0. \tag{1.1.4}$$

The steps to solve the NLS equation through IST are the following:

1. Given the initial data $\psi_0(x)$, we solve the *scattering problem* (1.1.2) and we find the scattering data $\Sigma(z)$;
2. From the equation (1.1.3), we determine how the scattering data evolves in time ($\Sigma(x, t)$);
3. We reconstruct the solution $\psi(x, t)$ of NLS (1.1.1) from the scattering data $\Sigma(x, t)$ by solving a Riemann-Hilbert Problem (RHP).

From now on, we focus on the Focusing NLS equation (+ case)

$$i\psi_t + \frac{\psi_{xx}}{2} + |\psi|^2\psi = 0, \tag{1.1.5}$$

and we consider the case where the initial data decays rapidly at $|x| \to +\infty$.

The Scattering problem

The Scattering problem consists in solving the equation (1.1.2)

$$\partial_x \vec{w} = U\vec{w} := \begin{bmatrix} -iz & \psi \\ -\overline{\psi} & iz \end{bmatrix} \vec{w} \tag{1.1.6}$$

with a potential $\psi_0(x)$ that decays to zero as $|x| \to +\infty$, i.e. when $|x|$ is large the solution can be approximated by

$$\vec{w}_0(z,x) = c_1 \begin{bmatrix} \mathrm{e}^{-izx} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{izx} \end{bmatrix}$$

where c_1 and c_2 are arbitrary constants.

We can consider the behaviour for $x \sim \pm\infty$ separately and looking for solutions $\vec{j}^{(1)}_{\pm}(z,x), \vec{j}^{(2)}_{\pm}(z,x)$ such that:

$$\vec{j}^{(1)}_{\pm}(z,x) = \begin{bmatrix} \mathrm{e}^{-izx} \\ 0 \end{bmatrix} + o(1) \qquad \vec{j}^{(2)}_{\pm}(z,x) = \begin{bmatrix} 0 \\ \mathrm{e}^{izx} \end{bmatrix} + o(1) \text{ as } x \to \pm\infty. \tag{1.1.7}$$

These solutions are called *Jost solution* and we indicate it with the 2×2 matrix $\mathbf{J}_{\pm}(z,x) = [\vec{j}^{(1)}_{\pm}(z,x), \vec{j}^{(2)}_{\pm}(z,x)]$.

For $z \in \mathbb{R}$ the equation (1.1.6) has two linearly independent column vector solutions. Since $\mathbf{J}_{\pm}(z,x)$ form a basis of the space of solutions, the matrices $\mathbf{J}_{\pm}(z,x)$ are linked by the transformation:

$$\mathbf{J}_{+}(z,x) = \mathbf{J}_{-}(z,x)\Lambda(z) \text{ for } z \in \mathbb{R}, \tag{1.1.8}$$

where $\Lambda(z)$ is called *Scattering matrix.*

Theorem 1.1.1. *The Jost solutions (1.1.7) exists and satisfy the following properties:*

1. *for* $z \in \mathbb{R}$ $\lim_{x\to\pm\infty} \mathbf{J}_{\pm}(z,x)\mathrm{e}^{izx\sigma_3} = \mathbf{1}$, *where* $\mathbf{1}$ *is the identity matrix and* $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;
2. *for* $z \in \mathbb{R}$ *they satisfy the Schwartz symmetry*

$$\overline{\mathbf{J}_{\pm}(z,x)} = \sigma_2 \mathbf{J}_{\pm}(z,x)\sigma_2, \tag{1.1.9}$$

   where $\sigma_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$;

3. *the functions* $\vec{j}^{(2)}_{+}(z,x)$ *and* $\vec{j}^{(1)}_{-}(z,x)$ *are continuous for* $\mathrm{Im}\,(z) \geq 0$ *and analytic in* $\mathrm{Im}\,(z) > 0$ *and satisfy*

$$\begin{aligned} \vec{j}^{(2)}_{+}(z,x) &= \mathcal{O}(\mathrm{e}^{-\mathrm{Im}\,(z)x}) \text{ as } x \to +\infty \\ \vec{j}^{(1)}_{-}(z,x) &= \mathcal{O}(\mathrm{e}^{\mathrm{Im}\,(z)x}) \text{ as } x \to -\infty \end{aligned}$$

   for $\mathrm{Im}\,(z) < 0$ *and*

$$\lim_{\mathrm{Im}\,(z)\to\infty} \mathrm{e}^{-izx}\vec{j}^{(2)}_{+}(z,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \lim_{\mathrm{Im}\,(z)\to\infty} \mathrm{e}^{izx}\vec{j}^{(1)}_{-}(z,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   for all $x \in \mathbb{R}$;

4. *the functions* $\vec{j}_-^{(2)}(z,x)$ *and* $\vec{j}_+^{(1)}(z,x)$ *are continuous for* $\mathrm{Im}\,(z) \leq 0$ *and analytic in* $\mathrm{Im}\,(z) < 0$ *and satisfy*

$$\vec{j}_-^{(2)}(z,x) = \mathcal{O}(\mathrm{e}^{-\mathrm{Im}\,(z)x}) \text{ as } x \to -\infty$$
$$\vec{j}_+^{(1)}(z,x) = \mathcal{O}(\mathrm{e}^{\mathrm{Im}\,(z)x}) \text{ as } x \to +\infty$$

for $\mathrm{Im}\,(z) < 0$ *and*

$$\lim_{\mathrm{Im}\,(z)\to-\infty} \mathrm{e}^{-izx}\vec{j}_-^{(2)}(z,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \lim_{\mathrm{Im}\,(z)\to-\infty} \mathrm{e}^{izx}\vec{j}_+^{(1)}(z,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for all $x \in \mathbb{R}$.

The proof of this theorem is standard and can be found in any book or lecture notes about Inverse Scattering Theory (see [3]).

From the points (1) (2) of Theorem 1.1.1 we can state the corollary:

Corollary 1.1.2. *For* $z \in \mathbb{R}$, *the scattering matrix* $\Lambda(z)$ *satisfy the following properties:*

1. $\det\Lambda(z) = 1$;
2. $\Lambda(z)$ *satisfy the Schwartz symmetry*

$$\overline{\Lambda(z)} = \sigma_2\Lambda(z)\sigma_2. \tag{1.1.10}$$

This means that $\Lambda(z)$ has the form

$$\Lambda(z) = \begin{bmatrix} \overline{a(z)} & \overline{b(z)} \\ -b(z) & a(z) \end{bmatrix}$$

where

$$a(z) = \det([\vec{j}_-^{(1)}(z,x), \vec{j}_+^{(2)}(z,x)]) \qquad b(z) = \det([\vec{j}_+^{(1)}(z,x), \vec{j}_-^{(1)}(z,x)]), \tag{1.1.11}$$

and they satisfy the condition

$$|a(z)|^2 + |b(z)|^2 = 1.$$

From the points (3) (4) of Theorem 1.1.1 and the equation (1.1.11) we can analytically continued the function $a(z)$ in the upper half plane, while the function $b(z)$ is only defined on the real line. Moreover, from the asymptotic behavior of Jost solutions we get

$$\lim_{\mathrm{Im}\,(z)\to+\infty} a(z) = 1.$$

Let us assume that $a(z)$ has N zeroes in the upper half plane $\{z_j\}_{j=1}^N$. From the determinantal form of $a(z)$, at those points the vectors $\vec{j}_-^{(1)}(z,x)$ and $\vec{j}_+^{(2)}(z,x)$ are linearly dependent

$$\vec{j}_-^{(1)}(z_j,x) = \tilde{\gamma}_j\,\vec{j}_+^{(2)}(z,x) \qquad \forall j = 1,\ldots,N; \tag{1.1.12}$$

with $\tilde{\gamma}_j \in \mathbb{C}$ constant, which is called *norming constant*. Since $\vec{j}_-^{(1)}(z,x)$ and $\vec{j}_+^{(2)}(z,x)$ decays at zero respectively for $x \to \mp\infty$, the points $\{z_j\}_{j=1}^N$ are the *eigenvalues* of the scattering problem.

The Scattering matrix $\Lambda(z)$, the points $\{z_j\}_{j=1}^N$ and the constant $\{\tilde{\gamma}_j\}_{j=1}^N$ are called *scattering data* of our problem.

Time evolution of the scattering data

To find how the scattering data evolves with respect the time parameter t we need to find the factors $c_\pm^{(1)}(z,t)$ and $c_\pm^{(2)}(z,t)$ such that the Jost functions:

$$c_\pm^{(1)}(z,t)\vec{j}_\pm^{(1)}(z,x), \qquad c_\pm^{(2)}(z,t)\vec{j}_\pm^{(2)}(z,x)$$

solves both the equation (1.1.6) and

$$\partial_t \vec{w} = V\vec{w} := \begin{bmatrix} -iz^2 + \frac{i}{2}|\psi|^2 & z\psi + \frac{i}{2}\psi_x \\ -z\overline{\psi} + \frac{i}{2}\overline{\psi_x} & iz - \frac{i}{2}|\psi|^2 \end{bmatrix} \vec{w}. \tag{1.1.13}$$

Proposition 1.1.3. *For $z \in \mathbb{R}$, the matrix functions*

$$\mathbf{J}_\pm(z,x,t)\mathrm{e}^{-iz^2 t\sigma_3} \tag{1.1.14}$$

solves both the equations of the Lax pairs (1.1.6) (1.1.13).

By applying (1.1.14) in the equation (1.1.13) and the using the relation (1.1.8), we arrive to a Lax-type equation for $\Lambda(z,t)$

$$\partial_t \Lambda(z,t) = iz^2 \left[\Lambda(z,t);\sigma_3\right]. \tag{1.1.15}$$

From (1.1.15) we deduce the following facts:

- the function $a(z,t) = a(z)$ in independent on t;
- the function $b(z,t)$ satisfies $b(z,t) = b(z,0)\mathrm{e}^{2iz^2 t}$.

Since $a(z,t) = a(z,0)$ for $z \in \mathbb{R}$ does not change in time, also its analytic continuation in the upper half plane does not depend in time. In particular, its zeros $\{z_j\}_{j=1}^N$ are constants of motion. To find how the norming constant depends on time we need to consider the equation (1.2.3)

$$\vec{j}_-^{(1)}(z_j,x,t) = \tilde{\gamma}_j(t)\,\vec{j}_+^{(2)}(z_j,x,t) \qquad \forall j = 1,\dots,N$$

and derive it respect to time. Then, by considering how $\vec{j}_-^{(1)}(z_j,x,t)$ and $\vec{j}_+^{(2)}(z,x,t)$ depends on time, we arrive at the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}_j = 2iz_j^2\tilde{\gamma}_j,$$

so $\tilde{\gamma}_j(t) = \tilde{\gamma}_j(0)\mathrm{e}^{2iz_j^2 t}$.

The Inverse scattering: Riemann-Hilbert problem

Let us define the matrix function

$$Y(z,x,t) := \begin{cases} [\frac{\mathrm{e}^{\theta(z,x,t)}}{a(z)}\vec{j}_-^{(1)}(z,x,t), \mathrm{e}^{-\theta(z,x,t)}\vec{j}_+^{(2)}(z,x,t)], & \mathrm{Im}\,(z) > 0, \\ [\mathrm{e}^{i(zx+z^2t)}\vec{j}_+^{(2)}(z,x,t), \frac{\mathrm{e}^{-i(zx+z^2t)}}{\overline{a(\overline{z})}}\vec{j}_+^{(2)}(z,x,t)], & \mathrm{Im}\,(z) < 0, \end{cases} \tag{1.1.16}$$

where

$$\theta(z,x,t) = i(zx + z^2t). \tag{1.1.17}$$

We notice that $\overline{a(\overline{z})}$ is analytic function in the lower half-plane, tends to 1 as $z \to \infty$ and vanishes at the complex conjugate of the roots of $a(z)$.

From the properties of the Jost solutions, $Y(z,x,t)$ is analytic in all the complex plane $\mathbb{C}$ except in the real line and at the points $\{z_1,\ldots,z_n,\overline{z_1},\ldots,\overline{z_n}\}$ and has to solve a *Riemann-Hilbert problem*:

Problem 1.1.4. *We are looking for a matrix $Y(z,x,t)$ analytic in $\mathbb{C}\backslash(\mathbb{R}\cup\{z_1,\ldots,z_n,\overline{z_1},\ldots,\overline{z_n}\})$ such that:*

1. **Jump condition:** *for $z \in \mathbb{R}$, $Y(z,x,t)$ takes continuous boundary values and must satisfy the equation*

$$Y(z,x,t)_+ = Y(z,x,t)_- V(z,x,t) \tag{1.1.18}$$

where $Y(z,x,t)_\pm = \lim_{\varepsilon\to 0} Y(z \pm i\varepsilon, x,t)$ and $V(z,x,t)$ is

$$V(z,x,t) := \begin{bmatrix} 1 + |\tilde{r}(z)|^2 & \mathrm{e}^{-2\theta(z,x,t)}\overline{\tilde{r}(z)} \\ \mathrm{e}^{2\theta(z,x,t)}\tilde{r}(z) & 1 \end{bmatrix}, \qquad \tilde{r}(z) := \frac{b(z)}{a(z)}. \tag{1.1.19}$$

2. **Residues:** *at $z = z_j$ and $z = \overline{z_j}$, $Y(z,x,t)$ has simple poles and satisfies*

$$\mathrm{Res}_{z=z_j} Y(z,x,t) = \lim_{z\to z_j} Y(z,x,t) \begin{bmatrix} 0 & 0 \\ c_j e^{2\theta(z,x,t)} & 0 \end{bmatrix} \tag{1.1.20}$$

and

$$\mathrm{Res}_{z=\overline{z_j}} Y(z,x,t) = \lim_{z\to \overline{z_j}} Y(z,x,t) \begin{bmatrix} 0 & -\overline{c_j} e^{-2\theta(z,x,t)} \\ 0 & 0 \end{bmatrix}, \tag{1.1.21}$$

where $c_j = \tilde{\gamma}_j / a'(z_j)$ and $a'(z) = \partial_z a(z)$.

3. **Symmetry:** *$Y(z,x,t)$ satisfies the Schwartz symmetry*

$$\overline{Y(\overline{z},x,t)} = \sigma_2 Y(z,x,t)\sigma_2.$$

4. **Normalization:** *as $z \to \infty$, $Y(z,x,t) \to \mathbf{1}$.*

From the solution of the RHP 1.1.4 given the scattering data, one can reconstruct the solution of the NLS equation (1.1.5) $\psi(x,t)$:

Proposition 1.1.5. *Given a matrix $Y(z,x,t)$ which solves the Riemann-Hilbert problem 1.1.4, then the function*

$$\psi(x,t) := 2i \lim_{z\to\infty} z(Y(z,x,t))_{12} \tag{1.1.22}$$

solves the NLS equation (1.1.5).

1.2 The soliton gas limit: from the RHP to the $\overline{\partial}$-problem

We consider the case where the initial datum $\psi_0(x)$ gives a purely discrete spectrum, i.e. $\hat{r}(z) \equiv 0$. Under this assumption, the RHP 1.1.4 consists just in the residue conditions at the points $\{z_j\}_{j=1}^N$. Our aim is to understand what happen in the case where $N \to +\infty$, so we need a way to rewrite those residue conditions.

We define γ_+ a simple loop oriented anticlockwise that encircles all the poles in $\mathbb{C}^+$ and let D_{γ_+} be the bounded domain with boundary γ_+. In the lower half plane, we define $\gamma_- := -\overline{\gamma_+}$ a simple loop oriented anticlockwise that encircles all the poles in $\mathbb{C}^-$ and let D_{γ_-} be the bounded domain with boundary γ_-.

We apply the transformation

$$\widetilde{Y}(z;x,t) = Y(z;x,t)T_N(z;x,t) \tag{1.2.1}$$

where

$$T_N(z;x,t) := \begin{cases} \begin{bmatrix} 1 & \sum\limits_{j=1}^N \frac{\bar{c}_j e^{-2\theta(z,x,t)}}{z-\overline{z}_j} \\ 0 & 1 \end{bmatrix} & \text{for } z \text{ inside the curve } \gamma_-, \\ \begin{bmatrix} 1 & 0 \\ -\sum\limits_{j=1}^N \frac{c_j e^{2\theta(z,x,t)}}{z-z_j} & 1 \end{bmatrix} & \text{for } z \text{ inside the curve } \gamma_+, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

The new matrix $\widetilde{Y}$ does not have any poles in $\mathbb{C}$, but it has a jump condition for $z \in \gamma_+ \cup \gamma_-$

$$\widetilde{Y}_+(z;x,t) = \widetilde{Y}_-(z;x,t) \begin{bmatrix} 1 & \sum\limits_{j=1}^N \frac{\bar{c}_j e^{-2\theta(z,x,t)}}{z-\overline{z}_j}\chi_{\gamma_-}(z) \\ -\sum\limits_{j=1}^N \frac{c_j e^{2\theta(z,x,t)}}{z-z_j}\chi_{\gamma_+}(z) & 1 \end{bmatrix} \tag{1.2.2}$$

where $\chi_{\gamma_\pm}$ are the characteristic function of the curves $\gamma_\pm$.This new RHP (1.2.2) has the same boundary conditions at infinity as the original RHP 1.1.4.

We now consider the limit the limit for $N \to +\infty$. Let $\mathscr{D}$ (or $\mathscr{D}^*$) be a simply connected domain such that $\mathscr{D} \subset D_{\gamma_+}$ (or $\mathscr{D}^* \subset D_{\gamma_-}$). We assume that the poles z_j lies in the upper half plane and, as N grows, they accumulate uniformly in $\mathscr{D}$. Same situation happen also for their complex conjugate $\overline{z_j}$. We assume also that the norming constants c_j scales as N^{-1} and are interpolated by a smooth function $\beta(z, \overline{z})$, bounded on $\mathscr{D}$, as follows:

$$c_j = \frac{\mathcal{A}}{\pi N}\beta(z_j, \overline{z}_j) \tag{1.2.3}$$

where $\mathcal{A}$ is the Area of the domain $\mathscr{D}$.

Proposition 1.2.1. *Let $c_j \in \mathbb{C}$ be defined as in (1.2.3) and the points $\{z_j\}_{j=1}^N \in \mathscr{D}$ such that, as N grows, they accumulate uniformly in $\mathscr{D}$. Then, for $z \notin \mathscr{D}$ the series*

$$\sum_{j=1}^{N} \frac{c_j}{z - z_j} \tag{1.2.4}$$

converges weakly for $N \to \infty$ to the integral

$$\iint_{\mathscr{D}} \frac{\beta(w, \overline{w})}{z - w}\frac{\mathrm{d}^2 w}{\pi}, \tag{1.2.5}$$

where $\beta(z, \overline{z})$ is a smooth function bounded in $\mathscr{D}$.

Proof We rewrite the quantities in (1.2.4) and (1.2.5) as

$$\sum_{j=1}^{N} \frac{c_j}{z - z_j} = \mathcal{A}\iint_{\mathbb{C}} \frac{\beta(w, \overline{w})}{z - w}\frac{\mathrm{d}\mu_N(w)}{\pi},$$

where μ_N is a discrete measure defined as

$$\mathrm{d}\mu_N(w) := \sum_{j=1}^{N} \frac{\delta_{z_j}}{N}\mathrm{d}^2 w, \qquad \delta_{z_j} = \begin{cases} 1 & z = z_j, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\iint_{\mathscr{D}} \frac{\beta(w, \overline{w})}{z - w}\frac{\mathrm{d}^2 w}{\pi} = \mathcal{A}\iint_{\mathbb{C}} \frac{\beta(w, \overline{w})}{z - w}\frac{\mathrm{d}\mu(w)}{\pi},$$

where μ is the measure defined as

$$\mathrm{d}\mu(w) := \frac{\chi_{\mathscr{D}}(w)}{\mathcal{A}}\mathrm{d}^2 w.$$

The points z_j of the discrete measure $\mathrm{d}\mu_N(w)$ are chosen in such a way that for any $\mathscr{C} \subset \mathbb{C}$ closed

$$N\int_{\mathscr{C}} \mathrm{d}\mu_N(w) = N_{\mathscr{C}}, \quad N_{\mathscr{C}} = \#\{z_j \in \mathscr{D} \text{ s.t. } z_j \in \mathscr{C} \cap \mathscr{D}\}.$$

Then, by Portmanteau's Theorem [15], if $\limsup_N \mu_N(C) \leq \mu(C)$, with C a closed set in $\mathbb{C}$, then the measure μ_N converges weakly to μ. We apply this theorem to our case:

$$\limsup_N \mu_N(\mathscr{C}) = \limsup_N \frac{N_{\mathscr{C}},}{N} \leq \frac{\mathcal{C}}{\mathcal{A}} = \mu(\mathscr{C}),$$

where we denote with $\mathcal{C}$ the area of the set $\mathscr{C} \cap \mathscr{D}$. So the measure μ_N converges to μ, and also the integrals converge

$$\mathcal{A} \iint_{\mathbb{C}} \frac{\beta(w,\overline{w})}{z-w} \frac{\mathrm{d}\mu_N(w)}{\pi} \overset{N\to\infty}{\to} \mathcal{A} \iint_{\mathbb{C}} \frac{\beta(w,\overline{w})}{z-w} \frac{\mathrm{d}\mu(w)}{\pi} = \iint_{\mathscr{D}} \frac{\beta(w,\overline{w})}{z-w} \frac{\mathrm{d}^2 w}{\pi}.$$

■

So, in the limit $N \to +\infty$, the RHP (1.2.2) becomes

$$\begin{aligned} &\widetilde{Y}_+(z;x,t) = \widetilde{Y}_-(z;x,t) J(z;x,t), \\ &\widetilde{Y}_+(z;x,t) \sim \mathbf{1} + \mathcal{O}\left(z^{-1}\right), \text{ as } z \to \infty \end{aligned} \tag{1.2.6}$$

with jump matrix

$$J(z;x,t) = \begin{bmatrix} 1 & \iint\limits_{\mathscr{D}^*} \frac{e^{-2\theta(z,x,t)}\beta^*(w,\bar{w}) d^2 w}{\pi(z-w)} \chi_{\gamma_-}(z) \\ -\iint\limits_{\mathscr{D}} \frac{e^{2\theta(z,x,t)}\beta(w,\bar{w}) d^2 w}{\pi(z-w)} \chi_{\gamma_+}(z) & 1 \end{bmatrix} \tag{1.2.7}$$

where $\beta^*(z,\overline{z}) = \overline{\beta(\overline{z},z)}$.

As last step we want to cancel the jumps across the paths in γ_+ and γ_-. We apply the transformation:

$$\Gamma(z;x,t) = \widetilde{Y}(z;x,t) T_\infty(z,\overline{z}) \tag{1.2.8}$$

where the matrix $T_\infty(z,\overline{z})$ is a smooth complex valued matrix , defined as

$$T_\infty(z,\overline{z}) = \begin{cases} \begin{bmatrix} 1 & -\iint\limits_{\mathscr{D}^*} \frac{e^{-2\theta(z,x,t)}\beta^*(w,\bar{w}) d^2 w}{\pi(z-w)} \\ 0 & 1 \end{bmatrix} & \text{for } z \text{ inside the loop } \gamma_-, \\ \begin{bmatrix} 1 & 0 \\ \iint\limits_{\mathscr{D}} \frac{e^{2\theta(z,x,t)}\beta(w,\bar{w}) d^2 w}{\pi(z-w)} & 1 \end{bmatrix} & \text{for } z \text{ inside the loop } \gamma_+, \\ \mathbf{1} \text{ otherwise .} \end{cases},$$

which tends at the identity matrix as $z \to \infty$.

The new matrix Γ has no jump through the contours γ_+ and γ_- but we do not have an analytic matrix function anymore. Indeed, by applying the $\partial_{\overline{z}}$ operator on (1.2.8) we obtain

$$\partial_{\overline{z}} \Gamma(z,\overline{z}) = \partial_{\overline{z}}(\widetilde{Y}(z) T_\infty(z,\overline{z})) = \widetilde{Y}(z) \partial_{\overline{z}} T_\infty(z,\overline{z}) = \Gamma(z,\overline{z}) M(z,\overline{z})$$

where we define $M(z,\overline{z})$ as

$$M(z,\overline{z}) := T_\infty^{-1}(z,\overline{z})\partial_{\overline{z}}T_\infty(z,\overline{z}). \tag{1.2.9}$$

Since the anti-holomorphic derivative $\partial_{\overline{z}}$ gives a non zero value only on the off-diagonal terms [1], $\partial_{\overline{z}}T_\infty(z,\overline{z})$ is given by

$$\partial_{\overline{z}}T_\infty(z,\overline{z}) = \begin{bmatrix} 0 & -\beta^*(z,\overline{z})e^{-2\theta(z,x,t)}\chi_{\mathscr{D}^*}(z) \\ \beta(z,\overline{z})e^{2\theta(z,x,t)}\chi_{\mathscr{D}}(z) & 0 \end{bmatrix}, \tag{1.2.10}$$

which is nil-potent. So the matrix $M(z,\overline{z})$ (1.2.9) is simply given by

$$M(z,\overline{z}) = T_\infty^{-1}(z,\overline{z})\partial_{\overline{z}}T_\infty(z,\overline{z}) = \begin{bmatrix} 0 & -\beta^*(z,\overline{z})e^{-2\theta(z,x,t)}\chi_{\mathscr{D}^*}(z) \\ \beta(z,\overline{z})e^{2\theta(z,x,t)}\chi_{\mathscr{D}}(z) & 0 \end{bmatrix}.$$

We have shown that from the RHP (1.2.6) with a non-standard jump matrix, we have derived a $\overline{\partial}$-problem

$$\begin{aligned} &\partial_{\overline{z}}\Gamma(z,\overline{z}) = \Gamma(z,\overline{z})M(z,\overline{z}) \\ &\overline{\Gamma(z,\overline{z})} = \sigma_2\Gamma(z,\overline{z})\sigma_2 \qquad \Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z\to\infty. \end{aligned} \tag{1.2.11}$$

Since all the Transformations from $Y(z,x,t)$ to $\Gamma(z,\overline{z})$ tends to the identity as $z\to\infty$, we recover the solution of the NLS equation $\psi(x,t)$ as follows:

$$\psi(x,t) = 2i\lim_{z\to\infty} z(\Gamma(z,x,t))_{12}. \tag{1.2.12}$$

1.2.1 The $\overline{\partial}$-problem

We focus on the study of the $\overline{\partial}$-problem (1.2.11). For particular choices of the function β and of the domain $\mathscr{D}$, we can reduce the $\overline{\partial}$-problem to a standard Riemann-Hilbert problem (Remark 1.2.2). We start by solving the $\overline{\partial}$-problem by splitting it in the components

$$\Gamma(z,\overline{z}) = [\vec{A}(z,\overline{z}) \quad \vec{B}(z,\overline{z})]$$

and then the problem (1.2.11) becomes a systems of two $\overline{\partial}$-problems:

$$\begin{aligned} &\partial_{\overline{z}}\vec{A}(z,\overline{z}) = 0 \\ &\partial_{\overline{z}}\vec{B}(z,\overline{z}) = -\beta^*(z,\overline{z})e^{-2\theta(z;x,t)}\vec{A} \end{aligned} \quad \text{for } z\in\mathscr{D}^* \tag{1.2.13}$$

$$\begin{aligned} &\partial_{\overline{z}}\vec{A}(z,\overline{z}) = \beta(z,\overline{z})e^{2\theta(z;x,t)}\vec{B} \\ &\partial_{\overline{z}}\vec{B}(z,\overline{z}) = 0 \end{aligned} \quad \text{for } z\in\mathscr{D} \tag{1.2.14}$$

with the boundary condition

$$\vec{A} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{B} \sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{as } z \to \infty. \tag{1.2.15}$$

The equations (1.2.13) and (1.2.14) are much more easier to solve and already give us two facts:

- $\vec{A}(z, \overline{z})$ is analytic in $\mathbb{C} \setminus \mathscr{D}$;
- $\vec{B}(z, \overline{z})$ is analytic in $\mathbb{C} \setminus \mathscr{D}^*$.

So, from the Cauchy-Pompeiu formula, we can rewrite the equations (1.2.13) and (1.2.14) as a system of two integral equations

$$\begin{aligned} \vec{A}(z, \overline{z}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \iint_{\mathscr{D}} \frac{\vec{B}(w)\beta(w, \bar{w})e^{2\theta(z;x,t)}}{w - z} \frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{2\pi i} \quad z \in \mathscr{D}, \\ \vec{B}(z, \overline{z}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \iint_{\mathscr{D}^*} \frac{\vec{A}(w)\beta^*(w, \bar{w})e^{-2\theta(z;x,t)}}{w - z} \frac{\mathrm{d}\bar{w} \wedge \mathrm{d}w}{2\pi i} \quad z \in \mathscr{D}^*. \end{aligned} \tag{1.2.16}$$

In general, for $z \notin \mathscr{D} \cup \mathscr{D}^*$, we should have additional analytic terms but in this case, from the smoothness of the problems, those terms are identically zeroes and the equations (1.2.16) hold also for $z \notin \mathscr{D} \cup \mathscr{D}^*$.

With the proper choice of the domain $\mathscr{D}$ and the function $\beta(z, \overline{z})$, this system can be solved exactly.

Let us assume now that $\beta(z)$ is analytic in $\mathscr{D}$ simply connected and the boundary of $\mathscr{D}$ is given by a equation

$$\widetilde{\phi}(z, \overline{z}) = 0. \tag{1.2.17}$$

We also assume that the equation (1.2.17) can be solved for $\overline{z}$ in terms of z, i.e

$$\overline{z} = S(z).$$

The function $S(z)$ is the so–called Schwarz function [30, 57] of the domain $\mathscr{D}$.

The Schwarz function admits an analytic extension to a maximal domain $\mathscr{D}^0 \subset \mathscr{D}$. Here we assume that $\mathscr{L} := \mathscr{D} \setminus \mathscr{D}^0$ consist of a *mother-body*, i.e., a collection of smooth arcs. An example of this is the ellipse where $\mathscr{L}$ is the segment connecting the foci.

For $z \notin \mathscr{D} \cup \mathscr{D}^*$, the integrands of the equations (1.2.16) are analytic, so we can use the Stokes' theorem and the Schwarz function of the domain to reduce the area integral to a contour integral, namely

$$\begin{aligned} \vec{A}(z, x, t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \oint_{\partial\mathscr{D}} \frac{\vec{B}(w, x, t)S(w)\beta(w)e^{2i\theta(z,x,t)}}{w - z} \frac{\mathrm{d}w}{2\pi i} \\ \vec{B}(z, x, t) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \oint_{\partial\mathscr{D}^*} \frac{\vec{A}(w, x, t)S^*(w)\beta^*(w)e^{-2i\theta(z,x,t)}}{w - z} \frac{\mathrm{d}w}{2\pi i}, \ z \notin \mathscr{D} \cup \mathscr{D}^*, \end{aligned} \tag{1.2.18}$$

where the boundary $\partial\mathscr{D}$ is oriented anticlockwise. Since the integrand is analytic, we shrink the contour integral to the mother-body $\mathscr{L}$, namely

$$\begin{aligned}\vec{A}(z,x,t) &= \begin{bmatrix}1\\0\end{bmatrix} - \oint_{\mathscr{L}} \frac{\vec{B}(w,x,t)\Delta S(w)\beta(w)e^{2i\theta(z,x,t)}}{w-z}\frac{dw}{2\pi i}\\ \vec{B}(z,x,t) &= \begin{bmatrix}0\\1\end{bmatrix} + \oint_{\mathscr{L}^*} \frac{\vec{A}(w,x,t)\Delta S^*(w)\beta^*(w)e^{-2i\theta(z,x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i},\ z\notin\mathscr{D}\cup\mathscr{D}^*,\end{aligned} \tag{1.2.19}$$

where $\Delta S(w) = S_-(w) - S_+(w)$, with $S_\pm(z)$ the boundary values of S on the oriented mother-body $\mathscr{L}$. The orientation of $\mathscr{L}$ is inherited by the orientation of $\partial\mathscr{D}$. We can express the system (1.4.1) in matrix form

$$\widetilde{\Gamma}(z,x,t) = \mathbf{1} + \int_{\mathscr{L}\cup\mathscr{L}^*} \frac{(\widetilde{\Gamma}(w,x,t))_-\mathrm{e}^{-i\theta(z,x,t)\sigma_3}\widetilde{M}(w,x,t)\mathrm{e}^{i\theta(z,x,t)\sigma_3}}{w-z}\frac{dw}{2\pi i} \tag{1.2.20}$$

where $\widetilde{\Gamma}(z,x,t)$ coincides with $\Gamma(z,x,t)$ for z outside $\mathscr{D}$ and $\mathscr{D}^*$ and where

$$\widetilde{M}(z,x,t) = \begin{bmatrix}0 & \Delta S^*(z)\beta^*(z)\chi_{\mathscr{L}^*}(z)\\ -\Delta S(z)\beta(z)\chi_{\mathscr{L}}(z) & 0\end{bmatrix} \tag{1.2.21}$$

Using then the Sokhotski-Plemelj formula we can rewrite the above integral equation as a Riemann-Hilbert problem for a matrix function $\widetilde{\Gamma}(z,x,t)$ analytic in $\mathbb{C}\backslash\{\mathscr{L}\cup\overline{\mathscr{L}}\}$ such that

$$\begin{aligned}\widetilde{\Gamma}_+(z,x,t) &= \widetilde{\Gamma}_-(z)e^{-i\theta(z,x,t)\sigma_3}(\mathbf{1}+\widetilde{M}(z,x,t))\mathrm{e}^{i\theta(z,x,t)\sigma_3}, \quad z\in\mathscr{L}\cup\overline{\mathscr{L}},\\ \widetilde{\Gamma}(z,x,t) &= \mathbf{1} + \mathcal{O}(z^{-1}), \quad \text{as } z\to\infty.\end{aligned} \tag{1.2.22}$$

In Chapter 2, we study the above RHP when when the domain $\mathscr{D}$ is an ellipse and the mother-body $\mathscr{L}$ is the segment connecting the foci of the ellipse. In this case, we are able to describe the asymptotic properties of the NLS solution for $t=0$ and large x and for $t\to\infty$.

Remark 1.2.2. *Under the same assumptions on the function $\beta(z,\overline{z})$ and the domain $\mathscr{D}$, the RHP (1.2.22) can be obtained from the RHP (1.2.6) in this way:*

1. *we apply the Stokes' Theorem at the off-diagonal terms of the jump matrix (1.2.7), obtaining a contour integral over the boundary $\partial\mathscr{D}$ (or $\partial\mathscr{D}^*$);*
2. *since the Schwarz function admits an analytic extension on the domain $\mathscr{D}$, we can shrink the contour integral to the mother-body $\mathscr{L}$;*
3. *we apply a transformation*

$$\widetilde{\Gamma}(z,x,t) = \widetilde{Y}(z,x,t)\begin{bmatrix}1 & -\int_{\mathscr{L}^*}\frac{\Delta S^*(w)\beta^*(w)e^{-2i\theta(z,x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i}\chi_{\mathscr{D}^*}(z)\\ \int_{\mathscr{L}^*}\frac{\Delta S(w)\beta(w)e^{2i\theta(z,x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i}\chi_{\mathscr{D}}(z) & 1\end{bmatrix}.$$

In this way, the matrix $\widetilde{\Gamma}$ does not have a jump condition in $z\in\gamma_+\cup\gamma_-$ but on the mother-body $\mathscr{L}\cup\mathscr{L}^*$. The new jump matrix is exactly the one in the RHP (1.2.22).

1.3 Fredholm determinant for the soliton gas

Let us take in consideration the RHP 1.1.4, with $\mathrm{Im}\,(z_j) > 0,\ \forall\ j = 1, \dots, N$. The solution $\widetilde{Y}(z,x,t)$ has the form

$$\widetilde{Y}(z,x,t) = \mathbf{1} + \sum_{j=1}^{N} \frac{\begin{bmatrix} u_j(x,t) & 0 \\ v_j(x,t) & 0 \end{bmatrix}}{z - z_j} + \sum_{j=1}^{N} \frac{\begin{bmatrix} 0 & -\overline{v_j(x,t)} \\ 0 & \overline{u_j(x,t)} \end{bmatrix}}{z - \overline{z_j}}.$$

When we apply the residue conditions, the functions $u_j(x,t), v_j(x,t)$ are determined by the algebraic systems:

$$\begin{bmatrix} u_j(x,t) \\ v_j(x,t) \end{bmatrix} = c_j e^{\theta(z_j,x,t)} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sum_{k=1}^{N} (z_j - \overline{z_k})^{-1} \begin{bmatrix} -\overline{v_k(x,t)} \\ \overline{u_k(x,t)} \end{bmatrix} \right),$$

and

$$\begin{bmatrix} -\overline{v_j(x,t)} \\ \overline{u_j(x,t)} \end{bmatrix} = -\overline{c_j} e^{-\theta(\overline{z_j},x,t)} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sum_{k=1}^{N} (\overline{z_j} - z_k)^{-1} \begin{bmatrix} u_k(x,t) \\ v_k(x,t) \end{bmatrix} \right).$$

By solving this equations and using the result (1.1.22), we obtain a formula for the solution $\psi_N(x,t)$:

$$\psi_N(x,t) = -2i \sum_{j=1}^{N} \overline{v_j(x,t)}. \tag{1.3.1}$$

Another way to recover the N-soliton solution from the scattering data is the Kay-Moses formula [67]

$$|\psi_N(x,t)|^2 = \partial_x^2 \log(\det_{\mathbb{C}^N}(\mathbf{1} + O_N(x,t)\overline{O_N(x,t)})), \tag{1.3.2}$$

where $O_N(x,t)$ is an $N \times N$ matrix with elements given by the scattering data as:

$$(O_N(x,t))_{jk} := \frac{\sqrt{c_j \overline{c_k}} \mathrm{e}^{\theta(z_j,x,t) - \theta(\overline{z_k},x,t)}}{i(z_j - \overline{z_k})}, \qquad O_N(x,t)^T = \overline{O_N(x,t)}. \tag{1.3.3}$$

The determinant in (1.3.2) actually define a τ-function for the NLS equation:

$$\tau_N(x,t) := \det(\mathbf{1} + O_N(x,t)\overline{O_N(x,t)}). \tag{1.3.4}$$

The existence of this determinant was proved by Borghese, Jenkins and Mclaughlin in [20].

We now state the following theorem:

Theorem 1.3.1. *Let $\{z_j\}_{j=1}^N \in \mathbb{C}^+$ be a set of N points such that, as N grows, they accumulates uniformly on a domain $\mathscr{D}$ and $\{c_j\}_{j=1}^N \in \mathbb{C}$ some constants defined in (1.2.3). Then the function $\tau_N(x,t)$ defined in (1.3.4) converges to a Fredholm determinant*

$$\tau(x,t) = \det(\mathrm{Id}_{\,L^2(\mathscr{D} \cup \mathscr{D}^*)} - \mathcal{K}), \tag{1.3.5}$$

where $\mathcal{K}$ is an integrable operator in $L^2(\mathscr{D}\cup\mathscr{D}^*)$ with kernel $\hat{\mathcal{K}}$ given by

$$\hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) = \frac{\sqrt{\beta(z,\overline{z})\beta^*(w,\overline{w})}e^{\theta(z,x,t)-\theta(w,x,t)}\chi_{\mathscr{D}}(z)\chi_{\mathscr{D}^*}(w)}{z-w} - \frac{\sqrt{\beta^*(z,\overline{z})\beta(w,\overline{w})}e^{\theta(w,x,t)-\theta(z,x,t)}\chi_{\mathscr{D}}(w)\chi_{\mathscr{D}^*}(z)}{z-w}. \tag{1.3.6}$$

Proof

Since the matrix $O_N(x,t)$ is an Hermitian matrix, we can represent it as the composition of an operator and its adjoint. We define $\mathcal{M}_N : L^2(x,+\infty)\to\mathbb{C}^N$ and $\mathcal{M}_N^\dagger : \mathbb{C}^N \to L^2(x,+\infty)$ as

$$\mathcal{M}_N[f]_j := \int_x^{+\infty}\sqrt{c_j}e^{\theta(z_j,s,t)}f(s)\mathrm{d}s, \tag{1.3.7}$$

$$\mathcal{M}_N^\dagger[\vec{v}](x) = \sum_{k=1}^N \sqrt{\overline{c_k}}e^{-\theta(\overline{z_k},x,t)}v_k, \tag{1.3.8}$$

then is easy to check that $O_N(x,t)\vec{v} = -(\mathcal{M}_N\circ\mathcal{M}_N^\dagger)[\vec{v}]$. In the same way, we define $\overline{\mathcal{M}}_N : L^2(x,+\infty)\to\mathbb{C}^N$ and $\overline{\mathcal{M}}_N^\dagger : \mathbb{C}^N\to L^2(x,+\infty)$

$$\overline{\mathcal{M}}_N[f]_j := \int_x^{+\infty}\sqrt{\overline{c_j}}e^{-\theta(\overline{z_j},s,t)}f(s)\mathrm{d}s, \tag{1.3.9}$$

$$\overline{\mathcal{M}}_N^\dagger[\vec{v}](x) = \sum_{k=1}^N \sqrt{c_k}e^{\theta(z_k,x,t)}v_k, \tag{1.3.10}$$

such that $\overline{O_N(x,t)}\vec{v} = -(\overline{\mathcal{M}}_N\circ\overline{\mathcal{M}}_N^\dagger)[\vec{v}]$.

From the definition of $\tau_N(x)$ and the permutation symmetry of the determinant we get

$$\begin{aligned}\tau_N(x,t) &= \det(\mathbf{1}+O_N(x,t)\overline{O_N(x,t)}) = \\ &= \det(\mathbf{1}+(\mathcal{M}_N\circ\mathcal{M}_N^\dagger)\circ(\overline{\mathcal{M}}_N\circ\overline{\mathcal{M}}_N^\dagger)) \\ &= \det(id_{L^2(x,+\infty)}+(\overline{\mathcal{M}}_N^\dagger\circ\mathcal{M}_N)\circ(\mathcal{M}_N^\dagger\circ\overline{\mathcal{M}}_N)) \\ &= \det(id_{L^2(x,+\infty)}+\mathcal{B}_N\circ\overline{\mathcal{B}}_N),\end{aligned}$$

with two new operators $\mathcal{B}_N := (\overline{\mathcal{M}}_N^\dagger\circ\mathcal{M}_N)$ and $\overline{\mathcal{B}}_N := (\mathcal{M}_N^\dagger\circ\overline{\mathcal{M}}_N))$ acting on $L^2(x,+\infty)$ that have the form

$$\begin{aligned}\mathcal{B}_N[f] &= \int_x^{+\infty}\hat{\mathcal{B}}_N(x+s)f(s)\mathrm{d}s, \qquad \hat{\mathcal{B}}_N(s) = \sum_{k=1}^N c_k e^{\theta(z_k,s,t)} \\ \overline{\mathcal{B}}_N[f] &= \int_x^{+\infty}\overline{\hat{\mathcal{B}}_N(x+s)}f(s)\mathrm{d}s, \qquad \overline{\hat{\mathcal{B}}_N(s)} = \sum_{k=1}^N \overline{c}_k e^{-\theta(\overline{z_k},s,t)}\end{aligned} \tag{1.3.11}$$

We now rescale the constants in the same way we saw in section 1.2 and let $N \to +\infty$. We state the following lemma. The proof is similar to the one presented in [54].

Lemma 1.3.2. *For (x,t) in compact set and $N \to +\infty$, the operators $\mathcal{B}_N[u]$ and $\overline{\mathcal{B}}_N[u]$, defined in (1.3.11), converge to the operators $\mathcal{B}$ and $\overline{\mathcal{B}}$ acting on $L^2(x,+\infty)$ and defined as*

$$\mathcal{B}[f] := \int_x^{+\infty} \hat{\mathcal{B}}(x+s)f(s)\mathrm{d}s, \qquad \hat{\mathcal{B}}(s) := \iint_{\mathscr{D}} \beta(w,\overline{w})e^{\theta(w,s,t)}\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2\pi i}. \tag{1.3.12}$$

This means that the τ-function $\tau_N(x,t)$ given by a $N\times N$ determinant converges to a τ-function given by the Fredholm determinant of the operator $\mathcal{B}\circ\overline{\mathcal{B}}$ acting on $L^2(x,+\infty)$:

$$\tau(x,t) = \det(\mathrm{Id}_{\,L^2(x,+\infty)} + \mathcal{B}\circ\overline{\mathcal{B}}). \tag{1.3.13}$$

The final part of the proof of the theorem consists in showing that the above Fredholm determinant can be written in the form (1.3.5). The operator $\mathcal{B}$ can be decomposed as the composition of two operators $\mathcal{L}: L^2(\mathscr{D}) \to L^2(x,+\infty)$ and $\mathcal{L}^\dagger : L^2(x,+\infty) \to L^2(\mathscr{D})$ defined in the following way:

$$\mathcal{L}[f] := \iint_{\mathscr{D}} \sqrt{\beta(w,\overline{w})}e^{\theta(w,x,t)}f(w)\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2\pi i}, \tag{1.3.14}$$

$$\mathcal{L}^\dagger[f] := \int_x^{+\infty} \sqrt{\beta(w,\overline{w})}e^{\theta(w,s,t)}f(s)\mathrm{d}s, \quad w\in\mathscr{D} \tag{1.3.15}$$

namely $\mathcal{B} = \mathcal{L}\circ\mathcal{L}^\dagger$. We can do the same also for $\overline{\mathcal{B}}$, decomposing it in two operators $\overline{\mathcal{L}} : L^2(\mathscr{D}^*) \to L^2(x,+\infty)$ and $\overline{\mathcal{L}}^\dagger : L^2(x,+\infty) \to L^2(\mathscr{D}^*)$ defined as:

$$\overline{\mathcal{L}}[f] := \iint_{\mathscr{D}^*} \sqrt{\beta^*(w,\overline{w})}e^{-\theta(w,x,t)}f(w)\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2\pi i}, \tag{1.3.16}$$

$$\overline{\mathcal{L}}^\dagger[f] := \int_x^{+\infty} \sqrt{\beta^*(w,\overline{w})}e^{-\theta(w,s,t)}f(s)\mathrm{d}s, \quad w\in\mathscr{D}^*. \tag{1.3.17}$$

Then the τ-function (1.3.13) becomes

$$\begin{aligned}\tau(x,t) &= \det(\mathrm{Id}_{\,L^2(x,+\infty)} + \mathcal{B}\circ\overline{\mathcal{B}}) = \det(\mathrm{Id}_{\,L^2(x,+\infty)} + (\mathcal{L}\circ\mathcal{L}^\dagger)\circ(\overline{\mathcal{L}}\circ\overline{\mathcal{L}}^\dagger)) =\\ &= \det(\mathrm{Id}_{\,L^2(\mathscr{D}^*)} + (\overline{\mathcal{L}}^\dagger\circ\mathcal{L})\circ(\mathcal{L}^\dagger\circ\overline{\mathcal{L}})) =\\ &= \det(\mathrm{Id}_{\,L^2(\mathscr{D}^*)} + \mathcal{P}\circ\overline{\mathcal{P}}),\end{aligned}$$

where $\mathcal{P} : L^2(\mathscr{D}) \to L^2(\mathscr{D}^*)$ and $\overline{\mathcal{P}} : L^2(\mathscr{D}^*) \to L^2(\mathscr{D})$ are defined as

$$\mathcal{P}[f] = i\iint_{\mathscr{D}} \frac{\sqrt{\beta^*(z,\overline{z})\beta(w,\overline{w})}}{(w-z)}\mathrm{e}^{\theta(w,x,t)-\theta(z,x,t)}f(w)\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2\pi i}, \quad z\in\mathscr{D}^*, \tag{1.3.18}$$

$$\overline{\mathcal{P}}[f] = -i\iint_{\mathscr{D}^*} \frac{\sqrt{\beta(z,\overline{z})\beta^*(w,\overline{w})}}{(w-z)}\mathrm{e}^{\theta(z,x,t)-\theta(w,x,t)}f(w)\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2\pi i}, \quad z\in\mathscr{D}. \tag{1.3.19}$$

We now define the operator $\mathcal{K}$ acting on the space $L^2(\mathscr{D} \cup \mathscr{D}^*) \cong L^2(\mathscr{D}) \bigoplus L^2(\mathscr{D}^*)$ as:

$$\mathcal{K}[f] := -i(\mathcal{P}[f]\chi_{\mathscr{D}^*}(z) + \overline{\mathcal{P}}[f]\chi_{\mathscr{D}}(z)). \tag{1.3.20}$$

From the properties of the determinant of a direct sum, we can write its Fredholm determinant as:

$$\begin{aligned}\det(\mathrm{Id}_{L^2(\mathscr{D}\cup\mathscr{D}^*)} - \mathcal{K}) &= \det(\mathrm{Id}_{L^2(\mathscr{D}\cup\mathscr{D}^*)} + i(\mathcal{P}[u]\chi_{\mathscr{D}^*}(z) + \overline{\mathcal{P}}[u]\chi_{\mathscr{D}}(z))) \\ &= \det_{L^2}\left(\det_{\mathbb{C}^2}\begin{bmatrix}\mathrm{Id}_{L^2(\mathscr{D})} & i\overline{\mathcal{P}} \\ i\mathcal{P} & \mathrm{Id}_{L^2(\mathscr{D}^*)}\end{bmatrix}\right) = \det(\mathrm{Id}_{L^2(\mathscr{D}^*)} + \mathcal{P}\circ\overline{\mathcal{P}}).\end{aligned} \tag{1.3.21}$$

We finally need to show that $\mathcal{K}$ is an integrable operator. This is achieved by rewriting (1.3.20) $\mathcal{K}$ in the form

$$\mathcal{K}[f] = \iint_{\mathscr{D}\cup\mathscr{D}^*} \hat{\mathcal{K}}(z,\overline{z},w,\overline{w})f(w)\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}, \tag{1.3.22}$$

where $\hat{\mathcal{K}}(z,\overline{z},w,\overline{w})$ is given by:

$$\begin{aligned}\hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) = &\frac{\sqrt{\beta(z,\overline{z})\beta^*(w,\overline{w})}e^{\theta(z,x,t)-\theta(w,x,t)}\chi_{\mathscr{D}}(z)\chi_{\mathscr{D}^*}(w)}{z-w} \\ &- \frac{\sqrt{\beta^*(z,\overline{z})\beta(w,\overline{w})}e^{\theta(w,x,t)-\theta(z,x,t)}\chi_{\mathscr{D}}(w)\chi_{\mathscr{D}^*}(z)}{z-w}.\end{aligned} \tag{1.3.23}$$

Then we introduce vectors $\vec{p}(z,\overline{z})$ and $\vec{q}(w,\overline{w})$ defined as:

$$\vec{p}(z,\overline{z}) = e^{-\theta(z,x,t)\sigma_3}\begin{bmatrix}-\sqrt{\beta^*(z,\overline{z})}\chi_{\mathscr{D}^*}(z) \\ \sqrt{\beta(z,\overline{z})}\chi_{\mathscr{D}}(z)\end{bmatrix}, \tag{1.3.24}$$

$$\vec{q}(z,\overline{z}) = \begin{bmatrix}\sqrt{\beta(z,\overline{z})}\chi_{\mathscr{D}}(z) \\ \sqrt{\beta^*(z,\overline{z})}\chi_{\mathscr{D}^*}(z)\end{bmatrix} e^{\theta(z,x,t)\sigma_3} \tag{1.3.25}$$

which satisfies the following properties:

$$\vec{p}(z,\overline{z})^T\,\vec{q}(z,\overline{z}) \equiv 0 \equiv (\partial_{\overline{z}}\vec{p}(z,\overline{z}))^T\,\vec{q}(z,\overline{z}) \qquad \vec{p},\vec{q} \in \mathcal{C}^\infty(\mathscr{D}, Mat(2\times 1,\mathbb{C})). \tag{1.3.26}$$

Then the kernel $\hat{\mathcal{K}}$ can be written as

$$\hat{\mathcal{K}}(z,\overline{z},w,\overline{w}) = \frac{\vec{p}(z,\overline{z})^T\vec{q}(w,\overline{w})}{z-w}$$

which coincides with the definition of an integrable operator.

■

1.4 Exact solution for the $\overline{\partial}$-problem: the soliton shielding phenomenon

In the section 1.2 we presented a way to solve the $\overline{\partial}$-problem (1.2.11) for a generic domain $\mathscr{D}$ and function $\beta(z,\overline{z})$. In this section we choose on $\mathscr{D}$ and $\beta(z,\overline{z})$ such that the system (1.2.16) can be solved exactly and then find which kind of solution of NLS they represent. The remarkable emerging feature is that as $N \to \infty$, for certain types of domains and densities, we have a "soliton shielding", namely, the gas behaves as a *finite* number of solitons.

1.4.1 Soliton gas in a disk

Consider the case where our domain $\mathscr{D}$ is a disk $\mathbb{D}_\rho(z_0)$ with ray ρ and centred in z_0. In particular, we choose ρ such that all the disk is in the upper half plane. We assume that the function $\beta(z,\overline{z})$ is analytic in $\mathbb{D}_\rho(z_0)$.

We analyze the case when $z \notin \mathbb{D}_\rho(z_0) \cup \mathbb{D}_\rho(z_0)^*$.

As we showed in section 1.2, for $\beta(z)$ analytic we can apply Stokes' Theorem and the integral equation (1.2.16) becomes

$$\begin{aligned}\vec{A}(z,x,t) &= \begin{bmatrix}1\\0\end{bmatrix} - \oint_{\partial\mathbb{D}_\rho(z_0)} \frac{\vec{B}(w,x,t)S(w)\beta(w)e^{2i\theta(z,x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i}\\ \vec{B}(z,x,t) &= \begin{bmatrix}0\\1\end{bmatrix} + \oint_{\partial\mathbb{D}_\rho(z_0)^*} \frac{\vec{A}(w,x,t)S^*(w)\beta^*(w)e^{-2i\theta(z,x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i},\end{aligned} \tag{1.4.1}$$

where now the Schwartz function is well define

$$S(z) = \frac{\rho^2}{w-z_0} + \overline{z_0}. \tag{1.4.2}$$

Choosing $z \notin \mathbb{D}_\rho(z_0) \cup \mathbb{D}_\rho(z_0)^*$, those integrals in are given by the residue at z_0. We get that the problem (1.4.1) becomes an algebraic system of four equations

$$\begin{cases}\vec{A}(z,z) = \begin{pmatrix}1\\0\end{pmatrix} + \rho^2 \dfrac{\vec{B}(z_0)\beta(z_0)e^{2\theta(z_0;x,t)}}{z-z_0}\\ \vec{B}(z,z) = \begin{pmatrix}0\\1\end{pmatrix} - \rho^2 \dfrac{\vec{A}(\overline{z_0})\beta^*(\overline{z_0})e^{-2\theta(\overline{z_0};x,t)}}{z-\overline{z_0}}\end{cases} \tag{1.4.3}$$

where the vectors $\vec{A}(\overline{z_0})$ and $\vec{B}(z_0)$ are a solution of the same system when $\vec{A}(z)|_{z=\overline{z_0}}$ and $\vec{B}(z)|_{z=z_0}$.

$$\begin{cases}\vec{A}(\overline{z_0}) = \begin{pmatrix}1\\0\end{pmatrix} + \rho^2 \dfrac{\vec{B}(z_0)\beta(z_0)e^{2\theta(z_0;x,t)}}{\overline{z_0}-z_0}\\ \vec{B}(z_0) = \begin{pmatrix}0\\1\end{pmatrix} - \rho^2 \dfrac{\vec{A}(\overline{z_0})\beta^*(\overline{z_0})e^{-2\theta(\overline{z_0};x,t)}}{z_0-\overline{z_0}}\end{cases} \tag{1.4.4}$$

For $z \in \mathbb{D}_\rho(z_0) \cup \mathbb{D}_\rho(z_0)^*$ we integrate directly the $\overline{\partial}$-problem (1.2.13) and (1.2.14) and we obtain another algebraic system of the form

$$\vec{A}(z,z) = \begin{pmatrix}1\\0\end{pmatrix} + \rho^2 \frac{\vec{B}(z_0)\beta(z_0)e^{2\theta(z_0;x,t)}}{z-z_0} + \left[\overline{z}\vec{B}(z)\overline{\beta(z)}e^{2\theta(z,x,t)} + \vec{X}_1(z)\right]\chi_{\mathbb{D}_\rho(z_0)}$$
$$\vec{B}(z,\overline{z}) = \begin{pmatrix}0\\1\end{pmatrix} - \rho^2 \frac{\vec{A}(\overline{z_0})\beta(\overline{z_0})e^{-2\theta(\overline{z_0};x,t)}}{z-\overline{z_0}} - \left[\overline{z}\beta(z)e^{2\theta(z,x,t)}\vec{A}(z) + \vec{X}_2(z)\right]\chi_{\mathbb{D}_\rho(z_0)^*} \tag{1.4.5}$$

where $\vec{X}_1(z)$ and $\vec{X}_2(z)$ are analytic functions which are determined by imposing that the solutions of the systems (1.4.3) and (1.4.5) coincides for $z \in \partial\mathbb{D}_\rho(z_0) \cup \partial\mathbb{D}_\rho(z_0)^*$. So we have that:

$$\vec{X}_1(z) = -\overline{z_0}\vec{B}(z)\overline{\beta(z)}e^{2\theta(z,x,t)} - \rho^2 \frac{\vec{B}(z)\overline{\beta(z)}e^{2\theta(z,x,t)}}{z-z_0},$$
$$\vec{X}_2(z) = -z_0\beta^*(z)e^{-2\theta(z,x,t)}\vec{A}(z) - \rho^2 \frac{\beta^*(z)e^{-2\theta(z,x,t)}\vec{A}(z)}{z-\overline{z_0}}.$$

For $z \in \mathbb{D}_\rho(z_0)$ or $z \in \mathbb{D}_\rho(z_0)^*$, the solution of system (1.4.5) exists if and only if the vectors $\vec{A}(\overline{z_0}), \vec{B}(z_0)$ solve the system (1.4.4), which is an linear system of 4 equations in 4 variables.

The Solution $\psi(x,t)$ for the focusing NLS

To find out which kind of soliton gas solution the matrix $\Gamma(z,\overline{z})$ yields, we need only the explicit expression of $\vec{B}(z,z)$ in the region $z \notin \mathbb{D}_\rho(z_0) \cup \mathbb{D}_\rho(z_0)^*$. By explicitly solving the system (1.4.4) we find the constants $\vec{A}(z_0)$ and $\vec{B}(\overline{z_0})$

$$\vec{A}(z_0) = \begin{bmatrix} \left(1 - |\beta(z_0)|^2\rho^4 \frac{e^{2\theta(z_0)-2\theta(\overline{z_0})}}{(\overline{z_0}-z_0)^2}\right)^{-1} \\ \rho^2\beta(z_0)\frac{e^{2\theta(z_0)}(\overline{z_0}-z_0)}{(\overline{z_0}-z_0)^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}} \end{bmatrix} \qquad \vec{B}(z_0) = \begin{bmatrix} \rho^2\beta^*(\overline{z_0})\frac{e^{-2\theta(\overline{z_0})}(\overline{z_0}-z_0)}{(\overline{z_0}-z_0)^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}} \\ \left(1 - |\beta(z_0)|^2\rho^4 \frac{e^{2\theta(z_0)-2\theta(\overline{z_0})}}{(z_0-\overline{z_0})^2}\right)^{-1} \end{bmatrix}, \tag{1.4.6}$$

and by substituting $\vec{A}(\overline{z_0})$ in the system (1.4.3) we obtain the explicit expression for $\vec{B}(z,\overline{z})$

$$\vec{B}(z,z) = \begin{bmatrix} \rho^2\frac{\beta^*(\overline{z_0})e^{-2\theta(\overline{z_0})}}{(z-\overline{z_0})} \\ 1 \end{bmatrix} + \frac{|\beta(z_0)|^2\rho^2}{z-\overline{z_0}} \begin{bmatrix} \rho^2\beta^*(\overline{z_0})\frac{e^{-2\theta(\overline{z_0})}}{(z_0-\overline{z_0})^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}} \\ \frac{(\overline{z_0}-z_0)}{(\overline{z_0}-z_0)^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}}. \end{bmatrix} e^{2\theta(z_0)-2\theta(\overline{z_0})}.$$

Then we apply the formula (1.2.12)

$$\begin{aligned}\psi(x,t) &= 2i\lim_{z\to\infty} z\left(\Gamma(z,\overline{z})\right)_{12} = 2i\lim_{z\to\infty} z(\vec{B}(z,\overline{z}))_1\\ &= 2i\rho^2\beta^*(\overline{z_0})e^{-2\theta(\overline{z_0},x,t)}\left(1+\frac{|\beta(z_0)|^2\rho^4 e^{2\theta(z_0)-2\theta(z_0)}}{(z_0-\overline{z_0})^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}}\right)\\ &= 2i\rho^2\beta^*(\overline{z_0})e^{-2\theta(\overline{z_0},x,t)}\left(\frac{(z_0-\overline{z_0})^2}{(z_0-\overline{z_0})^2-\rho^4|\beta(z_0)|^2e^{2\theta(z_0)-2\theta(\overline{z_0})}}\right). \qquad (1.4.7)\end{aligned}$$

we rewrite $z_0 = a+ib$ and $\beta^*(\overline{z_0}) = |\beta(z_0)|\mathrm{e}^{-2i\varphi}$, with $a,b\in\mathbb{R}$ and $\varphi\in[o,\pi]$, then we recover the one soliton solution from (1.4.7) through simply algebraic steps, namely

$$\begin{aligned}\psi(x,t) &= 2ib\left(\frac{2\mathrm{e}^{-2i[(a^2-b^2)t+ax+\varphi]}}{\frac{2b}{\rho^2|\beta(z_0)|}\mathrm{e}^{2b(x+2at)}+\frac{\rho^2|\beta(z_0)|}{2b}}\mathrm{e}^{-2b(x+2at)}\right)\\ &= 2b\mathrm{e}^{-2i[(a^2-b^2)t+ax+\varphi-\frac{\pi}{4}]}\mathrm{sech}(2b[(x-x_0)+2at]), \qquad (1.4.8)\end{aligned}$$

where $x_0 := \frac{1}{2b}\log\left(\frac{\rho^2|\beta(z_0)|}{2b}\right)$.

1.4.2 Soliton gas for quadrature domain

We now consider a new class of domains

$$\mathscr{D} := \left\{z\in\mathbb{C}\text{ s.t. }\left|(z-d_0)^m-d_1\right|<\rho\right\},\quad m\in\mathbb{N}, \qquad (1.4.9)$$

with $d_0\in\mathbb{C}^+$ and $|d_1|,\rho>0$ sufficiently small so that $\mathscr{D}\in\mathbb{C}^+$. When $m=1$ such domain coincides with the disk $\mathbb{D}_\rho(\lambda_0)$ of radius $\rho>0$ centred at $\lambda_0 = d_0+d_1$. For $m>1$ the domain $\mathscr{D}$ has a m–fold symmetry about d_0 and is simply connected if $|d_1|\le\rho$, and otherwise it has m connected components [8].The boundary of $\mathscr{D}$ is described by

$$\overline{z} = S(z),\quad S(z) = \overline{d_0}+\left(\overline{d}_1+\frac{\rho^2}{(z-d_0)^m-d_1}\right)^{\frac{1}{m}}. \qquad (1.4.10)$$

We also assume that $\beta(z,\overline{z}) := n(\overline{z}-\overline{d_0})^n\hat{\beta}(z)$, with $n\in\mathbb{N}$ and $\hat{\beta}(z)$ analytic in $\mathscr{D}$.

The n-soliton solution

Let us choose $n=m$. We then substitute $\overline{w} = S(w)$ and $\beta(z,\overline{z}) := n(\overline{z}-\overline{d_0})^n\hat{\beta}(z)$ in the contour integral (2.1.1) and use the residue theorem at the n poles given by the solution $\{\lambda_0,\dots,\lambda_{n-1}\}$ of the equation $(z-d_0)^n = d_1$. Then, for $z\notin\mathscr{D}\cup\mathscr{D}^*$, we get that:

$$\begin{aligned}\vec{A}(z,x,t) &= \begin{bmatrix}1\\0\end{bmatrix}+\rho^2\sum_{j=0}^{n-1}\frac{\vec{B}_j(x,t)\hat{\beta}(\lambda_j)}{\prod_{k\neq j}(\lambda_j-\lambda_k)}\frac{\mathrm{e}^{2\theta(\lambda_j)}}{z-\lambda_j}\\ \vec{B}(z,x,t) &= \begin{bmatrix}0\\1\end{bmatrix}-\rho^2\sum_{j=0}^{n-1}\frac{\vec{A}_j(x,t)\hat{\beta}(\overline{\lambda_j})}{\prod_{k\neq j}(\overline{\lambda_j}-\overline{\lambda_k})}\frac{\mathrm{e}^{-2\theta(\overline{\lambda_j})}}{z-\overline{\lambda_j}}\end{aligned} \qquad (1.4.11)$$

where $\vec{A}_j(x,t) = \vec{A}(\overline{\lambda_j}, x, t)$ and $\vec{B}_j(x,t) = \vec{B}(\lambda_j, x, t)$, that are given by solving the (1.4.11) with $\vec{A}(z,x,t)|_{z=\overline{\lambda_j}}$ and $\vec{B}(z,x,t)|_{z=\lambda_j}$.

It is easy to notice that, for $z \notin \mathscr{D} \cup \mathscr{D}^*$,the matrix $\Gamma(z,\overline{z})$, given by the system (1.4.11), is actually equal to a matrix $\Gamma_n(z)$, analytic in $\mathbb{C} \setminus \{\lambda_0, \ldots, \lambda_{n-1}, \overline{\lambda_0, \ldots, \lambda_{n-1}}\}$, which solves the residue conditions of the RHP 1.1.4 with $c_j = \rho^2 \hat{\beta}(\lambda_j)/\prod_{k\neq j}(\lambda_j - \lambda_k)$ for $j = 0, \ldots, n-1$. Since only the analytic part of $\Gamma(z,\overline{z})$ contributes to the solution of NLS, $\psi(x,t)$ coincides with the n-soliton solution with spectrum $\{\lambda_0, \ldots, \lambda_{n-1}, \bar{\lambda}_0, , \ldots, \bar{\lambda}_{n-1}, \}$.

The soliton solution of order n

We consider the case where $m = 1$, i.e. $\mathscr{D} = \mathbb{D}_\rho(z_0)$, and $\beta(z,\overline{z}) := n(\overline{z} - \overline{z_0})^n \hat{\beta}(z)$. Then the system (1.4.1) becomes

$$\begin{aligned} \vec{A}(z,x,t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \oint_{\partial \mathbb{D}_\rho(z_0)} \frac{\vec{B}(w,x,t)(S(w) - \overline{z_0})^n \beta(\hat{w}) e^{2i\theta(z,x,t)}}{w - z} \frac{\mathrm{d}w}{2\pi i} \\ \vec{B}(z,x,t) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \oint_{\partial \mathbb{D}_\rho(z_0)^*} \frac{\vec{A}(w,x,t)(S^*(w) - z_0)^n \hat{\beta}^*(w) e^{-2i\theta(z,x,t)}}{w - z} \frac{\mathrm{d}w}{2\pi i}. \end{aligned} \tag{1.4.12}$$

Since the function $(S(w) - \overline{z_0})^n$ and its complex conjugate have poles of order n in z_0 and $\overline{z_0}$ respectively, then the two integrals are given by the residue Theorem

$$\begin{aligned} \vec{A}(z,\overline{z}) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lim_{w\to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}w^{n-1}} \left(\frac{\vec{B}(w)\hat{\beta}(w) e^{2\theta(w,x,t)}}{(w - z)} \right) \\ \vec{B}(z,\overline{z}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \lim_{w\to \bar{z}_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}w^{n-1}} \left(\frac{\vec{A}(w)\hat{\beta}^*(w) e^{-2\theta(w,x,t)}}{(w - z)} \right). \end{aligned} \tag{1.4.13}$$

If we evaluate (1.4.13) respectively for z_0 and $\overline{z}_0$ and we calculate the derivatives:

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}z}(z_0); \ldots; \frac{d^{n-1}\vec{A}}{\mathrm{d}z^{n-1}}(z_0); \frac{\mathrm{d}\vec{B}}{\mathrm{d}z}(\overline{z}_0); \ldots; \frac{\mathrm{d}^{n-1}\vec{B}}{\mathrm{d}z^{n-1}}(\overline{z}_0), \tag{1.4.14}$$

we obtain a linear system of $4n$ equations in $4n$ variables.

Solving this system with the software *"Mathematica"* and using the formula (1.2.12) we obtain a solution of NLS which coincides with the n-degenerate soliton solution, as shown in Figure 1.1.

This kind of solutions has been studied by Bilman and Buckingham in [16, 17]. Specifically, they shown that, for $n \to +\infty$,its near field structure is described by the Painlevé III equation. An analogous asymptotically study has been performed for breathers in [18].

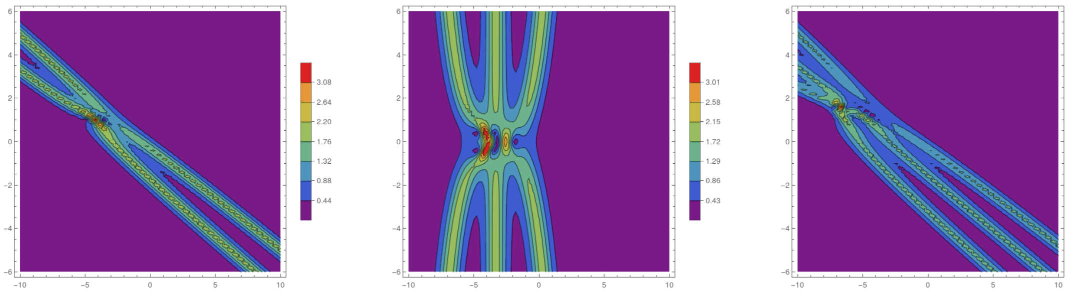


Figure 1.1: On the left the 2-degenerate soliton with $z_0 = 1 + i$. In the center the 3-degenerate soliton with $z_0 = i$. On the right the 3-degenerate soliton with $z_0 = 1 + i$.

1.4.3 Numerical simulations and Random Soliton gas

We present some numerical simulations about the phenomenon of "soliton shielding". In particular, we consider the case where the domain is the disk $\mathbb{D}_\rho(z_0)$, with $\rho = 1/10$, $z_0 = i$ and $\beta(z) = \pi/\rho^2$,and the points $\{z_j\}_{j=1}^N$ sampled in three different way:

1. a deterministic sample, where $\{z_j\}_{j=1}^N$ solves the *"Fekete Problem"* [46],
2. a random sample, where $\{z_j\}_{j=1}^N$ follow an uniform distribution,
3. a random sample, where $\{z_j\}_{j=1}^N$ are distributed according to the *"Ginibre ensemble"* [52].

The deterministic soliton gas

In the deterministic case, we choose a set of N points described by the vector $\boldsymbol{w} = (w_0, \ldots, w_{N-1})$ that minimizes the energy

$$E(\boldsymbol{w}) = -2 \sum_{0 \le j < k \le N-1} \log |w_j - w_k| + \frac{N}{2} \sum_{j=0}^{N-1} |w_j|^2, \tag{1.4.15}$$

(suitably translated/rescaled) over all possible configurations. The points that minimize the function $E(\boldsymbol{w})$ are called *"Fekete points"* and they are the ones who best approximate the uniform distributions on the unit disk [79]. The point $\{z_j\}_{j=1}^N$ are obtained by translating and rescaling the points $\{w_j\}_{j=1}^N$ as follows:

$$z_j = \rho(w_j - z_0) \qquad \text{for } j = 1, \ldots, j.$$

We generate various sets of Fekete points for different values of N and then we construct the solution $\psi_N(x,t)$ by using a Dressing Method algorithm introduced by Gelash et al. in [49]. The results are presented in Figure 1.2. Specifically, we show that the solution $\psi_N(x,t)$ is composed by a train of solitons that is pushed to $x = -\infty$ as $\mathcal{O}(\log N)$ and a limiting soliton centred in $x = 0.226$ which is stable as $N \to \infty$.

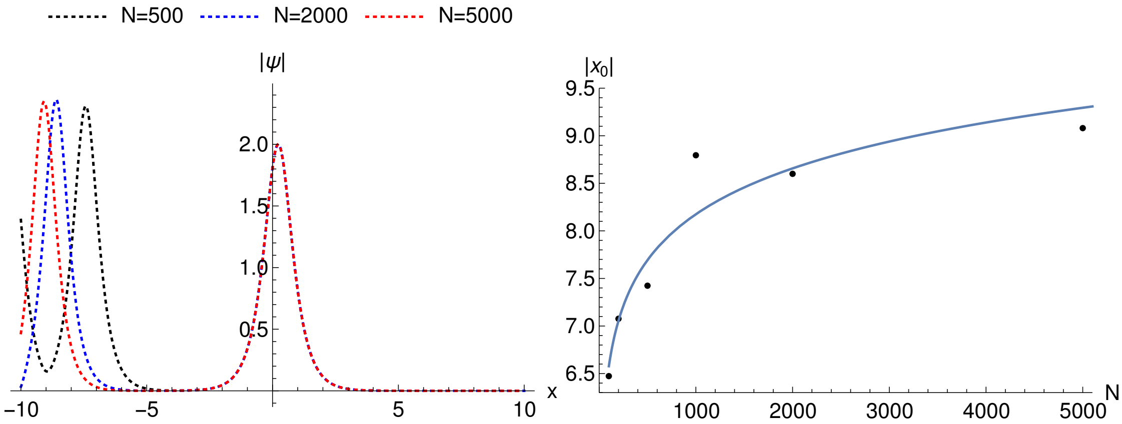


Figure 1.2: On the left, the plot of the gas that approximates the area measure using N Fekete points for $N = 500, 2000, 5000$, all centred in a disk of ray $1/10$ and center $\mathbf{z}_0 = i$ (with $\beta(z) = \pi/\rho^2$), and the emerging limiting (one-soliton) solution $\psi_\infty(x, t)$ centred at $x = 0.226$. On the right, a fit (with a curve of the form $q + p\log(N)$) of the distance between the peak of the limiting soliton solution $\psi_\infty(x, t)$ and the first peak of the remaining part of the solution that is going to infinity as $N \to \infty$.

Random soliton gas with Ginibre and uniform statistics

Let us now introduce randomness in the system by choosing the points $z_j = \rho(w_j - z_0)\chi_{\mathbb{C}^+}$ with $(w_0, \ldots, w_{N-1}) \in \mathbb{C}^N$ distributed according to the probability density (*Ginibre ensemble*)

$$\mu_N = \frac{1}{Z_N} e^{-E(w_0,\ldots,w_{N-1})} d^2w_0 \ldots d^2w_{N-1}, \tag{1.4.16}$$

where Z_N is the normalizing constant and $E(w_0, \ldots, w_{N-1})$ is the energy defined in (1.4.15). In the limit $N \to \infty$ the random points $\{w_0, \ldots, w_{N-1}\}$ fill uniformly the unit disk centered at zero (see e.g. [52]). For any smooth function $h : \mathbb{C} \to \mathbb{C}$, let us consider the random variable $X_h^N := \sum_{j=1}^N h(w_j)$. It is known [78] that

$$\frac{1}{N}\mathbb{E}[X_h] \underset{N\to\infty}{\longrightarrow} \int_{|w|\le 1} h(w) \mathrm{d}^2 w, \tag{1.4.17}$$

where $\mathbb{E}$ is the expectation with respect to the probability measure μ_N. Actually more is true [7] [78]: the limit of the random variable $X_h - \mathbb{E}[X_h]$ converges to a normal random variable $\mathcal{N}(0, \sigma)$ centred at zero and with finite variance σ^2 depending on h.

From the above arguments it is expected that the jump of the RHP (1.2.7), in probability, satisfies

$$\mathbb{P}\left(\left| \sum_{j=0}^{N-1} \frac{\mathcal{A}}{N} \frac{\beta(z_j, \overline{z}_j)}{z - z_j} - \iint_{\mathscr{D}} \frac{\beta(w, \bar{w})}{z - w} \mathrm{d}^2 w \right| > \epsilon \right) = \mathcal{O}(N^{-1}),$$

for $z \notin \mathscr{D}$. Using small norm arguments on the RHP [34], one may argue that the random N soliton solution $\psi_N(x,t,z_0,\ldots,z_{N-1})$ converges as $N\to\infty$ in probability to the one-soliton solution $\psi_\infty(x,t)$. Similar arguments can be used also when the soliton spectrum is sampled according to the uniform distribution on the unit disk. The complete mathematical proof would require a more elaborated argument, which is postponed to a subsequent publication. From numerical simulations, the fluctuations of $\psi_N(x,t,z_0,\ldots,z_{N-1})$ around the limiting value $\psi_\infty(x,t)$ are Gaussian with error that decreases at the rate $\mathcal{O}(N^{-1})$, when the random points $\{z_0,\ldots,z_{N-1}\}$ are sampled from the Ginibre ensemble while the rate is $\mathcal{O}(N^{-1/2})$ for the uniform distribution on the disk, see Figure 1.3.

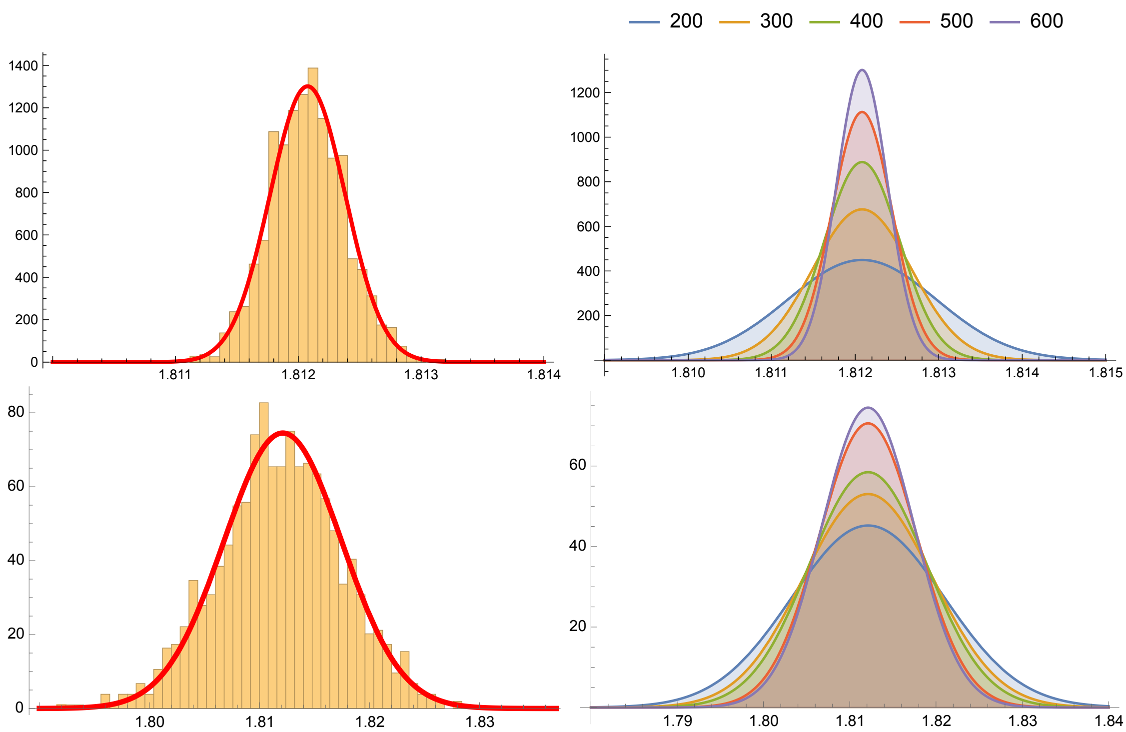


Figure 1.3: Left: the Gaussian fitting of the fluctuations of the $N=600$ soliton solution $\psi_N(0,0)$ with respect to the limiting solution $\psi_\infty(0,0)\simeq 1.812$ and 1000 trials. The point spectrum is sampled in the disk $\mathbb{D}_{1/10}(i)$ according to the Ginibre ensemble (top) and the uniform distribution (bottom). On the right figure the corresponding Gaussian fitting for $N=200,300,400,500,600$. The Gaussian distribution is centred at $\psi_\infty(0,0)$ and the error σ scales numerically as $0.178/N$ (Ginibre) and $0.129/N^{\frac{1}{2}}$ (uniform distribution). The scaling does not depend on the point $x=0, t=0$ chosen to make the statistics.

Chapter 2

Soliton gas on an elliptic domain: step-like oscillatory data

In this chapter we examine another example of the RHP for the soliton gas (1.2.6), where the domain $\mathscr{D} \subset \mathbb{C}^+$ has an elliptic shape and the function $\beta(z, \overline{z})$ is analytic. For simplicity, we assume that the foci $\{E_1, E_2\}$ of the ellipse on the imaginary axis, with $\mathrm{Im}\,(E_2) > \mathrm{Im}\,(E_1) > 0$.

2.1 The Riemann-Hilbert Problem

We consider the RHP (1.2.6) defined in a domain $\mathscr{D}$ with contour given by the equation of the ellipse

$$\sqrt{\mathrm{Re}\,(z)^2 + (\mathrm{Im}\,(z) - \mathrm{Im}\,(E_1))^2} + \sqrt{\mathrm{Re}\,(z)^2 + (\mathrm{Im}\,(z) - \mathrm{Im}\,(E_2))^2} = 2\rho,$$

where $\rho > 0$ is chosen sufficiently small such that $\mathscr{D}$ lies in the upper half plane. We assume also that $\beta(z, \overline{z})$ is analytic in $\mathscr{D}$.

As we already saw in section 1.2, we can apply Stokes' Theorem for $z \notin \mathscr{D}$ and the off-diagonal terms of the jump matrices (1.2.7) becomes

$$\iint\limits_{\mathscr{D}} \frac{\mathrm{e}^{2\theta(z,x,t)}\beta(w)\mathrm{d}^2 w}{\pi(z-w)} = \oint_{\partial\mathscr{D}} \frac{\beta(w)\overline{w}\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i} \tag{2.1.1}$$

and similarly for the integral over $\mathscr{D}^*$. The variable $\overline{w}$ can be expressed via Schwartz function $S(w)$

$$S(w) = 2\frac{\rho^2}{c^2}(w - iy_0) - z + 2\frac{\rho}{c^2}\sqrt{\rho^2 - c^2}R(w), \tag{2.1.2}$$

with $R(w) := \sqrt{(w-E_1)(w-E_2)}$, $y_0 := \frac{\mathrm{Im}\,(E_1)+\mathrm{Im}\,(E_2)}{2}$ and $c := \frac{\mathrm{Im}\,(E_2)-\mathrm{Im}\,(E_1)}{2}$.

This Schwartz function is analytic in $\mathbb{C}$ away from the segment $[E_1, E_2]$, with left/right boundary values $S_\pm(w)$ as $w \in [E_1, E_2]$. Then, for $z \notin \mathscr{D} \cup \mathscr{D}^*$, the integral along the boundary $\partial\mathscr{D}$ (or $\partial\mathscr{D}^*$) can be deformed to a line integral on the segment $[E_1, E_2]$ (or $[\overline{E_2}, \overline{E_1}]$), namely

$$\oint_{\partial\mathscr{D}} \frac{\beta(w)\overline{w}\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i} = \oint_{\partial\mathscr{D}} \frac{\beta(w)S(w)\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}$$
$$= \int_{E_1}^{E_2} \frac{\beta(w)\Delta S(w)\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}$$

where $\Delta S(z) := S_+(z) - S_-(z)$. With an abuse of notation, we define the matrix $\Gamma(z)$ as

$$\Gamma(z) := \begin{cases} \widetilde{Y}(z), & z \in \mathbb{C}\backslash\{D_{\gamma_+} \cup D_{\gamma_-}\} \\ \widetilde{Y}(z)T(z), & z \in D_{\gamma_+} \cup D_{\gamma_-} \end{cases} \tag{2.1.3}$$

where $T(z) =$

$$\begin{bmatrix} 1 & \displaystyle\int_{\overline{E_2}}^{\overline{E_1}} \frac{\beta^*(w)\Delta S^*(w)\mathrm{e}^{-2\theta(z;x,t)}}{w-z}\frac{\mathrm{d}w}{2\pi i}\chi_{\overline{D}_{\gamma_-}}(z) \\ \displaystyle\int_{E_1}^{E_2} \frac{\beta(w)\Delta S(w)\mathrm{e}^{2\theta(z;x,t)}}{z-w}\frac{\mathrm{d}w}{2\pi i}\chi_{\overline{D}_{\gamma_+}}(z) & 1 \end{bmatrix},$$

where $\overline{D}_{\gamma_\pm}$ is the closure of the sets $D_{\gamma_\pm}$. The matrix $\Gamma(z)$ does not have a jump on $\gamma_+ \cup \gamma_-$. Since $T(z)$ has a jump in $[E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]$ it follows that $\Gamma(z)$ is analytic in $\mathbb{C} \setminus \{[E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]\}$ with jump conditions

$$\begin{aligned} &\Gamma_+(z) = \Gamma_-(z)T_-^{-1}T_+ = \Gamma_-(z)\mathrm{e}^{-\theta(z;x,t)\sigma_3}V_0(z)\mathrm{e}^{\theta(z;x,t)\sigma_3},\ z \in [E_1, E_2] \cup [\overline{E_2}, \overline{E_1}] \\ &V_0(z) = \begin{bmatrix} 1 & \chi_{[\overline{E_2},\overline{E_1}]}(z)\Delta S^*(z)\beta^*(z) \\ -\chi_{[E_1,E_2]}(z)\Delta S(z)\beta(z) & 1 \end{bmatrix}, \end{aligned} \tag{2.1.4}$$

and $\Gamma(z) = \mathbf{1} + O(z^{-1})$, as $z \to \infty$. It is possible to find a similar RHP (without the term $\Delta S^*(z)$) also while studying the infinite soliton limit when the spectral points are distributed uniformly along the segments $[E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]$.

The primary challenge of the RHP (2.1.4) lies in the presence of highly oscillatory exponential in the jump matrices, which could cause divergences for $x \to \pm\infty$ and/or $t \to \pm\infty$. The Nonlinear Steepest Descent technique, developed by Deift and Zhou in [35], offers a method to study the behaviour of the matrix $\Gamma(z)$ and, consequently, the solution $\psi(x,t)$ of (1.1.5) for large values of the parameters x and/or t.

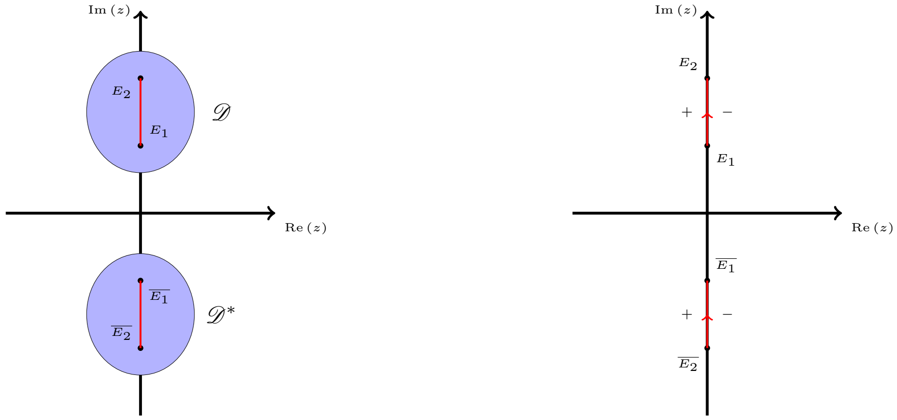


Figure 2.1: On the left the domains $\mathscr{D}$ and $\mathscr{D}^*$. On the right the jump contours of the RHP (2.2.1).

2.2 Asymptotics of the initial data $\psi_0(x)$

We consider the RHP (2.1.4) for $t = 0$. We define $r(z) := \Delta S(z)\beta(z)$, then the RHP becomes

$$\Gamma_+(z) = \Gamma_-(z)\begin{bmatrix} 1 & r^*(z)\mathrm{e}^{-2izx}\chi_{[\overline{E_2},\overline{E_1}]}(z) \\ -r(z)\mathrm{e}^{2izx}\chi_{[E_1,E_2]}(z) & 1 \end{bmatrix},\ z \in [E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]$$
$$\overline{\Gamma(\overline{z})} = \Gamma(z), \qquad \Gamma(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \tag{2.2.1}$$

We notice that, for $z \in [E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]$ the jump matrices has two different behaviours at $x \sim \pm\infty$:

1. for $x \to +\infty$ the jump matrices tends to the identity $\mathbf{1}$. Consequentially, the solution $\Gamma(z)$ tends to the identity exponentially fast as $x \to +\infty$ and, by using the relation (1.2.12), the initial datum $\psi_0(x)$ must have the same exponential behaviour at $x \sim +\infty$:
$$\psi_0(x) \sim \mathcal{O}(e^{-cx})$$
with $c > 0$.
2. for $x \to -\infty$ the jump matrices diverges. So we have to use the Nonlinear steepest descent method to obtain some information about the initial datum $\psi_0(x)$ as $x \sim -\infty$.

We now focus on the second case. The RHP and the analysis are similar to the one studied by Girotti, Tamara, Jenkins and McLaughlin in [53] for the KdV case.

We start the nonlinear steepest descent analysis by apply the following transformation

$$\hat{\Gamma}^{(1)}(z) = F_\infty^{\sigma_3}\Gamma(z)e^{ig(z)x\sigma_3}F(z)^{\sigma_3}, \tag{2.2.2}$$

where $g(z)$,$F(z)$ are two unknown function, analytic in $\mathbb{C}\setminus[\overline{E_2},E_2]$, and they satisfy the conditions:

$$\overline{g(\overline{z})} = g(z),\ \ \overline{F(\overline{z})} = (F(z))^{-1}, \tag{2.2.3}$$
$$g(z) \sim \mathcal{O}\left(z^{-1}\right),\ \ F(z) \sim F_\infty + \mathcal{O}\left(z^{-1}\right), \qquad \text{as } z \to \infty, \tag{2.2.4}$$
$$g(z) \sim |z-E_j|^{1/2}\text{or } g(z) \sim |z-\overline{E_j}|^{1/2} \qquad \text{at the endpoints.} \tag{2.2.5}$$

The matrix $\hat{\Gamma}^{(1)}(z)$ has jump conditions not only in $[E_1,E_2]\cup[\overline{E_2},\overline{E_1}]$ but also in $[\overline{E_1},E_1]$:

$$\hat{\Gamma}^{(1)}_+(z) = \hat{\Gamma}^{(1)}_-(z)\hat{V}^{(1)}(z,x,t), \tag{2.2.6}$$

where

$$\hat{V}^{(1)}(z,x) = \begin{cases} \begin{bmatrix} e^{ix(g_+-g_-)}\frac{F_+}{F_-} & r^*(z)e^{-2izx-ix(g_++g_-)}(F_+F_-)^{-1}\chi_{[\overline{E_2},\overline{E_1}]}(z) \\ -r(z)e^{2izx+ix(g_++g_-)}(F_+F_-)\chi_{[E_1,E_2]}(z) & e^{-ix(g_+-g_-)}\frac{F_-}{F_+} \end{bmatrix}, & z\in[E_1,E_2]\cup[\overline{E_2},\overline{E_1}], \\ \begin{bmatrix} e^{ix(g_+-g_-)}\frac{F_+}{F_-} & 0 \\ 0 & e^{-ix(g_+-g_-)}\frac{F_-}{F_+} \end{bmatrix} & z\in[\overline{E_1},E_1]. \end{cases} \tag{2.2.7}$$

We are looking for two function $g(z),F(z)$ such that the jump matrices $\hat{V}^{(1)}$ have constant off-diagonal elements in the jump matrices at $z\in[E_1,E_2]\cup[\overline{E_2},\overline{E_1}]$ and a constant jump matrix in the gap $z\in[\overline{E_1},E_1]$. This results in the condition that $g(z),F(z)$ solve the scalar Riemann-Hilbert problems:

$$\begin{aligned} &g_+(z)+g_-(z) = -2z \text{ for } z\in[E_1,E_2]\cup[\overline{E_2},\overline{E_1}], \\ &g_+(z)-g_-(z) = \Omega \text{ for } z\in[\overline{E_1},E_1], \end{aligned} \tag{2.2.8}$$

and

$$\begin{aligned} &F_-(z)F_+(z) = r^{-1}(z) \text{ for } z\in[E_1,E_2], \\ &F_-(z)F_+(z) = r^*(z) \text{ for } z\in[\overline{E_2},\overline{E_1}], \\ &\frac{F_+(z)}{F_-(z)} = e^{\Delta} \text{ for } z\in[\overline{E_1},E_1] \end{aligned} \tag{2.2.9}$$

with Ω and Δ constants, and boundary conditions given by (2.2.3), (2.2.4) and (2.2.5).

The problems (2.2.8) and (2.2.9) are solved by the functions:

$$g(z) = -z + \int_{E_2}^{z}\frac{\zeta^2+\kappa}{P_1(\zeta)}d\zeta \tag{2.2.10}$$

$$F(z)=\exp\Big\{\frac{P_1(z)}{2\pi i}\Big(-\int_{E_1}^{E_2}\frac{\log\hat\beta(\zeta)}{P_1(\zeta)_+(\zeta-z)}d\zeta+\int_{\overline{E_2}}^{\overline{E_1}}\frac{\log\hat\beta^*(\zeta)}{P_1(\zeta)_+(\zeta-z)}d\zeta+ \\ +\int_{\overline{E_1}}^{E_1}\frac{\Delta}{P_1(\zeta)(\zeta-z)}d\zeta\Big)\Big\} \tag{2.2.11}$$

where $P_1(z):=\sqrt{(z-E_1)(z-E_2)(z-\overline{E_1})(z-\overline{E_2})}$ is a multivalued complex function, analytic in $\mathbb{C}\setminus[\overline{E_2},\overline{E_1}]\cup[E_1,E_2]$, and with the constants κ, Δ and Ω defined as

$$\kappa:=\mathrm{Im}\,(E_2)^2\left(\frac{E(m)}{K(m)}-1\right),\quad \Omega:=-\frac{\pi\mathrm{Im}\,(E_2)}{2K(m)} \tag{2.2.12}$$

$$\Delta:=\frac{i\mathrm{Im}\,(E_2)}{2K(m)}\left[\int_{E_1}^{E_2}\frac{\log r(\zeta)}{P_1(\zeta)_+}d\zeta-\int_{\overline{E_1}}^{\overline{E_2}}\frac{\log r^*(\zeta)}{P_1(\zeta)_+}d\zeta\right] \tag{2.2.13}$$

and $K(m)$ and $E(m)$ are complete elliptic integral of the first and second type, and $m:=\frac{\mathrm{Im}\,(E_1)^2}{\mathrm{Im}\,(E_2)^2}\in(0,1\,]$ is the elliptic moduli.

Substituting (2.2.10) and (2.2.11) in the jump matrices (2.2.7) we get

$$\hat V^{(1)}(z,x)=\begin{cases}\begin{bmatrix}e^{ix(g_+-g_-)}\frac{F_+}{F_-} & \chi_{[\overline{E_2},\overline{E_1}]}(z)\\ -\chi_{[E_1,E_2]}(z) & e^{-ix(g_+-g_-)}\frac{F_-}{F_+}\end{bmatrix} \text{ for } z\in[E_1,E_2]\cup[\overline{E_2},\overline{E_1}],\\ \mathrm{e}^{i(x\Omega-\Delta)\sigma_3} \text{ for } z\in[\overline{E_1},E_1].\end{cases} \tag{2.2.14}$$

As next step, we factorize the jump matrices $\hat V^{(1)}(z,x)$ along the segments $[E_1,E_2]$ and $[\overline{E_1},\overline{E_2}]$. Indeed, for $z\in[E_1,E_2]$ we have:

$$\hat V^{(1)}(z,x)=\begin{bmatrix}1 & \frac{\mathrm{e}^{-2ix(g_-+z)}}{(F_-)^2r(z)}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & \frac{\mathrm{e}^{-2ix(g_++z)}}{(F_+)^2r(z)}\\ 0 & 1\end{bmatrix}, \tag{2.2.15}$$

while for $z\in[\overline{E_2},\overline{E_1}]$ we have

$$\hat V^{(1)}(z,x)=\begin{bmatrix}1 & 0\\ \frac{(F_-)^2}{r^*(z)}\mathrm{e}^{2ix(g_-+z)} & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{(F_+)^2}{r^*(z)}\mathrm{e}^{2ix(g_++z)} & 1\end{bmatrix}. \tag{2.2.16}$$

We analytically extend the first and the third matrices of the factorizations (3.1.19) and (3.1.20) in a neighborhood outside the segments $[E_1,E_2]$ and $[\overline{E_1},\overline{E_2}]$. We denote with $\mathscr{U}_\pm([E_1,E_2])$ the open set on the left $(+)$ and the right $(-)$ of the segment $[E_1,E_2]$ as it's shown in Figure 2.2, and with $\mathscr{U}_\pm([\overline{E_1},\overline{E_2}])$ their complex conjugate.

Then we introduce a new transformation

$$\hat\Gamma^{(2)}(z)=\hat\Gamma^{(1)}(z)T^{(2)}(z,t), \tag{2.2.17}$$

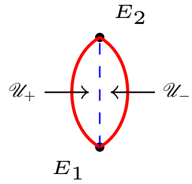


Figure 2.2: The lenses $\mathscr{U}_+([E_1, E_2])$ and $\mathscr{U}_-([E_1, E_2])$.

where

$$T^{(2)}(z,x) = \begin{cases} \begin{bmatrix} 1 & \frac{e^{-2ix(g(z)+z)}}{(F(z))^2 r(z)} \\ 0 & 1 \end{bmatrix} & z \in \mathscr{U}_-([E_1, E_2]), \\ \begin{bmatrix} 1 & -\frac{e^{-2ix(g(z)+z)}}{(F(z))^2 r(z)} \\ 0 & 1 \end{bmatrix} & z \in \mathscr{U}_+([E_1, E_2]), \\ \begin{bmatrix} 1 & 0 \\ \frac{(F(z))^2}{r^*(z)} e^{2ix(g(z)+z)} & 1 \end{bmatrix} & z \in \mathscr{U}_-([\overline{E_1}, \overline{E_2}]), \\ \begin{bmatrix} 1 & 0 \\ -\frac{(F(z))^2}{r^*(z)} e^{2ix(g(z)+z)} & 1 \end{bmatrix} & z \in \mathscr{U}_+([\overline{E_1}, \overline{E_2}]), \\ \mathbf{1} & \text{otherwise.} \end{cases} \tag{2.2.18}$$

The new matrix $\hat{\Gamma}^{(2)}(z)$ solves the RHP

$$\hat{\Gamma}^{(2)}_+(z) = \hat{\Gamma}^{(2)}_-(z) \hat{V}^{(2)}(z,x), \tag{2.2.19}$$

with the jump matrices also defined in the boundary of $\mathscr{U}_\pm([E_1, E_2])$ and $\mathscr{U}_\pm([\overline{E_1}, \overline{E_2}])$, which we denote with $L_\pm([E_1, E_2])$ and $L_\pm([\overline{E_2}, \overline{E_2}])$

$$\hat{V}^{(2)}(z,x) = \begin{cases} \begin{bmatrix} 1 & \frac{e^{-2ix(g(z)+z)}}{(F(z))^2 r(z)} \\ 0 & 1 \end{bmatrix} & z \in L_-([E_1, E_2]) \cup L_+([E_1, E_2]), \\ \begin{bmatrix} 1 & 0 \\ \frac{(F(z))^2}{r^*(z)} e^{2ix(g(z)+z)} & 1 \end{bmatrix} & z \in L_-([\overline{E_1}, \overline{E_2}]) \cup L_-([\overline{E_1}, \overline{E_2}]), \end{cases} \tag{2.2.20}$$

while in the original jump contours we have

$$\hat{V}^{(2)}(z,x) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & z \in [E_1, E_2] \cup [\overline{E_1}, \overline{E_2}] \\ e^{i(x\Omega - i\Delta)\sigma_3} & z \in [\overline{E_1}, E_1]. \end{cases} \tag{2.2.21}$$

The orientations of the contours it's shown in the figure 2.3.

We now analyze how the jump matrices $\hat{V}^{(2)}(z,x)$, at the boundaries of the lenses, behaves as $x \sim -\infty$. Specifically, we study the sign of $\operatorname{Im}(g(z)+z)$.

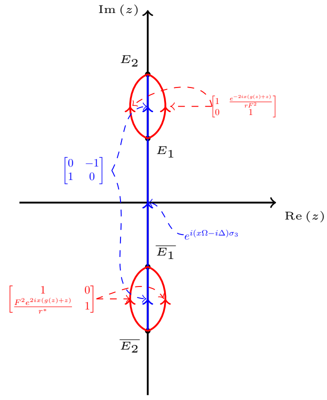


Figure 2.3: Jump contours and jump matrices $\hat{V}^{(2)}(z,x)$.

Lemma 2.2.1. *The function $g(z)$, defined in (2.2.10) satisfy the following inequalities:*

$$\begin{aligned} &\mathrm{Im}\,(g(z)+z) > 0 \textit{ for } z \in (\mathscr{U}_+([E_1,E_2]) \cup \mathscr{U}_-([E_1,E_2])) \setminus \{E_1,E_2\} \\ &\mathrm{Im}\,(g(z)+z) < 0 \textit{ for } z \in (\mathscr{U}_+([\overline{E_2},\overline{E_1}]) \cup \mathscr{U}_-([\overline{E_2},\overline{E_1}])) \setminus \{\overline{E_1},\overline{E_2}\} \end{aligned} \tag{2.2.22}$$

Proof The strategy is to recover the sign of the imaginary part of a function by studying the imaginary part of it's derivative. This means that we have to study

$$g'_\pm(z) + 1 = \frac{z^2+\kappa}{P_1(z)_\pm}. \tag{2.2.23}$$

Let us focus on the case where $z \in [E_1,E_2]$. We fix an orientation for $P_1(z)$. We choose the one such that $P_-(0) \in \mathbb{R}_+$

$$\begin{aligned} P_1(z)_- &= |P_1(z)|e^{i\pi/2} \\ P_1(z)_+ &= |P_1(z)|e^{-i\pi/2} \end{aligned} \text{ for } z \in [E_1,E_2].$$

We write $z = i\lambda$, with $\lambda \in \mathbb{R}_+$, and in the right side of $[E_1,E_2]$ we have

$$\mathrm{Im}\,(g'_-(z)+1) = \mathrm{Im}\left(\frac{z^2+\kappa}{P_1(z)_-}\right) = -\frac{-\lambda^2+\kappa}{|P_1(z)|} \tag{2.2.24}$$

since $\kappa \in \mathbb{R}_-$, we rewrite it as

$$(4.27) = \frac{\lambda^2+|\kappa|}{|P_1(z)|} > 0. \tag{2.2.25}$$

Meanwhile, in the left side of $[E_1,E_2]$

$$\mathrm{Im}\,(g'_+(z)+1) = \mathrm{Im}\left(\frac{z^2+\kappa}{P_1(z)_+}\right) = -\frac{\lambda^2+|\kappa|}{|P_1(z)|} < 0. \tag{2.2.26}$$

We integrate (2.2.23)

$$g(z) + z = \int_{E_2}^{z} \frac{\zeta^2 + \kappa}{P_1(z)} d\zeta \tag{2.2.27}$$

and it is trivial to see that the integral give us a pure real number for in $z \in [E_1, E_2]$. So, considering also (2.2.25) and (2.2.26), we have that

$$\operatorname{Im}(g(z) + z) > 0 \text{ for } z \in (\mathscr{U}_+([E_1, E_2]) \cup \mathscr{U}_-([E_1, E_2])) \setminus \{E_1, E_2\} \tag{2.2.28}$$

We follow the same ideas also for $z \in [\overline{E_2}, \overline{E_1}]$, with the sign of $\operatorname{Im}(g'_\pm + 1)$ that in this case are swapped

$$\begin{aligned} &\operatorname{Im}(g'_+ + 1) > 0, \\ &\operatorname{Im}(g'_- + 1) < 0, \end{aligned} \tag{2.2.29}$$

and so we get that $\operatorname{Im}(g(z) + z)$ is negative in the lenses $(\mathscr{U}_+([\overline{E_2}, \overline{E_1}]) \cup \mathscr{U}_-([\overline{E_2}, \overline{E_1}])) \setminus \{\overline{E_1}, \overline{E_2}\}$. ■

From Lemma 2.2.1, we find out that the jump matrices $\hat{V}^{(2)}(z, x)$, for $z \in L_\pm([E_1, E_2])$ and $z \in L_\pm([\overline{E_2}, \overline{E_1}])$, tends to the identity as $x \to -\infty$. Then the RHP (2.2.19) reduces to a model problem

$$\begin{aligned} &X_+(z) = X_-(z) V_X(z, x), \\ &\overline{X(\overline{z})} = X(z), \qquad X(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty, \end{aligned} \tag{2.2.30}$$

where

$$V_X(z, x) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z \in [E_1, E_2] \cup [\overline{E_1}, \overline{E_2}] \\ \mathrm{e}^{i(x\Omega - i\Delta)\sigma_3} z \in [\overline{E_1}, E_1]. \end{cases} \tag{2.2.31}$$

2.2.1 The solution of the model problem

The model problem (2.2.30) was already studied by Bertola and Tovbis in [14], and they solve it by using special functions living on a Riemann surface. We follow their same procedure.

We introduce a Riemann surface $\mathscr{P}_1$ of genus 1 defined as:

$$\mathscr{P}_1 = \{(w, z) \in \mathbb{C}^2 | w^2 = P_1(z)^2\}. \tag{2.2.32}$$

We introduce the homological basis α, β on it; with the β cycle encircling the segment $[E_1, E_2]$ counter-clock wise and the α cycle going from $\overline{E_1}$ to E_1 in the fist sheet and coming back in the second sheet. Then we define the *Abel map*

$$u(z, z_0) := -\frac{E_2}{2K(m)} \int_{z_0}^{z} \frac{d\zeta}{P_1(\zeta)}, \tag{2.2.33}$$

and the *Jacobi Theta Function*

$$\vartheta(z,\tau) := \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau+2i\pi nz}; \tag{2.2.34}$$

with τ a modular parameter that, in our case, is given by

$$\tau = \frac{iK(m')}{2K(m)}, \qquad m' = \sqrt{1-m^2}.$$

This function possesses the following properties:

Proposition 2.2.2. *For $l, j \in \mathbb{Z}$, the Jacobi Theta Function (2.2.34) satisfy the following conditions:*

1. $\vartheta(z+l\tau) = \vartheta(z)\mathrm{e}^{-2i\pi zl - i\pi l^2\tau}$;
2. $\vartheta(z+j) = \vartheta(z)$;
3. $\vartheta(-z) = \vartheta(z)$.

We now have all the instruments to solve the model problem (2.2.30):

Step 1 We solve the homogeneous RHP

$$\begin{aligned} &X_+^{(0)}(z) = X_-^{(0)}(z)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, && \text{for } z \in [E_1,E_2] \cup [\overline{E_1},\overline{E_2}] \\ &\overline{X^{(0)}(\overline{z})} = X^{(0)}(z), && X^{(0)}(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty. \end{aligned} \tag{2.2.35}$$

The solution of this RHP is given by:

$$X^{(0)}(z) = \frac{1}{2}\begin{bmatrix} \left(\phi_1(z)+\phi_1(z)^{-1}\right) & -i\left(\phi_1(z)-\phi_1(z)^{-1}\right) \\ i\left(\phi_1(z)-\phi_1(z)^{-1}\right) & \left(\phi_1(z)+\phi_1(z)^{-1}\right) \end{bmatrix} \tag{2.2.36}$$

where

$$\phi_1(z) := \left(\frac{(z-\overline{E_1})(z-E_2)}{(z-\overline{E_2})(z-E_1)}\right)^{\frac{1}{4}} \tag{2.2.37}$$

is analytic in $\mathbb{C} \setminus [\overline{E_2},\overline{E_1}] \cup [E_1,E_2]$, with the jump condition along the branch cuts

$$\phi_1(z)_+ = i\phi_1(z)_-.$$

Step 2 We consider the Abel map (2.2.33) with $z_0 = \infty$ and we check the jump conditions along the segment $[\overline{E_2},E_2]$:

$$u_+ + u_- = 0 \text{ for } z \in [E_1,E_2]; \tag{2.2.38}$$

$$u_+ - u_- = -\tau \text{ for } z \in [\overline{E_1},E_1]; \tag{2.2.39}$$

$$u_+ + u_- = -1 \text{ for } z \in [\overline{E_2},\overline{E_1}]. \tag{2.2.40}$$

Step 3 We introduce an ansatz for the RHP (2.2.30)

$$X(z) = \frac{C}{2}\begin{bmatrix} \left(\phi_1(z)+\phi_1(z)^{-1}\right)\varphi_1(z) & -i\left(\phi_1(z)-\phi_1(z)^{-1}\right)\psi_1(z) \\ i\left(\phi_1(z)-\phi_1(z)^{-1}\right)\varphi_2(z) & \left(\phi_1(z)+\phi_1(z)^{-1}\right)\psi_2(z) \end{bmatrix}, \tag{2.2.41}$$

then we need to find the functions $\varphi_j(z), \psi_j(z)$; with $j = 1,2$; such that they solve the scalar RHP

$$\begin{aligned} &(\varphi_j(z))_+ = (\psi_j(z))_- && z \in [E_1, E_2] \cup [\overline{E_2}, \overline{E_1}] \\ &(\varphi_j(z))_+ = \mathrm{e}^{i(x\Omega - i\Delta)}(\varphi_j(z))_- && \\ &(\psi_j(z))_+ = \mathrm{e}^{i(x\Omega - i\Delta)}(\psi_j(z))_- && z \in [\overline{E_1}, E_1]. \end{aligned} \tag{2.2.42}$$

Lemma 2.2.3. *The functions $\varphi_j(z,x), \psi_j(z,x)$; with $j = 1,2$; defined as*

$$\varphi_1(z,x) = \frac{\vartheta(u(z_1,\infty_1) + \frac{x\Omega - i\Delta}{2\pi})}{\vartheta(u(z_1,\infty_1))} \quad \psi_1(z,x) = \frac{\vartheta(u(z_2,\infty_1) + \frac{x\Omega - i\Delta}{2\pi})}{\vartheta(u(z_2,\infty_1))}, \tag{2.2.43}$$

$$\varphi_2(z,x) = \frac{\vartheta(u(z_1,\infty_2) + \frac{x\Omega - i\Delta}{2\pi})}{\vartheta(u(z_1,\infty_2))} \quad \psi_2(z,x) = \frac{\vartheta(u(z_2,\infty_2) + \frac{x\Omega - i\Delta}{2\pi})}{\vartheta(u(z_2,\infty_2))}, \tag{2.2.44}$$

where Ω and Δ are defined in (2.2.12) and (2.2.13); and z_1, z_2 indicates the point z respectively in the first and second sheet of the Riemann surface $\mathscr{P}_1$, solves the Riemann-Hilbert problem (2.2.42).

Proof The lemma is easily proved by using the jump conditions on the Abel map (2.2.38) and the properties of the Proposition 2.2.2. ■

The constant C is give by imposing the boundary condition $X(z) \sim \mathbf{1}$ as $z \to \infty$, and we obtain

$$C = \frac{\vartheta(0)}{\vartheta(\frac{x\Omega - i\Delta}{2\pi})}.$$

2.2.2 The error parametrix around $E_1, E_2, \overline{E_1}, \overline{E_2}$

Before analyzing the error parametrix around the endpoint of the segments $[E_1, E_2]$ and $[\overline{E_2}, \overline{E_1}]$, we should examine the behaviour of the function $r(z)$ near those points. Indeed, Girotti et al. in [54] studied a RHP similar to (2.2.19) for mKdV and they proved that if the function $r(z)$ has a local behaviour near the endpoints $E_j, \overline{E_j}$ of the form $r(z) \sim |z - E_j|^{\pm 1/2}\tilde{r}(z)$, with $\tilde{r}(z)$ locally bounded and non-zero , then it is possible to modify the lens opening factorization so that local parametrix near the points E_1 and E_2 are not needed. Specifically, we use the following assumption:

Assumption 2.2.4. *When $r(z)|z-E_j|^{\pm 1/2}$ is bounded and non zero on $[E_1,E_2]$, we assume that $r(z)$ admits an analytical continuation analytically outside $[E_1,E_2]$:*

$$\hat{r}(z) \text{ analytic in } \mathscr{U}_{h,-}\cup\mathscr{U}_{h,+}, \qquad \hat{r}(z)|_{z\in[E_1,E_2]} = r(z), \tag{2.2.45}$$

$$\hat{r}_+(z)+\hat{r}_-(z) = 0 \qquad z\in[E_2,E_2+ih]\cup[E_1-ih,E_1] \tag{2.2.46}$$

where $\mathscr{U}_{h,+}$ is define as

$$\mathscr{U}_{h,+} := \{z\in\mathbb{C}|\mathrm{Re}\,(z)\in(0\,,\,h] \\ \text{and } \mathrm{Im}\,(E_1)-\sqrt{h^2-\mathrm{Re}\,(z)^2}\le Im(z)\le \mathrm{Im}\,(E_2)+\sqrt{h^2-\mathrm{Re}\,(z)^2}\Big\}, \tag{2.2.47}$$

with some $0<h<\mathrm{Im}\,(E_1)$ and where $\mathscr{U}_{h,-}$ is defined by symmetry, $\mathscr{U}_{h,-}=\{z|-\overline{z}\in\mathscr{U}_{h,+}\}$.

It seems from this assumption that, after we apply the transformation (2.2.17), we have other jumps in the segments $[E_2,E_2+ih]$ and $[E_1-ih,E_1]$, but actually is not true. For example, for $z\in[E_2,E_2+ih]$ we have:

$$(\hat{\Gamma}^{(2)}_-(z))^{-1}\hat{\Gamma}^{(2)}_+(z) = \begin{bmatrix}1 & (\hat{r_+}^{-1}(z)+\hat{r_-}^{-1}(z))\frac{\mathrm{e}^{-2i(g(z)+z)}}{(F(z))^2}\\ 0 & 1\end{bmatrix} = \mathbf{1}, \tag{2.2.48}$$

while for $z\in[E_1-ih,E_1]$ we have

$$(\hat{\Gamma}^{(2)}_-(z))^{-1}\hat{\Gamma}^{(2)}_+(z) = \begin{bmatrix}\mathrm{e}^{ix\Omega+\Delta} & (\hat{r_+}^{-1}(z)+\hat{r_-}^{-1}(z))\frac{\mathrm{e}^{-2i(g(z)+z)}}{(F(z))^2}\\ 0 & \mathrm{e}^{-ix\Omega-\Delta}\end{bmatrix} = \mathrm{e}^{i(x\Omega-i\Delta)\sigma_3}. \tag{2.2.49}$$

So the lenses detaches from the E_1 and E_2 and fully enclose the the band. This means that solution of the model problem is an exponentially accurate model uniformly in $\mathbb{C}$.

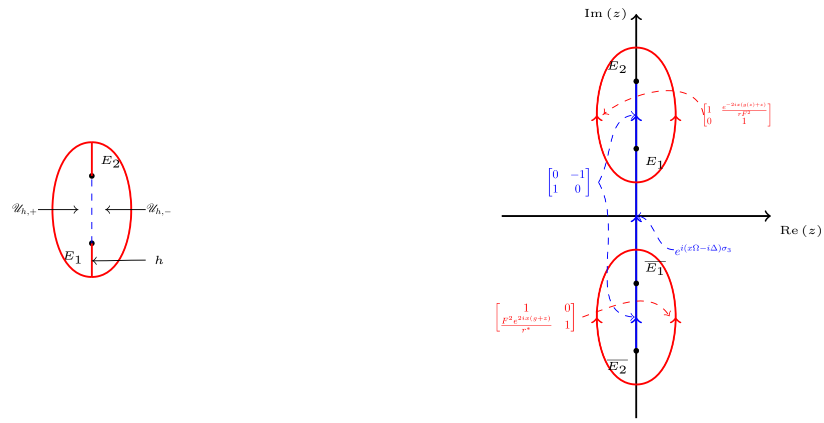


Figure 2.4: On the left the new lenses. On the right the Figure 2.3 with the new lenses.

In our case we have that $r(z)=\Delta S(z)\beta(z)$, with $\beta(z)$ bounded in the original $\mathscr{D}$ and $\Delta S(z)$ defined in the segment $[E_1,E_2]$ with behaviour at the end point of the for

$\Delta S(z) \sim |z - E_j|^{1/2}$. So, we are in the main hypothesis of the assumption 2.2.4 and the model problem $X(z, x)$ is an exponentially accurate approximation of our solution $\hat{\Gamma}^{(2)}(z)$, i.e.

$$X(z,x)(\hat{\Gamma}^{(2)}(z))^{-1} = \mathbf{1} + \mathcal{O}(e^{+cx}), \text{ as } x \to \infty, \tag{2.2.50}$$

with $c > 0$.

2.2.3 Asymptotic of initial datum for $x \sim -\infty$

We can now enunciate the following theorem:

Theorem 2.2.5. *The solution of the Riemann-Hilbert Problem (2.2.1) generates a step-like oscillatory initial datum for the focusing NLS equation (1.1.5) $\psi_0(x)$ with the following behaviours at $x \to \pm\infty$*

$$\psi_0(x) = \begin{cases} \mathcal{O}(e^{-cx}) \text{ as } x \to +\infty \\ -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1)) \dfrac{\vartheta(0)\vartheta(u_\infty + \frac{x\Omega - i\Delta}{2\pi})}{\vartheta(\frac{x\Omega-i\Delta}{2\pi})\vartheta(u_\infty)} + \mathcal{O}(e^{+cx}) \text{ as } x \to -\infty \end{cases} \tag{2.2.51}$$

with Ω and Δ defined in (2.2.12) and (2.2.13), $u_\infty := u(\infty_2, \infty_1)$ and $c > 0$.

Proof If $x \to +\infty$, then the jump matrices of the RHP (2.2.1) tends to the identity in a exponentially, i.e. $\Gamma_{12}(z, x) \sim e^{-cx}$, with $c > 0$. From the equation (1.2.12) we have that

$$\psi_0(x) = \mathcal{O}(e^{-cx}).$$

As we shown in the previews sections, for $x \to -\infty$ the solution $\hat{\Gamma}^{(2)}(z)$ is approximated by the matrix $X(z, x)$ with an exponentially small error. So the equation (1.2.12) becomes:

$$\begin{aligned}\psi_0(x) &= 2i \lim_{z\to\infty} z\Gamma_{12}(z) = 2i \lim_{z\to\infty} z(F_\infty^{-\sigma_3} X(z,x) \mathrm{e}^{-ig(z)x\sigma_3} F(z)^{-\sigma_3})_{12} + \mathcal{O}(e^{+cx}) \\ &= 2i \lim_{z\to\infty} zF_\infty^{-1} X_{12}(z,x)\mathrm{e}^{ig(z)x} F(z) + \mathcal{O}(e^{+cx})\end{aligned}$$

As $z \to \infty$, the behaviours of the functions $F(z), g(z)$ are given by the conditions (2.2.4), while the function $X_{12}(z, x)$ has the following expansion

$$X_{12}(z,x) = \frac{\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1)}{2z} \frac{\vartheta(0)\vartheta(u_\infty + \frac{x\Omega-i\Delta}{2\pi})}{\vartheta(\frac{x\Omega-i\Delta}{2\pi})\vartheta(u_\infty)} + \mathcal{O}(z^{-2}), \tag{2.2.52}$$

where $u_\infty := u(\infty_2, \infty_1)$. This implies that the initial datum $\psi_0(x)$ has the following behaviour at $x \to -\infty$:

$$\psi_0(x) = -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1)) \frac{\vartheta(0)\vartheta(u_\infty + \frac{x\Omega-i\Delta}{2\pi})}{\vartheta(\frac{x\Omega-i\Delta}{2\pi})\vartheta(u_\infty)} + \mathcal{O}(e^{+cx}). \tag{2.2.53}$$

■

Remark 2.2.6. *The equation (2.2.53) can be rewritten in terms of "Jacobi delta amplitude function"* $\mathrm{dn}(z)$*. Indeed, by using algebraic relation of the Theta function (2.2.34), such as*

$$\vartheta(u_\infty, \tau) = \vartheta_3\left(\frac{1}{2}; \tau\right) = \vartheta_4(0; \tau), \tag{2.2.54}$$

where $\vartheta_3(z;\tau)$ *and* $\vartheta_4(z;\tau)$ *are defined as:*

$$\vartheta_3(z; \tau) := \vartheta(z + u_\infty, \tau), \tag{2.2.55}$$

$$\vartheta_4(z; \tau) := \vartheta\left(z + u_\infty + \frac{1}{2}, \tau\right). \tag{2.2.56}$$

Then we have that:

$$\begin{aligned}
\psi_0(x) &= -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\frac{\theta_3(0;\tau)}{\theta_3(\frac{1}{2};\tau)}\frac{\theta_3(\frac{1}{2}+\frac{\Omega+\Delta}{2\pi};\tau)}{\theta_3(\frac{\Omega+\Delta}{2\pi};\tau)} = -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\frac{\theta_3(0;\tau)}{\theta_4(0;\tau)}\frac{\theta_4(\frac{\Omega+\Delta}{2\pi};\tau)}{\theta_3(\frac{\Omega+\Delta}{2\pi};\tau)} \\
&= -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\frac{\theta_4(\frac{\Omega+\Delta}{2\pi};2\tau)^2 + \theta_1(\frac{\Omega+\Delta}{2\pi};2\tau)^2}{\theta_4(\frac{\Omega+\Delta}{2\pi};2\tau)^2 - \theta_1(\frac{\Omega+\Delta}{2\pi};2\tau)^2} \\
&= -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\frac{1+\sqrt{m}\,\mathrm{sn}^2\left(\frac{K(m)}{\pi}(\Omega+\Delta), m\right)}{1-\sqrt{m}\,\mathrm{sn}^2\left(\frac{K(m)}{\pi}(\Omega+\Delta), m\right)} \\
&= -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\,\mathrm{nd}\left(\frac{K(m_1)}{\pi}(\Omega+\Delta), m_1\right) \\
&= -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1))\frac{1+\sqrt{m}}{1-\sqrt{m}}\,\mathrm{dn}\left(\frac{K(m_1)}{\pi}(\Omega+\Delta+\pi), m_1\right) \\
&= -i(\mathrm{Im}\,(E_2) + \mathrm{Im}\,(E_1))\,\mathrm{dn}\left(\frac{K(m_1)}{\pi}(\Omega+\Delta+\pi), m_1\right),
\end{aligned} \tag{2.2.57}$$

where we have used the Landen transformation to simplify the above expression

$$m_1 = \frac{4\sqrt{m}}{(1+\sqrt{m})^2} = \frac{4\mathrm{Im}\,(E_2)\mathrm{Im}\,(E_1)}{(\mathrm{Im}\,(E_2)+\mathrm{Im}\,(E_1))^2}, \qquad (1+\sqrt{m})K(m) = K(m_1) \tag{2.2.58}$$

so that

$$\psi_0(x) = -i(\mathrm{Im}\,(E_2) + \mathrm{Im}\,(E_1))\,\mathrm{dn}\left((\mathrm{Im}\,(E_2) + \mathrm{Im}\,(E_1))(x - x_0) + K(m_1), m_1\right),$$

where

$$x_0 = \frac{i}{2}\left[\int_{E_1}^{E_2}\frac{\log r(\zeta)}{R_+(\zeta)}d\zeta - \int_{\overline{E_1}}^{\overline{E_2}}\frac{\log r^*(\zeta)}{R_+(\zeta)}d\zeta\right]$$

In the limit $m_1 \to 1$ *or* $\mathrm{Im}\,(E_2) \to \mathrm{Im}\,(E_1)$ *we have* $x_0 \to -\frac{\pi}{2\mathrm{Im}\,(E_1)}\arg(r(i\mathrm{Im}\,(E_1)))$ *and that* $\psi_0(x)$ *tends to the one soliton solution.*

2.3 The zeros of the g-functions and phase transitions

We consider the scenario where the time parameter $t > 0$. In particular, we perform the long-time asymptotic ($t \to +\infty$) and long-space asymptotic analysis($x \to \pm\infty$) with $\eta = x/t \in \mathbb{R}$ fixed.

By applying the Nonlinear steepest descend analysis, we have to look at other function $g(z,\eta)$ analytic in $\mathbb{C} \setminus \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]} \cup [\overline{E_1}, E_1]$, where $\upsilon_{[z_1,z_2]}$ is the oriented contour from z_1 to z_2 on which $\mathrm{Im}\,(g(z,\eta)) = 0$, and have the following properties:

$$\overline{g(\overline{z},\eta)} = g(z,\eta), \tag{2.3.1}$$

$$g(z,\eta) \sim 2(\eta z + z^2)\mathcal{O}\left(z^{-1}\right) \text{,as } z \to \infty, \tag{2.3.2}$$

$$g(z,\eta) \sim |z - E_j|^{1/2} \text{or } g(z,\eta) \sim |z - \overline{E_j}|^{1/2} \qquad \text{at the endpoints.} \tag{2.3.3}$$

The conditions (2.3.1) and (2.3.2) imply that the level set $\mathrm{Im}\,(g(z,\eta)) = 0$ has two infinite branches: the real axis and another branch υ_∞ which is asymptotic to the vertical line $\mathrm{Re}\,(z) = -\eta/2$.

For $z \in \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]} \cup [\overline{E_1}, E_1]$ the g-function has those jump conditions along the bands

$$\begin{aligned} &g_+(z) + g_-(z) = 2(\eta z + z^2) \text{ for } z \in \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]}, \\ &g_+(z) - g_-(z) = \Omega \text{ for } z \in [\overline{E_1}, E_1], \end{aligned} \tag{2.3.4}$$

From solving the RHP (2.3.4) we get that the g-function is still of genus 1 and has the following form:

$$g(z,\eta) = 2\int_{E_2}^{z} \frac{(\zeta - \mu(\eta))(\zeta - d(\eta))(\zeta - \overline{d(\eta)})}{\sqrt{(\zeta - E_1)(\zeta - E_2)(\zeta - \overline{E}_1)(\zeta - \overline{E}_2)}} \mathrm{d}\zeta, \tag{2.3.5}$$

with $\mu(\eta) \in \mathbb{R}$ and $d(\eta) \in \mathbb{C}^+$. Fixing η and applying the conditions (2.3.1) and (2.3.2) to the new $g-$function, the critical points $\mu(\eta), d(\eta)$ satisfy the system of equations

$$\begin{cases} \mu + 2\mathrm{Re}\,(d) = -\dfrac{\eta}{2} \\ 2\mathrm{Re}\,(d)\mu + \mathrm{Re}\,(d)^2 + \mathrm{Im}\,(d)^2 = \dfrac{\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2}{2} \\ \mu(\mathrm{Re}\,(d)^2 + \mathrm{Im}\,(d)^2) = -\dfrac{\eta}{2}\mathrm{Im}\,(E_2)^2(1 - Q(m)) \end{cases} \tag{2.3.6}$$

where $Q(m) := E(m)/K(m)$ and $0 < m \leq 1$ is the elliptic moduli. The existence of the bands $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$ is guaranteed since they are trajectories of the quadratic differential $(g'(z,\eta))^2\mathrm{d}z^2$. Indeed, the local and global behaviour of the quadratic trajectories depends on the nature of the zeros and the poles of $(g'(z))^2$ and they obey the following rules [64, 85]:

- at a simple pole emerges exactly one critical trajectory;
- at a zero of multiplicity k emerges $k+2$ critical trajectories spacing under equal angles $\frac{2\pi}{(k+2)}$;
- at a pole of order $k > 2$ there are $k-2$ asymptotic directions spacing under equal angle $\frac{2\pi}{k-2}$ in a neighborhood $\mathscr{U}$ such that each trajectory entering $\mathscr{U}$ stays in $\mathscr{U}$ and tends to the pole in one of the critical directions;
- at a double pole the local behaviour of the trajectories depends on the real and imaginary part of the residue. There are three possible behaviours:
  - radial;
  - circular;
  - log-spiral.

In our case, $(g'(z))^2$ has four simple poles $E_1, E_2, \overline{E_1}, \overline{E_2}$ and three double zeroes $\mu(\eta)$, $d(\eta), \overline{d(\eta)}$. From the properties of the g-function (2.3.2) and (2.3.3), the points that are inside the critical trajectories $\mathrm{Im}\,(g(z)) = 0$ are $E_1, E_2, \overline{E_1}, \overline{E_2}$ and $\mu(\eta) \quad \forall\, \eta \in \mathbb{R}_-$. Since every critical trajectory should end either on a pole or on a zero [85], then we have four different critical trajectories: two given by the infinite branches that intersect at $\mu(\eta)$, one short trajectory $\upsilon_{[E_1,E_2]}$ which start at E_1 and ends at E_2 and its complex conjugate.

Changing the parameters η and m along their domains, the points $\mu(\eta), d(\eta)$ will move along the complex plane, changing also the trajectories in the level set $\mathrm{Im}\,(g) = 0$. Indeed, for some values of the η and m, the points $d(\eta), \overline{d(\eta)}$ can merge in the real axis or touch the branch υ_∞. In those specific situations, the branches of the leves sets $\mathrm{Im}\,(g(z,\eta)) = 0$ intersect into each other and cause a change in the nature of the points $\mu(\eta), d(\eta), \overline{d(\eta)}$. This means that the system (2.3.6), and consequentially the g-function (2.3.5), is not well defined for some values of η and m.

According how the level set $\mathrm{Im}\,(g(z,\eta)) = 0$ changes varying η and m, also the jumps of the model problem (2.2.30) change.

This implies that the solution $\psi(x,t)$ of the NLS eq (1.1.5), once we fix the elliptic domain $\mathscr{D}$, could have different asymptotic behaviour in different sectors of (x,t)-plane. The aim of this section is to look for the values of η and/or m such that we have this change of configurations, also called *phase transitions*.

We consider the imaginary part of $d(\eta)$. From the system (2.3.6), we get that

$$\mathrm{Im}\,(d) = \frac{\eta^2 - 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2) - (W(\eta, E_1, E_2))^{\frac{2}{3}}}{4\sqrt{3}(W(\eta, E_1, E_2))^{\frac{1}{3}}}, \tag{2.3.7}$$

with

$$
\begin{aligned}
W(\eta, E_1, E_2) &:= \eta[54\mathrm{Im}\,(E_2)^2 Q(m) + 9(\mathrm{Im}\,(E_1)^2 - 5\mathrm{Im}\,(E_2)^2) - \eta^2] \\
&\quad + \big(\, [6(\mathrm{Im}\,(E_2)^2 + \mathrm{Im}\,(E_1)^2) - \eta^2]^3 \\
&\quad + \eta^2[\eta^2 - 9(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2(6Q(m) - 5))]^2 \,\big)^{\frac{1}{2}} .
\end{aligned} \tag{2.3.8}
$$

We are looking for a configuration where we can have a double (or a triple) real zero of the function $g'(z)$, i.e. $d(\eta) \in \mathbb{R}$ and/or $\mu(\eta) = d(\eta)$. Indeed, setting $\mathrm{Im}\,(d) = 0$ we find the equation

$$
W(\eta, E_1, E_2) = [\eta^2 - 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2)]^{\frac{3}{2}}. \tag{2.3.9}
$$

Fixing m, we find that one solution of (2.3.9) is given by $\eta_c(m) := -\mathrm{Im}\,(E_2)\sqrt{6(m+1)}$ but, as $\eta \to \eta_c$, the function $W(\eta, E_1, E_2)$ has two values:

$$
W_+(\eta_c, E_1, E_2) = 0 \tag{2.3.10}
$$

$$
W_-(\eta_c, E_1, E_2) = 6\eta_c(m)\mathrm{Im}\,(E_2)^2\,(18Q(m) + m - 17)\,. \tag{2.3.11}
$$

The value of $W_\pm$ is determined by the sign of the function

$$
\mathcal{Q}(m) := 18Q(m) + m - 17. \tag{2.3.12}
$$

From implicit methods, we find out that $\mathcal{Q}(m)$ has a zero in $m_c \sim 0.12274$ and has signs:

$$
\begin{aligned}
&\mathcal{Q}(m) > 0 \quad \text{when } m < m_c \text{ and } \eta = \eta_c \\
&\mathcal{Q}(m) < 0 \quad \text{when } m > m_c \text{ and } \eta = \eta_c
\end{aligned}
$$

So, for $m < m_c$ we take $W_+(\eta_c, E_1, E_2) = 0$, while for $m \geq m_c$ we take $W_-(E_1, E_2)$.

For W_+ we are in the case where $\mathrm{Im}\,(d) = 0\ \forall\, m$. Indeed:

$$
\begin{aligned}
\lim_{\eta\to\eta_c} \mathrm{Im}\,(d) &= \lim_{\eta\to\eta_c} \left(\frac{\eta^2 - 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2)}{4\sqrt{3}(W_+(\eta, E_1, E_2))^{\frac{1}{3}}} - 4\sqrt{3}(W_+(\eta, E_1, E_2))^{\frac{1}{3}} \right) \\
&= \frac{0}{0} \text{ by using d'Hopital rule} \\
&= \lim_{\eta\to\eta_c} \frac{2\eta(W_+(\eta, E_1, E_2))^{\frac{2}{3}}}{4\sqrt{3}W'_+(\eta, E_1, E_2)} = 0
\end{aligned}
$$

since $W'_+(\eta_c, E_1, E_2) \neq 0$.

From (2.3.11), we notice that $W_-(E_1, E_2)$ has a zero in m_c. Then the point (η_c, m_c) is a critical point for the system (2.3.6), now we need to understand what configuration of the points $\mu, d, \overline{d}$ it represents.

Let us consider $\mathrm{Re}\,(d)$ and μ:

$$\mathrm{Re}\,(d) = -\frac{\eta}{6} - \frac{\eta^2 - 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2)}{12(W(\eta, E_1, E_2))^{\frac{1}{3}}} - \frac{(W(\eta, E_1, E_2))^{\frac{1}{3}}}{12}, \qquad (2.3.13)$$

$$\mu = -\frac{\eta}{6} + \frac{\eta^2 - 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2)}{6(W(\eta, E_1, E_2))^{\frac{1}{3}}} + \frac{(W(\eta, E_1, E_2))^{\frac{1}{3}}}{6}. \qquad (2.3.14)$$

If we impose $\mathrm{Re}\,(d) = \mu(\eta)$ we obtain an equation similar to (2.3.9):

$$W(\eta, E_1, E_2) = [-\eta^2 + 6(\mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2)]^{\frac{3}{2}}. \qquad (2.3.15)$$

Both (2.3.9) and (2.3.15) are satisfied at the point (η_c, m_c). This means that $g(z)$ has one critical point of degree 3, i.e. $d(\eta_c, m_c) = \overline{d(\eta_c.m_c)} = \mu$.

Let us fix the parameter m in one of the cases $m > m_c$ or $m < m_c$ and analyze the problem for η. For $m > m_c$, the g-function has still three distinct critical points $\mu(\eta), d(\eta), \overline{d(\eta)}$ as $\eta \to \eta_c$, but we can have a phase transition when the trajectories in $\mathrm{Im}\,(g) = 0$ intersect each other, i.e the points d and $\overline{d}$ touch respectively the curves $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$ of the level set $\mathrm{Im}\,(g) = 0$. This happen at $\eta = \eta^*$, with η^* given by the condition:

$$\mathrm{Im}\,(g(d(\eta^*))) = 0. \qquad (2.3.16)$$

Then, for $\eta^* < \eta < 0$, $\mathrm{Im}\,(g(d(\eta^*))) \neq 0$ with the point d and $\overline{d}$ who pass trough the bands $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$. Due to the local nature of the trajectories of $(g'(z, \eta))^2 \mathrm{d}z^2$ near $d(\eta)$ and $\overline{d(\eta)}$, they split the trajectories $\upsilon_{[E_1,E_2]}$, $\upsilon_{[\overline{E_2},\overline{E_1}]}$ in the three different ones:

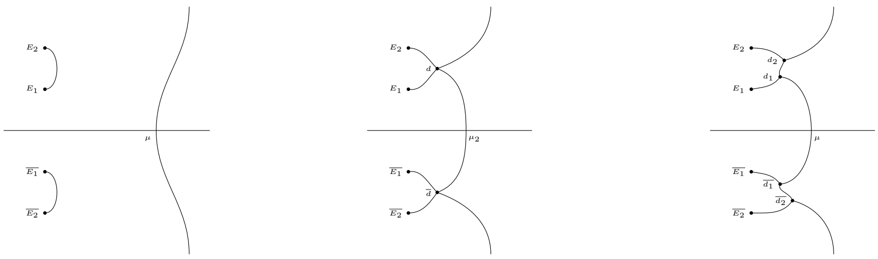
- one critical trajectory in the upper half plane, starting at E_2 and following asymptotically the infinite branch $\mathrm{Re}\, z = -\eta/2$;
- one critical trajectory in the lower half plane, starting at $\overline{E_2}$ and following asymptotically the infinite branch $\mathrm{Re}\, z = -\eta/2$;
- one short trajectory from E_1 to $\overline{E_1}$, which pass trough the point μ.

So the bands $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$ does not exist anymore and the g-function is not well defined. We need to define a new g-function which satisfies two additional properties:

1. $g'(z)$ has five zeroes: $d_1, d_2, \overline{d_1}, \overline{d_2}$ of degree $1/2$ and $\mu \in \mathbb{R}$ of degree 1;
2. $\mathrm{Im}\,(g(d_j, \eta)) = 0, \mathrm{Im}\,(g(\overline{d_j}, \eta)) = 0$ with $j = 1, 2$.

In this way, the existence of a short trajectory connecting the point E_1 with E_2 and its complex conjugate is guaranteed.

As $m \to m_c$, we reach the point (η_c, m_c); where $\mu, d, \overline{d}$ collapse in one point. In this case, we still have the phase transition described before but with a critical behaviour along the characteristic $x = \eta_c(m_c)t$. We study the case when $m < m_c$.



(a) $-\infty < \eta \leq \eta^*$ (b) $\eta = \eta^*(m)$ (c) $\eta^* < \eta < 0$

Figure 2.5: The level set $\text{Im}\,(g) = 0$ for $m > m_c$.

We fix $m < m_c$, we solve (2.3.9) for η. We define $c(m) := \text{Im}\,(E_2)^2[54Q(m) + 9(m-1)]$, then we rewrite the equation (2.3.9) as

$$(\eta_c(m)^2 - \eta^2) + \eta^2(c(m) - \eta^2)^2 = 0. \tag{2.3.17}$$

By expanding the two monomials, we obtain a fourth degree equation in η

$$\eta^4(3\eta_c(m)^2 - 2c(m)) - \eta^2(3\eta_c(m)^4 - c(m)^2) + \eta_c(m)^3 = 0, \tag{2.3.18}$$

which it is solved by $\eta_\pm(m)$

$$\eta_\pm(m) = -\left(\frac{3\eta_c(m)^4 - c(m)^2 \pm \sqrt{\Delta(m)}}{2(3\eta_c(m)^2 - 2c(m))}\right)^{\frac{1}{2}}, \tag{2.3.19}$$

where $\Delta(m)$ is the discriminant of (2.3.18), with $\Delta(m) > 0$ for $m < m_c$.

Remark 2.3.1. *For $m = m_c$, $c(m_c) = \eta_c(m_c)^2$ and $\Delta(m_c) = 0$, so we have only one solution of (2.3.18) which is exactly $\eta_c(m_c)$.*

For $\eta = \eta_+$, $\text{Im}\,(d) = 0$ and $\text{Re}\,(d) < \mu$, so the g-function has two real critical points, one simple and one double. For $\eta_+ < \eta < \eta_-$, the function $W(\eta, E_1, E_2)$ defined in (2.3.8) has values in the complex plane, which implies that the function (2.3.7) also have values in the complex plane. This means that now the points μ, d and $\overline{d}$ are real and the g-function (2.3.5) becomes

$$g(z) = 2\int_{E_2}^{z} \frac{(\zeta - \mu_1)(\zeta - \mu_2)(\zeta - \mu_3)}{\sqrt{(z - E_1)(z - E_2)(z - \overline{E}_1)(z - \overline{E}_2)}}\text{d}\zeta, \tag{2.3.20}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are solution of the system

$$\begin{cases} \mu_1 + \mu_2 + \mu_3 = -\dfrac{\eta}{2}; \\ (\mu_1 + \mu_2)\mu_3 + \mu_1\mu_2 = \dfrac{\text{Im}\,(E_1)^2 + \text{Im}\,(E_2)^2}{2}; \\ \mu_1\mu_2\mu_3 = -\dfrac{\eta}{2}\text{Im}\,(E_2)^2(1 - Q(m)). \end{cases} \tag{2.3.21}$$

In this case, our g-function is analytic in $\mathbb{C} \setminus \upsilon_{[E_2,\overline{E_2}]} \cup \upsilon_{[E_1,\overline{E_1}]}$, where $\upsilon_{[E_2,\overline{E_2}]} \cup \upsilon_{[E_1,\overline{E_1}]}$ are the oriented contours on which $\mathrm{Im}\,(g(z)) = 0$. Their existence is guaranteed by since they are trajectories of the quadratic differential $(g'(z))^2 dz^2$.

For $\eta = \eta_-$ the bands $\upsilon_{[E_2,\overline{E_2}]}$ and υ_∞ intersect at the point $\mu_2 = \mu_3$ and for $\eta > \eta_-$ the function $W(\eta, E_1, E_2) \in \mathbb{R}$, which implies that our g-function has still three zeroes, but only one (μ_1) is real while the other two are complex (with $\mu_2 = \overline{\mu_3}$). Due to the local behaviour of the trajectories of $(g'(z))^2 dz^2$ near the zeroes μ_1, μ_2 and μ_3 and since $\mu_1 < \mathrm{Re}\,(\mu_2)$, the short trajectory $\upsilon_{[E_2,\overline{E_2}]}$ is splitted in two trajectory , one starting at E_2 and following asymptotically the line $\mathrm{Re}\,(z) = -\eta/2$ and it's complex conjugate. This implies that both the g-functions (2.3.5) and (2.3.20) are not well define.

We need to define another g-function, still analytic in $\mathbb{C} \setminus \upsilon_{[E_2,\overline{E_2}]} \cup \upsilon_{[E_1,\overline{E_1}]}$, such that it satisfy the hypotesis (2.3.1),(2.3.2),(2.3.3) and the following conditions:

1. $g'(z)$ has 4 zeroes, $\mu_1, \mu_2 \in \mathbb{R}$ of degree 1 and a pair of complex conjugate points $d_1, \overline{d_1}$ of degree 1/2.
2. $\mathrm{Im}\,(g(d_1)) = 0$ and $\mathrm{Im}\,(g(\overline{d_1})) = 0$.

In this way, the existence of the short trajectory $\upsilon_{[E_2,\overline{E_2}]}$ is guaranteed.

Another phase transition emerges when the points μ_1 and μ_2 collide in one real point μ. In this case the existence of the short trajecyory $\upsilon_{[E_1,\overline{E_1}]}$ is not guaranteed and we need to define a new g-function, which satisfies the conditions (2.3.1),(2.3.2),(2.3.3) and (1),(2). We call η_* the value of η where $\mu_1(\eta_*) = \mu_2(\eta_*)$.

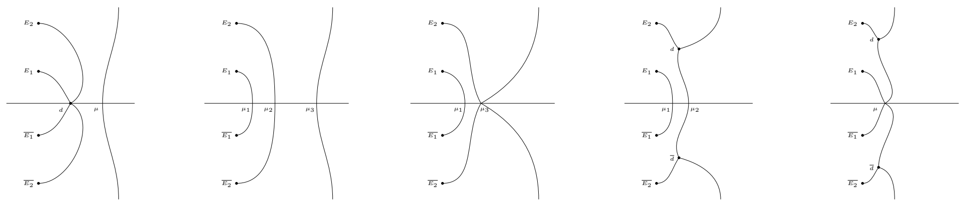


Figure 2.6: The level set $\mathrm{Im}\,(g) = 0$ for $m < m_c$ at η varing from $\eta = \eta_+(m)$ (left) to $\eta = \eta_*(m)$ (right)

In the next chapters, we will focus on the study of the long-time behaviour of the NLS solution $\psi(x,t)$ for the various cases we have shown before. Specifically, in chapter 3 we will study the case when $m > m_c$, while in chapter 4 we will treat the case when $m < m_c$, studying the sectors of the (x,t)-plane $\eta_+ < \eta \leq \eta_-$ and $\eta_- < \eta \leq \eta_*$.

Chapter 3

Elliptic domain with $m > m_c$: the Genus 1 and Genus 3 sectors

In this chapter, we examine the case where the elliptic parameter m is greater then m_c. As we already shown in section 2.3, the level set $\mathrm{Im}\,(g) = 0$ has one phase transitions for $-\infty < \eta < 0$, where the g-function increases his genus from one (for $-\infty < \eta < \eta^*$) to three (for $\eta^* < \eta < 0$). This implies that in this two sectors of the (x, t)-plane we have two different RHP. Consequently, the solution $\psi(x, t)$ has two different behaviours in those sectors as $t \sim +\infty$. In the next sections we will study the long-time asymptotic of the original RHP (2.1.4) and we will prove the following theorem:

Theorem 3.0.1. *Let $\mathscr{D} \subset \mathbb{C}^+$ be an elliptic domain with foci E_1, E_2. Suppose $r : [E_1, E_2] \to \mathbb{C}$ is an analytic and bounded function that vanishes at the foci as $r(E_j) \sim (z - E_j)^{1/2}$ for $j = 1, 2$. Suppose also that $E_1, E_2 \in i\mathbb{R}_+$, with $\mathrm{Im}\,(E_1) < \mathrm{Im}\,(E_2)$, and the parameter $m = \frac{\mathrm{Im}\,(E_1)^2}{\mathrm{Im}\,(E_2)^2} > m_c$, with m_c solution of the equation $\mathcal{Q}(m) = 0$, where $\mathcal{Q}(m)$ is define in (2.3.12). Then the large-time asymptotic of the soliton gas solution of NLS, recovered from the solution of the RHP (2.1.4), with spectrum in the domain $\mathscr{D}$ has the following form, according to the values of $\eta. = \frac{x}{t}$:*

- *for $-\infty < \eta < \eta^*$, with η^* a critical parameter depending on $\mathrm{Im}\,(E_1)$ and $\mathrm{Im}\,(E_2)$*

$$\psi(x, t) = i(\mathrm{Im}\,(E_1) + \mathrm{Im}\,(E_2))\times \\ \times \mathrm{dn}((\mathrm{Im}\,(E_1) + \mathrm{Im}\,(E_2))(x - x_0(\eta)) + K(m_1), m_1) + \mathcal{O}(\mathrm{e}^{-ct}); \tag{3.0.1}$$

where $c > 0$, $\mathrm{dn}(x)$ is the Jacobi delta amplitude, x_0 is determine by $r(z)$ and

$$K(m_1) := \sqrt{1 - m}K(m) \quad m_1 := \frac{4\sqrt{m}}{(1 + \sqrt{m})^2}; \tag{3.0.2}$$

- *for $\eta^* < \eta < 0$*

$$\psi(x,t) = i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1) + \mathrm{Im}\,(d_2(\eta)) - \mathrm{Im}\,(d_1(\eta)))\times$$
$$\times \frac{\Theta(0)\Theta(\vec{u}_\infty - \frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})}{\Theta(\frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})\Theta(\vec{u}_\infty)} + \mathcal{O}(t^{-1}); \tag{3.0.3}$$

where $\vec{\Omega}(\eta), \vec{\Delta}(\eta)$ are 3-dimensional real vectorswith components defined in (3.2.2) and (3.2.10), $\vec{u}_\infty$ is the Abel map (3.2.27) valued at $z=\infty$ and d_1 and d_2 are points in $\mathbb{C}^+$.

- *for $\eta > 0$ the solution decays exponentially*

$$\psi(x,t) \sim \mathcal{O}(e^{-ct})$$

with $c > 0$.

3.1 The Genus 1 sector

From the original RHP (2.1.4), we gather the time variable t from the argument of the exponential and we get

$$\Gamma(z)_+ = \Gamma(z)_- \begin{bmatrix} 1 & -r^*(z)\mathrm{e}^{-2it\hat{\theta}(z,\eta)}\chi_{[\overline{E_2},\overline{E_1}]} \\ r(z)\mathrm{e}^{2it\hat{\theta}(z,\eta)}\chi_{[E_1,E_2]} & 1 \end{bmatrix} \tag{3.1.1}$$

where $\eta = \frac{x}{t}$ and $\hat{\theta}(z,\eta) := z^2 + \eta z$.

We study the sign of $\mathrm{Im}\,(\hat{\theta}(z,\eta))$ to understand in which sectors of the (x,t)-plane the jump matrices in (3.1.1) has an exponential growing behaviour:

$$\mathrm{Im}\,(\hat{\theta}(z,\eta)) = \mathrm{Im}\,(z)(\eta + 2\mathrm{Re}\,(z)) = 0 \implies \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) = 0\} \cup \left\{z \in \mathbb{C} \mid \mathrm{Re}\,(z) = -\frac{\eta}{2}\right\}. \tag{3.1.2}$$

Then $\mathrm{Im}\,(\hat{\theta}) > 0$ for $\{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0,\ \mathrm{Re}\,(z) > -\frac{\eta}{2}\}$ and $\{z \in \mathbb{C} \mid \mathrm{Im}\,(z) < 0,\ \mathrm{Re}\,(z) < -\frac{\eta}{2}\}$, while $\mathrm{Im}\,(\hat{\theta}) < 0$ for $\{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0,\ \mathrm{Re}\,(z) < -\frac{\eta}{2}\}$ and $\{z \in \mathbb{C} \mid \mathrm{Im}\,(z) < 0,\ \mathrm{Re}\,(z) > -\frac{\eta}{2}\}$. The distributions of the signs its shown in Figure 3.1. This implies that, for $\eta > 0$, the jump matrices in (3.1.1) tends to the identity matrix as $t \to +\infty$, and the solution $\psi(x,t)$ decays exponentially at $t \to +\infty$. While for $\eta < 0$, jump has an exponential growing behavior as t tends to infinity. To proceed with the analysis of this scenario we need to apply the Nonlinear Steepest Descent technique.

We consider the sector $-\infty < \eta < \eta^*$. As we anticipated in section 2.3, we define a

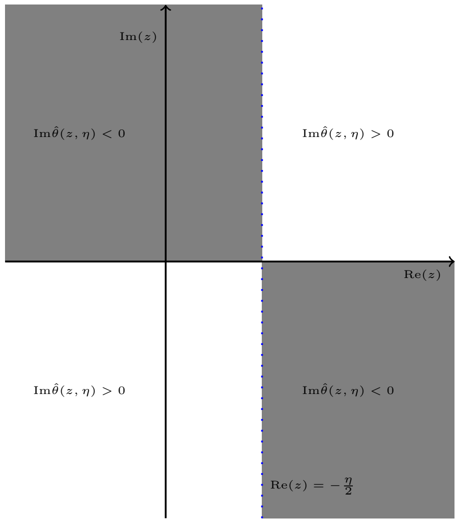


Figure 3.1: The sign of Im $(\hat{\theta}(z))$

g-function $g(z,\eta)$, analytic in $\mathbb{C} \setminus \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]}$, which satisfy the RHP

$$g(z,\eta)_+ + g(z,\eta)_- = 0 \text{ for } \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]};$$

$$g(z,\eta)_+ - g(z,\eta)_- = 2g(E_1) := \hat{\Omega}(\eta) \text{ for } [\overline{E_1}, E_1]; \tag{3.1.3}$$

$$g^*(z,\eta) = g(z,\eta); \quad g(z,\eta) = \hat{\theta}(z,\eta) + \mathcal{O}(z^{-1}) \text{ as } z \to \infty; \tag{3.1.4}$$

$$g(z,\eta) \sim |z - E_j|^{1/2}, \quad \text{or } g(z,\eta) \sim |z - \overline{E_j}|^{1/2} \text{ at the endpoints}; \tag{3.1.5}$$

where $\hat{\Omega}(\eta) \in \mathbb{R}$. The solution of the RHP (3.1.3) is given by the g-function (2.3.5) and the constant $\hat{\Omega}(\eta)$ is defined as

$$\hat{\Omega}(\eta) := \frac{\pi \text{Im}\,(E_2)}{2K(m)}\eta. \tag{3.1.6}$$

We now describe the application of the Nonlinear Steepest Descent step by step.

Step 1 we move the jump contours from the segments $[E_1, E_2] \cup [\overline{E_2}, \overline{E_1}]$ to the level set Im $(g) = 0$ connecting the endpoints. We denote this level sets as $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$. We define with Σ_1 and Σ_2 as the regions enclosed by the loop $\upsilon_{[E_1,E_2]} \cup [E_2, E_1]$ and its complex conjugate respectively. Subsequently, we apply the transformation

$$\Gamma^{(1)}(z) = \Gamma(z) G^{(1)}(z,\eta,t) \tag{3.1.7}$$

where

$$G^{(1)}(z,\eta,t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -r(z)e^{2it\hat{\theta}(z,\eta)} & 0 \end{bmatrix} \text{ for } z \in \Sigma_1 \\ \begin{bmatrix} 1 & r^*(z)e^{2it\hat{\theta}(z,\eta)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \Sigma_2 \\ \mathbf{1} \text{ otherwise} \end{cases} \tag{3.1.8}$$

In this way, the new jump contours are the one showed in Figure 3.2.

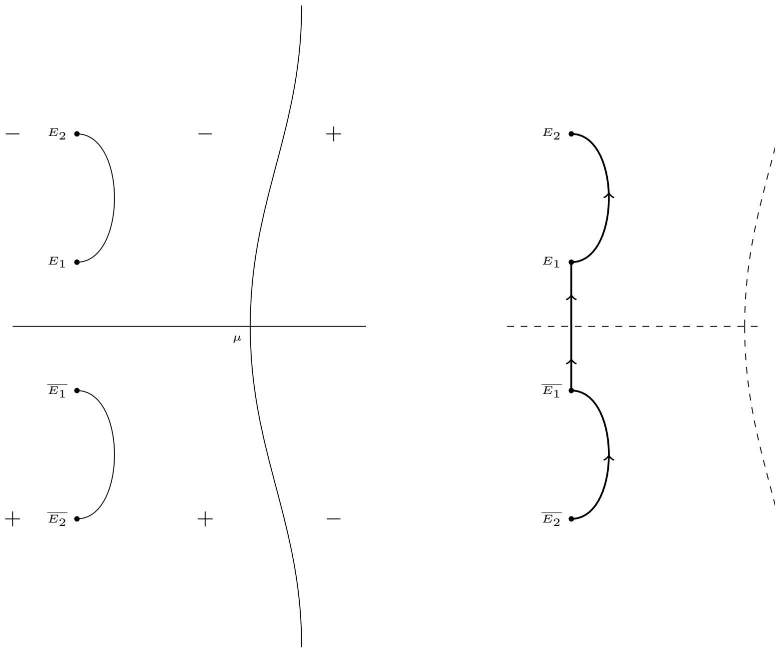


Figure 3.2: On the left: the sign distribution of Im (g). On the right: jump contour of the matrix $\Gamma^{(1)}(z)$.

Step 2 we absorb the exponent $2i\hat{\theta}(z,\eta)$ by applying the transformation

$$\Gamma^{(2)}(z) = \Gamma^{(1)}(z)e^{it(g(z,\eta)-\hat{\theta}(z,\eta))\sigma_3}. \tag{3.1.9}$$

where $g(z,\eta)$ is the g-function (2.3.5). Then the RHP (3.1.1) becomes:

$$\Gamma^{(2)}(z)_+ = \Gamma^{(2)}(z)_- \begin{bmatrix} e^{it(g_+-g_-)} & r^*(z)e^{-it(g_++g_-)}\chi_{v_{[\overline{E_2},\overline{E_1}]}} \\ -r(z)e^{it(g_++g_-)}\chi_{v_{[E_1,E_2]}} & e^{-it(g_+-g_-)} \end{bmatrix} \tag{3.1.10}$$

$$\Gamma^{(2)}(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty.$$

From the jump conditions (3.1.3) of the g-function, the RHP (3.1.10) becomes:

$$\Gamma^{(2)}(z)_+ = \Gamma^{(2)}(z)_- V^{(2)}(z,\eta,t) \tag{3.1.11}$$

$$\Gamma^{(2)}(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty,$$

where $V^{(2)}(z,\eta,t)$ is given by

$$V^{(2)}(z,\eta,t) = \begin{cases} \begin{bmatrix} \mathrm{e}^{it(g_+-g_-)} & 0 \\ -r(z) & \mathrm{e}^{-it(g_+-g_-)} \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,E_2]}, \\ \begin{bmatrix} \mathrm{e}^{it(g_+-g_-)} & r^*(z) \\ 0 & \mathrm{e}^{-it(g_+-g_-)} \end{bmatrix} \text{ for } z \in \upsilon_{[\overline{E_2},\overline{E_1}]}, \\ e^{it\hat{\Omega}\sigma_3} \text{ for } z \in [\overline{E_1}, E_1]. \end{cases} \tag{3.1.12}$$

Step 3 we absorb the function $r(z)$ inside the matrix $\Gamma^{(2)}(z)$ trough the transformation

$$\Gamma^{(3)}(z) = F_\infty(\eta)^{\sigma_3}\Gamma^{(2)}(z)F(z,\eta)^{\sigma_3}. \tag{3.1.13}$$

The jump matrices of the RHP (3.1.11) transforms as

$$F_-(z,\eta)^{-\sigma_3}V^{(2)}(z,\eta,t)F_+(z,\eta)^{\sigma_3} = \begin{cases} \begin{bmatrix} \mathrm{e}^{it(g_+-g_-)} & 0 \\ -F_+F_-r(z) & \mathrm{e}^{-it(g_+-g_-)} \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,E_2]}, \\ \begin{bmatrix} \mathrm{e}^{it(g_+-g_-)} & \frac{r^*(z)}{F_+F_-} \\ 0 & \mathrm{e}^{-it(g_+-g_-)} \end{bmatrix} \text{ for } z \in \upsilon_{[\overline{E_2},\overline{E_1}]}, \\ e^{it\hat{\Omega}\sigma_3}\left(\dfrac{F_+}{F_-}\right)^{\sigma_3} \text{ for } z \in [\overline{E_1}, E_1]. \end{cases}$$

We are looking for a function $F(z)$, analytic in $\mathbb{C} \setminus \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]} \cup [\overline{E_1}, E_1]$, which satisfy the scalar RHP

$$\begin{aligned} &F_+(z)F_-(z) = (r(z))^{-1} \text{ for } z \in \upsilon_{[E_1,E_2]}; \\ &F_+(z)F_-(z) = r^*(z) \text{ for } z \in \upsilon_{[\overline{E_2},\overline{E_1}]}; \\ &\frac{F_+(z)}{F_-(z)} = e^{i\Delta} \text{ for } z \in [\overline{E_1}, E_1]; \end{aligned} \tag{3.1.14}$$

$$F^*(z) = (F(z))^{-1}; \qquad F(z) = F_\infty + \mathcal{O}(z^{-1}). \tag{3.1.15}$$

The RHP (3.1.14) is solved by

$$\begin{aligned} F(z,\eta) = \exp\Bigg\{ \frac{P_1(z)}{2\pi i}\Bigg[&-\int_{\upsilon_{[E_1,E_2]}} \frac{\log(r(\zeta))}{(\zeta - z)(P_1(\zeta))_+}\mathrm{d}\zeta + \int_{\upsilon_{[\overline{E_2},\overline{E_1}]}} \frac{\log(r^*(\zeta))}{(\zeta - z)(P_1(\zeta))_+}\mathrm{d}\zeta \\ &+i\Delta \int_{\overline{E_1}}^{E_1} \frac{\mathrm{d}\zeta}{(\zeta - z)P_1(\zeta)}\Bigg]\Bigg\} \end{aligned} \tag{3.1.16}$$

and Δ is defined in (2.2.13).

So, the new matrix $\Gamma^{(3)}(z,\eta,t)$ satisfy a new RHP, with jump condition

$$\Gamma^{(3)}_+(z) = \Gamma^{(3)}_-(z)V^{(3)}(z,\eta,t), \tag{3.1.17}$$

where

$$V^{(3)}(z,\eta,t)=\begin{cases}\begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & 0\\ -1 & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[E_1,E_2]}\\ \begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & 1\\ 0 & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{E_2},\overline{E_1}]}\\ e^{it(\hat{\Omega}+\Delta)\sigma_3} \text{ for } z\in[\overline{E_1},E_1]\end{cases} \tag{3.1.18}$$

and same boundary conditions at infinity of (3.1.11).

Step 4 we proceed to factorize the jump matrices $V^{(3)}(z,t,\eta)$ and open the lenses around the jump contours. Notably,since the matrices $V^{(3)}(z,t,\eta)$ are upper or lower triangular in $\upsilon_{[E_1,E_2]}\cup\upsilon_{[\overline{E_2},\overline{E_1}]}$, we can express the factorization as:

$$V^{(3)}(z,\eta,t)=\begin{bmatrix}1 & \frac{\mathrm{e}^{-2itg_-}}{(F_-)^2r(z)}\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & \frac{\mathrm{e}^{-2itg_+}}{(F_+)^2r(z)}\\ 0 & 1\end{bmatrix} \text{ for } z\in\upsilon_{[E_1,E_2]}; \tag{3.1.19}$$

$$V^{(3)}(z,\eta,t)=\begin{bmatrix}1 & 0\\ -\frac{(F_-)^2}{r^*(z)}\mathrm{e}^{2itg_-} & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{(F_+)^2}{r^*(z)}\mathrm{e}^{2itg_+} & 1\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{E_2},\overline{E_1}]}. \tag{3.1.20}$$

From the boundness of the function $F(z)$ and $r(z)$ in a neighbourhood of the jump contours, we analytically extend the first and the third matrices of the factorizations (3.1.19) and (3.1.20).

We denote with $\mathscr{U}_\pm(\upsilon)$ the left $(+)$ or the right $(-)$ lens of υ and with $L_\pm(\upsilon)$ the boundary of $\mathscr{U}_\pm(\upsilon)$ without the curve υ , as it is displayed in the figure 3.3.

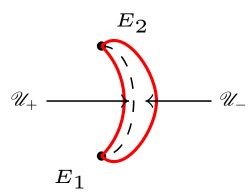


Figure 3.3: The lenses $\mathscr{U}_+(\upsilon_{[E_1,E_2]})$ and $\mathscr{U}_-(\upsilon_{[E_1,E_2]})$.

Then we apply the following transformation:

$$\Gamma^{(4)}(z):=\Gamma^{(3)}(z)G^{(4)}(z,\eta,t) \tag{3.1.21}$$

where

$$G^{(4)}(z,\eta,t) := \begin{cases} \begin{bmatrix} 1 & \frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{[E_1,E_2]}), \\ \begin{bmatrix} 1 & -\frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_+(\upsilon_{[E_1,E_2]}), \\ \begin{bmatrix} 1 & 0 \\ -\frac{(F(z,\eta))^2}{r^*(z)} e^{2itg(z,\eta)} & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{[\overline{E_2},\overline{E_1}]}), \\ \begin{bmatrix} 1 & 0 \\ -\frac{(F(z,\eta))^2}{r^*(z)} e^{2itg(z,\eta)} & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_+(\upsilon_{[\overline{E_2},\overline{E_1}]}) \end{cases} \tag{3.1.22}$$

and $G^{(4)}(z) = \mathbf{1}$ otherwise. In this way, we enlarge the jump contours by opening lenses around our jump contours $\upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]}$, as it si displayed in Figure 3.4, and the RHP (3.1.17) becomes

$$\Gamma_+^{(4)}(z) = \Gamma_+^{(4)}(z) V^{(4)}(z,\eta,t), \tag{3.1.23}$$

with $V^{(4)}(z,\eta,t)$ define also in the curves $L_\pm(\upsilon_{[E_1,E_2]})$, $L_\pm(\upsilon_{[\overline{E_2},\overline{E_1}]})$:

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in L_+(\upsilon_{[E_1,E_2]}) \cup L_-(\upsilon_{[E_1,E_2]}), \tag{3.1.24}$$

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ \frac{(F(z,\eta))^2 e^{2itg(z,\eta)}}{r^*(z)} & 1 \end{bmatrix} \text{ for } z \in L_+(\upsilon_{[\overline{E_2},\overline{E_1}]}) \cup L_+(\upsilon_{[\overline{E_2},\overline{E_1}]}), \tag{3.1.25}$$

while in the other jump contours it takes values:

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]}, \tag{3.1.26}$$

$$V^{(4)}(z,\eta,t) = e^{i(t\hat{\Omega}+\Delta)\sigma_3} \text{ for } z \in [\overline{E_1}, E_1]. \tag{3.1.27}$$

Looking at the distribution of the sign of $\operatorname{Im}(g)$ around the jump contours and at the results (3.1.24) and (3.1.25), we have that $V^{(4)}(z,\eta,t)$ tends to the identity matrix $\mathbf{1}$ as $t \to +\infty$ around the lenses.

3.1.1 The Model problem for the Genus 1 case

After removing the jumps which goes to the identity as $t \to +\infty$, the RHP (3.1.23) results in the following model problem

$$X_+(z,\eta,t) = X_-(z,\eta,t) V_X(z,\eta,t) \text{ for } z \in \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]} \cup [\overline{E_1}, E_1], \tag{3.1.28}$$

$$\overline{X(\overline{z})} = \sigma_2 X(z) \sigma_2, \quad X(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty,$$

with $V_X(z,\eta,t)$ which is given by

$$V_X(z,\eta,t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{for } z \in \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]}, \\ \mathrm{e}^{i(t\hat{\Omega}+\Delta)\sigma_3} & \text{for } z \in [\overline{E_1}, E_1]. \end{cases} \tag{3.1.29}$$

This model problem is equal to the one that we solved in section 2.2, with the same homological basis; as we can see in Figure (3.5). Then the solution is the same

$$X(z,\eta,t) = \begin{bmatrix} \frac{\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)+\frac{1}{\phi_1(z)}\right)\varphi_1(z,\eta,t) & -\frac{i\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)-\frac{1}{\phi_1(z)}\right)\psi_1(z,\eta,t) \\ \frac{i\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)-\frac{1}{\phi_1(z)}\right)\varphi_2(z,\eta,t) & \frac{\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)+\frac{1}{\phi_1(z)}\right)\psi_2(z,\eta,t) \end{bmatrix} \tag{3.1.30}$$

where $\phi_1(z), \psi_j(z,\eta,t), \varphi_j(z,\eta,t)$; with $j = 1,2$; are defined in Lemma 2.2.3.

3.1.2 Error Parametrix and long-time behaviour of $\psi(x,t)$

For the same reason explained in section 2.2, the parametrix at the end points $E_j, \overline{E_j}$, with $j = 1,2$, of this problem are exponentially neat to the identity, which means that for $t \to +\infty$ the error function goes like

$$\mathcal{E}_{E_j}(z,\eta,t) := X(z,\eta,t)(\Gamma^{(4)}(z,\eta,t))^{-1} = \mathbf{1} + \mathcal{O}(e^{-ct})$$

with $c > 0$ and uniformly in z.

This means that the long-time asymptotic behavior of the NLS solution $\psi(x,t)$ is the same of the initial datum $\psi_0(x)$ (2.2.53).

3.2 The Genus 3 sector

We consider the sector $\eta^* < \eta < 0$. As we shown in section 2.3, at $\eta = \eta^*$ the critical points $d(\eta^*), \overline{d(\eta^*)}$ hit the bands $\upsilon_{[E_1,E_2]}$ and $\upsilon_{[\overline{E_2},\overline{E_1}]}$ respectively. For $\eta > \eta^*$ we need to define a new g-function, still analytic in $\mathbb{C} \setminus \upsilon_{[E_1,E_2]} \cup \upsilon_{[\overline{E_2},\overline{E_1}]} \cup \upsilon_{[\overline{E_1},E_1]}$, which satisfy the conditions (3.1.4), (1) and (2), and solve the scalar RHP:

$$\begin{aligned} &g_+ + g_- = 0 \quad \text{for } z \in \upsilon_{[d_2(\eta);E_2]} \cup \upsilon_{[\overline{E_2},\overline{d_2(\eta)}]}; \\ &g_+ + g_- = \Omega_1(\eta) \quad \text{for } z \in \upsilon_{[E_1,d_1(\eta)]} \cup \upsilon_{[\overline{d_1(\eta)},\overline{E_1}]}; \\ &g_+ - g_- = \Omega_1(\eta) + \Omega_2(\eta) \quad \text{for } z \in \upsilon_{[d_1(\eta),d_2(\eta)]} \cup \upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]}; \\ &g_+ - g_- = \Omega_1(\eta) + \Omega_2(\eta) + \Omega_3(\eta) \quad \text{for } z \in \upsilon_{[\overline{d_1(\eta)};d_1(\eta)]}; \end{aligned} \tag{3.2.1}$$

where $\Omega_1(\eta), \Omega_2(\eta), \Omega_3(\eta)$ are defined as

$$\begin{aligned}
&\Omega_1(\eta) := g_+(d_1(\eta)) + g_+(d_1(\eta)),\\
&\Omega_2(\eta) := 2g_+(d_2(\eta)) - \Omega_1(\eta),\\
&\Omega_3(\eta) := g_+(E_1) + g_-(E_1) - \Omega_2(\eta)
\end{aligned} \tag{3.2.2}$$

The solution of the RHP (3.2.1) is the following:

$$g(z,\eta) = 2\int_{E_2}^{z} \frac{(\zeta-\mu(\eta))\sqrt{(\zeta-d_1(\eta))(\zeta-d_2(\eta))(\zeta-\overline{d_1(\eta)})(\zeta-\overline{d_2(\eta)})}}{\sqrt{(\zeta-E_1)(\zeta-E_2)(\zeta-\overline{E}_1)(\zeta-\overline{E}_2)}}\mathrm{d}\zeta \tag{3.2.3}$$

with $\mu(\eta) \in \mathbb{R}$.

The parameters $\mu(\eta), \mathrm{Re}\,(d_j(\eta)), \mathrm{Im}\,(d_j(\eta))$, with $j = 1, 2$, are still given by the conditions (3.1.4) (1) and (2); which they translate in the system of equations for η fix

$$\begin{aligned}
&2\mu(\eta) = -\eta - 2(\mathrm{Re}\,(d_1) + \mathrm{Re}\,(d_2));\\
&\mathrm{Im}\,(d_1(\eta))^2 + \mathrm{Im}\,(d_2(\eta))^2\\
&\qquad + 2\mathrm{Re}\,(d_1(\eta))\mathrm{Re}\,(d_2(\eta)) = \mathrm{Im}\,(E_1)^2 + \mathrm{Im}\,(E_2)^2;\\
&\int_{\overline{E_2}}^{E_2} dg = \int_{\overline{d_2}}^{d_2} dg = \int_{\overline{d_1}}^{d_1} dg = 0.
\end{aligned} \tag{3.2.4}$$

We can now proceed with the Nonlinear Steepest Descend analysis. First of all, we move the jump contours from the segments $[E_1, E_2]$ and its complex conjugate to the level sets of $\mathrm{Im}\,(g) = 0$, which is display in Figure 3.6. We apply the transformation (3.1.7), but in this case the sets Σ_1 and Σ_2 are encircled by the loop $\upsilon_{[E_1,d_1(\eta)]} \cup \upsilon_{[d_1(\eta),d_2(\eta)]} \cup \upsilon_{[d_2(\eta),E_2]} \cup [E_2, E_1]$ and its complex conjugate respectively.

The next step is to apply the transformation (3.1.9), where now the g-function given by the formula (3.2.3).

The matrix $\Gamma^{(2)}(z)$ must solve the RHP

$$\Gamma^{(2)}_+(z) = \Gamma^{(2)}_-(z)V^{(2)}(z,\eta,t), \tag{3.2.5}$$

$$\overline{\Gamma^{(2)}(\overline{z})} = \sigma_2\Gamma^{(2)}(z)\sigma_2, \quad \Gamma^{(2)}(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \qquad \text{as } z \to \infty, \tag{3.2.6}$$

where

$$V^{(2)}(z,\eta,t)=\begin{cases}\begin{bmatrix}\mathrm{e}^{it(g_+-g_-)} & 0\\ -r(z) & \mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[d_2(\eta),E_2]},\\ \begin{bmatrix}\mathrm{e}^{it(g_+-g_-)} & r^*(z)\\ 0 & \mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]},\\ \begin{bmatrix}\mathrm{e}^{it(g_+-g_-)} & 0\\ -r(z)\mathrm{e}^{it\Omega_1} & \mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[E_1,d_1(\eta)]},\\ \begin{bmatrix}\mathrm{e}^{it(g_+-g_-)} & r^*(z)\mathrm{e}^{-it\Omega_1}\\ 0 & \mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]},\\ \begin{bmatrix}\mathrm{e}^{it(\Omega_1+\Omega_2)} & 0\\ -r(z)\mathrm{e}^{it(g_++g_-)} & \mathrm{e}^{-it(\Omega_1+\Omega_2)}\end{bmatrix} \text{ for } z\in\upsilon_{[d_1(\eta),d_2(\eta)]},\\ \begin{bmatrix}\mathrm{e}^{it(\Omega_1+\Omega_2)} & r^*(z)\mathrm{e}^{-it(g_++g_-)}\\ 0 & \mathrm{e}^{-it(\Omega_1+\Omega_2)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]},\\ \mathrm{e}^{it(\Omega_1+\Omega_2+\Omega_3)\sigma_3} \text{ for } z\in\upsilon_{[\overline{d_1(\eta)},d_1(\eta)]}.\end{cases} \tag{3.2.7}$$

We apply the transformation (3.1.13), where the function $F(z,\eta)$ solve the scalar RHP

$$\begin{aligned}F_+F_-=(r(z))^{-1}\quad &\text{for } z\in\upsilon_{[d_2(\eta);E_2]};\quad F_+F_-=\frac{\mathrm{e}^{i\Delta_1}}{r(z)}\quad \text{for } z\in\upsilon_{[E_1;d_1(\eta)]};\\ F_+F_-=r^*(z)\quad &\text{for } z\in\upsilon_{[\overline{E_2};\overline{d_2}]};\quad F_+F_-=-\mathrm{e}^{i\Delta_1}r^*(z)\quad \text{for } z\in\upsilon_{[\overline{E_1},\overline{d_1(\eta)}]};\\ &\frac{F_+}{F_-}=e^{i(\Delta_1+\Delta_2)}\quad \text{for } z\in\upsilon_{[d_1(\eta);d_2(\eta)]}\cup\upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]};\\ &\frac{F_+}{F_-}=e^{i(\Delta_1+\Delta_2+\Delta_3)}\quad \text{for } z\in\upsilon_{[\overline{d_1(\eta)},d_1(\eta)]};\end{aligned} \tag{3.2.8}$$

with boundary condition given by (3.1.15).

This RHP is solved by:

$$\begin{aligned}F(z,\eta)=\exp\Bigg\{\frac{P_3(z)}{2\pi i}\Bigg[&-\int_{\upsilon_{[d_2,E_2]}}\frac{\log(r(\zeta))}{(\zeta-z)(P_3(\zeta))_+}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{E_2},\overline{d_2}]}}\frac{\log(r(\zeta))}{(\zeta-z)(P_3(\zeta))}\mathrm{d}\zeta\\ &+\int_{\upsilon_{[E_1,d_1]}}\frac{-\log(r(z))+i\Delta_1}{(\zeta-z)(P_3(\zeta))_+}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{d_1},\overline{E_1}]}}\frac{\log(r^*(z))+i\Delta_1}{(\zeta-z)(P_3(\zeta))_+}\mathrm{d}\zeta\\ &+\int_{\upsilon_{[d_1,d_2]}}\frac{i(\Delta_1+\Delta_2)}{(\zeta-z)P_3(\zeta)}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{d_2},\overline{d_1}]}}\frac{i(\Delta_1+\Delta_2)}{(\zeta-z)P_3(\zeta)}\mathrm{d}\zeta\\ &+\int_{\upsilon_{[\overline{d_1},d_1]}}\frac{i(\Delta_1+\Delta_2+\Delta_3)}{(\zeta-z)P_3(\zeta)}\mathrm{d}\zeta\Bigg]\Bigg\}\end{aligned} \tag{3.2.9}$$

where

$$P_3(z)=\sqrt{(\zeta-E_1)(\zeta-E_2)(\zeta-\overline{E}_1)(\zeta-\overline{E}_2)(\zeta-d_1(\eta))(\zeta-d_2(\eta))(\zeta-\overline{d_1(\eta)})(\zeta-\overline{d_2(\eta)})}$$

is a multivalued complex function, analytic in $\mathbb{C}\backslash \upsilon_{[d_2(\eta);E_2]}\cup\upsilon_{[E_1;d_1(\eta)]}\cup\upsilon_{[\overline{E_1},\overline{d_1(\eta)}]}\cup\upsilon_{[\overline{E_2};\overline{d_2}]}$, and $\Delta_1,\Delta_2,\Delta_3$ are determined by the system of equations:

$$
\begin{aligned}
&-\int_{\upsilon_{[d_2,E_2]}}\zeta^j\frac{\log(r(\zeta))}{(P_3(\zeta))}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{E_2},\overline{d_2}]}}\zeta^j\frac{\log(r(\zeta))}{(P_3(\zeta))}\mathrm{d}\zeta\\
&+\int_{\upsilon_{[E_1,d_1]}}\frac{\zeta^j(-\log(r(z))+i\Delta_1)}{(P_3(\zeta))_+}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{d_1},\overline{E_1}]}}\frac{\zeta^j(\log(r^*(z))+i\Delta_1)}{(P_3(\zeta))_+}\mathrm{d}\zeta\\
&+\int_{\upsilon_{[d_1,d_2]}}\frac{i\zeta^j(\Delta_1+\Delta_2)}{P_3(\zeta)}\mathrm{d}\zeta+\int_{\upsilon_{[\overline{d_2},\overline{d_1}]}}\frac{i\zeta^j(\Delta_1+\Delta_2)}{P_3(\zeta)}\mathrm{d}\zeta\\
&+\int_{\upsilon_{[\overline{d_1},d_1]}}\frac{i\zeta^j(\Delta_1+\Delta_2+\Delta_3)}{P_3(\zeta)}\mathrm{d}\zeta=0 \text{ with } j=0,1,2.
\end{aligned}
\tag{3.2.10}
$$

Then we have that $\Gamma^{(3)}(z)$ has new jump conditions

$$
\Gamma^{(3)}_+(z)=\Gamma^{(3)}_-(z)V^{(3)}(z,\eta,t),
\tag{3.2.11}
$$

where $V^3(z,\eta,t)=F_-^{-\sigma_3}(z,\eta)V^{(2)}(z,\eta,t)F_+^{\sigma_3}(z,\eta)$. With $F(z,\eta)$ given by (3.2.9), the jump matrices $V^{(3)}(z,\eta,t)$ have the following form:

$$
V^{(3)}(z,\eta,t)=\begin{cases}
\begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & 0\\ -1 & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[d_2(\eta),E_2]},\\
\begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & 1\\ 0 & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]},\\
\begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & 0\\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[E_1,d_1(\eta)]},\\
\begin{bmatrix}\frac{F_+}{F_-}\mathrm{e}^{it(g_+-g_-)} & -\mathrm{e}^{-i(t\Omega_1+\Delta_1)}\\ 0 & \frac{F_-}{F_+}\mathrm{e}^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]},\\
\begin{bmatrix}\mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]} & 0\\ -r(z)F_+F_-\mathrm{e}^{it(g_++g_-)} & \mathrm{e}^{-i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]}\end{bmatrix} \text{ for } z\in\upsilon_{[d_1(\eta),d_2(\eta)]},\\
\begin{bmatrix}\mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]} & r^*(z)(F_+F_-)^{-1}\mathrm{e}^{-it(g_++g_-)}\\ 0 & \mathrm{e}^{-i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]},\\
\mathrm{e}^{i[t(\Omega_1+\Omega_2+\Omega_3)+\Delta_1+\Delta_2+\Delta_3]\sigma_3} \text{ for } z\in\upsilon_{[\overline{d_1(\eta)},d_1(\eta)]}.
\end{cases}
\tag{3.2.12}
$$

We proceed with the factorization of the matrices $V^{(3)}(z,\eta,t)$. According to where they are defined, they factorize in two or three matrices. Indeed, for $z\in\upsilon_{[d_2(\eta),E_2]}\cup\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]}$ they have the same factorization of (3.1.19) and (3.1.20), while for $z\in\upsilon_{[E_1,d_1(\eta)]}\cup$

$\upsilon_{\overline{[d_1(\eta)},\overline{E_1]}}$ the matrices factorize as:

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{1}{(F_-)^2 r(z)} e^{-2itg_-} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathrm{e}^{-i(t\Omega_1+\Delta_1)} \\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & 0 \end{bmatrix} \times$$
$$\times \begin{bmatrix} 1 & \frac{1}{(F_+)^2 r(z)} e^{-2itg_+} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,d_1(\eta)]}, \tag{3.2.13}$$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ -\frac{(F_-)^2}{r^*(z)} \mathrm{e}^{2itg_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathrm{e}^{-i(t\Omega_1+\Delta_1)} \\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & 0 \end{bmatrix} \times$$
$$\times \begin{bmatrix} 1 & 0 \\ -\frac{(F_+)}{r^*(z)} e^{2itg_+} & 1 \end{bmatrix} \text{ for } z \in \upsilon_{\overline{[d_1(\eta)},\overline{E_1]}}. \tag{3.2.14}$$

In the contours $\upsilon_{[d_1(\eta),d_2(\eta)]}$ and $\upsilon_{\overline{[d_2(\eta)},\overline{d_1(\eta)]}}$, the matrices factorize as follows:

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ -r(z)(F_-)^2 e^{2itg_-} & 1 \end{bmatrix} \mathrm{e}^{+i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} \text{ for } z \in \upsilon_{[d_1(\eta),d_2(\eta)]}, \tag{3.2.15}$$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{r^*(z)}{(F_-)^2} \mathrm{e}^{-2itg_-} \\ 0 & 1 \end{bmatrix} \mathrm{e}^{+i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} \text{ for } z \in \upsilon_{\overline{[d_2(\eta)},\overline{d_1(\eta)]}}. \tag{3.2.16}$$

We open the lenses around the jump contour

$$\mathscr{U}_\pm(\upsilon_{[d_2(\eta),E_2]}), \mathscr{U}_\pm(\upsilon_{[E_1,d_1(\eta)]}), \mathscr{U}_-(\upsilon_{[d_1(\eta),d_2(\eta)]}),$$

and their complex conjugates. Then, defining the transformation

$$\Gamma^{(4)}(z) = \Gamma^{(3)}(z) G^{(4)}(z,\eta,t), \tag{3.2.17}$$

where

$$G^{(4)}(z,\eta,t) = \begin{cases} \begin{bmatrix} 1 & \frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{[d_2(\eta),E_2]}) \cup \mathscr{U}_-(\upsilon_{[E_1,d_1(\eta)]}) \\ \begin{bmatrix} 1 & -\frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_+(\upsilon_{[d_2(\eta),E_2]}) \cup \mathscr{U}_+(\upsilon_{[E_1,d_1(\eta)]}) \\ \begin{bmatrix} 1 & 0 \\ -\frac{(F(z,\eta))^2}{r^*(z)} \mathrm{e}^{2itg(z,\eta)} & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{\overline{[E_2},\overline{d_2(\eta)]}}) \cup \mathscr{U}_-(\upsilon_{\overline{[d_1(\eta)},\overline{E_1]}}) \\ \begin{bmatrix} 1 & 0 \\ \frac{(F(z,\eta))^2}{r^*(z)} \mathrm{e}^{2itg(z,\eta)} & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_+(\upsilon_{\overline{[E_2},\overline{d_2(\eta)]}}) \cup \mathscr{U}_+(\upsilon_{\overline{[d_1(\eta)},\overline{E_1]}}) \\ \begin{bmatrix} 1 & 0 \\ -r(z)(F(z,\eta))^2 e^{2itg(z,\eta)} & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{[d_1(\eta),d_2(\eta)]}) \\ \begin{bmatrix} 1 & \frac{r^*(z)}{(F(z,\eta))^2} \mathrm{e}^{-2itg(z,\eta)} \\ 0 & 1 \end{bmatrix} & \text{for } z \in \mathscr{U}_-(\upsilon_{\overline{[d_2(\eta)},\overline{d_1(\eta)]}}), \\ \mathbf{1} \text{ otherwise,} \end{cases} \tag{3.2.18}$$

we have that the matrix $\Gamma^{(4)}(z)$ satisfy a new RHP problem

$$\Gamma^{(4)}_{+}(z) = \Gamma^{(4)}_{-}(z)V^{(4)}(z,\eta,t), \tag{3.2.19}$$

with the jump matrices define also in the boundary of the lenses (see Figure 3.7)

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{e^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} \qquad \text{for } z \in L_{+}(\upsilon_{[d_2(\eta),E_2]}) \cup L_{-}(\upsilon_{[d_2(\eta),E_2]})$$
$$\cup L_{+}(\upsilon_{[E_1,d_1(\eta)]}) \cup L_{-}(\upsilon_{[E_1,d_1(\eta)]})$$

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ -\frac{(F(z,\eta))^2}{r^*(z)} \mathrm{e}^{2itg(z,\eta)} & 1 \end{bmatrix} \qquad \text{for } z \in L_{+}(\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]}) \cup L_{-}(\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]})$$
$$\cup L_{+}(\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]}) \cup L_{-}(\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]}),$$

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ -r(z)(F(z,\eta))^2 e^{2itg(z,\eta)} & 1 \end{bmatrix} \qquad \text{for } z \in L_{-}(\upsilon_{[d_1(\eta),d_2(\eta)]}),$$

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{r^*(z)}{(F(z,\eta))^2} \mathrm{e}^{-2itg(z,\eta)} \\ 0 & 1 \end{bmatrix} \qquad \text{for } z \in L_{-}(\upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]}),$$

and in the other jump contours $V^{(4)}(z,\eta,t)$ have values

$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{for } z \in \upsilon_{[d_2(\eta),E_2]} \cup \upsilon_{[\overline{E_2},\overline{d_2(\eta)}]},$$
$$V^{(4)}(z,\eta,t) = \begin{bmatrix} 0 & \mathrm{e}^{-i(t\Omega_1+\Delta_1)} \\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & 0 \end{bmatrix} \qquad \text{for } z \in \upsilon_{[E_1,d_1(\eta)]} \cup \upsilon_{[\overline{d_1(\eta)},\overline{E_1}]},$$

$$V^{(4)}(z,\eta,t) = \mathrm{e}^{+i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} \qquad \text{for } z \in \upsilon_{[d_1(\eta),d_2(\eta)]} \cup \upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]},$$
$$V^{(4)}(z,\eta,t) = \mathrm{e}^{i[t(\Omega_1+\Omega_2+\Omega_3)+\Delta_1+\Delta_2+\Delta_3]\sigma_3} \qquad \text{for } z \in \upsilon_{[\overline{d_1(\eta)},d_1(\eta)]}.$$

From the distribution of signs of $\mathrm{Im}\,(g)$ around the jump contours, we get that the jump matrices defined in the border of the lenses tends to the identity matrix $\mathbf{1}$ as $t \to +\infty$.

3.2.1 The Model problem in the Genus 3 case

In the limit $t \to +\infty$, the RHP (3.2.19) results in a new model problem

$$X_{+}(z,\eta,t) = X_{-}(z,\eta,t)V_X(z,\eta,t), \tag{3.2.20}$$
$$\overline{X(\overline{z})} = \sigma_2 X(z)\sigma_2 \qquad X(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty,$$

with $V_X(z,\eta,t)$ given by

$$V_X(z,\eta,t)=\begin{cases}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \text{ for } z\in \upsilon_{[d_2(\eta),E_2]}\cup\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]},\\ \begin{bmatrix}0 & \mathrm{e}^{-i(t\Omega_1+\Delta_1)}\\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & 0\end{bmatrix} \text{ for } z\in \upsilon_{[E_1,d_1(\eta)]}\cup\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]},\\ \mathrm{e}^{+i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} \text{ for } z\in \upsilon_{[d_1(\eta),d_2(\eta)]}\cup\upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]},\\ \mathrm{e}^{i[t(\Omega_1+\Omega_2+\Omega_3)+\Delta_1+\Delta_2+\Delta_3]\sigma_3} \text{ for } z\in \upsilon_{[\overline{d_1(\eta)},d_1(\eta)]}.\end{cases} \tag{3.2.21}$$

This kind of RHP is solved in a similar way to the previews problems (2.2.30) and (3.1.28).

Step 1 we solve the homogeneous RHP: For $z\in \upsilon_{[d_2(\eta),E_2]}\cup\upsilon_{[\overline{E_2},\overline{d_2(\eta)}]}\cup\upsilon_{[E_1,d_1(\eta)]}\cup\upsilon_{[\overline{d_1(\eta)},\overline{E_1}]}$

$$X_+^{(0)}(z,\eta)=X_-^{(0)}(z,\eta)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \tag{3.2.22}$$

$$\overline{X^{(0)}(\overline{z})}=\sigma_2X^{(0)}(z)\sigma_2, \qquad X^{(0)}(z)=\mathbf{1}+\mathcal{O}(z^{-1}) \text{ as } z\to\infty. \tag{3.2.23}$$

The solution of this RHP is given by:

$$X^{(0)}(z,\eta)=\begin{bmatrix}\left(\phi_3(z,\eta)+\frac{1}{\phi_3(z,\eta)}\right) & -i\left(\phi_3(z,\eta)-\frac{1}{\phi_3(z,\eta)}\right)\\ i\left(\phi_3(z,\eta)-\frac{1}{\phi_3(z,\eta)}\right) & \left(\phi_3(z,\eta)+\frac{1}{\phi_3(z,\eta)}\right)\end{bmatrix}, \tag{3.2.24}$$

with

$$\phi_3(z,\eta):=\left(\frac{(z-E_2)(z-d_1(\eta))(z-\overline{E_1})(z-\overline{d_2(\eta)})}{(z-\overline{E_2})(z-\overline{d_1(\eta)})(z-E_1)(z-d_2)}\right)^{1/4} \tag{3.2.25}$$

is analytic in $\mathbb{C}\setminus\upsilon_{[d_2(\eta);E_2]}\cup\upsilon_{[E_1;d_1(\eta)]}\cup\upsilon_{[\overline{E_1},\overline{d_1(\eta)}]}\cup\upsilon_{[\overline{E_2};\overline{d_2}]}$.

Step 2 we introduce a Riemann Surface $\mathscr{P}_3$ of genus 3

$$\mathscr{P}_3=\{(w,z)\in\mathbb{C}^2|w^2=P_3(z)^2\}, \tag{3.2.26}$$

with brunch cut along the curves $\upsilon_{[d_2,E_2]},\upsilon_{[E_1,d_1]},\upsilon_{[\overline{d_1},\overline{E_1}]},\upsilon_{[\overline{E_2},\overline{d_2}]}$. We also introduce a canonical homological basis α_j,β_j, with $j=1,2,3$ on it, which is shown in Figure 3.9, and we define the Abel map $\vec{u}(z,z_0)$ as

$$\vec{u}(z,z_0):=\int_{z_0}^{z}\vec{\varpi}; \tag{3.2.27}$$

where $\vec{\varpi}=(\varpi_1,\varpi_2,\varpi_3)^T$ is the vector of holomorphic differential on the Riemann Surface $\mathscr{P}_3$ and normalized as

$$\oint_{\alpha_j}\varpi_k=\delta_{jk} \text{ with } j,k=1,2,3, \tag{3.2.28}$$

and the integral of the holomorphic differentials ϖ_k over the beta cycles β_j defines the elements of the period matrix B

$$B_{jk} = \oint_{\beta_j} \varpi_k. \tag{3.2.29}$$

Fixing the base point $z_0 = \infty$, the Abel map satisfy the jump conditions:

$$\begin{aligned}
&u_+ + u_- = 0 \;\; z \in \upsilon_{[d_2(\eta),E_2]}, \quad u_+ + u_- = \vec{e}_1 \;\; z \in \upsilon_{[\overline{E_2},\overline{d_2(\eta)}]} \\
&u_+ + u_- = B\vec{e}_1 \;\; z \in \upsilon_{[E_1,d_1(\eta)]}, \\
&u_+ + u_- = \vec{e}_1 - \vec{e}_2 + \vec{e}_3 + B\vec{e}_1 \;\; z \in \upsilon_{[\overline{d_1(\eta)},\overline{E_1}]} \\
&u_+ - u_- = \vec{e}_3 - \vec{e}_1 + B(\vec{e}_1 + \vec{e}_2) \;\; z \in \upsilon_{[d_1(\eta),d_2(\eta)]} \\
&u_+ - u_- = \vec{e}_3 - \vec{e}_2 + B(\vec{e}_1 + \vec{e}_2) \;\; z \in \upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]} \\
&u_+ - u_- = \vec{e}_3 + B(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \;\; z \in \upsilon_{[\overline{d_1(\eta)},d_1(\eta)]}.
\end{aligned} \tag{3.2.30}$$

where $\vec{e}_1 = (1,0,0)^T, \vec{e}_2 = (0,1,0)^T, \vec{e}_3 = (0,0,1)^T$.

Step 3 we assume that the matrix $X(z,\eta,t)$ which solve the RHP (3.2.20) has the form

$$X(z,\eta,t) = \begin{bmatrix} \frac{C}{2}\left(\phi_3(z,\eta) + \frac{1}{\phi_3(z,\eta)}\right)\varphi_1(z,\eta,t) & -i\frac{C}{2}\left(\phi_3(z,\eta) - \frac{1}{\phi_3(z,\eta)}\right)\psi_1(z,\eta,t) \\ i\frac{C}{2}\left(\phi_3(z,\eta) - \frac{1}{\phi_3(z,\eta)}\right)\varphi_2(z,\eta,t) & \frac{C}{2}\left(\phi_3(z,\eta) + \frac{1}{\phi_3(z,\eta)}\right)\psi_2(z,\eta,t) \end{bmatrix}. \tag{3.2.31}$$

So we are looking for the functions $\varphi_1(z,\eta,t), \varphi_2(z,\eta,t), \psi_1(z,\eta,t), \psi_2(z,\eta,t)$ such that they solve the RHP:

$$\begin{aligned}
&(\varphi_j(z))_+ = (\psi_j(z))_- \;\; z \in \upsilon_{[d_2(\eta),E_2]} \cup \upsilon_{[\overline{E_2},\overline{d_2(\eta)}]}; \\
&(\varphi_j(z))_+ = \mathrm{e}^{i(t\Omega_1+\Delta_1)}(\psi_j(z))_- \;\; z \in \upsilon_{[E_1,d_1(\eta)]} \cup \upsilon_{[\overline{d_1(\eta)},\overline{E_1}]}; \\
&(\varphi_j(z))_+ = \mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]}(\varphi_j(z))_-, \\
&(\psi_j(z))_+ = \mathrm{e}^{-i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]}(\psi_j(z))_- \;\; z \in \upsilon_{[d_1(\eta),d_2(\eta)]} \cup \upsilon_{[\overline{d_2(\eta)},\overline{d_1(\eta)}]}; \\
&(\varphi_j(z))_+ = \mathrm{e}^{i[t(\Omega_1+\Omega_2+\Omega_3)+\Delta_1+\Delta_2+\Delta_3]}(\varphi_j(z))_-, \\
&(\psi_j(z))_+ = \mathrm{e}^{-i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]}(\psi_j(z))_- \;\; z \in \upsilon_{[\overline{d_1(\eta)},d_1(\eta)]};
\end{aligned} \tag{3.2.32}$$

with $j = 1, 2$. As we already seen in (2.2.42), this kind of RHP can be solved by using the Jacobi theta function of genus 3.

In general, the Jacobi theta function of genus $g > 1$ is defined as

$$\Theta(z, B) := \sum_{\vec{m}\in\mathbb{Z}^g} \exp[\pi i(B\vec{m},\vec{m}) + 2\pi i(z,\vec{m})], \tag{3.2.33}$$

with B the $g \times g$ period matrix and $z \in \mathbb{C}^g$. This function possesses the following properties:

Proposition 3.2.1. *For $M, N \in \mathbb{Z}^g$, then the Jacobi theta function satisfy the following conditions*

1. $\Theta(z + BM) = \Theta(z)\exp[-\pi i(BM, M) - 2\pi i(z, M)]$,
2. $\Theta(z + N) = \Theta(z)$,
3. $\Theta(-z) = \Theta(z)$.

With an abuse of notation, we will refer with $\Theta(z)$ the function (3.2.33) in the case $g = 3$ and $g = 2$.

We prove the following Lemma:

Lemma 3.2.2. *The functions*

$$\varphi_1(z) = \frac{\Theta(\vec{u}(z_1;\infty_1) - \frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_1;\infty_1))}, \quad \psi_1(z) = \frac{\Theta(\vec{u}(z_2;\infty_2) - \frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_2;\infty_1))}; \tag{3.2.34}$$

$$\varphi_2(z) = \frac{\Theta(-\vec{u}(z_1;\infty_2) - \frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_1;\infty_2))}, \quad \psi_2(z) = \frac{\Theta(-\vec{u}(z_2;\infty_2) - \frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_2;\infty_2))}; \tag{3.2.35}$$

where $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$, $\vec{\Delta} = (\Delta_1, \Delta_2, \Delta_3)^T$, z_1 and z_2 indicates the point z in the first or second sheet of the Riemann surface $\mathscr{P}_3$ respectively, satisfy the RHP (3.2.32).

Proof The lemma is easily proved by using the jump conditions on the Abel map (3.2.30) and the properties (1), (2), and (3) of Proposition 3.2.1. ■

The normalization constants C in (3.2.31) is derived by imposing the boundary condition $X(z) \sim \mathbf{1}$ as $z \to \infty$

$$C(\eta, t) = \frac{\Theta(0)}{\Theta(\frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}. \tag{3.2.36}$$

3.2.2 Error Parametrix for the Genus 3 case

The solution of the model problem $X(z, \eta, t)$ (3.2.20) is a good approximation of $\Gamma^{(4)}(z)$, which solve the RHP (3.2.19), in all the complex plane except the end points of the bands, $\{E_1, E_2, d_1(\eta), d_2(\eta)\}$ and their complex conjugates. However, we need to estimates the error function $\mathcal{E}(z)$ around those points, by constructing local parametrix around the end points of the bands.

For the points E_1, E_2 and their complex conjugate, since the function $r(z) \sim (z - E_j)^{1/2}$ near E_j with $j = 1, 2$, then the error function $\mathcal{E}_{E_j}(z)$ is exponentially near the identity as $t \to +\infty$. which means that the solution of the model problem $X(z, \eta, t)$, defined in (3.2.31), is an exponentially accurate approximation of $\Gamma^{(4)}(z)$. We need to study the local parametrix only for the points $d_1(\eta)$, $d_2(\eta)$ and their complex conjugate.

We define $\mathbb{D}_\varepsilon(z_0)$ the disk centered in z_0 and ray $0 < \varepsilon \ll 1$ and with Ξ the set of jump contours of the RHP (3.2.19).

Local parametrix around $d_1(\eta)$ and $d_2(\eta)$

We are looking for the matrices $\Gamma^{d_1}(z,\eta,t)$ and $\Gamma^{d_2}(z,\eta,t)$ that approximate $\Gamma^{(4)}(z,\eta,t)$ in $\mathbb{D}_\varepsilon(d_1)$ and $\mathbb{D}_\varepsilon(d_2)$ respectively. We consider the point $d_2(\eta)$ first.

We define $\Gamma^{(d_2,0)}(z)$ from the transformation

$$\Gamma^{(d_2,0)}(z,\eta,t) = \Gamma^{(4)}(z,\eta,t)F(z)^{-\sigma_3}, \quad z \in \mathbb{D}(d_2) \setminus \Xi. \tag{3.2.37}$$

Let $\{\mathcal{T}_j\}_1^5$ denote the open subset of $\mathbb{D}_\varepsilon(d_1)$, as it's show in the Figure 3.10. Let $\mathcal{Z}_j := \overline{\mathcal{T}_j} \cap \overline{\mathcal{T}_{j-1}}$, $j = 1,\dots,5$, $\overline{\mathcal{T}_0} \equiv \overline{\mathcal{T}_5}$, denote the curves separating the $\mathcal{T}_j$, oriented as in Figure 3.10.

From (3.2.19) and (3.2.37), the matrix $\Gamma^{(d_2,0)}(z,\eta,t)$ satisfy the jump conditions:

$$\Gamma_+^{(d_2,0)}(z;x,t) = \Gamma_-^{(d_2,0)}(z;x,t)V^{(d_2,0)}(z,x,t), \tag{3.2.38}$$

where

$$V^{(d_2,0)} = \begin{cases} \begin{pmatrix} 0 & r^{-1}(z) \\ -r(z) & 0 \end{pmatrix}, & z \in \mathcal{Z}_1 \\ \begin{pmatrix} 1 & \frac{\mathrm{e}^{-2itg(z,\eta)}}{r(z)} \\ 0 & 1 \end{pmatrix}, & z \in \mathcal{Z}_2 \cup \mathcal{Z}_5 \\ \begin{pmatrix} 1 & 0 \\ r(z)\mathrm{e}^{2itg(z,\eta)} & 1 \end{pmatrix}, & z \in \mathcal{Z}_3 \\ \mathrm{e}^{it(g_+(z,\xi)-g_-(z,\xi))\sigma_3}, & z \in \mathcal{Z}_4 \end{cases} \tag{3.2.39}$$

We define the function $g_{d_2}(z,\eta)$ as

$$g_{d_2}(z,\eta) := -\int_{d_2(\eta)}^{z} dg = \begin{cases} g(z,\eta) - g_-(d_2,\eta) & z \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3, \\ g(z,\eta) - g_+(d_2,\eta) & z \in \mathcal{T}_4 \cup \mathcal{T}_5, \end{cases}$$

and we apply another transformation

$$\Gamma^{(d_2,1)}(z,\eta,t) := \Gamma^{(d_2,0)}(z,\eta,t)A(z,\eta,t), \tag{3.2.40}$$

where $A(z,\eta,t)$ is defined as

$$A(z;x,t) := r(z)^{-\frac{\sigma_3}{2}} \begin{cases} e^{-itg_-(d_2)\sigma_3} & z \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3, \\ e^{-itg_+(d_2)\sigma_3} & z \in \mathcal{T}_4 \cup \mathcal{T}_5. \end{cases}$$

The new matrix function (3.2.40) satisfy a new RHP

$$\begin{aligned} &\Gamma_+^{(d_2,1)}(z,\eta,t) = \Gamma_-^{(d_2,1)}(z,\eta,t)V^{(d_2,1)}(z,\eta,t) \\ &\overline{\Gamma^{(d_2,1)}(\overline{z})} = \sigma_2\Gamma^{(d_2,1)}(z)\sigma_2 \\ &\Gamma^{(d_2,1)}(z) \sim \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } z \to \infty \end{aligned} \tag{3.2.41}$$

where $V^{(d_2,1)}(z)$ are the jump matrices (3.2.39) under the transform (3.2.40):

$$V^{(d_2,1)}(z,\eta,t) := \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathcal{Z}_1; \\ \begin{pmatrix} 1 & e^{-2itg_{d_2}(z,\xi)} \\ 0 & 1 \end{pmatrix} & z \in \mathcal{Z}_2 \cup \mathcal{Z}_5; \\ \begin{pmatrix} 1 & 0 \\ e^{2itg_{d_2}(z,\xi)} & 1 \end{pmatrix} & z \in \mathcal{Z}_3; \\ \mathbf{1} & z \in \mathcal{Z}_4. \end{cases} \tag{3.2.42}$$

We introduce a local change of coordinate

$$\zeta(z) := e^{-\frac{2}{3}\pi i}\left(\frac{3}{2}itg_{d_2}(z,\xi)\right)^{\frac{2}{3}} \tag{3.2.43}$$

such that the map $z \to \zeta$ maps $\mathcal{Z}_3$ into $\mathbb{R}_-$, and deforming the contour $\mathcal{Z}_j$ such that (3.2.43) maps $\mathcal{Z}_j$ into the rays Y_j in Figure 3.11.

The RHP (3.2.41) is a well known problem in complex analysis (see Appendix B of [74]) and it is solved by the matrix $\Gamma^{\text{Ai}}(z,\eta,t)$

$$\Gamma^{\text{Ai}}(\zeta(z)) := M^{(\text{Ai})}(\zeta(z)) \times \begin{cases} e^{-2itg_{d_2}(z)\sigma_3}, & z \in \mathcal{T}_1 \cup \mathcal{T}_4 \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} e^{-2itg_{d_2}(z)\sigma_3}, & z \in \mathcal{T}_2 \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} e^{-2itg_{d_2}(z)\sigma_3}, & z \in \mathcal{T}_3 \end{cases} \tag{3.2.44}$$

where

$$M^{(\text{Ai})}(z) := \begin{cases} \begin{pmatrix} \text{Ai}(z) & \omega^2\text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & -\omega\text{Ai}'(\omega^2 z) \end{pmatrix}, & z \in \mathbb{C}_+ \\ \begin{pmatrix} \text{Ai}(z) & -\omega\text{Ai}(\omega z) \\ \text{Ai}'(z) & -\omega^2\text{Ai}'(\omega z) \end{pmatrix}, & z \in \mathbb{C}_- \end{cases} \tag{3.2.45}$$

and $\text{Ai}(z)$ is the Airy function.

We define the local parametrix at the point $d_2(\eta)$ as

$$\Gamma_{d_2}(z,\eta,t) := B_{\text{Ai}}(z,\eta,t)\Gamma^{\text{Ai}}(z,\eta,t)A(z,\eta,t)^{-1}F(z,\eta)^{-\sigma_3}, \tag{3.2.46}$$

where

$$B_{\mathrm{Ai}}(z,\eta,t) := X(z,\eta,t)F(z)^{\sigma_3}A(z,\eta,t)\Gamma^{\mathrm{Ai}}_{inv,0}(z,\eta,t) \tag{3.2.47}$$

and

$$\Gamma^{\mathrm{Ai}}_{inv,0}(z;x,t) = i\sqrt{\pi}\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}(\zeta(z))^{\frac{\sigma_3}{4}}.$$

It is important to notice that $\Gamma^{\mathrm{Ai}}_{inv,0}$ has a jump condition in $z \in \mathcal{Z}_1$

$$\Gamma^{\mathrm{Ai}}_{inv,0}(z;x,t)_+ = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Gamma^{\mathrm{Ai}}_{inv,0}(z;x,t)_- \tag{3.2.48}$$

Lemma 3.2.3. *The matrix $B_{Ai}(z,\eta,t)$ defined in (3.2.47) is an analytic and bounded function in $\mathbb{D}_\varepsilon(d_2(\eta))$.*

Proof Since $X(z,\eta,t)$ doesn't have jumps in $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_5$, we need to check only the case where $z \in \mathcal{Z}_1 \cup \mathcal{Z}_4$.

For $z \in \mathcal{Z}_1$, by using the jump condition (3.2.48) of $\Gamma^{\mathrm{Ai}}_{inv,0}(z)$, we get

$$\begin{aligned}
B_{\mathrm{Ai}}(z;x,t)_+ &= X(z;x,t)_+F(z)_+^{\sigma_3}A(z;x,t)_+\Gamma^{Ai}_{inv,0}(z;x,t)_+\\
&= X(z;x,t)_-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}F(z)_-^{-\sigma_3}A(z;x,t)_-^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Gamma^{Ai}_{inv,0}(z;x,t)_-\\
&= X(z;x,t)_-F(z)_-^{\sigma_3}A(z;x,t)_-\Gamma^{Ai}_{inv,0}(z;x,t)_-,
\end{aligned}$$

while for $z \in \mathcal{Z}_4$

$$\begin{aligned}
B_{\mathrm{Ai}}(z;x,t)_+ &= X(z;x,t)_+F(z)_+^{\sigma_3}A(z;x,t)_+\Gamma^{Ai}_{inv,0}(z;x,t)_+ =\\
&= X(z;x,t)_-e^{-i(t\Omega_1+\Delta_1)}F(z)_-^{\sigma_3}e^{i\Delta_1\sigma_3}e^{it\Omega_1\sigma_3}A(z;x,t)_-\Gamma^{Ai}_{inv,0}(z;x,t)_- =\\
&= X(z;x,t)_-F(z)_-^{\sigma_3}A(z;x,t)_-\Gamma^{Ai}_{inv,0}(z;x,t)_-.
\end{aligned}$$

■

Then we have to prove the following proposition

Proposition 3.2.4. *The function $\Gamma_{d_2}(z)$ defined in (3.2.46) is analytic in $\mathbb{D}_\varepsilon(d_2) \setminus \Xi$, with the same jump conditions as $\Gamma^{(4)}(z)$. Also, for $t \to +\infty$ then*

$$\Gamma_{d_2}(z)(\Gamma^{(4)}(z))^{-1} = \mathbf{1} + \mathcal{O}(z^{-1}) \text{ as } t \to +\infty,\ z \in \partial\mathbb{D}_\varepsilon(d_2). \tag{3.2.49}$$

Proof From the definition (3.2.46) and Lemma 3.2.3, the matrix $\Gamma_{d_2}(z)$ has the same jump conditions of $\Gamma^{(4)}(z)$. We have now to check the property (3.2.49). For $z \in \partial\mathbb{D}_\varepsilon(d_2)$, then $\Gamma^{(4)}(z) \sim X(z)$ and the $\Gamma^{\mathrm{Ai}}(z)$ admits an asymptotic expansion at $\zeta \sim \infty$

$$\Gamma^{\mathrm{Ai}}(z;x,t) \sim \frac{(\zeta(z))^{-\frac{\sigma_3}{4}}}{2\sqrt{\pi}}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}(\mathbf{1} + \mathcal{O}(\zeta^{-\frac{3}{2}})). \tag{3.2.50}$$

Then we have that for $t \to +\infty$

$$\Gamma_{d_2}(\Gamma')^{-1} \sim X(z;x,t)F(z)^{\sigma_3}A(z;x,t)(\mathbf{1}+\mathcal{O}(\zeta^{-\frac{3}{2}}))A(z,x,t)^{-1}F(z)^{-\sigma_3}X(z;x,t)^{-1}.$$

Since both A and X are bounded in t, they doesn't contribute to the error $\mathcal{O}(\zeta^{-\frac{3}{2}}) \sim \mathcal{O}(t^{-1})$, and at the end we get

$$\Gamma_{d_2}(z)(\Gamma^{(4)}(z))^{-1} \sim \mathbf{1}+\mathcal{O}(t^{-1}).$$

■

We repeat the same strategy also for the point $d_1(\eta)$. Let $z \in \mathbb{D}_\varepsilon(d_1) \setminus \Xi$, we define

$$\Gamma^{(d_1,0)}(z) := \Gamma^{(4)}(z,\eta,t)F(z,\eta)^{-\sigma_3}. \tag{3.2.51}$$

We denote with $\{\mathcal{S}_j\}_{j=1}^5$ the open subsets of $\mathbb{D}_\varepsilon(d_1)$, as it is show in Figure 3.12, and with $\mathcal{Y}_j := \overline{\mathcal{S}_{j-1}} \cap \overline{\mathcal{S}_j}$. The matrix $\Gamma^{(d_1,0)}(z)$ has jump conditions

$$\Gamma_+^{(d_1,0)}(z) = \Gamma_-^{(d_1,0)}(z)V^{(d_1,0)}(z,\eta,t), \tag{3.2.52}$$

with $V^{(d_1,0)}(z,\eta,t) = F_-(z,\eta)^{\sigma_3}V^{(4)}(z,\eta,t)F_+(z,\eta)^{-\sigma_3}$ for $z \in \mathbb{D}_\varepsilon(d_1) \cap \Xi$.

We define the function $g_{d_1}(z,\eta)$

$$g_{d_1}(z,\eta) := \begin{cases} g(z,\eta) - g_-(d_1,\eta) & z \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3, \\ g(z,\eta) - g_+(d_1,\eta) & z \in \mathcal{S}_4 \cup \mathcal{S}_5, \end{cases} \tag{3.2.53}$$

and the transformation

$$\Gamma^{(d_1,1)}(z,\eta,t) := \Gamma^{(d_1,0)}(z,\eta,t)A_{d_1}(z,\eta,t), \tag{3.2.54}$$

with $A_{d_1}(z,\eta,t)$ defined as

$$A_{d_1}(z,\eta,t) := (-r(z))^{-\frac{\sigma_3}{2}} \begin{cases} e^{-itg_-(d_1)\sigma_3} & z \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3, \\ e^{-itg_+(d_1)\sigma_3} & z \in \mathcal{T}_4 \cup \mathcal{T}_5. \end{cases}$$

Then the new matrix function (3.2.54) satisfy the RHP

$$\begin{aligned} &\Gamma_+^{(d_1,1)}(z,\eta,t) = \Gamma_-^{(d_1,1)}(z,\eta,t)V^{(d_1,1)}(z,\eta,t) \\ &\overline{\Gamma^{(d_1,1)}(\overline{z})} = \sigma_2\Gamma^{(d_1,1)}(z)\sigma_2 \\ &\Gamma^{(d_1,1)}(z) \sim \mathbf{1}+\mathcal{O}(z^{-1}) \text{ as } z \to \infty \end{aligned} \tag{3.2.55}$$

with $V^{(d_1,1)}(z,\eta,t)$ give by

$$V^{(d_1,1)}(z,\eta,t) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in \mathcal{Y}_4; \\ \begin{pmatrix} 1 & -e^{-2itg_{d_2}(z,\xi)} \\ 0 & 1 \end{pmatrix} & z \in \mathcal{Y}_3 \cup \mathcal{Y}_5; \\ \begin{pmatrix} 1 & 0 \\ -e^{2itg_{d_2}(z,\xi)} & 1 \end{pmatrix} & z \in \mathcal{Y}_2; \\ \mathbf{1} & z \in \mathcal{Y}_1. \end{cases} \tag{3.2.56}$$

We apply the local change of variable (3.2.43) also in this case

$$\zeta_{d_1}(z) := \mathrm{e}^{-\frac{2}{3}\pi i}\left(\frac{3}{2}itg_{d_1}(z,\eta)\right)^{\frac{2}{3}}, \tag{3.2.57}$$

such that it maps $\mathcal{Y}_j$ in straight lines as in Figure 3.13. Adjusting the orientation of the jump contours, we find out the RHP (3.2.55) is still mapped in the RHP for Airy function. This means that the parametrix $\Gamma_{d_1}(z)$ is defined as

$$\Gamma_{d_1}(z,\eta,t) := B^{d_1}_{\mathrm{Ai}}(z,\eta,t)\Gamma^{\mathrm{Ai}}(z,\eta,t)A_{d_1}(z,\eta,t)^{-1}F(z,\eta)^{-\sigma_3}, \tag{3.2.58}$$

with

$$B_{\mathrm{Ai}}(z,\eta,t) := X(z,\eta,t)F(z)^{\sigma_3}A_{d_1}(z,\eta,t)\Gamma^{\mathrm{Ai}}_{inv,0}(z,\eta,t), \tag{3.2.59}$$

and that Lemma 3.2.3 and Proposition 3.2.4 are still valid also in this case.

By symmetry, we also have that the local parametrix at the points $\overline{d_1(\eta)}, \overline{d_2(\eta)}$ are still of the Airy type, so the parametrix contributes in the same way to the local error function $\mathcal{E}_{\overline{d_j}}(z)$.

3.2.3 Long-time asymptotic for the Genus 3 case

Let $\Gamma^{app}(z)$ be the matrix such that it is equal to $\Gamma_{d_j}(z)$ inside the disks $\mathbb{D}_\varepsilon(d_j)$, with $j=1,2$, and can be extended by symmetry also in the other domains $\mathbb{D}_\varepsilon(\overline{d_j})$. We define the error function

$$\mathcal{E}(z,\eta,t) = \Gamma^{(4)}(z,\eta,t)(\Gamma^{app}(z,\eta,t))^{-1}. \tag{3.2.60}$$

From proposition 3.2.4 we can conclude that, as $t\to+\infty$, the

$$\mathcal{E}(z,\eta,t) = \mathbf{1} + \mathcal{O}(t^{-1})$$

uniformly in z. Tracking back the chain of transformation from the reminder problem to the original RHP (2.1.4) for $\Gamma(z)$, we find that

$$\begin{aligned}\psi(x,t) &= 2i\lim_{z\to\infty} z(\Gamma^{(4)}(z,\eta,t))_{12}\\ &= 2i(\lim_{z\to\infty} z(X(z,\eta,t))_{12} + \lim_{z\to\infty} z(\mathcal{E}(z,\eta,t))_{12}).\end{aligned} \tag{3.2.61}$$

The second term in (3.2.61) is of order $\mathcal{O}(t^{-1})$ while the first term is given from the leading order of expansion at $z\to\infty$ of

$$(X(z,\eta,t))_{12} = -i\frac{\Theta(0)}{2\Theta(\frac{\vec{\Omega}t+\vec{\Delta}}{2\pi})}\left(\phi_3(z,\eta) - \frac{1}{\phi_3(z,\eta)}\right)\psi_1(z,\eta,t) \tag{3.2.62}$$

Indeed, by expanding in Taylor' series the functions $\phi_3(z,\eta)$ and $(\phi_3(z))^{-1}$ for $z\sim\infty$ we get

$$\left(\phi_3(z,\eta) - \frac{1}{\phi_3(z,\eta)}\right) = -\frac{i}{2z}[\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1) + \mathrm{Im}\,(d_2(\eta)) - \mathrm{Im}\,(d_1(\eta))] + \mathcal{O}(z^{-2}).$$

So the equation (3.2.61) becomes

$$\psi(x,t) = -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1) + \mathrm{Im}\,(d_2(\eta)) - \mathrm{Im}\,(d_1(\eta)))\times$$
$$\times\, \frac{\Theta(0)\Theta(\vec{u}_\infty - \frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})}{\Theta(\frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})\Theta(\vec{u}_\infty)} + \mathcal{O}(t^{-1}); \tag{3.2.63}$$

where $\vec{u}_\infty = \vec{u}(\infty_2, \infty_1)$.

This ends our proof of Theorem 3.0.1.

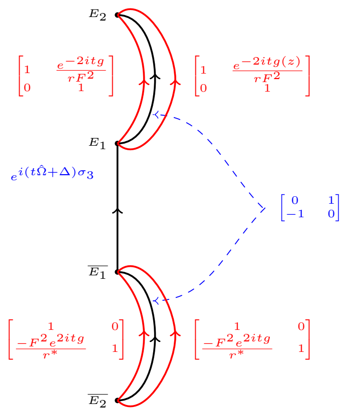


Figure 3.4: Jump contours and jump matrices $V^{(4)}(z, \eta, t)$

Figure 3.5: On the left: the homological basis of the Riemann Surface $w^2 = P_1(z)$. On the right: jump contours of the model problem (3.1.28).

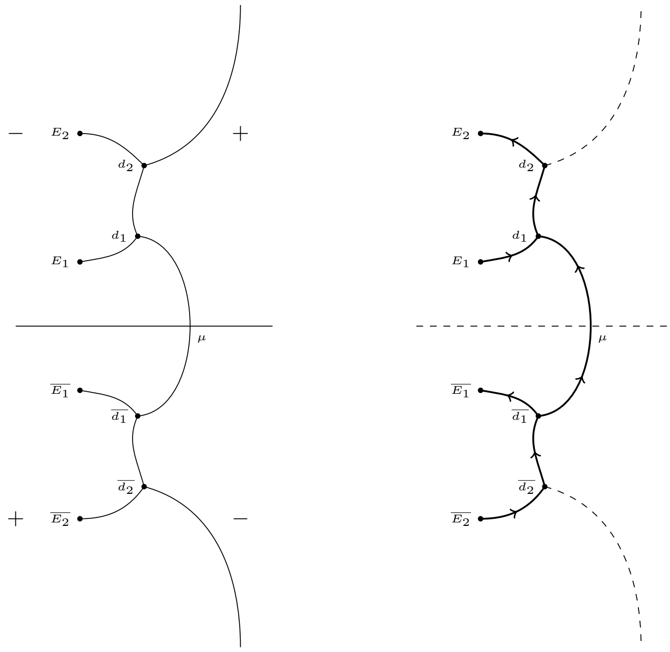


Figure 3.6: On the left: the distribution of signs of $\mathrm{Im}\,(g)$ in the genus 3 case. On the right: jump contours of the matrix $\Gamma^{(1)}(z)$ in the genus 3 case.

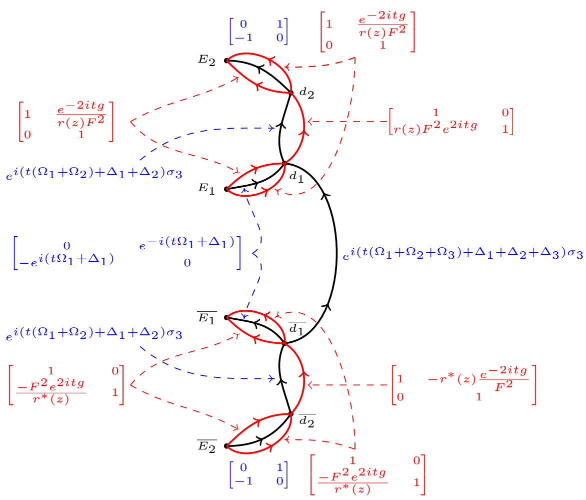


Figure 3.7: Jump contours and jump matrices of $V^{(4)}(z, \eta, t)$ for the genus 3 case

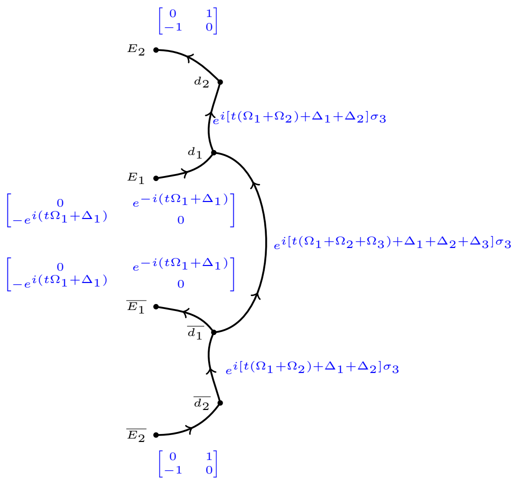


Figure 3.8: Jump contours and jump matrices of the model problem

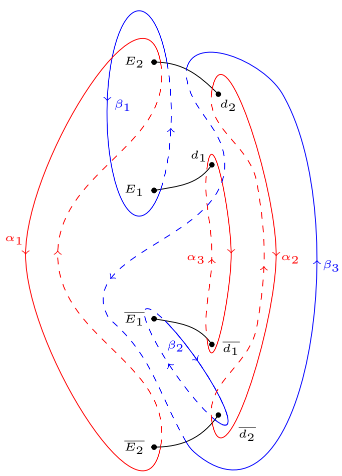


Figure 3.9: Homological basis of the Riemann surface $\mathscr{P}_3$

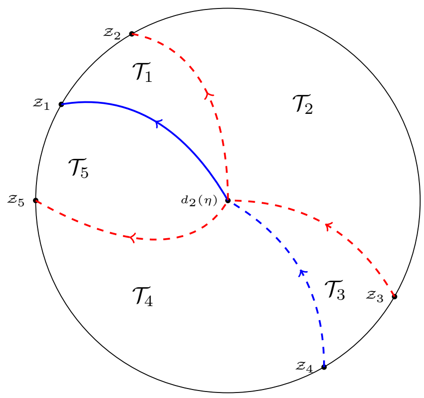


Figure 3.10: the disk $\mathbb{D}_\varepsilon(d_2)$

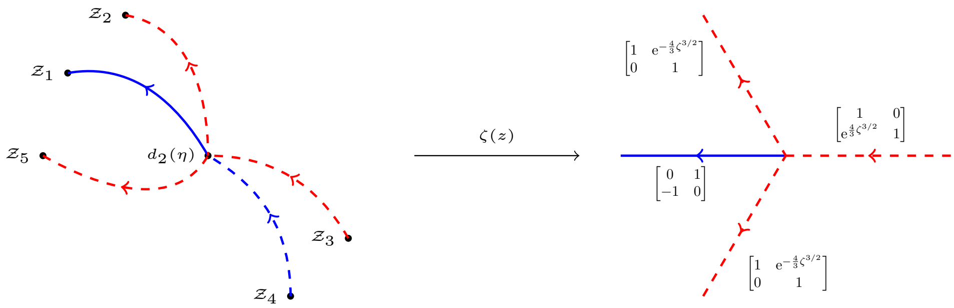


Figure 3.11: The RHP (3.2.41) in $\mathbb{D}_\varepsilon(d_2)$ mapped in the Airy RHP by $\zeta(z)$.

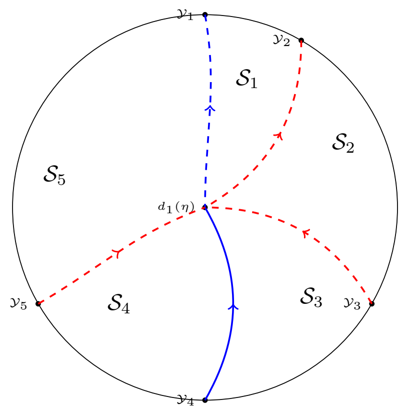


Figure 3.12: The disk $\mathbb{D}_\varepsilon(d_1)$

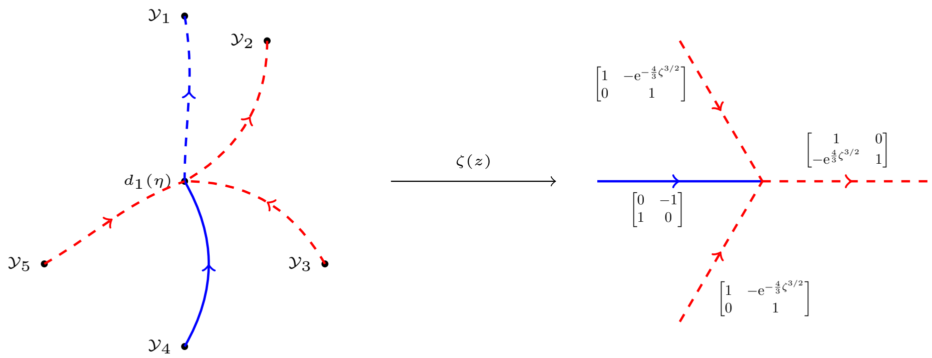


Figure 3.13: The RHP (3.2.55) in $\mathbb{D}_\varepsilon(d_1)$ mapped in the Airy RHP by $\zeta(z)$.

Chapter 4

Elliptic domain with $m < m_c$: the $(\text{Genus } 1)_s$ and Genus 2 sectors

In this chapter we examine the other scenario, where m is smaller thenm_c. As we shown in section 2.3, the level set $\operatorname{Im}(g) = 0$ has three phase transitions :

- For $\eta_+ < \eta \leq \eta_-$ the g-function is still of genus one, but with 3 real zeros (the "(Genus 1)$_s$" sector);
- For $\eta_- < \eta \leq \eta_*$ the g-function increases his genus from one to two (the "Genus 2" sector);
- For $\eta_* < \eta < 0$ the g-function increases his genus from two to three.

We will prove the following theorem:

Theorem 4.0.1. *Let $\mathscr{D} \subset \mathbb{C}^+$ be an elliptic domain with foci E_1, E_2. Suppose $r : [E_1, E_2] \to \mathbb{C}$ is an analytic and bounded function that vanishes at the foci as $r(E_j) \sim (z - E_j)^{1/2}$ for $j = 1, 2$. Suppose also that $E_1, E_2 \in i\mathbb{R}_+$, with $\operatorname{Im}(E_1) < \operatorname{Im}(E_2)$ and the parameter $m = \frac{\operatorname{Im}(E_1)^2}{\operatorname{Im}(E_2)^2} < m_c$, with m_c solution of the equation $\mathcal{Q}(m) = 0$, where $\mathcal{Q}(m)$ is define in (2.3.12). Then the leading order of the large-time asymptotic of the soliton gas solution of NLS, recovered from the solution of the RHP (2.1.4), with spectrum in the domain $\mathscr{D}$ has the following form, according to the values of $\eta := \frac{x}{t}$:*

- *for $-\infty < \eta < \eta_+$, with η_+ defined by the formula (2.3.19)*

$$\begin{aligned}\psi(x,t) &= i(\operatorname{Im}(E_1) + \operatorname{Im}(E_2))\times \\ &\times \operatorname{dn}((\operatorname{Im}(E_1) + \operatorname{Im}(E_2))(x - x_0(\eta)) + K(m_1), m_1) + \mathcal{O}(\mathrm{e}^{-ct});\end{aligned} \tag{4.0.1}$$

where $c > 0$, $\operatorname{dn}(x)$ is the Jacobi delta amplitude, x_0 is determine by $r(z)$ and

$$K(m_1) := \sqrt{1-m}K(m) \quad m_1 := \frac{4\sqrt{m}}{(1+\sqrt{m})^2}; \tag{4.0.2}$$

- *for $\eta_+ < \eta \le \eta_-$, with η_+ defined by the formula (2.3.19), the leading order of $\psi(x,t)$ is still a genus 1 wave, given by the equation (2.2.53);*

- *for $\eta_- < \eta \le \eta_*$, with η_* depending on $\operatorname{Im}(E_1)$ and $\operatorname{Im}(E_2)$, the leading order of $\psi(x,t)$ is*

$$\psi(x,t) \sim -i(\operatorname{Im}(E_2) - \operatorname{Im}(E_1) - \operatorname{Im}(d)) \frac{\Theta(0)\Theta(\vec{u}_\infty - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})\Theta(\vec{u}_\infty)}. \tag{4.0.3}$$

where $\vec{\Omega}(\eta), \vec{\Delta}(\eta)$ are 2-dimensional real vectors with components defined in (4.2.2), (4.2.3) and (4.2.15); $\vec{u}_\infty$ is the Abel map (4.2.23) evaluated at $z = \infty$ and $d \in \mathbb{C}^+$;

- *for $\eta_* < \eta < 0$*

$$\begin{aligned}\psi(x,t) = i(\operatorname{Im}(E_2) - \operatorname{Im}(E_1) + \operatorname{Im}(d_2(\eta)) - \operatorname{Im}(d_1(\eta)))\times \\ \times \frac{\Theta(0)\Theta(\vec{u}_\infty - \frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})}{\Theta(\frac{\vec{\Omega}(\eta)t+\vec{\Delta}(\eta)}{2\pi})\Theta(\vec{u}_\infty)} + \mathcal{O}(t^{-1});\end{aligned} \tag{4.0.4}$$

where $\vec{\Omega}(\eta), \vec{\Delta}(\eta)$ are 3-dimensional real vectors with components defined in (3.2.2) and (3.2.10), $\vec{u}_\infty$ is the Abel map (3.2.27) valued at $z = \infty$ and d_1 and d_2 are points in $\mathbb{C}^+$.

- *for $\eta > 0$ the solution decays exponentially*

$$\psi(x,t) \sim \mathcal{O}(e^{-ct})$$

with $c > 0$.

Since the strategy and the results for the sectors $-\infty < \eta < \eta_+$ and $\eta_* < \eta < 0$ resemble those described in chapter 3, we analyze only the long-time asymptotic of $\psi(x,t)$ in the (Genus $1)_s$ and in the Genus 2 sectors.

4.1 The (Genus $1)_s$ sector

As we shown in section 2.3, for $\eta_+ < \eta \le \eta_-$ the g-function has three real critical points and it's given by the formula (2.3.20). In Figure 4.1 we show the level set $\operatorname{Im}(g) = 0$. We denote with Σ_1 and Σ_2 the regions encircled respectively by loops $v_{[E_1,\mu_1]} \cup [\mu_1,\mu_2] \cup v_{[\mu_2,E_2]} \cup [E_2,E_1]$ and its complex conjugate.

Now we study the long-time asymptotic of the RHP (2.1.4) in this particular case.

The first step is to move the jumps from the segments $[E_1,E_2]$ and $[\overline{E_2},\overline{E_1}]$ to the border of Σ_1 and Σ_2 respectively.

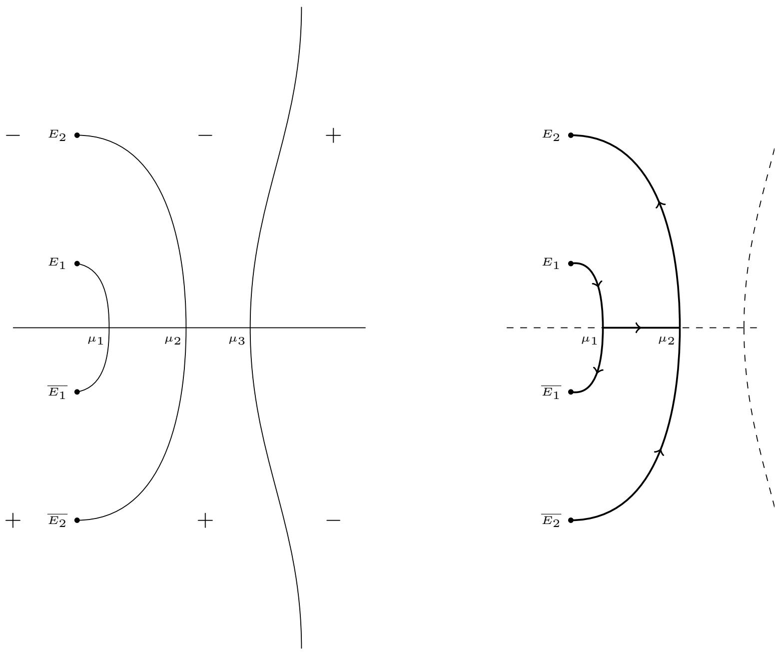


Figure 4.1: On the left: the distribution of signs for $\mathrm{Im}\,(g)$ for the (Genus 1)$_s$ case. On the right: jump contours of $\Gamma^{(1)}(z)$.

We apply the transformation $\Gamma^{(1)}(z) = \Gamma(z)G^{(1)}(z)$, where

$$G^{(1)}(z,x,t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ -r(z)e^{2it\hat{\theta}(z,\eta)} & 1 \end{bmatrix} \text{ for } z \in \Sigma_1 \\ \begin{bmatrix} 1 & r^*(z)e^{-2it\hat{\theta}(z,\eta)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \Sigma_2 \\ \mathbf{1} \text{ otherwise} \end{cases}. \tag{4.1.1}$$

The new matrix $\Gamma^{(1)}(z)$ not only has jumps both in the curves $\upsilon_{[E_1,\overline{E_1}]}$ and $\upsilon_{[\overline{E_2},E_2]}$, but also in the segment $[\mu_1,\mu_2]$:

$$\Gamma^{(1)}(z)_+ = \Gamma^{(1)}(z)_- V^{(1)}(z,t,\eta) \tag{4.1.2}$$

where

$$V^{(1)}(z,x,t) = \begin{cases} \begin{bmatrix} 1 & r^*(z)e^{-2it\hat{\theta}(z,\eta)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \upsilon_{[\mu_1,\overline{E_1}]} \cup \upsilon_{[\overline{E_2},\mu_2]} \\ \begin{bmatrix} 1 & 0 \\ -r(z)e^{2it\hat{\theta}(z,\eta)} & 1 \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,\mu_1]} \cup \upsilon_{[\mu_2,E_2]} \\ \begin{bmatrix} 1 & r^*(z)e^{-2it\hat{\theta}(z,\eta)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ r(z)e^{2it\hat{\theta}(z,\eta)} & 1 \end{bmatrix} \text{ for } z \in [\mu_1,\mu_2] \end{cases} \tag{4.1.3}$$

The next step involves the inclusion of the phase $\hat{\theta}(z,\eta)$ in the matrix $\Gamma^{(1)}(z)$. We apply the transformation

$$\Gamma^{(2)}(z) = \Gamma^{(2)}(z) \mathrm{e}^{it(g(z,\eta)-\hat{\theta}(z,\eta))\sigma_3}. \tag{4.1.4}$$

where $g(z,\eta)$ is given by (2.3.20). Then the RHP (4.1.2) becomes

$$\Gamma^{(2)}(z)_+ = \Gamma^{(2)}(z)_- V^{(2)}(z,t,\eta) \tag{4.1.5}$$

where

$$V^{(2)}(z,x,t) = \begin{cases} \begin{bmatrix} e^{it(g_+-g_-)} & r^*(z)e^{-it(g_++g_-)} \\ 0 & e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[\mu_1,\overline{E_1}]} \cup \upsilon_{[\overline{E_2},\mu_2]} \\ \begin{bmatrix} e^{it(g_+-g_-)} & 0 \\ -r(z)e^{it(g_++g_-)} & e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[E_1,\mu_1]} \cup \upsilon_{[\mu_2,E_2]} \\ \begin{bmatrix} (1+|r(z)|^2)e^{it(g_+-g_-)} & r^*(z)e^{-it(g_++g_-)} \\ r(z)e^{it(g_++g_-)} & e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in [\mu_1,\mu_2] \end{cases} \tag{4.1.6}$$

with the g-function (2.3.20) that satisfy the jump conditions:

$$g_+(z) + g_-(z) = 0 \qquad \text{for } z \in \upsilon_{[\overline{E_2},E_2]} \tag{4.1.7}$$

$$g_+(z) + g_-(z) = 2g(E_1) = \Omega \qquad \text{for } z \in \upsilon_{[E_1,\overline{E_1}]} \tag{4.1.8}$$

$$g_+(z) - g_-(z) = 0 \qquad \text{for } z \in [\mu_1,\mu_2] \tag{4.1.9}$$

with $\Omega \in \mathbb{R}$.

We introduce another transformation to handle the function $r(z)$

$$\Gamma^{(3)}(z) = F_\infty^{\sigma_3}(\eta)\Gamma^{(2)}(z)F(z)^{\sigma_3}, \tag{4.1.10}$$

where $F_\infty(\eta) := \lim_{z\to\infty} F(z)$ and $F(z)$ satisfy the conditions:

$$F^*(z) = (F(z))^{-1}, \qquad F(z) \sim F_\infty(\eta) + \mathcal{O}(z^{-1}) \text{ for } z \sim \infty. \tag{4.1.11}$$

Then the jump matrices (4.1.6) transform as $V^{(3)}(z,t,\eta) = F_-(z)^{-\sigma_3}V^{(2)}(z,t,\eta)F_+(z)^{\sigma_3}$

$$V^{(3)}(z,t,\eta) = \begin{cases} \begin{bmatrix} \frac{F_+}{F_-}e^{it(g_+-g_-)} & \frac{r^*(z)}{F_+F_-}\chi_{\upsilon_{[\overline{E_2},\mu_2]}} \\ -r(z)F_+F_-\chi_{\upsilon_{[\mu_2,E_2]}} & \frac{F_-}{F_+}e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[\overline{E_2},E_2]} \\ \begin{bmatrix} \frac{F_+}{F_-}e^{it(g_+-g_-)} & \frac{r^*(z)}{F_+F_-}e^{-it\Omega}\chi_{\upsilon_{[\mu_1,\overline{E_1}]}} \\ -r(z)F_+F_-e^{it\Omega}\chi_{\upsilon_{[E_1,\mu_1]}} & \frac{F_-}{F_+}e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[E_1,\overline{E_1}]} \\ \begin{bmatrix} \frac{F_+}{F_-}(1+|r(z)|^2) & \frac{r^*(z)}{F_+F_-}e^{-2itg} \\ r(z)F_+F_-e^{2itg} & \frac{F_-}{F_+} \end{bmatrix} & \text{for } z \in [\mu_1,\mu_2] \end{cases} \tag{4.1.12}$$

We are looking for a function $F(z,\eta)$ which solves this scalar RHP

$$F_+F_- = r(z)^{-1} \text{ for } z \in \upsilon_{[\mu_2,E_2]}, \quad F_+F_- = r^*(z) \text{ for } z \in \upsilon_{[\overline{E_2},\mu_2]}, \tag{4.1.13}$$

$$F_+F_- = r(z)^{-1}\mathrm{e}^{i\Delta} \text{ for } z \in \upsilon_{[E_1,\mu_1]},\ F_+F_- = r^*(z)\mathrm{e}^{i\Delta} \text{ for } z \in \upsilon_{[\mu_1,\overline{E_1}]} \tag{4.1.14}$$

$$F_+ = F_-(1+|r(z)|^2)^{-1} \text{ for } z \in [\mu_1,\mu_2] \tag{4.1.15}$$

with boundary conditions given by (4.1.11). This problem is solved by $F(z,\eta) = \mathrm{e}^{\Phi(z,\eta)}$, where $\Phi(z,\eta)$ has the form

$$\begin{aligned}\Phi(z,\eta) = \frac{P_1(z)}{2\pi i}\Bigg[&-\left(\int_{\upsilon_{[\mu_2,E_2]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+(\zeta-z)}\mathrm{d}\zeta + \left(\int_{\upsilon_{[\mu_2,E_2]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+(\zeta-z)}\mathrm{d}\zeta\right)^*\right)\\
&-\left(\int_{\upsilon_{[E_1,\mu_1]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+(\zeta-z)}\mathrm{d}\zeta + \left(\int_{\upsilon_{[E_1,\mu_1]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+(\zeta-z)}\mathrm{d}\zeta\right)^*\right)\\
&+i\Delta\left(\int_{\upsilon_{[E_1,\overline{E_1}]}} \frac{\mathrm{d}\zeta}{P_1(\zeta)_+(\zeta-z)}\right) - \int_{\mu_1}^{\mu_2} \frac{\log(1+|r(\zeta)|^2)}{P_1(\zeta)(\zeta-z)}\mathrm{d}\zeta\Bigg]\end{aligned} \tag{4.1.16}$$

where $P_1(z) := \sqrt{(z-E_1)(z-E_2)(z-\overline{E}_1)(z-\overline{E}_2)}$ is a multivalued complex function, analytic in $\mathbb{C}\setminus\upsilon_{[E_1,\overline{E_1}]}\cup\upsilon_{[E_2,\overline{E_2}]}$. The constant Δ is fixed by the behaviour of $F(z)$ for $z\to\infty$ and it is given by the equation

$$\Delta = \frac{\mathrm{Re}\left(\int_{\upsilon_{[\mu_2,E_2]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+}\mathrm{d}\zeta\right) + \mathrm{Re}\left(\int_{\upsilon_{[E_1,\mu_1]}} \frac{\log(r(\zeta))}{P_1(\zeta)_+}\mathrm{d}\zeta\right) - \int_{\mu_1}^{\mu_2} \frac{\log(1+|r(\zeta)|^2)}{P_1(\zeta)}\mathrm{d}\zeta}{i\int_{\upsilon_{[E_1,\overline{E_1}]}} \frac{\mathrm{d}\zeta}{P_1(\zeta)_+}\mathrm{d}\zeta} \in \mathbb{R}. \tag{4.1.17}$$

Then the jump matrices $V^{(3)}(z,\eta,t)$ become:

$$V^{(3)}(z,\eta,t) = \begin{cases} \begin{bmatrix} \frac{F_+}{F_-}e^{it(g_+-g_-)} & \chi\upsilon_{[\overline{E_2},\mu_2]} \\ -\chi\upsilon_{[\mu_2,E_2]} & \frac{F_-}{F_+}e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[\overline{E_2},E_2]} \\ \begin{bmatrix} \frac{F_+}{F_-}e^{it(g_+-g_-)} & e^{-i(t\Omega+\Delta)}\chi\upsilon_{[\mu_1,\overline{E_1}]} \\ -e^{i(t\Omega+\Delta)}\chi\upsilon_{[E_1,\mu_1]} & \frac{F_-}{F_+}e^{-it(g_+-g_-)} \end{bmatrix} & \text{for } z \in \upsilon_{[E_1,\overline{E_1}]} \\ \begin{bmatrix} 1 & \frac{r^*(z)}{F_+F_-}e^{-2itg} \\ r(z)F_+F_-e^{2itg} & (1+|r(z)|^2) \end{bmatrix} & \text{for } z \in [\mu_1,\mu_2] \end{cases} \tag{4.1.18}$$

We factorize the jump matrices and open lenses around the curves.

For $z \in [\mu_2, E_2]$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ \frac{(F_+)^2}{r^*(z)}\mathrm{e}^{2itg_+} & 1 \end{bmatrix}. \tag{4.1.19}$$

For $z \in [E_1, \mu_1]$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & \frac{\mathrm{e}^{-2itg_-}}{(F_-)^2 r(z)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-i(t\Omega+\Delta)} \\ -e^{i(t\Omega+\Delta)} & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{\mathrm{e}^{-2ig_+}}{(F_+)^2 r(z)} \\ 0 & 1 \end{bmatrix}. \tag{4.1.20}$$

For $z \in [\mu_1, \overline{E_1}]$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ \frac{(F_-)^2}{r^*(z)}\mathrm{e}^{2itg_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-i(t\Omega+\Delta)} \\ -e^{i(t\Omega+\Delta)} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{(F_+)^2}{r^*(z)}\mathrm{e}^{2itg_+} & 1 \end{bmatrix}. \tag{4.1.21}$$

For $z \in [\mu_1, \mu_2]$

$$V^{(3)}(z,\eta,t) = \begin{bmatrix} 1 & 0 \\ \frac{r(z)}{(1+|r(z)|^2)(F_-)^2}\mathrm{e}^{2itg_-} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{r^*(z)}{(1+|r(z)|^2)(F_+)^2}\mathrm{e}^{-2itg_+} \\ 0 & 1 \end{bmatrix}. \tag{4.1.22}$$

The function $(1 + r(z)r^*(z))$ is bounded in a neighbour of the segment $[\mu_1, \mu_2]$ while, as shown in [24, 31, 74], the function $(F(z))^2$ is bounded in all the domains Σ_1 and Σ_2. This means that we can analytically extend the factorization of the jump matrices and opening the lenses around the curves $\upsilon_{[E_1,\overline{E_1}]}$, $\upsilon_{[\overline{E_2},E_2]}$ and $[\mu_1, \mu_2]$. We apply a transformation similar to (3.1.21),

$$\Gamma^{(4)}(z) := \Gamma^{(3)}(z) G^{(4)}(z,\eta,t) \tag{4.1.23}$$

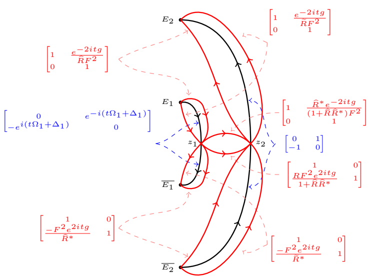
where

$$G^{(4)}(z,\eta,t) := \begin{cases} \begin{bmatrix} 1 & \frac{\mathrm{e}^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_-(\upsilon_{[\mu_2,E_2]}) \cup \mathscr{U}_+(\upsilon_{[E_1,\mu_1]}), \\ \begin{bmatrix} 1 & -\frac{\mathrm{e}^{-2itg(z,\eta)}}{(F(z,\eta))^2 r(z)} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_+(\upsilon_{[\mu_2,E_2]}) \cup \mathscr{U}_-(\upsilon_{[E_1,\mu_1]}), \\ \begin{bmatrix} 1 & 0 \\ -\frac{(F(z,\eta))^2}{r^*(z)}\mathrm{e}^{2itg(z,\eta)} & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_-(\upsilon_{[\overline{E_2},\mu_2]}) \cup \mathscr{U}_+(\upsilon_{[\mu_1,\overline{E_1}]}), \\ \begin{bmatrix} 1 & 0 \\ \frac{(F(z,\eta))^2}{r^*(z)}\mathrm{e}^{2itg(z,\eta)} & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_+(\upsilon_{[\overline{E_2},\overline{E_1}]}) \cup \mathscr{U}_-(\upsilon_{[\mu_1,\overline{E_1}]}), \\ \begin{bmatrix} 1 & 0 \\ \frac{r(z)}{(1+|r(z)|^2)(F_-)^2}\mathrm{e}^{2itg_-} & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_-([\mu_1,\mu_2]), \\ \begin{bmatrix} 1 & -\frac{r^*(z)}{(1+|r(z)|^2)(F_+)^2}\mathrm{e}^{-2itg_+} \\ 0 & 1 \end{bmatrix} \text{ for } z \in \mathscr{U}_-([\mu_1,\mu_2]), \\ \mathbf{1} \text{ otherwise.} \end{cases} \tag{4.1.24}$$

The new matrix $\Gamma^{(4)}(z)$ satisfies a RHP

$$\Gamma^{(4)}(z)_+ = \Gamma^{(4)}(z)_- V^{(4)}(z,\eta,t) \tag{4.1.25}$$

with jump matrices define in the boundary of the lenses, as we can see in Figure 4.2.



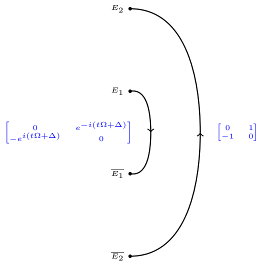


Figure 4.2: On the left: The Riemann-Hilbert problem for $\Gamma^{(4)}(z)$. On the right: The model problem of $X(z)$.

By studying the signs of $\mathrm{Im}\,(g)$ around the level sets $\mathrm{Im}\,(g) = 0$, we have that the jump matrices on the lenses tends to the identity as $t \to +\infty$ and the RHP (4.1.25) resolves in the model problem

$$X(z)_+ = X(z)_- V_X(z,\eta,t) \tag{4.1.26}$$

where

$$V_X(z,\eta,t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ for } z \in \upsilon_{[\overline{E_2},E_2]} \\ \begin{bmatrix} 0 & e^{-i(t\Omega(\eta)+\Delta(\eta))} \\ -e^{i(t\Omega(\eta)+\Delta(\eta))} & 0 \end{bmatrix} \text{ for } z \in \upsilon_{[E_1,\overline{E_1}]} \end{cases}. \tag{4.1.27}$$

This problem is solved in the same way as the model problem in the genus 1 sector, described in chapter 3, with the only difference given by the branch cut of the Riemann surface $\mathscr{P}_1$. Indeed, in this scenario the branch cut are given by the curves $\upsilon_{[E_1,\overline{E_1}]}$ and $\upsilon_{[E_2,\overline{E_2}]}$. We introduce the homological basis α, β as in Figure 4.3 and Abel map $u(z,z_0)$, already defined in (2.2.33).

The function $u(z,\infty)$ has the following jumps along the contours $\upsilon_{[E_1,\overline{E_1}]}, \upsilon_{[E_2,\overline{E_2}]}$ and $[\mu_1,\mu_2]$:

$$\begin{aligned} &u_+ + u_- = 0 \text{ for } z \in \upsilon_{[\mu_2,E_2]} \quad u_+ + u_- = 1 \text{ for } z \in \upsilon_{[\overline{E_2},\mu_2]} \\ &u_+ + u_- = \tau \text{ for } z \in \upsilon_{[E_1,\mu_1]} \quad u_+ + u_- = 1+\tau \text{ for } z \in \upsilon_{[\mu_1,\overline{E_1}]} \\ &u_+ - u_- = -1 \text{ for } z \in [\mu_1,\mu_2] \end{aligned} \tag{4.1.28}$$

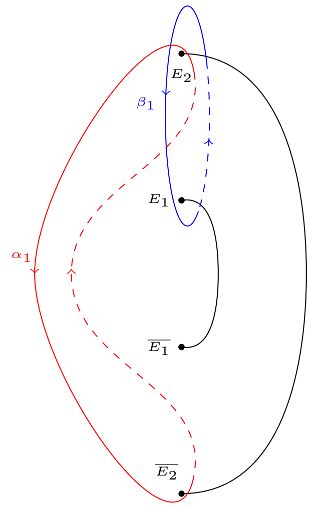


Figure 4.3: The new homological basis for the Riemann surface $\mathscr{P}_1$.

where $\tau := \oint_\beta \frac{d\zeta}{c_\alpha P_1(\zeta)}$ and $c_\alpha := \oint_\alpha \frac{d\zeta}{P_1(\zeta)}$.

Subsequently, we solve the model problem (4.1.26) in the same way of the problem (3.1.28) and the solution $X(z,\eta,t)$ of (4.1.26) has form:

$$X(z,\eta,t) = \begin{bmatrix} \frac{\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)+\frac{1}{\phi_1(z)}\right)\varphi_1(z,\eta,t) & -\frac{i\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)-\frac{1}{\phi_1(z)}\right)\psi_1(z,\eta,t) \\ \frac{i\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)-\frac{1}{\phi_1(z)}\right)\varphi_2(z,\eta,t) & \frac{\vartheta(0)}{2\vartheta(\frac{t\Omega+\Delta}{2\pi})}\left(\phi_1(z)+\frac{1}{\phi_1(z)}\right)\psi_2(z,\eta,t) \end{bmatrix}, \tag{4.1.29}$$

with $\phi_1(z)$ defined in (2.2.37) and

$$\varphi_1(z,\eta,t) := \frac{\vartheta(u(z_1,\infty_1)-\frac{t\Omega+\Delta}{2\pi})}{\vartheta(u(z_1,\infty_1))} \quad \psi_1(z,\eta,t) := \frac{\vartheta(u(z_2,\infty_1)-\frac{t\Omega+\Delta}{2\pi})}{\vartheta(u(z_2,\infty_1))} \tag{4.1.30}$$

$$\varphi_2(z,\eta,t) := \frac{\vartheta(u(z_1,\infty_2)-\frac{t\Omega+\Delta}{2\pi})}{\vartheta(u(z_1,\infty_1))} \quad \psi_2(z,\eta,t) := \frac{\vartheta(u(z_2,\infty_2)-\frac{t\Omega+\Delta}{2\pi})}{\vartheta(u(z_1,\infty_2))} \tag{4.1.31}$$

where z_1 and z_2 indicates the point z respectively in the first and in the second sheet of the Riemann surface $\mathscr{P}_1$, defined in (2.2.32). Since $\Gamma^{(3)}(z,\eta,t) \sim X(z,\eta,t)$ for $t \sim +\infty$, we obtain the long time asymptotic of the NLS solution $\psi(x,t)$ from the equation

$$\psi(x,t) \sim 2i \lim_{z\to\infty} z(X(z,\eta,t)e^{it(\hat{\theta}(z,\eta)-g(z,\eta))\sigma_3}F(z)^{-\sigma_3})_{12}, \tag{4.1.32}$$

and we get that the behaviour of $\psi(x,t)$ at $t\sim+\infty$ is still a genus 1 wave given by the formula (2.2.53).

4.2 The Genus 2 sector

We study the case where $\eta_+<\eta<\eta_*$. As we shown in section 2.3, the g-function (2.3.20) is not well defined anymore and we need to define a new g-function, analytic in $\mathbb{C}\setminus \upsilon_{[E_2,\overline{E_2}]}\cup\upsilon_{[E_1,\overline{E_1}]}$, which satisfy some additional conditions:

1. $g'(z)$ has 4 zeroes, $\mu_1,\mu_2\in\mathbb{R}$ of degree 1 and a pair of complex conjugate points $d,\overline{d}$ of degree 1/2;
2. $\mathrm{Im}\,(g(d))=0$ and $\mathrm{Im}\,(g(\overline{d}))=0$.

Then the level set $\mathrm{Im}\,(g(z))=0$ is descried by the Figure 4.4 and the bands $\upsilon_{[E_2,\overline{E_2}]}\cup\upsilon_{[E_1,\overline{E_1}]}$ are now well defined.

In this case, the g-function has different jumps along the level set $\mathrm{Im}\,(g)=0$:

$$g(z)_++g(z)_-=0 \qquad \text{for } z\in\upsilon_{[\overline{E_2},\overline{d}]}\cup\upsilon_{[d,E_2]}, \tag{4.2.1}$$
$$g(z)_+-g(z)_-=2g(\overline{d})=:\Omega_1+\Omega_2 \qquad \text{for } z\in\upsilon_{[\overline{d},d]}, \tag{4.2.2}$$
$$g(z)_++g(z)_-=2g(E_1)=:\Omega_1 \qquad \text{for } z\in\upsilon_{[E_1,\overline{E_1}]}, \tag{4.2.3}$$
$$g(z)_+-g(z)_-=0 \qquad \text{for } z\in[\mu_1,\mu_2]. \tag{4.2.4}$$

Then the new g-function has the following form:

$$g(z)=2\int_{E_2}^{z}\frac{(\zeta-\mu_1)(\zeta-\mu_2)\sqrt{(\zeta-d_1)(\zeta-\overline{d_1})}}{\sqrt{(\zeta-E_1)(\zeta-E_2)(\zeta-\overline{E}_1)(\zeta-\overline{E}_2)}}\mathrm{d}\zeta, \tag{4.2.5}$$

where $\mu_1,\mu_2,\mathrm{Re}\,(d_1)$ and $\mathrm{Im}\,(d_1)$ are given by the following conditions:

$$\begin{aligned}&4\mu_1\mu_2+4\mathrm{Re}\,(d_1)(\mu_1+\mu_2)+\\&2(\mathrm{Im}\,(d_1))^2-2(\mathrm{Im}\,(E_1)^2+\mathrm{Im}\,(E_2)^2)=0,\end{aligned} \tag{4.2.6}$$
$$\frac{\eta}{2}=-\mu_1-\mu_2-\mathrm{Re}\,(b), \tag{4.2.7}$$
$$\int_{\overline{E_1}}^{E_1}\mathrm{d}g=0,\qquad \int_{\overline{d_1}}^{d_1}\mathrm{d}g=0. \tag{4.2.8}$$

With an abuse of notation, we denote with Σ_1 and Σ_2 the regions encircled respectively by the loops $\upsilon_{[E_1,\mu_1]}\cup[\mu_1,\mu_2]\cup\upsilon_{[\mu_2,d]}\cup\upsilon_{[d,E_2]}\cup[E_2,E_1]$ and its complex conjugate.

Having moved the jump contours in the level set $\mathrm{Im}\,(g)=0$ that encircled Σ_1 and Σ_2 (see Figure 4.4), we study the long time asymptotic of the RHP (4.1.2), with jumps also in the curves $\upsilon_{[d,E_2]}$, $\upsilon_{[\mu_2,d]}$ and their complex conjugate.

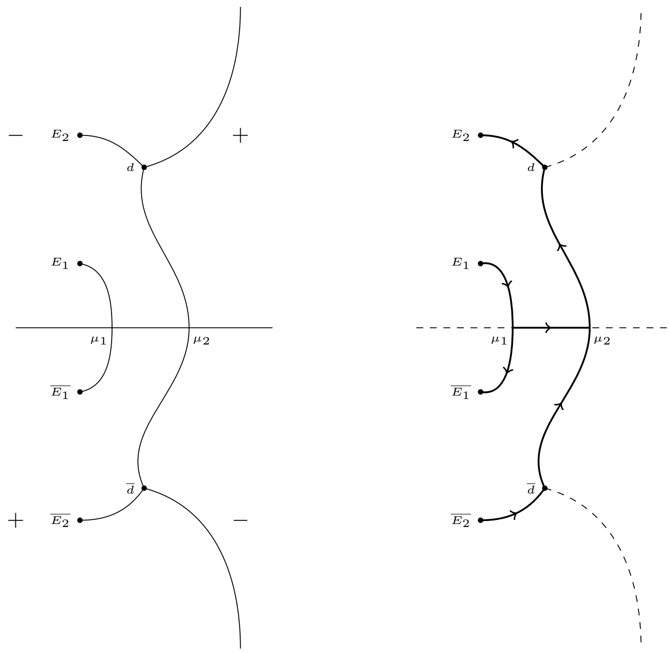


Figure 4.4: On the left: the distribution of signs for $\mathrm{Im}\,(g)$ for the Genus 2 case. On the right: jump contours of $\Gamma^{(1)}(z)$.

We apply the transformation (4.1.4), with the g-function given by (4.2.5). So the new RHP (4.1.5) has jump matrices

$$V^{(2)}(z,\eta,t)=\begin{cases}\begin{bmatrix} e^{it(g_+-g_-)} & r^*(z)\chi_{\upsilon_{[\overline{E_2},\overline{d}]}} \\ -r(z)\chi_{\upsilon_{[d,E_2]}} & e^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[d,E_2]}\cup\upsilon_{[\overline{E_2},\overline{d}]} \\ \begin{bmatrix} e^{it(g_+-g_-)} & r(z)e^{-it\Omega_1}\chi_{\upsilon_{[\mu_1,\overline{E_1}]}} \\ -r(z)e^{it\Omega_1}\chi_{\upsilon_{[E_1,\mu_1]}} & e^{-it(g_+-g_-)}\end{bmatrix} \text{ for } z\in\upsilon_{[E_1,\overline{E_1}]} \\ \begin{bmatrix} e^{it\Omega_2} & r^*(z)e^{-it(g_++g_-)}\chi_{\upsilon_{[\overline{d},\mu_2]}} \\ -r(z)e^{it(g_++g_-)}\chi_{\upsilon_{[\mu_2,d]}} & e^{-it\Omega_2}\end{bmatrix} \text{ for } z\in\upsilon_{[\overline{d},d]} \\ \begin{bmatrix} (1+|r(z)|^2) & r^*(z)\mathrm{e}^{-2itg(z)} \\ r(z)\mathrm{e}^{2itg(z)} & 1\end{bmatrix} \text{ for } z\in[\mu_1,\mu_2]\end{cases} \tag{4.2.9}$$

We adapt the transformation (4.1.10) in this new scenario.

Whit the new jump contours, the function $F(z)$ should satisfy a new scalar RHP

$$F_+F_- = r(z)^{-1} \text{ for } z \in \upsilon_{[d,E_2]}, \quad F_+F_- = r^*(z) \text{ for } z \in \upsilon_{[\overline{E_2},\overline{d}]}, \tag{4.2.10}$$

$$F_+F_- = r(z)^{-1}e^{i\Delta_1} \text{ for } z \in \upsilon_{[E_1,\mu_1]}, \; F_+F_- = r^*(z)e^{i\Delta_1} \text{ for } z \in \upsilon_{[\mu_1,\overline{E_1}]} \tag{4.2.11}$$

$$\frac{F_+}{F_-} = e^{i(\Delta_2+\Delta_1)} \text{ for } z \in \upsilon_{[\mu_2,d]}, \quad \frac{F_+}{F_-} = e^{i(\Delta_2+\Delta_2)} \text{ for } z \in \upsilon_{[\overline{d},\mu_2]} \tag{4.2.12}$$

$$F_+ = F_-(1+|r(z)|^2)^{-1} \text{ for } z \in [\mu_1,\mu_2], \tag{4.2.13}$$

with $\Delta_1, \Delta_2 \in \mathbb{R}$ and has the same conditions (4.1.11) of before. The solution of this RHP is similar to the one find in section 3.2

$$\begin{aligned} F(z,\eta) =&\exp\left\{\frac{P_2(z)}{2\pi i}\left[-\left(\int_{\upsilon_{[d,E_2]}}\frac{\log(r(\zeta))}{(P_2(\zeta))_+(\zeta-z)}d\zeta+\left(\int_{\upsilon_{[d,E_2]}}\frac{\log(r(\zeta))}{(P_2(\zeta))_+(\zeta-z)}d\zeta\right)^*\right)\right.\right.\\ &-\left(\int_{\upsilon_{[E_1,\mu_1]}}\frac{\log(r(\zeta))}{(P_2(\zeta))_+(\zeta-z)}d\zeta+\left(\int_{\upsilon_{[E_1,\mu_1]}}\frac{\log(r(\zeta))}{(P_2(\zeta))_+(\zeta-z)}d\zeta\right)^*\right)\\ &+i\Delta_1\left(\int_{\upsilon_{[E_1,\overline{E_1}]}}\frac{d\zeta}{(P_2(\zeta))_+(\zeta-z)}\right)+i(\Delta_2+\Delta_1)\left(\int_{\upsilon_{[\overline{d},d]}}\frac{d\zeta}{(P_2(\zeta))_+(\zeta-z)}\right)\\ &\left.\left.-\int_{\mu_1}^{\mu_2}\frac{\log(1+|r(\zeta)|^2)}{P_2(\zeta)(\zeta-z)}d\zeta\right]\right\}, \end{aligned} \tag{4.2.14}$$

where $P_2(z) = \sqrt{(z-E_1)(z-E_2)(z-\overline{E_1})(z-\overline{E_2})(z-d)(z-\overline{d})}$ is a multivalued complex function, analytic in $\mathbb{C}\setminus\upsilon_{[d,E_2]}\cup\upsilon_{[\overline{E_2},\overline{d}]}\cup\upsilon_{E_1,\overline{E_1}}$. The constants Δ_1 and Δ_2 are given by the condition $F(z) \sim F_\infty + \mathcal{O}(z^{-1})$ as $z \to \infty$

$$\begin{aligned} &\int_{\upsilon_{[d,E_2]}}\frac{\zeta^j\log(r(\zeta))}{(P_2(\zeta))_+}d\zeta+\left(\int_{\upsilon_{[d,E_2]}}\frac{\zeta^j\log(r(\zeta))}{(P_2(\zeta))_+}d\zeta\right)^*\\ &+\int_{\upsilon_{[E_1,\mu_1]}}\frac{\zeta^j\log(r(\zeta))}{(P_2(\zeta))_+}d\zeta+\left(\int_{\upsilon_{[E_1,\mu_1]}}\frac{\zeta^j\log(r(\zeta))}{(P_2(\zeta))_+}d\zeta\right)^*\\ &-i\Delta_1\left(\int_{\upsilon_{[E_1,\overline{E_1}]}}\frac{\zeta^j d\zeta}{(P_2(\zeta))_+}\right)+i(\Delta_2+\Delta_1)\left(\int_{\upsilon_{[\overline{d},d]}}\frac{\zeta^j d\zeta}{(P_2(\zeta))_+}\right)\\ &+\int_{\mu_1}^{\mu_2}\frac{\zeta^j\log(1+|r(\zeta)|^2)}{P_2(\zeta)}d\zeta = 0 \text{ for } j=0,1. \end{aligned} \tag{4.2.15}$$

By applying the transformation (4.1.10) in this case, the new RHP has the following jumps:

$$V^{(3)}(z,\eta,t)=\begin{cases}\begin{bmatrix}\frac{F_+}{F_-}e^{it(g_+-g_-)} & \chi_{\upsilon_{[\overline{E_2},\overline{d}]}}\\ -\chi_{\upsilon_{[d,E_2]}} & \frac{F_-}{F_+}e^{-it(g_+-g_-)}\end{bmatrix} & \text{for } z\in\upsilon_{[d,E_2]}\cup\upsilon_{[\overline{E_2},\overline{d}]}\\ \begin{bmatrix}\frac{F_+}{F_-}e^{it(g_+-g_-)} & e^{-i(t\Omega_1+\Delta_1)}\chi_{\upsilon_{[\mu_1,\overline{E_1}]}}\\ -e^{i(t\Omega_1+\Delta_1)}\chi_{\upsilon_{[E_1,\mu_1]}} & \frac{F_-}{F_+}e^{-it(g_+-g_-)}\end{bmatrix} & \text{for } z\in\upsilon_{[E_1,\overline{E_1}]}\\ \begin{bmatrix}e^{i(t\Omega_2+\Delta_2)} & \frac{r^*(z)}{F_+F_-}e^{-it(g_++g_-)}\chi_{\upsilon_{[\overline{d},\mu_2]}}\\ -r(z)F_+F_-e^{it(g_++g_-)}\chi_{\upsilon_{[\mu_2,d]}} & e^{-i(t\Omega_2+\Delta_2)}\end{bmatrix} & \text{for } z\in\upsilon_{[\overline{d},d]}\\ \begin{bmatrix}1 & \frac{r^*(z)}{F_+F_-}e^{-2itg}\\ r(z)F_+F_-e^{2itg} & (1+|r(z)|^2)\end{bmatrix} & \text{for } z\in[\mu_1,\mu_2]\end{cases}\tag{4.2.16}$$

The next step is the factorization matrices $V^{(3)}(z,\eta,t)$ and the introduction of the lenses around the jump contours. This process is similar to the one we shown in section 4.1, the only difference is given by the jumps in $\upsilon_{[\overline{d},d]}$, where

$$V^{(3)}(z,\eta,t)=\begin{cases}\begin{bmatrix}1 & \frac{r^*(z)}{(F_-)^2}\mathrm{e}^{-2ig_-(z)}\\ 0 & 1\end{bmatrix}\mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} & \text{for } z\in\upsilon_{[\overline{d},\mu_2]},\\ \begin{bmatrix}1 & 0\\ -r(z)(F_-)^2\mathrm{e}^{2ig_-(z)} & 1\end{bmatrix}\mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} & \text{for } z\in\upsilon_{[\mu_2,d]}.\end{cases}\tag{4.2.17}$$

We apply a final transformation, similar to (3.1.21)

$$\Gamma^{(4)}(z)=\Gamma^{(3)}(z)G^{(4)}(z),\tag{4.2.18}$$

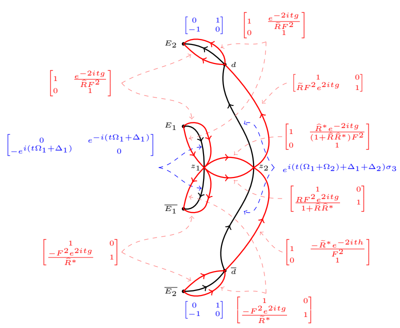
where $G^{(4)}(z)$ is similar to the one in the (Genus 1)$_s$ case but defined in the new lenses $\mathscr{U}_\pm(\upsilon)$. The main difference arise in the lenses $\mathscr{U}_-(\upsilon_{[\mu_2,d(\eta)]})$ and $\mathscr{U}_-(\upsilon_{[\overline{d(\eta)},\mu_2]})$, where $G^{(4)}(z)$ takes values:

$$G^{(4)}(z)=\begin{cases}\begin{bmatrix}1 & 0\\ -r(z)(F(z,\eta))^2\mathrm{e}^{2ig(z,\eta)} & 1\end{bmatrix} & \text{for } z\in\mathscr{U}_-(\upsilon_{[\mu_2,d(\eta)]}),\\ \begin{bmatrix}1 & \frac{r^*(z)}{(F(z,\eta))^2}\mathrm{e}^{-2ig(z,\eta)}\\ 0 & 1\end{bmatrix} & \text{for } z\in\mathscr{U}_-(\upsilon_{[\overline{d},\mu_2]}).\end{cases}\tag{4.2.19}$$

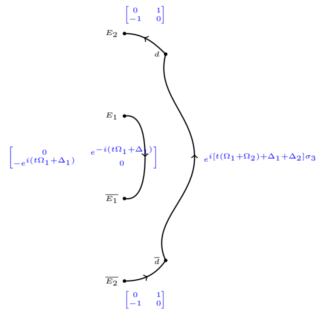
The matrix $\Gamma^{(4)}(z)$ has jump conditions also in the boundary of the lenses, as it is shown in Figure 4.5a.

From the distribution of the signs of $\mathrm{Im}\,(g)$, the jump matrices in the border to the lenses $L_\pm(\upsilon)$ tend to the identity as $t\to+\infty$. This implies that $\Gamma^{(4)}(z)$ has a leading order $X(z)$ in the long-time asymptotic which satisfy the model problem

$$X(z)_+=X(z)_-V_X(z,\eta,t),\tag{4.2.20}$$



(a) Jump contours for $V^{(3)}(z,\eta,t)$



(b) Jump matrices of model problem for Genus 2

where

$$V_X(z,\eta,t)=\begin{cases}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} \text{ for } z\in \upsilon_{[\overline{E_2},\overline{d}]}\cup \upsilon_{[d,E_2]}\\ \begin{bmatrix}0 & \mathrm{e}^{-i(t\Omega_1+\Delta_1)}\\ -\mathrm{e}^{i(t\Omega_1+\Delta_1)} & 0\end{bmatrix} \text{ for } z\in \upsilon_{[E_1,\overline{E_1}]}\\ \mathrm{e}^{i[t(\Omega_1+\Omega_2)+\Delta_1+\Delta_2]\sigma_3} \text{ for } z\in \upsilon_{[\overline{d},d]}.\end{cases} \tag{4.2.21}$$

The procedure to solve the model problem (4.2.20) is similar to the one we follow in Chapter 3 for the genus three sector. In this case, we introduce the Riemann surface of genus 2 $\mathscr{P}_2$, defined as

$$\mathscr{P}_2 := \{(w,z)\in\mathbb{C}^2 |\ w^2 = P_2(z)^2\}, \tag{4.2.22}$$

and the 2-dimensional Abel map

$$\vec{u}(z,z_0) := \int_{z_0}^{z}\vec{\varpi}, \tag{4.2.23}$$

where $\vec{\varpi}=(\varpi_1,\varpi_2)^T$ is the 2-dimensional vector of holomorphic differentials in the Riemann surface $\mathscr{P}_2$ and normalized such that

$$\oint_{\alpha_j}\varpi_k=\delta_{jk} \text{ for } j,k=1,2, \tag{4.2.24}$$

where α_j,β_j is the homological basis defined as in Figure 4.6.

Let $z_0=\infty$, the vector $\vec{u}(z,\infty)$ satisfy the jump conditions:

$$\begin{aligned}\vec{u}_++\vec{u}_-&=0 \text{ for } z\in\upsilon_{[d,E_2]} & \vec{u}_++\vec{u}_-&=e_1 \text{ for } z\in\upsilon_{[\overline{E_2},\overline{d}]}\\ \vec{u}_++\vec{u}_-&=Be_1 \text{ for } z\in\upsilon_{[E_1,\mu_1]} & \vec{u}_++\vec{u}_-&=e_1-e_2+Be_1 \text{ for } z\in\upsilon_{[\mu_1,\overline{E_1}]}\\ \vec{u}_+-\vec{u}_-&=B(e_1+e_2)-e_1 \text{ for } z\in\upsilon_{[\mu_2,d]} & \vec{u}_+-\vec{u}_-&=B(e_1+e_2)-e_2 \text{ for } z\in\upsilon_{[\overline{d},\mu_2]}\\ \vec{u}_+-\vec{u}_-&=e_2-e_1 \text{ for } z\in[\mu_1,\mu_2]\end{aligned}$$

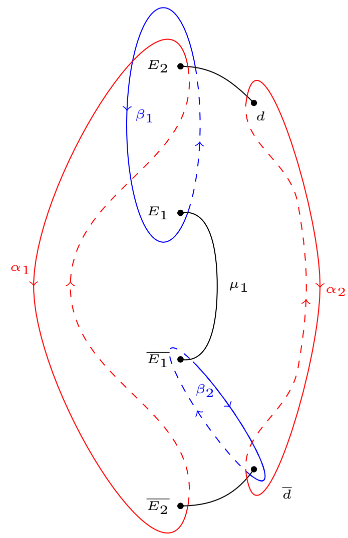


Figure 4.6: The homological basis for the Riemann surface $\mathscr{P}_2$.

where $e_1 = (1,0)^T$,$e_2 = (0,1)^T$ and B is the 2×2 periodic matrix with elements define as

$$B_{jk} = \oint_{\beta_j} \varpi_k \quad j,k = 1,2. \tag{4.2.25}$$

Following the same ideas of section 3.2.1, we find out that the matrix $X(z)$ has form

$$X(z,\eta,t) = \begin{bmatrix} \frac{\Theta(0)}{2\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}\left(\phi_2(z,\eta)+\frac{1}{\phi_2(z,\eta)}\right)\varphi_1(z,\eta,t) & -\frac{i\Theta(0)}{2\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}\left(\phi_2(z,\eta)-\frac{1}{\phi_2(z,\eta)}\right)\psi_1(z,\eta,t) \\ \frac{i\Theta(0)}{2\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}\left(\phi_2(z,\eta)-\frac{1}{\phi_2(z,\eta)}\right)\varphi_2(z,\eta,t) & \frac{\Theta(0)}{2\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}\left(\phi_2(z,\eta)+\frac{1}{\phi_2(z,\eta)}\right)\psi_2(z,\eta,t) \end{bmatrix}, \tag{4.2.26}$$

where $\phi_2(z,\eta)$ is define as

$$\phi_2(z,\eta) = \left(\frac{(z-\overline{E_1})(z-E_2)(z-\overline{d})}{(z-E_1)(z-\overline{E_2})(z-d)}\right)^{\frac{1}{4}}, \tag{4.2.27}$$

and the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are given by

$$\varphi_1(z,\eta,t) := \frac{\Theta(\vec{u}(z_1,\infty_1) - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_1,\infty_1))} \quad \psi_1(z,\eta,t) := \frac{\Theta(\vec{u}(z_2,\infty_1) - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_2,\infty_1))} \tag{4.2.28}$$

$$\varphi_2(z,\eta,t) := \frac{\Theta(\vec{u}(z_1,\infty_2) - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_1,\infty_1))} \quad \psi_2(z,\eta,t) := \frac{\Theta(\vec{u}(z_2,\infty_2) - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\vec{u}(z_2,\infty_2))} \tag{4.2.29}$$

with $\vec{\Omega}(\eta) = (\Omega_1(\eta), \Omega_2(\eta))^T, \vec{\Delta}(\eta) = (\Delta_1(\eta), \Delta_2(\eta))^T$.

Since $\Gamma^{(4)}(z) \sim X(z)$ as $t \to +\infty$, the long-time asymptotic of the solution of NLS $\psi(x,t)$ is still given by the equation (4.1.32), with $X(z)$ instead of $X(z)$, and the behaviour is given by a genus 2 wave

$$\psi(x,t) \sim -i(\mathrm{Im}\,(E_2) - \mathrm{Im}\,(E_1) - \mathrm{Im}\,(d))\frac{\Theta(0)\Theta(\vec{u}_\infty - \frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})}{\Theta(\frac{t\vec{\Omega}+\vec{\Delta}}{2\pi})\Theta(\vec{u}_\infty)}. \tag{4.2.30}$$

This ends our proof of Theorem 4.0.1.

Chapter 5

IIKS theory for the $\overline{\partial}$-problem: τ-function and KP hierarchy

In this last chapter we develop the Its-Izergin-Korepin-Slavnov (IIKS) theory of integrable operators $\mathcal{K}$ acting on a domain of the complex plane with smooth boundary, in analogy with the theory of integrable operators acting on contours of the complex plane [62]. We show how the resolvent operator $\mathcal{R}$ is obtained from the solution of a $\overline{\partial}$-problem in the complex plane and, in the case where the problem depends on other auxiliary parameters, that we can define its Malgrange one form, in analogy with the theory of isomonodromic problems [10, 27, 36, 48, 59, 66]. We show that this for is closed and coincides with the exterior differential of an *Hilbert-Carleman* determinant of the operator $\mathcal{K}$ and that, for a particular of $\overline{\partial}$-problem (in which also (1.2.11) is in), this determinant is a τ-function of the Kadomtsev-Petviashvili type.

5.1 Integrable operators and $\overline{\partial}$-problems

Let $\mathscr{D} \subset \mathbb{C}$ be a compact union of domains with smooth boundary and denote by $\mathcal{K}$ the integral operator acting on the space $L^2(\mathscr{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$ with a kernel $\hat{\mathcal{K}}(z, w)$ of the form

$$\hat{\mathcal{K}}(z, \overline{z}, w, \overline{w}) := \frac{p^T(z, \overline{z}) q(w, \overline{w})}{z - w}, \quad p(z, \overline{z}), q(z, \overline{z}) \in \mathrm{Mat}(r \times n, \mathbb{C}), \tag{5.1.1}$$

$$p^T(z, \overline{z})\, q(z, \overline{z}) \equiv 0 \quad \text{and} \quad (\partial_{\overline{z}} p(z, \overline{z}))^T\, q(z, \overline{z}) \equiv 0, \quad z, \overline{z} \in \mathscr{D}. \tag{5.1.2}$$

Here the matrix-valued functions f, g are assumed to be sufficiently smooth on $\mathscr{D}$ but no analyticity is required and for this reason we indicate the dependence on both variables z and $\overline{z}$. The vanishing requirements along the locus $z = w$ are sufficient to guarantee that the kernel $\hat{\mathcal{K}}$ admits a well-defined value on the diagonal and it is continuous on $\mathscr{D} \times \mathscr{D}$

$$\lim_{w \to z} \hat{\mathcal{K}}(z, \overline{z}, w, \overline{w}) = \hat{\mathcal{K}}(z, \overline{z}, z, \overline{z}) = \partial_z p^T(z, \overline{z})\, q(z, \overline{z}). \tag{5.1.3}$$

We have emphasized that the kernel and the functions are not holomorphically dependent on the variables; that said, from now on we omit the explicit dependence on $\overline{z}$, trusting that the class of functions we are dealing with will be clear by the context each time. The operator $\mathcal{K}$ acts as follows on functions

$$\mathcal{K}[\varphi](z) := \iint_{\mathscr{D}} \hat{\mathcal{K}}(z,w)\varphi(w)\frac{\mathrm{d}\overline{w}\wedge \mathrm{d}w}{2i}, \quad \varphi \in L^2(\mathscr{D}, \mathrm{d}^2 z)\otimes \mathbb{C}^n. \tag{5.1.4}$$

The kernel $\hat{\mathcal{K}}(z,\overline{z},w,\overline{w})$ and the corresponding integral operator $\mathcal{K}$ is a Hilbert–Schmidt operator with a well–defined and continuous value on the diagonal in $\mathscr{D}\times\mathscr{D}$ and therefore its trace and Fredholm determinant are well defined (Proposition 3.11.2 and Theorem 3.11.5 [82]) as limits of finite rank operators, namely

$$\lim_{n\to\infty} \mathrm{Tr}(P_n \mathcal{K} P_n) = \iint_{\mathscr{D}} \hat{\mathcal{K}}(z,\overline{z},z,\overline{z},)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i}, \tag{5.1.5}$$

where P_n is a finite rank projection such that $\lim_{n\to\infty} P_n = \mathrm{Id}$. Furthermore the limit

$$\lim_{n\to\infty} \det(\mathrm{Id} - P_n\mathcal{K}P_n) \tag{5.1.6}$$

exists. If $\mathcal{K}$ is trace class, then $\mathrm{Tr}(\mathcal{K})$ and $\det(\mathrm{Id} - \mathcal{K})$ coincide with the limits (5.1.5) and (5.1.6) respectively.

We introduce the following $\overline{\partial}$-problem for an $r\times r$ matrix-valued function $\Gamma(z,\overline{z})$.

Problem 5.1.1. *Find a matrix-valued function $\Gamma(z,\overline{z}) \in GL_r(\mathbb{C})$ such that*

$$\partial_{\overline{z}}\Gamma(z) = \Gamma(z)M(z); \qquad \Gamma(z) \underset{z\to\infty}{\to} \mathbf{1} \tag{5.1.7}$$

where $\mathbf{1}$ is the identity in $GL_r(\mathbb{C})$ and

$$M(z) := \begin{cases} \pi p(z) q^T(z), & \text{for } z\in\mathscr{D}, \\ 0 & \text{for } z \in \mathbb{C}\backslash\mathscr{D}. \end{cases} \tag{5.1.8}$$

We first show that

Lemma 5.1.2. *If a solution of the $\overline{\partial}$-problem 5.1.1 exists, it is unique. Furthermore* $\det\Gamma(z)\equiv 1$.

Proof. If Γ is a solution of the $\overline{\partial}$-problem 5.1.1 then

$$\partial_{\overline{z}} \det\Gamma = \mathrm{Tr}\Big(\mathrm{adj}(\Gamma)\partial_{\overline{z}}\Gamma\Big) = \mathrm{Tr}\Big(\mathrm{adj}(\Gamma)\Gamma M\Big) \tag{5.1.9}$$

where $\mathrm{adj}(\Gamma)$ denotes the adjugate matrix (the transposed of the co-factor matrix). Here Tr denotes the matrix trace. Now the product in the last formula yields $\mathrm{adj}(\Gamma)\Gamma = (\det\Gamma)\mathbf{1}$, so that

$$\partial_{\overline{z}}\det\Gamma = \det(\Gamma)\mathrm{Tr}(M) = 0 \tag{5.1.10}$$

where the last identity follows from the fact that M is traceless because $\mathrm{Tr}(M) = \mathrm{Tr}(M^T) = 0$. Thus $\det\Gamma$ is an entire function which tends to $\mathbf{1}$ at infinity, and hence it is identically equal to 1 by Liouville's theorem.

Now, if Γ_1, Γ_2 are two solutions, it follows easily that $R(z) := \Gamma_1\Gamma_2^{-1}$ is an entire matrix-valued function which tends to the identity matrix $\mathbf{1}$ at infinity and hence, by Liouville's theorem $R(z) \equiv \mathbf{1}$, thus proving the uniqueness. ■

Theorem 5.1.3. *The operator* $\mathrm{Id} - \mathcal{K}$ *with* $\mathcal{K}$ *as in (5.1.4) is invertible in* $L^2(\mathscr{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$ *if and only if the* $\overline{\partial}$*-problem 5.1.1 admits a solution. The resolvent* $\mathcal{R}$ *of* $\mathcal{K}$ *has kernel given by:*

$$\hat{\mathcal{R}}(z,w) := \frac{p^T(z)\Gamma^T(z)\left(\Gamma^T(w)\right)^{-1}q(w)}{z-w}, \quad (z,w) \in \mathscr{D} \times \mathscr{D}^* \tag{5.1.11}$$

where $\Gamma(z)$ *is a* $r \times r$ *matrix that solves the* $\overline{\partial}$*-problem 5.1.1.*

Proof. Suppose that the $\overline{\partial}$-problem 5.1.1 is solved by $\Gamma(z)$; we now show that the operator $(\mathrm{Id} - \mathcal{K})$ is invertible. Let us define the operator

$$\mathcal{R} : L^2(\mathscr{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n \to L^2(\mathscr{D}, \mathrm{d}^2 z) \otimes \mathbb{C}^n$$

with kernel $\hat{\mathcal{R}}(z,w)$ given by (5.1.11). To verify that $\mathcal{R}$ is the resolvent of the operator $\mathcal{K}$ we need to check the following condition

$$\begin{gathered}(\mathrm{Id} + \mathcal{R})\circ(\mathrm{Id} - \mathcal{K}) = \mathrm{Id} \\ \Downarrow \\ \mathcal{R}\circ\mathcal{K} = \mathcal{R} - \mathcal{K}.\end{gathered} \tag{5.1.12}$$

To this end we compute the kernel of $\mathcal{R}\circ\mathcal{K}$ namely

$$\begin{aligned}(\hat{\mathcal{R}}\circ\hat{\mathcal{K}})(z,w) &:= \iint_{\mathscr{D}} \hat{\mathcal{R}}(z,\zeta)\hat{\mathcal{K}}(\zeta,w)\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i} \\ &= \iint_{\mathscr{D}} \frac{p^T(z)\Gamma^T(z)\overbrace{(\Gamma^T(\zeta))^{-1}q(\zeta)p^T(\zeta)}^{=-\frac{1}{\pi}\partial_{\bar\zeta}(\Gamma^T(\zeta))^{-1}}q(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i} \\ &= -\frac{p^T(z)\Gamma^T(z)}{z-w}\iint_{\mathscr{D}} \partial_{\bar\zeta}(\Gamma^T(\zeta))^{-1}\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i\pi}q(w).\end{aligned} \tag{5.1.13}$$

If we consider the generalized Cauchy-Pompeiu formula for the matrix $(\Gamma^T(z))^{-1}$ we can express it in integral form as

$$(\Gamma^T(z))^{-1} = \mathbf{1} - \iint_{\mathscr{D}} \frac{\partial_{\bar\zeta}(\Gamma^T(\zeta))^{-1}}{\zeta - z}\frac{\mathrm{d}\overline{\zeta}\wedge d\zeta}{2\pi i}, \quad z \in \mathbb{C}. \tag{5.1.14}$$

We substitute (5.1.14) into (5.1.13):

$$
\begin{aligned}
(\hat{\mathcal{R}}\circ\hat{\mathcal{K}})(z,w) &= -\frac{p^T(z)\Gamma^T(z)}{z-w}\Big(\big((\Gamma^T(z))^{-1}-\mathbf{1}\big)-\big((\Gamma^T(w))^{-1}-\mathbf{1}\big)\Big)q(w)\\
&= -\frac{p^T(z)\Gamma^T(z)}{z-w}\Big((\Gamma^T(z))^{-1}-(\Gamma^T(w))^{-1}\Big)q(w)\\
&= \frac{p^T(z)\Gamma^T(z)(\Gamma^T(w))^{-1}q(w)}{z-w}-\frac{p^T(z)q(w)}{z-w}=\hat{\mathcal{R}}(z,w)-\hat{\mathcal{K}}(z,w).
\end{aligned}
\tag{5.1.15}
$$

This shows that indeed $\mathcal{R}$ satisfies the resolvent equation (5.1.12) and hence the operator $\mathrm{Id}-\mathcal{K}$ is invertible.

Vice versa, let us now suppose that the operator $\mathrm{Id}-\mathcal{K}$ is invertible and denote

$$
\mathcal{R}=\Big(\mathrm{Id}-\mathcal{K}\Big)^{-1}-\mathrm{Id}\,.
$$

We now verify that $\mathcal{R}$ has kernel

$$
\hat{\mathcal{R}}(z,w)=\frac{P(z)^T\,Q(w)}{z-w} \tag{5.1.16}
$$

where the matrices $P(z)$ and $Q(z)$ are defined as

$$
\begin{aligned}
P(z) &:= (\mathrm{Id}-\mathcal{K}^T)^{-1}[p](z)\\
Q(z) &:= (\mathrm{Id}-\mathcal{K})^{-1}[q](z),
\end{aligned}
\tag{5.1.17}
$$

with the inverse applied to each entry (and the transposition T acts on the matrix indices). Indeed we verify the condition (5.1.12) with R given by (5.1.16):

$$
\begin{aligned}
(\hat{\mathcal{R}}\circ\hat{\mathcal{K}})(z,w) &= \iint_{\mathscr{D}}\frac{P^T(z)Q(\zeta)p^T(\zeta)q(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i}=\frac{1}{z-w}\Bigg(\iint_{\mathscr{D}}\frac{P^T(z)Q(\zeta)p^T(\zeta)q(w)}{(z-\zeta)}\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i}+\\
&\quad+\iint_{\mathscr{D}}\frac{P^T(z)Q(\zeta)p^T(\zeta)q(w)}{(\zeta-w)}\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{2i}\Bigg)\\
&= \frac{1}{z-w}\Big(\mathcal{R}[p^T](z)q(w)+P^T(z)\mathcal{K}[Q](w)\Big).
\end{aligned}
$$

Adding and subtracting the kernels $\hat{\mathcal{K}}(z,w)$ and $\hat{\mathcal{R}}(z,w)$, we obtain

$$
\begin{aligned}
(\hat{\mathcal{R}}\circ\hat{\mathcal{K}})(z,w) = \frac{1}{z-w}\Big((\mathrm{Id}+\mathcal{R})[p^T](z)q(w)-P^T(z)(\mathrm{Id}-\mathcal{K})[Q](w)\Big)+\\
+\hat{\mathcal{R}}(z,w)-\hat{\mathcal{K}}(z,w).
\end{aligned}
\tag{5.1.18}
$$

With the definitions (5.1.17) the contributions in the first line of (5.1.18) cancel out and the condition (5.1.12) is satisfied. To conclude the proof we need to verify that

$$
P(z)=\Gamma(z)p(z),\quad Q(z)=\Gamma^{-1}(z)q(z) \tag{5.1.19}
$$

where the matrix Γ solves the $\overline{\partial}$-problem 5.1.1. To this end, let us define the matrix $\widetilde{\Gamma}(z)$

$$\widetilde{\Gamma}(z) := \mathbf{1} - \iint_{\mathscr{D}} \frac{P(\zeta)\, q^T(\zeta)}{\zeta - z} \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{2i}, \quad z \in \mathbb{C}. \tag{5.1.20}$$

From this definition it follows that

$$\begin{aligned} p^T(z)\widetilde{\Gamma}^T(z) &= p^T(z) - \iint_{\mathscr{D}} \frac{p^T(z) q(\zeta) P^T(\zeta)}{\zeta - z} \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{2i} \\ &= p^T(z) + \mathcal{K}[P^T](z) \\ &= p^T(z) + P^T(z) - (\mathrm{Id} - \mathcal{K})[P^T](z) \\ &= P^T(z) \end{aligned} \tag{5.1.21}$$

which implies

$$P(z) = \widetilde{\Gamma}(z) p(z). \tag{5.1.22}$$

We now substitute (5.1.22) in the definition (5.1.20):

$$\widetilde{\Gamma}(z) = \mathbf{1} - \iint_{\mathscr{D}} \frac{\widetilde{\Gamma}(\zeta) p(\zeta) q^T(\zeta)}{\zeta - z} \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{2i}. \tag{5.1.23}$$

Then, following the general Cauchy formula (5.1.14), we find that the matrix $\tilde{\Gamma}(z)$ satisfies

$$\partial_{\bar{z}} \tilde{\Gamma}(z) = \pi \widetilde{\Gamma}(z) p(z) q^T(z). \tag{5.1.24}$$

Finally, since the support $\mathscr{D}$ of M is compact, the equation (5.1.20) implies that $\widetilde{\Gamma}$ is analytic outside of $\mathscr{D}$ and tends to $\mathbf{1}$ as $|z| \to \infty$. Thus $\widetilde{\Gamma}$ solves the same $\overline{\partial}$-problem 5.1.1 and since the solution is unique, it must coincide with Γ. ■

5.2 The Hilbert-Carleman determinant

In Section 5.1 we have linked the solution of the $\overline{\partial}$-problem 5.1.1 to the existence of the inverse of $\mathrm{Id} - \mathcal{K}$. From the conditions (5.1.1) we conclude that $\mathcal{K}$ is a Hilbert-Schmidt operator with a well-defined and continuous diagonal in $\mathscr{D} \times \mathscr{D}$: according to [82] this is sufficient to define the Fredholm determinant for the operator $\mathrm{Id} - \mathcal{K}$, as explained in the following remark.

Remark 5.2.1. *In general, for a Hilbert-Schmidt operator $\mathcal{A}$, the Fredholm determinant is not defined but we can still define a regularization of it, called the Hilbert-Carleman determinant*

$$\det{}_2(\mathrm{Id} - \mathcal{A}) := \det\left((\mathrm{Id} - \mathcal{A})\mathrm{e}^{\mathcal{A}}\right). \tag{5.2.1}$$

We observe that $\det\left((\mathrm{Id}-\mathcal{A})\mathrm{e}^{\mathcal{A}}\right)=\det\left(\mathrm{Id}-\mathcal{T}_{\mathcal{A}}\right)$ *with* $\mathcal{T}_{\mathcal{A}}:=\mathrm{Id}-(\mathrm{Id}-\mathcal{A})\mathrm{e}^{\mathcal{A}}$ *and the operator* $\mathcal{T}_{\mathcal{A}}$ *is trace class because it has the representation*

$$\mathcal{T}_{\mathcal{A}}=-\sum_{n=2}^{\infty}\frac{n-1}{n!}\mathcal{A}^n.$$

If $\mathcal{A}$ *is of trace class we can rewrite the Hilbert-Carleman determinant as*

$$\det{}_2(\mathrm{Id}-\mathcal{A})=\det(\mathrm{Id}-\mathcal{A})\mathrm{e}^{\mathrm{Tr}(\mathcal{A})}. \tag{5.2.2}$$

Moreover, as for the Fredholm determinant, the Hilbert-Carleman determinant can be represented by a series

$$\det{}_2(\mathrm{Id}-\mathcal{A})=1+\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\Psi_n(\mathcal{A}) \tag{5.2.3}$$

where $\Psi_n(\mathcal{A})$ *is given by the Plemelj-Smithies formula*

$$\Psi_n(\mathcal{A})=\det\begin{bmatrix}0 & n-1 & 0 & \dots & 0 & 0\\ \mathrm{Tr}(\mathcal{A}^2) & 0 & n-2 & \dots & 0 & 0\\ \mathrm{Tr}(\mathcal{A}^3) & \mathrm{Tr}(\mathcal{A}^2) & 0 & \dots & 0 & 0\\ . & . & . & . & . & .\\ \mathrm{Tr}(\mathcal{A}^n) & \mathrm{Tr}(\mathcal{A}^{n-1}) & \mathrm{Tr}(\mathcal{A}^{n-2}) & \dots & \mathrm{Tr}(\mathcal{A}^2) & 0\end{bmatrix}.$$

It is shown that if $\mathcal{A}$ *is Hilbert-Schmidt then (5.2.3) converges ([55], Chapter 10, Theorem 3.1).*

Let us now assume that $\mathcal{K}$ depends smoothly on parameters $\boldsymbol{t}=(t_1,t_2,\dots,t_j,\dots)$ with $t_j\in\mathbb{C}$, $\forall\, j\geq 1$: we want to relate solutions of the $\overline{\partial}$-problem 5.1.1 with the variational equations for the determinant.

Proposition 5.2.2. *Let us suppose that the matrix* $M(z,\overline{z})$ *in the* $\overline{\partial}$*-Problem 5.1.1, depends smoothly on some parameters* $\boldsymbol{t}$*, while remaining identically nilpotent. Then the solution* $\Gamma(z)$ *of the* $\overline{\partial}$*-problem 5.1.1 is related to the logarithmic derivative of the Hilbert-Carleman determinant of* $\mathrm{Id}-\mathcal{K}$ *as follows:*

$$\delta\log\left[\det{}_2(\mathrm{Id}-\mathcal{K})\right]=-\iint_{\mathscr{D}}\mathrm{Tr}\left(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta M(z)\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}, \tag{5.2.4}$$

where δ *stands for the total differential in the space of parameters* $\boldsymbol{t}$*.*

Proof. Using the Jacobi variational formula

$$\delta\log\left[\det(\mathrm{Id}-\mathcal{K})\right]=-\mathrm{Tr}\left((\mathrm{Id}+\mathcal{R})\circ\delta\mathcal{K}\right), \tag{5.2.5}$$

we can rewrite the LHS of (5.2.4) as

$$\delta\log\left[\det{}_2(\mathrm{Id}-\mathcal{K})\right]=\delta\log\left[\det\left((\mathrm{Id}-\mathcal{K})\mathrm{e}^{\mathcal{K}}\right)\right]=-\mathrm{Tr}\left(\mathcal{R}\circ\delta\mathcal{K}\right), \tag{5.2.6}$$

where $\mathcal{R}\circ\delta\mathcal{K}$ is a trace class operator, since it is the composition of two Hilbert-Schmidt operators. Here Tr denotes the trace on the Hilbert space $L^2(\mathscr{D}, \mathrm{d}^2 z)\otimes\mathbb{C}^n$. The composition of the two operators $\mathcal{R}\circ\delta\mathcal{K}$ produces the kernel

$$
\begin{aligned}
(\hat{\mathcal{R}}\circ\delta\hat{\mathcal{K}})(z,w) &= \iint_{\mathscr{D}} \frac{p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)\delta(p^T(x)q(w))}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i} \\
&= \iint_{\mathscr{D}} \frac{p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)p^T(\zeta)\delta q(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}+ \qquad (5.2.7)\\
&+ \iint_{\mathscr{D}} \frac{p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)\delta p^T(\zeta)q(w)}{(z-\zeta)(\zeta-w)}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i} \qquad (5.2.8)
\end{aligned}
$$

where we have omitted explicit notation of the dependence on $\boldsymbol{t}$ of the functions f, g, F, G, Γ.

We focus on the term in (5.2.7). Using the identity $\frac{1}{(z-\zeta)(\zeta-w)} = \frac{1}{z-w}\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)$, we obtain

$$
(5.2.7) = \frac{p^T(z)\Gamma^T(z)}{z-w}\left(\iint_{\mathscr{D}}(\Gamma^T(\zeta))^{-1}q(\zeta)p^T(\zeta)\left(\frac{1}{z-\zeta}+\frac{1}{\zeta-w}\right)\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\right)\delta q(w) \qquad (5.2.9)
$$

In order to compute the trace we need to compute the kernel (5.2.9) along the diagonal $z=w$ and hence we consider $\lim_{w\to z}(5.2.9)$. Observe that $(\Gamma^T(\zeta))^{-1}q(\zeta)p^T(\zeta) = -\frac{1}{\pi}\partial_{\bar{\zeta}}(\Gamma^T(\zeta))^{-1}$, and hence we can apply the formula (5.1.14) to eliminate the integral and rewrite (5.2.9) as follows

$$
(5.2.9) = -\frac{p^T(z)\Gamma^T(z)\Big((\Gamma^T(z))^{-1}-\mathbf{1}\Big)\delta q(w)}{z-w} + \frac{p^T(z)\Gamma^T(z)\Big((\Gamma^T(w))^{-1}-\mathbf{1}\Big)\delta q(w)}{z-w} \qquad (5.2.10)
$$

$$
= \frac{p^T(z)\Big(\Gamma^T(z)(\Gamma^T(w))^{-1}-\mathbf{1}\Big)\delta q(w)}{z-w}. \qquad (5.2.11)
$$

We can now easily compute the expansion of (5.2.11) along the diagonal $w\to z$ by Taylor's formula, keeping in mind that Γ is not a holomorphic function inside $\mathscr{D}$:

$$
(5.2.11) = p^T(z)\partial_z\Gamma^T(z)(\Gamma^T(z))^{-1}\delta q(z) + \frac{\bar{z}-\bar{w}}{z-w}\overbrace{p^T(z)q(z)}^{\equiv 0}p^T(z)\delta q(w) + \mathcal{O}(|z-w|) \qquad (5.2.12)
$$

$$
= p^T(z)\partial_z\Gamma^T(z)(\Gamma^T(z))^{-1}\delta q(z) + \mathcal{O}(|z-w|). \qquad (5.2.13)
$$

Using the above expression we conclude that the trace in $L^2(\mathscr{D}, \mathrm{d}^2 z)\otimes\mathbb{C}^n$ of (5.2.7) is

$$
\mathrm{Tr}((5.2.7)) = \iint_{\mathscr{D}} \mathrm{Tr}\big(p^T(z)\partial_z\Gamma(z)^T(\Gamma^T(z))^{-1}\delta q(z)\big)\frac{\mathrm{d}\bar{z}\wedge\mathrm{d}z}{2i}. \qquad (5.2.14)
$$

Using the cyclicality of the trace and its invariance under transposition of the arguments, we reorder the terms (5.2.14) to the form

$$\mathrm{Tr}((5.2.7)) = \iint_{\mathscr{D}} \mathrm{Tr}\big(\Gamma^{-1}(z)\partial_z\Gamma(z)p(z)\delta q^T(z)\big)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i}. \tag{5.2.15}$$

We now consider the term (5.2.8). Taking its trace yields:

$$\begin{aligned}
&\mathrm{Tr}((5.2.8)) = \\
&= -\iint_{\mathscr{D}}\iint_{\mathscr{D}} \frac{\mathrm{Tr}\big(p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)\delta p^T(\zeta)q(z)\big)}{(z-\zeta)^2}\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i} \\
&= -\iint_{\mathscr{D}}\iint_{\mathscr{D}} \frac{\mathrm{Tr}\big(q(z)p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)\delta p^T(\zeta)\big)}{(z-\zeta)^2}\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i}
\end{aligned} \tag{5.2.16}$$

We observe that the integrand is in L^2_{loc} because the numerator vanishes to order $\mathcal{O}(|z-\zeta|)$ along the diagonal

$$\mathrm{Tr}\Big(q(z)p^T(z)\Gamma^T(z)(\Gamma^T(\zeta))^{-1}q(\zeta)\delta p^T(\zeta)\Big) = \mathrm{Tr}\Big(q(\zeta)\overbrace{p^T(\zeta)q(\zeta)}^{=0}\delta p^T(\zeta)\Big) + \mathcal{O}(|z-\zeta|), \tag{5.2.17}$$

and hence the integrand is $\mathcal{O}(|z-\zeta|^{-1})$ which is locally integrable with respect to the area measure. We can now relate this integral to $\partial_z\Gamma$ as follows. Using the formula (5.1.14) and the $\overline{\partial}$-problem 5.1.1 we can rewrite $\Gamma^T(\zeta)$ as

$$\Gamma^T(\zeta) = \mathbf{1} - \iint_{\mathscr{D}} \frac{\partial_{\bar z}(\Gamma^T(z))}{z-\zeta}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i} = \mathbf{1} - \iint_{\mathscr{D}} \frac{M^T(z)\Gamma^T(z)}{z-\zeta}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i} \tag{5.2.18}$$

Taking the holomorphic derivative with respect to ζ we get

$$\partial_\zeta\Gamma^T(\zeta) = -\iint_{\mathscr{D}} \frac{q(z)p^T(z)\Gamma^T(z)}{(z-\zeta)^2}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i}$$

Plugging the result into (5.2.16) we obtain

$$\begin{aligned}
(5.2.16) &= -\iint_{\mathscr{D}} \mathrm{Tr}\left((\Gamma^T(\zeta))^{-1}q(\zeta)\delta p(\zeta)\left(\iint_{\mathscr{D}} \frac{q(z)p^T(z)\Gamma^T(z)}{(z-\zeta)^2}\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2i}\right)\right)\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i} \\
&= \iint_{\mathscr{D}} \mathrm{Tr}\big((\Gamma^T(\zeta))^{-1}q(\zeta)\delta p^T(\zeta)\partial_\zeta(\Gamma^T(\zeta))\big)\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i} \\
&= \iint_{\mathscr{D}} \mathrm{Tr}\big(q(\zeta)\delta p^T(\zeta)\partial_\zeta\Gamma^T(\zeta)(\Gamma^{-1}(\zeta))^T\big)\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i} \\
&= \iint_{\mathscr{D}} \mathrm{Tr}\big(\Gamma^{-1}(\zeta)\partial_\zeta\Gamma(\zeta)\delta p(\zeta)q^T(\zeta)\big)\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i},
\end{aligned} \tag{5.2.19}$$

so that

$$\mathrm{Tr}((5.2.8)) = \iint_{\mathscr{D}} \mathrm{Tr}\big(\Gamma^{-1}(\zeta)\partial_\zeta\Gamma(\zeta)\delta p(\zeta) q^T(\zeta)\big)\frac{\mathrm{d}\overline{\zeta}\wedge \mathrm{d}\zeta}{2i}, \tag{5.2.20}$$

Combining (5.2.15) and (5.2.20) we have obtained that

$$\begin{aligned}
&-\mathrm{Tr}(\mathcal{R}\circ\delta\mathcal{K}) = -\mathrm{Tr}((5.2.7)+(5.2.8)) = \\
&= -\iint_{\mathscr{D}} \mathrm{Tr}(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta(p(z)q^T(z)))\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2i} = \\
&= -\iint_{\mathscr{D}} \mathrm{Tr}(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta M(z))\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{5.2.21}$$

This concludes the proof of Proposition 5.2.2. ■

5.2.1 Malgrange one form and τ-function

From Proposition 5.2.2 we define the following one form on the space of deformations, which we call *Malgrange one form* following the terminology in [9]:

$$\omega := -\iint_{\mathscr{D}} \mathrm{Tr}\Big(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta M(z)\Big)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}, \tag{5.2.22}$$

where $\Gamma(z)$ is the solution of the $\overline{\partial}$-problem 5.1.1 and $M(z)$ is defined in (5.1.8). For the operator $\mathcal{K}$ defined in (5.1.4), the Proposition 5.2.2 implies that

$$\omega = \delta \log \det\nolimits_2(\mathrm{Id} - \mathcal{K}), \tag{5.2.23}$$

and hence ω is an exact (and hence closed) one form in the space of deformation parameters the operator $\mathcal{K}$ may depend upon. The form ω can be shown to be closed under weaker assumptions on the matrix M than the ones that appears in the $\overline{\partial}$-problem 5.1.1 as the following theorem shows.

Theorem 5.2.3. *Suppose that the $r\times r$ matrix $M = M(z,\overline{z};\boldsymbol{t})$ is smooth and compactly supported in $\mathscr{D}$ (uniformly with respect to the parameters $\boldsymbol{t}$), depends smoothly on $\boldsymbol{t}$ and the matrix trace $\mathrm{Tr}(M)\equiv 0$. Let $\Gamma(z,\overline{z};\boldsymbol{t})$ be the solution of the $\overline{\partial}$-problem 0.0.27. Then the exterior differential of the one-form ω defined in (5.2.22) vanishes:*

$$\delta\omega = 0. \tag{5.2.24}$$

Proof. From the $\overline{\partial}$-problem we obtain

$$\delta(\partial_{\overline{z}}\Gamma) = \Gamma\delta M + \delta\Gamma M \quad \Rightarrow \quad \delta\Gamma(z) = \iint_{\mathscr{D}} \frac{\Gamma(w)\delta M(w)\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\Gamma(z) \tag{5.2.25}$$

Using (5.2.25) we can compute

$$\begin{aligned}
\delta\omega &= -\iint_{\mathscr{D}} \mathrm{Tr}\Big(\delta(\Gamma^{-1}\partial_z\Gamma\wedge\delta M)\Big)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i} = \\
&= \iint_{\mathscr{D}} \mathrm{Tr}\Big(\Gamma^{-1}\delta\Gamma\Gamma^{-1}\partial_z\Gamma\wedge\delta M\Big)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i} - \iint_{\mathscr{D}} \mathrm{Tr}\Big(\Gamma^{-1}\delta\partial_z\Gamma\wedge\delta M\Big)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}
\end{aligned} \tag{5.2.26}$$

From (5.2.25) we deduce

$$\delta\partial_z\Gamma(z) = -\iint_{\mathscr{D}} \frac{\Gamma(w)\delta M(w)\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge\mathrm{d}w}{2\pi i}\Gamma(z) + \delta\Gamma(z)\Gamma(z)^{-1}\partial_z\Gamma(z). \tag{5.2.27}$$

Substituting (5.2.27) in the equation (5.2.26) we obtain:

$$\delta\omega = \iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)\left(\iint_{\mathscr{D}} \frac{\Gamma(w)\delta M(w)\Gamma^{-1}(w)}{(w-z)^2}\frac{\mathrm{d}\overline{w}\wedge dw}{2\pi i}\right)\Gamma(z)\wedge\delta M(z)\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}. \tag{5.2.28}$$

The crux of the proof is now the correct evaluation of the iterated integral:

$$\delta\omega = \iint_{\mathscr{D}}\frac{\mathrm{d}^2 z}{\pi}\iint_{\mathscr{D}}\frac{\mathrm{d}^2 w}{\pi}\frac{F(z,w)}{(z-w)^2},$$
$$F(z,w) := \mathrm{Tr}\Big(\Gamma(w)\delta M(w)\Gamma^{-1}(w)\wedge\Gamma(z)\delta M(z)\Gamma^{-1}(z)\Big) \tag{5.2.29}$$

By applying Fubini's theorem, since the integrand is anti-symmetric in the exchange of the variables $z \leftrightarrow w$, we quickly conclude that the integral is zero. However the integrand is singular along the diagonal $\Delta := \{z = w\} \subset \mathscr{D}\times\mathscr{D}$ and we need to make sure that the integrand is absolutely summable.

Recalling that $F(z,w) = -F(w,z)$, so that $F(z,z) \equiv 0$, we now compute the Taylor expansion of $F(z,w)$ with respect to w near z;

$$F(z,w) = 0 + \partial_w F(z,z)(w-z) + \partial_{\overline{w}}F(z,z)(\overline{w}-\overline{z}) + \mathcal{O}(|z-w|^2). \tag{5.2.30}$$

Thus $\frac{|F(z,w)|}{|z-w|^2} = \mathcal{O}(|z-w|^{-1})$ which is integrable with respect to the area measure. Hence application of Fubini's theorem is justified. ■

From this theorem, we can define a τ-function associated to the the $\overline{\partial}$-problem 0.0.27 by

$$\tau(\boldsymbol{t}) = \exp\left(\int\omega\right). \tag{5.2.31}$$

In general the above τ-function is defined only up to scalar multiplication and hence should be rather thought of as a section of an appropriate line bundle over the space of deformation parameters, depending on the context. However, for M in the form specified in (5.1.8) we know from Proposition 5.2.2 that we can *identify* the τ-function with the regularized Hilbert-Carleman determinant:

$$\tau(\boldsymbol{t}) = \exp\left(\int\omega\right) = \det{}_2(\mathrm{Id} - \mathcal{K}). \tag{5.2.32}$$

In the next section, by choosing a specific dependence on the parameters $\boldsymbol{t}$ in the more general setting of M as in Thm. 5.2.3 we are going to show that $\tau(\boldsymbol{t})$ is a KP τ-function in the sense that it satisfies Hirota bilinear relations [60].

5.3 $\tau(\boldsymbol{t})$ as a KP τ-function

In this section we consider a specific type of dependence of M on the "times": let $M(z,\boldsymbol{t})$ be a 2×2 matrix that depends on $\boldsymbol{t}$ in the following form

$$M(z,\boldsymbol{t}) = \mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3} M_0(z) \mathrm{e}^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}\ , \tag{5.3.1}$$

with

$$\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j \tag{5.3.2}$$

and $M_0(z,\bar{z})$ a traceless matrix compactly supported on $\mathscr{D}$. A τ-function of the Kadomtsev-Petviashvili hierarchy, $\tau(\boldsymbol{t})$, can be characterized as a function of (formally) an infinite number of variable which satisfies the Hirota Bilinear relation

$$\mathrm{Res}_{z=\infty}(\tau(\boldsymbol{t}-[z^{-1}])\tau(\boldsymbol{s}+[z^{-1}])\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})} = 0 \tag{5.3.3}$$

where $\boldsymbol{t}\pm[z^{-1}]$ is the *Miwa Shift*, defined as:

$$\boldsymbol{t}\pm[z^{-1}] := \left(t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, \dots, t_j \pm \frac{1}{jz^j}, \dots\right). \tag{5.3.4}$$

The residue in (5.3.3) is meant in the formal sense, namely by considering the coefficient of z^{-1} in the expansion at infinity and can be thought of as the limit of $\oint_{|z|=R}$ as $R\to+\infty$. If the functions of z intervening in (5.3.3) can be written as analytic functions in a deleted neighbourhood of ∞, then the residue is a genuine integral; this is the case of interest below.

As described in [73], the equation (5.3.3) implies that the tau function satisfy an equation of the Hirota type

$$P(\mathcal{D}_1,\mathcal{D}_2,\dots)\,\tau^2 = 0 \tag{5.3.5}$$

where $\mathcal{D}_j$ is the Hirota derivative respect to t_j, defined as

$$\mathcal{D}_j\, p(\boldsymbol{t})q(\boldsymbol{t}) := (\partial_{t_j} - \partial_{t'_j})(p(\boldsymbol{t})q(\boldsymbol{t}'))|_{\boldsymbol{t}=\boldsymbol{t}'}, \tag{5.3.6}$$

and $P(\mathcal{D}_1,\mathcal{D}_2,\dots)$ is a polynomial in $(\mathcal{D}_1,\mathcal{D}_2,\dots)$. In particular, if we consider the first three times t_1,t_2 and t_3, and $t_k=0$ for $k\geq 3$ the equation (5.3.3) is equivalent to the KP equation in Hirota's form

$$(3\mathcal{D}_2^2 - 4\mathcal{D}_1\mathcal{D}_3 + \mathcal{D}_1^4)\,\tau^2 = 0. \tag{5.3.7}$$

Putting

$$\partial_{t_1}^2 \log\tau(t_1,t_2,t_3) = \frac{1}{2}u(t_1,t_2,t_3)$$

one obtains the celebrated KP equation

$$3\partial_{t_2}^2 u = \partial_{t_1}(4\partial_{t_3}u - \partial_{t_1}^3 u - 6u\partial_{t_1}u) \tag{5.3.8}$$

The rest of this section is devoted to the verification of the Hirota bilinear relation (5.3.3) for the KP tau function.

5.3.1 Hirota bilinear relation for the KP hierarchy

The main result is the following.

Theorem 5.3.1. *Let $\Gamma(z,\boldsymbol{t})$ be the solution of the $\overline{\partial}$-problem*

$$\partial_{\bar{z}}\Gamma(z,\boldsymbol{t}) = \Gamma(z,\boldsymbol{t})\mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}M_0(z)\mathrm{e}^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}, \qquad \Gamma(z,\boldsymbol{t}) \underset{z\to\infty}{\to} \mathbf{1}$$

with the traceless matrix $M_0(z)$ compactly supported on a bounded domain $\mathscr{D}$ of the complex plane and the function ξ given by the formal sum $\xi(z,\boldsymbol{t}) = \sum_{j=1}^{+\infty} z^j t_j$. Then the function

$$\tau(\boldsymbol{t}) = \exp\left(\int \omega\right), \tag{5.3.9}$$

with ω defined in (5.2.22) is a KP τ-function; i.e. it satisfies the Hirota Bilinear relation (5.3.3).

Remark 5.3.2. *In this setting the KP τ-function is in general complex–valued. Under appropriate additional symmetry constraints for the matrix M_0 and the domain $\mathscr{D}$ we can obtain a real–valued τ-function.*

We prove the theorem in several steps. We first analyse the effect of the Miwa shifts on the τ-function. For this purpose we need to determine how the Miwa shift acts on the matrices $\Gamma(z,\bar{z},t)$ and $M(z,\bar{z},\boldsymbol{t})$. We consider $M(z,\bar{z},\boldsymbol{t}\pm[\zeta^{-1}])$ first.

$$M(z,\boldsymbol{t}\pm[\zeta^{-1}]) = \mathrm{e}^{\xi(z,\boldsymbol{t}\pm[\zeta^{-1}])\sigma_3}M_0(z)\mathrm{e}^{-\xi(z,\boldsymbol{t}\pm[\zeta^{-1}])\sigma_3}$$

from the definition of $\xi(z,t)$ (5.3.2)

$$\xi(z,\boldsymbol{t}\pm[\zeta^{-1}]) = \sum_{j=1}^{+\infty} z^j\left(t_j \pm \frac{1}{j\zeta^j}\right) = \sum_{j=1}^{+\infty} z^j t_j \pm \sum_{j=1}^{+\infty}\frac{z^j}{j\zeta^j} = \xi(z,\boldsymbol{t}) \mp \ln\left(1-\frac{z}{\zeta}\right)$$

and we have that

$$M(z,\boldsymbol{t}\pm[\zeta^{-1}]) = \left(1-\frac{z}{\zeta}\right)^{\mp\frac{\sigma_3}{2}} M(z,\boldsymbol{t})\left(1-\frac{z}{\zeta}\right)^{\pm\frac{\sigma_3}{2}}. \tag{5.3.10}$$

For the matrices $\Gamma(z,\overline{z},\boldsymbol{t}\pm[\zeta^{-1}])$ we need to consider the two case separately. Let us start with the negative shift $\Gamma(z,\overline{z},\boldsymbol{t}-[\zeta^{-1}])$.

$$\begin{aligned}\partial_{\bar{z}}\Gamma(z,\boldsymbol{t}-[\zeta^{-1}]) &= \Gamma(z,\boldsymbol{t}-[\zeta^{-1}])M(z,\boldsymbol{t}-[\zeta^{-1}])\\ &= \Gamma(z,\boldsymbol{t}-[\zeta^{-1}])\left(1-\frac{z}{\zeta}\right)^{+\frac{\sigma_3}{2}} M(z,\boldsymbol{t})\left(1-\frac{z}{\zeta}\right)^{-\frac{\sigma_3}{2}}\\ &= \Gamma(z,\boldsymbol{t}-[\zeta^{-1}])D(z,\zeta)M(z,\boldsymbol{t})D^{-1}(z,\zeta)\end{aligned} \tag{5.3.11}$$

where

$$D(z,\zeta) = \begin{bmatrix} 1-\frac{z}{\zeta} & 0\\ 0 & 1\end{bmatrix}.$$

From (5.3.11), we notice that the matrix $\Gamma(z, t-[\zeta^{-1}])D(z,\zeta)$ satisfies the $\overline{\partial}$-problem 5.1.1, i.e. there exists a connection matrix $C(z)$ such that

$$\Gamma(z,\boldsymbol{t}-[\zeta^{-1}]) = C(z)\Gamma(z,\boldsymbol{t})D(z,\zeta)^{-1}, \tag{5.3.12}$$

where obviously $C(z)$ depends also on ζ and $\boldsymbol{t}$.

The matrix $C(z)$ is determined by the conditions that both $\Gamma(z,\boldsymbol{t})$ and $\Gamma(z,\overline{z},\boldsymbol{t}-[\zeta^{-1}])$ must tend to $\mathbf{1}$ for $z\to\infty$ and are regular at $z=\zeta$

$$\begin{gathered}\lim_{z\to\infty}\left(1-\frac{z}{\zeta}\right)^{-1}C(z)\begin{bmatrix}\Gamma_{11}(z,\boldsymbol{t})\\ \Gamma_{12}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\quad \lim_{z\to\infty}C(z)\begin{bmatrix}\Gamma_{21}(z,\boldsymbol{t})\\ \Gamma_{22}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\\ \lim_{z\to\zeta}\left(1-\frac{z}{\zeta}\right)^{-1}C(z)\begin{bmatrix}\Gamma_{11}(z,\boldsymbol{t})\\ \Gamma_{12}(z,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}\Gamma_{11}(\zeta,\boldsymbol{t})\\ 0\end{bmatrix}\end{gathered} \tag{5.3.13}$$

Solving the system (5.3.13), we obtain that the matrix $C(z)$ has the following form

$$C(z)=\begin{bmatrix}\left(1-\frac{z}{\zeta}\right)+\frac{\partial_z\Gamma_{12}(\infty)\Gamma_{21}(\zeta)}{\zeta\Gamma_{11}(\zeta)} & -\frac{\partial_z\Gamma_{12}(\infty)}{\zeta}\\ -\frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)} & 1\end{bmatrix}. \tag{5.3.14}$$

Following the same ideas, we can find a similar formula for $\Gamma(z,\overline{z},t+[\zeta^{-1}])$

$$\Gamma(z,\boldsymbol{t}+[\zeta^{-1}]) = \tilde{C}(z)\Gamma(z,\boldsymbol{t})\tilde{D}(z,\zeta)^{-1} \tag{5.3.15}$$

with

$$\tilde{D}(z,\zeta)=\begin{bmatrix}1 & 0\\ 0 & 1-\frac{z}{\zeta}\end{bmatrix}.$$

Also in this case, we have three conditions similar to (5.3.13) :

$$\begin{gathered}\lim_{z\to\infty}\tilde{C}(z)\begin{bmatrix}\Gamma_{11}(x,\boldsymbol{t})\\ \Gamma_{12}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\quad \lim_{z\to\infty}\left(1-\frac{z}{\zeta}\right)^{-1}\tilde{R}(z,\zeta)\begin{bmatrix}\Gamma_{21}(x,\boldsymbol{t})\\ \Gamma_{22}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\\ \lim_{z\to x}\left(1-\frac{z}{\zeta}\right)^{-1}\tilde{C}(z)\begin{bmatrix}\Gamma_{21}(x,\boldsymbol{t})\\ \Gamma_{22}(x,\boldsymbol{t})\end{bmatrix}=\begin{bmatrix}0\\ \Gamma_{22}(\zeta,\boldsymbol{t})\end{bmatrix}\end{gathered} \tag{5.3.16}$$

and we find out that $\tilde{C}(z)$ has the following form:

$$\tilde{C}(z)=\begin{bmatrix}1 & -\frac{\Gamma_{12}(\zeta)}{\Gamma_{22}(\zeta)}\\ -\frac{\partial_z\Gamma_{21}(\infty)}{\zeta} & \left(1-\frac{z}{\zeta}\right)+\frac{\partial_z\Gamma_{21}(\infty)\Gamma_{12}(\zeta)}{\zeta\Gamma_{22}(\zeta)}\end{bmatrix}. \tag{5.3.17}$$

We need to show how the Miwa shift acts on the Malgrange one form. We define $\delta_{[\zeta]}$ the differential deformed including the external parameter ζ

$$\delta_{[\zeta]} := \sum_{j=1}^{+\infty}\mathrm{d}t_j\partial_{t_j}+\mathrm{d}\zeta\partial_\zeta = \delta+\delta_\zeta. \tag{5.3.18}$$

Lemma 5.3.3. *When $\zeta \notin \mathscr{D}$ the Miwa shift (5.3.4) acts on the Malgrange one form (5.2.22) in the following way:*

$$\omega(\boldsymbol{t} \pm [\zeta^{-1}]) = \omega(\boldsymbol{t}) + \delta_{[\zeta]} \ln\left((\Gamma^{\mp 1}(\zeta))_{11}\right) \mp \delta_{[\zeta]}\gamma(\zeta)\,, \tag{5.3.19}$$

where $\Gamma(z)$ solves the $\overline{\partial}$-problem 5.1.1 and $\gamma(\zeta)$ is a $\boldsymbol{t}$ independent function defined as

$$\gamma(\zeta) := \iint_{\mathscr{D}} \log\left(\frac{\zeta}{\zeta - z}\right)(\partial_z M_0(z))_{11} \frac{\mathrm{d}\overline{z} \wedge \mathrm{d}z}{2\pi i}\,,\ \zeta \in \mathbb{C}\backslash\mathscr{D}, \tag{5.3.20}$$

that is is analytic (for $\zeta \notin \mathscr{D}$) and goes to zero as $\zeta \to \infty$.

Observe that since $\mathrm{Tr}M_0 = 0$ we may express the formula in terms of the $(2,2)$ entry instead. The proof of this lemma is presented in the Appendix A. Now we can state the following proposition:

Proposition 5.3.4. *For $\zeta \notin \mathscr{D}$ the following relations holds:*

$$\frac{\tau(\boldsymbol{t} - [\zeta^{-1}])}{\tau(\boldsymbol{t})} = \Gamma_{11}(\zeta, \boldsymbol{t})\mathrm{e}^{\gamma(\zeta)} \qquad \frac{\tau(\boldsymbol{t} + [\zeta^{-1}])}{\tau(\boldsymbol{t})} = \Gamma_{11}^{-1}(\zeta, \boldsymbol{t})\mathrm{e}^{-\gamma(\zeta)}\,, \tag{5.3.21}$$

where $\tau(t)$ is defined in (5.2.31), $\Gamma(z)$ solves the $\overline{\partial}$-problem 5.1.1 and $\gamma(\zeta)$ is defined in (5.3.20)

Proof. From Lemma 5.3.3 and the equation (5.2.31), we rewrite (5.3.19) as

$$\delta_{[\zeta]} \ln \tau(\boldsymbol{t} \pm [\zeta^{-1}]) = \delta_{[\zeta]} \ln \tau(\boldsymbol{t}) + \delta_{[\zeta]} \ln((\Gamma^{\mp 1}(\zeta))_{11}) \mp \delta_{[\zeta]}\gamma(\zeta) \tag{5.3.22}$$

an then, from the properties of the logarithm the statement (5.3.21) is proved. ■

Remark 5.3.5. *The exponential term $\mathrm{e}^{\gamma(\zeta)}$ could be absorbed by a gauge transformation in the formalism of the infinite dimensional Grassmannian manifold of Segal-Wilson ([59], Chapter 4). Such gauge transformations have no effect on the Hirota bilinear relation (5.3.3).*

Let us now define the matrix $H(z)$ as

$$H(z) := H(z; \boldsymbol{t}, \boldsymbol{s}) := \Gamma(z, \boldsymbol{t})\mathrm{e}^{(\xi(z,\boldsymbol{t}) - \xi(z,\boldsymbol{s}))E_{11}}\Gamma^{-1}(z, \boldsymbol{s}) \tag{5.3.23}$$

where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma(z, \overline{z}, \boldsymbol{t})$ solves the $\overline{\partial}$-problem 5.1.1 and $\boldsymbol{s} = (s_1, s_2, \ldots, s_j, \ldots)$ denotes another set of values for the deformation parameters.

Lemma 5.3.6. *The matrix $H(z)$ defined in (5.3.23) is analytic for all $z \in \mathbb{C}$.*

Proof. For $z \notin \mathscr{D}$ the statement is trivial, so we consider the case of $z \in \mathscr{D}$.

We apply the operator $\partial_{\bar{z}}$ to the matrix (5.3.23)

$$
\begin{aligned}
\partial_{\bar{z}} H(z) &= \partial_{\bar{z}}\Gamma(z,\boldsymbol{t})\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}}\Gamma^{-1}(z,\boldsymbol{s}) + \Gamma(z,\boldsymbol{t})\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}}\partial_{\bar{z}}\Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t})M(z,\boldsymbol{t})\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}}\Gamma^{-1}(z,\boldsymbol{s}) + \\
&\quad - \Gamma(z,\boldsymbol{t})\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}}M(z,\boldsymbol{s})\Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t})\Big(\mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}M_0(z)\mathrm{e}^{-\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}} + \\
&\quad -\mathrm{e}^{(\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s}))E_{11}}\mathrm{e}^{\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3}M_0(z)\mathrm{e}^{-\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3}\Big)\Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t})\Big(\mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\sigma_3}\mathrm{e}^{\frac{\xi(z,\boldsymbol{t})}{2}\mathbb{I}}M_0(z)\mathrm{e}^{-\xi(z,\boldsymbol{s})E_{11}} + \\
&\quad -\mathrm{e}^{\xi(z,\boldsymbol{t})E_{11}}M_0(z)\mathrm{e}^{\frac{\xi(z,\boldsymbol{s})}{2}\mathbb{I}}\mathrm{e}^{-\frac{\xi(z,\boldsymbol{s})}{2}\sigma_3}\Big)\Gamma^{-1}(z,\boldsymbol{s}) \\
&= \Gamma(z,\boldsymbol{t})\Big(\mathrm{e}^{\xi(z,\boldsymbol{t})E_{11}}M_0(z)\mathrm{e}^{-\xi(z,\boldsymbol{s})E_{11}} - \mathrm{e}^{\xi(z,\boldsymbol{t})E_{11}}M_0(z)\mathrm{e}^{-\xi(z,\boldsymbol{s})E_{11}}\Big)\Gamma^{-1}(z,\boldsymbol{s}) \\
&= 0
\end{aligned}
$$

and this proves the statement. ■

We are now ready to prove the main result of the section, namely Theorem 5.3.1.

Proof of Theorem 5.3.1. Let us compute the residue

$$
\begin{aligned}
&\mathrm{Res}_{z=\infty}(\tau(\boldsymbol{t}-[z^{-1}])\tau(\boldsymbol{s}+[z^{-1}])\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})}) \\
&= \tau(\boldsymbol{t})\tau(\boldsymbol{s})\mathrm{Res}_{z=\infty}(\frac{\tau(\boldsymbol{t}-[z^{-1}])}{\tau(\boldsymbol{t})}\frac{\tau(\boldsymbol{s}+[z^{-1}])}{\tau(\boldsymbol{s})}\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})}) \\
&= \tau(\boldsymbol{t})\tau(\boldsymbol{s})\mathrm{Res}_{z=\infty}(\Gamma_{11}(z,\boldsymbol{t})(\Gamma^{-1}(z,\boldsymbol{s}))_{11}\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})}) \\
&= \tau(\boldsymbol{t})\tau(\boldsymbol{s})\lim_{R\to\infty}\oint_{|z|=R}\Gamma_{11}(z,\boldsymbol{t})\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})}(\Gamma^{-1}(z,\boldsymbol{s}))_{11}\frac{\mathrm{d}z}{2\pi i}
\end{aligned} \tag{5.3.24}
$$

Consider the first diagonal element of the matrix $H(z)$. From the analyticity of $H(z)$ proved in Lemma 5.3.6 we get

$$
\begin{aligned}
0 = \left(\oint_{|z|=R}H(z)\frac{\mathrm{d}z}{2\pi i}\right)_{11} &= \oint_{|z|=R}\Gamma_{11}(z,\boldsymbol{t})\mathrm{e}^{\xi(z,\boldsymbol{t})-\xi(z,\boldsymbol{s})}(\Gamma^{-1}(z,\boldsymbol{s}))_{11}\frac{\mathrm{d}z}{2\pi i} + \\
&\quad - \oint_{|z|=R}\Gamma_{12}(z,\boldsymbol{t})\Gamma_{21}(z,\boldsymbol{s})\frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{5.3.25}
$$

So, we can rewrite (5.3.24) as

$$
(5.3.24) = \tau(\boldsymbol{t})\tau(\boldsymbol{s})\lim_{R\to\infty}\oint_{|z|=R}\Gamma_{12}(z,\boldsymbol{t})\Gamma_{21}(z,\boldsymbol{s})\frac{\mathrm{d}z}{2\pi i} \tag{5.3.26}
$$

Since both $\Gamma_{12}(z,\boldsymbol{t})$ and $\Gamma_{21}(z,\boldsymbol{s})$ are analytic for $|z|$ sufficiently large (given that $\mathscr{D}$ is compact) and

$$
\Gamma(z,\boldsymbol{t}) \sim \mathbb{I} + \mathcal{O}(z^{-1}) \quad \text{for } z\to\infty,
$$

it follows that (5.3.26) is zero because the integrand is $\mathcal{O}(z^{-2})$, and the statement is proved. ■

5.3.2 The focusing Nonlinear Schrödinger equation

In this subsection we make a specific choice of the matrix M_0 of the form

$$M_0(z) = \begin{bmatrix} 0 & \beta(z)\chi_{\mathscr{D}} \\ -\overline{\beta(\overline{z})}\chi_{\overline{\mathscr{D}}} & 0 \end{bmatrix},$$

where $\beta(z) = \beta(z, \overline{z})$ is a smooth function on $\mathscr{D} \subset \mathbb{C}_+$ and $\chi_{\mathscr{D}}$ ($\chi_{\overline{\mathscr{D}}}$) is the characteristic function of $\mathscr{D}$ ($\overline{\mathscr{D}}$). We observe that M_0 satisfies the Schwarz symmetry

$$\overline{M_0(\overline{z})} = \sigma_2 M_0(z) \sigma_2, \qquad \text{where } \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{5.3.27}$$

Let us consider the $\overline{\partial}$-problem

$$\begin{aligned} &\partial_{\overline{z}} \Gamma(z, \boldsymbol{t}) = \Gamma(z, \boldsymbol{t}) \mathrm{e}^{-i\xi(z,\boldsymbol{t})\sigma_3} M_0(z) \mathrm{e}^{i\xi(z,\boldsymbol{t})\sigma_3} \qquad \text{for } z \in \mathscr{D} \cup \overline{\mathscr{D}} \\ &\Gamma(z, \boldsymbol{t}) \underset{z\to\infty}{\to} \mathbf{1} \end{aligned} \tag{5.3.28}$$

with $\xi(z, \boldsymbol{t})$ as in (5.3.2). Here we have re-defined $t_j \to -2it_j$ to respect the customary normalization of times in the KP-hierarchy.

Theorem 5.3.7. *Let $\Gamma(z, \boldsymbol{t})$ be the solution of the $\overline{\partial}$-problem (B.0.1) and let*

$$\psi(\boldsymbol{t}) := 2i \lim_{z\to\infty} z(\Gamma(z, \boldsymbol{t}) - \mathbf{1})_{12}\,.$$

Then the function $\psi = \psi(\boldsymbol{t})$ satisfies the nonlinear Schrödinger hierarchy [45, 71] written in the recursive form

$$i\partial_{t_m}\psi_1 = 2\psi_{m+1}, \quad \psi_1 := \psi, \;\; m \geq 1, \tag{5.3.29}$$

$$\psi_m = \frac{i}{2}\partial_{t_1}\psi_{m-1} + \psi_1 h_{m-1}, \quad \partial_{t_1} h_m = 2\,\mathrm{Im}(\psi_1 \overline{\psi}_m), \tag{5.3.30}$$

where ψ_m and h_m are functions of $\boldsymbol{t}$ and $h_1 := 0$.

The proof of this theorem is classical and is deferred to Appendix B. In particular the second flow gives the focusing NLS equation

$$i\partial_{t_2}\psi + \frac{1}{2}\partial_{t_1}^2\psi + |\psi|^2\psi = 0,$$

where comparing with the notation in the introduction $t_2 = t$ and $t_1 = x$. The third flows gives the so called complex modified KdV equation

$$\partial_{t_3}\psi + \frac{\partial_{t_1}^3\psi}{4} + \frac{3}{2}|\psi|^2\partial_{t_1}\psi = 0.$$

Setting $t_k = 0$ for $k \geq 4$ one obtains that $v(t_1, t_2, t_3) := 2|\psi_1(t_1, t_2, t_3)|^2$ satisfies the KP equation (5.3.8) after the rescalings $v = -4u$ and $t_j \to \frac{i}{2}t_j$.

Appendix A

Proof of Lemma 5.3.3

In this section we give the proof of Lemma 5.3.3. Since the computations of $\omega(\boldsymbol{t} \pm [\zeta^{-1}])$ are the same, we give the proof only for $\omega(\boldsymbol{t} - [\zeta^{-1}])$.

From (5.2.22), (5.3.10) and (5.3.12), we get

$$\omega(\boldsymbol{t} - [\zeta^{-1}]) = -\iint_{\mathscr{D}} \operatorname{Tr}\left(\Gamma^{-1}\left(z, \boldsymbol{t} - [\zeta^{-1}]\right) \partial_z \Gamma\left(z, \boldsymbol{t} - [\zeta^{-1}]\right) \delta_{[\zeta]} M\left(z, \boldsymbol{t} - [\zeta^{-1}]\right)\right) \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i}$$

$$= -\iint_{\mathscr{D}} \operatorname{Tr}\left(D(z)\Gamma^{-1}(z)C^{-1}(z)\partial_z\left(C(z)\Gamma(z)D^{-1}(z)\right)\delta_{[\zeta]}\big(D(z)M(z,\boldsymbol{t})D^{-1}(z)\big)\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i}$$

$$= -\iint_{\mathscr{D}} \operatorname{Tr}\left(D(z)\Gamma^{-1}(z)\partial_z\Gamma(z)D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i} + \tag{A.0.1}$$

$$-\iint_{\mathscr{D}} \operatorname{Tr}\left(D(z)\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i} + \tag{A.0.2}$$

$$-\iint_{\mathscr{D}} \operatorname{Tr}\left(D(z)\partial_z D^{-1}(z)\delta_{[\zeta]}\left(D(z)M(z,\boldsymbol{t})D^{-1}(z)\right)\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i} \tag{A.0.3}$$

We now consider the three parts (A.0.1), (A.0.2), (A.0.3), separately.

Computation of (A.0.1). We find:

$$\begin{aligned}
(A.0.1) &= -\iint_{\mathscr{D}} \operatorname{Tr}\left(\Gamma^{-1}(z)\partial_z\Gamma(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i} + \\
&\quad -\iint_{\mathscr{D}} \operatorname{Tr}\left(\Gamma^{-1}(z)\partial_z\Gamma(z)\left[D^{-1}(z)\delta_\zeta D(z), M(z,\boldsymbol{t})\right]\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i} \\
&= \omega(\boldsymbol{t}) + \iint_{\mathscr{D}} \operatorname{Tr}\left(D^{-1}(z)\delta_\zeta D(z)\left[\Gamma^{-1}(z)\partial_z\Gamma(z), M(z,\boldsymbol{t})\right]\right)\frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{2\pi i}.
\end{aligned}$$

Since $\zeta \notin \mathscr{D}$, the matrix $D^{-1}(z)$ in (5.3.11) is analytic in $\mathscr{D}$ and using the $\overline{\partial}$-problem for Γ we can rewrite the two integrals as

$$\begin{aligned}(A.0.1) = \omega(\boldsymbol{t}) + \iint_{\mathscr{D}} \partial_{\bar{z}} \mathrm{Tr}\Big(\Gamma^{-1}(z)\partial_z\Gamma(z)D^{-1}(z)\delta_\zeta D(z)\Big)\frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i}+ \\ - \iint_{\mathscr{D}} \mathrm{Tr}\left(\partial_z M(z,\boldsymbol{t})D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i}.\end{aligned} \tag{A.0.4}$$

We now observe that the last integral is independent of $\boldsymbol{t}$, due to the fact that $D(z)$ is diagonal. Moreover, using

$$D^{-1}(z)\delta_\zeta D(z) = -\frac{z}{\zeta(z-\zeta)}E_{11}\mathrm{d}\zeta,$$

where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we find

$$-\iint_{\mathscr{D}} \mathrm{Tr}\left(\partial_z M(z,\boldsymbol{t})D^{-1}(z)\partial_\zeta D(z)\right)\frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i} = \iint_{\mathscr{D}} \frac{z}{\zeta(z-\zeta)}(\partial_z M_0(z))_{11}\frac{\mathrm{d}\bar{z}\wedge \mathrm{d}z}{2\pi i}. \tag{A.0.5}$$

The RHS of (A.0.5) equals $\partial_\zeta\gamma(\zeta)$. Now, the integrand of the remaining integral in (A.0.4) does not have a pole in $\mathscr{D}$ and we can use Stokes' Theorem

$$\oint_{\partial\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)\partial_z\Gamma(z)D^{-1}(z)\partial_\zeta D(z)\right)\frac{\mathrm{d}z}{2\pi i} = \oint_{-\partial\mathscr{D}} \frac{z}{\zeta(z-\zeta)}(\Gamma^{-1}(z)\partial_z\Gamma(z))_{11}\frac{\mathrm{d}z}{2\pi i}$$

where $-\partial\mathscr{D}$ is the border of $\mathscr{D}$ oriented clockwise. Since $\Gamma(z)$ is analytic outside $\mathscr{D}$, we can apply Cauchy's residue Theorem and pick up the residues at $z = \zeta$ (there is no residue at $z = \infty$ because the integrand is $\mathcal{O}(z^{-2})$):

$$\oint_{-\partial\mathscr{D}} \frac{z}{\zeta(z-\zeta)}\left(\Gamma_{22}(z)\partial_z\Gamma_{11}(z) - \partial_z\Gamma_{21}(z)\Gamma_{12}(z)\right)\frac{\mathrm{d}z}{2\pi i} = \Gamma_{22}(\zeta)\partial_\zeta\Gamma_{11}(\zeta) - \partial_\zeta\Gamma_{21}(\zeta)\Gamma_{12}(\zeta)$$

so that

$$(A.0.1) = \omega(\boldsymbol{t}) + \left(\Gamma_{22}(\zeta)\partial_\zeta\Gamma_{11}(\zeta) - \partial_\zeta\Gamma_{21}(\zeta)\Gamma_{12}(\zeta)\right)\mathrm{d}\zeta + \delta_\zeta\gamma(\zeta). \tag{A.0.6}$$

Computation of (A.0.2). Let us consider (A.0.2):

$$
\begin{aligned}
(A.0.2) = &-\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}+\\
&-\iint_{\mathscr{D}} \mathrm{Tr}\left(M(z,\boldsymbol{t})\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}+\\
&+\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)M(z,\boldsymbol{t})D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}\\
= &-\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}+\\
&+\iint_{\mathscr{D}} \mathrm{Tr}\left(\partial_{\overline{z}}\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}+\\
&+\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\overline{\partial}\Gamma(z)D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}.
\end{aligned}
\tag{A.0.7}
$$

Since the only singularity is at $z = \zeta$, which is outside the domain $\mathscr{D}$, we can apply Stokes' Theorem to the integration and we get

$$
\begin{aligned}
(A.0.2) = &-\iint_{\mathscr{D}} \mathrm{Tr}\left(C^{-1}(z)\partial_z C(z)\Gamma(z)\delta M(z,\boldsymbol{t})\Gamma^{-1}(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}+\\
&+\oint_{\partial\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\delta_\zeta D(z)\right)\frac{\mathrm{d}z}{2\pi i}.
\end{aligned}
\tag{A.0.8}
$$

Now observe that

$$
\Gamma(z,\boldsymbol{t})\delta M(z,\boldsymbol{t})\Gamma^{-1}(z,\boldsymbol{t}) = \partial_{\overline{z}}[(\delta\Gamma(z,\boldsymbol{t}))\Gamma^{-1}(z,\boldsymbol{t})]
\tag{A.0.9}
$$

Using (A.0.9) in the first integral of (A.0.8), we can rewrite it as a contour integral

$$
\begin{aligned}
&-\iint_{\mathscr{D}} \mathrm{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}\\
&= -\iint_{\mathscr{D}} \partial_{\overline{z}}\mathrm{Tr}\left(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\right)\frac{\mathrm{d}\overline{z}\wedge \mathrm{d}z}{2\pi i}\\
&= \oint_{-\partial\mathscr{D}} \mathrm{Tr}\left(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\right)\frac{\mathrm{d}z}{2\pi i}.
\end{aligned}
$$

From the explicit expression of C in (5.3.14) we obtain

$$
C^{-1}(z) = \frac{1}{\det C(z)}\mathrm{adj}(C(z)) = \frac{1}{\left(1-\frac{z}{\zeta}\right)}\begin{bmatrix} 1 & -\frac{\partial_z\Gamma_{12}(\infty)}{\zeta} \\ \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)} & \left(1-\frac{z}{\zeta}\right) - \frac{\partial_z\Gamma_{12}(\infty)\Gamma_{12}(\zeta)}{\zeta\Gamma_{11}(\zeta)}\end{bmatrix}
$$

$$
\partial_z C(z) = -\frac{1}{\zeta}E_{11}
$$

and

$$\operatorname{Tr}\left(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\right) = \frac{1}{(z-\zeta)}\left((\delta\Gamma(z)\Gamma^{-1}(z))_{11} + \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}(\delta\Gamma(z)\Gamma^{-1}(z))_{12}\right)$$
$$= \frac{\delta\Gamma_{11}(z)\Gamma_{22}(z) - \delta\Gamma_{12}(z)\Gamma_{21}(z)}{(z-\zeta)} + \frac{\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\left(\frac{\delta\Gamma_{12}(z)\Gamma_{11}(z) - \delta\Gamma_{11}(z)\Gamma_{12}(z)}{(z-\zeta)}\right).$$

We thus conclude that the first integral in (A.0.8) is given by

$$\oint_{-\partial\mathscr{D}} \operatorname{Tr}\bigl(C^{-1}(z)\partial_z C(z)\delta\Gamma(z)\Gamma^{-1}(z)\bigr)\frac{\mathrm{d}z}{2\pi i} = \delta\Gamma_{11}(\zeta)\Gamma_{22}(\zeta) - \frac{\delta\Gamma_{11}(\zeta)}{\Gamma_{11}(\zeta)}\Gamma_{12}(\zeta)\Gamma_{21}(\zeta)$$
$$= \frac{\delta\Gamma_{11}(\zeta)}{\Gamma_{11}(\zeta)} = \delta\ln\Gamma_{11}(\zeta). \tag{A.0.10}$$

To compute the second integral in (A.0.8) we expand the trace and obtain

$$\operatorname{Tr}\bigl(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\partial_\zeta D(z)\bigr) = -\frac{z}{\zeta(z-\zeta)^2}\Gamma_{11}(z)\left(\Gamma_{22}(z) - \frac{\Gamma_{12}(z)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\right).$$

So we are left with a contour integral with a double pole at $z=\zeta$ and a simple pole at $z=\infty$. Using the explicit expression (5.3.14) for the matrix C we obtain:

$$\oint_{\partial\mathscr{D}} \operatorname{Tr}\left(\Gamma^{-1}(z)C^{-1}(z)\partial_z C(z)\Gamma(z)D^{-1}(z)\partial_\zeta D(z)\right)\frac{\mathrm{d}z}{2\pi i}$$
$$= \oint_{-\partial\mathscr{D}} \frac{z}{\zeta(z-\zeta)^2}\Gamma_{11}(z)\left(\Gamma_{22}(z) - \frac{\Gamma_{12}(z)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}\right)\frac{\mathrm{d}z}{2\pi i}$$
$$= -\frac{1}{\zeta} + \frac{\det\Gamma(\zeta)}{\zeta} + \partial_\zeta(\Gamma_{11}(\zeta)\Gamma_{22}(\zeta)) - \frac{\partial_\zeta\bigl(\Gamma_{11}(\zeta)\Gamma_{12}(\zeta)\bigr)\Gamma_{21}(\zeta)}{\Gamma_{11}(\zeta)}$$
$$= \partial_\zeta\ln\Gamma_{11}(\zeta) + \Gamma_{11}(\zeta)\partial_\zeta\Gamma_{22}(\zeta) - \partial_\zeta\Gamma_{12}(\zeta)\Gamma_{21}(\zeta). \tag{A.0.11}$$

Combining (A.0.10) with (A.0.11) we have

$$(A.0.2) = \delta_{[\zeta]}(\ln(\Gamma_{11}(\zeta))) + (\Gamma_{11}(\zeta)\partial_\zeta\Gamma_{22}(\zeta) - \Gamma_{21}(\zeta)\partial_\zeta\Gamma_{12}(\zeta))\,\mathrm{d}\zeta. \tag{A.0.12}$$

Computation of (A.0.3). This term turns out to vanish; indeed

$$(A.0.3) = \iint_{\mathscr{D}} \operatorname{Tr}\left(D^{-1}(z)\partial_z D(z)\delta M(z,\boldsymbol{t})\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i} +$$
$$- \iint_{\mathscr{D}} \operatorname{Tr}\left(D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}), D^{-1}(z)\delta_\zeta D(z)\right]\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}$$
$$= -\iint_{\mathscr{D}} \operatorname{Tr}\left(\delta\xi(z,t)D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}),\sigma_3\right]\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i} +$$
$$- \iint_{\mathscr{D}} \operatorname{Tr}\left(D^{-1}(z)\partial_z D(z)\left[M(z,\boldsymbol{t}), D^{-1}(z)\delta_\zeta D(z)\right]\right)\frac{\mathrm{d}\overline{z}\wedge\mathrm{d}z}{2\pi i}$$

and the integrand vanishes identically because of the cyclicity of the trace and the fact that D is a diagonal matrix. In conclusion, adding the equations (A.0.6) and (A.0.12), we obtain

$$\omega(\boldsymbol{t} - [\zeta^{-1}]) = \omega(\boldsymbol{t}) + \delta_{[\zeta]} \ln(\Gamma_{11}(\zeta)) + \delta_{[\zeta]}\gamma(\zeta). \tag{A.0.13}$$

Substituting $C(z)$ and $D(z)$ with $\tilde{C}(z)$ and $\tilde{D}(z)$ respectively and using the nonsingular condition for K (5.1.1), we find $\omega(\boldsymbol{t} + [\zeta^{-1}])$ with similar calculations and we get the following result

$$\omega(\boldsymbol{t} + [\zeta^{-1}]) = \omega(\boldsymbol{t}) + \delta_{[\zeta]} \ln(\Gamma_{11}^{-1}(\zeta)) - \delta_{[\zeta]}\gamma(\zeta). \tag{A.0.14}$$

and this proves the Lemma 5.3.3. ■

Appendix B

The Focusing NLS hierarchy

We now consider a particular example of $\overline{\partial}$-problem 5.1.1, with $s = 2$ and a $M(z, \boldsymbol{t})$ similar to (5.3.1)

$$\begin{aligned} &\partial_{\bar{z}}\Gamma(z) = \Gamma(z)\mathrm{e}^{-i\xi(z,\boldsymbol{t})\sigma_3} M_0(z)\mathrm{e}^{i\xi(z,\boldsymbol{t})\sigma_3} && \text{for } z \in \mathscr{D} \cup \overline{\mathscr{D}} \\ &\Gamma(z,\bar{z}) \underset{z\to\infty}{\to} \mathbf{1} \end{aligned} \tag{B.0.1}$$

with $\overline{\mathscr{D}}$ the complex conjugate of $\mathscr{D}$ and $M_0(z)$ that satisfy the Schwarz symmetry

$$\overline{M_0(\bar{z}, z)} = \sigma_2 M_0(z, \bar{z})\sigma_2, \quad \text{where } \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{B.0.2}$$

Theorem B.0.1. *The solution $\Gamma(z)$ of the $\overline{\partial}$-problem (B.0.1) generates the Focusing Nonlinear Schrödinger hierarchy [56]*

$$i\partial_{t_m}\psi_1 = 2\psi_{m+1} \tag{B.0.3}$$

$$\psi_m = \frac{i}{2}\partial_{t_1}\psi_{m-1} + \psi_1 h_{m-1} \quad \partial_{t_1} h_m = -2\mathrm{Im}\,(\psi_1\overline{\psi}_m) \tag{B.0.4}$$

where ψ_m and h_m are functions of $\boldsymbol{t}$.

Proof

We define the matrix $\Psi(z)$ as

$$\Psi(z) = \Gamma(z)\mathrm{e}^{-i\xi(z,\boldsymbol{t})\sigma_3} \tag{B.0.5}$$

and if $\Gamma(z)$ solves the problem (B.0.1) then $\Psi(z, \bar{z})$ solves

$$\begin{aligned} &\partial_{\bar{z}}\Psi(z) = \Psi(z)M_0(z) && \text{for } z \in \mathscr{D} \cup \overline{\mathscr{D}}, \\ &\Psi(z,\bar{z}) \underset{z\to\infty}{\to} (\mathbf{1} + \mathcal{O}(z^{-1}))\mathrm{e}^{-i\xi(z,\boldsymbol{t})\sigma_3}. \end{aligned} \tag{B.0.6}$$

Since the derivative $\partial_{\bar{z}}$ and ∂_{t_j} commutes $\forall\, j$, both Ψ and $\partial_{t_j}\Psi$ satisfy the problem (B.0.6), and for Lemma 5.1.2 Ψ and $\partial_{t_j}\Psi$ are linked by an entire matrix, called B_j

$$\partial_{t_j}\Psi(z) = B_j\Psi(z,\bar{z}) \tag{B.0.7}$$

Consider the first parameter $t_1 = x$. We want look for the matrix B_1 from the equation $\partial_x\Psi(\Psi)^{-1}$.

$$\begin{aligned}
&\partial_x\Psi(\Psi)^{-1} = \partial_x(\Gamma(z)\mathrm{e}^{-i\xi(z,\boldsymbol{t})\sigma_3})\mathrm{e}^{i\xi(z,\boldsymbol{t})\sigma_3}(\Gamma(z))^{-1} = \\
&= -iz\Gamma(z)\sigma_3(\Gamma(z))^{-1} + \partial_x(\Gamma(z))(\Gamma(z))^{-1} \sim \\
&\sim -iz\left(\mathbf{1} + \sum_{k=1}^{\infty}\frac{\Gamma_k(\boldsymbol{t})}{z^k}\right)\sigma_3\left(\mathbf{1} + \sum_{k=1}^{\infty}\frac{\tilde{\Gamma}_k(\boldsymbol{t})}{z^k}\right) + \left(\sum_{k=1}^{\infty}\frac{\partial_x\Gamma_k(\boldsymbol{t})}{z^k}\right)\left(\mathbf{1} + \sum_{k=1}^{\infty}\frac{\tilde{\Gamma}_k(\boldsymbol{t})}{z^k}\right)
\end{aligned} \tag{B.0.8}$$

where $\Gamma_k(\boldsymbol{t})$ is a 2×2 matrix with the following form

$$\Gamma_k(\boldsymbol{t}) = \begin{bmatrix} a_k(\boldsymbol{t}) & b_k(\boldsymbol{t}) \\ -\bar{b}_k(\boldsymbol{t}) & \bar{a}_k(\boldsymbol{t}) \end{bmatrix} \tag{B.0.9}$$

and $\tilde{\Gamma}_k(\boldsymbol{t})$ is still a 2×2 matrix given by the recursive formula

$$\tilde{\Gamma}_k = -\Gamma_k - \sum_{j=1}^{k-1}\Gamma_j\tilde{\Gamma}_{k-j}. \tag{B.0.10}$$

Remark B.0.2. *We can rewrite (B.0.10) as an algebraic equation in $\mathbb{C}^n \otimes Mat(2,\mathbb{C})$, with $n > k$, and we can find an explicit form for the $\tilde{\Gamma}_k$:*

$$\begin{aligned}
\tilde{\Gamma}_k(\boldsymbol{t}) = &-\Gamma_k(\boldsymbol{t}) + \sum_{m=1}^{k-1}\Gamma_{k-m}\Gamma_m + \\
&+ \sum_{m=1}^{k-1}\sum_{l=2}^{p=k-m}(-1)^{l+1}\left[\sum_{k_1=1}^{p+1-l}\cdots\sum_{k_{l-1}=1}^{p-1-\kappa}\Gamma_{p-\kappa-k_{l-1}}\Gamma_{k_{l-1}}\ldots\Gamma_{k_1}\right]\Gamma_m,
\end{aligned} \tag{B.0.11}$$

with $\kappa = \sum_{j=1}^{l-2}k_j$.

Expanding (B.0.8) we find the laurent series for B_1:

$$\begin{aligned}
B_1 \sim &-iz\sigma_3 - i[\Gamma_1(\boldsymbol{t}),\sigma_3] + \\
&+ \sum_{k=2}^{\infty}z^{-(k-1)}\left[\partial_x\Gamma_{k-1} + \sum_{l=1}^{k-2}\partial_x\Gamma_l\tilde{\Gamma}_{k-l-1} - i\left([\Gamma_k,\sigma_3] + \sum_{j=1}^{k-1}[\Gamma_j,\sigma_3]\tilde{\Gamma}_{k-j}\right)\right]
\end{aligned} \tag{B.0.12}$$

and since B_1 must be an entire function then all the terms $z^{-(k-1)}$ must be zero $\forall\, k$.

Lemma B.0.3. *B_1 is entire iff the matricies $\Gamma_k(\boldsymbol{t})$ satisfy the equation*

$$i\partial_x\Gamma_{k-1} + [\Gamma_k, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_{k-1} = 0 \quad \forall\, k \tag{B.0.13}$$

Proof of the lemma It's a proof by induction.

From the equation (B.0.12), we can see that for $k = 2$ we get the equation (B.0.13). We now check for $k = 3$:

$k = 3$ $\quad \partial_x\Gamma_2 + \partial_x\Gamma_1\tilde{\Gamma}_1 - i\left[[\Gamma_3, \sigma_3] + [\Gamma_1, \sigma_3]\tilde{\Gamma}_2 + [\Gamma_2, \sigma_3]\tilde{\Gamma}_1\right] =$

from the equation (B.0.11) we get that $\tilde{\Gamma_1} = -\Gamma_1$ and $\tilde{\Gamma}_2 = -\Gamma_2 + (\Gamma_1)^2$, so we have that

$$\partial_x\Gamma_2 - i([\Gamma_3, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_2) - (\partial_x\Gamma_1 - i[\Gamma_2, \sigma_3] + i[\Gamma_1, \sigma_3]\Gamma_1) =$$
$$= \partial_x\Gamma_2 - i([\Gamma_3, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_2) \tag{B.0.14}$$

and this is the equation (B.0.13) for $k = 3$. We can do the same also for $k = 4$:

$k = 4$
$$\partial_x\Gamma_3 - i([\Gamma_4, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_3) + (\partial_x\Gamma_1 - i[\Gamma_2, \sigma_3] + i[\Gamma_1, \sigma_3]\Gamma_1)((\Gamma_1)^2 - \Gamma_2) +$$
$$- (\partial_x\Gamma_2 - i([\Gamma_3, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_2))\Gamma_1 =$$
$$= \partial_x\Gamma_3 - i([\Gamma_4, \sigma_3] - [\Gamma_1, \sigma_3]\Gamma_3) \tag{B.0.15}$$

and this is the equation (B.0.13) for $k = 3$.

We suppose, by induction, that (B.0.13) is valid for $k = m$ and we want to show that it still valid also for $k = m+1$. Consider the $k = m+1$ term of the laurent series (B.0.12):

$k = m + 1$
$$\partial_x\Gamma_m + \sum_{l=1}^{m-1} \partial_x\Gamma_l\tilde{\Gamma}_{m-l} - i\left([\Gamma_{m+1}, \sigma_3] + \sum_{j=1}^{m}[\Gamma_j, \sigma_3]\tilde{\Gamma}_{m+1-j}\right) =$$
$$= \partial_x\Gamma_m + \sum_{l=1}^{m-1} \partial_x\Gamma_l\tilde{\Gamma}_{m-l} - i[\Gamma_{m+1}, \sigma_3] - i[\Gamma_1, \sigma_3]\tilde{\Gamma}_m - i\sum_{j=2}^{m}[\Gamma_j, \sigma_3]\tilde{\Gamma}_{m+1-j} =$$
$$= \partial_x\Gamma_m - i[\Gamma_{m+1}, \sigma_3] + i[\Gamma_1, \sigma_3]\Gamma_m + \sum_{l=1}^{m-1} \partial_x\Gamma_l\tilde{\Gamma}_{m-l} + i\sum_{j=1}^{m-1}[\Gamma_1, \sigma_3]\Gamma_j\tilde{\Gamma}_{m-j} +$$
$$- i\sum_{j=1}^{m-1}[\Gamma_{j+1}, \sigma_3]\tilde{\Gamma}_{m-j} =$$
$$= \partial_x\Gamma_m - i[\Gamma_{m+1}, \sigma_3] + i[\Gamma_1, \sigma_3]\Gamma_m + \sum_{l=1}^{m-1}\left[(\partial_x\Gamma_l + i[\Gamma_1, \sigma_3]\Gamma_l - i[\Gamma_{l+1}, \sigma_3])\tilde{\Gamma}_{m-l}\right]$$
$$= \partial_x\Gamma_m - i[\Gamma_{m+1}, \sigma_3] + i[\Gamma_1, \sigma_3]\Gamma_m = 0$$

and this ends our proof. ■

Since also the other matricies B_m must be entire, the following equation holds $\forall\, m, k \in \mathbb{N}$:

$$\partial_{t_m}\Gamma_k + \sum_{l=1}^{k-1} \partial_{t_m}\Gamma_l\tilde{\Gamma}_{k-l} - i[\Gamma_{k+m}, \sigma_3] - \sum_{j=1}^{k+m-1} i[\Gamma_j, \sigma_3]\tilde{\Gamma}_{k+m-j} = 0. \tag{B.0.16}$$

The equation (B.0.13) gives us a relation between the coefficients $a_k(\boldsymbol{t}), b_k(\boldsymbol{t})$:

$$\partial_x a_k = -2ib_1\bar{b}_k \tag{B.0.17}$$

$$b_k = \frac{i}{2}\partial_x b_{k-1} + b_1\bar{a}_{k-1} \tag{B.0.18}$$

So the matrix B_1 is given by

$$B_1 = -iz\sigma_3 - i[\Gamma_1(\boldsymbol{t}), \sigma_3] = -iz\sigma_3 + B_1^0 = \begin{bmatrix} -iz & 2ib_1(\boldsymbol{t}) \\ 2i\bar{b}_1(\boldsymbol{t}) & iz \end{bmatrix}. \tag{B.0.19}$$

We now consider the m-component of $\boldsymbol{t}$. The matrix B_m is given from

$$\begin{aligned}
\partial_{t_m}\Psi(\Psi)^{-1} &= -iz^m \left[\sigma_3 + z^{-1}[\Gamma_1, \sigma_3] + \sum_{l=2}^{m} z^{-m}([\Gamma_k, \sigma_3] + \sum_{j=1}^{l-2}[\Gamma_j, \sigma_3]\tilde{\Gamma}_{l-j})\right] = \\
&= -iz^m \left[\sigma_3 + z^{-1}[\Gamma_1, \sigma_3] + \sum_{l=2}^{m} z^{-m}(\partial_x\Gamma_{l-1} + \sum_{j=1}^{l-2}\partial_x\Gamma_l\tilde{\Gamma}_{l-j-1})\right] = \\
&= z^{m-1}B_1 - \sum_{l=2}^{m-1} z^{m-l}(\partial_x\Gamma_{l-1} + \sum_{j=1}^{l-2}\partial_x\Gamma_l\tilde{\Gamma}_{l-j-1}) - (\partial_x\Gamma_{m-1}\sum_{j=1}^{m-2}\partial_x\Gamma_j\tilde{\Gamma}_{m-j-1}) = \\
&= zB_{m-1} + B_m^0 = B_m
\end{aligned} \tag{B.0.20}$$

this give us a recursive formula to find the Lax matricies B_m for $m = 2, \dots, n$ with

$$B_m^0 := -(\partial_x\Gamma_{m-1} + \sum_{j=1}^{m-2}\partial_x\Gamma_j\tilde{\Gamma}_{m-j-1}). \tag{B.0.21}$$

Now we need to prove that B_m^0 satisfy the recursive equations:

$$(B_m^0)_{12} = \frac{i}{2}\partial_x(B_{m-1}^0)_{12} - 2(B_{m-1}^0)_{11}b_1; \tag{B.0.22}$$

$$\partial_x(B_m^0)_{11} = 2i[(B_m^0)_{21}b_1 - (B_m^0)_{12}\bar{b}_1]. \tag{B.0.23}$$

Indeed, consider the case for $m = 2$ and $m = 3$:

$m = 2$ $B_2^0 = -\partial_x \Gamma_1$

from equations (B.0.17) and (B.0.19), we get that

$$(B_2^0)_{12} = -\partial_x b_1 = \frac{i}{2}(B_1^0)_{12} - 2(B_1^0)_{11} b_1 \quad \partial_x (B_2^0)_{11} = -\partial_x^2 a_1 = 2i\partial_x |b_1|^2$$

$m = 3$ $B_3^0 = -\partial_x \Gamma_2 + \partial_x \Gamma_1 \Gamma_1$

from (B.0.17) and (B.0.18) we get that

$$(B_3^0)_{12} = -\partial_x b_2 + \partial_x a_1 b_1 + \overline{a}_1 \partial_x b_1 = -\frac{i}{2}\partial_x^2 b_1 - 4i|b_1|^2 b_1 = \frac{i}{2}(B_2^0)_{12} - 2(B_2^0)_{11} b_1$$

$$\begin{aligned}
&\partial_x (B_3^0)_{11} = 2i\partial_x [b_1 \overline{b}_2 + \overline{b}_1 b_2 - |b_1|^2 (a_1 + \overline{a}_1)] = \\
&2i[\partial_x \overline{b}_2 b_1 + \partial_x b_2 \overline{b}_1 - \overline{a}_1 \partial_x b_1 \overline{b}_1 - a_1 \partial_x \overline{b}_1 b_1] = \\
&= 2i[(B_3^0)_{21} b_1 - (B_3^0)_{12} \overline{b}_1],
\end{aligned} \tag{B.0.24}$$

If we move one step further with $m = 4$ we notice that not only the equations (B.0.22) and (B.0.23) holds, but also that we can rewrite (B.0.21) in a recursive way:

$$B_m^0 = -\partial_x \Gamma_{n-1} - \sum_{j=1}^{m-2} B_{m-j} \Gamma_j. \tag{B.0.25}$$

It is easy to check that, from Schwarz symmetry, the following conditions holds:

$$(B_m^0)_{11} = -(B_m^0)_{22} \quad (B_m^0)_{12} = -\overline{(B_m^0)_{21}}. \tag{B.0.26}$$

Let us suppose, by induction, that (B.0.22) and (B.0.23) holds for m, now we want check that they are true for $m + 1$:

$$\begin{aligned}
m+1 \quad (B_{m+1}^0)_{12} &= -\partial_x b_m - \sum_{j=1}^{m-1} [(B_{m+1-j}^0)_{11} b_j + (B_{m+1-j}^0)_{12} \overline{a}_j] = \\
&= -\partial_x \left(\frac{i}{2} \partial_x b_{m-1} + b_1 \overline{a}_{m-1} \right) - \sum_{j=1}^{m-1} (B_{m+1-j}^0)_{11} b_j - \\
&\sum_{j=1}^{m-1} [\frac{i}{2} \partial_x (B_{m-j}^0)_{12} - 2(B_{m-j}^0)_{11} b_1] \overline{a}_j =
\end{aligned}$$

$$= -\partial_x^2 b_{m-1} - 4i|b_1|^2 b_{m-1} - \sum_{j=1}^{m-2} \frac{i}{2}\partial_x (B^0_{m-j})_{12}\overline{a}_j + 2\sum_{j=1}^{m-2}(B_{m-j})_{11} b_1 \overline{a}_j - \sum_{j=1}^{m-2}(B^0_{m+1-j})_{11} b_j =$$

$$= \frac{i}{2}\left[\partial_x^2 b_{m-1} + \sum_{j=1}^{m-2}\partial_x\left((B^0_{m-j})_{12}\overline{a}_j\right) + \sum_{j=1}^{m-2}\partial_x\left((B^0_{m-j})_{11}\overline{b}_j\right)\right] +$$

$$- \sum_{j=1}^{m-2}(B^0_{m-j})_{12}\overline{b}_1 b_{j-1} - \sum_{j=1}^{m-2}[(B^0_{m-j})_{21} b_1 - (B^0_{m-j})_{12}\overline{b}_1] b_j$$

$$+ \sum_{j=1}^{m-2}(B^0_{m-j})_{11} b_{j+1} - \sum_{j=1}^{m-2}(B^0_{m-j})_{11} b_1 \overline{a}_j +$$

$$- \sum_{j=1}^{m-1}(B^0_{m+1-j})_{11} b_j + 2\sum_{j=1}^{m-2}(B_{m-j})_{11} b_1 \overline{a}_j - 2i|b_1| b_{m-1} =$$

$$= \frac{i}{2}\partial_x (B^0_m)_{12} - (B^0_m)_{11} b_1 - \left[\partial_x \overline{a}_{m-1} + \sum_{j=1}^{m-2}((B^0_{m-j})_{21} b_j - (B^0_{m-j})_{11})\overline{a}_j\right] =$$

then, from the conditions (B.0.26)

$$= \frac{i}{2}\partial_x (B^0_m)_{12} - 2(B^0_m)_{11} b_1$$

we need to check the other condition:

$$\partial_x (B^0_{m+1})_{11} = 2i\partial_x (b_1 \overline{b}_m) - \sum_{j=1}^{m-1}\partial_x[(B^0_{m+1-j})_{11} a_j - (B^0_{m+1-j})_{12}\overline{b}_j] =$$

$$= 2i[\partial_x b_1 \overline{b}_m + b_1 \partial_x \overline{b}_m] - \sum_{j=1}^{m-1}[\partial_x (B^0_{m+1-j})_{11} a_j - 2i(B^0_{m+1-j})_{11} b_1 \overline{b}_j] +$$

$$+ \sum_{j=1}^{m-1}[\partial_x (B^0_{m+1-j})_{12}\overline{b}_j + (B^0_{m+1-j})_{12}\partial_x \overline{b}_j] =$$

$$= 2i[\partial_x b_1 \overline{b}_m + b_1 \partial_x \overline{b}_m] - 2i\sum_{j=1}^{m-1}[(B^0_{m+1-j})_{21} b_1 - (B^0_{m+1-j})_{12}\overline{b}_1] a_j +$$

$$+ 2i\sum_{j=1}^{m-1}(B^0_{m+1-j})_{11} b_1 \overline{b}_j + \partial_x (B^0_m)_{12}\overline{b}_1 + \sum_{j=2}^{m-1}\partial_x (B^0_{m+1-j})_{12}\overline{b}_j + \sum_{j=1}^{m-1}(B^0_{m+1-j})_{12}\partial_x \overline{b}_j =$$

$$= 2i[\partial_x b_1 \overline{b}_m + b_1 \partial_x \overline{b}_m] - 2i\sum_{j=1}^{m-1}[(B^0_{m+1-j})_{21} b_1 - (B^0_{m+1-j})_{12}\overline{b}_1] a_j + \tag{B.0.27}$$

$$+ 2i \sum_{j=1}^{m-1} (B^0_{m+1-j})_{11} b_1 \bar{b}_j - 2i((B^0_{m+1})_{21} + 2(B^0_m)_{11} b_1)\bar{b}_1 +$$

$$+ \sum_{j=1}^{m-2} \partial_x (B^0_{m-j})_{12} \bar{b}_{j+1} + 2i \sum_{j=1}^{m-1} (B^0_{m+1-j})_{12} (\bar{b}_{j+1} - \bar{b}_1 a_j) =$$

$$= 2i[\partial_x \bar{b}_m - \sum_{j=1}^{m-1} (B^0_{m+1-j})_{21} a_j + \sum_{j=1}^{m-1} (B^0_{m+1-j})_{11} \bar{b}_j] b_1 +$$

$$- 2i(B^0_{m+1})_{12} \bar{b}_1 - 4i(B^0_m)_{12} |b_1|^2 - 4i \sum_{j=1}^{m-2} (B^0_{m-j})_{11} \bar{b}_{j+1} b_1 =$$

$$= 2i[\partial_x \bar{b}_m - \sum_{j=1}^{m-1} (B^0_{m+1-j})_{21} a_j + \sum_{j=1}^{m-1} (B^0_{m+1-j})_{22} \bar{b}_j] b_1 - 2i(B^0_{m+1})_{12} \bar{b}_1 =$$

$$= 2i[(B^0_{m+1})_{21} b_1 - (B_{m+1})_{12} \bar{b}_1] = -4\mathrm{Im}\,(\overline{(B^0_{m+1})_{12}} b_1).$$

Then, by defining the functions:

$$\psi_1(\boldsymbol{t}) := 2ib_1(\boldsymbol{t}), \quad \psi_m(\boldsymbol{t}) := (B^0_m)_{12}(\boldsymbol{t}) \quad h_m(\boldsymbol{t}) = i(B^0_m)_{11}(\boldsymbol{t}) \tag{B.0.28}$$

we get that equations (B.0.22) and (B.0.23) are exactly the equations (B.0.4).

Let us consider the equation (B.0.16) for $k = 1$:

$$\partial_{t_m} \Gamma_1 - i[\Gamma_{m+1}, \sigma_3] - \sum_{j=1}^{m} i[\Gamma_j, \sigma_3] \tilde{\Gamma}_{m-j+1} = 0. \tag{B.0.29}$$

From equation (B.0.13) and (B.0.21), we can rewrite it as

$$\partial_{t_m} \Gamma_1 + B^0_{m+1} - i[\Gamma_1, \sigma_3](\Gamma_m + \tilde{\Gamma}_m + \sum_{j=1}^{m-1} \Gamma_j \tilde{\Gamma}_{m-j}) = 0, \tag{B.0.30}$$

then, from (B.0.10), the last term is identically zero and at the end we get:

$$\partial_{t_m} \Gamma_1 = -B^0_{m+1}. \tag{B.0.31}$$

By taking the element 12 of (B.0.31) and the definitions (B.0.28), we get the equation (B.0.3).

This ends the proof of the theorem. ■

Remark B.0.4. *If the matrix $M_0(z)$ has a symmetry of this kind:*

$$\overline{M_0(\bar{z}, z)} = \sigma_1 M_0(z) \sigma_1, \quad \text{with } \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{B.0.32}$$

then, by substituting $\bar{b}_k$ with $-\bar{b}_k$ in the proof of theorem, we can show that the solution $\Gamma(z, \bar{z})$ generates the Defocusing Nonlinear Schrödinger hierarchy.

Remark B.0.5. *Exist a generalization for $n \times n$ matricies of the NLS hierarchy, called the $\mathfrak{g}$ANKS hierarchy [38, 91]. The extension of Theorem B.0.1 for this hierarchy, and also the result 5.3.1 about the τ-function be treated in future works.*

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