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Constructions of exotic 4-manifolds with non-trivial fundamental group

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Abstract

In this thesis, we investigate exotic 4-manifolds.

The existence of infinite exotic families of 4-manifolds highlights the discrepancy between the topological and the smooth category in the 4-dimensional realm. By an exotic family we mean a collection of smooth manifolds whose elements are pairwise homeomorphic but not diffeomorphic. After Michael Freedman's celebrated result on the topological classification of closed simply connected 4-manifolds in 1982, several examples of infinite families of exotic simply connected 4-manifolds have been constructed. In recent times, there has been increasing interest in building new exotic 4-manifolds with non-trivial fundamental group. In my thesis, I focus on the construction of such examples.

The thesis is arranged into five main chapters and an appendix.

In Chapter 1 we collect all the background notions that are needed to construct the new exotic 4-manifolds in the thesis. In particular, we list the cut-and-paste construction tools and we survey criteria to identify the homeomorphism types, as well as invariants to distinguish the diffeomorphism types.

Chapter 2 is based on a joint work with Rafael Torres and Daniele Zuddas [25], in which we pin down the homeomorphism type of some closed oriented 4-manifolds with fundamental group $\mathbb{Z}$ and $\mathbb{Z}^2$. Such examples come in infinite families and are produced via torus-surgery-based constructions starting from some closed simply connected 4-manifolds with small Euler characteristics appearing in the work of Akhmedov–Park, Baldridge–Kirk and Torres. These diffeomorphism types are distinguished by their Seiberg–Witten invariants and any two homeomorphic 4-manifolds in these collections become diffeomorphic after connected summing with a single copy of a product of two 2-spheres. The homeomorphism types of the examples in our main result are pinned down by the study of the equivariant intersection form, invoking classification results by Freedmann–Quinn, Stong–Wang and Hambleton–Kreck–Teichner. Our main contribution is a mechanism to compute the equivariant intersection form of 4-manifolds that are obtained via torus surgeries along embedded 2-tori with trivial normal bundle.

The content of Chapter 3 is taken from a joint work with Rafael Torres [23]. We introduce a simple cut-and-paste mechanism that, given a pair (M_1, M_2) of homeomorphic

but not diffeomorphic 4-manifolds, produces for every integer p a pair $(M_1(p), M_2(p))$ of homeomorphic and potentially not diffeomorphic 4-manifolds. Such a mechanism alters the fundamental group of the initial pair and we show that it unveils new exotic irreducible smooth structures on closed 4-manifolds with finite cyclic fundamental group, which include $\mathbb{Q}$-homology real projective 4-spaces.

Chapter 4 is based on a joint work with Rafael Torres [24], where we build an exotic structure Y on the total space $\mathbb{RP}^2 \tilde{\times} S^2$ of the non-trivial 2-sphere bundle over the real projective plane. Such smooth structure is produced by Gluck twist on a smoothly embedded 2-sphere S inside $\mathbb{RP}^2 \times S^2$. We also show that there is a 5-dimensional cobordism with a single 2-handle between the 4-manifold Y and a mapping torus that was used by Cappell–Shaneson to construct an exotic $\mathbb{RP}^4$. We further show that twisting an embedded real projective plane inside Y produces a manifold that is homeomorphic but not diffeomorphic to the circle sum of two copies of $\mathbb{RP}^4$.

The results in Chapter 5 are taken from [22], a single author paper. It contains a discussion of $\mathrm{Pin}^{\pm}$-structures on non-orientable Lefschetz fibrations. More precisely, we study necessary and sufficient conditions for a 4-dimensional Lefschetz fibration over the 2-disk to admit a Pin^{+} or a Pin^{-}-structure, extending the work of Stipsicz in the oriented setting. As a corollary, we get existence results of $\mathrm{Pin}^{\pm}$-structures on closed nonorientable 4-manifolds and on Lefschetz fibrations over the 2-sphere. In particular, we show via three explicit examples how to read-off $\mathrm{Pin}^{\pm}$-structures from the Kirby diagram of a 4-manifold. We also provide a proof of the well-known fact that any closed 3-manifold admits a Pin^{-}-structure and we find a criterion to check whether or not it admits a Pin^{+}-structure in terms of a given handlebody decomposition. In particular, the main motivation for studying Pin^{+}-structures on 4-manifolds in this chapter is the fact that in some cases they might help to detect exotic phenomena in the nonorientable realm. For example, this is the case for the smooth structures produced in Chapter 3 and Chapter 4.

We conclude with an appendix, based on a short survey article [21] containing a proof of Dold–Whitney criterion for the parallelizability of a smooth closed connected orientable 4-manifold. This follows from a stronger result due to Dold and Whitney on the classification of oriented sphere bundles over a 4-complex. Our main contribution is to outline in detail an argument due to R. Kirby, using the classification of $SO(4)$-bundles over the 4-sphere by means of their Euler and first Pontryagin classes as a main tool.

La volta che mi venne da lasciar cadere quasi distrattamente la domanda: - Arsenal - Real Madrid, in semifinale, Arsenal gioca in casa, chi vince? - in un attimo compresi che con questo che pareva un casuale accozzo di parole avevo toccato una riserva infinita di nuove combinazioni tra i segni di cui la realtà compatta e opaca e uniforme si sarebbe servita per travestire la sua monotonia, e forse la corsa verso il futuro, quella corsa che io per primo avevo previsto e auspicato, non tendeva ad altro attraverso il tempo e lo spazio che a uno sbriciolarsi in alternative come queste, fino a dissolversi in una geometria d'invisibili triangoli e rimbalzi come il percorso del pallone tra le linee bianche del campo quali io cercavo d'immaginarmi tracciate in fondo al vortice luminoso del sistema planetario, decifrando i numeri segnati sul petto e la schiena di giocatori notturni irriconoscibili in lontananza.

Italo Calvino, *Le cosmicomiche* [34]

Contents

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Introduction

The main objects under investigation in this thesis are exotic smooth structures on 4-manifolds. Recall that a family of smooth n-manifolds is called *exotic* if its elements are pairwise homeomorphic but not diffeomorphic. It is known that such families do not exist in lower dimensions, i.e. every topological n-manifold supports a unique smooth structure up to diffeomorphisms for $n \leq 3$, see Munkres [117] for the case $n = 2$ and Moise [114] for $n = 3$. On the other hand, exotic phenomena are known to exist in a "controlled way" in higher dimensions. More precisely, it was shown in [98] that, given any integer $n \geq 5$, every closed oriented topological n-manifold supports at most finitely many pairwise nondiffeomorphic smooth structures.

A short historical overview

The main tool to understand smooth structures on higher dimensional simply connected manifolds is Smale's h-cobordism theorem [130], for which its author was awarded the Fields medal. More precisely, Smale showed the following: for $n \geq 5$, if two closed smooth simply connected n-manifolds X_1, X_2 are related by a cobordism W such that the inclusion maps $X_i \hookrightarrow W$ are homotopy equivalences, then there is a diffeomorphism $X_1 \cong_{\mathcal{C}^\infty} X_2$. Twenty years later, another Fields medal was awarded to Freedman for proving the topological analogue of Smale's result in dimension 4, see [58]. A consequence of his result is that the homeomorphism type of a closed simply connected topological 4-manifold is determined by its intersection form and Kirby–Siebenmann invariant, that we define in Subsection 1.2.1. While, at this point, the topological classification of simply connected 4-manifolds was completed, in the same years Donaldson showed that the smooth classification was (and still is) far from being completely understood. More precisely, he showed in [43] that any topological 4-manifold with non-diagonalizable definite intersection form is not smoothable and provided in [45] the first known example of simply connected 4-manifolds which are homeomorphic but not diffeomorphic. For his results, Donaldson was also awarded the Fields medal. This was the fertile soil that allowed for the development of the industry of exotic 4-manifolds.

A "recipe" for building new exotic 4-manifolds

Over the last decades, mathematicians have been looking for new examples of exotic 4-manifolds. The following procedure summarizes how exotic families are (in general) produced. This involves three main steps, which we will follow for the construction of the exotic families in Chapter 2, Chapter 3 and Chapter 4.

1. Build a family of 4-manifolds $\{X_i\}_{i\in I}$. Usually, one starts from some 4-manifold X and constructs the candidate exotic family $\{X_i\}_{i\in I}$ by performing cut-and-paste operations on X, see Section 1.1.

2. Show that the X_i's share the same homeomorphism type. In the simply connected case, this amounts to the study of their intersection form, see [59]. In general, one has to appeal to some classification result, which will depend on the fundamental group of the X_i's, see Section 1.2.

3. Show that X_i is not diffeomorphic to X_j for $i \neq j$. Smooth structures are generally distinguished thanks to gauge-theoretical invariants, for example the Seiberg-Witten invariants or the Ozsváth-Szabó mixed invariant from Heegaard Floer homology, see Section 1.3.

Some known examples with small Euler characteristic

After the celebrated results by Freedman and Donaldson, several examples of infinite exotic families of simply connected 4-manifolds have been constructed and studied by mathematicians. The key tool in such constructions was the computation of some smooth invariants developed by Donaldson, which allowed to unveil distinct smooth structures supported by smooth algebraic surfaces with the same topological type. These include the following examples.

- In [44] (1987), Donaldson uses Yang–Mills theory to show that the Dolgachev surface $E(1)_{2,3}$ is an exotic copy of the 4-manifold $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$.

- In [61] (1988), Friedman and Morgan strengthen Donaldson's result and show that $\mathbb{CP}^2 \#_k \overline{\mathbb{CP}}^2$ has infinitely many nonequivalent smooth structures underlying algebraic surfaces if $k \geq 9$.

- In [100] (1989), Kotschick shows that the Barlow surface is an exotic copy of $\mathbb{CP}^2 \#_8 \overline{\mathbb{CP}}^2$.

  A major breakthrough in the study of exotic smooth structures on 4-manifolds came from the study of the Seiberg–Witten invariants, which provided new tools for detecting new smooth structures on 4-manifolds. This led to the following results.

- In [53] (1998), Fintushel and Stern construct an infinite family of exotic $K3$.
- In [125] (2004), Park constructs an exotic $\mathbb{CP}\,\#_7\,\overline{\mathbb{CP}}^2$, using the rational blow-down operation announced by Fintushel–Stern in [54].
- In [134] (2005), Stipsicz and Szabó construct an exotic smooth structure on $\mathbb{CP}^2\,\#_6\,\overline{\mathbb{CP}}^2$.
- In [126] (2005), Park, Stipsicz and Szabó find a new exotic structure on $\mathbb{CP}^2\,\#_5\,\overline{\mathbb{CP}}^2$.
- In [55] (2007), Fintushel, Park and Stern construct infinitely many exotic copies of $\mathbb{CP}^2\,\#_3\,\overline{\mathbb{CP}}^2$ as an application of their "reverse engineering" procedure.
- In [15] (2008), Akhmedov and Park construct irreducible symplectic 4-manifolds with the homeomorphism type of $\mathbb{CP}^2\,\#_3\,\overline{\mathbb{CP}}^2$ and $\#_3\,\mathbb{CP}^2\,\#_5\,\overline{\mathbb{CP}}^2$.
- In [14] (2008), Akhmedov, Baykur and Park (as a follow up of [15]) build several infinite exotic families of small irreducible 4-manifolds, including exotic copies of $\mathbb{CP}^2\,\#_3\,\overline{\mathbb{CP}}^2$, $\#_3\,\mathbb{CP}^2\,\#_5\,\overline{\mathbb{CP}}^2$ and $\#_3\,\mathbb{CP}^2\,\#_7\,\overline{\mathbb{CP}}^2$.
- In [27] (2008) and [28] (2009), Baldridge and Kirk build other exotic copies of $\mathbb{CP}^2\,\#_3\,\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\,\#_5\,\overline{\mathbb{CP}}^2$, together with several other examples of simply connected and non-simply connected minimal symplectic 4-manifolds.
- In [16] (2010), Akhmedov and Park build infinite exotic families of 4-manifolds with the homeomorphism type of $\mathbb{CP}^2\,\#_2\,\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\,\#_4\,\overline{\mathbb{CP}}^2$, $\#_3\,\mathbb{CP}^2\,\#_4\,\overline{\mathbb{CP}}^2$, $\#_{2n-1}\,\mathbb{CP}^2\,\#_{2n}\,\overline{\mathbb{CP}}^2$ for every $n \geq 3$ and $\#_3\,\mathbb{CP}^2\,\#_k\,\overline{\mathbb{CP}}^2$ for $k = 6, 8, 10$.

Many of the known constructions of exotic simply connected 4-manifolds [15, 16, 14, 27, 28, 55] can be modified to produce families of irreducible pairwise nondiffeomorphic 4-manifolds with nontrivial fundamental group, as shown by [16, 28, 146, 148]. However, pinning down the homeomorphism type of such 4-manifolds is, in general, not an easy task. For example, if we restrict to $\mathbb{Z}$-manifolds (that is, manifolds with fundamental group isomorphic to the integers), studying their intersection form is not enough to determine their topological type; one has to deal with the equivariant intersection form, see Subsection 1.2.2 for more details. More precisely, we have the following results.

- In [80] (1997), Hambleton and Teichner prove the existence of two closed oriented topological 4-manifolds with infinite cyclic fundamental group and trivial Kirby–Siebenmann invariant which are not homotopy equivalent, despite having isometric intersection forms.
- In [60] (2007), Friedl, Hambleton, Melvin and Teichner show that the one of the two 4-manifolds constructed in [80] is smoothable, while the other one is not.

We remark that recent progress in the construction of exotic 4-manifolds with infinite cyclic fundamental group has been made by Levine, Lidman and Piccirillo in [107]. There, the authors develop a new Heegaard Floer based invariant, called the α-invariant, to detect exotic structures on 4-manifolds with $\mathbb{Z}$-fundamental group; they also provide the first known example of exotic smooth structure on a 4-manifold with definite intersection form.

It is worth to mention that, recently, there has been increasing interest in the construction of exotic structures on smooth 4-manifolds with non-trivial fundamental group, see for example [18], [19], [30], [32], [35], [104], [135], [136] and [140].

We now move to the nonorientable realm and give a quick survey of the main known results.

- In [37] (1976), Cappell and Shaneson construct a smooth 4-manifold that is simple homotopy equivalent but not smoothly h-cobordant to $\mathbb{RP}^4$, which is in particular an exotic $\mathbb{RP}^4$ in light of Freedman–Quinn [59].
- In [3] (1982), Akbulut constructs an exotic copy of $(S^1 \widetilde{\times} S^3)\,\#(S^2 \times S^2)$, building on work of Cappell and Shaneson [37].
- In [147] (2017), Torres constructs new exotic structures on the nonorientable 4-manifolds $(\mathbb{RP}^2 \widetilde{\times} S^2)\,\#_k(S^2 \times S^2)$ and $\#_r\,\mathbb{RP}^4\,\#_k(S^2 \times S^2)$ for every $k \geq 0$ and $1 \leq r \leq 3$, the diffeomorphism types are distinguished by an invariant associated to Pin^+-structures.

Outline

We now give a short overview of the main results of this thesis. Additional results are presented in the following chapters and are labeled with letters. Some of these are related to the results stated below, while others are independent.

The structure of Chapter 1 follows the three main steps of the "recipe" for building new exotic 4-manifolds. Therein, we list and describe the construction tools, the classification results and the invariants which are used to manufacture the exotic families in the following chapters.

In Chapter 2, we construct infinite exotic families of small irreducible 4-manifolds with fundamental group $\mathbb{Z}$ and $\mathbb{Z}^2$. These are produced via torus surgery on 4-manifolds already present in the literature [15, 16, 14, 27, 28, 55, 56, 139]. In particular, such examples support Conjecture 0.0.1 of Friedl–Hambleton–Melvin–Teichner and satisfy the equality $b_2 = |\sigma| + 4$, where b_2 and σ respectively denote the second Betti number and the signature. Our main contribution corresponds to the second step of the above described "recipe". In particular, we compute the *equivariant intersection form* of the 4-manifolds in the constructed families; the homeomorphism types are then pinned down by classification results due to Freedman–Quinn [59], Hambleton–Kreck–Teichner [78] and Stong–Wang

[138]. We remark that computing equivariant intersection forms is (in general) a highly non-trivial task, since this is a much more complicated object than the integral intersection form defined on the second homology modulo torsion, see Subsection 1.2.2. The diffeomorphism types are then distinguished by the Seiberg-Witten invariants, following an approach due to Fintushel-Park-Stern [55] based on the Morgan-Mrowka-Szabó formula [116]. In particular, we have the following main result.

Theorem 1 (cf. Theorem A). *For any $k \in \{2,3,4,5\}$, there are exotic families*

$$\{M_{p,k} \mid p \in \mathbb{N}\} \text{ and } \{N_{p,k} \mid p \in \mathbb{N}\}$$

of closed oriented irreducible 4-manifolds that satisfy the following properties.

1. *There are homeomorphisms*

$$M_{p,k} \cong_{\mathcal{C}^0} \#_2\, \mathbb{CP}^2 \#_{k+1}\, \overline{\mathbb{CP}}^2 \# (S^1 \times S^3)$$
$$N_{p,k} \cong_{\mathcal{C}^0} \#_2\, \mathbb{CP}^2 \#_{k+1}\, \overline{\mathbb{CP}}^2 \# (T^2 \times S^2)$$

   for every $p \in \mathbb{N}$.

2. *There are diffeomorphisms*

$$M_{p,k} \# (S^2 \times S^2) \cong_{\mathcal{C}^\infty} M_{0,k} \# (S^2 \times S^2)$$
$$N_{p,k} \# (S^2 \times S^2) \cong_{\mathcal{C}^\infty} N_{0,k} \# (S^2 \times S^2)$$

   for every $p \in \mathbb{N}$.

Moreover, the 4-manifolds $M_{1,k}$ and $N_{1,k}$ admit a symplectic structure of symplectic Kodaira dimension two.

Chapter 3 is also devoted to the construction of new exotic 4-manifolds, both in the oriented and in the nonorientable setting. Our main contribution here is a cut-and-paste procedure to "add fundamental group" to exotic pairs of 4-manifolds. We work out explicit examples from the work of [107] and [37], where the diffeomorphism types can be distinguished by the Oszváth–Szabó mixed invariant for closed 4-manifolds and by the η-invariant of Pin^+-structures (see Subsection 1.3.3), respectively. More precisely, we get the following results.

Theorem 2 (cf. Theorem C). *For every odd $p \geq 1$, there is a closed smooth oriented 4-manifold $\mathcal{R}_{2p}$ that is homeomorphic but not diffeomorphic to the connected sum $\Sigma_{2p} \#_4 \overline{\mathbb{CP}}^2$, where Σ_{2p} is a $\mathbb{Q}$-homology 4-sphere with cyclic fundamental group of order $2p$. The diffeomorphism types are distinsguished by the Ozsváth–Szabó mixed invariant.*

Theorem 3 (cf. Theorem D). *For every odd $p \geq 1$, there are closed smooth irreducible nonorientable 4-manifolds $A(2p)$ and $B(2p)$ with Euler characteristic one and cyclic fundamental group of order $2p$ that satisfy the following properties.*

- *There is a homeomorphism* $A(2p) \cong_{\mathcal{C}^0} B(2p)$.
- *The orientation double covering of* $A(2p)$ *is diffeomorphic to the orientation double covering of* $B(2p)$.
- *There is no diffeomorphism between* $A(2p)\,\#_k(S^2 \times S^2)$ *and* $B(2p)\,\#_k(S^2 \times S^2)$ *for any* $k \geq 0$.

The nonorientable exotic 4-manifolds in Chapter 4 are constructed by "twisting" embedded 2-spheres and real projective planes in standard 4-manifolds. In particular, we have the following results.

Theorem 4 (cf. Theorem E and Theorem G)**.** *There is a smoothly embedded* 2*-sphere* $S \subseteq \mathbb{RP}^2 \times S^2$ *with trivial normal bundle* $\nu(S) \cong D^2 \times S^2$ *such that the complements*

$$(\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(\{pt\} \times S^2) \text{ and } (\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(S)$$

are not homotopy equivalent, and doing a Gluck twist on S *yields a* 4*-manifold* Y *that is homeomorphic but not diffeomorphic to* $\mathbb{RP}^2 \mathbin{\widetilde{\times}} S^2$*, the nonorientable total space of the non-trivial* S^2*-bundle over* $\mathbb{RP}^2$*.*

Moreover, there is a smoothly embedded real projective plane $\mathbb{RP}^2 \subseteq Y$ *such that twisting it produces a* 4*-manifold* Z *that is homeomorphic but not diffeomorphic to the circle sum* $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$ *of two copies of the real projective* 4*-space along the standard real projective lines.*

We refer the reader to Definition 1.1.29 for the notion of circle sum. As in Chapter 3, the diffeomorphism types are distinguished by the η-invariant modulo $2\mathbb{Z}$ of the twisted Dirac operator associated to the Pin^+-structures.

Motivated by the results on nonorientable exotica in Chapter 3 and 4, in Chapter 5 we provide a purely combinatorial description of Pin^+- (and Pin^--) structures on nonorientable Lefschetz fibrations over the 2-disk.

Theorem 5 (cf. Theorem H and Theorem I)**.** *Let* X *be a smooth* 4*-manifold and* $f\colon X \to D^2$ *a Lefschetz fibration with regular fiber* Σ*.*

- *There is no* Pin^-*-structure on* X *if and only if there are* $k+1$ *vanishing cycles* $c_0, c_1, \ldots, c_k \subseteq \Sigma$ *such that* $[c_0] = \sum_{i=1}^k [c_i] \in H_1(\Sigma;\mathbb{Z}_2)$ *and* $k + \sum_{1 \leq i < j \leq k} c_i \cdot c_j \equiv 0 \mod 2$.
- *Suppose that* f *has vanishing cycles* $c_1, \ldots, c_n \subseteq \Sigma$*. Then* X *supports a* Pin^+*-structure if and only if* Σ *supports a* Pin^+*-structure and*

$$\mathit{rank}(C) = \mathit{rank}(C \mid A),$$

where C *is the* $\mathbb{Z}_2$*-reduction of the* $n \times r$ *matrix whose rows are given by the components of* $[c_1], \ldots, [c_n] \in H_1(\Sigma;\mathbb{Z}_4) \cong \mathbb{Z}_4^r$ *with respect to a fixed basis* $e_1, \ldots, e_r$ *and*

A is the column vector with entries $A_i = 1 + q_0^+([c_i]) \in \mathbb{Z}_2$ for $i = 1, \ldots, n$. Here, $q_0^+ \colon H_1(\Sigma; \mathbb{Z}_4) \to \mathbb{Z}_2$ is the enhancement corresponding to one of the Pin$^+$-structures on Σ, see Section 5.3.

We believe that a better understanding of such structures and of the η-invariant could potentially lead to the discovery of new nonorientable exotic 4-manifolds. We also provide a new proof of the well-known fact that any closed 3-manifold admits a Pin$^-$-structure and we find a criterion to check whether or not it admits a Pin$^+$-structure in terms of a handlebody decomposition. A characterization of Pin$^+$-structures on vector bundles á la Milnor is also discussed.

The appendix is based on a short survey article on the following theorem by Dold and Whitney on the parallelizability of 4-manifolds [21].

Theorem 6 (Dold–Whitney, cf. Theorem L)**.** *A closed orientable 4-manifold X is parallelizable if and only if its second Stiefel-Whitney class $w_2(X)$, first Pontryagin class $p_1(X)$ and Euler characteristic $\chi(X)$ are all vanishing.*

Open problems

Simply connected realm

It is still unknown whether or not every closed oriented smooth (simply connected) 4-manifold supports nonequivalent smooth structures. In particular, the existence of exotic smooth structures on the 4-sphere is one of the most famous and intriguing open problems in low dimensional topology. The difficulty in dealing with this question boils down to the fact that finding a non-trivial invariant for smooth structures on a 4-manifold with such a simple topology is a highly non-trivial task. We point out that there are many other "small" 4-manifolds on which the existence of exotic smooth structures is still unknown; some other examples are the complex projective plane $\mathbb{CP}^2$, its blow-up $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, the product manifolds $S^2 \times S^2$, $S^1 \times S^3$, $T^2 \times S^2$ and many more.

Infinite cyclic fundamental group

Hambleton and Teichner show in [80] that closed oriented topological 4-manifolds with $\mathbb{Z}$-fundamental group are classified up to homeomorphism by their intersection form when the inequality $b_2 \geq |\sigma| + 6$ is satisfied. On the other hand, as already mentioned, they provide a counterexample for when such a bound is not achieved. In particular, Friedl–Hambleton–Melvin–Teichner show in [60] that one of the 4-manifolds in the example by Hambleton and Teichner is not smoothable. As a consequence, they formulate the following conjecture, which is still open.

Conjecture 0.0.1 (Friedl–Hambleton–Melvin–Teichner). *Given a closed oriented smooth 4-manifold with $\mathbb{Z}$-fundamental group, is its homeomorphism type determined by its intersection form?*

Nonorientable realm

Moving to the nonorientable realm, the existence of infinite exotic families of nonorientable 4-manifolds is still an open question. Indeed, all the known examples of nonorientable exotica known to date come in finite families.

Notation and conventions

- We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, while the ring of integers is denoted by $\mathbb{Z}$.
- The complex projective n-dimensional space with its canonical and reversed orientation respectively is denoted by $\mathbb{CP}^n$ and $\overline{\mathbb{CP}}^n$, the real projective n-dimensional space is $\mathbb{RP}^n$, the n-dimensional torus is T^n, S^n is the standard n-sphere and D^n is the n-dimensional closed disk.
- Given two closed n-manifolds X and Y, we denote by $X \# Y$ their connected sum, i.e. the n-manifold
$$X \# Y = (X \setminus \operatorname{Int} D^n) \cup_\partial (Y \setminus \operatorname{Int} D^n)$$
obtained by gluing along their boundary the complements of two open n-dimensional balls inside X and Y respectively.
- Given two smoothly embedded transverse surfaces $A, B \subseteq X$ in a 4-manifold, we denote by $A \cdot B$ the algebraic intersection number.
- If $A \subseteq X$ is a (smooth) submanifold, we denote by $\nu(A) \subseteq X$ one of its closed tubular neighbourhoods.
- For a given n-manifold X, we denote by $\widetilde{X}$ its universal covering.
- All manifolds are assumed to be smooth, unless stated otherwise.
- Unless otherwise specified, homology groups are taken with integer coefficients.
- We denote by b_i the i^{th} Betti number of a space and by χ its Euler characteristic.
- Given a n-manifold X, we denote by $w_i(M) \in H^i(X; \mathbb{Z}_2)$ its i^{th} Stiefel–Whitney class and, in the case M is complex or almost-complex, by $c_i(M) \in H^{2i}(M)$ its i^{th} Chern class. If $n = 4$, its signature is indicated by $\sigma(M)$.

- A pair $(X, \mathfrak{t})$ given by a manifold X with a fixed Spin^c-structure $\mathfrak{t}$ is called a Spin^c-manifold.

Chapter 1

Background and preliminary results

1.1 Construction tools

We now survey all the cut-and-paste operations employed in this thesis to construct new 4-manifolds. We start with torus surgeries, which will be largely employed in Chapter 2.

1.1.1 Torus surgeries and framings

Let X be a smooth 4-manifold and let $T \subseteq X$ be a smoothly embedded 2-torus with trivial normal bundle. Fix a *framing* of T, i.e. a D^2-bundle identification

$$\phi \colon \nu(T) \longrightarrow T^2 \times D^2 \tag{1.1.1}$$

under which $T \subseteq \nu(T)$ corresponds to the zero section $T^2 \times \{0\} \subseteq T^2 \times D^2$. Denote by $p_1 \colon T^2 \times D^2 \to T^2$ the projection onto the first factor and let $a, b \subseteq T$ be simple loops whose homotopy classes generate $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$. The *push-offs* of a and b under the framing (1.1.1) are the simple loops

$$S^1_a = \phi^{-1}(a \times \{*\}) \quad \text{and} \quad S^1_b = \phi^{-1}(b \times \{*\}),$$

where $* \in \partial D^2$. Notice that S^1_a and S^1_b are homologous to a and b respectively inside $\nu(T)$. In particular,

$$H_1(\partial \nu(T)) \cong \mathbb{Z}^3 \cong \langle S^1_a, S^1_b, \mu_T \rangle$$

where μ_T denotes a meridian of T, i.e. a simple loop corresponding to $\{*\} \times \partial D^2 \subseteq T^2 \times D^2$ under the framing 1.1.1 [56, Section 3]. We also fix a *surgery curve* $\gamma \subseteq \partial(X \setminus \operatorname{Int} \nu(T))$, given by the push-off of a primitive curve $\gamma_T \subseteq T$.

The framings used in the sequel correspond to one of the following two canonical choices.

- If the 4-manifold X has a symplectic structure for which T is Lagrangian [1], we will choose (1.1.1) to be the *Lagrangian framing* (see [85]).

- If $[T] = 0 \in H_2(X)$, there is preferred framing known as the null-homologous framing [85, Section 5.3]. Under the *null-homologous framing* of T, we have that the surgery curve γ is such that

$$[\gamma] = 0 \in H_1(X \setminus \operatorname{Int} \nu(T)). \tag{1.1.2}$$

Definition 1.1.1. [Torus surgery] Given a pair of co-prime integers $p, q \in \mathbb{Z}$, the result of (p/q)*-torus surgery on T along the surgery curve* γ is the smooth 4-manifold

$$X_{T,\gamma}(p/q) = (X \setminus \operatorname{Int} \nu(T)) \cup_{\varphi} (T^2 \times D^2), \tag{1.1.3}$$

where the diffeomorphism $\varphi\colon T^2 \times \partial D^2 \to \partial(X \setminus \operatorname{Int} \nu(T))$ is required to satisfy

$$\varphi_*([\{pt\} \times \partial D^2]) = q[\gamma] + p[\mu_T] \in H_1(\partial(X \setminus \operatorname{Int} \nu(T))). \tag{1.1.4}$$

Remark 1.1.2. Recall that ± 1 are the only integers that are co-prime with 0. In particular, if we take $p = 0$ in Definition 1.1.1, then necessarily $q = \pm 1$ and vice versa.

Remark 1.1.3. The diffeomorphism type of $X_{T,\gamma}(p/q)$ is fully determined by the condition 1.1.4, since any two gluing maps satisfying 1.1.4 differ by an automorphism of $T^2 \times \partial D^2$ extending to $T^2 \times D^2$.

The submanifold

$$\widehat{T}_{p/q} = T^2 \times \{(0,0)\} \subseteq T^2 \times D^2 \tag{1.1.5}$$

of $X_{T,\gamma}(p/q)$ is called the *core* 2*-torus* of the surgery. Notice that

$$X \setminus \operatorname{Int} \nu(T) = X_{T,\gamma}(p/q) \setminus \operatorname{Int} \nu(\widehat{T}_{p/q}). \tag{1.1.6}$$

In the sequel, we will set

$$X_{T,\gamma}(p) = X_{T,\gamma}(p/1)$$

and call such a surgery a p*-torus surgery*. On the other hand, a $(\pm 1/q)$-torus surgery on a symplectic 4-manifold along a Lagrangian 2-torus with respect to the Lagrangian framing is called a *Luttinger surgery* [20, 108]. For further details on this cut-and-paste operation, we refer the reader to [28, Section 2.1], [85], [73, p. 310].

The following proposition summarizes several basic topological properties of this surgical operation.

---

[1] Given a symplectic 4-manifold (X, ω), a smoothly embedded 2-torus $T \subset X$ is called *Lagrangian* if the restriction $\omega|_T$ is trivial.

Proposition 1.1.4. *The signature and the Euler characteristic of a closed oriented* 4*-manifold* X *are invariant under torus surgeries. The fundamental group of a* 4*-manifold obtained by* (p/q)*-torus surgery is given by*

$$\pi_1(X_{T,\gamma}(p/q)) = \frac{\pi_1(X \setminus \operatorname{Int} \nu(T))}{\lhd\, [\mu_T]^p[\gamma]^q \,\rhd},$$

where $\lhd\, [\mu_T]^p[\gamma]^q \,\rhd$ *is the normal subgroup generated by* $[\mu_T]^p[\gamma]^q$*.*

Let $T \subseteq X$ *be a homologically essential embedded* 2*-torus with trivial normal bundle and* $\gamma_T \subseteq T$ *a curve whose homology class is primitive in* $H_1(X)$*. If* $q \neq 0$*, the following facts hold.*

1. *The first Betti numbers satisfy the relation*

$$b_1(X_{T,\gamma}(p/q)) = b_1(X) - 1.$$

2. *The second Betti numbers satisfy the relation*

$$b_2(X_{T,\gamma}(p/q)) = b_2(X) - 2.$$

Furthermore, if $q = 1$*, the core* 2*-torus* $\widehat{T}_p \subseteq X_{T,\gamma}(p)$ *is null-homologous.*

Remark 1.1.5. Torus surgeries can drastically change the topology of a given 4-manifold X. Indeed, work of Iwase [86] and Baykur–Morgan [29] implies that any closed, connected, smooth 4-manifold X is the result of multiple torus surgeries on disjointly embedded 2-tori inside a universal 4-manifold

$$Z(a,b,c,d,e) = \#_a(\mathbb{RP}^2 \times S^2)\,\#_b\,\mathbb{RP}^4\,\#_c(S^1 \times S^3)\,\#_d\,\mathbb{CP}^2\,\#_e\,\overline{\mathbb{CP}}^2$$

where the coefficients $a,b,c,d,e \in \mathbb{N}$ depend on the orientability, the signature and the Euler characteristic of X.

Torus surgeries can be reversed in the following sense: the original 4-manifold X can be obtained from $X_{T,\gamma}(p/q)$ by surgery on the core 2-torus $\widehat{T}_{p/q}$. A detailed description of this procedure for p-torus surgeries is given in the following lemma, which will be used in Section 2.4. For the convenience of the reader, we provide two proofs of this fact.

Lemma 1.1.6. *Let* $X_{T,\gamma}(p)$ *be the* 4*-manifold defined in (1.1.3). The core* 2*-torus* $\widehat{T}_p$ *is framed by identifying one of its tubular neighbourhoods with the copy of* $T^2 \times D^2$ *which is glued to* $X \setminus \operatorname{Int} \nu(T)$ *during the surgery procedure. The* 4*-manifold* X *is the result of* 0*-torus surgery on the core* 2*-torus* $\widehat{T}_p \subseteq X_{T,\gamma}(p)$ *along a curve whose push-off corresponds to the meridian* μ_T *of the surgery* 2*-torus* $T \subseteq X$ *under the identification*

$$X_{T,\gamma}(p) \setminus \operatorname{Int} \nu(\widehat{T}_p) = X \setminus \operatorname{Int} \nu(T).$$

First proof of Lemma 1.1.6. Fix a framing $\phi\colon \nu(T) \to T^2 \times D^2$ of the surgery 2-torus $T \subseteq X$. Without loss of generality, we can assume that the surgery curve γ corresponds to the second S^1-factor of $T^2 \cong S^1 \times S^1$. Following [2, Section 6.3], we choose the gluing map $\varphi\colon T^2 \times \partial D^2 \to \partial(X \setminus \operatorname{Int} \nu(T))$ defining $X_{T,\gamma}(p)$ (1.1.3) in such a way that the induced map in first homology is given by the matrix

$$\varphi_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & p \end{pmatrix} \tag{1.1.7}$$

with respect to the basis $\{[S^1 \times \{*\} \times \{*\}], [\{*\} \times S^1 \times \{*\}], [\{*\} \times \{*\} \times \partial D^2]\}$ of $H_1(S^1 \times S^1 \times \partial D^2)$, where the 2-torus T^2 is written as the product of two S^1-factors.

The identification

$$X = (X \setminus \operatorname{Int} \nu(T)) \cup_{\mathrm{Id}} \nu(T) \cong (X_{T,\gamma}(p) \setminus \operatorname{Int} \nu(\widehat{T}_p)) \cup_\psi (T^2 \times D^2) \tag{1.1.8}$$

shows that X is obtained by surgery on $X_{T,\gamma}(p)$ along the core 2-torus $\widehat{T}_p$, where the gluing map

$$\psi\colon T^2 \times \partial D^2 \to \partial(X_{T,\gamma}(p) \setminus \operatorname{Int} \nu(\widehat{T}_p)) \tag{1.1.9}$$

sends the curve $\{*\} \times \partial D^2$ to a loop α which under the identification $\partial(X_{T,\gamma}(p) \setminus \operatorname{Int} \nu(\widehat{T}_p)) = \partial(X \setminus \operatorname{Int} \nu(T))$ coincides with the meridian μ_T of the surgery 2-torus $T \subseteq X$. Notice that α is also the push-off of an essential curve $\alpha_{\widehat{T}_p}$ on the core 2-torus $\widehat{T}_p$. Indeed, by inverting the matrix (1.1.7), one can check that the homology class of $\mu_T \subseteq \partial(X \setminus \operatorname{Int} \nu(T))$ is mapped to the one defined by the second S^1-factor of $S^1 \times S^1 \times D^2$, which in turn is identified with $\nu(\widehat{T}_p) \subseteq X_{T,\gamma}(p)$ by our choice of the framing. In particular, the loop $\alpha_{\widehat{T}_p} \subseteq \widehat{T}_p$ giving α as a push-off corresponds to the second S^1-factor under the identification $\widehat{T}_p \cong S^1 \times S^1 \times \{0\} \subseteq S^1 \times S^1 \times D^2$. Since $\psi_*([\{*\} \times \partial D^2]) = [\alpha]$ the conclusion follows. □

Our second proof of Lemma 1.1.6 employs the useful technology of round 2-handles.

Second Proof of Lemma 1.1.6. A result of Baykur–Sunukjian [31, Lemma 1] implies that there is a cobordism

$$V = (I \times X) \cup R_2 \tag{1.1.10}$$

such that $\partial_- V = X$ and $\partial_+ V = X_{T,\gamma}(p)$, where $R_2 = S^1 \times D^2 \times D^2$ is a five-dimensional round 2-handle glued to $\partial_+(I \times X)$ via a map

$$\phi\colon R_2 = S^1 \times D^2 \times D^2 \to \partial_+(I \times X) \tag{1.1.11}$$

sending $S^1 \times \partial D^2 \times \{0\}$, $S^1 \times \partial D^2 \times D^2$ and $\{*\} \times \partial D^2 \times \{0\}$ homeomorphically onto $\{1\} \times T$, $\{1\} \times \nu(T)$ and $\{1\} \times \gamma$ respectively. In particular, the curve $\{*\} \times \{*\} \times \partial D^2 \subseteq$

$S^1 \times \partial D^2 \times D^2$ is identified with a meridian of the surgery 2-torus $T \subseteq X$. Turning the cobordism V upside down, we get a cobordism

$$W = (I \times X_{T,\gamma}(p)) \cup R_2' \tag{1.1.12}$$

where the dual round 2-handle $R_2' = R_2$ is glued to $\partial_+(I \times X_{T,\gamma}(p))$ via a map

$$\psi\colon S^1 \times D^2 \times \partial D^2 \to \partial_+(I \times X_{T,\gamma}(p)) \tag{1.1.13}$$

sending $S^1 \times \{0\} \times \partial D^2$, $S^1 \times D^2 \times \partial D^2$ and $\{*\} \times \{0\} \times \partial D^2$ homeomorphically onto $\{1\} \times \widehat{T}_p$, $\{1\} \times \nu(\widehat{T}_p)$ and $\{1\} \times \alpha$ respectively, where $\alpha \subseteq \widehat{T}_p$ is an essential curve. [31, Lemma 1] by Baykur-Sunukjian implies that X is obtained from $X_{T,\gamma}(p)$ via a torus surgery on the core 2-torus $\widehat{T}_p$ along α. Since the image of $\{*\} \times \{*\} \times \partial D^2 \subseteq S^1 \times \partial D^2 \times D^2$ under (1.1.13) is a push-off of α which has been identified with a meridian of T by (1.1.11), the result follows. $\square$

Luttinger surgery, irreducibility, and the symplectic Kodaira dimension

In this subsection we survey several results of Auroux–Donaldson–Katzarkov [20], Kotschick [101], Ho–Li [85] and Luttinger [108] that are used to conclude on the properties of the symplectic 4-manifolds of Theorem A.

Definition 1.1.7. A 4-manifold X is said to be *minimal* if it does not contain any smoothly embedded 2-spheres of self-intersection -1.

Theorem 1.1.8. *Let (X, ω) be a symplectic 4-manifold containing a Lagrangian 2-torus $T \subseteq X$ of self-intersection 0. We equip T with the Lagrangian framing.*

- (Luttinger [108], Auroux–Donaldson–Katzarkov [20]) *The 4-manifold $X_{T,\gamma}(\pm 1/q)$ admits a symplectic structure.*
- (Ho-Li [85, Proposition 3.1]) *If (X, ω) is minimal, then $X_{T,\gamma}(\pm 1/q)$ is minimal.*

Let (X, ω) be a minimal symplectic 4-manifold with Chern class $K_\omega = c_1(X, J)$, where J is an ω-compatible almost-complex structure on X.

Definition 1.1.9. (Ho–Li [85, Definition 2.2]) The *symplectic Kodaira dimension* $\mathrm{Kod}(X, \omega)$ of a minimal symplectic 4-manifold (X, ω) is defined by

$$\mathrm{Kod}(X,\omega) = \begin{cases} -\infty & \text{if } K_\omega \cdot [\omega] < 0 \text{ or } K_\omega \cdot K_\omega < 0 \\ 0 & \text{if } K_\omega \cdot [\omega] = 0 \text{ and } K_\omega \cdot K_\omega = 0 \\ 1 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega = 0 \\ 2 & \text{if } K_\omega \cdot [\omega] > 0 \text{ and } K_\omega \cdot K_\omega > 0 \end{cases}.$$

The symplectic Kodaira dimension of a non-minimal symplectic 4-manifold is defined as the symplectic Kodaira dimension of any of its minimal models.

Ho–Li's study of the effect that Luttinger surgery has on the Chern class K_ω and on the symplectic class $[\omega]$ of a symplectic 4-manifold (X, ω) yielded the following result.

Theorem 1.1.10 (Ho–Li [85]). *The symplectic Kodaira dimension is preserved under Luttinger surgery.*

The reader is directed towards [85, Section 3] for further details.

Besides minimality, we are interested in 4-manifolds with the following property.

Definition 1.1.11 (Definition 10.1.17, [73]). A smooth 4-manifold X is *irreducible* if for every smooth connected sum decomposition $X = X_1 \# X_2$ either X_1 or X_2 is a homotopy 4-sphere.

In order to conclude on the irreducibility of our examples, we will employ the following result.

Theorem 1.1.12 (Kotschick [101]). *Every minimal symplectic* 4*-manifold* (X, ω) *with residually finite fundamental group and* $b_2^+(X) > 1$ *is irreducible.*

Remark 1.1.13. Free abelian groups are residually finite.

1.1.2 A cut-and-paste-procedure to increase the fundamental group

In the following, we describe the cut-and-paste procedure employed in the construction of the 4-manifolds in Chapter 3.

Oriented setting

Let $\alpha \subseteq S^1 \times S^3$ be a simple loop whose homotopy class generates $\pi_1(S^1 \times S^3) \cong \mathbb{Z}$. Denote by $\alpha_p \subseteq S^1 \times S^3$ a simple loop representing the homology class $[\alpha]^p \in H_1(S^1 \times S^3) \cong \mathbb{Z}$. The main building block for our cut-and-paste construction is the compact 4-manifold

$$S_p = (S^1 \times S^3) \setminus \operatorname{Int} \nu(\alpha_p) \tag{1.1.14}$$

with boundary $\partial S_p = S^1 \times S^2$.

Definition 1.1.14. Given a smooth 4-manifold X and a simple loop $\alpha_X \subseteq X$, we define

$$X(p) = (X \setminus \operatorname{Int} \nu(\alpha_X)) \cup_\varphi S_p, \tag{1.1.15}$$

where $\varphi \colon \partial(X \setminus \operatorname{Int} \nu(\alpha_X)) \to \partial S_p$ is a chosen diffeomorphism.

Example 1.1.15. The closed smooth oriented 4-manifold

$$\Sigma_p = (D^2 \times S^2) \cup_\varphi S_p = S^4(p) \tag{1.1.16}$$

is a $\mathbb{Q}$-homology 4-sphere with $\pi_1(\Sigma_p) = \mathbb{Z}_p$.

A description of the cyclic coverings of the building block S_p is given in our next lemma.

Lemma 1.1.16. *Let S_p be the 4-manifold with boundary defined in (1.1.14).*

- *For every $p \in \mathbb{Z}_{>0}$, there is a double covering*

$$D_p \xrightarrow{2:1} S_p$$

corresponding to the index-two subgroup $2\mathbb{Z} \subseteq \mathbb{Z} = \pi_1(S_p)$.

- *If p is odd, then there is a diffeomorphism $D_p \cong_{\mathcal{C}^\infty} S_p$.*
- *If $p = 2q$, then there is a diffeomorphism*

$$D_p \cong_{\mathcal{C}^\infty} (S^1 \times S^3) \setminus \operatorname{Int} \nu(\alpha_q \sqcup \alpha'_q)$$

where α'_q is a parallel copy of α_q.

Proof. For any $p > 0$, the universal covering of S_p is

$$\widetilde{S}_p = \mathbb{R} \times (S^3 \setminus \{B_1, \dots, B_p\})$$

where $\{B_1, \dots, B_p\} \subseteq S^3$ are disjointly embedded smooth open 3-balls. The $\mathbb{Z}$-action on $\widetilde{S}_p$ is generated by the map

$$\begin{aligned} \phi \colon \widetilde{S}_p &\longrightarrow \widetilde{S}_p \\ (t, x) &\mapsto (t+1, f_p(x)) \end{aligned}$$

where f_p is the restriction of a smooth automorphism of S^3 which cyclically permutes the 3-balls $B_1, \dots, B_p$. The 2-fold covering of S_p associated to the subgroup $2\mathbb{Z} \subseteq \mathbb{Z} = \pi_1(S_p)$ is given by the quotient of $\widetilde{S}_p$ by the $\mathbb{Z}$-action generated by the map ϕ^2. The conclusion follows from the fact that the square of a cyclic permutation of odd length is still cyclic, while the square of a cyclic permutation of even length factors as a product of two cycles of half the original length. □

Remark 1.1.17. In the following, it will be useful to consider the genus one Heegaard splitting of the 3-sphere

$$S^3 = (S^1 \times D^2) \cup (D^2 \times S^1). \tag{1.1.17}$$

We can assume that the map f_p in the proof of Lemma 1.1.16 is a rotation of S^3 of angle $2\pi/p$, which restricts to a rotation along the D^2-factor on the solid torus on the left side of (1.1.17) and to a rotation along the S^1-factor on the solid torus on the right side. Moreover, we can choose the 3-balls $B_1, \dots, B_p$ to be symmetrically embedded in the solid torus on the right side of (1.1.17), where they are permuted by the action of f_p.

The simplicity of the construction makes it straight-forward to compute several invariants of the new 4-manifold (1.1.15).

Lemma 1.1.18. *The closed oriented* 4*-manifold* $X(p)$ *defined in (1.1.15) has the following topological invariants for every* $p > 0$*.*

1. $\chi(X(p)) = \chi(X)$;
2. $\sigma(X(p)) = \sigma(X)$.
3. $w_2(X(p)) = w_2(X)$.
4. *If* $\pi_1(X) = \mathbb{Z}_q$ *and the homotopy class of the simple loop* $\alpha_X \subseteq X$ *is a generator, then the fundamental group of* $X(p)$ *is* $\pi_1(X(p)) = \mathbb{Z}_{pq}$.

Proof. The proof is a consequence of straight-forward homological computations, Novikov's additivity and the Seifert–van Kampen theorem. □

Lemma 1.1.19. *Let* X *be a closed oriented* 4*-manifold and let* $\alpha_X \subseteq X$ *be a simple framed loop such that* $[\alpha_X] = 0 \in \pi_1(X)$. *There is a diffeomorphism*

$$X(p) = (X \setminus \operatorname{Int} \nu(\alpha_X)) \cup S_p \cong_{C^\infty} X \# \Sigma_p, \tag{1.1.18}$$

where Σ_p *is a* $\mathbb{Q}$*-homology* 4*-sphere with* $\pi_1(\Sigma_p) \cong \mathbb{Z}_p$.

Proof. The existence of the diffeomorphism (1.1.18) can be seen from the decomposition (1.1.16) in Example 1.1.15 and by noticing that

$$X = X \# S^4 = X \# ((D^2 \times S^2) \cup (S^1 \times D^3)). \tag{1.1.19}$$

□

Lemma 1.1.16 and Lemma 1.1.19 have the following consequence.

Lemma 1.1.20. *Let* X *be a closed connected oriented* 4*-manifold with universal covering* $\widetilde{X}$ *and let* $\alpha \subseteq X$ *be a simple framed loop generating* $\pi_1(X) = \mathbb{Z}_2$. *If* $p > 0$ *is odd, then the* 2*-fold covering* $\widehat{X}(2)$ *of the closed oriented* 4*-manifold*

$$X(p) = (X \setminus \operatorname{Int} \nu(\alpha)) \cup S_p$$

is diffeomorphic to

$$\widetilde{X}(p) = \widetilde{X} \# \Sigma_p.$$

We finish this subsection with the following observation regarding lifts of diffeomorphisms to coverings, whose proof is left to the reader.

Lemma 1.1.21. *Let*

$$f\colon X_1 \longrightarrow X_2 \tag{1.1.20}$$

be a diffeomorphism between two 4-manifolds X_1 and X_2 with fundamental group $\pi_1(X_i) = \mathbb{Z}_{2p}$ and let $\widetilde{X}_i$ be the 2-fold covering of X_i for $i = 1, 2$. The diffeomorphism (1.1.20) lifts to a diffeomorphism

$$\tilde{f}\colon \widetilde{X}_1 \longrightarrow \widetilde{X}_2.$$

Nonorientable setting

There are two ways to adapt the procedure described in the first part of this subsection to the nonorientable case. Notice that the framed simple loop $\alpha_X \subseteq X$ used in the construction (1.1.15) can be assumed to be orientation-preserving even if the ambient 4-manifold X is nonorientable and the construction described in that section results in a nonorientable 4-manifold $X(p)$. We now describe the second version of this construction.

Denote the total space of the nonorientable three-sphere bundle over the circle by $S^3 \widetilde{\times} S^1$ and let $\alpha \subseteq S^3 \widetilde{\times} S^1$ be a simple loop generating $\pi_1(S^3 \widetilde{\times} S^1) = \mathbb{Z}$. Let p be an odd integer and denote by $\alpha_p \subseteq S^3 \widetilde{\times} S^1$ the orientation-reversing simple loop representing $[\alpha]^p \in H_1(S^3 \widetilde{\times} S^1) = \mathbb{Z}$. Its tubular neighbourhood is diffeomorphic to the total space of the nonorientable 3-disk bundle over the circle, i.e. $\nu(\alpha_p) \cong D^3 \widetilde{\times} S^1$.

The main building block of out cut-and-paste construction in the nonorientable setting is the compact nonorientable 4-manifold

$$N_p = (S^3 \widetilde{\times} S^1) \setminus \operatorname{Int} \nu(\alpha_p) \tag{1.1.21}$$

with boundary $\partial N_p = S^2 \widetilde{\times} S^1$. Much like it was done in the orientable setting, remove the tubular neighbourhood of an orientation-reversing simple loop $\alpha_Y \subseteq Y$ from a closed nonorientable 4-manifold Y to obtain a compact nonorientable 4-manifold $Y \setminus \operatorname{Int} \nu(\alpha_Y)$ with boundary $\partial(Y \setminus \operatorname{Int} \nu(\alpha_Y)) = S^2 \widetilde{\times} S^1$. Cap it off with the compact 4-manifold (1.1.21) to construct the closed nonorientable 4-manifold

$$Y(p) = (Y \setminus \operatorname{Int} \nu(\alpha_Y)) \cup_\varphi N_p. \tag{1.1.22}$$

In the following, it will be useful to understand the universal covering of N_p, which we now describe via the following result.

Lemma 1.1.22. *For any odd $p > 0$, the compact nonorientable 4-manifold N_p defined in (1.1.21) is doubly covered by the oriented 4-manifold S_p (1.1.14):*

$$S_p \xrightarrow{2:1} N_p.$$

Proof. The two 4-manifolds S_p and N_p share the same universal covering. The deck transformations of N_p are generated by the map

$$\psi\colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i) \longrightarrow \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i)$$

given by

$$(t, x) \mapsto (t+1, g_p(x)),$$

where g_p is the restriction of an orientation reversing automorphism of S^3 that cyclically permutes $B_1, \ldots, B_p$. The conclusion follows from the existence of a diffeomorphism

$$S_p \longrightarrow (\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i))/ \sim_{\psi^2} .$$

□

Remark 1.1.23. We can view S^3 by means of its genus one Heegaard splitting (1.1.17) and assume that the map g_p in the proof of Lemma 1.1.22 is the composition of the map f_p of Remark 1.1.17 and a reflection ρ with respect to a two-sphere that intersects the first solid torus of (1.1.17) along an annulus $S^1 \times D^1$ and the second one in the disjoint union of two D^2-fibers.

The simplicity of the construction makes it straightforward to compute several basic invariants of the new 4-manifold (1.1.22) as we now sample; see [78] for details on these invariants.

Lemma 1.1.24. *The closed nonorientable* 4*-manifold* $Y(p)$ *defined in (1.1.22) has the following topological invariants for every* $p > 0$*.*

1. $\chi(Y(p)) = \chi(Y)$.
2. $w_2(Y(p)) = w_2(Y)$.
3. *If* $\pi_1(Y) = \mathbb{Z}_q$ *is generated by the homotopy class of the simple loop* $\alpha_Y \subseteq Y$*, then the fundamental group of* $Y(p)$ *is* $\pi_1(Y(p)) = \mathbb{Z}_{pq}$.

Much like in the case of Lemma 1.1.18, a proof of Lemma 1.1.24 is obtained through straightforward (co)homological computations, cobordism arguments and the Seifert–van Kampen theorem.

1.1.3 Gluck twists

Let X be a smooth 4-manifold and $S \subseteq X$ be a smoothly embedded 2-sphere with trivial normal bundle $\nu(S) \cong D^2 \times S^2$. Notice that there is a unique framing for S in this setting, see [73, Section 4.1]. The result of performing a Gluck twist to X on S is the 4-manifold

$$X_S = (X \setminus \operatorname{Int} \nu(S)) \cup_{\varphi} (D^2 \times S^2) \tag{1.1.23}$$

where the gluing diffeomorphism

$$\varphi\colon \partial(X \setminus \operatorname{Int}\nu(S)) \longrightarrow \partial(D^2 \times S^2) = S^1 \times S^2 \tag{1.1.24}$$

is given by

$$\varphi(\theta, x) = (\theta, \rho_\theta(x)) \tag{1.1.25}$$

for every $(\theta, x) \in S^1 \times S^2$. The map $\rho_\theta\colon S^2 \to S^2$ is a rotation of the 2-sphere through an angle $\theta \in \partial D^2 = S^1$ [66, Section 6].

Remark 1.1.25. For any smoothly embedded 2-sphere $S \subseteq S^4$, the Gluck twist S^4_S is a homotopy 4-sphere and hence a topological S^4 by Freedman–Quinn [59]. However, the smooth structure on S^4_S could a priori be non-standard. Indeed, several authors studied this cut-and-paste operation with the hope of finding exotic 4-spheres, see e.g. [118], [119] and [92].

1.1.4 Pin-structures and embedded surfaces

In order to introduce the remaining cut-and-paste operations in this section, we need to first discuss Pin^+ and Pin^--structures. These can be thought of as the non-oriented analogue of Spin-structures. We refer the reader to [78, 99, 137] for a precise definition of such Lie groups and for the background on $\mathrm{Pin}^\pm(n)$-structures on low dimensional manifolds. We recall some basic properties of these geometric structures in the present section.

In the oriented setting, it is known that the existence of a Spin-structure on a real vector bundle ξ on a manifold X is equivalent to the vanishing of $w_2(\xi) \in H^2(X, \mathbb{Z}_2)$, where w_i is the i^{th} Stiefel-Whitney class of ξ. In the case in which X is 4-dimensional, this is equivalent to having an even intersection form, provided that $H^2(X;\mathbb{Z})$ has no 2-torsion [73, Corollary 5.7.6]. This can be shown via the Wu formula, which states that for any $a \in H_2(X;\mathbb{Z}_2)$ one has

$$\langle w_2(X), a\rangle = a^2 \text{ modulo } 2 \tag{1.1.26}$$

where a^2 denotes the algebraic count of self-intersections of a surface in X representing a, see [73, Proposition 1.4.18]. In a similar way, there are cohomological obstructions to the existence of Pin^+ and Pin^--structures on the tangent bundle of a manifold, namely the vanishing of $w_2(X)$ and of $(w_2 + w_1^2)(X)$ respectively, see [99, Lemma 1.3]. Moreover, both Pin^+ and Pin^--structures (when they do exist) are torsors over $H^1(X, \mathbb{Z}_2)$, meaning that there is a free and transitive action of $H^1(X;\mathbb{Z}_2)$ over the sets of such geometric structures. However, the Wu formula does not hold when X is nonorientable, but one can still interpret the vanishing of $w_2(X)$ and $(w_2 + w_1^2)(X)$ in terms of properties of embedded surfaces. Indeed, for any 4-manifold X and $a \in H_2(X;\mathbb{Z}_2)$ there is a (possibly nonorientable) smoothly embedded surface $\Sigma \subseteq X$ representing a (see [73, Remark 1.2.4]) and one can show that

$$\langle w_2(X), a\rangle = \chi(\Sigma) + (w_1(\Sigma) \smile w_1(\nu(\Sigma)))([\Sigma]) + [\Sigma]^2 \text{ modulo } 2. \tag{1.1.27}$$

Since the tangent bundle of a 4-manifold supports a Pin^+-structure if and only if $w_2(X)$ vanishes, this is equivalent to checking that the evaluation (1.1.27) is trivial on any homology class $a \in H_2(X;\mathbb{Z}_2)$.

On the other hand, the obstruction for Pin^--structures to exist is $(w_2 + w_1^2)(X)$ [99, Lemma 1.3] so, in order to understand when such structures exist, we need to compute $w_1^2(X)$ and one can easily show that

$$\langle w_1^2(X), [\Sigma]\rangle = w_1^2(\Sigma) + w_1^2(\nu(\Sigma)). \tag{1.1.28}$$

Example 1.1.26 ($\mathbb{RP}^4$)**.** The 4-dimensional real projective space $\mathbb{RP}^4$ is an example of nonorientable 4-manifold supporting two distinct Pin^+-structures but no Pin^--structure. We have that $H_2(\mathbb{RP}^4;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by the homology class of $\mathbb{RP}^2$. Moreover, the tubular neighbourhood $\nu(\mathbb{RP}^2) \subseteq \mathbb{RP}^4$ is diffeomorphic to the twisted 2-disk bundle $D^2 \tilde{\times} \mathbb{RP}^2$ defined as the quotient of $D^2 \times S^2$ via the involution

$$\begin{aligned} D^2 \times S^2 &\longrightarrow D^2 \times S^2 \\ (x, y) &\mapsto (\rho_\pi(x), -y) \end{aligned}$$

where ρ_π denotes the rotation of S^2 of π radians about a fixed axis. One can show that such a quotient is diffeomorphic to

$$D^2 \tilde{\times} \mathbb{RP}^2 \cong (D^2 \times D^2) \cup_\phi (D^2 \times \mathrm{Mb}) \tag{1.1.29}$$

where Mb denotes the Möbius strip and ϕ is the map

$$\phi\colon D^2 \times S^1 \longrightarrow D^2 \times S^1, \quad (x, \theta) \mapsto (\rho_\theta(x), \theta)$$

and $\rho_\theta\colon D^2 \to D^2$ denotes the rotation of the 2-disk of angle θ with respect to the origin, see [2, Section 0] and [24, Example 7].

By (1.1.27) we have that

$$\langle w_2(\mathbb{RP}^4), [\mathbb{RP}^2]\rangle = \chi(\mathbb{RP})^2 + (w_1(\mathbb{RP}^2) \cup w_1(\nu(\mathbb{RP}^2)))([\mathbb{RP}^2]) + [\mathbb{RP}^2]^2 = 0 \in \mathbb{Z}_2$$

since $\chi(\mathbb{RP}^2) = 1 = [\mathbb{RP}^2]^2$ and $w_1(\nu(\mathbb{RP}^2)) = 0$, while (1.1.28) implies that

$$\langle w_1(\mathbb{RP}^4)^2, [\mathbb{RP}^2]\rangle = w_1^2(\mathbb{RP}^2) + w_1^2(\nu(\mathbb{RP}^2)) = 1 + 0 = 1 \in \mathbb{Z}_2$$

where $w_1^2(\mathbb{RP}^2)$ is computed as the self-intersection of a loop in $\mathbb{RP}^2$ whose $\mathbb{Z}_2$-homology class generates $H_1(\mathbb{RP}^2;\mathbb{Z}_2)$.

Remark 1.1.27. In this thesis we consider $\mathrm{Pin}^\pm$-structures on the tangent bundles of the manifolds under consideration, while in [79] they are defined on the normal bundles of embedded submanifolds.

In analogy to the Spin case, one can define the Pin^+ and Pin^- cobordism groups in each dimension, see [99, Section 1]. As sets, they consist of the equivalence classes of closed n-dimensional smooth manifolds with a fixed Pin^+ (resp. Pin^-)-structure, which are identified whenever they co-bound a Pin^+ (resp. Pin^-) $(n+1)$-manifold. The group operation is the one induced by the disjoint union of manifolds. We remark that Pin^+ and Pin^- cobordism groups are drastically different. In particular, the 4-dimensional Pin^+-cobordism group is

$$\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$$

(see [64, Section 2] and [99, Section 3]) and the η-invariant modulo $2\mathbb{Z}$ of the twisted Dirac operator associated to a Pin^+-structure on a 4-manifold is a complete invariant for Pin^+-cobordism classes, see [137]. In particular, there are instances in which such invariant can detect exotic behaviors in the nonorientable 4-dimensional realm, in the sense that there are examples of nonorientable Pin^+ 4-manifolds which share the same homeomorphism type but define two different classes in $\Omega_4^{\mathrm{Pin}^+}$ and hence cannot be diffeomorphic. To the best of our knowledge, following a suggestion in [65], this was shown for the first time in [137], where the η-invariant of Pin^+-structures is used to detect the exotic $\mathbb{RP}^4$ constructed in [37]. We remark that another exotic $\mathbb{RP}^4$ is constructed in [50] and its η-invariant is computed in [120]. An analogous approach has then been used also in [147] and in [24]. Chapter 4 is based on this latter work, so in this thesis we will see this strategy in action (see also Subsection 1.3.3).

On the other hand

$$\Omega_4^{\mathrm{Pin}^-} = 0$$

(see again [17, Theorem 5.1] and [99, Section 3]) and hence it is not possible to find Pin^- exotic 4-manifolds using this strategy. In particular, the study of Pin^+-structures on 4-manifolds could potentially lead to better understanding of exotic behaviors in the nonorientable realm.

1.1.5 Twisting an embedded real projective plane

Let X be a smooth 4-manifold and let $R \subseteq X$ be a smoothly embedded real projective plane with tubular neighbourhood $\nu(R)$ diffeomorphic to the non-trivial and nonorientable 2-disk bundle

$$D^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2 = \frac{D^2 \times S^2}{(x,y) \sim (-x,-y)} \tag{1.1.30}$$

see [76, Section 5]. The result of twisting $\nu(R) \subseteq X$ is the 4-manifold

$$X_R = (X \setminus \operatorname{Int} \nu(R)) \cup_{\varphi'} (D^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2), \tag{1.1.31}$$

where the gluing diffeomorphism

$$\varphi' \colon \partial(X \setminus \operatorname{Int} \nu(R)) \longrightarrow \partial(D^2 \mathbin{\widetilde{\times}} \mathbb{RP}^2) = S^1 \mathbin{\widetilde{\times}} S^2$$

is induced on the quotient $S^1 \tilde{\times} S^2 \cong \frac{S^1\times S^2}{(\theta,x)\sim(-\theta,-x)}$ by the Gluck twist map

$$\varphi\colon S^1 \times S^2 \longrightarrow S^1 \times S^2$$

defined in Equation (1.1.24), see [38, Section 2], [96].

There are instances where twisting an $\mathbb{RP}^2$ does not change the diffeomorphism type of X and it is equivalent to twisting a nonorientable 1-handle cf. [3, p. 82], [73, Figure 5.42]. In such a scenario, X is diffeomorphic to X_R and the result of this self-diffeomorphism is to relate the Pin^+-structures on X as we now exemplify.

Example 1.1.28. Deconstruct the real projective 4-space as

$$\mathbb{RP}^4 = (D^2 \tilde{\times} \mathbb{RP}^2) \cup (D^3 \tilde{\times} S^1) \tag{1.1.32}$$

and twist the obvious $\mathbb{RP}^2$. Since any self-diffeomorphism of $S^2 \tilde{\times} S^1$ extends to a self-diffeomorphism of $D^3 \tilde{\times} S^1$, twisting the $\mathbb{RP}^2$ in (1.1.32) does not change the diffeomorphism type of $\mathbb{RP}^4$. There are two Pin^+-structures $(\mathbb{RP}^4, \pm\phi^+_{\mathbb{RP}^4})$ and they are mutual inverses in the fourth Pin^+-cobordism group $\Omega_4^{\mathrm{Pin}^+} = \mathbb{Z}_{16}$, see Theorem 3.2.2 and [99]. The twist of the nonorientable 1-handle gives a self-diffeomorphism of $\mathbb{RP}^4$ that maps $\pm\phi^+_{\mathbb{RP}^4}$ to $\mp\phi^+_{\mathbb{RP}^4}$. We use the following convention to distinguish the Pin^+-structures:

$$\eta(\mathbb{RP}^4, \pm\phi^+_{\mathbb{RP}^4}) = \pm\frac{1}{8} \mod 2\mathbb{Z}. \tag{1.1.33}$$

1.1.6 Circle sums

Let $(X_1, \phi^+_{X_1})$ and $(X_2, \phi^+_{X_2})$ be closed Pin^+ 4-manifolds with $H^1(X_i;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and let $D^3 \tilde{\times} S^1$ be the non-trivial 3-disk bundle over S^1 endowed with one of its two Pin^+-structures. Recall that, if $\gamma_i \subseteq X_i$ is an orientation-reversing simple loop, there is a diffeomorphism $\nu(\gamma_i) \cong D^3 \tilde{\times} S^1$.

Definition 1.1.29. The *circle sum* of X_1 and X_2 (cf. [78]) is the smooth 4-manifold

$$X_1 \#_{S^1} X_2 = (X_1 \setminus \mathrm{Int}\, \nu(i_1(D^3 \tilde{\times} S^1))) \cup_\varphi (X_2 \setminus \mathrm{Int}\, \nu(i_2(D^3 \tilde{\times} S^1))) \tag{1.1.34}$$

where the gluing diffeomorphism is

$$\varphi = i_2| \cdot (i_1|)^{-1} \colon \partial(X_1 \setminus \mathrm{Int}\, \nu(i_1(D^3 \tilde{\times} S^1))) \longrightarrow \partial(X_2 \setminus \mathrm{Int}\, \nu(i_2(D^3 \tilde{\times} S^1))) \tag{1.1.35}$$

and the embeddings $i_i\colon D^3 \tilde{\times} S^1 \to X_i$ represent the generators of $H^1(X_i;\mathbb{Z}/2)$ via the universal coefficient theorem for $i = 1, 2$. Moreover, we ask i_1 to preserve the Pin^+-structure and i_2 to reverse it. This last condition implies that one can glue $\phi^+_{X_1}$ and $\phi^+_{X_2}$ to get a Pin^+-structure on $X_1 \#_{S^1} X_2$, that we shall denote by $\phi^+_{X_1 \#_{S^1} X_2}$.

Lemma 1.1.30. *Let (X_1, ϕ_1^+) and (X_2, ϕ_2^+) be two closed smooth nonorientable* Pin^+ *4-manifolds with $H^1(X_i; \mathbb{Z}_2) = \mathbb{Z}_2$ and let $\gamma_i \subseteq X_i$ be an orientation-reversing simple loop for $i = 1, 2$. There is a* Pin^+*-cobordism between the disjoint union $(X_1 \sqcup X_2, \phi_1^+ \sqcup \phi_2^+)$ and their circle sum $(X_1 \#_{S^1} X_2, \phi^+_{X_1 \#_{S^1} X_2})$ (1.1.34).*

Proof. (cf. [137, p. 160]). A cobordism W is obtained from $(X_1 \sqcup X_2) \times [0,1]$ by identifying its upper lid with the boundary of $(S^2 \mathbin{\widetilde{\times}} S^1) \times [0,1]$ as

$$(X_1 \setminus \operatorname{Int} \nu(i_1(D^3 \mathbin{\widetilde{\times}} S^1)) \times \{1\}) \cup ((S^2 \mathbin{\widetilde{\times}} S^1) \times [0,1]) \cup (X_2 \setminus \operatorname{Int} \nu(i_2(D^3 \mathbin{\widetilde{\times}} S^1)) \times \{1\}) \tag{1.1.36}$$

using the map (1.1.35). A Pin^+-structure on W is obtained by gluing together the matching Pin^+-structures on the building blocks in (1.1.36) and it induces a Pin^+-structure on $X_1 \#_{S^1} X_2$ [97, 99]. □

1.2 Detecting the homeomorphism type of a 4-manifold

1.2.1 The intersection form of a 4-manifold

We start this subsection by recalling the following definition.

Definition 1.2.1 (Intersection form)**.** Given a closed oriented smooth 4-manifold X, its *intersection form* is the symmetric bilinear pairing

$$I_X \colon (H_2(X)/\text{Torsion}) \times (H_2(X)/\text{Torsion}) \longrightarrow \mathbb{Z}$$

defined by

$$I_X(a,b) = A \cdot B$$

where A and B are closed oriented transversely embedded surfaces in X representing a and b respectively and $\cdot$ denotes the algebraic intersection number.

For further background and details, see [73, Section 1.2].

Remark 1.2.2. More generally, one can define the intersection form of a topological 4-manifold with possibly non-empty boundary. This is induced by the cup product pairing on the second cohomology via Poincaré (resp. Poincaré–Lefschetz) duality (see [73, Definition 1.2.1]). However, for the purpose of this note, it is enough to restrict to the closed smooth setting.

In order to apply the homeomorphism classification results, we are interested in studying the intersection form of a 4-manifold up to the following equivalence relation.

Definition 1.2.3. Let X and Y be two closed oriented 4-manifolds. An *isometry* between their intersection forms I_X and I_Y is a group isomorphism

$$h \colon H_2(X) \longrightarrow H_2(Y)$$

such that

$$I_X(a,b) = I_Y(h(a),h(b))$$

for all $a, b \in H_2(X)$. If such an isometry exists, we will say that the two intersection forms are *isometric* and write $I_X \cong I_Y$.

The following result explains the importance of the role of the intersection form of a 4-manifold in the construction of simply connected exotic 4-manifolds.

Theorem 1.2.4 (Freedman–Quinn ([58],[59])). *The homeomorphism type of a closed smooth oriented simply connected* 4*-manifold is determined by the isometry class of its intersection form.*

If the 4-manifold under consideration is not simply connected, in general there is no classification result available to pin down its homeomorphism type. Merely knowing its intersection form is usually not enough, as displayed by the following result, see [80].

Theorem 1.2.5 (Hambleton–Teichner). *There exist two non-homotopy equivalent closed topological* 4*-manifolds with isometric intersection forms, infinite cyclic fundamental group and vanishing Kirby–Siebenmann invariant.*

In particular, one of the two 4-manifolds in Theorem 1.2.5 topologically splits as a connected sum with a copy of $S^1 \times S^3$, while the other one does not and has been proven to be non-smoothable by Friedl–Hambleton–Melvin–Teichner in [60].

1.2.2 The equivariant intersection form of a 4-manifold

In the following, we are going to define the equivariant intersection form of a 4-manifold, describing its role in the detection of the homeomorphism type of closed 4-manifolds with free abelian fundamental group of rank $k \in \{1,2\}$.

Let X be a closed smooth oriented 4-manifold and let $\pi = \pi_1(X)$ be its fundamental group. From now on, we will always suppose that a base-point of X is fixed, although it will be omitted from our notation if it is not explicitly needed. The identification of π with the group of deck transformations of the universal covering $\widetilde{X} \to X$ makes $H_2(\widetilde{X})$ into a $\mathbb{Z}[\pi]$-module, where the group ring $\mathbb{Z}[\pi]$ is made up by formal linear combinations of elements of π with integer coefficients and is endowed with an involution that linearly extends the inverse map automorphism of π.

Definition 1.2.6 (Equivariant intersection form). Given a closed oriented 4-manifold X, its *equivariant intersection form* is the hermitian pairing

$$\widetilde{I}_X \colon H_2(\widetilde{X}) \times H_2(\widetilde{X}) \longrightarrow \mathbb{Z}[\pi]$$

given by

$$\widetilde{I}_X(a,b) = \sum_{g \in \pi_1(X)} I_{\widetilde{X}}(a, g_*(b)) g^{-1} \tag{1.2.1}$$

for any $a, b \in H_2(\widetilde{X})$, where $I_{\widetilde{X}}$ denotes the intersection form of $\widetilde{X}$ over the integers and g_* is the image of $g \in \pi$ under the identification with the group of deck transformations of the universal covering.

The *radical* of the equivariant intersection form is the subset

$$R(\widetilde{I}_X) = \{a \in H_2(\widetilde{X}) \mid \widetilde{I}_X(a,b) = 0 \;\forall b \in H_2(\widetilde{X})\}.$$

In the following, the identification $H_2(\widetilde{X}) \cong \pi_2(X)$ induced by the Hurewicz isomorphism is understood.

Remark 1.2.7. Notice that when $\pi_1(X)$ is trivial, then $\widetilde{X} = X$ and the equivariant intersection form of X is just the intersection form I_X defined on its second homology group modulo torsion.

In order to apply the homeomorphism classification results, we are interested in the study of the equivariant intersection form of a 4-manifold up to the following equivalence relation.

Definition 1.2.8. Let X and Y be two closed oriented 4-manifolds. An *isometry* between their equivariant intersection forms $\widetilde{I}_X$ and $\widetilde{I}_Y$ is a pair (g, h), where

$$g\colon \pi_1(X) \longrightarrow \pi_1(Y)$$

is an isomorphism of fundamental groups and

$$h\colon H_2(\widetilde{X}) \longrightarrow H_2(\widetilde{Y})$$

is a g-invariant isomorphism satisfying

$$\widetilde{I}_X(a,b) = \widetilde{I}_Y(h(a), h(b))$$

for all $a, b \in H_2(\widetilde{X})$. If such an isometry exists, we will say that the two equivariant intersection forms are *isometric* and write $\widetilde{I}_X \cong \widetilde{I}_Y$.

Remark 1.2.9. The hermitian forms induced by the equivariant intersection forms of two closed oriented 4-manifolds X and Y on the quotients $H_2(\widetilde{X})/R(\widetilde{I}_X)$ and $H_2(\widetilde{Y})/R(\widetilde{I}_Y)$ are non-singular (provided $H^3(\pi_1(X), \pi_2(X)) = 0 = H^3(\pi_1(Y), \pi_2(Y))$) and they are isometric if and only if $\widetilde{I}_X \cong \widetilde{I}_Y$ (see [79, Remark 1.4]). In particular, this implies that there is no loss in generality in studying the isometry class of the equivariant intersection form of a 4-manifold over the quotient $H_2(\widetilde{X})/R(\widetilde{I}_X)$.

The topological prototypes of our main Theorem A in Chapter 2 can be realized as a connected sum $X = M \# X_0$, where M is a closed simply connected 4-manifold and X_0 is either $S^1 \times S^3$ or $T^2 \times S^2$. The equivariant intersection form of such a connected sum is said to be extended from the integers. In the following, we recall the general definition [75].

Definition 1.2.10. The equivariant intersection form $\widetilde{I}_X$ of X is *extended from the integers* if

$$\widetilde{I}_X = Q \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1(X)]$$

for some integral unimodular form Q.

Remark 1.2.11. Let $U \subseteq \widetilde{X}$ be an open subset with the property that the universal covering map $\widetilde{p}\colon \widetilde{X} \to X$ sends it homeomorphically onto an open subset $\widetilde{p}(U) \subseteq X$. If U contains a collection of embedded surfaces generating the quotient $H_2(\widetilde{X})/R(\widetilde{I}_X)$ as a $\mathbb{Z}[\pi_1(X)]$-module, then $\widetilde{I}_X$ is extended from the integers (1.2.10).

Remark 1.2.12. A closed oriented 4-manifold X with infinite cyclic fundamental group that satisfies the inequality $b_2(X) - |\sigma(X)| \geq 6$ is within the stable range. It is known that the equivariant intersection form of such 4-manifolds is extended from the integers (see Definition 1.2.10) [80, Corollary 3 (2)]. This implies that such 4-manifolds are automatically homeomorphic to a connected sum with a copy of $S^1 \times S^3$. The topological prototypes of Theorem A in Chapter 2, on the other hand, satisfy

$$b_2(X) - |\sigma(X)| = 4.$$

Example 1.2.13. Let X be a closed smooth oriented 4-manifold with fundamental group $\pi_1(X) = \mathbb{Z}$ and a fixed handlebody decomposition. Without loss of generality, we can assume that just one among the 1-handles gives a non-trivial generator of $\pi_1(X)$. Let X^2 be the union of all handles up to index 2 and h^1 the non-trivial 1-handle in $\pi_1(X)$. From the recipe for building handlebody decompositions of cyclic coverings (see e.g. [73, Section 6.3]) it follows that the pre-image of the open subset $V = \mathrm{Int}(X^2 \setminus h^1) \subseteq X$ via the universal covering map is a disjoint union of copies of V. By Remark 1.2.11, the equivariant intersection form of X is extended from the integers and coincides with the intersection form I_X of X whenever the pre-image of $V \subseteq X$ under the universal covering map is $H_2(\widetilde{X})$-surjective. A similar argument can be applied to the case in which $\pi_1(X) = \mathbb{Z}^2$.

Classification up to homeomorphism

Results of Freedman–Quinn [59] and Hambleton–Kreck–Teichner [79] indicate that the equivariant intersection form encodes enough topological information of a closed smooth 4-manifold in order to pin down its homeomorphism class under some restrictions of the fundamental group. We summarize their results in the following statement. Recall that an oriented 4-manifold X has

1. type (I) if its universal covering is non-spin,
2. type (II) if it is spin,
3. type (III) if it is non-spin but with a spin universal covering.

Theorem 1.2.14. *Let X and Y be closed smooth oriented 4-manifolds.*

- (Freedmann–Quinn [59], Stong–Wang [138]) *If the fundamental group of X is infinite cyclic and there is an isometry $h\colon \widetilde{I}_X \to \widetilde{I}_Y$ between the equivariant intersection forms of X and Y as in Definition 1.2.8, then h is induced by a homeomorphism $X \to Y$.*

- (Hambleton–Kreck–Teichner [79, Theorem A]) *If the fundamental group of X is free abelian of rank two and there is an isometry $h\colon \widetilde{I}_X \to \widetilde{I}_Y$ between the equivariant intersection forms of X and Y as in Definition 1.2.8, then h is induced by a homeomorphism $X \to Y$ provided X and Y are of the same type.*

A proof of the first clause of Theorem 1.2.14 can be found in [127, Theorem 2.1].

Remark 1.2.15. Notice that, for a closed oriented 4-manifold X with infinite cyclic fundamental group, the type is determined by its equivariant intersection form. This follows from the fact that, as we will see in Section 1.2.2, the universal covering $\widetilde{X} \to X$ induces a surjective map in second homology. This does not hold when $\pi_1(X) = \mathbb{Z}^2$. For example, if Y is a Spin simply connected 4-manifold, then the two 4-manifolds $Y\#(T^2 \times S^2)$ and $Y\#(T^2 \widetilde{\times} S^2)$ have distinct type but still share the same equivariant intersection form.

Computations of second homotopy groups

The aim of this subsection is to characterize the $\mathbb{Z}[\pi_1(X)]$-module structure of the second homotopy group of a closed oriented 4-manifold X with $\pi_1(X) \cong \mathbb{Z}^k$, with $k \in \{1,2\}$. We start by recalling the following technical result.

Theorem 1.2.16 (Cartan–Leray spectral sequence, see [40]). *Let X be a connected topological space and π a group acting on X freely and properly. There is a spectral sequence of homological type with second page*

$$E^2_{p,q} = H_p(\pi; H_q(X))$$

strongly converging to the homology ring with integer coefficients of the quotient space X/π.

Corollary 1.2.17. *Let X be a closed connected oriented 4-manifold with $\pi_1(X) \cong \mathbb{Z}$ and denote by $\widetilde{X}$ and by π its universal covering and its fundamental group respectively. Endow $\mathbb{Z}$ with the unique $\mathbb{Z}[\pi]$-module structure for which every element $g \in \pi$ acts on $\mathbb{Z}$ via the identity. Then $\pi_2(X) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}$ is a free abelian group and its rank is equal to the second Betti number $b_2(X)$ of X.*

Proof. It is a direct application of Theorem 1.2.16, since π acts freely and properly on $\widetilde{X}$ by deck transformations. In particular, the second page of the Cartan–Leray spectral sequence of the pair $(\widetilde{X}, \pi)$ reads as follows, with differential d^2 with double grading equal to $(-2, 1)$.

$$\begin{tikzcd}
H_0(\mathbb{Z}; H_2(\widetilde{X}; \mathbb{Z})) & H_1(\mathbb{Z}; H_2(\widetilde{X}; \mathbb{Z})) & H_2(\mathbb{Z}; H_2(\widetilde{X}; \mathbb{Z})) \\
H_0(\mathbb{Z}; 0) & H_1(\mathbb{Z}; 0) = 0 & H_2(\mathbb{Z}; 0) = 0 \arrow[ull, "d^2_{2,1}" description] \\
H_0(\mathbb{Z}; \mathbb{Z}) & H_1(\mathbb{Z}; \mathbb{Z}) & H_2(\mathbb{Z}; \mathbb{Z}) = 0 \arrow[ull, "d^2_{2,0}" description]
\end{tikzcd}$$

By strong convergence, we hence have that

$$H_2(X; \mathbb{Z}) \cong H_0(\mathbb{Z}; H_2(\widetilde{X})).$$

The conclusion follows from the fact that the right hand side of the above identification is isomorphic to $H_2(\widetilde{X}; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}$, which can be in turn identified with $\pi_2(X) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}$ by the Hurewicz's theorem. $\square$

We are now ready to state and prove the main result of this section, in which we characterize the second homotopy group of 4-manifolds with certain free abelian fundamental groups. The main ideas used in the proof of the following result were kindly provided by Daniel Kasprowski.

Theorem 1.2.18. *Let X be a closed oriented 4-manifold with Euler characteristic $\chi = \chi(X)$ and second Betti number $b_2 = b_2(X)$.*

- *If the fundamental group of X is infinite cyclic, then its second homotopy group is*

$$\pi_2(X) \cong \mathbb{Z}[\pi_1(X)]^{b_2}.$$

- *If the fundamental group of X is $\pi_1(X) = \mathbb{Z}^2$, then its second homotopy group is*

$$\pi_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}[\pi_1(X)]^{\chi}.$$

In both of these cases, the equivariant intersection form of X is unimodular on the quotient of $\pi_2(X)$ with the radical of the equivariant intersection form (see Definition 1.2.6).

Proof. We start with the identity in the first clause and assume that $\pi_1(X) = \mathbb{Z}$. The universal coefficient spectral theorem [79, Section 3] implies that $\pi_2(X)$ is a free $\mathbb{Z}[\pi_1(X)]$-module, whose rank can be computed by using the Cartan–Leray spectral sequence ([40]), as shown in Corollary 1.2.17.

We now argue the identity of the second clause. If $\pi_1(X) = \mathbb{Z}^2$, the universal coefficient spectral theorem [79, Section 3] implies that there is an injective homomorphism

$$i\colon H^2(\pi_1(X); \mathbb{Z}[\pi_1(X)]) \cong \mathbb{Z} \hookrightarrow \pi_2(X)$$

and [79, Corollary 4.4] implies that the quotient of $\pi_2(X)$ by the image of such map is a stably-free $\mathbb{Z}[\pi_1(X)]$-module. Since every stably-free $\mathbb{Z}[\pi_1(X)]$-module is free whenever $\pi_1(X)$ is a finitely generated abelian group (see [87] and [89]) we have that $\pi_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}[\pi_1(X)]^k$ for some non-negative integer k. To conclude that $k = \chi(X)$, we apply a result of Kasprowski–Powell–Teichner [93, Theorem A] that says that X and $T^2 \times S^2$ become diffeomorphic after taking connected sums with copies of $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ on both sides. Since two diffeomorphic 4-manifolds have the same equivariant intersection form and connected summing with a copy of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ increases both the Euler characteristic and the rank of the second homotopy group of a 4-manifold by one, the conclusion follows.

The last part of the statement of Theorem 1.2.18 is proven in [79, Corollary 4.4]. □

1.2.3 Pinning down the homeomorphism type of the 4-manifolds produced in 1.1.2

The aim of this subsection is to provide a proof of the following result, of which we will largely make use in Chapter 3.

Theorem 1.2.19. *Let X_1 and X_2 be topological 4-manifolds such that there are locally flat simple loops*
$$\{\alpha_{X_i} \subseteq M_i \mid i = 1, 2\}$$
for which there is a homeomorphism
$$X_1 \setminus \operatorname{Int} \nu(\alpha_{X_1}) \longrightarrow X_2 \setminus \operatorname{Int} \nu(\alpha_{X_2}).$$
If $\alpha_{X_i} \subseteq X_i$ is an orientation-preserving loop, set $C_p = S_p$. Otherwise, let $C_p = N_p$. For $i = 1, 2$, define
$$X_i(p) = (X_i \setminus \operatorname{Int} \nu(\alpha_{X_i})) \cup C_p.$$
The 4-manifolds $X_1(p)$ and $X_2(p)$ are homeomorphic.

Oriented setting

A foundational result of Laudenbach–Poénaru [105] implies that any self-diffeomorphism of $S^1 \times S^2$ extends to a self-diffeomorphism of $S^1 \times D^3$. In the following, we extend this result to the building blocks S_p defined in Section 1.1.2.

Theorem 1.2.20. *For every odd integer $p \geq 1$ and for any self-diffeomorphism*
$$h \colon S^1 \times S^2 \longrightarrow S^1 \times S^2,$$
there is a diffeomorphism
$$H_p \colon S_p \longrightarrow S_p$$
such that $H_p|_\partial = h$.

Proof. A result of Gluck [66] states that the diffeotopy group of $S^1 \times S^2$ is isomorphic to $\mathbb{Z}_2$ and it is generated by the self-diffeomorphism

$$\varphi\colon S^1 \times S^2 \longrightarrow S^1 \times S^2 \tag{1.2.2}$$

given by

$$(\theta, x) \mapsto (\theta, \rho_\theta(x)),$$

where ρ_θ is the rotation of S^2 of angle θ around a fixed axis [66], [73, p. 156], cf. Subsection 1.1.3. In order to prove Theorem 1.2.20, it is enough to show that the diffeomorphism φ admits an extension $\Phi_p\colon S_p \to S_p$ for any odd $p > 0$. We explicitly build the diffeomorphism Φ_p by using the identification

$$S_p \cong (\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_\phi$$

coming from Lemma 1.1.16 as follows. Without loss of generality, we work in the setting of Remark 1.1.17. In particular, the deck transformation ϕ and the collection of 3-balls $B_1, \ldots, B_p \subseteq D^2 \times S^1$ are defined as in the proof of Lemma 1.1.16.

Consider the map

$$F\colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i) \longrightarrow S^3 \setminus \sqcup_{i=1}^p B_i$$

defined by

$$(t, x) \mapsto r_t(x),$$

where r_t denotes the rotation of S^3 of an angle $2\pi t$ that restricts to a rotation along the D^2-factor on the solid torus $D^2 \times S^1$ of the Heegaard decomposition of S^3 in Remark 1.1.17. Such solid torus contains the 3-balls $B_1, \ldots, B_p$. Notice that for every $t \in [0,1]$, the map $F(t, \cdot)$ is an f_p-equivariant self-diffeomorphism of $S^3 \setminus \sqcup_{i=1}^p B_i$, where f_p can be chosen to act on the genus one Heegaard splitting of S^3 described in Remark 1.1.17. We can now define the automorphism

$$\Phi\colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i) \longrightarrow \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i)$$

$$(t, x) \mapsto (t, F(t, x)).$$

It is immediate to check that Φ is ϕ-equivariant and, thus, it induces a self-diffeomorphism $\Phi_p\colon S_p \to S_p$. In particular, the restriction of Φ_p to $\partial S_p = S^1 \times S^2$ is a composition of p Gluck twists, which is again a Gluck twist since p is odd. This concludes the proof. $\square$

As a corollary, we get the following result, which will help us pin down the homeomorphism type of the 4-manifolds constructed in Chapter 3.

Corollary 1.2.21. *Let X_1 and X_2 be topological 4-manifolds and let $\alpha_{X_i} \subseteq X_i$ be a locally flat orientation-preserving simple framed loop for $i = 1, 2$. Suppose there is a homeomorphism*

$$f\colon X_1 \setminus \operatorname{Int} \nu(\alpha_{X_1}) \longrightarrow X_2 \setminus \operatorname{Int} \nu(\alpha_{X_2}).$$

The 4*-manifolds*

$$X_1(p) = (X_1 \setminus \operatorname{Int} \nu(\alpha_{X_1})) \cup S_p \quad \textit{and} \quad X_2(p) = (X_2 \setminus \operatorname{Int} \nu(\alpha_{X_2})) \cup S_p$$

are homeomorphic.

The reader is directed to [62, Chapter 5] regarding existence and uniqueness results of the tubular neighbourhood of a locally flat simple loop in a topological 4-manifold.

Nonorientable setting

We start this subsection by extending a foundational result due to César de Sá [38] and Miller–Naylor [110], which states that any self-diffeomorphism of $S^2 \widetilde{\times} S^1$ extends to a self-diffeomorphism of $D^3 \widetilde{\times} S^1$.

Theorem 1.2.22. *For every odd integer $p \geq 1$ and for any self-diffeomorphism*

$$f\colon S^2 \widetilde{\times} S^1 \longrightarrow S^2 \widetilde{\times} S^1,$$

there is a diffeomorphism

$$F_p\colon N_p \longrightarrow N_p$$

such that $F_p|_\partial = f$.

Proof. Kim–Raymond [96] (cf. [110]) showed that the diffeotopy group of $S^2 \widetilde{\times} S^1$ is a quotient of the one of $S^1 \times S^2$. This is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and a set of generators is given by the Gluck twist (1.2.2) and the product maps $\mathrm{Id}_{S^1} \times a$ and $r \times \mathrm{Id}_{S^2}$, where a is the antipodal involution of S^2 and r is a orientation-reversing automorphism of S^1. We prove the extension result for the map

$$f\colon S^2 \widetilde{\times} S^1 \to S^2 \widetilde{\times} S^1 \tag{1.2.3}$$

given by the Gluck twist and leave the cases of the other generators of the diffeotopy group of $S^2 \widetilde{\times} S^1$ to the interested reader.

Without loss of generality, we can work in the context of Remark 1.1.23. Consider the map

$$G\colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i) \longrightarrow S^3 \setminus \sqcup_{i=1}^{p} B_i$$
$$(t, x) \mapsto r_t(x)$$

where r_t is a rotation of S^3 of angle πt, restricting to a rotation along the D^2-factor of $D^2 \times S^1$ in the genus 1 Heegaard splitting (1.1.17). One can check that the smooth automorphism

$$T\colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i) \longrightarrow \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^{p} B_i)$$
$$(t, x) \mapsto (t, G(t, x))$$

is ψ-equivariant (where ψ is the map generating the deck transformations of N_p in Lemma 1.1.22) and, therefore, it induces a smooth automorphism

$$F_p \colon N_p \longrightarrow N_p.$$

To conclude, it is enough to show that the smooth map T_p induced by T on the quotient $(\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_{\psi^2} \cong S_p$ (the orientable double covering of N_p, see Lemma 1.1.22) restricts to a Gluck twist on the boundary. This follows from the existence of a commutative diagram

$$\begin{tikzcd}
(\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_\phi \arrow[r, "\Phi_p"] \arrow[d, "m_2"] & (\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_\phi \arrow[d, "m_2"] \\
(\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_{\psi^2} \arrow[r, "T_p"] & (\mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i))/\sim_{\psi^2} .
\end{tikzcd}$$

Here Φ_p is the map in Theorem 1.2.20, ϕ has been defined in the proof of Lemma 1.1.16, while $m_2 \colon \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i) \to \mathbb{R} \times (S^3 \setminus \sqcup_{i=1}^p B_i)$ is defined by $m_2(t, x) = (2t, x)$. □

A useful consequence of Theorem 1.2.22 is recorded in the following corollary.

Corollary 1.2.23. *Let Y_1 and Y_2 be topological 4-manifolds and let $\alpha_{Y_i} \subseteq Y_i$ be locally flat orientation-reversing simple loops, for $i = 1, 2$. Suppose there is a homeomorphism*

$$f \colon Y_1 \setminus \operatorname{Int} \nu(\alpha_{Y_1}) \longrightarrow Y_2 \setminus \operatorname{Int} \nu(\alpha_{Y_2}).$$

The 4-manifolds

$$Y_1(p) = (Y_1 \setminus \operatorname{Int} \nu(\alpha_{Y_1})) \cup N_p \quad \text{and} \quad Y_2(p) = (Y_2 \setminus \operatorname{Int} \nu(\alpha_{Y_2})) \cup N_p$$

are homeomorphic.

Proof of Theorem 1.2.19. It is a direct consequence of Corollaries 1.2.21 and 1.2.23. □

1.3 Smooth invariants of 4-manifolds

1.3.1 Seiberg–Witten invariants

We regard the Seiberg–Witten invariant of a closed smooth oriented 4-manifold X as a function

$$\mathrm{SW}_X \colon \{k \in H_2(X) \mid \mathrm{PD}(k) \equiv w_2(X) \mod 2\} \longrightarrow \mathbb{Z},$$

where $\mathrm{PD}(k)$ denotes the Poincaré dual of k. We refer the reader to [14], [55] and the references therein for the necessary background on Seiberg–Witten theory.

We will make use of work of Fintushel–Park–Stern [55] based on the Morgan–Mrowka–Szabó formula [116] to distinguish the diffeomorphism types of the 4-manifolds constructed in Chapter 2 via their Seiberg–Witten invariants [55, Section 2].

Infinitely many diffeomorphism types and irreducibility

The following result summarizes the results of Fintushel–Park–Stern [55, Theorem 1, Corollary 1] that we will need in the sequel. We use the notation introduced in Section 1.1.1.

Theorem 1.3.1 (Fintushel–Park–Stern [55])**.** *Let X' be a closed smooth 4-manifold that contains a smoothly embedded null-homologous 2-torus $T' \subseteq X'$ and let $\gamma' \subseteq T'$ be a simple loop with null-homologous push-off in $X' \setminus \operatorname{Int} \nu(T')$. Furthermore, suppose that the 4-manifold $X'_{T',\gamma'}(0)$ has exactly one Seiberg–Witten basic class up to sign. Then the collection $\{X'_{T',\gamma'}(1/p) \mid p \in \mathbb{N}\}$ consists of pairwise non-diffeomorphic elements.*

The gauge-theoretical background on the usage of Theorem 1.3.1 has already been well-documented in the literature [14, 28, 55, 56, 148]. The mechanism to apply Theorem 1.3.1 to our examples is summarized in the following commutative diagram

$$\begin{tikzcd}
X = X'_{T',\gamma'}(0) \arrow[r, "(1)"] \arrow[d, "(2)"'] & X_{T,\gamma}(1) = X' \arrow[d, "(3)"] \\
X_{T,\gamma}(p) \arrow[r, "\mathrm{id}"] & X'_{T',\gamma'}(1/(p-1))
\end{tikzcd}$$

where the numbered arrows indicate the following steps in the construction; cf. Proposition 1.1.4.

(1) The 4-manifold X in the top left hand corner of the diagram is irreducible, symplectic and it contains a homologically essential Lagrangian 2-torus T with trivial self-intersection. Using the Lagrangian framing, perform a (± 1)-torus surgery on $T \subseteq X$ along a given surgery curve $\gamma \subseteq T$ that is primitive in $H_1(X)$ to obtain the symplectic 4-manifold $X_{T,\gamma}(\pm 1)$ on the upper right side of the diagram. Set $X' = X_{T,\gamma}(\pm 1)$ and let $T' \subseteq X'$ be the null-homologous core 2-torus of the surgery; see Proposition 1.1.4. Notice that $X = X'_{T',\gamma'}(0)$ as it was proven in Lemma 1.1.6. A result of Taubes [142] implies that the canonical classes $k = -c_1(X, J)$ and $k' = -c_1(X', J')$ are basic classes of X and X' respectively, and their Seiberg–Witten

invariants are $\mathrm{SW}_X(\pm k) = \pm 1 = \mathrm{SW}_{X'}(\pm k')$. An argument due to Akhmedov–Baykur–Park implies that, under the circumstances considered in our construction, these are the only basic classes of X and X' [14, Section 4.1].

(2) Perform a p-torus surgery on $T \subseteq X$ along $\gamma \subseteq X$ using the Lagrangian framing to obtain a 4-manifold $X_{T,\gamma}(p)$.

(3) Perform a $(1/(p-1))$-torus surgery on $T' \subseteq X'$ along the surgery curve $\gamma' \subseteq T'$ using the null-homologous framing to produce the 4-manifold $X'_{T'.\gamma'}(p)$. Notice that

$$X_{T,\gamma}(p) = X'_{T',\gamma'}(1/(p-1)).$$

The Morgan–Mrowka–Szabó formula yields

$$\begin{aligned}\mathrm{SW}_{X_{T',\gamma'}(1/(p-1))}(k_{1/p}) &= \mathrm{SW}_{X'}(k') + (p-1)\cdot \mathrm{SW}_X(k)\\ &= 1 + (p-1) = p\end{aligned}$$

where the basic class $k_{1/p}$ of $X_{1/(p-1)}$ corresponds to the basic class k_X of X. Thus, the collection $\{X_{T,\gamma}(p) \mid p \in \mathbb{N}\}$ consists of pairwise non-diffeomorphic 4-manifolds given that their Seiberg–Witten invariants are pairwise different.

Remark 1.3.2. We point out that an argument due to Szabó [139] that uses a special case of the Morgan–Mrowka–Szabó formula [139, Theorem 3.4] can also be used to reach the same conclusion on the Seiberg–Witten invariant of the collection $\{X_{T,\gamma}(p) \mid p \in \mathbb{N}\}$.

The following result will allow us to conclude that the collection $\{X_{T,\gamma}(p) \mid p \in \mathbb{N}\}$ consists of irreducible 4-manifolds. It generalizes a result of Park on simply connected 4-manifolds [124, Theorem 1] and its proof follows Park's argument to a great degree.

Theorem 1.3.3. *Let Z be a closed minimal symplectic 4-manifold with $b_2^+(Z) > 1$ and let M be a closed smooth 4-manifold with $\pi_1(M) = \mathbb{Z}^k$ for $k = 1, 2$. Suppose there is an isomorphism*

$$\xi\colon H^2(Z) \longrightarrow H^2(M) \tag{1.3.1}$$

that preserves the cup product and that restricts to a one-to-one correspondence between the basic classes of Z and M. Then the 4-manifold M is irreducible.

Proof. The sole contribution of this proof to the argument due to Park [124, p. 574] (that uses work of Kotschick [101]) is a modification to covering the cases of free abelian fundamental group in the statement of Theorem 1.3.3. A famous result of Taubes's [142, Main Theorem] says that, since Z is symplectic, there is a basic class $k_Z = -c_1(Z, J) \in \mathcal{B}_Z$. In particular, the hypothesis on the isomorphism (1.3.1) implies that there is at least one basic class $k_M \in \mathcal{B}_M$.

Suppose now that $M = X_1 \# X_2$. Since $\pi_1(M) = \mathbb{Z}^k$ and $\mathbb{Z}^k$ does not split as a free product in a non-trivial way, we can assume without loss of generality that $\pi_1(X_2) = \mathbb{Z}^k$.

Given that the Seiberg–Witten invariant of M satisfies $\mathrm{SW}_M(k_M) \neq 0$, we have that $b_2^+(X_1)$ and $b_2^+(X_2)$ cannot both be positive [73, Theorem 2.4.6]. In particular, we have that either $b_2^+(X_1) = 0$ or $b_2^+(X_2) = 0$. Suppose first that $b_2^+(X_1) = 0$. If the second Betti number satisfies $b_2(X_1) = 0$, then X_1 is homeomorphic to the 4-sphere by Freedman's theorem [73, Corollary 1.2.28] and the claim follows. We need to rule out the subcase $b_2(X_1) = b_2^-(X_1) > 0$ and, to do so, we proceed by contradiction. The intersection form I_{X_1} of X_1 is negative definite. Donaldson's theorem [46] implies that $I_{X_1} \cong n\langle -1 \rangle$, where $n = b_2(X_1) > 0$. Let $\{e_1, \dots, e_{b_2(X_1)}\}$ be a basis for the second cohomology group $H^2(X_1)$ with $e_i \cdot e_j = -\delta_{i,j}$ for each $i, j = 1, \dots, b_2(X_1)$, where $\delta_{i,j}$ denotes the Kronecker delta. The neck pinching argument [46, 101] implies that there is at least one basic class $k_{X_2} \in \mathcal{B}_{X_2}$, for which $\mathrm{SW}_{X_2}(k_{X_2}) \neq 0$. Moreover, for any Seiberg–Witten basic class L of $M = X_1 \# X_2$, the cohomology class

$$L + \sum_{j=1}^{b_2(X_1)} a_i e_i \tag{1.3.2}$$

is basic for M, where $a_i = \pm 1$. Moreover, all the basic classes of M can be written as in Equation 1.3.2. Consider now the basic class $k_M = \xi(k_Z) \in \mathcal{B}_M$. Up to changing e_i with $-e_i$ if necessary, we can write

$$k_M = L' - \sum_{j=1}^{b_2(X_1)} e_j$$

for some other basic class $L' \in \mathcal{B}_M$. As Park observed [124, p. 574], the combination $k_M + 2e_1 = L + e_1 - \sum_{j=2}^{b_2(X_1)} e_j$ is also a basic class of M and $k_Z + 2\xi^{-1}(e_1) \in \mathcal{B}_Z$. A result of Taubes [144] (see [73, Remark 10.1.16 (b)]) implies that the Poincaré dual of $\xi^{-1}(e_1)$ is represented by a symplectically embedded 2-sphere $S \subseteq Z$ with $[S]^2 = -1$. This is, however, impossible since Z is minimal by hypothesis and we have reached a contradiction. We conclude that $b_2(X_1) = b_2^-(X_1) = 0$ and X_1 is homeomorphic to S^4. This concludes the proof of the first case.

We now work on the second case $b_2^+(X_2) = 0$. Notice that the previous argument applies verbatim if the intersection form over the integers of X_2 is negative definite and, consequently, it rules out the case $b_2^-(X_2) > 0$. We need to rule out the subcase $b_2^-(X_2) = b_2(X_2) = 0$. In this scenario, the Euler characteristic of X_2 is zero by our assumption on its fundamental group. We apply the neck pinching argument [46, 101] once more to conclude that there is at least one basic class $k_{X_1} \in \mathcal{B}_{X_1}$ and $\mathrm{SW}_{X_1}(k_{X_1}) \neq 0$. This, however, is impossible since $\chi(X_2) = 0$ implies that $b_2(X_1)$ is even. The dimension of the moduli space of solutions to the Seiberg–Witten equations and $b_2^+(X_1)$ have opposite parity. In particular, in the case $b_2^+(X_1) = 0 \mod 2$, the Seiberg–Witten invariant of X_1 is zero [143, Section 2], [73, Section 2.4]. We conclude that the 4-manifold M is irreducible. □

1.3.2 Ozsváth–Szabó mixed invariant

In this subsection, we briefly recall the definition of the Ozsváth–Szabó mixed invariant for closed smooth 4-manifolds. From now on, we denote by $\mathbb{F}$ the field with two elements.

Given a closed oriented 3-manifold Y with a Spin^c-structure $\mathfrak{s}$, we denote by $HF^-(Y, \mathfrak{s})$ the minus version of its Heegaard Floer homology. This is a finitely generated graded module over $\mathbb{F}[U]$, whose U-torsion part is denoted by $HF_{\text{red}}(Y, \mathfrak{s})$. We refer the reader to the foundational paper [121] for further details on these invariants.

Example 1.3.4. The 3-sphere has a unique Spin^c-structure $\mathfrak{s}$ and

$$HF^-(S^3, \mathfrak{s}) \cong \mathbb{F}[U],$$

where the degree of the unit element is $\deg(1) = 0$.

Given a Spin^c-cobordism $(W, \mathfrak{t})$ between $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$, there is an induced map

$$F^-_{(W,\mathfrak{t})} \colon HF^-(Y_1, \mathfrak{s}_1) \longrightarrow HF^-(Y_2, \mathfrak{s}_2).$$

Moreover, there is a canonical projection

$$\Pi \colon HF^-(Y, \mathfrak{s}) \cong HF_{\text{red}}(Y, \mathfrak{s}) \oplus \mathbb{F}[U]^k \longrightarrow HF_{\text{red}}(Y, \mathfrak{s}),$$

see [122, Section 2.3]. For more details on this invariant, we refer the reader to [109].

Definition 1.3.5. Let $(W, \mathfrak{t})$ be a compact Spin^c 4-manifold with non-empty boundary $\partial(W, \mathfrak{t}) = (Y, \mathfrak{s})$, where $\mathfrak{s} = \mathfrak{t}|_Y$ is non-torsion (i.e. its first Chern class $c_1(\mathfrak{s})$ is not a torsion element in $H_2(Y; \mathbb{Z})$). The relative invariant of $(W, \mathfrak{t})$ is

$$\Psi_{W,\mathfrak{t}} = \Pi(F^-_{(W \setminus D^4, \mathfrak{t}|)}(1)) \in HF^-_{\text{red}}(Y, \mathfrak{s}).$$

The duality between $HF_{\text{red}}(Y)$ and $HF_{\text{red}}(\overline{Y})$ [123, page 376] allows to define a pairing

$$\langle \cdot, \cdot \rangle_Y \colon HF_{\text{red}}(Y, \mathfrak{s}) \otimes HF_{\text{red}}(\overline{Y}, \mathfrak{s}) \longrightarrow \mathbb{F}.$$

With this, we are ready to define the Ozsváth–Szabó mixed invariant for smooth 4-manifolds.

Let X be a closed smooth 4-manifold with $b^+ = 1$ and indefinite intersection form, and let $L \subseteq H_2(X; \mathbb{Q})$ be a 1-dimensional subspace on which the intersection form vanishes. We say that a Spin^c-structure $\mathfrak{t}$ on X is allowable if $c_1(\mathfrak{t}|_L)$ is non-vanishing. An embedded, oriented, separating 3-manifold $Y \subseteq X$ is an admissible cut if the image in $H_2(X; \mathbb{Q})$ of the map induced by the inclusion $Y \to X$ is equal to L. At this point, we can write

$$X = W_1 \cup_Y W_2$$

where W_1 and W_2 are oriented in such a way that $\partial W_1 = Y = \overline{\partial W_2}$.

Definition 1.3.6. The closed 4-manifold invariant of the Spin^c-manifold $(X, \mathfrak{s})$ with respect to the 1-dimensional subspace $L \subseteq H_2(X; \mathbb{Q})$ is

$$\Phi_{X,\mathfrak{t},L} = \langle \Psi_{(W_1,\mathfrak{t}_1)}, \Psi_{(W_2,\mathfrak{t}_2)} \rangle_Y,$$

where $\mathfrak{t}_1 = \mathfrak{t}|_{W_1}$, $\mathfrak{t}_2 = \mathfrak{t}|_{W_2}$ and $\mathfrak{s} = \mathfrak{t}|_Y$.

We refer the reader to [122] and to [107, Section 6.1] for more background on such invariants.

1.3.3 A Pin$^+$-invariant

We collect results of Stolz [137] that are employed in Chapter 3 and Chapter 4 to distinguish two smooth structures. Such results employ a spectral invariant associated to Pin$^+$-manifolds, as we will now see.

Let (X, ϕ^+_X) be a closed smooth 4-manifold with Pin$^+$-structure ϕ^+_X and let $\eta(X, \phi^+_X) \in \frac{1}{8}\mathbb{Z}$ be the η-invariant of the twisted Dirac operator on X, see [137, Section 2].

Theorem 1.3.7 (Stolz [137])**.** *Let (X, ϕ^+_X) be a closed smooth 4-manifold with Pin$^+$-structure ϕ^+_X.*

- *The invariant $\eta(X, \phi^+_X) \mod 2\mathbb{Z}$ is a complete Pin$^+$-bordism invariant.*
- *Given two closed smooth nonorientable Pin$^+$- 4-manifolds X_1 and X_2, if*

$$\eta(X_1, \phi^+_{X_1}) \neq \eta(X_2, \phi^+_{X_2}) \mod 2\mathbb{Z} \tag{1.3.3}$$

for every choice of Pin$^+$-structures $\phi^+_{X_1}$ and $\phi^+_{X_2}$ on X_1 and X_2 respectively, then X_1 and X_2 are not stably diffeomorphic.

Here we say that two 4-manifolds X_1 and X_2 are *stably diffeomorphic* if there exists $k \geq 0$ such that there is a diffeomorphism between the connected sums

$$X_1 \#_k (S^2 \times S^2) \text{ and } X_2 \#_k (S^2 \times S^2).$$

Remark 1.3.8. Stolz showed that the condition (1.3.3) is an obstruction to the existence of a diffeomorphism $f\colon X_1 \to X_2$ such that $f^*\phi^+_{X_2} = \phi^+_{X_1}$.

We now exhibit our main computational tools to keep track of the η-invariant of a Pin$^+$-manifold undergoing cut-and-paste operations.

Proposition 1.3.9. *Let X be a closed smooth 4-manifold with first cohomology group $H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2$ and that admits a Pin$^+$-structure (X, ϕ^+_X). Let $\alpha_X \subseteq X$ be an orientation-reversing simple loop such that $[\alpha_X] \neq 0 \in H_1(X; \mathbb{Z}_2)$ and set*

$$X(p) = (X \setminus \operatorname{Int} \nu(\alpha_X)) \cup N_p, \tag{1.3.4}$$

where $p \neq 0 \pmod 2$*, cf. Equation (1.1.22). There are* Pin^+*-structures*

$$(S^3 \widetilde{\times} S^1, \phi^+) \quad \text{and} \quad (X(p), \phi^+_{X(p)})$$

along with a Pin^+*-cobordism*

$$((V, \phi^+_V); (X, \phi^+_X) \sqcup (S^3 \widetilde{\times} S^1, \phi^+), (X(p), \phi^+_{X(p)})). \tag{1.3.5}$$

In particular, the identity

$$[(X, \phi^+_X)] = [(X(p), \phi^+_{X(p)})] \in \Omega_4^{\mathrm{Pin}^+} \tag{1.3.6}$$

holds.

A similar statement to Proposition 1.3.9 continues to hold if the simple loop $\alpha_X \subseteq X$ is assumed to be orientation-preserving.

Proof. We first assemble the Pin^+-structure $\phi^+_{X(p)}$ on $X(p)$ as follows. The Pin^+-structure ϕ^+_X on X restricts to a Pin^+-structure $(X \setminus \mathrm{Int}\,\nu(\alpha_X), \phi^+_X|)$ [99, Construction 1.5], which in turn restricts to a Pin^+-structure

$$(\partial(X \setminus \mathrm{Int}\,\nu(\alpha_X)), \phi^+_X|_\partial) = (S^2 \widetilde{\times} S^1, \phi^+_\partial). \tag{1.3.7}$$

Since

$$H^1(X; \mathbb{Z}_2) = \mathbb{Z}_2 = H^1(X \setminus \mathrm{Int}\,\nu(\alpha_X); \mathbb{Z}_2) = H^1(\partial(X \setminus \mathrm{Int}\,\nu(\alpha_X)); \mathbb{Z}_2),$$

there are two choices of such structures [137, Corollary 6.4] and there is a one-to-one correspondence between the set of Pin^+-structures on X and the one of Pin^+-structures on $\partial(X \setminus \mathrm{Int}\,\nu(\alpha_X)) \cong S^2 \widetilde{\times} S^1$ (see [97, Proposition 1 Chapter IV]). We now argue that there is a Pin^+-structure $\phi^+_{N_p}$ on N_p that induces the Pin^+-structure (1.3.7) on $\partial N_p = S^2 \widetilde{\times} S^1$. Since $w_2(S^3 \widetilde{\times} S^1) = 0$ and $H^1(S^3 \widetilde{\times} S^1; \mathbb{Z}_2) = \mathbb{Z}_2$, there are exactly two Pin^+-structures on $S^3 \widetilde{\times} S^1$ [137, Corollary 6.4]. Since $p \neq 0 \mod 2$ by hypothesis, there is a Pin^+-structure $(S^3 \widetilde{\times} S^1, \phi^+)$ whose restriction to (N_p, ϕ_{N_p}) matches (1.3.7) on the boundary. We conclude that we can glue $\phi^+_X|$ and $\phi^+_{N_p}$ to get a Pin^+-structure $\phi^+_{X(p)}$ on $X(p)$. Next, we put together the compact 5-manifold V by gluing the product cobordisms

$$X \times [0,1] \text{ and } (S^3 \widetilde{\times} S^1) \times [0,1]$$

along the upper lids $X \times \{1\}$ and $(S^3 \widetilde{\times} S^1) \times \{1\}$ according to the construction of $X(p)$ in (1.3.4). The Pin^+-structure (V, ϕ^+_V) is assembled analogously to the assembly of (X, ϕ^+_X) that was described before. Given that

$$[(S^3 \widetilde{\times} S^1, \pm\phi^+)] = 0 \in \Omega_4^{\mathrm{Pin}^+},$$

the identity (1.3.6) follows from the existence of the cobordism (1.3.5). □

We now record the changes of the η-invariant under the cut-and-paste constructions discussed in Section 1.1.5. Let $X_0 = (X \setminus \operatorname{Int}\nu(R)) \cup (D^3 \tilde{\times} S^1)$ be the 4-manifold obtained by blowing down a smoothly embedded real projective plane $R \subseteq X$ as in [3, p. 94] and let X_R be the 4-manifold obtained by twisting it as in Section 1.1.5.

Proposition 1.3.10. *Let X be a closed smooth 4-manifold that contains a smoothly embedded real projective plane $R \subseteq X$ with $\nu(R) = D^2 \tilde{\times} \mathbb{RP}^2$. Suppose that there is a Pin^+-structure (X, ϕ^+_X) inducing the Pin^+-structure on $(\nu(R), \phi^+_{\nu(R)})$ that extends to $(\mathbb{RP}^4, \pm\phi^+_{\mathbb{RP}^4})$ with the convention introduced in Example 1.1.28. There are Pin^+-structures $(X_R, \phi^+_{X_R})$ and $(X_0, \phi^+_{X_0})$ such that the identities*

$$\eta(X_0, \phi^+_{X_0}) = \eta(X, \phi^+_X) \mp \frac{1}{8} \mod 2\mathbb{Z}, \tag{1.3.8}$$

$$\eta(X_0, \phi^+_{X_0}) = \eta(X_R, \phi^+_{X_R}) \pm \frac{1}{8} \mod 2\mathbb{Z} \tag{1.3.9}$$

and

$$\eta(X_R, \phi^+_{X_R}) = \eta(X, \phi^+_X) \mp \frac{2}{8} \mod 2\mathbb{Z} \tag{1.3.10}$$

hold.

Proof. Assemble together $(X_0, \phi^+_{X_0})$ and $(X_R, \phi^+_{X_R})$ from Pin^+-structures on the building blocks that match on the common boundaries. The Pin^+-structure ϕ^+_X induces Pin^+-structures $(X \setminus \operatorname{Int}\nu(R), \phi^+_X|)$ and $(\nu(R), \phi^+_{\nu(R)})$ since these are codimension zero submanifolds of X [99], and both these Pin^+-structures coincide on $\partial(X \setminus \operatorname{Int}\nu(R)) = \partial\nu(R)$. Recall that the self-diffeomorphism (1.2.3) of $S^2 \tilde{\times} S^1$ identifies the two possible choices of Pin^+-structures cf. [97, p. 35]. The existence of the Pin^+-structure $(X_R, \phi^+_{X_R})$ follows. Moreover, either choice of Pin^+-structure $(D^3 \tilde{\times} S^1, \pm\phi^+)$ yields a Pin^+-structure $(X_0, \phi^+_{X_0})$. To see that the identities (1.3.8) and (1.3.9) hold, one constructs a Pin^+-cobordism as in the proof of Lemma 1.1.30. Such cobordism is obtained by patching together $X_0 \times [0,1]$ and $\mathbb{RP}^4 \times [0,1]$ along $X_0 \times \{1\}$ and $\mathbb{RP}^4 \times \{1\}$. We obtain Pin^+ cobordisms $(V; X, X_0 \sqcup \mathbb{RP}^4)$ and $(V_R; X_R, X_0 \sqcup \mathbb{RP}^4)$ inducing different Pin^+-structures on $\mathbb{RP}^4$. The η-invariant is a complete Pin^+-cobordism invariant and it is additive with respect to disjoint unions. It follows that (1.3.8) and (1.3.9) hold. These two values imply (1.3.10).

□

Chapter 2

Some small irreducible exotic 4-manifolds with free abelian fundamental group

It has been noticed by several authors [16, 28, 146, 148] that torus surgery based constructions of closed simply connected 4-manifolds of small Euler characteristic [15, 16, 14, 27, 28, 55, 56, 139] can be modified in order to produce small 4-manifolds with multiple choices of fundamental group, including free abelian groups. The homeomorphism class of the 4-manifolds that have been obtained in this way, however, had not been previously determined. Merely knowing the isometry class of the intersection form over the integers of a non-simply connected 4-manifold does not suffice to determine its topological type. Instead, one must determine its equivariant intersection form up to isometry.

The computation of the equivariant intersection form of a 4-manifold can be a rather challenging task, especially when the size of the 4-manifold is outside the stable range and Bass's cancellation results do not apply as in Hambleton–Teichner [80, Corollary 3 (2)]; see Remark 1.2.12. The main novel contribution of this chapter is to develop a mechanism to compute the equivariant intersection form of 4-manifolds that are constructed by using torus surgeries, and show that it is extended from the integers. The mechanism can be briefly described as follows.

- Take a closed oriented 4-manifold X with $\pi_1(X) \cong \mathbb{Z}$.

- There is an isomorphism

$$H_2(\widetilde{X}) \cong H_2(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1(X)]$$

of $\mathbb{Z}[\pi_1(X)]$-modules (see 1.2.18), where $\widetilde{X}$ denotes the universal covering of X. In particular, a basis of the $\mathbb{Z}[\pi_1(X)]$-module $H_2(\widetilde{X})$ can always be obtained by lifts of embedded surfaces in X whose homology classes form a basis of $H_2(X)$ as a free abelian group.

- We consider representatives of a basis of $H_2(X)$ given by smoothly embedded surfaces and, using cut-and-paste operations, we trade each surface for a new one in the same homology class and with trivial π_1-inclusion.

- In this way, we get a collection of embedded surfaces in X and the equivariant intersection form can be computed on their lifts in the universal covering $\widetilde{X}$.

In our examples, the fact that our 4-manifolds come from torus surgery operations allows us to perform the cut-and-paste operations mentioned in Step (3) in such a way that the equivariant intersection form turns out to be extended from the integers. The construction is explained in full generality in Section 2.2 and it is displayed in action in Section 2.3. Besides the utility that our mechanism most likely will find in future research on the existence of exotic smooth structures, our contribution is of interest with respect to a conjecture due to Friedl–Hambleton–Melvin–Teichner [60, Conjecture 1.3] (cf. [80]). Their conjecture states that the equivariant intersection of any closed smooth 4-manifold with infinite cyclic fundamental group is extended from the integers. Once we have computed the equivariant intersection form, results of Freedman–Quinn [59], Stong–Wang [138] and Hambleton–Kreck–Teichner [79] are then used to pin down the homeomorphism type of several 4-manifolds with free abelian fundamental groups. The following theorem is a result of these contributions.

Theorem A. *Let $k \in \{2,3,4,5\}$. There are infinite collections*

$$\{M_{p,k} \mid p \in \mathbb{N}\} \text{ and } \{N_{p,k} \mid p \in \mathbb{N}\}$$

of closed smooth oriented pairwise non-diffeomorphic 4-manifolds that satisfy the following properties.

1. *The 4-manifolds $M_{p,k}$ and $N_{p,k}$ are irreducible for all $p \neq 0$.*

2. *There are homeomorphisms*

$$M_{p,k} \cong_{\mathcal{C}^0} \#_2\, \mathbb{CP}^2 \,\#_{k+1}\, \overline{\mathbb{CP}}^2 \,\#\, (S^1 \times S^3)$$
$$N_{p,k} \cong_{\mathcal{C}^0} \#_2\, \mathbb{CP}^2 \,\#_{k+1}\, \overline{\mathbb{CP}}^2 \,\#\, (T^2 \times S^2)$$

   for every $p \in \mathbb{N}$.

3. *There are diffeomorphisms*

$$M_{p,k} \,\#\, (S^2 \times S^2) \cong_{\mathcal{C}^\infty} M_{0,k} \,\#\, (S^2 \times S^2)$$
$$N_{p,k} \,\#\, (S^2 \times S^2) \cong_{\mathcal{C}^\infty} N_{0,k} \,\#\, (S^2 \times S^2)$$

   for every $p \in \mathbb{N}$.

Moreover, the 4*-manifolds* $M_{1,k}$ *and* $N_{1,k}$ *admit a symplectic structure of symplectic Kodaira dimension two.*

Another contribution of this Chapter is an extension of recent work of Baykur–Sunukjian [31] on stabilizations of smooth structures on simply connected 4-manifolds to arbitrary fundamental group as we sample in the third clause of Theorem A. Indeed, results of Gompf [71] and Wall [151] say that two closed smooth oriented homeomorphic 4-manifolds become diffeomorphic after taking a connected sum with n copies of $S^2 \times S^2$ for some $n \in \mathbb{N}$. Baykur–Sunukjian [31] showed that the value $n = 1$ suffices for any pair of closed smooth simply connected non-spin 4-manifolds that are related by a torus surgery. Our extension of their work in Section 2.4 might be of independent interest in addition to our main result. In particular, the technical tool we use to stabilize the examples produced in Theorem A is the following.

Proposition B. *Let* X *be a closed smooth oriented* 4*-manifold that contains a smoothly embedded* 2*-torus* $T \subseteq X$ *with trivial normal bundle and let* $X_{T,\gamma}(p)$ *be the result of* p*-torus surgery on* T *along a simple closed curve* $\gamma \subseteq T$*, as defined in (1.1.3). Suppose that the complement* $X \setminus \operatorname{Int} \nu(T)$ *is non-spin and that the inclusion* $X \setminus \operatorname{Int} \nu(T) \to X$ *induces an isomorphism of fundamental groups* $\pi_1(X \setminus \operatorname{Int} \nu(T)) \to \pi_1(X)$*. There is a diffeomorphism*

$$X_{T,\gamma}(p) \,\#\, (S^2 \times S^2) \cong_{\mathcal{C}^\infty} X_{T,\gamma}(0) \,\#\, (S^2 \times S^2)$$

for every $p \in \mathbb{N}$*.*

Chapter 2 is structured as follows. The key component of the procedure to compute the equivariant intersection forms of 4-manifolds that are obtained via torus surgeries is detailed in Section 2.2; another method to compute intersections of surfaces is outlined in [39]. Section 2.1 contains a description of a construction of small symplectic 4-manifolds due to Baldridge–Kirk [28], where we also investigate further their fundamental group computations. The construction of the examples of Theorem A is described in Section 2.1.1. Their homeomorphism classes are identified in Section 2.3 by computing the equivariant intersection form through the intersection of the lifts of surfaces to the corresponding universal covering. A procedure to obtain such surfaces in 4-manifolds that are obtained from torus surgeries is described in Section 2.2. Our result on stabilizations with a single copy of $S^2 \times S^2$, as previously mentioned, is presented in Section 2.4. The proof of Theorem A is given in Section 2.5.

2.1 A construction of irreducible 4-manifolds.

2.1.1 An irreducible symplectic 4-manifold with free abelian fundamental group

We describe a construction due to Baldridge–Kirk [27, Section 4], [28, Section 4.3] of a symplectic 4-manifold M (this 4-manifold is denoted by Z in [28]) that serves as a raw material to produce an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds in the homeomorphism type of $\mathbb{CP}^2 \#_3 \overline{\mathbb{CP}}^2$; such examples were also constructed by Akhmedov–Park [15], Fintushel–Park–Stern [55] and Akhmedov–Baykur–Park [14]; see Remark 2.1.3. Such 4-manifold is a generalized fiber sum

$$M = X_1 \#_{\Sigma_2} X_2 \tag{2.1.1}$$

of two symplectic building blocks

$$X_1 = (T^2 \times T^2) \#_2 \overline{\mathbb{CP}}^2$$

and

$$X_2 = T^2 \times \Sigma_2$$

along a symplectically embedded surface Σ_2 of genus two with trivial normal bundle. A result of Gompf [67] (see [73, Theorem 10.2.1]) implies that M admits a symplectic structure. Moreover, a result of Usher [149] allows us to conclude that the symplectic 4-manifold M is minimal and Theorem 1.1.12 implies M is irreducible since $\pi_1(M) = \mathbb{Z}^6$ is residually finite and $b_2^+(M) > 1$. Its symplectic Kodaira dimension is immediately computed as follows.

Lemma 2.1.1. *The symplectic Kodaira dimension of the irreducible symplectic* 4*-manifold* (M, ω) *defined in Formula 2.1.1 is* $\mathrm{Kod}(M, \omega) = 2$.

Proof. Let $K_\omega \in H^2(M)$ be the Chern class of the minimal symplectic 4-manifold (M, ω). We have that

$$K_\omega \cdot K_\omega = 2\chi(M) + 3\sigma(M) > 0$$

and the claim follows from Definition 1.1.9. □

We now recall some homotopy theoretical traits of the 4-manifold M that are useful for our purposes. Baldridge–Kirk [28, Theorem 5] showed that the group $\pi_1(M) = \mathbb{Z}^6$ is generated by the based homotopy classes of six loops $x, y, a_1, a_2, b_1, b_2 \subseteq M$. Moreover, they described smooth representatives for a basis of the second singular homology group with integer coefficients $H_2(M) = \mathbb{Z}^{16}$ in [28, Proposition 12]. We refer the reader to their paper for the description of such embedded surfaces, which we will denote by

$$H_1, H_2, H_3, F, T_1, R_1, T_2, R_2, T_3, R_3, T_4, R_4, T_1', R_1', T_2', R_2'$$

using the same notation. With respect to such basis, the intersection form of M is represented by the matrix

$$Q_M = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \bigoplus_{i=1}^{6} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1.2}$$

In particular, the six hyperbolic summands correspond to geometrically dual pairs of smoothly embedded 2-tori. Baldridge and Kirk show that performing torus surgery on the collection of pairwise disjoint 2-tori

$$\mathcal{T} = \{T_1, T_2, T_3, T_4, T_1', T_2'\} \tag{2.1.3}$$

with an appropriate choice of loops and coefficients yields an infinite family of exotic 4-manifolds with the same homeomorphism type of $\mathbb{CP}^2 \#_3 \overline{\mathbb{CP}}^2$ [28, Corollary 14].

2.1.2 Some 4-manifolds obtained from torus surgeries on M and their fundamental groups

Baldridge–Kirk perform torus surgeries on the 4-manifold M defined in 2.1.1 to produce several interesting symplectic 4-manifolds of Euler characteristic six and signature minus two, which include examples with infinite cyclic fundamental group [28, Theorem 22] and free abelian fundamental group of rank three in [28, Corollary 28]. A modest tweak in their construction yields a myriad of other examples, which include the 4-manifolds of Theorem A for $k = 3, 5$ [28, 146].

The following result is an extension of some fundamental group computations by Baldridge–Kirk. More precisely, we study the fundamental group of the 4-manifolds obtained by torus surgery on the elements of all the possible subcollections of $\mathcal{T}$. We show that (-1)-torus surgery on M on such subfamilies yields a 4-manifold with either a free abelian or a non-abelian fundamental group.

Theorem 2.1.2 (c.f. Baldridge–Kirk [28])**.** *Let M be the smooth closed oriented 4-manifold defined in formula 2.1.1. The following facts hold.*

1. *Performing (-1)-torus surgery on a 2-torus in $\mathcal{T}$ yields a closed irreducible symplectic 4-manifold X of symplectic Kodaira dimension two and with non-abelian fundamental group*

$$\pi_1(X) \cong \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_3, [x_i, x_j] = 1 \text{ if } \{i, j\} \neq \{1, 2\} \rangle.$$

2. *Performing (-1)-torus surgery on a collection of two distinct 2-tori $\mathcal{T}_2 \subseteq \mathcal{T}$ produces a closed irreducible symplectic 4-manifold $M_{\mathcal{T}_1}$ of symplectic Kodaira dimension two*

and with non-abelian fundamental group. Moreover, if

$$\mathcal{T}_2 \in \{(T_1,', T_1), (T_3, T_1'), (T_2', T_4)\}$$

performing (-1)*-torus surgery and* p*-torus surgery respectively on the two* 2*-tori in* $\mathcal{T}_2$ *for all* $p \in \mathbb{Z}$ *yields an infinite family of closed oriented* 4*-manifolds with the same non-abelian fundamental group.*

3. *Performing* (-1)*-torus surgery on a collection of three distinct* 2*-tori* $\mathcal{T}_3 \subseteq \mathcal{T}$ *produces a closed irreducible symplectic* 4*-manifold* $M_{\mathcal{T}_3}$ *of symplectic Kodaira dimension two and with fundamental group* $\pi_1(M_{\mathcal{T}_3}) \cong \mathbb{Z}^3$ *provided* $\mathcal{T}_3$ *is among the collections in the following table, where the second column contains the generators of* $\pi_1(M_{\mathcal{T}_3}) \cong \mathbb{Z}^3$.

$\mathcal{T}_3$	Generators of $\pi_1(M_{\mathcal{T}_3})$
T_1, T_2, T_4	a_2, b_1, b_2
T_1, T_2, T_1'	y, a_2, b_2
T_1, T_3, T_4	a_1, b_1, b_2
T_2, T_4, T_2'	x, a_2, b_1
T_2, T_1', T_2'	x, y, a_2
T_3, T_4, T_1'	x, a_1, b_2
T_3, T_4, T_2'	x, a_1, b_1
T_3, T_1', T_2'	x, y, a_1

In all the other cases, $\pi_1(M_{\mathcal{T}_3})$ *is non-abelian.*

4. *Performing* (-1)*-torus surgery on a collection of four distinct* 2*-tori* $\mathcal{T}_4 \subseteq \mathcal{T}$ *produces a closed irreducible symplectic* 4*-manifold* $M_{\mathcal{T}_4}$ *of symplectic Kodaira dimension two and with fundamental group* $\pi_1(M_{\mathcal{T}_4})$ *as reported in the following table, where* $G(x_1, x_2)$ *denotes the non-abelian group*

$$G(x_1, x_2) = \langle x_1, x_2 \mid [x_1, x_2] = [x_2, x_1^{-1}] = [x_2^{-1}, x_1] \rangle.$$

Surgered 2-*tori* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$	$\pi_1(X_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_4})$
T_1, T_2, T_3, T_4	$\mathbb{Z}^2 \cong \langle b_1, b_2 \rangle$
T_1, T_2, T_3, T_1'	$G(y, b_2)$
T_1, T_2, T_4, T_1'	$\mathbb{Z}^2 \cong \langle a_2, b_2 \rangle$
T_1, T_2, T_4, T_2'	$G(a_2, b_1)$
T_1, T_2, T_1', T_2'	$\mathbb{Z}^2 \cong \langle y, a_2 \rangle$
T_1, T_3, T_4, T_1'	$\mathbb{Z}^2 \cong \langle a_1, b_2 \rangle$
T_1, T_3, T_4, T_2'	$\mathbb{Z}^2 \cong \langle a_1, b_1 \rangle$
T_1, T_3, T_1', T_2'	$\mathbb{Z}^2 \cong \langle y, a_2 \rangle$
T_2, T_3, T_4, T_2'	$G(x, b_1)$
T_2, T_3, T_1', T_2'	$\mathbb{Z}^2 \cong \langle x, y \rangle$
T_2, T_4, T_1', T_2'	$\mathbb{Z}^2 \cong \langle x, a_2 \rangle$
T_3, T_4, T_1', T_2'	$\mathbb{Z}^2 \cong \langle x, a_1 \rangle$

In all the cases not reported in the table, the fundamental group of the surgery is non-abelian.

The 4*-manifold that is obtained if we perform three* (-1)*-torus surgeries and a* p*-torus surgery with* $p \neq \pm 1$ *on a collection of four distinct* 2*-tori* $\mathcal{T}_4 \subseteq \mathcal{T}$ *has still fundamental group* $\mathbb{Z}^2$*.*

5. *Performing* (-1)*-torus surgery on a collection of five distinct* 2*-tori* $\mathcal{T}_5 \subseteq \mathcal{T}$ *produces a closed irreducible symplectic* 4*-manifold of symplectic Kodaira dimension two and with infinite cyclic fundamental group. A generator of such group is given by a loop among* x, y, a_1, a_2, b_1, b_2 *that is contained in* $\mathcal{T} \setminus \mathcal{T}_5$*.*
   The 4*-manifold that is obtained if we perform four* (-1)*-torus surgeries and a* p*-torus surgery with* $p \neq \pm 1$ *on a collection of five distinct* 2*-tori* $\mathcal{T}_5 \subseteq \mathcal{T}$ *has infinite cyclic fundamental group.*

Proof. The claims regarding irreducibility and the existence of a symplectic structure of symplectic Kodaira dimension two follow from Theorem 1.1.8, Theorem 1.1.12 and Lemma 2.1.1. The proof of the other claims is based heavily on Baldridge–Kirk's computation of the fundamental group of of the complement C of the disjoint union of the 2-tori in the collection $\mathcal{T}$ in [28, Theorem 11] and of the based homotopy classes of their meridians, together with Seifert–van Kampen's theorem. In particular, [28, Theorem 11] tells us that the $\pi_1(C)$ is generated by the based homotopy classes of the loops x, y, a_1, a_2, b_1, b_2 with relations

$$\begin{aligned} &[x, a_1] = 1, [x, a_2] = 1, [y, a_1] = 1, [y, a_2] = 1, [x, y] = 1 \\ &[b_1, b_2] = 1, [a_1, b_1] = 1, [a_1, b_2] = 1, [a_2, b_2] = 1. \end{aligned} \tag{2.1.4}$$

2-torus	Meridian	Surgery curve	p-torus surgery relation
T_1	$[b_1^{-1}, y^{-1}]$	x	$x = [b_1^{-1}, y^{-1}]^{-p}$
T_2	$[x^{-1}, b_1]$	a_1	$a_1 = [x^{-1}, b_1]^{-p}$
T_3	$[b_2^{-1}, y^{-1}]$	a_2	$a_2 = [b_2^{-1}, y^{-1}]^{-p}$
T_4	$[x^{-1}, b_2]$	y	$y = [x^{-1}, b_2]^{-p}$
T_1'	$[a_2^{-1}, a_1^{-1}]$	b_1	$b_1 = [a_2^{-1}, a_1^{-1}]^{-p}$
T_2'	$[b_1, a_2]$	b_2	$b_2 = [b_1, a_2]^{-p}$

Table 2.1: 2-tori, meridians, surgery curves and relations introduced to π_1.

The rows of Table 2.1 contain the 2-tori of the collection $\mathcal{T}$ together with the based homotopy classes of their meridians, the surgery curves and the relations imposed on the generators of $\pi_1(C)$ by gluing back the 2-tori while performing a p-torus surgery.

One can compute the free abelian fundamental groups in the statement by following the strategy employed in [28, Theorem 13]. We outline the computations for the case $\mathcal{T}_4 = \{T_3, T_4, T_1', T_2'\}$ for the convenience of the reader. The fundamental group of the 4-manifold $M_{T_3,T_4,T_1',T_2'}$ is generated by the loops x, y, a_1, a_2, b_1, b_2 with relations 2.1.4 and

$$\begin{aligned} &a_2 = [b_2^{-1}, y^{-1}], y = [x^{-1}, b_2], b_1 = [a_2^{-1}, a_1^{-1}], b_2 = [b_1, a_2], \\ &[b_1, y] = 1, [b_1, x] = 1. \end{aligned} \tag{2.1.5}$$

Since x commutes both with b_1 and a_2 by 2.1.4 and 2.1.5, one gets that $y = 1$ by substituting the expression for b_2 into the one for y in 2.1.5. This in turn implies that $a_2 = 1, b_1 = 1$ and $b_2 = 1$, while x and a_1 commute by 2.1.4. It hence follows that

$$\pi_1(M_{T_3,T_4,T_1',T_2'}) \cong \langle x, a_1 \mid [x, a_1] = 1 \rangle \cong \mathbb{Z}^2.$$

The non-abelian cases are carried out by first finding a presentation of the fundamental group of the 4-manifold obtained by torus surgery on M by means of generators and relations using Seifert–van Kampen's theorem and then building a surjective group homomorphism onto the quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbb{H}.$$

We now report the computations of the non-abelian fundamental group of the manifold $M_{T_1,T_2,T_3,T_1'}$ as a leading example. Using the Seifert–van Kampen's theorem we conclude that the fundamental group $\pi_1(M_{T_1,T_2,T_3,T_1'})$ is generated by the based homotopy classes of the loops x, y, a_1, a_2, b_1, b_2. The relations are the ones in (2.1.4), characterizing the fundamental group of the complement of the union 2-tori, plus the following ones:

$$\begin{aligned} &x = [b_1^{-1}, y^{-1}], a_1 = [x^{-1}, b_1], a_2 = [b_2^{-1}, y^{-1}] \\ &b_1 = [a_2^{-1}, a_1^{-1}], [b_1, a_2] = 1, [x, b_2] = 1. \end{aligned} \tag{2.1.6}$$

Substituting the expression for b_1 in (2.1.6) into the one for a_1 and using the fact that x commutes both with a_1 and a_2 by (2.1.4), we get that $a_1 = 1$. By substituting a_1 into the relations (2.1.6), we get that also $b_1 = 1$ and hence $x = 1$. In particular, this implies that the fundamental group $\pi_1(M_{T_1,T_2,T_3,T_1'})$ is generated by the based homotopy classes of y and b_2. The non-trivial relations are

$$[y, [b_2^{-1}, y^{-1}]] = 1, [[b_2^{-1}, y^{-1}], b_2] = 1 \tag{2.1.7}$$

and one can check that they are equivalent to the conditions

$$[y, b_2] = [b_2, y^{-1}] = [b_2^{-1}, y]. \tag{2.1.8}$$

In particular, we have just proved that

$$\pi_1(M_{T_1,T_2,T_3,T_1'}) \cong \langle y, b_2 \mid [y, b_2] = [b_2, y^{-1}] = [b_2^{-1}, y] \rangle.$$

This group is non-abelian, since it is possible to define a group homomorphism

$$\phi \colon \pi_1(M_{T_1,T_2,T_3,T_1'}) \longrightarrow Q_8 \tag{2.1.9}$$

by setting $\phi(y) = i$ and $\phi(b_2) = j$. □

Remark 2.1.3. Exotic irreducible smooth structures on $\mathbb{CP}^2 \#_3 \overline{\mathbb{CP}}^2$ were constructed by Akhmedov–Park [15] and Baldridge–Kirk [27, 28] by performing six torus surgeries to the 4-manifold described in formula (2.1.1). Akhmedov–Baykur–Park [14] showed that the difference in these two constructions is the choice of three out of the six 2-tori involved in the surgeries. In particular, the mechanism described in Section 1.3.1 applies verbatim to the 4-manifolds of Theorem 2.1.2.

In each item of Theorem 2.1.2 there are various choices for the sub-collection of tori to perform the surgery. We conclude this subsection by stating the following open problem.

Question 2.1.4. Are the resulting manifolds diffeomorphic to each other?

2.2 Getting rid of essential loops on surfaces via surgery

We work in the following setting throughout this section.

- The 4-manifold $X' = X_{T,\gamma}(p)$ is the result of a p-torus surgery to a closed smooth oriented 4-manifold X on a homologically essential 2-torus $T \subseteq X$ along a primitive curve $\gamma \subseteq T$ and $T' \subseteq X'$ is the core 2-torus after the surgery.

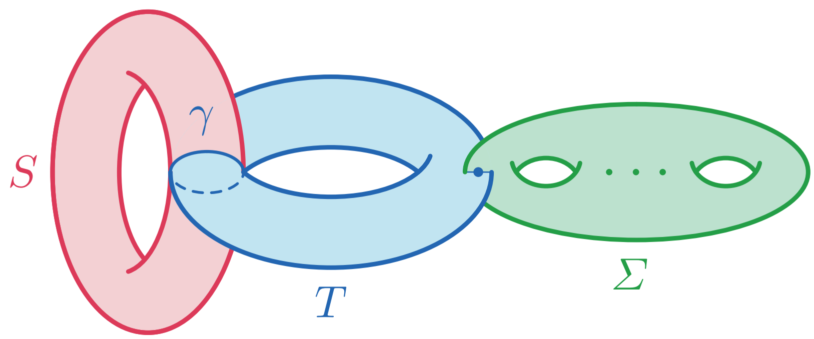

Figure 2.1: Reciprocal positions of T, S and Σ under the hypothesis of Section 2.2.

- There is a smoothly embedded 2-torus $S \subseteq X$ that is nowhere tangent to T satisfying

$$\gamma = S \cap T = \sigma_1,$$

  where $\sigma_1, \sigma_2 \subseteq S$ are two simple loops that generate $\pi_1(S)$.

- For a given framing $\nu(T) \cong \gamma \times \eta \times D^2$, the intersection $S \cap \nu(T)$ is of the form $\gamma \times \{p_\eta\} \times D^1$, where $D^1 \subseteq D^2$ is the standard interval;

- There is a smoothly embedded closed oriented genus g surface $\Sigma \subseteq X$ with trivial normal bundle and that is geometrically dual to the 2-torus T. Moreover, $[\sigma_2] \notin (i_\Sigma)_*(\pi_1(\Sigma))$, where $i_\Sigma \colon \Sigma \hookrightarrow X$ is the inclusion map.

Notice that the assumption on the intersection $S \cap \nu(T)$ implies that, up to an isotopy inside X, we can make S disjoint from the surgery 2-torus T, so that it defines a smoothly embedded surface

$$S' \subseteq X \setminus \operatorname{Int} \nu(T) = X' \setminus \operatorname{Int} \nu(T') \subseteq X'$$

with curves $\sigma_1', \sigma_2' \subseteq S'$ corresponding to $\sigma_1, \sigma_2 \subseteq S$. In particular, since $S \cap \nu(T) = \gamma \times \{p_\eta\} \times D^1$, one can do so by moving the standard interval $D^1 \subseteq D^2$ along one normal direction, until it degenerates to one point and then disappears.

Lemma 2.2.1. *Under the above assumptions, there is a smoothly embedded closed connected oriented surface $S^* \subseteq X'$ of genus $2pg$ such that $[S^*] = [S'] \in H_2(X')$ and $[\sigma_2'] \notin i_*(\pi_1(S^*)) \subseteq \pi_1(X')$, where $i_* \colon \pi_1(S^*) \to \pi_1(X')$ is the homomorphism induced by the inclusion.*

Proof. Consider a properly embedded cylinder $C = S \setminus \operatorname{Int} \nu(T) \subseteq X \setminus \operatorname{Int} \nu(T)$ that bounds two parallel push-offs of γ. Up to a slight perturbation, we can suppose that such push-offs are of the form

$$\gamma^i = \gamma \times \{p_{\eta,i}\} \times \{p_{\mu,i}\} \subseteq \gamma \times \eta \times \partial D^2$$

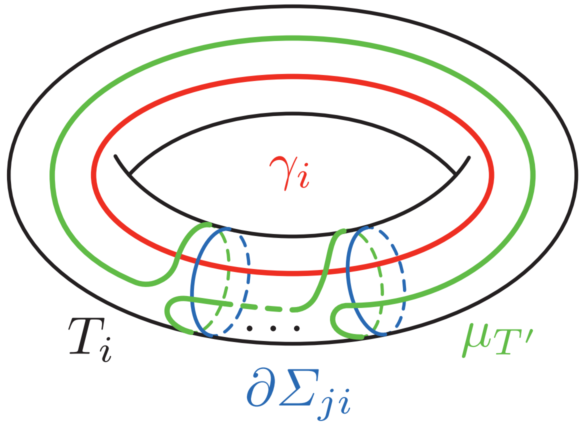

Figure 2.2: Curves in the 2-tori T_1 and T_2.

inside $\partial\nu(T) \cong \gamma \times \eta \times \partial D^2$, where $p_{\eta,1} \neq p_{\eta,2}$. Since $X \setminus \text{Int}\, \nu(T) = X' \setminus \text{Int}\, \nu(T')$, such cylinder is also contained in X', where it is denoted by $C' \subseteq X'$. Let $\{\Sigma_j^1, \Sigma_j^2 \mid j = 1, \ldots, p\}$ be $2p$ distinct parallel copies of the properly embedded surface

$$\Sigma \setminus \text{Int}\, \nu(T) \subseteq X \setminus \text{Int}\, \nu(T).$$

In particular, we can suppose without loss of generality that

$$\partial\Sigma_j^i = \{p_{\gamma,j}\} \times \{p_{\eta,i}\} \times \partial D^2 \subseteq \gamma \times \eta \times \partial D^2$$

for $i = 1, 2$, $j = 1, \ldots, p$, where $p_{\gamma,h} \neq p_{\gamma,k}$ for any $h \neq k$. The union of the cylinder C with the surfaces Σ_j^i gives a properly immersed surface

$$\tilde{S} = C \cup_{i,j} \Sigma_j^i \subseteq X \setminus \text{Int}\, \nu(T)$$

with $\partial\tilde{S} \subseteq T_1 \sqcup T_2$, where for $i = 1, 2$ we set

$$T_i = \gamma \times \{p_{\eta,i}\} \times \partial D^2 \subseteq \partial\nu(T).$$

In particular, $\partial\tilde{S}$ is the union of the push-offs γ^i and $\partial\Sigma_j^i$ for $j = 1, \ldots, p$, which inside the 2-tori T_1, T_2 respectively look like the union of a longitude with p disjoint parallel copies of a meridian curve, given by $\partial\Sigma_j^i$ for $j = 1, \ldots, p$, as in Figure 2.2. We orient γ_i as a boundary component of $C \subseteq S$, while an orientation on Σ_j^i is defined by imposing that γ_i and $\partial\Sigma_j^i$ have both the same or the opposite orientation as the curves γ and μ_T inside T_i. Notice that in this way we have that $[\Sigma_j^1] = -[\Sigma_k^2] \in H_2(X, \nu(T))$ for all $j, k = 1, \ldots, p$.

There are hence $2p$ points of self-intersection between the cylinder C and the surfaces Σ_j^i inside the two 2-tori T_1 and T_2. We can resolve such singularities by gluing bands in the

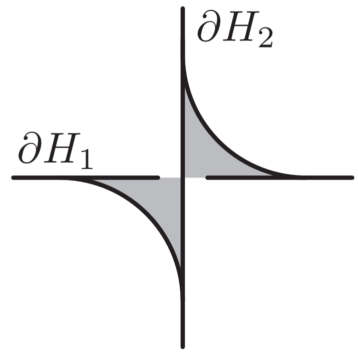

Figure 2.3: Band desingularization.

following way. Locally around the intersection point between C and Σ^i_j, we picture the two surfaces in a 3-dimensional chart with coordinates x, y, z in such a way that C is locally sent to the the half-plane $H_1 = \{x = 0, y \geq 0\}$, Σ^i_j goes onto $H_2 = \{z = 0, y \leq 0\}$ and the intersection point $C \cap \Sigma^i_j$ is sent to the origin. Apply a small translation to H_1 along the y-axis to make it disjoint from H_2, e.g. by mapping it to the subset $H'_1 = \{x = 0, y \geq \epsilon\}$ with $\epsilon > 0$, and glue H'_1 and H_2 by means of a 2-dimensional band doing a $\frac{\pi}{2}$-twist in the z-direction, where the sense of rotation is determined by respecting the local orientations of C and Σ^i_j. Up to composing with an ambient isotopy of $\mathbb{R}^3$, we now get a surface with boundary contained in the plane of equation $y = 0$ as in Figure 2.3.

By locally substituting the intersection point of C with Σ^i_j with this local model for every $j = 1, \ldots, p$, we get a new properly embedded surface

$$\widehat{S} \subseteq X \setminus \operatorname{Int} \nu(T) = X' \setminus \operatorname{Int} \nu(T')$$

of genus $2gp$ with two boundary components which are isotopic inside each T_i to an oriented closed curve which is the connected sum of γ^i with the $\partial\Sigma^i_j$'s. In particular, such closed curves are isotopic to the meridian of the core 2-torus T' and have opposite orientations. This allows us to cap-off the boundary components of $\widehat{S} \subseteq X'$ with two 2-disks bounding two parallel copies of the meridian of T' inside $\nu(T')$ and finally get the desired submanifold $S^* \subseteq X'$.

Let us now consider the homology class $[S^*] \in H_2(X')$. Since the core 2-torus $T' \subseteq X'$ is null-homologous by Proposition 1.1.4, the natural map $H_2(X') \to H_2(X', \nu(T'))$ is injective and $[S^*] = [S'] \in H_2(X')$ if and only if $[S^*] = [S'] \in H_2(X', \nu(T'))$. This last identity holds since

$$[S^*] = [\widehat{S} \cup D^2_1 \cup -D^2_2] = [\widehat{S}] = [C' \cup_{i,j} \Sigma^i_j] = [C'] = [S'] \in H_2(X', \nu(T'))$$

given that we have oriented each surface Σ^i_j in such a way that $[\Sigma^1_j] = -[\Sigma^2_j] \in H_2(X', \nu(T'))$

for every $j = 1, \ldots, p$, while D_1^2, D_2^2 are the two disks used to cap-off the boundary components of $\widehat{S}$. □

2.3 Topological prototypes.

We now pin down the homeomorphism type of the 4-manifolds that were constructed in Section 2.1.2 and begin with those with infinite cyclic fundamental group. For 2-tori $T \in \mathcal{T} = \{T_1, T_2, T_3, T_4, T_1', T_2'\}$ and $p \in \mathbb{N}$, denote by $M_T(p)$ the 4-manifold that is obtained from M defined in formula (2.1.1) by performing (-1)-torus surgery on four distinct elements of the set $\mathcal{T} \setminus \{T\}$ and a p-torus surgery on the remaining 2-torus, with the choices surgery curves explained at the beginning of Section 2.1.2. Theorem 2.1.2 says that $\pi_1(M_T(p)) \cong \mathbb{Z}$ for every $p \in \mathbb{N}$. As it was mentioned in the introduction, these 4-manifolds need not be new [28, 146, 148].

Theorem 2.3.1. *For every $p \in \mathbb{N}$, there exists a homeomorphism*

$$M_T(p) \cong_{C^0} \#_2\, \mathbb{CP}^2 \#_4\, \overline{\mathbb{CP}}^2 \#(S^1 \times S^3).$$

Proof. For the sake of clarity in the exposition, we argue in detail the case where $T = T_2$ and the p-torus surgery is performed along the 2-torus T_4. The proof for all other choices is carried out in a similar way. Theorem 2.1.2 implies that

$$\pi_1(M_{T_2}(p)) = \langle a_1 \rangle \cong \mathbb{Z}.$$

In order to apply Theorem 1.2.14 to pin down the homeomorphism type of $M_{T_2}(p)$, we compute its equivariant intersection form up to isometries.

We first consider the intersection form of $M_{T_2}(p)$ over the integers. Proposition 1.1.4 implies that $b_2(M_{T_2}(p)) = b_2(M) - 2 \cdot 5 = 6$. In particular, the second homology group

$$H_2(M_{T_2}(p)) \cong \mathbb{Z}^6$$

is generated by the surfaces $T_2, R_2, H_1, H_2, H_3, F$ (see [28, Proposition 12]). Notice that these surfaces do not intersect the 2-tori in M that are used in the surgeries to produce $M_{T_2}(p)$ and, hence, they define six embedded surfaces inside $M_{T_2}(p)$ for every $p \in \mathbb{N}$. It follows that the intersection form of $M_{T_2}(p)$ over the integers coincides with the one of $\#_2\, \mathbb{CP}^2 \#_4\, \overline{\mathbb{CP}}^2 \#(S^1 \times S^3)$ for every $p \in \mathbb{N}$. To finish the proof of Theorem 2.3.1, we show that the equivariant intersection form of $M_{T_2}(p)$ is extended from the integers.

The surfaces H_1, H_2, H_3 and R_2 admit a lift to the universal covering of $M_{T_2}(p)$, since the image of their fundamental groups via the inclusion induced homomorphism is trivial. Moreover, for a fixed choice of a lifting, these surfaces define elements of the second homology group of the universal covering of $M_{T_2}(p)$, on which the equivariant intersection form can be computed. In the following, we will build two new surfaces that are homologous to T_2 and F, respectively, and which also admitting a lift to the universal covering of $M_{T_2}(p)$. We start with T_2. Apply Lemma 2.2.1 with the choices $S = T_2$, $\Sigma = R_4$, $T = T_4$, $\gamma = y$, $X' = M_{T_2}(p)$ and X equal to the 4-manifold obtained by performing torus surgery on the 2-tori T_1, T_3, T_1', T_2' to obtain a smoothly embedded oriented surface $T_2^{*p} \subseteq M_{T_2}(p)$ such that $i_*(\pi_1(T_2^{*p})) = 0 \subseteq \pi_1(M_{T_2}(p))$, where i_* is the inclusion-induced homomorphism. In order to find a new representative for $[F] \in H_2(M_{T_2}(p))$, we consider the connected sum decomposition of F into two 2-tori F_1 and F_2 as in [28, Section 4.3], where $\pi_1(F_i) \cong \langle a_i, b_i \mid [a_i, b_i] = 1\rangle \cong \mathbb{Z}^2$ for $i = 1, 2$. We will replace F_1 in such decomposition with a surface that admits a lift to the universal covering of the ambient manifold. By the definition of the map used in (2.1.1) for gluing the two building blocks X_1 and X_2, we have that (up to an ambient isotopy of $M_{T_2}(p)$) F_1 intersects the tubular neighbourhood $\nu(T_1')$ along a cylinder bounding two parallel push-offs of the loop b_1, which is killed at the level of the fundamental group by the torus surgery we perform on T_1'. Notice that one might first think to set $S = F_1$, $\Sigma = R_1'$, $T = T_1'$, and $\gamma = b_1$ in order to apply Lemma 2.2.1, and get a new liftable surface representing the same homology class as F_1. However, in this case the procedure described in Lemma 2.2.1 would build a surface that cannot be lifted to the universal covering of $M_{T_2}(p)$, since the inclusion $R_1' \subseteq M_{T_2}(p)$ does not induce the trivial homomorphism between fundamental groups. We can get around this issue by iterating the process and applying Lemma 2.2.1 with a different choice of Σ. In particular, for our purposes we can set Σ to be the geometrically dual surface to T_1' obtained by applying Lemma 2.2.1 to R_1' with choices $S = R_1'$, $\Sigma = R_3$ and $T = T_3$. Indeed, a basis of $\pi_1(R_1')$ is given by a_1, a_2 and a_2 is null-homotopic in $M_{T_2}(p)$ thanks to the torus surgery on T_3. Moreover, T_3 has a geometrically dual 2-torus R_3 with the desired properties. In this way, we get out of F_1 a new surface F_1^{*p} in the same homology class. The surface F^{*p} obtained by substituting F_1 with F_1^{*p} in the connected sum decomposition of F gives the desired new representative of $[F]$.

We conclude that the equivariant intersection form of $M_{T_2}(p)$ can be computed by using homology classes in $H_2(\widetilde{X}_{T_2}(p)) \cong \pi_2(M_{T_2}(p))$ that are represented by lifts of the surfaces H_1, H_2, H_3, F^{*p}, T_2^{*p} and R_2. In particular, we point out that there is a handlebody decomposition of $M_{T_2}(p)$ for which the hypothesis of Example 1.2.13 are satisfied and hence the equivariant intersection form if $M_{T_2}(p)$ is extended from the integers. Indeed, starting from the fiber sum decomposition (2.1.1) of $M = X_1 \#_{\Sigma_2} X_2$, one can also see $M_{T_2}(p)$ as a generalized fiber sum

$$M_{T_2}(p) = M_{1,T_2}(p) \#_{\Sigma_2} M_{2,T_2}(p)$$

along a surface of genus two, where $M_{1,T_2}(p)$ is obtained from X_1 by (-1)-torus surgery on T_1' and T_2' and $M_{2,T_2}(p)$ is defined by simultaneously performing two (-1)-torus surgeries on T_1 and T_3 and a p-torus surgery on T_4 to X_2. The above decomposition implies that $M_{T_2}(p)$ is obtained by performing four loop surgeries and a 2-sphere surgery to the connected sum $M_{1,T_2}(p) \# M_{2,T_2}(p)$. Moreover, all such surgery operations happen away from the six embedded surfaces we want to use to compute the equivariant intersection form. We can hence build a handlebody decomposition of $M_{T_2}(p)$ starting from the standard ones of $X_1 = T^4 \#_2 \overline{\mathbb{CP}}^2$ and $X_2 = T^2 \times \Sigma_2$ (see [8, Chapter 4]), by keeping track of the result of the loop surgeries, the sphere surgery and all the torus surgery needed to construct $M_{T_2}(p)$. In particular, the single 1-handle giving a non-trivial element of $\pi_1(M_{T_2}(p)) \cong \langle a_1 \rangle$ will come from the 1-handle in the handlebody decomposition of $T^2 \times \Sigma_2$ corresponding to the loop a_1 in the symplectic basis $\{a_1, b_1, a_2, b_2\}$ of $\pi_1(\Sigma_2)$. It is then not hard to verify that $H_1, H_2, H_3, F^{*p}, T_2^{*p}$ and R_2 are contained in the complement of such 1-handle inside the 2-skeleton of $M_{T_2}(p)$ and the conclusion follows. □

Remark 2.3.2. If in the construction in the proof of Theorem 2.3.1, we perform a p-torus surgery with $p \neq 1$ instead of $p = 1$, then the modified representatives of the homology classes under consideration will have larger genus. However, their algebraic intersection will not change.

We spend the remaining part of this section working out the homeomorphism class of the 4-manifolds with free abelian group of rank two that are under consideration. Let us now pick a pair of distinct 2-tori

$$T, T' \in \mathcal{T} = \{T_1, T_2, T_3, T_4, T'_1, T'_2\}$$

and let $M_{T,T'}(p)$ be the 4-manifold obtained by performing (-1)-torus surgery on three distinct 2-tori in $\mathcal{T} \setminus \{T, T'\}$ and a p-torus surgery on the remaining 2-torus to the 4-manifold constructed in formula (2.1.1) (cf. [15, 28, 146, 148]). Theorem 2.1.2 implies that we can make these choices in such a way that $\pi_1(M_{T,T'}(p)) \cong \mathbb{Z}^2$.

Theorem 2.3.3. *For every $p \in \mathbb{N}$, there exists a homeomorphism*

$$M_{T,T'}(p) \cong_{C^0} \#_2 \mathbb{CP}^2 \#_4 \overline{\mathbb{CP}}^2 \# (T^2 \times S^2).$$

Proof. The strategy of the proof is the same as the one of Theorem 2.3.1: we will show that the equivariant intersection form of $M_{T,T'}(p)$ is extended from the integers. For the sake of clarity, we consider the case with choices $T = T_1$ and $T' = T_2$, and perform a p-torus surgery on T_4. The other cases follow from a similar argument. Theorem 2.1.2 implies that the group $\pi_1(M_{T_1,T_2}(p)) \cong \mathbb{Z}^2$ is generated by the based homotopy classes of x and a_1. On the other hand, as already done in the proof of Theorem 2.3.1, one can find new representatives for the second homology classes given by the six surfaces $H_1, H_2, H_3, F, T_2, R_2$ that do lift to the universal covering. On such elements, the equivariant intersection form is given by the matrix

$$Q_p = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3.1}$$

Since $H_2(\widetilde{X}_{T_1,T_2}(p))/R(\widetilde{I}_{M_{T_1,T_2}(p)})$ is a free $\mathbb{Z}[\pi_1(M_{T_1,T_2}(p))]$-module of rank $\chi(M_{T_1,T_2}(p)) = 6$ by Theorem, 1.2.18, the fact that Q_p is unimodular together with Remark 1.2.9 implies that

$$\widetilde{I}_{M_{T_1,T_2}(p)} \cong Q_p \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_1(M_{T_1,T_2}(p))]$$

for every $p \in \mathbb{N}$. Since both $M_{T_1,T_2}(p)$ and its universal covering are non-spin for any $p \in \mathbb{N}$, Theorem 2.3.3 follows from Theorem 1.2.14. $\square$

2.4 Stabilization of p-torus surgeries.

This section is devoted to the proof of Proposition B, which is an extension of results of Baykur–Sunukjian [31] to the non simply connected realm. This tool will be used in the proof of Item (3) in Theorem A. Our extension is based on coupling Baykur–Sunukjian's work with a well-known trick due to Moishezon's [115] that describes a relation between 0-torus surgeries and surgery along loops.

Proof of Proposition B. There is a cobordism

$$V = (I \times X) \cup R_2 \tag{2.4.1}$$

for a 5-dimensional round 2-handle $R_2 = S^1 \times D^2 \times D^2$ such that $\partial_- V = X$ and $\partial_+ V = X_{T,\gamma}(p)$ by a result of Baykur–Sunukjian [31, Lemma 1]. The round 2-handle R_2 decomposes into a 2-handle $h_2 = D^2 \times D^3$ and a 3-handle $h_2 = D^3 \times D^2$ and the cobordism (2.4.1) can be expressed as

$$V = (I \times X) \cup h_2 \cup h_3. \tag{2.4.2}$$

We now look in further detail at the middle level $V_{1/2}$ of (2.4.2) in order to prove Proposition B as it was done by Baykur–Sunukjian in [31]. The attaching circle of the 5-dimensional 2-handle in (2.4.2) is the surgery curve $\gamma \subseteq X$ (cf. [31, Proof of Lemma 4]). In particular, we have

$$V_{1/2} = (X \setminus \text{Int } \nu(\gamma)) \cup (D^2 \times S^2), \tag{2.4.3}$$

where the loop $\gamma \subseteq T \subseteq X$ is framed using the product framing of the 2-torus T. Apply Moisheson's trick [115] (see [72]) to describe (2.4.3) as the result of doing surgery along a null-homotopic loop $\gamma' \subseteq X_{T,\gamma}(0)$ and conclude that

$$V_{1/2} = X_{T,\gamma}(0) \# (S^2 \times S^2) \tag{2.4.4}$$

by using the hypothesis that the complement $X \setminus \text{Int } \nu(T)$ is non-spin. Notice that there is a diffeomorphism

$$X_{T,\gamma}(0) \# (S^2 \times S^2) \cong_{\mathcal{C}^\infty} X_{T,\gamma}(0) \# (S^2 \widetilde{\times} S^2),$$

where $S^2 \widetilde{\times} S^2$ is the twisted S^2-bundle over S^2.

Turn the cobordism (2.4.2) upside down, to see that $V_{1/2}$ is also obtained by attaching a 5-dimensional 2-handle to $X_{T,\gamma}(p)$ along a null-homotopic loop: this loop is null-homotopic since we have assumed that the meridian μ_T of the 2-torus $T \subseteq X$ is null-homotopic in $X \setminus \operatorname{Int} \nu(T)$. Given that $X_{T,\gamma}(p) \setminus \operatorname{Int} \nu(\widehat{T}_p) = X \setminus \operatorname{Int} \nu(T)$ is non-spin, we conclude that

$$V_{1/2} = X_{T,\gamma}(p) \# (S^2 \times S^2) \tag{2.4.5}$$

and the proposition follows.

□

We state the following generalization of Proposition B for the convenience of the reader.

Corollary 2.4.1. *Let*

$$\bigsqcup_{i=1}^{k} T_i \subseteq X$$

be a collection of pairwise disjointly embedded 2-tori in a closed oriented smooth 4-manifold X and let X_m be the result of m simultaneous p-torus surgeries on the 2-tori $T_1, \ldots, T_m$ along embedded loops $\gamma_1, \ldots, \gamma_m$ respectively, for $m = 1, \ldots, k$. Assume that γ_m is null-homotopic in X_{m-1} and that the meridian μ_{T_m} of each 2-torus T_m is null-homotopic in the complement $X_{m-1} \setminus \operatorname{Int} \nu(T_m)$ for every $m = 1, \ldots, k$, where we set $X_0 = X$. Then there is a diffeomorphism

$$X_{m-1} \# (S^2 \times S^2) \cong_{C^\infty} X_m \# (S^2 \times S^2)$$

for every $m = 1, \ldots, k$.

Proof. By induction on k, using Proposition B. □

2.5 Proof of Theorem A.

We provide a detailed proof of the statement in the case $k = 3$. Set $M_{p,3} = M_T(p)$ and $N_{p,3} = M_{T,T'}(p)$, where $M_T(p)$ is the 4-manifold defined in the proof of Theorem 2.3.1, and $M_{T,T'}(p)$ is the 4-manifold defined the proof of Theorem 2.3.3; cf. Theorem 2.1.2. Theorem 1.3.1 along with the discussion in Section 1.3.1 and Theorem 1.3.3 imply that the two collections

$$\{M_{p,3} \mid p \in \mathbb{N}\} \text{ and } \{N_{p,3} \mid p \in \mathbb{N}\}$$

consist of pairwise non-diffeomorphic elements that are irreducible provided $p \neq 0$; see Remark 2.1.3. Their homeomorphism classes have been determined in Theorem 2.3.1 and Theorem 2.3.3. The third clause of Theorem A follows from Proposition B. The 4-manifolds $M_{1,3}$ and $N_{1,3}$ admit a symplectic structure of symplectic Kodaira dimension two by 1.1.10, in view of 2.1.1.

k	χ	σ	Raw material	Number of surgeries	Source
2	5	-1	$(T^4 \# \overline{\mathbb{CP}}^2) \#_{\Sigma_2} (T^2 \times \Sigma_2)$	$(4 \text{ or } 3) + 1$	[16, 146]
3	6	-2	$(T^4 \#_2 \overline{\mathbb{CP}}^2) \#_{\Sigma_2} (T^2 \times \Sigma_2)$	$(4 \text{ or } 3) + 1$	[15, 28, 146]
4	7	-3	$(T^4 \# \overline{\mathbb{CP}}^2) \#_{\Sigma_2} (T^4 \#_2 \overline{\mathbb{CP}}^2)$	$(2 \text{ or } 1) + 1$	[16, 146]
5	8	-4	$(T^2 \times S^2 \#_4 \overline{\mathbb{CP}}^2) \#_{\Sigma_2} (T^2 \times \Sigma_2)$	$(2 \text{ or } 1) + 1$	[28, 146]

Table 2.2: Raw materials for the collections with $k \in \{2, 3, 4, 5\}$

The infinite collections for the values $k \in \{2, 4, 5\}$ are readily available in the literature [16, 28, 146] as we have summarized in Table 2.5. From left to right, the columns of such table contain the value of $k \in \{2, 3, 4, 5\}$, the Euler characteristic and the signature of the raw material in the fourth column. Starting from such 4-manifold, we define $M_{p,k}$ and $N_{p,k}$ as the result of some number of (-1)-torus surgeries and a p-torus surgery. This piece of information appears in the fifth column as a sum, where the first addendum refers to the number of (-1)-surgeries in the construction of $M_{p,k}$ and $N_{p,k}$ respectively. The bibliographic references containing the precise definition and the properties of the raw material and of the 2-tori involved in the surgical operations are listed in the last column. By construction, these families satisfy the hypothesis of Proposition B, which proves the third item. It is straight-forward to verify that the mechanism of Section 2.2 applies to these examples and their equivariant intersection form is extended from the integers. The details are left to the interested reader.

□

Chapter 3

A cut-and-paste mechanism to introduce fundamental group and construct new 4-manifolds

The main contribution of this chapter is to record a simple construction mechanism and to sample its efficiency by building new irreducible smooth structures on closed 4-manifolds. The procedure consists of two steps.

1. Start with a pair (X_1, X_2) of smooth 4-manifolds that are homeomorphic, but not diffeomorphic. For $i = 1, 2$, let $\alpha_{X_i} \subseteq X_i$ be locally flat loops which correspond under a homeomorphism $X_1 \cong_{C^0} X_2$. Produce smooth 4-manifolds $X_1(p)$ and $X_2(p)$ by the procedure described in Section 1.1.2, using the building block S_p if α_{X_i} is orientation-preserving and N_p otherwise. This cut-and-paste procedure introduces fundamental group, in the sense that
$$\pi_1(X_i(p)) \neq \pi_1(X_i),$$
and its codimension three nature avoids cumbersome fundamental group computations that are unavoidable in the codimension two case: see [27].

   Theorem 1.2.19 implies that $X_1(p)$ and $X_2(p)$ are still homeomorphic.

2. Find an invariant that distinguishes the diffeomorphism types of the new 4-manifolds $X_1(p)$ and $X_2(p)$.

In principle, one would hope that the diffeomorphism invariant that discerns X_1 from X_2 tells apart the new 4-manifolds $X_1(p)$ and $X_2(p)$ as well. This is indeed the case in all the examples under consideration in this chapter.

The first application of this new cut-and-paste construction is an extension of a recent result of Levine–Lidman–Piccirillo. In [107], the authors constructed the first example of a nonequivalent smooth structure on a closed smooth oriented 4-manifold with definite intersection form. More precisely, they built a 4-manifold $\mathcal{R}$ that is homeomorphic but not diffeomorphic to the connected sum $\Sigma_2 \#_4 \overline{\mathbb{CP}}^2$, where Σ_2 is a $\mathbb{Q}$-homology 4-sphere with $\pi_1(\Sigma_2) = \mathbb{Z}_2$. Using the pair $(\Sigma_2 \#_4 \overline{\mathbb{CP}}^2, \mathcal{R})$ in the first step of our mechanism, we obtain the following generalization of their result.

Theorem C. *For every odd $p > 0$, there is a closed smooth oriented 4-manifold $\mathcal{R}_{2p}$ that is homeomorphic but not diffeomorphic to the connected sum $\Sigma_{2p} \#_4 \overline{\mathbb{CP}}^2$, where Σ_{2p} is a $\mathbb{Q}$-homology 4-sphere with cyclic fundamental group of order $2p$. The diffeomorphism types are distinguished by the Ozsváth–Szabó mixed invariant.*

To distinguish the smooth structures in [107, Theorem 1.2], Levine–Lidman–Piccirillo show that the universal covers $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$ and $\mathcal{E}$ of the 4-manifolds $\Sigma_2 \#_4 \overline{\mathbb{CP}}^2$ and $\mathcal{R}$ are non-diffeomorphic by means of their Ozsváth–Szabó invariant [122]. For every odd $p > 0$, we show that the 2-fold coverings of our 4-manifolds are obtained from $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$ and $\mathcal{E}$, respectively, by taking a connected sum with a $\mathbb{Q}$-homology 4-sphere. This observation, together with the properties of the Heegaard Floer smooth 4-dimensional invariant, allows us to adapt the proof of [107, Theorem 1.2] to distinguish such coverings and, hence, also the 4-manifolds $\Sigma_{2p} \#_4 \overline{\mathbb{CP}}^2$ and $\mathcal{R}_{2p}$.

Concerning the nonorientable realm, we construct new smooth structures on $\mathbb{Q}$-homology real projective 4-spaces and non-smoothable copies of these 4-manifolds. In order to build the novel 4-manifolds, we start with the Cappell–Shaneson's exotic $\mathbb{RP}^4$ [37]. We address the second step of our construction mechanism by computing the η-invariant [137] of the smooth 4-manifolds involved in Section 1.3.3. Our main result in this case is the following theorem.

Theorem D. *For every odd $p > 0$, there are closed smooth irreducible nonorientable 4-manifolds $A(2p)$ and $B(2p)$ that satisfy the following properties.*

1. *The Euler characteristics of these 4-manifolds are equal to*

$$\chi(A(2p)) = 1 = \chi(B(2p))$$

and their fundamental groups are isomorphic to

$$\pi_1(A(2p)) = \mathbb{Z}_{2p} = \pi_1(B(2p)).$$

2. *There is a homeomorphism $A(2p) \cong_{C^0} B(2p)$.*

3. *The orientation double covering $\widehat{A}(2p)$ is diffeomorphic to the orientation double covering $\widehat{B}(2p)$.*

4. *The universal covers $\widetilde{A}(2p)$ and $\widetilde{B}(2p)$ are diffeomorphic to the connected sum $\#_{2p-1}(S^2 \times S^2)$.*

5. *There is no diffeomorphism between* $A(2p)\#_k(S^2\times S^2)$ *and* $B(2p)\#_k(S^2\times S^2)$ *for any* $k\geq 0$.

6. *The connected sums* $A(2p)\#\mathbb{CP}^2$ *and* $B(2p)\#\mathbb{CP}^2$ *are diffeomorphic.*

Theorem D provides a pair of smooth orientation-reversing free $\mathbb{Z}_{2p}$-actions on a connected sum $\#_{p-1}(S^2\times S^2)$ that are equivariantly homeomorphic, but not equivariantly diffeomorphic.

We prove Theorem C and D in Section 3.1 and 3.2.2 respectively.

3.1 Orientable examples

In order to construct the 4-manifolds $\mathcal{R}_{2p}$ of Theorem C, we make heavy use of the tools developed by Levine–Lidman–Piccirillo in [107].

3.1.1 Raw materials for Theorem C

We will employ as a raw material the exotic 4-manifolds $\mathcal{E}$ and $\mathcal{R}$ constructed in [107, Theorem 1.2].

Theorem 3.1.1 (Levine–Lidman–Piccirillo [107])**.**

1. *There exists a smooth* 4*-manifold* $\mathcal{E}$*, which is homeomorphic but not diffeomorphic to* $\mathbb{CP}^2\#_9\overline{\mathbb{CP}^2}$.

2. *There exists a smooth* 4*-manifold* $\mathcal{R}$*, which is homeomorphic but not diffeomorphic to* $\Sigma_2\#_4\overline{\mathbb{CP}^2}$*, where* Σ_2 *is a* $\mathbb{Q}$*-homology* 4*-sphere with* $\pi_1(\Sigma_2)\cong\mathbb{Z}_2$.

We refer the reader to [107, Section 6.2] for the precise definition of the 4-manifolds $\mathcal{E}$ and $\mathcal{R}$. The 4-manifolds of Theorem C are defined by applying the construction described in Section 1.1.2 to the exotic 4-manifold $\mathcal{R}$ in the second item of Theorem 3.1.1.

More precisely, any integer $p\geq 1$, we define the closed oriented smooth 4-manifold

$$\mathcal{R}_{2p}=(\mathcal{R}\setminus\operatorname{Int}\nu(\alpha_{\mathcal{R}}))\cup S_p \tag{3.1.1}$$

where $\alpha_{\mathcal{R}}\subseteq\mathcal{R}$ is a simple closed curve whose based homotopy class generates the group $\pi_1(\mathcal{R})=\mathbb{Z}_2$. The homeomorphism type of each 4-manifold (3.1.1) is pinned down in the following lemma.

Lemma 3.1.2. *For every integer* $p\geq 1$*, there is a homeomorphism*

$$\mathcal{R}_{2p}\xrightarrow{\mathcal{C}^0}\Sigma_{2p}\#_4\overline{\mathbb{CP}^2}, \tag{3.1.2}$$

where Σ_{2p} *is a* $\mathbb{Q}$*-homology* 4*-sphere with fundamental group* $\mathbb{Z}_{2p}$.

Proof. The $\mathbb{Q}$-homology 4-sphere on the right side of (3.1.2) can be deconstructed as

$$\Sigma_{2p} = (\Sigma_2 \setminus \operatorname{Int} \nu(\alpha_{\Sigma_2})) \cup S_p, \tag{3.1.3}$$

where Σ_2 is the $\mathbb{Q}$-homology 4-sphere appearing in Theorem 3.1.1 and $\alpha_{\Sigma_2} \subseteq \Sigma_2$ is a simple loop whose homotopy class generates the fundamental group $\pi_1(\Sigma_2) \cong \mathbb{Z}_2 \cong \pi_1(\Sigma_2 \#_4 \overline{\mathbb{CP}^2})$. The second clause of Theorem 3.1.1 implies that there is a homeomorphism

$$(\Sigma_2 \#_4 \overline{\mathbb{CP}^2}) \setminus \operatorname{Int} \nu(\alpha_{\Sigma_2}) \xrightarrow{\mathcal{C}^0} \mathcal{R} \setminus \operatorname{Int} \nu(\alpha_{\mathcal{R}}).$$

Such homeomorphism can be extended to a homeomorphism as in (3.1.2) by the decomposition of Σ_{2p} in (3.1.3) and Corollary 1.2.21. $\square$

Remark 3.1.3. Alternatively, the existence of a homeomorphism as in Lemma 3.1.2 follows from work of Hambleton–Kreck [77, Theorem C]; cf. [136, Theorem 2.1].

To distinguish the diffeomorphism class of $\mathcal{R}_{2p}$ from the one of the connected sum $\Sigma_{2p} \#_4 \overline{\mathbb{CP}^2}$, we look at the double covering

$$\mathcal{E}_p \xrightarrow{2:1} \mathcal{R}_{2p}$$

corresponding to the index-two subgroup $\mathbb{Z}_p \subseteq \mathbb{Z}_{2p} = \pi_1(\mathcal{R}_{2p})$.

Proposition 3.1.4. *If $p > 0$ is odd, the 2-fold covering of $\mathcal{R}_{2p}$ corresponding to the index-two subgroup $\mathbb{Z}_p \subseteq \mathbb{Z}_{2p}$ is diffeomorphic to the 4-manifold*

$$\mathcal{E}_p = (\mathcal{E} \setminus \operatorname{Int} \nu(\gamma)) \cup S_p = \mathcal{E} \# \Sigma_p,$$

where $\mathcal{E}$ is the exotic 4-manifold of the first clause of Theorem 3.1.1, $\gamma \subseteq \mathcal{E}$ is a simple loop and S_p is the 4-manifold defined in (1.1.14).

Proof. This follows from the definition of $\mathcal{R}$ in [107] as the quotient of $\mathcal{E}$ by a fixed-point-free orientation-reversing involution, together with Lemma 1.1.20. $\square$

3.1.2 Diffeomorphism obstruction

In light of the definition of $\mathcal{E}$ in [107, Section 6.2], Proposition 3.1.4 implies the existence of a decomposition

$$\mathcal{E}_p = V \cup_\zeta V_p, \tag{3.1.4}$$

where $V = C_0 \cup_{\phi_0 \circ \sigma} Z$ is the building block defined in [107, Construction 6.4]. In particular, C_0 is a contractible 4-manifold, Z consists of five -1-framed 2-handles attached to ∂C_0, ∂V

is the result of 0-surgery on S^3 along the Stevedore knot and ζ is any of its orientation-reversing fixed-point-free involutions. On the other hand, for every $p \geq 1$, V_p is defined to be the 4-manifold

$$V_p = C_0^p \cup_{\phi_0 \circ \sigma} Z$$

where C_0^p is the 4-manifold obtained from C_0 by removing a neighbourhood of a simple closed curve and gluing back a copy of S_p. Moreover, we set

$$\mathcal{E}'_p = V' \cup_\zeta V'_p$$

where $V' = C_0 \cup_{\phi_0} Z$ and $V'_p = C_0^p \cup_{\phi_0} Z$ are made up by the same building blocks as V and V_p with different gluing maps. We refer the reader to [107, Construction 6.4] for further details.

The aim of this section is to provide a proof of the following proposition.

Proposition 3.1.5. *For every odd $p \in \mathbb{N}$, the following holds.*

1. *The 4-manifold $\mathcal{E}'_p$ is diffeomorphic to*

$$X_p = ((\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}) \setminus \operatorname{Int} \nu(\gamma)) \cup S_p = (\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}) \# \Sigma_p,$$

   where $\gamma \subseteq \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}$ is a simple loop.

2. *The 4-manifold $\mathcal{E}_p$ is homeomorphic but not diffeomorphic to X_p.*

We proceed to provide some technical lemmas that are needed to distinguish the smooth structures of Theorem C. These results are generalizations of [107, Proposition 6.9] and [107, Lemma 6.3] respectively. In particular, we make use of the Ozsváth–Szabó 4-manifold invariants defined in [122, Section 2.4].

In the following, we adopt the notational conventions introduced in Subsection 1.3.2.

Lemma 3.1.6. *For any $p \geq 1$, there is a Spin^c-structure $(V_p, \mathfrak{t}_p)$ such that $\Psi_{V_p,\mathfrak{t}_p} \neq 0$. Moreover, all such Spin^c-structures can be chosen to coincide on $Z \subseteq V_p$. In particular, V_p is homeomorphic but not diffeomorphic to $X_0(Q_0)'_p \#_4 \overline{\mathbb{CP}^2}$ for*

$$X_0(Q_0)'_p = (X_0(Q_0) \setminus \operatorname{Int} \nu(\gamma)) \cup S_p,$$

where $X_0(Q_0)$ is the 0-trace of the Stevedore knot and $\gamma \subseteq X_0(Q_0)$ is a null-homotopic simple loop.

Proof. The case $p = 1$ is [107, Proposition 6.9]. Since $C_0^p \setminus B^4$ is a rational homology cobordism between S^3 and ∂C_0^p, the same proof holds for $p > 1$ by replacing C_0 with C_0^p in the argument. □

The proof of the following result is a straightforward adaptation of the one of [107, Lemma 6.3].

Lemma 3.1.7. *Let K be a fibered amphichiral knot of genus g. Let W_1 and W_2 be compact oriented 4-manifolds with positive second Betti number and boundary $S^3_0(K)$. Suppose there are* Spin^c *structures $\{(W_i, \mathfrak{u}_i) \mid i = 1, 2\}$ such that $c_1(\mathfrak{u}_i)$ evaluates to $2g-2$ on a generator of $H_2(S^3_0(K))$ and for which $\Psi_{W_i,\mathfrak{u}_i} \neq 0$. Let X be the result of gluing together W_1 and W_2 via any choice of orientation-reversing diffeomorphism of $S^3_0(K)$, and let $L \subseteq H_2(X;\mathbb{Q})$ be the line corresponding to the inclusion*

$$H_2(S^3_0(K)) \longrightarrow H_2(X).$$

Then there is a Spin^c*-structure $(X, \mathfrak{t})$ for which $\Phi_{X,\mathfrak{t},L} \neq 0$.*

Proof of Proposition 3.1.5. Item (1) follows from the construction and [107, Theorem 6.7]. By Lemma 1.1.19, there is a 2-fold covering

$$X_p \cong \mathcal{E}'_p \xrightarrow{2:1} \Sigma_{2p} \#_4 \overline{\mathbb{CP}^2}.$$

Let us now prove Item (2). We see that $\mathcal{E}_p$ and $\mathcal{E}'_p$ are homeomorphic by construction, since $\mathcal{E}$ and $\mathcal{E}'$ are homeomorphic by [107, Theorem 6.7]. To conclude the proof, we need to show that the 2-fold coverings $\mathcal{E}_p \to \mathcal{R}_{2p}$ and $\mathcal{E}'_p \to \Sigma_2 \#_4 \overline{\mathbb{CP}^2}$ are not diffeomorphic. Consider the decomposition

$$\mathcal{E}_p = V_a \cup T_{Q_0} \cup V^p_b \tag{3.1.5}$$

where V_a and V^p_b are copies of V and V_p respectively without the $T^-_{Q_0}$ component, see [107, Construction 6.4]. As for the case of $\mathcal{E}$, we see that $H_2(\mathcal{E}_p)$ has a basis

$$(E_1, \ldots, E_4, E^p_5, \ldots, E^p_8, \Xi, \Theta)$$

where:

- $(E_1, \ldots, E_4)$ is a diagonal basis for V_a:
- $(E^p_5, \ldots, E^p_8)$ is a diagonal basis for V^p_b;
- (Ξ, Θ) is the basis for $H_2(T_{Q_0})$ from [107, Remark 3.11]. In particular, Ξ is represented by a capped-off Seifert surface for Q_0 in $S^3(Q_0)$ and Θ is represented by a 2-sphere of self-intersection -2 and $\Xi \cdot \Theta = 1$.

As a consequence of Lemma 3.1.6 and 3.1.7, there is a Spin^c-structure $\mathfrak{t}$ on $\mathcal{E}_p$ such that $\Phi_{\mathcal{E}_p,\mathfrak{t},L} \neq 0$, where $L = \mathrm{Span}(\Xi) \subseteq H_2(\mathcal{E}_p;\mathbb{Q})$. By [107, Lemma 6.2], we have that Ξ cannot be represented by any surface of genus less than 2. It is possible to show that if there was a diffeomorphism

$$\mathcal{E}_p \xrightarrow{\mathcal{C}^\infty} X_p$$

then one could build a spherical representative for Ξ as in the proof of [107, Theorem 6.7], which would lead to a contradiction. □

Remark 3.1.8. When p is even, the 4-manifold D_p of Lemma 1.1.16 has two boundary components and the double covering $\mathcal{E}_p$ of $\mathcal{R}_{2p}$ is

$$\mathcal{E}_p = (\mathcal{R} \setminus \operatorname{Int} \nu(\alpha_{\mathcal{R}})) \cup D_p \cup (\mathcal{R} \setminus \operatorname{Int} \nu(\alpha_{\mathcal{R}})).$$

In particular, we have not been able to show that $\mathcal{R}_{2p}$ is exotic by reducing to the study of the Heegaard Floer invariants of $\mathcal{E}_p$ as in [107].

3.1.3 Proof of Theorem C

The 4-manifold $\mathcal{R}_{2p}$ was constructed in (3.1.1) for any odd $p > 0$ and its homeomorphism type was pinned down in Lemma 3.1.2; see Remark 3.1.3. The claim that there does not exist a diffeomorphism $\mathcal{R}_{2p} \to \Sigma_{2p} \#_4 \overline{\mathbb{CP}^2}$ follows from Proposition 3.1.5. Indeed, if any such diffeomorphism between $\mathcal{R}_{2p}$ and $\Sigma_{2p} \#_4 \overline{\mathbb{CP}^2}$ were to exist, it would induce a diffeomorphism between the 2-fold coverings

$$\mathcal{E}_p \xrightarrow{\mathcal{C}^\infty} ((\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}) \setminus \operatorname{Int} \nu(\gamma)) \cup S_p = (\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}^2}) \# \Sigma_p = X_p$$

by Lemma 1.1.20 and Lemma 1.1.21. However, Proposition 3.1.5 indicates that these 4-manifolds are not diffeomorphic. □

3.2 Nonorientable examples

3.2.1 Nonorientable raw material and its properties

With the purpose of streamlining the exposition, we encapsule the background results we build on to prove Theorem D in the next triad of results. We first recall the building block that serves as raw material of the procedure we described in the introduction, along with several of its basic properties.

Theorem 3.2.1.

- (Cappell–Shaneson [37]) *There is a 4-manifold Q that is homeomorphic, but not diffeomorphic to $\mathbb{RP}^4$. These two 4-manifolds remain non-diffeomorphic under connect sums with arbitrarily many copies of $S^2 \times S^2$.*

- (Gompf [68]) *The universal covering $\widetilde{Q}$ is diffeomorphic to S^4.*

- (Akbulut [8, Theorem 2]) *There is a diffeomorphism*

$$Q \# \mathbb{CP}^2 \xrightarrow{\mathcal{C}^\infty} \mathbb{RP}^4 \# \mathbb{CP}^2.$$

We now add a few words on the foundational results summarized in the previous theorem in order to put them into the context of our proof of Theorem D. The analysis done by Akbulut [8, Section 1] on the 4-manifold Q built by Cappell–Shaneson [37] reveals that the compact 4-manifold $Z = Q \setminus \operatorname{Int} \nu(\alpha_Q)$ (where $\alpha_Q \subseteq Q$ is a simple loop that generates $\pi_1(Q) = \mathbb{Z}_2$ is homeomorphic but not diffeomorphic to $D^2 \tilde{\times} \mathbb{RP}^2$. The second clause of Theorem 3.2.1 encapsules work of Gompf, where he showed that the orientation double covering of Z is diffeomorphic to $D^2 \times S^2$. The third clause of Theorem 3.2.1 follows from the property $Z \# \mathbb{CP}^2 = (D^2 \tilde{\times} \mathbb{RP}^2) \# \mathbb{CP}^2$ [8, Section 3].

We conclude this section by recalling the value of the η-invariant on the Pin^+-structures of the 4-manifolds $\mathbb{RP}^4$ and Q.

Theorem 3.2.2.

- (Kirby–Taylor [99, Theorem 5.2]) *There is a group isomorphism*

$$\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$$

  and the class of the real projective 4*-space*

$$[(\mathbb{RP}^4, \pm\phi^+_{\mathbb{RP}^4})] = \pm 1 \in \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}.$$

  is a generator.

- (Stolz [137])

$$[(Q, \pm\phi^+_Q)] = \pm 9 \in \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}.$$

3.2.2 Proof of Theorem D

Fix an odd $p \geq 0$ and consider the compact 4-manifold N_p that was defined in (1.1.21). Assemble the nonorientable 4-manifolds

$$A(2p) = (\mathbb{RP}^4 \setminus \operatorname{Int} \nu(\alpha_{\mathbb{RP}^4})) \cup N_p = (D^2 \tilde{\times} \mathbb{RP}^2) \cup N_p \tag{3.2.1}$$

and

$$B(2p) = (Q \setminus \operatorname{Int} \nu(\alpha_Q)) \cup N_p = Z \cup N_p, \tag{3.2.2}$$

where Q is Cappell–Shaneson's exotic copy of the real projective 4-space [37] discussed in Theorem 3.2.1. Lemma 1.1.24 implies that the 4-manifolds (3.2.1) and (3.2.2) are $\mathbb{Q}$-homology real projective 4-spaces whose fundamental group is a finite cyclic group of order $2p$. This establishes the claim of Item (1) of Theorem D. Moreover, the 4-manifolds $A(2p)$ and $B(2p)$ are homeomorphic by Corollary 1.2.23 since Q is homeomorphic to $\mathbb{RP}^4$. This settles the claim of Item (2). Since there are Pin^+-structures $(\mathbb{RP}^4, \pm\phi^+_{\mathbb{RP}^4})$ and $(Q, \pm\phi^+_Q)$, Proposition 1.3.9 implies that the 4-manifolds (3.2.1) and (3.2.2) admit a Pin^+-structure. There are actually two such structures. Indeed, the first cohomology groups with $\mathbb{Z}_2$-coefficients of the 4-manifolds (3.2.1) and (3.2.2) are

$$H^1(A(2p);\mathbb{Z}_2) = \mathbb{Z}_2 = H^1(B(2p);\mathbb{Z}_2)$$

and it follows that there are two Pin$^+$-structures $(A(2p), \pm\phi^+_{A(2p)})$ and $(B(2p), \pm\phi^+_{B(2p)})$ [137, Corollary 6.4]. Moreover, the Pin$^+$-structures on the same underlying smooth 4-manifold are mutual inverses in the fourth Pin$^+$-cobordism group $\Omega_4^{\mathrm{Pin}^+}$ and the values

$$\eta(A(2p), \pm\phi^+_{A(2p)}) = \pm\frac{1}{8} \mod 2\mathbb{Z} \tag{3.2.3}$$

and

$$\eta(B(2p), \pm\phi^+_{B(2p)}) = \pm\frac{7}{8} \mod 2\mathbb{Z} \tag{3.2.4}$$

hold by Proposition 1.3.9 and Theorem 3.2.2. Using Theorem 1.3.7, we conclude that the connected sums

$$A(2p)\,\#_k(S^2 \times S^2) \text{ and } B(2p)\,\#_k(S^2 \times S^2)$$

belong to different classes in $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}_{16}$ for any $k \geq 0$ and that the 4-manifolds $A(2p)$ and $B(2p)$ are not stably-diffeomorphic. This settles the claim of Item (5). Item (6) follows from the third clause of Theorem 3.2.1 and the corresponding coverings of the decompositions (3.2.1) and (3.2.2).

□

Chapter 4

Nonorientable exotica via twisting operations

Akbulut [7, 5], [2, Section 9.5] showed the existence of 2-spheres

$$S \subseteq (S^3 \widetilde{\times} S^1)\,\#(S^2 \times S^2) \text{ and } S' \subseteq (S^3 \widetilde{\times} S^1)\,\#\,\mathbb{CP}^2\,\#\,\overline{\mathbb{CP}}^2$$

such that performing a Gluck twist to either of them yields a 4-manifold that is homeomorphic but not diffeomorphic to the connected sum of the non-trivial 3-sphere bundle over the circle $S^3 \widetilde{\times} S^1$ with a copy of $S^2 \times S^2$. The first main result of this chapter records such examples in 4-manifolds that are known to be irreducible.

Theorem E. *There is a smoothly embedded* 2*-sphere* $S \subseteq \mathbb{RP}^2 \times S^2$ *with trivial normal bundle* $\nu(S) = D^2 \times S^2$ *such that the complements*

$$(\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(\{pt\} \times S^2) \text{ and } (\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(S)$$

are not homotopy equivalent, and doing a Gluck twist on S *yields a* 4*-manifold* Y *that is homeomorphic but not diffeomorphic to* $\mathbb{RP}^2 \widetilde{\times} S^2$*, the nonorientable total space of the nontrivial* S^2*-bundle over* $\mathbb{RP}^2$*.*

Performing a Gluck twist on the 2-sphere $\{pt\} \times S^2 \subseteq \mathbb{RP}^2 \times S^2$ yields $\mathbb{RP}^2 \widetilde{\times} S^2$ as explained in Section 1.1.3. A key ingredient in our proof of Theorem E is a result due to Akbulut [3, Section 4], which we state in Theorem 4.8.1. The nonequivalent smooth structure Y of Theorem E was constructed in [147, Theorem 1]; cf. Proposition 4.6.1. Our next result provides a different perspective on the production of the 4-manifold Y of Theorem E.

Theorem F. *Let* M_A *be the Cappell–Shaneson's nonorientable* 3*-torus bundle over* S^1 *with monodromy* A *as in (4.5.2), and let* $\alpha_0 \subseteq M_A$ *be a cross-section. Let* $\alpha_0^2 \subseteq M_A$ *be an orientation-preserving simple loop that represents the element* $[\alpha_0]^2 \in \pi_1(M_A)$*. Surgery on* α_0^2 *with the canonical framing yields* $\mathbb{RP}^2 \times S^2$*, while surgery on* $\alpha_0^2 \subseteq M_A$ *with the* *non-canonical framing yields the 4-manifold Y of Theorem E, which is homeomorphic but not diffeomorphic to* $\mathbb{RP}^2 \mathbin{\tilde{\times}} S^2$.

In particular, Y is a boundary component of a Pin^+*-cobordism that is obtained by adding a 5-dimensional 2-handle to the product cobordism $M_A \times I$*

To distinguish the smooth structures of Theorem E and Theorem F, we use the η-invariant and results of Stolz [137]. The main property of this spectral invariant that we exploit is that it is a complete Pin^+-cobordism invariant $\mod 2\mathbb{Z}$ (see Theorem 1.3.7).

The choice of framing of the loops in Theorem F is taken following Gompf [69, Section 2]; see Remark 4.10.1. The monodromy of the mapping torus M_A is described in Section 4.5. The first clause of Theorem F is essentially due to Akbulut [3, Section 4]; see Theorem 4.8.1. The simple construction of the 4-manifold Y in terms of adding one single 2-handle resembles the construction of the Cappell–Shaneson homotopy 4-spheres; see Remark 4.10.1. Theorem F is proven in Section 4.10.2.

The third main result exemplifies constructions of nonequivalent smooth structures obtained by twisting an embedded projective plane in the 4-manifold Y of Theorem E; see Section 1.1.5 for a precise definition of this cut-and-paste operation.

Theorem G. *There is a smoothly embedded real projective plane $\mathbb{RP}^2 \subseteq Y$ such that twisting it produces a 4-manifold Z that is homeomorphic but not diffeomorphic to the circle sum $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$ of two copies of the real projective 4-space.*

To the best of our knowledge, twisting an $\mathbb{RP}^2$ had not been previously used to construct nonequivalent smooth structures. We discuss in Section 4.10.3 how the examples of Theorem G are made of the same two compact pieces, and the isotopy class of the diffeomorphism used to identify their boundaries determines the homeomorphism type [38, 96]. Handlebodies of the 4-manifolds of Theorem G are drawn in Figure 4.4.

Let us now describe the organization of Chapter 4. In Section 4.1 we explain the conventions we adopt to draw Kirby diagrams of nonorientable 4-manifolds, while in Section 4.2 we relate two S^2-bundles over $\mathbb{RP}^2$ using Gluck twists. In Section 4.3 we show via an example that circle sums do not preserve irreducibility. In Section 4.4 we describe a procedure to draw Kirby diagrams of 4-manifolds obtained by twisting a real projective plane. The raw materials employed in the construction of the smooth structures are described in Section 4.5, and a quick survey of some construction is given in Section 4.7. Key ingredients in the proofs of our theorems are collected in Section 4.6 and Section 4.8. The former contains a description of the 4-manifold Y, while in the latter we recall a theorem of Akbulut. Stabilizations with a copy of $\mathbb{CP}^2$ are discussed in Section 4.9. The proofs of our main results are found in Section 4.10.

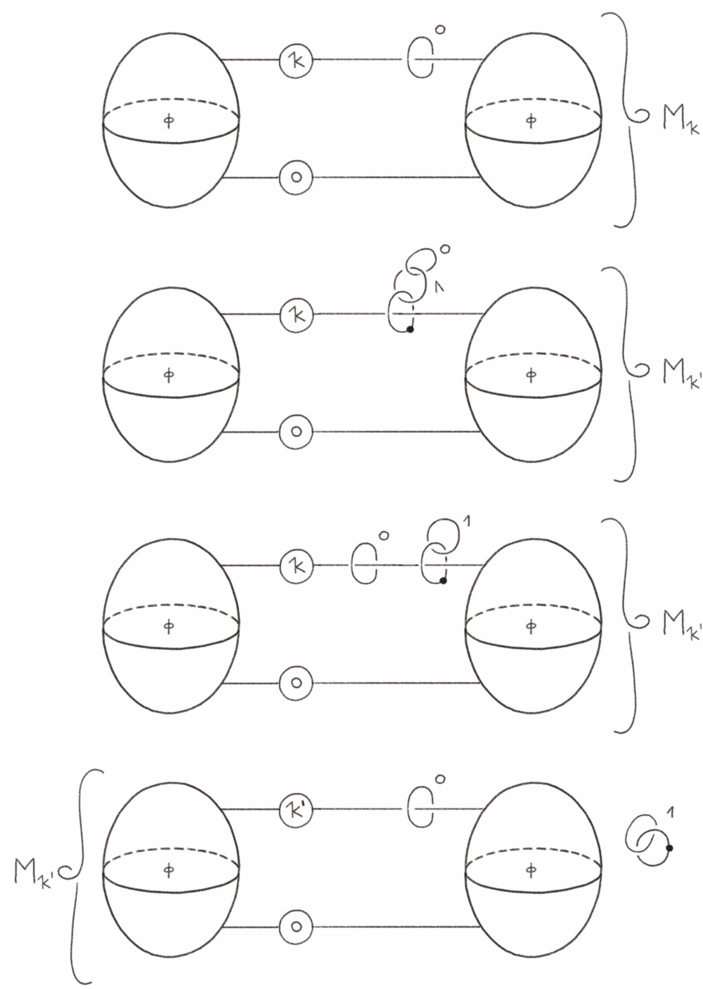

Figure 4.1: Kirby diagrams describing the proof of Lemma 4.2.1

4.1 Kirby diagrams of nonorientable 4-manifolds

Foundational results concerning handlebodies of nonorientable 4-manifolds have been obtained by Akbulut [3, 2], César de Sá [38] and Miller–Naylor [110]. We refer the reader to these sources for background material, and proceed to briefly recall the conventions of Akbulut [3], [2, Section 1.5] that we follow to draw handlebodies in this thesis. The pair of round balls representing the attaching region of the orientation-reversing 1-handles are drawn with small arcs at their centers to denote that the standard oriented 1-handle identification ([73, Figure 4.15]) is composed with a reflection across the plane orthogonal to the arcs [2, Figure 1.25]. In the handlebody depicted in Figure 4.2, one of the round balls is at ∞. In this case, the identification is the composition of such reflection with the radial map that identifies a point on the sphere centered at the origin with a point on the sphere centered at ∞ [2, Figure 1.25]. The dotted circle notation will be used for any orientable 1-handle [6, 38].

The canonical framing of a 2-handle is the blackboard framing [73, Section 4.5], which depends on the chosen projection. This is the framing coming from the normal vector field to the surface of the page of the paper, and the framing of any 2-handle is taken with respect to this framing [73, p. 117]. The notation used to record the framing of the attaching circle of a 2-handle in a nonorientable handlebody depends on whether the 2-handle runs or does not run over a nonorientable 1-handle as follows [3, Section 0]. When the 2-handle does not run over any nonorientable 1-handle, the framing is specified with a single integer number. For example, this is the case of the 0-framed circle in any of the handlebodies of Figure 4.1. If the 2-handle runs over a nonorientable 1-handle, an integer number is assigned to each arc of the attaching circle minus the nonorientable 1-handles. Following [2, Section 1.5], we record the framing of such a 2-handle as a pair of circled integer numbers $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. For example, in the handlebody on top of Figure 4.1, the framing of the 2-handle that runs over the nonorientable 1-handle is $(k, 0)$; it indicates that one adds k twists to the blackboard framing. Moreover, the sign of the framing of a 2-handle flips when going across an orientation-reversing 1-handle [3, p. 78 - 79]. The 3- and 4-handles do not need to be drawn to determine the diffeomorphism type of a nonorientable 4-manifold due to a foundational result of César de Sá [38] and Miller–Naylor [110].

4.2 2-sphere bundles over the real projective plane

There are two nonorientable total spaces of S^2-bundles over $\mathbb{RP}^2$ [84, Chapter 12]. The non-trivial bundle is given as the quotient

$$\mathbb{RP}^2 \widetilde{\times} S^2 = \frac{S^2 \times S^2}{(x, y) \sim (-x, \rho_\pi(y))}, \tag{4.2.1}$$

where $\rho_\pi\colon S^2 \to S^2$ is a rotation of π radians [84]. The bundle (4.2.1) can also be regarded as the unit sphere bundle of the rank 3 vector bundle over $\mathbb{RP}^2$ given by the Whitney sum of two copies of the canonical bundle γ with one copy of the trivial line bundle. The latter description explains the notation (4.2.1).

Both of these bundles are obtained by adding a copy of $D^2 \times S^2$ to

$$(\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(\Sigma') = Mb \times S^2 = \mathbb{RP}^2 \tilde{\times} S^2 \setminus \operatorname{Int} \nu(\Sigma), \tag{4.2.2}$$

where Mb is the Möbius band, $\Sigma' = \{pt\} \times S^2 \subseteq \mathbb{RP}^2 \times S^2$ and $\Sigma \subseteq \mathbb{RP}^2 \tilde{\times} S^2$ is a fiber. The diffeomorphism class of the resulting 4-manifold depends on the isotopy class of the diffeomorphism $\phi\colon \partial Mb \times S^2 \to \partial D^2 \times S^2 = S^1 \times S^2$ that is used to glue in $D^2 \times S^2$ to (4.2.2). Using $\phi = \mathrm{id}$ yields $\mathbb{RP}^2 \times S^2$, while using the diffeomorphism (1.1.25) yields $\mathbb{RP}^2 \tilde{\times} S^2$. The next lemma is a summary of this discussion.

Lemma 4.2.1.

- *The non-trivial bundle* $\mathbb{RP}^2 \tilde{\times} S^2$ *is obtained by performing a Gluck twist on* $\Sigma' \subseteq \mathbb{RP}^2 \times S^2$.

- *The trivial bundle* $\mathbb{RP}^2 \times S^2$ *is obtained by performing a Gluck twist on* $\Sigma \subseteq \mathbb{RP}^2 \tilde{\times} S^2$.

Proof. Let M_k be the 4-manifold given by the handlebody on the top level of Figure 4.1 along with a 3- and a 4-handle. We have $M_k = \mathbb{RP}^2 \times S^2$ if $k = 0 \mod 2$, and $M_k = \mathbb{RP}^2 \tilde{\times} S^2$ if $k = 1 \mod 2$. This can be seen as follows. There are two nonorientable 2-sphere bundles over the real projective plane [84, Section 12.3] and both of them arise as the double of a nonorientable 2-disk bundle over $\mathbb{RP}^2$ [73, Example 4.6.3]. Handlebodies of these 2-disk bundles are obtained by removing the 0-framed 2-handle, and the 3- and 4-handles from the handlebody of M_K [3, p. 79] (cf. [110]). The 0-framed 2-handle (coming from the double construction) in the handlebody of M_K of Figure 4.1 represents the tubular neighbourhood $\nu(S) = D^2 \times S^2$ of an embedded 2-sphere $S \subseteq M_k$ [73, Exercise 5.4.3. (d)]. Let $M_{k'}$ be the 4-manifold obtained by doing a Gluck twist on this 2-sphere. Its handlebody is drawn on the second stage from top to bottom in Figure 4.1 following Akbulut–Yasui's method [12, Figure 1]. Take the 0-framed 2-handle that links the 1-framed 2-handle, and slide it under the 1-handle represented by the dotted circle to obtain the third stage of Figure 4.1. Sliding the k-framed 2-handle over the 1-framed 2-handle and then under the dotted circle results in the handlebody drawn at the bottom of Figure 4.1: the framing k' is an odd integer if and only if k is an even integer. The bottom level of Figure 4.1 is the handlebody of M'_k, where the Hopf link of the dotted circle and the 1-framed unknot is a cancelling pair of 1- and 2-handles. $\square$

4.3 Circles sums and irreducibility

We point out that the circle sum of two irreducible 4-manifolds need not result in an irreducible 4-manifold.

Lemma 4.3.1. *The circle sum* $\mathbb{RP}^2 \tilde{\times} S^2 \#_{S^1} \mathbb{RP}^4$ *is diffeomorphic to* $\mathbb{RP}^4 \# (S^2 \times S^2)$.

Proof. A handlebody of $\mathbb{RP}^4 \# (S^2 \times S^2)$ is obtained by modifying the handlebody on the third stage from top to bottom of Figure 4.1 as follows: erase the 0-framed circle, substitute the dotted circle with a 2-framed circle, change the framing of the linking circle from 1 to 0, and set $k = 1$. The 3-handle and the 4-handle do not need to be drawn [38, 110]. A handlebody for $\mathbb{RP}^2 \tilde{\times} S^2 \#_{S^1} \mathbb{RP}^4$ is obtained by adding a $(1, 0)$-framed 2-handle to the first stage from bottom to top of 4.3. Slide the $(-1, 0)$-framed 2-handle over the $(1, 0)$-handle and then slide the latter over the other $(1, 0)$-handle to obtain the handlebody of $\mathbb{RP}^4 \# (S^2 \times S^2)$. □

4.4 Twisting real projective planes

We describe a method to draw a handlebody of the construction (1.1.31) by building heavily on the operation of blowing down an $\mathbb{RP}^2$ that was introduced by Akbulut in [3, p. 79 - 80]. The method is described in Figure 4.2, where one of the round balls of the attaching region of the 1-handle is at the point of infinity and a single round ball is drawn. The handlebody of X is depicted by the diagram at the top level of Figure 4.2, where the union of the 1-handle and the 2-handle h with strands drawn horizontally with framing $(1, 0)$ is the tubular neighbourhood $\nu(R) = D^2 \tilde{\times} \mathbb{RP}^2$. We first argue that this is the only case that needs to be considered. If the 2-handle in the handle decomposition of $\nu(R)$ has framing $(-1, 0)$, first rotate the round ball drawn 360 degrees around the axis indicated by the arc ([3, p. 79]) to change its framing to $(0, -1)$, and then transfer the framing through the orientation-reversing 1-handle to obtain a 2-handle with framing $(1, 0)$ as in the top level of Figure 4.2 (see [2, Figure 1.26]). Denote the 2-handles that link h by $\{K_1, \ldots, K_i\}$ with their respective framings given by $\{k_1, \ldots, k_i\}$. Slide the attaching circles of all the 2-handles $\{K_r \mid r = 1, \ldots, i\}$ over h until they do not link it anymore and obtain the second level of Figure 4.2 (cf. [3, p. 80]). The framings are now $\{k_1 - 1, \ldots, k_i - 1\}$ and the strands have a -1 full twist as indicated in the handlebody of X drawn in the second level of Figure 4.2. At this point we perform the twist (1.1.31) on $\nu(R)$ and draw the handlebody of X_R at the bottom level of Figure 4.2. This is done by first blowing down the real projective plane R by erasing the 2-handle h from the handlebody of X drawn in the second level of Figure 4.2, and then adding a 2-handle h' with framing $(-1, 0)$ that replaces h.

So far we have described a method that produces a 4-manifold X', which is obtained by carving $\nu(R)$ out of X and gluing it back in using a self-diffeomorphism of the boundary $\partial(X \setminus \text{Int}\, \nu(R)) = S^2 \tilde{\times} S^1$. To conclude that X' is diffeomorphic to (1.1.31), we argue as

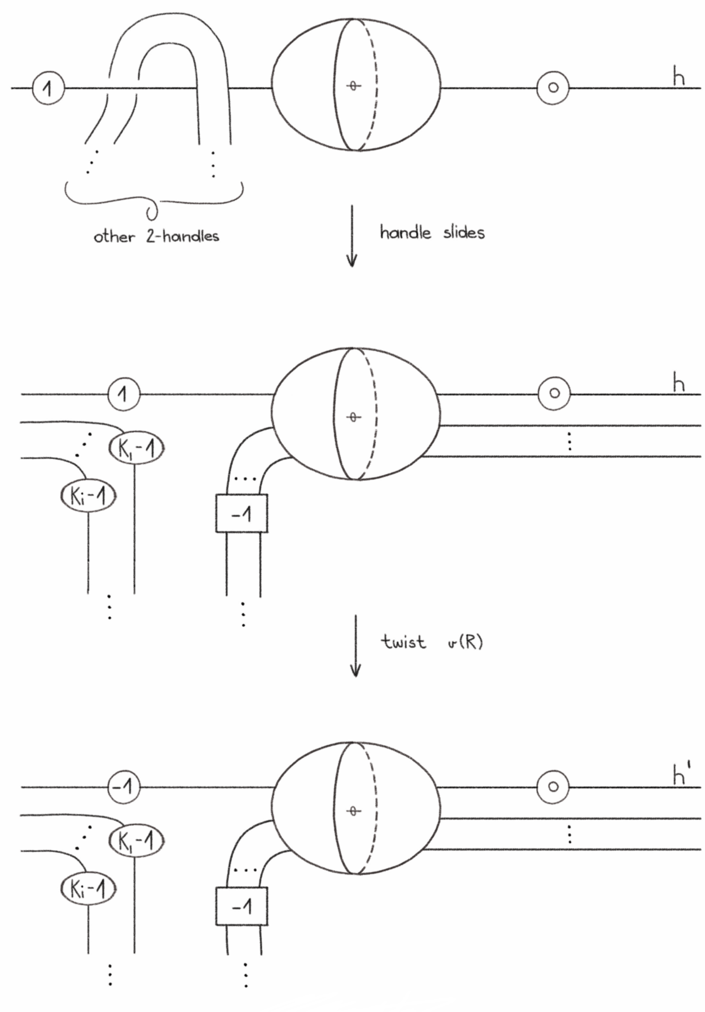

Figure 4.2: Twisting an embedded real projective plane

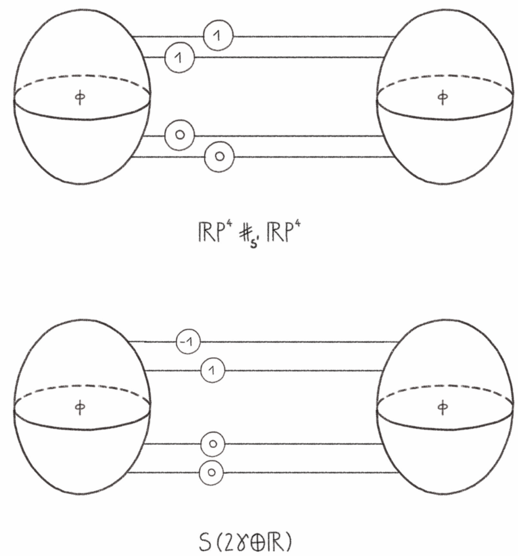

Figure 4.3: Kirby diagrams of $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$ and $S(2\gamma \oplus \mathbb{R})$

follows (cf. [12, Proof Lemma 2.2]). A result of Kim–Raymond [96] on the diffeotopy group of $S^2 \tilde{\times} S^1$ (cf. [110, Remark 3.12]) implies that we need to check that the gluing diffeomorphism (1.2.3) does not extend to a self-diffeomorphism of $D^2 \tilde{\times} \mathbb{RP}^2$. This is done in the following example.

Example 4.4.1. The homotopy equivalence classes of the double

$$\mathbb{RP}^2 \tilde{\times} S^2 = (D^2 \tilde{\times} \mathbb{RP}^2) \cup_{\mathrm{id}} (D^2 \tilde{\times} \mathbb{RP}^2) \tag{4.4.1}$$

and of the circle sum

$$\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4 = (D^2 \tilde{\times} \mathbb{RP}^2) \cup_{\varphi'} (D^2 \tilde{\times} \mathbb{RP}^2) \tag{4.4.2}$$

do not coincide [95]. Handlebodies of these two nonorientable 4-manifolds are drawn in 4.3 with an additional 3- and 4-handle in each, making visible the procedure of twisting an $\mathbb{RP}^2$.

Hambleton–Hillman [76] studied the homotopy and homeomorphism types of closed 4-manifolds whose universal covering is $S^2 \times S^2$. A representative of a homotopy equivalence class in the quadratic 2-type of PD_4-complexes with Euler characteristic one and fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a 4-manifold W, which arises as the quotient space of $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$ by a free involution [76, Definition 7.1]. In order to exemplify the utility of the construction (1.1.31), we now remark that W can be obtained from the non-trivial bundle $\mathbb{RP}^2 \tilde{\times} \mathbb{RP}^2$ by twisting an $\mathbb{RP}^2$.

Example 4.4.2. Use the decomposition $\mathbb{RP}^2 = D^2 \cup Mb$ to deconstruct the bundle as

$$\mathbb{RP}^2 \tilde{\times} \mathbb{RP}^2 = (D^2 \tilde{\times} \mathbb{RP}^2) \cup_\varphi (Mb \tilde{\times} \mathbb{RP}^2), \tag{4.4.3}$$

where $Mb \tilde{\times} \mathbb{RP}^2$ is a non-trivial Möbius band bundle over $\mathbb{RP}^2$ with boundary $\partial(Mb \tilde{\times} \mathbb{RP}^2) = S^2 \tilde{\times} S^1$. In the previous equality, there is a choice of diffeomorphism $f\colon \partial(Mb \tilde{\times} \mathbb{RP}^2) \to S^2 \tilde{\times} S^1$ that is made as follows. A Pin^+-structure on $\mathbb{RP}^2 \tilde{\times} \mathbb{RP}^2$ induces a Pin^+-structure on each of the building blocks in (4.4.3) as codimension zero submanifolds [99, Construction 1.5]. The induced Pin^+-structures restrict to the same Pin^+-structure $\phi^+_{S^2 \tilde{\times} S^1}$ on each of the $S^2 \tilde{\times} S^1$ boundary components $\partial(D^2 \tilde{\times} \mathbb{RP}^2)$ and $\partial(Mb \tilde{\times} \mathbb{RP}^2)$. The diffeomorphism f is chosen so that it preserves the bundle structure and satisfies $f^*\phi^+_{S^2 \tilde{\times} S^1} = \phi^+_{S^2 \tilde{\times} S^1}$. A result of Kim–Raymond [96] says that there are two isotopy classes of self-diffeomorphisms of $S^2 \tilde{\times} S^1$ that respect the bundle structure. Taking into account the induced Pin^+-structures, we conclude that the gluing diffeomorphism φ in (4.4.3) is isotopic to the identity map. From (4.4.3), we see that $\mathbb{RP}^2 \tilde{\times} S^2$ is obtained by taking a double covering (fiberwise) of $\mathbb{RP}^2 \tilde{\times} \mathbb{RP}^2$. Using a self-diffeomorphism φ' of $S^2 \tilde{\times} S^1$ that preserves the bundle structure and that is not isotopic to the identity map, we obtain

$$W = (D^2 \tilde{\times} \mathbb{RP}^2) \cup_{\varphi'} (Mb \tilde{\times} \mathbb{RP}^2) \tag{4.4.4}$$

with $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$ as a double covering.

The 4-manifolds X and X_R are $\mathbb{CP}^2$-stable in the following sense.

Lemma 4.4.3. *There is a diffeomorphism*

$$X \# \mathbb{CP}^2 \xrightarrow{\mathcal{C}^\infty} X_R \# \mathbb{CP}^2. \tag{4.4.5}$$

Proof. Erase the 2-handle h from the handlebody in the middle level of Figure 4.2 to produce the compact 4-manifold $X_0' = X \setminus \operatorname{Int} \nu(R)$. To draw a handlebody of $X \# \overline{\mathbb{CP}}^2$, add a -1-framed unknot to the middle diagram of Figure 4.2, and slide the 2-handle h over it. The new 2-handle h now has $(0,0)$-framing and it is linked by a -1-framed circle. This exhibits a decomposition of the connected sum as

$$X \# \overline{\mathbb{CP}}^2 = X_0' \cup ((D^2 \tilde{\times} \mathbb{RP}^2) \# \overline{\mathbb{CP}}^2). \tag{4.4.6}$$

To see that there is a decomposition

$$X_R \# \mathbb{CP}^2 = X_0' \cup ((D^2 \tilde{\times} \mathbb{RP}^2) \# \mathbb{CP}^2), \tag{4.4.7}$$

draw a handlebody for this connected sum by adding a 1-framed unknot to the bottom diagram of Figure 4.2 and sliding the $(-1,0)$-framed 2-handle h' over it. These two handlebodies of $X \# \overline{\mathbb{CP}}^2$ and $X_R \# \mathbb{CP}^2$ differ solely on the sign of the circle that links the $(0,0)$-framed 2-handle and, in particular, the gluing maps in (4.4.6) and (4.4.7) coincide. Since $M \# \mathbb{CP}^2$ is diffeomorphic to $M \# \overline{\mathbb{CP}}^2$ for every nonorientable 4-manifold M, we conclude that the diffeomorphism (4.4.5) exists.

□

Although twisting an embedded $\mathbb{RP}^2$ is a cut-and-paste operation of local nature (in the sense that $X \setminus D^2 \tilde{\times} \mathbb{RP}^2 \subseteq X_R$ is a codimension zero submanifold) the topology of X_R can well be drastically different to that of X as exemplified in the following lemma.

Lemma 4.4.4. *Twisting an $\mathbb{RP}^2$ need not preserve irreducibility.*

Proof. The circle sum of three copies of the real projective 4-space $\#_{S^1} 3 \cdot \mathbb{RP}^4$ contains a copy of $D^2 \tilde{\times} \mathbb{RP}^2$. Twisting this $\mathbb{RP}^2$ yields $\mathbb{RP}^2 \tilde{\times} S^2 \#_{S^1} \mathbb{RP}^4$ (cf. 4.3), which is diffeomorphic to $\mathbb{RP}^4 \# (S^2 \times S^2)$ by Lemma 4.3.1. □

4.5 Pin-structures on a mapping torus and framings on its cross-section

As it was done by Cappell–Shaneson in [37, Section 2], consider the diffeomorphism

$$\varphi_A \colon T^3 \longrightarrow T^3 = \mathbb{R}^3/\mathbb{Z}^3 \tag{4.5.1}$$

whose induced map $(\varphi_A)_* \colon H_1(T^3) \to H_1(T^3) = \mathbb{Z}^3$ is given by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(3;\mathbb{Z}). \tag{4.5.2}$$

The mapping torus

$$M_A = (T^3 \times [0,1]) / \sim \tag{4.5.3}$$

of the diffeomorphism (4.5.1) under the identification $(x,0) \sim (\varphi_A(x),1)$ is the total space of a nonorientable T^3-bundle over S^1. Let ν_0 be the tubular neighbourhood of the loop

$$\alpha_0 = ((\{(x_0\} \times [0,1]) / \sim) \subseteq M_A, \tag{4.5.4}$$

where $\varphi_A(x_0) = x_0$ for $x_0 \in T^3$. The loop α_0 inherits a canonical framing from $\{x_0\} \times [0,1]$ [1], [69, Section 2]. We will call the other framing the non-canonical framing following Gompf [69, p. 1668]. The compact 4-manifold ν_0 is diffeomorphic to $D^3 \tilde{\times} S^1$ with $\partial\nu_0 = S^2 \tilde{\times} S^1$.

The 4-manifold (4.5.3) admits a pair of $\mathrm{Pin}^\dagger$-structures for $\dagger = +, -$ and we record some of their properties in the following proposition.

Proposition 4.5.1.

- *There is a pair of* $\mathrm{Pin}^\dagger$*-structures* $(M_A, \pm\phi^\dagger)$ *for* $\dagger = +, -$.
- *The* Pin^-*-structure* (M_A, ϕ^-) *induces the* Pin^-*-structure on* $M_A \setminus \mathrm{Int}\, \nu_0$ *that restricts to the* Pin^-*-structure on* α_0 *that extends to the* 2*-disk* D^2.
- *The* Pin^+*-structure* (M_A, ϕ^+) *induces the* Pin^+*-structure on* $M_A \setminus \mathrm{Int}\, \nu_0$ *that restricts to the* Pin^+*-structure on* α_0 *that extends to the Möbius band* Mb.

The proof of Proposition 4.5.1 follows from well-known results [99, 137]. We will write $(M_A, \phi^+) = (M_A, \phi^+_{M_A})$ for the Pin^+-structure on M_A in the sequel.

The circle sum $X \#_{S^1} M_A$ along an orientation-reversing simple loop in a 4-manifold X and the loop (4.5.4) is a potential exotic smooth structure on X. The homeomorphism type of these 4-manifolds is the same by the following result.

Theorem 4.5.2. (Cappell–Shaneson [37], Freedman [58]) *Let* X *be a closed smooth nonorientable* 4*-manifold and let* $\alpha \subseteq X$ *be a simple loop whose homotopy class generates* $\pi_1(X) = \mathbb{Z}_2$. *Let* $\alpha_0 \subseteq M_A$ *be the orientation-reversing simple loop in the mapping torus* M_A *that was defined in (4.5.4). The* 4*-manifolds* X *and* $X \#_{S^1} M_A = (X \setminus \mathrm{Int}\, \nu(\alpha)) \cup (M \setminus \mathrm{Int}\, \nu_0)$ *are homeomorphic.*

A proof of Theorem 4.5.2 is obtained by generalizing an argument due to Ruberman [128, Proof of Theorem 2]. It also follows as a corollary of work of Hambleton–Kreck–Teichner [78, Theorem 1]. We now provide a sketch of the argument for the convenience of the reader (cf. [137, p. 159]). A result of Cappell–Shaneson [37, Theorem 3.1] says that X is simple homotopy equivalent to the circle sum $X \#_{S^1} M_A$ via a simple homotopy equivalence whose normal invariant is the non-trivial element in the kernel of $[X, G/O] \to [X, G/TOP]$. Using the surgery exact sequence [152, Theorems 10.3 and 10.5]

$$\to L_5(\mathbb{Z}[\pi_1(X)], w_1(X)) \to \mathcal{S}^{TOP}(X) \to [X, G/TOP] \to \tag{4.5.5}$$

one concludes that there is a topological s-cobordism between these two 4-manifolds. Given that $\pi_1(X) = \mathbb{Z}_2$, the surgery group vanishes ([91, Section 4]) and results of Freedman imply that X is homeomorphic to $X \#_{S^1} M_A$ [59, Chapter 11] [58].

We end this section with the following result on the η-invariant of M_A and its circle sums.

Theorem 4.5.3 (Stolz [137])**.**

- *The value of the* η*-invariant of the mapping torus (4.5.3) is*

$$\eta(M_A, \pm\phi^+_{M_A}) = 1 \mod 2\mathbb{Z} \tag{4.5.6}$$

for both Pin^+*-structures* $(M_A, \pm\phi^+_{M_A})$.

- *Let $\alpha_1 \subseteq X_1$ be an orientation-reversing simple loop in a closed 4-manifold X_1 with Pin^+-structure $\phi^+_{X_1}$, which restricts to the Pin^+-structure on the submanifold α_1 that extends to the 2-disk. There is a Pin^+-structure ϕ^+_X on the circle sum*

$$X = X_1 \#_{S^1} M_A = (X_1 \setminus \mathrm{Int}\, \nu(\alpha_1)) \cup (M_A \setminus \mathrm{Int}\, \nu_0) \tag{4.5.7}$$

for which the η-invariant is

$$\eta(X, \phi^+_X) = \eta(X_1, \phi^+_{X_1}) + 1 \mod 2\mathbb{Z}. \tag{4.5.8}$$

4.6 An nonequivalent smooth structure and an embedded 2-sphere

With the purpose of unveiling some of its properties, we now describe in detail the construction of the 4-manifold Y of Theorem E given in [147].

Proposition 4.6.1. (cf. [147, Section 3.2]) *There is a smooth 4-manifold Y that satisfies the following properties.*

- *It is homeomorphic to $\mathbb{RP}^2 \tilde{\times} S^2$.*
- *The value of its η-invariant is*

$$\eta(Y, \phi^+_Y) = 1 \mod 2\mathbb{Z} \tag{4.6.1}$$

for every Pin^+-structure ϕ^+_Y on Y, and Y is neither diffeomorphic nor stably diffeomorphic to $\mathbb{RP}^2 \tilde{\times} S^2$.

- *There is a smoothly embedded 2-sphere $\Sigma \subseteq Y$ with trivial tubular neighbourhood $\nu(\Sigma) = D^2 \times S^2$ and non-trivial homology class $[\Sigma] \neq 0 \in H_2(Y; \mathbb{Z}_2)$.*

The computation of (4.6.1) appeared in [147, Proposition 4, Section 3.2]. We provide the argument in order to make this chapter more self-contained.

Proof. The 4-manifold in the statement of the proposition is the circle sum

$$Y = \mathbb{RP}^2 \tilde{\times} S^2 \#_{S^1} M_A = (\mathbb{RP}^2 \tilde{\times} S^2 \setminus \nu(\alpha)) \cup (M_A \setminus \mathrm{Int}\, \nu_0), \tag{4.6.2}$$

where $\alpha \subseteq \mathbb{RP}^2 \tilde{\times} S^2$ is a loop whose homotopy class generates the fundamental group $\pi_1(\mathbb{RP}^2 \tilde{\times} S^2) = \mathbb{Z}_2$, and $M_A \setminus \mathrm{Int}\, \nu_0$ is the compact 4-manifold constructed in Section 4.5. Theorem 4.5.2 implies that Y is homeomorphic to $\mathbb{RP}^2 \tilde{\times} S^2$. There are two Pin^+-structures related to the η-invariant that we need to consider since $H^1(Y; \mathbb{Z}_2) = \mathbb{Z}_2$ [137, Corollary 6.4]. We have that

$$\eta(\mathbb{RP}^2 \tilde{\times} S^2, \phi^+) = 0 \mod 2\mathbb{Z} \tag{4.6.3}$$

for every Pin$^+$-structure ϕ^+, given that $\mathbb{RP}^2 \tilde{\times} S^2$ is a Pin$^+$-boundary of a D^3-bundle over $\mathbb{RP}^2$. Applying Theorem 4.5.3 we determine the value

$$\eta(Y, \phi^+_Y) = 1 \mod 2\mathbb{Z}. \tag{4.6.4}$$

Since the values of the η-invariants are different for all Pin$^+$-structures, we conclude that Y is neither diffeomorphic nor stably diffeomorphic to $\mathbb{RP}^2 \tilde{\times} S^2$ by Theorem 1.3.7. The 2-sphere fiber $\Sigma \subseteq \mathbb{RP}^2 \tilde{\times} S^2$ has a tubular neighbourhood $\nu(\Sigma) = D^2 \times S^2$ and it is disjoint from the seam of the gluing along loops (4.6.2) in the assembly of Y. We abuse notation and conclude that there is an embedded 2-sphere $\Sigma \subseteq Y$ with the desired properties.

□

4.7 Nonequivalent smooth structures on nonorientable 4-manifolds

As already mentioned in Section 3.2.1, Cappell–Shaneson constructed a 4-manifold

$$Q = \mathbb{RP}^4 \#_{S^1} M_A = (D^2 \tilde{\times} \mathbb{RP}^2) \cup (M_A \setminus \mathrm{Int}\, \nu_0) \tag{4.7.1}$$

that is homeomorphic, but not diffeomorphic to $\mathbb{RP}^4$ [37, 128, 59]. Let $\gamma \subseteq Q$ be a loop whose homotopy class generates the group $\pi_1(Q)$. A close look to Cappell–Shaneson's proof reveals their construction of a compact 4-manifold

$$N = Q \setminus \mathrm{Int}\, \nu(\gamma) \tag{4.7.2}$$

that is homeomorphic but not diffeomorphic to $D^2 \tilde{\times} \mathbb{RP}^2$ and whose boundary is $\partial N = S^2 \tilde{\times} S^1$ [3, Section 2]. A handlebody of (4.7.2) was drawn by Akbulut in [3, Figures 1.26 and 1.33]. Torres observed in [147, Theorem 2] that (4.7.2) along with Theorems 4.5.2 and 1.3.7 can be used to produce an nonequivalent smooth structure on a myriad of closed nonorientable 4-manifolds, which include

$$Y = \mathbb{RP}^2 \tilde{\times} S^2 \#_{S^1} M_A. \tag{4.7.3}$$

and

$$Z = (\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4) \#_{S^1} M_A = \mathbb{RP}^4 \#_{S^1} Q. \tag{4.7.4}$$

4.8 A result of Akbulut

For nonorientable 4-manifolds with a Pin$^+$-structure, the building block (4.7.2) and the η-invariant are effective tools to produce nonequivalent smooth structures (cf. [50, 51, 103]). The situation where there is no Pin$^+$-structure available is strikingly different as the following theorem suggests.

Theorem 4.8.1. (Akbulut [3, Section 2]). *Let $\alpha \subseteq \mathbb{RP}^2 \times S^2$ be a simple loop whose homotopy class generates the group $\pi_1(\mathbb{RP}^2 \times S^2)$. The 4-manifold*

$$Y' = ((\mathbb{RP}^2 \times S^2) \setminus \operatorname{Int} \nu(\alpha)) \cup (M_A \setminus \operatorname{Int} \nu_0) \tag{4.8.1}$$

is diffeomorphic to $\mathbb{RP}^2 \times S^2$.

Theorem 4.8.1 is one of the two key ingredients in the proof of Theorem E.

4.9 Connected sums with complex projective planes

The closed smooth 4-manifolds with fundamental group $\mathbb{Z}_2$ and Euler characteristic equal to two that are considered in this note compose three homeomorphism classes and five diffeomorphism classes. All these 4-manifolds become diffeomorphic under the connected sum of a single copy of the complex projective plane. The following proposition addresses the situation regarding the representatives of the three homeomorphism classes.

Proposition 4.9.1. *There are diffeomorphisms*

$$f_1\colon \mathbb{RP}^2 \tilde{\times} S^2 \# \mathbb{CP}^2 \xrightarrow{C^\infty} (\mathbb{RP}^2 \times S^2) \# \mathbb{CP}^2 \tag{4.9.1}$$

and

$$f_2\colon (\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4) \# \mathbb{CP}^2 \xrightarrow{C^\infty} (\mathbb{RP}^2 \times S^2) \# \mathbb{CP}^2. \tag{4.9.2}$$

Proof. The existence of the diffeomorphism (4.9.1) follows from Lemma 4.2.1 and the known property that Gluck twists come undone by taking a connected sum with a copy of $\mathbb{CP}^2$ [73, Exercise 5.2.7 (b)], [12, Theorem 1.1]. The existence of the diffeomorphism (4.9.2) follows from Example 4.4.1, Lemma 4.4.3 and (4.9.1). □

The following result implies that the nonequivalent smooth structures that we have considered in this chapter become diffeomorphic after adding a copy of $\mathbb{CP}^2$.

Theorem 4.9.2. (Akbulut [3, Theorem 2]). *Let Q be Cappell–Shaneson's exotic real projective 4-space that was constructed in (4.7.1). There is a diffeomorphism between $Q \# \mathbb{CP}^2$ and $\mathbb{RP}^4 \# \mathbb{CP}^2$.*

The existence of topologically equivalent and smoothly nonequivalent 2- and 3-spheres in the non-trivial $\mathbb{RP}^2$-bundle over S^2 is an immediate consequence of Theorem 4.9.2 and work of Cappell–Shaneson [37].

In order to prove Theorem 4.9.2, Akbulut shows that there is a diffeomorphism

$$N \# \mathbb{CP}^2 \xrightarrow{C^\infty} (D^2 \tilde{\times} \mathbb{RP}^2) \# \mathbb{CP}^2 \tag{4.9.3}$$

for the compact 4-manifold (4.7.2). Corollary 4.9.3 follows immediately from Akbulut's Theorem 4.9.2 by noticing that both constructions (4.7.3) and (4.7.4) contain a copy of N. The latter is responsible for the nonequivalent smooth structure. For example,

$$N \subseteq Z = ((\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4) \#_{S^1} M_A) = \mathbb{RP}^4 \#_{S^1} Q. \tag{4.9.4}$$

We finish this section by stating a consequence of Theorem 4.9.2: the nonequivalent smooth structures (4.7.3) and (4.7.4) are $\mathbb{CP}^2$-stable in the following sense.

Corollary 4.9.3. *There are diffeomorphisms*

$$f_3\colon Y \# \mathbb{CP}^2 \xrightarrow{\mathcal{C}^\infty} (\mathbb{RP}^2 \times S^2) \# \mathbb{CP}^2 \tag{4.9.5}$$

and

$$f_4\colon Z \# \mathbb{CP}^2 \xrightarrow{\mathcal{C}^\infty} (\mathbb{RP}^2 \times S^2) \# \mathbb{CP}^2. \tag{4.9.6}$$

4.10 Proofs of main results

4.10.1 Proof of Theorem E

Construct

$$Y' = (Y \setminus \operatorname{Int} \nu(\Sigma)) \cup_\varphi (D^2 \times S^2) \tag{4.10.1}$$

by performing a Gluck twist to the 2-sphere $\Sigma \subseteq Y$ of Proposition 4.6.1. Let $\alpha \subseteq \mathbb{R}P^2 \times S^2$ be a simple loop that generates $\pi_1(\mathbb{R}P^2 \times S^2) = \mathbb{Z}_2$. Another construction of (4.10.1) is $Y' = ((\mathbb{R}P^2 \times S^2) \setminus \operatorname{Int} \nu(\alpha)) \cup (M_A \setminus \operatorname{Int} \nu_0)$, and Theorem 4.8.1 says that it is diffeomorphic to $\mathbb{R}P^2 \times S^2$. From (4.10.1), we see that there is a smoothly embedded 2-sphere $S \subseteq \mathbb{R}P^2 \times S^2$ such that one recovers Y by performing a Gluck twist on it. To conclude that the exteriors are not homotopy equivalent, we look at their fundamental groups. The exterior of the factor 2-sphere $(\mathbb{R}P^2 \times S^2) \setminus \operatorname{Int} \nu(\{pt\} \times S^2)$ is $Mb \times S^2$ and $(\mathbb{R}P^2 \times S^2) \setminus \operatorname{Int} \nu(S) = (Mb \times S^2) \#_{S^1} M_A$, where the circle sum is taken with matching Pin$^+$-structures. These two compact 4-manifolds have non-isomorphic fundamental groups [37, Proposition 2.4].

$\square$

4.10.2 Proof of Theorem F

Let $\alpha_0^2 \subseteq M_A$ be the simple loop that represents the element $[\alpha_0^2] \in \pi_1(M_A)$ where $\alpha_0 \subseteq M_A$ is the cross-section of the mapping torus (4.5.3) that was described in Section 4.5. Carve out a tubular neighbourhood $\nu(\alpha_0^2) = S^1 \times D^3$ of the loop $\alpha_0^2 \subseteq M_A$ and build the 4-manifold

$$Y(\psi) = (M_A \setminus \operatorname{Int} \nu(\alpha_0^2)) \cup_\psi (D^2 \times S^2), \tag{4.10.2}$$

where $\psi = \mathrm{id}$ corresponds to the canonical framing defined in Section 4.5, and $\psi = \varphi$ corresponds to the non-canonical one, where the diffeomorphism φ is described in (1.1.25). Notice that the 4-manifold (4.10.2) is a boundary component of the cobordism that is obtained from adding a 2-handle to $M_A \times I$. Rephrase (4.10.2) as

$$Y(\psi) = (M_A \setminus \mathrm{Int}\, \nu(\alpha_0^2)) \cup_\psi (D^2 \times S^2) = ((Mb \times S^2) \#_{S^1} M_A)) \cup_\psi (D^2 \times S^2). \quad (4.10.3)$$

Lemma 4.2.1 and Theorem 4.8.1 imply that $Y(\mathrm{id})$ is diffeomorphic to $\mathbb{RP}^2 \times S^2$; notice that $Y(\mathrm{id})$ is denoted by Y' in Theorem 4.8.1. The 4-manifold $Y(\varphi)$ is denoted by Y in Proposition 4.6.1 and Theorem E.

$\square$

Remark 4.10.1. A Cappell–Shaneson homotopy 4-sphere is constructed by doing surgery along a cross-section of a certain 3-torus bundle over the circle M_{A^2} with monodromy A^2 for $A \in \mathrm{GL}(3;\mathbb{Z})$ [37, 1]. A myriad of these homotopy 4-spheres are known to be diffeomorphic to S^4 by work of several authors [1, 4, 10, 11, 68, 70, 69]. Theorem F addresses the corresponding situation in the nonorientable realm.

4.10.3 Proof of Theorem G

Deconstruct the smooth structure on $\mathbb{RP}^2 \tilde{\times} S^2$ of Proposition 4.6.1 as

$$Y = N \cup_{\mathrm{id}} (D^2 \tilde{\times} \mathbb{RP}^2), \quad (4.10.4)$$

where N is the 4-manifold (4.7.2). A tubular neighbourhood $\nu(\mathbb{RP}^2) = D^2 \tilde{\times} \mathbb{RP}^2$ of a smoothly embedded $\mathbb{RP}^2 \subseteq Y$ is visible from (4.10.4). Twisting this $\mathbb{RP}^2$ yields

$$Z = N \cup_{\varphi'} (D^2 \tilde{\times} \mathbb{RP}^2). \quad (4.10.5)$$

It is immediate that (4.10.5) is homeomorphic to $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$. Using Theorem 1.3.7 and the additivity of the η-invariant under the operation of circle sum, we conclude that Z is neither diffeomorphic nor stably diffeomorphic to $\mathbb{RP}^4 \#_{S^1} \mathbb{RP}^4$. $\square$

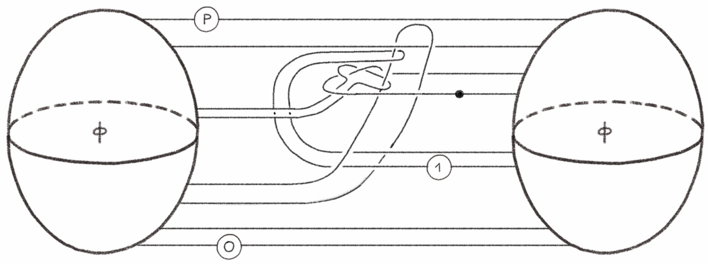

Figure 4.4: Handlebodies of (4.10.4) and (4.10.5) are drawn in Figure 4.4. The value $p = -1$ corresponds to Y, and $p = 1$ to Z. Figure 4.4 corrects [147, Fig 2].

Chapter 5

Pin-structures and Lefschetz fibrations

This project was motivated by the effectiveness of Pin^+-structures in detecting the nonorientable exotica in Chapters 3 and 4.

In this chapter, we find necessary and sufficient conditions for a (possibly nonorientable) Lefschetz fibration over the 2-disk to support a Pin^- or a Pin^+-structure. In particular, such conditions are expressed in terms of the homology classes of the vanishing cycles of the Lefschetz fibration, with coefficients taken in $\mathbb{Z}_2$ and $\mathbb{Z}_4$ respectively. As an application of this criterion, we provide explicit necessary and sufficient conditions for a given Lefschetz fibration over the 2-sphere to support a Pin^+ or a Pin^--structure. We refer the reader to Section 5.1 for a quick recap of basic facts about nonorientable Lefschetz fibrations. Our main reference for this subject is [111].

In Section 5.2 we prove the following statement, extending the work of A. Stipsicz on Spin-structures over Lefschetz fibrations in [133, Theorem 1.1].

Theorem H. *Let X be a smooth 4-manifold and $f\colon X \to D^2$ a Lefschetz fibration with regular fiber Σ. There is no Pin^--structure on X if and only if there are $k+1$ vanishing cycles $c_0, c_1, \dots, c_k$ such that $[c_0] = \sum_{i=1}^{k}[c_i] \in H_1(\Sigma;\mathbb{Z}_2)$ and $k + \sum_{1\le i<j\le k} c_i \cdot c_j \equiv 0 \mod 2$.*

As we will see in Section 5.2, Theorem H is a consequence of the fact that Pin^--structures on a Lefschetz fibration over the 2-disk with regular fiber Σ naturally correspond to maps

$$q^-\colon H_1(\Sigma;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x) + q^-(y) + 2x\cdot y$ for every $x, y \in H_1(\Sigma;\mathbb{Z}_2)$ and $q^-([c]) = 2$ on every vanishing cycle $c \subseteq \Sigma$, see Lemma 5.2.2. Moreover, such correspondence is equivariant with respect to the action of $H^1(X;\mathbb{Z}_2)$.

Section 5.3 is devoted to the study of Pin^+-structures on Lefschetz fibrations. By [41], a Pin^+-structure on the regular fiber Σ corresponds to a map

$$q_0^+\colon H_1(\Sigma;\mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

with the property that $q_0^+(x+y) = q_0^+(x) + q_0^+(y) + x \cdot y \in \mathbb{Z}_2$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_4)$. We show that we can choose q_0^+ to be the restriction of a Pin^+-structure defined on the whole fibration if and only if the following holds.

Theorem I. *The total space of the Lefschetz fibration $f\colon X \to D^2$ with vanishing cycles $c_1, \ldots, c_n \subseteq \Sigma$ supports a Pin^+-structure if and only if Σ supports a Pin^+-structure and*

$$\mathit{rank}(C) = \mathit{rank}(C \mid A)$$

where C is the $\mathbb{Z}_2$-reduction of the $n \times r$ matrix (c_{ij}) whose rows are given by the components of $[c_1], \ldots, [c_n] \in H_1(\Sigma; \mathbb{Z}_4) \cong \mathbb{Z}_4^r$ with respect to a fixed basis $e_1, \ldots, e_r$ of the free $\mathbb{Z}_4$-module $H_1(\Sigma; \mathbb{Z}_4)$ and A is the column vector with entries $A_i = 1 + q_0^+([c_i]) \in \mathbb{Z}_2$ for $i = 1, \ldots, n$.

A key tool in the proof of the above result is Lemma 5.3.1, in which we show that Pin^+-structures on X correspond to maps

$$q^+ \colon H_1(\Sigma; \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for every $x, y \in H_1(\Sigma; \mathbb{Z}_4)$ and $q^+([c]) = 1$ on every vanishing cycle $c \subseteq \Sigma$. The correspondence is also in this case equivariant with respect to the $H^1(X; \mathbb{Z}_2)$-action.

We remark that the study of Pin^+ and Pin^--structures on nonorientable 4-manifolds is essentially reduced to the case of nonorientable Lefschetz fibrations over the 2-disk due to the following result, which is a straightforward consequence of [111, Theorem 1.1].

Theorem J (Miller–Özbağcı [111])**.** *Let X be a closed nonorientable smooth 4-manifold. There is a decomposition*

$$X = L \cup_\partial H$$

where L is a nonorientable Lefschetz fibration over the 2-disk and H is a nonorientable 1-handlebody.

In particular, the proof of [111, Theorem 1.1] shows an explicit way of endowing a (possibly nonorientable) 2-handlebody with the structure of a Lefschetz fibration over the 2-disk, see also [81] and [48]. Moreover, a 4-manifold $X = L \cup_\partial H$ as in Theorem J admits a $\mathrm{Pin}^\pm$-structure if and only if L does. This latter conditions can be checked using Theorem H and Theorem I.

In Section 5.4 we provide explicit examples in which, starting from a handlebody decomposition of a nonorientable 4-manifold, we endow its 2-handlebody with the structure of a Lefschetz fibration over the 2-disk. We then apply our results in order to check whether such 4-manifolds support a Pin^- or a Pin^+-structure. In the case they do, we find all the possible structures by looking at the $H^1(X; \mathbb{Z}_2)$-action on the associated quadratic enhancements on the regular fibers. The nonorientable handlebodies under consideration describe the three 4-manifolds $\mathbb{RP}^4$, $\mathbb{RP}^2 \tilde{\times} S^2$ and $\mathbb{RP}^2 \times S^2$ and can be found in [3], [24], [110] and [147].

Section 5.5 is devoted of the study of necessary and sufficient conditions for the existence of a $\mathrm{Pin}^{\pm}$-structure on a Lefschetz fibration over the 2-sphere. This is the nonorientable version of [133, Theorem 1.3].

Theorem K. *Let $f\colon X \to S^2$ be a (possibly nonorientable) Lefschetz fibration with regular fiber Σ. Then X supports a Pin^--structure if and only if $X \setminus \mathrm{Int}\,\nu(\Sigma)$ does and there exists a smoothly embedded surface $\sigma \subseteq X$ which is dual to Σ in $H_2(X;\mathbb{Z}_2)$ and such that*

$$[\sigma]^2 + (w_1(\sigma) \cup w_1(\nu(\sigma))([\sigma]) + w_1^2(\nu(\sigma)) = 0 \in \mathbb{Z}_2.$$

Analogously, X supports a Pin^+-structure if and only if $X \setminus \mathrm{Int}\,\nu(\Sigma)$ does and there exists a smoothly embedded surface $\sigma \subseteq X$ which is dual to Σ in $H_2(X;\mathbb{Z}_2)$ and such that

$$\chi(\sigma) + [\sigma]^2 + (w_1(\sigma) \cup w_1(\nu(\sigma)))([\sigma]) = 0 \in \mathbb{Z}_2.$$

In Section 5.3 we show that if $X = L \cup H$ is a closed 4-manifold with a decomposition as in the statement of Theorem J, then the datum of a Pin^+-structure on X is equivalent to the datum of a map

$$q^+\colon H_1(\Sigma;\mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\Sigma;\mathbb{Z}_4)$ taking value $1 \in \mathbb{Z}_2$ on all the vanishing cycles of L, where Σ denotes a regular fiber. We leave the reader with the following question.

Question: Is there a formula for the η-invariant modulo $2\mathbb{Z}$ of a Pin^+-structure on X in terms of the corresponding map $q^+\colon H_1(\Sigma;\mathbb{Z}_4) \to \mathbb{Z}_2$?

In Section 5.5 we study Lefschetz fibrations over the 2-sphere, 5.6 is devoted to a brief discussion on the 3-dimensional case, while Section 5.7 contains a characterization of Pin^+-structures on vector bundle which resembles Milnor's characterization of Spin-structures, see [112, Alternative definitions 2].

5.1 Nonorientable Lefschetz fibrations

Since the conditions 1.1.27 and 1.1.28 are not immediate to check in the nonorientable setting and depend heavily on the embeddings of the homologically essential surfaces, in the following sections we develop another way to combinatorially understand when a 4-manifold is Pin^+ or Pin^- by means of a specific kind of decomposition. To do this, we will need the notion of nonorientable Lefschetz fibration over the 2-disk.

We start by recalling the definition of Lefschetz fibration. Our main reference for this topic in the nonorientable realm is [111].

Definition 5.1.1. Let X be a compact, connected 4-manifold and let B be a compact, connected surface, both with possibly non-empty boundary. A *Lefschetz fibration* is a smooth submersion

$$f\colon X \longrightarrow B$$

away from finitely many points in the interior of B such that each fiber contains at most one critical point and f is a fiber bundle with surface fiber on the complement of the critical values. Moreover, around each critical point, we require f to conform to the local complex model

$$(z_1, z_2) \mapsto z_1 z_2.$$

Remark 5.1.2. If X is nonorientable and B is oriented, each regular fiber is a nonorientable surface. In this case it makes no sense to ask for the orientation of the local model around a critical point to be compatible to the one of X.

In the following, we will just consider the case in which $B = D^2$ and Σ is the regular fiber. It is possible to show that every Lefschetz fibration

$$f\colon X \longrightarrow D^2$$

is obtained from the product one

$$\Sigma \times D^2 \longrightarrow D^2$$

by gluing 4-dimensional 2-handles along $\Sigma \times \partial D^2$ with framing ± 1 with respect to the fiber framing. The attaching curves of such 2-handles are push-offs in distinct fibers of simple closed curves inside Σ, which are called the vanishing cycles of the fibration and are denoted in the following by $c_1, \ldots, c_n \subseteq \Sigma$. Moreover, $\langle w_1(X), [c_i] \rangle = 0$ for all $i = 1, \ldots, n$ and hence the tubular neighbourhood of these attaching regions is necessarily diffeomorphic to a product of S^1 with the unit interval. We will call *two-sided* all curves with this property.

5.2 Pin$^-$-structures on 4-manifolds via Lefschetz fibrations

It is well known that the datum of a Spin-structure over an oriented surface is equivalent to the one of a quadratic enhancement

$$s\colon H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$$

satisfying the condition

$$s(x+y) = s(x) + s(y) + x \cdot y$$

for all $x, y \in H_1(\Sigma; \mathbb{Z}_2)$, where $\cdot$ denotes the $\mathbb{Z}_2$-intersection number between cycles, see [88] and [133]. A geometric interpretation of this algebraic object can be given as follows.

The 1-dimensional Spin-cobordism group is $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}_2$ [112] and every closed simple curve $\gamma \subseteq \Sigma$ in a Spin surface inherits a Spin-structure. In particular, if s is the quadratic enhancement associated to the fixed Spin-structure on Σ, then $s([\gamma]) = 0$ if and only if the induced Spin-structure on γ is the one bounding the unique Spin-structure on the 2-disk.

It is possible to give a similar description for Pin^- structures on (not necessarily orientable) surfaces, as shown by the following result.

Theorem 5.2.1 (Kirby-Taylor, [99])**.** *There is a* $1:1$ *correspondence between* Pin^-*-structures on a surface* Σ *and quadratic enhancements*

$$q^-\colon H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_4$$

with the property that

$$q^-(x+y) = q^-(x) + q^-(y) + 2\, x \cdot y$$

for any $x, y \in H_1(\Sigma;\mathbb{Z}_2)$.

In particular, if $\gamma \subseteq \Sigma$ is a simple closed curve, then $q^-([\gamma])$ is even if and only if γ is two-sided and this is the case whenever γ is a vanishing cycle for a Lefschetz fibration. Moreover, recall that the 1-dimensional Pin^--cobordism group is $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}_2$ (see [17, Theorem 5.1] and [99, Section 0]) and every closed simple curve $\gamma \subseteq \Sigma$ in a Pin^- surface inherits a Pin^- structure. We have that $q^-([\gamma]) = 0$ if and only if γ inherits the bounding Pin^--structure.

In order to prove Theorem H, we will need the following Lemma.

Lemma 5.2.2. *Let* $f\colon X \to D^2$ *be a Lefschetz fibration with regular fiber* Σ *and vanishing cycles* $c_1, \dots, c_n \subseteq \Sigma$. *There is a natural one to one correspondence between the set of* Pin^-*-structures on* X *and the set of quadratic enhancements*

$$q^-\colon H_1(\Sigma;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x) + q^-(y) + 2x \cdot y$ *for all* $x, y \in H_1(\Sigma;\mathbb{Z}_2)$ *and* $q^-([c_i]) = 2$ *for all* $i = 1, \dots, n$. *Moreover, such correspondence is equivariant with respect to the free and transitive action of* $H^1(X;\mathbb{Z}_2)$.

Proof. The existence of a Pin^--structure on X is equivalent to the existence of a Pin^--structure on Σ that extends to the 2-handles. This follows from the fact that any Pin^--structure on X induces by restriction a Pin^- structure on a (trivial) tubular neighbourhood $\nu(\Sigma) \cong \Sigma \times D^2$ and all the Pin^--structures on $\Sigma \times D^2$ are pull-backs of the ones on Σ via the projection map $\Sigma \times D^2 \to \Sigma$, see also the proof of [133, Theorem 1.1]. Since such 2-handles are attached to $\Sigma \times \partial D^2$ with relative odd framing, this corresponds to a Pin^--structure on Σ restricting to the non-bounding one on every vanishing cycle. The conclusion follows from Theorem 5.2.1.

The free and transitive action of $H^1(X;\mathbb{Z}_2)$ on the set of Pin$^-$-structures on X can be seen as follows. In [99, Section 3] it is shown that $H^1(\Sigma;\mathbb{Z}_2)$ acts on the set of Pin$^-$-structures of the surface Σ by

$$q_\gamma^-(x) = q^-(x) + 2 \cdot \gamma(x) \tag{5.2.1}$$

for all $\gamma \in H^1(\Sigma;\mathbb{Z}_2)$ and $x \in H_1(\Sigma;\mathbb{Z}_2)$. In particular, Pin$^-$-structures on Σ equivariantly correspond to quadratic enhancements and the action (5.2.1) is well defined on $H^1(X;\mathbb{Z}_2) \subseteq H^1(\Sigma;\mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2) \oplus K$, where $K \subseteq H^1(\Sigma;\mathbb{Z}_2)$ is the kernel of the map induced by the inclusion $\Sigma \subseteq X$. $\square$

In light of this fact, it is possible to characterize the existence of Pin$^-$-structures on nonorientable Lefschetz fibrations over the 2-disk in terms of the $\mathbb{Z}_2$-homology classes of the vanishing cycles.

Proof of Theorem H. By Lemma 5.2.2, X does not support any Pin$^-$-structure if and only if it is not possible to build a map

$$q^- \colon H_1(\Sigma;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

such that $q^-(x+y) = q^-(x) + q^-(y) + 2x \cdot y$ for all $x, y \in H_1(\Sigma;\mathbb{Z}_2)$ and such that $q^+([c_i]) = 2$ for all $i = 1, \dots, n$. The remaining part of the proof is essentially the same as the proof of [133, Theorem 1.1]. We will now sketch it for the convenience of the reader. Let $V \subseteq H_1(\Sigma;\mathbb{Z}_2)$ be the vector subspace generated by the classes of the vanishing cycles. Notice that any given enhancement q^- on V can be extended to the whole $H_1(\Sigma;\mathbb{Z}_2)$ by setting $q^- = 0$ on a basis $u_1, \dots, u_k$ of an orthogonal vector subspace and using the relation (5.2.1) to define it on the vectors of the form $\sum_j u_{i_j} + v$, where $v \in V$. In particular, in order to define a Pin$^-$-structure on X, one needs to find a quadratic enhancement $q^- \colon V \to \mathbb{Z}_4$ which takes value 2 on the class of any vanishing cycle. Let $v_1, \dots, v_r$ be a basis of V represented by vanishing cycles and let $V_i = \langle v_1, \dots, v_i \rangle$ for $i = 1, \dots, r$. We then want to define q^- inductively on each V_i as in the proof of [133, Theorem 1.1], setting $q^-(v_i) = 2$ at each step. One then notices that things go wrong if and only if there are k classes $v_{i_1}, \dots, v_{i_k}$ such that their sum $v_0 = \sum_{j=1}^k v_{i_j} \in H_1(\Sigma;\mathbb{Z}_2)$ is again represented by a vanishing cycle and $q^-(v_0) \neq 2 \in \mathbb{Z}_4$. This implies that $q^-(v_0) = 0 \in \mathbb{Z}_4$, being v_0 represented by a curve with trivial tubular neighbourhood. Since by construction we had that $q^-(v_{i_j}) = 2$ for $j = 1, \dots, k$, using (5.2.1) the condition $q^-(q_0) = 0 \in Z_4$ translates into

$$q^-(v_0) = \sum_{j=1}^k q^-(v_{i_j}) + 2 \cdot \Big(\sum_{1 \le i_n \le i_m \le k} v_{i_n} \cdot v_{i_m} \Big) = 2 \cdot \Big(k + \sum_{1 \le i_n \le i_m \le k} v_{i_n} \cdot v_{i_m} \Big) = 0 \in \mathbb{Z}_4.$$

The conclusion then follows by noticing that $2 \cdot x = 0 \in \mathbb{Z}_4$ if and only if $x = 0 \in \mathbb{Z}_2$. $\square$

Note that, if we restrict to orientable surfaces, there is a natural map

$$\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Pin}^-}$$

giving a group isomorphism, see [99, Theorem 2.1]. At the level of the associated quadratic forms, the enhancement

$$s\colon H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_2$$

corresponds to

$$q^-\colon H_1(\Sigma;\mathbb{Z}_2) \to \mathbb{Z}_4$$

given by

$$q^-(x) = 2 \cdot s(x)$$

for every $x \in H_1(\Sigma;\mathbb{Z}_2)$, where $2\cdot$ denotes the inclusion $\mathbb{Z}_2 \subseteq \mathbb{Z}_4$. In particular, the condition on the vanishing cycles we found in Theorem H coincides with the one in [133, Theorem 1.1] and this is due to the fact that, when restricting to orientable Lesfschetz fibrations over D^2, being Spin coincides with being Pin^-.

5.3 Pin^+-structures on Lefschetz fibrations over the 2-disk

As a consequence of [41, Theorem A], there is a canonical affine bijective correspondence between Pin^+-structures on a surface Σ and the set of maps

$$q^+\colon H_1(\Sigma;\mathbb{Z}_4) \to \mathbb{Z}_2 \tag{5.3.1}$$

with the property that

$$q^+(x+y) = q^+(x) + q^+(y) + x\cdot y \tag{5.3.2}$$

for every $x, y \in H_1(\Sigma;\mathbb{Z}_4)$, where $x \cdot y \in \mathbb{Z}_2$ denotes the algebraic intersection modulo 2 between representatives of the $\mathbb{Z}_2$-reductions of the classes $x, y \in H_1(\Sigma;\mathbb{Z}_4)$.

Recall that the 1-dimensional Pin^+-cobordism group is

$$\Omega_1^{\mathrm{Pin}^+} \cong \{0\}$$

see [99, Section 3]. However, S^1 has two distinct Pin^+-structures. These correspond to two different trivializations of the direct sum $TS^1 \oplus \epsilon$ of the tangent bundle of S^1 with a trivial line bundle, where the first one is given by the restriction to S^1 of a trivialization of the tangent bundle TD^2 of the 2-disk and the second one comes from the Lie group framing of S^1. In particular, one can show that such structures have the property that they respectively bound the 2-disk and the Möbius strip, see [99, Section 1]. A close look at the construction of q^+ starting from a Pin^+-structure on Σ shows that $q^+([\gamma]) = 0$ if the induced Pin^+-structure on γ is the one bounding a 2-disk, while $q^+([\gamma]) = 1$ if it is the one bounding the Möbius strip, see [99]. Note that, given any two maps q_1^+, q_2^+ corresponding to Pin^+-structures on Σ, their difference can be regarded as an element

$$q_1^+ - q_2^+ \in \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}_4),\mathbb{Z}_2).$$

The proof of Theorem I is based on the following Lemma.

Lemma 5.3.1. *Let $f\colon X \to D^2$ be a Lefschetz fibration with regular fiber Σ and vanishing cycles $c_1,\dots,c_n \subseteq \Sigma$. There is a natural bijection between the set of Pin^+-structures on X and the set of quadratic enhancements*

$$q^+\colon H_1(\Sigma;\mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x\cdot y$ for all $x, y \in H_1(\Sigma;\mathbb{Z}_4)$ and $q^+([c_i]) = 1$ for all $i = 1,\dots,n$. Moreover, such correspondence is equivariant with respect to the free and transitive action of $H^1(X;\mathbb{Z}_2)$.

Proof. As is the Pin^- case, the existence of a Pin^+-structure on X is equivalent to the existence of a Pin^+-structure on Σ that extends to the 2-handles. Since such 2-handles are attached to $\Sigma \times \partial D^2$ with relative odd framing, this corresponds to a Pin^+-structure on Σ restricting to the one bounding the Möbius strip on every vanishing cycle. The conclusion follows from [41].

In the Pin^+-case, $H^1(\Sigma;\mathbb{Z}_2)$ acts on the set of Pin^+-structures of the surface Σ by

$$q^+_\gamma(x) = q^+(x) + \gamma(x) \tag{5.3.3}$$

for all $\gamma \in H^1(\Sigma;\mathbb{Z}_2)$ and $x \in H_1(\Sigma;\mathbb{Z}_4)$, see [41]. In particular, Pin^+-structures on Σ equivariantly correspond to quadratic enhancements. The action of $\alpha \in H^1(X;\mathbb{Z}_2)$ on the Pin^+-structure on X described by the enhancement q^+ corresponds to the action of the pull-back $i^*(\alpha) \in H^1(\Sigma;\mathbb{Z}_2)$ on q^+, as described in Equation 5.3.3. Here $\iota\colon \Sigma \to X$ denotes the inclusion map. □

Let $f\colon X \to D^2$ be a Lefschetz fibration over the 2-disk with surface fiber Σ. Let $c_1,\dots,c_n \subseteq \Sigma$ be its vanishing cycles. Note that one can find simple closed curves $e_1,\dots,e_g$ inducing a basis of $H_1(\Sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2^g$. From now on, we will assume that Σ has a Pin^+-structure, since this is a necessary condition for X to be a Pin^+-manifold. In particular, this means that Σ is not a closed nonorientable surface of odd Euler characteristic. Let

$$q_0^+\colon H_1(\Sigma;\mathbb{Z}_4) \to \mathbb{Z}_2 \tag{5.3.4}$$

be the quadratic enhancement associated to a fixed Pin^+-structure on Σ and let $c_j = \sum_{i=1}^r c_{ji}[e_i]$. This gives the setting of Theorem I, of which we now provide a proof.

Proof of Theorem I. Lemma 5.3.1 implies that X has a Pin^+-structure if and only if there is a map

$$q^+ \colon H_1(\Sigma, \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

such that $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\Sigma; \mathbb{Z}_4)$ and $q^+([c_i]) = 1$ for all $i = 1, \ldots, n$. In particular, every such q^+ is necessarily of the form $q^+ = q_0^+ + l$ for some linear map $l = \sum_{i=1}^r \lambda_i [e^i]$, where $[e^i] \in H^1(\Sigma; \mathbb{Z}_2)$ denotes the dual element to $[e_i] \in H_1(\Sigma; \mathbb{Z}_2)$. In particular, this reduces the problem to finding $l = \sum_{i=1}^r \lambda_i e^i$ such that for every $j = 1, \ldots, n$ we have

$$q^+([c_j]) = q_0^+([c_j]) + l([c_j]) = q_0^+([c_j]) + \sum_{i=1}^r c_{ji}\lambda_i = 1.$$

The conclusion follows from Rouché-Capelli's theorem [129, Theorem 2.38]. □

Remark 5.3.2. The proof of Theorem H as well as the one of [133, Theorem 1.1] rely on the fact that $H_1(\Sigma; \mathbb{Z}_2)$ is a vector space. This is why we can not adopt the same strategy for Theorem I, since $H_1(\Sigma; \mathbb{Z}_4)$ is just a $\mathbb{Z}_4$-module.

5.4 Some examples

In this section, we show how to apply the results proven so far. In particular, starting from the handlebody decomposition of some small nonorientable 4-manifold M, we endow the union of handles up to index 2 with the structure of a Lefschetz fibration over the 2-disk, following the procedure explained in the proof of [111, Theorem 1.1]. At this point, we apply Theorem H and Theorem I to understand whether or not M admits a Pin^+ or a Pin^--structure, describing the action of $H^1(M; \mathbb{Z}_2)$ in terms of the action of a subgroup of $H^1(\Sigma; \mathbb{Z}_2)$ on the quadratic enhancements on the non-singular fiber Σ of the Lefschetz fibration. We refer the reader to [2, Section 1.5] for the conventions and background on Kirby diagrams of nonorientable 4-manifolds, see also [3], [24, Section 2.1], [38] and [110]. The procedure we describe in this section applies to any closed nonorientable 4-manifold represented by means of a Kirby diagram.

Example 5.4.1 ($\mathbb{RP}^4$)**.** Figure 5.1 represents a Kirby diagram for the standard handlebody decomposition of $\mathbb{RP}^4$ with a single k-handle for each $k = 0, \ldots, 4$. In particular, the union of all handles up to index 2 corresponds to a tubular neighbourhood $\nu(\mathbb{RP}^2) \cong D^2 \tilde{\times} \mathbb{RP}^2$ of $\mathbb{RP}^2$ (see [3, Section 0] and [24, Section 2.5]) and $\mathbb{RP}^4$ is the result of attaching a 3-handle and a 4-handle to this handlebody. Recall that 3-handles and 4-handles need not be drawn by [38] and [110], see [24, Example 7]. The union of handles up to index 2 can be seen as a Lefschetz fibration over the 2-disk with the Möbius band Mb as fiber and a single vanishing cycle c_1, given by the projection of the $(1,0)$-framed 2-handle, see Figure 5.2.

A Pin^+-structure on Mb extending to $\mathbb{RP}^4$ is given by the quadratic enhancement

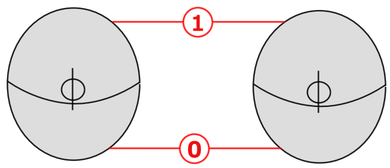

Figure 5.1: A Kirby diagram of $\mathbb{RP}^4$.

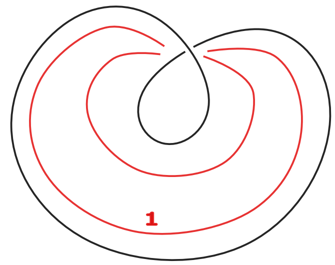

Figure 5.2: The Lefschetz fibration associated to the 2-handlebody of $\mathbb{RP}^4$.

$$q^+ \colon H_1(\mathrm{Mb};\mathbb{Z}_4) \longrightarrow \mathbb{Z}_2 \tag{5.4.1}$$

defined by the condition $q^+([e_1]) = 1$, where $[e_1]$ is the generator of $H_1(\mathrm{Mb};\mathbb{Z}_4) \cong \mathbb{Z}_4$ represented by the core of the band. Indeed, $[c_1] = 2[e_1]$ and hence

$$q^+([c_1]) = q^+(2[e_1]) = q^+([e_1]) + q^+([e_1]) + e_1^2 = 1 \in \mathbb{Z}_2.$$

The other Pin^+-structure

$$q_x^+(y) = q^+(y) + x \cdot y \quad \text{for all } y \in H_1(\mathrm{Mb};\mathbb{Z}_4)$$

is obtained by acting on q^+ by the cohomology class dual to $[e_1]$, where $x \cdot y$ indicates the intersection number between x and the $\mathbb{Z}_2$-reduction of y. Note that the condition $q_x^+([c_1]) = 1$ is still satisfied.

On the other hand, if $\mathbb{RP}^4$ supported a Pin^--structure, then there would be a quadratic enhancement

$$q^- \colon H_1(\mathrm{Mb};\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

satisfying $q^-([c_1]) = 2$. But this is impossible, since it would imply that

$$2 = q^-([c_1]) = q^-(2[e_1]) = q^-(0) = 0 \in \mathbb{Z}_4.$$

Example 5.4.2 ($\mathbb{RP}^2 \mathbin{\widetilde{\times}} S^2$)**.** The nonorientable 4-manifold $\mathbb{RP}^2 \mathbin{\widetilde{\times}} S^2$ is obtained as a quotient of $S^2 \times S^2$ under the orientation-reversing involution

$$\begin{aligned} \iota \colon S^2 \times S^2 &\longrightarrow S^2 \times S^2 \\ (x,y) &\mapsto (-x, \rho_\pi(y)) \end{aligned}$$

where $\rho_\theta \colon S^2 \to S^2$ is the rotation of S^2 of angle θ about a fixed axis for any $\theta \in S^1$, see [84, Chapter 12]. Moreover, it can also be seen as the result of a Gluck twist on the product $\mathbb{RP}^2 \times S^2$ along a 2-sphere fiber, see [24, Lemma 1]. In particular, there is a decomposition

$$\mathbb{RP}^2 \mathbin{\widetilde{\times}} S^2 = (D^2 \times S^2) \cup_\phi (\mathrm{Mb} \times S^2)$$

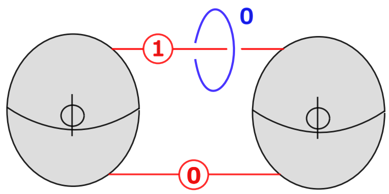

Figure 5.3: A Kirby diagram of $\mathbb{RP}^2 \tilde{\times} S^2$.

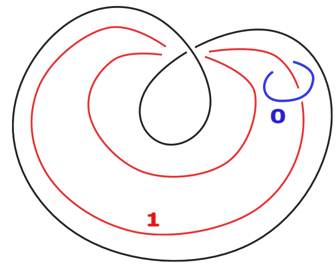

Figure 5.4: Attaching circles of the 2-handles of $\mathbb{RP}^2 \tilde{\times} S^2$ onto the page of the trivial Lefschetz fibration given by the union of handles up to index one.

where the gluing map is

$$\begin{aligned} \phi\colon S^1 \times S^2 &\longrightarrow S^1 \times S^2 \\ (\theta, x) &\mapsto (\theta, \rho_\theta(x)). \end{aligned}$$

A Kirby diagram of this manifold is given in Figure 5.3, see [3, Section 0], [24, Figure 1], [110] and [147, Section 4.1]. It is obtained by considering $S^2 \tilde{\times} \mathbb{RP}^2$ as the double of $D^2 \tilde{\times} \mathbb{RP}^2$. Figure 5.4 represents the projection of the attaching circles of the 2-handles onto the page of the trivial Lefschetz fibration given by the union of handles up to index one. In order to endow the 2-handlebody of $S^2 \tilde{\times} \mathbb{RP}^2$ with the structure of a Lefschetz fibration over D^2, we need to stabilize the page and add vanishing cycles as in Figure 5.5, so that the attaching spheres of the 2-handles can be isotoped into different pages and have the right framing. This is Harer's trick and is explained in the nonorientable setting in [111, Proof of Theorem 1.1], see also [81] and [48, Theorem 2.1]. At the level of Kirby diagrams, this corresponds to adding canceling pairs of 1 and 2-handles.

In Figure 5.6 we fix a set of simple closed loops $e_1, \ldots, e_6$ inducing a homology basis of the page Σ of this new fibration. The quadratic enhancement

$$q^+\colon H_1(\Sigma; \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

defined by the condition $q^+([e_i]) = 1$ for every $i = 1, \ldots, 6$ has the property that $q^+([c_i]) = 1$ for all the vanishing cycles $c_1, \ldots, c_7$ and hence it determines a Pin^+-structure on $\mathbb{RP}^2 \tilde{\times} S^2$. The other Pin^+-structure can be found by acting on q^+ with $[e_1] \in H_1(\Sigma; \mathbb{Z}_2)$.

On the other hand, $\mathbb{RP}^2 \tilde{\times} S^2$ does not support a Pin^--structure. Indeed, if this was the case, we could find a quadratic enhancement

$$q^-\colon H_1(\Sigma; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

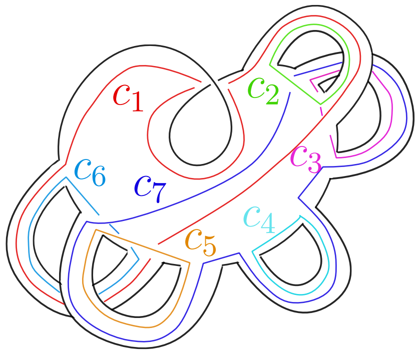

Figure 5.5: The 2-handlebody of $\mathbb{RP}^2 \widetilde{\times} S^2$ as a Lefschetz fibration over D^2.

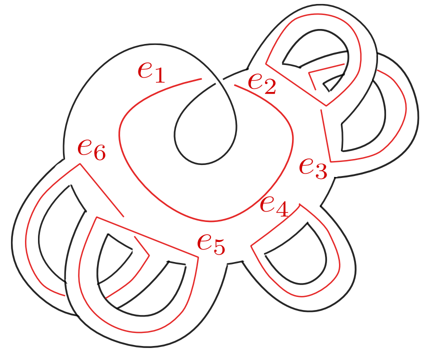

Figure 5.6: Curves representing a basis of $H_1(\Sigma)$.

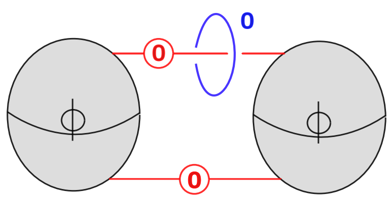

Figure 5.7: A Kirby diagram of $\mathbb{RP}^2 \times S^2$.

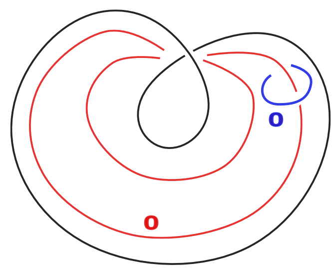

Figure 5.8: Attaching circles of the 2-handles of $\mathbb{RP}^2 \times S^2$ onto the page of the trivial Lefschetz fibration given by the union of handles up to index one.

such that $q^-([e_i]) = 2$ for all $i = 1, \ldots, 7$ and this would imply that

$$2 = q^-([c_1]) = q^-([e_2 + e_6]) = q^-([e_2]) + q^-([e_6]) = q^-([c_2]) + q^-([c_6]) = 2 + 2 = 0 \in \mathbb{Z}_4$$

a contradiction.

Example 5.4.3 ($\mathbb{RP}^2 \times S^2$)**.** A Kirby diagram of the product 4-manifold $\mathbb{RP}^2 \times S^2$ is given in Figure 5.7, see [3, Section 0], [24, Figure 1], [110] and [147, Section 4.1]. It is obtained by considering $\mathbb{RP}^2 \times S^2$ as the double of $\mathbb{RP}^2 \times D^2$. Figure 5.8 shows the projection of the attaching circles of the 2-handles onto the page of the trivial Lefschetz fibration given by the union of handles up to index one.

We use Harer's trick again (see [111, Proof of Theorem 1.1], [81] and [48, Theorem 2.1]) and show that the union of the handles up to index 2 in the fixed decomposition of

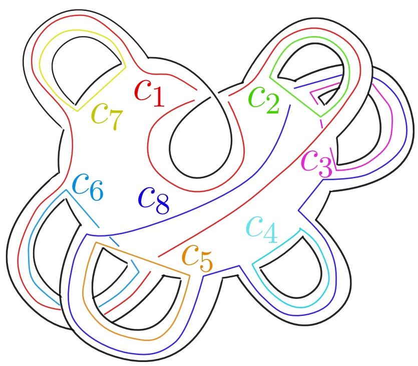

Figure 5.9: The 2-handlebody of $\mathbb{RP}^2 \times S^2$ as a Lefschetz fibration over D^2.

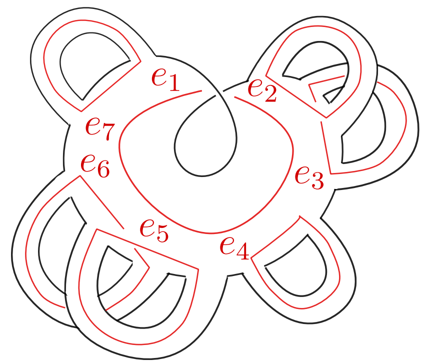

Figure 5.10: Curves representing a basis of $H_1(\Sigma)$.

$\mathbb{RP}^2 \times S^2$ is given by the surface Σ and by the vanishing cycles $c_1, \ldots, c_8$ in Figure 5.9. Note that, with respect to the case of $\mathbb{RP}^2 \widetilde{\times} S^2$, we need to stabilize the page one additional time in order to adjust the framing of the red colored 2-handle.

In Figure 5.10 we again fix a set of loops $e_1, \ldots, e_7 \subseteq \Sigma$ inducing a first homology basis of the page.

The quadratic enhancement

$$q^- \colon H_1(\Sigma; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

defined by the condition $q^-([e_i]) = 2$ for every $i = 1, \ldots, 7$ has the property that $q^-([c_i]) = 2$ for all the vanishing cycles $c_1, \ldots, c_8$ and hence it determines a Pin^--structure on $S^2 \times \mathbb{RP}^2$. The other Pin^--structure can be found by acting on q^- with $[e_1] \in H_1(\Sigma; \mathbb{Z}_2)$.

On the other hand, $\mathbb{RP}^2 \times S^2$ can not support a Pin^+-structure. Indeed, if this was the case we could find a quadratic enhancement

$$q^+ \colon H_1(\Sigma; \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2$$

such that $q^+([c_i]) = 1$ for all $i = 1, \ldots, 8$ and this would imply that

$$\begin{aligned} 1 = q^+([c_1]) = q^+([2e_1 + e_2 + e_6 + e_7]) = \\ 2q^+([e_1]) + e_1^2 + q^+([e_2]) + q^+([e_6]) + q^+([e_7]) = \\ 1 + q^+([c_2]) + q^+([c_6]) + q^+([c_7]) = 1 + 1 + 1 + 1 = 0 \in \mathbb{Z}_2 \end{aligned}$$

a contradiction.

5.5 Pin-structures on Lefschetz fibrations over the 2-sphere

In this section, we sketch a proof of Theorem K.

Proof of Theorem K. The proof is a $\mathbb{Z}_2$-coefficients version of the one of [133, Theorem 1.3] using the two formulas (1.1.28) and (1.1.27) in Section 1.1.4. We hence leave it to the interested reader as an exercise. □

Remark 5.5.1. One can check whether $X \setminus \operatorname{Int} \nu(\Sigma)$ supports a $\mathrm{Pin}^{\pm}$-structure by applying Theorem H and Theorem I respectively.

5.6 Pin-structures on 3-manifolds

The following well-known fact can be easily proven by adapting to the Pin^- case the one of [133, Theorem 1.4], see [99].

Theorem 5.6.1. *Any closed 3-manifold M admits a Pin^--structure.*

Sketch of Proof. We reduce to the case in which M is nonorientable, since if M is orientable then it is automatically Spin and hence also Pin^+ and Pin^-. The idea is to write M as the union of a nonorientable 3-dimensional handlebody H of genus g with g 2-handles and a single 3-handle. Then [99, Corollary 1.12] implies that M admits a Pin^--structure if and only if the closed surface ∂H supports a Pin^--structure which restricts to the one bounding the 2-disk on all the attaching spheres of the 2-handles and on all the belt spheres of the 1-handles. Such a Pin^--structure can always be constructed by means of a quadratic enhancement $q^- \colon H_1(\partial H; \mathbb{Z}_2) \to \mathbb{Z}_4$ which vanishes on all such curves. The proof follows the same lines of the one of [133, Theorem 1.4] and we leave the details to the interested reader. □

On the other hand, it is not true that any closed 3-manifold admits a Pin^+-structure, since the condition $w_2(M) = 0$ is not always satisfied in the nonorientable setting. $\mathbb{RP}^2 \times S^1 \# N$ for any 3-manifold N is such an example. However, we now show that it is still possible to check this condition by considering a handlebody decomposition of M.

Theorem 5.6.2. *Let $M = H \cup H'$ be a closed nonorientable 3-manifold obtained by adding g 2-handles and a 3-handle to a genus-g nonorientable handlebody H and let $q_0^+ \colon H_1(\partial H; \mathbb{Z}_4) \to \mathbb{Z}_2$ be a quadratic enhancement corresponding to a Pin^+ structure on ∂H. Let $a_1, \dots, a_g \in H_1(\partial H; \mathbb{Z}_4)$ and $b_1, \dots, b_g \in H_1(\partial H; \mathbb{Z}_4)$ be the homology classes of the attaching circles of the 2-handles and of the belt circles of the 1-handles of H respectively and set*

$$q_0^+([a_j]) = \alpha_j \quad \text{and} \quad q_0^+([b_j]) = \beta_j$$

for $j = 1, \dots, g$. If for a fixed basis $e_1, \dots, e_{2g}$ of $H_1(\partial H; \mathbb{Z}_2)$ we have

$$\tilde{a}_j = \sum_{i=1}^{2g} a_{j,i} e_i \quad \text{and} \quad \tilde{b}_j = \sum_{i=1}^{2g} b_{j,i} e_i$$

for $j = 1, \dots, g$, where $\tilde{a}_j$ and $\tilde{b}_j$ are the $\mathbb{Z}_2$-reduction of a_j and b_j respectively, then M has a Pin$^+$-structure if and only if

$$\mathit{rank}(C|A) = \mathit{rank}(C)$$

where C and A are respectively the $2g \times 2g$ matrix and the column vector

$$C = \begin{pmatrix} a_{1,1} & \dots & a_{1,2g} \\ \vdots & \ddots & \vdots \\ a_{g,1} & \dots & a_{g,2g} \\ b_{1,1} & \dots & b_{1,2g} \\ \vdots & \ddots & \vdots \\ b_{2g,1} & \dots & b_{g,2g} \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \\ \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}. \tag{5.6.1}$$

Proof. There is a Pin$^+$-structure on M if and only if the nonorientable closed surface ∂H has a Pin$^+$-structure which extends to the attaching circles of the 2-handles and to the belt spheres of the 1-handles in H. In particular, this is equivalent to check whether or not there exists a map

$$q^+ \colon H_1(\partial H; \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2 \tag{5.6.2}$$

satisfying the condition $q^+(x+y) = q^+(x) + q^+(y) + x \cdot y$ for all $x, y \in H_1(\partial H; \mathbb{Z}_4)$ and which vanishes on $a_1, \dots, a_g$ and on $b_1, \dots, b_g$.

If we fix a quadratic enhancement

$$q_0^+ \colon H_1(\partial H; \mathbb{Z}_4) \longrightarrow \mathbb{Z}_2 \tag{5.6.3}$$

corresponding to a Pin$^+$-structure on ∂H, then all the other Pin$^+$-structures on ∂H will be of the form $q^+ = q_0^+ + l$ for some linear map $l \in \mathrm{Hom}(H_1(\partial H; \mathbb{Z}_4), \mathbb{Z}_2)$. It follows that M supports a Pin$^+$-structure if and only if there is $l = \sum_{i=1}^{2g} x_i e^i \in \mathrm{Hom}(H_1(\partial H; \mathbb{Z}_4), \mathbb{Z}_2)$ such that

$$\sum_{i=1}^{2g} a_{j,i} x_i = \alpha_j \quad \text{and} \quad \sum_{i=1}^{2g} b_{j,i} x_i = \beta_j$$

in $\mathbb{Z}_2$ for $j = 1, \dots, g$, where $e^1, \dots, e^{2g}$ is the dual basis to $e_1, \dots, e_{2g} \in H_1(\partial H; \mathbb{Z}_2)$. The conclusion follows from Rouché–Capelli's theorem.

□

5.7 Interpretation of Pin$^+$-structures à la Milnor

Kirby and Taylor showed in [99] that the sets of Pin$^+$ and Pin$^-$-structures on a fixed vector bundle ξ are in one to one correspondence with Spin-structures on $\xi \oplus 3 \cdot \det(\xi)$ and

$\xi \oplus \det(\xi)$ respectively. In this section, we remark that for Pin^+-structures one can get another equivalent characterization, which recalls the following one of Spin-structures by Milnor.

Theorem 5.7.1 (Milnor [112]). *There is a canonical bijection between the set of* Spin*-structures on a real rank-k vector bundle $\xi\colon E \to M$ and the set of homotopy classes of trivializations of $\xi|_{M^1}$ extending to trivializations of $\xi|_{M^2}$, where M^i denotes the i^{th} skeleton of M. Moreover, such correspondence is equivariant with respect to the action of $H^1(M;\mathbb{Z}_2)$.*

Recall that $x \in H^1(M;\mathbb{Z}_2)$ acts on a trivialization of $\xi|_{M^1}$ by flipping the framing of any loop $\gamma \subseteq M^1$ such that $\langle x, [\gamma]\rangle = 1 \in \mathbb{Z}_2$.

In a similar fashion, we prove the following result. We remark that this follows essentially from the discussion in [99].

Theorem 5.7.2. *Let $\xi\colon E \to M$ be a real rank-k vector bundle. There is a canonical bijection between the set of Pin^+-structures on ξ and the set of homotopy classes of $(k-1)$-tuples of everywhere linearly independent sections of $\xi|_{M^1}$ extending to $\xi|_{M^2}$, where M^i denotes the i^{th} skeleton of M. Moreover, such correspondence is equivariant with respect to the action of $H^1(M;\mathbb{Z}_2)$.*

We remark that the definition of the action of $H^1(M;\mathbb{Z}_2)$ on the set of homotopy classes of linearly independent sections of $\xi|_{M^1}$ is completely analogous to the one described after the statement of Theorem 5.7.1.

Proof. The vanishing of $w_2(\xi)$ is a necessary and sufficient condition for the existence of both a $(k-1)$-tuple of linearly independent sections of $\xi|_{M^2}$ and a Pin^+-structure on ξ. We now suppose that $w_2(\xi)$ is trivial and we endow $\det(\xi)$ with its canonical Pin^+-structure, see [99, Addendum to 1.2]. Let $v_1, \dots, v_{k-1}$ be a $(k-1)$-tuple of linearly independent sections of $\xi|_{M^1}$ extending to $\xi|_{M^2}$. We are now going to associate to such a $(k-1)$-tuple a Pin^+-structure on $\xi|_{M^2}$. This will uniquely determine a Pin^+-structure on ξ, as in the case of Spin-structures. Using $v_1, \dots, v_{k-1}$ one can define a vector bundle isomorphism

$$\xi|_{M^2} \cong \epsilon^{k-1} \oplus \det(\xi) \tag{5.7.1}$$

where ϵ^{k-1} denotes the trivial rank-$(k-1)$ real vector bundle. The associated Pin^+-structure on $\xi|_{M^2}$ is given by the pull-back via (5.7.1) of the Pin^+-structure on $\epsilon^{k-1} \oplus \det(\xi)$ given by the direct sum of the trivial Spin-structure on ϵ^{k-1} with the fixed Pin^+-structure on $\det(\xi)$. It is now easy to verify that this correspondence satisfies all the desired properties. □

Chapter 6

Appendix

Dold–Whitney's parallelizability of 4-manifolds

Stiefel's Parallelizability Theorem asserts that any closed orientable 3-manifold has trivial tangent bundle. In addition to the original proof [132], one can find several elementary proofs of this fact, see for example Bais and Zuddas [26], Benedetti and Lisca [33], Durst, Geiges, Gonzalo and Kegel [47], Gonzalo [74], Kirby [97, Chapter VII], Geiges [63, Section 4.2], Fomenko and Matveev [57, Section 9.4] and Whitehead [153]. Motivated by this, one could ask whether it is possible to provide a simple proof of the following result.

Theorem L (Dold–Whitney)**.** *A closed orientable* 4*-manifold* X *is parallelizable if and only if its second Stiefel-Whitney class* $w_2(X)$*, first Pontryagin class* $p_1(X)$ *and Euler characteristic* $\chi(X)$ *are all vanishing.*

Recall that an n-manifold X is *parallelizable* if its tangent bundle TX is trivial, i.e., if there is an isomorphism of vector bundles $TX \cong X \times \mathbb{R}^n$. One can easily verify that this is equivalent to ask for the existence of n vector fields on X which are everywhere linearly independent. Such n-tuple is usually called a *framing*.

Theorem L is due to Dold and Whitney's classification of $SO(4)$-bundles over closed, orientable 4-manifolds by means of their second Stiefel-Whitney class, first Pontryagin class and Euler characteristics, see [42]. A proof of this result can be also found in [145]. The argument we present in this appendix is elementary and builds up on the discussion in Kirby's book [97, Chapter VI]. It is worth to remark that parellizable 4-manifolds are still not fully understood, for example it is still unknown which finitely presented fundamental groups can occur as the fundamental group of a closed parallelizable 4-manifold, see [90].

Moreover, we recall that a closed orientable 4-manifold supports an Engel structure if and only if its tangent bundle is trivial, as shown in [150].

We start by recalling the following argument on the classification of $SO(4)$-principal bundles over the 4-sphere. This can be found in [97, Chapter VI].

Lemma 6.0.1. *$SO(4)$-principal bundles over S^4 are determined by their Euler and first Pontryagin classes.*

Proof. Since $SO(4)$-principal bundles over S^4 are in one to one correspondence with homotopy classes of functions $f\colon S^3 \to SO(4)$, it is enough to show that the map

$$\Phi\colon \pi_3(SO(4)) \xrightarrow{(\chi,-(p_1+2\chi)/4)} \mathbb{Z} \oplus \mathbb{Z} \tag{6.0.1}$$

given by

$$f \mapsto (\chi(f), -(p_1(f) + 2\chi(f))/4)$$

is an isomorphism. Here $\chi(f)$ and $p_1(f)$ denote the Euler characteristics and the first Pontryagin class of the $SO(4)$-principal bundle associated to f respectively. Note that we already know that $\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$, since the universal covering of $SO(4)$ is $\mathrm{Spin}(4) \cong S^3 \times S^3$.

In the following, we view S^3 as the group of unit quaternions and work under the identifications

$$\mathbb{R}^3 \cong \langle i, j, k \rangle \subseteq \mathbb{H} \cong \mathbb{R}^4.$$

Let

$$\eta\colon S^3 \to SO(4) \tag{6.0.2}$$

be the map defined by

$$\eta(q)(q') = q \cdot q'$$

where $\cdot$ denotes the quaternionic multiplication. The homotopy class of this map determines the quaternionic Hopf bundle over S^4, which has Euler number $\chi(\eta) = 1$. Its first Chern class $c_1(\eta) \in H^2(S^4; \mathbb{Z}) = 0$ is necessarily trivial and this implies that the first Pontryagin class is

$$p_1(\eta) = (c_1^2 - 2c_2)(\eta) = -2.$$

We also define

$$\nu\colon S^3 \to SO(3) \subseteq SO(4) \tag{6.0.3}$$

to be the map

$$\nu(q)(q') = q \cdot q' \cdot q^{-1}$$

and we compute the Euler and first Pontryagin class of the associated $SO(4)$-bundle. This can be done by noticing that the tangent bundle τ of S^4 is determined by the clutching function $(\eta)^2 \cdot (\nu)^{-1}$, where $\cdot$ and $(\,\cdot\,)^{-1}$ denote the point-wise multiplication and inversion in the Lie group $SO(4)$ respectively, see [131, §23.6]. Since the characteristic class p_1 does not change under Whitney sums with trivial bundles, we also have that

$$p_1(\tau) = p_1(\tau \oplus \epsilon^1) = p_1(\epsilon^5) = 0.$$

Since $\chi(\tau) = 2$, this implies that $\chi(\nu) = 0$, $p_1(\nu) = -4$.

In particular, the map (6.0.1) is such that $\Phi(\eta) = (1, 0)$ and $\Phi(\nu) = (0, 1)$. Moreover, one can check that

$$\Phi(f \cdot g) = \Phi(f) + \Phi(g)$$

for every $f, g \in \pi_3(SO(4))$, where $\cdot$ denotes the point-wise multiplication in the Lie group $SO(4)$. This concludes the proof, since any surjective homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a group isomorphism. □

Remark 6.0.2. The proof of Lemma 6.0.1 also implies that, up to homotopy, the clutching functions of all possible $SO(4)$-bundles over the 4-sphere are obtained by multiplying copies of η, ν and their point-wise inverses.

We are now ready to prove Theorem L.

Proof. Let X be a closed orientable 4-manifold. It is straightforward to verify that the triviality of its tangent bundle implies the vanishing of $w_2(X)$, $p_1(X)$ and $\chi(X)$.

Conversely, fix a handlebody decomposition of X with just one 0-handle and one 4-handle. The vanishing of the second Stiefel-Whitney class implies the existence of a trivialization of the tangent bundle up to the 2-handles. Since $\pi_2(SO(4)) = 0$, one can always extend such trivialization over the 3-handles. The problem is now finding an extension over the remaining 4-handle h^4. In order to tackle this problem, we consider the change of basis map

$$f : \partial(X \setminus h^4) \cong S^3 \to SO(4)$$

between a fixed framing on $X \setminus h^4$ and the one of ∂h^4 that extends to the 4-ball. Note that, if f is homotopic to the constant map, we can find a global framing for X, which is hence parallelizable.

Consider the lift

$$\widetilde{f} = (f_1, f_2) : S^3 \to S^3 \times S^3 \cong \mathrm{Spin}(4)$$

of f to the universal covering of $SO(4)$ and let f_1, f_2 be its two components. Since these are both continuous maps from the 3-sphere to itself, their degrees $\deg(f_i)$ are defined for $i = 1, 2$. In particular, we will now show that

$$\deg(f_1) - \deg(f_2) = \chi(X) \tag{6.0.4}$$

and

$$\deg(f_1) + \deg(f_2) = -p_1(X)/2, \tag{6.0.5}$$

which will allow us to conclude our proof. Indeed, one has that f is null-homotopic if and only if the degrees of both f_1 and f_2 are trivial, which by (6.0.4) and (6.0.5) happens only in the case in which both $\chi(X)$ and $p_1(X)$ vanish.

We now show that equations 6.0.4 and 6.0.5 hold for the lifts

$$(\eta_1, \eta_2), (\nu_1, \nu_2) \colon S^3 \to S^3 \times S^3$$

of η and ν respectively. The conclusion will then follow from linearity. Indeed, it is an exercise to show that

$$\deg(a \cdot b) = \deg(a) + \deg(b)$$

for any two continuous maps $a, b \colon S^3 \to S^3$ and, up to homotopy, every continuous function from the 3-sphere to $SO(4)$ can be expressed as a composition of copies of η, ν and the functions given by their point-wise quaternionic inverses.

By definition, ν lifts to a map which is the identity of S^3 on both components and hence

$$\deg(\nu_1) - \deg(\nu_2) = 1 - 1 = 0 = \chi(\nu)$$

and

$$\deg(\nu_1) + \deg(\nu_2) = 1 + 1 = 2 = -p_1(\nu)/2.$$

On the other hand, η lifts to a map which is the identity on the first component and which is constantly equal to the unit quaternion on the second one. In particular,

$$\deg(\eta_1) - \deg(\eta_2) = 1 - 0 = 1 = \chi(\eta)$$

and

$$\deg(\eta_1) + \deg(\eta_2) = 1 + 0 = 1 = -p_1(\eta)/2$$

which concludes the argument. □

Remark 6.0.3. One can also prove equation (6.0.4) by noticing that

$$\chi(X) = \deg(f_1 \cdot f_2^{-1}). \tag{6.0.6}$$

Indeed, recall that the double covering map

$$\pi \colon S^3 \times S^3 \to SO(3)$$

is defined by

$$\pi(q_1, q_2)(q) = q_1 \cdot q \cdot q_2^{-1}.$$

This implies that the map

$$f_1 \cdot f_2^{-1} \colon S^3 \to S^3$$

is recording the coordinates of a non-vanishing vector field on $X \setminus h^4$ with respect to the standard framing of the unit 4-ball h^4. The conclusion then follows from Poincaré-Hopf's theorem (see e.g. [113, Chapter 6]) and from the fact that

$$\deg(f_1 \cdot f_2^{-1}) = \deg(f_1) - \deg(f_2).$$

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