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Some realization problems in Smooth Ergodic Theory

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Chapter 1

Introduction

1.1 Background and results.

A realization problem in dynamical systems is asking whether there is always a system (with a given regularity and structure) which satisfies some ergodic properties, for example: does every manifold admit an ergodic diffeomorphism? The goal of this thesis is to study some smooth realization problems for uniformly expanding maps on the circle with a given regularity. Uniformly expanding maps lie within a broader class of dynamical systems which exhibit geometric properties which give rise to chaotic behaviour. The study of the statistical properties of such systems in there modern context traces back to Henri Poincaré with the qualitative study of ordinary differential equations which uses probabilistic approach to understand the behaviour of a typical orbit of the system rather than having a full description of orbits point wise.

Many results are already established in this direction and in particular it is known that regular enough uniformly expanding maps (say of class $C^{1+\alpha},\ \alpha > 0$) preserve a unique absolutely continuous (with respect to Lebesgue) probability measure which is also exponentially mixing. The tools involved in the proofs of these properties limit however the results for maps with lower regularity and in fact results known about such maps are more in the negative direction rather than the positive direction.

In this thesis we will prove three results which already have been published or accepted for publication, where we study some properties of uniformly expanding circle maps with low regularity.

In Chapter one the main goal is to see whether the techniques used in the study of the maps of higher regularities are somewhat intrinsic to the ergodic properties rather than the map itself, that is, whether in the class of uniformly expanding maps proving for example ergodicity and invariance of an acip is equivalent to proving that the classical techniques hold. It turns out that answering this question boils down to study the relationship between the property of bounded distortion and ergodicity as the other parts of the argument used for proving existence and ergodicity of an acip are known to hold for all uniformly expanding maps regardless of regularity, Our result in this direction proves that bounded distortion is not necessary, in fact it is generically not the case in the C^1 class. This result has been published in [12]: *H. Ounesli. C^1 genericity of unbounded distortion for Lebesgue preserving uniformly expanding circle maps. Indag. Math, volume 35(3), pages:523-530*

In Chapter two we study the properties of the space of Lebesgue-preserving C^1 uniformly expanding circle maps. Our main goal is to prove that this space is locally path-connected with respect to the C^1 topology and moreover the connected components are determined by the degree of the maps considered. We also establish the homotopy type of the space and prove that its fundamental group is infinite-cyclic. This is curious from dynamics point of view as it suggests that even though C^1 uniformly expanding maps preserving Lebesgue can be deformed to each other, these deformations are not necessarily equivalent. This result has been published in [8]: *H. Boukhecham and H. Ounesli. Topology of the space of measure-preserving transformations of the circle. Rend. Mat. Univ. Trieste, Volume 55, 2023*

Finally, in Chapter three we study the existence of an acip for arbitrarily low regular C^1 uniformly expanding maps We mean here by low regularity, a map whose derivative has a canonical modulus of continuity which is not Dini-integrable. We prove that for any given modulus of continuity there exists a C^1 uniformly expanding map on the circle where the derivative has a modulus of continuity equivalent to the given modulus and yet admits an acip. we also prove that in that case we can both construct a map which preserves an acip with density as regular as the derivative and one that preserves exactly the Lebesgue measure. This result has been accepted for publications [13]: *H. Ounesli. On the existence of absolutely continuous invariant probability measures for C^1 expanding maps. Journal of Dynamical and Control Systems (to appear, 2024).*

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Chapter 2

Genericity of unbounded distortion.

This result has been published in [12]: *H. Ounesli. C^1 generecity of unbounded distortion for Lebesgue presving uniformly expanding circle maps. Indag. Math, volume 35(3), pages:523-530.*

2.1 Introduction and statement of results.

Let $E^1(\mathbb{S}^1)$ be the space of C^1 orientation-preserving uniformly expanding maps on the circle, i.e. C^1 maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ for which there exists some uniform constant $\sigma > 1$ such that $f'(x) \geq \sigma$. We let λ denotes Lebesgue measure on $\mathbb{S}^1$ and recall that a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is said to preserve Lebesgue measure if $\lambda(f^{-1}(A)) = \lambda(A)$ for any measurable set $A \subseteq \mathbb{S}^1$. We let

$$\Gamma_\lambda(\mathbb{S}^1) \coloneqq \{f \in E^1(\mathbb{S}^1) : f \text{ preserves Lebesgue measure}\}.$$

The simplest and well known examples of maps in $\Gamma_\lambda(\mathbb{S}^1)$ are the maps of the form $f(x) = \kappa x \bmod 1$, for $\kappa \in \mathbb{N}, \kappa \geq 2$. These maps are locally affine and are therefore easily seen to preserve Lebesgue measure λ. Although it is not immediately intuitive, it turns out there are many other maps in $E^1(\mathbb{S}^1)$ which also preserve Lebesgue measure without being piecewise affine, for instance, in our previous work [13] and [8] we have proved that the space of Lebesgue preserving uniformly expanding maps is a locally connected space, and if we consider only degree 2 maps then we showed that this space is homeomorphic to an infinite dimensional Lie group, later on in the paper we will prove a general result which makes explicit the remarkable flexibility in their construction. On the other hand, a more challenging property to prove for a given map in $E^1(\mathbb{S}^1)$ is ergodicity of Lebesgue measure, recall that λ is ergodic if and only if any measurable set $A \subset \mathbb{S}^1$ which is invariant by f has either 0 measure or full measure (A set A is invariant if $f^{-1}A = A$). Remarkably, there are uniformly expanding maps for which Lebesgue measure is invariant but not ergodic [16] and therefore

$$\Gamma_{\lambda,er}(\mathbb{S}^1) = \{f \in \Gamma_\lambda(\mathbb{S}^1) : \lambda \text{ is ergodic}\} \text{ is strictly contained in } \Gamma_\lambda(\mathbb{S}^1).$$

but we know that such examples are rare since by [14] the space $\Gamma_{\lambda,er}(\mathbb{S}^1)$ is residual in $\Gamma_\lambda(\mathbb{S}^1)$. On the positive side, it has been shown that these maps must have considerable low regularity, for example, a well known result dating back to the 1950s states that any $C^{1+\alpha}$ uniformly expanding map has a unique absolutely continuous invariant measure equivalent to Lebesgue and hence is ergodic with respect to Lebesgue. The main technique used in the prove of ergodicity in such settings is the property of having bounded distortion. There are many refernces that provides a prove of the result, look for instance at [1]. We recall that every $f \in E^1(\mathbb{S}^1)$ admits a family of partitions $\mathcal{P}_n := \{\omega_{n,i}\}$ which are injectivity domains of f^n.

Definition 2.1. We say that f has *bounded distortion* if

$$\mathcal{D} := \sup_{n\geq 1} \sup_{\omega_{n,i}\in\mathcal{P}_n} \sup_{x,y\in\omega_{n,i}} \log \frac{|(f^n)'(x)|}{|(f^n)'(y)|} < \infty.$$

In a sense bounded distortion is a way of saying how much a map is close to a locally affine one, interestingly enough, although there are maps for which Lebesgue measure is invariant and ergodic and at the same time of arbitrarily low regularity, the prove is quite non constructive and one can not check if the bounded distortion property is satisfied for most cases where the regularity is lower than $C^{1+\alpha}$ which leads us to ask the following question: *Is bounded distortion necessary for Lebesgue measure to be ergodic?*. We will give a negative answer to this question, in fact, we have:

Theorem 1. *The set of maps for which Lebesgue measure is invariant and ergodic with unbounded distortion is a residual subset in $\Gamma_{\lambda,er}(\mathbb{S}^1)$ with respect to the C^1-topology.*

This results suggest that other techniques, which do not rely on bounded distortion, need to be developed to prove the ergodicity of Lebesgue measure in specific classes of maps.

2.2 Overview of the proof

We first introduce some notation. For any $n \geq 2$, let $\{I_i\}_{1\leq i\leq n}$ be a partition of the unit interval into non-trivial adjacent closed subintervals $I_i = [x_i^-, x_i^+]$ which intersect only at the endpoints, so that $x_0^- = 0$ and $x_n^+ = 1$. Let $1 \leq i_0 \leq n$ and suppose that for each $i \neq i_0$ there is given a C^1 expanding diffeomorphism $f_i : I_i \to [0,1]$. We are interested in whether we can construct a C^1 expanding diffeomorphism $f_{i_0} : I_{i_0} \to [0,1]$ which defines a piecewise C^1 full branch map preserving Lebesgue measure, and further more whether this can actually be constructed in such a way as to represent a a C^1 uniformly expanding circle map. The following result give some natural and explicitly verifiable necessary and sufficient conditions for this to be the case.

Proposition 2 (Missing Branch Extension Proposition)**.** *There exists a (unique) extension to a Lebesgue-preserving uniformly expanding full branch map if and only if for all $x \in [0,1]$ we have:*

$$\sum_{\substack{1\leq i\leq n\\ i\neq i_0}} \frac{1}{f_i' \circ f_i^{-1}(x)} < 1 \tag{2.1}$$

Moreover, this extension represents a C^1 uniformly expanding circle map of degree n if and only if

$$f_{i-1}'(x_{i-1}^+) = f_i'(x_i^-) \quad \text{and} \quad f_{i+1}'(x_{i+1}^-) = f_i'(x_i^+) \tag{2.2}$$

for all $i \neq i_0$ (if $i = 0$ we replace $i-1$ by n in the first equality, and if $i = n$ we replace $i+1$ by 1 in the second inequality), and

$$f_{i_0-1}'(x_{i_0-1}^+) = \frac{1}{1 - \sum_{i\neq i_0} f_i'(x_i^-)} \tag{2.3}$$

and

$$f_{i_0+1}'(x_{i_0+1}^-) = \frac{1}{1 - \sum_{i\neq i_0} f_i'(x_i^+)} \tag{2.4}$$

(where also, if $i_0 = 1$ we replace $i_0 - 1$ by n in the left hand side of (2.3) *and if $i_0 = n$ we replace $i_0 + 1$ by 1 in the left hand side of* (2.4)*).*

Remark 2.2.1. We remark that (2.3) is a very mild and very "open" condition that allows a huge amount of flexibility in the choice of the given branches, thus indicating that the space of such Lebesgue preserving maps is very large. Condition (2.2) gives essentially trivial matching conditions on the derivatives at the left and right endpoints of the domains of each branch which ensure that the map is C^1 when considered as a circle map. Conditions (2.3) and (2.4) are less intuitively immediate but essentially ensure that the unique extension given in the first part of the proposition also satisfies such matching conditions.

In Section 2.3 we prove the Missing Branch Extension Proposition 2 which is the key ingredient in the proof of the following result. Consider the set $\Gamma_\lambda(\mathbb{S}^1)$ of uniformly expanding C^1 circle maps which preserve Lebesgue measure but for which Lebesgue measure is not necessarily ergodic. Letting

$$\mathcal{B} = \{f \in \Gamma_\lambda(\mathbb{S}^1) : f \text{ has bounded distortion}\} \quad \text{and} \quad \mathcal{B}^c := \Gamma_\lambda(\mathbb{S}^1) \setminus \mathcal{B}$$

we will prove the following in Section 2.4.

Proposition 3. *$\mathcal{B}^c$ is C^1 dense in $\Gamma_\lambda(\mathbb{S}^1)$.*

Then, in Section 2.5 we will prove the following.

Proposition 4. *The space $\Gamma_\lambda(\mathbb{S}^1)$ is C^1 residual in its completion $\Gamma^\star_\lambda(\mathbb{S}^1)$.*

Now we will give a proof of the Theorem assuming the previous propositions.

Proof of Theorem 1. Let us denote by d_k the map:

$$d_k : \Gamma_\lambda(\mathbb{S}^1) \to \mathbb{R}_+ \tag{2.5}$$

defined by

$$d_k(f) = \sup_{\substack{x,y \in \omega_i^k \\ 1 \le i \le deg(f)^k}} |\log \frac{f^k(x)}{f^k(y)}|.$$

For every $k \in \mathbb{N}$, d_k is continuous in the C^1 topology, to see that, first notice that for $\epsilon > 0$ small enough and $f, g \in \Gamma_\lambda(\mathbb{S}^1)$ such that $d(f,g) \le \epsilon$, where d denotes the C^1-distance, then $deg(f) = deg(g)$. On the other hand k being fixed, $d(f^k, g^k) \le C_k \epsilon$ where C_k is a positive constant depending only on k. This two remarks are enough to conclude the continuity of d_k. Now notice that $\mathcal{B}$ is equal to the following set:

$$\bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k \ge m} D_{n,k} \tag{2.6}$$

where $D_{n,k}$ is the set of elements of $\Gamma_\lambda(\mathbb{S}^1)$ whose distortion at level k is less or equal to n, in more precise terms $D_{n,k} = d_k^{-1}([0,n])$. This sets are closed in the C^1-topology since for every $k \in \mathbb{N}$ the map d_k is continuous and so we conclude also that the sets is $\bigcap_{k \geq m} D_{n,k}$ are closed, hence $\mathcal{B}$ is an F_σ set and so $\mathcal{B}^c$ is a G_δ set, by proposition 3 we conclude it is residual. Now by [14] we know that $\Gamma_{\lambda,er}(\mathbb{S}^1)$ is residual in $\Gamma_\lambda(\mathbb{S}^1)$ and by proposition 4 we conclude that the intersection $\mathcal{B}^c \cap \Gamma_{\lambda,er}(\mathbb{S}^1)$ is residual in $\Gamma_{\lambda,er}(\mathbb{S}^1)$. This finishes the proof of the theorem. □

2.3 Proof of Proposition 2

Proof. We will start by recalling one of the classical tools to show that a measure is invariant. Let $f \in E^1(S^1)$ and, for all $h \in L^1_\lambda(S^1)$ and $\mu_h := h \cdot \lambda$, we define the transfer operator associated to f and acting on $L^1_\lambda(S^1)$ as

$$Ph = \frac{d\big(f_*\mu_h\big)}{d\lambda}. \tag{2.7}$$

This operator can be interpreted as the density of the push-forward of measures in respect to Lebesgue. It is well known that the fixed points of P corresponds to the densities of f-invariant measures and that the transfer operator for maps of degree n has an explicit formula given by

$$Ph(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)}. \tag{2.8}$$

Let us now consider f to be an expanding circle map of degree n, represented as a full branch map of the unit interval with n branches $\{f_i\}_{1 \leq i \leq n}$ defined on adjacent intervals $\{I_i = [x_i^-, x_i^+]\}_{1 \leq i \leq n}$. Let $h : [0,1] \to \mathbb{R}_+$ be an L^1_λ function. h is the density of an f-invariant measure if and only if the relation 2.8 is satisfied, which can be written as:

$$h(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)} = \sum_{1 \leq y_i \leq n} \frac{h(y_i)}{f'(y_i)}. \tag{2.9}$$

where each y_i represents the pre-image of x by the i-th branch of f. Now take h to be identically equal to 1 i,e the density of Lebesgue measure, and suppose that $n-1$ branches are known, we want to show we can construct the missing n-th branch such that the resulting map is a circle expanding map preserving Lebesgue measure. Equation 2.9 is equivalent to

$$\frac{1}{f'(y_{i_0})} = 1 - \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \frac{1}{f'(y_i)}, \tag{2.10}$$

where i_0 is the index of the missing branch. Notice that $x_i = f_i^{-1}(x)$, equation 2.10 then becomes:

$$\frac{1}{f'(f_{i_0}^{-1}(x))} = 1 - \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \frac{1}{f'(f_i^{-1}(x))}, \tag{2.11}$$

If condition 2.1 holds then we obtain the following first order ODE:

$$f_{i_0}'(x) = \frac{1}{1 - \sum_{\substack{1 \le i \le n \\ i \ne i_0}} \frac{1}{f'(f_i^{-1}(f_{i_0(x)}))}}, \tag{2.12}$$

this is a continuous first order ODE defined on a compact rectangular domain, by Peano's existence theorem, there must exists a maximal solution defined on I_{i_0} with the initial condition that $f_{i_0}(x_{i_0}^-) = 0$.

We will show that this solution is an expanding diffeomorphism of I_{i_0} onto the interval $[0, 1]$ and that along the other fixed branches it defines a circle map, we will also show the solution is unique using the dynamics since Peano's existence theorem fails to ensure existence under only a continuity assumption.

First, notice that by (2.1) we get $f_{i_0}' > 1$, hence it remains only to prove its surjective, which means $f_{i_0}(x_{i_0}^+) = 1$, indeed, by contradiction, suppose that $f_{i_0}(x_{i_0}^+) < 1$ and let $I^\star = [0, f_{i_0}(x_{i_0}^+)]$. Notice that on $I^\star$ equation (12) becomes:

$$\sum_{y \in f^{-1}(x)} \frac{1}{f'(y)} = 1 \tag{2.13}$$

this yields:

$$\lambda(f^{-1}(I^\star)) = \int_{I^\star} \sum_{y \in f^{-1}(x)} \frac{1}{f'(y)} dx = \lambda(I^\star), \tag{2.14}$$

On the other hand, since $f_{i_0}^{-1}(I^\star) = \emptyset$ we obtain that

$$\lambda(f^{-1}([0, 1] \setminus I^\star) < \lambda(f^{-1}([0, 1] \setminus I^\star), \tag{2.15}$$

which leads to the following contradiction

$$\lambda(f^{-1}([0,1])) = \lambda(f^{-1}(I^\star)) + \lambda(f^{-1}([0,1] \setminus I^\star) < 1$$

We finally obtain that f defines a full branch map of the interval, since also we have that (13) is satisfied on all the unit interval then Lebesgue measure is preserved.

It remains to show that f represents a circle map, indeed we need to check that $f'(x_i^-) = f'(x_i^+)$ as well as $f'(0) = f'(1)$. This follows directly my the assumption on the derivative of the branches at the end points of the partition elements. □

2.4 Proof of Proposition 3

We will prove that there exists a dense set of Lebesgue reserving maps of the circle with unbounded distortion in $\Gamma_\lambda(\mathbb{S}^1)$. Our idea is to take an element of $\Gamma_\lambda(\mathbb{S}^1)$ with bounded distortion and prove it can be approximated arbitrarily by ones whitch have unbounded distortion. We recall the following useful definitions.

Definition 2.2. We say that a map $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity if $\omega(0) = 0$, is continuous and concave.

Definition 2.3. We say that a modulus of continuity ω is Dini-integrable if the following condition holds:

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

Proof. Let $f \in \Gamma_\lambda(\mathbb{S}^1) \backslash \mathcal{B}$ and Let $\epsilon > 0$, on a small enough neighborhood V_0 of the unique fixed point of f which we are assuming to be 0 let $\tilde{f}'|_{V_0} = f' + \epsilon\omega$ where ω is a non Dini-integrable modulus of continuity. Let us extend the frst branch in a way that $d_1(f_1, \tilde{f}_1) \le \epsilon$ while keeping the remaining $n-2$ unchanged, this is possible by normality of the circle.

Lemma 5. *The $(n-1)$-branches of f obtained after perturbing the first branch extend to a Lebesgue preserving map $\tilde{f}$ of degree n which is ϵ-close to f.*

Proof. The extension to a Lebesgue preserving map of degree n follows by proposition 2. To see that it is ϵ-close to f let us consider f_n and $\tilde{f}_n$ to be the last branches of both maps, since they preserve Lebesgue measure we have that for a every interval $I \subset [0,1]$:

$$\lambda(I) = \sum_{1 \le i \le n} \lambda(f_i^{-1}(I)) = \sum_{1 \le i \le n} \lambda(\tilde{f}_i^{-1}(I))$$

By construction this is equivalent to

$$\lambda(f_1^{-1}(I)) - \lambda(\tilde{f}_1^{-1}(I)) = \lambda(f_n^{-1}(I)) - \lambda(\tilde{f}_n^{-1}(I)) \tag{2.16}$$

since $|\lambda(f_1^{-1}(I)) - \lambda(\tilde{f}_1^{-1}(I))| \leq \epsilon$, by Lebesgue density theorem we obtain that $d_1(f_n, \tilde{f}_n) \leq \epsilon$. This finishes the proof. □

To finish the proof, it remain to show that these perturbations yield element which have unbounded distortion.

Lemma 6. *For every $\epsilon > 0$ and $f \in \Gamma_\lambda(\mathbb{S}^1)\backslash\mathcal{B}$, the perturbed map $\tilde{f}$ has unbounded distortion.*

Proof. By the formula given in the introduction of the definition of bunded distortion we have

$$|\log \frac{(\tilde{f}^k)'(x)}{(\tilde{f}^k)'(y)}| = |\sum_{0\leq i\leq k-1} (\log(\tilde{f}'(\tilde{f}^i(x)) - \log \tilde{f}'(\tilde{f}^i(y))|$$

Using mean value theorem, for every $0 \leq i \leq k-1$ there exists

$$\lambda = \min_{x\in S^1} \leq z_i \leq \sigma = \max_{x\in S^1} |\tilde{f}'(x)| > 1$$

such that

$$\sum_{0\leq i\leq k-1} \log \tilde{f}'(\tilde{f}^i(x)) - \log \tilde{f}'(\tilde{f}^i(y)) = \sum_{0\leq i\leq k-1} \frac{1}{z_i}(\tilde{f}'(\tilde{f}^i(x)) - \tilde{f}'(\tilde{f}^i(y))).$$

Now for every $k \in \mathbb{N}$, let us take the first partition element of order k, i.e. $\omega_1^k = [0, r_k]$ where $\sigma^{-k} \leq r_k \leq \lambda^{-k}$. Let us take $y = 0$ and $x_k \in \omega_1^k$ such that $\tilde{f}^k(x_k) \in V_0\backslash\{0\}$, this possible by taking the pre-image of a point in $V_0\backslash\{0\}$ by the first branch $\tilde{f}_1^k$, that is, we consider $x_k = f_1^{-k}(x_0)$ We get

$$\left|\log \frac{(\tilde{f}^k)'(x_k)}{(\tilde{f}^k)'(0)}\right| \geq \frac{1}{\sigma}|\sum_{0\leq i\leq k-1} (\tilde{f}'(\tilde{f}^i(x_k)) - \tilde{f}'(0))|$$

but since $\tilde{f}'(\tilde{f}^i(x_k) = f'(\tilde{f}^i(x_k) + \omega(\tilde{f}^i(x_k)$ we obtain

$$\left|\log \frac{(\tilde{f}^k)'(x_k)}{(\tilde{f}^k)'(0)}\right| \geq \frac{1}{\sigma}|\sum_{0\leq i\leq k-1} (f'(\tilde{f}^i(x_k)) - f'(0)) + \sum_{0\leq i\leq k-1} \omega(\tilde{f}^i(x_k))| \tag{2.17}$$

by choice of x_k we obtain that

$$\left|\log\frac{(\tilde{f}^k)'(x_k)}{(\tilde{f}^k)'(0)}\right| \geq |\sum_{0\leq i\leq k-1}(f'(C\sigma^{i-k})-f'(0))+\sum_{0\leq i\leq k-1}\omega(C\sigma^{i-k})|,$$

Where $C>0$ is a constant. Since f has bounded distortion the first term is bounded, and since ω is not Dini-integrable by [5] we deduce that the second sum diverges hence we obtain unbounded distortion. □

This finishes the proof of the proposition. □

2.5 Proof of Proposition 4

We recall the definition of a residual set.

Definition 2.4. A subset $R\subset X$ of a metric space is said to be residual if it is a dense G_δ set, we say that elements of R are generic in X.

Proof. We want to prove that the space $\Gamma_\lambda(\mathbb{S}^1)$ is residual in its completion. Its clear that the completion is the following space:

$$\Gamma^\star_\lambda(\mathbb{S}^1)=\{f:\mathbb{S}^1\to\mathbb{S}^1 \text{ such that } f'(x)\geq 1 \text{ for all } x\in[0,1]\}$$

Now consider a countable basis of the topology of $\mathbb{S}^1$ by closed intervals $\{I_n\}$ and define the set:

$$S_{I_n}=\{f\in\Gamma^\star_\lambda(\mathbb{S}^1) \text{ such that } f'|_{I_n}>1\}.$$

Clearly we have:

$$\Gamma_\lambda(\mathbb{S}^1)=\bigcap_{n\in\mathbb{N}}S_{I_n}$$

and that S_{I_n} are open sets in the C^1 topology and hence $\Gamma_\lambda(\mathbb{S}^1)$ is a G_δ set, on the other hand, every element in $\Gamma^\star_\lambda(\mathbb{S}^1)$ is clearly arbitrarily close to an element of $\Gamma_\lambda(\mathbb{S}^1)$ and hence $\Gamma_\lambda(\mathbb{S}^1)$ is a residual set of its completion. □

Chapter 3

Topological properties of the space of conservative expanding maps.

This result has been published in [8]: *H. Boukhecham and H. Ounesli. Topology of the space of measure-preserving transformations of the circle Rend. Mat. Univ. Trieste, Volume 55, 2023*

3.1 Introduction and statement of results.

One of the classical problems in topology, dynamics, and geometry is studying properties of the group of *diffeomorphisms* of a closed manifold M, preserving a given smooth volume form ω. Questions about the topology of this space, dynamics-rigidity phenomenons, and algebraic properties can be addressed. There has been extensive work in this direction, as in [10, 18]. In particular, in [11] J.Moser has shown that these groups are locally arc-connected. In this paper, we generalize Moser's result on arc-conectedness to a space of non-invertible volume preserving maps in dimension 1. More precisely, we consider our manifold to be the circle, and we study the space of C^1 orientation preserving uniformly expanding maps of degree 2, preserving the natural volume form on the circle i.e Lebesgue measure. We denote this space by Λ_{Leb}. Our results suggest that the facts known for volume preserving *diffeomorphism* groups can be extended to spaces of non-invertible volume preserving maps. The only topological information we know about Λ_{Leb} is that it is of first category in the space $C^1(S^1, S^1)$ of all C^1 maps of the circle, this was shown in [9].
Our result shows that Λ_{Leb} is indeed arc-connected, with fundamental group $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$. Moreover, we show that this space is homeomorphic to a natural infinite dimensional Lie group.

Remark: We always denote by $D_+(S^1)$ the group of circle diffeomorphisms which preserves the orientation and $D_+(I,J)$ for the space of orientation preserving interval diffeomorphisms and $D_{+,exp}(I,J)$ for the expanding ones (i.e $f' \geq \gamma > 1$). T^2 denotes the torus $S^1 \times S^1$.

Theorem 7. *The space Λ_{Leb} endowed with the C^1-topology is homeomorphic to $T^2 \setminus diag(T^2) \times D_+(S^1, 0$ is fixed), in particular, Λ_{Leb} is arc-connected, and $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$.*

This theorem, as mentioned before, is an extension of Moser result on local arc-connectedness of the group of volume preserving diffeomorphisms. However, our result extends it only in dimension one. Intuitively the result says that for any two Lebesgue preserving uniformly expanding circle maps f, g there exists a deformation between each other $\gamma(t) : [0,1] \to \Lambda_{Leb}$ which preserves Lebesgue along the deformation. The fact that the fundamental group is isomorphic to $\mathbb{Z}$ signifies that any deformation is generated by a fixed deformation in Λ_{Leb}. On the other hand, we show that the space Λ_{Leb} is huge in a sense albeit being meagre in $C^1(S^1, S^1)$, as we have partially proven in [13]. We conjecture that our result can be extended to arbitrary dimensions.

Conjecture. Let (M, g) be a closed Riemannian manifold and ω its volume form. The space $\Lambda^r_\omega(M)$ of C^1 expanding r-folds of M, preserving the volume form, is locally arc-connected.

3.2 Proof of the Theorem.

3.2.1 Uniformly expanding circle maps

Denote by $E^1(S^1)$ the space of uniformly expanding maps of the circle, and by Λ_{Leb} the sub-space of maps f of degree 2 and preserving the Lebesgue measure λ (i.e $f_*\lambda = \lambda$) and the orientation. We endow this space with the C^1-topology. The circle is seen as the natural quotient space $[0,1]/(0 \sim 1)$. Circle maps of degree 2 which are orientation preserving, up to conjugacy with a rotation, can be regarded as interval maps with two full branches (see figure 3.1).

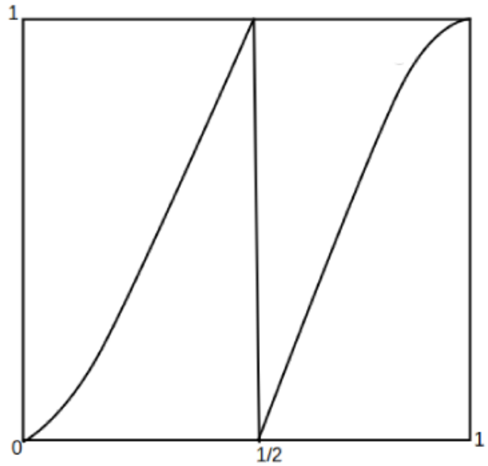

Figure 3.1: A representation of a circle map of degree 2 on the unit interval.

We recall that uniformly expanding circle maps of degree 2 have two main characteristics: a unique fixed point $p \in S^1$ and two branch-arcs determined by two distinct points $x_1 \neq x_2 \in S^1$.

3.2.2 The transfer operator.

Let $f \in E^1(S^1)$. We define the transfer operator P associated to f, and acting on $L^1_\lambda(S^1)$ as: if $h \in L^1_\lambda(S^1)$ then:

$$Ph = \frac{d\big(f_* \mu_h\big)}{d\lambda}. \tag{3.1}$$

where $\mu_h = h \cdot \lambda$. The transfer operator provides the density of the push-forward of a given absolutely continuous measure with respect to Lebesgue. The transfer operator for maps of degree 2 has an explicit formula:

$$Ph(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)}. \tag{3.2}$$

The main property of this operator is the following Folklore proposition:

Proposition 8. *The set of absolutely continuous invariant measures of f correspond to the non-negative fixed points of the operator P.*

3.2.3 Proof of Theorem 1.

The proof of the theorem will be based on the following proposition, which we consider to be of independent interest:

Proposition 9. *Let $a \in (0,1)$ and $f_1 : [0,a] \to [0,1]$ be an expanding C^1-diffeomorphism, then there exists a unique extension of f_1 to a Lebesgue-preserving full branch expanding transformation of the unit interval.*

Proof. Consider the differential equation

$$f_2'(x) = \frac{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big)}{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big) - 1}, \; x \in [a,1], \tag{3.3}$$

Since f_1 is C^1, by Peano's existence theorem the Cauchy problem with the initial condition $f_2(a) = 0$ admits a maximal solution f_2 defined on the interval $[a,1]$. Let's show that f_2 maps diffeomorphically onto $[0,1]$. Notice that $f_2'(x) > 1$ for all $x \in [a,1]$, therefore it only remains to show that $f_2(1) = 1$. Assume that $f_2(1) < 1$ and consider $I = [0,b]$ where $b = f_2(1)$. We notice that for every $y \in I$ we get:

$$\frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} = 1, \tag{3.4}$$

This implies in particular:

$$f_\star\lambda([0,b]) = \lambda(f_1^{-1}([0,b])) + \lambda(f_2^{-1}([0,b]))$$

$$- \int_{[0,b]} \frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} d\lambda = \lambda([0,b])$$

On the other hand, we know that $f_\star\lambda([b,1]) = \lambda(f_1^{-1}([b,1])) < \lambda([b,1])$ which implies that $\lambda(f_\star([0,1])) < \lambda([0,1])$, resulting in a contradiction. The case $b > 1$ results in the same contradiction, hence $b = 1$, this implies in particular that (4.10) is satisfied for every $x \in [0,1]$ and hence the Lebesgue measure is preserved. Since $b = 1$, we also get that (5) is satisfied on all the interval and hence f preserves λ.

Uniqueness cannot be deduced directly from the equation (4.9), because Peano's existence theorem provides only existence, we will deduce it using the fact that the solution preserves λ. Let $f, g : [0,1] \to [0,1]$ be two full branch interval maps which preserve Lebesgue measure, assume they have the same first branches (i,e $f_1 = g_1$) on an interval $[0,a]$, then for every $y \in [0,1]$ we have

$$\lambda([0, y]) = \lambda(f^{-1}([0, y])) = \lambda(g^{-1}([0, y])),$$

which implies by assumption that

$$\lambda([a, f_2^{-1}(y)]) = \lambda([a, g_2^{-1}(y)]),$$

this implies that $f_2^{-1}(y) = g_2^{-1}(y)$, thus $f = g$.

□

Lemma 10. *The extension of an expanding diffeomorphism $f_1 : [0, a] \to [0, 1]$ to a full branch interval map preserving Lebesgue is a C^1 circle map, if and only if the following holds:*

$$f_1'(0) = \frac{f_1'(a)}{f_1'(a) - 1} \tag{3.5}$$

Proof. This is because for a full branch map to lift to a circle map, the derivatives at the end points must coincide, as well as the left and right derivatives at the point a, and so by equation (4), we need (4.2) to hold. □

We will use the previous results to show that Λ_{Leb} is arc connected.

Corollary 11. *Λ_{Leb} is arc connected.*

Proof. Let f be the doubling map of the circle, and $g \in \Lambda_{Leb}$. Up to composing g with a rotation, we can assume that g and f have the same fixed point 0. Denote by x_g the point in S^1 such that $\int_0^{x_g} g'(t)\, dt = 1$, we will construct a homotopy between g and $\tilde{g}$ in Λ_{Leb}, such that $x_{\tilde{g}} = \frac{1}{2}$. Without loss of generality, let us assume that $x_g > \frac{1}{2}$. For $x_g > \epsilon > \frac{1}{2}$, translate horizontally the graph of $g|_{(\epsilon, x_g)}$ to $(\frac{1}{2} - x_g + \epsilon, \frac{1}{2})$ by a linear homotopy $T(t, \cdot)$. Now let z close enough to 0, more precisely, chose $z < \frac{1}{2} - x_g + \epsilon$. Construct a homotopy $H(t, x)$ as follows: for every t define $H(t, \cdot)|_{[0,z]} = g$ and $H(t, \cdot)|_{[\epsilon - t, x_g - t]} = T(t, \cdot)$, and for every t extend it in a C^1 and expanding way to the whole interval $[0, x_g - t]$, as represented on the figure below. This yields a homotopy between g and $\tilde{g}$ in Λ_{Leb}, because condition (4.2) is satisfied for every t, also $\tilde{g}$ satisfies $x_{\tilde{g}} = \frac{1}{2}$.

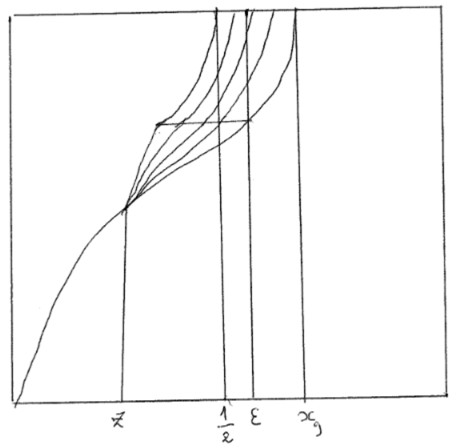

Figure 3.2: A representation of the homotopies H and T.

The second step is to construct an appropriate homotopy between $\tilde{g}$ and f. This is straight forward by considering a continuous family of expanding C^1 maps $(h_c : [0,\frac{1}{2}] \to [0,1])_{c\in[2,g'(0)]\text{or}[g'(0),2]}$ with $h_c'(0) = c$ and $h_c'(\frac{1}{2}) = \frac{c}{c-1}$. Notice in this case that $\tilde{g}|_{[0,\frac{1}{2}]}$ is homotopic to $h_{g'(0)}$ by simply taking $H(t,x) = t\tilde{g}|_{[0,\frac{1}{2}]}(x)+(1-t)h_{g'(0)}(x)$ and same for $f|_{[0,\frac{1}{2}]}$ and h_2 by $G(t,x) = tf|_{[0,\frac{1}{2}]}(x) + (1-t)h_2(x)$, this homotopies satisfy (4.2), and so they extend to a homotopy in Λ_{Leb} between $\tilde{g}$ and f by concatenating the extension of the homotopy H with the extenstion of the family $(h_c)_c$ and the extension of G in Λ_{Leb}, this finishes the proof of arc-connectedness. □

Proposition 12. *The space Λ_{Leb} is homeomorphic to the infinite dimensional Lie group $\mathbb{T}^2 \setminus diag(\mathbb{T}^2) \times D(S^1, 0)$.*

Proof. Let Γ be the space:

$$\Gamma = \bigcup_{0\leq x-y<1} \{f \in \mathrm{D}^1_{+,exp}([x,y],[0,1]) \text{ such that } f'(x) = \frac{f'(y)}{f'(y)-1}\}.$$

Proposition 2.2 results naturally in a map $\mathcal{F}$:

$$\mathcal{F} : \Gamma \to \Lambda_{Leb},$$

defined by sending an element $f \in \Gamma$ to a Lebesgue preserving circle map, by extension after translating $[x,y]$ to $[0,x-y]$, and translating the solution back.

Proposition 13. *The map $\mathcal{F}$ is a homeomorphism (in the C^1-topology).*

Proof. By proposition 2.2 and lemma 2.3, the map is well defined and for every $f \in \Gamma$, there exists a unique extension of f to a circle expanding map preserving Lebesgue measure. Continuity follows from the fact that the unique solutions to a continuous family of Cauchy problems $(ODE_t)_{t\in I}$, with a continuous family of initial conditions form a continuous family $(f_t)_{t\in I}$ in the C^1-topology and this shows that $\mathcal{F}$ is a continuous injection.
The image of the operator $\mathcal{F}$ covers all Lebesgue preserving circle maps f, whose fixed point p_f is inside the branch interval $[x, y]$ of the specific element, hence it is surjective, the inverse is clearly continuous and hence is a homeomorphism.

□

to finish the proof, notice that Γ is homeomorphic to

$$\mathbb{T}^2 \setminus diag(\mathbb{T}^2) \times \{f \in D_+([0, \frac{1}{2}], [0,1]) \text{ such that } f(0) = \frac{f(\frac{1}{2})}{f(\frac{1}{2}) - 1}\}$$

and that:

$$\{f \in D_+([0, \frac{1}{2}], [0,1]) \text{ such that } f'(0) = \frac{f'(\frac{1}{2})}{f'(\frac{1}{2}) - 1}\}$$
$$\simeq D_+([0,1], [0,1] \text{ such that } f'(0) = f'(1)) \simeq D_+(S^1, 0 \text{ is fixed}).$$

Now remark that $\mathbb{T}^2 \setminus diag(\mathbb{T}^2)$ inherits the Lie group structure of $\mathbb{C} \setminus \{0\}$ and $D_+(S^1, 0 \text{ is fixed})$ is an infinite dimensional Lie group. □

Corollary 14. $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$.

Proof. First, notice that $\pi_1(\mathbb{T}^2 \setminus diag(\mathbb{T}^2)) = \pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$, on the other hand, by results of [3], we know that the injection of $SO(2)$ in $D_+(S^1)$ induces a splitting of the fundamental group $\pi_1(D_+(S^1)) = \pi_1(SO(2)) \oplus \pi_1(D_+([0,1], \partial[0,1]))$, and since we know that $\pi_1(SO(2)) = \mathbb{Z}$, and that $D_+([0,1], \partial[0,1])$ is contractible, we deduce that $\pi_1(D_+(S^1)) = \mathbb{Z}$ and that $D_+(S^1, 0 \text{ is fixed})$ is simply connected. So we have $\pi_1(\Lambda_{Leb}) = \mathbb{Z}$.

□

Remark. Arc-connectedness can be deduced again by the fact that our space is homeomorphisc to an infinite dimensional Lie group. However, we consider our prove of arc-cnnectedness to be of independent interest since we believe the idea can be generalized to higher dimensions as we conjectured in the statement of results.

Chapter 4

Existence of conservative expanding maps of any given regularity.

This result has been accepted for publications [13]: *H. Ounesli. On the existence of absolutely continuous invariant probability measures for C^1 expanding maps. Journal of Dynamical and Control Systems (to appear, 2024).*

4.1 Introduction and statement of results

Let $E^1(\mathbb{S}^1)$ be the space of C^1 uniformly expanding maps on the circle. It is essentially a Folklore Theorem dating back to the 1950s that if $f \in E^1(\mathbb{S}^1)$ is $C^{1+\alpha}$, i.e if the derivative is Hölder continuous, then f admits a unique ergodic invariant probability measure equivalent to Lebesgue. This result, together with the techniques involved in the proof, have led to a huge area of research and many generalizations to uniformly and non-uniformly expanding maps on manifolds of arbitrary dimension as well as to more general hyperbolic and non-uniformly hyperbolic systems.

However, even in this simplest setting of uniformly expanding circle maps there are still open problems for maps with lower degrees of regularity. Indeed Góra and Schmitt [7] constructed an example of a map $f \in E^1(\mathbb{S}^1)$ which does not admit any invariant probability measure absolutely continuous with respect to Lebesgue (*acip*). Quas [15] then showed that this is not an isolated example by proving that *generically* in the C^1-topology, maps in $E^1(\mathbb{S}^1)$ have no *acip* and, more recently Avila and Bochi [2] even showed that generically in the C^1-topology, maps in $E^1(\mathbb{S}^1)$ do not even have an absolutely continuous invariant *σ-finite* measure.

On the more "positive" side, it is possible to relax the condition on the Hölder continuity of the derivative to some extent. Recall that the *modulus of continuity* of a continuous map $\rho : X \to Y$ between two metric spaces is a continuous map $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ vanishing at 0 and satisfying

$$d_Y(\rho(x), \rho(y)) \leq \omega(d_X(x, y)) \tag{4.1}$$

for every $x, y \in X$. We say that ω is *Dini-integrable* if

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

Notice that saying that ρ is Hölder continuous is exactly equivalent to saying that ρ has a modulus of continuity of the form $\omega(t) = Ct^\alpha$ for some $\alpha \in (0, 1)$ and that this implies in particular that ω is Dini-integrable. Fan and Jiang [4] showed that if the derivative of $f \in E^1(\mathbb{S}^1)$ has a modulus of continuity which is Dini-integrable then f admits a unique ergodic invariant probability measure equivalent to Lebesgue, thus extending the Folklore Theorem to a lower degree of regularity of the map.

All the counterexamples mentioned above must therefore have modulus of continuity for the derivative which is not Dini-integrable and a natural question is whether Dini-integrability defines a precise *cut-off* between C^1 uniformly expanding maps which admit and which do not admit an *acip*. In this paper we explore this "underground" world of maps in $E^1(\mathbb{S}^1)$ with very low regularity, in particular whose derivative have modulus of continuity which is not Dini-integrable. We show that for *any* given modulus of continuity ω there are (uncountably many) maps in $E^1(\mathbb{S}^1)$ whose derivative has a modulus of continuity equivalent to ω but nevertheless still admit an *acip*. In particular there is no specific *cut-off* based on the regularity of the derivative, which means that other characteristics of the map somehow come into play.

Existence of acip

To state our results we define the *canonical modulus of continuity* of ρ by

$$\omega_\rho(t) := \sup\{|\rho(x) - \rho(y)| : d(x, y) < t\}.$$

Notice that ω_ρ always exists if X is compact since every continuous function is uniformly continuous. It is also easy to check that ω_ρ is increasing, concave, and satisfies (4.1). We define the space of all potential moduli of continuity by

$$K := \{\omega \in C^0(\mathbb{R}^+, \mathbb{R}^+) : \text{continuous, increasing, concave, } \omega(0) = 0\},$$

and define an equivalence relation on K by letting $\omega \simeq \tilde{\omega}$ if the ratio $\omega/\tilde{\omega}$ is uniformly bounded above and below. Then, following [7], we say that $\omega \in K$ is an *optimal modulus of continuity* for ρ if it is *equivalent to* ω_ρ.

Remark 4.1.1. Despite its name, the optimal modulus of continuity is not unique but rather defines a class of functions of which ω_ρ is, in some sense, a canonical representative and such that all the moduli in this class have essentially the same behaviour near 0. For example if ρ is Hölder continuous and its canonical modulus is $\omega_\rho(t) = Ct^\alpha$, for some $C, \alpha > 0$, then any optimal modulus of continuity for ρ will have the form $\omega(t) = \nu(t)t^\alpha$ where $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded away of 0.

The equivalence relation on K defined above induces an equivalence relation on the space $E^1(\mathbb{S}^1)$ by letting $f \sim g$ whenever $\omega_{f'} \simeq \omega_{g'}$, i.e. whenever the corresponding canonical moduli of the derivatives f', g' are equivalent. The equivalence classes associated to this equivalence relation are of the form

$$E^1_\omega(\mathbb{S}^1) := \{f \in E^1(\mathbb{S}^1) : \omega_{f'} \simeq \omega\}$$

for $\omega \in K$. Indeed, notice that for $\omega, \tilde{\omega} \in K$ we have that $E^1_\omega(\mathbb{S}^1) = E^1_{\tilde{\omega}}(\mathbb{S}^1)$ if $\omega \simeq \tilde{\omega}$ and $E^1_\omega(\mathbb{S}^1) \cap E^1_{\tilde{\omega}}(\mathbb{S}^1) = \emptyset$ otherwise. Notice that $E^1_\omega(\mathbb{S}^1)$ contains a large number of maps, as specifying only the modulus of continuity of f' leaves a lot of freedom in the definition of f. We are interested in the sets

$$\Gamma^1_\omega(\mathbb{S}^1) := \{f \in E^1_\omega(\mathbb{S}^1) : f \text{ admits an } acip \text{ equivalent to Lebesgue}\}.$$

By [4], as mentioned above, if ω is Dini-integrable, and therefore in particular if ω is Hölder continuous, every $f \in E^1_\omega(\mathbb{S}^1)$ admits an *acip* equivalent to Lebesgue and therefore $\Gamma^1_\omega(\mathbb{S}^1) = E^1_\omega(\mathbb{S}^1)$. On the other hand, if ω is not Dini-integrable then by [7] there exist examples of $\omega \in K$ such that $E^1_\omega(\mathbb{S}^1) \neq \Gamma^1_\omega(\mathbb{S}^1)$, and [15, 2] even seem to suggest that there may be examples of $\omega \in K$ for which $\Gamma^1_\omega(\mathbb{S}^1) = \emptyset$. Our main result shows that this is not the case and that, *on the contrary,* $\Gamma^1_\omega(\mathbb{S}^1) \neq \emptyset$ *for every* $\omega \in K$. Moreover, our arguments are quite constructive and yield additional information about the possible regularities of the densities of the *acip*, and in particular show that their regularity may be as low as that of f itself, i.e. have ω as an optimal modulus of continuity, or very smooth, including cases in which *Lebesgue measure itself is invariant.* For every $\omega \in K$ and $f \in \Gamma^1_\omega(\mathbb{S}^1)$, we let μ_f denote the *acip* equivalent to Lebesgue, let $\rho_f = d\mu_f/dm$ denote its (continuous) density with respect to Lebesgue, and let ω_{ρ_f} denote the canonical modulus of continuity of ρ_f.

Theorem 15. *For every $\omega \in K$ there exists an uncountable set in $\Gamma^1_\omega(\mathbb{S}^1)$ for which $\omega_{\rho_f} \simeq \omega$ and which can be given in a relatively explicit way, see* (4.3).

The main point of Theorem 15 is the fact that $\Gamma^1_\omega(\mathbb{S}^1) \neq \emptyset$, which means that even maps in $E^1(\mathbb{S}^1)$ with *arbitrarily low regularity* can admit an *acip* and also implies that distinct maps with equivalent moduli of continuity can have quite different ergodic properties. Indeed it implies that $\Gamma^1_\omega(\mathbb{S}^1) \neq \emptyset$ in particular for the specific modulus of continuity ω of the counterexample constructed in [7] which however does not admit an *acip*. The additional statements about the densities of the *acip* highlight the fact that $\Gamma^1_\omega(\mathbb{S}^1)$ is in fact quite a large set and that there is a remarkable flexibility in the construction of examples with various kinds of densities. The fact that $f \in \Gamma^1_\omega(\mathbb{S}^1)$ can preserve a density whose modulus of continuity is equivalent to the modulus of f' seems quite natural but turns out to be somewhat coincidental as we show that there exists also maps $f \in \Gamma^1_\omega(\mathbb{S}^1)$ which preserve densities which are much more regular than that of f', even Lebesgue measure itself.

Theorem 16. *Let $a \in (0,1)$ and let $f_1 : [0,a] \to [0,1]$ be an expanding C^1-diffeomorphism. Then there exists a unique extension of f_1 to a Lebesgue-preserving full branch expanding transformation of the unit interval. This extension represents a C^1 map on the circle if and only if the following holds:*

$$f_1'(0) = \frac{f_1'(a)}{f_1'(a) - 1} \tag{4.2}$$

In particular, for every $\omega \in K$ there exists an uncountable set in $\Gamma^1_\omega(\mathbb{S}^1)$ for which μ_f is Lebesgue.

For for future reference, for every $\omega \in K$ we let

$$\Gamma_{\omega,\lambda}(\mathbb{S}^1) := \{f \in \Gamma^1_\omega(\mathbb{S}^1) : \text{Lebesgue measure is invariant}\}$$

Bounded and unbounded distortion

One of the main techniques for proving the existence of an *acip* is through a bounded distortion property. For $f \in E^1(\mathbb{S}^1)$ we let $\{\omega_i^{(n)}\}$ denote the injectivity domains of f^n and say that f has *bounded distortion* if

$$\mathcal{D} := \sup_{n\geq 1} \sup_{\omega_i^{(n)}} \sup_{x,y\in\omega_i^{(n)}} \log \frac{(f^n)'(x)}{(f^n)'(y)} < \infty.$$

It is possible to show that if ω is Dini-integrable then every $f \in E^1_\omega(\mathbb{S}^1)$ has bounded distortion and therefore, since by classical arguments bounded distortion implies the existence of an *acip* equivalent to Lebesgue, this implies that $\Gamma^1_\omega(\mathbb{S}^1) = E^1_\omega(\mathbb{S}^1)$, as mentioned above. If ω is not Dini-integrable then bounded distortion cannot be guaranteed and indeed our construction of the *acip* for maps for maps $f \in E^1_\omega(\mathbb{S}^1)$ in this setting does not explicitly use any distortion estimates. An interesting question therefore is whether Dini-integrability is a necessary as well as a sufficient condition for uniformly bounded distortion and, if not, whether there is actually is any underlying bounded distortion property which is implicitly responsible for the existence of an *acip* in the cases given by Theorem 15.

Conjecture. $\forall\ \omega \in K$ non Dini-integrable, unbounded distortion is C^1-generic in $\Gamma^1_\omega(\mathbb{S}^1)$

While we cannot give a full answer to the conjecture we can show that many maps have an *acip* despite not having bounded distortion. For $\omega \in K$ we consider a subset of the family $\Gamma_{\omega,\lambda}(\mathbb{S}^1)$ defined above for which the derivative has an explicit form near 0.

$$\mathcal{F}_\omega \coloneqq \{f \in \Gamma_{\omega,\lambda}(\mathbb{S}^1) : f_1'(x) = 2 + 2\omega(x) \text{ on a small enough interval } [0, t_\omega]\}$$

It is clear by the statement in Theorem 16 that $\mathcal{F}_\omega$ is an uncountable set.

Theorem 17. *Every map in $\mathcal{F}_\omega$ has bounded distortion if and only if the optimal modulus of continuity $\omega_{f'}$ of f' is Dini-integrable.*

Finally, also in the direction of the Conjecture above, we show that unbounded distortion is generic in a somewhat different sense. More precisely, we define on $E^1(S^1)$ the C^{1+mod}-topology induced by the metric

$$d_{1+mod}(f, g) = d_1(f, g) + d_0(\omega_{f'}, \omega_{g'}),$$

where d_1 is the C^1 distance, and d_0 is the C^0-distance. In the distance d_{1+mod}, maps are close if they are C^1-close, and their moduli of continuity $\omega_{f'}$ and $\omega_{g'}$ of their derivatives are close in the C^0-topology. Notice that this is a natural metric on the space of C^1 maps and stronger than the usual C^1 metric.

Theorem 18. *There exists a subset $\Gamma \subset E^1(S^1)$ which contains exactly one element from each equivalence class $E^1_\omega(S^1)$ which preserves Lebesgue measure for $\omega \in K$, such that C^{1+mod} generic maps $f \in \Gamma$ have unbounded distortion.*

Remark 4.1.2. This theorem implies, in particular, that most maps in Γ have unbounded distortion and still preserve a continuous probability measure equivalent to Lebesgue. Such examples are rare to find in the literature. The only example we know of is the Quas example in [17] where he constructed an expanding map of the circle preserving Lebesgue but not ergodic and hence has unbounded distortion.

Proof of part 1 of Theorem 1

Let $\omega \in K$, we will construct uncountably many maps in $\Gamma^1_\omega(\mathbb{S}^1)$ for which the density they preserve has ω as an optimal modulus of continuity. We will construct these as maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 2, orientation-preserving which we represent as full branch map of the unit interval $[0, 1]$ with two C^1 branches f_1 and f_2 defined respectively on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ satisfying $f_1'(0) = f_2'(1)$ and $f'_{l,1}(\frac{1}{2}) = f'_{r,2}(\frac{1}{2})$ where l and r denote the left and right derivatives at $x = \frac{1}{2}$.
We will first give an overview of the proof and reduce it to a number of technical propositions which we will prove in the subsequent sections.

Overview of the proof

Our idea is to fix a continuous density satisfying certain conditions and prove that under those conditions we can construct a uniformly expanding map of the circle preserving the measure defined by that density and for which the regularity of the derivative is the same as that of the density.

Lemma 19. *For every $\omega \in K$ there exists $\rho : [0, 1] \to \mathbb{R}$ continuous, having ω as an optimal modulus of continuity, strictly greater than $1/2$, satisfying:*

$$\int_0^{\frac{1}{2}} \rho(t)dt = \int_{\frac{1}{2}}^1 \rho(t)dt = \frac{1}{2}, \tag{P1}$$

$$\max_{[0,1]} \rho - \min_{[0,1]} \rho < \frac{1}{2}, \tag{P2}$$

and

$$\rho(0) = \rho(1) = 1. \tag{P3}$$

Now, assuming the conditions of the previous lemma, for $x \in [0, 1]$ let:

$$g(x) = \int_0^x \rho(t)dt$$

and define $f_\rho : [0, 1] \to [0, 1]$ by

$$f_\rho(x) = \begin{cases} 2g & \text{if } x \in [0, \frac{1}{2}] \\ (g - \frac{1}{2}I)^{-1} \circ (g - \frac{1}{2}) & \text{if } x \in [\frac{1}{2}, 1] \end{cases} \tag{4.3}$$

We will show that the map f_ρ is a well defined C^1 expanding circle map which preserves the density ρ and whose derivative f'_ρ has ω as an optimal modulus of continuity, thus proving part 1 of Theorem 1.

We split the proof into the following propositions. First of all let $\rho : [0,1] \to \mathbb{R}$ be a continuous map such that $\rho > 1/2$ and consider the following system of ordinary differential equations:

$$\begin{cases} f_1' = 2\rho & \text{on } [0,\frac{1}{2}] \text{ with } f_1(0) = 0, \\ f_2' = \dfrac{2\rho}{2\rho \circ f_2 - 1} & \text{on } [\frac{1}{2},1] \text{with } f_2(\frac{1}{2}) = 0. \end{cases} \tag{S}$$

Proposition 20. *If ρ satisfies $(P1)$ then the system (S) has a solution that defines a full branch map f of the unit interval $[0,1]$ which preserves the measure μ defined by the density ρ.*

Proposition 21. *If ρ satisfies $(P1) - (P3)$ then the map previously constructed coincides with f_ρ and represents a C^1 uniformly expanding map of the circle.*

Proposition 22. *Let $\omega \in K$ and ρ be the function given by Lemma 19. Then f_ρ' has ω as an optimal modulus of continuity, and in particular $f_\rho \in \Gamma^1_\omega(\mathbb{S}^1)$.*

Proof of Proposition 20

We will split the proof of the proposition to 3 lemmas.

Lemma 23. *If $\rho > 1/2$ and satisfies $(P1)$ then f_ρ is a well defined full branch map of the interval.*

Proof. First, notice that by definition $g(0) = 0$ and by $(P1)$ we have $g(\frac{1}{2}) = 1/2$, so we obtain that f_ρ maps diffeomorphically $[0, \frac{1}{2}]$ to $[0,1]$. Now notice that $g - \frac{1}{2}$ maps $[\frac{1}{2}, 1]$ to $[0, \frac{1}{2}]$ and since $g' = \rho > \frac{1}{2}$ then $(g - \frac{1}{2}I)$ is a diffeomorphism which maps $[0,1]$ to $[0, \frac{1}{2}]$ and hence f_ρ maps diffeomorphically $[\frac{1}{2}, 1]$ to $[0,1]$. We conclude that our map is well defined and full branch on the interval $[0,1]$. □

Lemma 24. *Under the previous conditions, f_ρ is a solution to the system (S)*

Proof. Let us recall that g is the map defined on $[0,1]$ by:

$$g(x) = \int_0^x \rho(t)dt.$$

Clearly, $f_{\rho,1}'(x) = 2\rho(x)$, now we have:

to show the other equality, notice that

$$f_2' = \frac{2\rho}{2\rho \circ f_2 - 1}$$

is equivalent to:

$$2f_2'\rho\circ f_2 - f_2' = 2\rho \iff 2(g\circ f_2 - \frac{1}{2}f_2)' = 2g'$$

after integrating over $\left[\frac{1}{2}, x\right]$ we obtain:

$$\left(g - \frac{1}{2}I\right)\circ f_2(x) = g(x) - \frac{1}{2}.$$

where I denotes the identity map, notice that $(g - \frac{1}{2}I)' > 0$ and hence $g - \frac{1}{2}I$ is invertible, we obtain finally:

$$f_2 = f_{\rho,2} = (g - \frac{1}{2}I)^{-1}\circ(g - \frac{1}{2}).$$

and so we conclude that the system (S) admits f_ρ as a solution. □

Lemma 25. *If a solution of (S) is full branch then it preserves the measure μ defined by the density ρ.*

Proof. We start by recalling the following sublemma:

Sublemma 26. *If $f_\star\mu([0,y]) = \mu([0,y])$ for every $y\in[0,1]$ then μ is f-ivariant.*

Proof. The σ-algebra of Lebesgue measurable sets is generated by intervals of the form $[0,y]$ and all subsets of Borel sets of zero measure, since f is a C^1 local diffeomorphism then it already preserves sets of measure zero, and so if the assumption of the lemma is satisfied then μ if f-invariant. □

Now let $y\in[0,1]$ and consider f to be a full branch map solution to (S) on the unit interval $[0,1]$, since the derivative is everywhere positive, the branches are injective and so every pre-image contains exactly two points, therefore, we have that $f^{-1}(\{y\}) = \{f_1^{-1}(y), f_2^{-1}(y)\}$ such that $f_1^{-1}(y)\in[0,\frac{1}{2}]$ and $f_2^{-1}(y)\in[\frac{1}{2},1]$, of course, we are assuming for simplicity here that the middle point of the interval is the end point of the first branch, we obtain:

$$f_\star\mu([0,y]) = \mu(f^{-1}([0,y])) = \mu([0,f_1^{-1}(y)]) + \mu([\frac{1}{2}, f_2^{-1}(y)]). \tag{4.4}$$

$$\phi_1(x) = \frac{1}{2}x \text{ and } \phi_2(x) = \int_0^x(\rho(t) - \frac{1}{2})dt.$$

Clearly $\mu([0,y]) = \phi_1(y) + \phi_2(y)$ and ϕ_1 maps $[0,1]$ to $[0,\frac{1}{2}]$ and ϕ_2 maps $[0,1]$ to $[0,\frac{1}{2}]$ because $\phi_2(2)$ is increasing since $\rho > \frac{1}{2}$, by definition also $\phi_2(0) = 0$ and by $(P1)$

$$\phi_2(1) = \int_0^1 \rho(t) - \frac{1}{2}dt = \int_0^1 \rho(t)dt - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Now we want to solve the following equations:

$$\mu([0, f_1^{-1}(y)]) = \phi_1(y) \text{ and } \mu([\frac{1}{2}, f_2^{-1}(y)]) = \phi_2(y) \tag{4.5}$$

Which is equivalent to:

$$\int_0^{f_1^{-1}(y)} \rho(t)dt = \phi_1(y) \text{ and } \int_{\frac{1}{2}}^{f_2^{-1}(y)} \rho(t)dt = \phi_2(y).$$

by differentiating both sides of the previous equations and using the formula:

$$\frac{d}{dy}\int_\alpha^{u(y)} v(x)dx = u'(y)v(u(y)). \tag{4.6}$$

we obtain the following two equations:

$$(f_1^{-1})'\rho \circ f_1^{-1} = \frac{1}{2} \text{ and } (f_2^{-1})'\rho \circ f_2^{-1} = \rho - \frac{1}{2}.$$

using that $(f^{-1})' = 1/f' \circ f^{-1}$ we obtain:

$$\frac{\rho}{f_1'} \circ f_1^{-1} = \frac{1}{2} \text{ and } \frac{\rho}{f_2'} \circ f_2^{-1} = \rho - \frac{1}{2}.$$

compositing the first equation of the previous equation by f_1 and the second equation by f_2 we obtain exactly the system (S) but without particular initial conditions due to the differentiation step prior to obtaining (4) and so we are not sure the solution corresponds exactly to equation (4.5), we will show that the initial conditions of system (S) are sufficient to obtain (4.5) and hence complete the proof. Notice that

$$\frac{d}{dy}\int_0^{f_1^{-1}(y)} \rho(t)dt = \phi_1'(y) \text{ and } \frac{d}{dy}\int_{\frac{1}{2}}^{f_2^{-1}(y)} \rho(t)dt = \phi_2'(y).$$

Let us integrate the left and right hand side on $[0,z]$, precisely:

$$\int_0^z (\frac{d}{dy}\int_0^{f_1^{-1}(y)} \rho(t)dt)dy = \int_0^z \phi_1'(y)dy,$$

and

$$\int_0^z \left(\frac{d}{dy}\int_{\frac{1}{2}}^{f_2^{-1}(y)} \rho(t)dt\right)dy = \int_0^z \phi_2'(y)dy.$$

this is equivalent to:

$$\int_0^{f_1^{-1}(z)} \rho(t)dt - \int_0^{f_1^{-1}(0)} \rho(t)dt = \phi_1(z) - \phi_1(0) = \phi_1(z).$$

and

$$\int_{\frac{1}{2}}^{f_2^{-1}(z)} \rho(t)dt - \int_{\frac{1}{2}}^{f_2^{-1}(0)} \rho(t)dt = \phi_2(z) - \phi_2(0) = \phi_2(z).$$

since the initial conditions are $f_1^{-1}(0) = 0$ and $f_2^{-1}(0) = \frac{1}{2}$ we obtain finally the following:

$$\mu([0, f_1^{-1}(z)]) = \phi_1(z)$$

and

$$\mu([\frac{1}{2}, f_2^{-1}(z)]) = \phi_2(z)$$

. This shows that (3) is satisfied and hence finishes the proof. □

By the previous lemmas we proved, the map f_ρ is a full branch map which preserves the measure μ defined by ρ, hence finishing the proof of proposition 6.

Proof of Proposition 21

Proof. Since $\rho > \frac{1}{2}$ we have that $f'(x) = 2\rho(x) > 1$ for every $x \in [0, \frac{1}{2}]$, now by $(P2)$ we also have that:

$$\frac{2\min\limits_{[0,1]}\rho}{2\max\limits_{[0,1]}\rho - 1} > \frac{2\min\limits_{[0,1]}\rho}{2(\min\limits_{[0,1]}\rho + \frac{1}{2}) - 1} = 1$$

and hence we obtain for every $x \in [\frac{1}{2}, 1]$ that $f_2'(x) > 1$ and so f is uniformly expanding, it remains to prove that f represents a C^1 map of the circle as explained at the beginning of the proof of the theorem, namely, we have that $f_{l,1}'(\frac{1}{2}) = f_{r,2}'(\frac{1}{2})$ because by $(P3)$:

$$f_{r,2}'(\frac{1}{2}) = \frac{2\rho(\frac{1}{2})}{2\rho(0) - 1} = 2\rho(\frac{1}{2}) = f_{l,1}'(\frac{1}{2}).$$

Finally, we have $f'(0) = 2\rho(0) = 2$ and $f'(1) = \dfrac{2\rho(1)}{2\rho(1)-1} = 2$. This shows indeed that the solution f defines a C^1 uniformly expanding map on the circle and preserves μ. □

4.1.1 Proof of Proposition 22

Proof. Lets start by proving that ω is a modulus of continuity for f_2' (not necessarily optimal).

$$\omega_{f_2'}(t) = \sup_{0\leq|x-y|<t} |f_2'(x) - f_2'(y)|$$

$$= \sup \frac{1}{(2\rho\circ f_2(x)-1)(2\rho\circ f_2(y)-1)} |4\rho(x)\rho\circ f_2(y) - 4\rho(y)\rho\circ f(x) + 2\rho(y) - 2\rho(x)|$$

since $\rho > 1/2$ we obtain that $\dfrac{1}{(2\rho\circ f_2(x)-1)(2\rho\circ f_2(y)-1)}$ is uniformly bounded above by a constant $C > 0$ and therefore:

$$\omega_{f_2'}(t) \leq 4C|\rho(x)\rho\circ f_2(y) - \rho(y)\rho\circ f_2(x)| + 2C|\rho(y) - \rho(x)|$$

taking $M = \max\rho$ and by adding and substituting $\rho(x)\rho\circ f_2(x)$ on the first term we get

$$\omega_{f_2'}(t) \leq 4CM|\rho\circ f_2(x) - \rho\circ f_2(y)| + (4CM + 2C)|\rho(y) - \rho(x)|$$

since f_2 is C^1 and hence Lipschitz (with a constant L) we obtain:

$$\omega_{f_2'}(t) \leq 4CM\omega(Lt) + (4CM + 2C + 1)\omega(t)$$

by sub-additivity of ω we obtain finally that there is $\alpha > 0$ such that

$$\omega_{f_2'}(t) \leq \alpha\omega(t).$$

Now, since on $[0, \frac{1}{2}]$ $f_1' = 2\rho$ then $\omega_{f_1'} = \omega$, this finishes the proof. □

4.1.2 Proof of Lemma 19

To finish the proof of the theorem, it clearly only remains to prove Lemma 19.

Proof. The existence of ρ is guaranteed because on a small enough interval $[0, t_\omega]$ on which $\omega(t) << \frac{3}{2}$ we can chose $\rho(t)$ to be equal to $1 + \omega(t)$ for every $t \in [0, t_\rho]$, since the translation of an element in K admits itself as an optimal modulus of continuity, it is enough to extend it to the rest of the interval in a C^1 way while satisfying the other properties which do not depend on how we define ρ on a small enough neighborhood of 0 as far as on that neighborhood $(P2)$ and $(P3)$ are satisfied there. We will in fact consider in the rest of the paper ρ being defined on some neighborhood $[0, t_\omega]$ as $1 + \omega(t)$. □

Proof of Theorem 16

We split the proof into two parts. We first prove the first statement concerning the extension of an arbitrary diffeomorphism to a Lebesgue measure preserving circle map, and then show that this map has optimal modulus of continuity ω

The philosophy of the proof will be similar to the proof of Theorem 15 but we will introduce a new ordinary differential equation that arises naturally from the transfer operator. Let $f \in E^1(S^1)$ and, for all Lebesgue-integrable $h \in L^1_\lambda(S^1)$ and $\mu_h := h \cdot \lambda$, we define the transfer operator associated to f and acting on $L^1_\lambda(S^1)$ as

$$Ph = \frac{d\big(f_* \mu_h\big)}{d\lambda}. \tag{4.7}$$

This operator takes the density of an absolutely continuous measure to the density of its push-forward by f in respect to Lebesgue. It is well known that the fixed points of P correspond to the densities of f-invariant measures and that the transfer operator for maps of degree 2 has an explicit formula given by

$$Ph(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{f'(y)}. \tag{4.8}$$

Consider the differential equation

$$f_2'(x) = \frac{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big)}{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big) - 1}, \; x \in [a, 1], \tag{4.9}$$

Since f_1 is C^1, by Peano's existence theorem the Cauchy problem with the initial condition $f_2(a) = 0$ admits a maximal solution f_2 defined on the interval $[a, 1]$. Let's show that f_2 maps diffeomorphically onto $[0, 1]$. Notice that $f_2'(x) > 1$ for all $x \in [a, 1]$, therefore it only remains to show that $f_2(1) = 1$. Assume that $f_2(1) < 1$ and consider $I = [0, b]$ where $b = f_2(1)$. Equation 9 implies that

$$\frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} = 1, \tag{4.10}$$

This implies in particular:

$$f_\star\lambda([0,b]) = \lambda(f_1^{-1}([0,b])) + \lambda(f_2^{-1}([0,b])) =$$

$$\int_{[0,b]} \frac{1}{f_1'\big(f_1^{-1}(y)\big)} + \frac{1}{f_2'\big(f_2^{-1}(y)\big)} d\lambda = \lambda([0,b])$$

On the other hand, we know that $f_\star\lambda([b,1]) = \lambda(f_1^{-1}([b,1])) < \lambda([b,1])$ which implies that $\lambda(f_\star([0,1])) < \lambda([0,1])$, resulting in a contradiction. The case $b > 1$ results in the same contradiction, hence $b = 1$, this implies in particular that (4.10) is satisfied for every $x \in [0,1]$ and hence the Lebesgue measure is preserved (taking $h = 1$ on [0,1])

Uniqueness cannot be deduced directly from the equation (4.9), because Peano's existence theorem provides only existence, we will deduce it using the fact that the solution preserves λ. Let $f, g : [0,1] \to [0,1]$ be two full branch interval maps which preserve Lebesgue measure, assume they have the same first branches (i,e $f_1 = g_1$) on an interval $[0,a]$, then for every $y \in [0,1]$ we have

$$\lambda([0,y]) = \lambda(f^{-1}([0,y])) = \lambda(g^{-1}([0,y])),$$

which implies by assumption that

$$\lambda([a, f_2^{-1}(y)]) = \lambda([a, g_2^{-1}(y)]),$$

this implies that $f_2^{-1}(y) = g_2^{-1}(y)$, thus uniqueness of solutions.

For the second part of the proposition, we want to show that the full branch map obtained represents a circle map if and only if (4.2) holds. This is because for a full branch map to lift to a circle map we need that the derivatives at the end points to coincide, as well as the left and right derivatives at the point a and so by equation (9) we need (4.2) to hold.

It just remains to show that f has ω as an optimal modulus of continuity. Take $a \in (0,1)$ and consider a C^1 expanding diffeomorphism $f_1 : [0,a] \to [0,1]$ admitting $\omega \in K$ as an optimal modulus of continuity and satisfying condition 4.2. By the previous section, this extends to a Lebesgue preserving circle expanding map f, the regularity of the derivative on the first branch is by choice ω-continuous, for the second branch f_2 we know that:

$$f_2'(x) = \frac{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big)}{f_1'\Big(f_1^{-1}\big(f_2(x)\big)\Big) - 1}.$$

Consider the map $\varphi = f_1^{-1} \circ f_2$. For $x, y \in [a, 1]$ we have:

$$|f_2'(x) - f_2'(y)| = |\frac{f_1'(\varphi(x))}{f_1'(\varphi(x)) - 1} - \frac{f_1'(\varphi(y))}{f_1'(\varphi(y)) - 1}| = |\frac{f_1'(\varphi(x)) - f_1'(\varphi(y))}{(f_1'(\varphi(x)) - 1)(f_1'(\varphi(y)) - 1)}|.$$

Since $(f_1'(\varphi(x)) - 1)(f_1'(\varphi(y)) - 1)$ is bounded away from 0 because $f' > 1$ and since φ is Lipschitz (since it is C^1 on a compact interval) we obtain:

$$\sup_{|x-y|\leq t} |f_2'(x) - f_2'(y)| \simeq \sup_{|x-y|\leq t} |f_1'(\varphi(x)) - f_1'(\varphi(y))| \simeq \omega(t).$$

We conclude that $f \in \Gamma^1_\omega(\mathbb{S}^1)$. Notice that in our construction, the choices we made to construct an example allow to construct uncountably many such element. This finishes the proof of the theorem.

Proof of Theorem 17

Proof. Let $\omega \in K$ and $f \in \mathcal{F}_\omega$. For $k \in \mathbb{N}$ and by the chain rule we have that:

$$|\log \frac{(f^k)'(x)}{(f^k)'(y)}| = |\sum_{0\leq i\leq k-1} (\log(f'(f^i(x)) - \log f'(f^i(y))|$$

Using mean value theorem, for every $0 \leq i \leq k-1$ there exists

$$\lambda = \min_{x\in S^1} |f'(x)| \leq z_i \leq \sigma = \max_{x\in S^1} |f'(x)| > 1$$

such that

$$\sum_{0\leq i\leq k-1} \log f'(f^i(x)) - \log f'(f^i(y)) = \sum_{0\leq i\leq k-1} \frac{1}{z_i}(f'(f^i(x)) - f'(f^i(y))).$$

Now for every $k \in \mathbb{N}$, let us take the first partition element of order k, i.e. $\omega_1^k = [0, r_k]$ where $\sigma^{-k} \leq r_k \leq \lambda^{-k}$. Let us take $y = 0$ and $x_k \in \omega_1^k$ such that $f^k(x_k) \leq t_\omega$, for instance we can take $x_k = f_1^{-k}(t_\omega)$ for k large enough, where f_1^k denotes the first branch of the *k-th* iterate of f. From this we obtain

$$\left|\log \frac{(f^k)'(x_k)}{(f^k)'(0)}\right| \geq \frac{1}{\sigma} \sum_{0\leq i\leq k-1} (f'(f^i(x_k)) - f'(0))$$

and we have that $f'(f^i(x_k)) = 2\rho(f^i(x_k)) = 2 + 2\omega_{f'}(f^i(x_k))$ and $f'(0) = 2$ and so we obtain

$$\left|\log \frac{(f^k)'(x_k)}{(f^k)'(0)}\right| \geq \frac{2}{\sigma} \sum_{0 \leq i \leq k-1} \omega_{f'}(f^i(x_k)).$$

We have that $f^i(x_k) = f^{i-k}(t_\omega)$ and so we obtain

$$f^i(x_k) \geq C\sigma^{i-k}$$

and hence we get

$$\left|\log \frac{(f^k)'(x_k)}{(f^k)'(0)}\right| \geq \frac{2}{\sigma} \sum_{0 \leq i \leq k-1} \omega_{f'}(C\sigma^{i-k}). \tag{4.11}$$

We can now apply a Lemma from [6].

Lemma 27 ([6])**.** *$\omega \in K$ is not Dini-integrable if and only if for every $\sigma > 1$ we have*

$$\lim_{k\to\infty} \sum_{1 \leq i \leq k} \omega(\sigma^{-i}) = \infty.$$

Applying Lemma 27 to the inequality in (4.11) we get that if if ω is not Dini-integrable we get

$$|\log \frac{(f^k)'(x_k)}{(f^k)'(0)}| \to \infty$$

and so f has unbounded distortion. Conversely, if ω is Dini-integrable then f has bounded distortion by [4] and so this finishes the proof. □

Proof of Theorem 18

To prove Theorem 18 we first prove that the set of moduli which are not Dini-integrable are generic. First of all, for every $r \in \mathbb{N}$ let $K_r \subset K$ be the space of moduli of continuity satisfying

$$\int_0^1 \frac{\omega(t)}{t} dt \leq r$$

and let

$$K_\infty := \bigcup_{r\in\mathbb{N}} K_r \quad \text{and} \quad K_* := K \setminus K_\infty$$

be the set of Dini integrable and non-Dini-integrable moduli of continuity respectively.

Proposition 28. *K_* is a residual (dense G_δ) set in the C^0-topology.*

Before proving the proposition we prove two lemmas.

Lemma 29. *The spaces K_r are closed subspaces of K in the C^0 topology.*

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence in K_r which converges uniformly to a map $\omega \in K$. For every $\epsilon > 0$ the sequence $\omega_n(t)/t$ converges uniformly to $\omega(t)/t$ on $[\epsilon, 1]$ and therefore

$$\int_\epsilon^1 \frac{\omega(t)}{t} dt = \lim \int_\epsilon^1 \frac{\omega_n(t)}{t} dt \leq r. \tag{4.12}$$

Since (4.12) holds for every $\epsilon > 0$ we deduce that $\omega \in K_r$ and hence K_r is closed. □

Lemma 30. *The spaces K_r have empty interior in K.*

Proof. We will show that K_* is dense in K, which clearly implies the statement. Let $\omega \in K_\infty$ and $\omega_0 \in K_*$ such that $|\omega_0|_\infty = 1$. For every ϵ we have that $\omega_\epsilon = \omega + \epsilon\omega_0 \in K_*$ and $|\omega_\epsilon - \omega|_\infty = \epsilon$. This implies that K_* intersects every open set in K and hence is dense. □

Proof of Proposition 28. Since K is complete in the uniform topology and K_∞ is a countable union of closed sets with empty interior we conclude by Baire's category theorem that K_* is a dense G_δ set. □

Proof of Theorem 18. By Theorem 17 for every $\omega \in K$ there exists an uncountable family $\mathcal{F}_\omega$ whose elements have unbounded distortion if and only if $\omega \in K_*$. In particular, for every $\omega \in K$ we have that

$$\mathcal{F}_\omega \cap \Gamma^1_{\omega,\lambda}(\mathbb{S}^1) \neq \emptyset. \tag{4.13}$$

Proposition 31. *There exists a continuous map $\varphi : K \to E^1(\mathbb{S}^1)$ such that $\forall \omega \in K$: $\varphi(\omega) \in \mathcal{F}_\omega \cap \Gamma^1_{\omega,\lambda}(\mathbb{S}^1)$*

Proof. Consider the map $\epsilon : K \to \mathbb{R}_+$ defined by $\epsilon(\omega) = \omega^{-1}(\frac{1}{2})$, the map ϵ is C^{1+mod} continuous. Consider now for every $\omega \in K$ a neighbourhood $V_\omega = (0, \frac{1}{4}\epsilon(\omega))$. On each V_ω define $\varphi'(\omega)|_{V_\omega} = 2 + 2\omega$ then, extend it to $[0, \frac{1}{2}]$ in a way that the collection of maps $\varphi|_{[0,\frac{1}{2}]}(\omega)$ forms a continuous family of C^1 function in the C^{1+mod} topology and satisfying conditions of Theorem 2, this is possible by the flexibility we have on the complement of V_ω in $[0, \frac{1}{2}]$ and since the maps are already continuous on the family of intervals $\{V_\omega\}$. By Theorem 2 these extend to Lebesgue preserving uniformly expanding maps on the circle, whose modulus of continuity is ω and it is also straightforward to check that the map $\varphi(\omega)$ depend continuously on ω in the C^{1+mod} topology. □

We chose $\Gamma = \varphi(K)$. Let $\mathcal{M} : \Gamma \to K$ denote the map which assigns to each $f \in \Gamma$ the modulus $\omega_{f'} \in K$, clearly $\mathcal{M}$ is continuous and so $\mathcal{M}^{-1}(K_\star)$ is a G_δ set, it remains to show that this set is dense, let $U \subset \Gamma$ be a non empty open set, since φ is continuous we have $\varphi^{-1}(U)$ is open in K and hence by density of $K_\star$ in K we obtain that $\mathcal{M}^{-1}(K_\star) \cap U \neq \emptyset$ which proves that $\mathcal{M}^{-1}(K_\star)$ is a residual set which finishes the proof. □

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