European Journal of Mathematics (2023) 9:108
https://doi.org/10.1007/s40879-023-00702-4

RESEARCH ARTICLE

Deformation of pairs and Noether–Lefschetz loci in toric varieties

Ugo Bruzzo[1,2,3,4] · **William D. Montoya**[5]

Received: 6 March 2022 / Revised: 7 August 2023 / Accepted: 11 October 2023

Abstract

We continue our study of the Noether–Lefschetz loci in toric varieties and investigate deformation of pairs (V, X) where V is a complete intersection subvariety and X a quasi-smooth hypersurface in a simplicial projective toric variety $\mathbb{P}^{2k+1}_{\Sigma}$, with $V \subset X$. The hypersurface X is supposed to be of *Macaulay type*, which means that its toric Jacobian ideal is Cox–Gorenstein, a property that generalizes the notion of Gorenstein ideal in the standard polynomial ring. Under some assumptions, we prove that the class $\lambda_V \in H^{k,k}(X)$ deforms to an algebraic class if and only if it remains of type (k, k). Actually we prove that locally the Noether–Lefschetz locus is an irreducible component of a suitable Hilbert scheme. This generalizes Theorem 4.2 in our previous work (Bruzzo and Montoya 15(2):682–694, 2021) and the main theorem proved by Dan (in: Analytic and Algebraic Geometry. Hindustan Book Agency, New Delhi, pp 107–115, 2017).

Keywords Noether–Lefschetz locus · Hodge locus · Toric varieties

---

The first author's research was partially supported by the CNPq "Bolsa de Produtividade em Pesquisa" 313333/2020, by the PRIN project "Moduli Theory and Birational Classification" and by GNSAGA-INdAM. The second author acknowledges support from FAPESP postdoctoral grant no. 2019/23499-7.

✉ Ugo Bruzzo
bruzzo@sissa.it

William D. Montoya
montoya@unicamp.br

1 Departamento de Matemática, Universidad Federal da Paraíba, Campus I, 58051-900 João Pessoa, PB, Brazil

2 SISSA (International School for Advanced Studies), Via Bonomea 265, 34136 Trieste, Italy

3 INFN (Istituto Nazionale di Fisica Nucleare), Sezione di Trieste, Trieste, Italy

4 IGAP (Institute for Geometry and Physics), Trieste, Italy

5 Instituto de Matemática, Estatística e Computação Científica, Universidad Estadual de Campinas, Rua Sérgio Buarque, de Holanda 651, 13083-859 Campinas, SP, Brazil

Published online: 09 November 2023 Springer

Mathematics Subject Classification 14C22 · 14C30 · 14J40 · 14J70 · 14M25

1 Introduction

In this short note we continue our study of the Noether–Lefschetz loci in toric varieties and investigate the deformation of pairs (V, X) where V is a k-dimensional complete intersection subvariety and X a quasi-smooth ample hypersurface in a simplicial projective toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ of odd dimension $2k + 1 \geqslant 3$, with $V \subset X$. We make two assumptions:

- The hypersurface X is supposed to be of *Macaulay type,* which means that its toric Jacobian ideal is Cox–Gorenstein, a property that generalizes the notion of Gorenstein ideal in a standard polynomial ring. This will be discussed in Sect. 3. Cox–Gorenstein ideals are studied in some detail in [2].
- The local Noether–Lefschetz locus $\mathrm{NL}^{k,\beta}_{\lambda_V,U}$, also called "Hodge locus" in the literature when $\mathbb{P}^{2k+1}_{\Sigma}$ is a projective space, as defined in Sect. 5, is not empty (a condition for this to happen is for instance given in [3, Lemma 3.7]). Here λ_V is the cohomology class of V, and β is the class of X in $\mathrm{Pic}(\mathbb{P}^{2k+1}_{\Sigma})$. Then the full Noether–Lefschetz locus NL_{β}, defined as the locus in the linear system $|\beta|$ of the points corresponding to quasi-smooth hypersurfaces whose (k, k)-cohomology does not come entirely from the ambient variety $\mathbb{P}^{2k+1}_{\Sigma}$, is locally analytically a finite union of Hodge loci [5].

Moreover, under the further assumption that β satisfies $\beta = q\eta + \beta'$, $n \in \mathbb{N}$, where $q \in \mathbb{Q}_{>0}$, η is a primitive ample class in $\mathbb{P}^{2k+1}_{\Sigma}$, and β' is a nef Cartier class, if X contains a k-dimensional complete intersection subvariety with $\deg_{\eta} V < qm_{k+1}$, where m_{k+1} is a rational number only depending on $\mathbb{P}^{2k+1}_{\Sigma}$ and the choice of a polarization, we will show that its associated cohomology class λ_V deforms to an algebraic class if and only it remains of type (k, k).

This extends the work of Dan in [10] and the last result of [4, Theorem 4.2] for toric varieties with higher Picard rank (there the Picard number was assumed to be one, and moreover, the result is asymptotic).

2 Infinitesimal variation of the Hodge structure

According to Batyrev and Cox in [1], the cohomology of hypersurfaces in projective simplicial toric varieties has a pure Hodge structure. In this section, we introduce its infinitesimal variation following the notions due to Carlson, Green, Griffiths and Harris in [7].

Definition 2.1 A *polarized Hodge structure of weight n*, denoted by $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$, is a Hodge structure together with a bilinear form $Q\colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ satisfying

$$\begin{aligned} &Q(\psi,\phi)=(-1)^n Q(\phi,\psi), && \\ &Q(\psi,\phi)=0, && \psi\in H^{p,q},\ \phi\in H^{p',q'} \text{ and } p\neq q', \\ &i^{p-q}Q(\psi,\overline{\psi})>0, && 0\neq\psi\in H^{p,q}. \end{aligned}$$

Definition 2.2 An *infinitesimal variation of Hodge structure* $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ is given by a polarized Hodge structure together with a vector space T and linear map

$$\delta\colon T \to \bigoplus_{1\leqslant p\leqslant n} \operatorname{Hom}\left(H^{p,q}, H^{p-1,q+1}\right)$$

that satisfies the following two conditions:

$$\begin{aligned} &\delta(\xi_1)\delta(\xi_2)=\delta(\xi_2)\delta(\xi_1), && \xi_1,\xi_2\in T, \\ &Q(\delta(\xi)\phi,\psi)+Q(\phi,\delta(\xi)\psi)=0 && \text{for } \xi\in T \text{ and } \phi\in F^p,\ \ \psi\in F^{n-p+1}. \end{aligned}$$

Here $F^\bullet$ is the filtration of H^n given by

$$F^p=\bigoplus_{i=0}^{p} H^{n-i,i}.$$

If $X \xrightarrow{i} \mathbb{P}^d_\Sigma$ is a quasi-smooth hypersurface in a simplicial projective toric variety $\mathbb{P}^d_\Sigma$ of dimension d, its primitive cohomology of degree $d-1$ is defined by the exact sequence [1]

$$0\to i^*H^{d-1}(\mathbb{P}^d_\Sigma,\mathbb{C})\to H^{d-1}(X,\mathbb{C})\to H^{d-1}_{\text{prim}}(X,\mathbb{C})\to 0.$$

The pullback i^* is compatible with the Hodge structures so that the primitive cohomology has a pure Hodge structure as well.

For a quasi-smooth hypersurface X in a simplicial projective toric variety, δ is the morphism associated via tensor-hom adjuction to $\gamma=\sum_p \gamma_p$, where

$$\gamma_p\colon T_X\mathcal{M}_\beta\otimes H^{p,d-1-p}_{\text{prim}}(X)\to H^{p,d-1-p}_{\text{prim}}(X)$$

is the natural multiplication map; for more details see [3, Section 3.3]. Given an infinitesimal variation of Hodge structure of weight $2k$, there is an invariant associated to $\gamma\in H^{k,k}_{\mathbb{Z}}$.

Definition 2.3 The *third invariant* associated to $\gamma\in H^{k,k}_{\mathbb{Z}}$ is

$$H^{k,k}(-\gamma):=\left\{\psi\in H^{k,k}\mid \langle\delta^0(\xi)\psi,\gamma\rangle=0 \text{ for all } \xi\in T\right\}.$$

Let us assume γ is the primitive part of the class of k-codimensional algebraic cycle $V = \sum_i n_i V_i$ in X with support $\sigma(V)$. Let $I_{\sigma(V)}$ be the ideal associated to $\sigma(V)$ and denote by $H^k(\Omega^k_X(-V))$ the image of the composed map

$$H^k\big(X, \Omega^k_X \otimes I_{\sigma(V)}\big) \to H^k(X, \Omega^k_X) \to H^k_{\text{prim}}(\Omega^k_X).$$

One has the following fact [11, Observation 4.a.4].

Lemma 2.4 $H^k(\Omega^k_X(-V)) \subseteq H^{k,k}(-\gamma)$.

This is the result we shall need later on.

3 Macaulay-type hypersurfaces

In this section we characterize a class of hypersurfaces in toric varieties that satisfy a generalization of the Macaulay theorem which holds for projective spaces. As we shall see, these are hypersurfaces whose toric Jacobian ideal (whose definition will be recalled later in this section) has a property which generalizes the notion of Gorenstein ideal in a polynomial ring.

The Cox ring S of a complete simplicial toric variety $\mathbb{P}^d_\Sigma$ is graded over the effective classes in the class group $\mathrm{Cl}(\mathbb{P}^d_\Sigma)$

$$S = \sum_{\alpha \in \mathrm{Cl}(\mathbb{P}^d_\Sigma)} S^\alpha, \quad S^\alpha = H^0\big(\mathbb{P}^d_\Sigma, \mathcal{O}_{\mathbb{P}^d_\Sigma}(\alpha)\big)$$

(see e.g. [8]). Following [2], we give a definition of *Cox–Gorenstein ideal* of the Cox rings which generalizes to toric varieties the definition given by Otwinowska in [12] for projective spaces.

Definition 3.1 A graded ideal I of S is said to be a *Cox–Gorentstein ideal of socle degree* $N \in \mathrm{Cl}(\mathbb{P}^d_\Sigma)$ if

- the quotient $R = S/I$ is Artinian;
- $\dim_{\mathbb{C}} R^N = 1$;
- for every homogeneous class $\alpha \in \mathrm{Cl}(\mathbb{P}^d_\Sigma)$, either the natural bilinear morphism (called "Poincaré duality")

$$R^\alpha \times R^{N-\alpha} \to R^N \simeq \mathbb{C}$$

  is nondegenerate, or $R^\alpha = R^{N-\alpha} = 0$.

Example 3.2 We give here some examples of Cox–Gorenstein ideals. In all cases the proof that the relevant ideal is Cox–Gorenstein is done by direct computation.
1. $\mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates (x, y, u, v), and

$$I = \big(x^2u - y^2v, xv, yu, x^3, y^3, u^2, v^2, xy\big).$$

I is Cox–Gorenstein of socle degree (2, 1).

2. $\mathbb{P}^1 \times \mathbb{P}^2$ with homogeneous coordinates (x, y, u, v, w); the annihilator of $f = xu^2 + uvw$ in the ring of polynomial operators $\mathbb{C}[\partial_x, \partial_y, \partial_u, \partial_v, \partial_w]$ is a Cox–Gorenstein ideal of socle degree (2, 1).

3. A singular example is provided by the fake weighted projective space associated with the fan generated by $v_1 = (-3, -2)$, $v_2 = (1, 2)$, $v_3 = (1, 0)$ in $\mathbb{R}^3$. The resulting variety has class group $\mathbb{Z} \oplus \mathbb{Z}_2$ and is a quotient $\mathbb{P}[1, 1, 2]/\mathbb{Z}_2$. The divisors D_1, D_2, D_3 associated with the rays have bidegree (1, 1), (1, 0) and (2, 1), respectively. Write the Cox ring as $S = \mathbb{C}[x, y, z]$ and consider the ideal $I = (x, y^2, z^3)$; its socle degree is $N = (5, 0)$. Indeed $R^{5,0}$ is generated by the class of the monomial yz^2. The other nonzero graded pieces of R are

$$R^{0,0} = \mathbb{C}, \quad R^{1,0} = \mathbb{C}[y], \quad R^{2,1} = \mathbb{C}[z], \quad R^{3,1} = \mathbb{C}[yz], \quad R^{4,0} = \mathbb{C}[z^2]$$

which clearly satisfy the Poincaré duality.

Examples of Cox–Gorenstein ideals may be given in terms of *toric Jacobian ideals*. For every ray $\rho \in \Sigma(1)$ denote by v_ρ its rational generator, and by x_ρ the corresponding variable in the Cox ring. Recall that d is the dimension of the toric variety $\mathbb{P}^d_\Sigma$, while we denote by $r = \#\,\Sigma(1)$ the number of rays. Given $f \in S^\beta$, one defines its *toric Jacobian ideal* as

$$J_0(f) = \left(x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_r} \frac{\partial f}{\partial x_{\rho_r}} \right).$$

We recall from [1] the definition of nondegenerate hypersurface and some properties (Definition 4.13 and Proposition 4.15).

Definition 3.3 Let $f \in S^\beta$, with β an ample Cartier class. The associated hypersurface $X_f \subset \mathbb{P}^d_\Sigma$ is *nondegenerate* if for all $\sigma \in \Sigma$ the affine hypersurface $X_f \cap O(\sigma)$ is a smooth codimension one subvariety of the orbit $O(\sigma)$ of the action of the torus $\mathbb{T}^d$.

Proposition 3.4 (1) *Every nondegenerate hypersurface is quasi-smooth.*
(2) *If f is generic then X_f is nondegenerate.*

We collect here, with some changes in the terminology, some results that are already contained in [9, Proposition 5.3].

Proposition 3.5 *Let $f \in S^\beta$, and let $\{\rho_1, \ldots, \rho_d\} \subset \Sigma(1)$ be such that $v_{\rho_1}, \ldots, v_{\rho_d}$ are linearly independent.*

(1) *The toric Jacobian ideal of f coincides with the ideal*

$$\left(f, x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_d} \frac{\partial f}{\partial x_{\rho_d}} \right).$$

(2) *The following conditions are equivalent:*

(a) *f is nondegenerate;*
(b) *the polynomials $x_{\rho_i}\frac{\partial f}{\partial x_{\rho_i}}$, $i=1,\dots,r$, do not vanish simultaneously on X_f;*
(c) *the polynomials f and $x_{\rho_i}\frac{\partial f}{\partial x_{\rho_i}}$, $i=1,\dots,d$, do not vanish simultaneously on X_f.*

Now we define the notion of *hypersurface of Macaulay type*.

Definition 3.6 Let $f\in S^\beta$ be nondegenerate, with β an ample Cartier class. f is said to be of the *Macaulay type* if its toric Jacobian ideal $J_0(f)$ is a Cox–Gorenstein ideal of socle degree $N=(d+1)\beta-\beta_0$, where β_0 is the anticanonical class of $\mathbb{P}^d_\Sigma$.

Example 3.7 1. According to this definition, any generic smooth hypersurface in $\mathbb{P}^d$ is of Macaulay type.

2. Macaulay-type hypersurfaces in singular toric varieties do exist; a simple example is the curve $x+y^2+z^2=0$ in $\mathbb{P}[1,1,2]$, where $\deg x=2$ and $\deg y=\deg z=1$.

3. Another singular example, this time with class group different from $\mathbb{Z}$, is provided by the fake weighted projective space of Example 3.2.3 by letting $f=x^4+y^2+z^2$. The toric Jacobian ideal is $I=(x^4,y^4,z^2)$ and the socle degree is $N=(8,0)$.

Actually a result in [2] shows that every nondegenerate ample Cartier hypersurface in a simplicial projective toric variety with Picard number 1 is of Macaulay type.

4 The tangent space to the Noether–Lefschetz locus

From now on we assume $d=2k+1$. Let $f\in S^\beta$ define a nondegenerate quasi-smooth hypersurface X in $\mathbb{P}^{2k+1}_\Sigma$ and suppose β is ample. Moreover, we assume that the hypersurface X is of Macaulay type. Let $N=(k+1)\beta-\beta_0$ and let $J_0(f)$ be the toric Jacobian ideal associated to f, which is Cox–Gorenstein of socle degree $2N+\beta_0$. Then there is a perfect pairing $R_0^\alpha\times R_0^{2N+\beta_0-\alpha}\to R_0^{2N+\beta_0}$ for $\alpha\leqslant 2N+\beta_0$. Let us denote by T_0' the subspace of R_0^N which is the kernel of the multiplication map $\cdot x_1,\dots,x_rP\colon R_0^N\to R_0^{2N+\beta_0}$ and by T_0 its inverse image in S^N, where P is a preimage of γ under the natural map

$$\begin{array}{ccccc} S^N & \longrightarrow & S^N/J^N & \xrightarrow{\sim} & H^{k,k}_{\mathrm{prim}}(X) \\ P & \longmapsto & \overline{P} & \longmapsto & \gamma. \end{array}$$

Definition 4.1 Let $T\subset S$ be the $\mathrm{Cl}(\Sigma)$-graded module such that T^α is the largest subspace where $T^\alpha\otimes S^{N-\alpha}$ is contained in T_0 for $\alpha\leqslant N$, $T^N=T_0$ and $E^{N+\alpha}=T_0\otimes S^\alpha$ for $\alpha\geqslant 0$.

Remark 4.2 Note that T is a Cox–Gorentein ideal with socle degree N.

Actually T^β is the tangent space of the local Noether–Lefschetz locus at f.[1]

[1] For ease of notation we write f but we mean its class modulo a nonzero constant factor.

Lemma 4.3 *$T_f \,\mathrm{NL}_{\lambda,\beta} \cong T^\beta$, where λ is a primitive class in $H^{k,k}(X_f, \mathbb{Q})$.*

Proof An overbar will denote the class in $R = S/J$ of an element in S. Now, $H \in T^\beta$ if and only if $\overline{H} \otimes R^{N-\beta}$ is contained in T_0', which is equivalent to

$$\overline{x_0 \ldots x_r}\,\overline{PH} \otimes R^{N-\beta} = 0 \quad \text{in} \quad R^{N+\beta_0};$$

using Poincaré duality that means $\overline{x_0 \ldots x_r}\,\overline{PH} = 0$ in $R^{N+\beta+\beta_0}$ and equivalently $\overline{PH} = 0$ in $R^{N+\beta}$ if and only if $H \in T_f \,\mathrm{NL}_{\lambda,\beta}$ (see [6, Theorem 6.2]). □

Let us suppose that V is the zero locus of $\langle A_1, \ldots, A_{k+1}\rangle$ and since $V \subset X_f$ there exist polynomials $K_1, \ldots, K_{k+1}$ of degree $\beta - \deg(A_i)$ such that $f = A_1K_1 + \cdots + A_{k+1}K_{k+1}$. Let $I = \langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1}\rangle$.

Proposition 4.4 *$T^\alpha = I^\alpha$ for $\alpha \leqslant N$.*

Proof Let W_1 be the zero locus of $\langle K_1, A_2, \ldots, A_{k+1}\rangle$. Since $V \cup W_1$ is equal to $X_f \cap \{A_2 = \cdots = A_{k+1} = 0\}$, λ_V is equal to $-\lambda_{W_1}$ in the primitive cohomology. Now, let us denote by W_2 the zero locus of $K_1, \ldots, K_{k+1}$ then, as before, $[\lambda_V]_{\mathrm{prim}} = [a\lambda_{W_2}]_{\mathrm{prim}}$, $a \in \mathbb{Z}$. By Lemma 2.4 we have $\langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1}\rangle \subset T$. Since X is quasi-smooth, the ideal $\langle A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1}\rangle$ is Cox–Gorenstein with socle degree N, the socle degree of T, so that I and T coincide in degree $\alpha \leqslant N$. □

5 Main theorem

In this section we prove our main result. We start by recalling the construction of the local Noether–Lefschetz locus [6]. Given an ample class β in $\mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma)$, let

$$\mathcal{U}_\beta \subset \mathbb{P}\big(H^0(\mathbb{P}^{2k+1}_\Sigma), \mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)\big)$$

be the open subset parameterizing quasi-smooth hypersurfaces and let $\pi : \chi_\beta \to \mathcal{U}_\beta$ be the tautological family. One considers the local system $\mathcal{H}^{2k} = R^{2k}\pi_\star\mathbb{C} \otimes \mathcal{O}_{\mathcal{U}_\beta}$ over $\mathcal{U}_\beta$.

If $f \in \mathcal{U}_\beta$, let $\lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q}))$ be a nonzero class, and let $U \subset \mathcal{U}_\beta$ be a contractible open subset around f. Finally, let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by λ_f and let $\overline{\lambda}$ be its image in $(\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U)$, where

$$F^k\mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}.$$

Definition 5.1 (*Local Noether–Lefschetz Locus*) $\mathrm{NL}^{k,\beta}_{\lambda,U} = \{G \in U \mid \overline{\lambda}_G = 0\}$.

Let η be a polarization for $\mathbb{P}^{2k+1}_\Sigma$, that we assume to be primitive in the Picard group. Given the Hilbert polynomial P of a subscheme V, computed with respect to η, we denote by Hilb_P the Hilbert scheme of closed subschemes of $\mathbb{P}^{2k+1}_\Sigma$ with Hilbert polynomial P. We denote by Q the Hilbert polynomial of quasi-smooth hypersurface in $\mathbb{P}^{2k+1}_\Sigma$ whose class in the Picard group is β. The flag Hilbert scheme $\mathrm{Hilb}_{P,Q}$

parametrizes all pairs (V, X) where $V \in \mathrm{Hilb}_P$ and X is a quasi-smooth hypersurface in $\mathbb{P}^{2k+1}_\Sigma$ of class β containing V. Let pr_1 be the projection to the first component and $\mathrm{pr}_2\colon \mathrm{Hilb}_{P,Q} \to \mathcal{U}_\beta$ the natural projection to the open set which parametrizes quasi-smooth hypersurfaces in $\mathbb{P}^{2k+1}_\Sigma$. Note that $\mathrm{pr}_1(\mathrm{Hilb}_{P,Q})$ is irreducible, so that there exists a unique component in $\mathrm{Hilb}_{P,Q}$ such that $\mathrm{pr}_1(\mathrm{Hilb}_{P,Q})$ coincides with the parameter space for complete intersection subschemes in $\mathbb{P}^{2k+1}_\Sigma$.

For Z a d-dimensional closed subvariety of $\mathbb{P}^{2k+1}_\Sigma$ we define its degree as $\deg_\eta Z = [Z]\cdot\eta^d$.

Lemma 5.2 *There is a positive rational number m_{k+1} such that $\deg_\eta W \geqslant m_{k+1}$ for all $(k+1)$-dimensional closed subvarieties W of $\mathbb{P}^{2k+1}_\Sigma$.*

Proof Let a be the smallest integer such that $a\eta$ is very ample. Then $a\eta$ defines a closed embedding $j\colon \mathbb{P}^{2k+1}_\Sigma \to \mathbb{P}^N$ for some N. Denoting by H the hyperplane class in $\mathbb{P}^N$, one has

$$\deg_\eta W = \frac{1}{a^k}\, j^*H^k\cdot[W] = \frac{1}{a^k}\, H^k\cdot j_*[W] \geqslant \frac{1}{a^k}$$

and one sets $m_{k+1} = 1/a^k$. □

The next lemma is a version of the Bézout theorem in the present context.

Lemma 5.3 *If X is an ample Cartier hypersurface in $\mathbb{P}^{2k+1}_\Sigma$ whose class in $\mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma)$ satisfies $\beta = q\eta + \beta'$, where $q \in \mathbb{N}_{>0}$ and β' is a nef Cartier class, and $V = X\cap W$ is a k-dimensional subvariety contained in X, where W is a $(k+1)$-dimensional closed subvariety $W \subset \mathbb{P}^{2k+1}_\Sigma$, then $\deg_\eta V \geqslant qm_{k+1}$.*

Proof We shall denote by (Z) the class in $A_d(\mathbb{P}^{2k+1}_\Sigma)$ of a d-dimensional closed subvariety Z of $\mathrm{Pic}(\mathbb{P}^{2k+1}_\Sigma)$, and by $[Z]$ its class in $A^{2k+1-d}(\mathbb{P}^{2k+1}_\Sigma)$. Thus we have

$$\begin{aligned}\deg_\eta V &= \langle \eta^k, (W)\cap[X]\rangle = \langle \eta^k\cup[X], (W)\rangle = \langle \eta^k\cup(q\eta+\beta'), (W)\rangle\\ &= q\deg_\eta W + \langle \eta^k\cup\beta', [W]\rangle \geqslant qm_{k+1} + \langle \eta^k\cup\beta', [W]\rangle.\end{aligned}$$

Since β' is nef we have $\langle \eta^k\cup\beta', [W]\rangle \geqslant 0$, hence the claim follows. □

Now we state and prove the main result of this paper.

Theorem 5.4 *Assume that β is as in Lemma 5.3. Let V be a quasi-smooth complete intersection in $\mathbb{P}^{2k+1}_\Sigma$ of codimension $k+1$ and let X be a quasi-smooth hypersurface of class β containing V such that $\deg_\eta V < qm_{k+1}$. Assume also that X is of the Macaulay type. Then,*

λ_V deforms to a (k,k) class if and only if $\lambda_{[V]}$ deforms to an algebraic cycle.

In particular, for a suitable open subset U, $\mathrm{NL}^{k,\beta}_{\lambda_V,U}$ is isomorphic to an irreducible component of $U\cap\mathrm{pr}_2(\mathrm{Hilb}_{P,Q})$, where P and Q are the Hilbert polynomials of V and X, respectively.

Proof By the assumption on the degree of V, one has $\mathrm{pr}_2(\mathrm{Hilb}_{P,Q}) \subset \mathrm{NL}^{k,\beta}_{\lambda_V,U}$. Then,

$$\mathrm{codim}_U\, \mathrm{pr}_2(\mathrm{Hilb}_{P,Q}) \geqslant \mathrm{codim}_U\, \mathrm{NL}^{k,\beta}_{\lambda_V,U} \geqslant \mathrm{codim}_{T_X U}\, T_X \mathrm{NL}^{k,\beta}_{\lambda_V,U}.$$

On the other hand, keeping in mind that $T^\beta = I^\beta \subset I^\beta_V$, we have a natural map ϕ from T_β to $\mathrm{Hilb}_{P,Q}$, which sends a homogeneous polynomial of degree β to its zero locus. One has $\overline{\mathrm{Im}\,(\phi)} \subset \overline{\mathrm{pr}_2(\mathrm{Hilb}_{P,Q})}$ and since the zero locus is invariant under the torus action, $\dim T^\beta > \dim \overline{\mathrm{Im}\,(\phi)}$. Hence,

$$\mathrm{codim}\, \mathrm{pr}_2(\mathrm{Hilb}_{P,Q}) \leqslant \mathrm{codim}\, \overline{\mathrm{Im}\,(\phi)} \leqslant \mathrm{codim}\, T^\beta = \mathrm{codim}\, T_X \mathrm{NL}^{k,\beta}_{\lambda_V,U}.$$

So $\mathrm{pr}_2(\mathrm{Hilb}_{P,Q})$ and $\mathrm{NL}^{k,\beta}_{\lambda_V,U}$ have the same dimension, which implies the claim. □

Note that the Noether-Lefschetz locus $\mathrm{NL}_{\lambda_V \beta}$ is nonempty as V is primitive due to Lemma 5.3.

Acknowledgements We thank Ananyo Dan for interesting correspondence about the variational Hodge conjecture in projective spaces, Rafael Holanda and Rodrigo Godim for useful discussions about Gorenstein ideals, and the referee for their careful reading of the paper and for suggesting several improvements.

Funding Open access funding provided by Scuola Internazionale Superiore di Studi Avanzati - SISSA within the CRUI-CARE Agreement.

References

1. Batyrev, V.V., Cox, D.A.: On the Hodge structure of projective hypersurfaces in toric varieties. Duke Math. J. **75**(2), 293–338 (1994)
2. Bruzzo, U., Gondim, R., Holanda, R., Montoya, W.D.: Artinian Cox–Gorenstein graded $\mathbb{C}$-algebras (work in progress)
3. Bruzzo, U., Grassi, A.: Picard group of hypersurfaces in toric 3-folds. Internat. J. Math. **23**(2), Art. No. 1250028 (2012)
4. Bruzzo, U., Montoya, W.D.: On the Hodge conjecture for quasi-smooth intersections in toric varieties. São Paulo J. Math. Sci. **15**(2), 682–694 (2021)
5. Bruzzo, U., Montoya, W.D.: Codimension bounds for the Noether–Lefschetz components for toric varieties. Eur. J. Math. **8**(3), 806–814 (2022)
6. Bruzzo, U., Montoya, W.D.: An asymptotic description of the Noether–Lefschetz components in toric varieties (2022). arXiv:1905.01570v3
7. Carlson, J., Green, M., Griffiths, P., Harris, J.: Infinitesimal variations of Hodge structure (I). Compositio Math. **50**(2–3), 109–205 (1983)
8. Cox, D.A.: The homogeneous coordinate ring of a toric variety. J. Algebraic Geom. **4**(1), 17–50 (1995)
9. Cox, D.A.: Toric residues. Ark. Mat. **34**(1), 73–96 (1996)
10. Dan, A.: Noether–Lefschetz locus and a special case of the variational Hodge conjecture: Using elementary techniques. In: Aryasomayajula, A., et al. (eds.) Analytic and Algebraic Geometry, pp. 107–115. Hindustan Book Agency, New Delhi (2017)

11. Griffiths, P., Harris, J.: Infinitesimal variations of Hodge structure (II): an infinitesimal invariant of Hodge classes. Compositio Math. **50**(2–3), 207–265 (1983)
12. Otwinowska, A.: Composantes de petite codimension du lieu de Noether–Lefschetz: un argument asymptotique en faveur de la conjecture de Hodge pour les hypersurfaces. J. Algebraic Geom. **12**(2), 307–320 (2003)