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Pure gravity traveling quasi-periodic water waves with constant vorticity

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Abstract

We prove the existence of small amplitude time quasi-periodic solutions of the *pure gravity* water waves equations with *constant vorticity*, for a bidimensional fluid over a flat bottom delimited by a space periodic free interface. Using a Nash-Moser implicit function iterative scheme we construct traveling nonlinear waves which pass through each other slightly deforming and retaining forever a quasiperiodic structure. These solutions exist for any fixed value of depth and gravity and restricting the vorticity parameter to a Borel set of asymptotically full Lebesgue measure.

Contents

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1 Introduction

A problem of fundamental importance in fluid mechanics regards the search for traveling surface waves. Since the pioneering work of Stokes [32] in 1847, a huge literature has established the existence of steady traveling waves, namely solutions (either periodic or localized in space) which look stationary in a moving frame. The majority of the results concern bidimensional fluids. At the end of the section we shortly report on the vast literature on this problem.

In the recent work [7] we proved the first bifurcation result of *time quasi-periodic traveling* solutions of the water waves equations under the effects of gravity, constant vorticity, and exploiting the capillarity effects at the free surface. For pure gravity irrotational water waves with infinite depth, quasi-periodic traveling waves has been obtained in Feola-Giuliani [16].

The goal of this paper is to prove the existence of time quasi-periodic traveling water waves, also in the physically important case of the *pure gravity* equations with non zero *constant vorticity*, for any value of the *depth* of the water, finite or infinite. These solutions are a nonlinear superposition of multiple Stokes waves traveling with rationally independent speeds, and can not be reduced to steady solutions in any moving frame. We are able to use the vorticity as a parameter: the solutions that we construct exist for any value of gravity and depth of the fluid, provided the vorticity is restricted to a Borel set of asymptotically full measure, see Theorem 1.2. We also remark that, in case of non zero vorticity, bifurcation of standing waves can not be expected, as they are not allowed by the linear theory.

It is well known that this is a subtle small divisor problem. Major difficulties are that: (i) the vorticity parameter enters in the dispersion relation only at the zero order; (ii) there are resonances among the linear frequencies which can be avoided only for traveling waves; (iii) the dispersion relation of the pure gravity equations is sublinear at infinity; (iv) the nonlinear transport term is a singular perturbation of the unperturbed linear water waves vector field. Related difficulties appear in the search of pure gravity time periodic *standing* waves which have been constructed in the last years for irrotational fluids by Iooss, Plotnikov, Toland [30, 24, 21], extended to time quasi-periodic standing waves in Baldi-Berti-Haus-Montalto [2]. In presence of surface tension, time periodic and quasi-periodic standing waves were constructed respectively by Alazard-Baldi

[1] and Berti-Montalto [9]. We mention that also the construction of gravity steady traveling waves periodic in space presents small divisor difficulties for three dimensional fluids. These solutions, in a moving frame, look steady bi-periodic waves and have been constructed for irrotational fluids by Iooss-Plotnikov [22, 23] using the speed as a bidimensional parameter (for gravity-capillary waves in [13], this is not a small divisor problem).

We first recall the equations. We consider the Euler equations of hydrodynamics for a 2-dimensional incompressible and inviscid fluid with constant vorticity γ, under the action of *pure gravity*. The fluid occupies the region $\mathcal{D}_{\eta,\mathtt{h}} := \big\{(x,y) \in \mathbb{T}\times\mathbb{R} \, : \, -\mathtt{h} < y < \eta(t,x)\big\}, \mathbb{T} := \mathbb{T}_x := \mathbb{R}/(2\pi\mathbb{Z})$, with a (possibly infinite) depth $\mathtt{h} > 0$ and space periodic boundary conditions. The unknowns of the problem are the free surface $y = \eta(t,x)$ of the time dependent domain $\mathcal{D}_{\eta,\mathtt{h}}$ and the divergence free velocity field $\big(\begin{smallmatrix} u(t,x,y) \\ v(t,x,y)\end{smallmatrix}\big)$. If the fluid has constant vorticity $v_x - u_y = \gamma$, the velocity field is the sum of the Couette flow $\big(\begin{smallmatrix} -\gamma y \\ 0\end{smallmatrix}\big)$ (recently studied in [5, 37] and references therein), which carries all the vorticity γ of the fluid, and an irrotational field, expressed as the gradient of a harmonic function Φ, called the generalized velocity potential. Denoting $\psi(t,x) := \Phi(t,x,\eta(t,x))$ the evaluation of the generalized velocity potential at the free interface, one recovers Φ by solving the elliptic problem $\Delta\Phi = 0$ in $\mathcal{D}_{\eta,\mathtt{h}}$, $\Phi = \psi$ at $y = \eta(t,x)$ and $\Phi_y \to 0$ as $y \to -\mathtt{h}$. The last condition means the impermeability of the bottom: $\Phi_y(t,x,-\mathtt{h}) = 0$ if $\mathtt{h} < \infty$, and $\lim_{y\to-\infty} \Phi_y(t,x,y) = 0$, if $\mathtt{h} = +\infty$. Imposing that the fluid particles at the free surface evolve onto it (kinematic boundary condition) and that the pressure of the fluid equals the constant atmospheric pressure at the free surface (dynamic boundary condition), the time evolution of the fluid is determined by the following system of equations

$$
\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \, . \end{cases} \tag{1.1}
$$

Here g is the gravity and $G(\eta)$ is the Dirichlet-Neumann operator

$$
G(\eta)\psi := G(\eta,\mathtt{h})\psi := (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)} \, .
$$

As observed in the irrotational case by Zakharov [39], and in presence of constant vorticity by Wahlén [36], the water waves equations (1.1) are the Hamiltonian system $\eta_t = \nabla_\psi H$, $\psi_t = (-\nabla_\eta + \gamma\partial_x^{-1}\nabla_\psi)H$, where ∇

denotes the L^2-gradient, with Hamiltonian (cfr. Section 2)

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \Big(\psi\, G(\eta)\psi + g\eta^2\Big)\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}} \Big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\Big)\,\mathrm{d}x\,. \tag{1.2}$$

The equations (1.1) enjoy two important symmetries. First of all, they are time reversible. We say that a solution of (1.1) is *reversible* if $\eta(-t,-x) = \eta(t,x)$, $\psi(-t,-x) = -\psi(t,x)$. Second, since the bottom of the fluid domain is flat, they are *invariant by space translations*.

The variables (η,ψ) of system (1.1) belong to some Sobolev space $H^s_0(\mathbb{T}) \times \dot{H}^s(\mathbb{T})$ for some s large. Here $H^s_0(\mathbb{T})$ is the Sobolev space of functions with zero average $H^s_0(\mathbb{T}) := \big\{u \in H^s(\mathbb{T}) : \int_{\mathbb{T}} u(x)\mathrm{d}x = 0\big\}$ and $\dot{H}^s(\mathbb{T})$ the corresponding homogeneous Sobolev space, obtained by identifying the functions in $H^s(\mathbb{T})$ which differ by a constant. This choice of the phase space is allowed because $\int_{\mathbb{T}} \eta(t,x)\,\mathrm{d}x$ is a prime integral of (1.1) and the right hand side of (1.1) depends only on η and $\psi - \frac{1}{2\pi}\int_{\mathbb{T}}\psi\,\mathrm{d}x$.

Linearizing (1.1) at the equilibrium $(\eta,\psi) = (0,0)$ gives the system

$$\partial_t\eta = G(0)\psi\,, \quad \partial_t\psi = -g\eta + \gamma\partial_x^{-1}G(0)\psi\,, \tag{1.3}$$

where $G(0)$ is the Dirichlet-Neumann operator at the flat surface $\eta = 0$. A direct computation reveals that $G(0) := G(0,\mathtt{h})$ is the Fourier multiplier operator

$$G(0,\mathtt{h}) := D\,\tanh(\mathtt{h}D) \text{ if } \mathtt{h} < \infty\,, \quad G(0,\mathtt{h}) := |D| \text{ if } \mathtt{h} = +\infty\,, \tag{1.4}$$

where $D := \frac{1}{\mathrm{i}}\partial_x$. Thus its symbol $G_j(0) := G_j(0,\mathtt{h})$ is, for any $j \in \mathbb{Z}$,

$$G_j(0,\mathtt{h}) := j\tanh(\mathtt{h}j) \;\text{ if } \mathtt{h} < \infty\,, \quad G_j(0,\mathtt{h}) := |j| \;\text{ if } \mathtt{h} = +\infty\,. \tag{1.5}$$

As showed in Section 2, all reversible solutions of (1.3) are the linear superposition of plane waves, traveling either to the right or to the left, given by

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} = \sum_{n\in\mathbb{N}} &\begin{pmatrix} M_n\rho_n\cos(nx - \Omega_n(\gamma)t)\\ P_n\rho_n\sin(nx - \Omega_n(\gamma)t)\end{pmatrix}\\ &+ \begin{pmatrix} M_n\rho_{-n}\cos(nx + \Omega_{-n}(\gamma)t)\\ P_{-n}\rho_{-n}\sin(nx + \Omega_{-n}(\gamma)t)\end{pmatrix}\,,\end{aligned} \tag{1.6}$$

where $\rho_n \geqslant 0$ are arbitrary amplitudes and M_n, $P_{\pm n}$ are the real coefficients

$$M_j := \left(\frac{G_j(0)}{g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}}\right)^{\frac{1}{4}},\ j \in \mathbb{Z}\setminus\{0\},\ \ P_{\pm n} := \frac{\gamma}{2}\frac{M_n}{n} \pm M_n^{-1},\ n \in \mathbb{N}\,. \tag{1.7}$$

The frequencies $\Omega_{\pm n}(\gamma)$ in (1.6) are

$$\Omega_j(\gamma) := \sqrt{\Big(g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}\Big)G_j(0)} + \frac{\gamma}{2}\frac{G_j(0)}{j}\,, \quad j \in \mathbb{Z}\backslash\{0\}\,. \tag{1.8}$$

Note that the map $j \mapsto \Omega_j(\gamma)$ is not even due to the vorticity term $\gamma G_j(0)/j$, which is odd in j.

All the linear solutions (1.6) are either time periodic, quasi-periodic or almost-periodic, depending on the irrationality properties of the frequencies $\Omega_{\pm n}(\gamma)$ and the number of non zero amplitudes $\rho_{\pm n}$. The problem of the existence of traveling quasi-periodic in time water waves is formulated as follows.

Definition 1.1. (Quasi-periodic traveling wave) We call $(\eta(t,x),\psi(t,x))$ a time quasi-periodic *traveling* wave with irrational frequency vector $\omega = (\omega_1,\ldots,\omega_\nu) \in \mathbb{R}^\nu$, $\nu \in \mathbb{N}$, i.e. $\omega\cdot\ell \neq 0$ for any $\ell \in \mathbb{Z}^\nu\backslash\{0\}$, and "wave vectors" $(j_1,\ldots,j_\nu) \in \mathbb{Z}^\nu$, if there exist functions $(\breve\eta,\breve\psi) : \mathbb{T}^\nu \to \mathbb{R}^2$ such that $(\eta(t,x),\psi(t,x)) = (\breve\eta(\omega_1 t - j_1 x,\ldots,\omega_\nu t - j_\nu x), \breve\psi(\omega_1 t - j_1 x,\ldots,\omega_\nu t - j_\nu x))$.

Note that, if $\nu = 1$, such functions are time periodic and indeed stationary in a moving frame with speed ω_1/j_1. If the number of the irrational frequencies in greater or equal than 2, such waves cannot be reduced to steady waves by any choice of the moving frame.

We construct traveling quasi-periodic solutions of the nonlinear equations (1.1) with a diophantine frequency vector ω belonging to an open bounded subset Ω in $\mathbb{R}^\nu$, namely, for some $\upsilon \in (0,1)$, $\tau > \nu - 1$,

$$\mathtt{DC}(\upsilon,\tau) := \Big\{\omega \in \Omega \subset \mathbb{R}^\nu \ : \ |\omega\cdot\ell| \geqslant \upsilon\langle\ell\rangle^{-\tau}\,,\ \forall\,\ell \in \mathbb{Z}^\nu\backslash\{0\}\Big\}\,,$$

where $\langle\ell\rangle := \max\{1,|\ell|\}$, and with $(\breve\eta,\breve\psi)$ in some Sobolev space

$$H^s(\mathbb{T}^\nu,\mathbb{R}^2) = \Big\{\breve f(\varphi) = \sum_{\ell\in\mathbb{Z}^\nu} f_\ell\, e^{\mathrm{i}\ell\cdot\varphi} \in \mathbb{R}^2 \ : \ \|\breve f\|_s^2 := \sum_{\ell\in\mathbb{Z}^\nu} |f_\ell|^2\langle\ell\rangle^{2s} < \infty\Big\}\,.$$

Fixed finitely many arbitrary *distinct* natural numbers

$$\mathbb{S}^+ := \{\overline{n}_1,\ldots,\overline{n}_\nu\} \subset \mathbb{N}\,, \quad 1 \leqslant \overline{n}_1 < \ldots < \overline{n}_\nu\,, \tag{1.9}$$

and signs

$$\Sigma := \{\sigma_1,\ldots,\sigma_\nu\}, \quad \sigma_a \in \{-1,1\}\,, \quad a = 1,\ldots,\nu\,, \tag{1.10}$$

we consider reversible quasi-periodic traveling wave solutions of the linear system (1.3), given by

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} &= \sum_{a\in\{1,\dots,\nu\colon \sigma_a=+1\}}\begin{pmatrix}M_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\cos(\overline{n}_a x-\Omega_{\overline{n}_a}(\gamma)t)\\ P_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\sin(\overline{n}_a x-\Omega_{\overline{n}_a}(\gamma)t)\end{pmatrix}\\ &+\sum_{a\in\{1,\dots,\nu\colon \sigma_a=-1\}}\begin{pmatrix}M_{\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\cos(\overline{n}_a x+\Omega_{-\overline{n}_a}(\gamma)t)\\ P_{-\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\sin(\overline{n}_a x+\Omega_{-\overline{n}_a}(\gamma)t)\end{pmatrix}\end{aligned} \tag{1.11}$$

where $\xi_{\pm\overline{n}_a}>0$, $a=1,\dots,\nu$. The frequency vector of (1.11) is given by

$$\vec{\Omega}(\gamma) := (\Omega_{\sigma_a\overline{n}_a}(\gamma))_{a=1,\dots,\nu}\in\mathbb{R}^\nu\,. \tag{1.12}$$

Theorem 1.2 proves the existence of quasi-periodic traveling waves of (1.1), close to the linear solutions (1.11), for most values of the vorticity $\gamma\in[\gamma_1,\gamma_2]$, with a frequency vector $\widetilde{\Omega}:=(\widetilde{\Omega}_{\sigma_a\overline{n}_a})_{a=1,\dots,\nu}$, close to $\vec{\Omega}(\gamma):=(\Omega_{\sigma_a\overline{n}_a}(\gamma))_{a=1,\dots,\nu}$.

Theorem 1.2. (KAM for traveling gravity water waves with constant vorticity) *Consider finitely many tangential sites* $\mathbb{S}^+\subset\mathbb{N}$ *as in* (1.9) *and signs* Σ *as in* (1.10). *Fix a subset* $[\gamma_1,\gamma_2]\subset\mathbb{R}$. *Then there exist* $\overline{s}>0$, $\varepsilon_0\in(0,1)$ *such that, for any* $|\xi|\leqslant\varepsilon_0^2$, $\xi:=(\xi_{\sigma_a\overline{n}_a})_{a=1,\dots,\nu}\in\mathbb{R}^\nu_+$, *the following hold:*

1) There exists a Borel set $\mathcal{G}_\xi\subset[\gamma_1,\gamma_2]$ *with density one at* $\xi=0$, *i.e.* $\lim_{\xi\to0}|\mathcal{G}_\xi|=\gamma_2-\gamma_1$;

2) For any $\gamma\in\mathcal{G}_\xi$, *the gravity water waves equations* (1.1) *have a reversible quasi-periodic traveling wave solution (according to Definition 1.1) of the form*

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} &= \sum_{a\in\{1,\dots,\nu\}\colon \sigma_a=+1}\begin{pmatrix}M_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\cos(\overline{n}_a x-\widetilde{\Omega}_{\overline{n}_a}(\gamma)t)\\ P_{\overline{n}_a}\sqrt{\xi_{\overline{n}_a}}\sin(\overline{n}_a x-\widetilde{\Omega}_{\overline{n}_a}(\gamma)t)\end{pmatrix}\\ &+\sum_{a\in\{1,\dots,\nu\}\colon \sigma_a=-1}\begin{pmatrix}M_{\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\cos(\overline{n}_a x+\widetilde{\Omega}_{-\overline{n}_a}(\gamma)t)\\ P_{-\overline{n}_a}\sqrt{\xi_{-\overline{n}_a}}\sin(\overline{n}_a x+\widetilde{\Omega}_{-\overline{n}_a}(\gamma)t)\end{pmatrix}+r(t,x)\end{aligned} \tag{1.13}$$

where $r(t,x)=\breve{r}(\widetilde{\Omega}_{\sigma_1\overline{n}_1}(\gamma)t-\sigma_1\overline{n}_1x,\dots,\widetilde{\Omega}_{\sigma_\nu\overline{n}_\nu}(\gamma)t-\sigma_\nu\overline{n}_\nu x)$, *for* $\breve{r}\in H^{\overline{s}}(\mathbb{T}^\nu,\mathbb{R}^2)$, *satisfying* $\lim_{\xi\to0}\frac{\|\breve{r}\|_{\overline{s}}}{\sqrt{|\xi|}}=0$, *with a Diophantine frequency vector* $\widetilde{\Omega}:=(\widetilde{\Omega}_{\sigma_a\overline{n}_a})_{a=1,\dots,\nu}\in\mathbb{R}^\nu$, *depending on* γ,ξ, *and satisfying* $\lim_{\xi\to0}\widetilde{\Omega}=\vec{\Omega}(\gamma)$. *In addition these quasi-periodic solutions are linearly stable.*

The solutions (1.13) are a slight deformation of the quasi-periodic linear traveling waves (1.11). Thus, for $\xi \neq \xi'$ small enough, and $\gamma \in \mathcal{G}_\xi \cap \mathcal{G}_{\xi'}$ the quasi-periodic solutions (1.13) are different. The solutions (1.13) are linearly stable in the sense that the linearized vector field at the quasi-periodic traveling wave solutions (1.13) has purely imaginary Floquet exponents, see (5.8). This is a byproduct of the KAM reducibility of section 7. In particular, arguing as in [9, pages 6-7], the Sobolev norms of the solutions of the linearized equations at the solutions (1.13) are uniformly bounded in t.

Let us make some comments about the result.

1) *Vorticity as parameter and irrotational quasi-periodic traveling waves.* We are able to use the vorticity γ as a parameter, even though the dependence of the linear frequencies $\Omega_j(\gamma)$ in (1.8) with respect to γ affects only the order 0. If $\gamma_1 < 0 < \gamma_2$ we do not know if the value $\gamma = 0$ belongs to the set $\mathcal{G}_\xi$ for which the quasi periodic solutions (1.13) exist. Nevertheless, irrotational quasi-periodic traveling solutions of (1.1) with $\gamma = 0$ exist for most values of the depth $\mathtt{h}$, see Remark 4.6. These traveling waves do not clearly reduce to the standing waves constructed in [2], which are even in the space variable.

2) *More general traveling solutions.* The Diophantine condition (5.10) could be weakened requiring only $|\omega \cdot \ell| \geqslant \upsilon \langle \ell \rangle^{-\tau}$ for any $\ell \in \mathbb{Z}^\nu \backslash \{0\}$ with $\ell_1 \sigma_1 \overline{n}_1 + ... + \ell_\nu \sigma_\nu \overline{n}_\nu = 0$, so that ω could admit one non-trivial resonance. This is the natural minimal requirement to look for traveling solutions of the form $U(\omega t - \vec{\jmath} x)$, see Definition 3.1 and Remark 5.2. For $\nu = 2$ solutions of these kind could be time periodic, with clearly a completely different shape with respect to the classical Stokes traveling waves [32].

Let us make some comments about the proof.

3) *Symmetrization and reduction in order of the linearized operator.* The leading order of the linearization of the water waves system (1.1) at any quasi-periodic traveling wave is given by the Hamiltonian transport operator (see (6.15)) $\mathcal{L}_{\mathrm{TR}} := \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & 0 \\ 0 & \widetilde{V} \partial_x \end{pmatrix}$ where $\widetilde{V}(\varphi, x)$ is a small quasi-periodic traveling wave. By the almost-straightening Lemma 6.3 (cfr. Appendix A), for any (ω, γ) satisfying non-resonance conditions as in (5.11), we conjugate $\mathcal{L}_{\mathrm{TR}}$ via a symplectic transformation induced by a diffeomorphism of the torus $y = x + \beta(\varphi, x)$ to a transport operator $\omega \cdot \partial_\varphi + \begin{pmatrix} \mathtt{m}_1 \partial_y & 0 \\ 0 & \mathtt{m}_1 \partial_y \end{pmatrix} + \begin{pmatrix} \partial_y p_{\overline{\mathtt{n}}} & 0 \\ 0 & p_{\overline{\mathtt{n}}} \partial_y \end{pmatrix}$, for some constant $\mathtt{m}_1 \in \mathbb{R}$ and an exponentially small function $p_{\overline{\mathtt{n}}}(\varphi, x)$, see (6.24). For standing waves [2]

we have $\mathtt{m}_1 = 0$ and the complete conjugation of $\mathcal{L}_{\rm TR}$ is proved for any ω diophantine. Here we do not perform the full straightening of the transport operator $\mathcal{L}_{\rm TR}$ (i.e. we have $\overline{\mathtt{n}} < \infty$) in order to formulate a simple non-resonance condition as in (5.11). The KAM algebraic reduction scheme is like in [17, 3] (the estimates in [17] after finitely many iterative steps are not sufficient for our purposes). We also perform in a symplectic way other steps of the reduction to constant coefficients of the lower order terms of the linearized operator. This prevents the appearance of unstable operators. Since Section 6.4 we shall preserve only the reversible structure.

4) *Traveling waves and Melnikov non-resonance conditions.* We strongly use the invariance under space translations of the Hamiltonian nonlinear water waves vector field (1.1), i.e. the "momentum conservation", in the construction of the traveling quasi-periodic waves. We list the main points:

(*i*) The Floquet exponents (5.8) of the quasi-periodic solutions (1.13) are a singular perturbation of the unperturbed linear frequencies in (1.8), with leading terms of order 1. The Melnikov non-resonance conditions formulated in the Cantor-like set $\mathcal{C}^{\upsilon}_{\infty}$ in (5.10)-(5.13) hold on a set of large measure only thanks to the conservation of the momentum, see Section 5.2.

(*ii*) We can impose Melnikov conditions that *do not lose* space derivatives, see (5.12), simplifying the reduction in decreasing orders of Section 6 and the KAM reducibility scheme of Section 7. Indeed, it is enough to reduce to constant coefficients the linearized vector operator until the order 0 (included) in order to have a sufficiently good asymptotic expansion of the perturbed frequencies to prove the inclusion Lemma 5.6. Conversely, in [2] the second order Melnikov conditions verified for the standing pure gravity waves lose several space derivatives and many more steps of regularization are needed.

(iii) The invariance by space translations allows to avoid resonances between the linear frequencies in the construction of the quasiperiodic *traveling* waves. For example, with infinite depth $\mathtt{h} = +\infty$, these are given by $\Omega_j(\gamma) = \omega_j(\gamma) + \frac{\gamma}{2}{\rm sign}(j)$, and there are $\ell \in \mathbb{Z}^\nu \setminus \{0\}$, $j, j' \notin \{\sigma_a \overline{n}_a\}_{a=1,\dots,\nu}$, with $j \neq j'$, such that $\sum_{a=1}^{\nu} \ell_a \, \Omega_{\sigma_a \overline{n}_a}(\gamma) + \Omega_j(\gamma) - \Omega_{j'}(\gamma) \equiv 0$ for all γ. For example if $\sigma_1 = \sigma_2$, it is enough to take $\ell = (\ell_1, \ell_2, 0, \dots, 0) = (-1, 1, 0, \dots, 0)$ and $j = -\sigma_1 \overline{n}_1$, $j' = -\sigma_2 \overline{n}_2$. To exclude this resonance we exploit the momentum condition $\sum_{a=1}^{\nu} \ell_a \, \sigma_a \overline{n}_a + j - j' = 0$. The indexes above violate this constraint, as $\overline{n}_1 \neq \overline{n}_2$ by (1.9). We shall systematically use this kind of arguments to exclude nontrivial resonances.

Before concluding this introduction, we shortly describe the huge literature regarding time periodic traveling waves, which are steady in a moving frame, and refer to [7] for a wider overview.

Literature about time periodic traveling wave solutions. After the work of Stokes [32], the first rigorous construction of small amplitude space periodic steady traveling waves goes back to the 1920's with the papers of Nekrasov [29], Levi-Civita [26] and Struik [33], in case of irrotational bidimensional flows under the action of pure gravity. In the presence of vorticity, Gerstner [19] in 1802 gave an explicit example of periodic traveling wave, in infinite depth, and non-zero vorticity, while Dubreil-Jacotin [15] in 1934 proved the first bifurcation result of periodic traveling waves with small vorticity, extended later by Goyon [20] and Zeidler [40] for large vorticity. We point out the recent works of Wahlén [35] for capillary-gravity waves and non-constant vorticity, and of Martin [28], Walhén [36] for constant vorticity. They all deal with 2d water waves and can ultimately be deduced by the classical Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue. We also mention that these local bifurcation results can be extended to global branches of steady traveling waves by the theory of global analytic, or topological, bifurcation, see e.g. Keady-Norbury [27], Toland [34], for irrotational flows and Constantin-Strauss [12] with nonconstant vorticity. We suggest the reading of [10] for further results. We finally quote the recent numerical work of Wilkening-Zhao [38] about spatially quasi-periodic gravity-capillary 1d-water waves.

2 Hamiltonian structure and linearization at the origin

The Hamiltonian formulation of the water waves equations (1.1) was obtained by Constantin-Ivanov-Prodanov [11] and Wahlén [36]. It reduces to the Craig-Sulem-Zakharov formulation in [39, 14] if $\gamma = 0$. On the phase space $H^1_0(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$, endowed with the non canonical Poisson tensor $J_M(\gamma) := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix}$, we consider the Hamiltonian H defined in (1.2). Such Hamiltonian is well defined on $H^1_0(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$ since $G(\eta)[1] = 0$ and $\int_{\mathbb{T}} G(\eta)\psi \, \mathrm{d}x = 0$. It turns out [11, 36] that equations (1.1) are the Hamiltonian system generated by $H(\eta, \psi)$ with respect to the Poisson tensor $J_M(\gamma)$.

Reversible structure. Defining on the phase space $H^1_0(\mathbb{T}) \times \dot{H}^1(\mathbb{T})$ the involution

$$\mathcal{S}\begin{pmatrix}\eta\\ \psi\end{pmatrix} := \begin{pmatrix}\eta^\vee\\ -\psi^\vee\end{pmatrix}, \quad \eta^\vee(x) := \eta(-x), \tag{2.1}$$

the Hamiltonian (1.2) is invariant under $\mathcal{S}$, that is $H \circ \mathcal{S} = H$. This follows as the Dirichlet-Neumann operator satisfies $G(\eta^\vee)[\psi^\vee] = (G(\eta)[\psi])^\vee$. Equivalently, since the involution $\mathcal{S}$ is anti-symplectic, the water waves vector field X in the right hand side on (1.1) satisfies $X \circ \mathcal{S} = -\mathcal{S} \circ X$.

Translation invariance. Since the bottom of the domain occupied by the fluid is flat, the water waves equations (1.1) are invariant under space translations. Specifically, defining the translation operator

$$\tau_\varsigma \colon u(x) \mapsto u(x+\varsigma), \quad \varsigma \in \mathbb{R}, \tag{2.2}$$

the Hamiltonian (1.2) satisfies $H \circ \tau_\varsigma = H$ for any $\varsigma \in \mathbb{R}$. Equivalently, the water waves vector field satisfies $X \circ \tau_\varsigma = \tau_\varsigma \circ X$, for all $\varsigma \in \mathbb{R}$. This property follows since $\tau_\varsigma \circ G(\eta) = G(\tau_\varsigma \eta) \circ \tau_\varsigma$.

Wahlén coordinates. We introduce the Wahlén [36] coordinates (η, ζ) via the map

$$\begin{pmatrix}\eta\\ \psi\end{pmatrix} = W\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, W := \begin{pmatrix}\mathrm{Id} & 0\\ \frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}, W^{-1} := \begin{pmatrix}\mathrm{Id} & 0\\ -\frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}. \tag{2.3}$$

The change of coordinates W maps the phase space $H_0^1 \times \dot{H}^1$ into itself and conjugates the Poisson tensor $J_M(\gamma)$ to $W^{-1}J_M(\gamma)(W^{-1})^* = J$, where $J := \left(\begin{smallmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{smallmatrix}\right)$ is the canonical one. The Hamiltonian (1.2) becomes

$$\mathcal{H} := H \circ W, \quad \text{i.e.} \quad \mathcal{H}(\eta,\zeta) := H\big(\eta, \zeta + \frac{\gamma}{2}\partial_x^{-1}\eta\big), \tag{2.4}$$

and the Hamiltonian equations are transformed into

$$\partial_t \eta = \nabla_\zeta \mathcal{H}, \quad \partial_t \zeta = -\nabla_\eta \mathcal{H}. \tag{2.5}$$

The symplectic form of (2.5) is the standard one,

$$\mathcal{W}\Big(\left(\begin{smallmatrix}\eta_1\\ \zeta_1\end{smallmatrix}\right), \left(\begin{smallmatrix}\eta_2\\ \zeta_2\end{smallmatrix}\right)\Big) := (-\zeta_1, \eta_2)_{L^2} + (\eta_1, \zeta_2)_{L^2}. \tag{2.6}$$

The transformation W defined in (2.3) is reversibility preserving, namely it commutes with the involution $\mathcal{S}$ in (2.1) (see Definition 3.14 below), and commutes with the translation operator τ_ς. Thus also the Hamiltonian $\mathcal{H}$ in (2.4) is invariant under the involution $\mathcal{S}$ and the translation operator τ_ς.

Linearization at the equilibrium. We now show that the reversible solutions of the linear system (1.3) have the form (1.6). In the Wahlén coordinates (2.3), the linear Hamiltonian system (1.3) is transformed into the

Hamiltonian system

(2.7)

$$\partial_t \begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\boldsymbol{\Omega}_W \begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \ \boldsymbol{\Omega}_W := \begin{pmatrix} g - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0)\partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1}G(0) \\ \frac{\gamma}{2}G(0)\partial_x^{-1} & G(0)\end{pmatrix}$$

generated by the quadratic Hamiltonian

$$\mathcal{H}_L(\eta,\zeta) = \frac{1}{2}\left(\boldsymbol{\Omega}_W \begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \begin{pmatrix}\eta\\ \zeta\end{pmatrix}\right)_{L^2}. \tag{2.8}$$

We first conjugate (2.7) under the symplectic transformation $\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \mathcal{M}\begin{pmatrix}u\\ v\end{pmatrix}$ where $\mathcal{M}$ is the diagonal matrix of self-adjoint Fourier multipliers

$$\mathcal{M} := \begin{pmatrix} M(D) & 0 \\ 0 & M(D)^{-1}\end{pmatrix}, \ M(D) := \Big(\frac{G(0)}{g - \frac{\gamma^2}{4}\partial_x^{-1}G(0)\partial_x^{-1}}\Big)^{1/4}, \tag{2.9}$$

with the real valued symbol M_j as in (1.7). The map $\mathcal{M}$ is reversibility preserving. By a direct computation, system (2.7) assumes the symmetric form

(2.10)

$$\partial_t \begin{pmatrix}u\\ v\end{pmatrix} = J\boldsymbol{\Omega}_S \begin{pmatrix}u\\ v\end{pmatrix}, \ \boldsymbol{\Omega}_S := \mathcal{M}^*\boldsymbol{\Omega}_W\mathcal{M} = \begin{pmatrix}\omega(\gamma,D) & -\frac{\gamma}{2}\partial_x^{-1}G(0) \\ \frac{\gamma}{2}G(0)\partial_x^{-1} & \omega(\gamma,D)\end{pmatrix},$$

where

$$\omega(\gamma,D) := \sqrt{g\,G(0) - \Big(\frac{\gamma}{2}\partial_x^{-1}G(0)\Big)^2}\,. \tag{2.11}$$

Now we introduce complex coordinates by the transformation

$$\begin{pmatrix}u\\ v\end{pmatrix} = \mathcal{C}\begin{pmatrix}z\\ \overline{z}\end{pmatrix}, \ \mathcal{C} := \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{Id} & \mathrm{Id}\\ -\mathrm{i} & \mathrm{i}\end{pmatrix}, \ \mathcal{C}^{-1} := \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{Id} & \mathrm{i}\\ \mathrm{Id} & -\mathrm{i}\end{pmatrix}. \tag{2.12}$$

In these variables, the Hamiltonian system (2.10) becomes the diagonal system

(2.13)

$$\partial_t \begin{pmatrix}z\\ \overline{z}\end{pmatrix} = \begin{pmatrix}-\mathrm{i} & 0\\ 0 & \mathrm{i}\end{pmatrix}\boldsymbol{\Omega}_D \begin{pmatrix}z\\ \overline{z}\end{pmatrix}, \ \boldsymbol{\Omega}_D := \mathcal{C}^*\boldsymbol{\Omega}_S\mathcal{C} = \begin{pmatrix}\Omega(\gamma,D) & 0\\ 0 & \overline{\Omega}(\gamma,D)\end{pmatrix},$$

where

$$\Omega(\gamma,D) := \omega(\gamma,D) + \mathrm{i}\,\frac{\gamma}{2}\partial_x^{-1}G(0)\,. \tag{2.14}$$

We regard the system (2.13) in $\dot{H}^1 \times \dot{H}^1$. The diagonal system (2.13) amounts to the scalar equation

$$\partial_t z = -\mathrm{i}\Omega(\gamma, D) z\,, \quad z(x) = \sum\nolimits_{j\in\mathbb{Z}\setminus\{0\}} z_j e^{\mathrm{i}jx}\,, \tag{2.15}$$

which, written in the exponential Fourier basis, amounts to $\dot{z}_j = -\mathrm{i}\Omega_j(\gamma) z_j$, $j \in \mathbb{Z}\setminus\{0\}$. Note that, in these complex coordinates, the involution $\mathcal{S}$ in (2.1) reads as

$$\begin{pmatrix} z(x) \\ \overline{z(x)} \end{pmatrix} \mapsto \begin{pmatrix} \overline{z(-x)} \\ z(-x) \end{pmatrix}, \quad i.e. \quad z_j \mapsto \overline{z_j}\,, \quad \forall j \in \mathbb{Z}\setminus\{0\}\,. \tag{2.16}$$

Any *reversible* solution of (2.15) has the form

$$z(t,x) := \frac{1}{\sqrt{2}} \sum\nolimits_{j\in\mathbb{Z}\setminus\{0\}} \rho_j\, e^{-\mathrm{i}\,(\Omega_j(\gamma)t - j\,x)} \quad \text{with} \;\; \rho_j \in \mathbb{R}\,.$$

Back in the variables (η, ψ) defined in (2.3), using that by (2.9), (2.12),

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \mathcal{M}\,\mathcal{C} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} M(D) & M(D) \\ -\mathrm{i}M(D)^{-1} & \mathrm{i}M(D)^{-1} \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix},$$

these solutions assume the form (1.6).

We finally express the Fourier coefficients $z_j \in \mathbb{C}$ in (2.15) as $z_j = \frac{\alpha_j + \mathrm{i}\beta_j}{\sqrt{2}}$, where $(\alpha_j, \beta_j) \in \mathbb{R}^2$, for any $j \in \mathbb{Z}\setminus\{0\}$. In the new coordinates $(\alpha_j, \beta_j)_{j\in\mathbb{Z}\setminus\{0\}}$, the symplectic form (2.6) becomes $2\pi \sum_{j\in\mathbb{Z}\setminus\{0\}} \mathrm{d}\alpha_j \wedge \mathrm{d}\beta_j$. The quadratic Hamiltonian $\mathcal{H}_L$ in (2.8) reads $2\pi \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{\Omega_j(\gamma)}{2}(\alpha_j^2 + \beta_j^2)$, and the involution $\mathcal{S}$ in (2.1) reads $(\alpha_j, \beta_j) \mapsto (\alpha_j, -\beta_j)$, $j \in \mathbb{Z}\setminus\{0\}$. We may also enumerate these independent variables as $\big(\alpha_{-n}, \beta_{-n}, \alpha_n, \beta_n\big)$, $n \in \mathbb{N}$. Thus the phase space $\mathfrak{H} := L^2_0 \times \dot{L}^2$ of (2.5) decomposes as the direct sum $\mathfrak{H} = \sum_{n\in\mathbb{N}} V_{n,+} \oplus V_{n,-}$ of 2-dimensional symplectic subspaces

$$V_{n,+} := \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} M_n(\alpha_n \cos(nx) - \beta_n \sin(nx)) \\ M_n^{-1}(\beta_n \cos(nx) + \alpha_n \sin(nx)) \end{pmatrix}, (\alpha_n, \beta_n) \in \mathbb{R}^2 \right\},$$

$$V_{n,-} := \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} M_n(\alpha_{-n} \cos(nx) + \beta_{-n} \sin(nx)) \\ M_n^{-1}(\beta_{-n} \cos(nx) - \alpha_{-n} \sin(nx)) \end{pmatrix}, (\alpha_{-n}, \beta_{-n}) \in \mathbb{R}^2 \right\},$$

which are invariant for the linear Hamiltonian system (2.7). The involution $\mathcal{S}$ defined in (2.1) and the translation operator τ_ς in (2.2) leave the subspaces $V_{n,\sigma}$, $\sigma \in \{\pm\}$, invariant.

Tangential and normal subspaces of the phase space. We split the phase space $\mathfrak{H} = \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma} \oplus \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$, where $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ is the finite dimensional *tangential subspace*

$$\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma} := \sum\nolimits_{a=1}^{\nu} V_{\overline{n}_a,\sigma_a} \tag{2.17}$$

and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is the *normal subspace*

$$\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} := \sum\nolimits_{a=1}^{\nu} V_{\overline{n}_a,-\sigma_a} \oplus \sum\nolimits_{n\in\mathbb{N}\setminus\mathbb{S}^+} \big(V_{n,+} \oplus V_{n,-} \big) \,. \tag{2.18}$$

Both the subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ are symplectic. We denote by $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ the symplectic projections on the subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$, respectively. The restricted symplectic form $\mathcal{W}|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}$ is represented by the symplectic structure $J_{\angle}^{-1} := \Pi_{\angle}^{L^2} J^{-1}_{|\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}$ where $\Pi_{\angle}^{L^2}$ is the L^2-projector on the subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. Its associated Poisson tensor is $J_{\angle} := \Pi^{\angle}_{\mathbb{S}^+,\Sigma} J_{|\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}$. By Lemma 2.6 in [7], we have that $J_{\angle}^{-1} J_{\angle} = J_{\angle} J_{\angle}^{-1} = \mathrm{Id}_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}$.

Action-angle coordinates. We introduce action-angle coordinates on the tangential subspace $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ defined in (2.17). Given the sets $\mathbb{S}^+$ and Σ defined respectively in (1.9) and (1.10), we define the set

$$\mathbb{S} := \{\overline{\jmath}_1, \ldots, \overline{\jmath}_\nu\} \subset \mathbb{Z}\setminus\{0\}\,, \quad \overline{\jmath}_a := \sigma_a \overline{n}_a\,, \quad a = 1, \ldots, \nu\,, \tag{2.19}$$

and the action-angle coordinates $(\theta_j, I_j)_{j\in\mathbb{S}}$, by the relations, for any $j \in \mathbb{S}$, for any $0 < |I_j| < \xi_j$,

$$\alpha_j = \sqrt{\frac{1}{\pi}(I_j + \xi_j)}\cos(\theta_j)\,,\ \beta_j = -\sqrt{\frac{1}{\pi}(I_j + \xi_j)}\sin(\theta_j)\,. \tag{2.20}$$

In view of (2.17)-(2.18), we represent any function of the phase space $\mathfrak{H}$ as

$$\begin{aligned} A(\theta, I, w) &:= v^{\intercal}(\theta, I) + w \\ &= \frac{1}{\sqrt{\pi}} \sum_{j\in\mathbb{S}} \left[\begin{pmatrix} M_j \sqrt{I_j + \xi_j}\cos(\theta_j - jx) \\ -M_j^{-1}\sqrt{I_j + \xi_j}\sin(\theta_j - jx) \end{pmatrix} \right] + w\,, \end{aligned} \tag{2.21}$$

where $\theta := (\theta_j)_{j\in\mathbb{S}} \in \mathbb{T}^\nu$, $I := (I_j)_{j\in\mathbb{S}} \in \mathbb{R}^\nu$ and $w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. In view of (2.21), the involution $\mathcal{S}$ in (2.1) reads

$$\vec{\mathcal{S}} : (\theta, I, w) \mapsto (-\theta, I, \mathcal{S}w)\ , \tag{2.22}$$

the translation operator τ_ς in (2.2) reads

$$\vec{\tau}_\varsigma : (\theta,\, I,\, w) \mapsto (\theta - \vec{\jmath}\varsigma,\, I,\, \tau_\varsigma w), \quad \forall \varsigma \in \mathbb{R}\,, \tag{2.23}$$

where

$$\vec{\jmath} := (j)_{j\in\mathbb{S}} = (\bar{\jmath}_1,\dots,\bar{\jmath}_\nu) \in \mathbb{Z}^\nu\backslash\{0\}\,, \tag{2.24}$$

and the symplectic 2-form (2.6) becomes

$$\mathcal{W} = \sum\nolimits_{j\in\mathbb{S}} (\mathrm{d}\theta_j \wedge \mathrm{d}I_j) \oplus \mathcal{W}|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}}\,. \tag{2.25}$$

Given a Hamiltonian $K\colon \mathbb{T}^\nu\times\mathbb{R}^\nu\times\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma} \to \mathbb{R}$, the associated Hamiltonian vector field is $X_K := \big(\partial_I K, -\partial_\theta K, J_\angle \nabla_w K\big)$ where $\nabla_w K$ denotes the L^2 gradient of K with respect to $w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

Tangential and normal subspaces in complex variables. The linear map $\mathcal{MC}$ is an isomorphism between the tangential subspace $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ defined in (2.17) and

$$\mathbf{H}_{\mathbb{S}} := \Big\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \,:\, z(x) = \sum\nolimits_{j\in\mathbb{S}} z_j e^{\mathrm{i}jx} \Big\}\,,$$

and between the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ defined in (2.18) and

$$\mathbf{H}^{\perp}_{\mathbb{S}_0} := \Big\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} : z(x) = \sum\nolimits_{j\in\mathbb{S}_0^c} z_j e^{\mathrm{i}jx} \in L^2 \Big\}\,,\ \mathbb{S}_0^c := \mathbb{Z}\backslash(\mathbb{S}\cup\{0\})\,. \tag{2.26}$$

Denoting by $\Pi^{\intercal}_{\mathbb{S}}$, $\Pi^{\perp}_{\mathbb{S}_0}$, the L^2-orthogonal projections on the subspaces $\mathbf{H}_{\mathbb{S}}$ and $\mathbf{H}^{\perp}_{\mathbb{S}_0}$, we have that

$$\Pi^{\intercal}_{\mathbb{S}^+,\Sigma} = \mathcal{MC}\,\Pi^{\intercal}_{\mathbb{S}}\,(\mathcal{MC})^{-1}\,, \quad \Pi^{\angle}_{\mathbb{S}^+,\Sigma} = \mathcal{MC}\,\Pi^{\perp}_{\mathbb{S}_0}\,(\mathcal{MC})^{-1}\,. \tag{2.27}$$

Moreover (cfr. Lemma 2.9 in [7])

$$(v^{\intercal}, \mathbf{\Omega}_W w)_{L^2} = 0\,, \quad \forall v^{\intercal} \in \mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}\,, \ \ \forall w \in \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}\,. \tag{2.28}$$

Notation. For $a \lesssim_s b$ we mean $a \leqslant C(s)b$ for a constant $C(s) > 0$. Let $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{N}_0 := \{0\}\cup\mathbb{N}$.

3 Functional setting

We report basic notation, definitions, and results used along the paper, concerning traveling waves, pseudo-differential operators, tame operators, and the algebraic properties of Hamiltonian, reversible and momentum preserving operators.

Definition 3.1. (Quasi-periodic traveling waves) Let $\vec{\jmath} := (\bar{\jmath}_1,\dots,\bar{\jmath}_\nu) \in \mathbb{Z}^\nu$ be the vector defined in (2.24). A function $u(\varphi,x)$ is a *quasi-periodic traveling* wave if it has the form $u(\varphi,x) = U(\varphi - \vec{\jmath}x)$ where $U : \mathbb{T}^\nu \to \mathbb{C}^K$, $K \in \mathbb{N}$, is a $(2\pi)^\nu$-periodic function.

Comparing with Definition 1.1, we call *quasi-periodic traveling* wave both $u(\varphi, x) = U(\varphi - \vec{\jmath}x)$ and the function of time $u(\omega t, x) = U(\omega t - \vec{\jmath}x)$. Quasi-periodic traveling waves are characterized by $u(\varphi - \vec{\jmath}\varsigma, \cdot) = \tau_\varsigma u$ for any $\varsigma \in \mathbb{R}$, where τ_ς is the translation operator in (2.2). Product and composition of quasi-periodic traveling waves are quasi-periodic traveling waves. Expanded in Fourier series, a quasi-periodic traveling wave has the form $u(\varphi, x) = \sum_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}, j + \vec{\jmath}\cdot\ell = 0} u_{\ell,j} e^{\mathrm{i}(\ell\cdot\varphi + jx)}$. For $K \geqslant 1$ we define

$$\Pi_K u := \sum_{\langle \ell \rangle \leqslant K,\, j \in \mathbb{S}_0^c,\, j + \vec{\jmath}\cdot\ell = 0} u_{\ell,j} e^{\mathrm{i}(\ell\cdot\varphi + jx)}, \tag{3.1}$$

and $\Pi_K^\perp := \mathrm{Id} - \Pi_K$. For a function $u(\varphi, x)$ we define the averages

$$\begin{aligned} \langle u \rangle_{\varphi,x} &:= \frac{1}{(2\pi)^{\nu+1}} \int_{\mathbb{T}^{\nu+1}} u(\varphi, x)\, \mathrm{d}\varphi\, \mathrm{d}x, \\ \langle u \rangle_\varphi(x) &:= \frac{1}{(2\pi)^\nu} \int_{\mathbb{T}^{\nu+1}} u(\varphi, x)\, \mathrm{d}\varphi\,; \end{aligned} \tag{3.2}$$

we note that $\langle u \rangle_\varphi = \langle u \rangle_{\varphi,x}$ when $u(\varphi, x)$ is a quasi-periodic traveling wave.

Whitney-Sobolev functions. We consider families of Sobolev functions $\lambda \mapsto u(\lambda) \in H^s(\mathbb{T}^{\nu+1})$ which are k_0-times differentiable in the sense of Whitney with respect to the parameter $\lambda := (\omega, \gamma) \in F \subset \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$ where $F \subset \mathbb{R}^{\nu+1}$ is a closed set. We refer to Definition 2.1 in [2], for the definition of Whitney-Sobolev functions. Given $\upsilon \in (0, 1)$, by the Whitney extension theorem, we have the equivalence $\|u\|_{s,F}^{k_0,\upsilon} \sim_{\nu,k_0} \sum_{|\alpha| \leqslant k_0} \upsilon^{|\alpha|} \|\partial_\lambda^\alpha u\|_{L^\infty(\mathbb{R}^{\nu+1}, H^s)}$. For simplicity we denote $\| \ \|_{s,F}^{k_0,\upsilon} = \| \ \|_s^{k_0,\upsilon}$. Classical tame estimates for the product hold (see e.g. Lemma 2.4 in [2]): for all $s \geqslant s_0 > (\nu + 1)/2$,

$$\|uv\|_s^{k_0,\upsilon} \leqslant C(s, k_0) \|u\|_s^{k_0,\upsilon} \|v\|_{s_0}^{k_0,\upsilon} + C(s_0, k_0) \|u\|_{s_0}^{k_0,\upsilon} \|v\|_s^{k_0,\upsilon}, \tag{3.3}$$

and

$$\begin{aligned} \|\Pi_K u\|_s^{k_0,\upsilon} &\leqslant K^\alpha \|u\|_{s-\alpha}^{k_0,\upsilon}, \quad 0 \leqslant \alpha \leqslant s, \\ \|\Pi_K^\perp u\|_s^{k_0,\upsilon} &\leqslant K^{-\alpha} \|u\|_{s+\alpha}^{k_0,\upsilon}, \quad \alpha \geqslant 0\,. \end{aligned} \tag{3.4}$$

The composition operator $u(\varphi, x) \mapsto \mathtt{f}(u)(\varphi, x) := f(\varphi, x, u(\varphi, x))$ satisfies the following lemma.

Lemma 3.2. (Lemma 2.6 in [2]**)** *Let $f \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{R}, \mathbb{R})$. If $u(\lambda) \in H^s(\mathbb{T}^d)$ is a family of Sobolev functions with $\|u\|_{s_0}^{k_0,\upsilon} \leqslant 1$, then, for all $s \geqslant s_0 :=$*

$(d+1)/2$, $\|\mathtt{f}(u)\|_s^{k_0,\upsilon} \leqslant C(s,k_0,f)\big(1+\|u\|_s^{k_0,\upsilon}\big)$. *If* $f(\varphi,x,0)=0$ *then* $\|\mathtt{f}(u)\|_s^{k_0,\upsilon} \leqslant C(s,k_0,f)\|u\|_s^{k_0,\upsilon}$.

Consider a φ-dependent diffeomorphism of $\mathbb{T}_x$ given by $y = x+\beta(\varphi,x)$.

Lemma 3.3. *Let* $\|\beta\|_{2s_0+k_0+2}^{k_0,\upsilon} \leqslant \delta(s_0,k_0)$ *small enough. Then the composition operator* $(\mathcal{B}u)(\varphi,x) := u(\varphi,x+\beta(\varphi,x))$ *satisfies* $\|\mathcal{B}u\|_s^{k_0,\upsilon} \lesssim_{s,k_0} \|u\|_{s+k_0}^{k_0,\upsilon} + \|\beta\|_s^{k_0,\upsilon}\|u\|_{s_0+k_0+1}^{k_0,\upsilon}$*, for any* $s \geqslant s_0$*, and the function* $\breve{\beta}$ *defined by the inverse diffeomorphism* $x = y+\breve{\beta}(\varphi,y)$*, satisfies* $\|\breve{\beta}\|_s^{k_0,\upsilon} \lesssim_{s,k_0} \|\beta\|_{s+k_0}^{k_0,\upsilon}$.

Constant transport equation on quasi-periodic traveling waves. Let $\mathtt{m} \in \mathbb{R}$. For any (ω,γ) satisfying $|\omega\cdot\ell + \mathtt{m}\, j| \geqslant \upsilon\langle \ell\rangle^{-\tau}$ for all $(\ell,j) \in \mathbb{Z}^{\nu+1}\setminus\{0\}$ with $\vec{\jmath}\cdot\ell + j = 0$, given a quasi-periodic traveling wave $u(\varphi,x)$ with zero average with respect to φ the transport equation $(\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)v = u$ has the quasi-periodic traveling wave solution $(\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)^{-1}u := \sum_{\substack{(\ell,j)\in\mathbb{Z}^{\nu+1}\setminus\{0\} \\ \vec{\jmath}\cdot\ell+j=0}} \frac{u_{\ell,j}}{\mathrm{i}(\omega\cdot\ell+\mathtt{m}\, j)} e^{\mathrm{i}(\ell\cdot\varphi+jx)}$. For any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, we define its extension

$$(\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)^{-1}_{\mathrm{ext}} u(\varphi,x) := \sum_{\substack{(\ell,j)\in\mathbb{Z}^{\nu+1} \\ \vec{\jmath}\cdot\ell+j=0}} \frac{\chi((\omega\cdot\ell+\mathtt{m}\, j)\upsilon^{-1}\langle\ell\rangle^{\tau})}{\mathrm{i}(\omega\cdot\ell+\mathtt{m}\, j)} u_{\ell,j} e^{\mathrm{i}(\ell\cdot\varphi+jx)}\,, \tag{3.5}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ is an even positive $\mathcal{C}^\infty$ cut-off function such that

$$\chi(\xi) = 0 \text{ if } |\xi| \leqslant \tfrac{1}{3},\ \chi(\xi) = 1 \text{ if } |\xi| \geqslant \tfrac{2}{3},\ \partial_\xi\chi(\xi) > 0, \forall\, \xi \in (\tfrac{1}{3},\tfrac{2}{3}). \tag{3.6}$$

Note that $(\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)^{-1}_{\mathrm{ext}} u = (\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)^{-1}u$ for all $(\omega,\gamma) \in \mathtt{TC}(\mathtt{m};\upsilon,\tau)$. If $\mathtt{m} : \mathbb{R}^\nu \times [\gamma_1,\gamma_2] \to \mathbb{R}$, $(\omega,\gamma) \mapsto \mathtt{m}(\omega,\gamma)$ is a function with $|\mathtt{m}|^{k_0,\upsilon} \leqslant C$, then, for $\mu := k_0 + \tau(k_0+1)$,

$$\|(\omega\cdot\partial_\varphi + \mathtt{m}\,\partial_x)^{-1}_{\mathrm{ext}} u\|_{s,\mathbb{R}^{\nu+1}}^{k_0,\upsilon} \leqslant C(k_0)\upsilon^{-1}\|u\|_{s+\mu,\mathbb{R}^{\nu+1}}^{k_0,\upsilon}\,. \tag{3.7}$$

Furthermore, for any $\omega \in \mathbb{R}^\nu$, $\mathtt{m}_1,\mathtt{m}_2 \in \mathbb{R}$ and $s \geqslant 0$

$$\|\big((\omega\cdot\partial_\varphi + \mathtt{m}_1\,\partial_x)^{-1}_{\mathrm{ext}} - (\omega\cdot\partial_\varphi + \mathtt{m}_2\,\partial_x)^{-1}_{\mathrm{ext}}\big)u\|_s \leqslant C\upsilon^{-2}\,|\mathtt{m}_1 - \mathtt{m}_2|\,\|u\|_{s+2\tau+1}\,. \tag{3.8}$$

Linear operators. We consider φ-dependent families of linear operators $A : \mathbb{T}^\nu \mapsto \mathcal{L}(L^2(\mathbb{T}_x))$, $\varphi \mapsto A(\varphi)$, acting on subspaces of $L^2(\mathbb{T}_x)$. We also regard A as an operator $(Au)(\varphi,x) := (A(\varphi)u(\varphi,\,\cdot\,))(x)$. Expanding $u(\varphi,x)$ in Fourier,

$$Au(\varphi,x) = \sum_{j,j'\in\mathbb{Z}}\sum_{\ell,\ell'\in\mathbb{Z}^\nu} A_j^{j'}(\ell-\ell')u_{\ell',j'}e^{\mathrm{i}(\ell\cdot\varphi+jx)}\,. \tag{3.9}$$

We identify an operator A with its matrix $\big(A_j^{j'}(\ell-\ell')\big)_{j,j'\in\mathbb{Z},\ell,\ell'\in\mathbb{Z}^\nu}$.

Real operators. A linear operator A is *real* if $A = \overline{A}$, where $\overline{A}$ is defined by $\overline{A}(u) := \overline{A(\overline{u})}$. We represent a real operator acting on (η, ζ) by a matrix $\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are real operators acting on the scalar valued components $\eta, \zeta \in L^2(\mathbb{T}_x, \mathbb{R})$. The change of coordinates (2.12) transforms a real operator $\mathcal{R}$ into a complex one acting on the variables $(z, \overline{z})$, given by the matrix

$$\mathbf{R} := \mathcal{C}^{-1}\mathcal{R}\mathcal{C} = \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \overline{\mathcal{R}}_2 & \overline{\mathcal{R}}_1 \end{pmatrix}, \begin{array}{l} \mathcal{R}_1 := \{(A+D) - \mathrm{i}(B-C)\}/2 \,, \\ \mathcal{R}_2 := \{(A-D) + \mathrm{i}(B+C)\}/2 \,. \end{array} \tag{3.10}$$

We call *real* a matrix operator acting on the complex variables $(z, \overline{z})$ of this form.

Pseudodifferential calculus. We report basic notions of pseudodifferential calculus, following [9].

Definition 3.4. **(ΨDO)** A *pseudodifferential* symbol $a(x, j)$ of order m is the restriction to $\mathbb{R} \times \mathbb{Z}$ of a function $a(x, \xi)$ which is $\mathcal{C}^\infty$-smooth on $\mathbb{R} \times \mathbb{R}$, 2π-periodic in x, and satisfies, $\forall \alpha, \beta \in \mathbb{N}_0$, $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\beta}$. We denote by S^m the class of symbols of order m and $S^{-\infty} := \cap_{m \geqslant 0} S^m$. To a symbol $a(x, \xi)$ in S^m we associate its quantization acting on a 2π-periodic function $u(x) = \sum_{j \in \mathbb{Z}} u_j \, e^{\mathrm{i} j x}$ as $[\mathrm{Op}(a)u](x) := \sum_{j \in \mathbb{Z}} a(x, j) u_j \, e^{\mathrm{i} j x}$. We denote by $\mathrm{OP}S^m$ the set of pseudodifferential operators of order m and $\mathrm{OP}S^{-\infty} := \bigcap_{m \in \mathbb{R}} \mathrm{OP}S^m$. For a matrix of pseudodifferential operators $\mathbf{A} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $A_i \in \mathrm{OP}S^m$, $i = 1, \ldots, 4$, we say that $\mathbf{A} \in \mathrm{OP}S^m$.

When the symbol $a(x)$ is independent of ξ, the operator $\mathrm{Op}(a)$ is the multiplication operator by the function $a(x)$, i.e. $\mathrm{Op}(a) : u(x) \mapsto a(x)u(x)$. In such a case we also denote $\mathrm{Op}(a) = a(x)$.

For any $m \in \mathbb{R} \backslash \{0\}$, we set $|D|^m := \mathrm{Op}\big(\chi(\xi)|\xi|^m\big)$, where χ is an even, positive $\mathcal{C}^\infty$ cut-off satisfying (3.6). We identify the Hilbert transform $\mathcal{H}$, acting on the 2π-periodic functions, defined by

$$\mathcal{H}(e^{\mathrm{i}jx}) := -\mathrm{i}\,\mathrm{sign}\,(j) e^{\mathrm{i}jx} \quad \forall j \neq 0 \,, \quad \mathcal{H}(1) := 0 \,, \tag{3.11}$$

with the Fourier multiplier $\mathrm{Op}(-\mathrm{i}\,\mathrm{sign}\,(\xi)\chi(\xi))$. Similarly we regard the operator

$$\partial_x^{-1} \left[e^{\mathrm{i}jx} \right] := -\,\mathrm{i}\, j^{-1} \, e^{\mathrm{i}jx} \quad \forall\, j \neq 0 \,, \quad \partial_x^{-1}[1] := 0 \,, \tag{3.12}$$

as the Fourier multiplier $\partial_x^{-1} = \mathrm{Op}\left(-\mathrm{i}\,\chi(\xi)\xi^{-1}\right)$ and the projector π_0 as

$$\pi_0 u := \frac{1}{2\pi}\int_{\mathbb{T}} u(x)\,dx\,, \tag{3.13}$$

with the Fourier multiplier $\mathrm{Op}\big(1-\chi(\xi)\big)$. Finally we define, for any $m \in \mathbb{R}\backslash\{0\}$, $\langle D\rangle^m := \pi_0 + |D|^m$.

We consider families of pseudodifferential operators having symbols $a(\lambda;\varphi,x,\xi)$ which are k_0-times differentiable with respect to a parameter $\lambda := (\omega,\gamma)$ in an open subset $\Lambda_0 \subset \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$. Note that $\partial_\lambda^k A = \mathrm{Op}\left(\partial_\lambda^k a\right)$ for any $k \in \mathbb{N}_0^{\nu+1}$. We recall the pseudodifferential norm as in Definition 2.11 in [9].

Definition 3.5. (Weighted ΨDO norm) Let $A(\lambda) := a(\lambda;\varphi,x,D) \in \mathrm{OP}S^m$ be a pseudodifferential operator with symbol $a(\lambda;\varphi,x,\xi) \in S^m$, $m \in \mathbb{R}$, k_0-times differentiable with respect to $\lambda \in \Lambda_0 \subset \mathbb{R}^{\nu+1}$. For $\upsilon \in (0,1)$, $\alpha \in \mathbb{N}_0$, $s \geqslant 0$, we define

$$\|A\|_{m,s,\alpha}^{k_0,\upsilon} := \sum\nolimits_{|k|\leqslant k_0} \upsilon^{|k|} \sup_{\lambda\in\Lambda_0} \left\|\partial_\lambda^k A(\lambda)\right\|_{m,s,\alpha}\,,$$

where $\|A(\lambda)\|_{m,s,\alpha} := \max_{0\leqslant\beta\leqslant\alpha} \sup_{\xi\in\mathbb{R}} \|\partial_\xi^\beta a(\lambda,\cdot,\cdot,\xi)\|_s \langle\xi\rangle^{-m+\beta}$. For a matrix $\mathbf{A} \in \mathrm{OP}S^m$, we define $\|\mathbf{A}\|_{m,s,\alpha}^{k_0,\upsilon} := \max_{i=1,\ldots,4} \|A_i\|_{m,s,\alpha}^{k_0,\upsilon}$.

If $\mathrm{Op}(a)$, $\mathrm{Op}(b)$ are pseudodifferential operators with symbols $a \in S^m$, $b \in S^{m'}$, $m,m' \in \mathbb{R}$, then the composition operator $\mathrm{Op}(a)\mathrm{Op}(b)$ is a pseudodifferential operator $\mathrm{Op}(a\#b)$ with symbol $a\#b \in S^{m+m'}$. It admits the asymptotic expansion: for any $N \geqslant 1$

$$(a\#b)(x,\xi) = \sum_{\beta=0}^{N-1} \frac{1}{\mathrm{i}^\beta \beta!}\partial_\xi^\beta a(x,\xi)\partial_x^\beta b(x,\xi) + (r_N(a,b))(x,\xi)\,, \tag{3.14}$$

where $r_N(a,b) \in S^{m+m'-N}$. The commutator between two pseudodifferential operators $\mathrm{Op}(a) \in \mathrm{OP}S^m$ and $\mathrm{Op}(b) \in \mathrm{OP}S^{m'}$ is a pseudodifferential operator in $\mathrm{OP}S^{m+m'-1}$ with symbol $a \star b \in S^{m+m'-1}$, that admits, by (3.14), the expansion $a\star b = -\mathrm{i}\,\{a,b\} + \widetilde{r}_2(a,b)$, where $\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$ is the Poisson bracket between $a(x,\xi)$ and $b(x,\xi)$, and

$$\widetilde{r}_2(a,b) := r_2(a,b) - r_2(b,a) \in S^{m+m'-2}\,. \tag{3.15}$$

The following quantitative estimates are proved in Lemma 2.13 in [9].

Lemma 3.6. (Composition and Commutator) *Let* $A = a(\lambda;\varphi,x,D)$, $B = b(\lambda;\varphi,x,D)$ *be pseudodifferential operators with* $a(\lambda;\varphi,x,\xi) \in S^m$, $b(\lambda;\varphi,x,\xi) \in S^{m'}$, $m,m' \in \mathbb{R}$. *Then* $A \circ B \in \mathrm{OP}S^{m+m'}$ *satisfies, for any* $\alpha \in \mathbb{N}_0$, $s \geqslant s_0$,

$$\|AB\|^{k_0,\upsilon}_{m+m',s,\alpha} \lesssim_{m,\alpha,k_0} C(s)\, \|A\|^{k_0,\upsilon}_{m,s,\alpha}\, \|B\|^{k_0,\upsilon}_{m',s_0+|m|+\alpha,\alpha} + C(s_0)\, \|A\|^{k_0,\upsilon}_{m,s_0,\alpha}\, \|B\|^{k_0,\upsilon}_{m',s+|m|+\alpha,\alpha} \, .$$

Moreover, for any integer $N \geqslant 1$, *the remainder* $R_N := \mathrm{Op}(r_N)$ *in* (3.14) *satisfies*

$$\|\mathrm{Op}(r_N(a,b))\|^{k_0,\upsilon}_{m+m'-N,s,\alpha} \lesssim_{m,N,\alpha,k_0} C(s)\, \|A\|^{k_0,\upsilon}_{m,s,N+\alpha}\, \|B\|^{k_0,\upsilon}_{m',s_0+|m|+2N+\alpha,N+\alpha} + C(s_0)\, \|A\|^{k_0,\upsilon}_{m,s_0,N+\alpha}\, \|B\|^{k_0,\upsilon}_{m',s+|m|+2N+\alpha,N+\alpha} \, . \tag{3.16}$$

As a consequence the commutator $[A,B] := AB - BA \in \mathrm{OP}S^{m+m'-1}$ *satisfies*

$$\|[A,B]\|^{k_0,\upsilon}_{m+m'-1,s,\alpha} \lesssim_{m,m',\alpha,k_0} C(s)\, \|A\|^{k_0,\upsilon}_{m,s+|m'|+\alpha+2,\alpha+1}\, \|B\|^{k_0,\upsilon}_{m',s_0+|m|+\alpha+2,\alpha+1} + C(s_0)\, \|A\|^{k_0,\upsilon}_{m,s_0+|m'|+\alpha+2,\alpha+1}\, \|B\|^{k_0,\upsilon}_{m',s+|m|+\alpha+2,\alpha+1} \, . \tag{3.17}$$

Finally we consider the exponential of pseudodifferential operators of order 0, see Lemma 2.12 in [8].

Lemma 3.7. (Exponential map) *If* $A := \mathrm{Op}(a(\lambda;\varphi,x,\xi))$ *is in* OPS^0, *then* e^A *is in* OPS^0 *and for any* $s \geqslant s_0$, $\alpha \in \mathbb{N}_0$, *there exists* $C(s,\alpha) > 0$ *so that* $\|e^A - \mathrm{Id}\|^{k_0,\upsilon}_{0,s,\alpha} \leqslant \|A\|^{k_0,\upsilon}_{0,s+\alpha,\alpha} \exp\big(C(s,\alpha)\|A\|^{k_0,\upsilon}_{0,s_0+\alpha,\alpha}\big)$.

$\mathcal{D}^{k_0}$-tame and -$(-\frac{1}{2})$-modulo-tame operators. Let $A := A(\lambda)$ be a linear operator k_0-times differentiable with respect to the parameter λ in an open set $\Lambda_0 \subset \mathbb{R}^{\nu+1}$.

Definition 3.8. **($\mathcal{D}^{k_0}$-σ-tame,** [9]**)** Let $\sigma \geqslant 0$. A linear operator $A := A(\lambda)$ is $\mathcal{D}^{k_0}$-σ-tame if there exists a non-decreasing function $[s_0, S] \to [0,+\infty)$, $s \mapsto \mathfrak{M}_A(s)$, with possibly $S = +\infty$, such that, for all $s_0 \leqslant s \leqslant S$ and $u \in H^{s+\sigma}$, $\sup_{|k|\leqslant k_0} \sup_{\lambda\in\Lambda_0} \upsilon^{|k|} \|(\partial_\lambda^k A(\lambda))u\|_s \leqslant \mathfrak{M}_A(s_0)\, \|u\|_{s+\sigma} + \mathfrak{M}_A(s)\, \|u\|_{s_0+\sigma}$. We say that $\mathfrak{M}_A(s)$ is a *tame constant* of the operator A. The constant $\mathfrak{M}_A(s) = \mathfrak{M}_A(k_0,\sigma,s)$ may also depend on k_0,σ but we omit to write them. When the "loss of derivatives" σ is zero, we simply write $\mathcal{D}^{k_0}$-tame instead of $\mathcal{D}^{k_0}$-0-tame. For a matrix as in (3.10), we denote $\mathfrak{M}_{\mathbf{R}}(s) := \max\{\mathfrak{M}_{\mathcal{R}_1}(s), \mathfrak{M}_{\mathcal{R}_2}(s)\}$.

The class of $\mathcal{D}^{k_0}$-σ-tame operators is closed under composition.

Lemma 3.9. (Composition, Lemma 2.20 in [9]**)** *Let A, B be respectively $\mathcal{D}^{k_0}$-σ_A-tame and $\mathcal{D}^{k_0}$-σ_B-tame operators with tame constants respectively $\mathfrak{M}_A(s)$ and $\mathfrak{M}_B(s)$. Then the composed operator $A \circ B$ is $\mathcal{D}^{k_0}$-$(\sigma_A + \sigma_B)$-tame with*

$$\mathfrak{M}_{AB}(s) \leqslant C(k_0) \left(\mathfrak{M}_A(s) \mathfrak{M}_B(s_0 + \sigma_A) + \mathfrak{M}_A(s_0) \mathfrak{M}_B(s + \sigma_A) \right) .$$

The action of a $\mathcal{D}^{k_0}$-σ-tame operator $A(\lambda)$ on a Sobolev function $u = u(\lambda) \in H^{s+\sigma}$ is bounded by $\|Au\|_s^{k_0,\upsilon} \lesssim_{k_0} \mathfrak{M}_A(s_0) \|u\|_{s+\sigma}^{k_0,\upsilon} + \mathfrak{M}_A(s) \|u\|_{s_0+\sigma}^{k_0,\upsilon}$ (see Lemma 2.22 in [9]) and pseudodifferential operators are tame operators. In particular, we use the following lemma, see Lemma 2.21 in [9].

Lemma 3.10. *Let $A = a(\lambda; \varphi, x, D) \in \mathrm{OP}S^0$ be a family of pseudodifferential operators satisfying $\|A\|_{0,s,0}^{k_0,\upsilon} < \infty$ for $s \geqslant s_0$. Then A is $\mathcal{D}^{k_0}$-tame, with $\mathfrak{M}_A(s) \leqslant C(s) \|A\|_{0,s,0}^{k_0,\upsilon}$, for any $s \geqslant s_0$.*

In view of the KAM reducibility scheme of Section 7 we also consider the notion of $\mathcal{D}^{k_0}$-$(-\frac{1}{2})$-modulo-tame operator. Given a linear operator A acting as in (3.9), the majorant operator $|A|$ is defined to have the matrix elements $(|A_j^{j'}(\ell - \ell')|)_{\ell, \ell' \in \mathbb{Z}^\nu, j, j' \in \mathbb{Z}}$.

Definition 3.11. **($\mathcal{D}^{k_0}$-$(-\frac{1}{2})$-modulo-tame)** A linear operator $A = A(\lambda)$ is $\mathcal{D}^{k_0}$-$(-\frac{1}{2})$*-modulo-tame* if there exists a non-decreasing function $[s_0, S] \to [0, +\infty]$, $s \mapsto \mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} A \langle D \rangle^{\frac14}}(s)$, such that for all $k \in \mathbb{N}_0^{\nu+1}$, $|k| \leqslant k_0$, the majorant operator $\langle D \rangle^{\frac14} |\partial_\lambda^k A| \langle D \rangle^{\frac14}$ satisfies, for all $s_0 \leqslant s \leqslant S$ and $u \in H^s$,

$$\sup_{|k| \leqslant k_0} \sup_{\lambda \in \Lambda_0} \upsilon^{|k|} \| \langle D \rangle^{\frac14} |\partial_\lambda^k A| \langle D \rangle^{\frac14} u \|_s \leqslant \mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} A \langle D \rangle^{\frac14}}(s_0) \|u\|_s + \mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} A \langle D \rangle^{\frac14}}(s) \|u\|_{s_0} .$$

For a matrix as in (3.10), we denote

$$\mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} \mathbf{R} \langle D \rangle^{\frac14}}(s) := \max \left\{ \mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} \mathcal{R}_1 \langle D \rangle^{\frac14}}(s), \mathfrak{M}^\sharp_{\langle D \rangle^{\frac14} \mathcal{R}_2 \langle D \rangle^{\frac14}}(s) \right\} .$$

Given a linear operator A acting as in (3.9), we define the operator $\langle \partial_\varphi \rangle^{\mathtt{b}} A$, $\mathtt{b} \in \mathbb{R}$, whose matrix elements are $\langle \ell - \ell' \rangle^{\mathtt{b}} A_j^{j'}(\ell - \ell')$ and the *smoothed operator* $\Pi_N A$, $N \in \mathbb{N}$ whose matrix elements are

$$(\Pi_N A)_j^{j'}(\ell - \ell') := \begin{cases} A_j^{j'}(\ell - \ell') & \text{if } \langle \ell - \ell' \rangle \leqslant N \\ 0 & \text{otherwise} . \end{cases} \tag{3.18}$$

We also denote $\Pi_N^\perp := \mathrm{Id} - \Pi_N$. Arguing as in Lemma 2.27 in [9], we have that

$$\begin{aligned} &\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\Pi_N^\perp A\langle D\rangle^{\frac14}}(s) \leqslant N^{-\mathtt{b}}\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}A\langle D\rangle^{\frac14}}(s)\,, \\ &\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\Pi_N^\perp A\langle D\rangle^{\frac14}}(s) \leqslant \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s)\,. \end{aligned} \tag{3.19}$$

From Lemma A.5-(iv) in [18] and the proof of Lemma 2.22 in [8], we deduce the following lemma.

Lemma 3.12. *Let A, B, $\langle\partial_\varphi\rangle^{\mathtt{b}} A$, $\langle\partial_\varphi\rangle^{\mathtt{b}} B$ be $\mathcal{D}^{k_0}$-$(-\frac12)$-modulo-tame operators. Then $A+B$, $A\circ B$ and $\langle\partial_\varphi\rangle^{\mathtt{b}}(AB)$ are $\mathcal{D}^{k_0}$-$(-\frac12)$-modulo-tame with*

$$\begin{aligned} \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}(A+B)\langle D\rangle^{\frac14}}(s) &\leqslant \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s) + \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}B\langle D\rangle^{\frac14}}(s) \\ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}AB\langle D\rangle^{\frac14}}(s) &\lesssim_{k_0} \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}B\langle D\rangle^{\frac14}}(s_0) \\ &\quad + \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s_0)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}B\langle D\rangle^{\frac14}}(s) \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}(AB)\langle D\rangle^{\frac14}}(s) \lesssim_{\mathtt{b},k_0} \\ &\ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}A\langle D\rangle^{\frac14}}(s)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}B\langle D\rangle^{\frac14}}(s_0) + \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}A\langle D\rangle^{\frac14}}(s_0)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}B\langle D\rangle^{\frac14}}(s) \\ &+ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}B\langle D\rangle^{\frac14}}(s_0) + \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s_0)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}B\langle D\rangle^{\frac14}}(s). \end{aligned}$$

If $\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s_0) \leqslant 1$, then $e^{\pm A} - \mathrm{Id}$ and $\langle\partial_\varphi\rangle^{\mathtt{b}}(e^{\pm A} - \mathrm{Id})$ are $\mathcal{D}^{k_0}$-$(-\frac12)$-modulo-tame with

$$\begin{aligned} &\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}(e^{\pm A}-\mathrm{Id})\langle D\rangle^{\frac14}}(s) \lesssim_{k_0} \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s)\,, \\ &\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}(e^{\pm A}-\mathrm{Id})\langle D\rangle^{\frac14}}(s) \lesssim_{k_0,\mathtt{b}} \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}A\langle D\rangle^{\frac14}}(s) \\ &\qquad\qquad + \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}A\langle D\rangle^{\frac14}}(s)\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}A\langle D\rangle^{\frac14}}(s_0)\,. \end{aligned}$$

The next inequality provides a sufficient condition for an operator R to be $\mathcal{D}^{k_0}$-$(-\frac12)$-modulo-tame: it results (cfr. with Lemma 7.6 in [9])

$$\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}R\langle D\rangle^{\frac14}}(s),\ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}R\langle D\rangle^{\frac14}}(s) \lesssim_{s_0} \max\big\{\widetilde{\mathbb{M}}(s), \widetilde{\mathbb{M}}(s,\mathtt{b})\big\} \tag{3.20}$$

where $\widetilde{\mathbb{M}}(s,\mathtt{b}) := \max_{m=1,\ldots,\nu} \big\{ \mathfrak{M}_{\langle D\rangle^{\frac14} \partial_{\varphi_m}^{s_0+\mathtt{b}} R \langle D\rangle^{\frac14}}(s), \mathfrak{M}_{\langle D\rangle^{\frac14} [\partial_{\varphi_m}^{s_0+\mathtt{b}} R, \partial_x] \langle D\rangle^{\frac14}}(s) \big\}$ and

$$\widetilde{\mathbb{M}}(s) := \max_{m=1,\ldots,\nu} \big\{ \mathfrak{M}_{\langle D\rangle^{\frac14} R \langle D\rangle^{\frac14}}(s), \mathfrak{M}_{\langle D\rangle^{\frac14} [R,\partial_x] \langle D\rangle^{\frac14}}(s), \mathfrak{M}_{\langle D\rangle^{\frac14} \partial_{\varphi_m}^{s_0} R \langle D\rangle^{\frac14}}(s), \mathfrak{M}_{\langle D\rangle^{\frac14} [\partial_{\varphi_m}^{s_0} R, \partial_x] \langle D\rangle^{\frac14}}(s) \big\} .$$

Hamiltonian, Reversible and Momentum preserving operators. We shall exploit the Hamiltonian and reversible structure of the water waves equations as well as their invariance under space translations.

Definition 3.13. (Hamiltonian and Symplectic operators) A real matrix operator $\mathcal{R}$ on $L^2(\mathbb{T}_x, \mathbb{R}^2)$ is *Hamiltonian* if $J^{-1}\mathcal{R}$ is self-adjoint, namely $B^* = B$, $C^* = C$, $A^* = -D$ and A, B, C, D are real. It is *symplectic* if $\mathcal{W}(\mathcal{R}u, \mathcal{R}v) = \mathcal{W}(u,v)$, $\forall u, v \in L^2(\mathbb{T}_x, \mathbb{R}^2)$, where $\mathcal{W}$ is the symplectic 2-form in (2.6).

Let $\mathcal{S}$ be the involution (2.1) acting on the variables $(\eta,\zeta) \in \mathbb{R}^2$, or (2.22) acting on the action-angle-normal variables (θ, I, w), or (2.16) acting in the $(z, \overline{z})$ complex variables introduced in (2.12).

Definition 3.14. (Reversible/reversibility preserving op.) The operator $\mathcal{R}(\varphi)$ is *reversible* if $\mathcal{R}(-\varphi) \circ \mathcal{S} = -\mathcal{S} \circ \mathcal{R}(\varphi)$ for all $\varphi \in \mathbb{T}^\nu$. It is *reversibility preserving* if $\mathcal{R}(-\varphi) \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}(\varphi)$ for all $\varphi \in \mathbb{T}^\nu$.

By (2.16), an operator $\mathbf{R}(\varphi)$ as in (3.10) is reversible, respectively anti-reversible, if, for any $i = 1, 2$, $\mathcal{R}_i(-\varphi) \circ \mathcal{S} = -\mathcal{S} \circ \mathcal{R}_i(\varphi)$, resp. $\mathcal{R}_i(-\varphi) \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}_i(\varphi)$, where, with a small abuse of notation, we denote $(\mathcal{S}u)(x) = \overline{u(-x)}$. Moreover we have the following lemma (cfr. Lemmata 3.18 and 3.19 of [7]).

Lemma 3.15. *An operator $\mathbf{R}(\varphi)$, $\varphi \in \mathbb{T}^\nu$, as in (3.10) is reversible, respectively reversibility preserving, if, for any $i = 1, 2$, $(\mathcal{R}_i)_j^{j'}(-\varphi) = -\overline{(\mathcal{R}_i)_j^{j'}(\varphi)}$, resp. $(\mathcal{R}_i)_j^{j'}(-\varphi) = \overline{(\mathcal{R}_i)_j^{j'}(\varphi)}$, $\forall\, \varphi \in \mathbb{T}^\nu$, i.e. $(\mathcal{R}_i)_j^{j'}(\ell) = -\overline{(\mathcal{R}_i)_j^{j'}(\ell)}$, respectively $(\mathcal{R}_i)_j^{j'}(\ell) = \overline{(\mathcal{R}_i)_j^{j'}(\ell)}$, $\forall\, \ell \in \mathbb{Z}^\nu$. A pseudo-differential operator $\mathrm{Op}(a(\varphi, x, \xi))$ is reversible, respectively reversibility preserving, if and only if its symbol satisfies $a(-\varphi, -x, \xi) = -\overline{a(\varphi, x, \xi)}$, resp. $a(-\varphi, -x, \xi) = \overline{a(\varphi, x, \xi)}$.*

The composition of a reversible operator with a reversibility preserving operator is reversible. The flow generated by a reversibility preserving operator is reversibility preserving. If $\mathcal{R}(\varphi)$ is reversibility preserving, then

$(\omega \cdot \partial_\varphi \mathcal{R})(\varphi)$ is reversible. We shall say that a linear operator of the form $\omega \cdot \partial_\varphi + A(\varphi)$ is reversible if $A(\varphi)$ is reversible. Conjugating the reversible operator $\omega \cdot \partial_\varphi + A(\varphi)$ by a family of invertible reversibility preserving maps $\Phi(\varphi)$, we get the transformed reversible operator

$$\begin{aligned} &\Phi^{-1}(\varphi) \circ \big(\omega \cdot \partial_\varphi + A(\varphi)\big) \circ \Phi(\varphi) = \omega \cdot \partial_\varphi + A_+(\varphi)\,, \\ &A_+(\varphi) := \Phi^{-1}(\varphi)\,(\omega \cdot \partial_\varphi \Phi(\varphi)) + \Phi^{-1}(\varphi) A(\varphi) \Phi(\varphi)\,. \end{aligned} \tag{3.21}$$

A function $u(\varphi, \cdot)$ is *reversible* if $\mathcal{S}u(\varphi, \cdot) = u(-\varphi, \cdot)$ and *antireversible* if $-\mathcal{S}u(\varphi, \cdot) = u(-\varphi, \cdot)$. The same definition holds in the action-angle-normal variables (θ, I, w) with the involution $\vec{\mathcal{S}}$ defined in (2.22) and in the $(z, \overline{z})$ complex variables with the involution in (2.16). A reversibility preserving operator maps reversible, respectively anti-reversible, functions into reversible, respectively anti-reversible, functions, see Lemma 3.22 in [7]. If X is a reversible vector field, namely $X \circ \mathcal{S} = -\mathcal{S} \circ X$, and $u(\varphi, x)$ is a reversible quasi-periodic function, then the linearized operator $\mathrm{d}_u X(u(\varphi, \cdot))$ is reversible, see Lemma 3.22 in [7]. Finally we recall that the projections $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$, $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ defined below (2.18) commute with the involution $\mathcal{S}$ in (2.1) and the orthogonal projectors $\Pi_{\mathbb{S}}$ and $\Pi^{\perp}_{\mathbb{S}_0}$ commute with the involution in (2.16).

Definition 3.16. (Momentum preserving operators) A φ-dependent family of linear operators $A(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is *momentum preserving* if $A(\varphi - \vec{\jmath}\varsigma) \circ \tau_\varsigma = \tau_\varsigma \circ A(\varphi)$, $\forall\, \varphi \in \mathbb{T}^\nu$, $\varsigma \in \mathbb{R}$, where the translation operator τ_ς is defined in (2.2). A linear matrix operator $\mathbf{A}(\varphi)$ is *momentum preserving* if each of its components is momentum preserving.

If X is a translation invariant vector field, i.e. $X \circ \tau_\varsigma = \tau_\varsigma \circ X$, for all $\varsigma \in \mathbb{R}$, and u is a quasi-periodic traveling wave, then the linearized operator $\mathrm{d}_u X(u(\varphi, \cdot))$ is momentum preserving. If $A(\varphi), B(\varphi)$ are momentum preserving operators then the composition $A(\varphi) \circ B(\varphi)$ and the adjoint $(A(\varphi))^*$ are momentum preserving, cfr. Lemma 3.25 in [7]. Moreover, if $A(\varphi)$ is invertible, then $A(\varphi)^{-1}$ is momentum preserving. Assume that $\partial_t \Phi^t(\varphi) = A(\varphi)\Phi^t(\varphi)$, $\Phi^0(\varphi) = \mathrm{Id}$, has a unique propagator $\Phi^t(\varphi)$, $t \in [0, 1]$. Then $\Phi^t(\varphi)$ is momentum preserving.

We shall say that a linear operator of the form $\omega \cdot \partial_\varphi + A(\varphi)$ is momentum preserving if $A(\varphi)$ is momentum preserving. If $\omega \cdot \partial_\varphi + A(\varphi)$ and $\Phi(\varphi)$ are momentum preserving, the transformed operator $\omega \cdot \partial_\varphi + A_+(\varphi)$ in (3.21) is momentum preserving as well. Given a momentum preserving linear operator $A(\varphi)$ and a quasi-periodic traveling wave u, according to

Definition 3.1, then $A(\varphi)u$ is a quasi-periodic traveling wave. The characterizations of the momentum preserving property, in Fourier space and for a pseudo-differential operator, is given below (see Lemmata 3.28 and 3.29 in [7]).

Lemma 3.17. *Let φ-dependent family of operators $A(\varphi)$, $\varphi \in \mathbb{T}^\nu$, is momentum preserving if and only if the matrix elements $A_j^{j'}(\ell)$ of $A(\varphi)$, defined by* (3.9)*, are different from zero if $\vec{\jmath}\cdot\ell + j - j' = 0$, $\forall\, \ell \in \mathbb{Z}^\nu$, $j, j' \in \mathbb{Z}$. A pseudodifferential operator* $\mathrm{Op}(a(\varphi, x, \xi))$ *is momentum preserving if and only if its symbol satisfies $a(\varphi - \vec{\jmath}\varsigma, x, \xi) = a(\varphi, x + \varsigma, \xi)$ for any $\varsigma \in \mathbb{R}$.*

The symplectic projections $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$, $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$, defined below (2.18), the L^2-projections $\Pi^{L^2}_{\angle}$, $\Pi_{\mathbb{S}}$, $\Pi^{\perp}_{\mathbb{S}_0}$ defined below (2.26) are momentum preserving, cfr. Lemma 3.31 in [7].

Quasi-periodic traveling waves in action-angle-normal coordinates. Recalling (2.23), if $u(\varphi, x)$ is a quasi-periodic traveling wave with action-angle-normal components $(\theta(\varphi), I(\varphi), w(\varphi, x))$, the condition $\tau_\varsigma u = u(\varphi - \vec{\jmath}\varsigma, \cdot)$ becomes $\begin{pmatrix} \theta(\varphi)-\vec{\jmath}\varsigma \\ I(\varphi) \\ \tau_\varsigma w(\varphi,\cdot) \end{pmatrix} = \begin{pmatrix} \theta(\varphi-\vec{\jmath}\varsigma) \\ I(\varphi-\vec{\jmath}\varsigma) \\ w(\varphi-\vec{\jmath}\varsigma,\cdot) \end{pmatrix}$, for any $\varsigma \in \mathbb{R}$. Since $\theta(\varphi) = \varphi + \Theta(\varphi)$, with a $(2\pi)^\nu$-periodic function $\Theta : \mathbb{R}^\nu \mapsto \mathbb{R}^\nu$, $\varphi \mapsto \Theta(\varphi)$, the traveling wave condition becomes

$$\begin{pmatrix} \Theta(\varphi) \\ I(\varphi) \\ \tau_\varsigma w(\varphi,\cdot) \end{pmatrix} = \begin{pmatrix} \Theta(\varphi-\vec{\jmath}\varsigma) \\ I(\varphi-\vec{\jmath}\varsigma) \\ w(\varphi-\vec{\jmath}\varsigma,\cdot) \end{pmatrix}, \quad \forall\, \varsigma \in \mathbb{R}\,. \tag{3.22}$$

Definition 3.18. (Traveling wave variation) A traveling wave variation $g(\varphi) = (g_1(\varphi), g_2(\varphi), g_3(\varphi,\cdot)) \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is a function satisfying (3.22). or equivalently $D\vec{\tau}_\varsigma g(\varphi) = g(\varphi - \vec{\jmath}\varsigma)$ for any $\varsigma \in \mathbb{R}$, where $D\vec{\tau}_\varsigma$ is the differential of $\vec{\tau}_\varsigma$, namely $D\vec{\tau}_\varsigma(\Theta, I, w)^\top = (\Theta, I, \tau_\varsigma w)^\top$.

According to Definition 3.16, a linear operator acting in $\mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is momentum preserving if $A(\varphi - \vec{\jmath}\varsigma) \circ D\vec{\tau}_\varsigma = D\vec{\tau}_\varsigma \circ A(\varphi)$ for any $\varsigma \in \mathbb{R}$. In this case if $g \in \mathbb{R}^\nu \times \mathbb{R}^\nu \times \mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ is a traveling wave variation, then $A(\varphi)g(\varphi)$ is a traveling wave variation.

4 Transversality of linear frequencies

In this section we extend the KAM theory approach in [4, 9, 2, 7] to deal with the linear frequencies $\Omega_j(\gamma)$ defined in (1.8), using the vorticity as a parameter.

Definition 4.1. A function $f = (f_1, \ldots, f_N) : [\gamma_1, \gamma_2] \to \mathbb{R}^N$ is *non-degenerate* if, for any $c \in \mathbb{R}^N \backslash \{0\}$, the scalar function $f \cdot c$ is not identically zero on the whole interval $[\gamma_1, \gamma_2]$.

From a geometric point of view, the function f is non-degenerate if and only if the image curve $f([\gamma_1, \gamma_2]) \subset \mathbb{R}^N$ is not contained in any hyperplane of $\mathbb{R}^N$.

We shall use that the maps $\gamma \mapsto \Omega_j(\gamma)$ are analytic in $[\gamma_1, \gamma_2]$. For any $j \in \mathbb{Z}\backslash\{0\}$, we decompose the linear frequencies $\Omega_j(\gamma)$ as

$$\Omega_j(\gamma) = \omega_j(\gamma) + \frac{\gamma}{2}\frac{G_j(0)}{j}, \quad \omega_j(\gamma) := \sqrt{g\, G_j(0) + \Big(\frac{\gamma}{2}\frac{G_j(0)}{j}\Big)^2}, \tag{4.1}$$

where $G_j(0)$ is the Dirichlet-Neumann operator defined in (1.5).

Lemma 4.2. (Non-degeneracy-I) *The following frequency vectors are non-degenerate on* $[\gamma_1, \gamma_2]$*: (1)* $\vec{\Omega}(\gamma) := (\Omega_j(\gamma))_{j \in \mathbb{S}} \in \mathbb{R}^\nu$*; (2)* $\big(\vec{\Omega}(\gamma), 1\big) \in \mathbb{R}^{\nu+1}$*; (3)* $\big(\vec{\Omega}(\gamma), \Omega_j(\gamma)\big) \in \mathbb{R}^{\nu+1}$ *for any* $j \in \mathbb{Z}\backslash(\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$*; (4)* $\big(\vec{\Omega}(\gamma), \Omega_j(\gamma), \Omega_{j'}(\gamma)\big) \in \mathbb{R}^{\nu+2}$*, for any* $j, j' \in \mathbb{Z}\backslash(\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$ *and* $|j| \neq |j'|$.

Proof. We prove items 1, 3, 4 of the Lemma. We first compute the jets of the functions $\gamma \mapsto \Omega_j(\gamma)$ at $\gamma = 0$. Using that $G_j(0) = G_{|j|}(0) > 0$, see (1.5), we write (4.1) as

$$\Omega_j(\gamma) = \sqrt{g\, G_{|j|}(0)}\Big(\sqrt{1 + \gamma^2 \mathsf{c}_j^2} + \gamma \mathrm{sgn}(j) \mathsf{c}_j\Big), \quad \mathsf{c}_j := \frac{1}{2|j|}\sqrt{G_{|j|}(0)\, g^{-1}},$$

for any $j \in \mathbb{Z}\backslash\{0\}$. Each function $\gamma \mapsto (1 + \gamma^2 \mathsf{c}_j^2)^{1/2} + \gamma\, \mathrm{sgn}(j) \mathsf{c}_j$ is real analytic on the whole real line $\mathbb{R}$, and in a neighborhood of $\gamma = 0$, it admits the power series expansion

$$\begin{aligned} \Omega_j(\gamma) &= \sqrt{g\, G_{|j|}(0)}\Big(1 + \sum_{n \geqslant 1} a_n (\gamma^2 \mathsf{c}_j^2)^n + \gamma\, \mathrm{sgn}(j) \mathsf{c}_j\Big) \\ &= \sqrt{g\, G_{|j|}(0)} + \frac{\mathrm{sgn}(j)}{2}\frac{G_{|j|}(0)}{|j|}\gamma + \sum_{n \geqslant 1} \frac{a_n}{g^{n-\frac{1}{2}} 2^{2n}} \frac{(G_{|j|}(0))^{n+\frac{1}{2}}}{|j|^{2n}} \gamma^{2n} \end{aligned} \tag{4.2}$$

where $a_n := \binom{1/2}{n} \neq 0$ for any $n \geqslant 1$. From (4.2), we deduce that, for any $j \in \mathbb{Z}\backslash\{0\}$, for any $n \geqslant 1$,

$$\partial_\gamma^{2n} \Omega_j(0) = b_{2n} g_j \Big(\frac{G_{|j|}(0)}{|j|^2}\Big)^n, \ g_j := \sqrt{g\, G_{|j|}(0)} > 0\,, \ b_{2n} := \frac{(2n)!\, a_n}{g^n 2^{2n}} \neq 0\,. \tag{4.3}$$

We now prove that, for any N and integers $1 \leqslant |j_1| < |j_2| < \ldots < |j_N|$, the function $[\gamma_1, \gamma_2] \ni \gamma \mapsto (\Omega_{j_1}(\gamma), ..., \Omega_{j_N}(\gamma)) \in \mathbb{R}^N$ is non-degenerate according to Definition 4.1. Suppose, by contradiction, that $(\Omega_{j_1}(\gamma), ..., \Omega_{j_N}(\gamma))$ is degenerate, i.e. there exists $c \in \mathbb{R}^N \backslash \{0\}$ such that $c_1 \Omega_{j_1}(\gamma) + ... + c_N \Omega_{j_N}(\gamma) = 0$, $\forall \gamma \in [\gamma_1, \gamma_2]$, hence, by analyticity, it is identically zero for any $\gamma \in \mathbb{R}$. By differentiation we get $c_1 (\partial_\gamma^2 \Omega_{j_1})(\gamma) + ... + c_N (\partial_\gamma^2 \Omega_{j_N})(\gamma) = 0, \ldots, c_1 (\partial_\gamma^{2N} \Omega_{j_1})(\gamma) + ... + c_N (\partial_\gamma^{2N} \Omega_{j_N})(\gamma) = 0$. As a consequence the $N \times N$ matrix $\mathcal{A}(\gamma) := \begin{pmatrix} (\partial_\gamma^2 \Omega_{j_1})(\gamma) & \cdots & (\partial_\gamma^2 \Omega_{j_N})(\gamma) \\ \vdots & \ddots & \vdots \\ (\partial_\gamma^{2N} \Omega_{j_1})(\gamma) & \cdots & (\partial_\gamma^{2N} \Omega_{j_N})(\gamma) \end{pmatrix}$ is singular for any $\gamma \in \mathbb{R}$ and $\det \mathcal{A}(\gamma) = 0$, for all $\gamma \in \mathbb{R}$. In particular, at $\gamma = 0$ we have $\det \mathcal{A}(0) = 0$. On the other hand, by (4.3) and the multi-linearity of the determinant, we compute $\det \mathcal{A}(0) = b_2 ... b_{2N} \prod_{a=1}^N g_{j_a} f(j_a) \det \mathcal{V}(f)$, where $\mathcal{V}(f) := \begin{pmatrix} 1 & \cdots & 1 \\ f(j_1) & \cdots & f(j_N) \\ \vdots & \ddots & \vdots \\ f(j_1)^{N-1} & \cdots & f(j_N)^{N-1} \end{pmatrix}$ and $f(j) := |j|^{-2} G_{|j|}(0)$. This Vandermonde determinant is

$$\det \mathcal{A}(0) = b_2 ... b_{2N} \prod_{a=1}^N g_{j_a} f(j_a) \prod_{1 \leqslant p < q \leqslant N} (f(j_q) - f(j_p)) \, .$$

Note that $f(j) = |j|^{-2} G_{|j|}(0) > 0$ is even in $j \in \mathbb{Z} \backslash \{0\}$. We claim that the function $f(j)$ is monotone for any $j > 0$, from which, together with (4.3) and the assumption $1 \leqslant |j_1| < ... < |j_N|$, we obtain $\det \mathcal{A}(0) \neq 0$. This is a contradiction.

We now prove the monotonicity of the function $f : (0, +\infty) \to (0, +\infty)$, $f(y) := y^{-1} \tanh(\mathtt{h} y)$. For $\mathtt{h} = +\infty$, it is trivial. If $\mathtt{h} < +\infty$, we compute $\partial_y f(y) = y^{-2} g(\mathtt{h} y)$ where $g(x) := -\tanh(x) + x(1 - \tanh^2(x))$. Then $\partial_y f(y) < 0$ for any $y > 0$ if and only if $g(x) < 0$ for any $x > 0$. We note that $\lim_{x \to 0^+} g(x) = 0$, $\lim_{x \to +\infty} g(x) = -1$ and $g(x)$ is monotone decreasing for $x > 0$ because $\partial_x g(x) = -2x \tanh(x)(1 - \tanh^2(x)) < 0$ for any $x > 0$. The proof of item 2 is similar. □

Note that in items 3 and 4 of Lemma 4.2 we require that j and j' do not belong to $\{0\} \cup \mathbb{S} \cup (-\mathbb{S})$. In order to deal in Proposition 4.5 when j and j' belong to $-\mathbb{S}$, we need also the following lemma. It is a direct consequence of the proof of Lemma 4.2, noting that $\Omega_j(\gamma) - \omega_j(\gamma)$ is linear in γ (cfr. (4.1)) and its derivatives of order higher than two identically vanish.

Lemma 4.3. (Non-degeneracy-II) *Let* $\vec{\omega}(\gamma) := \big(\omega_{\bar{\jmath}_1}(\gamma), \ldots, \omega_{\bar{\jmath}_\nu}(\gamma)\big)$*. The following vectors are non-degenerate on* $[\gamma_1, \gamma_2]$*:* $(\vec{\omega}(\gamma), \gamma) \in \mathbb{R}^{\nu+1}$ *and* $(\vec{\omega}(\gamma), \omega_j(\gamma), \gamma) \in \mathbb{R}^{\nu+2}$ *for any* $j \in \mathbb{Z}\backslash(\{0\} \cup \mathbb{S} \cup (-\mathbb{S}))$.

We provide the following asymptotic estimate of the linear frequencies.

Lemma 4.4. (Asymptotics) *For any* $j \in \mathbb{Z}\backslash\{0\}$ *we have*

$$\omega_j(\gamma) = \sqrt{g}|j|^{\frac{1}{2}} + \frac{c_j(\gamma)}{\sqrt{g}|j|^{\frac{1}{2}}}\,, \quad \textit{where} \sup_{j\in\mathbb{Z}\backslash\{0\},\gamma\in[\gamma_1,\gamma_2]} |\partial_\gamma^n c_j(\gamma)| \leqslant C_{n,\mathtt{h}} \tag{4.4}$$

for any $n \in \mathbb{N}_0$ *and for some finite constant* $C_{n,\mathtt{h}} > 0$.

Proof. By (4.1), we deduce (4.4) with

$$c_j(\gamma) := \frac{g|j|\big(\frac{G_{|j|}(0)}{|j|} - 1\big) + \big(\frac{\gamma}{2}\frac{G_{|j|}(0)}{|j|}\big)^2}{1 + \sqrt{\frac{G_{|j|}(0)}{|j|} + \frac{1}{g|j|}\Big(\frac{\gamma}{2}\frac{G_{|j|}(0)}{|j|}\Big)^2}} \tag{4.5}$$

and using that $\frac{G_{|j|}(0)}{|j|} - 1 = -\frac{2}{1+e^{2\mathtt{h}|j|}}$, cfr. (1.5). □

The next proposition is the main result of the section. We remind that $\vec{\jmath} = (\bar{\jmath}_1, \ldots, \bar{\jmath}_\nu)$ denotes the vector in $\mathbb{Z}^\nu\backslash\{0\}$ of tangential sites, cfr. (2.24) and (2.19). We also recall that $\mathbb{S}_0^c = \mathbb{Z}\backslash(\mathbb{S} \cup \{0\})$.

Proposition 4.5. (Transversality) *There exist* $m_0 \in \mathbb{N}$ *and* $\rho_0 > 0$ *such that, for any* $\gamma \in [\gamma_1, \gamma_2]$*, the following hold:*

$$\max_{0\leqslant n\leqslant m_0} |\partial_\gamma^n \vec{\Omega}(\gamma)\cdot\ell| \geqslant \rho_0\langle\ell\rangle\,, \quad \forall\, \ell \in \mathbb{Z}^\nu\backslash\{0\}\,; \tag{4.6}$$

$$\begin{cases} \max_{0\leqslant n\leqslant m_0} |\partial_\gamma^n \,(\vec{\Omega}(\gamma)\cdot\ell + \Omega_j(\gamma))| \geqslant \rho_0\langle\ell\rangle\,, \\ \vec{\jmath}\cdot\ell + j = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j \in \mathbb{S}_0^c\,; \end{cases} \tag{4.7}$$

$$\begin{cases} \max_{0\leqslant n\leqslant m_0} |\partial_\gamma^n \,(\vec{\Omega}(\gamma)\cdot\ell + \Omega_j(\gamma) - \Omega_{j'}(\gamma))| \geqslant \rho_0\langle\ell\rangle \\ \vec{\jmath}\cdot\ell + j - j' = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j, j' \in \mathbb{S}_0^c\,,\ (\ell, j, j') \neq (0, j, j)\,; \end{cases} \tag{4.8}$$

$$\begin{cases} \max_{0\leqslant n\leqslant m_0} |\partial_\gamma^n \,(\vec{\Omega}(\gamma)\cdot\ell + \Omega_j(\gamma) + \Omega_{j'}(\gamma))| \geqslant \rho_0\langle\ell\rangle \\ \vec{\jmath}\cdot\ell + j + j' = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j, j' \in \mathbb{S}_0^c\,. \end{cases} \tag{4.9}$$

We call ρ_0 *the* amount of non-degeneracy, m_0 *the* index of non-degeneracy.

Proof. We now prove (4.8). The proof of (4.6), (4.7), (4.9) follows similarly. We set for brevity $\Gamma := [\gamma_1, \gamma_2]$. We assume $j_m \neq j'_m$ because the case $j_m = j'_m$ is included in (4.6). By contradiction, we assume that, for any $m \in \mathbb{N}$, there exist $\gamma_m \in \Gamma$, $\ell_m \in \mathbb{Z}^\nu$ and $j_m, j'_m \in \mathbb{S}_0^c$, $(\ell_m, j_m, j'_m) \neq (0, j_m, j_m)$, such that, for any $0 \leqslant n \leqslant m$, satisfying $\vec{\jmath} \cdot \ell_m + j_m - j'_m = 0$,

$$\left|\partial_\gamma^n \big(\vec{\Omega}(\gamma) \cdot \tfrac{\ell_m}{\langle \ell_m \rangle} + \tfrac{1}{\langle \ell_m \rangle}\big(\Omega_{j_m}(\gamma) - \Omega_{j'_m}(\gamma)\big)\big)_{|\gamma=\gamma_m}\right| < \tfrac{1}{\langle m \rangle}\,. \tag{4.10}$$

We have that $\ell_m \neq 0$, otherwise, by the momentum condition $j_m = j'_m$. Up to subsequences $\gamma_m \to \overline{\gamma} \in \Gamma$ and $\ell_m / \langle \ell_m \rangle \to \overline{c} \in \mathbb{R}^\nu \setminus \{0\}$.

STEP 1. We start with the case when $(\ell_m)_{m \in \mathbb{N}} \subset \mathbb{Z}^\nu$ *is bounded.* Up to subsequences, we have definitively that $\ell_m = \overline{\ell} \in \mathbb{Z}^\nu \setminus \{0\}$. The sequences $(j_m)_{m\in\mathbb{N}}$ and $(j'_m)_{m\in\mathbb{N}}$ may be bounded or unbounded. Up to subsequences, we consider the different cases:

Case (a). $|j_m|, |j'_m| \to +\infty$ *for* $m \to \infty$. We have that $j_m \cdot j'_m > 0$, because, otherwise, $|j_m - j'_m| = |j_m| + |j'_m| \to +\infty$ contradicting that $|j_m - j'_m| = |\vec{\jmath} \cdot \ell_m| \leqslant C$. Recalling (1.5) we have, for any $j \cdot j' > 0$, that

$$\left|\frac{G_j(0)}{j} - \frac{G_{j'}(0)}{j'}\right| \leqslant C_{\mathtt{h}} \Big(\frac{1}{|j|^{\frac12}} + \frac{1}{|j'|^{\frac12}}\Big)\,. \tag{4.11}$$

Moreover, by the momentum condition $\vec{\jmath} \cdot \ell_m + j_m - j'_m = 0$, we deduce

$$|\sqrt{|j_m|} - \sqrt{|j'_m|}| \leqslant \frac{|j_m - j'_m|}{\sqrt{|j_m|} + \sqrt{|j'_m|}} \leqslant \frac{C|\ell_m|}{\sqrt{|j_m|} + \sqrt{|j'_m|}}\,. \tag{4.12}$$

By (4.1), Lemma 4.4, $j_m \cdot j'_m > 0$, (4.11), (4.12), we conclude that

$$\begin{aligned}\partial_\gamma^n(\Omega_{j_m}(\gamma) - \Omega_{j'_m}(\gamma)) &= \sqrt{g}\,\partial_\gamma^n\big(\sqrt{|j_m|} - \sqrt{|j'_m|}\big) \\ &\quad + \partial_\gamma^n\Big(\frac{c_{j_m}(\gamma)}{\sqrt{g}|j_m|^{\frac12}} - \frac{c_{j'_m}(\gamma)}{\sqrt{g}|j'_m|^{\frac12}} + \frac{\gamma}{2}\Big(\frac{G_{j_m}(0)}{j_m} - \frac{G_{j'_m}(0)}{j'_m}\Big)\Big) \to 0\end{aligned}$$

as $m \to +\infty$. Passing to the limit in (4.10), we obtain $\partial_\gamma^n \{\vec{\Omega}(\gamma) \cdot \overline{\ell}\}_{|\gamma=\overline{\gamma}} = 0$ for any $n \in \mathbb{N}_0$. Hence the analytic function $\gamma \mapsto \vec{\Omega}(\gamma) \cdot \overline{\ell}$ is identically zero, contradicting Lemma 4.2-1, since $\overline{\ell} \neq 0$.

Case (b). $(j_m)_{m\in\mathbb{N}}$ *is bounded and* $|j'_m| \to \infty$ (or viceversa): this case is excluded by the momentum condition $\vec{\jmath} \cdot \ell_m + j_m - j'_m = 0$ in (4.10) and since (ℓ_m) is bounded.

Case (c). Both $(j_m)_{m\in\mathbb{N}}$, $(j'_m)_{m\in\mathbb{N}}$ *are bounded:* we have definitively that $j_m = \overline{\jmath}$ and $j'_m = \overline{\jmath}'$, with $\overline{\jmath}, \overline{\jmath}' \in \mathbb{S}_0^c$ and, since $j_m \neq j'_m$, we have $\overline{\jmath} \neq \overline{\jmath}'$. Therefore (4.10) becomes, in the limit $m \to \infty$, $\partial_\gamma^n\big(\vec{\Omega}(\gamma) \cdot \overline{\ell} +$

$\Omega_{\overline{\jmath}}(\gamma) - \Omega_{\overline{\jmath}'}(\gamma)\big)_{|\gamma=\overline{\gamma}} = 0$, $\forall\, n \in \mathbb{N}_0$, $\vec{\jmath}\cdot\overline{\ell} + \overline{\jmath} - \overline{\jmath}' = 0$. By analyticity, we obtain that

$$\vec{\Omega}(\gamma)\cdot\overline{\ell} + \Omega_{\overline{\jmath}}(\gamma) - \Omega_{\overline{\jmath}'}(\gamma) = 0 \quad \forall\,\gamma \in \Gamma\,, \quad \vec{\jmath}\cdot\overline{\ell} + \overline{\jmath} - \overline{\jmath}' = 0\,. \tag{4.13}$$

We distinguish several cases:

• Let $\overline{\jmath}, \overline{\jmath}' \notin -\mathbb{S}$ and $|\overline{\jmath}| \neq |\overline{\jmath}'|$. By (4.13) the vector $(\vec{\Omega}(\gamma), \Omega_{\overline{\jmath}}(\gamma), \Omega_{\overline{\jmath}'}(\gamma))$ is degenerate with $c := (\overline{\ell}, 1, -1) \neq 0$, contradicting Lemma 4.2-4.

• Let $\overline{\jmath}, \overline{\jmath}' \notin -\mathbb{S}$ and $\overline{\jmath}' = -\overline{\jmath}$. In view of (4.1), the first equation in (4.13) becomes $\vec{\omega}(\gamma)\cdot\overline{\ell} + \frac{\gamma}{2}\big(\sum_{a=1}^{\nu}\overline{\ell}_a \frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a} + 2\frac{G_{\overline{\jmath}}(0)}{\overline{\jmath}}\big) = 0$, $\forall \gamma \in \Gamma$. By Lemma 4.3 the vector $(\vec{\omega}(\gamma), \gamma)$ is non-degenerate, thus $\overline{\ell} = 0$ and $2\frac{G_{\overline{\jmath}}(0)}{\overline{\jmath}} = 0$, which is a contradiction.

• Let $\overline{\jmath}' \notin -\mathbb{S}$ and $\overline{\jmath} \in -\mathbb{S}$. With no loss of generality suppose $\overline{\jmath} = -\overline{\jmath}_1$. In view of (4.1), the first equation in (4.13) implies that, for any $\gamma \in \Gamma$,

$$\begin{aligned}(\overline{\ell}_1 + 1)\omega_{\overline{\jmath}_1}(\gamma) &+ \sum_{a=2}^{\nu}\overline{\ell}_a\omega_{\overline{\jmath}_a}(\gamma) - \omega_{\overline{\jmath}'}(\gamma)\\ &+ \frac{\gamma}{2}\Big((\overline{\ell}_1 - 1)\frac{G_{\overline{\jmath}_1}(0)}{\overline{\jmath}_1} + \sum_{a=2}^{\nu}\overline{\ell}_a\frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a} - \frac{G_{\overline{\jmath}'}(0)}{\overline{\jmath}'}\Big) = 0\,.\end{aligned}$$

By Lemma 4.3 the vector $\big(\vec{\omega}(\gamma), \omega_{\overline{\jmath}'}(\gamma), \gamma\big)$ is non-degenerate, which is a contradiction.

• Last, let $\overline{\jmath}, \overline{\jmath}' \in -\mathbb{S}$ and $\overline{\jmath} \neq \overline{\jmath}'$. With no loss of generality suppose $\overline{\jmath} = -\overline{\jmath}_1$ and $\overline{\jmath}' = -\overline{\jmath}_2$. Then the first equation in (4.13) reads, for any $\gamma \in \Gamma$, $(\overline{\ell}_1 + 1)\omega_{\overline{\jmath}_1}(\gamma) + \big(\overline{\ell}_2 - 1\big)\,\omega_{\overline{\jmath}_2} + \sum_{a=3}^{\nu}\overline{\ell}_a\omega_{\overline{\jmath}_a}(\gamma) + \frac{\gamma}{2}\big((\overline{\ell}_1 - 1)\frac{G_{\overline{\jmath}_1}(0)}{\overline{\jmath}_1} + (\overline{\ell}_2 + 1)\frac{G_{\overline{\jmath}_2}(0)}{\overline{\jmath}_2} + \sum_{a=3}^{\nu}\overline{\ell}_a\frac{G_{\overline{\jmath}_a}(0)}{\overline{\jmath}_a}\big) = 0$. Since the vector $(\vec{\omega}(\gamma), \gamma)$ is non-degenerate by Lemma 4.3, it implies $\overline{\ell}_1 = -1$, $\overline{\ell}_2 = 1$, $\overline{\ell}_3 = \ldots = \overline{\ell}_\nu = 0$. Inserting these values in (4.13) we obtain $-2\overline{\jmath}_1 + 2\overline{\jmath}_2 = 0$. This contradicts $\overline{\jmath} \neq \overline{\jmath}'$.

STEP 2. We finally consider the case when $(\ell_m)_{m\in\mathbb{N}}$ *is unbounded*. Up to subsequences $\ell_m \to \infty$ as $m \to \infty$ and $\lim_{m\to\infty}\ell_m/\langle\ell_m\rangle =: \overline{c} \neq 0$. By (4.1), Lemma 4.4, (4.11), we have, for any $n \geqslant 1$,

$$\begin{aligned}\partial_\gamma^n \frac{1}{\langle\ell_m\rangle}\big(\Omega_{j_m}(\gamma) - \Omega_{j_m'}(\gamma)\big)_{|\gamma=\gamma_m} &= \partial_\gamma^n\Big(\frac{1}{\langle\ell_m\rangle\sqrt{g}}\Big(\frac{c_{j_m}(\gamma)}{|j_m|^{\frac12}} - \frac{c_{j_m'}(\gamma)}{|j_m'|^{\frac12}}\Big)\\ &\quad + \frac{\gamma}{2\langle\ell_m\rangle}\Big(\frac{G_{j_m}(0)}{j_m} - \frac{G_{j_m'}(0)}{j_m'}\Big)_{|\gamma=\gamma_m}\Big) \to 0\end{aligned}$$

as $m \to \infty$. Therefore, for any $n \geqslant 1$, taking $m \to \infty$ in (4.10) we get $\partial_\gamma^n \big(\vec{\Omega}(\gamma) \cdot \overline{c}\big)_{|\gamma=\overline{\gamma}} = 0$. By analyticity this implies $\vec{\Omega}(\gamma) \cdot \overline{c} = \overline{d}$, for all $\gamma \in \Gamma$, contradicting Lemma 4.2-2, since $\overline{c} \neq 0$. □

Remark 4.6. *For the irrotational case $\gamma = 0$, quasi-periodic traveling waves exist for most values of the* depth $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$. *In detail, the non-degeneracy property of the linear frequencies with respect to* $\mathtt{h}$ *as in Lemma 4.2 is proved in Lemma 3.2 in* [2], *whereas the transversality properties hold by restricting the bounds in Lemma 3.4 in* [2] *to the Fourier sites satisfying the momentum conditions.*

5 Proof of Theorem 1.2

Under the rescaling $(\eta, \zeta) \mapsto (\varepsilon\eta, \varepsilon\zeta)$, the Hamiltonian system (2.5) transforms into the Hamiltonian system generated by

$$\mathcal{H}_\varepsilon(\eta, \zeta) := \varepsilon^{-2}\mathcal{H}(\varepsilon\eta, \varepsilon\zeta) = \mathcal{H}_L(\eta, \zeta) + \varepsilon P_\varepsilon(\eta, \zeta)\,,$$

where $\mathcal{H}$ is the water waves Hamiltonian (2.4) expressed in the Wahlén coordinates (2.3), $\mathcal{H}_L$ is as in (2.8) and $P_\varepsilon(\eta, \zeta) := \varepsilon^{-3}\mathcal{H}_{\geqslant 3}(\varepsilon\eta, \varepsilon\zeta)$, denoting $\mathcal{H}_{\geqslant 3} := \mathcal{H} - \mathcal{H}_L$ the cubic part of the Hamiltonian. We study this Hamiltonian system in the action-angle and normal coordinates (θ, I, w), considering the Hamiltonian $H_\varepsilon(\theta, I, w)$ defined by

$$H_\varepsilon := \mathcal{H}_\varepsilon \circ A = \varepsilon^{-2}\mathcal{H} \circ \varepsilon A \tag{5.1}$$

where A is the map defined in (2.21). The associated symplectic form is given in (2.25). By (2.28) (see also (2.20)), in the variables (θ, I, w) the quadratic Hamiltonian $\mathcal{H}_L$ defined in (2.8) simply reads, up to a constant, $\mathcal{N} := \mathcal{H}_L \circ A = \vec{\Omega}(\gamma) \cdot I + \frac{1}{2}\left(\mathbf{\Omega}_W w, w\right)_{L^2}$, where $\vec{\Omega}(\gamma) \in \mathbb{R}^\nu$ is defined in (1.12) and $\mathbf{\Omega}_W$ in (2.7). Thus the Hamiltonian H_ε in (5.1) is

$$H_\varepsilon = \mathcal{N} + \varepsilon P \qquad \text{with} \qquad P := P_\varepsilon \circ A\,. \tag{5.2}$$

5.1 Nash-Moser theorem of hypothetical conjugation

Instead of looking directly for quasi-periodic solutions of the Hamiltonian system generated by H_ε, we look for quasi-periodic solutions of the modified Hamiltonians, where $\alpha \in \mathbb{R}^\nu$ are additional parameters,

$$H_\alpha := \mathcal{N}_\alpha + \varepsilon P\,, \quad \mathcal{N}_\alpha := \alpha \cdot I + \tfrac{1}{2}\left(w, \mathbf{\Omega}_W w\right)_{L^2}\,. \tag{5.3}$$

We consider the nonlinear operator $\mathcal{F}(i, \alpha) := \mathcal{F}(\omega, \gamma, \varepsilon; i, \alpha) := \omega \cdot \partial_\varphi i(\varphi) - X_{H_\alpha}(i(\varphi))$. If $\mathcal{F}(i, \alpha) = 0$, then $i(\varphi)$ is an invariant torus for the Hamiltonian vector field X_{H_α}, filled with quasi-periodic solutions with frequency

ω. Each Hamiltonian H_α in (5.3) is invariant under the involution $\vec{\mathcal{S}}$ and the translations $\vec{\tau}_\varsigma$, $\varsigma \in \mathbb{R}$, defined respectively in (2.22) and in (2.23): $H_\alpha \circ \vec{\mathcal{S}} = H_\alpha$, $H_\alpha \circ \vec{\tau}_\varsigma = H_\alpha$, $\forall \varsigma \in \mathbb{R}$. We look for a reversible traveling torus embedding $i(\varphi) = (\theta(\varphi), I(\varphi), w(\varphi))$, namely satisfying

$$\vec{\mathcal{S}} i(\varphi) = i(-\varphi)\,, \qquad \vec{\tau}_\varsigma i(\varphi) = i(\varphi - \vec{\jmath}\varsigma)\,, \quad \forall \varsigma \in \mathbb{R}\,. \tag{5.4}$$

The operator $\mathcal{F}(\cdot, \alpha)$ maps a reversible, respectively traveling, wave into an anti-reversible, respectively traveling, wave variation, according to Definition 3.18.

The norm of the periodic components of the embedded torus

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0, 0) := \big(\Theta(\varphi), I(\varphi), w(\varphi)\big), \ \Theta(\varphi) := \theta(\varphi) - \varphi\,, \tag{5.5}$$

is $\|\mathfrak{I}\|_s^{k_0,\upsilon} := \|\Theta\|_{H^s_\varphi}^{k_0,\upsilon} + \|I\|_{H^s_\varphi}^{k_0,\upsilon} + \|w\|_s^{k_0,\upsilon}$, where $k_0 := m_0 + 2$ and $m_0 \in \mathbb{N}$ is the index of non-degeneracy provided by Proposition 4.5. We will omit to write the dependence of the various constants with respect to k_0. We look for quasi-periodic solutions of frequency ω in a δ-neighborhood

$$\Omega := \big\{\omega \in \mathbb{R}^\nu \ : \ \mathrm{dist}\big(\omega, \vec{\Omega}[\gamma_1, \gamma_2]\big) < \delta\big\}$$

with $\delta > 0$ (independent of ε) of the curve $\vec{\Omega}[\gamma_1, \gamma_2]$ defined by (1.12).

Theorem 5.1. (Theorem of hypothetical conjugation) *There exist positive constants* $\mathrm{a}_0, \varepsilon_0, C$ *depending on* $\mathbb{S}$, k_0 *and* $\tau \geqslant 1$ *such that, for all* $\upsilon = \varepsilon^{\mathrm{a}}$, $\mathrm{a} \in (0, \mathrm{a}_0)$ *and for all* $\varepsilon \in (0, \varepsilon_0)$, *there exist:*

1. *a* k_0*-times differentiable function of the form* $\alpha_\infty : \Omega \times [\gamma_1, \gamma_2] \mapsto \mathbb{R}^\nu$,

$$\alpha_\infty(\omega, \gamma) := \omega + r_\varepsilon(\omega, \gamma) \quad \textit{with} \quad |r_\varepsilon|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,; \tag{5.6}$$

2. *embedded reversible traveling tori* $i_\infty(\varphi)$ *(cfr.* (5.4)*), defined for all* $(\omega, \gamma) \in \Omega \times [\gamma_1, \gamma_2]$, *satisfying*

$$\|i_\infty(\varphi) - (\varphi, 0, 0)\|_{s_0}^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,; \tag{5.7}$$

3. k_0*-times differentiable functions* $\mu_j^\infty : \mathbb{R}^\nu \times [\gamma_1, \gamma_2] \to \mathbb{R}$, $j \in \mathbb{S}_0^c = \mathbb{Z} \setminus (\mathbb{S} \cup \{0\})$, *of the form*

$$\mu_j^\infty(\omega, \gamma) = \mathtt{m}_1^\infty(\omega, \gamma) j + \mathtt{m}_{\frac12}^\infty(\omega, \gamma)\Omega_j(\gamma) - \mathtt{m}_0^\infty(\omega, \gamma)\mathrm{sgn}(j) + \mathfrak{r}_j^\infty(\omega, \gamma)\,, \tag{5.8}$$

with $\Omega_j(\gamma)$ *defined in* (1.8)*, satisfying*

$$\begin{aligned} &|\mathtt{m}_1^\infty|^{k_0,\upsilon} \leqslant C\varepsilon\,, \ |\mathtt{m}_{\frac12}^\infty - 1|^{k_0,\upsilon} + |\mathtt{m}_0^\infty|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1}\,, \\ &\sup_{j \in \mathbb{S}_0^c} |j|^{\frac12} |\mathfrak{r}_j^\infty|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-3}\,, \end{aligned} \tag{5.9}$$

such that, for all (ω,γ) *in the Cantor-like set*

$$\mathcal{C}^{\upsilon}_{\infty} := \Big\{ (\omega,\gamma) \in \Omega \times [\gamma_1,\gamma_2] \, : \, |\omega \cdot \ell| \geqslant 8\upsilon \langle \ell \rangle^{-\tau}, \ \ \forall\, \ell \in \mathbb{Z}^{\nu} \backslash \{0\} \, ; \tag{5.10}$$

$$|\omega \cdot \ell - \mathtt{m}_1^{\infty}(\omega,\gamma) j| \geqslant 8\upsilon \langle \ell \rangle^{-\tau}, \ \forall\, \ell \in \mathbb{Z}^{\nu}, \ j \in \mathbb{S}_0^c \ \textit{with} \ \vec{\jmath} \cdot \ell + j = 0; \tag{5.11}$$

$$|\omega \cdot \ell + \mu_j^{\infty}(\omega,\gamma)| \geqslant 4\upsilon \, |j|^{\frac{1}{2}} \, \langle \ell \rangle^{-\tau} \, , \ \forall\, \ell \in \mathbb{Z}^{\nu}, \ j \in \mathbb{S}_0^c \ \textit{with} \ \vec{\jmath} \cdot \ell + j = 0 \, ;$$

$$\big|\omega \cdot \ell + \mu_j^{\infty}(\omega,\gamma) - \mu_{j'}^{\infty}(\omega,\gamma)\big| \geqslant 4\upsilon \, \langle \ell \rangle^{-\tau} \, , \tag{5.12}$$
$$\forall \ell \in \mathbb{Z}^{\nu}, \ j, j' \in \mathbb{S}_0^c, \ (\ell, j, j') \neq (0, j, j) \ \textit{with} \ \vec{\jmath} \cdot \ell + j - j' = 0 \, ;$$

$$\big|\omega \cdot \ell + \mu_j^{\infty}(\omega,\gamma) + \mu_{j'}^{\infty}(\omega,\gamma)\big| \geqslant 4\upsilon \, \big(|j|^{\frac{1}{2}} + |j'|^{\frac{1}{2}} \big) \langle \ell \rangle^{-\tau} \, , \tag{5.13}$$
$$\forall\, \ell \in \mathbb{Z}^{\nu}, \ j, j' \in \mathbb{S}_0^c \, , \ \textit{with} \ \vec{\jmath} \cdot \ell + j + j' = 0 \Big\} \, ,$$

the function $i_\infty(\varphi) := i_\infty(\omega,\gamma,\varepsilon;\varphi)$ *solves* $\mathcal{F}(\omega,\gamma,\varepsilon;(i_\infty,\alpha_\infty)(\omega,\gamma)) = 0$*. As a consequence, the embedded torus* $\varphi \mapsto i_\infty(\varphi)$ *is invariant for the Hamiltonian vector field* $X_{H_{\alpha_\infty(\omega,\gamma)}}$ *as it is filled by quasi-periodic reversible traveling wave solutions with frequency* ω*.*

Theorem 5.1 is deduced by a Nash-Moser iteration scheme at the end of Section 7.

Remark 5.2. *The Diophantine condition* (5.10) *could be weakened requiring only* $|\omega \cdot \ell| \geqslant \upsilon \langle \ell \rangle^{-\tau}$ *for any* $\ell \cdot \vec{\jmath} = 0$*. If so, the vector* ω *could admit one non-trivial resonance, i.e.* $\overline{\ell} \in \mathbb{Z}^{\nu} \backslash \{0\}$ *such that* $\omega \cdot \overline{\ell} = 0$*, and the orbit* $\{\omega t\}_{t \in \mathbb{R}}$ *would densely fill a (*$\nu - 1$*)-dimensional torus, orthogonal to* $\overline{\ell}$*. In any case* $\vec{\jmath} \cdot \overline{\ell} \neq 0$ *(otherwise* $|\omega \cdot \overline{\ell}| \geqslant \upsilon \langle \overline{\ell} \rangle^{-\tau} > 0$*, contradicting* $\omega \cdot \overline{\ell} = 0$*) and then* $\overline{\{\omega t - \vec{\jmath} x\}_{t \in \mathbb{R}, x \in \mathbb{R}}} = \mathbb{T}^{\nu}$*. This is the natural minimal requirement to look for traveling quasi-periodic solutions* $U(\omega t - \vec{\jmath} x)$ *(Definition 3.1).*

5.2 Measure estimates: proof of Theorem 1.2

Now we deduce from Theorem 5.1 the existence of quasi-periodic solutions of the original Hamiltonian system generated by H_ε in (5.2) and not of just H_{α_∞}. By (5.6), the function $\alpha_\infty(\,\cdot\,,\gamma)$ from Ω into its image $\alpha_\infty(\Omega,\gamma)$ is invertible and

$$\begin{aligned} &\beta = \alpha_\infty(\omega,\gamma) = \omega + r_\varepsilon(\omega,\gamma) \ \Leftrightarrow \\ &\omega = \alpha_\infty^{-1}(\beta,\gamma) = \beta + \breve{r}_\varepsilon(\beta,\gamma) \, , \quad |\breve{r}_\varepsilon|^{k_0,\upsilon} \leqslant C\varepsilon\upsilon^{-1} \, . \end{aligned} \tag{5.14}$$

Then, for any $\beta \in \alpha_\infty(\mathcal{C}^{\upsilon}_{\infty})$, Theorem 5.1 proves the existence of an embedded invariant torus filled by quasi-periodic solutions with Diophantine frequency $\omega = \alpha_\infty^{-1}(\beta,\gamma)$ for the Hamiltonian $H_\beta = \beta \cdot I + \frac{1}{2}(w, \mathbf{\Omega}_W w)_{L^2} + \varepsilon P$. Consider the curve of the unperturbed tangential frequency vector

$\vec{\Omega}(\gamma)$ in (1.12). In Theorem 5.3 below we prove that for "most" values of $\gamma \in [\gamma_1, \gamma_2]$ the vector $(\alpha_\infty^{-1}(\vec{\Omega}(\gamma), \gamma), \gamma)$ is in $\mathcal{C}_\infty^\upsilon$, obtaining an embedded torus for the Hamiltonian H_ε in (5.1), filled by quasi-periodic solutions with Diophantine frequency vector $\omega = \alpha_\infty^{-1}(\vec{\Omega}(\gamma), \gamma)$, denoted $\widetilde{\Omega}$ in Theorem 1.2. Thus $\varepsilon A(i_\infty(\widetilde{\Omega} t))$, where A is defined in (2.21), is a quasi-periodic traveling wave solution of the water waves equations (2.5) written in the Wahlén variables. Finally, going back to the original Zakharov variables via (2.3) we obtain solutions of (1.1). This proves Theorem 1.2 together with the following measure estimates.

Theorem 5.3. (Measure estimates) *Let*

$$\upsilon = \varepsilon^{\mathrm{a}}\,,\ 0 < \mathrm{a} < \min\{\mathrm{a}_0, 1/(4m_0^2)\}\,,\ \tau > m_0(2m_0\nu + \nu + 2)\,, \tag{5.15}$$

where m_0 is given in Proposition 4.5 and $k_0 := m_0 + 2$. Then, for $\varepsilon \in (0, \varepsilon_0)$ small enough, the measure of the set

$$\mathcal{G}_\varepsilon := \left\{\gamma \in [\gamma_1, \gamma_2] \ : \ \left(\alpha_\infty^{-1}(\vec{\Omega}(\gamma), \gamma), \gamma\right) \in \mathcal{C}_\infty^\upsilon\right\}$$

satisfies $|\mathcal{G}_\varepsilon| \to \gamma_2 - \gamma_1$ as $\varepsilon \to 0$.

The rest of this section is devoted to prove Theorem 5.3. By (5.14) we have

$$\vec{\Omega}_\varepsilon(\gamma) := \alpha_\infty^{-1}(\vec{\Omega}(\gamma), \gamma) = \vec{\Omega}(\gamma) + \vec{r}_\varepsilon\,, \tag{5.16}$$

where $\vec{r}_\varepsilon(\gamma) := \breve{r}_\varepsilon(\vec{\Omega}(\gamma), \gamma)$ satisfies

$$|\partial_\gamma^k \vec{r}_\varepsilon(\gamma)| \leqslant C\varepsilon\upsilon^{-(1+k)}\,, \quad \forall\ |k| \leqslant k_0\,,\ \text{uniformly on } [\gamma_1, \gamma_2]\,. \tag{5.17}$$

We also denote, with a small abuse of notation, for all $j \in \mathbb{S}_0^c$,

$$\begin{aligned} \mu_j^\infty(\gamma) :=& \mu_j^\infty\left(\vec{\Omega}_\varepsilon(\gamma), \gamma\right) \\ :=& \mathtt{m}_1^\infty(\gamma) j + \mathtt{m}_{\frac12}^\infty(\gamma)\Omega_j(\gamma) - \mathtt{m}_0^\infty(\gamma)\mathrm{sgn}(j) + \mathfrak{r}_j^\infty(\gamma)\,, \end{aligned} \tag{5.18}$$

where, for sake of simplicity in the notation, $\mathtt{m}_1^\infty(\gamma) := \mathtt{m}_1^\infty(\vec{\Omega}_\varepsilon(\gamma), \gamma)$, $\mathtt{m}_{\frac12}^\infty(\gamma) := \mathtt{m}_{\frac12}^\infty(\vec{\Omega}_\varepsilon(\gamma), \gamma)$, $\mathtt{m}_0^\infty(\gamma) := \mathtt{m}_0^\infty(\vec{\Omega}_\varepsilon(\gamma), \gamma)$, $\mathfrak{r}_j^\infty(\gamma) := \mathfrak{r}_j^\infty(\vec{\Omega}_\varepsilon(\gamma), \gamma)$. By (5.9) and (5.17) we get the estimates

$$|\partial_\gamma^k \mathtt{m}_1^\infty(\gamma)| \leqslant C\varepsilon\upsilon^{-k},\ \left|\partial_\gamma^k\left(\mathtt{m}_{\frac12}^\infty(\gamma) - 1\right)\right| + |\partial_\gamma^k \mathtt{m}_0^\infty(\gamma)| \leqslant C\varepsilon\upsilon^{-k-1}, \tag{5.19}$$

$$\sup_{j\in\mathbb{S}_0^c} |j|^{\frac12}\left|\partial_\gamma^k \mathfrak{r}_j^\infty(\gamma)\right| \leqslant C\varepsilon\upsilon^{-3-k}\,, \quad \forall\, 0 \leqslant k \leqslant k_0\,. \tag{5.20}$$

Recalling (5.10)-(5.13), we estimate the measure of the complementary set

$$\mathcal{G}_\varepsilon^c := [\gamma_1,\gamma_2]\backslash\mathcal{G}_\varepsilon = \Big(\bigcup_{\ell\neq 0} R_\ell^{(0)} \cup R_\ell^{(T)}\Big) \cup \Big(\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,\, j\in\mathbb{S}_0^c \\ \vec{\jmath}\cdot\ell+j=0}} R_{\ell,j}^{(I)}\Big)$$

$$\cup \Big(\bigcup_{\substack{(\ell,j,j')\neq(0,j,j),\, j\neq j' \\ \vec{\jmath}\cdot\ell+j-j'=0}} R_{\ell,j,j'}^{(II)}\Big) \cup \Big(\bigcup_{\substack{\ell\in\mathbb{Z}^\nu,\, j,j'\in\mathbb{S}_0^c, \\ \vec{\jmath}\cdot\ell+j+j'=0}} Q_{\ell,j,j'}^{(II)}\Big), \tag{5.21}$$

where the “nearly-resonant sets" are, recalling the notation $\Gamma = [\gamma_1,\gamma_2]$,

$$R_\ell^{(0)} := R_\ell^{(0)}(\upsilon,\tau) := \Big\{\gamma\in\Gamma \,:\, |\vec{\Omega}_\varepsilon(\gamma)\cdot\ell| < 8\upsilon\langle\ell\rangle^{-\tau}\Big\}, \tag{5.22}$$

$$R_\ell^{(T)} := R_\ell^{(T)}(\upsilon,\tau) := \Big\{\gamma\in\Gamma \,:\, |(\vec{\Omega}_\varepsilon(\gamma) - \mathtt{m}_1^\infty(\gamma)\vec{\jmath})\cdot\ell| < 8\upsilon\langle\ell\rangle^{-\tau}\Big\},$$

$$R_{\ell,j}^{(I)} := R_{\ell,j}^{(I)}(\upsilon,\tau) := \Big\{\gamma\in\Gamma \,:\, |\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mu_j^\infty(\gamma)| < 4\upsilon|j|^{\frac12}\langle\ell\rangle^{-\tau}\Big\}, \tag{5.23}$$

and the sets $R_{\ell,j,j'}^{(II)} := R_{\ell,j,j'}^{(II)}(\upsilon,\tau)$, $Q_{\ell,j,j'}^{(II)} := Q_{\ell,j,j'}^{(II)}(\upsilon,\tau)$ are

$$R_{\ell,j,j'}^{(II)} := \Big\{\gamma\in\Gamma \,:\, |\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mu_j^\infty(\gamma) - \mu_{j'}^\infty(\gamma)| < 4\upsilon\,\langle\ell\rangle^{-\tau}\Big\}, \tag{5.24}$$

$$Q_{\ell,j,j'}^{(II)} := \Big\{\gamma\in\Gamma \,:\, |\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mu_j^\infty(\gamma) + \mu_{j'}^\infty(\gamma)| < \tfrac{4\upsilon\left(|j|^{\frac12}+|j'|^{\frac12}\right)}{\langle\ell\rangle^\tau}\Big\}. \tag{5.25}$$

The third union in (5.21) may require $j\neq j'$ because $R_{\ell,j,j}^{(II)}\subset R_\ell^{(0)}$. In the sequel we shall always suppose the momentum conditions on the indexes ℓ, j, j' in (5.21). Some of the above sets are empty.

Lemma 5.4. *For $\varepsilon\in(0,\varepsilon_0)$ small enough, if $Q_{\ell,j,j'}^{(II)}\neq\emptyset$ then $|j|^{\frac12}+|j'|^{\frac12}\leqslant C\langle\ell\rangle$.*

Proof. If $Q_{\ell,j,j'}^{(II)}\neq\emptyset$ then there is $\gamma\in[\gamma_1,\gamma_2]$ such that

$$|\mu_j^\infty(\gamma)+\mu_{j'}^\infty(\gamma)| < \frac{4\upsilon(|j|^{\frac12}+|j'|^{\frac12})}{\langle\ell\rangle^\tau} + C|\ell|.$$

By (5.18) we have $\mu_j^\infty(\gamma)+\mu_{j'}^\infty(\gamma) = \mathtt{m}_1^\infty(\gamma)(j+j') + \mathtt{m}_{\frac12}^\infty(\gamma)(\Omega_j(\gamma)+\Omega_{j'}(\gamma)) - \mathtt{m}_0^\infty(\gamma)(\mathrm{sgn}(j)+\mathrm{sgn}(j')) + \mathfrak{r}_j^\infty(\gamma)+\mathfrak{r}_{j'}^\infty(\gamma)$. Then, by (5.19)-(5.20) with $k=0$, Lemma 4.4 and the momentum condition $j+j'=-\vec{\jmath}\cdot\ell$, we deduce, for ε small enough, $|\mu_j^\infty(\gamma)+\mu_{j'}^\infty(\gamma)| \geqslant -C\varepsilon|\ell| + \frac{\sqrt{g}}{2}\big||j|^{\frac12}+|j'|^{\frac12}\big| - C' - C\varepsilon\upsilon^{-3}$. The above bounds imply $||j|^{\frac12}+|j'|^{\frac12}|\leqslant C\langle\ell\rangle$, for ε small enough. ∎

In order to estimate the measure of the sets (5.22)-(5.25), the key point is to prove that the perturbed frequencies satisfy transversality properties similar to the ones (4.6)-(4.9) satisfied by the unperturbed frequencies. By Proposition 4.5, (5.16), and the estimates (5.17), (5.19)-(5.20) we deduce the following lemma (cfr. Lemma 5.5 in [7]).

Lemma 5.5. (Perturbed transversality) *For $\varepsilon \in (0, \varepsilon_0)$ small enough and for all $\gamma \in [\gamma_1, \gamma_2]$,*

$$\max_{0 \leqslant n \leqslant m_0} |\partial_\gamma^n \vec{\Omega}_\varepsilon(\gamma) \cdot \ell| \geqslant \frac{\rho_0}{2}\langle \ell \rangle, \quad \forall\, \ell \in \mathbb{Z}^\nu \backslash \{0\}\,;$$

$$\max_{0 \leqslant n \leqslant m_0} |\partial_\gamma^n (\vec{\Omega}_\varepsilon(\gamma) - \mathtt{m}_1^\infty(\gamma)\vec{\jmath}) \cdot \ell| \geqslant \frac{\rho_0}{2}\langle \ell \rangle, \quad \forall \ell \in \mathbb{Z}^\nu \backslash \{0\}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\gamma^n (\vec{\Omega}_\varepsilon(\gamma) \cdot \ell + \mu_j^\infty(\gamma))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle\,, \\ \vec{\jmath} \cdot \ell + j = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j \in \mathbb{S}_0^c\,; \end{cases}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\gamma^n (\vec{\Omega}_\varepsilon(\gamma) \cdot \ell + \mu_j^\infty(\gamma) - \mu_{j'}^\infty(\gamma))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j - j' = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j, j' \in \mathbb{S}_0^c\,,\ (\ell, j, j') \neq (0, j, j)\,; \end{cases}$$

$$\begin{cases} \max_{0 \leqslant n \leqslant m_0} |\partial_\gamma^n (\vec{\Omega}_\varepsilon(\gamma) \cdot \ell + \mu_j^\infty(\gamma) + \mu_{j'}^\infty(\gamma))| \geqslant \frac{\rho_0}{2}\langle \ell \rangle \\ \vec{\jmath} \cdot \ell + j + j' = 0\,,\ \ell \in \mathbb{Z}^\nu\,,\ j, j' \in \mathbb{S}_0^c\,. \end{cases}$$

The transversality estimates of Lemma 5.5 and an application of Rüssmann Theorem 17.1 in [31] (which applies as the functions $\vec{\Omega}_\varepsilon(\gamma)$, $\mathtt{m}_1^\infty(\gamma)$ and $\mu_j^\infty(\gamma)$ are bounded in the $\mathcal{C}^{m_0+1}$-topology thanks to (5.16)–(5.20)) directly imply the following bounds for the sets in (5.21): we have (cfr. Lemma 5.6 in [7]).

$$|R_\ell^{(0)}|, |R_\ell^{(T)}| \lesssim (\upsilon \langle \ell \rangle^{-(\tau+1)})^{\frac{1}{m_0}}\,,\ |R_{\ell,j}^{(I)}| \lesssim \big(\upsilon |j|^{\frac12} \langle \ell \rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,, \tag{5.26}$$
$$|R_{\ell,j,j'}^{(II)}| \lesssim \big(\upsilon \langle \ell \rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,, \quad |Q_{\ell,j,j'}^{(II)}| \lesssim \big(\upsilon \big(|j|^{\frac12} + |j'|^{\frac12}\big) \langle \ell \rangle^{-(\tau+1)}\big)^{\frac{1}{m_0}}\,.$$

By (5.26), and the choice of τ in (5.15), we have

$$\Big|\bigcup_{\ell \neq 0} R_\ell^{(0)} \cup R_\ell^{(T)}\Big| \leqslant \sum_{\ell \neq 0} |R_\ell^{(0)}| + |R_\ell^{(T)}| \lesssim \sum_{\ell \neq 0} \Big(\frac{\upsilon}{\langle \ell \rangle^{\tau+1}}\Big)^{\frac{1}{m_0}} \lesssim \upsilon^{\frac{1}{m_0}}\,, \tag{5.27}$$

$$\Big|\bigcup_{\ell \neq 0, j = -\vec{\jmath} \cdot \ell} R_{\ell,j}^{(I)}\Big| \leqslant \sum_{\ell \neq 0} |R_{\ell, -\vec{\jmath}\cdot\ell}^{(I)}| \lesssim \sum_{\ell} \Big(\frac{\upsilon}{\langle \ell \rangle^{\tau+\frac12}}\Big)^{\frac{1}{m_0}} \lesssim \upsilon^{\frac{1}{m_0}}\,, \tag{5.28}$$

and using also Lemma 5.4,

$$\Big|\bigcup_{\substack{\ell,\, j, j' \in \mathbb{S}_0^c \\ \vec{\jmath}\cdot\ell + j + j' = 0}} Q_{\ell,j,j'}^{(II)}\Big| \leqslant \sum_{\substack{\ell, |j| \leqslant C\langle \ell \rangle^2, \\ j' = -\vec{\jmath}\cdot\ell - j}} |Q_{\ell,j,j'}^{(II)}| \lesssim \sum_{\ell, |j| \leqslant C\langle \ell \rangle^2} \Big(\frac{\upsilon}{\langle \ell \rangle^{\tau}}\Big)^{\frac{1}{m_0}} \lesssim \upsilon^{\frac{1}{m_0}}\,. \tag{5.29}$$

We are left with estimating the measure of

$$\bigcup_{\substack{(\ell,j,j')\neq(0,j,j), j\neq j' \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} = \Big(\bigcup_{\substack{j\neq j'\,,\ j\cdot j'<0 \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} \Big) \cup \Big(\bigcup_{\substack{j\neq j'\,,\ j\cdot j'>0 \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'} \Big) := \mathtt{I}_1 \cup \mathtt{I}_2\,. \tag{5.30}$$

We first estimate the measure of $\mathtt{I}_1$. For $j\cdot j' < 0$, the momentum condition reads $j - j' = \mathrm{sgn}(j)(|j| + |j'|) = -\vec{\jmath}\cdot\ell$, thus $|j|, |j'| \leqslant C\langle\ell\rangle$. Hence, by (5.26) and the choice of τ in (5.15), we have

$$|\mathtt{I}_1| \leqslant \sum_{\ell, |j|\leqslant C\langle\ell\rangle, j'=j+\vec{\jmath}\cdot\ell} |R^{(II)}_{\ell,j,j'}| \lesssim \sum_{\ell,|j|\leqslant C\langle\ell\rangle} \Big(\frac{\upsilon}{\langle\ell\rangle^{\tau+1}}\Big)^{\frac{1}{m_0}} \lesssim \upsilon^{\frac{1}{m_0}}\,. \tag{5.31}$$

Then we estimate the measure of $\mathtt{I}_2$ in (5.30). The key step is given in the next lemma. Remind the definition of the sets $R^{(II)}_{\ell,j,j'}$ and $R^{(T)}_{\ell}$ in (5.22)-(5.24).

Lemma 5.6. *Let $\upsilon_0 \geqslant \upsilon$ and $\tau \geqslant \tau_0 \geqslant 1$. There is a constant $C_1 > 0$ such that, for ε small enough, for any $\vec{\jmath}\cdot\ell + j - j' = 0$, $j\cdot j' > 0$, if $\min\{|j|,|j'|\} \geqslant C_1\upsilon_0^{-2}\langle\ell\rangle^{2(\tau_0+1)}$, then $R^{(II)}_{\ell,j,j'}(\upsilon,\tau) \subset \bigcup_{\ell\neq0} R^{(T)}_{\ell}(\upsilon_0,\tau_0)$.*

Proof. If $\gamma \in [\gamma_1,\gamma_2]\backslash \bigcup_{\ell\neq 0} R^{(T)}_{\ell}(\upsilon_0,\tau_0)$, then $|(\vec{\Omega}_\varepsilon(\gamma) - \mathtt{m}_1^\infty(\gamma)\vec{\jmath})\cdot\ell| \geqslant 8\upsilon_0\langle\ell\rangle^{-\tau_0}$ for any $\ell \in \mathbb{Z}\backslash\{0\}$. By (5.18), the condition $j - j' = -\vec{\jmath}\cdot\ell$, (5.19), (5.20), Lemma 4.4 and $j\cdot j' > 0$, (4.11), we deduce that

$$\begin{aligned}
&|\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mu_j^\infty(\gamma) - \mu_{j'}^\infty(\gamma)| \\
&\geqslant |\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mathtt{m}_1^\infty(j-j')| - |\mathtt{m}^\infty_{\frac12}||\Omega_j(\gamma) - \Omega_{j'}(\gamma)| - |\mathfrak{r}_j^\infty(\gamma) - \mathfrak{r}_{j'}^\infty(\gamma)| \\
&\geqslant |(\vec{\Omega}_\varepsilon(\gamma) - \mathtt{m}_1^\infty\vec{\jmath})\cdot\ell| - (1 - C\varepsilon\upsilon^{-1})\big||j|^{\frac12} - |j'|^{\frac12}\big| \\
&\quad - C\Big(\frac{1}{|j|^{\frac12}} + \frac{1}{|j'|^{\frac12}}\Big) - C\frac{\varepsilon}{\upsilon^3}\Big(\frac{1}{|j|^{\frac12}} + \frac{1}{|j'|^{\frac12}}\Big) \\
&\geqslant \frac{8\upsilon_0}{\langle\ell\rangle^{\tau_0}} - \frac12\frac{|j-j'|}{|j|^{\frac12}+|j'|^{\frac12}} - \Big(\frac{C}{|j|^{\frac12}} + \frac{C}{|j'|^{\frac12}}\Big) \geqslant \frac{8\upsilon_0}{\langle\ell\rangle^{\tau_0}} - C\Big(\frac{\langle\ell\rangle}{|j|^{\frac12}} + \frac{\langle\ell\rangle}{|j'|^{\frac12}}\Big) \geqslant \frac{4\upsilon_0}{\langle\ell\rangle^{\tau_0}}
\end{aligned}$$

for any $|j|,|j'| > C_1\upsilon_0^{-2}\langle\ell\rangle^{2(\tau_0+1)}$, for $C_1 > C^2/64$. Since $\upsilon_0 \geqslant \upsilon$ and $\tau \geqslant \tau_0$ we deduce that $|\vec{\Omega}_\varepsilon(\gamma)\cdot\ell + \mu_j^\infty(\gamma) - \mu_{j'}^\infty(\gamma)| \geqslant 4\upsilon\langle\ell\rangle^{-\tau}$, namely $\gamma \notin R^{(II)}_{\ell,j,j'}(\upsilon,\tau)$. □

Note that the set of indexes (ℓ,j,j') such that $\vec{\jmath}\cdot\ell + j - j' = 0$ and $\min\{|j|,|j'|\} < C_1\upsilon_0^{-2}\langle\ell\rangle^{2(\tau_0+1)}$ is included, for υ_0 small enough, into the set

$$\mathcal{I}_\ell := \Big\{(\ell,j,j')\ :\ \vec{\jmath}\cdot\ell + j - j' = 0\,,\ |j|,|j'| \leqslant \upsilon_0^{-3}\langle\ell\rangle^{2(\tau_0+1)}\Big\} \tag{5.32}$$

because $\max\{|j|,|j'|\} \leqslant \min\{|j|,|j'|\} + |j-j'| < C_1 \upsilon_0^{-2}\langle \ell \rangle^{2(\tau_0+1)} + C\langle \ell \rangle \leqslant \upsilon_0^{-3}\langle \ell \rangle^{2(\tau_0+1)}$. As a consequence, by Lemma 5.6 we deduce that

$$\mathtt{I}_2 = \bigcup_{\substack{j\neq j'\,,\, j\cdot j'>0 \\ \vec{\jmath}\cdot\ell+j-j'=0}} R^{(II)}_{\ell,j,j'}(\upsilon,\tau) \subset \Big(\bigcup_{\ell\neq 0} R^{(T)}_{\ell}(\upsilon_0,\tau_0)\Big) \bigcup \Big(\bigcup_{(\ell,j,j')\in\mathcal{I}_\ell} R^{(II)}_{\ell,j,j'}(\upsilon,\tau)\Big). \tag{5.33}$$

Lemma 5.7. *Let* $\tau_0 := m_0\nu$ *and* $\upsilon_0 = \upsilon^{\frac{1}{4m_0}}$. *Then* $|\mathtt{I}_2| \leqslant C\upsilon^{\frac{1}{4m_0^2}}$.

Proof. By (5.27) (applied with υ_0, τ_0 instead of υ, τ), and $\tau_0 = m_0\nu$, we have

$$\Big|\bigcup_{\ell\neq 0} R^{(T)}_{\ell}(\upsilon_0,\tau_0)\Big| \lesssim \upsilon_0^{\frac{1}{m_0}} \lesssim \upsilon^{\frac{1}{4m_0^2}}\,. \tag{5.34}$$

Moreover, recalling (5.32),

$$\begin{aligned} \Big|\bigcup_{(\ell,j,j')\in\mathcal{I}_\ell} R^{(II)}_{\ell,j,j'}(\upsilon,\tau)\Big| &\lesssim \sum_{\substack{\ell\in\mathbb{Z}^\nu \\ |j|\leqslant C_1\upsilon_0^{-3}\langle\ell\rangle^{2(\tau_0+1)}}} \Big(\frac{\upsilon}{\langle\ell\rangle^{\tau+1}}\Big)^{\frac{1}{m_0}} \\ &\lesssim \sum_{\ell\in\mathbb{Z}^\nu} \frac{\upsilon^{\frac{1}{m_0}}\upsilon_0^{-3}}{\langle\ell\rangle^{\frac{\tau+1}{m_0}-2(\tau_0+1)}} \lesssim \upsilon^{\frac{1}{4m_0}}\,, \end{aligned} \tag{5.35}$$

by the choice of τ in (5.15) and υ_0. The lemma follows by (5.33), (5.34) and (5.35). □

Proof of Theorem 5.3 completed. By (5.21), (5.27), (5.28), (5.29), (5.30), (5.31) and Lemma 5.7, we deduce that $|\mathcal{G}_\varepsilon^c| \leqslant C\upsilon^{\frac{1}{4m_0^2}}$. For $\upsilon = \varepsilon^{\mathtt{a}}$ as in (5.15), we get $|\mathcal{G}_\varepsilon| \geqslant \gamma_2 - \gamma_1 - C\varepsilon^{\mathtt{a}/4m_0^2}$. □

5.3 Approximate inverse

The key step to prove Theorem 5.1 via a Nash-Moser iterative scheme is the construction of an *almost approximate right inverse* of the linearized operator $\mathrm{d}_{i,\alpha}\mathcal{F}(i_0,\alpha_0)[\widehat{\imath},\widehat{\alpha}] = \mathrm{d}_{i,\alpha}\mathcal{F}(i_0) = \omega\cdot\partial_\varphi\widehat{\imath} - \mathrm{d}_i X_{H_\alpha}(i_0(\varphi))[\widehat{\imath}] - (\widehat{\alpha},0,0)$. We follow closely the strategy in [6], implemented for the water waves equations in [9, 2, 7]. Thus we shall be short. With this approach we are reduced to construct an almost inverse for the linear operator $\mathcal{L}_\omega$, defined in (5.42) below, acting on the normal directions.

We assume the smallness condition, for some $\mathtt{k} := \mathtt{k}(\tau,\nu) > 0$, $\varepsilon\upsilon^{-\mathtt{k}} \ll 1$, and the following hypothesis, which is verified by the approximate solutions obtained in the Nash-Moser Theorem 7.7.

- ANSATZ. The map $(\omega,\gamma) \mapsto \mathfrak{I}_0(\omega,\gamma) = i_0(\varphi;\omega,\gamma) - (\varphi,0,0)$ is

k_0-times differentiable with respect to the parameters $(\omega,\gamma)\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$ and, for some $\mu:=\mu(\tau,\nu)>0$, $\upsilon\in(0,1)$,

$$\|\mathfrak{I}_0\|_{s_0+\mu}^{k_0,\upsilon}+|\alpha_0-\omega|^{k_0,\upsilon}\leqslant C\varepsilon\upsilon^{-1}\,. \tag{5.36}$$

The torus $i_0(\varphi)=(\theta_0(\varphi),I_0(\varphi),w_0(\varphi))$ is reversible and traveling, according to (5.4).

We first modify $i_0(\varphi)$ to a nearby isotropic torus $i_\delta(\varphi)$. The next lemma follows as in Lemma 5.3 in [2] and Lemma 6.2 in [7]. Let $Z(\varphi):=\mathcal{F}(i_0,\alpha_0)(\varphi)=\omega\cdot\partial_\varphi i_0(\varphi)-X_{H_{\alpha_0}}(i_0(\varphi))$.

Lemma 5.8. (Isotropic torus) *There exists an isotropic torus $i_\delta(\varphi):=(\theta_0(\varphi),I_\delta(\varphi),w_0(\varphi))$ satisfying, for some $\sigma:=\sigma(\nu,\tau)$ and for all $s\geqslant s_0$,*

$$\|I_\delta-I_0\|_s^{k_0,\upsilon}\lesssim_s\|\mathfrak{I}_0\|_{s+1}^{k_0,\upsilon}\,,\ \|I_\delta-I_0\|_s^{k_0,\upsilon}\lesssim_s\upsilon^{-1}\big(\|Z\|_{s+\sigma}^{k_0,\upsilon}+\|Z\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big) \tag{5.37}$$

$$\|\mathcal{F}(i_\delta,\alpha_0)\|_s^{k_0,\upsilon}\lesssim_s\|Z\|_{s+\sigma}^{k_0,\upsilon}+\|Z\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\,,\ \|\mathrm{d}_i(i_\delta)[\widehat{\imath}]\|_{s_1}\lesssim_{s_1}\|\widehat{\imath}\|_{s_1+1}\,, \tag{5.38}$$

for $s_1\leqslant s_0+\mu$ (cfr. (5.36)*). Furthermore $i_\delta(\varphi)$ is a reversible and traveling torus, cfr.* (5.4)*.*

We introduce the diffeomorphism $G_\delta:(\phi,y,\mathtt{w})\to(\theta,I,w)$ of the phase space $\mathbb{T}^\nu\times\mathbb{R}^\nu\times\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$,

$$\begin{pmatrix}\theta\\ I\\ w\end{pmatrix}:=G_\delta\begin{pmatrix}\phi\\ y\\ \mathtt{w}\end{pmatrix}:=\begin{pmatrix}\theta_0(\phi)\\ I_\delta(\phi)+\big[\partial_\phi\theta_0(\phi)\big]^{-\top}y+[(\partial_\theta\widetilde{w}_0)(\theta_0(\phi))]^\top J_\angle^{-1}\mathtt{w}\\ w_0(\phi)+\mathtt{w}\end{pmatrix} \tag{5.39}$$

where $\widetilde{w}_0(\theta):=w_0(\theta_0^{-1}(\theta))$. It is proved in Lemma 2 of [6] that G_δ is symplectic, because the torus i_δ is isotropic (Lemma 5.8). In the new coordinates, i_δ is the trivial embedded torus $(\phi,y,\mathtt{w})=(\phi,0,0)$. The diffeomorphism G_δ in (5.39) is reversibility and momentum preserving, in the sense that (Lemma 6.3 in [7]) $\vec{\mathcal{S}}\circ G_\delta=G_\delta\circ\vec{\mathcal{S}}$, $\vec{\tau}_\varsigma\circ G_\delta=G_\delta\circ\vec{\tau}_\varsigma$ for any $\varsigma\in\mathbb{R}$, where $\vec{\mathcal{S}}$ and $\vec{\tau}_\varsigma$ are defined respectively in (2.22), (2.23). Under the symplectic diffeomorphism G_δ, the Hamiltonian vector field X_{H_α} changes into $X_{K_\alpha}=(DG_\delta)^{-1}X_{H_\alpha}\circ G_\delta$, where $K_\alpha:=H_\alpha\circ G_\delta$ is reversible and momentum preserving. The Taylor expansion of K_α at the trivial torus $(\phi,0,0)$ is

$$\begin{aligned}K_\alpha(\phi,y,\mathtt{w})=&K_{00}(\phi,\alpha)+K_{10}(\phi,\alpha)\cdot y+(K_{01}(\phi,\alpha),\mathtt{w})_{L^2}+\tfrac12K_{20}(\phi)y\cdot y\\&+(K_{11}(\phi)y,\mathtt{w})_{L^2}+\tfrac12(K_{02}(\phi)\mathtt{w},\mathtt{w})_{L^2}+K_{\geqslant3}(\phi,y,\mathtt{w})\,,\end{aligned}$$

where $K_{\geqslant3}$ collects all terms at least cubic in $(y,\mathtt{w})$. Here K_{02} is a self-adjoint operator on $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

The key step concerns the construction of an "almost approximate" inverse of

$$\mathcal{L}_\omega := \Pi^{\angle}_{\mathbb{S}^+,\Sigma}\left(\omega\cdot\partial_\varphi - JK_{02}(\varphi)\right)|_{\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}} \tag{5.40}$$

is "almost invertible" (on traveling waves) up to remainders of size $O(N_{\mathtt{n}-1}^{-\mathtt{a}})$, where, for $\mathtt{n}\in\mathbb{N}_0$

$$N_{\mathtt{n}} := K_{\mathtt{n}}^p\,, \quad K_{\mathtt{n}} := K_0^{\chi^{\mathtt{n}}}\,, \quad \chi = 3/2\,. \tag{5.41}$$

The $(K_{\mathtt{n}})_{\mathtt{n}\geqslant 0}$ is the scale used in the nonlinear Nash-Moser iteration at the end of Section 7 and $(N_{\mathtt{n}})_{\mathtt{n}\geqslant 0}$ is the one in the almost-straightening Lemma 6.3 and in the almost-diagonalization Theorem 7.1. Let $H^s_{\angle}(\mathbb{T}^{\nu+1}) := H^s(\mathbb{T}^{\nu+1})\cap\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$.

(AI) **Almost invertibility of** $\mathcal{L}_\omega$: *There exist positive real numbers σ, $\mu(\mathtt{b})$, $\mathtt{a}$, p, K_0 and a subset $\Lambda_o \subset \mathtt{DC}(\upsilon,\tau)\times[\gamma_1,\gamma_2]$ such that, for all $(\omega,\gamma)\in\Lambda_o$, the operator $\mathcal{L}_\omega$ may be decomposed as*

$$\mathcal{L}_\omega = \mathcal{L}_\omega^< + \mathcal{R}_\omega + \mathcal{R}_\omega^\perp\,, \tag{5.42}$$

where, for any traveling wave function $g\in H^{s+\sigma}_{\angle}(\mathbb{T}^{\nu+1},\mathbb{R}^2)$ and for any $(\omega,\gamma)\in\Lambda_o$, there is a traveling wave solution $h\in H^s_{\angle}(\mathbb{T}^{\nu+1},\mathbb{R}^2)$ of $\mathcal{L}_\omega^< h = g$ satisfying, for all $s_0\leqslant s\leqslant S-\mu(\mathtt{b})-\sigma$, $\|(\mathcal{L}_\omega^<)^{-1}g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\sigma}^{k_0,\upsilon} + \|g\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon}\big)$. In addition, if g is anti-reversible, then h is reversible. Moreover, for any $s_0\leqslant s\leqslant S-\mu(\mathtt{b})-\sigma$, for any traveling wave $h\in\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ and for any $\mathrm{b}>0$, *the operators $\mathcal{R}_\omega,\mathcal{R}_\omega^\perp$ satisfy the estimates*

$$\|\mathcal{R}_\omega h\|_s^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-3}N_{\mathtt{n}-1}^{-\mathtt{a}}\big(\|h\|_{s+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon}\big)\,,$$

$$\|\mathcal{R}_\omega^\perp h\|_{s_0}^{k_0,\upsilon} \lesssim_S K_{\mathtt{n}}^{-\mathrm{b}}\big(\|h\|_{s_0+\mathrm{b}+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s_0+\mu(\mathtt{b})+\sigma+\mathrm{b}}^{k_0,\upsilon}\big)\,,$$

$$\|\mathcal{R}_\omega^\perp h\|_s^{k_0,\upsilon} \lesssim_S \|h\|_{s+\sigma}^{k_0,\upsilon} + \|h\|_{s_0+\sigma}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon}\,.$$

The goal of Sections 6 and 7 is the proof of the above assumption (AI), see Theorem 7.6. By (AI), arguing as in Proposition 6.5 and Theorem 6.6 in [7], we deduce the following.

Theorem 5.9. (Almost approximate inverse) *Assume* (AI). *There is $\overline{\sigma} := \overline{\sigma}(\tau,\nu,k_0) > 0$ such that, if* (5.36) *holds with $\mu = \mu(\mathtt{b}) + \overline{\sigma}$, there exists an operator $\mathbf{T}_0$, defined for all $(\omega,\gamma)\in\Lambda_o$, that is an almost approximate right inverse of* $\mathrm{d}_{i,\alpha}\mathcal{F}(i_0)$, *namely*

$$\mathrm{d}_{i,\alpha}\mathcal{F}(i_0)\circ\mathbf{T}_0 - \mathrm{Id} = \mathcal{P}(i_0) + \mathcal{P}_\omega(i_0) + \mathcal{P}_\omega^\perp(i_0).$$

More precisely, for any anti-reversible traveling wave variation $g := (g_1, g_2, g_3)$*, for all* $s_0 \leqslant s \leqslant S - \mu(\mathtt{b}) - \overline{\sigma}$*,*

$$\|\mathbf{T}_0 g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big),$$

and, for any $\mathrm{b} > 0$*, the following estimates hold:*

$$\begin{aligned}\|\mathcal{P}g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\Big(&\|\mathcal{F}(i_0,\alpha_0)\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon}\\ &+ \big(\|\mathcal{F}(i_0,\alpha_0)\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathcal{F}(i_0,\alpha_0)\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\big)\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\Big),\end{aligned} \tag{5.43}$$

$$\|\mathcal{P}_\omega g\|_s^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-4}N_{\mathtt{n}-1}^{-\mathtt{a}}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big), \tag{5.44}$$

$$\|\mathcal{P}_\omega^\perp g\|_{s_0}^{k_0,\upsilon} \lesssim_{S,\mathrm{b}} \upsilon^{-1}K_{\mathtt{n}}^{-\mathrm{b}}\left(\|g\|_{s_0+\overline{\sigma}+b}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s_0+\mu(\mathtt{b})+b+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\right), \tag{5.45}$$

$$\|\mathcal{P}_\omega^\perp g\|_s^{k_0,\upsilon} \lesssim_S \upsilon^{-1}\big(\|g\|_{s+\overline{\sigma}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\overline{\sigma}}^{k_0,\upsilon}\|g\|_{s_0+\overline{\sigma}}^{k_0,\upsilon}\big). \tag{5.46}$$

6 The linearized operator in the normal subspace

The Hamiltonian operator $\mathcal{L}_\omega$ defined in (5.40) has the form (cfr. Lemma 7.1 in [7])

$$\mathcal{L}_\omega = \Pi_{\mathbb{S}^+,\Sigma}^{\angle}(\mathcal{L} - \varepsilon JR)|_{\mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\angle}}. \tag{6.1}$$

Here, $\mathcal{L}$ is the Hamiltonian operator $\mathcal{L} := \omega\cdot\partial_\varphi - J\partial_u\nabla_u\mathcal{H}(T_\delta(\varphi))$, where $\mathcal{H}$ is the water waves Hamiltonian in the Wahlén variables defined in (2.4), evaluated at the reversible traveling wave

$$\begin{aligned}T_\delta(\phi) := \varepsilon A(i_\delta(\phi)) &= \varepsilon A\left(\theta_0(\phi), I_\delta(\phi), w_0(\phi)\right)\\ &= \varepsilon v^{\intercal}\left(\theta_0(\phi), I_\delta(\phi)\right) + \varepsilon w_0(\phi),\end{aligned} \tag{6.2}$$

the torus $i_\delta(\varphi) := (\theta_0(\varphi), I_\delta(\varphi), w_0(\varphi))$ is defined in Lemma 5.8 and $A(\theta, I, w)$, $v^{\intercal}(\theta, I)$ in (2.21), whereas $R(\phi)$ has the 'finite rank" form

$$R(\phi)[h] = \sum\nolimits_{j=1}^{\nu}(h, g_j)_{L^2}\,\chi_j, \quad \forall\, h \in \mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\angle}, \tag{6.3}$$

for functions $g_j, \chi_j \in \mathfrak{H}_{\mathbb{S}^+,\Sigma}^{\angle}$ satisfying, for some $\sigma := \sigma(\tau,\nu,k_0) > 0$, any $j = 1,\dots,\nu$, for all $s \geqslant s_0$,

$$\begin{aligned}&\|g_j\|_s^{k_0,\upsilon} + \|\chi_j\|_s^{k_0,\upsilon} \lesssim_s 1 + \|\mathfrak{I}_\delta\|_{s+\sigma}^{k_0,\upsilon},\\ &\|\mathrm{d}_i g_j[\widehat{\imath}]\|_s + \|\mathrm{d}_i\chi_j[\widehat{\imath}]\|_s \lesssim_s \|\widehat{\imath}\|_{s+\sigma} + \|\widehat{\imath}\|_{s_0+\sigma}\|\mathfrak{I}_\delta\|_{s+\sigma}.\end{aligned} \tag{6.4}$$

In order to compute $\mathcal{L}$ we use the "shape derivative" formula, see e.g. [25], $G'(\eta)[\widehat{\eta}]\psi = -G(\eta)(B\widehat{\eta}) - \partial_x(V\widehat{\eta})$, where

$$B(\eta,\psi) := \frac{G(\eta)\psi + \eta_x\psi_x}{1+\eta_x^2}, \quad V(\eta,\psi) := \psi_x - B(\eta,\psi)\eta_x. \tag{6.5}$$

Then, recalling (2.4), (2.3), (1.2) the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \omega\cdot\partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} + G(\eta)B & -G(\eta) \\ g + B\widetilde{V}_x + BG(\eta)B & \widetilde{V}\partial_x - BG(\eta) \end{pmatrix} + \frac{\gamma}{2}\begin{pmatrix} -G(\eta)\partial_x^{-1} & 0 \\ \partial_x^{-1}G(\eta)B - BG(\eta)\partial_x^{-1} - \frac{\gamma}{2}\partial_x^{-1}G(\eta)\partial_x^{-1} & -\partial_x^{-1}G(\eta) \end{pmatrix}, \tag{6.6}$$

where

$$\widetilde{V} := V - \gamma\eta\,, \tag{6.7}$$

and the functions $B := B(\eta,\psi)$, $V := V(\eta,\psi)$ in (6.6)-(6.7) are evaluated at the reversible traveling wave $(\eta,\psi) := WT_\delta(\varphi)$ where $T_\delta(\varphi)$ is defined in (6.2).

Notation. In (6.6) and hereafter the function B is identified with the multiplication operators $h \mapsto Bh$. If there is no parenthesis, composition of operators is understood, for example $BG(\eta)B$ means $B\circ G(\eta)\circ B$.

We consider the operator $\mathcal{L}$ in (6.6) acting on (a dense subspace of) the whole $L^2(\mathbb{T})\times L^2(\mathbb{T})$. In particular we extend the operator ∂_x^{-1} to act on the whole $L^2(\mathbb{T})$ as in (3.12).

By the reversible and space-invariance properties of the water waves equations explained in Section 2 and since $(\eta,\zeta) = T_\delta(\varphi)$ is a reversible traveling wave, $(\mathrm{even}(\varphi,x), \mathrm{odd}(\varphi,x))$, we deduce that (cfr. Lemma 7.3 in [7]) the functions $B, \widetilde{V}$ defined in (6.5), (6.7) are quasi-periodic traveling waves, B is $\mathrm{odd}(\varphi,x)$ and $\widetilde{V}$ is $\mathrm{even}(\varphi,x)$. The Hamiltonian operator $\mathcal{L}$ is reversible and momentum preserving.

We shall always assume the following ansatz (satisfied by the approximate solutions along the nonlinear Nash-Moser iteration): for some constants $\mu_0 := \mu_0(\tau,\nu) > 0$ (cfr. Lemma 5.8)

$$\|\mathfrak{I}_0\|_{s_0+\mu_0}^{k_0,\upsilon}\,,\ \|\mathfrak{I}_\delta\|_{s_0+\mu_0}^{k_0,\upsilon} \leqslant 1\,. \tag{6.8}$$

It is sufficient to estimate the variation of operators, functions, etc, with respect to the approximate torus $i(\varphi)$ in a low norm $\|\ \|_{s_1}$ for all Sobolev indexes s_1 such that

$$s_1 + \sigma_0 \leqslant s_0 + \mu_0\,, \quad \text{for some } \sigma_0 := \sigma_0(\tau,\nu) > 0\,. \tag{6.9}$$

Thus, by (6.8), we have $\|\mathfrak{I}_0\|_{s_1+\sigma_0}^{k_0,\upsilon}$, $\|\mathfrak{I}_\delta\|_{s_1+\sigma_0}^{k_0,\upsilon} \leqslant 1$. The constants μ_0 and σ_0 represent the *loss of derivatives* accumulated along the reduction procedure of the next sections. They are independent of the Sobolev index s. In the next sections $\mu_0 := \mu_0(\tau,\nu,M,\alpha) > 0$ will depend also on indexes

M, α, whose maximal values will be fixed depending only on τ and ν. In particular M is fixed in (7.2), whereas the maximal value of α depends on M, as explained in Remark 6.10.

As a consequence of Lemma 3.2 and (5.37), the Sobolev norm of the function $u = T_\delta(\varphi)$ defined in (6.2) satisfies $\|u\|_s^{k_0,\upsilon} = \|\eta\|_s^{k_0,\upsilon} + \|\zeta\|_s^{k_0,\upsilon} \leqslant \varepsilon C(s)\big(1 + \|\mathfrak{I}_0\|_s^{k_0,\upsilon}\big)$ for all $s \geqslant s_0$. Similarly, using (5.38), $\|\Delta_{12} u\|_{s_1} \lesssim_{s_1} \varepsilon \|i_2 - i_1\|_{s_1}$ where $\Delta_{12} u := u(i_2) - u(i_1)$.

In Sections 6.1-6.6 we make several transformations to conjugate the operator $\mathcal{L}$ in (6.6) to a constant coefficients Fourier multiplier, up to a pseudo-differential operator of order $-1/2$ and a remainder that satisfies tame estimates, see $\mathcal{L}_8$ in (6.113). In Section 6.7 we shall conjugate the operator $\mathcal{L}_\omega$ in (6.1).

6.1 Linearized good unknown of Alinhac

The first step is to conjugate the linear operator $\mathcal{L}$ in (6.6) by the symplectic (Definition 3.13) multiplication matrix operator $\mathcal{Z} := \begin{pmatrix} \mathrm{Id} & 0 \\ B & \mathrm{Id} \end{pmatrix}$. Since $\mathcal{Z}^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ -B & \mathrm{Id} \end{pmatrix}$ we obtain

$$\mathcal{L}_1 := \mathcal{Z}^{-1} \mathcal{L} \mathcal{Z} = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & -G(\eta) \\ a & \widetilde{V} \partial_x \end{pmatrix} - \frac{\gamma}{2} \begin{pmatrix} G(\eta)\partial_x^{-1} & 0 \\ \frac{\gamma}{2} \partial_x^{-1} G(\eta) \partial_x^{-1} & \partial_x^{-1} G(\eta) \end{pmatrix}, \tag{6.10}$$

where a is the function

$$a := g + \widetilde{V} B_x + \omega \cdot \partial_\varphi B \, . \tag{6.11}$$

As in [25] and [9, 2], the matrix $\mathcal{Z}$ amounts to a linear version of the “good unknown of Alinhac".

Lemma 6.1. *The maps* $\mathcal{Z}^{\pm 1} - \mathrm{Id}$ *are* $\mathcal{D}^{k_0}$*-tame with tame constants satisfying, for some* $\sigma := \sigma(\tau, \nu, k_0) > 0$*, for all* $s \geqslant s_0$*,* $\mathfrak{M}_{\mathcal{Z}^{\pm 1} - \mathrm{Id}}(s)$*,* $\mathfrak{M}_{(\mathcal{Z}^{\pm 1} - \mathrm{Id})^*}(s) \lesssim_s \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big)$*. The function* a *in* (6.11) *is a quasi-periodic traveling wave* even(φ, x)*. There is* $\sigma := \sigma(\tau, \nu, k_0) > 0$ *such that, for all* $s \geqslant s_0$*,*

$$\|a - g\|_s^{k_0,\upsilon} + \|\widetilde{V}\|_s^{k_0,\upsilon} + \|B\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\big(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\big) \, . \tag{6.12}$$

Moreover, for any s_1 *as in* (6.9)*,*

$$\|\Delta_{12} a\|_{s_1} + \|\Delta_{12} \widetilde{V}\|_{s_1} + \|\Delta_{12} B\|_{s_1} \lesssim_{s_1} \varepsilon \|i_1 - i_2\|_{s_1+\sigma} \, , \tag{6.13}$$

$$\|\Delta_{12}(\mathcal{Z}^{\pm 1}) h\|_{s_1}, \|\Delta_{12}(\mathcal{Z}^{\pm 1})^* h\|_{s_1} \lesssim_{s_1} \varepsilon \|i_1 - i_2\|_{s_1+\sigma} \|h\|_{s_1} \, . \tag{6.14}$$

The operator $\mathcal{L}_1$ *is Hamiltonian, reversible and momentum preserving.*

Proof. The estimates for $B, \widetilde{V}, a$ follow by their expressions in (6.5), (6.7), (6.11), Lemma 3.2, (3.3) and the bounds for the Dirichlet-Neumann operator in Lemma 3.10 in [7]. Since B is a quasi-periodic traveling wave, $\mathrm{odd}(\varphi, x)$, $\mathcal{Z}$ is reversibility and momentum preserving (Definitions 3.14 and 3.16). □

6.2 Almost-straightening of the first order transport operator

We now write the operator $\mathcal{L}_1$ in (6.10) as

$$\mathcal{L}_1 = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & 0 \\ 0 & \widetilde{V}\partial_x \end{pmatrix} + \begin{pmatrix} -\frac{\gamma}{2} G(0)\partial_x^{-1} & -G(0) \\ a - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0) \partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1} G(0) \end{pmatrix} + \mathbf{R}_1, \tag{6.15}$$

where, by the decomposition of the Dirichlet-Neumann operator in Lemma 3.10 in [7],

$$\mathbf{R}_1 := - \begin{pmatrix} \frac{\gamma}{2}\mathcal{R}_G(\eta)\partial_x^{-1} & \mathcal{R}_G(\eta) \\ \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1}\mathcal{R}_G(\eta)\partial_x^{-1} & \frac{\gamma}{2}\partial_x^{-1}\mathcal{R}_G(\eta) \end{pmatrix} \tag{6.16}$$

is a small remainder in $\mathrm{OP}S^{-\infty}$. The aim of this section is to conjugate the variable coefficients quasi-periodic transport operator $\mathcal{L}_{\mathrm{TR}} := \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & 0 \\ 0 & \widetilde{V}\partial_x \end{pmatrix}$ to a constant coefficients transport operator $\omega \cdot \partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}}\, \partial_y$, up to an exponentially small remainder, see (6.23)-(6.24), where $\mathtt{n} \in \mathbb{N}_0$ and

$$N_{\mathtt{n}} := N_0^{\chi^{\mathtt{n}}}, \ N_0 > 1, \quad \chi = 3/2, \quad N_{-1} := 1. \tag{6.17}$$

Such small remainder is left because we assume only finitely many non-resonance conditions, see (6.22). In the next lemma we conjugate $\mathcal{L}_{\mathrm{TR}}$ by a *symplectic* (Definition 3.13) transformation

$$\mathcal{E} := \begin{pmatrix} (1 + \beta_x(\varphi, x)) \circ \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{pmatrix}, \tag{6.18}$$

where the composition operator

$$(\mathcal{B}u)(\varphi, x) := u\,(\varphi, x + \beta(\varphi, x)) \tag{6.19}$$

is induced by a φ-dependent diffeomorphism $y = x + \beta(\varphi, x)$ of the torus $\mathbb{T}_x$, for some small quasi-periodic traveling wave $\beta : \mathbb{T}^\nu_\varphi \times \mathbb{T}_x \to \mathbb{R}$, $\mathrm{odd}(\varphi, x)$.

Remark 6.2. *We denote ∂_y the derivative operator in the new variable $y = x + \beta(\varphi, x)$, see Lemmata 6.3 and 6.5, and Appendix A. For simplicity of notation, at the beginning of Section 6.3, the variable y is relabelled back with x.*

Let

$$\mathtt{b} := [\mathtt{a}] + 2 \in \mathbb{N}\,, \quad \mathtt{a} := 3(\tau_1 + 1) \geqslant 1\,, \quad \tau_1 := k_0 + (k_0+1)\tau\,. \tag{6.20}$$

Lemma 6.3. (Almost-Straightening of the transport operator) *There exists $\tau_2(\tau,\nu) > \tau_1(\tau,\nu) + 1 + \mathtt{a}$ such that, for all $S > s_0 + k_0$, there are $N_0 := N_0(S,\mathtt{b}) \in \mathbb{N}$ and $\delta := \delta(S,\mathtt{b}) \in (0,1)$ such that, if $N_0^{\tau_2}\varepsilon\upsilon^{-1} < \delta$ the following holds true. For any $\overline{\mathtt{n}} \in \mathbb{N}_0$:*

1. There exist a constant $\mathtt{m}_{1,\overline{\mathtt{n}}} := \mathtt{m}_{1,\overline{\mathtt{n}}}(\omega,\gamma) \in \mathbb{R}$, where $\mathtt{m}_{1,0} = 0$, defined for any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, and a quasi-periodic traveling wave $\beta(\varphi,x) := \beta_{\overline{\mathtt{n}}}(\varphi,x)$, $\mathrm{odd}(\varphi,x)$, satisfying, for some $\sigma = \sigma(\tau,\nu,k_0) > 0$, the estimates

$$|\mathtt{m}_{1,\overline{\mathtt{n}}}|^{k_0,\upsilon} \lesssim \varepsilon\,, \ \|\beta\|_s^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma+\mathtt{b}}^{k_0,\upsilon})\,, \ \forall\, s_0 \leqslant s \leqslant S\,, \tag{6.21}$$

independently of $\overline{\mathtt{n}}$;

2. For any (ω,γ) in $\mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau) := \mathtt{TC}_{\overline{\mathtt{n}}+1}(\mathtt{m}_{1,\overline{\mathtt{n}}},2\upsilon,\tau)$ defined as

$$\mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau) := \Big\{(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2] : |(\omega - \mathtt{m}_{1,\overline{\mathtt{n}}}\vec{\jmath}) \cdot \ell| \geqslant \frac{2\upsilon}{\langle \ell \rangle^\tau}, \forall\, 0 < |\ell| \leqslant N_{\overline{\mathtt{n}}}\Big\} \tag{6.22}$$

the operator $\mathcal{L}_{\mathrm{TR}} = \omega\cdot\partial_\varphi + \begin{pmatrix} \partial_x \widetilde{V} & 0 \\ 0 & \widetilde{V}\partial_x \end{pmatrix}$ is conjugated to

$$\mathcal{E}^{-1}\mathcal{L}_{\mathrm{TR}}\mathcal{E} = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\,\partial_y + \mathbf{P}_2^\perp\,, \quad \mathbf{P}_2^\perp := \begin{pmatrix} \partial_y p_{\overline{\mathtt{n}}} & 0 \\ 0 & p_{\overline{\mathtt{n}}}\partial_y \end{pmatrix}\,, \tag{6.23}$$

and the real quasi-periodic traveling wave function $p_{\overline{\mathtt{n}}}(\varphi,y)$, $\mathrm{even}(\varphi,y)$, satisfies, for some $\sigma = \sigma(\tau,\nu,k_0)$, $\sigma > 0$, and for any $s_0 \leqslant s \leqslant S$,

$$\|p_{\overline{\mathtt{n}}}\|_s^{k_0,\upsilon} \lesssim_{s,\mathtt{b}} \varepsilon\, N_{\overline{\mathtt{n}}-1}^{-\mathtt{a}}(1 + \|\mathfrak{I}_0\|_{s+\sigma+\mathtt{b}}^{k_0,\upsilon})\,; \tag{6.24}$$

3. The operators $\mathcal{E}^{\pm}$ are $\mathcal{D}^{k_0}$-(k_0+1)-tame, the operators $\mathcal{E}^{\pm1} - \mathrm{Id}$, $(\mathcal{E}^{\pm1} - \mathrm{Id})^*$ are $\mathcal{D}^{k_0}$-(k_0+2)-tame with tame constants satisfying, for some $\sigma := \sigma(\tau,\nu,k_0) > 0$ and for all $s_0 \leqslant s \leqslant S - \sigma$,

$$\mathfrak{M}_{\mathcal{E}^{\pm1}}(s) \lesssim_S 1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}, \ \ \mathfrak{M}_{\mathcal{E}^{\pm1}-\mathrm{Id}}(s) + \mathfrak{M}_{(\mathcal{E}^{\pm1}-\mathrm{Id})^*}(s) \lesssim_S \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma+\mathtt{b}}^{k_0,\upsilon})\,. \tag{6.25}$$

4. Furthermore, for any s_1 as in (6.9)*,*

$$|\Delta_{12}\mathtt{m}_{1,\overline{\mathtt{n}}}| \lesssim \varepsilon\,\|i_1 - i_2\|_{s_1+\sigma}\,, \quad \|\Delta_{12}\beta\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-1}\|i_1 - i_2\|_{s_1+\sigma+\mathtt{b}}\,, \tag{6.26}$$

$$\|\Delta_{12}(\mathcal{A})h\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma+\mathtt{b}}\,\|h\|_{s_1+\sigma+\mathtt{b}}\,, \quad \mathcal{A} \in \{\mathcal{E}^{\pm1}, (\mathcal{E}^{\pm1})^*\}\,. \tag{6.27}$$

Proof. We apply Theorem A.2 and Corollary A.4 to the transport operator $X_0 = \omega\cdot\partial_\varphi + \widetilde{V}\partial_x$, which has the form (A.1) with $p_0 = \widetilde{V}$. By

(6.12) and (6.8), the smallness condition (A.3) holds for $N_0^{\tau_2}\varepsilon\upsilon^{-1}$ sufficiently small. Therefore there exist a constant $\mathtt{m}_{1,\bar{\mathtt{n}}} \in \mathbb{R}$ and a quasi-periodic traveling wave $\beta(\varphi, x) := \beta_{\bar{\mathtt{n}}}(\varphi, x)$, $\mathrm{odd}(\varphi, x)$, such that, for any (ω, γ) in $\mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon, \tau) \subseteq \Lambda_{\bar{\mathtt{n}}+1}^{\upsilon,\mathrm{T}} \subseteq \Lambda_{\bar{\mathtt{n}}}^{\upsilon,\mathrm{T}}$ (see Corollary A.3) we have $\mathcal{B}_{\bar{\mathtt{n}}}^{-1}(\omega \cdot \partial_\varphi + \widetilde{V}\partial_x)\mathcal{B}_{\bar{\mathtt{n}}} = \omega \cdot \partial_\varphi + (\mathtt{m}_{1,\bar{\mathtt{n}}} + p_{\bar{\mathtt{n}}}(\varphi, y))\partial_y$ where the function $p_{\bar{\mathtt{n}}}$ satisfies (6.24) by (A.5) and (6.12). The estimates (A.6), (A.12), (6.12) imply (6.21), (6.25). The conjugated operator $\mathcal{E}^{-1}\mathcal{L}_{\mathrm{TR}}\mathcal{E} = \omega \cdot \partial_\varphi + \begin{pmatrix} A_1 & 0 \\ 0 & (\mathtt{m}_{1,\bar{\mathtt{n}}}+p_{\bar{\mathtt{n}}})\partial_y \end{pmatrix}$, where $\omega \cdot \partial_\varphi + A_1 = \mathcal{B}^{-1}(1+\beta_x)^{-1}\big(\omega \cdot \partial_\varphi + \partial_x \widetilde{V}\big)(1+\beta_x)\mathcal{B}$. Since $\mathcal{L}_{\mathrm{TR}}$ is Hamiltonian (Definition 3.13), and the map $\mathcal{E}$ is symplectic, $\mathcal{E}^{-1}\mathcal{L}_{\mathrm{TR}}\mathcal{E}$ is Hamiltonian as well. In particular $A_1 = -((\mathtt{m}_{1,\bar{\mathtt{n}}}+p_{\bar{\mathtt{n}}})\partial_y)^* = \mathtt{m}_{1,\bar{\mathtt{n}}}\partial_y + \partial_y p_{\bar{\mathtt{n}}}$. This proves (6.23). The estimates (6.26)-(6.27) follow by (A.10)-(A.11), the bound for $\|\Delta_{12}\beta_{\bar{\mathtt{n}}}\|_{s_1}$ in Corollary A.4 and (6.13)-(6.14). □

The next lemma is used to prove the inclusion of the Cantor sets associated to two approximate solutions.

Lemma 6.4. *Let* i_1, i_2 *be close enough and* $0 < 2\upsilon - \rho < 2\upsilon < 1$. *Then*

$$\varepsilon C(s_1) N_{\bar{\mathtt{n}}}^{\tau+1}\|i_1 - i_2\|_{s_1+\sigma} \leqslant \rho \ \Rightarrow\ \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon, \tau)(i_1) \subseteq \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon - \rho, \tau)(i_2)\,.$$

Proof. For any $(\omega, \gamma) \in \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon, \tau)(i_1)$, using also (6.26), we have, for any $\ell \in \mathbb{Z}^\nu \backslash \{0\}$, $|\ell| \leqslant N_{\bar{\mathtt{n}}}$,

$$\begin{aligned} |(\omega - \mathtt{m}_{1,\bar{\mathtt{n}}}(i_2)\vec{\jmath}) \cdot \ell| &\geqslant |(\omega - \mathtt{m}_{1,\bar{\mathtt{n}}}(i_1)\vec{\jmath}) \cdot \ell| - C|\Delta_{12}\mathtt{m}_{1,\bar{\mathtt{n}}}||\ell| \\ &\geqslant \frac{2\upsilon}{\langle \ell \rangle^\tau} - C(s_1)\varepsilon N_{\bar{\mathtt{n}}}\|i_1 - i_2\|_{s_1+\sigma} \geqslant \frac{2\upsilon - \rho}{\langle \ell \rangle^\tau}. \end{aligned}$$

We conclude that $(\omega, \gamma) \in \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon - \rho, \tau)(i_2)$. □

We now conjugate the whole operator $\mathcal{L}_1$ in (6.15)-(6.16) by the operator $\mathcal{E}$ in (6.18). We first compute the conjugation of the matrix

$$\begin{aligned} &\mathcal{E}^{-1}\begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -G(0) \\ a - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1}G(0)\partial_x^{-1} & -\frac{\gamma}{2}\partial_x^{-1}G(0) \end{pmatrix}\mathcal{E} \\ &= \begin{pmatrix} -\frac{\gamma}{2}\mathcal{B}^{-1}(1+\beta_x)^{-1}G(0)\partial_x^{-1}(1+\beta_x)\mathcal{B} & -\mathcal{B}^{-1}(1+\beta_x)^{-1}G(0)\mathcal{B} \\ \mathcal{B}^{-1}\big(a - \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1}G(0)\partial_x^{-1}\big)(1+\beta_x)\mathcal{B} & -\frac{\gamma}{2}\mathcal{B}^{-1}\partial_x^{-1}G(0)\mathcal{B} \end{pmatrix}. \end{aligned}$$

The multiplication operator for $a(\varphi, x)$ is transformed into the multiplication operator for the function

$$\mathcal{B}^{-1}a(1+\beta_x)\mathcal{B} = \mathcal{B}^{-1}\big(a(1+\beta_x)\big)\,. \tag{6.28}$$

We write the Dirichlet-Neumann operator $G(0)$ in (1.4) as

$$G(0) = G(0,\mathtt{h}) = \partial_x \mathcal{H} T(\mathtt{h})\,, \tag{6.29}$$

where $\mathcal{H}$ is the Hilbert transform defined in (3.11) and

$$T(\mathtt{h}) := \begin{cases} \tanh(\mathtt{h}|D|) & = \mathrm{Id} + \mathrm{Op}(r_{\mathtt{h}}) \quad \text{if } \mathtt{h} < +\infty\,, \\ & \quad r_{\mathtt{h}}(\xi) := -\frac{2}{1+e^{2\mathtt{h}|\xi|\chi(\xi)}} \in S^{-\infty}\,, \\ \mathrm{Id} & \quad \text{if } \mathtt{h} = \infty\,. \end{cases} \tag{6.30}$$

We have the conjugation formula (see formula (7.42) in [2])

$$\mathcal{B}^{-1} G(0) \mathcal{B} = \big\{\mathcal{B}^{-1}(1+\beta_x)\big\}\, G(0) + \mathcal{R}_1\,, \tag{6.31}$$

where

$$\mathcal{R}_1 := \big\{\mathcal{B}^{-1}(1+\beta_x)\big\}\, \partial_y \big(\mathcal{H}\,\big(\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\big) + \big(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}\big)\,(\mathcal{B}^{-1}T(\mathtt{h})\mathcal{B})\big)\,.$$

The operator $\mathcal{R}_1$ is in $\mathrm{OP}S^{-\infty}$ because both $\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})$ and $\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}$ are in $\mathrm{OP}S^{-\infty}$ and there is $\sigma > 0$ such that, for any $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0$ and $s \geqslant s_0$,

$$\begin{aligned} &\|\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|\beta\|_{s+m+\alpha+\sigma}^{k_0,\upsilon}\,, \\ &\|\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|\beta\|_{s+m+\alpha+\sigma}^{k_0,\upsilon}\,. \end{aligned} \tag{6.32}$$

The first estimate is given in Lemmata 2.36 and 2.32 in [9], whereas the second one follows because $r_{\mathtt{h}} \in S^{-\infty}$ (see (6.30)), Lemma 2.18 in [2] and Lemmata 2.34, 2.32 in [9]. Therefore by (6.31) we obtain

$$\mathcal{B}^{-1}(1+\beta_x)^{-1} G(0)\mathcal{B} = \{\mathcal{B}^{-1}(1+\beta_x)^{-1}\}\mathcal{B}^{-1}G(0)\mathcal{B} = G(0) + \mathcal{R}_B\,, \tag{6.33}$$

where

$$\mathcal{R}_B := \{\mathcal{B}^{-1}(1+\beta_x)^{-1}\}\, \mathcal{R}_1\,. \tag{6.34}$$

Next we transform $G(0)\partial_x^{-1}$. By (6.29) and using the identities $\mathcal{H}\partial_x\partial_x^{-1} = \mathcal{H}$ and $\mathcal{H}T(\mathtt{h}) = \partial_y^{-1}G(0)$ on the periodic functions, we have that

$$\begin{aligned} &\mathcal{B}^{-1}(1+\beta_x)^{-1} G(0)\partial_x^{-1}(1+\beta_x)\mathcal{B} = G(0)\partial_y^{-1} + \mathcal{R}_A \\ &\mathcal{B}^{-1}\partial_x^{-1}G(0)\mathcal{B} = \partial_y^{-1}G(0) + \mathcal{R}_D\,, \end{aligned} \tag{6.35}$$

where

$$\begin{aligned} \mathcal{R}_D &= (\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})(\mathcal{B}^{-1}T(\mathtt{h})\mathcal{B}) + \mathcal{H}\big(\mathcal{B}^{-1}\mathrm{Op}(r_{\mathtt{h}})\mathcal{B} - \mathrm{Op}(r_{\mathtt{h}})\big)\,, \\ \mathcal{R}_A &= \{\mathcal{B}^{-1}(1+\beta_x)^{-1}\}\big[\mathcal{H}T(\mathtt{h}), \{\mathcal{B}^{-1}(1+\beta_x)\} - 1\big] \\ &\quad + \{\mathcal{B}^{-1}(1+\beta_x)^{-1}\}\mathcal{R}_D\{\mathcal{B}^{-1}(1+\beta_x)\}\,. \end{aligned} \tag{6.36}$$

The operator $\mathcal{R}_D$ is in $\mathrm{OP}S^{-\infty}$ by (6.32), (6.30). Also $\mathcal{R}_A$ is in $\mathrm{OP}S^{-\infty}$ using that, by Lemma 2.35 of [9] and (6.30), there is $\sigma > 0$ such that, for any $m \in \mathbb{N}$, $s \geqslant s_0$, and $\alpha \in \mathbb{N}_0$,

$$\|[\mathcal{H}T(\mathtt{h}), \widetilde{a}]\|_{-m,s,\alpha}^{k_0,\upsilon} \lesssim_{m,s,\alpha,k_0} \|\widetilde{a}\|_{s+m+\alpha+\sigma}^{k_0,\upsilon} . \tag{6.37}$$

Finally we conjugate $\partial_x^{-1} G(0) \partial_x^{-1}$. By the Egorov Proposition 3.9 in [7] to ∂_x^{-1}, for any $N \in \mathbb{N}$, we have

$$\mathcal{B}^{-1}\partial_x^{-1}(1+\beta_x)\mathcal{B} = \mathcal{B}^{-1}\partial_x^{-1}\mathcal{B}\{\mathcal{B}^{-1}(1+\beta_x)\} = \partial_y^{-1} + P_{-2,N}^{(1)}(\varphi,x,D) + \mathtt{R}_N , \tag{6.38}$$

where $P_{-2,N}^{(1)}(\varphi, x, D) \in \mathrm{OP}S^{-2}$ is given by

$$\begin{aligned} P_{-2,N}^{(1)}(\varphi,x,D) &:= \big[\{\mathcal{B}^{-1}(1+\beta_x)^{-1}\}, \partial_y^{-1}\big]\{\mathcal{B}^{-1}(1+\beta_x)\} \\ &\quad + \sum_{j=1}^{N} p_{-1-j}\partial_y^{-1-j}\{\mathcal{B}^{-1}(1+\beta_x)\} \end{aligned}$$

for some functions $p_{-1-j}(\lambda; \varphi, y)$, $j = 0, \ldots, N$, and a regularizing operator $\mathtt{R}_N$ satisfying the estimates (3.30)-(3.31) of Proposition 3.9 in [7]. By (6.35), (6.38), we obtain

$$\mathcal{B}^{-1}\partial_x^{-1}G(0)\partial_x^{-1}(1+\beta_x)\mathcal{B} = \partial_y^{-1}G(0)\partial_y^{-1} + P_{-2,N}^{(2)} + \mathtt{R}_{2,N} \tag{6.39}$$

where

$$P_{-2,N}^{(2)} := \partial_y^{-1}G(0)P_{-2,N}^{(1)}(\varphi,x,D) \in \mathrm{OP}S^{-2} \tag{6.40}$$

$$\mathtt{R}_{2,N} := \mathcal{R}_D(\mathcal{B}^{-1}\partial_x^{-1}(1+\beta_x)\mathcal{B}) + G(0)\partial_y^{-1}\mathtt{R}_N . \tag{6.41}$$

In conclusion, by Lemma 6.3, (6.28), (6.33), (6.35) and (6.39) we obtain the following lemma, which summarizes the main result of this section.

Lemma 6.5. *Let $N \in \mathbb{N}$. For any $\overline{\mathtt{n}} \in \mathbb{N}_0$ and for all $(\omega,\gamma) \in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$, the operator $\mathcal{L}_1$ in* (6.15) *is conjugated to the real, Hamiltonian, reversible and momentum preserving operator*

$$\begin{aligned} \mathcal{L}_2 &:= \mathcal{E}^{-1}\mathcal{L}_1\mathcal{E} \\ &= \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_y + \begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_y^{-1} & -G(0) \\ a_1 - \left(\frac{\gamma}{2}\right)^2 \partial_y^{-1}G(0)\partial_y^{-1} & -\frac{\gamma}{2}\partial_y^{-1}G(0) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ -\left(\frac{\gamma}{2}\right)^2 P_{-2,N}^{(2)} & 0 \end{pmatrix} + \mathbf{R}_2^{\Psi} + \mathbf{T}_{2,N} + \mathbf{P}_2^{\perp} , \end{aligned} \tag{6.42}$$

defined for any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, where:

1. The constant $\mathtt{m}_{1,\overline{\mathtt{n}}} = \mathtt{m}_{1,\overline{\mathtt{n}}}(\omega,\gamma) \in \mathbb{R}$ satisfies $|\mathtt{m}_{1,\overline{\mathtt{n}}}|^{k_0,\upsilon} \lesssim \varepsilon$, independently on $\overline{\mathtt{n}}$;

2. The real quasi-periodic traveling wave $a_1 := \mathcal{B}^{-1}\big(a(1+\beta_x)\big)$, $\mathrm{even}(\varphi, x)$, *satisfies, for some* $\sigma := \sigma(k_0, \tau, \nu) > 0$ *and for all* $s_0 \leqslant s \leqslant S - \sigma$,

$$\|a_1 - g\|_s^{k_0,\upsilon} \lesssim_s \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}) ; \tag{6.43}$$

3. The operator $P_{-2,N}^{(2)}$ *is a pseudodifferential operator in* $\mathrm{OP}S^{-2}$, *reversibility and momentum preserving, and, for some* $\sigma_N := \sigma_N(\tau, \nu, N) > 0$, *for finitely many* $0 \leqslant \alpha \leqslant \alpha(M)$ *(fixed in Remark 6.10) and for all* $s_0 \leqslant s \leqslant S - \sigma_N - \alpha$, *satisfies*

$$\|P_{-2,N}^{(2)}\|_{-2,s,\alpha}^{k_0,\upsilon} \lesssim_{s,N,\alpha} \varepsilon \upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+\alpha}^{k_0,\upsilon}) ; \tag{6.44}$$

4. For any $\mathtt{q} \in \mathbb{N}_0^\nu$ *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$, $n_1, n_2 \in \mathbb{N}_0$ *with* $n_1 + n_2 \leqslant N - (k_0 + \mathtt{q}_0) + 2$, *the operator* $\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}}(\mathbf{R}_2^\Psi(\varphi) + \mathbf{T}_{2,N}(\varphi))\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with tame constant satisfying, for some* $\sigma_N(\mathtt{q}_0) = \sigma_N(\mathtt{q}_0, k_0, \tau, \nu) > 0$, *for any* $s_0 \leqslant s \leqslant S - \sigma_N(\mathtt{q}_0)$,

$$\mathfrak{M}_{\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}}(\mathbf{R}_2^\Psi(\varphi) + \mathbf{T}_{2,N}(\varphi))\langle D\rangle^{n_2}}(s) \lesssim_{S,N,\mathtt{q}_0} \varepsilon \upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N(\mathtt{q}_0)}^{k_0,\upsilon}\big); \tag{6.45}$$

5. The operator $\mathbf{P}_2^\perp$ *is defined in* (6.23) *and the function* $p_{\mathtt{n}}$ *satisfies* (6.24)*;*

6. Furthermore, for any s_1 *as in* (6.9), *finitely many* $0 \leqslant \alpha \leqslant \alpha(M)$, $\mathtt{q} \in \mathbb{N}_0^\nu$, *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$, *and* $n_1, n_2 \in \mathbb{N}_0$, *with* $n_1 + n_2 \leqslant N - \mathtt{q}_0 + 1$,

$$\begin{aligned} &|\Delta_{12}\mathtt{m}_{1,\overline{\mathtt{n}}}| \lesssim_{s_1} \varepsilon \|i_1 - i_2\|_{s_1+\sigma} , \ \|\Delta_{12} a_1\|_{s_1} \lesssim \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma} , \\ &\|\Delta_{12} P_{-2,N}^{(2)}\|_{-2,s_1,\alpha} \lesssim_{s_1,N,\alpha} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma_N+\alpha} , \end{aligned} \tag{6.46}$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}(\mathbf{R}_2^\Psi + \mathbf{T}_{2,N})\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,N,\mathtt{q}_0} \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_1+\sigma_N(\mathtt{q}_0)} . \tag{6.47}$$

Proof. Item 1 follows by Lemma 6.3. The function a_1 satisfies (6.43) by (6.11), (3.3), (6.12), (6.25), (6.21). The estimate (6.44) follows by (6.40), Lemmata 3.6, 6.3 and Lemma 3.8, Propositions 3.9 in [7]. The operators $\mathbf{R}_2^\Psi$, $\mathbf{T}_{2,N}$ in (6.42) are $\mathbf{R}_2^\Psi := -\begin{pmatrix} \frac{\gamma}{2}\mathcal{R}_A & \mathcal{R}_B \\ 0 & \frac{\gamma}{2}\mathcal{R}_D \end{pmatrix} + \mathcal{E}^{-1}\mathbf{R}_1\mathcal{E}$, $\mathbf{T}_{2,N} := -\left(\frac{\gamma}{2}\right)^2 \begin{pmatrix} 0 & 0 \\ \mathtt{R}_{2,N} & 0 \end{pmatrix}$ where $\mathcal{R}_B$, $\mathcal{R}_A$, $\mathcal{R}_D$, are defined in (6.34), (6.36), and $\mathbf{R}_1$, $\mathtt{R}_{2,N}$ in (6.16), (6.41). Thus the estimate (6.45) holds by Lemmata 3.9, 3.10, 6.3, 3.3, (6.32), (6.37), Lemma (6.21), Proposition 3.9 in [7], Lemma 3.10 in [7] and Lemmata 2.34, 2.32 in [9]. The estimates (6.46)-(6.47) are proved similarly. □

6.3 Symmetrization of the order $1/2$

The goal of this section is to symmetrize the order $1/2$ of the quasi-periodic Hamiltonian operator $\mathcal{L}_2$ in (6.42). From now on, we neglect the contribution of the operator $\mathbf{P}_2^\perp$, which will be conjugated in Section 6.7.

For simplicity of notation we denote such operator $\mathcal{L}_2$ as well.

Step 1: We first conjugate the operator $\mathcal{L}_2$ in (6.42), where we relabel the space variable $y \rightsquigarrow x$, by the real, symplectic, reversibility preserving and momentum preserving transformations $\widetilde{\mathcal{M}} := \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$, $\widetilde{\mathcal{M}}^{-1} := \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda \end{pmatrix}$, where $\Lambda \in {\rm OP}S^{\frac14}$ is the Fourier multiplier

$$\Lambda := \tfrac{1}{\sqrt{g}}\pi_0 + M(D)\,, \ \ \Lambda^{-1} := \sqrt{g}\pi_0 + M(D)^{-1} \in {\rm OP}S^{-\frac14}\,, \tag{6.48}$$

with π_0 defined in (3.13) and (cfr. (2.9))

$$M(D) := G(0)^{\frac14}\big(g - (\tfrac{\gamma}{2})^2\partial_x^{-1}G(0)\partial_x^{-1}\big)^{-\frac14} \in {\rm OP}S^{\frac14}\,. \tag{6.49}$$

We have the identities $\Lambda^{-1}G(0)\Lambda^{-1} = \omega(\gamma,D)$ and

$$\Lambda\big(g - \big(\tfrac{\gamma}{2}\big)^2\partial_x^{-1}G(0)\partial_x^{-1}\big)\Lambda = \Lambda^{-1}G(0)\Lambda^{-1} + \pi_0 = \omega(\gamma,D) + \pi_0\,, \tag{6.50}$$

where $\omega(\gamma,D) \in {\rm OP}S^{\frac12}$ is defined in (2.11). By (6.42) we compute

$$\begin{aligned} \mathcal{L}_3 &:= \widetilde{\mathcal{M}}^{-1}\mathcal{L}_2\widetilde{\mathcal{M}} \\ &= \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -\Lambda^{-1}G(0)\Lambda^{-1} \\ \Lambda\big(a_1 - (\frac{\gamma}{2})^2\partial_x^{-1}G(0)\partial_x^{-1}\big)\Lambda & -\frac{\gamma}{2}G(0)\partial_x^{-1} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ -(\frac{\gamma}{2})^2\Lambda P_{-2,N}^{(2)}\Lambda & 0 \end{pmatrix} + \widetilde{\mathcal{M}}^{-1}\mathbf{R}_2^{\Psi}\widetilde{\mathcal{M}} + \widetilde{\mathcal{M}}^{-1}\mathbf{T}_{2,N}\widetilde{\mathcal{M}}\,. \end{aligned} \tag{6.51}$$

By (6.50), (6.48) and (6.49), we get

$$\begin{aligned} &\Lambda\big(a_1 - (\tfrac{\gamma}{2})^2\partial_x^{-1}G(0)\partial_x^{-1}\big)\Lambda = \omega(\gamma,D) + (a_1 - g)\Lambda^2 + [\Lambda, a_1]\Lambda + \pi_0 \\ &= a_2^2\omega(\gamma,D) + \tfrac{a_1-g}{g}(\tfrac{\gamma}{2})^2M(D)^2\partial_x^{-1}G(0)\partial_x^{-1} + [\Lambda,a_1]\Lambda + \pi_0 + \tfrac{a_1-g}{g}\pi_0 \end{aligned}$$

where a_2 is the real quasi-periodic traveling wave function (with a_1 defined in Lemma 6.5)

$$a_2 := \sqrt{\tfrac{a_1}{g}} = \sqrt{1 + \tfrac{a_1-g}{g}}\,, \quad {\rm even}(\varphi,x)\,. \tag{6.52}$$

Therefore, by (6.51), (6.50) and the above computation we obtain

$$\begin{aligned} \mathcal{L}_3 &= \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -\omega(\gamma,D) \\ a_2\omega(\gamma,D)a_2 & -\frac{\gamma}{2}\partial_x^{-1}G(0) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ \pi_0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C_3 & 0 \end{pmatrix} + \mathbf{R}_3^{\Psi} + \mathbf{T}_{3,N}\,, \end{aligned} \tag{6.53}$$

where

$$\begin{aligned} C_3 &:= a_2[a_2,\omega(\gamma,D)] + \tfrac{a_1-g}{g}(\tfrac{\gamma}{2})^2M(D)^2\partial_x^{-1}G(0)\partial_x^{-1} \\ &\quad + [\Lambda,a_1]\Lambda - (\tfrac{\gamma}{2})^2\Lambda P_{-2,N}^{(2)}\Lambda \end{aligned} \tag{6.54}$$

is in $\mathrm{OP}S^{-\frac{1}{2}}$ and

$$\mathbf{R}_3^{\Psi} := \widetilde{\mathcal{M}}^{-1}\mathbf{R}_2^{\Psi}\widetilde{\mathcal{M}} + \begin{pmatrix} 0 & 0 \\ (\frac{a_1}{g} - 1)\pi_0 & 0 \end{pmatrix}, \ \mathbf{T}_{3,N} := \widetilde{\mathcal{M}}^{-1}\mathbf{T}_{2,N}\widetilde{\mathcal{M}}\,. \tag{6.55}$$

The operator $\mathcal{L}_3$ in (6.53) is Hamiltonian, reversible and momentum preserving.

Step 2: We now conjugate the operator $\mathcal{L}_3$ in (6.53) with the symplectic matrix of multiplication operators $\mathcal{Q} := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, $\mathcal{Q}^{-1} := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$, where q is a real function, close to 1, to be determined, see (6.59). We have that

$$\mathcal{L}_4 := \mathcal{Q}^{-1}\mathcal{L}_3\mathcal{Q} = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x + \begin{pmatrix} A & B \\ C & D \end{pmatrix} + \mathcal{Q}^{-1}(\mathbf{R}_3^{\Psi} + \mathbf{T}_{3,N})\mathcal{Q}\,, \tag{6.56}$$

where (see Definition 3.13)

$$A := -D^* = -\tfrac{\gamma}{2}q^{-1}G(0)\partial_x^{-1}q + \mathtt{m}_{1,\bar{\mathtt{n}}}q^{-1}q_x + q^{-1}(\omega\cdot\partial_\varphi q)\,, \tag{6.57}$$

$$B := -q^{-1}\omega(\gamma, D)q^{-1}\,, \ C := qa_2\omega(\gamma, D)a_2 q + q\pi_0 q + qC_3 q\,. \tag{6.58}$$

We choose the function q so that the coefficients of the highest order terms of the off-diagonal self-adjoint operators B and C satisfy $q^{-1} = qa_2$, namely as the real quasi-periodic traveling wave, $\mathrm{even}(\varphi, x)$

$$q(\varphi, x) := a_2(\varphi, x)^{-\frac{1}{2}}\,. \tag{6.59}$$

Thus $\mathcal{Q}$ is reversibility and momentum preserving. In view of (6.57)-(6.58) and (6.59) the operator $\mathcal{L}_4$ in (6.56) becomes

$$\begin{aligned}\mathcal{L}_4 = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x &+ \begin{pmatrix} -\frac{\gamma}{2}G(0)\partial_x^{-1} & -a_2^{\frac{1}{2}}\omega(\gamma, D)a_2^{\frac{1}{2}} \\ a_2^{\frac{1}{2}}\omega(\gamma, D)a_2^{\frac{1}{2}} & -\frac{\gamma}{2}\partial_x^{-1}G(0) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ \pi_0 & 0 \end{pmatrix} + \begin{pmatrix} a_3 & 0 \\ C_4 & -a_3 \end{pmatrix} + \mathbf{R}_4^{\Psi} + \mathbf{T}_{4,N}\,,\end{aligned} \tag{6.60}$$

where a_3 is the real quasi-periodic traveling wave function, $\mathrm{odd}(\varphi, x)$,

$$a_3 := \mathtt{m}_{1,\bar{\mathtt{n}}}q^{-1}q_x + q^{-1}(\omega\cdot\partial_\varphi q)\,, \quad C_4 := qC_3 q \in \mathrm{OP}S^{-\frac{1}{2}}\,, \tag{6.61}$$

and $\mathbf{R}_4^{\Psi}$, $\mathbf{T}_{4,N}$ are the smoothing remainders (recall that $G(0)\partial_x^{-1} = \mathcal{H}T(\mathtt{h})$)

$$\begin{aligned}\mathbf{R}_4^{\Psi} &:= \begin{pmatrix} -\frac{\gamma}{2}q^{-1}[\mathcal{H}T(\mathtt{h}), q-1] & 0 \\ q\pi_0 q - \pi_0 & -\frac{\gamma}{2}[q-1, \mathcal{H}T(\mathtt{h})]q^{-1} \end{pmatrix} + \mathcal{Q}^{-1}\mathbf{R}_3^{\Psi}\mathcal{Q} \in \mathrm{OP}S^{-\infty}\,, \\ \mathbf{T}_{4,N} &:= \mathcal{Q}^{-1}\mathbf{T}_{3,N}\mathcal{Q}\,.\end{aligned} \tag{6.62}$$

The operator $\mathcal{L}_4$ in (6.60) is Hamiltonian, reversible and momentum preserving.

Step 3: We finally move in complex coordinates, conjugating the operator $\mathcal{L}_4$ in (6.60) via the transformation $\mathcal{C}$ defined in (2.12). The main result of this section is the following lemma.

Lemma 6.6. *Let $N \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. We have that*

$$\begin{aligned}\mathcal{L}_5 := (\widetilde{\mathcal{M}}\mathcal{Q}\mathcal{C})^{-1}\mathcal{L}_2\widetilde{\mathcal{M}}\mathcal{Q}\mathcal{C} = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x + \mathrm{i}\, a_2\,\mathbf{\Omega}(\gamma, D) + a_4\mathcal{H} \\ + \mathrm{i}\,\mathbf{\Pi}_0 + \mathbf{R}_5^{(-\frac{1}{2},d)} + \mathbf{R}_5^{(0,o)} + \mathbf{T}_{5,N}\,,\end{aligned} \tag{6.63}$$

where:

1. The real quasi-periodic traveling wave $a_2(\varphi, x)$ in (6.52)*,* $\mathrm{even}(\varphi, x)$*, satisfies, for some $\sigma = \sigma(k_0, \tau, \nu) >$ and for any $s_0 \leqslant s \leqslant S - \sigma$,*

$$\|a_2 - 1\|_s^{k_0,\upsilon} \lesssim \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,; \tag{6.64}$$

2. $\mathbf{\Omega}(\gamma, D)$ is the matrix of Fourier multipliers (see (2.13), (2.14)*)*

$$\mathbf{\Omega}(\gamma, D) = \begin{pmatrix} \Omega(\gamma,D) & 0 \\ 0 & -\overline{\Omega(\gamma,D)} \end{pmatrix}, \quad \Omega(\gamma, D) = \omega(\gamma, D) + \mathrm{i}\,\frac{\gamma}{2}\partial_x^{-1}G(0)\,; \tag{6.65}$$

3. The operator $\mathbf{\Pi}_0 := \frac{1}{2}\begin{pmatrix} \pi_0 & \pi_0 \\ -\pi_0 & -\pi_0 \end{pmatrix}$;

4. The real quasi-periodic traveling wave $a_4(\varphi, x) := \frac{\gamma}{2}(a_2(\varphi, x) - 1)$, $\mathrm{even}(\varphi, x)$*, satisfies, for some $\sigma := \sigma(k_0, \tau, \nu) > 0$,*

$$\|a_4\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,, \quad \forall s_0 \leqslant s \leqslant S - \sigma\,; \tag{6.66}$$

5. $\mathbf{R}_5^{(-\frac{1}{2},d)} \in \mathrm{OP}S^{-\frac{1}{2}}$ and $\mathbf{R}_5^{(0,o)} \in \mathrm{OP}S^0$ are pseudodifferential operators of the form

$$\mathbf{R}_5^{(-\frac{1}{2},d)} := \begin{pmatrix} r_5^{(d)}(\varphi, x, D) & 0 \\ 0 & \overline{r_5^{(d)}(\varphi, x, D)} \end{pmatrix},$$
$$\mathbf{R}_5^{(0,o)} := \begin{pmatrix} 0 & r_5^{(o)}(\varphi, x, D) \\ \overline{r_5^{(o)}(\varphi, x, D)} & 0 \end{pmatrix},$$

reversibility and momentum preserving and, for some $\sigma_N := \sigma(\tau, \nu, N) > 0$, for finitely many $0 \leqslant \alpha \leqslant \alpha(M)$ (fixed in Remark 6.10), and for all $s_0 \leqslant s \leqslant S - \sigma_N - 3\alpha$, satisfies

$$\|\mathbf{R}_5^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s,\alpha}^{k_0,\upsilon} + \|\mathbf{R}_5^{(0,o)}\|_{0,s,\alpha}^{k_0,\upsilon} \lesssim_{s,N,\alpha} \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+3\alpha}^{k_0,\upsilon})\,; \tag{6.67}$$

6. For any $\mathtt{q} \in \mathbb{N}_0^\nu$ with $|\mathtt{q}| \leqslant \mathtt{q}_0$, $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 \leqslant N - (k_0 + \mathtt{q}_0) + \frac{3}{2}$, the operator $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{5,N}(\varphi)\langle D\rangle^{n_2}$ is $\mathcal{D}^{k_0}$-tame with tame *constant satisfying, for some* $\sigma_N(\mathtt{q}_0) = \sigma_N(\mathtt{q}_0, k_0, \tau, \nu) > 0$ *and for any* $s_0 \leqslant s \leqslant S - \sigma_N(\mathtt{q}_0)$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{5,N}(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,N,\mathtt{q}_0} \varepsilon\upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N(\mathtt{q}_0)}^{k_0,\upsilon}\big)\,; \tag{6.68}$$

7. The operators $\mathcal{Q}^{\pm 1}$*,* $\mathcal{Q}^{\pm 1} - \mathrm{Id}$*,* $(\mathcal{Q}^{\pm 1} - \mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-tame with tame constants satisfying, for some* $\sigma := \sigma(\tau, \nu, k_0) > 0$ *and for all* $s_0 \leqslant s \leqslant S - \sigma$*,*

$$\mathfrak{M}_{\mathcal{Q}^{\pm1}}(s) \lesssim_S 1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}\,,\ \mathfrak{M}_{\mathcal{Q}^{\pm1}-\mathrm{Id}}(s) + \mathfrak{M}_{(\mathcal{Q}^{\pm1}-\mathrm{Id})^*}(s) \lesssim_S \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,. \tag{6.69}$$

8. Furthermore, for any s_1 *as in* (6.9)*, finitely many* $0 \leqslant \alpha \leqslant \alpha(M)$*,* $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, and* $n_1, n_2 \in \mathbb{N}_0$*, with* $n_1 + n_2 \leqslant N - \mathtt{q}_0 + \frac{1}{2}$*,*

$$\|\Delta_{12}(\mathcal{A})h\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma}\,\|h\|_{s_1+\sigma}\,, \quad \mathcal{A} \in \{\mathcal{Q}^{\pm1} = (\mathcal{Q}^{\pm1})^*\}\,, \tag{6.70}$$
$$\|\Delta_{12}a_2\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma}\,,\ \|\Delta_{12}a_4\|_{s_1} \lesssim \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma}\,,$$
$$\|\Delta_{12}\mathbf{R}_5^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s_1,\alpha} + \|\Delta_{12}\mathbf{R}_5^{(0,o)}\|_{0,s_1,\alpha} \lesssim_{s_1,N,\alpha} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma_N+2\alpha}\,,$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{5,N}(\varphi)\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,N,\mathtt{q}_0} \varepsilon\upsilon^{-1}\,\|i_1 - i_2\|_{s_1+\sigma_N(\mathtt{q}_0)}\,. \tag{6.71}$$

The real operator $\mathcal{L}_5$ *is Hamiltonian, reversible and momentum preserving.*

Proof. By the expression of $\mathcal{L}_4$ in (6.60) and (3.10) we obtain that $\mathcal{L}_5$ has the form (6.63) with $r_5^{(d)} := \frac{\gamma}{2}(a_2 - 1)\mathcal{H}(T(\mathtt{h}) - 1) + \mathrm{i}\big(\frac{1}{2}C_4 + a_2^{\frac{1}{2}}[\omega(\gamma, D), a_2^{\frac{1}{2}}]\big) \in \mathrm{OP}S^{-\frac{1}{2}}$, $r_5^{(o)} := a_3 + \frac{\mathrm{i}}{2}C_4 \in \mathrm{OP}S^0$ (with C_4 given in (6.61)) and $\mathbf{T}_{5,N} := \mathcal{C}^{-1}(\mathbf{R}_4^\Psi + \mathbf{T}_{4,N})\mathcal{C}$. The function q defined in (6.59), with a_2 in (6.52), satisfies, by (6.43) and Lemma 3.2, for all $s_0 \leqslant s \leqslant S-\sigma$, $\|q^{\pm1} - 1\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\upsilon^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})$. Therefore (6.64) and (6.66) follow by (6.52). The estimate (6.67) follows by the above definitions of $r_5^{(o)}$ and $r_5^{(d)}$, (6.64), (6.59), (6.54), (6.52), (6.43), (6.44), (6.61), (6.48), (2.9), Lemma 6.5. The estimate (6.68) follows by (6.62), (6.55), (6.37), (6.45), (6.43) Lemmata 3.9, 3.10. The estimates (6.69) follow by Lemma 3.10. The estimates (6.70)- (6.71) are proved similarly. □

6.4 Symmetrization up to smoothing remainders

We now transform the operator $\mathcal{L}_5$ in (6.63) into the operator $\mathcal{L}_6$ in (6.72) which is block diagonal up to a regularizing remainder. From this step we do not preserve any further the Hamiltonian structure, but only the reversible and momentum preserving one (it is sufficient for proving Theorem 5.1).

Lemma 6.7. *Fix $\mathtt{m}, N \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. There exist real, reversibility and momentum preserving operator matrices $\{\mathbf{X}_k\}_{k=1}^{\mathtt{m}}$ of the form $\mathbf{X}_k := \begin{pmatrix} 0 & \chi_k(\varphi,x,D) \\ \overline{\chi_k(\varphi,x,D)} & 0 \end{pmatrix}$, with $\chi_k(\varphi,x,\xi) \in S^{-\frac{k}{2}}$, such that, conjugating $\mathcal{L}_5$ in* (6.63) *via the map $\mathbf{\Phi}_{\mathtt{m}} := e^{\mathbf{X}_1} \circ \cdots \circ e^{\mathbf{X}_{\mathtt{m}}}$, we obtain the real, reversible and momentum preserving operator*

$$
\begin{aligned}
\mathcal{L}_6 := \mathcal{L}_6^{(\mathtt{m})} := \mathbf{\Phi}_{\mathtt{m}}^{-1}\, \mathcal{L}_5\, \mathbf{\Phi}_{\mathtt{m}} = \omega \cdot \partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}} \partial_x + \mathrm{i}\, a_2 \mathbf{\Omega}(\gamma, D) \\
+ a_4 \mathcal{H} + \mathrm{i} \mathbf{\Pi}_0 + \mathbf{R}_6^{(-\frac{1}{2},d)} + \mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)} + \mathbf{T}_{6,N}\,,
\end{aligned} \tag{6.72}
$$

where:

1. $\mathbf{R}_6^{(-\frac{1}{2},d)} := \mathbf{R}_{6,\mathtt{m}}^{(-\frac{1}{2},d)} := \begin{pmatrix} r_6^{(d)}(\varphi,x,D) & 0 \\ 0 & \overline{r_6^{(d)}(\varphi,x,D)} \end{pmatrix} \in \mathrm{OP}S^{-\frac{1}{2}}$ *is block-diagonal, $\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)}$ is a smoothing off-diagonal remainder*

$$
\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)} := \mathbf{R}_{6,\mathtt{m}}^{(-\frac{\mathtt{m}}{2},o)} := \begin{pmatrix} 0 & r_6^{(o)}(\varphi,x,D) \\ \overline{r_6^{(o)}(\varphi,x,D)} & 0 \end{pmatrix} \in \mathrm{OP}S^{-\frac{\mathtt{m}}{2}}\,, \tag{6.73}
$$

satisfying, for finitely many $0 \leqslant \alpha \leqslant \alpha(\mathtt{m})$ (fixed in Remark 6.10), for some $\sigma_N := \sigma_N(k_0,\tau,\nu,N) > 0$, $\aleph_{\mathtt{m}}(\alpha) > 0$ and for all $s_0 \leqslant s \leqslant S - \sigma_N - \aleph_{\mathtt{m}}(\alpha)$,

$$
\|\mathbf{R}_6^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s,\alpha}^{k_0,\upsilon} + \|\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)}\|_{-\frac{\mathtt{m}}{2},s,\alpha}^{k_0,\upsilon} \lesssim_{s,\mathtt{m},N,\alpha} \varepsilon\upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+\aleph_{\mathtt{m}}(\alpha)}^{k_0,\upsilon}\big). \tag{6.74}
$$

Both $\mathbf{R}_6^{(-\frac{1}{2},d)}$ and $\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)}$ are reversible and momentum preserving;

2. For any $\mathtt{q} \in \mathbb{N}_0^\nu$ with $|\mathtt{q}| \leqslant \mathtt{q}_0$, $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 \leqslant N - (k_0 + \mathtt{q}_0) + \frac{3}{2}$, the operator $\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}} \mathbf{T}_{6,N}(\varphi) \langle D\rangle^{n_2}$ is $\mathcal{D}^{k_0}$-tame with a tame constant satisfying, for some $\sigma_N(\mathtt{q}_0) := \sigma_N(k_0,\tau,\nu,\mathtt{q}_0)$, for any $s_0 \leqslant s \leqslant S - \sigma_N(\mathtt{q}_0) - \aleph_{\mathtt{m}}(0)$,

$$
\mathfrak{M}_{\langle D\rangle^{n_1} \partial_\varphi^{\mathtt{q}} \mathbf{T}_{6,N}(\varphi) \langle D\rangle^{n_2}}(s) \lesssim_{S,\mathtt{m},N,\mathtt{q}_0} \varepsilon\upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N(\mathtt{q}_0)+\aleph_{\mathtt{m}}(0)}^{k_0,\upsilon}\big)\,. \tag{6.75}
$$

3. The conjugation map $\mathbf{\Phi}_{\mathtt{m}}$ satisfies, for all $s_0 \leqslant s \leqslant S - \sigma_N - \aleph_{\mathtt{m}}(0)$,

$$
\|\mathbf{\Phi}_{\mathtt{m}}^{\pm 1} - \mathrm{Id}\|_{0,s,0}^{k_0,\upsilon} + \|\big(\mathbf{\Phi}_{\mathtt{m}}^{\pm 1} - \mathrm{Id}\big)^*\|_{0,s,0}^{k_0,\upsilon} \lesssim_{s,\mathtt{m},N} \varepsilon\upsilon^{-1}\big(1 + \|\mathfrak{I}_0\|_{s+\sigma_N+\aleph_{\mathtt{m}}(0)}^{k_0,\upsilon}\big)\,. \tag{6.76}
$$

4. Furthermore, for any s_1 as in (6.9)*, finitely many $0 \leqslant \alpha \leqslant \alpha(\mathtt{m})$, $\mathtt{q} \in \mathbb{N}_0^\nu$, with $|\mathtt{q}| \leqslant \mathtt{q}_0$, and $n_1, n_2 \in \mathbb{N}_0$, with $n_1 + n_2 \leqslant N - \mathtt{q}_0 + \frac{1}{2}$, we have*

$$
\|\Delta_{12}\mathbf{R}_6^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s_1,\alpha} + \|\Delta_{12}\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)}\|_{-\frac{\mathtt{m}}{2},s_1,\alpha} \lesssim_{s_1,\mathtt{m},N,\alpha} \varepsilon\upsilon^{-1}\|i_1 - i_2\|_{s_1+\sigma_N+\aleph_{\mathtt{m}}(\alpha)}\,,
$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{6,N}\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})}\lesssim_{s_1,\mathtt{m},N,\mathtt{q}_0}\varepsilon\upsilon^{-1}\|i_1-i_2\|_{s_1+\sigma_N(\mathtt{q}_0)+\aleph_{\mathtt{m}}(0)}\,,$$
$$\|\Delta_{12}\mathbf{\Phi}_{\mathtt{m}}^{\pm 1}\|_{0,s_1,0}+\|\Delta_{12}(\mathbf{\Phi}_{\mathtt{m}}^{\pm 1})^*\|_{0,s_1,0}\lesssim_{s_1,\mathtt{m},N}\varepsilon\upsilon^{-1}\|i_1-i_2\|_{s_1+\sigma_N+\aleph_{\mathtt{m}}(0)}\,.$$

Proof. The proof is inductive. The operator $\mathcal{L}_6^{(0)}:=\mathcal{L}_5$ satisfies (6.74)-(6.75) with $\aleph_0(\alpha):=3\alpha$, by (6.67)-(6.68). Suppose we have done already $\mathtt{m}$ steps obtaining an operator $\mathcal{L}_6^{(\mathtt{m})}$ as in (6.72) with $\mathbf{R}_{6,\mathtt{m}}^{(-\frac12,d)}:=\mathbf{R}_6^{(-\frac12,d)}$ and $\mathbf{R}_{6,\mathtt{m}}^{(-\frac12,o)}:=\mathbf{R}_6^{(-\frac{\mathtt{m}}{2},o)}$ and the remainder $\mathbf{\Phi}_{\mathtt{m}}^{-1}\mathbf{T}_{5,N}\mathbf{\Phi}_{\mathtt{m}}$, instead of $\mathbf{T}_{6,N}$. We now show how to define $\mathcal{L}_6^{(\mathtt{m}+1)}$. Let

$$\chi_{\mathtt{m}+1}(\varphi,x,\xi):=-\big(2\mathrm{i}\,a_2(\varphi,x)\omega(\gamma,\xi)\big)^{-1}r_{6,\mathtt{m}}^{(o)}(\varphi,x,\xi)\chi(\xi)\in S^{-\frac{\mathtt{m}}{2}-\frac12}\,, \tag{6.77}$$

where χ is the cut-off function defined in (3.6) and $\omega(\gamma,\xi)$ is the symbol (cfr. (2.11))

$$\omega(\gamma,\xi):=\sqrt{G(0;\xi)\Big(g+\frac{\gamma^2}{4}\frac{G(0;\xi)}{\xi^2}\Big)}\in S^{\frac12},\ G(0;\xi):=\begin{cases}\chi(\xi)|\xi|\tanh(\mathtt{h}|\xi|)\,, & \mathtt{h}<+\infty\\ \chi(\xi)|\xi|\,, & \mathtt{h}=+\infty\,.\end{cases}$$

Note that $\chi_{\mathtt{m}+1}$ in (6.77) is well defined because $\omega(\gamma,\xi)$ is positive on the support of $\chi(\xi)$ and a_2 is close to 1. We conjugate $\mathcal{L}_6^{(\mathtt{m})}$ in (6.72) by the flow generated by $\mathbf{X}_{\mathtt{m}+1}$ with $\chi_{\mathtt{m}+1}(\varphi,x,\xi)$ defined in (6.77). By (6.74) and (6.65), for suitable constants $\aleph_{\mathtt{m}+1}(\alpha)>\aleph_{\mathtt{m}}(\alpha)$, for finitely many $\alpha\in\mathbb{N}_0$ and for any $s_0\leqslant s\leqslant S-\sigma_N-\aleph_{\mathtt{m}+1}(\alpha)$,

$$\|\mathbf{X}_{\mathtt{m}+1}\|_{-\frac{\mathtt{m}}{2}-\frac12,s,\alpha}^{k_0,\upsilon}\lesssim_{s,\mathtt{m},\alpha}\varepsilon\upsilon^{-1}\big(1+\|\mathfrak{I}_0\|_{s+\sigma_N+\aleph_{\mathtt{m}+1}(\alpha)}^{k_0,\upsilon}\big)\,. \tag{6.78}$$

Therefore, by Lemmata 3.7, 3.6 and the induction assumption (6.76) for $\mathbf{\Phi}_{\mathtt{m}}$, the conjugation map $\mathbf{\Phi}_{\mathtt{m}+1}:=\mathbf{\Phi}_{\mathtt{m}}e^{\mathbf{X}_{\mathtt{m}+1}}$ is well defined and satisfies estimate (6.76) with $\mathtt{m}+1$. By the Lie expansion (see (3.16)-(3.17) in [7]), we have that $\mathcal{L}_6^{(\mathtt{m}+1)}:=e^{-\mathbf{X}_{\mathtt{m}+1}}\mathcal{L}_6^{(\mathtt{m})}e^{\mathbf{X}_{\mathtt{m}+1}}$ is equal to

$$\begin{aligned}\mathcal{L}_6^{(\mathtt{m}+1)}&=\omega\cdot\partial_\varphi+\mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x+\mathrm{i}a_2\mathbf{\Omega}(\gamma,D)+\mathrm{i}\mathbf{\Pi}_0+a_4\mathcal{H}+\mathbf{R}_{6,\mathtt{m}}^{(-\frac12,d)}\\&\quad-\big[\mathbf{X}_{\mathtt{m}+1},\mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x+\mathrm{i}\,a_2\mathbf{\Omega}(\gamma,D)\big]+\mathbf{R}_{6,\mathtt{m}}^{(-\frac{\mathtt{m}}{2},o)}+\mathbf{\Phi}_{\mathtt{m}+1}^{-1}\mathbf{T}_{5,N}\mathbf{\Phi}_{\mathtt{m}+1}\end{aligned} \tag{6.79}$$

$$-\int_0^1 e^{-\tau\mathbf{X}_{\mathtt{m}+1}}\big[\mathbf{X}_{\mathtt{m}+1}\,,\,\omega\cdot\partial_\varphi+\mathrm{i}\mathbf{\Pi}_0+a_4\mathcal{H}+\mathbf{R}_{6,\mathtt{m}}^{(-\frac12,d)}\big]e^{\tau\mathbf{X}_{\mathtt{m}+1}}\mathrm{d}\tau \tag{6.80}$$

$$-\int_0^1 e^{-\tau\mathbf{X}_{\mathtt{m}+1}}\big[\mathbf{X}_{\mathtt{m}+1},\mathbf{R}_{6,\mathtt{m}}^{(-\frac{\mathtt{m}}{2},o)}\big]e^{\tau\mathbf{X}_{\mathtt{m}+1}}\mathrm{d}\tau \tag{6.81}$$

$$+\int_0^1(1-\tau)e^{-\tau\mathbf{X}_{\mathtt{m}+1}}\big[\mathbf{X}_{\mathtt{m}+1},\big[\mathbf{X}_{\mathtt{m}+1},\mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x+\mathrm{i}\,a_2\mathbf{\Omega}(\gamma,D)\big]\big]e^{\tau\mathbf{X}_{\mathtt{m}+1}}\mathrm{d}\tau\,. \tag{6.82}$$

In view of (6.65), (6.73) and the form of $\mathbf{X}_{\mathtt{m}+1}$, we have that

$$-\big[\mathbf{X}_{\mathtt{m}+1}, \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i}\, a_2 \mathbf{\Omega}(\gamma, D)\big] + \mathbf{R}_{6,\mathtt{m}}^{(-\frac{\mathtt{m}}{2},o)} = \begin{pmatrix} 0 & Z_{\mathtt{m}+1} \\ \overline{Z_{\mathtt{m}+1}} & 0 \end{pmatrix} =: \mathbf{Z}_{\mathtt{m}+1}\,,$$

where, denoting for brevity $\chi_{\mathtt{m}+1} := \chi_{\mathtt{m}+1}(\varphi, x, \xi)$, it results

$$\begin{aligned} Z_{\mathtt{m}+1} &= \mathrm{i}\,(\mathrm{Op}(\chi_{\mathtt{m}+1}) a_2\, \omega(\gamma, D) + a_2\, \omega(\gamma, D)\mathrm{Op}(\chi_{\mathtt{m}+1})) \\ &\quad + \big[\mathrm{Op}(\chi_{\mathtt{m}+1}), -\mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + a_2\, \tfrac{\gamma}{2}\partial_x^{-1} G(0)\big] + \mathrm{Op}(r_{6,\mathtt{m}}^{(o)})\,. \end{aligned}$$

By (3.14), (3.16) and $\chi_{\mathtt{m}+1} \in S^{-\frac{\mathtt{m}}{2}-\frac{1}{2}}$ by (6.77), we get

$$\mathrm{Op}(\chi_{\mathtt{m}+1}) a_2 \omega(\gamma, D) + a_2 \omega(\gamma, D)\mathrm{Op}(\chi_{\mathtt{m}+1}) = \mathrm{Op}\big(2 a_2 \omega(\gamma, \xi)\chi_{\mathtt{m}+1}\big) + \mathtt{r}_{\mathtt{m}+1},$$

where $\mathtt{r}_{\mathtt{m}+1}$ is in $\mathrm{OP}S^{-\frac{\mathtt{m}}{2}-1}$. By (6.77) and (6.83)

$$\begin{aligned} Z_{\mathtt{m}+1} &= \mathrm{i}\mathtt{r}_{\mathtt{m}+1} + \big[\mathrm{Op}(\chi_{\mathtt{m}+1}), -\mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + a_2 \tfrac{\gamma}{2}\partial_x^{-1} G(0)\big] \\ &\quad + \mathrm{Op}(r_{6,\mathtt{m}}^{(o)}(1 - \chi(\xi))) \in \mathrm{OP}S^{-\frac{\mathtt{m}}{2}-\frac{1}{2}}\,. \end{aligned}$$

The remaining operators in (6.80)-(6.82) are in $\mathrm{OP}S^{-\frac{\mathtt{m}+1}{2}}$. Thus the operator $\mathcal{L}_6^{(\mathtt{m}+1)}$ in (6.79) has the form (6.72) at $\mathtt{m}+1$ with

$$\mathbf{R}_{6,\mathtt{m}+1}^{(-\frac{1}{2},d)} + \mathbf{R}_{6,\mathtt{m}+1}^{(-\frac{\mathtt{m}+1}{2},o)} := \mathbf{R}_{6,\mathtt{m}}^{(-\frac{1}{2},d)} + \mathbf{Z}_{\mathtt{m}+1} + (6.80) + (6.81) + (6.82)$$

and a smoothing remainder $\mathbf{\Phi}_{\mathtt{m}+1}^{-1}\mathbf{T}_{5,N}\mathbf{\Phi}_{\mathtt{m}+1}$. By Lemma 3.6, (6.74), (6.78), (6.66), we have that $\mathbf{R}_{6,\mathtt{m}+1}^{(-\frac{1}{2},d)}$ and $\mathbf{R}_{6,\mathtt{m}+1}^{(-\frac{\mathtt{m}+1}{2},o)}$ satisfy (6.74) at order $\mathtt{m}+1$ for suitable constants $\aleph_{\mathtt{m}+1}(\alpha) > \aleph_{\mathtt{m}}(\alpha)$. The operator $\mathbf{\Phi}_{\mathtt{m}+1}^{-1}\mathbf{T}_{5,N}\mathbf{\Phi}_{\mathtt{m}+1}$ satisfies (6.75) at order $\mathtt{m}+1$ by Lemmata 3.9, 3.10 and (6.68), (6.76). Item 4 follows similarly. □

So far the operator $\mathcal{L}_6$ of Lemma 6.7 depends on the two “regularizing" indexes $\mathtt{m}, N$. We now fix

$$\mathtt{m} := 2M\,,\ M \in \mathbb{N}\,, \quad N = M\,. \tag{6.83}$$

6.5 Reduction of the order 1/2

The goal of this section is to transform the operator $\mathcal{L}_6$ in (6.72) with $\mathtt{m} := 2M$, $N = M$ (cfr. (6.83)), into the operator $\mathcal{L}_7$ in (6.95) whose coefficient in front of $\mathbf{\Omega}(\gamma, D)$ is constant. We write $\mathcal{L}_6 = \omega\cdot\partial_\varphi + \begin{pmatrix} P_6 & 0 \\ 0 & \overline{P_6}\end{pmatrix} + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_6^{(-M,o)} + \mathbf{T}_{6,M}$, where $P_6 := P_6(\varphi, x, D)$ is

$$P_6 := \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i} a_2(\varphi, x)\Omega(\gamma, D) + a_4\mathcal{H} + r_6^{(d)}(\varphi, x, D)\,. \tag{6.84}$$

We conjugate $\mathcal{L}_6$ through the real operator $\mathbf{\Phi}(\varphi) := \begin{pmatrix} \Phi(\varphi) & 0 \\ 0 & \overline{\Phi}(\varphi) \end{pmatrix}$ where $\Phi(\varphi) := \Phi^\tau(\varphi)|_{\tau=1}$ is the time 1-flow of the PDE

$$\partial_\tau \Phi^\tau(\varphi) = \mathrm{i} A(\varphi)\Phi^\tau(\varphi)\,,\ \Phi^0(\varphi) = \mathrm{Id}\,,\ A(\varphi) := b(\varphi,x)|D|^{\frac{1}{2}}\,, \tag{6.85}$$

and $b(\varphi,x)$ is a real quasi-periodic traveling wave, $\mathrm{odd}(\varphi,x)$, chosen later, see (6.92). Thus $\mathrm{i}b(\varphi,x)|D|^{\frac{1}{2}}$ is reversibility and momentum preserving as well as $\mathbf{\Phi}(\varphi)$. Moreover $\Phi\pi_0 = \pi_0 = \Phi^{-1}\pi_0$, which implies $\mathbf{\Phi}^{-1}\mathbf{\Pi}_0\mathbf{\Phi} = \mathbf{\Pi}_0\mathbf{\Phi}$. By the Lie expansion (see e.g. (3.16)-(3.17) in [7]), we have

$$\begin{aligned}\Phi^{-1}P_6\Phi &= P_6 - \mathrm{i}[A,P_6] - \frac{1}{2}[A,[A,P_6]] + \sum_{n=3}^{2M+1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^n_{A(\varphi)}(P_6) + T_M\,,\\ T_M &:= \frac{(-\mathrm{i})^{2M+2}}{(2M+1)!}\int_0^1 (1-\tau)^{2M+1}\Phi^{-\tau}(\varphi)\,\mathrm{ad}^{2M+2}_{A(\varphi)}(P_6)\,\Phi^\tau(\varphi)\mathrm{d}\tau\,,\end{aligned} \tag{6.86}$$

and

$$\begin{aligned}\Phi^{-1}\circ\omega\cdot\partial_\varphi\circ\Phi &= \omega\cdot\partial_\varphi + \mathrm{i}(\omega\cdot\partial_\varphi A) + \frac{1}{2}[A,\omega\cdot\partial_\varphi A]\\ &\quad - \sum_{n=3}^{2M+1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^{n-1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi)) + T'_M\,,\\ T'_M &:= -\frac{(-\mathrm{i})^{2M+2}}{(2M+1)!}\int_0^1 (1-\tau)^{2M+1}\Phi^{-\tau}(\varphi)\,\mathrm{ad}^{2M+1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi))\,\Phi^\tau(\varphi)\mathrm{d}\tau\,.\end{aligned} \tag{6.87}$$

Note that $\mathrm{ad}^{2M+2}_{A(\varphi)}(P_6)$ and $\mathrm{ad}^{2M+1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi))$ are in $\mathrm{OP}S^{-M}$. We now determine the pseudo-differential term of order $1/2$ in (6.86)-(6.87). We use the expansion of the linear dispersion operator $\Omega(\gamma,D)$, defined by (4.1), (1.5), and, since $j \to c_j(\gamma) \in S^0$ (see (4.5)),

$$\Omega(\gamma,D) = \sqrt{g}|D|^{\frac{1}{2}} + \mathrm{i}\,\tfrac{\gamma}{2}\mathcal{H} + r_{-\frac{1}{2}}(\gamma,D)\,,\ r_{-\frac{1}{2}}(\gamma,D) \in \mathrm{OP}S^{-\frac{1}{2}}\,, \tag{6.88}$$

where $\mathcal{H}$ is the Hilbert transform in (3.11). By (6.84), that $A = b|D|^{\frac{1}{2}}$, (3.15), (6.88) we get

$$\begin{aligned}[A,P_6] &= \big[b|D|^{\frac{1}{2}}, \mathtt{m}_{1,\mathtt{n}}\partial_x + \mathrm{i}\sqrt{g}a_2|D|^{\frac{1}{2}}\\ &\qquad + (a_4 - \tfrac{\gamma}{2}a_2)\mathcal{H} + r_6^{(d)}(x,D) + \mathrm{i}\,a_2 r_{-\frac{1}{2}}(\gamma,D)\big]\\ &= -\mathtt{m}_{1,\bar{\mathtt{n}}}b_x|D|^{\frac{1}{2}} - \mathrm{i}\tfrac{\sqrt{g}}{2}(b_x a_2 - (a_2)_x b)\mathcal{H} + \mathrm{Op}(r_{b,-\frac{1}{2}})\,,\end{aligned} \tag{6.89}$$

where $r_{b,-\frac{1}{2}} \in S^{-\frac{1}{2}}$ is small with b. As a consequence, the contribution at order $\frac{1}{2}$ of the operator $\mathrm{i}\,\omega\cdot\partial_\varphi A + P_6 - \mathrm{i}[A,P_6]$ is $\mathrm{i}\big(\omega\cdot\partial_\varphi b + \mathtt{m}_{1,\bar{\mathtt{n}}}b_x + \sqrt{g}\,a_2\big)|D|^{\frac{1}{2}}$. We choose $b(\varphi,x)$ as the solution of

$$(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\bar{\mathtt{n}}}\partial_x)b + \sqrt{g}\,\Pi_{N_{\bar{\mathtt{n}}}}\,a_2 = \sqrt{g}\,\mathtt{m}_{\frac{1}{2}} \tag{6.90}$$

where $\mathtt{m}_{\frac12}$ is the average (see (3.2))

$$\mathtt{m}_{\frac12} := \langle a_2 \rangle_{\varphi,x} \,. \tag{6.91}$$

We define $b(\varphi,x)$ to be the real, $\mathrm{odd}(\varphi,x)$, quasi-periodic traveling wave

$$b(\varphi,x) := -\sqrt{g}(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x)^{-1}_{\mathrm{ext}}\big(\Pi_{N_{\overline{\mathtt{n}}}} a_2(\varphi,x) - \mathtt{m}_{\frac12}\big) \tag{6.92}$$

recall (3.5). Note that $b(\varphi,x)$ and $\mathtt{m}_{\frac12}$ are defined for any $(\omega,\gamma)\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$ and that, for any $(\omega,\gamma)\in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$ defined in (6.22), it solves (6.90). We deduce by (6.86), (6.87), (6.84), (6.89)-(6.92), that

$$L_7 := \Phi^{-1}(\varphi)\,(\omega\cdot\partial_\varphi + P_6)\,\Phi(\varphi)$$

is, for any $(\omega,\gamma)\in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$,

$$\begin{aligned} L_7 &= \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i}\,\mathtt{m}_{\frac12}\Omega(\gamma,D) + a_5\mathcal{H} \\ &\quad + \mathrm{Op}(r_7^{(d)}) + T_M + T_M' + \mathrm{i}\sqrt{g}(\Pi^\perp_{N_{\overline{\mathtt{n}}}} a_2)|D|^{\frac12}, \end{aligned}$$

where $a_5(\varphi,x)$ is the real function (using that $a_4 = \frac{\gamma}{2}(a_2-1)$)

$$\begin{aligned} a_5 &:= \tfrac{\gamma}{2}(\mathtt{m}_{\frac12}-1) - \tfrac{\sqrt{g}}{2}(b_x a_2 - (a_2)_x b) \\ &\quad + \tfrac{\mathtt{m}_{1,\overline{\mathtt{n}}}}{4}\big(b_{xx}b - b_x^2\big) + \tfrac14\big(b(\omega\cdot\partial_\varphi b)_x - (\omega\cdot\partial_\varphi b)b_x\big)\,, \end{aligned} \tag{6.93}$$

and

$$\begin{aligned} \mathrm{Op}(r_7^{(d)}) &:= \mathrm{Op}(-\mathrm{i} r_{b,-\frac12} + \mathrm{i}\,(a_2 - \mathtt{m}_{\frac12})r_{-\frac12}(\gamma,D) + r_6^{(d)}) \\ &+ \tfrac12\big[b|D|^{\frac12}, \mathrm{i}\tfrac{\sqrt{g}}{2}(b_x a_2 - (a_2)_x b)\mathcal{H} - \mathrm{Op}(r_{b,-\frac12})\big] \\ &+ \tfrac12\mathrm{Op}(\widetilde{r}_2(b|\xi|^{\frac12}, (\mathtt{m}_{1,\overline{\mathtt{n}}}b_x + \omega\cdot\partial_\varphi b)|\xi|^{\frac12})) + \sum_{n=3}^{2M+1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^n_{A(\varphi)}(P_6) \\ &- \sum_{n=3}^{2M+1}\frac{(-\mathrm{i})^n}{n!}\mathrm{ad}^{n-1}_{A(\varphi)}(\omega\cdot\partial_\varphi A(\varphi)) \in \mathrm{OP}S^{-\frac12}\,, \end{aligned} \tag{6.94}$$

with $\widetilde{r}_2(\,\cdot\,,\,\cdot\,)$ defined in (3.15). In conclusion we have the following lemma.

Lemma 6.8. *Let* $M\in\mathbb{N}$, $\mathtt{q}_0\in\mathbb{N}_0$. *Let* $b(\varphi,x)$ *be the quasi-periodic traveling wave function* $\mathrm{odd}(\varphi,x)$ *in* (6.92). *Then, for any* $\overline{\mathtt{n}}\in\mathbb{N}_0$, *conjugating* $\mathcal{L}_6$ *in* (6.72) *via the invertible, real, reversibility and momentum preserving map* $\mathbf{\Phi}$ *(cfr.* (6.85)*), we obtain, for any* $(\omega,\gamma)\in\mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$, *the real, reversible and momentum preserving operator*

$$\begin{aligned} \mathcal{L}_7 := \mathbf{\Phi}^{-1}\mathcal{L}_6\mathbf{\Phi} &= \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\,\partial_x + \mathrm{i}\,\mathtt{m}_{\frac12}\mathbf{\Omega}(\gamma,D) + a_5\mathcal{H} \\ &\qquad + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{R}_7^{(-\frac12,d)} + \mathbf{T}_{7,M} + \mathbf{Q}_7^\perp\,, \end{aligned} \tag{6.95}$$

defined for any $(\omega,\gamma)\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$, *where:*

1. The real constant $\mathtt{m}_{\frac{1}{2}}$ *defined in* (6.91) *satisfies* $|\mathtt{m}_{\frac{1}{2}} - 1|^{k_0,\upsilon} \lesssim \varepsilon\upsilon^{-1}$*;*

2. The real, quasi-periodic traveling wave function $a_5(\varphi, x)$ *defined in* (6.93)*,* $\mathrm{even}(\varphi, x)$*, satisfies, for some* $\sigma = \sigma(\tau, \nu, k_0) > 0$*, for all* $s_0 \leqslant s \leqslant S - \sigma$*,*

$$\|a_5\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\upsilon^{-2}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}), \quad |\langle a_5\rangle_{\varphi,x}|^{k_0,\upsilon} \lesssim \varepsilon\upsilon^{-1}; \tag{6.96}$$

3. The block-diagonal operator $\mathbf{R}_7^{(-\frac{1}{2},d)} := \begin{pmatrix} r_7^{(d)}(\varphi,x,D) & 0 \\ 0 & \overline{r_7^{(d)}(\varphi,x,D)} \end{pmatrix} \in \mathrm{OP}S^{-\frac{1}{2}}$*, with* $r_7^{(d)}(\varphi, x, D)$ *defined in* (6.94)*, satisfies for finitely many* $0 \leqslant \alpha \leqslant \alpha(M)$ *(fixed in Remark 6.10), for some* $\sigma_M(\alpha) := \sigma_M(k_0, \tau, \nu, \alpha) > 0$ *and for all* $s_0 \leqslant s \leqslant S - \sigma_M(\alpha)$*,*

$$\|\mathbf{R}_7^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s,\alpha}^{k_0,\upsilon} \lesssim_{s,M,\alpha} \varepsilon\upsilon^{-2}(1 + \|\mathfrak{I}_0\|_{s+\sigma_M(\alpha)}^{k_0,\upsilon}); \tag{6.97}$$

4. For any $\mathtt{q} \in \mathbb{N}_0^\nu$ *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*,* $n_1, n_2 \in \mathbb{N}_0$ *with* $n_1 + n_2 \leqslant M - \frac{3}{2}(k_0 + \mathtt{q}_0) + \frac{3}{2}$*, the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{7,M}(\varphi)\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with tame constant satisfying, for some* $\sigma_M(\mathtt{q}_0) := \sigma_M(k_0, \tau, \nu, \mathtt{q}_0)$*, for any* $s_0 \leqslant s \leqslant S - \sigma_M(\mathtt{q}_0)$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{7,M}(\varphi)\langle D\rangle^{n_2}}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon\upsilon^{-2}(1 + \|\mathfrak{I}_0\|_{s+\sigma_M(\mathtt{q}_0)}^{k_0,\upsilon}); \tag{6.98}$$

5. The operator $\mathbf{Q}_7^\perp := \mathrm{i}\sqrt{g}(\Pi_{N_{\mathtt{n}}}^\perp a_2)|D|^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *where* $a_2(\varphi, x)$ *is defined in* (6.52) *and satisfies* (6.64)*;*

6. The operators $\mathbf{\Phi}^{\pm 1} - \mathrm{Id}$*,* $(\mathbf{\Phi}^{\pm 1} - \mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-*$\frac{1}{2}(k_0 + 1)$*-tame, with tame constants satisfying, for some* $\sigma > 0$ *and for all* $s_0 \leqslant s \leqslant S - \sigma$*,*

$$\mathfrak{M}_{\mathbf{\Phi}^{\pm 1} - \mathrm{Id}}(s) + \mathfrak{M}_{(\mathbf{\Phi}^{\pm 1} - \mathrm{Id})^*}(s) \lesssim_S \varepsilon\upsilon^{-2}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon}). \tag{6.99}$$

7. Furthermore, for any s_1 *as in* (6.9)*, finitely many* $0 \leqslant \alpha \leqslant \alpha(M)$*,* $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, and* $n_1, n_2 \in \mathbb{N}_0$*, with* $n_1 + n_2 \leqslant M - \frac{3}{2}\mathtt{q}_0$*, we have*

$$\|\Delta_{12} a_5\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-2}\|i_1 - i_2\|_{s_1+\sigma}, \quad |\Delta_{12}\mathtt{m}_{\frac{1}{2}}| \lesssim \varepsilon\upsilon^{-1}\|i_1 - i_2\|_{s_0+\sigma}, \tag{6.100}$$

$$\|\Delta_{12}\mathbf{R}_7^{(-\frac{1}{2},d)}\|_{-\frac{1}{2},s_1,\alpha} \lesssim_{s_1,M,\alpha} \varepsilon\upsilon^{-2}\|i_1 - i_2\|_{s_1+\sigma_M(\alpha)}, \tag{6.101}$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{7,M}\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,\mathtt{q}_0} \varepsilon\upsilon^{-2}\|i_1 - i_2\|_{s_1+\sigma_M(\mathtt{q}_0)}, \tag{6.102}$$

$$\|\Delta_{12}(\mathcal{A})h\|_{s_1} \lesssim_{s_1} \varepsilon\upsilon^{-2}\|i_1 - i_2\|_{s_1+\sigma}\|h\|_{s_1+\sigma}, \quad \mathcal{A} \in \{\mathbf{\Phi}^{\pm 1}, (\mathbf{\Phi}^{\pm 1})^*\}. \tag{6.103}$$

Proof. The estimate $|\mathtt{m}_{\frac{1}{2}} - 1|^{k_0,\upsilon} \lesssim \varepsilon\upsilon^{-1}$ follows by (6.91) and (6.64). The function $b(\varphi, x)$ defined in (6.92) satisfies, by (3.7) and (6.64), $\|b\|_s^{k_0,\upsilon} \lesssim_s \varepsilon\upsilon^{-2}(1 + \|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})$, for some $\sigma > 0$ and for all $s_0 \leqslant s \leqslant S - \sigma$. Thus, the estimate (6.96) is deduced by (6.93), $|\mathtt{m}_{\frac{1}{2}} - 1|^{k_0,\upsilon} \lesssim \varepsilon\upsilon^{-1}$, (6.64), (6.8). The estimate (6.97) follows by (6.94), (6.84), Lemma 3.6, the estimate for $\|b\|_s^{k_0,\upsilon}$, and (6.74), (6.64), (6.66). The smoothing term $\mathbf{T}_{7,M}$ in (6.95) is, using that $\mathbf{\Phi}^{-1}\mathbf{\Pi}_0\mathbf{\Phi} = \mathbf{\Pi}_0\mathbf{\Phi}$, $\mathbf{T}_{7,M} := \mathbf{\Phi}^{-1}\mathbf{T}_{6,M}\mathbf{\Phi} + \mathrm{i}\mathbf{\Pi}_0(\mathbf{\Phi} - \mathrm{Id}) + \mathbf{\Phi}^{-1}\mathbf{R}_6^{(-M,o)}\mathbf{\Phi} + \begin{pmatrix} T_M+T_M' & 0 \\ 0 & \overline{T_M+T_M'} \end{pmatrix}$ with T_M and T_M' defined in (6.86), (6.87). The estimate (6.99) follows by Lemma 2.38 in [2] and the estimate for $\|b\|_s^{k_0,\upsilon}$. The estimate (6.98) follows by (6.84), Lemmata 3.9, 3.10, the tame estimates of $\mathbf{\Phi}$ in Proposition 2.37 in [2], and (6.66), (6.99), (6.75). The estimates (6.100), (6.101), (6.102), (6.103) are proved similarly, using also (3.8). □

6.6 Reduction of the order 0

The goal of this section is to transform the operator $\mathcal{L}_7$ in (6.95) into the operator $\mathcal{L}_8$ in (6.113) whose coefficient in front of the Hilbert transform $\mathcal{H}$ is a real constant. From now on, we neglect the contribution of $\mathbf{Q}_7^{\perp}$ in (6.95) which will be conjugated in Section 6.7. For simplicity of notation we denote such operator $\mathcal{L}_7$ as well. We first write $\mathcal{L}_7 = \omega\cdot\partial_\varphi + \begin{pmatrix} P_7 & 0 \\ 0 & \overline{P_7} \end{pmatrix} + \mathrm{i}\mathbf{\Pi}_0 + \mathbf{T}_{7,M}$, where

$$P_7 := \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i}\mathtt{m}_{\frac{1}{2}}\Omega(\gamma, D) + a_5(\varphi,x)\mathcal{H} + \mathrm{Op}(r_7^{(d)}) . \tag{6.104}$$

We conjugate $\mathcal{L}_7$ through the time-1 flow $\Psi(\varphi) := \Psi^\tau(\varphi)|_{\tau=1}$ generated by

$$\partial_\tau\Psi^\tau(\varphi) = B(\varphi)\Psi^\tau(\varphi)\,,\ \Psi^0(\varphi) = \mathrm{Id}\,, \quad B(\varphi) := b_1(\varphi,x)\mathcal{H}\,, \tag{6.105}$$

where $b_1(\varphi,x)$ is a real quasi-periodic traveling wave $\mathrm{odd}(\varphi,x)$ chosen later (see (6.111)) and $\mathcal{H}$ is the Hilbert transform in (3.11). Thus by Lemmata 3.15, 3.17 the operator $b_1(\varphi,x)\mathcal{H}$ is reversibility and momentum preserving and so is its flow $\Psi^\tau(\varphi)$. Note that, since $\mathcal{H}(1) = 0$, we have $\Psi(\varphi)\pi_0 = \pi_0 = \Psi^{-1}(\varphi)\pi_0$. By the Lie expansion (see (3.16)-(3.17) in [7]), we have

$$\begin{aligned} \Psi^{-1}P_7\Psi &= P_7 - [B,P_7] + \sum_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}^n_{B(\varphi)}(P_7) + L_M\,, \\ L_M &:= \frac{(-1)^{M+1}}{M!}\int_0^1(1-\tau)^M\Psi^{-\tau}(\varphi)\,\mathrm{ad}^{M+1}_{B(\varphi)}(P_7)\,\Psi^\tau(\varphi)\mathrm{d}\tau\,, \end{aligned} \tag{6.106}$$

and

$$\Psi^{-1}\circ \omega\cdot\partial_\varphi \circ\Psi{=}\omega\cdot\partial_\varphi + (\omega\cdot\partial_\varphi B(\varphi)) - \sum\nolimits_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}_{B(\varphi)}^{n-1}(\omega\cdot\partial_\varphi B(\varphi)) + L_M' \,,$$

$$L_M' {:=} \frac{(-1)^M}{M!}\int_0^1 (1-\tau)^M \Psi^{-\tau}(\varphi)\,\mathrm{ad}_{B(\varphi)}^{M}(\omega\cdot\partial_\varphi B(\varphi))\,\Psi^{\tau}(\varphi)\mathrm{d}\tau \,. \tag{6.107}$$

The number M will be fixed in (7.2). The contributions at order 0 come from $(\omega\cdot\partial_\varphi B) + P_7 - [B, P_7]$. Since $B = b_1\mathcal{H}$, by (6.104), (3.15) and (6.88) we have

$$[B, P_7] = -\mathtt{m}_{1,\overline{\mathtt{n}}}(b_1)_x\mathcal{H} + \mathrm{Op}(r_{b_1,-\frac12}) \,, \tag{6.108}$$

where $\mathrm{Op}(r_{b_1,-\frac12}) \in \mathrm{OP}S^{-\frac12}$ is small with b_1. As a consequence, the 0 order term of the operator $\omega\cdot\partial_\varphi B + P_7 - [B, P_7]$ is $\big(\omega\cdot\partial_\varphi b_1 + \mathtt{m}_{1,\overline{\mathtt{n}}}(b_1)_x + a_5\big)\mathcal{H}$. We choose b_1 as the solution of

$$(\omega\cdot\partial_\varphi b_1 + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x) b_1 + \Pi_{N_{\overline{\mathtt{n}}}} a_5 = \mathtt{m}_0 \tag{6.109}$$

where $\mathtt{m}_0$ is the average (see (3.2))

$$\mathtt{m}_0 := \langle a_5\rangle_{\varphi,x} \,. \tag{6.110}$$

We define $b_1(\varphi,x)$ to be the real, $\mathrm{odd}(\varphi,x)$, quasi-periodic traveling wave

$$b_1(\varphi,x) := -(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x)_{\mathrm{ext}}^{-1}\big(\Pi_{N_{\overline{\mathtt{n}}}} a_5(\varphi,x) - \mathtt{m}_0\big) \,, \tag{6.111}$$

recall (3.5). Note that $b_1(\varphi,x)$ is defined for any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$ and that, for any $(\omega,\gamma) \in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$ defined in (6.22), it solves (6.109).

We deduce by (6.106)-(6.107) and (6.108), (6.111), that

$$L_8 {:=} \Psi^{-1}(\varphi)\big(\omega\cdot\partial_\varphi + P_7\big)\Psi(\varphi)$$

is, for any $(\omega,\gamma) \in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$,

$$L_8 = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i}\,\mathtt{m}_{\frac12}\Omega(\gamma,D) + \mathtt{m}_0\mathcal{H} + \mathrm{Op}(r_8^{(d)}) + L_M + L_M' + (\Pi_{N_{\overline{\mathtt{n}}}}^{\perp} a_5)\mathcal{H},$$

where

$$\begin{aligned}\mathrm{Op}(r_8^{(d)}) := \mathrm{Op}(-r_{b_1,-\frac12} + r_7^{(d)}) + \sum\nolimits_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}_{B(\varphi)}^{n}(P_7)\\ - \sum\nolimits_{n=2}^{M}\frac{(-1)^n}{n!}\mathrm{ad}_{B(\varphi)}^{n-1}(\omega\cdot\partial_\varphi B(\varphi)) \in \mathrm{OP}S^{-\frac12} \,.\end{aligned} \tag{6.112}$$

In conclusion we have the following lemma.

Lemma 6.9. *Let $M \in \mathbb{N}$, $\mathtt{q}_0 \in \mathbb{N}_0$. Let b_1 be the quasi-periodic traveling wave defined in* (6.111). *Then, for any $\overline{\mathtt{n}} \in \mathbb{N}_0$, conjugating the operator $\mathcal{L}_7$ in* (6.95) *via the invertible, real, reversibility and momentum preserving* *map* $\Psi(\varphi)$ *(cfr.* (6.105)*), we obtain, for any* $(\omega,\gamma)\in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon,\tau)$*, the real, reversible and momentum preserving operator*

$$\begin{aligned}\mathcal{L}_8 &:= \Psi^{-1}\mathcal{L}_7\Psi = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\overline{\mathtt{n}}}\partial_x + \mathrm{i}\,\mathtt{m}_{\frac12}\boldsymbol{\Omega}(\gamma,D) + \mathtt{m}_0\mathcal{H} \\ &\quad + \mathrm{i}\boldsymbol{\Pi}_0 + \mathbf{R}_8^{(-\frac12,d)} + \mathbf{T}_{8,M} + \mathbf{Q}_8^{\perp}\,,\end{aligned} \tag{6.113}$$

defined for any $(\omega,\gamma)\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$*, where*

1. The constant $\mathtt{m}_0$ *defined in* (6.110) *satisfies* $|\mathtt{m}_0|^{k_0,\upsilon}\lesssim \varepsilon\upsilon^{-1}$*;*

2. The block-diagonal operator $\mathbf{R}_8^{(-\frac12,d)} = \begin{pmatrix} r_8^{(d)}(\varphi,x,D) & 0 \\ 0 & \overline{r_8^{(d)}(\varphi,x,D)}\end{pmatrix}\in \mathrm{OP}S^{-\frac12}$*, with* $r_8^{(d)}(\varphi,x,D)$ *defined in* (6.112) *and, for some* $\sigma_M:=\sigma_M(k_0,\tau,\nu)>0$ *and for all* $s_0\leqslant s\leqslant S-\sigma_M$*, satisfies*

$$\|\mathbf{R}_8^{(-\frac12,d)}\|_{-\frac12,s,1}^{k_0,\upsilon}\lesssim_{s,M}\varepsilon\upsilon^{-3}(1+\|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon})\,; \tag{6.114}$$

3. For any $\mathtt{q}\in\mathbb{N}_0^\nu$ *with* $|\mathtt{q}|\leqslant \mathtt{q}_0$*,* $n_1,n_2\in\mathbb{N}_0$ *with* $n_1+n_2\leqslant M-\frac32(k_0+\mathtt{q}_0)+\frac32$*, the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{8,M}(\varphi)\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame with tame constant satisfying, for some* $\sigma_M(\mathtt{q}_0):=\sigma_M(k_0,\tau,\nu,\mathtt{q}_0)$*, for any* $s_0\leqslant s\leqslant S-\sigma_M(\mathtt{q}_0)$*,*

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathbf{T}_{8,M}(\varphi)\langle D\rangle^{n_2}}(s)\lesssim_{S,M,\mathtt{q}_0}\varepsilon\upsilon^{-3}(1+\|\mathfrak{I}_0\|_{s+\sigma_M(\mathtt{q}_0)}^{k_0,\upsilon})\,; \tag{6.115}$$

4. The operator $\mathbf{Q}_8^\perp = (\Pi_{N_{\overline{\mathtt{n}}}}^\perp a_5)\mathcal{H}\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ *where* $a_5(\varphi,x)$ *is defined in* (6.93) *and satisfies* (6.96)*;*

5. The operators $\Psi^{\pm1}-\mathrm{Id}$*,* $(\Psi^{\pm1}-\mathrm{Id})^*$ *are* $\mathcal{D}^{k_0}$*-tame, with tame constants satisfying, for some* $\sigma:=\sigma(k_0,\tau,\nu)>0$ *and for all* $s_0\leqslant s\leqslant S-\sigma$*,*

$$\mathfrak{M}_{\Psi^{\pm1}-\mathrm{Id}}(s)+\mathfrak{M}_{(\Psi^{\pm1}-\mathrm{Id})^*}(s)\lesssim_s\varepsilon\upsilon^{-3}(1+\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})\,; \tag{6.116}$$

6. Furthermore, for any s_1 *as in* (6.9)*,* $\mathtt{q}\in\mathbb{N}_0^\nu$*, with* $|\mathtt{q}|\leqslant\mathtt{q}_0$*, and* $n_1,n_2\in\mathbb{N}_0$*, with* $n_1+n_2\leqslant M-\frac32\mathtt{q}_0$*,*

$$\|\Delta_{12}\mathbf{R}_8^{(-\frac12,d)}\|_{-\frac12,s_1,1}\lesssim_{s_1,M}\frac{\varepsilon}{\upsilon^3}\|i_1-i_2\|_{s_1+\sigma_M}\,,\ |\Delta_{12}\mathtt{m}_0|\lesssim\frac{\varepsilon}{\upsilon}\|i_1-i_2\|_{s_0+\sigma}\,, \tag{6.117}$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathbf{T}_{8,M}\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})}\lesssim_{s_1,M,\mathtt{q}_0}\varepsilon\upsilon^{-3}\|i_1-i_2\|_{s_1+\sigma_M(\mathtt{q}_0)}\,, \tag{6.118}$$

$$\|\Delta_{12}(\Psi^{\pm1})h\|_{s_1}+\|\Delta_{12}(\Psi^{\pm1})^*h\|_{s_1}\lesssim_{s_1}\varepsilon\upsilon^{-3}\|i_1-i_2\|_{s_1+\sigma}\|h\|_{s_1+\sigma}\,. \tag{6.119}$$

Proof. The estimate for $\mathtt{m}_0$ follows by (6.110) and (6.96). The function $b_1(\varphi,x)$ defined in (6.111), satisfies, by (6.96), (3.7), $\|b_1\|_s^{k_0,\upsilon}\lesssim_s\varepsilon\upsilon^{-3}(1+\|\mathfrak{I}_0\|_{s+\sigma}^{k_0,\upsilon})$ for some $\sigma>0$ and for any $s_0\leqslant s\leqslant S-\sigma$. The estimate (6.114) follows by (6.112), (6.104), Lemma 3.6, and (6.96), (6.97) and the estimate for $\|b_1\|_s^{k_0,\upsilon}$. Using that $\Psi(\varphi)\pi_0=\pi_0=\Psi^{-1}(\varphi)\pi_0$, the smoothing term $\mathbf{T}_{8,M}$ in (6.113) is $\mathbf{T}_{8,M} := \Psi^{-1}\mathbf{T}_{7,M}\Psi + \mathrm{i}\mathbf{\Pi}_0(\Psi - \mathrm{Id}) + \begin{pmatrix} L_M + L'_M & 0 \\ 0 & \overline{L_M + L'_M} \end{pmatrix}$ with L_M and L'_M introduced in (6.106), (6.107). The estimate (6.115) follows by Lemmata 3.9, 3.10, 3.7, (6.104), (6.96), (6.98), (6.116) and the estimate for $\|b_1\|_s^{k_0,\upsilon}$. The estimate (6.116) follows by Lemmata 3.7, 3.10 and the estimate for $\|b_1\|_s^{k_0,\upsilon}$. The estimates (6.117), (6.118), (6.119) are proved in the same fashion. □

Remark 6.10. *In Proposition 6.13 we shall estimate* $\|[\partial_x, \mathbf{R}_8^{(-\frac{1}{2},d)}]\|^{k_0,\upsilon}_{-\frac{1}{2},s,0}$ *using* (6.114) *and* (3.17). *In order to control* $\|\mathbf{R}_8^{(-\frac{1}{2},d)}\|^{k_0,\upsilon}_{-\frac{1}{2},s,1}$ *we used the estimates* (6.97) *for finitely many* $\alpha \in \mathbb{N}_0$, $\alpha \leqslant \alpha(M)$, *depending on* M, *as well similar estimates for* $\mathbf{R}_6^{(-\frac{1}{2},d)}$, $\mathbf{R}_5^{(-\frac{1}{2},d)}$, *etc. In Proposition 6.13 we shall use* (6.117)-(6.118) *only for* $s_1 = s_0$.

6.7 Conclusion: reduction of $\mathcal{L}_\omega$

By Sections 6.1-6.6, the linear operator $\mathcal{L}$ in (6.6) is conjugated, under the map

$$\mathcal{W} := \mathcal{Z}\mathcal{E}\widetilde{\mathcal{M}}\mathcal{Q}\mathcal{C}\mathbf{\Phi}_{2M}\mathbf{\Phi}\Psi\,, \tag{6.120}$$

for any $(\omega,\gamma) \in \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon,\tau)$, $\bar{\mathtt{n}} \in \mathbb{N}_0$, into the real, reversible and momentum preserving operator

$$\mathcal{W}^{-1}\mathcal{L}\mathcal{W} = \mathcal{L}_8 - \mathbf{Q}_8^{\perp} + \mathbf{P}_{\bar{\mathtt{n}}}^{\perp} + \mathbf{Q}_{\bar{\mathtt{n}}}^{\perp}\,, \tag{6.121}$$

where $\mathcal{L}_8$ is defined in (6.113), and

$$\mathbf{P}_{\bar{\mathtt{n}}}^{\perp} := \big(\widetilde{\mathcal{M}}\mathcal{Q}\mathcal{C}\mathbf{\Phi}_{2M}\mathbf{\Phi}\Psi\big)^{-1}\mathbf{P}_2^{\perp}\widetilde{\mathcal{M}}\mathcal{Q}\mathcal{C}\mathbf{\Phi}_{2M}\mathbf{\Phi}\Psi, \ \ \mathbf{Q}_{\bar{\mathtt{n}}}^{\perp} := \Psi^{-1}\mathbf{Q}_7^{\perp}\Psi + \mathbf{Q}_8^{\perp}\,, \tag{6.122}$$

with $\mathbf{P}_2^{\perp}$, $\mathbf{Q}_7^{\perp}$ and $\mathbf{Q}_8^{\perp}$ defined respectively in (6.23), Lemmata 6.8, 6.9. The operator $\mathcal{L}_8$ is defined for any $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$.

A similar conjugation result holds for the projected operator $\mathcal{L}_\omega$ in (5.40), i.e. (6.1), which acts in the normal subspace $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$. We denote by $\Pi^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\Pi^{\angle}_{\mathbb{S}^+,\Sigma}$ the projections on the subspaces $\mathfrak{H}^{\intercal}_{\mathbb{S}^+,\Sigma}$ and $\mathfrak{H}^{\angle}_{\mathbb{S}^+,\Sigma}$ and $\Pi^{\intercal}_{\mathbb{S}_0^+,\Sigma} := \Pi^{\intercal}_{\mathbb{S}^+,\Sigma} + \pi_0$, so that $\Pi^{\intercal}_{\mathbb{S}_0^+,\Sigma} + \Pi^{\angle}_{\mathbb{S}^+,\Sigma} = \mathrm{Id}$ on the whole $L^2 \times L^2$. We remind that $\mathbb{S}_0 = \mathbb{S} \cup \{0\}$, where $\mathbb{S}$ is the set defined in (2.19). We denote by $\Pi_{\mathbb{S}_0} := \Pi^{\intercal}_{\mathbb{S}} + \pi_0$, where $\Pi^{\intercal}_{\mathbb{S}}$ is defined below (2.26). We have $\Pi_{\mathbb{S}_0} + \Pi^{\perp}_{\mathbb{S}_0} = \mathrm{Id}$. Arguing as in Lemma 7.15 in [7] we have the following.

Lemma 6.11. *Let* $M > 0$. *There is* $\sigma_M > 0$ *(depending also on* k_0, τ, ν*) such that, assuming* (6.8) *with* $\mu_0 \geqslant \sigma_M$, *the following holds: the map* $\mathcal{W}$

defined in (6.120) *has the form* $\mathcal{W} = \widetilde{\mathcal{M}}\mathcal{C} + \mathcal{R}(\varepsilon)$ *where, for all* $s_0 \leqslant s \leqslant S - \sigma_M$, $\|\mathcal{R}(\varepsilon)h\|_s^{k_0,\upsilon} \lesssim_{S,M} \varepsilon\upsilon^{-3}\big(\|h\|_{s+\sigma_M}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon}\|h\|_{s_0+\sigma_M}^{k_0,\upsilon}\big)$. *Moreover* $\mathcal{W}^\perp := \Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{W}\Pi_{\mathbb{S}_0}^{\perp}$ *is invertible and, for all* $s_0 \leqslant s \leqslant S - \sigma_M$,

$$\begin{aligned} \|(\mathcal{W}^\perp)^{\pm1}h\|_s^{k_0,\upsilon} &\lesssim_{S,M} \|h\|_{s+\sigma_M}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon}\|h\|_{s_0+\sigma_M}^{k_0,\upsilon}\,, \\ \|\Delta_{12}(\mathcal{W}^\perp)^{\pm1}h\|_{s_1} &\lesssim_{s_1,M} \varepsilon\upsilon^{-3}\,\|i_1 - i_2\|_{s_1+\sigma_M}\,\|h\|_{s_1+\sigma_M}\,. \end{aligned} \tag{6.123}$$

The operator $\mathcal{W}^\perp$ *maps (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves.*

For any $(\omega,\gamma) \in \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon,\tau)$, $\bar{\mathtt{n}} \in \mathbb{N}_0$, the operator $\mathcal{L}_\omega$ in (5.40) (i.e. (6.1)) is conjugated via $\mathcal{W}^\perp$ to

$$\mathcal{L}_\perp := (\mathcal{W}^\perp)^{-1}\mathcal{L}_\omega\mathcal{W}^\perp = \Pi_{\mathbb{S}_0}^\perp\,(\mathcal{L}_8 - \mathbf{Q}_8^\perp)\,\Pi_{\mathbb{S}_0}^\perp + \mathbf{P}_{\perp,\bar{\mathtt{n}}} + \mathbf{Q}_{\perp,\bar{\mathtt{n}}} + \mathcal{R}^f, \tag{6.124}$$

where

$$\mathbf{P}_{\perp,\bar{\mathtt{n}}} := \Pi_{\mathbb{S}_0}^\perp\mathbf{P}_{\bar{\mathtt{n}}}^\perp\Pi_{\mathbb{S}_0}^\perp\,, \quad \mathbf{Q}_{\perp,\bar{\mathtt{n}}} := \Pi_{\mathbb{S}_0}^\perp\mathbf{Q}_{\bar{\mathtt{n}}}^\perp\Pi_{\mathbb{S}_0}^\perp, \tag{6.125}$$

and $\mathcal{R}^f$ is, by (6.121), Lemma 6.11 and (2.27), the finite rank operator

$$\begin{aligned} \mathcal{R}^f &:= (\mathcal{W}^\perp)^{-1}\Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{R}(\varepsilon)\Pi_{\mathbb{S}_0}\big(\mathcal{L}_8 - \mathbf{Q}_8^\perp + \mathbf{P}_{\bar{\mathtt{n}}}^\perp + \mathbf{Q}_{\bar{\mathtt{n}}}^\perp\big)\Pi_{\mathbb{S}_0}^\perp \\ &\quad - (\mathcal{W}^\perp)^{-1}\Pi_{\mathbb{S}^+,\Sigma}^{\angle}\mathcal{L}\Pi_{\mathbb{S}_0^+,\Sigma}^{\intercal}\mathcal{R}(\varepsilon)\Pi_{\mathbb{S}_0}^\perp - \varepsilon(\mathcal{W}^\perp)^{-1}\Pi_{\mathbb{S}^+,\Sigma}^{\angle}JR\mathcal{W}^\perp\,. \end{aligned} \tag{6.126}$$

Lemma 6.12. (Estimates of the remainders) *The operator* $\mathcal{R}^f$ *in* (6.126) *has the finite rank form* (6.3), (6.4). *Let* $\mathtt{q}_0 \in \mathbb{N}_0$ *and* $M \geqslant \frac{3}{2}(k_0 + \mathtt{q}_0) + \frac{3}{2}$. *There exists* $\aleph(M,\mathtt{q}_0) > 0$ *(depending also on* k_0, τ, ν*) such that, for any* $n_1, n_2 \in \mathbb{N}_0$, *with* $n_1 + n_2 \leqslant M - \frac{3}{2}(k_0 + \mathtt{q}_0) + \frac{3}{2}$, *and any* $\mathtt{q} \in \mathbb{N}_0^\nu$, *with* $|\mathtt{q}| \leqslant \mathtt{q}_0$, *the operator* $\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}^f\langle D\rangle^{n_2}$ *is* $\mathcal{D}^{k_0}$*-tame, with a tame constant satisfying, for any* $s_0 \leqslant s \leqslant S - \aleph(M,\mathtt{q}_0)$ *and any* s_1 *as in* (6.9),

$$\mathfrak{M}_{\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\mathcal{R}^f\langle D\rangle^{n_2}}(s) \lesssim_{S,M,\mathtt{q}_0} \varepsilon\upsilon^{-3}\big(1 + \|\mathfrak{I}_0\|_{s+\aleph(M,\mathtt{q}_0)}^{k_0,\upsilon}\big)\,, \tag{6.127}$$

$$\|\langle D\rangle^{n_1}\partial_\varphi^{\mathtt{q}}\Delta_{12}\mathcal{R}^f\langle D\rangle^{n_2}\|_{\mathcal{L}(H^{s_1})} \lesssim_{s_1,M,\mathtt{q}_0} \frac{\varepsilon}{\upsilon^3}\|i_1 - i_2\|_{s_1+\aleph(M,\mathtt{q}_0)}\,. \tag{6.128}$$

The operators $\mathbf{P}_{\perp,\bar{\mathtt{n}}}$ *and* $\mathbf{Q}_{\perp,\bar{\mathtt{n}}}$ *in* (6.125), (6.122) *satisfy, for some* $\sigma_M = \sigma_M(k_0,\tau,\nu) > 0$, *for all* $s_0 \leqslant s \leqslant S - \sigma_M$,

$$\|\mathbf{P}_{\perp,\bar{\mathtt{n}}}h\|_s^{k_0,\upsilon} \lesssim_S \varepsilon N_{\bar{\mathtt{n}}-1}^{-\mathtt{a}}\big(\|h\|_{s+\sigma_M}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M+\mathtt{b}}^{k_0,\upsilon}\|h\|_{s_0+\sigma_M}^{k_0,\upsilon}\big)\,, \tag{6.129}$$

$$\|\mathbf{Q}_{\perp,\bar{\mathtt{n}}}h\|_{s_0}^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-2}N_{\bar{\mathtt{n}}}^{-\mathrm{b}}\big(1 + \|\mathfrak{I}_0\|_{s_0+\sigma_M+\mathrm{b}}^{k_0,\upsilon}\big)\|h\|_{s_0+\frac{1}{2}}^{k_0,\upsilon}\,, \ \forall\, \mathrm{b} > 0, \tag{6.130}$$

$$\|\mathbf{Q}_{\perp,\bar{\mathtt{n}}}h\|_s^{k_0,\upsilon} \lesssim_S \varepsilon\upsilon^{-2}\big(\|h\|_{s+\frac{1}{2}}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\sigma_M}^{k_0,\upsilon}\|h\|_{s_0+\frac{1}{2}}^{k_0,\upsilon}\big)\,. \tag{6.131}$$

Proof. The estimate (6.127) follows by (6.126), (6.120), Lemma 6.11, (6.113), (6.3), (3.3), (6.123), (6.114), (6.115), (6.4). The estimate (6.128) follows similarly. The estimates (6.129), (6.130), (6.131) follow from (6.125), (6.122), (6.23), the definitions of $\mathbf{Q}_7^\perp$, $\mathbf{Q}_8^\perp$ using the estimates (6.24), (6.64), (6.96), (3.4), (6.123), (6.116), (6.99), (6.76), (6.69). □

The next proposition summarizes the main result of this section.

Proposition 6.13. (Reduction of $\mathcal{L}_\omega$ up to smoothing operators) *For any* $\overline{\mathtt{n}} \in \mathbb{N}_0$ *and for all* $(\omega, \gamma) \in \mathtt{TC}_{\overline{\mathtt{n}}+1}(2\upsilon, \tau)$ *(cfr.* (6.22)*), the operator* $\mathcal{L}_\omega$ *in* (5.40) *(i.e.* (6.1)*) is conjugated as in* (6.124) *to the real, reversible and momentum preserving operator* $\mathcal{L}_\perp$*. For all* $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$ *the extended operator defined by the right hand side in* (6.124)*, has the form*

$$\mathcal{L}_\perp = \omega \cdot \partial_\varphi \mathbb{I}_\perp + \mathrm{i}\, \mathbf{D}_\perp + \mathbf{R}_\perp + \mathbf{P}_{\perp,\overline{\mathtt{n}}} + \mathbf{Q}_{\perp,\overline{\mathtt{n}}}\,, \tag{6.132}$$

where $\mathbb{I}_\perp$ *denotes the identity map of* $\mathbf{H}^\perp_{\mathbb{S}_0}$ *(cfr.* (2.26)*) and:*

1. $\mathbf{D}_\perp$ *is the diagonal operator*

$$\mathbf{D}_\perp := \begin{pmatrix} \mathcal{D}_\perp & 0 \\ 0 & -\overline{\mathcal{D}_\perp} \end{pmatrix}, \ \mathcal{D}_\perp := \operatorname{diag}_{j \in \mathbb{S}_0^c} \mu_j\,, \ \mathbb{S}_0^c := \mathbb{Z} \backslash (\mathbb{S} \cup \{0\})\,, \tag{6.133}$$

with eigenvalues $\mu_j := \mathtt{m}_{1,\overline{\mathtt{n}}} j + \mathtt{m}_{\frac12} \Omega_j(\gamma) - \mathtt{m}_0 \operatorname{sgn}(j) \in \mathbb{R}$*, where* $\Omega_j(\gamma)$ *is the dispersion relation* (1.8) *and the real constants* $\mathtt{m}_{1,\overline{\mathtt{n}}}, \mathtt{m}_{\frac12}, \mathtt{m}_0$*, defined respectively in Lemma 6.3,* (6.91)*,* (6.110)*, satisfy*

$$|\mathtt{m}_{1,\overline{\mathtt{n}}}|^{k_0,\upsilon} \lesssim \varepsilon\,, \quad |\mathtt{m}_{\frac12} - 1|^{k_0,\upsilon} + |\mathtt{m}_0|^{k_0,\upsilon} \lesssim \varepsilon \upsilon^{-1}\,. \tag{6.134}$$

In addition, for some $\sigma > 0$*,*

$$|\Delta_{12} \mathtt{m}_{1,\overline{\mathtt{n}}}| \lesssim \varepsilon \|i_1 - i_2\|_{s_0+\sigma}\,, \ |\Delta_{12} \mathtt{m}_{\frac12}| + |\Delta_{12} \mathtt{m}_0| \lesssim \varepsilon \upsilon^{-1} \|i_1 - i_2\|_{s_0+\sigma}\,; \tag{6.135}$$

2. For any $\mathtt{q}_0 \in \mathbb{N}_0$*,* $M > \frac32 (k_0 + \mathtt{q}_0) + \frac32$*, there is a constant* $\aleph(M, \mathtt{q}_0) > 0$ *(depending also on* k_0*,* τ*,* ν*) such that, assuming* (6.8) *with* $\mu_0 \geqslant \aleph(M, \mathtt{q}_0)$*, for any* $s_0 \leqslant s \leqslant S - \aleph(M, \mathtt{q}_0)$*,* $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*, the operators* $\partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp$*,* $[\partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp, \partial_x]$ *are* $\mathcal{D}^{k_0}$*-tame with tame constants satisfying*

$$\mathfrak{M}_{\langle D\rangle^{\frac14} \partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp \langle D\rangle^{\frac14}}(s),\ \mathfrak{M}_{\langle D\rangle^{\frac14} [\partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp, \partial_x] \langle D\rangle^{\frac14}}(s) \lesssim_{S,M,\mathtt{q}_0} \frac{\varepsilon}{\upsilon^3} (1 + \|\mathfrak{I}_0\|^{k_0,\upsilon}_{s+\aleph(M,\mathtt{q}_0)}). \tag{6.136}$$

Moreover, for any $\mathtt{q} \in \mathbb{N}_0^\nu$*, with* $|\mathtt{q}| \leqslant \mathtt{q}_0$*,*

$$\begin{aligned} &\|\langle D\rangle^{\frac14} \partial_\varphi^{\mathtt{q}} \Delta_{12} \mathbf{R}_\perp \langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} + \|\langle D\rangle^{\frac14} \partial_\varphi^{\mathtt{q}} \Delta_{12} [\mathbf{R}_\perp, \partial_x] \langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} \\ &\lesssim_M \varepsilon \upsilon^{-3} \|i_1 - i_2\|_{s_0+\aleph(M,\mathtt{q}_0)}\,. \end{aligned} \tag{6.137}$$

The operator $\mathbf{R}_\perp := \mathbf{R}_\perp(\varphi)$ *is real, reversible and momentum preserving.*

3. The remainders $\mathbf{P}_{\perp,\mathtt{n}}, \mathbf{Q}_{\perp,\mathtt{n}}$ *are defined in* (6.125) *and satisfy the estimates* (6.129)-(6.131).

Proof. By (6.124) and (6.113) we deduce (6.132) with

$$\mathbf{R}_\perp := \Pi_{\mathbb{S}_0}^\perp (\mathbf{R}_8^{(-\frac{1}{2},d)} + \mathbf{T}_{8,M})\Pi_{\mathbb{S}_0}^\perp + \mathcal{R}^f .$$

The estimates (6.134)-(6.135) follow by Lemmata 6.6, 6.8, 6.9. The estimate (6.136) follows by Lemmata 3.6, 3.10 and (6.114), (6.115), (6.127), choosing $(n_1, n_2) = (1,2),(2,1)$. The estimate (6.137) follows similarly. □

7 Almost-invertibility of $\mathcal{L}_\omega$ and proof of Theorem 5.1

In this section we almost-diagonalize the operator $\omega \cdot \partial_\varphi \mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_\perp + \mathbf{R}_\perp(\varphi)$ obtained neglecting from $\mathcal{L}_\perp$ in (6.132) the remainders $\mathbf{P}_{\perp,\mathtt{n}}$ and $\mathbf{Q}_{\perp,\mathtt{n}}$, by a KAM iterative scheme, see Theorem 7.2. Then we deduce the decomposition (5.42) of the operator $\mathcal{L}_\omega$ in the almost-invertibility assumption (AI) of Section 5.3. Finally, we state Theorem 7.7, which implies Theorem 5.1.

Almost-diagonalization

We start with the real, reversible and momentum preserving operator $\mathcal{L}_\perp =: \mathbf{L}_0 := \mathbf{L}_0(i) := \omega \cdot \partial_\varphi \mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_0 + \mathbf{R}_\perp^{(0)}$, acting in $\mathbf{H}_{\mathbb{S}_0}^\perp$ and defined for all $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, with $\mathcal{D}_0 := \mathcal{D}_\perp$ as in (6.133) and

$$\mathbf{R}_\perp^{(0)} := \mathbf{R}_\perp := \begin{pmatrix} R_\perp^{(0,d)} & R_\perp^{(0,o)} \\ \overline{R_\perp^{(0,o)}} & \overline{R_\perp^{(0,d)}} \end{pmatrix}, \qquad \begin{matrix} R_\perp^{(0,d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp , \\ R_\perp^{(0,o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp , \end{matrix} \tag{7.1}$$

which satisfies (6.136), (6.137). We denote

$$H_{\pm\mathbb{S}_0}^\perp = \{h(x) = \textstyle\sum_{j\notin\pm\mathbb{S}_0} h_j e^{\pm\mathrm{i}jx} \in L^2\}.$$

Note that $\overline{\mathcal{D}_0} : H_{-\mathbb{S}_0}^\perp \to H_{-\mathbb{S}_0}^\perp$, where $\overline{\mathcal{D}_0} = \overline{\mathcal{D}_\perp} = \mathrm{diag}_{j\in -\mathbb{S}_0^c}(\mu_{-j}^{(0)})$ as in (6.133). Proposition 6.13 implies that $\mathbf{R}_\perp^{(0)}$ satisfies the estimates (7.4)-(7.5) below by fixing the constant M large enough, namely

$$M := \left[\tfrac{3}{2}(k_0 + s_0 + \mathtt{b}) + \tfrac{3}{2}\right] + 1 \in \mathbb{N}, \tag{7.2}$$

where $\mathtt{b}$ is defined in (6.20). We also set

$$\mu(\mathtt{b}) := \aleph(M, s_0 + \mathtt{b}), \tag{7.3}$$

where the constant $\aleph(M, \mathtt{q}_0)$ is given in Proposition 6.13, with $\mathtt{q}_0 = s_0 + \mathtt{b}$. We define

$$\begin{aligned}\mathbb{M}_0(s) &:= \max_{m=1,\ldots,\nu} \big\{ \mathfrak{M}_{\langle D\rangle^{\frac14} \mathbf{R}_\perp^{(0)} \langle D\rangle^{\frac14}}(s),\ \mathfrak{M}_{\langle D\rangle^{\frac14} [\mathbf{R}_\perp^{(0)}, \partial_x] \langle D\rangle^{\frac14}}(s), \\ &\qquad \mathfrak{M}_{\langle D\rangle^{\frac14} \partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)} \langle D\rangle^{\frac14}}(s),\ \mathfrak{M}_{\langle D\rangle^{\frac14} [\partial_{\varphi_m}^{s_0} \mathbf{R}_\perp^{(0)}, \partial_x] \langle D\rangle^{\frac14}}(s) \big\}, \\ \mathbb{M}_0(s,\mathtt{b}) &:= \max_{m=1,\ldots,\nu} \big\{ \mathfrak{M}_{\langle D\rangle^{\frac14} \partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)} \langle D\rangle^{\frac14}}(s),\ \mathfrak{M}_{\langle D\rangle^{\frac14} [\partial_{\varphi_m}^{s_0+\mathtt{b}} \mathbf{R}_\perp^{(0)}, \partial_x] \langle D\rangle^{\frac14}}(s) \big\}.\end{aligned}$$

Then, assuming (6.8) with $\mu_0 \geqslant \mu(\mathtt{b})$, by (6.136), (7.2), (7.3), (6.137), we have, for all $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$,

$$\begin{aligned}&\mathfrak{M}_0(s,\mathtt{b}) := \max\Big\{ \mathbb{M}_0(s), \mathbb{M}_0(s,\mathtt{b}) \Big\} \leqslant C(S)\varepsilon\upsilon^{-3}(1 + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})}^{k_0,\upsilon}), \\ &\mathfrak{M}_0(s_0,\mathtt{b}) \leqslant C(S)\varepsilon\upsilon^{-3}.\end{aligned} \tag{7.4}$$

Moreover, for all $\mathtt{q} \in \mathbb{N}_0^\nu$, with $|\mathtt{q}| \leqslant s_0 + \mathtt{b}$,

$$\begin{aligned}&\|\langle D\rangle^{\frac14} \partial_\varphi^{\mathtt{q}} \Delta_{12} \mathbf{R}_\perp^{(0)} \langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})},\ \|\langle D\rangle^{\frac14} \Delta_{12} [\partial_\varphi^{\mathtt{q}} \mathbf{R}_\perp^{(0)}, \partial_x] \langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} \\ &\qquad\qquad \leqslant C(S)\varepsilon\upsilon^{-3} \|i_1 - i_2\|_{s_0+\mu(\mathtt{b})}.\end{aligned} \tag{7.5}$$

We perform the almost-reducibility of $\mathbf{L}_0$ along the scale $(N_{\mathtt{n}})_{\mathtt{n}\in\mathbb{N}_0}$, see (6.17).

Theorem 7.1. (Almost-diagonalization of $\mathbf{L}_0$: KAM iteration) *There exists $\tau_2(\tau,\nu) > \tau_1(\tau,\nu) + 1 + \mathtt{a}$ (with $\tau_1, \mathtt{a}$ defined in* (6.20)*) such that, for all $S > s_0$, there is $N_0 := N_0(S,\mathtt{b}) \in \mathbb{N}$ such that, if*

$$N_0^{\tau_2} \mathfrak{M}_0(s_0,\mathtt{b}) \upsilon^{-1} \leqslant 1, \tag{7.6}$$

then, for all $\overline{\mathtt{n}} \in \mathbb{N}_0$, $\mathtt{n} = 0, 1, \ldots, \overline{\mathtt{n}}$:

$(\mathbf{S1})_{\mathtt{n}}$ *There exists a real, reversible and momentum preserving operator*

$$\mathbf{L}_{\mathtt{n}} := \omega\cdot\partial_\varphi \mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_{\mathtt{n}} + \mathbf{R}_\perp^{(\mathtt{n})}, \quad \mathbf{D}_{\mathtt{n}} := \begin{pmatrix} \mathcal{D}_{\mathtt{n}} & 0 \\ 0 & -\overline{\mathcal{D}_{\mathtt{n}}} \end{pmatrix}, \tag{7.7}$$

where $\mathcal{D}_{\mathtt{n}} := \mathrm{diag}_{j\in\mathbb{S}_0^c}\, \mu_j^{(\mathtt{n})}$, defined for all (ω,γ) in $\mathbb{R}^\nu \times [\gamma_1,\gamma_2]$, where $\mu_j^{(\mathtt{n})}$ are k_0-times differentiable real functions

$$\begin{aligned}\mu_j^{(\mathtt{n})}(\omega,\gamma) &:= \mu_j^{(0)}(\omega,\gamma) + \mathfrak{r}_j^{(\mathtt{n})}(\omega,\gamma), \\ \mu_j^{(0)} &= \mathtt{m}_{1,\overline{\mathtt{n}}}\, j + \mathtt{m}_{\frac12}\, \Omega_j(\gamma) - \mathtt{m}_0\, \mathrm{sgn}(j),\end{aligned} \tag{7.8}$$

satisfying $\mathfrak{r}_j^{(0)} = 0$ and, for $\mathtt{n} \geqslant 1$ and any $j \in \mathbb{S}_0^c$

$$|j|^{\frac12} |\mathfrak{r}_j^{(\mathtt{n})}|^{k_0,\upsilon} \leqslant C(S,\mathtt{b})\varepsilon\upsilon^{-3}, \quad |j|^{\frac12} |\mu_j^{(\mathtt{n})} - \mu_j^{(\mathtt{n}-1)}|^{k_0,\upsilon} \leqslant C(S,\mathtt{b})\varepsilon\upsilon^{-3} N_{\mathtt{n}-2}^{-\mathtt{a}}. \tag{7.9}$$

The remainder $\mathbf{R}_\perp^{(\mathtt{n})} := \begin{pmatrix} R_\perp^{(\mathtt{n},d)} & R_\perp^{(\mathtt{n},o)} \\ \overline{R_\perp^{(\mathtt{n},o)}} & \overline{R_\perp^{(\mathtt{n},d)}} \end{pmatrix}$ *with* $R_\perp^{(\mathtt{n},d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp$, $R_\perp^{(\mathtt{n},o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp$, *and the operator* $\langle \partial_\varphi \rangle^{\mathtt{b}} \mathbf{R}_\perp^{(\mathtt{b})}$ *are* $\mathcal{D}^{k_0}$-$(-\frac{1}{2})$*-modulo-tame, with modulo-tame constants*

$$\mathfrak{M}_{\mathtt{n}}^\sharp(s) := \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\mathbf{R}_\perp^{(\mathtt{n})}\langle D\rangle^{\frac14}}(s)\,, \quad \mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) := \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{R}_\perp^{(\mathtt{n})}\langle D\rangle^{\frac14}}(s)\,, \tag{7.10}$$

which satisfy, for some constant $C_*(s_0,\mathtt{b}) > 0$*, for all* $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$*,*

$$\begin{aligned} \mathfrak{M}_{\mathtt{n}}^\sharp(s) &\leqslant C_*(s_0,\mathtt{b})\mathfrak{M}_0(s,\mathtt{b}) N_{\mathtt{n}-1}^{-\mathtt{a}}\,, \\ \mathfrak{M}_{\mathtt{n}}^\sharp(s,\mathtt{b}) &\leqslant C_*(s_0,\mathtt{b})\mathfrak{M}_0(s,\mathtt{b}) N_{\mathtt{n}-1}\,. \end{aligned} \tag{7.11}$$

Define the sets $\Lambda_{\mathtt{n}}^\upsilon = \Lambda_{\mathtt{n}}^\upsilon(i)$ *by* $\Lambda_0^\upsilon := \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$ *and, for* $\mathtt{n} = 1,\ldots,\overline{\mathtt{n}}$*,*

$$\begin{aligned} \Lambda_{\mathtt{n}}^\upsilon := \big\{ & \lambda = (\omega,\gamma) \in \Lambda_{\mathtt{n}-1}^\upsilon \ : \ \big|\omega\cdot\ell + \mu_j^{(\mathtt{n}-1)} - \mu_{j'}^{(\mathtt{n}-1)}\big| \geqslant \upsilon\langle\ell\rangle^{-\tau} \\ & \forall\ |\ell| \leqslant N_{\mathtt{n}-1}\,,\ j,j' \notin \mathbb{S}_0\,,\ (\ell,j,j') \neq (0,j,j),\ \text{with } \vec{\jmath}\cdot\ell + j - j' = 0\,, \\ & \big|\omega\cdot\ell + \mu_j^{(\mathtt{n}-1)} + \mu_{j'}^{(\mathtt{n}-1)}\big| \geqslant \upsilon\,\big(|j|^{\frac12} + |j'|^{\frac12}\big)\langle\ell\rangle^{-\tau} \\ & \forall\ |\ell| \leqslant N_{\mathtt{n}-1}\,,\ j,j' \notin \mathbb{S}_0 \text{ with } \vec{\jmath}\cdot\ell + j + j' = 0 \big\}\,. \end{aligned} \tag{7.12}$$

For $\mathtt{n} \geqslant 1$ *there exists a real, reversibility and momentum preserving map, defined for all* $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$*, of the form* $\mathbf{\Phi}_{\mathtt{n}-1} = e^{\mathbf{X}_{\mathtt{n}-1}}$*, where* $\mathbf{X}_{\mathtt{n}-1} := \begin{pmatrix} X_{\mathtt{n}-1}^{(d)} & X_{\mathtt{n}-1}^{(o)} \\ \overline{X_{\mathtt{n}-1}^{(o)}} & \overline{X_{\mathtt{n}-1}^{(d)}} \end{pmatrix}$ *and the operators* $X_{\mathtt{n}-1}^{(d)} : H_{\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp$, $X_{\mathtt{n}-1}^{(o)} : H_{-\mathbb{S}_0}^\perp \to H_{\mathbb{S}_0}^\perp$*, such that, for all* $\lambda \in \Lambda_{\mathtt{n}}^\upsilon$*, the following conjugation formula holds:*

$$\mathbf{L}_{\mathtt{n}} = \mathbf{\Phi}_{\mathtt{n}-1}^{-1}\mathbf{L}_{\mathtt{n}-1}\mathbf{\Phi}_{\mathtt{n}-1}\,. \tag{7.13}$$

The operators $\mathbf{X}_{\mathtt{n}-1}$, $\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}_{\mathtt{n}-1}$ *are* $\mathcal{D}^{k_0}$-$(-\frac12)$*-modulo-tame satisfying, for all* $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$*,*

$$\begin{aligned} \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\mathbf{X}_{\mathtt{n}-1}\langle D\rangle^{\frac14}}(s) &\leqslant C(s_0,\mathtt{b})\upsilon^{-1}N_{\mathtt{n}-1}^{\tau_1}N_{\mathtt{n}-2}^{-\mathtt{a}}\mathfrak{M}_0(s,\mathtt{b})\,, \\ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}_{\mathtt{n}-1}\langle D\rangle^{\frac14}}(s) &\leqslant C(s_0,\mathtt{b})\upsilon^{-1}N_{\mathtt{n}-1}^{\tau_1}N_{\mathtt{n}-2}\mathfrak{M}_0(s,\mathtt{b})\,. \end{aligned} \tag{7.14}$$

$(\mathbf{S2})_{\mathtt{n}}$ *Let* $i_1(\omega,\gamma)$, $i_2(\omega,\gamma)$ *such that* $\mathbf{R}_\perp^{(\mathtt{n})}(i_1)$, $\mathbf{R}_\perp^{(\mathtt{n})}(i_2)$ *satisfy* (7.4), (7.5). *Then, for all* $(\omega,\gamma) \in \mathbb{R}^\nu \times \mathbb{R}$

$$\|\langle D\rangle^{\frac14}|\Delta_{12}\mathbf{R}_\perp^{(\mathtt{n})}|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} \lesssim_{S,\mathtt{b}} \varepsilon\upsilon^{-3}N_{\mathtt{n}-1}^{-\mathtt{a}}\|i_1 - i_2\|_{s_0+\mu(\mathtt{b})}\,, \tag{7.15}$$

$$\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\,\Delta_{12}\mathbf{R}_\perp^{(\mathtt{n})}|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} \lesssim_{S,\mathtt{b}} \varepsilon\upsilon^{-3}N_{\mathtt{n}-1}\|i_1 - i_2\|_{s_0+\mu(\mathtt{b})}\,. \tag{7.16}$$

Furthermore, for $\mathtt{n} \geqslant 1$*, for all* $j \in \mathbb{S}_0^c$*,*

$$|j|^{\frac{1}{2}}|\Delta_{12}(\mathfrak{r}_j^{(\mathtt{n})} - \mathfrak{r}_j^{(\mathtt{n}-1)})| \leqslant C\|\langle D\rangle^{\frac{1}{4}}|\Delta_{12}\mathbf{R}_\perp^{(\mathtt{n})}|\langle D\rangle^{\frac{1}{4}}\|_{\mathcal{L}(H^{s_0})}\,,$$
$$|j|^{\frac{1}{2}}|\Delta_{12}\mathfrak{r}_j^{(\mathtt{n})}| \leqslant C(S,\mathtt{b})\varepsilon\upsilon^{-3}\,\|i_1 - i_2\|_{s_0+\mu(\mathtt{b})}\,.$$

$(\mathbf{S3})_{\mathtt{n}}$ *Let* i_1, i_2 *be like in* $(\mathbf{S2})_{\mathtt{n}}$ *and* $0 < \rho < \upsilon/2$*. Then*

$$\varepsilon\upsilon^{-3}C(S)N_{\mathtt{n}-1}^{\tau+1}\,\|i_1 - i_2\|_{s_0+\mu(\mathtt{b})} \leqslant \rho \;\Rightarrow\; \Lambda_{\mathtt{n}}^{\upsilon}(i_1) \subseteq \Lambda_{\mathtt{n}}^{\upsilon-\rho}(i_2)\,. \tag{7.17}$$

Theorem 7.1 implies also that the invertible operator $\mathbf{U}_0 := \mathbb{I}_\perp$, $\mathbf{U}_{\overline{\mathtt{n}}} := \mathbf{\Phi}_0 \circ \ldots \circ \mathbf{\Phi}_{\overline{\mathtt{n}}-1}$ for $\overline{\mathtt{n}} \geqslant 1$, has almost diagonalized $\mathbf{L}_0$. We have indeed the following corollary.

Theorem 7.2. (Almost-diagonalization of $\mathbf{L}_0$) *Assume* (6.8) *with* $\mu_0 \geqslant \mu(\mathtt{b})$*. For all* $S > s_0$*, there exist* $N_0 = N_0(S,\mathtt{b}) > 0$ *and* $\delta_0 = \delta_0(S) > 0$ *such that, if the smallness condition* $N_0^{\tau_2}\varepsilon\upsilon^{-4} \leqslant \delta_0$ *holds, with* $\tau_2 = \tau_2(\tau,\nu)$ *as in in Theorem 7.1, then, for all* $\overline{\mathtt{n}} \in \mathbb{N}_0$ *and for all* $(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$ *the operators* $\mathbf{U}_{\overline{\mathtt{n}}}^{\pm 1} - \mathbb{I}_\perp$ *are* $\mathcal{D}^{k_0}$*-*$(-\frac{1}{2})$*-modulo-tame satisfying* $\mathfrak{M}^\sharp_{\mathbf{U}_{\overline{\mathtt{n}}}^{\pm1}-\mathbb{1}_\perp}(s) \lesssim_S \varepsilon\upsilon^{-4}N_0^{\tau_1}(1+\|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})}^{k_0,\upsilon})$ *for all* $s_0 \leqslant s \leqslant S-\mu(\mathtt{b})$*, with* τ_1 *as in* (6.20)*. Moreover* $\mathbf{U}_{\overline{\mathtt{n}}}$*,* $\mathbf{U}_{\overline{\mathtt{n}}}^{-1}$ *are real, reversibility and momentum preserving. The operator* $\mathbf{L}_{\overline{\mathtt{n}}} = \omega\cdot\partial_\varphi\mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_{\overline{\mathtt{n}}} + \mathbf{R}_\perp^{(\overline{\mathtt{n}})}$*, defined in* (7.7) *with* $\mathtt{n} = \overline{\mathtt{n}}$ *is real, reversible and momentum preserving. The operator* $\mathbf{R}_\perp^{(\overline{\mathtt{n}})}$ *is* $\mathcal{D}^{k_0}$*-*$(-\frac{1}{2})$*-modulo-tame and, for all* $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$*,*

$$\mathfrak{M}^\sharp_{\langle D\rangle^{\frac{1}{4}}\mathbf{R}_\perp^{(\overline{\mathtt{n}})}\langle D\rangle^{\frac{1}{4}}}(s) \lesssim_S \varepsilon\upsilon^{-3}N_{\overline{\mathtt{n}}-1}^{-\mathtt{a}}(1 + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})}^{k_0,\upsilon})\,. \tag{7.18}$$

Moreover, for all (ω,γ) *in* $\Lambda_{\overline{\mathtt{n}}}^{\upsilon} = \Lambda_{\overline{\mathtt{n}}}^{\upsilon}(i) = \bigcap_{\mathtt{n}=0}^{\overline{\mathtt{n}}}\Lambda_{\mathtt{n}}^{\upsilon}$*, where the sets* $\Lambda_{\mathtt{n}}^{\upsilon}$ *are defined in* (7.12)*, the conjugation formula* $\mathbf{L}_{\overline{\mathtt{n}}} := \mathbf{U}_{\overline{\mathtt{n}}}^{-1}\mathbf{L}_0\mathbf{U}_{\overline{\mathtt{n}}}$ *holds.*

Proof of Theorem 7.1.

The proof of Theorem 7.1 is inductive. We first show that $(\mathbf{S1})_{\mathtt{n}}$-$(\mathbf{S3})_{\mathtt{n}}$ hold when $\mathtt{n} = 0$.

Proof of $(\mathbf{S1})_0$-$(\mathbf{S3})_0$. Properties (7.7)-(7.8) for $\mathtt{n} = 0$ hold by (6.132), (6.133), (7.1) with $\mathfrak{r}_j^{(0)} = 0$. Moreover, by (3.20), we get, for any $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$, that $\mathfrak{M}_0^\sharp(s), \mathfrak{M}_0^\sharp(s,\mathtt{b}) \lesssim_{s_0,\mathtt{b}} \mathfrak{M}_0(s,\mathtt{b})$ and that (7.11) for $\mathtt{n} = 0$ holds. The estimates (7.15), (7.16) at $\mathtt{n} = 0$ follow similarly by (7.5). Finally $(\mathbf{S3})_0$ is trivial since $\Lambda_0^{\upsilon}(i_1) = \Lambda_0^{\upsilon-\rho}(i_2) = \mathbb{R}^\nu \times [\gamma_1,\gamma_2]$.

The reducibility step. We now describe the generic inductive step, showing how to transform $\mathbf{L}_{\mathtt{n}}$ into $\mathbf{L}_{\mathtt{n}+1}$ by the conjugation with $\mathbf{\Phi}_{\mathtt{n}}$. For simplicity we drop the index $\mathtt{n}$ and we write $+$ instead of $\mathtt{n}+1$, so that we write $\mathbf{L} := \mathbf{L}_{\mathtt{n}}$, $\mathbf{L}_+ := \mathbf{L}_{\mathtt{n}+1}$, $\mathbf{R}_\perp := \mathbf{R}_\perp^{(\mathtt{n})}$, $\mathbf{R}_\perp^{(+)} := \mathbf{R}_\perp^{(\mathtt{n}+1)}$, $N := N_{\mathtt{n}}$, etc.. Let

$$\mathbf{\Phi} := e^{\mathbf{X}}, \quad \mathbf{X} := \begin{pmatrix} X^{(d)} & X^{(o)} \\ \overline{X^{(o)}} & \overline{X^{(d)}} \end{pmatrix}, \quad \begin{aligned} X^{(d)} &: H^\perp_{\mathbb{S}_0} \to H^\perp_{\mathbb{S}_0}, \\ X^{(o)} &: H^\perp_{-\mathbb{S}_0} \to H^\perp_{\mathbb{S}_0}, \end{aligned} \tag{7.19}$$

where $\mathbf{X}$ is chosen below in (7.23), (7.24). We transform $\mathbf{L}$ in (7.7) into

$$\begin{aligned} \mathbf{L}_+ := \mathbf{\Phi}^{-1}\mathbf{L}\mathbf{\Phi} &= \omega\cdot\partial_\varphi \mathbb{I}_\perp + \mathrm{i}\,\mathbf{D} + ((\omega\cdot\partial_\varphi\mathbf{X}) - \mathrm{i}[\mathbf{X},\mathbf{D}] + \Pi_N\mathbf{R}_\perp) + \Pi_N^\perp\mathbf{R}_\perp \\ &- \int_0^1 e^{-\tau\mathbf{X}}[\mathbf{X},\mathbf{R}_\perp]e^{\tau\mathbf{X}}\,\mathrm{d}\tau - \int_0^1 (1-\tau)e^{-\tau\mathbf{X}}[\mathbf{X},(\omega\cdot\partial_\varphi\mathbf{X}) - \mathrm{i}[\mathbf{X},\mathbf{D}]]e^{\tau\mathbf{X}}\,\mathrm{d}\tau, \end{aligned} \tag{7.20}$$

with $\Pi_N\mathbf{R}_\perp$ defined as in (3.18) and $\Pi_N^\perp := \mathrm{Id} - \Pi_N$. We want to solve the homological equation

$$\omega\cdot\partial_\varphi\mathbf{X} - \mathrm{i}[\mathbf{X},\mathbf{D}] + \Pi_N\mathbf{R}_\perp = [\mathbf{R}_\perp] \tag{7.21}$$

where $[\mathbf{R}_\perp] := \begin{pmatrix} [R_\perp^{(d)}] & 0 \\ 0 & \overline{[R_\perp^{(d)}]} \end{pmatrix}$, with $[R_\perp^{(d)}] := \mathrm{diag}_{j\in\mathbb{S}_0^c}(R_\perp^{(d)})_j^j(0)$. By (7.7) and (7.19), the homological equation (7.21) is equivalent to the two scalar homological equations

$$\begin{aligned} &\omega\cdot\partial_\varphi X^{(d)} - \mathrm{i}(X^{(d)}\mathcal{D} - \mathcal{D}X^{(d)}) + \Pi_N R_\perp^{(d)} = [R_\perp^{(d)}] \\ &\omega\cdot\partial_\varphi X^{(o)} + \mathrm{i}(X^{(o)}\overline{\mathcal{D}} + \mathcal{D}X^{(o)}) + \Pi_N R_\perp^{(o)} = 0\,. \end{aligned} \tag{7.22}$$

The solutions of (7.22) are, for all $(\omega,\gamma)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$ (see (7.12) with $\mathtt{n}\rightsquigarrow\mathtt{n}+1$)

$$(X^{(d)})_j^{j'}(\ell) := \begin{cases} -\dfrac{(R_\perp^{(d)})_j^{j'}(\ell)}{\mathrm{i}(\omega\cdot\ell+\mu_j-\mu_{j'})} & \text{if } \begin{cases} (\ell,j,j')\neq(0,j,j),\ j,j'\in\mathbb{S}_0^c,\ \langle\ell\rangle\leqslant N \\ \ell\cdot\vec{\jmath}+j-j'=0 \end{cases} \\ 0 & \text{otherwise}, \end{cases} \tag{7.23}$$

$$(X^{(o)})_j^{j'}(\ell) := \begin{cases} -\dfrac{(R_\perp^{(o)})_j^{j'}(\ell)}{\mathrm{i}(\omega\cdot\ell+\mu_j+\mu_{-j'})} & \text{if } \begin{cases} \forall\,\ell\in\mathbb{Z}^\nu\ j,-j'\in\mathbb{S}_0^c,\ \langle\ell\rangle\leqslant N \\ \ell\cdot\vec{\jmath}+j-j'=0 \end{cases} \\ 0 & \text{otherwise}. \end{cases} \tag{7.24}$$

Note that, since $-j'\in\mathbb{S}_0^c$, we can apply the bounds (7.12) for $(\omega,\gamma)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$.

Lemma 7.3. **(Homological equations)** *The real operator* $\mathbf{X}$ *defined in* (7.19), (7.23), (7.24)*, (which for all* $(\omega,\gamma)\in\Lambda_{\mathtt{n}+1}^{\upsilon}$ *solves the homological equation* (7.21)*) admits an extension to* $\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$. *Such extended* *operator is $\mathcal{D}^{k_0}$-$(-\frac{1}{2})$-modulo-tame satisfying, for all $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$,*

$$\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\mathbf{X}\langle D\rangle^{\frac14}}(s) \lesssim_{k_0} \frac{N^{\tau_1}}{\upsilon}\mathfrak{M}^\sharp(s)\,,\ \mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{X}\langle D\rangle^{\frac14}}(s) \lesssim_{k_0} \frac{N^{\tau_1}}{\upsilon}\mathfrak{M}^\sharp(s,\mathtt{b}), \tag{7.25}$$

where $\tau_1 := \tau(k_0+1)+k_0$. For all $(\omega,\gamma) \in \mathbb{R}^\nu \times \mathbb{R}$,

$$\begin{aligned}\|\langle D\rangle^{\frac14}|\Delta_{12}\mathbf{X}|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} &\lesssim N^{2\tau+1}\upsilon^{-1}(\|\langle D\rangle^{\frac14}|\Delta_{12}\mathbf{R}_\perp|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})}\\ &\quad+\|\langle D\rangle^{\frac14}|\mathbf{R}_\perp(i_2)|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})}\|i_1-i_2\|_{s_0+\mu(\mathtt{b})})\,,\end{aligned} \tag{7.26}$$

$$\begin{aligned}\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\Delta_{12}\mathbf{X}|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})} &\lesssim N^{2\tau+1}\upsilon^{-1}(\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\Delta_{12}\mathbf{R}_\perp|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})}\\ &\quad+\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{R}_\perp(i_2)|\langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})}\|i_1-i_2\|_{s_0+\mu(\mathtt{b})})\,.\end{aligned} \tag{7.27}$$

The operator $\mathbf{X}$ is reversibility and momentum preserving.

Proof. We prove that (7.25) holds for $X^{(d)}$. The proof for $X^{(o)}$ holds analogously. First, we extend the solution in (7.23) to all λ in $\mathbb{R}^\nu \times [\gamma_1,\gamma_2]$ by setting $(X^{(d)})_j^{j'}(\ell) = \mathrm{i}\, g_{\ell,j,j'}(\lambda)(R_\perp^{(d)})_j^{j'}(\ell)$, where $g_{\ell,j,j'}(\lambda) := \frac{\chi(f(\lambda)\rho^{-1})}{f(\lambda)}$, with $f(\lambda) := \omega\cdot\ell + \mu_j - \mu_{j'}$, $\rho := \upsilon\langle\ell\rangle^{-\tau}$, and χ is the cut-off function (3.6). By (7.8), (7.9), (6.134), (7.12), Lemma 4.4, (3.6), we deduce that, for any $k_1 \in \mathbb{N}_0^\nu$, $|k_1| \leqslant k_0$, $\sup_{|k_1|\leqslant k_0}\left|\partial_\lambda^{k_1} g_{\ell,j,j'}\right| \lesssim_{k_0} \langle\ell\rangle^{\tau_1}\upsilon^{-1-|k_1|}$, $\tau_1 = \tau(k_0+1)+k_0$, and we deduce, for all $0 \leqslant |k| \leqslant k_0$,

$$|\partial_\lambda^k (X^{(d)})_j^{j'}(\ell)| \lesssim_{k_0} \langle\ell\rangle^{\tau_1}\upsilon^{-1-|k|}\sum\nolimits_{|k_2|\leqslant|k|}\upsilon^{|k_2|}|\partial_\lambda^{k_2}(R_\perp^{(d)})_j^{j'}(\ell)|.$$

By (7.23) we have $(X^{(d)})_j^{j'}(\ell) = 0$ for all $\langle\ell\rangle > N$. For all $|k| \leqslant k_0$, we get

$$\begin{aligned}&\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\,\partial_\lambda^k X^{(d)}|\langle D\rangle^{\frac14}h\|_s^2 \lesssim_{k_0} N^{2\tau_1}\\ &\upsilon^{-2(1+|k|)}\sum_{|k_2|\leqslant|k|}\upsilon^{2|k_2|}\|\langle D\rangle^{\frac14}|\langle\partial_\varphi\rangle^{\mathtt{b}}\,\partial_\lambda^{k_2}R_\perp^{(d)}|\langle D\rangle^{\frac14}|h|\|_s^2\\ &\overset{\text{Def.3.11,(7.10)}}{\lesssim_{k_0}} N^{2\tau_1}\upsilon^{-2(1+|k|)}\big(\mathfrak{M}^\sharp(s,\mathtt{b})^2\|h\|_{s_0}^2 + \mathfrak{M}^\sharp(s_0,\mathtt{b})^2\|h\|_s^2\big)\,,\end{aligned}$$

and, by Definition 3.11, we conclude that $\mathfrak{M}^\sharp_{\langle D\rangle^{\frac14}\langle\partial_\varphi\rangle^{\mathtt{b}}X^{(d)}\langle D\rangle^{\frac14}}(s) \lesssim_{k_0} N^{\tau_1}\upsilon^{-1}\mathfrak{M}^\sharp(s,\mathtt{b})$. The analogous estimates for $\langle\partial_\varphi\rangle^{\mathtt{b}}X^{(o)}$, $X^{(d)}$, $X^{(o)}$ and (7.26), (7.27) follow similarly. ☐

By (7.20), (7.21), for all $\lambda \in \Lambda_{\mathtt{n}+1}^\upsilon$, we have

$$\mathbf{L}_+ = \mathbf{\Phi}^{-1}\mathbf{L}\mathbf{\Phi} = \omega\cdot\partial_\varphi\mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_+ + \mathbf{R}_\perp^{(+)}\,, \tag{7.28}$$

where

$$
\begin{aligned}
&\mathbf{D}_+ := \mathbf{D} - \mathrm{i}[\mathbf{R}_\perp]\,,\\
&\mathbf{R}_\perp^{(+)} := \Pi_N^\perp \mathbf{R}_\perp - \int_0^1 e^{-\tau \mathbf{X}}[\mathbf{X}, \mathbf{R}_\perp] e^{\tau \mathbf{X}}\, \mathrm{d}\tau \\
&\quad + \int_0^1 (1-\tau) e^{-\tau \mathbf{X}}[\mathbf{X}, \Pi_N \mathbf{R}_\perp - [\mathbf{R}_\perp]] e^{\tau \mathbf{X}}\, \mathrm{d}\tau\,.
\end{aligned}
\tag{7.29}
$$

The right hand sides of (7.28)-(7.29) define an extension of $\mathbf{L}_+$ to the whole parameter space $\mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, since $\mathbf{R}_\perp$ and $\mathbf{X}$ are defined on $\mathbb{R}^\nu \times [\gamma_1, \gamma_2]$. The new operator $\mathbf{L}_+$ in (7.28) has the same form of $\mathbf{L}$ in (7.7) with the non-diagonal remainder $\mathbf{R}_\perp^{(+)}$, sum of a term $\Pi_N^\perp \mathbf{R}_\perp$ supported on high frequencies and of a quadratic function of $\mathbf{X}$ and $\mathbf{R}_\perp$. The new normal form $\mathbf{D}_+$ is diagonal:

Lemma 7.4. (New diagonal part) *For all $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, we have*

$$
\mathrm{i}\,\mathbf{D}_+ = \mathrm{i}\,\mathbf{D} + [\mathbf{R}_\perp] = \mathrm{i}\begin{pmatrix} \mathcal{D}_+ & 0 \\ 0 & -\overline{\mathcal{D}_+} \end{pmatrix}\,,\ \mathcal{D}_+ := \mathrm{diag}_{j \in \mathbb{S}_0^c}\, \mu_j^{(+)}\,,\ \mu_j^{(+)} := \mu_j + \mathtt{r}_j \in \mathbb{R}\,,
$$

where each $\mathtt{r}_j$ satisfies, on $\mathbb{R}^\nu \times [\gamma_1, \gamma_2]$,

$$
|j|^{\frac12} |\mathtt{r}_j|^{k_0, \upsilon} = |j|^{\frac12} |\mu_j^{(+)} - \mu_j|^{k_0,\upsilon} \lesssim \mathfrak{M}^\sharp(s_0)\,. \tag{7.30}
$$

Moreover $|j|^{\frac12} |\mathtt{r}_j(i_1) - \mathtt{r}_j(i_2)| \lesssim \|\langle D\rangle^{\frac14} |\Delta_{12} \mathbf{R}_\perp| \langle D\rangle^{\frac14}\|_{\mathcal{L}(H^{s_0})}$.

Proof. We have that $\mathtt{r}_j := -\mathrm{i}(R_\perp^{(d)})_j^j(0) \in \mathbb{R}$, by the reversibility of $R_\perp^{(d)}$ and Lemma 3.15. Recalling the definition of $\mathfrak{M}^\sharp(s_0)$ in (7.10) (with $s = s_0$) and Definition 3.11, we deduce that $|j|^{\frac12} |\partial_\lambda^k (R_\perp^{(d)})_j^j(0)| \lesssim \upsilon^{-|k|} \mathfrak{M}^\sharp(s_0)$, for all $0 \leqslant |k| \leqslant k_0$, and (7.30) follows. The bound for $|j|^{\frac12} |\mathtt{r}_j(i_1) - \mathtt{r}_j(i_2)|$ is similar. $\square$

The iterative step. Assume that the statements $(\mathbf{S1})_{\mathtt{n}}$-$(\mathbf{S3})_{\mathtt{n}}$ are true. We now prove $(\mathbf{S1})_{\mathtt{n}+1}$-$(\mathbf{S3})_{\mathtt{n}+1}$.

PROOF OF $(\mathbf{S1})_{\mathtt{n}+1}$. The real operator $\mathbf{X}_{\mathtt{n}}$ defined in Lemma 7.3 is defined for all $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$ and, by (7.25), (7.11), satisfies the estimates (7.14) at the step $\mathtt{n}+1$. By (7.28), for all $\lambda \in \Lambda_{\mathtt{n}+1}^\upsilon$, the conjugation formula (7.13) holds at the step $\mathtt{n}+1$. By Lemma 7.4, the operator $\mathbf{D}_{\mathtt{n}+1}$ is diagonal with eigenvalues $\mu_j^{(\mathtt{n}+1)} = \mu_j^{(0)} + \mathfrak{r}_j^{(\mathtt{n}+1)}$ with $\mathfrak{r}_j^{(\mathtt{n}+1)} := \mathfrak{r}_j^{(\mathtt{n})} + \mathtt{r}_j^{(\mathtt{n})}$ satisfying, using also (7.11), (7.9) at the step $\mathtt{n}+1$. The next lemma provides the estimates for $\mathbf{R}_\perp^{(\mathtt{n}+1)} = \mathbf{R}_\perp^{(+)}$ defined in (7.29).

Lemma 7.5. *The operators* $\mathbf{R}_\perp^{(\mathtt{n}+1)}$ *and* $\langle\partial_\varphi\rangle^{\mathtt{b}}\mathbf{R}_\perp^{(\mathtt{n}+1)}$ *are* $\mathcal{D}^{k_0}$*-*$(-\frac{1}{2})$*-modulo-tame with modulo-tame constants satisfying, for any* $s_0 \leqslant s \leqslant S - \mu(\mathtt{b})$*,*

$$\mathfrak{M}^\sharp_{\mathtt{n}+1}(s) \lesssim_s N_{\mathtt{n}}^{-\mathtt{b}}\mathfrak{M}^\sharp_{\mathtt{n}}(s,\mathtt{b}) + N_{\mathtt{n}}^{\tau_1}\upsilon^{-1}\mathfrak{M}^\sharp_{\mathtt{n}}(s)\mathfrak{M}^\sharp_{\mathtt{n}}(s_0)\,, \tag{7.31}$$

$$\mathfrak{M}^\sharp_{\mathtt{n}+1}(s,\mathtt{b}) \lesssim_{s,\mathtt{b}} \mathfrak{M}^\sharp_{\mathtt{n}}(s,\mathtt{b}) + N_{\mathtt{n}}^{\tau_1}\upsilon^{-1}\big(\mathfrak{M}^\sharp_{\mathtt{n}}(s,\mathtt{b})\mathfrak{M}^\sharp_{\mathtt{n}}(s_0) + \mathfrak{M}^\sharp_{\mathtt{n}}(s_0,\mathtt{b})\mathfrak{M}^\sharp_{\mathtt{n}}(s)\big)\,. \tag{7.32}$$

Moreover, the estimates (7.11) *hold at the step* $\mathtt{n}+1$.

Proof. The estimates (7.31), (7.32) follow by (7.29), (3.19), Lemma 3.12, and (7.25), (7.11), (6.20), (6.17), (7.6). The estimates (7.11) at the step $\mathtt{n}+1$ follow by (7.31), (7.32), (7.11) at the step $\mathtt{n}$, (6.20), the smallness condition (7.6) with $N_0 = N_0(S, s_0, \mathtt{b}) > 0$ large enough and $\tau_2 > \tau_1 + 1 + \mathtt{a}$. □

PROOF OF $(\mathbf{S2})_{\mathtt{n}+1}$. It follows by similar arguments and we omit it.
PROOF OF $(\mathbf{S3})_{\mathtt{n}+1}$. Use (7.8), (6.134)-(6.135), $(\mathbf{S2})_{\mathtt{n}}$, and the momentum conditions in (7.12).

Almost invertibility of $\mathcal{L}_\omega$

By (6.132), (6.124) and Theorem 7.2, we obtain

$$\mathcal{L}_\omega = \mathbf{W}_{\bar{\mathtt{n}}}\mathbf{L}_{\bar{\mathtt{n}}}\mathbf{W}_{\bar{\mathtt{n}}}^{-1} + \mathcal{W}^\perp\mathbf{P}_{\perp,\bar{\mathtt{n}}}(\mathcal{W}^\perp)^{-1} + \mathcal{W}^\perp\mathbf{Q}_{\perp,\bar{\mathtt{n}}}(\mathcal{W}^\perp)^{-1},\ \mathbf{W}_{\bar{\mathtt{n}}} := \mathcal{W}^\perp\mathbf{U}_{\bar{\mathtt{n}}}\,, \tag{7.33}$$

where the operator $\mathbf{L}_{\bar{\mathtt{n}}}$ is defined in (7.7) with $\mathtt{n} = \bar{\mathtt{n}}$ and $\mathbf{P}_{\perp,\bar{\mathtt{n}}}$, $\mathbf{Q}_{\perp,\bar{\mathtt{n}}}$ satisfy the estimates in Lemma 6.12. By (6.123) and Theorem 7.2, we have, for some $\sigma := \sigma(\tau,\nu,k_0) > 0$, for any $s_0 \leqslant s \leqslant S - \mu(\mathtt{b}) - \sigma$,

$$\|\mathbf{W}_{\bar{\mathtt{n}}}^{\pm 1}h\|_s^{k_0,\upsilon} \lesssim_S \|h\|_{s+\sigma}^{k_0,\upsilon} + \|\mathfrak{I}_0\|_{s+\mu(\mathtt{b})+\sigma}^{k_0,\upsilon}\|h\|_{s_0+\sigma}^{k_0,\upsilon}\,. \tag{7.34}$$

In order to prove the almost invertibility assumption (AI) of $\mathcal{L}_\omega$ in Section 5.3, we decompose the operator $\mathbf{L}_{\bar{\mathtt{n}}}$ in (7.7) (with $\bar{\mathtt{n}}$ instead of $\mathtt{n}$) as

$$\mathbf{L}_{\bar{\mathtt{n}}} = \mathbf{D}_{\bar{\mathtt{n}}}^{<} + \mathbf{Q}_\perp^{(\bar{\mathtt{n}})} + \mathbf{R}_\perp^{(\bar{\mathtt{n}})} \tag{7.35}$$

where $\mathbf{R}_\perp^{(\bar{\mathtt{n}})}$ satisfies (7.18), whereas

$$\begin{aligned} \mathbf{D}_{\bar{\mathtt{n}}}^{<} &:= \Pi_{K_{\bar{\mathtt{n}}}}(\omega\cdot\partial_\varphi\mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_{\bar{\mathtt{n}}})\Pi_{K_{\bar{\mathtt{n}}}} + \mathrm{i}\Pi_{K_{\bar{\mathtt{n}}}}^\perp\Sigma\,,\\ \mathbf{Q}_\perp^{(\bar{\mathtt{n}})} &:= \Pi_{K_{\bar{\mathtt{n}}}}^\perp(\omega\cdot\partial_\varphi\mathbb{I}_\perp + \mathrm{i}\,\mathbf{D}_{\bar{\mathtt{n}}})\Pi_{K_{\bar{\mathtt{n}}}}^\perp - \mathrm{i}\Pi_{K_{\bar{\mathtt{n}}}}^\perp\Sigma\,, \end{aligned} \tag{7.36}$$

the smoothing operator Π_K on the traveling waves is defined in (3.1), $\Pi_K^\perp := \mathrm{Id} - \Pi_K$ and $\Sigma := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. We have that $K_{\mathtt{n}} := K_0^{\chi^{\mathtt{n}}}$, $\chi = 3/2$ (cfr.

(5.41)), and K_0 will be fixed in (7.40). For all $\lambda = (\omega, \gamma)$ in the set

(7.37)
$$\Lambda^{\upsilon,I}_{\bar{\mathtt{n}}+1} := \Big\{ \lambda \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2] : |\omega \cdot \ell + \mu_j^{(\bar{\mathtt{n}})}| \geqslant \upsilon \frac{|j|^{\frac12}}{\langle \ell \rangle^\tau}, \ \forall |\ell| \leqslant K_{\bar{\mathtt{n}}}, \ j \in \mathbb{S}_0^c, j + \vec{\jmath} \cdot \ell = 0 \Big\},$$

the operator $\mathbf{D}^{\leqslant}_{\bar{\mathtt{n}}}$ in (7.36) is invertible on the subspace of the traveling waves $\tau_\varsigma g(\varphi) = g(\varphi - \vec{\jmath}\varsigma)$, $\varsigma \in \mathbb{R}$, such that $g(\varphi, \cdot) \in \mathbf{H}^\perp_{\mathbb{S}_0}$. More precisely there exists an extension of the inverse operator to the whole $\mathbb{R}^\nu \times [\gamma_1, \gamma_2]$ satisfying $\|(\mathbf{D}^{\leqslant}_{\bar{\mathtt{n}}})^{-1} g\|_s^{k_0,\upsilon} \lesssim_{k_0} \upsilon^{-1} \|g\|^{k_0,\upsilon}_{s+\tau_1}$, $\tau_1 = k_0 + \tau(k_0+1)$. Standard smoothing properties imply that the operator $\mathbf{Q}^{(\bar{\mathtt{n}})}_\perp$ in (7.36) satisfies, for any traveling wave $h \in \mathbf{H}^\perp_{\mathbb{S}_0}$, for all $b > 0$, $\|\mathbf{Q}^{(\bar{\mathtt{n}})}_\perp h\|^{k_0,\upsilon}_{s_0} \lesssim K_{\bar{\mathtt{n}}}^{-b} \|h\|^{k_0,\upsilon}_{s_0+b+1}$, $\|\mathbf{Q}^{(\bar{\mathtt{n}})}_\perp h\|^{k_0,\upsilon}_s \lesssim \|h\|^{k_0,\upsilon}_{s+1}$. Therefore, by the decompositions (7.33), (7.35), Theorem 7.2 (note that (5.36) and Lemma 5.8 imply (6.8)), Proposition 6.13, the fact that $\mathbf{W}_{\bar{\mathtt{n}}}$ maps (anti)-reversible, respectively traveling, waves, into (anti)-reversible, respectively traveling, waves (Lemma 6.11) and estimates (7.34), (3.4) we deduce the following theorem.

Theorem 7.6. **(Almost invertibility of $\mathcal{L}_\omega$)** *Assume* (5.36). *Let* $\mathtt{a}, \mathtt{b}$ *as in* (6.20) *and M as in* (7.2). *Let $S > s_0 + k_0$ and assume the smallness condition $N_0^{\tau_2} \varepsilon \upsilon^{-4} \leqslant \delta_0$ of Theorem 7.2. Then the almost invertibility assumption (AI) in Section 5.3 holds with Λ_o replaced by*

(7.38)
$$\mathbf{\Lambda}^{\upsilon}_{\bar{\mathtt{n}}+1} := \mathbf{\Lambda}^{\upsilon}_{\bar{\mathtt{n}}+1}(i) := \Lambda^{\upsilon}_{\bar{\mathtt{n}}+1} \cap \Lambda^{\upsilon,I}_{\bar{\mathtt{n}}+1} \cap \mathtt{TC}_{\bar{\mathtt{n}}+1}(2\upsilon, \tau),$$

(see (7.12)*,* (7.37)*,* (6.22)*), with $\mu(\mathtt{b})$ defined in* (7.3)*, and*

$$\begin{aligned}
\mathcal{L}^{<}_\omega &:= \mathbf{W}_{\bar{\mathtt{n}}} \mathbf{D}^{<}_{\bar{\mathtt{n}}} \mathbf{W}_{\bar{\mathtt{n}}}^{-1}, \\
\mathcal{R}_\omega &:= \mathbf{W}_{\bar{\mathtt{n}}} \mathbf{R}^{(\bar{\mathtt{n}})}_\perp \mathbf{W}_{\bar{\mathtt{n}}}^{-1} + \mathcal{W}^\perp \mathbf{P}_{\perp,\bar{\mathtt{n}}} (\mathcal{W}^\perp)^{-1}, \\
\mathcal{R}^\perp_\omega &:= \mathbf{W}_{\bar{\mathtt{n}}} \mathbf{Q}^{(\bar{\mathtt{n}})}_\perp \mathbf{W}_{\bar{\mathtt{n}}}^{-1} + \mathcal{W}^\perp \mathbf{Q}_{\perp,\bar{\mathtt{n}}} (\mathcal{W}^\perp)^{-1}.
\end{aligned}$$

Proof of Theorem 5.1

Theorem 7.7 is deduced, in a by now standard way, from the almost invertibility of $\mathcal{L}_\omega$ in Theorem 7.6, as in [9, 2, 7]. Note that the estimates (5.43), (5.44), (5.45), (5.46) coincide with (5.49)-(5.52) in [2] with $M = 1/2$. Thus we shall be short. Consider the finite dimensional subspaces of traveling wave variations

$$E_{\mathtt{n}} := \big\{ \mathfrak{I}(\varphi) = (\Theta, I, w)(\varphi) \text{ such that (3.22) holds} : \Theta = \Pi_{\mathtt{n}} \Theta, \ I = \Pi_{\mathtt{n}} I, \ w = \Pi_{\mathtt{n}} w \big\}$$

where $\Pi_{\mathtt{n}} w := \Pi_{K_{\mathtt{n}}} w$ as in (3.1) with $K_{\mathtt{n}}$ in (5.41), and $\Pi_{\mathtt{n}} g(\varphi) := \sum_{|\ell| \leqslant K_{\mathtt{n}}} g_\ell e^{\mathrm{i}\ell\cdot\varphi}$. Let

$$\begin{aligned} & \mathtt{a}_1 := \max\{6\sigma_1 + 13, \chi(p(\tau+1) + \mu(\mathtt{b}) + 2\sigma_1) + 1\}\,, \\ & \mathtt{a}_2 := \chi^{-1}\mathtt{a}_1 - \mu(\mathtt{b}) - 2\sigma_1\,, \quad \mu_1 := 3(\mu(\mathtt{b}) + 2\sigma_1) + 1\,, \\ & \mathtt{b}_1 := \mathtt{a}_1 + 2\mu(\mathtt{b}) + 4\sigma_1 + 3 + \chi^{-1}\mu_1\,, \quad \chi = 3/2\,, \\ & \sigma_1 := \max\{\overline{\sigma}, 2s_0 + 2k_0 + 5\}\,, \quad S - \mu(\mathtt{b}) - \overline{\sigma} = s_0 + \mathtt{b}_1\,, \end{aligned} \tag{7.39}$$

where $\overline{\sigma} = \overline{\sigma}(\tau, \nu, k_0) > 0$ is defined by Theorem 5.9, $\mu(\mathtt{b})$ is defined in (7.3), and $\mathtt{b} = [\mathtt{a}] + 2$ in (6.20). The exponent p in (5.41) is $p := 3\mathtt{a}^{-1}(\mu(\mathtt{b}) + 4\sigma_1 + 1)$. Given a function $W = (\mathfrak{I}, \beta)$ where $\mathfrak{I}$ is the periodic component of a torus as in (5.5) and $\beta \in \mathbb{R}^\nu$, we denote $\|W\|_s^{k_0,\upsilon} := \|\mathfrak{I}\|_s^{k_0,\upsilon} + |\beta|^{k_0,\upsilon}$.

Theorem 7.7. **(Nash-Moser)** *There exist $\delta_0, C_* > 0$ such that, if*

$$\begin{aligned} & K_0^{\tau_3}\varepsilon\upsilon^{-4} < \delta_0\,,\ \tau_3 := \max\{p\tau_2, 2\sigma_1 + \mathtt{a}_1 + 4\}\,, \\ & K_0 := \upsilon^{-1}\,,\ \upsilon := \varepsilon^{\mathrm{a}}\,,\ 0 < \mathrm{a} < (4 + \tau_3)^{-1}\,, \end{aligned} \tag{7.40}$$

where $\tau_2 = \tau_2(\tau, \nu)$ is given by Theorem 7.1, then, for all $\mathtt{n} \geqslant 0$:

$(\mathcal{P}1)_{\mathtt{n}}$ *There exists a k_0-times differentiable function*

$$\widetilde{W}_{\mathtt{n}} : \mathbb{R}^\nu \times [\gamma_1, \gamma_2] \to E_{\mathtt{n}-1} \times \mathbb{R}^\nu, \quad \lambda = (\omega, \gamma) \mapsto \widetilde{W}_{\mathtt{n}}(\lambda) := (\widetilde{\mathfrak{I}}_{\mathtt{n}}, \widetilde{\alpha}_{\mathtt{n}} - \omega),$$

for $\mathtt{n} \geqslant 1$, and $\widetilde{W}_0 := 0$, satisfying $\|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_*\varepsilon\upsilon^{-1}$. Let $\widetilde{U}_{\mathtt{n}} := U_0 + \widetilde{W}_{\mathtt{n}}$, where $U_0 := (\varphi, 0, 0, \omega)$. The difference $\widetilde{H}_{\mathtt{n}} := \widetilde{U}_{\mathtt{n}} - \widetilde{U}_{\mathtt{n}-1}$, for $\mathtt{n} \geqslant 1$, satisfies $\|\widetilde{H}_1\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_*\varepsilon\upsilon^{-1}$ and, for any $\mathtt{n} \geqslant 2$, $\|\widetilde{H}_{\mathtt{n}}\|_{s_0+\mu(\mathtt{b})+\sigma_1}^{k_0,\upsilon} \leqslant C_*\varepsilon\upsilon^{-1}K_{\mathtt{n}-1}^{-\mathtt{a}_2}$. The torus embedding $\widetilde{\imath}_{\mathtt{n}} := (\varphi, 0, 0) + \widetilde{\mathfrak{I}}_{\mathtt{n}}$ is reversible and traveling, i.e. (5.4) *holds;*

$(\mathcal{P}2)_{\mathtt{n}}$ *We define $\mathcal{G}_0 := \Omega \times [\gamma_1, \gamma_2]$, $\mathcal{G}_{\mathtt{n}+1} := \mathcal{G}_{\mathtt{n}} \cap \mathbf{\Lambda}_{\mathtt{n}+1}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}})$, $\mathtt{n} \geqslant 0$, where $\mathbf{\Lambda}_{\mathtt{n}+1}^{\upsilon}(\widetilde{\imath}_{\mathtt{n}})$ is in* (7.38). *Then, for any $\lambda \in \mathcal{G}_{\mathtt{n}}$, setting $K_{-1} := 1$, we have $\|\mathcal{F}(\widetilde{U}_{\mathtt{n}})\|_{s_0}^{k_0,\upsilon} \leqslant C_*\varepsilon K_{\mathtt{n}-1}^{-\mathtt{a}_1}$;*

$(\mathcal{P}3)_{\mathtt{n}}$ (HIGH NORM) *For all $\lambda \in \mathcal{G}_{\mathtt{n}}$, we have $\|\widetilde{W}_{\mathtt{n}}\|_{s_0+\mathtt{b}_1}^{k_0,\upsilon} \leqslant C_*\varepsilon\upsilon^{-1}K_{\mathtt{n}-1}^{\mu_1}$.*

Proof. The proof follows as in [9, 2]. The verification that each $\widetilde{\imath}_{\mathtt{n}}$ is reversible and traveling is in [7]. □

Theorem 5.1 is a standard corollary of Theorem 7.7, as in [9, 2, 7]. Let $\upsilon = \varepsilon^{\mathrm{a}}$, with $0 < \mathrm{a} < \mathrm{a}_0 := 1/(4 + \tau_3)$. Then, the smallness condition in (7.40) is verified for $0 < \varepsilon < \varepsilon_0$ small enough and Theorem 7.7 holds. By $(\mathcal{P}1)_{\mathtt{n}}$ the sequence $\widetilde{W}_{\mathtt{n}}$ converges to a function $W_\infty : \mathbb{R}^\nu \times [\gamma_1, \gamma_2] \to H^{s_0}_\varphi \times H^{s_0}_\varphi \times H^{s_0} \times \mathbb{R}^\nu$, and we define $U_\infty := (i_\infty, \alpha_\infty) := (\varphi, 0, 0, \omega) + W_\infty$. The torus i_∞ is reversible and traveling, i.e. (5.4) holds. By $(\mathcal{P}1)_{\mathtt{n}}$ we also deduce the bounds

$$
\begin{aligned}
&\|U_\infty - U_0\|^{k_0,\upsilon}_{s_0+\mu(\mathtt{b})+\sigma_1} \leqslant C_* \varepsilon \upsilon^{-1} , \\
&\|U_\infty - \widetilde{U}_{\mathtt{n}}\|^{k_0,\upsilon}_{s_0+\mu(\mathtt{b})+\sigma_1} \leqslant C \varepsilon \upsilon^{-1} K_{\mathtt{n}}^{-\mathtt{a}_2} , \ \forall \, \mathtt{n} \geqslant 1 .
\end{aligned} \tag{7.41}
$$

In particular (5.6)-(5.7) hold. By Theorem 7.7-$(\mathcal{P}2)_{\mathtt{n}}$, $\mathcal{F}(\lambda; U_\infty(\lambda)) = 0$ holds for any λ in the set

$$
\bigcap_{\mathtt{n}\in\mathbb{N}_0} \mathcal{G}_{\mathtt{n}} \overset{(7.38)}{=} \mathcal{G}_0 \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda^{\upsilon}_{\mathtt{n}}(\tilde{\imath}_{\mathtt{n}-1}) \Big] \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda^{\upsilon,I}_{\mathtt{n}}(\tilde{\imath}_{\mathtt{n}-1}) \Big] \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \mathtt{TC}_{\mathtt{n}}(2\upsilon, \tau)(\tilde{\imath}_{\mathtt{n}-1}) \Big] ,
$$

where $\mathcal{G}_0 := \Omega \times [\gamma_1, \gamma_2]$. To conclude the proof of Theorem 5.1 it remains only to define the μ_j^∞ in (5.8) and prove that the set $\mathcal{C}^\upsilon_\infty$ in (5.10)-(5.13) is contained in $\cap_{\mathtt{n}\geqslant 0} \mathcal{G}_{\mathtt{n}}$. We first define

$$
\mathcal{G}_\infty := \mathcal{G}_0 \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda^{2\upsilon}_{\mathtt{n}}(i_\infty) \Big] \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \Lambda^{2\upsilon,I}_{\mathtt{n}}(i_\infty) \Big] \cap \Big[\bigcap_{\mathtt{n}\geqslant 1} \mathtt{TC}_{\mathtt{n}}(4\upsilon, \tau)(i_\infty) \Big] . \tag{7.42}
$$

By (7.41), Lemma 6.4 and (7.17), one deduces that $\mathcal{G}_\infty \subseteq \cap_{\mathtt{n}\geqslant 0} \mathcal{G}_{\mathtt{n}}$, where $\mathcal{G}_{\mathtt{n}}$ are defined in $(\mathcal{P}2)_{\mathtt{n}}$ (cfr. e.g. Lemma 8.6 in [9]). We define μ_j^∞ in (5.8) with $\mathtt{m}^\infty_{1,\mathtt{n}} := \mathtt{m}_{1,\mathtt{n}}(i_\infty)$, $\mathtt{m}^\infty_{\frac12} = \mathtt{m}_{\frac12}(i_\infty)$, $\mathtt{m}^\infty_0 = \mathtt{m}_0(i_\infty)$, and $\mathtt{m}_{1,\mathtt{n}}, \mathtt{m}_{\frac12}, \mathtt{m}_0$ as in Proposition 6.13. By (7.9), $(\mathfrak{r}_j^{(\mathtt{n})}(i_\infty))_{\mathtt{n}\in\mathbb{N}}$, with $\mathfrak{r}_j^{(\mathtt{n})}$ given by Theorem 7.1-$(\mathbf{S1})_{\mathtt{n}}$ (evaluated at $i = i_\infty$), is a Cauchy sequence in $| \cdot |^{k_0,\upsilon}$. Let $\mathfrak{r}_j^\infty := \lim_{\mathtt{n}\to\infty} \mathfrak{r}_j^{(\mathtt{n})}(i_\infty)$, $j \in \mathbb{S}_0^c$. It results $|j|^{\frac12} |\mathfrak{r}_j^\infty - \mathfrak{r}_j^{(\mathtt{n})}(i_\infty)|^{k_0,\upsilon} \leqslant C \varepsilon \upsilon^{-3} N_{\mathtt{n}-1}^{-\mathtt{a}}$ for any $\mathtt{n} \geqslant 0$. Recalling that $\mathfrak{r}_j^{(0)}(i_\infty) = 0$ and (6.134), the estimates (5.9) hold. Finally one checks (see e.g. Lemma 8.7 in [9]) that the set $\mathcal{C}^\upsilon_\infty$ in (5.10)-(5.13) satisfies $\mathcal{C}^\upsilon_\infty \subseteq \mathcal{G}_\infty$, with $\mathcal{G}_\infty$ in (7.42), and so $\mathcal{C}^\upsilon_\infty \subseteq \cap_{\mathtt{n}\geqslant 0} \mathcal{G}_{\mathtt{n}}$. This concludes the proof of Theorem 5.1.

Appendix A: Almost straightening of a transport operator

The main results of this appendix are Theorem A.2 and Corollary A.4. The goal is to almost-straighten a linear quasi-periodic transport operator of the form

$$
X_0 := \omega \cdot \partial_\varphi + p_0(\varphi, x) \partial_x , \tag{A.1}
$$

to a constant coefficient one $\omega\cdot\partial_\varphi+\mathtt{m}_{1,\mathtt{n}}\partial_x$, up to a small term $p_{\mathtt{n}}\partial_x$, see (A.4) and (A.5). We follow the scheme of Section 4 in [3]. We first introduce the following norm: for any $u=u(\lambda)\in H^s(\mathbb{T}^{\nu+1})$, $s\in\mathbb{R}$, k_0-times differentiable with respect to $\lambda=(\omega,\gamma)\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]$, we define the norm

$$|u|_s^{k_0,\upsilon}:=\sum_{k\in\mathbb{N}^{\nu+1},0\leqslant|k|\leqslant k_0}\upsilon^{|k|}\sup_{\lambda\in\mathbb{R}^\nu\times[\gamma_1,\gamma_2]}\|\partial_\lambda^k u(\lambda)\|_{s-|k|}.$$

It satisfies $|u|_s^{k_0,\upsilon}\leqslant\|u\|_s^{k_0,\upsilon}\leqslant|u|_{s+k_0}^{k_0,\upsilon}$ for any $s\in\mathbb{R}$. Note the key estimate (A.2) for the composition where there is no loss of k_0-derivatives on the highest norm $|u|_s^{k_0,\upsilon}$, unlike the corresponding estimate in Lemma 3.3 with $\|\ \|_s^{k_0,\upsilon}$. This is crucial to prove (A.18) and then deduce the a-priori bound (A.5) for the divergence of the high norms of the functions $p_{\mathtt{n}}$. The following lemma follows as in [9]. Let $\mathfrak{s}_0:=s_0+k_0>\frac{1}{2}(\nu+1)+k_0$.

Lemma A.1. *The following hold:*

(i) For any $s\geqslant\mathfrak{s}_0$, we have

$$|uv|_s^{k_0,\upsilon}\leqslant C(s)|u|_s^{k_0,\upsilon}|v|_{\mathfrak{s}_0}^{k_0,\upsilon}+C(\mathfrak{s}_0)|u|_{\mathfrak{s}_0}^{k_0,\upsilon}|v|_s^{k_0,\upsilon}.$$

The tame constant $C(s):=C(s,k_0)$ is monotone in $s\geqslant\mathfrak{s}_0$.

(ii) For $N\geqslant1$ and $\alpha\geqslant0$ we have $|\Pi_N u|_s^{k_0,\upsilon}\leqslant N^\alpha|u|_{s-\alpha}^{k_0,\upsilon}$ and $|\Pi_N^\perp u|_s^{k_0,\upsilon}\leqslant N^{-\alpha}|u|_{s+\alpha}^{k_0,\upsilon}$ for any $s\in\mathbb{R}$.

(iii) Let $|\beta|_{2\mathfrak{s}_0+1}^{k_0,\upsilon}\leqslant\delta(\mathfrak{s}_0)$ small enough. Then the composition operator $\mathcal{B}$ defined as in (6.19) *satisfies the tame estimate, for any $s\geqslant\mathfrak{s}_0+1$,*

$$|\mathcal{B}u|_s^{k_0,\upsilon}\leqslant C(s)(|u|_s^{k_0,\upsilon}+|\beta|_s^{k_0,\upsilon}|u|_{\mathfrak{s}_0+1}^{k_0,\upsilon}). \tag{A.2}$$

The constant $C(s):=C(s,k_0)$ is monotone in $s\geqslant\mathfrak{s}_0$. Moreover, the diffeomorphism $x\mapsto x+\beta(\varphi,x)$ is invertible and its inverse $y\mapsto y+\breve\beta(\varphi,y)$ satisfies, for any $s\geqslant\mathfrak{s}_0$, $|\breve\beta|_s^{k_0,\upsilon}\leqslant C(s)|\beta|_s^{k_0,\upsilon}$.

(iv) For any $\epsilon>0$, $a_0,b_0\geqslant0$ and $p,q>0$, there exists $C_\epsilon=C_\epsilon(p,q)>0$, with $C_1<1$, such that

$$|u|_{a_0+p}^{k_0,\upsilon}|v|_{b_0+q}^{k_0,\upsilon}\leqslant\epsilon|u|_{a_0+p+q}^{k_0,\upsilon}|v|_{b_0}^{k_0,\upsilon}+C_\epsilon|u|_{a_0}^{k_0,\upsilon}|v|_{b_0+p+q}^{k_0,\upsilon}.$$

Remind that $N_{\mathtt{n}}:=N_0^{\chi^{\mathtt{n}}}$, $\chi=3/2$, $N_{-1}:=1$, see (6.17).

Theorem A.2 (Almost straightening). *Let X_0 be the quasi-periodic transport operator in* (A.1)*, where $p_0(\varphi,x)$ is a quasi-periodic traveling wave,* even(φ, x), *defined for all* $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$. *For any* $S > \mathfrak{s}_0$, *there exist* $\tau_2 > \tau_1 + 1 + \mathtt{a}$, $\delta := \delta(S, \mathfrak{s}_0, k_0, \mathtt{b}) > 0$ *and* $N_0 := N_0(S, \mathfrak{s}_0, k_0, \mathtt{b}) \in \mathbb{N}$ *(with* τ_1, $\mathtt{a}$, $\mathtt{b}$ *in* (6.20)*) such that, if*

$$N_0^{\tau_2} \, |p_0|_{2\mathfrak{s}_0+\mathtt{b}+1}^{k_0,\upsilon} \, \upsilon^{-1} \leqslant \delta < 1 \, , \tag{A.3}$$

then, for any $\overline{\mathtt{n}} \in \mathbb{N}_0$, *for any* $\mathtt{n} = 0, \ldots, \overline{\mathtt{n}}$, *the following holds true:*

$(\mathbf{S1})_\mathtt{n}$ *There exists a linear quasi-periodic transport operator*

$$X_\mathtt{n} := \omega \cdot \partial_\varphi + (\mathtt{m}_{1,\mathtt{n}} + p_\mathtt{n}(\varphi, x))\partial_x \, , \tag{A.4}$$

defined for all $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, *where* $p_\mathtt{n}(\varphi, x)$ *is a quasi-periodic traveling wave function,* even(φ, x), *such that, for any* $\mathfrak{s}_0 \leqslant s \leqslant S$,

$$|p_\mathtt{n}|_s^{k_0,\upsilon} \leqslant C(s, \mathtt{b}) N_{\mathtt{n}-1}^{-\mathtt{a}} |p_0|_{s+\mathtt{b}}^{k_0,\upsilon} \, , \ |p_\mathtt{n}|_{s+\mathtt{b}}^{k_0,\upsilon} \leqslant C(s, \mathtt{b}) N_{\mathtt{n}-1} |p_0|_{s+\mathtt{b}}^{k_0,\upsilon} \, , \tag{A.5}$$

for some constant $C(s, \mathtt{b}) \geqslant 1$ *monotone in* $s \in [\mathfrak{s}_0, S]$, *and* $\mathtt{m}_{1,\mathtt{n}}$ *is a real constant satisfying*

$$\begin{aligned} &|\mathtt{m}_{1,\mathtt{n}}|^{k_0,\upsilon} \leqslant 2 \, |p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon} \, , \\ &|\mathtt{m}_{1,\mathtt{n}} - \mathtt{m}_{1,\mathtt{n}-1}|^{k_0,\upsilon} \leqslant C(\mathfrak{s}_0, \mathtt{b}) N_{\mathtt{n}-2}^{-\mathtt{a}} |p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon} \, , \ \forall \mathtt{n} \geqslant 2 \, . \end{aligned} \tag{A.6}$$

Let $\Lambda_0^{\mathrm{T}} := \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, *and, for* $\mathtt{n} \geqslant 1$, $\Lambda_\mathtt{n}^{\mathrm{T}} := \Lambda_\mathtt{n}^{\upsilon,\mathrm{T}}(p_0)$ *defined as*

$$\Lambda_\mathtt{n}^{\mathrm{T}} := \Big\{ (\omega, \gamma) \in \Lambda_{\mathtt{n}-1}^{\mathrm{T}} : |(\omega - \mathtt{m}_{1,\mathtt{n}-1} \vec{\jmath}) \cdot \ell| \geqslant \frac{\upsilon}{\langle \ell \rangle^\tau} \ \forall \, \ell \in \mathbb{Z}^\nu \backslash \{0\}, \ |\ell| \leqslant N_{\mathtt{n}-1} \Big\} \, . \tag{A.7}$$

For $\mathtt{n} \geqslant 1$, *there exists a quasi-periodic traveling wave function* $g_{\mathtt{n}-1}(\varphi, x)$, odd$(\varphi, x)$, *defined for all* $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, *fulfilling*

$$|g_{\mathtt{n}-1}|_s^{k_0,\upsilon} \leqslant C(s) N_{\mathtt{n}-1}^{\tau_1} \upsilon^{-1} |\Pi_{N_{\mathtt{n}-1}} p_{\mathtt{n}-1}|_s^{k_0,\upsilon} \, , \ \forall \mathfrak{s}_0 \leqslant s \leqslant S \, , \tag{A.8}$$

for some constant $C(s) \geqslant 1$ *monotone in* $s \in [\mathfrak{s}_0, S]$, *such that, defining the composition operator* $(\mathcal{G}_{\mathtt{n}-1} u)(\varphi, x) := u(\varphi, x + g_{\mathtt{n}-1}(\varphi, x))$, *induced by the diffeomorphism* $x \mapsto x + g_{\mathtt{n}-1}(\varphi, x)$, *we have, for any* (ω, γ) *in the set* $\Lambda_\mathtt{n}^{\mathrm{T}}$ *(cfr.* (A.7)*), the following conjugation formula*

$$X_\mathtt{n} = \mathcal{G}_{\mathtt{n}-1}^{-1} \, X_{\mathtt{n}-1} \, \mathcal{G}_{\mathtt{n}-1} \, . \tag{A.9}$$

$(\mathbf{S2})_\mathtt{n}$ *Let* $\Delta_{12} p_0 := p_{0,1} - p_{0,2}$. *For any* $s_1 \in [s_0 + 1, S]$, *there exist* $C(s_1) > 0$ *and* $\delta'(s_1) \in (0, 1)$ *such that if*

$$N_0^{\tau_2} \sup_{(\omega,\gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]} \big(\|p_{0,1}\|_{s_1+\mathtt{b}} + \|p_{0,2}\|_{s_1+\mathtt{b}} \big) \upsilon^{-1} \leqslant \delta'(s_1),$$

then, for all $(\omega,\gamma)\in\mathbb{R}^\nu\times\mathbb{R}$,

(A.10)
$$\|\Delta_{12}p_{\mathtt{n}}\|_{s_1-1}\leqslant C(s_1)N_{\mathtt{n}-1}^{-\mathtt{a}}\|\Delta_{12}p_0\|_{s_1+\mathtt{b}},\quad \|\Delta_{12}p_{\mathtt{n}}\|_{s_1+\mathtt{b}}\leqslant C(s_1)N_{\mathtt{n}-1}\|\Delta_{12}p_0\|_{s_1+\mathtt{b}},$$

(A.11)
$$|\Delta_{12}(\mathtt{m}_{1,\mathtt{n}+1}-\mathtt{m}_{1,\mathtt{n}})|\leqslant\|\Delta_{12}p_{\mathtt{n}}\|_{s_0}\,,\quad |\Delta_{12}\mathtt{m}_{1,\mathtt{n}}|\leqslant C(s_1)\|\Delta_{12}p_0\|_{s_0}\,.$$

Moreover, for any $s\geqslant s_0$*, one has*

$$\|\Delta_{12}g_{\mathtt{n}}\|_s\lesssim_s \upsilon^{-1}\big(\|\Pi_{N_{\mathtt{n}}}\Delta_{12}p_{\mathtt{n}}\|_{s+\tau}+\upsilon^{-1}|\Delta_{12}\mathtt{m}_{1,\mathtt{n}}|\|\Pi_{N_{\mathtt{n}}}p_{\mathtt{n},2}\|_{s+2\tau+1}\big).$$

We deduce the following corollaries.

Corollary A.3. *For any* $\bar{\mathtt{n}}\in\mathbb{N}_0$ *we have* $\mathtt{TC}_{\bar{\mathtt{n}}+1}(\mathtt{m}_{1,\bar{\mathtt{n}}},2\upsilon,\tau)\subset\Lambda_{\bar{\mathtt{n}}+1}^{\upsilon,\mathrm{T}}$*, with* $\mathtt{TC}_{\bar{\mathtt{n}}+1}(\mathtt{m}_{1,\bar{\mathtt{n}}},2\upsilon,\tau)$ *as in* (6.22).

Proof. When $\bar{\mathtt{n}}=0$, by definition we have $\mathtt{TC}_1(2\upsilon,\tau)\subset\Lambda_1^{\upsilon,\mathrm{T}}$. Let $(\omega,\gamma)\in\mathtt{TC}_{\bar{\mathtt{n}}+1}(\mathtt{m}_{1,\bar{\mathtt{n}}},2\upsilon,\tau)$. For any $k=0,\ldots,\bar{\mathtt{n}}-1$ we have, by (A.6), $|\mathtt{m}_{1,\bar{\mathtt{n}}}-\mathtt{m}_{1,k}|\lesssim_{\mathfrak{s}_0,\mathtt{b}}N_{k-1}^{-\mathtt{a}}|p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon}$. Thus, recalling (6.22), for all $0<|\ell|\leqslant N_k$, we have $|(\omega-\mathtt{m}_{1,k}\vec{\jmath})\cdot\ell|\geqslant|(\omega-\mathtt{m}_{1,\bar{\mathtt{n}}}\vec{\jmath})\cdot\ell|-|\mathtt{m}_{1,\bar{\mathtt{n}}}-\mathtt{m}_{1,k}||\vec{\jmath}||\ell|\geqslant 2\upsilon\langle\ell\rangle^{-\tau}-CN_{k-1}^{-\mathtt{a}}|p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon}|\ell|\geqslant\upsilon\langle\ell\rangle^{-\tau}$ if $CN_k^{\tau+1}N_{k-1}^{-\mathtt{a}}|p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon}\upsilon^{-1}<1$, which is satisfied by (A.3) and (6.20). Thus, recalling (A.7), we have proved that $(\omega,\gamma)\in\Lambda_{\bar{\mathtt{n}}+1}^{\upsilon,\mathrm{T}}$. □

The composition operator $\mathcal{B}_{\mathtt{n}}$, defined inductively by $\mathcal{B}_{\mathtt{n}}:=\mathcal{B}_{\mathtt{n}-1}\circ\mathcal{G}_{\mathtt{n}-1}$, $\mathtt{n}\in\mathbb{N}$, $\mathcal{B}_0:=\mathrm{Id}$, provides the almost-straightening conjugation of the transport vector field X_0.

Corollary A.4. *For any* $\bar{\mathtt{n}}\in\mathbb{N}_0$ *and* $(\omega,\gamma)\in\mathtt{TC}_{\bar{\mathtt{n}}+1}(\mathtt{m}_{1,\bar{\mathtt{n}}},2\upsilon,\tau)$ *we have the conjugation formula* $X_{\bar{\mathtt{n}}}=\mathcal{B}_{\bar{\mathtt{n}}}^{-1}X_0\mathcal{B}_{\bar{\mathtt{n}}}$*, where* $X_{\bar{\mathtt{n}}}$ *is given in* (A.4) *with* $\mathtt{n}=\bar{\mathtt{n}}$*. Moreover, when* $\bar{\mathtt{n}}\geqslant 1$*, for any* $\mathtt{n}=1,\ldots,\bar{\mathtt{n}}$*, each* $\mathcal{B}_{\mathtt{n}}$ *is the composition operator induced by the diffeomorphism of the torus* $x\mapsto x+\beta_{\mathtt{n}}(\varphi,x)$*,* $(\mathcal{B}_{\mathtt{n}}u)(\varphi,x)=u(\varphi,x+\beta_{\mathtt{n}}(\varphi,x))$*, where the function* $\beta_{\mathtt{n}}$ *is a quasi-periodic traveling wave,* $\mathrm{odd}(\varphi,x)$*, satisfying, for any* $\mathfrak{s}_0\leqslant s\leqslant S$*, for some constant* $\underline{C}(S)\geqslant 1$*,*

(A.12)
$$|\beta_{\mathtt{n}}|_s^{k_0,\upsilon}\leqslant\underline{C}(S)\upsilon^{-1}N_0^{\tau_1}|p_0|_{s+\mathtt{b}}^{k_0,\upsilon}\,.$$

Furthermore, for $p_{0,1},p_{0,2}$ *as in* $(\mathbf{S2})_{\mathtt{n}}$*, we have*

$$\|\Delta_{12}\beta_{\bar{\mathtt{n}}}\|_{s_1}\leqslant\overline{C}(S)\upsilon^{-1}N_0^{\tau}\|\Delta_{12}p_0\|_{s_1+\mathtt{b}}.$$

Proof. We have $\beta_1=g_0$, and inductively $\beta_{\mathtt{n}}=\beta_{\mathtt{n}-1}+\mathcal{B}_{\mathtt{n}-1}g_{\mathtt{n}-1}$. Since $g_{\mathtt{n}}$ is a quasi-periodic traveling wave $\mathrm{odd}(\varphi,x)$, so is $\beta_{\mathtt{n}}$. The estimates follow by Theorem A.2 and Lemma A.1. □

Proof of Theorem A.2. The proof is inductive. In Lemma A.5 we prove that the norms $|p_{\mathtt{n}}|_s^{k_0,\upsilon}$ satisfy inequalities of a Nash-Moser iterative scheme, which converges under the smallness condition (A.3).

The step $\mathtt{n} = 0$. Items $(\mathbf{S1})_0$, $(\mathbf{S2})_0$, hold with $\mathtt{m}_{1,0} := 0$.

The reducibility step. We show now how to transform $X_{\mathtt{n}}$ in (A.4) into $X_{\mathtt{n}+1}$ by conjugating with the composition operator $\mathcal{G}_{\mathtt{n}}$ induced by the diffeomorphism $y := x + g_{\mathtt{n}}(\varphi, x)$ of $\mathbb{T}_x$ where $g_{\mathtt{n}}(\varphi, x)$ is a periodic function defined below, see (A.14). A direct computation gives (cfr. Remark 6.2)

$$\mathcal{G}_{\mathtt{n}}^{-1}\, X_{\mathtt{n}}\, \mathcal{G}_{\mathtt{n}} = \omega\cdot\partial_\varphi + \mathtt{m}_{1,\mathtt{n}}\partial_y + \{\mathcal{G}_{\mathtt{n}}^{-1}\big((\omega\cdot\partial_\varphi + \mathtt{m}_{1,\mathtt{n}}\partial_x)g_{\mathtt{n}} + p_{\mathtt{n}} + p_{\mathtt{n}}(g_{\mathtt{n}})_x\big)\}\partial_y\,.$$

We choose $g_{\mathtt{n}}(\varphi, x)$ as the solution of the homological equation

$$(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\mathtt{n}}\partial_x)g_{\mathtt{n}}(\varphi, x) + \Pi_{N_{\mathtt{n}}} p_{\mathtt{n}} = \langle p_{\mathtt{n}}\rangle_{\varphi,x} \tag{A.13}$$

where $\langle p_{\mathtt{n}}\rangle_{\varphi,x}$ is the average of $p_{\mathtt{n}}$ defined as in (3.2). So we define

$$g_{\mathtt{n}}(\varphi, x) := -(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\mathtt{n}}\partial_x)_{\rm ext}^{-1}(\Pi_{N_{\mathtt{n}}} p_{\mathtt{n}} - \langle p_{\mathtt{n}}\rangle_{\varphi,x}) \tag{A.14}$$

where the operator $(\omega\cdot\partial_\varphi + \mathtt{m}_{1,\mathtt{n}}\partial_x)_{\rm ext}^{-1}$ is introduced in (3.5). The function $g_{\mathtt{n}}(\varphi, x)$ is defined for all parameters $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$, it is a quasi-periodic traveling wave, $\mathrm{odd}(\varphi, x)$, fulfills (A.8) at the step $\mathtt{n}$ (by (3.7)), and for any (ω, γ) in the set $\Lambda_{\mathtt{n}+1}^{\mathrm{T}}$ defined in (A.7), it solves the homological equation (A.13). By (A.8) at the step $\mathtt{n}$, (A.5), (A.3), $\mathtt{a} \geqslant \chi\tau_1 + 3$ (see (6.20))

$$|g_{\mathtt{n}}|_{2\mathfrak{s}_0+1}^{k_0,\upsilon} \leqslant C(\mathfrak{s}_0) N_{\mathtt{n}}^{\tau_1} N_{\mathtt{n}-1}^{-\mathtt{a}} |p_0|_{2\mathfrak{s}_0+\mathtt{b}+1}^{k_0,\upsilon}\upsilon^{-1} < \delta(\mathfrak{s}_0) \tag{A.15}$$

provided N_0 is large enough. By Lemma A.1 the diffeomorphism $y = x + g_{\mathtt{n}}(\varphi, x)$ is invertible and its inverse $x = y + \breve{g}_{\mathtt{n}}(\varphi, y)$ (which induces the operator $\mathcal{G}_{\mathtt{n}}^{-1}$) satisfies $|\breve{g}_{\mathtt{n}}|_s^{k_0,\upsilon} \leqslant C(s)|g_{\mathtt{n}}|_s^{k_0,\upsilon}$. For any (ω, γ) in $\Lambda_{\mathtt{n}+1}^{\mathrm{T}}$, the operator $X_{\mathtt{n}+1} = \mathcal{G}_{\mathtt{n}}^{-1}\, X_{\mathtt{n}}\, \mathcal{G}_{\mathtt{n}}$ takes the form (A.4) at step $\mathtt{n}+1$ with

$$\begin{aligned} &\mathtt{m}_{1,\mathtt{n}+1} := \mathtt{m}_{1,\mathtt{n}} + \langle p_{\mathtt{n}}\rangle_{\varphi,x} \in \mathbb{R}\,, \\ &p_{\mathtt{n}+1}(\varphi, y) := \{\mathcal{G}_{\mathtt{n}}^{-1}\big(\Pi_{N_{\mathtt{n}}}^\perp p_{\mathtt{n}} + p_{\mathtt{n}}(g_{\mathtt{n}})_x\big)\}(\varphi, y)\,. \end{aligned} \tag{A.16}$$

This verifies (A.9) at step $\mathtt{n}+1$. Note that $\mathtt{m}_{1,\mathtt{n}+1} \in \mathbb{R}$ and $p_{\mathtt{n}+1}(\varphi, y)$ in (A.16) are defined for all $(\omega, \gamma) \in \mathbb{R}^\nu \times [\gamma_1, \gamma_2]$. We first show the following iterative estimates of Nash-Moser type.

Lemma A.5. *The function* $p_{\mathtt{n}+1}$ *in* (A.16) *satisfies, for any* $\mathfrak{s}_0 \leqslant s \leqslant S$,

$$|p_{\mathtt{n}+1}|_s^{k_0,\upsilon} \leqslant C_1(s)\big(N_{\mathtt{n}}^{-\mathtt{b}}|p_{\mathtt{n}}|_{s+\mathtt{b}}^{k_0,\upsilon} + N_{\mathtt{n}}^{\tau_1+1}\upsilon^{-1}|p_{\mathtt{n}}|_s^{k_0,\upsilon}|p_{\mathtt{n}}|_{\mathfrak{s}_0}^{k_0,\upsilon}\big) \tag{A.17}$$

$$|p_{\mathtt{n}+1}|_{s+\mathtt{b}}^{k_0,\upsilon} \leqslant C_2(s,\mathtt{b})\big(|p_{\mathtt{n}}|_{s+\mathtt{b}}^{k_0,\upsilon} + N_{\mathtt{n}}^{\tau_1+1}\upsilon^{-1}|p_{\mathtt{n}}|_{s+\mathtt{b}}^{k_0,\upsilon}|p_{\mathtt{n}}|_{\mathfrak{s}_0}^{k_0,\upsilon}\big) \tag{A.18}$$

where $C_1(s), C_2(s,\mathtt{b}) > 0$ *are monotone in* $\mathfrak{s}_0 \leqslant s \leqslant S$. *Moreover,* (A.5)-(A.6) *hold at the step* $\mathtt{n}+1$.

Proof. We write $p_{\mathtt{n}+1}$ in (A.16) as $p_{\mathtt{n}+1} := \mathcal{G}_{\mathtt{n}}^{-1} F_{\mathtt{n}}$ with $F_{\mathtt{n}} := \Pi_{N_{\mathtt{n}}}^{\perp} p_{\mathtt{n}} + p_{\mathtt{n}}(g_{\mathtt{n}})_x$. By Lemma A.1, we get

$$|F_{\mathtt{n}}|_s^{k_0,\upsilon} \leqslant |\Pi_{N_{\mathtt{n}}}^{\perp} p_{\mathtt{n}}|_s^{k_0,\upsilon} + C(s)|p_{\mathtt{n}}|_s^{k_0,\upsilon}|g_{\mathtt{n}}|_{\mathfrak{s}_0+1}^{k_0,\upsilon} + C(\mathfrak{s}_0)|p_{\mathtt{n}}|_{\mathfrak{s}_0}^{k_0,\upsilon}|g_{\mathtt{n}}|_{s+1}^{k_0,\upsilon}.$$

Therefore (A.17) follows by (A.2), (A.8) at step $\mathtt{n}$, Lemma A.1 and (A.15). The estimate (A.18) follows analogously.

By (A.17) and (A.5) we have, for any $\mathfrak{s}_0 \leqslant s \leqslant S$,

$$\begin{aligned} |p_{\mathtt{n}+1}|_s^{k_0,\upsilon} &\leqslant C_1(S)\, C(s,\mathtt{b}) \big(N_{\mathtt{n}}^{-\mathtt{b}} N_{\mathtt{n}-1} |p_0|_{s+\mathtt{b}}^{k_0,\upsilon} \\ &\qquad + C(\mathfrak{s}_0,\mathtt{b})\upsilon^{-1} N_{\mathtt{n}}^{\tau_1+1} N_{\mathtt{n}-1}^{-2\mathtt{a}} |p_0|_{s+\mathtt{b}}^{k_0,\upsilon} |p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon} \big) \\ &\leqslant C(s,\mathtt{b}) N_{\mathtt{n}}^{-\mathtt{a}} |p_0|_{s+\mathtt{b}}^{k_0,\upsilon}, \end{aligned}$$

asking $C_1(S) N_{\mathtt{n}}^{-\mathtt{b}} N_{\mathtt{n}-1} \leqslant \frac{1}{2} N_{\mathtt{n}}^{-\mathtt{a}}$ and

$$C_1(S) C(\mathfrak{s}_0,\mathtt{b})\upsilon^{-1} N_{\mathtt{n}}^{\tau_1+1} N_{\mathtt{n}-1}^{-2\mathtt{a}} |p_0|_{\mathfrak{s}_0+\mathtt{b}}^{k_0,\upsilon} \leqslant \tfrac{1}{2} N_{\mathtt{n}}^{-\mathtt{a}},$$

which both follow by (6.20), the smallness assumption (A.3) and with $N_0 := N_0(S) > 0$ sufficiently large. This proves the first estimate of (A.5) at step $\mathtt{n}+1$. The second follows similarly. By (A.16) and (A.5), we prove (A.6) at step $\mathtt{n}+1$. □

The proof of $(\mathbf{S1})_{\mathtt{n}+1}$ is complete. The item $(\mathbf{S2})_{\mathtt{n}+1}$ follows similarly. The proof of Theorem A.2 is concluded. □

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