SISSA
Scuola
Internazionale
Superiore di
Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Local Ehrhart Theory and Gale Duality

Candidate:
Vadym Kurylenko

Advisor:
Fernando Rodriguez Villegas

Academic Year 2023-24

Dedicated to my grandmother, Nesterenko Svetlana Andreevna.

Abstract

The central object of this dissertation is the local h^*-polynomial of a lattice polytope. This is an invariant that arises in the study of counting the lattice points inside polytopes and their dilations, the so-called Ehrhart theory. If a lattice polytope is spanning, it can be defined by the data of its Gale dual. In this thesis, we try to understand what we can say about the local h^*-polynomial from a given Gale dual.

The thesis comprises of three projects. In the first one, we introduce the shifted products of circuits and give arithmetic-flavoured expressions for the their (local) h^*-polynomial, extending the known results for circuits. In the second project, we prove that there exists a geometric construction which preserves the coefficients of the local h^*-polynomial up to an overall shift. We call it a Lawrence twist. This construction in particular helps to disprove a conjecture about polytopes with vanishing local h^*-polynomial. In the third project, we obtain the complete classification of the four-dimensional simplices with vanishing local h^*-polynomial. We show that any such simplex that is not a free join must belong either to a certain one-parameter family of simplices or it must be one of the six sporadic cases.

Acknowledgments

I would like to express my gratitude to my supervisor, Fernando Rodriguez Villegas, for for his generosity with problems and ideas, for suggesting in the very beginning to read that paper by Batyrev and Borisov that changed my mathematical interests, and for showing that coding can be an important part of mathematical research.

I am deeply grateful to Benjamin Nill for his genuine interest in my work, for inviting me for research visits to Magdeburg, and for reading the preliminary version of the last chapter of this thesis.

I would like to thank Jacopo Stoppa for agreeing to be my supervisor before my move to Trieste and for being the reason I wanted to move there in the first place, even though eventually I ended up not being his student.

I owe my gratitude to Murad Alim for the PhD year I spent in Hamburg with him before SISSA, and for him being OK with me moving to Trieste, though it didn't go exactly as planned.

I am thankful to Asem for being my academic brother, for countless discussions we had and many examples that we worked out together, and for helping me not to drown completely whenever we got too close to number theory.

I am indebted to Giulia for many helpful conversations, for her valuable support, and for sharing her notes on the $(1,1,1,1)$ hypergeometric motives.

My thanks go to Mykyta for listening to my numerous mathematical and not only complaints and for his help with multiple questions.

I also would like to thank Martín, Moha and Vasily Golyshev for being a part of our research group and for many helpful discussions. I am grateful to Danylo Radchenko for generously sharing with us his Sage implementation of the formula of Batyrev and Borisov.

I would like to thank Lorenzo, Riccardo and Sohaib for all the fun we had during these 4 years. Special thanks goes to Sohaib for being such a gentleman and a scholar.

My deepest thanks goes to Daria for her love and support during the last 10 years, especially during the last one, for multiple ways she helped me and for believing in me. I am thankful to my mother for being so curious, in particular so about the state of this thesis. A special thank you goes to Eveline for the immense joy that she brought and ensuring a minimal daily physical activity.

I express my profound gratitude to the Armed Forces of Ukraine for all the sacrifice and for protecting my family in Kharkiv.

Finally, I would like to thank all the SISSA and ICTP staff and community for their assistance and for providing such an amazing work atmosphere!

1 Introduction

The main object of study in this thesis is lattice polytopes. These are simple geometric objects, however, they hide a lot of mysteries. Lattice polytopes are fundamental objects that appear in multiple research areas such as toric geometry, representation theory and mathematical physics. At their core, lattice polytopes are number-theoretic objects, and this combination of geometry and number theory leads to a variety of non-trivial and fascinating problems that hide behind the apparent simplicity.

A classical direction of studying lattice polytopes is the so-called Ehrhart theory. It deals with counting the number of lattice points in a given lattice polytope and its dilations. Eugene Ehrhart showed [Ehr62] that the number of points in the k-th dilation of a d-dimensional lattice polytope P is given by evaluating at k a certain polynomial of degree d. This polynomial is now known as the Ehrhart Polynomial of P. This means that if we pack these numbers into a generating series, the outcome is going to be a rational function of particular form

$$1 + \sum_{k \geq 1} \mid kP \cap \mathbb{Z}^d \mid t^k = \frac{h^*(P,t)}{(1-t)^{d+1}}.$$

Its numerator, known as the h^*-polynomial, is probably the most studied object in Ehrhart theory today. There are numerous interesting questions about it. What are the restrictions on its coefficients? When do these coefficients form a unimodal sequence? What are the possible meaningful variations of $h^*(P,t)$, such as, for example, a natural q-version?

While the h^*-polynomial definitely plays a big role in this work, and some results are discussed in Chapter 3, the primary focus for us is really on its younger sibling, the local h^*-polynomial. This invariant of a lattice polytope was introduced by Betke and McMullen [BM85], and extended later by Stanley [Sta92]. The main reason for introducing this polynomial was to answer the following question. Given a triangulation of a lattice polytope, can we compute the h^*-polynomial of this polytope via the point counting of the cells of the triangulation? In other words, can one give a local expression for the h^*-polynomial? Betke and McMullen answered the question positively and more recently Katz and Stapledon [KS16a] extended the result to any polyhedral subdivision. Let us finally introduce the local h^*-polynomial. It is defined by the following weighted alternating sum of the h^*-polynomials of the faces of the polytope

$$l^*(P,t) := \sum_{F \subseteq P} (-1)^{\dim P - \dim F} h^*(F,t)\ g([F,P]^*,t).$$

The weights $g([F,P]^*,t)$ will be explained in Chapter 2.

The local h^*-polynomial appeared in various works before. Let us give a short overview. After its origin in the the aforementioned works of Betke, McMullen and Stanley, it appeared implicitly in the work of Batyrev and Borisov [BB96]. Later, it resurfaced independently in the paper of Borisov and Mavlyutov [BM03]. They, along with Karu [Kar08], showed that the coefficients of the local h^*-polynomial are non-negative. Other early works where the local h^*-polynomial came up are [BN08],[Sch12] and [NS13]. The important work of Katz and Stapledon [KS16a] discussed $l^*(P,t)$ in great detail, studying its interplay with polyhedral subdivisions and discussing its refinements. The local h^*-polynomials of the simplices corresponding to the weighted projective spaces were studied in [Sol19] and those of the s-lecture hall simplices in [GS20]. In [Vil19] and [RRV22] $l^*(P,t)$ appeared in the study of integrality of factorial ratios and hypergeometric motives, respectively. In [APPS22] the question of unimodality of the coefficients of $l^*(P,t)$ was investigated. The paper [BKN23] by Borger, Kretschmer and Nill investigates the class of polytopes with vanishing local h^*-polynomial. The work [BBC+23] discusses the local h^*-polynomial of simplices whose coordinates are in Hermitian normal form with one non-trivial row.

The projects discussed in this thesis arise essentially from the following simple observation. Let A be a spanning d-dimensional point configuration consisting of n lattice points. There exists a natural duality that puts in correspondence a spanning integral vector configuration of size n but in dimension $n-d-1$. This correspondence is known as Gale duality (see Chapter 2 for more details). Let P be a spanning d-dimensional lattice polytope and take A to be a spanning subset of its lattice points such that $\operatorname{conv}(A) = P$. The polytope P can now be defined by considering a Gale dual G of A. A few natural questions arise. How can we extract the Ehrhart-theoretic data of P directly from the given Gale dual G? Does this perspective offer any new insights or advantages?

For the simplest possible case of one-dimensional Gale duals, the answer to the first question was given by Rodriguez Villegas in [Vil19]. The corresponding point configurations are called circuits, also defined as minimally dependent point configurations. In this case, there exist elegant arithmetic formulae for both the h^*- and l^*-polynomials of P. Specifically, it is possible to express these polynomials as a certain sum over a finite set of positive integers determined by the Gale dual of A. We present the formula for the h^*-polynomial here, but postpone all the details until Chapter 3:

$$h^*(P,t) = \sum_{N\geq 1} \frac{t^{m_+(N)}-1}{t-1} \sum_{\substack{b=1\\ \gcd(b,N)=1}}^{N-1} t^{\sum_i \{\frac{bG_i}{N}\}}.$$

The **first objective** of this thesis is to generalize this result to more complex point configurations. We do so for point configurations that we call shifted products of circuits. They are defined by the property that their Gale dual G can be chosen in such a way that the vectors in G either belong to the positive parts of the coordinate axes or to the negative orthant of $\mathbb{R}^{n-d-1}$. Here is an example of a shifted product of circuits with a three-dimensional Gale dual given by the columns of

$$\begin{pmatrix} -2 & -1 & -2 & -2 & 7 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -2 & 4 & 0 & 0 \\ -3 & -4 & 0 & 0 & 0 & 0 & 0 & -1 & 8 \end{pmatrix}.$$

In Chapter 3 we achieve this objective and derive an expression for the h^*-polynomial of a shifted product of circuits as a finite sum over tuples of integers, generalizing the formula of Rodriguez Villegas, see Theorem 8. This is possible because of the existence of a rather convenient triangulation of these point configurations combined with the fact that we can express the h^*-polynomial in terms of the local contributions from the local h^*-polynomials. Moreover, if a shifted product of circuits is simplicial, Theorem 11 provides a similar formula for its local h^*-polynomial.

The **second project**, covered in Chapter 4, explores constructions that preserve the coefficients of the local h^*-polynomial. Suppose P is a spanning lattice polytope and G is a Gale dual of $P \cap \mathbb{Z}^d$. We perform the following operation. We add to the Gale dual G a centrally symmetric vector configuration S. This results in a vector configuration $\tilde{G}$ that can be thought as a Gale dual of the lattice points of another polytope $\tilde{P}$. We call the polytope $\tilde{P}$ the Lawrence twist of P by S after the Lawrence polytopes defined by the property that their Gale dual is centrally symmetric.

In Theorem 12 we show that the local h^*-polynomials of P and $\tilde{P}$ are identical up to a simple factor of $t^{\frac{1}{2}|S|}$. This result stems from the connection between the local h^*-polynomial and the geometry of the affine hypersurfaces in tori. Namely, we use the fact that the coefficients of the local h^*-polynomial are in fact certain Hodge numbers of the middle cohomology of a generic hypersurface defined by the Newton polytope P.

This construction enables us to provide a counter-example to one of the conjectures of Borger, Kretschmer and Nill [BKN23] regarding the possible degree of $h^*(P, t)$ for spanning polytopes with vanishing $l^*(P, t)$.

We finish Chapter 4 by conjecturing that for a wide class of spanning polytopes there exists another operation that preserves the coefficients of the local h^*-polynomial. In the case of one-dimensional Gale diagrams, it appeared first in in [GV24], where the name total twist was coined. This construction is dependent on the parity properties of the elements of a Gale dual.

What happens if the polytope P is not spanning? It is still possible to define a Gale dual, and it will give us useful information about the faces of P, but in terms of the lattice structures we lose the duality. However, if our polytope is a simplex Δ, there still exists a similar duality. Batyrev and Hofscheier [BH13] explained that to each simplex Δ we can associate a linear code C_Δ. When Δ is spanning, this linear code is equivalent to the Gale dual of $\Delta \cap \mathbb{Z}^d$. This duality between simplices and linear codes is the main engine of the **third project**, discussed in Chapter 5.

We use this correspondence between lattice simplices and certain linear codes to follow up the paper [BKN23] and try to answer the following question: what simplices have vanishing local h^*-polynomial? Such simplices are called thin. This is one of the simplest questions one can ask about any invariant. However, it does not seem that it has an easy answer for the local h^*-polynomial. This question in fact appeared first implicitly in the celebrated book [GKZ94] of Gel'fand, Kapranov and Zelevenskii. They were able to answer it for the two-dimensional triangles. Recently Borger, Kretschmer and Nill [BKN23] revived the question and managed to classify all three-dimensional thin simplices, namely, all such simplices are lattice pyramids.

Using the aforementioned duality, we can translate the vanishing of $l^*(\Delta, t)$ to the following property of the corresponding linear code C_Δ. The simplex Δ is thin if and only if the linear code C_Δ has no words of maximal weight, i.e. there is at least one zero in each of its words. The technical difficulty appearing here is the fact that we have to work with linear codes over finite rings $\mathbb{Z}_N$ of integers mod N and not just over finite fields.

The above translation allows us to answer the question in dimension four. We obtain the following classification. If Δ is a 4-dimensional thin simplex and it is not a free join, then it either belongs to a one-parameter family or it is one of the six sporadic cases. The details can be found in Theorem 15. The results of Chapter 5 are also recorded in the preprint [Kur24].

I would like to end this introduction by mentioning some other works where the interplay of Ehrhart theory and Gale duality was observed. In [NP15] Gale duality was used in the discussion about the degree of $h^*(P, t)$. Very recently, the usage of Gale duality for Ehrhart-theoretic problems appeared in [BBR24]. Given a vector configuration A and its Gale dual G, the authors were able to compute the Ehrhart polynomial of a lattice zonotope defined by G through certain arithmetic data of the matrix A.

Contents

2 Background material

2.1 Basic definitions

Generally speaking, this thesis is concerned with geometric objects in $\mathbb{R}^d$. The most basic ones are the following.

Definition 1. A **point configuration** in $\mathbb{R}^d$ is a finite collection of points $\{a_i\}_{i \in I}$. Similarly, a **vector configuration** is a finite collection of vectors $\{v_i\}_{i \in I}$.

Since it is usually more convenient to work with linear algebra than affine-linear algebra, we often want to use vector configurations instead of the point configurations. Let $A = \{a_1, \dots, a_n\}$ be a point configuration in $\mathbb{R}^d$. We define **homogenization** of A to be the vector configuration $\bar{A}$ in $\mathbb{R}^{d+1}$ given by $(a_1, 1), \dots, (a_n, 1)$. Let us denote by A also the matrix whose columns are the coordinates of the points $a_1, \dots, a_n$. Then the homogenization $\bar{A}$ can be represented by the matrix with a row of 1's added at the end.

We are particularly interested in the points of $\mathbb{R}^d$ that belong to a certain lattice, for example, the points whose coordinates are integral. The central object of this thesis is that of a lattice polytope.

Definition 2. Let $M \simeq \mathbb{Z}^d \subseteq \mathbb{R}^d$ be a lattice. A **lattice polytope** is a convex hull of a finite number of points $a_1, \dots, a_n$ belonging to the lattice M.

Definition 3. We say that two polytopes P and Q are **isomorphic** (or **unimodularly equivalent**) if there exists an invertible affine-linear transformation of the lattice M that maps P to Q.

Most of the times we take the lattice M to be equal to $\mathbb{Z}^d$.

Definition 4. A point configuration A is called **spanning** if the points of A affinely span the lattice $\mathbb{Z}^d$. Equivalently, the homogenization $\bar{A}$ spans linearly the lattice $\mathbb{Z}^{d+1}$.

We call a lattice polytope P **spanning** if the point configuration $P \cap \mathbb{Z}^d$ is spanning. A lattice polytope is called **hollow** if it does not contain any lattice points in its interior and it is called **empty** if its only lattice points are its vertices.

We use Δ_n to denote the standard n-dimensional **unimodular simplex**

$$\Delta_n = \operatorname{conv}(0, e_1, \ldots, e_d),$$

where $e_1, \ldots, e_d$ is the standard lattice basis of $\mathbb{Z}^d$.

Definition 5. Let $m \geq 2$ and $P_1, \ldots, P_m$ be lattice polytopes in $\mathbb{R}^d$. The **Cayley sum** of **length** m is

$$P_1 * \ldots * P_m = \operatorname{conv}\left((P_1 \times 0), (P_2 \times e_1), \ldots, (P_m \times e_{m-1})\right) \subseteq \mathbb{R}^d \times \mathbb{R}^{m-1}.$$

A polytope is called **Cayley polytope** if it is isomorphic to some Cayley sum.

There is another very nice characterization of the Cayley polytopes in terms of another invariant of polytopes.

Definition 6. Let $\langle -, - \rangle$ denote the standard scalar product in $\mathbb{R}^d$. The **lattice width** of a lattice polytope P is defined as the minimum of

$$\max_{x \in P} \langle u, x \rangle - \min_{x \in P} \langle u, x \rangle$$

over all non-zero integer linear forms u.

Proposition 1. A lattice polytope P is of lattice width 1 if and only if it is a Cayley polytope.

A particular class of Cayley polytopes that is particularly useful in the theory of lattice points counting is that of free joins.

Definition 7. Let $P_1 \subseteq \mathbb{R}^{d_1}$ $P_2 \subseteq \mathbb{R}^{d_2}$ be lattice polytopes. A **free join** of P_1 and P_2 is

$$P_1 \circ_{\mathbb{Z}} P_2 = \operatorname{conv}\left(P_1 \times \{0^{d_2}\} \times \{0\}, \{0^{d_1}\} \times P_2 \times \{1\}\right) \subseteq \mathbb{R}^{d_1+d_2+1}.$$

An important example of a free join is when one of the polytopes, say P_2, is a single point, so $d_2 = 0$. In this case, the resulting free join is called the **lattice pyramid** over P_1

$$\operatorname{Pyr}(P_1) = \operatorname{conv}\left((P_1 \times \{0\}, \{0^{d_1}\} \times \{1\}\right) \subseteq \mathbb{R}^{d_1+1}.$$

In this thesis we mostly use the normalized volume of a polytope defined for a d-dimensional polytope by

$$\operatorname{vol}_{\mathbb{Z}} P := d! \cdot \operatorname{vol}_{\mathbb{R}} P,$$

where $\operatorname{vol}_{\mathbb{R}}$ is the standard Euclidean volume in $\mathbb{R}^d$.

Finally let us mention a natural addition operation for convex bodies P an Q in $\mathbb{R}^d$. The **Minkowski sum** of P and Q is defined by

$$P + Q := \{p + q \ : \ p \in P, \ q \in Q\}.$$

2.2 Posets and associated polynomials

h-polynomials and g-polynomials

Given a d-dimensional polytope P, one of its basic combinatorial invariants is the number of i-dimensional faces f_i with convention that $f_{-1} = 1$. The tuple $(f_{-1}, f_0, f_1, \dots)$ is known as **f-vector** of P. When P is simplicial, it is convenient to introduce new numbers h_i by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}. \tag{2.1}$$

The tuple

$$(h_0, h_1, \dots, h_d)$$

is called the **h-vector** of P.

The entries of the f-vector satisfy certain linear relations called Dehn-Sommerville relations. They take particularly simple form if expressed using the h-vector

$$h_i = h_{d-i}. \tag{2.2}$$

If P is not simplicial and we try to define the numbers h_i in the same way, then this relation does not hold anymore. However, there is a way to fix it introduced by Stanley [Sta87]. To describe it, it is convenient to switch to the language of posets. We follow closely the presentation of [KS16a].

Let B be a finite poset. If it has a unique minimal element, we denote it by $\hat{0}_B$, and if there is a unique maximal element, we denote it by $\hat{1}_B$. We will omit the subscript if it is clear which poset we are talking about. A poset is called **locally graded** if in every interval $[x, y]$ all the maximal chains have the same length, denoted by $\rho(x, y)$. The **rank of** B is the longest length of a maximal chain in B. If B has a unique minimal element and is locally graded, then we call it **lower graded**.

Suppose B has unique minimal and maximal elements. We call B **Eulerian** if every interval of positive length has the same number of elements of odd and even rank. If B is locally graded, then it is **locally Eulerian** if every its interval is Eulerian. A locally Eulerian poset is called **lower Eulerian** if it contains a unique minimal element.

Example 1. Consider a two-dimensional lattice triangle with 4 lattice points. And consider its triangulation depicted on Figure 2.1. Let B be the face poset of this triangulation. It is shown on Figure 2.2. This poset has rank 3. It is not Eulerian, since there are three maximal elements. However, every interval of length k is just a face poset of a $k-1$-dimensional simplex. Therefore, B is lower Eulerian.

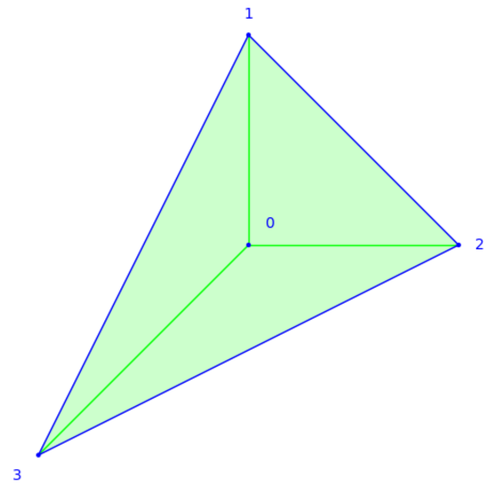


Figure 2.1: A triangulated triangle

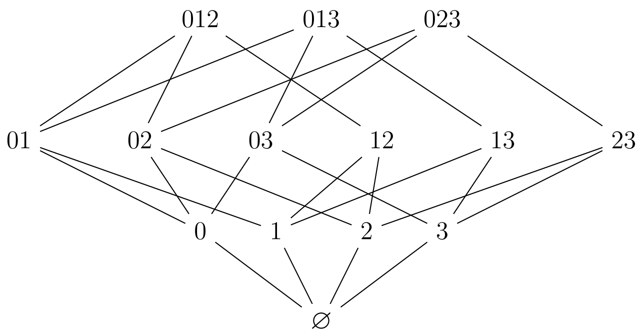


Figure 2.2: Face poset of the triangulation in Figure 2.1

To generalize the notion of h-vectors to the case of non-simplicial polytopes Stanley introduced the following polynomials defined recursively.

Definition 8. Let B be an Eulerian poset of rank n. If $n = 0$ set $g(B, t) = 1$. For $n \geq 1$ define $g(B, t)$ as the unique polynomial of degree $< n/2$ satisfying

$$t^n g(B, t^{-1}) = \sum_{x \in B} g([\hat{0}, x], t)(t-1)^{n-\rho(\hat{0}, x)}.$$

Remark 1 (Remark 4.2 in [BN08])**.** Here are a few nice properties of the g-polynomial.

- This polynomial can be understood as a certain measure of complexity of the poset B. In particular, $g(B, t) = 1$ if and only if B is isomorphic to the **Boolean** poset B_n, i.e. it is isomorphic to the face poset of an $(n-1)$-dimensional simplex.
- $g(B, t)$ has non-negative coefficients.
- Let C and D be posets. Their **product** $B = C \times D$ is the poset whose underlying set is the Cartesian product of the corresponding underlying sets of C and D with the order relation given by

  $$(x_1, y_1) \leq (x_2, y_2) \quad \text{iff} \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

  The g-polynomial is multiplicative with respect to this product, i.e.

  $$g(B, t) = g(C, t) \cdot g(D, t).$$

Analogously, for a lower Eulerian poset we define another polynomial.

Definition 9. Let B be a lower Eulerian poset of rank n. Define h**-polynomial** of B to be

$$t^n h(B, t^{-1}) = \sum_{x \in B} g([\hat{0}, x], t)(t-1)^{n-\rho(\hat{0}, x)}.$$

In particular, if B is Eulerian, then $g(B, t) = h(B, t)$.

Example 2. Consider again the triangulation from Example 1. Let B be its face poset. Let us compute its h-polynomial.

$$t^3 \; h(B, \; t^{-1}) = (t-1)^3 + 4(t-1)^2 + 6(t-1) + 3.$$

Thus,

$$h(B, \; t) = t^2 + t + 1.$$

The above poset is an example of a particularly nice class of lower Eulerian posets, that are just slightly more difficult than the Boolean ones. A poset B is called **simplicial** if for every $x \in B$ the interval $[\hat{0}, x]$ is Boolean. In this case, the computation of $h(B,t)$ is particularly easy, since for every x we have $g([\hat{0},x],t) = 1$. Thus,

$$h(B,t) = \sum_{x \in B} t^{\rho(\hat{0},x)} \, (1-t)^{\operatorname{rk} B - \rho(\hat{0},x)}. \tag{2.3}$$

Defined in this way the h-polynomial is not necessarily palindromic, however, for an Eulerian poset B we can write

$$(1-t)\; h(B \setminus \{\hat{1}\}, t) = g(B,t) - t^n g(B, t^{-1}).$$

From this we see that $h(B \setminus \{\hat{1}\}, t)$ is palindromic. If we take B to be the face lattice L_P of a polytope P, then $h(L_P \setminus \{\hat{1}\}, t)$ is called the **toric h-polynomial** of P. This is the generalization of the h-vector given by Stanley. In particular, if P is simplicial, then the lower Eulerian poset $L_P \setminus \{\hat{1}\}$ is simplicial, so we get

$$h(L_P \setminus \{\hat{1}\}, t) = \sum_{i=0}^{\dim P} h_i t^{\dim P - i} = \sum_{F \subsetneq P} (t-1)^{\dim P - \dim F - 1}$$

and thus h_i is exactly the same as in (2.1).

Local h-polynomial

Recall that a polyhedral subdivision of a polytope P is the subdivision of P into a finite number of polytopes, such that intersection of any two of them is a face of both. We will discuss this in a bit more detail in the next subsection. Abstracting the notion of polyhedral subdivisions of polytopal complexes, Katz and Stapledon [KS16a] introduced the notion of strong formal subdivisions for lower Eulerian posets.

Definition 10 (Definition 3.17 in [KS16a])**.** Let $\sigma : \Gamma \to B$ be an order-preserving, rank-increasing function between two locally Eulerian posets. Then σ is a **strong formal subdivision** if it is surjective and for all $y \in \Gamma$ and $x \in B$ with $\sigma(y) \leq x$

- $\sum_{\substack{y \leq y' \\ \sigma(y') = x}} (-1)^{\rho(\hat{0}_B, x) - \rho(\hat{0}_\Gamma, y')} = 0,$
- there exists $y \leq y' \in \Gamma$ such that $\rho(y') = \rho(x)$ and $\sigma(y') = x$.

In this thesis we deal in practice only with two examples of the strong formal subdivisions. First one is the identity map from B to B. This is a trivial example of a strongly formal subdivision. The other example comes directly from polyhedrdal subdivisions.

Lemma 1 (Lemma 3.25 in [KS16a])**.** Let $\mathcal{S}'$ and $\mathcal{S}$ be polyhedral subdivisions such that $\mathcal{S}'$ refines $\mathcal{S}$. Denote by the same symbols the corresponding face posets and define the function $\sigma : \mathcal{S}' \to \mathcal{S}$ by

$$\sigma(F') = \min_{\dim F} \left\{ F \in \mathcal{S} \ : \ F' \subseteq F \right\}.$$

Then this defines a strong formal subdivision $\sigma : \mathcal{S}' \to \mathcal{S}$.

Example 3. Let us again go back to Example 1. We can take $\mathcal{S}'$ to be the triangulation of the triangle and $\mathcal{S}$ to be the triangle itself. Therefore, the corresponding strong formal subdivision of posets is the subdivision of the Boolean poset of rank 3 by the poset depicted on Figure 2.2.

Definition 11 ([KS16a], Definition 4.1)**.** Let $\sigma : \Gamma \to B$ be a strong formal subdivision between a lower Eulerian poset Γ and an Eulerian poset B. The **local** h**-polynomial** $l_B(\Gamma, t)$ is defined by

$$l_B(\Gamma, t) \coloneqq \sum_{x \in B} (-1)^{\rho(x, \hat{1}_B)} \ g([x, \hat{1}_B]^*, t) \ h(\Gamma_x, t), \tag{2.4}$$

where

$$\Gamma_x \coloneqq \{ y \in \Gamma \ : \ \sigma(y) \leq x \}.$$

The polynomial constructed this way is now palindromic

$$l_B(\Gamma, t) = t^{\operatorname{rk} B} l_B(\Gamma, t^{-1}),$$

however a priori it might have some negative coefficients. We can reverse the definition (2.4) and express the "global" invariant $h(\Gamma, t)$ by

$$h(\Gamma, t) = \sum_{x \in B} l_{[\hat{0}_B, x]}(\Gamma_x, t) \ g([x, \hat{1}_B], t). \tag{2.5}$$

Here is a lemma that we will use in the next chapter for computation of certain local h-polynomials.

Lemma 2 (Example 4.8 in [KS16a])**.** If B is an Eulerian poset of rank $n > 0$ and $\sigma : B \to B$ is the trivial subdivision, then

$$l_B(B, t) = 0.$$

In this thesis we are particularly interested in polyhedral subdivisions. Let P be a polytope and $\mathcal{T}$ be its polyhedral subdivision. Let Δ be a cell of $\mathcal{T}$. We define a **link** of Δ to be the set of all the cells of $\mathcal{T}$ containing it

$$\operatorname{link}(\Delta, \mathcal{T}) = \{F \in \mathcal{T} \ : \ \Delta \subseteq F\}.$$

In particular, it can be viewed as a lower Eulerian poset. Consider $B = [\sigma(\Delta), P]$ and $\Gamma = \operatorname{link}(\Delta, \mathcal{T})$ with $\sigma_\Delta : \Gamma \to B$ given by the restriction of the strong formal subdivision from $\mathcal{T}$ to $\operatorname{link}(\Delta, \mathcal{T})$. Then the above definition specialises to the **relative local h-polynomial**:

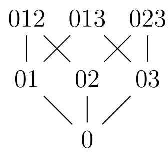
$$l_{\Delta,\mathcal{T}}(t) := \sum_{\Delta \subseteq F \leq P} (-1)^{\dim P - \dim F} h(\operatorname{link}(\Delta, \mathcal{T}_F), t) \ g([F, P]^*, t),$$

where $F \leq P$ means that F is a face of P. This polynomial was already considered in [Ath12] and [NS12].

Example 4. Let us continue Example 1 and compute a few of the relative local h-polynomials. Let us start with $\Delta = \{0\}$. In this case, the only face that contains Δ is the whole polytope P, thus

$$l_{\{0\},\mathcal{T}} = h\left(\operatorname{link}(\{0\}, \mathcal{T}),\ t\right) = t^2 + t + 1,$$

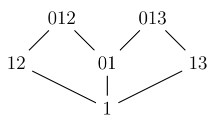
where $h(\operatorname{link}(\{0\}, \mathcal{T})$ is the simplicial poset



Let us also consider $\Delta = \{1\}$. In this case,

$$l_{\{1\},\mathcal{T}} = h(B_0, t) \cdot g(B_2, t) - 2h(B_1) \cdot g(B_1, t) + h(C, t) \cdot g(B_0, t) = t,$$

where C is the simplicial poset



2.3 Ehrhart theory

h^*-polynomial

Let us come back to lattice polytopes. By nowadays a classical problem is to study the number of lattice points in P and its dilations, i.e. to look at the numbers $| kP \cap M |$ for $k \in \mathbb{N}$. It was shown by Ehrhart [Ehr62] that

$$| kP \cap M | : \mathbb{N} \to \mathbb{N}$$

considered as a function of k is a polynomial of degree d. It is known by the name **Ehrhart polynomial**. This implies that the series

$$1 + \sum_{k \geq 1} | kP \cap M | \, t^k = \frac{h^*(P,t)}{(1-t)^{d+1}}$$

is a rational function. The polynomial

$$h^*(P,t) = \sum_{i=0}^{d} h_i^*(P) t^i$$

is called **h^*-polynomial** of P. From the result of Ehrhart it follows that the polynomial $h^*(P,t)$ is of degree at most d. We call $\deg h^*(P,t)$ the **degree** $\deg P$ of P. It also follows that $h^*(P,t)$ has integral coefficients. Stanley [Sta80] showed later that these coefficients are in fact non-negative. Here is a list of some other properties of the h^*-polynomial:

- $h_0^*(P) = 1$,
- $h_1^*(P) = |P \cap M| - d - 1$,
- $h_d^*(P) = |P^\circ \cap M|$,
- $\sum h_i^*(P) = \operatorname{vol}_{\mathbb{Z}} P$.

For a general polytope P usually it is not easy to give a nice combinatorial meaning to the coefficients $h_i^*(P)$. However, when P is a simplex the coefficients h_i^* have a particularly nice combinatorial interpretation. Let us denote the vertices of the simplex by $v_0, \ldots, v_d$ and let $(v_i, 1)$ be the lift of v_i to $M \oplus \mathbb{Z}$. Consider the set of points

$$\Pi_P = \left\{ \sum_{i=0}^{d} \lambda_i (v_i, 1), \quad 0 \leq \lambda_i < 1 \right\}.$$

It is called the half-open parallelepiped associated to P. The h^*-polynomial of P can be written in a nice way as a sum over integral points inside Π_P

$$h^*(P,t) = \sum_{x \in \Pi_P \cap M \oplus \mathbb{Z}} t^{x_{d+1}}.$$

In other words, h^*_k is the number of integral points in the half-open parallelepiped Π_P at height k.

Local h^*-polynomial

Analogously to the polynomial invariants of posets one can define a local variant of the h^*-polynomial. This is a less studied modification of the h^*-polynomial. It was introduced first by Betke and McMullen [BM85] in order to describe how one can reconstruct $h^*(P,t)$ from the Ehrhart-theoretic data of the cells of a triangulation of a polytope. It is worth noting, however, that this was done before Stanley's work [Sta92] on local h-vectors, and in fact, it served as a motivation for it.

Let P again be a simplex with vertices $v_0, \ldots, v_d$. The local h^*-polynomial, also known in this case as the **box polynomial**, is defined in a manner very similar to the usual h^*-polynomial. Specifically, it is a sum over the points of the interior of the half-open parallelepiped Π_P

$$l^*(P,t) := \sum_{x \in \Pi_P^\circ \cap M \oplus \mathbb{Z}} t^{x_{d+1}}. \tag{2.6}$$

From this definition, it is straightforward to deduce that for a simplex P, we have

$$l^*(P,t) = \sum_{F \subseteq P} (-1)^{\operatorname{codim} P} h^*(F,t)$$

and

$$h^*(P,t) = \sum_{F \subseteq P} l^*(F,t).$$

For any polytope P we have the following definition.

Definition 12 (Example 7.13 in [Sta92] and Definition 7.2 in [KS16a])**.** The **local h^*-polynomial** of P is

$$l^*(P,t) := \sum_{F \subseteq P} (-1)^{\dim P - \dim F} h^*(F,t)\; g([F,P]^*,t).$$

Let us list a few of the properties of the polynomial $l^*(P,t) = \sum l_i^*(P)t^i$

1. $l^*(P,t)$ is palindromic, namely $l_i^*(P) = l_{d+1-i}^*(P)$,
2. $l_i^*(P)$ are non-negative,
3. $l_1^*(P) = |P^\circ \cap M| = l_d^*(P) = h_d^*(P)$,
4. $l_1^*(P) \leq l_i^*(P), \quad i = 2, \ldots, d$.

The first two points above follow from the fact that $l^*(P,t)$ is a generating function of certain Hodge numbers, which we will see in the next subsection. Another proofs of the second point were given in [BM03] and [Kar08]. The fourth point was proven by Katz and Stapledon [KS16a].

The formula in the definition can be reversed to get

$$h^*(P,t) = \sum_{F \subseteq P} l^*(F,t)\ g([F,P],t). \tag{2.7}$$

A key property of the local h^*-polynomial is that it allows us to reconstruct the h^*- and l^*-polynomials of P if we know all the local h^*-polynomials of the cells of a polyhedral subdivision. Let $\mathcal{T}$ be a triangulation of P. In this case, the first formula from the next theorem was proved in [BM85] (Theorem 1) and the second one can be found in [NS12]. More recently, this result has been generalized to any polyhedral subdivision in [KS16a] (Lemma 7.12).

Theorem 2. Let $\sigma : \mathcal{T} \to [\emptyset, P]$ be a strong formal subdivison induced by a lattice polyhedral subdivision $\mathcal{T}$ of a lattice polytope P. We can express the (local) h^*-polynomial of P via

$$h^*(P,t) = \sum_{\Delta \in \mathcal{T}} h\left(\mathrm{link}_{\mathcal{T}}(\Delta),\ t\right)\ l^*(\Delta,t),$$

$$l^*(P,t) = \sum_{\Delta \in \mathcal{T}} l_{\Delta,\mathcal{T}}(t)\ l^*(\Delta,t).$$

Since both the local h^*-polynomial and the relative local h-polynomial have non-negative coefficients, this implies

$$l_{\varnothing,\mathcal{T}}(t) \leq l^*(P,t).$$

The (local) h^*-polynomial behaves particularly nicely under the operation of taking a free join.

Proposition 2. The h^*- and local h^*-polynomials are multiplicative with respect to free joins, namely

$$h^*(P \circ_{\mathbb{Z}} Q, t) = h^*(P, t) \cdot h^*(Q, t),$$

$$l^*(P \circ_{\mathbb{Z}} Q, t) = l^*(P, t) \cdot l^*(Q, t).$$

The first part of the above proposition can be found in [HT09], and the second part was shown in [NS13]. This proposition implies, that for a lattice pyramid $\mathrm{Pyr}(P)$ we have

$$h^*(\mathrm{Pyr}(P), t) = h^*(P, t), \quad \text{and} \quad l^*(\mathrm{Pyr}(P), t) = 0$$

The following class of polytopes is the main object of study in Chapter 5.

Definition 13. A lattice polytope P is called **thin** if $l^*(P, t) = 0$.

From (2.7) one can deduce that if $\deg P \leq \frac{d}{2}$, then $l^*(P, t)$ must vanish. Therefore, we call the polytopes P satisfying this condition **trivially thin**. Since the linear coefficient $l_1^*(P)$ counts the number of lattice points in the interior of P, this implies that a thin polytope must be hollow.

Remark 3. One can meet a lot of different notation for both h^*- and l^*-polynomials. Instead of h^* one can encounter $\delta(P, t)$, $W(P, t)$, $S(P, t)$, $\Phi_\Delta(t)$, etc., and instead of $l^*(P, t)$ one can see $\tilde{S}(P, t)$, $\delta^{\#}(P, t)$ and $B(P, t)$ (for simplices).

However, recently exactly the notation h^* and l^* seems to be dominant. This can be explained by the fact that $h^*(P, t)$ and $l^*(P, t)$ exhibit a behaviour very similar to that of the h- and local h-polynomials of posets as we have seen above.

2.4 Affine hypersurfaces in tori

Hodge-Deligne polynomials

Both h^*- and l^*-polynomials have a close relation to the geometry of the affine hypersurfaces defined by the Newton polytope P. Let P be a d-dimensional lattice polytope. Consider

$$Z_P \ : \ \left\{ x \in (\mathbb{C}^*)^d \ : \ \sum_{m \in P \cap \mathbb{Z}^d} c_m x^m = 0 \right\},$$

where a_m are in $\mathbb{C}$ and are non-zero if m is a vertex of P. We consider situations only when a_m are sufficiently generic, namely we require Z_P to be P-regular in the sense of Batyrev (see [Bat93], Definition 3.3).

An important topological invariant of an algebraic variety Z is its cohomology $H^\bullet(Z, \mathbb{Q})$. A very basic piece of information about cohomology is its dimension. Putting the corresponding numbers into a generating series gives us the **Poincaré polynomial** of Z

$$P_Z(t) = \sum_{k \geq 0} \dim H^k(Z, \mathbb{Q})\ t^k.$$

Its specialization is the classical topological invariant called Euler characteristic

$$\chi(Z) = P_Z(-1).$$

One can go one step further and refine the Poincaré polynomial. As was shown by Deligne [Del74] the cohomology of an algebraic variety carries an additional structure. It is called mixed Hodge structure. Let us explain what it is.

Definition 14. Let H be a finite-dimensional vector space over $\mathbb{Q}$. A **pure Hodge structure** of weight $n \in \mathbb{Z}$ on H is a direct sum decomposition of the complex vector space

$$H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q},$$

such that

$$\overline{H^{p,q}} = H^{q,p}.$$

Alternatively, we can define the Hodge structure with the use of a filtration. We say that H carries a Hodge structure if there is a decreasing filtration $F^\bullet$ on $H_{\mathbb{C}}$

$$0 \subset F^n \subset \cdots \subset F^0 = H_{\mathbb{C}},$$

such that $F^p \oplus \overline{F^{n-p+1}} \simeq H_{\mathbb{C}}$.

A typical example of and also the source of inspiration for the above definition is the cohomology vector spaces of a compact complex variety. The study of cohomology of noncompact algebraic varieties motivated the following definition.

Definition 15. Let H be as before. A **mixed Hodge structure** on H consists of the following two filtrations.

- An increasing (weight) filtration $W_\bullet$ on H;

- A decreasing (Hodge) filtration $F^\bullet$ on $H_{\mathbb{C}}$,

such that the Hodge filtration induces a pure Hodge structure on the graded pieces of the weight filtration, i.e. for each k we have a decomposition

$$(Gr_k^W)_{\mathbb{C}} = (W_k/W_{k-1})_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}.$$

Definition 16. The dimensions of the components in the above decomposition are called **Hodge numbers**

$$h^{p,q}(H) := \dim H^{p,q}.$$

The cohomology (with compact support) of an algebraic variety Z carries a mixed Hodge structure. Define

$$e(Z;u,v) := \sum_{k\geq 0}(-1)^k \sum_{p+q=k} h^{p,q}(H_c^k(Z,\mathbb{Q}))u^p v^q.$$

This is called the **Hodge-Deligne polynomial** of Z. We use here the cohomology with compact support $H_c^\bullet(Z,\mathbb{Q})$ because it gives $e(Z;u,v)$ with better properties compared with the one we would obtain if we used the ordinary cohomology. In particular, this way $e(Z,u,v)$ becomes a motivic invariant in the sense that the map $e(-;u,v) : \mathrm{Var}_{\mathbb{C}} \to \mathbb{Z}[u,v]$ factors through the Grothendieck group of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$. See [KS16b] for more details. In particular, it means that the following holds

- For varieties X and Y we have

$$e(X \times Y;u,v) = e(X;u,v) \cdot e(Y;u,v);$$

- if $X = \sqcup_i X_i$ is a disjoint union of a finite set of locally closed subvarieties X_i, then

$$e(X;u,v) = \sum_i e(X_i;u,v).$$

See [DK87] for the proofs of the above statements about the Hodge-Deligne polynomial.

Example 5. With these two properties we can easily compute the Hodge-Deligne polynomial of an algebraic torus $(\mathbb{C}^*)^n$. Let us start with the cohomology of the projective line $\mathbb{P}^1$. We know that $\dim H^0(\mathbb{P}^1, \mathbb{Q}) = \dim H^2(\mathbb{P}^1, \mathbb{Q}) = 1$ and zero otherwise. Moreover, since $\mathbb{P}^1$ is compact, the Hodge structures on its cohomology are pure. Therefore, we get

$$e(\mathbb{P}^1; u, v) = 1 + uv.$$

We can decompose $\mathbb{P}^1$ as

$$\mathbb{P}^1 = \mathbb{C}^* \sqcup \{p_0\} \sqcup \{p_\infty\}.$$

The Hodge-Deligne polynomial of a point is 1, thus we get

$$e(\mathbb{C}^*; u, v) = e(\mathbb{P}^1; u, v) - e(\{p_0\}; u, v) - e(\{p_\infty\}; u, v) = uv - 1$$

and

$$e((\mathbb{C}^*)^n; u, v) = (uv - 1)^n.$$

Moreover, this allows to actually get back the Hodge numbers

$$h^{p,p}(H^{n+p}((\mathbb{C}^*)^n, \mathbb{Q})) = \binom{n}{p}.$$

Out particular interest lies in affine hypersurfaces Z_P in algebraic tori. Let us follow [DK87] and discuss further basic properties of the cohomology of Z_P. First of all, we can limit ourselves to studying P of full dimension d. Otherwise, P would define a hypersurface Z'_P in a torus of lower dimension d' and we could write

$$e(Z_P; u, v) = (uv - 1)^{d-d'} e(Z'_P; u, v).$$

Since Z_P is an affine variety of dimensions $d-1$, it is a classical result in algebraic geometry (Grothendieck vanishing theorem and Poincaré duality) that

$$H^i_c(Z_P, \mathbb{Q}) = 0, \quad \text{for} \;\; i = 0, \ldots, d-2.$$

Moreover, the higher cohomology groups are well understood as well due to the following proposition.

Proposition 3 ([DK87], Proposition 3.9)**.** There exist homomorhisms

$$\phi_i : \; H^i_c(Z_P, \mathbb{C}) \to H^{i+2}_c((\mathbb{C}^*)^d, \mathbb{C})$$

that are isomorphisms for $i > d-1$ and surjective for $i = d-1$.

Therefore, we know all the cohomology of Z_P except for the middle one $H_c^{d-1}(Z_P, \mathbb{Q})$. This motivates the following definition.

Definition 17. Define the **primitive cohomology** of Z_P to be the kernel of ϕ_i

$$PH_c^i(Z_P) \coloneqq \ker \phi_i.$$

Since mixed Hodge structures form an abelian category, the cohomology $PH_c^i(Z_P)$ carries a mixed Hodge structure. We can easily relate its Hodge-Deligne polynomial to that of Z_P. However, let us multiply it with an overall sign, so that its coefficients are always positive, namely, define

$$e_{prim}(Z_P; u, v) \coloneqq (-1)^{d-1} \sum_{k \geq 0} (-1)^k \sum_{p+q=k} h^{p,q}(PH_c^k(Z_P, \mathbb{Q})) u^p v^q.$$

Now to get $e_{prim}(Z_P; u, v)$ we have to subtract from each $H^i(Z_P)$ the cohomology of the torus $H^{i+2}((\mathbb{C}^*)^d)$ except for $i = d-2$. On the level of Hodge-Deligne polynomials for a full-dimensional polytope P this amounts to subtracting $((uv-1)^d - (-1)^d)/(uv)$ from $e_{prim}(Z_P; u, v)$. After taking into account all the signs one arrives at

$$e_{\text{prim}}(Z_P; u, v) = (-1)^{d-1} e(Z_P; u, v) + \frac{(1-uv)^d - 1}{uv}. \tag{2.8}$$

Expressing $e(Z_P; u, v)$

The Hodge-Deligne polynomial $e(Z_P; u, v)$ has been studied extensively over the years. In [DK87] Danilov and Khovanskii obtained an algorithm for computing it. However, it was not very explicit. Later, Batyrev and Borisov [BB96] obtained a recursive formula for computing $e(Z_P; u, v)$ through the Hodge-Deligne polynomials of the faces of P. More recently, Borisov and Mavlyutov [BM03] gave an exact expression for $e(Z_P; u, v)$ using the Ehrhart-theoretic invariants of the faces of P and the invariants of poset of faces of P discussed in the beginning of this chapter. Here is their result.

Proposition 4 (Proposition 5.5 in [BM03])**.** Let P be a d-dimensional lattice polytope and $Z_P \subseteq (\mathbb{C}^*)^d$ be a generic hypersurface with Newton polytope P, then the Hodge-Deligne polynomial of Z_P is

$$e(Z_P; u, v) = \frac{(uv-1)^d}{uv} + \frac{(-1)^{d+1}}{uv} \sum_{F \subseteq P} u^{\dim F + 1} l^*(F, u^{-1}v)\ g([F, P], uv).$$

Note that comparing to [BM03] we use polytopes instead of Gorenstein cones and also we use poset $[F, P]$ and not its dual because of the different definition of $l^*(P, t)$ comparing to the definition of $\tilde{S}(P, t)$ in their paper. Rewriting this for the Hodge-Deligne polynomial of the primitive cohomology we obtain a nicer looking formula

$$u\ v\ e_{prim}(Z_P; u, v) + 1 = \sum_{F \subseteq P} u^{\dim F + 1} l^*(F, u^{-1}v)\ g([F, P], uv). \tag{2.9}$$

Note that the right-hand side is exactly what appears as **mixed h^*-polynomial** in [KS16a].

Lemma 3. The primitive Hodge-Deligne polynomial is invariant under taking pyramids

$$e_{prim}(Z_{\mathrm{Pyr}(P)}; u, v) = e_{prim}(Z_P; u, v).$$

Proof. We can write

$$u\ v\ e_{prim}(Z_{\mathrm{Pyr}(P)}; u, v) + 1 = \sum_{F \subseteq P \subset \mathrm{Pyr}(P)} u^{\dim F + 1} l^*(F, u^{-1}v)\ g([F, \mathrm{Pyr}(P)], uv) +$$

$$+ \sum_{\substack{F \subseteq \mathrm{Pyr}(P) \\ F \not\subseteq P}} u^{\dim F + 1} l^*(F, u^{-1}v)\ g([F, P], uv).$$

Note that in the second sum each face is in fact a pyramid over some face in P, therefore all the local h^*-polynomials vanish in this sum. In the first sum the only difference from the analogous sum for $e_{prim}(Z_P; u, v)$ is that we have $[F, \mathrm{Pyr}(P)]$ instead of $[F, P]$. Note that $[F, \mathrm{Pyr}(P)]$ is a product of $[F, P]$ with the Boolean poset of rank 1. Thus, by Remark 1 we have

$$g([F, \mathrm{Pyr}(P)], t) = g([F, P], t).$$

Therefore, we get the desired equality. □

However, the Hodge-Deligne polynomial $e_{prim}(Z_p; u, v)$ is not multiplicative under taking free joins, unlike the mixed h^*-polynomial of Katz and Stapledon.

Now let us explain how we can go back to the polynomials that we studied in the previous part. If we specialize to $u = 1$ we obtain

$$t\ e_{prim}(Z_P; 1, t) + 1 = h^*(P, t).$$

Moreover, $e_{prim}(Z_P; u, v)$ is a two-variable polynomial of degree $d-1$, and the highest degree part is almost exactly the local h^*-polynomial, namely we have

$$u\ v\ e_{prim}(Z; u, v)\ |_{\deg=d-1} = u^{d+1} l^* \left(P, \frac{v}{u}\right).$$

In other words, the local h^*-polynomial can be thought as the generating function of the top-weight Hodge numbers of $PH_c^{d-1}(Z_P)$

$$l^*(P, t) = t \cdot \sum_{p=0}^{d-1} h^{p,d-1-p}(PH_c^{d-1}(Z_P))\ t^p.$$

This way the Ehrhart-theoretic polynomials $h^*(P, t)$ and $l^*(P, t)$ can be connected to the geometry of the affine hypersurface Z_P.

Cayley polytopes and complete intersections

Consider a system of k equations in $(\mathbb{C}^*)^d$ given by $f_1 = f_2 = \ldots = f_k = 0$, where $f_i = \sum_{j=1}^{n_i} c_{i,j} x^{m_{i,j}}$ are generic Laurent polynomials. Let P_i be the corresponding Newton polytopes of f_i. Let Y be the complete intersection inside $(\mathbb{C}^*)^d$ defined as the vanishing locus of the above system.

Consider the Cayley polytope

$$P = P_1 * \ldots * P_k.$$

We assume that $\dim(\sum_i P_i) = d$, so $\dim P = d + k - 1$. Let $Z_P \subseteq (\mathbb{C}^*)^{n+k-1}$ be a generic hypersurface with Newton polytope P. It can be defined by the vanishing of

$$Z_P\ :\ f_1 + y_2 f_2 + \ldots + y_k f_k = 0.$$

Unfortunately, there is no simple relation between the Hodge-Deligne polynomials of Z_P and Y. However, if we consider the hypersurface $\tilde{Z} \subseteq (\mathbb{C}^*)^d \times \mathbb{C}^k$ instead

$$\tilde{Z}\ :\ 1 + y_1 f_1 + y_2 f_2 + \ldots + y_k f_k = 0,$$

then Danilov and Khovanskii [DK87] gave the following simple formula

$$e(\tilde{Z}; u, v) = (uv)^{k-1} \left((uv-1)^n - e(Y; u, v)\right).$$

Note that $\tilde{Z}$ is not a hypersurface in an algebraic torus. However, there is a stratification of $\tilde{Z}$ given by affine hypersurfaces in tori that was described in [DRHN16]. For $I \subseteq [k]$ define

$$Z_I = \tilde{Z} \cap \{y_j \neq 0\ :\ j \in I\} \cap \{y_j = 0\ :\ j \notin I\}.$$

Each of these is now an affine hypersurface in $(\mathbb{C}^*)^{d+|I|}$. Let us denote by P_I the corresponding Newton polytopes. Note, that in fact each of P_I is a lattice pyramid over a Cayley polytope

$$P_I = \mathrm{Pyr}\left(*_{i\in I} P_i\right).$$

In particular, $P_{[k]}$ is a lattice pyramid over P. Let us define the dimensions

$$d_I = \dim(\sum_{i\in I} P_i).$$

Then the dimensions of the corresponding Newton polytopes are

$$\dim P_I = d_I + |I|.$$

Using this stratification we can get for the Hodge-Deligne polynomials

$$e(Z_{[k]}; u, v) = (uv)^{k-1}\left((uv-1)^n - e(Y; u, v)\right) - \sum_{I\subsetneq[k]} e(Z_I; u, v). \tag{2.10}$$

It is convenient to go from $e(Z_I; u, v)$ to $e_{prim}(Z_I; u, v)$ but we have to be careful, because some of the polytopes P_I might be not full-dimensional. We have

$$e(Z_I; u, v) = (-1)^{d_I+|I|}(uv-1)^{d-d_I}\left[-e_{prim}(Z_I; u, v) + \frac{(1-uv)^{d_I+|I|}-1}{uv}\right].$$

The relation that we now have is

$$(-1)^{d+k}\left(-e_{prim}(Z_{[k]}; u, v) + \left(\frac{(1-uv)^{d+k}-1}{uv}\right)\right) = \tilde{R}_Y + \tilde{R}_1 - \tilde{R}_2, \tag{2.11}$$

where

$$\tilde{R}_Y = (uv)^{k-1}\left((uv-1)^d - e(Y; u, v)\right),$$

$$\tilde{R}_1 = \sum_{I\subsetneq[k]} (-1)^{d_I+|I|}(uv-1)^{d-d_I} e_{prim}(Z_I; u, v),$$

$$\tilde{R}_2 = \sum_{I\subsetneq[k]} (-1)^{d_I+|I|}(uv-1)^{d-d_I}\ \frac{(1-uv)^{d_I+|I|}-1}{uv}.$$

We would like to use this expression to derive a formula for the local h^*-polynomial of P through the Hodge-Deligne polynomials of Y and Z_I's. By Lemma 3 it holds that $e_{prim}(Z_{[k]}; u, v) = e(P; u, v)$. Therefore, restricting to the degree

$\dim P - 1 = d + k - 2$ would single out the terms corresponding to the local h^*-polynomial of P. Let us collect all the terms of this degree in (2.11). From the left-hand side we obtain

$$R_L = (-1)^{d+k+1} \, \frac{u^{d+k-1}}{v} \, l^*(P, \frac{v}{u}) + (-1)^{3/2(d+k)} \epsilon_{d+k} \binom{d+k}{\frac{d+k}{2}} (uv)^{\frac{d+k-2}{2}},$$

where ϵ_x is the function defined by

$$\epsilon_x = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

The terms $\tilde{R}_Y, \tilde{R}_1$ and $\tilde{R}_2$ produce the following expressions correspondingly

$$R_Y = (-1)^{\frac{d+k}{2}} \epsilon_{d+k} \binom{d}{\frac{d-k}{2}} (uv)^{\frac{d+k-2}{2}} - e(Y; u, v) \mid_{\deg = d-k} (uv)^{k-1},$$

$$R_1 = \sum_{I \subsetneq [k]} \epsilon_{d-d_I+k-|I|} \sum_{\substack{i,j \geq 0 \\ 2i+j=d+k-2}} (-1)^{d+|I|-i} \binom{d-d_I}{i} (uv)^i e_{prim}(Z_I; u, v) \mid_{\deg = j},$$

$$R_2 = \sum_{I \subsetneq [k]} \epsilon_{d+k} \sum_{\substack{i,j \geq 0 \\ 2i+2j=d+k-2}} (-1)^{d+|I|+i+j-1} \binom{d-d_I}{i} \binom{d_I+|I|}{j+1} (uv)^{\frac{d+k-2}{2}}.$$

The resulting equation, from which an expression for $l^*(P, t)$ can be derived, takes the form

$$R_L = R_Y + R_1 - R_2. \tag{2.12}$$

This formula is particularly nice in the case $d = k$ because in this situation the complete intersection Y is 0-dimensional and therefore $e(Y)$ is just the number of solutions to the corresponding system of equations $f_1 = \ldots = f_d = 0$. By the famous theorem of Bernstein-Khovanskii-Kushnirenko [Ber75, Kou76] this is equal to the mixed volume of the polytopes $P_1, \ldots, P_d$. Let us recall its definition.

Definition 18. Let $P_1, \ldots, P_d$ be convex bodies in $\mathbb{R}^d$. Their **mixed volume** $MV(P_1, \ldots, P_d)$ is the unique symmetric multilinear function such that

$$MV(P, P, \ldots, P) = \mathrm{vol}_{\mathbb{Z}}(P).$$

It can be also computed using

$$V(P_1,\dots,P_d)=\sum_{\varnothing\neq I\subseteq[d]}(-1)^{|I|}\operatorname{vol}_{\mathbb{R}}\left(\sum_{i\in I}P_i\right),$$

where $\operatorname{vol}_{\mathbb{R}}$ is the usual Euclidean volume.

Taking the case $d=k$ and specializing the formula (2.12) to $v=t$ and $u=1$, one can get the formula for the local h^*-polynomial of P, but here we write down only the case when all the polytopes P_i are of full dimension.

Theorem 4. Let $P_1,\dots,P_d$ be d-dimensional lattice polytopes in $\mathbb{R}^d$. The local h^*-polynomial of the Cayley sum $P=P_1*\dots*P_d$ is

$$l^*(P,t)=(MV(P_1,\dots,P_d)-1)\,t^d.$$

A similar theorem for the h^*-polynomial of P can be obtained from (2.11) by specializing $v=t,u=1$ again.

Theorem 5. Let $P_1,\dots,P_d$ be lattice polytopes in $\mathbb{R}^d$. The h^*-polynomial of the Cayley sum $P=P_1*\dots*P_d$ is given by

$$h^*(P,t)=MV(P_1,\dots,P_d)t^d-(t-1)^d-\sum_{I\subsetneq[k]}(-1)^{d_I+|I|}(t-1)^{d-d_I}h^*(*_{i\in I}P_i,t).$$

It also gives an expression for the mixed volume in terms of the volumes of the Cayley sums

$$MV(P_1,\dots,P_d)=\sum_{\substack{I\subseteq[d]\\ d_I=d}}(-1)^{d+|I|}\operatorname{vol}_{\mathbb{Z}}(*_{i\in I}P_i).$$

2.5 Triangulations

We have seen before that triangulations and more generally polyhedral subdivisions are commonplace ingredients of the research in Ehrhart theory of lattice polytopes. Let us have a closer look at them and consider a few results that will be useful for us in the next chapter. The main source is [DLRS10].

Let A be a d-dimensional point configuration with a set of labels I. Let us remind again that we often abuse notation and identify elements of I with the corresponding points of A.

Definition 19. A **triangulation** of A is a collection $\mathcal{T}$ of subsets of I such that for each $\Delta \in \mathcal{T}$ the convex hull $\mathrm{conv}(\Delta)$ is a simplex, and the following conditions are satisfied:

(CP) If $\Delta \in \mathcal{T}$ and $F \subseteq \Delta$, then also $F \in \mathcal{T}$ (closure property);

(IP) If $\Delta_1 \neq \Delta_2$ are cells in $\mathcal{T}$, then $\mathrm{relint}\Delta_1 \cap \mathrm{relint}\Delta_2 = \varnothing$ (intersection property);

(UP) $\bigcup_{\Delta \in \mathcal{T}} \mathrm{conv}(\Delta) \supseteq \mathrm{conv}(A)$ (union property).

A **polyhedral subdivision** of A is defined in the same way except we drop the requirement that $\mathrm{conv}(\Delta)$ is a simplex, and we only require that $\mathrm{conv}(\Delta)$ is a convex polytope.

This is one of the most common definitions of a triangulation. However, it is not always the most useful one. We are going to use different versions of intersection and union properties. But for this we require a little bit of preparation.

Consider a configuration Z consisting of exactly $d+2$ points $a_1, \ldots, a_{d+2}$. In this case, the homogenization $\bar{Z}$ has exactly one linear relation

$$\sum_{i=1}^{d+2} \gamma_i \bar{a}_i = 0.$$

This relation divides the set $[d+2]$ into three subsets

$$Z_+ = \{j \in [d+2] \;:\; \gamma_j > 0\}, \qquad Z_- = \{j \in [d+2] \;:\; \gamma_j < 0\},$$
$$Z_0 = \{j \in [d+2] \;:\; \gamma_j = 0\}.$$

Definition 20. A subset of I is called a **circuit** if it is a minimal dependent set, i.e. every proper subset is independent. In other words, the pair (Z_+, Z_-) is a circuit.

A closely related concept is that of dependence signatures. Let now $V = \{v_i\}_{i \in I}$ be a vector configuration, then every linear relation $\sum_{i \in I} \gamma_i v_i = 0$ again divides I into three subsets V_0, V_-, V_+. We call this partition a **dependence signature** on V. We call it **positive** if V_- is empty, and **negative** if V_+ is empty.

Proposition 5 (Theorems 4.1.14 and 4.1.15 [DLRS10])**.** Let $\mathcal{T}$ be a family of subsets such that the corresponding point configurations are affinely independent. Suppose $\mathcal{T}$ satisfies the closure property. Then the following are equivalent:

- Intersection property;
- There is no circuit (Z_+, Z_-) with $Z_+,\ Z_- \in \mathcal{T}$.

Proposition 6 (Lemma 4.1.16 in [DLRS10])**.** Let $\mathcal{T}$ be a collection of subsets of A. Assume that it satisfies the closure and intersection properties. Then the following are equivalent:

- Union property;
- For each face F of a maximal cell Δ, either F is a facet of some other maximal cell Δ', or F is contained in a facet of $\mathrm{conv}(A)$.

In the view of this proposition, it would be nice to have a convenient description of the faces of $\mathrm{conv}(A)$. This will be achieved in the next section, see Lemma 6.

2.6 Gale duality

Gale transform is an often used tool in the combinatorics of polytopes. It takes a point configuration of n points in d-dimensional space and transforms into a vector configuration of n-points in dimension $n - d - 1$. It is particularly useful when $n < 2d + 1$, so we get a point configuration of lower dimension which might help to find an easier way of solving a problem in the original d-dimensional setting.

The way it is usually defined, a Gale transform is an operation over the field $\mathbb{R}$. It means that we can transform the original configuration by $GL(d, \mathbb{R})$. This does not fit in the setting of Ehrhart theory, where the lattice structure is in the focus. Nevertheless, it can be easily fixed as it is done in the theory of toric varieties (see, for example, [CLS11], chapter 9).

Let $A \subseteq \mathbb{Z}^d$ be an integral spanning d-dimensional point configuration consisting of $n > d + 1$ points. Let $\bar{A}$ denote the corresponding homogeneous vector configuration in $\mathbb{Z}^{d+1}$. The following definition of Gale transforms is taken from [CLS11] and [CD22]. The matrix $\bar{A}$ defines us a linear map and a short exact sequence

$$0 \longrightarrow \ker \bar{A} \xrightarrow{\ i\ } \mathbb{Z}^n \xrightarrow{\ \bar{A}\cdot\ } \mathbb{Z}^{d+1} \longrightarrow 0.$$

Note, that since A is spanning, $\ker \bar{A}$ is a lattice as well.

Dualizing the above sequence we get

$$0 \longrightarrow (\mathbb{Z}^{d+1})^* \xrightarrow{\ (\bar{A}\cdot)^*\ } (\mathbb{Z}^n)^* \xrightarrow{\ i^*\ } (\ker \bar{A})^* \longrightarrow 0.$$

There exists an isomorphism ϕ that identifies $(\ker \bar{A})^*$ with $\mathbb{Z}^{n-d-1}$. Let e_i^* be the standard basis of $(\mathbb{Z}^n)^*$. Define

$$\{\gamma_i = \phi \circ i^*(e_i^*)\}_{i=1}^n$$

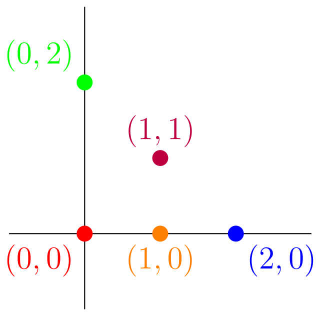
to be a **Gale transform** (or **Gale dual**) of the given point configuration A. This is a spanning vector configuration in $\mathbb{Z}^{n-d-1}$. Let us denote by G be the matrix whose columns are given by γ_i. One can see that are multiple choices for a Gale dual of A, but they are all $GL(n-d-1, \ \mathbb{Z})$ equivalent. Also note that since we work with the homogenization $\bar{A}$, it implies that

$$\sum_{i=1}^n \gamma_i = 0.$$

Moreover, we can reverse the above procedure. Suppose we are given a vector configuration in $\mathbb{Z}^{n-d-1}$. This defines us a map i^* and short exact sequence as above. Dualizing it we can arrive at a $d+1$-dimensional vector configuration. If we start with G such that its elements sum to zero, then this $d+1$-dimensional configuration is the homogenization of a d-dimensional point configuration. One can check, that this way one one obtains a point configuration that is $GL(d, \mathbb{Z})$ equivalent to the configuration A. Therefore, we talk about duality.

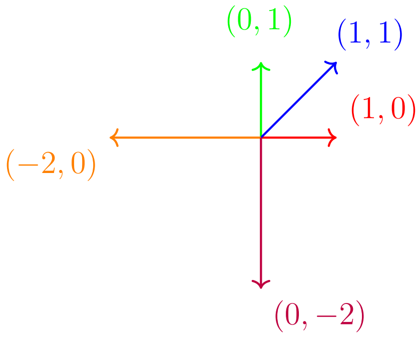
Example 6. Let us consider a two-dimensional point configuration given by

$$A = \begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \end{pmatrix}.$$



Note, that the convex hull of A is the twice dilated standard unimodular simplex $2\Delta_2$. In practice, to find a Gale dual, it is enough to find a matrix G such that $\bar{A} \cdot G^T = 0$ and the columns of G span $\mathbb{Z}^{n-d-1}$. A Gale dual for our given configuration can be chosen to be

$$G = \begin{pmatrix} 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix}$$



Since lattice pyramids exhibit rather trivial properties in terms of the Ehrhart-theoretic properties, we would like to avoid them. We can use the following lemma.

Lemma 4. Let $A \subseteq \mathbb{Z}^d$ be a spanning point configuration with a Gale dual G. The convex hull $\operatorname{conv}(A)$ is a lattice pyramid if and only if $0 \in G$.

Proof. The only if direction is clear. Suppose now $0 \in G$. Then the points corresponding to $G \setminus \{0\}$ must lie in a hyperplane. Since we assume that A is spanning, the point of A corresponding to 0 must lie at the distance 1 from this hyperplane, therefore the convex hull of A is a lattice pyramid. □

Therefore, from now on we assume that $0 \notin G$ to exclude lattice pyramids from consideration.

Let us finish this section with a useful lemma that allows one to compute volumes of simplices that are subconfigurations of A through a Gale dual G. This lemma can be found already in [GKZ94], but the most convenient formulation for us is in [MFR+16].

Lemma 5 (Lemma 2.10 in [MFR+16])**.** Suppose we are given a short exact sequence of vector spaces

$$0 \longrightarrow \mathbb{R}^{n-c} \xrightarrow{B\cdot} \mathbb{R}^n \xrightarrow{A\cdot} \mathbb{R}^c \longrightarrow 0,$$

where A is a $c \times n$ matrix and B is a $n \times (n-c)$ matrix. Then there is a non-zero constant $\delta \in \mathbb{R}$ such that for each set $I = \{i_1, \dots, i_c\} \subseteq [n]$ of indices we have the equality (up to a scalar) of the absolute values of complimentary minors

$$|B_{[n]\setminus I,[n-c]}| = \delta\ |A_{[c],I}|,$$

where $X_{K,L}$ denotes a minor of matrix X corresponding to the rows with the index set K and the columns with the index set L.

Corollary 1. We can restrict the above lemma to the situation of integral matrices A and B, if moreover we require that

$$\gcd\left(\left\{B_{[n]\setminus I,n-c}\right\}_{I\subseteq[n],|I|=c}\right) = \gcd\left(\left\{A_{c,I}\right\}_{I\subseteq[n],|I|=c}\right) = 1,$$

then we deduce that $\delta = 1$. Note that in our setup of point configurations and their Gale duals, the above equalities correspond exactly to the condition that A and G are spanning configurations. Therefore, this allows us to compute the volumes of simplices with vertices in A via the Gale dual G easily.

In this thesis we often define a spanning polytope using its Gale dual. A priori we should be able to read off all the information about the polytope $\mathrm{conv}(A)$ from G. One thing that is relatively easy to extract are the faces of $\mathrm{conv}(A)$. In particular, we can read off the facets, that are of interest in Proposition 6. There is the following result.

Lemma 6 (Lemma 4.1.38(iii) in [DLRS10])**.** The faces of $\mathrm{conv}(A)$ are complements of the positive dependence signatures of a Gale dual of A. In particular, the facets of $\mathrm{conv}(A)$ correspond to the minimal positive dependencies.

Example 7. Consider again the point configuration from Example 6. If we enumerate the columns of A from 1 to 5, then we see that the minimal positive dependencies of its Gale dual are $\{1,4\},\{3,5\}$ and $\{2,4,5\}$. Their complements correspondingly are $\{2,3,5\}$, $\{1,2,4\}$ and $\{1,3\}$. These are exactly the edges of the triangle $2\Delta_2$.

3 Ehrhart theory and Gale Transforms

3.1 Spanning circuits

The goal of this chapter is to try to express the Ehrhart-theoretic invariants of the polytope $\text{conv}(A)$ using its Gale dual G. We start with the simplest possible situation when the Gale transform of A is one dimensional. In other words $n = d+2$. Recall that such configurations are called circuits. Since we also require these $d+2$ points to span the lattice we call them **spanning circuits**. This case was already treated by Rodriguez Villegas in [Vil19]. Although this is a particular case of shifted products of circuits that we consider later in this chapter, it is still illustrative and didactic to consider the case of circuits separately. Therefore, here we provide a detailed proof of the results stated in [Vil19].

If G is 1-dimensional, it means that it is defined by a tuple of numbers

$$(\gamma_1, \gamma_2, \ldots, \gamma_{d+2})$$

that satisfy

1. $\gcd(\gamma_1, \ldots, \gamma_{d+2}) = 1$ (G is spanning),
2. $\sum_i \gamma_i = 0$,
3. $\gamma_i \neq 0$ (A is not a pyramid).

Let $a_1, a_2, \ldots, a_{d+2}$ be the points of A and

$$\bar{a}_i \coloneqq (a_i, 1) \in \mathbb{Z}^{d+1}$$

be the corresponding homogenizations. Moreover, denote $P \coloneqq \text{conv}(A)$.

One of the benefits of working with a configuration consisting of only $d+2$ points is the simplicity of its triangulations. There are exactly two triangulations, that are entirely determined by the signs of γ_i's, i.e. by the circuit (A_+, A_-). These triangulations are given as follows (see, for example, Lemma 2.4.2 in [DLRS10]).

Lemma 7. There are exactly two triangulations of a circuit $A = (A_+, A_-)$ given by

$$\mathcal{T}_+ = \{C \subseteq [d+2] \,:\, A_+ \not\subseteq C\}\,, \quad \mathcal{T}_- = \{C \subseteq [d+2] \,:\, A_- \not\subseteq C\}\,.$$

Knowing this we would like to use Theorem 2 to compute $h^*(P,t)$ and $l^*(P,t)$. In order to do this, we have to compute $l^*(\Delta,t)$, $h(\mathrm{link}_{\mathcal{T}}(\Delta),t)$ and $l_{\Delta,\mathcal{T}}(t)$ for each cell Δ of a chosen triangulation. We fix the triangulation $\mathcal{T} = \mathcal{T}_+$.

Computing $l^*(\Delta,t)$

Recall that we have

$$\sum_{i=1}^{d+2} \gamma_i \bar{a}_i = 0. \tag{3.1}$$

Let Δ be a k-dimensional cell in a triangulation $\mathcal{T}$ with vertices given by $a_{i_1}, \ldots, a_{i_{k+1}}$. Let us again abuse the notation and denote the set of indices $\{i_1, \ldots, i_{k+1}\}$ also by Δ. To compute $l^*(\Delta,t)$ we need to describe the integral points inside the open parallelepiped Π°_Δ. Let $z \in \Pi^\circ_\Delta \cap \mathbb{Z}^{d+1}$, then

$$z = \sum_{i\in\Delta} \lambda_i \bar{a}_i, \quad \lambda_i \in (0,1).$$

Since the point configuration A is spanning, we can also write

$$z = \sum_{i=1}^{d+2} c_i \bar{a}_i$$

for some $c_i \in \mathbb{Z}$ that are not unique, because there is the relation (3.1). Let us fix an $\alpha \in \Delta^c := [d+2] \setminus \Delta$. Then from (3.1) we obtain

$$\bar{a}_\alpha = -\frac{1}{\gamma_\alpha} \sum_{i\in[d+2]\setminus\{\alpha\}} \gamma_i \bar{a}_i$$

and

$$z = \sum_{i\in[d+2]\setminus\{\alpha\}} \left(c_i - \frac{c_\alpha}{\gamma_\alpha}\gamma_i\right) \bar{a}_i.$$

Since P is not a pyramid, any $d+1$ points of A do not lie in one hyperplane, any $d+1$ vectors $\bar{a}_i$ form a basis of the vector space $\mathbb{Q}^{d+1}$, in particular, this is true for $\{\bar{a}_i\}_{i\in[d+2]\setminus\{\alpha\}}$. Therefore, we have an equality

$$\lambda_i = c_i - \frac{c_\alpha}{\gamma_\alpha}\gamma_i$$

for all $i \in [d+2] \setminus \{\alpha\}$. Since $\lambda_i \in [0,1)$, we can apply the fractional part defined by $\{x\} = x - \lfloor x \rfloor$. It gives

$$\lambda_i = \left\{ -\frac{c_\alpha}{\gamma_\alpha} \gamma_i \right\}.$$

We want $\lambda_i = 0$ for $i \in \Delta^c$, so we need $\gamma_\alpha \mid c_\alpha \gamma_i$ for $i \in \Delta^c$,i.e.

$$\left\{ \frac{c_\alpha \gamma_\beta}{\gamma_a} \right\} = 0, \quad \forall\, \alpha, \beta \in \Delta^c. \tag{3.2}$$

Moreover, since the above construction should be a priori independent of α, we have

$$\left\{ \frac{c_\alpha \gamma_i}{\gamma_a} \right\} = \left\{ \frac{c_\beta \gamma_i}{\gamma_\beta} \right\}, \quad \forall \alpha, \beta \in \Delta^c, \quad \forall i \in [d+2]. \tag{3.3}$$

Lemma 8. Denote $g_\Delta = \gcd\left(\{\gamma_j\}_{j \in \Delta^c}\right)$, then (3.2) and (3.3) imply that there exists an integer $d_{\alpha,\Delta}$ and an integer $b_\Delta \in [0, g_\Delta - 1]$ such that

$$\frac{c_\alpha}{\gamma_\alpha} = d_{\alpha,\Delta} + \frac{b_\Delta}{g_\Delta}.$$

Proof. The equations (3.2) imply that for each $\alpha, \beta \in \Delta^c$ there exists an integer $n_{\alpha\beta}$ such that

$$\frac{c_\alpha}{\gamma_\alpha} = \frac{n_{\alpha\beta}}{\gamma_\beta}.$$

This in turn implies that there exists an integer $\tilde{b}_{\alpha,\Delta}$ and, thus, integers $d_{\alpha,\Delta}$ and $b_{\alpha,\Delta} \in [0, g_\Delta - 1]$ such that

$$\frac{c_\alpha}{\gamma_\alpha} = \frac{n_{\alpha\beta}}{\gamma_\beta} = \frac{\tilde{b}_{\alpha,\Delta}}{g_\Delta} = d_{\alpha,\Delta} + \frac{b_{\alpha,\Delta}}{g_\Delta}.$$

Due to (3.3) the integer $b_{\alpha,\Delta}$ does not in fact depend on α, thus, we arrive at the desired statement. □

Now, ignoring the minus sign, for the given point $z \in \Pi^\circ_\Delta$ we can express the corresponding coefficients as

$$\lambda_i = \left\{ \frac{b_\Delta}{g_\Delta} \gamma_i \right\}. \tag{3.4}$$

Since $z \in \Pi_\Delta^\circ$, all of the λ_i should be non-zero for $i \in \Delta$. To express all the possible $z \in \Pi_\Delta^\circ$ we need to consider all the possible λ_i of the form given by (3.4) that satisfy this. It means that we need to consider

$$\lambda_i \in \left\{ \left\{ \frac{b\gamma_i}{N} \right\} : N \mid g_\Delta,\ N \nmid \gamma_i, \quad b = 1, \dots, N-1, \text{ with } \gcd(b, N) = 1 \right\}.$$

This allows us to write for the local h^*-polynomial of a non-empty cell Δ

$$l^*(\Delta, t) = \sum_{\substack{N \mid g_\Delta \\ N \nmid \gamma_i\ \forall i \in \Delta}} \sum_{\substack{b=1 \\ \gcd(b,N)=1}}^{N-1} t^{\sum_{i\in\Delta} \left\{ \frac{b\gamma_i}{N} \right\}}. \tag{3.5}$$

For the empty cell we have $l^*(\varnothing, t) = 1$.

The following lemma is trivial but rather crucial. It will allow us to go from the sum over the cells of a triangulation in Theorem 2 to a sum over integers $N \geq 1$.

Lemma 9. If N appears in (3.5) for some cell Δ, then this is the only such cell in $\mathcal{T}$.

Proof. Suppose the opposite holds, so there are two simplices Δ_1 and Δ_2 such that $N \mid g_{\Delta_1}$ and $N \mid g_{\Delta_2}$, but $N \nmid \gamma_i$ for $i \in \Delta_1$ and $N \nmid \gamma_i$ for $i \in \Delta_2$. Then for any index j in the symmetric difference of Δ_1 and Δ_2 we have that N must simultaneously divide and not divide γ_j. □

Computing the combinatorial polynomials coming from posets.

For a cell $\Delta \in \mathcal{T}$ let us define

$$m_\pm(\Delta) = \#\{j \in \Delta^c \ : \ \mathrm{sign}(\gamma_j) = \pm 1\}.$$

Often we will omit Δ from the notation and write just $m_\pm$.

Let us describe the poset $\mathrm{link}(\Delta, \mathcal{T})$. The cells F that contain Δ must have $m_\pm(F) \leq m_\pm(\Delta)$, but also $m_+(F)$ can not reach 0 since $\mathcal{T} = \mathcal{T}_+$. Therefore the rank of $\mathrm{link}(\Delta, \mathcal{T})$ is $m_+ + m_- - 1$. For a cell F that contains Δ introduce

$$(i_-(\Delta, F),\ i_+(\Delta, F)) \coloneqq (m_-(\Delta) - m_-(F), m_+(\Delta) - m_+(F))$$

For a fixed $(i_-(\Delta, F),\ i_+(\Delta, F))$ there are

$$\binom{m_-(\Delta)}{i_-(\Delta, F)} \binom{m_+(\Delta)}{i_+(\Delta, F)}$$

cells in $\mathcal{T}$ containing Δ. This allows to easily compute $h(\mathrm{link}(\Delta, \mathcal{T}),\ t)$ using (2.3)

$$h(\mathrm{link}(\Delta, \mathcal{T}),\ t) = \sum_{i_-=0}^{m_-} \sum_{i_+=0}^{m_+-1} \binom{m_-}{i_-}\binom{m_+}{i_+} t^{i_-+i_+}(1-t)^{m_-+m_+-1-i_--i_+} = \frac{1-t^{m_+}}{1-t}.$$

Let us proceed with computation of the relative local h-polynomials. Recall that it is defined by

$$l_{\Delta,\mathcal{T}}(t) = \sum_{\Delta \subseteq F \leq P} (-1)^{\dim P - \dim F} h(\mathrm{link}(\Delta, \mathcal{T}_F), t)\ g([F,P]^*, t).$$

If $F \neq P$, then since P is simplicial, $\mathrm{link}(\Delta, \mathcal{T}_F)$ is an Eulerian poset which is actually Boolean, so $h(\mathrm{link}(\Delta, \mathcal{T}_F), t) = 1$. If $F = P$, then $[F, P]$ is a one-point poset, so $g([F,P]^*, t) = 1$, so we get

$$l_{\Delta,\mathcal{T}}(t) = \sum_{\substack{\Delta \subseteq F \leq P \\ F \neq P}} (-1)^{\dim P - \dim F} g([F,P]^*, t) + h(\mathrm{link}(\Delta, \mathcal{T}), t).$$

We already computed the last term. If Δ is not a face of P, the first summand is empty and we are done. If Δ is a face of P, it does not seem to be an easy task to compute the first summand term by term. Nevertheless, we may make use of Lemma 2. The poset $[\Delta, P]$ is always Eulerian, therefore

$$l_{[\Delta,P]}([\Delta,P], t) = \sum_{\substack{\Delta \subseteq F \leq P \\ F \neq P}} (-1)^{\dim P - \dim F} g([F,P]^*, t) + h([\Delta, P], t) = 0.$$

Thus, we obtain

$$l_{\Delta,\mathcal{T}}(t) = \begin{cases} h(\mathrm{link}(\Delta, \mathcal{T}), t) - h([\Delta, P], t), & \text{if } \Delta \text{ if a face of } P, \\ h(\mathrm{link}(\Delta, \mathcal{T}), t) & \text{else.} \end{cases} \tag{3.6}$$

In the case when Δ is a face, the poset $[\Delta, P]$ has a structure very similar to that of $\mathrm{link}(\Delta, \mathcal{T})$. It again has the rank $m_+ + m_- - 1$. Due to Proposition 6 the faces $F \in [\Delta, P]$ cannot have m_+ or m_- equal zero. The poset $[\Delta, P]$ might be not simplicial. However, for all the proper faces F the posets $[\Delta, F]$ are simplicial. Therefore, we get

$$t^{m_++m_--1}\ h([\Delta,P],\ t^{-1}) - h([\Delta,P],\ t) =$$

$$= \sum_{i_-=0}^{m_--1} \binom{m_-}{i_-} \sum_{i_+=0}^{m_+-1} \binom{m_+}{i_+} (t-1)^{m_-+m_+-1-i_--i_+} = \frac{1}{t-1}\left[t^{m_++m_-} - t^{m_+} - t^{m_-} + 1\right].$$

The degree of $h([\Delta, P], t)$ must be smaller than $\lfloor 1/2(m_+ + m_- - 1)\rfloor$. Thus we get

$$h([\Delta, P], t) = \begin{cases} \dfrac{t^{m_-} - 1}{t-1}, & \text{if } m_+ \geq m_-, \\ \dfrac{t^{m_+} - 1}{t-1}, & \text{if } m_+ \leq m_-. \end{cases}$$

And the relative local h-polynomial is

$$l_{\Delta, \mathcal{T}}(t) = \begin{cases} \dfrac{t^{m_+} - t^{m_-}}{t-1}, & \text{if } m_+ \geq m_-, \\ 0, & \text{if } m_+ \leq m_-. \end{cases}$$

Note that this also includes the case of Δ not being a face of P, since in that case $m_- = 0$.

Putting everything together.

From the above computations we arrive at

$$h^*(P, t) = \frac{t^{m_+(\varnothing)} - 1}{t-1} \cdot 1 + \sum_{\varnothing \neq \Delta \in \mathcal{T}_+} \frac{t^{m_+(\Delta)} - 1}{t-1} \sum_{\substack{N \mid g_\Delta \\ N \nmid \gamma_i \ \forall i \in \Delta}} \sum_{\substack{b=1 \\ \gcd(b,N)=1}}^{N-1} t^{\sum_i \{\frac{b\gamma_i}{N}\}}.$$

Let us introduce

$$m_\pm(N) = \#\{j \in [d+2] \ : \ N \mid \gamma_j, \ \operatorname{sign}(\gamma_j) = \pm 1\}.$$

Since each N corresponds to exactly one simplex, we can rewrite

$$h^*(P, t) = \sum_{N \geq 1} \frac{t^{m_+(N)} - 1}{t-1} l^*_N(t),$$

where

$$l^*_N(t) = \sum_{\substack{b=1 \\ \gcd(b,N)=1}}^{N-1} t^{\sum_i \{\frac{b\gamma_i}{N}\}}.$$

for $N \geq 2$ and $l^*_1(t) \coloneqq 1$ corresponding to the empty cell. Similarly we arrive at

$$l^*(P, t) = \sum_{\substack{N \geq 1, \\ m_+(N) > m_-(N)}} \frac{t^{m_+(N)} - t^{m_-(N)}}{t-1} l^*_N(t). \tag{3.7}$$

Note, that everything above can be done with the other triangulation $\mathcal{T}_-$, which would amount to exchanging m_+ and m_- in the above formulas.

Corti-Golyshev and zigzag diagrams

From the above results one can derive a celebrated formula conjectured by Corti and Golyshev [CG11] and proved by Fedorov [Fed18] for the Hodge numbers of a hypergeometric local system. The following lemma from [Bob09] is the key in going from (3.7) to the formula of Corti and Golyshev.

Lemma 10 (Remark 2.2, [Bob09])**.** Consider the function

$$f(x,\gamma) = \sum_{i:\gamma_i<0} \lfloor -\gamma_i x \rfloor - \sum_{i:\gamma_i>0} \lfloor \gamma_i x \rfloor .$$

Define the relatively prime polynomials $P(T)$ and $Q(T)$ by

$$\frac{P(T)}{Q(T)} = \frac{\prod_{i:\gamma_i<0}(T^{-\gamma_i}-1)}{\prod_{i:\gamma_i>0}(T^{\gamma_i}-1)}.$$

There exist a number $K \geq 1$ and tuples $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_K \leq 1$ and $0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_K \leq 1$ such that

$$P(T) = \prod_{i=1}^{K}(T - e^{2\pi i \alpha_i}), \quad Q(T) = \prod_{i=1}^{K}(T - e^{2\pi i \beta_i}).$$

Then for $x \in [0,1]$ we have

$$f(x,\gamma) = \#\{\alpha_i \mid \alpha_i \leq x\} - \#\{\beta_i \mid \beta_i \leq x\} .$$

Let us redefine the functions $m_\pm(-)$ for the interval $[0,1]$ by introducing functions $m_\pm : [0,1] \to \mathbb{N}$

$$m_\pm(x) := \#\{i \mid \{\gamma_i x\} = 0, \quad \operatorname{sign}\gamma_i = \pm 1\} .$$

In particular for any $x = b/N$ with $\gcd(b,N) = 1$ we have $m_\pm(x) = m_\pm(N)$. For $x \in [0,1]$ we have

$$\sum_{i=1}^{d+2}\{\gamma_i x\} = -\sum_{i=1}^{d+2}\lfloor \gamma_i x \rfloor = -\sum_{i:\gamma_i>0}\lfloor \gamma_i x \rfloor + \sum_{i:\gamma_i<0}\lfloor -\gamma_i x \rfloor + m_-(0) - m_-(x).$$

Consider the following set

$$B' = \left\{ \frac{b}{N} \mid N \geq 1, \quad b \in [1, N-1], \quad \gcd(b,N) = 1, \quad m_+(N) > m_-(N) \right\} \cup \{1\} .$$

Now we can write

$$l^*(P,t) = t^{m_-(0)} \sum_{x\in B'} \frac{t^{m_+(x)-m_-(x)}-1}{t-1}\, t^{\#\{\alpha_i|\alpha_i\le x\}-\#\{\beta_i|\beta_i\le x\}}. \tag{3.8}$$

Consider the tuple

$$\beta = [\beta_1, \beta_2, \dots, \beta_K]$$

and the corresponding set

$$B = \{x \mid x \in \beta\}\,.$$

Each summand in (3.8) is non-zero only if $x \in B$, otherwise $m_+(x) - m_-(x) = 0$. Moreover, for x in B the value $m_+(x) - m_-(x)$ is exactly the number of times x appears in the tuple β. Finally, we can rewrite

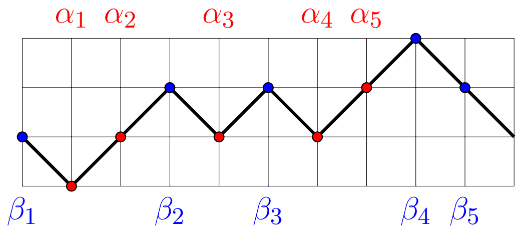
$$l^*(P,t) = t^{m_-(0)} \sum_{j=1}^{K} t^{\#\{\alpha_i|\alpha_i\le\beta_j\}-j} \tag{3.9}$$

or in other words

$$l^*_k = \#\left\{j \mid \#\left\{\alpha_i \mid \alpha_i \le \beta_j\right\} - j + m_-(0) = k\right\},$$

which is exactly the formula of Corti and Golyshev for the Hodge numbers of the corresponding hypergeometric local system. Therefore, we see that the Hodge numbers of the hypersurface Z_P coincide with the Hodge numbers of the hypergeometric local system H_γ defined as the local system of solutions of the differential equation on $\mathbb{C}^* \setminus \{1\}$ that annihilates the hypergeometric function ${}_KF_{K-1}(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_K; t)$. A priori, at the moment it is not formally known (though widely believed) that H_γ arises as a variation of Hodge structure on $H^{d-1}(Z_P)$, but the fact that the above Hodge numbers coincide serves as a strong evidence for this. See also the upcoming work of Abdelraouf and Gugiatti [AG].

The formula (3.9) gives rise to the method of zigzag diagrams described in [RRV22]. Suppose the jth term of the sum contributes to l^*_k. Let a be the number of α_i's between β_j and β_{j+1}, then the $(j+1)$th term contributes to l^*_{k+a-1}. This can be represented by the following diagrammatic rule. Choose the smallest number among all the alphas and betas. If it is an α_i, then put a red dot at the origin and proceed with a North-East move after. If it is a β_i, then put a blue dot at the origin and proceed with a South-East move. Then take the next smallest number of all the alphas and betas and continue the same way as above but from the already reached point. One can get the coefficients of $l^*(P,t)$ up to an overall shift by counting the number of blue or red dots at each height. Here is an example with $K = 5$ taken from [RRV22] with $(1,3,1)$ as the list of the coefficients of $l^*(P,t)$.



Internal zeroes.

Proposition 7. Let P be a spanning circuit, then the local h^*-polynomial has no internal zeroes. In other words, let n be the first non-zero coefficient $l_k^*(P)$, then for $n \leq i \leq d+1-n$ we have $l_i^*(P) \geq 1$.

Proof. It is easily seen from the zigzag diagrams. Since there is an equal amount of α's and β's, each zigzag returns at the same height from which it started. Suppose the starting height corresponds to j such that $n \leq j \leq d+1-n$, then there must be a blue dot at each height from n to $d+1-n$.

□

In [HKN18] it was proven that for a spanning P, the h^*-polynomial has no internal zeroes. The above proposition and numerical computations suggest that one should expect that there is a similar result in the local situation.

Conjecture 1. Let P be a spanning lattice polytope. The local h^*-polynomial of P has no internal zeroes.

3.2 Shifted products of circuits

There is a point configuration that contains more than $d+2$ points, but still resembles some properties of the circuits. This class of point configurations was inspired by the iterated circuits introduced by Esterov [Est10].

Let us consider point configurations whose Gale duals can be chosen in the following form

$$G = \left(\begin{array}{cccccccc} \gamma_{0,-}^{(1)} & \gamma_{-}^{(1)} & \gamma_{+}^{(1)} & 0 & 0 & \dots & 0 & 0 \\ \gamma_{0,-}^{(2)} & 0 & 0 & \gamma_{-}^{(2)} & \gamma_{+}^{(2)} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{0,-}^{(L)} & 0 & 0 & 0 & 0 & \dots & \gamma_{-}^{(L)} & \gamma_{+}^{(L)} \\ \hline I_{0,-} & I_{1,-} & I_{1,+} & I_{2,-} & I_{2,+} & \dots & I_{L,-} & I_{L,+} \end{array}\right). \tag{3.10}$$

Let us explain the notation. For $k \geq 1$ each $\gamma_{k,\pm}^{(j)}$ is a tuple of integers of signature $\pm$. The tuples $\gamma_{0,-}^{(j)}$ have non-positive entries, however, there has to be no columns of zeroes. The first $|I_{0,-}|$ columns have indices $I_{0,-}$ and they have only non-positive entries. The next $I_{1,-}$ columns have negative entries only in the first row and everything else is zero. The next $I_{1,+}$ columns have positive entries in the first row and zeroes elsewhere and so on. Suppose, the corresponding configuration A has dimension d, then there are $d + L + 1$ columns in G. Let us give a quick example.

Example 8. Consider

$$G = \begin{pmatrix} -5 & -2 & -3 & -1 & 4 & 4 & 3 & 0 & 0 \\ -11 & -1 & 0 & 0 & 0 & 0 & 0 & 6 & 6 \end{pmatrix}.$$

Here $I_{0,-} = \{1, 2\},\ I_{1,-} = \{3, 4\},\ I_{1,+} = \{5, 6, 7\},\ I_{2,-} = \varnothing,\ I_{2,+} = \{8, 9\}$.

Definition 21. We call a point configuration A with a Gale dual G of the form (3.10) a **shifted product of circuits**.

The reason for the name is as follows. Suppose A is the point configuration with a Gale dual G as above. For each $j = 1, \ldots, L$ the subconfiguration consisting of the points with indices $I_{0,-} \cup I_{j,-} \cup I_{j,+}$ is in fact a circuit of dimension $|I_{0,-} \cup I_{j,-} \cup I_{j,+}| - 2$. Therefore, A can be viewed as a collection of L circuits of different dimensions such that they have $|I_{0,-}|$ points in common. If $I_{0,-}$ is empty, then our configuration is just a product of circuits. Moreover, as we will see later, the volume of A is the product of the volumes of the circuits.

Shifted products of circuits look particularly nice if $m_{0,-} = 1$. In this case, the coordinates of the points of A can be chosen to be the columns of a matrix of the form

$$A = \begin{pmatrix} 0 & \tilde{A}_1 & 0 & \ldots & 0 \\ 0 & 0 & \tilde{A}_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \tilde{A}_L \end{pmatrix},$$

such that each configuration $A_i = (0, \tilde{A}_i)$ is a circuit. This is a particular example of an **iterated circuit**. As was explained in [For19] these are the configurations that correspond to the matrices of the form

$$A = \begin{pmatrix} 0 & \tilde{A}_1 & * & \ldots & * \\ 0 & 0 & \tilde{A}_2 & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \tilde{A}_L \end{pmatrix}.$$

Example 9. A standard example of an iterated circuit, that is also a shifted product of circuits, is the point configuration consisting of the vertices of the octahedron and the origin

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This is clearly spanning. A Gale dual can be chosen to be

$$G = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Theorem 6. There exists a triangulation $\mathcal{T}_0$ of a shifted product of circuits described by

$$\mathcal{T}_0 = \{C \subseteq [d+L+1] : I_{k,+} \not\subseteq C \quad \forall k \neq 0\}.$$

Proof. Recall that we have to show that $\mathcal{T}_0$ satisfies the three properties from Definition 19. The closure property is evident. To prove the intersection property we are going to make use of Proposition 5. For this we have to describe the circuits of A. Each circuit corresponds to a linear combination of the rows of G from (3.10). Therefore, for each circuit, either Z_+ or Z_- is going to contain a whole $I_{k,+}$ for at least one k. Therefore, we see that there is no circuit Z such that both Z_+ and Z_- are in $\mathcal{T}_0$.

Since we know that the closure and the intersection properties are satisfied, we can make use of Proposition 6. However, for this we have to describe the facets of conv(A) first.

Lemma 11. The facets of conv(A) are of the form

- $C \subseteq [d+L+1]$ such that $|C| = d+L-1$ and $I_{k,\pm} \not\subseteq C$ for some k;
- $C \subseteq [d+L+1]$ such that $|C| = d$ and $I_{0,-} \not\subseteq C$, $\quad I_{j,+} \not\subseteq C$ for $j \geq 1$.

Proof. According to Lemma 6 the faces of A are the complements of the positive dependence signatures of G. Therefore, the facets of A correspond to the minimal positive dependence signatures, i.e. to those from which we cannot delete an element and still be left a positive dependence signature. There are two types of these:

- A subconfiguration of G consisting of two points p_{i_+} and p_{i_-}, such that $i_+ \in I_{k,+}$ and $i_- \in I_{k,-}$ for some k;
- A subconfiguration of G consisting of L points $p_{i_0}, p_{i_1}, \ldots, p_{i_L}$ with $i_0 \in I_{0,-}$ and $i_j \in I_{j,+}$ for each $j \geq 1$.

The complements of these subsets of $[d+L+1]$ are exactly the sets described in the statement of the lemma.

□

Suppose that $C = \{i_1, i_2, \ldots, i_{d+1}\}$ is a maximal cell in $\mathcal{T}_0$. Let us consider its facet $F = \{i_2, i_3, \ldots, i_{d+1}\}$. There are a few possible scenarios.

- If $i_1 \in I_{k,+}$, then there must be some other index $j_1 \in I_{k,+}$ that is not in C, such that there is a maximal cell $\{j_1, i_2, \ldots, i_{d+1}\}$. Otherwise we would have $I_{k,+} \subseteq C$. Note that this cell is in fact full-dimensional, since there could be no dependencies between the points $j_1, i_2, \ldots, i_{d+1}$;
- If $i_1 \in I_{0,-}$, then F must belong to a facet of the second type from the previous lemma since by the definition of C it cannot contain any $I_{k,+}$.
- If $i_1 \in I_{k,-}$ for some $k = 1, \ldots, L$, then F is a facet of P of the first type from the previous lemma.

□

Remark 7. From Lemma 11 we see that in general, a shifted product of circuits is not simplicial. There are

$$|I_{0,-}| \cdot \prod_{k=1}^{L} |I_{k,+}|$$

facets that are always simplices and

$$\sum_{k=1}^{L} |I_{k,+}| \cdot |I_{k,-}|$$

facets with $d+L-1$ points that are not simplices in general. If among these $d+L-1$ points there are $d+1$ points that form a $(d-1)$ dimensional circuit such that $|Z_\pm| = 1$ and $|Z_\mp| = d$, then some of these points are not vertices of the facet. Note that $\operatorname{conv}(A)$ is guaranteed to be simplicial if all the $I_{k,-}$ are empty for $k \geq 1$.

Proposition 8. Volume of a shifted product of circuits A defined by the Gale dual (3.10) is

$$\operatorname{vol} P = \prod_{i=1}^{L} \sum_{j \in I_{j,+}} G_{ij}. \tag{3.11}$$

In particular, the volume of a shifted product of circuits could never be a prime number unless $L = 1$ and it is actually a circuit.

Proof. The maximal cells correspond to the sets of indices where exactly one index is missing from each I_{k_+}. Therefore by Corollary 1 we get

$$\operatorname{vol} P = \sum_{\substack{\Delta \in \mathcal{T}_0 \\ |\Delta| = d+1}} \operatorname{vol} \Delta = \sum_{(i_1,\dots,i_L) \in I_{1,+} \times \dots \times I_{L,+}} G_{1i_1} G_{2i_2} \dots G_{Li_L},$$

which can be brought to the product in (3.10). □

The h^*-polynomial

We are going to use the result of Betke and Mcmullen again, so we start by computing the local h^*-polynomials of the cells of the triangulation $\mathcal{T}_0$. Recall that for each $i = 1, \dots, L$ we have a relation

$$\sum_{j=1}^{d+L+1} G_{ij} \bar{a}_j = 0.$$

Let $\Delta = \{i_1, \dots, i_{s+1}\}$ be an s-dimensional cell in the triangulation $\mathcal{T}_0$. To compute $l^*(\Delta, t)$ we need to describe the integral points inside the open parallelepiped Π°_Δ. Let $z \in \Pi^\circ_\Delta \cap \mathbb{Z}^{d+1}$. These can be written as

$$z = \sum_{i \in \Delta} \lambda_i \bar{a}_i, \quad \lambda_i \in (0, 1).$$

For each $k = 1, \dots, L$ let us fix an index $n_k \in \Delta^c \cap I_{k,+}$. This is always possible because for each k and for each cell Δ we have $I_{k,+} \not\subseteq \Delta$. However, note that this choice might be not unique. Since the configuration A is spanning, we can write

$$z = \sum_{i=1}^{d+L+1} c_i \bar{a}_i, \quad c_i \in \mathbb{Z}\,.$$

Using the relations above we arrive at

$$z = \sum_{j \in I_{0,-}} \left(c_j - \sum_{k=1}^{L} c_{n_k} \frac{G_{kj}}{G_{kn_k}} \right) \bar{a}_j + \sum_{k=1}^{L} \sum_{j \in I_{k,-} \cup I_{k,+} \setminus \{n_k\}} \left(c_j - c_{n_k} \frac{G_{kj}}{G_{kn_k}} \right) \bar{a}_j. \tag{3.12}$$

Since $[d+L+1] \setminus \{n_1, n_2, \ldots, n_k\}$ corresponds to a maximal cell in $\mathcal{T}_0$, the corresponding points $\bar{a}_i$ linearly span $\mathbb{R}^{d+1}$. Therefore, we can equate the coefficients of $\bar{a}_i$ with λ_i and get the following conditions:

$$c_j - \sum_{k=1}^{L} c_{n_k} \frac{G_{kj}}{G_{kn_k}} = 0 \quad \text{for} \;\; j \in I_{0,-} \cap \Delta^c, \tag{3.13}$$

$$\lambda_j = c_j - \sum_{k=1}^{L} c_{n_k} \frac{G_{kj}}{G_{kn_k}} \neq 0 \quad \text{for} \;\; j \in I_{0,-} \cap \Delta \tag{3.14}$$

$$c_j - c_{n_k} \frac{G_{kj}}{G_{kn_k}} = 0 \quad \text{for} \;\; j \in \left(I_{k,-} \cup I_{k,+} \right) \cap \Delta^c, \tag{3.15}$$

$$\lambda_j = c_j - c_{n_k} \frac{G_{kj}}{G_{kn_k}} \neq 0 \quad \text{for} \;\; j \in \left(I_{k,-} \cup I_{k,+} \right) \cap \Delta. \tag{3.16}$$

For each $k = 1, \ldots, L$ introduce

$$g_{k,\Delta} := \gcd\left(\{ G_{kj} \}_{j \in (I_{k,-} \cup I_{k,+}) \cap \Delta^c} \right).$$

Using (3.15) and arguing in the same way as in Lemma 8 we can show that for each cell Δ there exists a tuple of integers $(b_{1,\Delta}, \ldots, b_{L,\Delta})$ such that we can write

$$\left\{ \sum_{k=1}^{L} b_{k,\Delta} \frac{G_{kj}}{g_{k,\Delta}} \right\} = 0 \quad \text{for} \;\; j \in I_{0,-} \cap \Delta^c, \tag{3.17}$$

$$\lambda_j = \left\{ \sum_{k=1}^{L} b_{k,\Delta} \frac{G_{kj}}{g_{k,\Delta}} \right\} \neq 0 \quad \text{for} \;\; j \in I_{0,-} \cap \Delta \tag{3.18}$$

$$\left\{ b_{k,\Delta} \frac{G_{kj}}{g_{k,\Delta}} \right\} = 0 \quad \text{for} \;\; j \in \big(I_{k,-} \cup I_{k,+} \big) \cap \Delta^c, \tag{3.19}$$

$$\lambda_j = \left\{ b_{k,\Delta} \frac{G_{kj}}{g_{k,\Delta}} \right\} \neq 0 \quad \text{for} \;\; j \in \big(I_{k,-} \cup I_{k,+} \big) \cap \Delta. \tag{3.20}$$

As before for the case of circuits, it is convenient to single out divisors of $g_{k,\Delta}$ and consider $b_{k,\Delta}$ that are relatively prime to a divisor. Let us introduce tuples

$$N_\Delta = \big(N_{1,\Delta}, N_{2,\Delta}, \ldots, N_{L,\Delta} \big)$$

where each $N_{i,L} \geq 1$ and $N_{i,\Delta} \mid g_{i,\Delta}$. For each N we can also consider tuples

$$b_\Delta^{(N)} = \big(b_{1,\Delta}, b_{2,\Delta}, \ldots, b_{L,\Delta} \big)$$

with $b_{i,\Delta} \geq 1$ and $\gcd(b_{i,\Delta}, N_{i,\Delta}) = 1$ unless $N_{i,\Delta} = 1$, then we take $b_{i,\Delta} = 0$. At last, let us consider the set of all the possible pairs of the tuples above for a given cell Δ

$$\Lambda_\Delta := \Big\{ (b, N) \in \mathbb{Z}^L \times \mathbb{Z}^L \;\; : \; \text{conditions 1.-5. below are satisfied} \Big\}$$

such that

1. $N_i \geq 1$ and $N_i \mid g_{i,\Delta}$,
2. $b_i \geq 1$ and $\gcd(b_i, N_i) = 1$ if $N_i \geq 2$,
3. $b_i = 0$ if $N_i = 1$,
4. $\left\{ \sum_{k=1}^L G_{kj} \frac{b_k}{N_k} \right\} = 0 \quad \text{for} \;\; j \cap \Delta^c$,
5. $\left\{ \sum_{k=1}^L G_{kj} \frac{b_k}{N_k} \right\} \neq 0 \quad \text{for} \;\; j \cap \Delta$.

We can now express the local h^*-polynomial of Δ as follows

$$l^*(\Delta, t) = \sum_{(b,N) \in \Lambda_\Delta} t^{\sum_{j \in [d+L+1]} \left\{ \sum_{k=1}^L G_{kj} \frac{b_k}{N_k} \right\}}. \tag{3.21}$$

Let $\Delta \in \mathcal{T}_0$ be a cell in the triangulation. Similar to the case of circuits we can describe the poset $\operatorname{link}(\Delta, \mathcal{T}_0)$. For each k let us introduce

$$m_{k,\pm}(\Delta) = |\{ j \in I_{k,\pm} \cap \Delta^c \}|.$$

Each $F \in \operatorname{link}(\Delta, \mathcal{T}_0)$ has $0 < m_{k,+}(F) \leq m_{k,+}(\Delta)$ and $0 \leq m_{k,-}(F) \leq m_{k,-}(\Delta)$. Thus, the rank of $\operatorname{link}(\Delta, \mathcal{T}_0)$ is $\sum_{k=1}^{L} m_{k,+} + \sum_{k=0}^{L} m_{k,-} - L$. Completely analogous to the case of circuits, using (2.3), we can compute

$$h(\operatorname{link}(\Delta, \mathcal{T}_0),\ t) = \sum_{i_{0,-}=0}^{m_{0,-}(\Delta)} \sum_{i_{1,-}=0}^{m_{1,-}(\Delta)} \sum_{i_{1,+}=0}^{m_{1,+}(\Delta)-1} \cdots \sum_{i_{L,-}=0}^{m_{L,-}(\Delta)} \sum_{i_{L,+}=0}^{m_{L,+}(\Delta)-1} \binom{m_{0,-}(\Delta)}{i_{0,-}} \binom{m_{1,-}(\Delta)}{i_{1,-}} \times$$
$$\times \binom{m_{1,+}(\Delta)}{i_{1,+}} \times \ldots \times \binom{m_{L,-}(\Delta)}{i_{L,-}} \binom{m_{L,+}(\Delta)}{i_{L,+}} \times$$
$$\times\, t^{\sum_{j=0}^{L} i_{j,-}+i_{j,+}}\, (1-t)^{\sum_{j=0}^{L} m_{j,-}+m_{j,+}-L-i_{j,-}-i_{j,+}} = \prod_{i=1}^{L} \frac{t^{m_{i,+}(\Delta)}-1}{t-1}.$$

Finally we arrive at

$$h^*(P,t) = \prod_{i=1}^{L} \frac{t^{m_{i,+}(\varnothing)}-1}{t-1} \cdot 1 + \sum_{\varnothing \neq \Delta \in \mathcal{T}_0} \prod_{i=1}^{L} \frac{t^{m_{i,+}(\Delta)}-1}{t-1} \sum_{(b,N)\in\Lambda_\Delta} t^{\sum_{j\in[d+L+1]} \left\{ \sum_{k=1}^{L} G_{kj} \frac{b_k}{N_k} \right\}}$$

Introduce

$$m_{i,+}(N_i) \coloneqq |\{j \in I_{i,+} \ :\ N_i \mid G_{ji}\}|$$

and note that whenever $(b, N) \in \Lambda_\Delta$, then $m_{i,+}(\Delta) = m_{i,+}(N_i)$ for each $i = 1, \ldots, L$. Next, we are going to get exchange the sum over the cells of $\mathcal{T}_0$ for a sum over tuples of integers with the help of the following lemma.

Lemma 12. If a pair of tuples $(b_1, \ldots, b_L)$ and $(N_1, \ldots, N_L)$ appears in the expression (3.21) for a cell Δ, then this is the only cell for which this is the case.

Proof. The proof is almost exactly the same as of Lemma 9. Namely, if we assume that there are two cells Δ_1 and Δ_2 satisfying the condition of the lemma, then there exists an index j in the symmetric difference of Δ_1 and Δ_2 such that $\left\{ \sum_{k=1}^{L} G_{kj} \frac{b_k}{N_k} \right\}$ is simultaneously zero and non-zero. □

Let $N_i^{\max} = \max\{G_{ij} \ :\ j = 1, \ldots, d+L+1,\ G_{ij} > 0\}$ be the biggest positive entry of the ith row of the matrix G defining Gale dual. Consider the set of pairs of tuples

$$\Lambda_G \coloneqq \{(b, N) \ :\ \text{conditions 1.-3. below are satisfied}\}$$

such that

1. $1 \leq N_i \leq N_i^{\max}$;

2. $b_i \geq 1$ and $\gcd(b_i, N_i) = 1$ if $N_i \geq 2$;

3. $b_i = 0$ if $N_i = 1$;

Then instead of summing over the cells of $\mathcal{T}_0$ we can sum over all the tuples in Λ_G. This gives us the main theorem of this section.

Theorem 8. Let A be a spanning interlocking circuit, i.e. it is a point configuration with a Gale dual G given by (3.10) and $P = \operatorname{conv}(A)$. The h^*-polynomial of P has the following expression:

$$h^*(P,t) = \sum_{(b,N)\in\Lambda_G} \prod_{i=1}^{L} \frac{t^{m_{i,+}(N_i)} - 1}{t-1} \, t^{F(b,N)},$$

where

$$F(b,N) = \sum_{i=1}^{d+L+1} \left\{ \sum_{j=1}^{L} \frac{b_i}{N_i} \, G_{ij} \right\}.$$

Example 10. Let us demonstrate how the above formula works with an example. Consider a point configuration defined by a Gale dual

$$G = \begin{pmatrix} -2 & -2 & 1 & 3 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The coordinates of A can be taken to be the columns of

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \end{pmatrix},$$

so P is a 3-dimensional polytope with 6 vertices. We have $N_1^{\max} = 3$ and $N_2^{\max} = 2$. Let us write down a table with the members of Λ_G and the corresponding values of $F(b,N)$ and $h(\operatorname{link}(\Delta, \mathcal{T}_0),\ t)$

The value $F((1,1),(3,2))$ is computed, for example, as follows:

$$F((1,1),(3,2)) = \left\{-2\cdot\frac{1}{3} - 1\cdot\frac{1}{2}\right\} + \left\{-2\cdot\frac{1}{3} - 2\cdot\frac{1}{2}\right\} + \left\{\frac{1}{3}\right\} + \left\{\frac{3}{3}\right\} + \left\{\frac{1}{2}\right\} + \left\{\frac{2}{2}\right\}.$$

Summing everything up, we get

$$h^*(P,t) = t^3 + 6t^2 + 4t + 1.$$

(b, N)	$h(\text{link}(\Delta, \mathcal{T}_0),\ t)$	$F(b, N)$
$(0,0),(1,1)$	$(t+1)^2$	0
$(0,1),(1,2)$	$t+1$	1
$(1,0),(3,1)$	$t+1$	1
$(2,0),(3,1)$	$t+1$	2
$(1,1),(3,2)$	1	2
$(2,1),(3,2)$	1	2

Remark 9. Just as a sanity check we can compute the volume of P from $h^*(P,t)$ by specializing $t = 1$. We get

$$\operatorname{vol} P = \sum_{N_1,\ldots,N_L \geq 1} \prod_{i=1}^{L} m_{i,+}(N_i)\ \varphi(N_i).$$

Using the Gauss identity $\sum_{d|n} \phi(d) = n$ we arrive exactly at the same result as in (3.11).

Remark 10. Despite looking somewhat difficult, the formula from Theorem 8 offers considerable computational advantage for this class of polytopes if $n-d-1$ is small with respect to d. For example, we considered small samples of randomly generated 8-dimensional shifted products of circuits with $L = 2$ and $L = 3$, and the computation with the above formula was on average a few hundred times faster compared to the standard methods from SageMath [The22].

Local h^*-polynomial of simplicial shifted products of circuits

The derivation of the formula (3.7) of the local h^*-polynomial of a spanning circuit relied heavily on the fact that the convex hull of a circuit is a simplicial polytope. As discussed earlier, shifted products of circuits are usually not simplicial. However, using Lemma 11 we can guarantee that it is simplicial if all the $I_{k,-}$ are empty for $k \geq 1$, i.e. we have a Gale dual of the form

$$G = \left(\begin{array}{ccccc} \gamma_{0,-}^{(1)} & \gamma_{+}^{(1)} & 0 & \ldots & 0 \\ \gamma_{0,-}^{(2)} & 0 & \gamma_{+}^{(2)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{0,-}^{(N)} & 0 & 0 & \ldots & \gamma_{+}^{(N)} \\ \hline I_{0,-} & I_{1,+} & I_{2,+} & \ldots & I_{N,+} \end{array}\right). \tag{3.22}$$

We can compute the local h-polynomial contributions using (3.6) again. For this we have to understand the structure of the poset $[\Delta, P]$ for a face Δ. As we know from Lemma 6, the faces of P should correspond to the complements of positive dependence signatures of the Gale dual G. These are very easy to describe since all $I_{k,-}$ are empty. Each positive signature must have at least one element from each of $I_{0,-}, I_{1,+}, I_{2,+}, \ldots$. Therefore, if Δ is a face, we can get the elements F of $[\Delta, P]$ by adding elements from $[d+L+1] \setminus \Delta$ as long as we have at least one element from each $I_{0,-}, I_{1,+}, \ldots, I_{L,+}$ in $[d+L+1] \setminus F$. The rank of the poset $[\Delta, P]$ is $m_{0,-}(\Delta) + \sum m_{k,+}(\Delta) - L$. Omitting Δ from the arguments of $m_{\pm}$, we compute

$$\begin{aligned}
&t^{m_{0,-}+\sum m_{k,+}-L}\ h([\Delta, P],\ t^{-1}) - h([\Delta, P],\ t) = \\
&= \sum_{i_{0,-}=0}^{m_{0,-}-1} \sum_{i_{1,+}=0}^{m_{1,+}-1} \cdots \sum_{i_{L,+}=0}^{m_{L,+}-1} \binom{m_{0,-}}{i_{0,-}} \times \\
&\times \binom{m_{1,+}}{i_{k,+}} \cdots \binom{m_{L,+}}{i_{L,+}} (t-1)^{m_{0,-}+\sum m_{k,+}-L-i_{0,-}-\sum i_{k,+}} = \\
&= (t^{m_{0,-}}-1) \prod_{i=1}^{L} \frac{t^{m_{i,+}}-1}{t-1}.
\end{aligned}$$

The degree of $h([\Delta, P], t)$ must be smaller than $\lfloor 1/2(m_{0,-} + \sum m_{k,+} - L) \rfloor$. Let us define an operator of truncation as

$$\begin{aligned}
\tau_{<k} : \mathbb{R}[t] &\to \mathbb{R}[t] \\
\sum_{i \geq 1} c_i t^i &\to \sum_{i=1}^{k-1} c_i t^i.
\end{aligned}$$

We can use it to obtain

$$h([\Delta, P], t) = -\tau_{\lfloor 1/2(m_{0,-}+\sum m_{k,+}-L) \rfloor} \left[(t^{m_{0,-}}-1) \prod_{i=1}^{L} \frac{t^{m_{i,+}}-1}{t-1} \right]. \tag{3.23}$$

Note, that if considered the above expression for a cell Δ that is not a face, we would obtain zero, because such a cell must contain all the points from $I_{0,-}$ by Lemma 6. Thus, for any cell Δ we can write

$$l_{\Delta,\mathcal{T}_0}(t) = \prod_{i=1}^{L} \frac{t^{m_{i,+}}-1}{t-1} + \tau_{\lfloor 1/2(m_{0,-}+\sum m_{k,+}-L) \rfloor} \left[(t^{m_{0,-}}-1) \prod_{i=1}^{L} \frac{t^{m_{i,+}}-1}{t-1} \right].$$

When we try to combine it with the local h^*-polynomials of the cells we get

$$l^*(P,t)\sum_{\Delta\in\mathcal{T}_0} l_{\Delta,\mathcal{T}_0}(t) \sum_{(b,N)\in\Lambda_\Delta} t^{\sum_{j\in[d+L+1]}\left\{\sum_{k=1}^{L} G_{kj}\frac{b_k}{N_k}\right\}},$$

we see that now it is not enough to introduce $m_{k,\pm}(N)$ to exchange the summation over the cells Δ to the summation over Λ_G. Let us define

$$m_-(b,N) := |\{j\in I_{0,-} \mid \left\{\sum_{i=1}^{L} G_{ji}\frac{b_i}{N_i}\right\} = 0\}|.$$

Theorem 11. Let A be a spanning and simplicial shifted product of circuits with a Gale dual G given by (3.22), and $P=\operatorname{conv}(A)$. The local h^*-polynomial of P has the following expression:

$$l^*(P,t) = \sum_{(b,N)\in\Lambda_G} l_{\mathcal{T}_0}(b,N,t)\ t^{F(b,N)},$$

where

$$l_{\mathcal{T}_0}(b,N,t) = \prod_{i=1}^{L}\frac{t^{m_{i,+}(N_i)}-1}{t-1} + \tau_{\lfloor 1/2(m_{0,-}(b,N)+\sum m_{k,+}(N_k)-L)\rfloor}\left[(t^{m_{0,-}(b,N)}-1)\prod_{i=1}^{L}\frac{t^{m_{i,+}(N_i)}-1}{t-1}\right]$$

and as before

$$F(b,N) = \sum_{i=1}^{d+L+1}\left\{\sum_{j=1}^{L}\frac{b_i}{N_i}\ G_{ij}\right\}.$$

Note that unlike the previous cases considered in this section, the part of the formula coming from the poset combinatorics now depends on (b,N) and not just on N.

Question 1. Is there an analogue of the Corti-Golyshev formula for the case of simplicial shifted products of circuits with $L\geq 2$? Although the formula for the local h^*-polynomial looks quite similar to the circuits case, it is very much not clear what could be a formula in the spirit of Corti and Golyshev. Moreover, it is not known what could play the role of the corresponding local systems.

Numerous computations suggest the following two conjectures.

Conjecture 2. Shifted products of circuits are never thin for $L\geq 2$.

Conjecture 3. If a shifted product of circuits P with $L \geq 2$ has

$$l^*(P,t) = a \cdot t^k$$

for some a and k, then it must fall into one of the following cases

- P is 1-dimensional
- P is 3-dimensional defined by a Gale dual of the form

$$\begin{pmatrix} -a_1 & -a_2 & a_1 & a_2 & 0 & 0 \\ -b_1 & -b_2 & 0 & 0 & b_1 & b_2 \end{pmatrix}.$$

This is quite different from the circuits, i.e. the case $L = 1$. As explained in [Vil19], the circuits having a monomial local h^*-polynomial correspond to the algebraic hypergeometric functions. These were classified by [BH89], and it is a rather non-trivial result. It seems that for shifted products of circuits with $L \geq 2$ there does not exist any sporadic cases and we have only very simple configurations with a monomial local h^*-polynomial.

4 Operations Preserving Hodge Vectors

Consider a lattice polytope P and let Δ_n be the n-dimensional unimodular simplex, then the free join $P \circ_{\mathbb{Z}} \Delta_n$ has the same h^*-polynomial as the polytope P

$$h^*(P \circ_{\mathbb{Z}} \Delta_n, t) = h^*(P, t).$$

This seems to be the only known construction that preserves the h^*-polynomial. What can we say about the local h^*-polynomial? The goal of this section is to discuss this question.

Suppose we are given a polytope P and out of it we construct some polytope $\tilde{P}$. If $\dim \tilde{P} \neq \dim P$, then it is not possible that they have the same local h^*-polynomial, because we have $t^{d+1} l^*(P, t^{-1}) = l^*(P, t)$ for a d-dimensional polytope P. So the question transforms to the following. Given a polytope P, is it possible to construct a non-isomorphic polytope $\tilde{P}$ of the same dimension with the same local h^*-polynomial? As far as I am aware, there is no such construction. Moreover, I believe, it likely does not exist.

Nevertheless, the section does not end here. The theory of hypergeometric motives [RRV22] suggests to consider not the local h^*-polynomial, but the tuple of its coefficients without the outside vanishing coefficients. For example, if we have a local h^*-polynomial $t^2 + t^4$, the corresponding tuple is $(1, 0, 1)$. We call such a tuple the **Hodge vector** of P. In the area of hypergeometric motives the following question arises. Suppose we have a $(d-1)$ dimensional hypersurface Z_P, that realizes some hypergeometric motive, with Hodge vector of length $l < d$. Does there exist some other variety V of dimension $l - 1$ realizing the same motive with the same Hodge vector? Somewhat simplifying this question and reversing the direction in terms of dimensions, we can ask the following.

Question 2. Given a d-dimensional polytope P, is there a general way of constructing a polytope Q of dimension bigger than d such that both polytopes share the same Hodge vector?

Since the local h^*-polynomial is multiplicative, there are numerous trivial constructions. Namely, let S be a polytope with $l^*(S,t) = t^m$ for some m. Then the free join $P \circ_{\mathbb{Z}} S$ has the same Hodge vector as P. Interestingly, this is not the only way to preserve the Hodge vector.

Lawrence twist

Let us go back to spanning circuits for a moment. Recall that it is defined by a tuple of integers $(\gamma_1, \dots, \gamma_{d+2})$. Moreover, we can associate to it a pair of tuples $(\alpha_1, \dots, \alpha_K)$ and $(\beta_1, \dots, \beta_K)$ defined by

$$\frac{\prod_{i:\gamma_i<0}(T^{-\gamma_i} - 1)}{\prod_{i:\gamma_i>0}(T^{\gamma_i} - 1)} = \frac{\prod_{i=1}^{K}(T - e^{2\pi i \alpha_i})}{\prod_{i=1}^{K}(T - e^{2\pi i \beta_i})}.$$

Recall that from this data Corti, Golyshev and Fedorov gave a formula for the coefficient of the local h^*-polynomial of the circuit

$$l_n^* = \#\left\{j \mid \#\left\{\alpha_i \mid \alpha_i \leq \beta_j\right\} - j + m_-(0) = n\right\}.$$

Now note the following. If we extend the tuple $(\gamma_1, \dots, \gamma_{d+2})$ by adjoining to it a tuple of integers of the form $(y_1, -y_1, y_2, -y_2, \dots, y_k, -y_k)$, then this does not affect the definition of alphas and betas. In the above formula for l_n^* it affects only $m_-(0)$, therefore, the local h^*-polynomial of the circuit extended this way is the local h^*-polynomial of the initial circuit multiplied with t^k. In other words, they have the same Hodge vector.

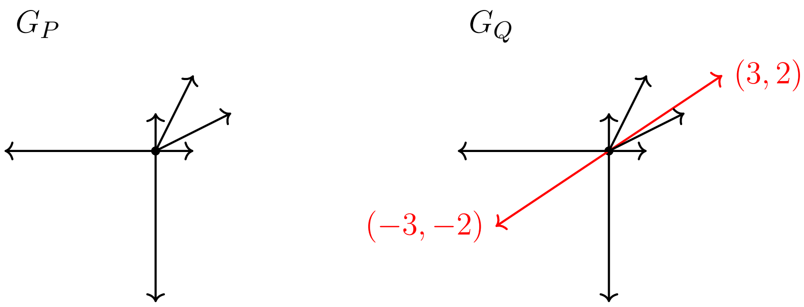
For a general Gale diagram G we do not know how one should introduce alphas and betas and do not know of a way to express the local h^*-polynomial in a way similar to the formula of Corti, Golyshev and Fedorov. Nevertheless, surprisingly, by adding pairs of opposite vectors to the Gale diagram G, the Hodge vector remains intact.

Definition 22. Let P and Q be spanning polytopes such that their Gale transforms can be chosen in such a way that $G_Q = G_P \sqcup S_k$, where S_k is a centrally symmetric configuration consisting of $2k$ vectors. Then we call Q the **Lawrence twist** of P by S_k.

This name is motivated by the Lawrence polytopes. These are the polytopes whose Gale dual is given by a centrally symmetric configuration of vectors. See, for example, [BS90].

Example 11. Consider the following two-dimensional Gale transform and its Lawrence twist by $[(3,2),(-3,-2)]$.

$$G_P = \begin{pmatrix} -4 & 0 & 1 & 0 & 2 & 1 \\ 0 & -4 & 0 & 1 & 1 & 2 \end{pmatrix} \implies \begin{pmatrix} -4 & 0 & 1 & 0 & 2 & 1 & 3 & -3 \\ 0 & -4 & 0 & 1 & 1 & 2 & 2 & -2 \end{pmatrix} = G_Q$$



One can compute $l^*(P,t) = 5t^2$ and $l^*(Q,t) = 5t^3$.

Theorem 12. Let Q be the Lawrence twist of P by S_k, then

$$l^*(Q,t) = t^k \, l^*(P,t),$$

i.e. the Hodge vectors of P and Q coincide.

Proof. This proof is inspired by the geometric proof given in [GV24] (see section 1.3.2 there) for the invariance of Hodge vectors of spanning circuits. We extend their proof to any spanning polytope. First of all, notice that it is enough to prove the theorem for the case of $k = 1$, i.e. the difference between G_P and G_Q is a pair of opposite vectors. Suppose d is the dimension of P, hence the dimension of Q is $d+2$. Suppose the point configuration corresponding to G_P is given by the columns $a_1, a_2, \dots, a_n$ of

$$A_P = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ a_{12} & \dots & a_{n2} \\ \vdots & \dots & \vdots \\ a_{1d} & \dots & a_{nd} \end{pmatrix}.$$

We can write the point configuration corresponding to $G_Q = G_P \cup \{v, -v\}$ as

$$A_Q = \begin{pmatrix} a_{11} & \dots & a_{n1} & 0 & 0 \\ a_{12} & \dots & a_{n2} & 0 & 0 \\ \vdots & \dots & \vdots & 0 & 0 \\ a_{1d} & \dots & a_{nd} & 0 & 0 \\ k_1 & \dots & k_n & 0 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix},$$

where $(k_1, k_2, \dots, k_n)$ is an integral solution of the system

$$\sum_{j=1}^{n} G_{P,ij} k_j = v_i.$$

It is always possible to find such a solution because by the definition Gale dual G_P forms a spanning vector configuration in $\mathbb{Z}^{n-d-1}$. Moreover, note that A_Q is spanning if and only if A_P is spanning.

Let us consider the corresponding affine hypersurface $Z_Q \subseteq (\mathbb{C})^{d+2}$.

$$Z_Q \ : \ \sum_{i=1}^{n} c_i \ x_1^{a_{i1}} \dots x_d^{a_{id}} x_{d+1}^{k_i} + x_{d+2} \cdot (c_{n+1} + c_{n+2} \ x_{d+1}) = 0.$$

Consider also the d-dimensional variety that is complete intersection

$$Y_Q : \begin{cases} \sum_{i=1}^{n} c_i \ x_1^{a_{i1}} \dots x_d^{a_{id}} x_{d+1}^{k_i} = 0 \\ c_{n+1} + c_{n+2} \ x_{d+1} = 0. \end{cases}$$

We see from the second equation that the variable x_{d+1} is constant and actually this complete intersection is isomorphic to a hypersurface $\tilde{Z}_Q$ in $(\mathbb{C}^*)^d$ (the torus of the first d variables)

$$\tilde{Z}_Q = \sum_{i=1}^{n} c_i \left(-\frac{c_{n+1}}{c_{n+2}} \right)^{k_i} \ x_1^{a_{i1}} \dots x_d^{a_{id}} = 0.$$

If the coefficients c_i are generic enough, the coefficients $c_i \left(-\frac{c_{n+1}}{c_{n+2}} \right)^{k_i}$ are generic as well. Moreover, the Newton polytope of $\tilde{Z}_Q$ is actually exactly P. Therefore we have the following relation between the Hodge-Deligne polynomials

$$e(Y_Q; u, v) = e(\tilde{Z}_Q; u, v) = e(Z_P; u, v).$$

We can relate the top weight parts of Hodge-Deligne polynomials of Z_Q and Y_Q with the help of the results from the first chapter. Namely, let us consider the equation (2.12) with the following setup: $k = 2$,

$$P_1 = \text{conv}\left((a_1, k_1), (a_2, k_2), \dots, (a_n, k_n)\right)$$

$$P_2 = \text{conv}\left((0, \dots, 0, 0), (0, \dots, 0, 1)\right),$$

so $d_{\{1\}} = d+1$ and $d_{\{2\}} = 1$ and $Q = P_1 * P_2$. We have to be careful and note that we have to use $d+1$ instead of d in (2.12). In this situation, we obtain

$$R_L = (-1)^d \frac{u^{d+2}}{v} l^* \left(Q, \frac{v}{u}\right) - (-1)^{\frac{3}{2}(d+1)} \epsilon_{d+1} \binom{d+3}{\frac{d+3}{2}} (uv)^{\frac{d+1}{2}},$$

$$R_Y = (-1)^d u^d l^* \left(P, \frac{v}{u}\right) + (-1)^{\frac{d-1}{2}} \epsilon_{d+1} \binom{d}{\frac{d-3}{2}} uv,$$

$$R_1 = 0,$$

$$R_2 = (-1)^{\frac{3}{2}(d+1)} \epsilon_{d+1} \left[\binom{d+2}{\frac{d+3}{2}} + \binom{d}{\frac{d-1}{2}} + 2\binom{d}{\frac{d+1}{2}} \right].$$

R_1 is zero because $\dim P_1 = \dim(P_1 + P_2)$ and because $e_{prim}(Z_{P_2}; u, v) = 0$. One can verify that the relation $R_L = R_Y + R_1 - R_2$ from (2.12) leads after specializing to $v = t, u = 1$ to

$$l^*(Q, t) = t \cdot l^*(P, t).$$

□

Remark 13. It is worth noting that taking a Lawrence twist of a polytope P increases the degree. By considering the same setup as in the above proof and plugging $u = 1, v = t$ in the equation (2.11) one obtains

$$h^*(Q, t) = h^*(P_1, t) + t \cdot h^*(P, t).$$

Thus,

$$\deg Q \geq \deg P + 1.$$

With the help of Lawrence twists we can answer the following interesting question from [BKN23].

Question 3. Suppose d-dimensional P is spanning and $l^*(P, t) = 0$. Is it true that if P is not a free join, then $\deg h^*(P, t) \leq d/2$, i.e. P is trivially thin?

Benjamin Nill suggested a way to construct a counterexample to the above question using the result of Theorem 12. It led to the following corollary.

Corollary 2. The answer to Question 3 is negative. There are infinitely-many non-isomorphic counter-examples, the smallest appearing in dimension 5. For instance, the polytope whose vertices are the columns of

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is thin and has $h^*(P,t) = 3t^3 + 4t^2 + 3t + 1$, so its degree is bigger than $d/2$. One can verify that it is not a free join.

Proof. Suppose d is odd. Consider a thin spanning d-dimensional polytope P such that $\deg P = \lfloor \frac{d}{2} \rfloor + 1$, so it is not trivially thin. There are plenty of such examples given by free joins. For instance, one can consider a free join of the 1-dimensional unit interval with any two-dimensional spanning polytope of degree 2. This will produce a thin 3-dimensional polytope with degree 2. Applying a Lawrence twist by some S_k will produce a thin spanning polytope of dimension $d + 2k$ of degree at least $\lfloor \frac{d}{2} \rfloor + 1 + k$. Therefore, it is not trivially thin as well. As the next lemma shows, there are infinitely many choices of S_k that lead to a Lawrence twist that is not a free join. □

Lemma 13. Let P be a lattice polytope. We can choose S_k in such a way that the Lawrence twist Q of P by S_k is not a free join.

Proof. Let T be a point configuration whose convex hull is a free join $\operatorname{conv}(T) = \operatorname{conv}(T_1) \circ_{\mathbb{Z}} \operatorname{conv}(T_2)$. Let $|T| = n$, $|T_i| = n_i$ and $\dim T = d = d_1 + d_2 + 1$. We can choose a Gale dual of T in the block diagonal form

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix},$$

where G_i is a $(n_i - d_i - 1) \times n_i$ matrix. This in particular implies that for any Gale dual of T all the $(n - d - 1)$-minors constructed from the first n_1 columns vanish. The same holds for the remaining n_2 columns.

If we have a Gale dual of some configuration of size n, then the necessary condition for it to be a free join is the existence of a partition of $[n]$ into the sets $I_1, \ldots, I_m$ such that for each set I_i the above minors condition is satisfied.

Now suppose that our configuration is a Lawrence twist by two opposite vectors $v, -v$. If this configuration is a free join, then v must belong to one of the sets I_j as above that subdivide $[n+2]$. If this is the case for a fixed I_j of a fixed subdivision of $[n+2]$, then at worst this gives a hyperplane inside of $\mathbb{Z}^{n-d-1}$ of the possible vectors v that could lead to a free join. Considering this for each subset I_j of every partition of $[n+2]$ defines a hyperplane arrangement in the space of possible v. This means, that in any case there are still infinitely many vectors, lying in the complement of this hyperplane arrangement, that result in a Lawrence twist that is not a free join.

□

Total twists

Let us return to the circuits again. The rational numbers α's and β's cannot be arbitrary. There must be the following symmetry. If x is in $(\alpha_1, \ldots, \alpha_K)$ or in $(\beta_1, \ldots, \beta_K)$, then $-x \mod 1$ must be in that tuple as well with the same multiplicity. If we add $1/2$ to all α_i and β_j, then the resulting tuples also satisfy the same symmetry. Is there a circuit that gives this new tuples? The answer is yes and it is constructed as follows. Suppose we start with

$$G = (\gamma_1, \gamma_2, \ldots, \gamma_{d+2}).$$

For each γ_i we do the following

- if γ_i is odd, then we remove it from G and add $2\gamma_i$ and $-\gamma_i$ to G,
- if γ_i is even, then we keep it in G.

One can check that on the corresponding α's and β's it has the desired effect

$$\alpha_i \to \alpha_i + \frac{1}{2}, \quad \beta_i \to \beta_i + \frac{1}{2}.$$

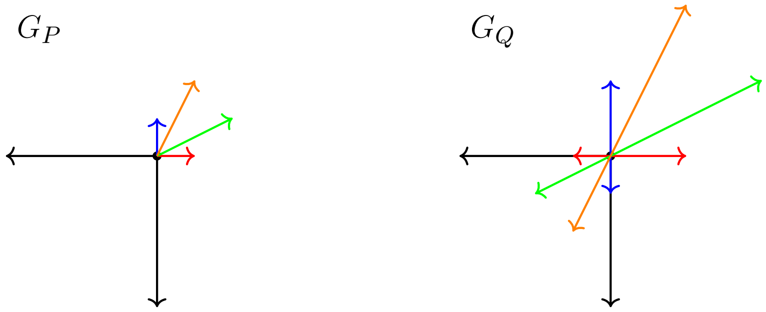
This transformation was called **total twist** in [GV24]. From the formula of Corti, Golyshev and Fedorov we see that the Hodge vector remains intact after performing the total and the corresponding local h^*-polynomial changes by the factor $t^{\frac{1}{2}\#\text{odd } \gamma_i}$.

Peculiarly, a similar transformation seems to be working for a big class of Gale diagrams. Let G be a Gale diagram of a spanning point configuration A. For $v \in G$ define $g_v = \gcd(v_1, \ldots, v_{n-d-1})$. Do the following

- if g_v is odd, then we remove v from G and add $2v$ and $-v$ to G,
- if g_v is even, then we keep v in G.

Example 12. Consider again the same Gale dual G_P as in Example 11. It has 4 elements with odd g_v. Therefore, we can perform the operation of total twist.

$$\begin{pmatrix} -4 & 0 & 1 & 0 & 2 & 1 \\ 0 & -4 & 0 & 1 & 1 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} -4 & 0 & 2 & -1 & 0 & 0 & 4 & -2 & 2 & -1 \\ 0 & -4 & 0 & 0 & 2 & -1 & 2 & -1 & 4 & -2 \end{pmatrix}$$



One can check that $l^*(Q,t) = t^2\, l^*(P,t) = 5t^4$.

Note how in the case of 1-dimensional G the number of odd γ_i is always an even number. This is important, since this implies that the dimension of our point configuration changes by an even number, which is necessary if we hope to preserve the Hodge vector. However, for higher-dimensional Gale diagrams this might be not the case. Nevertheless, if we require G to have even number of elements with odd g_v, then numerous computations for different Gale diagrams suggest the following conjecture.

Conjecture 4. Let G be a Gale diagram of the point configuration A, and B be the point configuration corresponding to the total twist of G. Then

$$l^*(\text{conv}(B), t) = t^{|\{v \in G : g_v \text{ is odd }\}|}\, l^*(\text{conv}(A), t).$$

Remark 14. Note that unlike the Lawrence twist, the total twist of a spanning polytope P does not necessarily has width 1. However, it cannot be of width higher than 2.

In [GV24] a way to construct the coordinates of B for the case of spanning circuits was suggested. It works well also for the higher-dimensional Gale diagrams. Suppose there are c elements of G with odd g_i. Consider A given by the columns of the following matrix

$$A = \begin{pmatrix} a^O_{1,1} & \dots & a^O_{c,1} & a^E_{c+1,1} & \dots & a^E_{n,1} \\ a^O_{1,2} & \dots & a^O_{c,2} & a^E_{c+1,2} & \dots & a^E_{n,2} \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ a^O_{1,d} & \dots & a^O_{c,d} & a^E_{c+1,d} & \dots & a^E_{n,d} \\ \end{pmatrix},$$

where we marked the points by O and E depending on the parity of the corresponding element of G. The coordinates for the points of B can be chosen to be

$$B = \begin{pmatrix} a^O_{1,1} & a^O_{1,1} & \dots & a^O_{c,1} & a^O_{c,1} & a^E_{c+1,1} & \dots & a^E_{n,1} \\ a^O_{1,2} & a^O_{1,2} & \dots & a^O_{c,2} & a^O_{c,2} & a^E_{c+1,2} & \dots & a^E_{n,2} \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^O_{1,d} & a^O_{1,d} & \dots & a^O_{c,d} & a^O_{c,d} & a^E_{c+1,d} & \dots & a^E_{n,d} \\ 1 & 2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 2 & 0 & \dots & 0 \end{pmatrix}.$$

Observe the following. In the construction of Lawrence twist, the resulting polytope Q admits a projection to the unimodular simplex Δ_k. In the construction of total twist the resulting polytope admits a projection to $2\Delta_m$ for some even m. Both the unimodular simplices and $2\Delta_m$ are examples of thin polytopes. We can ask ourselves the following question. Given a polytope P and a thin polytope S, is there a way to transform P into some polytope Q in such a way, that Q admits a projection to S and the local h^*-polynomials of P and Q coincide up to a trivial factor?

5 Thin Simplices

In this section we continue to investigate the local h^*-polynomial further. Recall, that a polytope Δ is called **thin** if $l^*(\Delta, t) = 0$. Can we classify thin polytopes? This problem appeared first in the context of simplices in the work of Gel'fand, Kapranov and Zelevinskii, see [GKZ94][11.4.B]. They coined the name thin and managed to classify two-dimensional thin simplices. Recently this question has been given a new life and extended to general polytopes by Borger, Kretschmer and Nill in [BKN23]. They were able to classify the three-dimensional thin lattice polytopes as well as to characterize the thin Gorenstein polytopes.

Since the local h^*-polynomial is multiplicative, having one thin polytope enables us to construct many thin polytopes in higher dimensions that are free joins with a thin factor. That is why a more meaningful question is the following.

Question 4. Can we classify the thin polytopes that are not free joins?

We restrict ourselves to the case of simplices. In dimension two, the only such simplex is the twice dilated standard unimodular simplex $2\Delta_2$. In dimension 3 there are no such examples and all thin simplices happen to be lattice pyramids. The main goal of this chapter is to answer this classification question in dimension 4.

In order to do so we make use of the point of view on lattice simplices presented in [BH13] which we review in Section 5.1. The crucial idea is that for each lattice simplex Δ we can construct an extended linear code C_Δ over $\mathbb{Z}_{N_\Delta}$ for some $N_\Delta \geq 1$. This correspondence is in fact one-to-one up to the corresponding isomorphisms. A linear code C_Δ of rank m and length $d+1$ can be generated by the rows of an $m \times (d+1)$ matrix g_Δ. Thus, we can reduce the study of a simplex Δ to the study of a matrix g_Δ with entries from $\mathbb{Z}_{N_\Delta}$.

The property of Δ being thin translates to a simple property of C_Δ. The simplex Δ is thin if and only if the corresponding linear code C_Δ has no words of maximal weight, i.e. each element of C_Δ contains a zero.

One can go further and also use the language of hyperplane arrangements. The columns of the matrix g_Δ define a hyperplane arrangement $\mathcal{H}_\Delta$ inside $\mathbb{Z}^m_{N_\Delta}$. The thin simplices then correspond to the hyperplane arrangements $\mathcal{H}_\Delta$ that have vanishing complement.

The above point of view allows us to obtain a classification of the four-dimensional thin lattice simplices that are not free joins. It is summarized in the following theorem proved in Section 5.2.

Theorem 15. Let Δ be a four-dimensional thin simplex that is not a free join. Then Δ is either one of the 6 sporadic cases in the Table 5.1 below or it belongs to the following one-parameter family of width 1 given for even $N_\Delta \geq 2$ by

$$g_\Delta = \begin{pmatrix} N_\Delta/2 & 0 & N_\Delta/2 & 0 & 0 \\ 0 & N_\Delta/2 & N_\Delta/2 & 1 & N_\Delta - 1 \end{pmatrix} \tag{5.1}$$

with

$$h^*(\Delta, t) = \left(\frac{3N_\Delta}{2} - 1\right) t^2 + \frac{N_\Delta}{2} t + 1.$$

Case	g_Δ	N_Δ	$h^*(\Delta, t)$	width	spanning
1	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	2	$5t^2 + 10t + 1$	2	yes
2	$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}$	3	$7t^2 + t + 1$	1	no
3	$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 1 & 3 & 0 \end{pmatrix}$	4	$5t^2 + 2t + 1$	1	no
4	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 2 & 3 \end{pmatrix}$	4	$t^3 + 11t^2 + 3t + 1$	2	no
5	$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 0 & 2 & 3 & 3 & 0 \end{pmatrix}$	4	$9t^2 + 6t + 1$	2	yes
6	$\begin{pmatrix} 4 & 0 & 1 & 2 & 1 \\ 4 & 4 & 0 & 4 & 4 \end{pmatrix}$	8	$t^3 + 11t^2 + 3t + 1$	2	no

Table 5.1: Sporadic thin simplices for $d = 4$.

Case	Vertices
1	$(0, 0, 0, 0), (2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2, 0), (0, 0, 0, 2)$
2	$(-1, -1, 0, 1), (0, 0, 0, 0), (0, 1, 1, 1), (-1, 0, 1, -1), (1, -1, 1, 0)$
3	$(-1, 1, 0, 0), (0, 0, 0, 0), (-1, -1, -1, 1), (1, 1, -1, 1), (0, 0, 1, 1)$
4	$(-1, 1, -1, -1), (-1, -1, 0, -1), (0, 0, 1, 1), (1, 1, 0, -1), (0, 0, -1, 1)$
5	$(-1, 0, 1, -1), (0, 2, -1, -1), (-1, 0, -1, 1), (1, 0, -1, 1), (0, 0, 1, 1)$
6	$(0, 0, 0, -1), (0, -1, -1, -1), (1, -1, 1, 1), (-1, 0, 0, 1), (1, 1, -1, 1)$
family	$(-1,0,0,0), (1,-1,-3,2), (1,0,0,0), (0,N/2,-N/2+1,0), (0,0,1,0)$

Table 5.2: Vertices of the thin simplices from Theorem 15.

We see that, contrary to the dimensions ≤ 3, in dimension 4 there are infinitely many thin simplices that are not free joins.

5.1 Lattice simplices and modular arithmetic

Linear codes

We are going to relate lattice simplices to linear codes, so here are the necessary definitions and facts from coding theory. Let $\mathbb{Z}_N$ be the ring of integers modulo N. **In this section we always identify the elements of $\mathbb{Z}_N$ with their representatives** $0, 1, \dots, N-1$. This allows us to speak about gcd of the elements of $\mathbb{Z}_N$, by which we mean the usual integer gcd of the corresponding representatives.

Definition 23. A **linear code** of length n over $\mathbb{Z}_N$ is a submodule C of the free module $\mathbb{Z}_N^n$.

A standard way to produce a linear code is to take an $m \times n$ matrix g over $\mathbb{Z}_N$ and consider the submodule of $\mathbb{Z}_N^n$ generated by the rows of g. The elements of a linear code are called **words**. For a word $c \in C$ its **weight** is the number of non-zero entries

$$w(c) := |\{i \,:\, c_i \neq 0\}|.$$

The generating function of weights is called **weight enumerator**

$$W_C(X) = \sum_{c \in C} X^{w(c)}.$$

We call two linear codes C_1 and C_2 of length n **isomorphic** if after a possible permutation of indices the sets of words coincide, that is, there exists a permutation

$\sigma \in S_n$ such that

$$\{c \,:\, c \in C_1\} = \{\sigma(c) \,:\, c \in C_2\}.$$

Clearly, isomorphic linear codes have the same weight enumerator. Moreover, if a linear code is defined by a generating matrix, then any permutation of its rows and columns defines an isomorphic linear code.

Definition 24. We call a linear code C **extended** if for every word $c \in C$ we have $\sum c_i = 0 \mod N$.

In this paper we use only extended linear codes. For an extended linear code C we can define the **height** of a word $c \in C$ to be

$$\mathrm{ht}(c) \coloneqq \frac{1}{N}\sum_{i=1}^{n} c_i.$$

From simplices to linear codes

Let $\Delta \subseteq M \simeq \mathbb{Z}^d$ be a d-dimensional lattice simplex with vertex set $v_0, \ldots, v_d$. Following [BH13] one can associate the following additive group to a simplex:

$$\Lambda_\Delta \coloneqq \left\{ (x_0, \ldots, x_d) \in (\mathbb{Q}/\mathbb{Z})^{d+1} \;:\; \sum_{i=0}^{d} \{x_i\}(v_i, 1) \in M \oplus \mathbb{Z} \right\},$$

where $\{\cdot\} : \mathbb{Q} \to [0, 1)$ is the fractional part function.

We can consider the group Λ_Δ as a linear code. Let N_Δ be the least common multiple of the denominators of x_i's of Λ_Δ. Since the order of Λ_Δ is the normalized volume of the simplex, N_Δ is a divisor of $\mathrm{vol}_{\mathbb{Z}}(\Delta)$. Given an element $(x_0, \ldots, x_d)$, we take the representative $(\{x_0\}, \ldots, \{x_d\})$ and multiply it by N_Δ. In this way we can promote each element of Λ_Δ to a word of an extended code over $\mathbb{Z}_{N_\Delta}$. Let us denote this linear code by C_Δ

$$C_\Delta \coloneqq \left\{ c = (c_0, \ldots, c_d) \in (\mathbb{Z}_{N_\Delta})^{d+1} \;:\; \sum c_i(v_i, 1) = 0 \mod N_\Delta \right\}.$$

Since N_Δ was chosen to be minimal, the greatest common divisor of all the entries of all the words with N_Δ is 1.

The following theorem is crucial since it allows us to talk interchangeably about simplices and linear codes.

Theorem 16 ([BH13], Theorem 2.3.)**.** Up to the corresponding isomorphisms there is a one-to-one correspondence between d-dimensional lattice simplices and extended linear codes C of length $d+1$ over $\mathbb{Z}_N$ for $N \geq 1$ such that the greatest common divisor of all the entries of all the words in C and N is 1.

So far, we only described how from a given simplex one constructs the corresponding linear code. Let us also describe the inverse construction. Let C be an extended code over $\mathbb{Z}_N$ of length $d+1$. Consider the natural projection map $\pi : \mathbb{Z}^{d+1} \to \mathbb{Z}_N^{d+1}$. The preimage $M := \pi^{-1}(C)$ is a sublattice of $\mathbb{Z}^{d+1}$. Let $e_0, \dots, e_d$ be the standard basis of $\mathbb{Z}^{d+1}$. Define the simplex

$$\Delta_C = \operatorname{conv}(Ne_0, \dots, Ne_d)$$

with respect to the affine lattice $\operatorname{aff}(Ne_0, \dots, Ne_d) \cap M$. This gives us a map $C \to \Delta_C$.

Example 13. Consider twice the standard two-dimensional simplex $\Delta = 2\Delta_2$. As was already mentioned, this is the only interesting thin simplex in dimension two. The coordinates of its vertices are $(0,0)$, $(2,0)$, $(0,2)$. The corresponding group Λ_Δ has $4 = \operatorname{vol}_{\mathbb{Z}}(\Delta)$ elements, namely

$$\left\{(0,0,0), \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right\}.$$

We see that $N_\Delta = 2$ and the corresponding linear code C_Δ is a linear code over $\mathbb{Z}_2$ with 4 words

$$\{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}.$$

This linear code can be generated by the following matrix

$$g_\Delta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let us describe how to go back to the simplex from this linear code. The lattice M is generated by $f_0 = (1,1,0), f_1 = (0,1,1)$ and $f_2 = (1,0,1)$. The affine lattice $\operatorname{aff}(2e_0, 2e_1, 2e_2)$ is the two-dimensional affine lattice given by $\{x \in \mathbb{Z}^3 : x_0 + x_1 + x_2 = 2\}$. Thus, $\operatorname{aff}(2e_0, 2e_1, 2e_2) \cap M$ is the affine sublattice of M given by

$$\left\{af_0 + bf_1 + cf_2 \;:\; (a,b,c) \in \mathbb{Z}^3, a+b+c = 1\right\} \subseteq M.$$

The vertices of the simplex are given by $(2,0,0), (0,2,0), (0,0,2)$ in M, and thus in $\operatorname{aff}(2e_0, 2e_1, 2e_2) \cap M$ they are given by $(1,-1,1), (1,1,-1), (-1,1,1)$. After rotating everything with the matrix $((1,1,0),(0,1,1),(1,1,1))$ we can project to the first two coordinates and obtain the simplex $2\Delta_2$ in its usual form.

Note that the height of a word $c \in C_\Delta$ corresponds to the height of the vector $\sum_{i=0}^{d} \frac{c_i}{N_\Delta}(v_i, 1)$ in the half-open parallelepiped of Δ. Therefore, we can calculate the h^* -polynomial of a simplex Δ via the corresponding linear code

$$h^*(\Delta, t) = \sum_{c \in C_\Delta} t^{\mathrm{ht}(c)}.$$

Moreover, the words that have no zeros correspond exactly to the points of Π°_Δ, giving us an expression for the local h^*-polynomial

$$l^*(\Delta, t) = \sum_{\substack{c \in C_\Delta \\ w(c)=d+1}} t^{\mathrm{ht}(c)}.$$

From this we see that the thinness of the simplex Δ translates into a nice property of the corresponding linear code. The sum defining $l^*(\Delta, t)$ is empty if and only if there are no words of weight $d + 1$ in C_Δ. This motivates the following definition.

Definition 25. An extended linear code C is called **thin** if it contains no words of maximal weight.

Corollary 3. A lattice simplex Δ is thin if and only if the corresponding linear code C_Δ is thin.

Note, that the linear code considered in the Example 13 above is thin.

It is easy to see [BH13] that the lattice simplex Δ is a lattice pyramid if and only if the corresponding linear code is **degenerate**. This means that there exists an index $i \in \{0, \ldots, d\}$ such that every word $c \in C_\Delta$ has $c_i = 0$. In other words, all the generating matrices of C_Δ have a column of zeros.

We also have to translate the construction of free joins to the language of linear codes. Consider first the following definition.

Definition 26. Suppose C_1 and C_2 are linear codes over $\mathbb{Z}_{N_1}$ and $\mathbb{Z}_{N_2}$ correspondingly. A **direct sum** of C_1 and C_2 is the linear code $C_1 \oplus C_2$ over $\mathbb{Z}_{\mathrm{lcm}(N_1,N_2)}$ whose words are given by

$$\left\{ \frac{\mathrm{lcm}(N_1, N_2)}{N_1} \cdot c_1 \mid \frac{\mathrm{lcm}(N_1, N_2)}{N_2} \cdot c_2 \ : \ c_1 \in C_1, \ c_2 \in C_2 \right\},$$

where | is the usual concatenation. This construction corresponds exactly to the free joins on the simplices side.

Lemma 14. A simplex Δ is a free join of the simplices Δ_1 and Δ_2 if and only if the corresponding linear code C_Δ is a direct sum of the linear codes C_{Δ_1} and C_{Δ_2}.

Proof. It is clear from the construction that if a simplex is a free join, then the corresponding linear code is a direct sum. Suppose we have a linear code C over $\mathbb{Z}_N$ which is a direct sum $C = C_1 \oplus C_2$. The corresponding simplices are $\Delta_C = \Delta_{C_1 \oplus C_2}$. We know that the simplex $\Delta_1 \circ_{\mathbb{Z}} \Delta_2$ is isomorphic to $\Delta_{C_1 \oplus C_2}$. By the fact that the correspondence in Theorem 16 is one-to-one we can conclude that $\Delta_C \simeq \Delta_1 \circ_{\mathbb{Z}} \Delta_2$. □

Remark 17. If a linear code C is a direct sum $C_1 \oplus C_2$, then the weight enumerator of C factorizes

$$W_C(X) = W_{C_1}(X) \cdot W_{C_2}(X).$$

This gives a useful necessary condition for deciding if the simplex C_Δ is a free join. Namely, if both the polynomials $W_C(X)$ and $h^*(\Delta_C)$ factorize into polynomials with non-negative coefficients, then Δ_C might be a free join, but a priori it is not guaranteed to be one. For example, consider the code C of length 7 over $\mathbb{Z}_4$ generated by

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 2 & 3 \\ 2 & 3 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

We have

$$W_C(X) = (X^2 + 1)(3X^4 + 1), \quad h^*(\Delta_C, t) = (t + 1)(3t + 1).$$

This code is non-degenerate, so if it is a direct sum, it must have two non-degenerate factors. In particular, one of the factors must correspond to a simplex that is not a lattice pyramid and has $h^*(\Delta, t) = t + 1$. Using the classification of degree 1 lattice polytopes in [BN07] we can deduce that this factor must correspond to the 1-dimensional simplex $[0, 2]$. Therefore, the second factor must correspond to a 5-dimensional simplex with $h^*(\Delta, t) = 3t + 1$ that is not a lattice pyramid. Using the same classification, one can deduce that this is not possible.

5.1.1 From simplices to hyperplane arrangements

Let g be an $m \times n$ generating matrix of a linear code C over $\mathbb{Z}_N$. Let g_i be the ith column of g. Define

$$H_i = \{x \in \mathbb{Z}_N^m \mid g_i \cdot x = 0 \mod N\}.$$

We call this set the **hyperplane** defined by g_i. This way from a linear code we get a hyperplane arrangement over $\mathbb{Z}_N$

$$\mathcal{H}_C = \{H_0, \ldots, H_d\}.$$

If the linear code is the code C_Δ coming from a lattice simplex, let us denote this hyperplane arrangement as $\mathcal{H}_\Delta$. The following proposition is going to be the central tool for classifying the four-dimensional thin simplices.

Corollary 4. A simplex Δ is thin if and only if the complement of $\mathcal{H}_\Delta$ is empty, i.e. $\mathbb{Z}_N^m = H_0 \cup H_1 \cup \ldots \cup H_d$.

Proof. Suppose g_Δ is a generating matrix of C_Δ and it has m rows. Then the words of C_Δ can be obtained by multiplying g_Δ from the left by all the possible $a = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}_N^m$. Now, clearly, if every a belongs to $\mathcal{H}_\Delta$, then every word has a zero at some place. Furthermore, if a word has a zero at the ith place, it means that the corresponding points a belong to the hyperplane H_i. Thus, if every word has a zero somewhere, then all the points a belong to some hyperplane. □

For a hyperplane H_i defined by the column g_i of the $m \times (d+1)$ generating matrix g_Δ define

$$\gcd_i := \gcd(g_{1i}, \ldots, g_{mi}, N). \tag{5.2}$$

Then the hyperplane H_i contains $\gcd_i \cdot N^{m-1}$ points (see e.g. [Smi61]).

We give a necessary condition for the code C to be thin.

Proposition 9. Let C be a linear code of length $d+1$ over $\mathbb{Z}_N$ and g be its generating matrix. If the linear code C is thin, then

$$\sum_{i=0}^{d} \gcd_i > N.$$

Proof. Suppose g has m rows. Let$\{H_0, \ldots, H_d\}$ be the corresponding hyperplane arrangement. All the hyperplanes contain at most $\sum_{i=0}^{d} \gcd_i \cdot N^{m-1}$ points. Since the point 0^m belongs to all of them, we arrive at the strict inequality

$$\sum_{i=0}^{d} \gcd_i \cdot N^{m-1} > N^m.$$

□

In principle, one can continue with the inclusion-exclusion procedure to account for the intersections of subsets of the hyperplanes. But the expressions for the number of points of these intersections become more difficult and it does not seem practical to proceed this way.

Corollary 5. If C is thin and every $\gcd_i = 1$ (in particular, if N is prime), then $d \geq N$.

Corollary 6. Let p be the smallest prime in the factorization of N. If C is thin, then $d \geq p$.

Proof. Suppose first that $N = p^k$ for some $k \geq 1$. Then at worst we have d columns with $\gcd_i = p^{k-1}$, and the remaining column must have $\gcd_i = 1$ since otherwise the whole generating matrix g has a common factor. Thus $d\,p^{k-1} + 1 > p^k$, from which it follows that $p \leq d$.

Now suppose that $p_1 < p_2$ are the two smallest distinct primes in the factorization of N. Then at worst there are d columns with $\gcd_i = N/p_1$ and one column with $\gcd_i = N/p_2$. Thus,

$$d > p_1 - \frac{p_1}{p_2}.$$

Since $0 < p_1/p_2 < 1$, we have $d \geq p_1$. □

5.1.2 Spanning thin simplices

Spanning lattice simplices have a nice characterization in terms of linear codes.

Proposition 10. A lattice simplex Δ is spanning if and only if the corresponding linear code C_Δ can be generated by a matrix g whose rows have height one.

Proof. The "only if" direction is straightforward. Since Δ is spanning, every word in C_Δ is a combination of words that correspond to the lattice points of the simplex, this gives a generating matrix with height one rows.

For the "if" direction consider the half-open parallelepiped. Since C_Δ is generated by the words of height one, it means that any point inside Π_Δ is a linear combination with integral coefficients of the points at height one. Since we can cover $M \oplus \mathbb{Z}$ by translating Π_Δ by multiples of $(v_i, 1)$ it follows that any lattice point is expressible through the points of $\Delta \times \{1\}$, i.e. Δ is spanning. □

Remark 18. Note that if Δ is spanning we might need a lot of rows to have a generating matrix with all rows having height 1. There is a universal bound on this number, namely one needs at most $(d+1)2^{(d+1)}$ rows, see [AHN23] and [ES06].

For example, the simplex corresponding to the linear code generated by

$$\begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 1 & 5 \end{pmatrix}$$

over $\mathbb{Z}_6$ is spanning since this code can also be generated by

$$\begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

However, this linear code cannot be generated by a matrix with two rows with height 1.

Nevertheless, when N is prime and a linear code C of dimension m corresponds to a spanning simplex, then this code can be generated by a matrix with exactly m rows with height 1.

It seems that spanning thin simplices are relatively rare. It is particularly so for the simplices with prime N_Δ. Computations in low dimensions ($d \leq 8$) suggest the following conjecture.

Conjecture 5. For prime $N \geq 3$ there are no non-degenerate thin codes corresponding to spanning simplices.

The following proposition shows that this conjecture is true for the linear codes of dimension 2. Moreover, it gives some weak bounds for the codes of higher dimension.

Theorem 19. Suppose N is prime. Let Δ be a spanning thin simplex and C be the corresponding linear code of dimension $m \geq 2$ over $\mathbb{Z}_N$. Then $N \leq N_m$ for N_m given by

$$2,\ 17,\ 83,\ 379, 1499,\ 5987$$

for $2 \leq m \leq 7$ respectively.

Proof. Since N is prime, every column has $\gcd_i = 1$. Since the code can be generated by a matrix g with m rows of height 1, the sum of all the entries of the generating matrix is mN. By Corollary 5 the arrangement $\mathcal{H}_\Delta$ might have empty complement only if there are at least $N+1$ different hyperplanes. We want to estimate the sum of the entries of the normals that define these hyperplanes, that is, the sum of the entries of g. We will use this estimate to show that for a fixed

m starting from some N the sum of the entries of g with $N+1$ different columns can only be larger than mN.

Let H be a hyperplane in $\mathbb{Z}_N^m$ defined by a normal h. Let us call h minimal and denote it h_{min} if the sum of its entries is the smallest possible among all the normals defining the same hyperplane. Note, that minimal h can be not unique. For example, consider $N = 7$, $m = 2$ and the hyperplane defined by the normal $h = (2,3)$. This h is not minimal and the minimal one is given by $h_{min} = (3,1) = 5 \cdot (2,3)$. If we consider the hyperplane defined by $(6,1)$, then all its normals are minimal.

Let us introduce a function that sends a hyperplane to the sum of the entries of its minimal normal

$$n: \ H \to \sum_{i=1}^{m} (h_{min})_i.$$

This function defines a total ordering on the set of hyperplanes in $\mathbb{Z}_N^m$ by $H \leq \tilde{H}$ if and only if $n(H) \leq n(\tilde{H})$. Let us index the hyperplanes in $\mathbb{Z}_N^m$ by $\mathbb{N}$ in a way compatible with the order above. For example, the hyperplane defined by $h = (1,0)$ is H_1, the one defined by $(0,1)$ is H_2, the one defined by $(1,1)$ is H_3, etc.. We are interested in evaluating the sum

$$S(m,N) = \sum_{i=1}^{N+1} n(H_i).$$

Note that this sum does not depend on the chosen indexing.

Let us denote by $c_i(m,N)$ the number of the hyperplanes H in $\mathbb{Z}_N^m$ with $n(H) = i$. Define

$$k_{m,N} := \min \left\{ k \in \mathbb{N} \ : \ \sum_{i=1}^{k+1} c_i(m,N) > N+1 \right\}.$$

Now we can write

$$S(m,N) = \sum_{i=1}^{k_{m,N}} i \, c_i(m,N) + \left(N + 1 - \sum_{i=1}^{k_{m,N}} c_i(m,N) \right) \left(k_{m,N} + 1 \right).$$

We can also consider the same situation in $\mathbb{Z}^m$. We can again put a total order on the set of hyperplanes and we can define $c_i(m)$ to be the number of hyperplanes in $\mathbb{Z}^m$ with $n(H) = i$, which can be computed as

$$c_i(m) = \left| \left\{ (x_1, \ldots, x_m) \in \mathbb{Z}_{\geq 0}^m \ : \ \sum_{j=1}^{m} x_i = i \text{ and } \gcd(x_1, \ldots, x_m) = 1 \right\} \right|.$$

This number is also known as the number of new colors that can be mixed with i units of m given colors, see [OEI24] for $m = 3$. For $m = 2$ it is exactly the value of the Euler's totient function $c_i(2) = \phi(i)$. The generating function of $c_i(m)$ can be given by

$$\sum_{i\geq 0} c_i(m)t^i = \sum_{k\geq 1} \frac{\mu(k)}{(1-t^k)^m}.$$

In the same manner we define

$$k^{\mathbb{Z}}_{m,N} = \min\left\{k \in \mathbb{N} \,:\, \sum_{i=1}^{k+1} c_i(m) > N+1\right\}$$

and

$$S^{\mathbb{Z}}(m,N) \coloneqq \sum_{i=1}^{k^{\mathbb{Z}}_{m,N}} i\, c_i(m) + \left(N+1-\sum_{i=1}^{k^{\mathbb{Z}}_{m,N}} c_i(m)\right)\left(k^{\mathbb{Z}}_{m,N}+1\right).$$

Note that this sum also makes sense for a non-prime N.

Lemma 15. For a prime N we have $S(m,N) \geq S^{\mathbb{Z}}(m,N)$.

Proof. First of all, notice that $c_i(m) \geq c_i(m,N)$. This implies that for a given N the value of $k^{\mathbb{Z}}_{m,N}$ cannot be larger than $k_{m,N}$. Suppose $k^{\mathbb{Z}}_{m,N} = k_{m,N} - x$ for some $x \geq 0$. Consider

$$S(m,N) - S^{\mathbb{Z}}(m,N) = \sum_{i=1}^{k_{m,N}-x} \left(i - \left(k_{m,N}+1\right)\right)\left(c_i(m,N) - c_i(m)\right) - \\ - \sum_{i=k_{m,N}-x+1}^{k_{m,N}} c_i(m,N)\left(k_{m,N}+1-i\right) + x\left((N+1) - \sum_{i=1}^{k_{m,N}-x} c_i(m)\right).$$

In the second sum $k_{m,N}+1-i$ is always positive and $k_{m,N}+1-i \leq x$, therefore if we substitute $k_{m,N}+1-i$ with x and use $c_i(m) \geq c_i(m,N)$ we get

$$S(m,N) - S^{\mathbb{Z}}(m,N) \geq \\ \sum_{i=1}^{k_{m,N}-x} \left(i - \left(k_{m,N}+1\right)\right)\left(c_i(m,N) - c_i(m)\right) + x\left((N+1) - \sum_{i=1}^{k_{m,N}} c_i(m)\right).$$

All the summands above are non-negative, thus $S(m,N) \geq S^{\mathbb{Z}}(m,N)$. □

Now we want to show that for each m there exists $N_m^{\mathbb{Z}}$ such that for $N > N_m^{\mathbb{Z}}$ we have $S^{\mathbb{Z}}(m,N) > mN$. This would imply that for a given m there cannot be any spanning thin simplices with $N > N_m^{\mathbb{Z}}$.

Lemma 16. The function $S^{\mathbb{Z}}(m,N)$ grows with N, namely

$$S^{\mathbb{Z}}(m,N+1) - S^{\mathbb{Z}}(m,N) = k_{m,N}^{\mathbb{Z}} + 1.$$

Moreover, this growth is non-decreasing.

Proof. Note that the consecutive values of $k_{m,N}^{\mathbb{Z}}$ can either be the same or differ by one, namely

$$k_{m,N+1}^{\mathbb{Z}} = k_{m,N}^{\mathbb{Z}} + \varepsilon_{m,N}$$

with $\varepsilon_{m,N} \in \{0,1\}$. Consider

$$S^{\mathbb{Z}}(m,N+1) - S^{\mathbb{Z}}(m,N) = k_{m,N}^{\mathbb{Z}} + 1 + \varepsilon_{m,N}\left(N + 2 - \sum_{i=1}^{k_{m,N}^{\mathbb{Z}}+\varepsilon_{m,N}} c_i(m)\right).$$

When $\varepsilon_{m,N} = 0$ the last summand does not appear. When $\varepsilon_{m,N} = 1$ it means that we actually have exactly $\sum_{i=1}^{k_{m,N}^{\mathbb{Z}}+1} c_i(m) = N+2$ and overall it gives

$$S^{\mathbb{Z}}(m,N+1) - S^{\mathbb{Z}}(m,N) = k_{m,N}^{\mathbb{Z}} + 1$$

This is always positive, so $S^{\mathbb{Z}}(m,N)$ always grows with increasing N. Moreover, this growth is non-decreasing since

$$\left(S^{\mathbb{Z}}(m,N+2) - S^{\mathbb{Z}}(m,N+1)\right) - \left(S^{\mathbb{Z}}(m,N+1) - S^{\mathbb{Z}}(m,N)\right)$$

can only take values 0 and 1. □

This lemma shows that there exists $N_m^{\mathbb{Z}}$ such that for every $N > N_m^{\mathbb{Z}}$ the difference $\left(S^{\mathbb{Z}}(m,N+1) - S^{\mathbb{Z}}(m,N)\right) \geq m+1$. We can find $N_m^{\mathbb{Z}}$ computationally for small values of m.

m	2	3	4	5	6	7
$N_m^{\mathbb{Z}}$	2	18	86	380	1502	5992

By considering for each $N_m^{\mathbb{Z}}$ the closest prime from below we get the values of N_m for $S(m,N)$ from the statement of the theorem. □

5.1.3 Lattice width

In this subsection we show that lattice simplices of width 1 have a useful description in terms of the corresponding linear codes.

A few notions similar to that of a Cayley polytope were considered by Arnau Padrol in his PhD thesis [Pad13]. Note that a d-dimensional Cayley polytope Δ of length m can be equivalently defined as a polytope such that there exists a lattice projection $\mathbb{Z}^d \to \mathbb{Z}^{m-1}$ that maps Δ onto the standard simplex Δ_{m-1}. For the first notion, instead of considering the standard simplex as the image of a projection, one can relax this condition and project to any simplex. One calls a point configuration **affine Cayley** of length m if there exists a projection $\mathbb{R}^d \to \mathbb{R}^{m-1}$ that maps Δ onto the vertex set of an $(m-1)$-dimensional simplex.

For the second notion consider the following. Let V be a vector configuration. It is called **affine Cayley*** of length m if there exists a partition

$$V = V_1 \sqcup V_2 \sqcup \ldots \sqcup V_m,$$

such that $\sum_{v \in V_i} v = 0$ for all $i = 1, \ldots, m$.

At first, the above two concepts seem not to be related. However, we may connect them using Gale duality.

Proposition 11 (Proposition 7.22 in [Pad13])**.** A point configuration is affine Cayley if and only if its Gale dual is affine Cayley*.

We are interested in the situations when the point configuration A is the configuration of the lattice points of a simplex Δ. In this case, we can read a Gale dual G from the linear code C_Δ. Suppose Δ has $d+1+m$ lattice points. Then there are m words $c_1, \ldots, c_m$ of height 1 in C_Δ. Each such word gives a row in the matrix G that takes the form

$$G = \begin{pmatrix} -N_\Delta & 0 & \ldots & 0 & c_{10} & c_{11} & \ldots & c_{1d} \\ 0 & -N_\Delta & \ldots & 0 & c_{20} & c_{21} & \ldots & c_{2d} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -N_\Delta & c_{m0} & c_{m1} & \ldots & c_{md} \end{pmatrix}.$$

The last $d+1$ columns correspond to the vertices of the simplex and the first m columns correspond to the m remaining lattice points. Clearly, the rows are linearly independent, so the columns of this matrix indeed give us a Gale dual.

For linear codes there exists a natural analog of affine Cayley* configurations.

Definition 27. We say that an extended linear code of length $d+1$ over $\mathbb{Z}_N$ **splits** into m parts if there is a set partition $I_1, \ldots, I_m$ of $\{0, 1, \ldots, d\}$ such that for any $j = 1, \ldots, m$ and any word c we have $\sum_{i \in I_j} c_i = 0 \mod N$.

This definition implies that, if the code C splits, then for each height 1 word c there exists k, such that $c_i = 0$ for all $i \in \{0, \ldots, d\} \setminus I_k$. This in turn implies that the above matrix G has a nice block-diagonal structure after permuting the columns. Therefore, if C splits, then the corresponding simplex is affine Cayley*. Moreover, the condition of splitting is stronger than just being affine Cayley* and it implies that the simplex must be Cayley, which we show in the next proposition.

Proposition 12. A lattice simplex Δ is Cayley of length m if and only if the corresponding linear code splits into m parts.

Proof. We will treat the case $m = 2$. One can easily generalize it to any m. If Δ is Cayley, it means that we can find an affine unimodular transformation of $\mathbb{Z}^d$ such that there is a coordinate where the set of vertices of I_1 has a 0 and the set of vertices I_2 has 1. Now it follows by construction that any word $c \in C_\Delta$ must have $\sum_{i \in I_2} c_i = 0 \mod N_\Delta$ and thus also $\sum_{i \in I_1} c_i = 0 \mod N_\Delta$. Therefore, the corresponding linear code splits into two pieces.

Now suppose that the linear code splits. As we saw above, it implies that the simplex Δ is affine Cayley*. Thus, after an affine unimodular transformation the set of the vertices of Δ splits into two parts I_1 and I_2. In the first part they have the form $v_i = (0, w_i)$ for $i \in I_1$ and for some $w_i \in \mathbb{Z}^{d-1}$ and in the second part they are of the form $v_i = (k, w_i)$ for $i \in I_2$ and for some $k \in \mathbb{N}$. In other words, there is a projection onto the vertices of the 1-dimensional simplex $[0, k]$.

We are going to show that k must equal 1. Consider two simplices Δ_1 with vertices $(0, w_i)$ for $i \in I_1$ and $(1, w_i)$ for $i \in I_2$ and Δ_k for some $k \geq 2$ with vertices $(0, w_i)$ for $i \in I_1$ and (k, w_i) for $i \in I_2$. Let C_1 and C_k be the corresponding linear codes. Since Δ_1 is Cayley, we know that the corresponding linear code splits. Its words are defined by the rational numbers $\lambda_i^{(1)}$ such that

$$\sum_{i \in I_1} \lambda_i^{(1)}(0, w_i, 1) + \sum_{i \in I_2} \lambda_i^{(1)}(1, w_i, 1) \in M \oplus \mathbb{Z}\,.$$

The code C_k is defined by the rational numbers $\lambda_i^{(k)}$ such that

$$\sum_{i \in I_1} \lambda_i^{(k)}(0, w_i, 1) + \sum_{i \in I_2} \lambda_i^{(k)}(k, w_i, 1) \in M \oplus \mathbb{Z}\,.$$

Note that $N_{\Delta_1} \mid N_{\Delta_k}$. All the tuples $(\lambda_0^{(1)}, \ldots, \lambda_d^{(1)})$ also give a tuple for the code C_k, therefore,

$$\frac{N_{\Delta_k}}{N_{\Delta_1}} C_1 \subseteq C_k$$

as sets. Suppose there are tuples $(\lambda_0^{(k)}, \ldots, \lambda_d^{(k)})$ not coming from C_1, then necessarily $\sum_{i \in I_2} \lambda_i^{(k)} \notin \mathbb{Z}$ since otherwise it would correspond to some word of C_1. This tuple then corresponds to a word that does not satisfy $\sum_{i \in I_2} c_i = 0 \mod N_{\Delta_k}$, i.e. the linear code C_k does not split, a contradiction. Therefore, k must be equal to 1 and the corresponding simplex Δ is Cayley. □

5.2 Four-dimensional thin simplices

5.2.1 General strategy of classification

This section is devoted to the proof of the classification of thin four-dimensional simplices presented in Theorem 15. In this subsection we outline the main steps that lead to this result. From the previous section, we know that the search for thin simplices is equivalent to the search of linear codes over $\mathbb{Z}_N$ without maximal weight words or to the search of hyperplane arrangements over $\mathbb{Z}_N$ with empty complement. This way instead of working with d-dimensional simplices we can work simply with the generating matrices of linear codes. Let C be a linear code over $\mathbb{Z}_N$ of length $d+1$ and let g be an $m \times (d+1)$ matrix that generates C. We require this code to be extended and require that the greatest common divisor of all the entries of g and N is 1, we write $\gcd(g, N) = 1$. Since the code is extended, the submodule C is in fact a subgroup of $\mathbb{Z}_N^d$, so it is enough to consider $1 \leq m \leq d$.

From now on let us fix the dimension to $d = 4$. From Proposition 9 it follows that if all the $\gcd_i$ satisfy $\gcd_i \leq N/5$, then the linear code C cannot be thin. Since each $\gcd_i$ is a divisor of N, they must be of the form N/α for α an integer. This leads to the following lemma.

Lemma 17. If C is thin, then at least one of the $\gcd_i$ is N/α with $\alpha \in \{2, 3, 4\}$.

For a fixed d Corollaries 5 and 6 suggest that we might have more thin simplices when N is small. To embark upon the classification and simplify the proof we classify the thin linear codes with $N \leq 8$ using a computer algebra. The following proposition is the outcome of a programme written in SageMath available online

[1].

Proposition 13. For $N \leq 8$ there are 10 non-isomorphic thin linear codes that are not direct sums. Six of them are presented in the Table 5.1 and the remaining four are members of the family (5.1).

Building on this classification of thin linear codes with $N \leq 8$, we will gradually cover all the possible remaining cases. We will start with $m = 1$, i.e. generating matrices with one row, and proceed to $m = 4$. It is helpful to note that if the linear code generated by g is thin, then also the linear codes generated by subsets of the rows of g are thin. Moreover, each row of g must contain at least one zero, otherwise we would immediately get a word of maximal weight.

The case when the generating matrix has only one row is quite trivial, since this row must have a zero.

Lemma 18. If $m = 1$ and C is thin, then it corresponds to a lattice pyramid.

For the $m = 2$ case we will proceed as follows. As we noted before, at least one of the columns of g must have $\gcd_i = N/\alpha$ with $\alpha \in \{2, 3, 4\}$. If all the five columns have the same $\gcd_i$, then $\gcd(g, N) \neq 1$. The same happens when four out of the five columns have the same $\gcd_i$ because C is extended. Let

$$M_\alpha := N/\alpha$$

be the maximal $\gcd_i$ of g, then we have to consider the cases when there are $1, 2$ or 3 columns with $\gcd_i = M_\alpha$. They are covered in Subsection 5.2.2 in the Lemmas 19, 20 and 21, respectively. The outcome of these lemmas can be summarized in the following proposition.

Proposition 14. For $m = 2$ and $N \geq 9$ a non-degenerate linear code C that is not a direct sum is thin if and only if N is even and the generating matrix can be chosen as

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & N-1 \end{pmatrix}. \tag{5.3}$$

Building on the $m = 2$ case we can deal with the $m = 3$ case using the fact that each pair of the rows of g must generate a thin linear code, thus each such pair is either a multiple of a generating matrix from the $m = 2$ situations or has a column of zeros. In a similar manner we can treat the $m = 4$ case. In Subsections 5.2.3 and 5.2.4 we will prove the following.

[1] https://github.com/VadymKurylenko/Thin-Simplices

Proposition 15. Suppose $N \geq 9$, g has three or four rows and it generates a non-degenerate linear code C that is not a direct sum. If C is thin, then it can be generated by a matrix with 2 rows of the form (5.3), i.e. there are no new interesting linear codes compared to the situation of $m = 2$.

All together, Propositions 13, 14 and 15 combine to Theorem 15 from the beginning of the chapter.

Note that from now on the equality sign will predominantly mean equality mod N. In the few cases when confusion is possible we will write $=_{\mathbb{Z}}$ for the equality that has to be understood over $\mathbb{Z}$.

5.2.2 The $m = 2$ case

In this part we are going to treat the generating matrices that have two rows. Recall that for $\alpha = 2, 3, 4$

$$M_\alpha := \frac{N}{\alpha}.$$

Lemma 19. Let $N \geq 9$. Suppose g has two rows and only one column of g has $\gcd_i = M_\alpha$ and other columns have $\gcd_i < M_\alpha$. In this case, g does not generate a thin linear code, unless g has a column of zeros.

Proof. No zeros in the column with $\gcd_i = M_\alpha$. Suppose at first that the column with $\gcd_i = M_\alpha$ does not have zeros. Then since every row must have at least one zero, we can write

$$g = \underset{\quad H_0 \qquad H_1 \quad H_2 \quad H_3 \quad H_4}{\begin{pmatrix} a_1 M_\alpha & b_1 & 0 & b_5 & b_7 \\ a_2 M_\alpha & 0 & b_4 & b_6 & b_8 \end{pmatrix}}$$

with $a_i \in \{0, 1, \ldots, \alpha - 1\}$ such that $\gcd(a_1, a_2, \alpha) = 1$ and $b_i \in \{0, 1, \ldots, N - 1\}$. *The indices H_i of the columns denote the corresponding hyperplanes from $\mathcal{H}_C$ and they are added for the reader's convenience.*

Consider a set of points

$$\{(\alpha, 1),\ (1, \alpha),\ (\alpha, -1),\ (-1, \alpha)\} \subseteq \mathbb{Z}_N^2\,. \tag{5.4}$$

None of these points can be contained in H_0, H_1 or H_2, unless $\alpha b_1 = 0$ or $\alpha b_4 = 0$, but this would violate the assumption that only one column has $\gcd_i = M_\alpha$. Thus, these points must be contained in the remaining two hyperplanes. Any triple of the above four points cannot lie in the same hyperplane without violating the assumptions of the lemma. Therefore, H_3 must contain a pair of the points and H_4 must contain the remaining pair.

Suppose that $\alpha \neq 2$, then we might also consider the points

$$\{(\alpha, 2),\ \ (2, \alpha)\}\,.$$

They are not in the hyperplanes H_1 or H_2. The hyperplane H_0 might contain one of these points if $\alpha = 4$. Suppose that $(\alpha, 2)$ is not in H_0. One can deduce that among the remaining two hyperplanes it can only be contained in the hyperplane that contains the points $(1, \alpha)$ and $(-1, \alpha)$. Suppose it is H_3, then we must have $2b_5 = 0$ and $4b_6 = 0$. If $\alpha = 4$, this would mean that $\gcd_3 = M_4$, a contradiction. Thus, we must have $\alpha = 3$ and so $(2, \alpha) \notin H_0$. This point may belong only to the hyperplane that contains $(\alpha, 1)$ and $(\alpha, -1)$, that is H_4, and this corresponds to $4b_7 = 0$ and $2b_8 = 0$. These restrictions give us

$$g = \begin{pmatrix} a_1 M_3 & b_1 & 0 & N_\Delta/2 & \tilde{b}_7 N_\Delta/4 \\ a_2 M_3 & 0 & b_4 & \tilde{b}_6 N_\Delta/4 & N_\Delta/2 \end{pmatrix}$$

with $\tilde{b}_6, \tilde{b}_7 \in \{1, 3\}$. Consider the points $(1, 1)$ and $(1, -1)$. One can check that now we cannot have both of these points in $\mathcal{H}_C$.

Consider now $\alpha = 2$. Recall that each H_3 and H_4 must contain exactly a pair of points from (5.4). Suppose $(\alpha, 1), (1, \alpha) \in H_3$, then we must have

$$g = \begin{pmatrix} M_2 & b_1 & 0 & b_5 & b_7 \\ M_2 & 0 & b_4 & b_6 & b_8 \end{pmatrix}$$
$$\quad\ \ H_0 \quad H_1 \quad H_2 \quad H_3 \quad H_4$$

with $3b_5 = 3b_6 = 3b_7 = 3b_8 = 0$. Consider the words corresponding to the points $(3, 0)$ and $(0, 3)$. They are $(M_2, 3b_1, 0, 0, 0)$ and $(M_2, 0, 3b_4, 0, 0)$. Since C is extended, we have $3b_1 = M_2$ and $3b_4 = M_2$. Therefore, the hyperplanes H_1 and H_2 do not contain all the points of the form $(3, x)$ and $(x, 3)$ with non-zero x. In particular, the point $(3, 2)$ has to be either in H_3 or in H_4. It is easy to check that this is not possible in this situation.

If $(\alpha, 1), (-1, \alpha) \in H_3$, the situation is similar. Now the corresponding conditions are $5b_5 = 5b_6 = 5b_7 = 5b_8 = 0$. It gives $5b_1 = M_2$ and $5b_4 = M_2$. This again does not allow H_1 or H_2 to contain the point $(3, 2)$. One can check that it also does not belong to H_3 or H_4.

If $(\alpha, 1), (\alpha, -1) \in H_3$, then we have $4b_5 = 4b_8 = 0$ and $2b_6 = 2b_7 = 0$. It gives $4b_1 = M_2$ and $4b_4 = M_2$. Therefore, the point $(3, 2)$ must be contained again in

H_3 or H_4. This is not possible unless, for example, $b_5 = 0$ and $2b_6 = 0$, but this would give $\gcd_3 = M_2$.

A zero in the column with $\gcd_i = M_\alpha$. Suppose the column with $\gcd_i = M_\alpha$ has a zero entry. Note that the points $(1, 1)$ and $(1, -1)$ must be in $\mathcal{H}_C$, thus, the generating matrix must be of the form

$$g = \begin{pmatrix} a_1 M_\alpha & 0 & b_3 & b_5 & b_7 \\ 0 & b_2 & -b_3 & b_5 & b_8 \end{pmatrix}. \\ \quad\; H_0 \qquad H_1 \quad H_2 \quad H_3 \quad H_4$$

Let us consider the following table, where the entries are the values (up to sign) on the corresponding points taken by the linear forms defining the hyperplanes H_2 and H_3. None of the points from the table can be contained in H_0 or H_1.

	$(1, \alpha)$	$(-1, \alpha)$	$(\alpha + 1, 1)$	$(\alpha + 1, 2)$	$(\alpha + 1, -1)$	$(\alpha + 1, -2)$
H_2	$(\alpha - 1)b_3$	$(\alpha + 1)b_3$	αb_3	$(\alpha - 1)b_3$	$(\alpha + 2)b_3$	$(\alpha + 3)b_3$
H_3	$(\alpha + 1)b_5$	$(\alpha - 1)b_5$	$(\alpha + 2)b_5$	$(\alpha + 3)b_5$	αb_5	$(\alpha - 1)b_5$

Consider the pair of points $(\alpha + 1, 1)$ and $(\alpha + 1, 2)$. None of them can be contained in H_2, since both of the conditions $\alpha b_3 = 0$ and $(\alpha - 1)b_3 = 0$ would imply $\gcd_i \geq M_\alpha$. Moreover, they cannot be both in H_3, since it would require $b_5 = 0$. Thus, at least one of them must be in H_4. The same logic applies to the pair of points $(\alpha + 1, -1)$ and $(\alpha + 1, -2)$.

Suppose one of the points $(\alpha + 1, 1)$ and $(\alpha + 1, 2)$ belongs to H_3, then the point $(1, \alpha)$ must be in H_4 since otherwise either $b_5 = 0$ or $2b_5 = 0$. On the contrary, if both of the points $(\alpha + 1, 1)$ and $(\alpha + 1, 2)$ belong to H_4, then $(1, \alpha) \notin H_4$, since it would require $b_7 = b_8 = 0$. Again, the same applies for the pair $(\alpha + 1, -1)$, $(\alpha + 1, -2)$ and the point $(-1, \alpha)$.

Therefore, one of the following options must be satisfied:

- $(\alpha + 1, 1)$, $(\alpha + 1, -1)$, $(1, \alpha)$, $(-1, \alpha) \in H_4$,
- $(\alpha + 1, 1)$, $(\alpha + 1, -2)$, $(1, \alpha)$, $(-1, \alpha) \in H_4$,
- $(\alpha + 1, 2)$, $(\alpha + 1, -1)$, $(1, \alpha)$, $(-1, \alpha) \in H_4$,
- $(\alpha + 1, 2)$, $(\alpha + 1, -2)$, $(1, \alpha)$, $(-1, \alpha) \in H_4$,
- $(\alpha + 1, 1)$, $(\alpha + 1, 2)$, $(\alpha + 1, -1)$, $(\alpha + 1, -2) \in H_4$.

It is a quick check that in the first four cases one obtains $2b_7 = 2b_8 = 0$ and hence $\gcd_4 = N/2$, a contradiction. The last case requires $b_8 = (\alpha+1)b_7 = 0$. Moreover, in this case, both of the points $(1,\alpha)$ and $(-1,\alpha)$ must be in $H_2 \cup H_3$, that is $(\alpha+1)b_3 = (\alpha+1)b_5 = 0$. Consider the first row of g and multiply it with $(\alpha+1)$, since C is extended we have $a_1 M_\alpha + (\alpha+1)(b_3+b_5+b_7) = a_1 M_\alpha = 0$, i.e. the first column is a zero column, so the code is degenerate. □

Lemma 20. Let $N \geq 9$. Suppose g has two rows and exactly two columns have $\gcd_i = M_\alpha$ and the other columns have $\gcd_i < M_\alpha$. In this situation g cannot generate a thin non-degenerate linear code.

Proof. No zeros in the columns with $\gcd_i = M_\alpha$. Suppose first that the columns with $\gcd_i = M_\alpha$ do not have any zeros, then

$$g = \underset{}{\begin{pmatrix} a_1 M_\alpha & a_3 M_\alpha & 0 & b_3 & b_5 \\ a_2 M_\alpha & a_4 M_\alpha & b_2 & 0 & b_6 \end{pmatrix}}.$$
$$\quad H_0 \qquad H_1 \qquad H_2 \quad H_3 \quad H_4$$

As in the previous lemma consider the points

$$\{(\alpha,1),\ (1,\alpha),\ (\alpha,-1),\ (-1,\alpha)\}.$$

Under the assumptions none of these points can be contained in the first four hyperplanes. They cannot all be contained in H_4 as well. Therefore, the complement of $\mathcal{H}_C$ is not empty.

A zero in one of the columns with $\gcd_i = M_\alpha$. Suppose that one of the columns with $\gcd_i = M_\alpha$ has a zero, hence we can write

$$g = \begin{pmatrix} a_1 M_\alpha & 0 & b_1 & b_3 & b_5 \\ a_2 M_\alpha & a_4 M_\alpha & 0 & b_4 & b_6 \end{pmatrix}.$$

Suppose the points $(1,1)$ and $(1,-1)$ do not belong to H_0, then we must have

$$g = \begin{pmatrix} a_1 M_\alpha & 0 & b_1 & b_3 & b_5 \\ a_2 M_\alpha & a_4 M_\alpha & 0 & -b_3 & b_5 \end{pmatrix}.$$
$$\quad H_0 \qquad H_1 \qquad H_2 \quad H_3 \quad H_4$$

Consider the points $(\alpha,1)$ and $(\alpha,\alpha+1)$. They cannot be in H_3 since it would correspond to $(\alpha-1)b_3 = 0$ (violates that $\gcd_3 < M_\alpha$) or $b_3 = 0$ (gives a column of zeros). Therefore, both of them must be in H_4, but this would give $\alpha b_5 = 0$, that is, $\gcd_4 = M_\alpha$.

Suppose now that the point $(1,1)$ is not in H_0, but $(1,-1)$ is. Note that in this case the hyperplane containing $(1,1)$ cannot contain any of $(\alpha,1)$ and $(\alpha,\alpha+1)$. Therefore, both of these points must lie in the remaining hyperplane H_4, so we can write

$$g=\begin{pmatrix} a_1M_\alpha & 0 & b_1 & b_3 & b_5 \\ a_1M_\alpha & a_4M_\alpha & 0 & -b_3 & -\alpha b_5 \end{pmatrix}$$

with $\alpha^2 b_5 = 0$. Consider the word corresponding to the point $(0,\alpha)$

$$(0,0,0,-\alpha b_3,0).$$

Since the linear code must be extended, it is necessary to have $\alpha b_3 = 0$ and $\gcd_3 = M_\alpha$.

Suppose that the point $(1,-1)$ is not in H_0, but $(1,1)$ is. The hyperplane containing $(1,-1)$ cannot contain $(\alpha,-1)$, otherwise it would have $\gcd_i > M_\alpha$. Therefore, we can write

$$g=\begin{pmatrix} a_1M_\alpha & 0 & b_1 & b_3 & b_5 \\ -a_1M_\alpha & a_4M_\alpha & 0 & b_3 & \alpha b_5 \end{pmatrix}.$$
$$\quad H_0 \qquad H_1 \qquad H_2 \quad H_3 \quad H_4$$

Since $(1,1) \in H_0$, $(1,-1) \notin H_0$ and the zeroth column has no zero entries, we must have $\alpha \neq 2$. So we can consider the point $(2,-1)$. Clearly, it is not contained in the first 4 hyperplanes, and the last one can contain it only if $(\alpha-2)b_5 = 0$. If $\alpha = 3$, then $b_5 = 0$, and the last column is a column of zeros. If $\alpha = 4$, then $2b_5 = 0$ and the last column is $(b_5, 0)^t$, but then $\gcd_4 = M_2$.

We have deduced that both $(1,1)$ and $(1,-1)$ must belong to H_0. This is possible only if $\alpha = 2$ and $a_1 = a_2 = 1$. Using the fact that C is extended we have

$$g=\begin{pmatrix} M_2 & 0 & b_1 & b_3 & M_2-b_1-b_3 \\ M_2 & M_2 & 0 & b_4 & -b_4 \end{pmatrix}.$$
$$H_0 \quad H_1 \quad H_2 \quad H_3 \qquad\quad H_4$$

Consider the points $(\alpha,1)$ and $(2\alpha,1)$. If both belonged to H_3 we would have $\alpha b_3 = 0$ and $b_4 = 0$, that is, $\gcd_3 = M_2$. The same is true for the pair of points $(\alpha,-1)$ and $(2\alpha,-1)$. We have to distribute the points in these pairs between H_3 and H_4. It is enough to consider only two cases out of four.

Suppose $(\alpha,1)$ and $(2\alpha,-1)$ belong to H_3. The other two points must be in H_4. It gives

$$\alpha b_3 + b_4 = 2\alpha b_3 - b_4 = -\alpha(b_1+b_3) + b_4 = -2\alpha(b_1+b_3) - b_4 = 0.$$

From this we deduce that $3b_4 = 0$ and $3\alpha b_3 = 3\alpha b_1 = 0$. Under this conditions none of H_3 and H_4 can contain the point $(\alpha, \alpha+1)$ unless $\alpha b_4 = 0$. Since $\alpha = 2$, this is equivalent to $2b_4 = 0$, which together with $3b_4 = 0$ gives $b_4 = 0$. Thin, in turn, together with $\alpha b_3 + b_4 = 0$ implies $2b_3 = 0$, that is $\gcd_3 = M_2$.

Now suppose $(\alpha, 1)$ and $(\alpha, -1)$ belong to H_3. It gives

$$\alpha b_3 + b_4 = \alpha b_3 - b_4 = -2\alpha(b_1+b_3) - b_4 = -2\alpha(b_1+b_3) + b_4 = 0.$$

It follows that $2b_4 = 0$. We cannot have $b_4 = 0$ since it would give $\gcd_3 = M_\alpha$, therefore we need $\alpha = 2$ and $b_4 = M_2$. Moreover, we must have $2\alpha b_3 = 4b_3 = 0$ and $4\alpha b_1 = 8b_1 = 0$. Therefore, we can write

$$g = \begin{pmatrix} M_2 & 0 & \tilde{b}_1 N/8 & \tilde{b}_3 N/4 & M_2 - \tilde{b}_1 N/8 - \tilde{b}_3 N/4 \\ M_2 & M_2 & 0 & M_2 & M_2 \end{pmatrix}.$$

for $\tilde{b}_1 \in \{1, 2, \ldots, 7\}$ and $\tilde{b}_3 \in \{1, 3\}$. We see that $N/8 \mid g$ and we can reduce to the existing classification of linear codes with $N \le 8$.

zeros in the columns with $\gcd_i = M_\alpha$. Suppose that both columns with $\gcd_i = M_\alpha$ have a zero. Suppose at first that these zeros are in the different rows. Since the first two hyperplanes do not contain $(1,1)$ and $(1,-1)$ we can write

$$g = \begin{pmatrix} a_1 M_\alpha & 0 & b_1 & b_3 & b_5 \\ 0 & a_4 M_\alpha & -b_1 & b_3 & b_6 \end{pmatrix}.$$
$$\quad H_0 \qquad H_1 \qquad H_2 \quad H_3 \quad H_4$$

Assume that $\alpha \neq 2$. Consider the points $(2,1)$, $(1,2)$, $(2,-1)$, $(-1,2)$. One can check that none of them can belong to the first four hyperplanes under the given assumptions, thus, they must be in H_4, but this is possible only if it is a zero column.

Now consider $\alpha = 2$. Since the code is extended, $b_5 = M_2 - b_1 - b_3$ and $b_6 = M_2 + b_1 - b_3$. Consider the following table that contains points and the values (up to a sign) achieved on these points by the linear functionals defining H_2, H_3 and H_4.

	$(3,1)$	$(3,-1)$	$(1,3)$	$(1,-3)$	$(5,1)$	$(5,-1)$
H_2	$2b_1$	$4b_1$	$2b_1$	$4b_1$	$4b_1$	$6b_1$
H_3	$4b_3$	$2b_3$	$4b_3$	$2b_3$	$6b_3$	$4b_3$
H_4	$2b_1+4b_3$	$4b_1+2b_3$	$2b_1-4b_3$	$4b_1-2b_3$	$4b_1+6b_3$	$6b_1+4b_3$

All of the above points will be in the hyperplane arrangement if $4b_1 = 4b_3 = 0$, but this would imply $N/4 \mid g$. If both $4b_1 \neq 0$ and $4b_3 \neq 0$ then some of these points are not in $\mathcal{H}_C$. Therefore, exactly one of $4b_1$ and $4b_3$ must be zero. We can choose $4b_1 = 0$ and $4b_3 \neq 0$. The other case clearly gives an equivalent linear code. Now the point $(3, 1)$ must belong to H_4, i.e. we must have $2b_1 + 4b_3 = 0$, which implies $8b_3 = 0$. Thus, $N/8 \mid g$ and we again reduced to the existing classification of codes with $N \leq 8$.

Now suppose that both of the zeros from the columns with $\gcd_i = M_\alpha$ are in the same row, so we have

$$g = \begin{pmatrix} a_1 M_\alpha & a_3 M_\alpha & b_1 & b_3 & b_5 \\ 0 & 0 & -b_1 & b_3 & b_6 \end{pmatrix}.$$

In the same way as in the previous case we deduce that we need $\alpha = 2$, thus

$$g = \begin{pmatrix} M_2 & M_2 & b_1 & b_3 & -b_1 - b_3 \\ 0 & 0 & -b_1 & b_3 & b_1 - b_3 \end{pmatrix}.$$

The last table remains unchanged for this situation, thus we reduce again to the situation $N \leq 8$. □

Lemma 21. Let $N \geq 9$. Suppose g has two rows and exactly three columns have $\gcd_i = M_\alpha$ and the other columns have $\gcd_i < M_\alpha$. In this case, for even N there exists a thin linear code that is not a direct sum and it is generated by

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & N-1 \end{pmatrix}. \tag{5.5}$$

Note that these linear codes correspond to simplices of width 1 due to Proposition 12.

Proof. No zeros in the columns with $\gcd_i = M_\alpha$. Suppose that the columns with $\gcd_i = M_\alpha$ do not have any zeros, then

$$g = \begin{pmatrix} a_1 M_\alpha & a_3 M_\alpha & a_5 M_\alpha & 0 & b_3 \\ a_2 M_\alpha & a_4 M_\alpha & a_6 M_\alpha & b_2 & 0 \end{pmatrix}.$$

In this case, $M_\alpha \mid g$ since the linear code is extended.

At least one zero in the columns with $\gcd_i = M_\alpha$ *but only in one row.* Consider

$$g = \begin{pmatrix} a_1 M_\alpha & a_3 M_\alpha & a_5 M_\alpha & 0 & b_3 \\ a_2 M_\alpha & a_4 M_\alpha & 0 & b_2 & b_4 \end{pmatrix}$$
$$\quad H_0 \qquad H_1 \qquad H_2 \quad H_3 \quad H_4$$

with possibly a_2 and a_4 being zero and other a_i being non-zero. Since the linear code is extended, b_3 is divisible by M_α. Consider the points $(1, \alpha)$ and $(-1, \alpha)$. They cannot be contained in the first four hyperplanes, therefore, both of them belong to H_4. This leads to $2\alpha b_4 = 0$ and $2b_3 = 0$, forcing $\alpha \neq 3$ since $M_\alpha \mid b_3$. If $\alpha = 2$, then $N/4$ divides g. So we need to consider only $\alpha = 4$.

Consider the points $(2, 1)$ and $(3, 1)$. Since $2b_3 = 0$ none of these points can be in H_4 unless it is a column of zeros or $\gcd_4 = N/2$. Thus, they must belong to the hyperplanes H_0 and H_1. Clearly, they cannot be in the same one of these hyperplanes. We can assume that $(2, 1) \in H_0$ and $(3, 1) \in H_1$. This implies that the generating matrix g is divisible by $N/8$ so we can reduce to the case $N = 8$.

Two columns with $\gcd_i = M_\alpha$ *have zeros in different rows.* Consider

$$g = \underset{\qquad H_0 \qquad\quad H_1 \qquad\quad H_2 \qquad H_3 \quad H_4}{\begin{pmatrix} a_1 M_\alpha & 0 & a_5 M_\alpha & b_1 & b_3 \\ a_2 M_\alpha & a_4 M_\alpha & 0 & b_2 & b_4 \end{pmatrix}}$$

with possibly one of a_1 or a_2 being zero. Suppose at first that $\alpha \neq 2$. The points $(1, 1)$ and $(1, -1)$ cannot belong to the same hyperplane unless its $\gcd_i = M_2$, therefore, we have to distribute them among two different hyperplanes. Suppose that $(1, 1) \in H_3$ and $(1, -1) \in H_4$. This implies that the points $(2, 1), (2, -1), (1, 2)$ and $(1, -2)$ do not belong to neither of these hyperplanes, unless one of them is a zero column or they have $\gcd_3 = \gcd_4 = M_3$. Thus, all these points must belong to H_0, but it is possible only if it is a zero column.

Another option is to have $(1, 1) \in H_0$ and $(1, -1) \in H_3$ (the situation of $(1, -1) \in H_0$ and $(1, 1) \in H_3$ is equivalent). We have

$$g = \underset{\qquad H_0 \qquad\quad H_1 \qquad\quad H_2 \qquad H_3 \quad H_4}{\begin{pmatrix} a_1 M_\alpha & 0 & a_5 M_\alpha & b_1 & b_3 \\ -a_1 M_\alpha & a_4 M_\alpha & 0 & b_1 & b_4 \end{pmatrix}}.$$

The points $(2, 1)$ and $(1, 2)$ cannot belong to H_0 and also they do not belong to H_3 since it would imply $\gcd_3 = M_3$. The only possibility left is $(2, 1), (1, 2) \in H_4$, but this implies $\gcd_4 = M_3$ as well.

Consider now $\alpha = 2$. The generating matrix takes form

$$g = \underset{\quad H_0 \qquad H_1 \quad H_2 \quad H_3 \qquad\qquad H_4}{\begin{pmatrix} a_1 M_2 & 0 & M_2 & b_1 & (-a_1 + 1) M_2 - b_1 \\ M_2 & M_2 & 0 & b_2 & -b_2 \end{pmatrix}}.$$

Suppose $a_1 = 0$. Then without loss of generality $(1, 1)$ must be in H_3. The point $(3, 1)$ then must be in H_4, which would imply $4b_1 = 0$ and $N/4 \mid g$.

So we have to consider only the case when $a_1 = 1$ and

$$g = \begin{pmatrix} M_2 & 0 & M_2 & b_1 & -b_1 \\ M_2 & M_2 & 0 & b_2 & -b_2 \end{pmatrix}. \tag{5.6}$$

It is easy to see that for any choice of b_1 and b_2 these linear codes are thin. The proof is concluded by the following lemma

Lemma 22. For any non-zero choice of (b_1, b_2) the linear code C generated by (5.6) is either a direct sum or it is isomorphic to the code generated by

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & N-1 \end{pmatrix}.$$

The corresponding simplex has

$$h^*(\Delta_C, t) = \left(\frac{3N}{2} - 1\right) t^2 + \frac{N}{2} t + 1.$$

Proof. Let us describe all the words in the linear code generated by (5.6). First, let us show that all the words of the form

$$(0, 0, 0, 2k, -2k)$$

for $k \in \{0, 1, \ldots, N/2 - 1\}$ appear in C. For this it is enough to show that there are coefficients (c_1, c_2) that give the word $(0, 0, 0, 2, N-2)$. If $\gcd(b_1, b_2, M_2) \neq 1$, then it divides g and we reduce to the same type of situation but for a lower N. Therefore, we can assume that $\gcd(b_1, b_2, M_2) = 1$. By Bezout's identity there exist integers x, y, z such that $xb_1 + yb_2 + zM =_{\mathbb{Z}} 1$ consequently $2xb_1 + 2yb_2 + zN =_{\mathbb{Z}} 2$. Considering this identity modulo N we see that we can take $(c_1, c_2) = (2x, 2y)$ mod N. Note that there are no words of the form $(0, 0, 0, k, -k)$ for odd k.

Suppose, b_1 and b_2 are odd. Since $\gcd(b_1, b_2, N) = 1$ there are integers x, y, z such that

$$xb_1 + yb_2 + zN =_{\mathbb{Z}} 1.$$

Since N is even and b_1, b_2 are odd, exactly one of x and y must be even. Considering this identity modulo N we deduce that there are such coefficients (c_1, c_2) that give us words $(M_2, M_2, 0, 1, -1)$ and $(M_2, 0, M_2, 1, -1)$. Multiplying these with odd numbers we get all the words of the form

$$(M_2, M_2, 0, 2k+1, -2k-1) \quad \text{and} \quad (M_2, 0, M_2, 2k+1, -2k-1).$$

We are left to consider the coefficients (c_1, c_2) with both c_i odd. Take $(c_1, c_2) = (b_2, -b_1)$. This gives us the word $(0, M_2, M_2, 0, 0)$. Adding to this word all the words of the form $(0, 0, 0, 2k, -2k)$ from before we can get all the words of the form

$$(0, M_2, M_2, 2k, -2k).$$

This way we exhausted all the possible coefficients (c_1, c_2).

Now suppose that b_1 is odd and b_2 is even. The same considerations as above allow one to deduce that the corresponding linear code C contains all the words of the form $(0, M_2, M_2, 2k+1, -1-2k)$, $(M_2, M_2, 0, 2k+1, -1-2k)$, $(M_2, 0, M_2, 2k, -2k)$. So C is not exactly the same as the codes with odd b_1 and b_2 but a permutation of the first three columns gives an isomorphism between them.

Finally, suppose that both b_1 and b_2 are even. This is possible only if $4 \nmid N$ since otherwise we would have $2 \mid g$. Since $4 \nmid N$, we have odd M_2. Consider the word $(M_2, M_2, 0, b_1, -b_1)$ and multiply it with M_2. Since b_1 is even, it gives $(M_2, M_2, 0, 0, 0)$. In a similar way we obtain $(M_2, 0, M_2, 0, 0)$ from $(M_2, 0, M_2, b_2, -b_2)$. We also have $(0, M_2, M_2, 0, 0)$ as their sum. Now adding the words $(0, 0, 0, 2k, -2k)$ to these, we obtain all the possible words in C. Having the list of all the words in C, we see that in the case of b_1 and b_2 being even the linear code can be generated by the matrix

$$\begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix}$$

as well. Thus, this is a direct sum and the corresponding simplex is a free join.

Since for different choices of b_1 and b_2 the codes are isomorphic, we can simply fix $(b_1, b_2) = (0, 1)$ and arrive so at the statement of the lemma. It is easy to compute the corresponding h^*-polynomials, since the words correspond to the integral points of the half-open parallelepiped. □

□

With this lemma we have covered all the possible cases of g with two rows. Now we can move on to the generating matrices with three rows.

5.2.3 $m = 3$ cases

Proposition 7 (Part 1)**.** Suppose $N \geq 9$, g has three rows and it generates a non-degenerate linear code C that is not a direct sum. If C is thin, then it can be generated by a matrix with 2 rows of the form (5.3).

Proof. If a matrix g with three rows generates a thin linear code, then matrices constructed from the pairs of rows of g must also generate thin linear codes. We can choose the first two rows to be a multiple of a generating matrix of a thin code and add a third row to it. For the first two rows we have six options: a matrix with a column of zeros, multiples of the cases 2,3,4,6 from Table 5.1 and multiples of the matrices from Lemma 22. We have to treat them one by one.

Case 2 from Table 5.1. We look at the generating matrices over $\mathbb{Z}_{3k}$ for $k \geq 3$ of the form

$$g = \begin{pmatrix} 0 & 0 & k & k & k \\ k & 2k & 0 & k & 2k \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

such that $\gcd(g) = 1$. If we want the first and the third rows to generate a thin code, then we need $a_0 = 0$ ($a_1 = 0$ is equivalent). Same argument applied to the second and third row gives $a_2 = 0$. We reduced to

$$g = \begin{pmatrix} 0 & 0 & k & k & k \\ k & 2k & 0 & k & 2k \\ 0 & a_1 & 0 & a_3 & -a_1 - a_3 \end{pmatrix}.$$

Consider the points $(1, 1, 1)$ and $(1, 2, 1)$. One can check that they cannot be both in $\mathcal{H}_C$ unless $k \mid g$.

Cases 3,4 and 6 from Table 5.1. All these cases behave quite similarly in this situation. We present here only the case 3. We look at the generating matrices over $\mathbb{Z}_{4k}$ for $k \geq 3$ of the form

$$g = \begin{pmatrix} 0 & 0 & k & k & 2k \\ 2k & 2k & k & 3k & 0 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}.$$

Since the first and the third row must generate a thin code, we need $a_0 = 0$ ($a_1 = 0$ is equivalent). The second and the third row generate a thin code under one of the following conditions:

1. $a_4 = 0$,
2. $a_1 = 2k$, $a_4 = 2k$.

In the first scenario we have

$$g = \begin{pmatrix} 0 & 0 & k & k & 2k \\ 2k & 2k & k & 3k & 0 \\ 0 & a_1 & a_2 & -a_1 - a_2 & 0 \end{pmatrix}.$$

Consider again the points $(1,1,1)$ and $(1,2,1)$. One can check, that we cannot cover both of them unless $k \mid g$.

In the second scenario we have

$$g = \begin{pmatrix} 0 & 0 & k & k & 2k \\ 2k & 2k & k & 3k & 0 \\ 0 & 2k & a_2 & -a_2 & 2k \end{pmatrix}.$$

The point $(1,1,2)$ can be in $\mathcal{H}_C$ if either $a_2 = 0$ or $2a_2 = 2k$, but both of these conditions lead to $k \mid g$.

First two rows given by a generating matrix belonging to the family (5.1). Let N be even. Consider generating matrices over $\mathbb{Z}_N$ of the form

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}.$$

One can show that if we considered instead the first two rows of g multiplied by an integer $k \geq 2$, then similar to the already treated situations we would arrive at $k \mid g$. Therefore, it is enough to consider the matrix above.

First of all, we need that the the first and the third rows

$$\begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

generate a thin linear code. It is possible if any of the following conditions is true:

1. $a_2 = 0$,
2. $a_3 = 0$,
3. $a_2 = M_2$, $a_0 = M_2$, $a_1 = 0$,
4. $a_2 = M_2$, $a_0 = 0$, $a_1 = M_2$.

We also need the same for the second and third rows

$$\begin{pmatrix} M_2 & 0 & M_2 & 1 & -1 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}.$$

It is possible if

5. $a_1 = 0$,

6. $a_1 = M_2$, $a_0 = M_2$, $a_2 = 0$,

7. $a_2 = M_2$, $a_0 = 0$, $a_1 = M_2$ (note, this is the same as condition (4)).

There are the following consistent pairs of the above conditions that do not trivially give a linear code equivalent to the one generated by the first two rows: (1) and (5), (1) and (6), (2) and (5), (3) and (5), (4) and (7).

Let us consider the above pairs one by one.

Conditions (1) and (5). In this case, we have

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ a_0 & 0 & 0 & a_3 & a_4 \end{pmatrix}_{\substack{\\ H_0 \quad H_1 \quad H_2 \quad H_3 \quad H_4}}.$$

If $(1,1,1) \in H_0$, then $a_0 = 0$ and $a_4 = -a_3$. If a_3 is even, the the third row is a linear combination of the first two as we have seen in Lemma 22. If a_3 is odd, then we can show that the resulting linear code is equivalent to the one generated by

$$g = \begin{pmatrix} M & M & 0 & 0 & 0 \\ M & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{5.7}$$

See Lemma 23 below.

If $(1,1,1) \notin H_0$, then up to a permutation of the last two columns it must lie in H_3. We have

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ a_0 & 0 & 0 & -1 & 1-a_0 \end{pmatrix}_{\substack{\\ H_0 \quad H_1 \quad H_2 \quad H_3 \quad H_4}}.$$

Consider the point $(1,1,-1)$. It cannot be in the hyperplane arrangement unless $a_0 = 0$ (which leads to the situation of Lemma 23) or $a_0 = 2$ in which case the point $(1,1,2)$ is neither in H_0 nor in H_4 as long as $N \geq 9$.

Lemma 23. The linear code over $\mathbb{Z}_N$ generated by

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ 0 & 0 & 0 & a & -a \end{pmatrix}$$

with odd a is isomorphic to the code generated by

$$g_0 = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This is a direct sum, and the corresponding simplex is a free join of $2\Delta_2$ and a 1-dimensional interval of length N.

Proof. Since a is odd, the following equalities hold

$$\begin{aligned} (M_2, 0, M_2, 0, 0) &= a\ (M_2, 0, M_2, 1, -1) - \ (0, 0, 0, a, -a), \\ (0, 0, 0, 1, -1) &= (1-a)\ (M_2, 0, M_2, 1, -1) + \ (0, 0, 0, a, -a) \end{aligned}$$

This is an invertible linear transformation between the last two rows of g and g_0. Therefore, the linear codes generated by g and g_0 are isomorphic. □

Conditions (1) and (6). In this case, the generating matrix is

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ M_2 & M_2 & 0 & a_3 & -a_3 \end{pmatrix}.$$

We can substitute the third row by the difference of the third and the first rows. Now again it is either the linear code generated by the first two rows if a_3 is even or it is the situation of the Lemma 23 if a_3 is odd.

Conditions (2) and (5). In this case, the generating matrix is

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ a_0 & 0 & a_2 & 0 & -a_2 - a_0 \end{pmatrix}.$$
$$\qquad H_0 \quad H_1 \quad H_2 \quad H_3 \qquad H_4$$

Consider the words corresponding to the points $(1, 2, 1)$ and $(1, 2, -1)$. They are

$$(a_0 + M_2, M_2, a_2, 2, -2 - a_2 - a_0), \quad (-a_0 + M_2, M_2, -a_2, 2, -2 + a_2 + a_0).$$

There are exactly three options how both of them can have a zero. The first one is $a_2 = 0$, i.e.

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ a_0 & 0 & 0 & 0 & -a_0 \end{pmatrix}.$$

The point $(1,1,1)$ can be contained only in H_4, i.e. $a_0 = -1$. Consider now the word corresponding to the point $(1,1,2)$. It is $(-2, M_2, M_2, 1, 1)$, so the code generated by g cannot be thin.

The second option is to have $a_2 = -a_0 - 2 = -a_0 + 2$. This implies $4 = 0$, so necessarily $N = 4$.

The only option left is to have $a_0 = M_2$, which gives

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ M_2 & 0 & a_2 & 0 & M_2 - a_2 \end{pmatrix}.$$

Consider the points $(1,1,1)$ and $(1,1,-1)$. They correspond to the words

$$(M_2, M_2, M_2 + a_2, 1, M_2 - 1 - a_2), \quad (M_2, M_2, M_2 - a_2, 1, M_2 - 1 + a_2).$$

There are two ways for both of these words to have a zero. It is possible if $a_2 = M_2 - 1 = M_2 + 1$, but it implies $2 = 0$, that is $N = 2$. The other option is $a_2 = M_2$. Now we can substitute the second row with the difference of the second and the third rows and we see that this linear code is a direct sum and the corresponding simplex is a free join of $2\Delta_2$ and the interval of length N.

Conditions (3) and (5). In this case, the generating matrix is

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ M_2 & 0 & M_2 & a_3 & -a_3 \end{pmatrix}.$$

If a_3 is odd, then the last row is already a word in the linear code generated by the first two rows, so we need to consider only even a_3. In this case, from the last two rows we can get the words

$$\begin{aligned} (M, 0, M, 0, 0) &= a_3\,(M, 0, M, 1, -1) - (M, 0, M, a_3, -a_3), \\ (0, 0, 0, 1, -1) &= (a_3 + 1)\,(M, 0, M, 1, -1) - (M, 0, M, a_3, -a_3). \end{aligned}$$

This is an invertible transformation, so the generating matrix

$$\begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

generates the same linear code.

Conditions (4) and (7). In this case, the generating matrix is

$$g = \begin{pmatrix} M_2 & M_2 & 0 & 0 & 0 \\ M_2 & 0 & M_2 & 1 & -1 \\ 0 & M_2 & M_2 & a_3 & -a_3 \end{pmatrix}.$$

Again we have either a direct sum or the third row is a combination of the first two. The proof is almost identical to the previous case.

A column of zeros in every pair of rows So far we have showed that if any pair of the rows of a generating matrix with three rows is non-degenerate, then there are no new interesting linear codes comparing to the case of just two rows. The only option left to consider is when all the restrictions from three rows to two rows have a column of zeros. In this case, the generating matrix takes the form

$$g = \begin{pmatrix} d_1 & 0 & 0 & b_1 & b_4 \\ 0 & d_2 & 0 & b_2 & b_5 \\ 0 & 0 & d_3 & b_3 & b_6 \end{pmatrix}.$$
$$\quad H_0 \quad H_1 \quad H_2 \quad H_3 \quad H_4$$

We will show that this matrix cannot generate a thin linear code for $N \geq 9$ unless this code is a direct sum.

Consider the points $(1,1,1)$, $(1,1,-1)$, $(1,-1,1)$, $(-1,1,1)$. They must be contained in the union $H_3 \cup H_4$. One can check that if one of these hyperplanes contains three out of the four points, then it necessarily contains the fourth one as well. Thus, up to permuting these hyperplanes there are two possible situations: either H_3 contains two out of four points or all of them.

Let us start with the case when all the four points are in H_3. It requires $2b_1 = 2b_2 = 2b_3 = 0$ and $b_1 + b_2 + b_3 = 0$. We can choose $b_1 = b_2 = M_2$ and $b_3 = 0$. Now we have

$$g = \begin{pmatrix} d_1 & 0 & 0 & M_2 & M_2 - d_1 \\ 0 & d_2 & 0 & M_2 & M_2 - d_2 \\ 0 & 0 & d_3 & 0 & -d_3 \end{pmatrix}.$$
$$\quad H_0 \quad H_1 \quad H_2 \quad H_3 \quad H_4$$

Consider the points of the form $(2, \pm 1, \pm 1)$. They can be either in H_0 or H_4. If $2d_1 \neq 0$, then all of them must be in H_4. In that case $2d_1 = 2d_2 = 4d_1 = 0$, which implies $N/4 \mid g$. Therefore, we must have $2d_1 = 0$. By considering the points $(\pm 1, 2, \pm 1)$ we arrive in the same way at $2d_2 = 0$. Now we have

$$g = \begin{pmatrix} M_2 & 0 & 0 & M_2 & 0 \\ 0 & M_2 & 0 & M_2 & 0 \\ 0 & 0 & d_3 & 0 & -d_3 \end{pmatrix},$$

which is a direct sum.

Now consider the situation when each of H_3 and H_4 contain only two out of the four points. Say, $(1,1,1)$, $(1,1,-1)$ are in H_3 and $(1,-1,1)$, $(-1,1,1)$ are in H_4. It implies the generating matrix must be of the form

$$g = \begin{pmatrix} d_1 & 0 & 0 & b_1 & b_4 \\ 0 & d_2 & 0 & \tilde{b}_3 M_2 - b_1 & \tilde{b}_6 M_2 + b_4 \\ 0 & 0 & d_3 & \tilde{b}_3 M_2 & \tilde{b}_6 M_2 \end{pmatrix}$$

with $\tilde{b}_i = 0$ or 1. We cannot have $\tilde{b}_3 = \tilde{b}_6$ since it would imply $d_3 = 0$. It is enough to consider $\tilde{b}_3 = 0$ and $\tilde{b}_6 = 1$, which gives $d_3 = M_2$. Since the code is extended we can write

$$g = \begin{pmatrix} d_1 & 0 & 0 & b_1 & -d_1 - b_1 \\ 0 & M_2 + d_1 + 2b_1 & 0 & -b_1 & M_2 - d_1 - b_1 \\ 0 & 0 & M_2 & 0 & M_2 \end{pmatrix}$$
$$\qquad H_0 \qquad\quad H_1 \qquad\quad H_2 \quad H_3 \qquad\quad H_4$$

Consider the point $(2,1,1)$. If it is in H_0, then $d_1 = M_2$. In this case, the point $(1,2,1)$ cannot belong to H_1 since it would give $4b_1 = 0$ and $N/4 \mid g$, so this point must belong to H_4 giving $3b_1 = 0$. Consider now the point $(3,2,1)$. The only option is $(3,2,1) \in H_4$, but now this would imply $b_1 = 0$ giving a few columns of zeros.

Suppose now that $(2,1,1) \in H_4$, i.e. $-3(d_1 + b_1) = 0$. Consider the point $(1,2,1)$. If it belongs to H_1, then $2d_1 + 4b_1 = 0$, implying $b_1 = d_1$ and $N/6 \mid g$. The other option is to have $(1,2,1) \in H_4$, which leads to $M_2 - 3(d_1 + b_1) = M_2 = 0$, a contradiction.

Now we have considered all the possible generating matrices of interest with three rows. □

5.2.4 $m = 4$ case

Proposition 7 (Part 2)**.** Suppose $N \geq 9$, g has four rows and it generates a non-degenerate linear code C that is not a direct sum. If C is thin, then it can be generated by a matrix with 2 rows of the form (5.3).

Proof. Each pair and triple of the rows of g should generate a thin linear code, therefore we have the following three options for the first three rows

- multiples of the case 5 from Table 5.1,

- a direct sum with a factor $C_{2\Delta_2}$,
- three rows with a column of zeros.

From the proof of Proposition 7 it follows that the direct sum case can only lead to a thin code that is again a direct sum.

Consider the situation when the first three rows is a multiple of the generating matrix of the case 5 from Table 5.1. We have

$$g = \begin{pmatrix} 0 & 0 & k & k & 2k \\ 2k & 2k & 0 & 2k & 2k \\ 0 & 2k & 3k & 3k & 3k \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

over $\mathbb{Z}_{4k}$ with $k \geq 3$. From the previous subsection we know that if g has three rows and $N \geq 9$, then either it is a member of the family (5.1) or it is a direct sum with a factor $C_{2\Delta_2}$. In both of these cases there should be three columns with $\gcd_i = M_2$. We see that that if consider the last three rows of g, it is not possible to choose a_i's in such a way, that there are three columns with $\gcd_i = M_2$. Therefore, the linear code generated by g cannot be thin.

The only option left now is when each triple of rows of g has a column of zeros, i.e.

$$g = \begin{pmatrix} d_1 & 0 & 0 & 0 & -d_1 \\ 0 & d_2 & 0 & 0 & -d_2 \\ 0 & 0 & d_3 & 0 & -d_3 \\ 0 & 0 & 0 & d_4 & -d_4 \end{pmatrix}.$$

The fourth hyperplane must contain all the points $(\pm 1, \pm 1, \pm 1, \pm 1)$. This leads to all $d_i = M_2$, i.e. g is just a multiple of the Case 1 from Table 5.1. □

This finishes the classification of the four-dimensional thin simplices.

5.3 Experiments and questions

By using linear codes instead of simplices, it becomes somewhat easier to produce examples of thin simplices. By considering generating matrices of extended linear codes over $\mathbb{Z}_{N_\Delta}$ for N_Δ small and with only a few rows, one can relatively quickly obtain a small database of thin simplices. The corresponding SageMath [The22] code is available on GitHub[2]. Using the criterion from Remark 17 we can quickly

[2] https://github.com/VadymKurylenko/Thin-Simplices

establish which simplices are definitely not free joins. In Table 5.3 the found data is presented. This is by far not a classification.

d	#	spanning	width 1	width 2	empty	non-trivially thin
5	69	0	69	0	5	67
6	704	4	655	49	35	541
7	1071	0	1071	0	130	1053

Table 5.3: Thin simplices that are not free joins found experimentally

Note the scarcity of spanning thin simplices in the above table. In dimension 4 we also have only 2 spanning thin simplices. Furthermore, it is somewhat surprising that we were not able to find any spanning thin simplices in dimensions 5 and 7. This naturally leads to the following questions.

Question 5. Are there finitely many spanning thin simplices that are not free joins in each dimension? Are there any in the odd-dimensional case?

Another thing that we can look at is lattice width. As discussed in [BKN23], it seems that thin lattice polytopes tend to have small width. To compute the lattice width we first make use of Proposition 12 from Subsection 5.1.3 to single out width 1 simplices and then we use Polymake [AGH+17] to compute the width of the remaining examples. One can see that the amount of thin simplices of width ≥ 2 is rather small and in odd dimensions we were not able to find any. We would like to state a few conjectures in the strongest form possible.

Conjecture 6. In each even dimension there are only finitely many thin simplices of lattice width ≥ 2.

Conjecture 7. In odd dimensions all thin simplices have width 1.

One can also ask whether these conjectures hold for thin polytopes, that are not simplices. As was shown in [BKN23], all three-dimensional thin polytopes have width 1.

In [BKN23] a question was asked whether there are thin empty simplices with non-trivial quotient group in dimensions ≥ 5. We answer this question positively. All the empty simplices considered in the table above have a non-trivial quotient group. Moreover, all of them have width 1. Here is an example.

Example 14. The 6-dimensional simplex, whose vertices are the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -3 & -6 & -1 \\ 0 & 0 & 0 & 0 & 2 & -6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 10 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix},$$

is thin and empty with $h^*(\Delta, t) = 3t^4 + 12t^3 + 16t^2 + 1$. Its quotient group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$. The corresponding linear code over $\mathbb{Z}_8$ can be generated by

$$\begin{pmatrix} 0 & 3 & 3 & 4 & 4 & 5 & 5 \\ 4 & 0 & 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 4 & 4 & 4 & 0 & 4 \end{pmatrix}.$$

Bibliography

[AG] Asem Abdelraouf and Giulia Gugiatti. Geometric realization of hypergeometric local systems, in preparation.

[AGH+17] Benjamin Assarf, Ewgenij Gawrilow, Katrin Herr, Michael Joswig, Benjamin Lorenz, Andreas Paffenholz, and Thomas Rehn. Computing convex hulls and counting integer points with \tt polymake. *Mathematical Programming Computation*, 9(1):1–38, 2017.

[AHN23] Gennadiy Averkov, Johannes Hofscheier, and Benjamin Nill. Generalized flatness constants, spanning lattice polytopes, and the Gromov width. *Manuscripta Mathematica*, 170(1-2):147–165, 2023.

[APPS22] Karim Alexander Adiprasito, Stavros Argyrios Papadakis, Vasiliki Petrotou, and Johanna Kristina Steinmeyer. Beyond positivity in Ehrhart Theory, October 2022. arXiv:2210.10734 [math].

[Ath12] Christos A. Athanasiadis. Flag subdivisions and -vectors. *Pacific Journal of Mathematics*, 259(2):257–278, 2012.

[Bal21] Gabriele Balletti. Enumeration of Lattice Polytopes by Their Volume. *Discrete & Computational Geometry*, 65(4):1087–1122, June 2021.

[Bat93] Victor V. Batyrev. Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori. *Duke Mathematical Journal*, 69(2):349–409, February 1993. Publisher: Duke University Press.

[BB96] Victor V. Batyrev and Lev A. Borisov. Mirror duality and string-theoretic Hodge numbers. *Inventiones Mathematicae*, 126(1):183–203, 1996.

[BBC+23] Esme Bajo, Benjamin Braun, Giulia Codenotti, Johannes Hofscheier, and Andrés R. Vindas-Meléndez. Local \$hˆ*\$-polynomials for one-row Hermite normal form simplices, September 2023. arXiv:2309.01186 [math].

[BBR24] Eleonore Bach, Matthias Beck, and Sophie Rehberg. Acyclotopes and Tocyclotopes. *arXiv preprint arXiv:2409.15227*, 2024.

[Ber75] D. N. Bernstein. The number of roots of a system of equations. *Akademija Nauk SSSR. Funkcional\cprime nyi Analiz i ego Priloženija*, 9(3):1–4, 1975.

[BH89] F. Beukers and G. Heckman. Monodromy for the hypergeometric function $_nf_{n-1}$. *Inventiones Mathematicae*, 95(2):325–354, 1989.

[BH13] Victor Batyrev and Johannes Hofscheier. Lattice polytopes, finite abelian subgroups in $\SL(n,\C)$ and coding theory, September 2013. arXiv:1309.5312 [math].

[BKN23] Christopher Borger, Andreas Kretschmer, and Benjamin Nill. Thin Polytopes: Lattice Polytopes With Vanishing Local h*-Polynomial. *International Mathematics Research Notices*, page rnad231, October 2023. _eprint: https://academic.oup.com/imrn/advance-article-pdf/doi/10.1093/imrn/rnad231/51824740/rnad231.pdf.

[BM85] U. Betke and P. McMullen. Lattice points in lattice polytopes. *Monatshefte für Mathematik*, 99(4):253–265, December 1985.

[BM03] Lev A. Borisov and Anvar R. Mavlyutov. String cohomology of Calabi–Yau hypersurfaces via mirror symmetry. *Advances in Mathematics*, 180(1):355–390, December 2003.

[BN07] Victor Batyrev and Benjamin Nill. Multiples of lattice polytopes without interior lattice points. *Moscow Mathematical Journal*, 7(2):195–207, 349, 2007.

[BN08] Victor Batyrev and Benjamin Nill. Combinatorial aspects of mirror symmetry. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 35–66. Amer. Math. Soc., Providence, RI, 2008.

[Bob09] Jonathan W. Bober. Factorial ratios, hypergeometric series, and a family of step functions. *Journal of the London Mathematical Society. Second Series*, 79(2):422–444, 2009.

[BS90] Margaret Bayer and Bernd Sturmfels. Lawrence polytopes. *Canadian Journal of Mathematics. Journal Canadien de Mathématiques*, 42(1):62–79, 1990.

[CD22] Eduardo Cattani and Alicia Dickenstein. Non-splitting flags, Iterated Circuits, $\underline{\mathbf \sigma}$-matrices and Cayley configurations, January 2022. arXiv:2105.00302 [math].

[CG11] Alessio Corti and Vasily Golyshev. Hypergeometric equations and weighted projective spaces. *Science China Mathematics*, 54(8):1577–1590, August 2011.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[Del74] Pierre Deligne. Théorie de hodge, III. *Publications mathématiques de l'IHÉS*, 44(1):5–77, January 1974.

[DK87] V I Danilov and A G Khovanskiĭ. NEWTON POLYHEDRA AND AN ALGORITHM FOR COMPUTING HODGE–DELIGNE NUMBERS. *Mathematics of the USSR-Izvestiya*, 29(2):279–298, April 1987.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *Triangulations*, volume 25 of *Algorithms and Computation in Mathematics*. Springer, Berlin, Heidelberg, 2010.

[DRHN16] Sandra Di Rocco, Christian Haase, and Benjamin Nill. Discrete Mixed Volume and Hodge-Deligne Numbers, September 2016. arXiv:1609.09115 [math].

[Ehr62] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 254:616–618, 1962.

[ES06] Friedrich Eisenbrand and Gennady Shmonin. Carathéodory bounds for integer cones. *Operations Research Letters*, 34(5):564–568, 2006.

[Est10] A. Esterov. Newton polyhedra of discriminants of projections. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 44(1):96–148, 2010.

[Fed18] Roman Fedorov. Variations of Hodge structures for hypergeometric differential operators and parabolic Higgs bundles. *International Mathematics Research Notices. IMRN*, (18):5583–5608, 2018.

[For19] Jens Forsgård. Defective dual varieties for real spectra. *Journal of Algebraic Combinatorics*, 49(1):49–67, February 2019.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[GS20] Nils Gustafsson and Liam Solus. Derangements, Ehrhart theory, and local h-polynomials. *Advances in Mathematics*, 369:107169, August 2020.

[GV24] Giulia Gugiatti and Fernando Rodriguez Villegas. Hypergeometric local systems over $\backslashmathbb {Q} with Hodge vector (1, 1, 1, 1) $. *arXiv preprint arXiv:2401.13529*, 2024.

[HKN18] Johannes Hofscheier, Lukas Katthän, and Benjamin Nill. Ehrhart Theory of Spanning Lattice Polytopes. *International Mathematics Research Notices*, 2018(19):5947–5973, October 2018.

[HT09] Martin Henk and Makoto Tagami. Lower bounds on the coefficients of Ehrhart polynomials. *European Journal of Combinatorics*, 30(1):70–83, 2009.

[Kar08] Kalle Karu. Ehrhart analogue of the h-vector. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 139–146. Amer. Math. Soc., Providence, RI, 2008.

[Kou76] A. G. Kouchnirenko. Polyèdres de Newton et nombres de Milnor. *Inventiones Mathematicae*, 32(1):1–31, 1976.

[KS16a] Eric Katz and Alan Stapledon. Local h-polynomials, invariants of subdivisions, and mixed Ehrhart theory. *Advances in Mathematics*, 286:181–239, January 2016.

[KS16b] Eric Katz and Alan Stapledon. Tropical geometry, the motivic nearby fiber, and limit mixed Hodge numbers of hypersurfaces. *Research in the Mathematical Sciences*, 3:Paper No. 10, 36, 2016.

[Kur24] Vadym Kurylenko. Thin Simplices via Modular Arithmetic. *arXiv preprint arXiv:2404.03975*, 2024.

[MFR+16] Stefan Müller, Elisenda Feliu, Georg Regensburger, Carsten Conradi, Anne Shiu, and Alicia Dickenstein. Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry. *Foundations of Computational Mathematics. The Journal of the Society for the Foundations of Computational Mathematics*, 16(1):69–97, 2016.

[NP15] Benjamin Nill and Arnau Padrol. The degree of point configurations: Ehrhart theory, Tverberg points and almost neighborly polytopes. *European Journal of Combinatorics*, 50:159–179, 2015.

[NS12] Benjamin Nill and Jan Schepers. Combinatorial questions related to stringy E-polynomials of Gorenstein polytopes. *Toric Geometry (K. Altmann et. al., eds.), Oberwolfach Reports*, (21):62–64, 2012.

[NS13] Benjamin Nill and Jan Schepers. Gorenstein polytopes and their stringy E-functions. *Mathematische Annalen*, 355(2):457–480, 2013.

[OEI24] OEIS Foundation Inc. Entry A048240 in The On-Line Encyclopedia of Integer Sequences, 2024.

[Pad13] Arnau Padrol. Neighborly and almost neighborly configurations, and their duals. *arXiv preprint arXiv:1304.7186*, 2013.

[RRV22] David P. Roberts and Fernando Rodriguez-Villegas. Hypergeometric Motives. *Notices of the American Mathematical Society*, 69(06):1, June 2022.

[Sch12] Jan Schepers. Stringy Hodge numbers of strictly canonical nondegenerate singularities. *Journal of Algebraic Geometry*, 21(2):273–297, 2012.

[Smi61] Henry John Stephen Smith. Xv. on systems of linear indeterminate equations and congruences. *Philosophical transactions of the royal society of london*, (151):293–326, 1861. Publisher: The Royal Society London.

[Sol19] Liam Solus. Local hˆ*-polynomials of some weighted projective spaces. In *Algebraic and geometric combinatorics on lattice polytopes*, pages 382–399. World Sci. Publ., Hackensack, NJ, 2019.

[Sta80] Richard P. Stanley. Decompositions of Rational Convex Polytopes. In J. Srivastava, editor, *Annals of Discrete Mathematics*, volume 6 of *Combinatorial Mathematics, Optimal Designs and Their Applications*, pages 333–342. Elsevier, January 1980.

[Sta87] Richard Stanley. Generalized H-Vectors, Intersection Cohomology of Toric Varieties, and Related Results. In *Commutative Algebra and Combinatorics*, volume 11, pages 187–214. Mathematical Society of Japan, January 1987.

[Sta92] Richard P. Stanley. Subdivisions and Local h-Vectors. *Journal of the American Mathematical Society*, 5(4):805–851, 1992. Publisher: American Mathematical Society.

[The22] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.7)*. 2022.

[Vil19] Fernando Rodriguez Villegas. Mixed Hodge numbers and factorial ratios, July 2019. arXiv:1907.02722 [math].