SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Geometry and Mathematical Physics

Black hole perturbations from supersymmetric gauge theory and analytic perturbative methods

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Academic Year 2023-24

Abstract

We study linear perturbations around different black hole geometries. We describe two methods that provide the quantization condition for the quasinormal mode frequencies of the perturbation field. The first method is based on techniques from supersymmetric gauge theory and conformal field theory that allow us to explicitly write the connection coefficients for the differential equation encoding the spectral problem. With a closely related analysis, we study one-loop effective actions of scalar fields in the black hole backgrounds by applying a version of the Gelfand-Yaglom theorem generalized to include regular singularities. In particular, the analytic properties of the final results are made explicit by the contributions of the quasinormal modes. The second method provides a perturbative expansion of the local solutions of the differential equation based on multiple polylogarithmic functions around regular singular points and a newly introduced set of special functions called multiple polyexponential integrals around irregular singularities. The convergence properties of Nekrasov's functions are also considered, because of their relevance to the connection formulae and the physical problems analyzed.

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. The discussion is based on the following works:

- **On the Convergence of Nekrasov Functions**, with Giulio Bonelli, and Alessandro Tanzini, Ann. Henri Poincaré (2023),
`https://doi.org/10.1007/s00023-023-01349-3`

- **Black hole perturbation theory and multiple polylogarithms**, with Gleb Aminov, Giulio Bonelli, Alba Grassi, and Alessandro Tanzini, J. High Energ. Phys. 2023, 59 (2023),
`https://doi.org/10.1007/JHEP11%282023%29059`

- **One loop effective actions in Kerr-(A)dS Black Holes**, with Giulio Bonelli, and Alessandro Tanzini, arXiv:2405.13830 [hep-th],
`https://doi.org/10.48550/arXiv.2405.13830`

- **Black hole scattering amplitudes via analytic small-frequency expansion and monodromy**, with Gleb Aminov, arXiv:2409.06681 [hep-th],
`https://doi.org/10.48550/arXiv.2409.06681`

- **Basics of Multiple Polyexponential Integrals**, with Gleb Aminov, arXiv:2409.06760 [math.CA],
`https://doi.org/10.48550/arXiv.2409.06760`

Contents

Chapter 1

Introduction

The research area of Mathematical Physics covers the wide spectrum of developments of mathematical methods suitable for application to problems in physics. Two important aspects are evident. First, the inspiration for studies identified under the label of Mathematical Physics arises from physical phenomena and takes strength from the need to formulate a mathematical model able to reproduce, understand, and explain physical results obtained through observations or numerical simulations. Second, the model itself is required to be rigorous from a purely mathematical viewpoint and is usually based on analytic and geometric grounds.

This thesis focuses on a corner of this massive subject. The physical motivation of the present analysis arises from the recent experimental verification of gravitational waves [1]. Thanks to the growing data from the observations, there has been a renewed interest in analytically computing gravitational quantities within the realm of General Relativity to test the validity of its predictions or to measure deviations from them. Among the relevant quantities that can be investigated, a crucial role is played by the black hole *quasinormal mode frequencies* (QNMs).

For a general physical system, QNMs are the characteristic modes of energy dissipation of a perturbed object or field. They compose a set of discrete and complex frequencies, the imaginary part being associated with the decay timescale of the perturbation. Within general relativity, QNMs naturally appear in the analysis of *linear perturbations* around fixed gravitational backgrounds. Among these, black hole geometries are intrinsically dissipative due to the presence of an *event horizon*, which acts as a one-way causal boundary surface, separating the communication of information at a classical level. For example, the QNMs are encoded in the damped oscillations appearing in the last phase of the merger of two colliding black holes, the so-called *ringdown phase*, and, hence, they have a direct connection to gravitational wave observations.

From a mathematical perspective, the perturbation fields obey second-order differential equations, whose symmetries are dictated by the symmetry properties of the geometric background. In most cases, these symmetries allow one to separate variables and reduce the problem to the study of a system of linear ordinary differential equations (ODE)s, or a single ODE, the radial one. The quantization condition for the QNMs is obtained by imposing two suitable boundary conditions on the perturbation fields. In a black hole background, one of the boundary conditions is imposed at the black hole horizon, which is a singular point of the relevant ODE. Here, the perturbation is required to be purely ingoing, matching the idea that nothing can escape from the region inside the black hole horizon. The second boundary condition, instead, is more specific to the background asymptotic geometry. In asymptotically flat spacetimes, the boundary condition is imposed at spatial infinity, in asymptotically de Sitter spacetimes, the boundary condition is imposed at the *cosmological horizon*, which has a radius greater than the event horizon's one, and in asymptotically anti-de Sitter spacetimes, the boundary condition is imposed at the anti-de Sitter boundary. Moreover, also the type of boundary condition to impose is not fixed but depends on the specific perturbation under scrutiny (for a recent review of the computation of QNMs in different black hole geometries and imposing different boundary conditions see [2]).

Denoting with Φ the perturbation field, using the separation of variables of the perturbation equation, the following decomposition in Fourier modes can be imposed

$$\Phi(t,r,\Omega)=\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{\ell,\vec{m}}e^{-i\omega t}S_{\omega,\ell,\vec{m}}(\Omega)R_{\omega,\ell,\vec{m}}(r). \tag{1.1.1}$$

In the spherically symmetric cases, the angular functions $S_{\omega,\ell,\vec{m}}(\Omega)$ coincide with the spherical harmonics $Y_{\ell,\vec{m}}(\Omega)$.

The analysis of the QNMs focuses on the radial differential equation, satisfied by $R_{\omega,\ell,\vec{m}}(r)$, and it depends on the separation constants, which can be found by solving the coupled angular problems.[1] Redefining appropriately the radial variable r and the wave function $R(r)$, it is possible to rewrite the radial ODE in normal form,

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\psi(z)+V(z)\,\psi(z)=0, \tag{1.1.2}$$

where the singularity structure of the problem is encoded in the potential $V(z)$. This is why the study of black hole perturbation theory is intimately related to the study of second-order linear ODEs on the Riemann sphere. To distinguish the second-order ODEs in terms of their singularity structure, the starting point is given by the ODEs in which all singular points are regular, according to the following definition [3]:

Definition 1.1.1. For a second-order differential equation of the form (1.1.2), a point z_0 is

- a *regular point* if $V(z)$ is analytic at $z=z_0$,
- a *regular singular point* if $V(z)$ has a pole of order up to 2 at $z=z_0$,
- an *irregular singular point* otherwise.

[1] In the spherically symmetric cases, the separation constants are given in terms of the eigenvalues of the spherical harmonics.

A linear differential equation in which every singular point, including the point at infinity, is a regular singularity is called *Fuchsian*.

The simplest case of Fuchsian ODE is the one in which the only two singular points are at $z = 0$ and $z = \infty$. In this case, the differential equation can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\,\psi(z) + \frac{\frac{1}{4} - a_0^2}{z^2}\,\psi(z) = 0, \tag{1.1.3}$$

where we call $a_0 \in \mathbb{C}$ the *index* of the regular singularity $z = 0$. If $a_0 \neq 0$, a basis of independent solutions of this ODE is given by $z^{\frac{1}{2} \pm a_0}$. The case $a_0 = 0$ is a special one, and a basis of solutions is given by $\sqrt{z}, \sqrt{z}\,\log(z)$.

The immediate following case is the *Hypergeometric differential equation*. This is a Fuchsian ODE with regular singular points at $z = 0, 1, \infty$. The main difference with respect to the previous case is that the solutions of the Hypergeometric differential equation do not admit a Taylor series expansion that is well-defined on the whole complex plane, but they admit local expansions centered at one of the singularities, which are convergent until they reach the next closest singular point. These local solutions are built out of the *Gaussian Hypergeometric function*

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\,z^k, \quad |z| < 1, \tag{1.1.4}$$

where $a, b, c \in \mathbb{C}$ parametrize the indices of the singular points, and where $(x)_n$ denotes the (rising) Pochhammer symbol.

Although the local solutions are given in terms of series expansions with a finite radius of convergence, thanks to the integral representation of the Hypergeometric functions, it is possible to analytically continue a specific solution on the whole Riemann sphere. Concretely, this is realized by the Hypergeometric connection formulae.

The connection formulae are particularly useful for the quantization of the QNMs since they make it possible to impose on the same wave solution both boundary conditions, which in principle are imposed in different local regions. However, in black hole perturbation problems, the differential equation is more complicated than a Hypergeometric differential equation, having a higher number of regular singularities or involving the presence of irregular ones. For these more involved differential equations, the connection matrices relating local solutions around different singularities were not known in closed form.

A crucial contribution in this direction has been made in [4], where connection formulae for the *Heun differential equation* and its confluence forms were obtained with techniques coming from Liouville conformal field theory (CFT). The Heun's differential equation [5, 6, 7, 8] is a Fuchsian differential equation with four regular singularities. By performing a suitable redefinition of the variable, realized by a PGL_2 transformation, it is always possible to send three of the singular points in $z = 0, 1, \infty$, and, once this is fixed, the position of the fourth singularity is determined and usually denoted with t, which acts as a new modulus for the problem.

The Heun differential equation arises in Liouville CFT from the *semiclassical limit* of the *BPZ equation* [9] satisfied by a five-point correlation function, where a *degenerate field* appears as one of the insertions.

Liouville CFT is an interacting CFT, with coupling usually denoted with b, and with central charge $c = 1 + 6Q^2$, where $Q = b + b^{-1}$ (see [10, 11] for detailed reviews of the theory). As for every 2d CFT, the symmetry algebra of Liouville CFT is the Virasoro algebra, whose generators L_n, $n \in \mathbb{Z}$ satisfy the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad m, n \in \mathbb{Z}. \tag{1.1.5}$$

These generators characterize the mode expansion of the energy-momentum tensor $T(z)$:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \tag{1.1.6}$$

In the representation theory of the Virasoro algebra, a crucial role is played by the *primary operators* V_Δ, where Δ is called its *conformal dimension*[2]. These satisfy a special transformation rule under conformal transformations $z \mapsto w$:

$$V_\Delta(z)\, \mathrm{d}z^\Delta \mapsto V_\Delta(w)\, \mathrm{d}w^\Delta. \tag{1.1.7}$$

The conformal dimension Δ of a primary field is usually parametrized by the *conformal momentum* α such that

$$\Delta = \Delta_\alpha = \frac{Q^2}{4} - \alpha^2. \tag{1.1.8}$$

We will usually denote a primary field as V_α, where the spectrum is parametrized in such a way that $\alpha \in i\mathbb{R}$. Starting from a primary operator V_α, thanks to the *state-operator correspondence* [12], it is possible to construct the *primary state* $|\Delta_\alpha\rangle$, which defines a lowest weight representations of the Virasoro algebra. The other states in the same representation are obtained by acting on the primary state with negative Virasoro generators and are called *descendants*. When considering correlation functions between primary operators, we will use the following notation on bra states denoting the insertion of a primary field V_α at $z = \infty$:

$$\langle \Delta_\alpha | \equiv \lim_{z \to \infty} z^{2\Delta_\alpha} \langle V_\alpha(z). \tag{1.1.9}$$

Liouville CFT allows the existence of reducible representations of the Virasoro algebra. The invariant submodules are generated by *null states*, which have the property of being annihilated by all the positive generators of the local conformal transformations L_n, $n > 0$. A primary state that admits a descendant which is a null state is

[2] In general, a CFT is covariant under two copies of the Virasoro algebra, a holomorphic part and an antiholomorphic one. Each primary operator is then characterized by the holomorphic dimension Δ, its antiholomorphic dimension $\bar{\Delta}$, and the *spin* $J = \Delta - \bar{\Delta}$. In Liouville CFT the spectrum is *diagonal*, and so $\Delta = \bar{\Delta}$ and $J = 0$ for all the operators.

called *degenerate*. At level 2, there exists a null state, which is a descendant of the state corresponding to the degenerate field $\Phi_{2,1}$ having conformal weight $\Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}b^2$,

$$\left(b^{-2}L_{-1}^2 + L_{-2}\right)|\Phi_{2,1}\rangle. \tag{1.1.10}$$

The null states decouple from every correlation function, that is, for every m-tuple of primary fields V_{α_i}, one has

$$\langle\prod_{i=1}^{m} V_{\alpha_i}(z_i)\left[b^{-2}L_{-1}^2 + L_{-2}\right]\Phi_{2,1}(z)\rangle = 0. \tag{1.1.11}$$

Using the *local Ward identities* [12], it is possible to express the $(m+1)$-point function involving the descendant field in terms of differential operators acting on $(m+1)$-point functions of primary fields. Indeed, since the operators L_{-n} can be represented as differential operators, we have that the $(m+1)$-point correlation function including the degenerate insertion,

$$\langle\prod_{i=1}^{m} V_{\alpha_i}(z_i)\Phi_{2,1}(z)\rangle, \tag{1.1.12}$$

satisfies a partial differential equation (PDE) known as *BPZ equation* [9]:

$$\left[\frac{1}{b^2}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{m}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z-z_i)^2}\right)\right]\langle\prod_{i=1}^{m} V_{\alpha_i}(z_i)\Phi_{2,1}(z)\rangle = 0. \tag{1.1.13}$$

If we perform the operator product expansion (OPE) [12] between the degenerate field $\Phi_{2,1}(z)$ and one of the primary fields $V_{\alpha_i}(z_i)$,

$$\Phi_{2,1}(z)V_{\alpha_i}(z_i) = \sum_{\pm} C_{\alpha_{2,1}\alpha_i}^{\alpha_i\pm\frac{b}{2}}(z-z_i)^{\frac{bQ}{2}\mp\alpha_i}V_{\alpha_i\mp}(z_i) + \mathcal{O}\left((z-z_i)^{\frac{bQ}{2}\mp\alpha_i+1}\right), \tag{1.1.14}$$

inside the correlator (1.1.12), where the OPE coefficients $C_{\alpha\beta}^{\gamma}$ are determined by the three-point functions of the theory[3], we obtain the *conformal block expansion* of a basis of local solutions of the BPZ equation for $z \sim z_i$. By considering the correlation function

$$\langle\Delta_\infty|V_{\alpha_1}(1)V_{\alpha_t}(t)\Phi_{2,1}(z)|\Delta_0\rangle, \tag{1.1.15}$$

where we assume $0 < |t| < 1$, we can introduce the conformal-block expansion

$$\begin{aligned}&\langle\Delta_\infty|V_{\alpha_1}(1)V_{\alpha_t}(t)\Phi_{2,1}(z)|\Delta_0\rangle = \\ &\sum_{\theta=\pm}\int \mathrm{d}\alpha\, C_{\alpha_{2,1}\alpha_0}^{\alpha_{0\theta}} C_{\alpha_t\alpha_{0\theta}}^{\alpha} C_{\alpha_\infty\alpha_1\alpha}\left|\mathfrak{F}\left({\alpha_1 \atop \alpha_\infty}\alpha^{\alpha_t}\alpha_{0\theta}{\alpha_{2,1} \atop \alpha_0};t,\frac{z}{t}\right)\right|^2,\end{aligned} \tag{1.1.16}$$

[3]It is commonly said thet Liouville CFT is solved, in the sense that the three-point functions admit explicit expressions known as *DOZZ formulae* [13, 14].

where

$$\alpha_\theta = \alpha - \theta\, \frac{b}{2}, \quad \theta = \pm 1, \tag{1.1.17}$$

and $\mathfrak{F}$ denotes the conformal blocks, and the norm squared reflects the fact that both holomorphic and anti-holomorphic contributions are considered. Since these give analogous contributions, we only consider the holomorphic ones in what follows. In the semiclassical limit,

$$b \to 0, \qquad \alpha_i \to \infty, \qquad b\,\alpha_i \equiv a_i \ \text{ finite}, \tag{1.1.18}$$

the divergence of the conformal blocks exponentiates [15] and the z-dependence becomes subleading:

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha^{\alpha_t}\alpha_{0\theta}\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\right) = t^{\Delta_\alpha - \Delta_{\alpha_t} - \Delta_{\alpha_{0\theta}}}\, z^{\frac{b^2+1}{2} + \theta\, b\, \alpha_0} \exp\left[\frac{1}{b^2}\left(F\left(t\right) + \mathcal{O}(b^2)\right)\right], \tag{1.1.19}$$

where F denotes the classical four-point conformal block and is related to the conformal block without the degenerate insertion by

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha\begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; t\right) = t^{\Delta_\alpha - \Delta_{\alpha_t} - \Delta_{\alpha_0}} \exp\left[\frac{1}{b^2}\left(F\left(t\right) + \mathcal{O}(b^2)\right)\right]. \tag{1.1.20}$$

The divergences in the conformal blocks can be cured by dividing the original conformal block by the one without the degenerate insertion. We denote the resulting finite, semiclassical conformal block by

$$\begin{aligned}&\mathcal{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha^{\alpha_t}\alpha_{0\theta}\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\right) = \\ &= \lim_{b\to 0} \frac{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha^{\alpha_t}\alpha_{0\theta}\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha\begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; t\right)} = t^{-\theta\, a_0} z^{\frac{1}{2} + \theta\, a_0} \exp\left[-\frac{\theta}{2}\partial_{a_0} F(t)\right] \cdot \left[1 + \mathcal{O}\left(t, \frac{z}{t}\right)\right].\end{aligned} \tag{1.1.21}$$

In the semiclassical limit, the BPZ equation satisfied by (1.1.15) becomes a Heun differential equation

$$\mathcal{D}_{\text{Heun}}\, \psi(z) = 0, \tag{1.1.22}$$

$$\mathcal{D}_{\text{Heun}} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} + \frac{-\frac{1}{2} - a_\infty^2 + a_0^2 + a_1^2 + a_t^2}{z(z-1)} + \frac{(t-1)\,u}{z(z-1)(z-t)},$$

where

$$u = t\, \frac{\partial\, F\left(t\right)}{\partial\, t}, \tag{1.1.23}$$

and a_0, a_1, a_t, a_∞ denote the indices of the singular points.

The Heun equation also enters the computation of surface operators in $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theory with $N_f \leq 4$ [16, 17, 18, 19]. This is not a coincidence. Indeed, there exists a correspondence between 2d Liouville CFT and 4d $\mathcal{N} = 2$ $SU(2)$

supersymmetric gauge theory in the so-called Ω-background, the *AGT correspondence* [20]. This makes a precise dictionary between quantities defined in the two theories, and, in particular, relates the 2d Liouville conformal blocks to the Nekrasov's instanton partition functions [21, 22].

Thanks to AGT correspondence, the connection formulae for the Heun equation and its confluence forms obtained in [4] have an explicit dependence on special functions, namely the *Nekrasov-Shatashvili* (NS) *functions* [23], which are obtained performing a suitable limit in the Ω-background parameters and admit a structure defined by a series expansion in t. We describe the conventions used for these functions in Appendix A. It has been shown that these functions are building blocks to compute quantum periods [24, 25, 26, 27, 28], eigenfunctions [29, 30, 31, 32, 33, 34], and Fredholm determinants [35, 36].

Being defined as series expansions, a natural question concerns the convergence properties of these special functions. These are currently unknown in full generality but several studies on the convergence of Nekrasov's instanton functions have been conducted with different methods and for specific ranges of the gauge parameters [37, 38, 39, 40, 41, 42]. We describe in Chapter 4 how estimates for the convergence radius under some genericity assumptions on the Ω-background parameters can be obtained from known combinatorial results.

The NS functions were initially applied to studying spectral problems describing black hole perturbation theory in [43], where QNMs in four-dimensional asymptotically flat black holes were considered. The approach has then been generalized to various gravitational backgrounds and extends beyond the QNMs computation [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72]. Also, a related approach based on Painlevé equations has been developed in [73, 74, 75, 76, 77, 78, 79, 80, 81, 82]. Other interesting related results have been elaborated in [83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105].

The techniques arising from supersymmetric gauge theory and Liouville CFT have a wide range of applicability in black hole perturbation theory. However, depending on the specific gravitational dictionary and the imposed boundary conditions, there are cases in which the method is less effective. For this reason, in Chapter 2 we also apply an alternative perturbative method [4]. This method is similar to Hamiltonian perturbation theory, and although the numerical implementation of this algorithm is well known (see e.g. [46, 110, 111]), the analytic computation becomes quickly quite complicated.

For the problems at hand, the following Ansatz for the eigenfunctions is introduced in local regions of the punctured sphere on which the differential equation is defined

$$\psi(z) = \psi_0(z) + \sum_{k\geq 1} \psi_k(z)\kappa^k, \tag{1.1.24}$$

where κ is some suitably chosen expansion parameter. At each order in κ, $\psi(z)$ is

[4] We remark that similar methods were already developed, for example, the *method of matched asymptotic expansions* (see [106] for a review). An important application to black hole problems can be found in [107, 108, 109], whose technique became known as the *MST method*.

determined by a second-order equation

$$\psi_k''(z) + \varphi(z)\,\psi_k'(z) + \nu(z)\,\psi_k(z) + \eta_k(z) = 0, \tag{1.1.25}$$

which we solve by using the method of variation of parameters. The functions φ and ν in (1.1.25) are known from the original differential equation,[5] and the non-homogeneous part of the differential equation $\eta_k(z)$ is fully determined by the solutions to the previous orders $\psi_m(z)$ with $m \leq k-1$. Let $\psi_0(z), g_0(z)$ be the two solutions of the homogeneous part of (1.1.25).[6] Then, we write the generic solution to (1.1.25) as[7]

$$\psi_k(z) = c_k\, g_0(z) - g_0(z) \int^z \psi_0(z')\, \frac{\eta_k(z')}{W_0(z')}\, \mathrm{d}z' + \psi_0(z) \int^z g_0(z')\, \frac{\eta_k(z')}{W_0(z')}\, \mathrm{d}z', \tag{1.1.26}$$

where $W_0(z)$ is the Wronskian between the two leading order solutions

$$W_0 \equiv \psi_0(z)\, g_0'(z) - \psi_0'(z)\, g_0(z). \tag{1.1.27}$$

Imposing the two boundary conditions and gluing the solutions in the different local regions fixes the integration constants c_k and provides the quantization of the frequency of the perturbation.

In the case of asymptotically (anti-)de Sitter spacetime, only regular singularities appear in the differential equation, with the number of singular points depending on the specific type of perturbation. For the spectral problems we are interested in and which we describe in Sec. 2.1-2.2-2.3, we focus on cases with four and five singularities. In all the analyzed cases, the leading order solutions are described in terms of rational or logarithmic functions, and their Wronskian is a rational function. Hence, the wave function at order κ^k is described in terms of *multiple polylogarithms* of weight k and lower [112, 113].

For the kind of perturbations studied in Sec. 2.1 and Sec. 2.2, the wave function can be determined in terms of multiple polylogarithms in a single variable

$$\mathrm{Li}_{s_1,\ldots,s_n}(z) = \sum_{k_1 > k_2 > \cdots > k_n \geq 1}^{\infty} \frac{z^{k_1}}{k_1^{s_1} \ldots k_n^{s_n}}. \tag{1.1.28}$$

The positive integer n is called *level*, and the sum $s_1 + \cdots + s_n$ is called *weight*.

The multiple polylogarithms satisfy an integral recurrence relation, that differs according to the value of the first weight s_1. For $s_1 \geq 2$, one has

$$z\,\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{s_1,\ldots,s_n}(z) = \mathrm{Li}_{s_1-1,\ldots,s_n}(z) \tag{1.1.29}$$

[5]The wave equation is understood to be Taylor expanded as $\psi''(z) + \sum_{k=0}^{\infty} \kappa^k (\varphi_k(z)\psi'(z) + \nu_k(z)\psi(z)) = 0$, so that one finds explicitly $\eta_k(z) = \sum_{m=1}^{k} (\varphi_m(z)\psi_{k-m}'(z) + \nu_m(z)\psi_{k-m}(z))$. In the text, $\varphi_0(z) = \varphi(z)$ and $\nu_0(z) = \nu(z)$.

[6]These are the two solutions of the leading order differential equation, $g_0(z)$ being the one that does not satisfy the relevant boundary condition.

[7]The integrals appearing in (1.1.26) are the indefinite ones.

and, for $s_1 = 1$,

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{1,s_2,\ldots,s_n}(z) = \mathrm{Li}_{s_2,\ldots,s_n}(z). \tag{1.1.30}$$

For the gravitational perturbation studied in Sec. 2.3, the wave function can be determined in terms of multiple polylogarithms in several variables

$$\mathrm{Li}_{s_1,\ldots,s_n}(z_1,\ldots,z_n) = \sum_{k_1>k_2>\cdots>k_n\geq 1}^{\infty} \frac{z_1^{k_1}\ldots z_n^{k_n}}{k_1^{s_1}\ldots k_n^{s_n}}. \tag{1.1.31}$$

The integral recurrence relations satisfied by these functions are

$$z_1\,\partial_{z_1}\mathrm{Li}_{s_1,\ldots,s_n}(z_1,\ldots,z_n) = \mathrm{Li}_{s_1-1,\ldots,s_n}(z_1,\ldots,z_n) \tag{1.1.32}$$

for $s_1 > 1$, and

$$(1-z_1)\partial_{z_1}\mathrm{Li}_{1,s_2,\ldots,s_n}(z_1,\ldots,z_n) = \mathrm{Li}_{s_2,\ldots,s_n}(z_1 z_2, z_3,\ldots,z_n). \tag{1.1.33}$$

for $s_1 = 1$.

We refer to Appendix B for the needed properties of these special functions.

In the case of asymptotically flat black holes, one of the boundary conditions is imposed at spatial infinity, which is an irregular singularity of the differential equation. Around this point, which in our conventions corresponds to $z = \infty$, we encounter leading order solutions of the ODE which are given by products of rational functions and exponential functions. In the first order of perturbation, the solution of the ODE involves the exponential integral

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{\mathrm{e}^t}{t}\mathrm{d}t. \tag{1.1.34}$$

This function can also be written as a Taylor series expansion around $z = 0$ plus a logarithm:

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty}\frac{z^k}{k!\,k}, \quad |\mathrm{Arg}(-z)| < \pi, \tag{1.1.35}$$

where γ denotes the Euler-Mascheroni constant. When computing the next orders of perturbation, we need to take further integrations, and an iterative structure similar to the one defining the multiple polylogarithms appears.

Therefore, in Appendix C, we define functions that generalize the exponential integral as multiple polylogarithms do for the logarithm. For studies on related functions see [114, 115, 116, 117, 118, 119, 120]. We are interested in the properties of the multiple polyexponential functions both for $z \to \infty$, where, in the black hole problem, the boundary condition is imposed, and for $z = 0$, where the local solution is required to glue continuously with the local solution of the nearby local region.

Taking inspiration from the recursive integral structure of the multiple polylogarithms, we define the set of *multiple polyexponential integrals* as

$$\begin{aligned} \mathrm{ELi}_{1,s_2,\ldots,s_n}(z) &= -\int_{-\infty}^{z}\frac{\mathrm{e}^t}{t}\,\mathrm{ELi}_{s_2,\ldots,s_n}(-t)\,\mathrm{d}t, \\ s_1 > 1: \quad \mathrm{ELi}_{s_1,\ldots,s_n}(z) &= \int_{-\infty}^{z}\frac{1}{t}\,\mathrm{ELi}_{s_1-1,s_2,\ldots,s_n}(t)\,\mathrm{d}t, \end{aligned} \tag{1.1.36}$$

where $n, s_1, \dots, s_n \in \mathbb{Z}_{>0}$ and the starting point is

$$\mathrm{ELi}_1(z) = \mathrm{Ei}(z). \tag{1.1.37}$$

The integer n is called *level* and the sum $s_1+\cdots+s_n$ is called *weight*. These functions have an explicit asymptotic behavior when $z \to \infty$ and satisfy the following simple recursive derivative relations:

$$z\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{ELi}_{s_1,s_2,\dots,s_n}(z) = \begin{cases} -\mathrm{e}^z\,\mathrm{ELi}_{s_2,\dots,s_n}(-z) & s_1 = 1, \\ \mathrm{ELi}_{s_1-1,s_2,\dots,s_n}(z) & s_1 > 1. \end{cases} \tag{1.1.38}$$

To make contact with the behavior at $z = 0$, we will generalize the Taylor series in (1.1.35) and define the set of *undressed multiple polyexponential functions*:

$$el_{s_1,s_2\dots,s_n}(z) = \sum_{k_1>k_2>\dots>k_n\geq 1} \frac{1}{k_1^{s_1}k_2^{s_2}\dots k_n^{s_n}}\frac{z^{k_1}}{k_1!}, \tag{1.1.39}$$

where again $n, s_1, \dots, s_n \in \mathbb{Z}_{>0}$.

From this definition, it can be proved that the recursive differential relation satisfied by the undressed multiple polyexponential functions is not as simple as (1.1.38) and involves sums over ordered partitions of integers.

As a consequence, it is not straightforward to find the relations between the undressed multiple polyexponential functions and the multiple polyexponential integrals. However, it is possible to define a new set of functions, that we call *dressed multiple polyexponential functions*, and we denote with $\mathrm{EL}_{s_1,\dots,s_n}(z)$, starting from (sums of) the undressed ones that satisfy the integral recurrence relation analogous to (1.1.38):

$$z\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{EL}_{s_1,s_2,\dots,s_n}(z) = \begin{cases} -\mathrm{e}^z\,\mathrm{EL}_{s_2,\dots,s_n}(-z) & s_1 = 1, \\ \mathrm{EL}_{s_1-1,s_2,\dots,s_n}(z) & s_1 > 1. \end{cases} \tag{1.1.40}$$

When the level is equal to 1, these simply coincide with the corresponding undressed functions

$$\mathrm{EL}_s(z) = el_s(z), \quad \text{for all } s \in \mathbb{Z}_{>0}. \tag{1.1.41}$$

The first difference appears in the definition of $\mathrm{EL}_{1,s}(z)$. By noticing that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{\mathrm{op}(s)} el_{1,\mathrm{op}(s)}(z) = -\sum_{\mathrm{op}(s)} el_{\mathrm{op}(s)}(z) - \mathrm{e}^z el_s(-z), \tag{1.1.42}$$

we define the functions

$$\mathrm{EL}_{1,s}(z) \equiv \sum_{\mathrm{op}(s+1)} el_{\mathrm{op}(s+1)}(z), \tag{1.1.43}$$

that satisfy

$$z\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{EL}_{1,s}(z) = -\mathrm{e}^z\mathrm{EL}_s(-z). \tag{1.1.44}$$

Defining the dressed functions with higher levels in such a way that the recursions (1.1.40) are satisfied, it is possible to determine the relations between the EL functions with the undressed ones analogous to (1.1.43). From these relations, it is possible to provide the Taylor series expansion of the dressed multiple polyexponential functions around $z = 0$.

Moreover, starting from the relation (1.1.35), which we can rewrite with the new notations as

$$\mathrm{ELi}_1(z) = \gamma + \log(-z) + \mathrm{EL}_1(z), \tag{1.1.45}$$

the validity of the integral recurrence relations (1.1.40) and (1.1.38) also makes it possible to write the relations between the dressed multiple polyexponential functions and the multiple polyexponential integrals. In particular, the general relation depends on coefficients whose structure is determined by multiple zeta values (MZVs). For useful properties of the MZVs, see [121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137]. We discuss the three different sets of functions in Appendix C. This will enable us to control the behavior of the solution of the original ODE both around $z = 0$ and in the asymptotic region $z \to \infty$.

The study of QNMs is strictly related to the computation of the one-loop Euclidean quantum actions. A general formalism for the study of the quantum effective actions in a black hole background in Euclidean quantum gravity [138] was settled in [139], where a formula for the computation of determinants in thermal spacetimes in terms of QNMs was proposed. The results were then interpreted in Lorentzian geometry in [140, 141]. So far, explicit calculations were restricted to three-dimensional black holes or more in general problems reducible to Hypergeometric operators. In Chapter 3, we devise a method to compute the one-loop determinants directly from the connection coefficients of the Heun equations by generalizing the Gelfand-Yaglom method [142] to differential operators with regular singularities.

We perform the full computations in the cases of scalar perturbations in Kerr-de Sitter black hole in four dimensions, in Schwarzschild-de Sitter black hole in four dimensions, and in Schwarzschild-anti-de Sitter in five dimensions, where the radial problem is encoded in a Heun differential equation and the boundary conditions are imposed in regular singular points of the problem. The application of the Gelfand-Yaglom method to higher dimensional differential operators admitting the separation of variables is not a novelty, as it has been studied in [143]. Similar formulae for the computation of determinants of differential operators with regular singularities appear also in [144].

Our analysis gives closed formulae for the determinants leading to the one-loop effective action and the rule to compute their spectrum from the NS function of the quantum integrable system associated with the specific Heun equation arising in the gravitational problem. The expressions we find make explicit the analytic properties of the one-loop effective actions with respect to the gravitational parameters and the precise contributions of the QNMs.

We conclude with Chapter 5 drawing the conclusions, highlighting the main results, and pointing to open problems and further outlook.

Chapter 2

Black hole linear perturbations and QNMs

In this chapter, we discuss linear perturbations around different four-dimensional spacetimes. We first study the Schwarzschild-de Sitter black holes, where we find the quasinormal mode frequencies by applying the two methods described in the introduction. We then discuss the Schwarzschild-anti-de Sitter case, for which the gauge theory method is less effective. Finally, we study the asymptotically flat Schwarzschild black hole in the small frequency expansion, comparing the results with the gauge theory method analyzed in [43].

2.1 Perturbations of de Sitter black holes in four dimensions

2.1.1 Schwarzschild de Sitter black hole

The metric describing the de Sitter Schwarzschild black hole in four dimensions (SdS_4) is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{2.1.1}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2, \tag{2.1.2}$$

where M is the mass of the black hole and $\Lambda > 0$ is the cosmological constant. In what follows, we fix $\Lambda = 3$, and then we suppose M to be in the range $0 < M^2 < 1/27$ to have three real roots for the equation $rf(r) = 0$, since otherwise we would have unphysical solutions. We denote these roots by

$$R_h, \quad R_\pm, \tag{2.1.3}$$

where $R_h \in]0, 1/\sqrt{3}[$ is the smallest positive real root, and $R_\pm$ are real and given in terms of R_h by

$$R_\pm = \frac{-R_h \pm \sqrt{4 - 3R_h^2}}{2}. \tag{2.1.4}$$

We study a class of linear perturbations of the SdS_4 geometry with spin $s \in \{0, 1, 2\}$, encoded in the following radial equation (see [145] and reference therein)

$$\left(\partial_r^2 + \frac{f'(r)}{f(r)} \partial_r + \frac{\omega^2 - V(r)}{f(r)^2} \right) R(r) = 0, \tag{2.1.5}$$

where the potential is

$$V(r) = f(r) \left[\frac{\ell(\ell+1)}{r^2} + (1 - s^2) \left(\frac{2M}{r^3} \right) \right]. \tag{2.1.6}$$

For $s = 0$, this equation describes conformally coupled scalar perturbations; for $s = 1$, electromagnetic perturbations; and for $s = 2$, odd (Regge–Wheeler or vector-type) gravitational perturbations.

The boundary conditions we impose on the wave function are the presence of only ingoing modes at the event horizon R_h and the presence of only outgoing modes at the cosmological horizon R_+. These conditions can be made explicit by introducing the *tortoise coordinate* r_* defined by

$$\mathrm{d}r_* = \frac{\mathrm{d}r}{f(r)}. \tag{2.1.7}$$

In terms of r_*, the behavior of $R(r)$ near R_h, R_+ is described by plane waves, so we ask that $R(r)$ behaves as $\exp(-i\omega r_*)$ for $r \sim R_h$ and as $\exp(i\omega r_*)$ for $r \sim R_+$. The latter radial equation apparently has five regular singular points located at $r = \{0, R_h, R_\pm, \infty\}$. However, as pointed out in [146], under the change of variable

$$z(r) = \frac{r(R_+ - R_-)}{R_+(r - R_-)}, \tag{2.1.8}$$

and redefinition of the wave function

$$\psi(z) = z^{-\gamma/2}(z-1)^{-\delta/2}(z-t)^{-\epsilon/2}\sqrt{f(r)}\,\frac{R_-(R_+ - R_-)}{R_+(r - R_-)}\,R(r), \tag{2.1.9}$$

where

$$\begin{aligned} t &= \frac{R_h(R_- - R_+)}{R_+(R_- - R_h)}, \\ \gamma &= 1 - 2s, \\ \delta &= 1 - \frac{2i\,\omega\,R_+}{(R_+ - R_h)(R_+ - R_-)}, \\ \epsilon &= 1 + \frac{2i\omega R_h}{1 - 3R_h^2}, \end{aligned} \tag{2.1.10}$$

the singularity at infinity is removed, and the equation becomes a Heun equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\,\beta\,z - q}{z\,(z-1)\,(z-t)} \right] \psi(z) = 0, \tag{2.1.11}$$

with

$$
\begin{aligned}
\alpha &= 1 - s + \frac{2i\,\omega\,R_-}{(R_- - R_h)(R_- - R_+)},\\
\beta &= 1 - s,\\
q &= \frac{\ell(\ell+1)}{R_+(R_- - R_h)} + \frac{(1-s)^2 R_h}{R_h - R_-} - \frac{s(1-s)R_-^2}{R_+(R_h - R_-)}.
\end{aligned}
\tag{2.1.12}
$$

In the z coordinate, the horizon $r = R_h$ is mapped to $z = t$, the cosmological horizons $r = R_\pm$ are mapped to $z = 1$ and $z = \infty$, respectively, while the origin, $r = 0$, is mapped to $z = 0$.

The boundary conditions described for $R(r)$ imply the following behaviors for the function ψ:

$$
\begin{aligned}
&\psi(z) \sim 1 && \text{for} \quad z \sim 1,\\
&\psi(z) \sim (z-t)^{1-\epsilon} && \text{for} \quad z \sim t.
\end{aligned}
\tag{2.1.13}
$$

We now want to obtain the analytic formula from which the quasinormal modes can be computed in the limit where t is small, $0 < t \ll 1$, or, equivalently, R_h is small, $R_h \ll 1$. For this purpose, we write the following dictionary for the gauge parameters in terms of Heun's parameters and gravitational quantities (see appendix A for the conventions used):

$$
\begin{aligned}
t &= \frac{R_h(R_- - R_+)}{R_+(R_- - R_h)} \ ,\\
a_0 &= \frac{1-\gamma}{2} = s \ ,\\
a_1 &= \frac{1-\delta}{2} = \frac{i\,\omega\,R_+}{(R_+ - R_h)(R_+ - R_-)} \ ,\\
a_t &= \frac{1-\epsilon}{2} = -\frac{i\omega R_h}{1 - 3R_h^2} \ ,\\
a_\infty &= \frac{\alpha - \beta}{2} = \frac{i\,\omega\,R_-}{(R_- - R_h)(R_- - R_+)} \ ,\\
u &= \frac{-2q + 2t\alpha\beta + \gamma\epsilon - t(\gamma+\delta)\epsilon}{2(t-1)} \ .
\end{aligned}
\tag{2.1.14}
$$

Connection Problem

The computation of quasinormal mode frequencies is obtained by imposing purely ingoing boundary conditions at the event horizon $z = t$ and purely outgoing at the positive cosmological horizon $z = 1$. The independent solutions of the Heun equation for $z \sim t$

are

$$
\begin{aligned}
\psi_-^{(t)}(z) &= \text{Heun}\left(\frac{t}{t-1}, \frac{q-t\alpha\beta}{1-t}, \alpha, \beta, \epsilon, \delta, \frac{z-t}{1-t}\right),\\
\psi_+^{(t)}(z) &= (z-t)^{1-\epsilon}\text{Heun}\left(\frac{t}{t-1}, \frac{q-(\beta-\gamma-\delta)(\alpha-\gamma-\delta)t-\gamma(\epsilon-1)}{1-t},\right.\\
&\qquad \left. -\alpha+\gamma+\delta, -\beta+\gamma+\delta, 2-\epsilon, \delta, \frac{z-t}{1-t}\right),
\end{aligned}
\tag{2.1.15}
$$

and the ones for $z \sim 1$ are

$$
\begin{aligned}
\psi_-^{(1)}(z) =& \left(\frac{z-t}{1-t}\right)^{-\alpha}\text{Heun}\left(t, q+\alpha(\delta-\beta), \alpha, \delta+\gamma-\beta, \delta, \gamma, t\frac{1-z}{t-z}\right),\\
\psi_+^{(1)}(z) =& (z-1)^{1-\delta}\left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta}\text{Heun}\left(t, q-(\delta-1)\gamma t-(\beta-1)(\alpha-\delta+1),\right.\\
&\left. -\beta+\gamma+1, \alpha-\delta+1, 2-\delta, \gamma, t\frac{1-z}{t-z}\right).
\end{aligned}
\tag{2.1.16}
$$

Taking into account the boundary conditions (2.1.13), the connection coefficient between $\psi_+^{(t)}$ and $\psi_+^{(1)}$ has to be set equal to zero. Indeed, after selecting the local solution $\psi_+^{(t)}$ around the black hole horizon, we can rewrite it around $z=1$ as

$$
\psi_+^{(t)} = \mathcal{C}_{t+}^{1+}\,\psi_+^{(1)} + \mathcal{C}_{t+}^{1-}\,\psi_-^{(1)}, \tag{2.1.17}
$$

where $\mathcal{C}_{t+}^{1+}, \mathcal{C}_{t+}^{1-}$ denote the connection coefficients. To select the local solution $\psi_-^{(1)}$ it is therefore sufficient to impose $\mathcal{C}_{t+}^{1+} = 0$.

In terms of the connection matrices of hypergeometric functions

$$
\mathcal{M}_{\theta\theta'}(a_1, a_2; a_3) = \frac{\Gamma(-2\theta' a_2)\Gamma(1+2\theta a_1)}{\Gamma\left(\frac{1}{2}+\theta a_1-\theta' a_2+a_3\right)\Gamma\left(\frac{1}{2}+\theta a_1-\theta' a_2-a_3\right)}, \tag{2.1.18}
$$

where $\theta, \theta' = \pm$, the connection formula for small t from $z \sim t$ to $z \sim 1$ is given by

$$
\begin{aligned}
&t^{-\frac{1}{2}+a_0-a_t}(1-t)^{-\frac{1}{2}+a_1}e^{-\frac{1}{2}\partial_{a_t}F(t)}\psi_+^{(t)}(z) =\\
&\sum_{\sigma=\pm}\mathcal{M}_{+\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)-}(a, a_1; a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F(t)}(1-t)^{a_t-\frac{1}{2}}e^{i\pi(a_1+a_t)+\frac{1}{2}\partial_{a_1}F(t)}\psi_-^{(1)}(z)+\\
&\sum_{\sigma=\pm}\mathcal{M}_{+\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)+}(a, a_1; a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F(t)}(1-t)^{a_t-\frac{1}{2}}e^{i\pi(-a_1+a_t)-\frac{1}{2}\partial_{a_1}F(t)}\psi_+^{(1)}(z).
\end{aligned}
\tag{2.1.19}
$$

This leads us to the quantization condition for the quasinormal modes in the form

$$
\sum_{\sigma=\pm}\mathcal{M}_{+\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)+}(a, a_1; a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F(t)} = 0, \tag{2.1.20}
$$

which can be rewritten as

$$t^{-2a}e^{\partial_a F(t)}\,\frac{\Gamma(1+2a)^2}{\Gamma(1-2a)^2}\prod_{\theta_1,\theta_2=\pm}\frac{\Gamma(\frac{1}{2}-a+a_t+\theta_1\,a_0)\Gamma(\frac{1}{2}-a-a_1+\theta_2\,a_\infty)}{\Gamma(\frac{1}{2}+a+a_t+\theta_1\,a_0)\Gamma(\frac{1}{2}+a-a_1+\theta_2\,a_\infty)}=1. \quad (2.1.21)$$

Note that this is nothing but

$$\exp\left(\partial_a F_{\text{full}}(t)\right)=1, \quad (2.1.22)$$

where $F_{\text{full}}(t)$ is the full NS free energy, since the ratio of Gamma functions in (2.1.22) represents the 1-loop corrections.

QNMs at large ℓ

The previous quantization condition gets simplified in the large ℓ limit, where we neglect non-perturbative effects in ℓ of the form R_h^ℓ. This regime was studied for AdS_5 black holes in [147, 59], since in this limit, the quasinormal mode frequencies become real, and, via the AdS/CFT correspondence, they compute the dimensions of certain operators in the holographic conformal field theory, see [148, 149, 150, 151, 152, 153, 154, 155] and references therein. In the de Sitter case, in this regime, the quasinormal mode frequencies are purely imaginary, and their interpretation from the point of view of holography is, at present, less clear.

In the leading order in R_h, $a \sim \pm\left(\ell+\frac{1}{2}\right)$. Choosing the plus sign, the quantization condition

$$\sum_{\sigma=\pm}\mathcal{M}_{+\sigma}(a_t,a;a_0)\mathcal{M}_{(-\sigma)+}(a,a_1;a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F(t)}=0 \quad (2.1.23)$$

simplifies to

$$\mathcal{M}_{+-}(a_t,a;a_0)\mathcal{M}_{++}(a,a_1;a_\infty)t^{-a}e^{\frac{1}{2}\partial_a F(t)}=0, \quad (2.1.24)$$

since the other term is exponentially suppressed. This condition is satisfied if and only if

$$\frac{\Gamma(2a)\Gamma(1-2a_t)\Gamma(1+2a)\Gamma(-2a_1)}{\Gamma(\frac{1}{2}+a+a_t+a_0)\Gamma(\frac{1}{2}+a+a_t-a_0)\Gamma(\frac{1}{2}+a-a_1-a_\infty)\Gamma(\frac{1}{2}+a-a_1+a_\infty)}=0, \quad (2.1.25)$$

which is solved at the poles of the Gamma functions in the denominator. Only the last one admits poles among the four Gamma functions in the denominator, consistently with our regime $R_h \ll 1$. These are given by the condition

$$\frac{1}{2}+a-a_1+a_\infty=-n, \quad \text{with} \quad n\in\mathbb{Z}_{\geq 0}. \quad (2.1.26)$$

Expanding the parameters in R_h and writing ω as

$$\omega=\sum_{k=0}^{\infty}\omega_k R_h^k, \quad (2.1.27)$$

we obtain from this condition

$$\begin{aligned}
&\omega_0 = i(-\ell - n - 1);\\
&\omega_1 = 0;\\
&\omega_2 = -\frac{i}{8\ell(\ell+1)(2\ell+1)(2\ell-1)(2\ell+3)}\Big\{\ell^4\left(60n^2+60n+22\right)+\ell^3\big(120n^2+48ns^2\\
&\qquad +122n+24s^2+45\big)+\ell^2\left[8n^2\left(3s^2+2\right)+n\left(96s^2+19\right)+8s^4+44s^2+8\right]\\
&\qquad +\ell\left[4n^2\left(6s^2-11\right)+n\left(24s^4-43\right)+20s^4-4s^2-15\right]+12(n+1)^2s^2\left(s^2-2\right)\Big\};\\
&\omega_3 = 0;\\
&\qquad\vdots
\end{aligned}\tag{2.1.28}$$

Higher orders can also be computed systematically, but their expressions are cumbersome; hence we do not write them explicitly. Notice that in this limit, all the odd orders ω_{2k+1} seem to vanish. Moreover, these formulas are correct for finite ℓ up to order $R_h^{2\ell+1}$, as will be shown in Sec. 2.1.2. We also note that we expect the series (2.1.27) to be convergent in R_h, the need for non-perturbative effects in ℓ can be inferred from the fact that at higher orders this series develops some apparent poles in ℓ. For instance, for $s = 0$ a pole at $\ell = 0$ appears in the expression of ω_4:

$$\omega_4|_{s=0} = \frac{i(n+1)^4}{\ell} + \mathcal{O}\left(\ell^0\right) \ . \tag{2.1.29}$$

2.1.2 Perturbation theory around dS_4

QNMs in pure dS_4

The pure de Sitter case can be obtained by taking the limit $t \to 0$ or, equivalently, $R_h \to 0$. As the event horizon disappears in this limit, it is enough to consider only the region near the cosmological horizon $r = R_+$. In this limit, the Heun equation becomes a Hypergeometric equation, whose solutions are

$$z^{s-\ell-1}{}_2F_1(-\ell,-\ell-i\omega_0;-2\ell;z), \quad z^{\ell+s}{}_2F_1(\ell+1,\ell+1-i\omega_0;2\ell+2;z), \tag{2.1.30}$$

where ω_0 is the leading order term in the R_h expansion of the frequency (2.1.27). Since ℓ is a non-negative integer, the hypergeometric functions get truncated to polynomials as

$$\begin{aligned}
{}_2F_1\left(-\ell,-\ell-i\omega_0;-2\ell;z\right) &= \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{(-\ell-i\omega_0)_k}{(-2\ell)_k}\,z^k,\\
{}_2F_1\left(\ell+1,\ell+1-i\omega_0;2\ell+2;z\right) &= (-1)^\ell\left(\frac{z}{2}\right)^{-2\ell-1}\frac{((2\ell+1)!!)^2}{2\,(2\ell+1)}\frac{\Gamma\left(i\omega_0-\ell\right)}{\Gamma\left(i\omega_0+\ell+1\right)}\times\\
&\quad\times\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{(-\ell-i\omega_0)_k}{(-2\ell)_k}\left(1-(1-z)^{i\omega_0}\frac{(-\ell+i\omega_0)_k}{(-\ell-i\omega_0)_k}\right)z^k.
\end{aligned}\tag{2.1.31}$$

The boundary conditions require that the radial part of the gravitational perturbation $R(r)$ is well-defined as $r \to 0$. Using the dictionary for the wave function (2.1.9), we rewrite the latter requirement in terms of $\psi(z)$:

$$z^{\gamma/2}\psi(z) = z^{-s+1/2}\psi(z) \sim 1 \quad \text{for} \quad z \sim 0. \tag{2.1.32}$$

Thus, we have to pick a regular solution at $z \sim 0$ and consider an additional factor of $z^{-s+1/2}$. Looking at the first solution from (2.1.30), we can see that $z^{-\ell-1/2}{}_2F_1(-\ell, -\ell - i\omega_0; -2\ell; z)$ is not regular at $z \sim 0$ for any allowed value of ℓ. Indeed, the other combination gives the solution, which is regular at $z \sim 0$:

$$z^{\ell+1/2}{}_2F_1(\ell+1, \ell+1-i\omega_0; 2\ell+2; z) \sim z^{\ell+1/2} \sim 0. \tag{2.1.33}$$

In addition, the boundary conditions at the cosmological horizon require the eigenfunction to be regular with a well-defined Taylor expansion at $z = 1$. This is possible only if $i\omega_0 \in \mathbb{Z}_{\geq 0}$ (due to the term $(1-z)^{i\omega_0}$ in (2.1.31)). Moreover, to avoid the poles in the Gamma functions in (2.1.31):

$$\frac{\Gamma(i\omega_0 - \ell)}{\Gamma(i\omega_0 + \ell + 1)} = \prod_{k=-\ell}^{\ell} (i\omega_0 - k)^{-1}, \tag{2.1.34}$$

we must exclude all the values of $i\omega_0$ that are smaller or equal to ℓ (these poles indicate that the second expression in (2.1.31) have to be rewritten in terms of $\log(z-1)$ for $i\omega_0 = \ell, \ell-1, \ldots, -\ell+1, -\ell$). This gives the well-known quantization condition for the QNM frequencies of the pure dS_4:

$$i\omega_0 = \ell + n + 1, \quad \text{with } n \in \mathbb{Z}_{\geq 0}. \tag{2.1.35}$$

The corresponding eigenfunction is

$$f_0^L(z) = z^{\ell+s} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\ell+1)_k}{(2\ell+2)_k} z^k . \tag{2.1.36}$$

We also note that the discarded solution is

$$g_0^L(z) = z^{s-\ell-1}(1-z)^{\ell+n+1} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \frac{(n+1)_k}{(-2\ell)_k} z^k . \tag{2.1.37}$$

The Wronskian between f_0^L and g_0^L is

$$W_0^L(z) = -(2\ell+1) z^{2s-2} (1-z)^{\ell+n}. \tag{2.1.38}$$

Left Region

Here we call the region near the cosmological horizon $r = R_+$ left region due to the analogy with the corresponding quantum mechanical problem on the complex plane. The local variable in this region is z, and the leading order solutions in R_h (and so in t) of the Heun equation (2.1.11) are given in (2.1.36), (2.1.37). Expanding in small R_h the solution and the frequencies we get for the outgoing solution $\psi_-^{(1)}$ at the cosmological horizon

$$\psi_-^{(1)}(z) = \frac{\ell!\,(2\ell+n+1)!}{(2\ell+1)!\,(\ell+n)!} f_0(z) + \frac{(-2)^\ell\, i\omega_1}{(i\omega_1+\ell+n+1)} \frac{n!\,(2\ell-1)!!}{(\ell+n)!} g_0(z) + \mathcal{O}(R_h)\,, \tag{2.1.39}$$

where ω_1 is a coefficient in the R_h expansion of the frequency (2.1.27). Since $g_0(z)$ blows up as $z \to 0$, it should not be present in the leading order of the wave function in the left region. Hence, we require $\omega_1 = 0$. On the other hand, the incoming wave solution at the cosmological horizon is

$$\psi_+^{(1)}(z) \sim (z-1)^{i\omega}\left(1 + i\omega\log(z-1)\,R_h\right) + \mathcal{O}\left(R_h^2\right). \tag{2.1.40}$$

After we fix $\omega_1 = 0$ and proceed with the perturbative method, the logarithm function $\log(z-1)$ appears in higher orders in R_h (and t). The only source of this function is the incoming wave solution (2.1.40), and we will be canceling any contributions of $\log(z-1)$ by fixing the coefficients c_K in the perturbative expansion of the wave function (1.1.24), (1.1.26).

After establishing the boundary condition, we compute the integrals in (1.1.26). As we show in Appendix B.2, these integrals are described in terms of the multiple polylogarithms in a single variable:

$$\mathrm{Li}_{s_1,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1}^{\infty} \frac{z^{k_1}}{k_1^{s_1}\ldots k_n^{s_n}}. \tag{2.1.41}$$

The latter admits for $s_1 \geq 2$:

$$z\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{s_1,\ldots,s_n}(z) = \mathrm{Li}_{s_1-1,\ldots,s_n}(z) \tag{2.1.42}$$

and for $s_1 = 1$, $n \geq 2$:

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{1,s_2,\ldots,s_n}(z) = \mathrm{Li}_{s_2,\ldots,s_n}(z)\,. \tag{2.1.43}$$

The *weight* of the multiple polylogarithm $\mathrm{Li}_{s_1,\ldots,s_n}(z)$ is $s_1 + \cdots + s_n$, and the *level* is n. At each order t^{k+1}, both integrands in (1.1.26) are linear combinations of the following terms with maximum weight k:

$$\frac{\sum_{m=0}^{r_1}\alpha_m\, z^m}{z^{i_1}\,(z-1)^{j_1}}\log(z)^{p_1}\,, \quad \frac{\sum_{m=0}^{r_2}\beta_m\, z^m}{z^{i_2}\,(z-1)^{j_2}}\mathrm{Li}_{s_1,\ldots,s_n}(1-z)\,, \tag{2.1.44}$$

where $r_{1,2}$, $i_{1,2}$, $j_{1,2}$, p_1 are some non-negative integers, and $0 \leq p_1 \leq k$, $s_1 + \cdots + s_n \leq k$. After taking the integrals, the only new functions that appear are multiple polylogarithms of maximum weight $k+1$. Moreover, both integrals in (1.1.26) are linear combinations of terms similar to (2.1.44):

$$\frac{\sum_{m=0}^{r_1+1} \gamma_m \, z^m}{z^{i_1-1} \left(z-1\right)^{j_1-1}} \log\left(z\right)^{p_1} , \quad \frac{\sum_{m=0}^{r_2+1} \delta_m \, z^m}{z^{i_2-1} \left(z-1\right)^{j_2-1}} \mathrm{Li}_{s_1,\ldots,s_n}\left(1-z\right) \tag{2.1.45}$$

and terms containing new combinations of logarithms and multiple polylogarithms that were not present in (2.1.44):

$$\log\left(z-1\right) , \quad \log\left(z\right)^{k+1} , \quad \mathrm{Li}_{\hat{s}_1,\ldots,\hat{s}_{\hat{n}}}\left(1-z\right) , \tag{2.1.46}$$

where the maximum weight is $k+1$:

$$\hat{s}_1 + \cdots + \hat{s}_{\hat{n}} \leq k+1. \tag{2.1.47}$$

One of the differences between (2.1.44) and (2.1.45) is that $r_{1,2}$, $i_{1,2}$, and $j_{1,2}$ are shifted by 1 or -1. These shifts are specific to the left region of the SdS_4 case (and even then may be subjected to reevaluation for some values of quantum numbers n, ℓ, and s that we did not consider). In the right region, the shifts are different but can be determined case by case. Even though the optimal choice of shifts depends on the case at hand, there is a choice of big enough shifts applicable to all quantum numbers for both regions.

To summarize, we reduced the problem of solving the initial ODE in a given order in t to a system of linear equations on the coefficients in front of the functions from (2.1.46) and γ_m, δ_m.[1] The resulting corrections $f_k^L\left(z\right)$ to the wave function in the left region are linear combinations of the following functions:

$$\sum_{m=-k_1}^{l_1} \zeta_m^L \, z^m \log\left(z\right)^{p_1} , \quad \sum_{m=-k_2}^{l_2} \xi_m^L \, z^m \, \mathrm{Li}_{s_1,\ldots,s_n}\left(1-z\right) , \tag{2.1.48}$$

where $k_{1,2}$, $l_{1,2}$, p_1 are some non-negative integers, $0 \leq p_1 \leq k$, $s_1 + \cdots + s_n \leq k$, and ζ_m^L, ξ_m^L are z-independent quantities.

Right Region

The right region is near the event horizon $r = R_h$, or $z = t$. We introduce the local variable $z^R = t/z$ so that the horizon is at $z^R = 1$. In the z^R variable, the equation (2.1.11) reads

$$\begin{aligned} &\frac{\mathrm{d}^2\psi(z^R)}{(\mathrm{d}z^R)^2} + \left(\frac{2-\gamma}{z^R} + \frac{\delta t}{z^R(z^R-t)} + \frac{\epsilon}{z^R(z^R-1)}\right)\frac{\mathrm{d}\psi(z^R)}{\mathrm{d}z^R} + \\ &+ \frac{\alpha\beta t - q z^R}{(z^R)^2(z^R-1)(z^R-t)}\psi(z^R) = 0. \end{aligned} \tag{2.1.49}$$

[1] Here we simplified the index structure of γ_m and δ_m, the full list of indices should be $\gamma_m(p_1)$ and $\delta_m(p_1, s_1, \ldots, s_k)$.

In the remaining part of this subsection, we will mostly omit the R index on the z variable (except for the cases where it could be confusing). We take as leading order solutions in R_h (and so in t) of this equation

$$\begin{aligned} f_0^R(z) =& z^{-\ell-s}{}_2F_1(-\ell-s,-\ell+s;-2\ell;z) = \\ =& z^{-\ell-s}\sum_{k=0}^{\ell-s}\frac{(s-\ell)_k\,(-\ell-s)_k}{(-2\ell)_k}\frac{z^k}{k!}, \\ g_0^R(z) =& z^{-s}\left\{\sum_{m=-s}^{\ell-1} a_{s\ell m}z^{-m} + \log(1-z)\sum_{m=s}^{\ell} b_{s\ell m}z^{-m}\right\}, \end{aligned} \tag{2.1.50}$$

with

$$\begin{aligned} b_{s\ell m} =& \frac{(-1)^{\ell+m+1}}{(m+s)!(m-s)!}\frac{(2\ell+1)!}{(\ell+s)!(\ell-s)!}\frac{(\ell+m)!}{(\ell-m)!}, \\ a_{s\ell m} =& -b_{s\ell m}(H_{\ell+s}+H_{\ell-s}-H_{m+s}-H_{m-s}). \end{aligned} \tag{2.1.51}$$

The Wronskian between f_0^R and g_0^R is

$$W_0^R(z) = \frac{2\ell+1}{z^{2s}(z-1)}. \tag{2.1.52}$$

Here we would like to comment on the choice of the logarithm function $\log\left(1-z^R\right)$ in the solution g_0^R. The other possible choice of the logarithm could be, for example, $\log\left(z^R-1\right)$. This choice dictates what functions will appear in higher orders in t and affects the R_h expansion of the frequency ω. Throughout the discussion, we work with the principal value of the complex logarithm, and thus the change in the argument affects the position of the branch cut on the complex z plane. Our wave function $\psi(z)$ can be viewed as an analytic continuation of the physical solution on half of the real line $r \geq 0$. In the de Sitter case, the coordinate transformation $z(r)$ is (2.1.8) with real parameters $R_\pm$. Since we want the solution to be continuous across the real slice $R_h < r < R_+$, the branch cut should not cross the interval $t < z^R < 1$, where t is small and positive. This leaves us with $\log\left(1-z^R\right)$, and the branch cut runs from $z^R = 1$ to $z^R = +\infty$. The other logarithm function that appears in higher orders in t is $\log\left(z^R\right)$, and the corresponding branch cut runs from $z^R = 0$ to $z^R = -\infty$ also avoiding the interval $t < z^R < 1$ (see Figure 2.1).

The boundary condition near the horizon requires us to keep the solution corresponding to the incoming wave and discard the one corresponding to the outgoing wave. According to (2.1.15), the two solutions behave like

$$\begin{aligned} &\text{outgoing wave}: \quad \psi_-^{(t)}\left(z^R\right) \sim 1, && z^R \sim 1, \\ &\text{incoming wave}: \quad \psi_+^{(t)}\left(z^R\right) \sim \left(1-z^R\right)^{1-\epsilon}, && z^R \sim 1. \end{aligned} \tag{2.1.53}$$

Since $1-\epsilon = \mathcal{O}(R_h)$, both waves in the right-hand side of (2.1.53) have Taylor expansions in R_h that start with 1. One must also consider the higher orders in R_h to distinguish

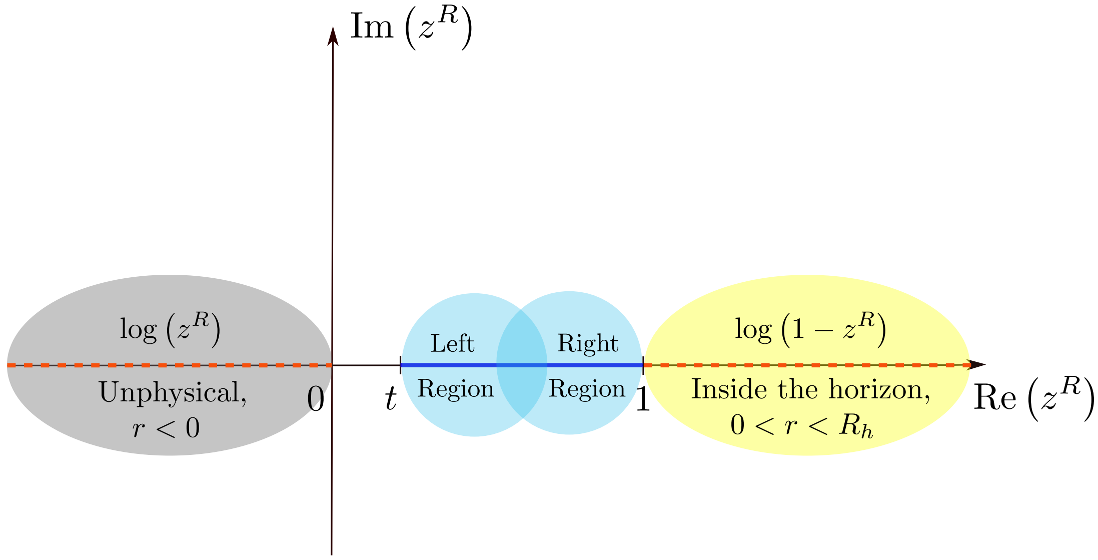

Figure 2.1: Branch cuts (dashed red lines) on the complex z^R plane for de Sitter black holes.

the two expansions. The incoming wave solution has a particular dependence on the logarithm function $\log(1-z)$ in each order in R_h (or t):

$$\psi_+^{(t)}(z) \sim 1 - 2i\omega_0 \log(1-z)\, R_h + \mathcal{O}\left(R_h^2\right), \quad z \sim 1. \tag{2.1.54}$$

In the leading order in R_h both $\psi_-^{(t)}$ and $\psi_+^{(t)}$ are given by the same function $f_0^R(z)$. Since the other function $g_0^R(z)$ contains the logarithm, it enters $\psi_+^{(t)}$ in the higher orders in R_h. The constants c_k from (1.1.26) are fixed by matching with the logarithmic behavior of the incoming wave solution (2.1.54) in each order in R_h.

The integrals in (1.1.26) are again described in terms of the multiple polylogarithms in a single variable (see Appendix B.2). We construct the linear basis of functions for each integral in the way it was done in the previous section for the left region. The only difference is that we need to add powers of the second logarithm function $\log(1-z)$ to formulas (2.1.44), (2.1.45) and (2.1.46). In particular, the second integrand from (1.1.26) at order t^k of the form

$$g_0^R(z)\,\frac{\eta_k^R(z)}{W_0^R(z)} \tag{2.1.55}$$

will have a maximum weight k because the logarithm function $\log(1-z)$ is present in the leading order solution $g_0^R(z)$. The resulting integral, however, will be of the same weight k due to the pole structure in (2.1.55). Eventually, the corrections $f_k^R(z)$ to the wave function in the right region are linear combinations of the following functions of maximum weight k:

$$
\begin{aligned}
&\sum_{m=-k_1}^{l_1} \zeta_m^R \, z^m \, \log\left(1-z\right)^{p_1} \log\left(z\right)^{p_2} \,, \\
&\sum_{m=-k_2}^{l_2} \xi_m^R \, z^m \, \log\left(1-z\right)^{p_3} \operatorname{Li}_{s_1,\ldots,s_n}\left(1-z\right) ,
\end{aligned}
\tag{2.1.56}
$$

where $k_{1,2}$, $l_{1,2}$, $p_{1,2,3}$ are some non-negative integers, and $0 \leq p_1 + p_2 \leq k$, $p_3 + s_1 + \cdots + s_n \leq k$.

Results for QNM frequencies

The final step in the perturbative procedure is to glue the local solutions by requiring that the wave function and its first derivative are continuous at the intersection of the two regions. There is a certain freedom in choosing the intersection point as long as it lies in the region of convergence of both local solutions. We choose the point $z = t^{1/2}$, which is the same as $z^R = t^{1/2}$. Note that the expansions of $\psi^{L,R}\left(z^{L,R}\right)$ are given as series expansions around $z^{L,R} = 1$ up to orders $t^{m_{L,R}}$:

$$
\begin{aligned}
\psi^L\left(z\right) =& f_0^L\left(z\right) + \sum_{k=1}^{m_L} f_k^L(z) t^k + O\left(t^{m_L+1}\right) , \\
\psi^R\left(z^R\right) =& f_0^R\left(z^R\right) + \sum_{k=1}^{m_R} f_k^R(z^R) t^k + O\left(t^{m_R+1}\right) .
\end{aligned}
\tag{2.1.57}
$$

What happens when we take $z^{L,R} \sim t^{1/2}$ and expand for a small t? Some terms $f_k^L(z)t^k$ in $\psi^L\left(z\right)$ will contribute to orders lower than t^k. This could lead to a reshuffling, where, for example, $f_1^L(z)t$ becomes the leading order contribution at $z \sim t^{1/2}$. This happens when $\ell \geq 1$, as seen from (2.1.36):

$$
\begin{aligned}
f_0^L\left(t^{1/2}\right) &\sim t^{(s+\ell)/2}, \\
f_1^L\left(t^{1/2}\right) t &\sim t^{(s-\ell+1)/2}.
\end{aligned}
\tag{2.1.58}
$$

However, since we are within the radius of convergence of $\psi^L\left(z\right)$, this reshuffling involves only a finite number of terms. For all values of quantum numbers we have considered, the reshuffling is superficial and goes away after the frequency is set to one of the quasinormal modes.

The continuity condition

$$
\partial_z \log\left(\frac{\psi^L(z)}{\psi^R(t/z)}\right)\Bigg|_{z=t^{1/2}} = 0 \tag{2.1.59}
$$

can be equivalently stated as

$$\begin{aligned} \psi^L\left(t^{1/2}\right) &= C(t)\,\psi^R\left(t^{1/2}\right), \\ \partial_z\psi^L(z)\Big|_{z=t^{1/2}} &= C(t)\,\partial_z\psi^R(t/z)\Big|_{z=t^{1/2}}, \end{aligned} \tag{2.1.60}$$

where $C(t)$ is a normalization factor. The advantage of (2.1.60) is that we can use one of the equations to understand which orders in t we can trust when expanding (2.1.57) at $z^{L,R} = t^{1/2}$, then use the other one to fix the frequencies.

Using `Mathematica`, we compute the local solutions up to orders $m_L = 10$ and $m_R = 7$. This allows us to determine the R_h expansion of the frequency up to order R_h^9 or less depending on the value of ℓ. In all computed orders, we find the real part of the quasinormal modes is zero, which agrees with the earlier observations made by numerical computations [156, 157, 158]. The results for the imaginary part of the quasinormal mode frequencies $\omega_{n,\ell,s}$, starting from $n = 0$, are

$$\begin{aligned} \mathrm{Im}\left(\omega_{0,0,0}\right) = &-1 - \frac{5}{8}R_h^2 - 3R_h^3 - \left[\frac{1287}{128} + 2\log\left(2R_h\right)\right]R_h^4 + \left[\pi^2 - \frac{119}{4} - 15\log(2R_h)\right]R_h^5 \\ &+ \left[\frac{25}{3}\pi^2 - \frac{102\,621}{1024} - \frac{271}{4}\log\left(2R_h\right) - 5\log^2(2R_h) + 6\,\zeta(3)\right]R_h^6 + \mathcal{O}\left(R_h^7\right), \\ \mathrm{Im}\left(\omega_{0,1,1}\right) = &-2 - \frac{7}{12}R_h^2 + \frac{7123}{1728}R_h^4 + 8\,R_h^5 + \left[\frac{2\,757\,809}{124\,416} + \frac{32}{3}\log\left(2R_h\right)\right]R_h^6 \\ &- \frac{4}{27}\left[13 + 72\,\pi^2 - 468\log(2R_h)\right]R_h^7 + \mathcal{O}\left(R_h^8\right), \\ \mathrm{Im}\left(\omega_{0,2,2}\right) = &-3 - \frac{27}{40}R_h^2 + \frac{51\,423}{16\,000}R_h^4 - \frac{72\,333\,747}{3\,200\,000}R_h^6 - \frac{72}{5}R_h^7 + \left[\frac{60\,278\,884\,503}{512\,000\,000}\right. \\ &\left. - \frac{144}{5}\log\left(2R_h\right)\right]R_h^8 + \frac{9}{50}\left[625 + 240\,\pi^2 - 1008\log\left(2R_h\right)\right]R_h^9 + \mathcal{O}\left(R_h^{10}\right). \end{aligned} \tag{2.1.61}$$

Let us also report the results for $n = 1$:

$$\begin{aligned}
\mathrm{Im}\left(\omega_{1,0,0}\right) =& -2 - \frac{17}{4}R_h^2 - 24\,R_h^3 - \left[\frac{9791}{64} + 32\log\left(2R_h\right)\right]R_h^4 \\
&+ \left[32\,\pi^2 - 654 - 384\log\left(2R_h\right)\right]R_h^5 + \left[\frac{1408}{3}\pi^2 - \frac{1\,770\,481}{512}\right. \\
&\left. - 3276\log\left(2R_h\right) - 256\log^2\left(2R_h\right) + 384\,\zeta\left(3\right)\right]R_h^6 + \mathcal{O}\left(R_h^7\right), \\
\mathrm{Im}\left(\omega_{1,1,1}\right) =& -3 - \frac{21}{8}R_h^2 + \frac{4137}{128}R_h^4 + 72\,R_h^5 + \left[\frac{249\,879}{1024} + 144\log\left(2R_h\right)\right]R_h^6 \\
&+ \left[303 - 216\,\pi^2 + 1188\log\left(2R_h\right)\right]R_h^7 + \mathcal{O}\left(R_h^8\right), \\
\mathrm{Im}\left(\omega_{1,2,2}\right) =& -4 - \frac{71}{30}R_h^2 + \frac{1\,910\,399}{108\,000}R_h^4 - \frac{44\,927\,058\,551}{194\,400\,000}R_h^6 - \frac{768}{5}R_h^7 \\
&+ \left[\frac{685\,871\,572\,615\,439}{279\,936\,000\,000} - \frac{2048}{5}\log\left(2R_h\right)\right]R_h^8 \\
&+ \frac{64}{225}\left[2880\,\pi^2 - 53 - 10\,656\log\left(2R_h\right)\right]R_h^9 + \mathcal{O}\left(R_h^{10}\right).
\end{aligned} \tag{2.1.62}$$

Some of the results presented above were shortened for the reader's convenience. The full expressions and more expansions of frequencies for other choices of ℓ and s can be found in the `Mathematica` files on `https://github.com/GlebAminov/BH_PolyLog`. The irrational numbers entering these QNM frequencies are $\log(2)$ and multiple zeta values.

2.2 Perturbations of anti-de Sitter black holes in four dimensions

The metric describing the AdS_4 Schwarzschild black hole is given by (2.1.1), with $\Lambda < 0$. We denote the roots of $rf(r) = 0$ by

$$R_h, \quad R_\pm, \tag{2.2.63}$$

where, for $\Lambda < 0$, $R_\pm$ are complex conjugate and given by

$$R_\pm = \frac{-R_h \pm i\sqrt{3R_h^2 - \frac{12}{\Lambda}}}{2}, \tag{2.2.64}$$

in terms of the BH horizon $R_h \in \mathbb{R}_{>0}$. We fix $\Lambda = -3$ and study the same perturbations of the Schwarzschild de Sitter case, described by equation (2.1.5). According to $\mathrm{AdS}_4/\mathrm{CFT}_3$ holography, the conformally coupled scalar field is dual to scalar operators of conformal dimension $\Delta = 1$ or $\Delta = 2$, from the relation $\mu^2 = \Delta(\Delta - 3)$. The main difference with the SdS_4 case lies in the boundary conditions we impose on the solution. Indeed, we still require the presence of only ingoing modes near the horizon but impose the vanishing Dirichlet boundary condition at the AdS boundary.[2]

[2]In the context of AdS/CFT, these are not always the more physically relevant boundary conditions. Alternatively, one often considers Robin boundary conditions, which we discuss in Sec. 2.3.

With the change of variables

$$z(r) = \frac{r(R_- - R_+)}{R_-(r - R_+)}, \tag{2.2.65}$$

and redefinition of the wave function

$$\psi(z) = z^{-\gamma/2}(z-1)^{-\delta/2}(z-t)^{-\epsilon/2}\sqrt{f(r)}\,\frac{R_+(R_- - R_+)}{R_-(r - R_+)}\,R(r), \tag{2.2.66}$$

with

$$\begin{aligned} t &= \frac{R_h(R_+ - R_-)}{R_-(R_+ - R_h)}, \\ \gamma &= 1 - 2s, \\ \delta &= 1 - \frac{2i\,\omega\,R_-}{(R_- - R_h)(R_- - R_+)}, \\ \epsilon &= 1 - \frac{2i\omega R_h}{1 + 3R_h^2}, \end{aligned} \tag{2.2.67}$$

the singularity at infinity is removed, and the equation (2.1.5) becomes a Heun equation (2.1.11) with

$$\begin{aligned} \alpha &= 1 - s + \frac{2i\,\omega\,R_+}{(R_+ - R_h)(R_+ - R_-)}, \\ \beta &= 1 - s, \\ q &= \frac{\ell(\ell+1)}{R_-(R_h - R_+)} + \frac{(1-s)^2 R_h}{R_h - R_+} - \frac{s(1-s)R_+^2}{R_-(R_h - R_+)}. \end{aligned} \tag{2.2.68}$$

In these coordinates, the horizon is at $z = t$ while the boundary is at

$$z_\infty = 1 - \frac{R_+}{R_-}. \tag{2.2.69}$$

We also consider the small black hole limit, $R_h \ll 1$.

The boundary conditions in terms of the ψ function are given by

$$\begin{aligned} &\psi(z) \sim 1 \quad \text{for } z \sim t, \\ &\psi(z_\infty) = 0. \end{aligned} \tag{2.2.70}$$

Notice that the AdS boundary ($z = z_\infty$) is not a singular point of the perturbation equation. This makes the approach based on the Seiberg-Witten theory less effective. One can write the quantization condition using the connection formulae between Heun functions, but in this case, an expansion of the Heun functions in R_h is needed.

2.2.1 QNMs in pure AdS_4

The pure AdS_4 case can be recovered in the limit $t \to 0$ or, equivalently, $R_h \to 0$. In this limit, the z variable is given by

$$z = \frac{2r}{r - i}, \tag{2.2.71}$$

and the AdS boundary is at $z = 2$. The leading order solutions in t of the Heun equation (2.1.11) are given by

$$z^{s-\ell-1} {}_2F_1(-\ell, -\ell+\omega_0; -2\ell; z), \quad z^{\ell+s} {}_2F_1(\ell+1, \ell+1+\omega_0; 2\ell+2; z), \tag{2.2.72}$$

where ω_0 is the leading order term in the R_h expansion of the frequency (2.1.27). As in the de Sitter case, these hypergeometric functions reduce to (2.1.31), where we replace $-i\omega_0$ by ω_0.

The first boundary condition from (2.2.70) tells us that the wave function $\psi(z)$ is regular at $z = 0$. This singles out the second solution from (2.2.72). Then, the second boundary condition at $z = 2$ requires the following expression to vanish:

$${}_2F_1(\ell+1, \ell+1+\omega_0; 2\ell+2; 2) = 4^{-\ell-1} \frac{(2\ell+1)!}{\ell!} \frac{\Gamma\left(\frac{-\omega_0-\ell}{2}\right)}{\Gamma\left(\frac{-\omega_0+\ell+2}{2}\right)} \left[1 + (-1)^{\ell-\omega_0+1}\right], \tag{2.2.73}$$

which gives the quantization condition for the QNM frequencies of the pure AdS_4

$$\omega_0 = \ell + 2n + 2, \quad n \in \mathbb{Z}_{\geq 0} \ \text{ or } \ \omega_0 = -\ell - 2n - 2, \quad n \in \mathbb{Z}_{\geq 0}. \tag{2.2.74}$$

Here we have two branches of frequencies, positive and negative, and one is related to another by the complex conjugation of the radial part of the perturbation $R(r)$.

In the following subsections, we will perturb around the pure AdS case to obtain the corrections for the Schwarzschild anti-de Sitter small black holes. Following the same logic as in the de Sitter case, we will divide the space into two regions: left (L) and right (R). The left region describes the physical space near the AdS boundary with $r \to \infty$, and the right one is the space near the horizon $r = R_h$. After having determined the expansion of the solution $\psi(z)$ in each region up to certain orders in the expansion parameter t, we require that the function $\psi(z)$ and its first derivative are continuous in a point in the intersection of two regions, which we can fix at $z = t^{1/2}$ (other values of z are possible as long as they lie inside the convergence radius of the two solutions).

2.2.2 Left Region

The local coordinate in the left region is z, and the AdS boundary is at z_∞, which has the following expansion in R_h:

$$z_\infty = \frac{3R_h^2 + 4 + i\,R_h\sqrt{3R_h^2+4}}{2R_h^2+2} = 2 + iR_h - \frac{R_h^2}{2} + \mathcal{O}\left(R_h^3\right). \tag{2.2.75}$$

The wave function in the left region $\psi^L(z)$ satisfies the same Heun equation (2.1.11). The form of the leading order solutions depends on which branch of frequencies we choose in (2.2.74). For the negative branch $\omega_0 = -\ell - 2n - 2$, we have

$$
\begin{aligned}
f_0^L(z) =& z^{\ell+s} \sum_{m=0}^{2n+1} (-1)^m \binom{2n+1}{m} \frac{(\ell+1)_m}{(2\ell+2)_m} z^m, \\
g_0^L(z) =& z^{s-\ell-1} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} \frac{(-2\ell-2n-2)_m}{(-2\ell)_m} z^m,
\end{aligned}
\tag{2.2.76}
$$

and for the positive branch $\omega_0 = \ell + 2n + 2$:

$$
\begin{aligned}
f_0^L(z) =& \frac{z^{\ell+s}}{(1-z)^{2n+\ell+2}} \sum_{m=0}^{2n+1} (-1)^m \binom{2n+1}{m} \frac{(\ell+1)_m}{(2\ell+2)_m} z^m, \\
g_0^L(z) =& \frac{z^{s-\ell-1}}{(1-z)^{2n+\ell+2}} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} \frac{(-2\ell-2n-2)_m}{(-2\ell)_m} z^m.
\end{aligned}
\tag{2.2.77}
$$

For both branches, the Wronskian can be written in terms of ω_0 as

$$
W_0^L(z) = -(2\ell+1) z^{2s-2} (1-z)^{-\omega_0-1}. \tag{2.2.78}
$$

We will apply the perturbative method to both positive and negative values of ω_0, but the final result is straightforward. The only difference between the two branches is the sign of the real part of the frequency expansion (2.1.27), which again corresponds to complex conjugation of $R(r)$.

The boundary condition in the left region is simply $\psi^L(z_\infty) = 0$. Since $f_0^L(2) = 0$ and $g_0^L(2) \neq 0$, we get the following perturbative expansion for the wave function in the left region:

$$
\psi^L(z) = f_0^L(z) + \sum_{k\geq 1} f_k^L(z) t^k, \tag{2.2.79}
$$

where $f_k^L(z)$ are given by (1.1.26). The constants c_k in (1.1.26) are fixed by expanding $\psi^L(z_\infty)$ in powers of t and requiring the coefficients in this expansion to vanish.

As we explain in Appendix B.2, the integrals in (1.1.26) are described in terms of the multiple polylogarithms in a single variable (2.1.41). Since the weights of the multiple polylogarithms appearing at order t^k are less or equal to k, we can construct a linear basis of functions in which the integrals in (1.1.26) can be expanded. We take the same steps (2.1.44)–(2.1.46) as we did in the SdS_4 case to do this. The only difference is that we add the second logarithm function $\log(z-1)$ to (2.1.44). To be more precise, the integrands in (1.1.26) at order t^{k+1} are given by the linear combination of the following functions:

$$
\begin{aligned}
&\frac{\sum_{m=0}^{r_1} \alpha_m z^m}{z^{i_1}(z-1)^{j_1}} \log(z-1)^{p_1} \log(z)^{p_2}, \\
&\frac{\sum_{m=0}^{r_2} \beta_m z^m}{z^{i_2}(z-1)^{j_2}} \log(z-1)^{p_3} \mathrm{Li}_{s_1,\ldots,s_n}(1-z),
\end{aligned}
\tag{2.2.80}
$$

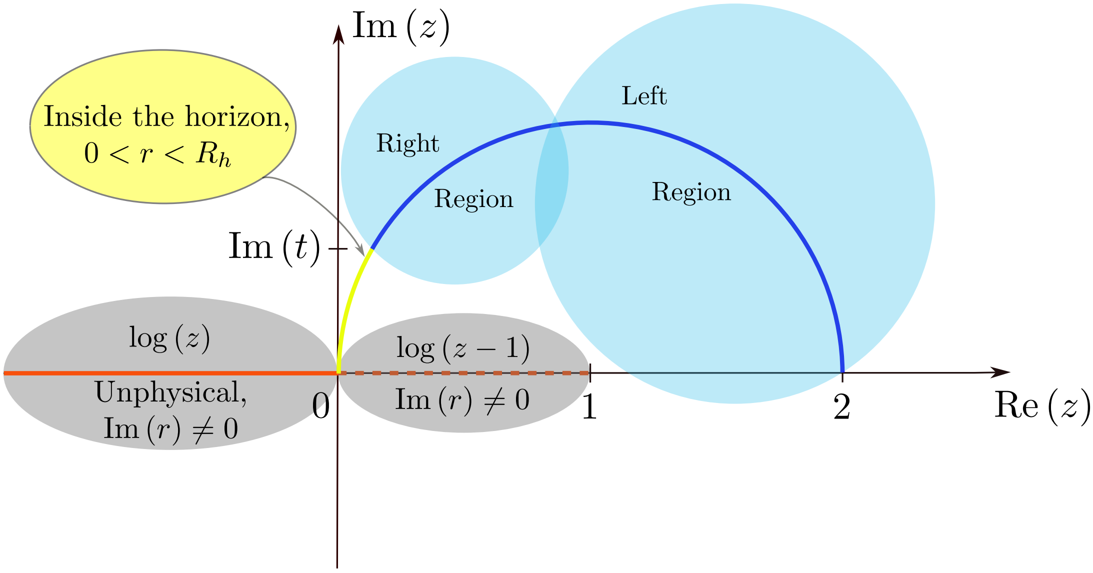

Figure 2.2: Branch cuts (red lines) on the complex z plane for anti-de Sitter black holes.

where $r_{1,2}$, $i_{1,2}$, $j_{1,2}$, $p_{1,2,3}$ are some non-negative integers and $p_1 + p_2 \leq k$, $p_3 + s_1 + \cdots + s_n \leq k$. The reasoning behind our choice of the branches of the logarithm functions $\log(z)$ and $\log(z-1)$ is the same as in Sec. 2.1.2. We want the wave function $\psi(z)$ to be continuous across the real slice $R_h < r < +\infty$. In the $SAdS_4$ case, the coordinate transformation $z(r)$ is given by (2.2.65) with complex parameters $R_\pm$. Taking into account that r and R_h are real, we have from (2.2.65):

$$(\mathrm{Re}(z) - 1)^2 + \mathrm{Im}(z)^2 = 1. \tag{2.2.81}$$

Thus, the real slice is approximately half the circle with the center in $z = 1$ on the complex z plane (see Figure 2.2). It starts at $z = t$ and ends at $z = z_\infty$. Simple analysis shows that $\mathrm{Im}(t) > 0$ and $\mathrm{Im}(z_\infty) > 0$ when $R_h > 0$. This justifies our choice of logarithm functions since both branch cuts do not cross the real slice. On the other hand, if one picks $\log(1-z)$ instead of $\log(z-1)$, the corresponding branch cut would touch the real slice at the point $z = 2$ when evaluating $\psi^L(z_\infty)$. This, in turn, would lead to incorrect results for QNM frequencies.

2.2.3 Right region

In the right region, we introduce the local coordinate

$$z^R = \frac{t}{z}. \tag{2.2.82}$$

The horizon is now situated at $z^R = 1$. The wave function in the right region $\psi^R\left(z^R\right)$ satisfies the following equation in terms of z^R:

$$\frac{\mathrm{d}^2\psi^R}{\left(\mathrm{d}z^R\right)^2} + \left(\frac{2-\gamma}{z^R} + \frac{\delta\, t}{z^R\left(z^R-t\right)} + \frac{\epsilon}{z^R\left(z^R-1\right)}\right)\frac{\mathrm{d}\,\psi^R}{\mathrm{d}z^R} + \frac{\alpha\beta\, t - q\, z^R}{\left(z^R\right)^2\left(z^R-1\right)\left(z^R-t\right)}\psi^R = 0. \tag{2.2.83}$$

Suppressing the R index on z^R, the two leading order solutions are given by

$$\begin{aligned} f_0^R(z) =& z^{-\ell-s}\sum_{m=0}^{\ell+s}(-1)^m\binom{\ell+s}{m}\frac{(s-\ell)_m}{(-2\,\ell)_m}\,z^m,\\ g_0^R(z) =& z^{-s}\left\{\sum_{m=-s}^{\ell-1}a_{s\ell m}\,z^{-m} + \log(1-z)\sum_{m=s}^{\ell}b_{s\ell m}\,z^{-m}\right\}, \end{aligned} \tag{2.2.84}$$

where the constants $a_{s\ell m}$, $b_{s\ell m}$ can be determined for any $\ell \geq s \geq 0$ as

$$a_{s\ell m} = -b_{s\ell m}\left(H_{\ell+s} + H_{\ell-s} - H_{m+s} - H_{m-s}\right),$$

$$b_{s\ell m} = \frac{(-1)^{\ell+m+1}}{(m+s)!\,(m-s)!}\,\frac{(2\,\ell+1)!}{(\ell+s)!\,(\ell-s)!}\,\frac{(\ell+m)!}{(\ell-m)!}.$$

The expressions in (2.2.84) are independent of which branch of frequencies we choose in (2.2.74) because the leading order of (2.2.83) does not contain ω_0. The Wronskian of f_0^R and g_0^R is given by

$$W_0^R(z) = \frac{2\,\ell+1}{z^{2s}(z-1)}. \tag{2.2.85}$$

The boundary condition in the right region tells us that ψ^R is regular at $z^R = 1$. Thus, we can write the following perturbative expansion:

$$\psi^R(z) = f_0^R(z) + \sum_{k\geq 1} f_k^R(z)t^k, \tag{2.2.86}$$

where $f_k^R(z)$ are computed using (1.1.26). Unlike in the left region, the choice of the logarithm function in $g_0^R(z)$ is unimportant. This is due to the boundary condition that requires canceling contributions of $\log(1-z)$ in each order t^k. The resulting corrections $f_k^R(z)$ are linear combinations of the following functions of maximum weight k:

$$\sum_{m=-k_1}^{l_1}\zeta_m^R\,z^m\,\log(z)^{p_1}, \qquad \sum_{m=-k_2}^{l_2}\xi_m^R\,z^m\,\mathrm{Li}_{s_1,\ldots,s_n}(1-z), \tag{2.2.87}$$

where $k_{1,2}$, $l_{1,2}$, p_1 are some non-negative integers, and $0 \leq p_1 \leq k$, $s_1 + \cdots + s_n \leq k$.

2.2.4 Results for QNM frequencies

To determine the QNM frequencies, we use the continuity condition in the form (2.1.60):

$$\psi^L\left(t^{1/2}\right) = C(t)\,\psi^R\left(t^{1/2}\right), \quad \partial_z\psi^L(z)\Big|_{z=t^{1/2}} = C(t)\,\partial_z\psi^R(t/z)\Big|_{z=t^{1/2}}, \tag{2.2.88}$$

where $\psi^{L,R}\left(z^{L,R}\right)$ are computed up to orders $m_{L,R}$ in t around $z^{L,R}=1$:

$$\begin{aligned}\psi^L(z) &= f_0^L(z) + \sum_{k=1}^{m_L} f_k^L(z)t^k + O\left(t^{m_L+1}\right), \\ \psi^R\left(z^R\right) &= f_0^R\left(z^R\right) + \sum_{k=1}^{m_R} f_k^R(z^R)t^k + O\left(t^{m_R+1}\right).\end{aligned} \tag{2.2.89}$$

Similarly to the SdS_4 case, the reshuffling of terms (2.1.58) occurs in $\psi^L(z)$ when we take $z\sim t^{1/2}$. For all values of quantum numbers we have considered, this reshuffling is superficial and goes away after the frequency is set to one of the quasinormal modes.

Using `Mathematica`, we compute the local solutions up to orders $m_L=7$ and $m_R=8$ (sometimes even up to $m_L=9$ and $m_R=10$). This allows us to determine the R_h expansion of the frequency up to order R_h^7 or less depending on the value of ℓ. In all computed cases, the imaginary part does not appear before order $2\ell+2$ in R_h:

$$\mathrm{Im}\left(\omega_{n,\ell,s}\right) \sim R_h^{2\ell+2}. \tag{2.2.90}$$

As mentioned, the results computed for negative and positive branches of ω_0 only differ by the sign in the real part of the frequency expansion. Below are the results for the real and imaginary parts of the quasinormal mode frequencies $\omega_{n,\ell,s}$ corresponding to the positive branch, starting from $n = 0$:

$$
\begin{aligned}
\mathrm{Re}\left(\omega_{0,0,0}\right) =& 2 - \frac{4}{\pi}\,R_h - \left(\frac{1}{4} + \frac{24}{\pi^2}\right) R_h^2 \\
& - \left(\frac{4\,\pi}{3} - \frac{94}{3\,\pi} - \frac{16}{\pi}\,\log\left(4\,R_h\right) + \frac{208}{\pi^3} - \frac{112}{\pi^3}\,\zeta\left(3\right)\right) R_h^3 + \mathcal{O}\left(R_h^4\right), \\
\mathrm{Im}\left(\omega_{0,0,0}\right) =& -\frac{8}{\pi}\,R_h^2 - \left(8 + \frac{16}{\pi^2}\right) R_h^3 \\
& - \left(\frac{40\,\pi}{3} - \frac{65}{\pi} - \frac{128}{\pi}\,\log\left(2\,R_h\right) + \frac{192}{\pi^3} - \frac{448}{\pi^3}\,\zeta\left(3\right)\right) R_h^4 + \mathcal{O}\left(R_h^5\right), \\
\mathrm{Re}\left(\omega_{0,1,1}\right) =& 3 - \frac{4}{\pi}\,R_h + \left(\frac{27}{8} - \frac{140}{3\,\pi^2}\right) R_h^2 \\
& - \left(3\,\pi - \frac{601}{12\,\pi} - \frac{18}{\pi}\,\log\left(2\right) + \frac{2020}{3\,\pi^3} - \frac{168}{\pi^3}\,\zeta\left(3\right)\right) R_h^3 + \mathcal{O}\left(R_h^4\right), \\
\mathrm{Im}\left(\omega_{0,1,1}\right) =& -\frac{16}{\pi}\,R_h^4 - \left(24 + \frac{96}{\pi^2}\right) R_h^5 - \left(60\,\pi + \frac{579}{\pi} - \frac{264}{\pi}\,\log\left(2\,R_h\right)\right. \\
& \left. + \frac{11\,536}{9\,\pi^3} - \frac{1344}{\pi^3}\,\zeta\left(3\right)\right) R_h^6 + \mathcal{O}\left(R_h^7\right), \\
\mathrm{Re}\left(\omega_{0,2,2}\right) =& 4 - \frac{64}{15\,\pi}\,R_h + \left(\frac{37}{6} - \frac{80\,896}{1125\,\pi^2}\right) R_h^2 - \left(\frac{256\,\pi}{45} - \frac{1\,536\,256}{10\,125\,\pi} - \frac{512}{45\,\pi}\,\log\left(2\right)\right. \\
& \left. + \frac{120\,946\,688}{84\,375\,\pi^3} - \frac{57\,344}{225\,\pi^3}\,\zeta\left(3\right)\right) R_h^3 + \mathcal{O}\left(R_h^4\right), \\
\mathrm{Im}\left(\omega_{0,2,2}\right) =& -\frac{128}{5\,\pi}\,R_h^6 - \left(\frac{256}{5} + \frac{6144}{25\,\pi^2}\right) R_h^7 + \mathcal{O}\left(R_h^8\right).
\end{aligned}
\tag{2.2.91}
$$

For $n = 1$ we have:

$$
\begin{aligned}
\mathrm{Re}\left(\omega_{1,0,0}\right) =& 4 - \frac{40}{3\,\pi}\,R_h + \left(\frac{25}{6} - \frac{5200}{27\,\pi^2}\right) R_h^2 - \left(\frac{160\,\pi}{9} - \frac{45\,064}{81\,\pi} - \frac{800}{9\,\pi}\,\log\left(2\right)\right. \\
& \left. - \frac{128}{\pi}\,\log\left(R_h\right) + \frac{1\,200\,800}{243\,\pi^3} - \frac{22\,400}{9\,\pi^3}\,\zeta\left(3\right)\right) R_h^3 + \mathcal{O}\left(R_h^4\right), \\
\mathrm{Im}\left(\omega_{1,0,0}\right) =& -\frac{32}{\pi}\,R_h^2 - \left(64 + \frac{2240}{9\,\pi^2}\right) R_h^3 - \left(\frac{640\,\pi}{3} - \frac{4252}{3\,\pi} - \frac{1920}{\pi}\,\log\left(2\,R_h\right)\right. \\
& \left. + \frac{101\,120}{9\,\pi^3} - \frac{35\,840}{3\,\pi^3}\,\zeta\left(3\right)\right) R_h^4 + \mathcal{O}\left(R_h^5\right), \\
\mathrm{Re}\left(\omega_{1,1,1}\right) =& 5 - \frac{172}{15\,\pi}\,R_h + \left(\frac{2071}{120} - \frac{791\,372}{3375\,\pi^2}\right) R_h^2 - \left(\frac{215\,\pi}{9} - \frac{27\,888\,631}{40\,500\,\pi}\right. \\
& \left. + \frac{40\,678}{225\,\pi}\,\log\left(2\right) + \frac{5\,269\,420\,724}{759\,375\,\pi^3} - \frac{103\,544}{45\,\pi^3}\,\zeta\left(3\right)\right) R_h^3 + \mathcal{O}\left(R_h^4\right),
\end{aligned}
\tag{2.2.92}
$$

$$
\begin{aligned}
\mathrm{Im}\left(\omega_{1,1,1}\right) =& -\frac{400}{3\,\pi}\,R_h^4-\left(\frac{1000}{3}+\frac{39\,904}{27\,\pi^2}\right)R_h^5-\left(\frac{12\,500\,\pi}{9}+\frac{328\,711}{27\,\pi}-\frac{49\,880}{9\,\pi}\,\log\left(2\,R_h\right)\right.\\
&\left.+\frac{14\,315\,216}{243\,\pi^3}-\frac{481\,600}{9\,\pi^3}\,\zeta\left(3\right)\right)R_h^6+\mathcal{O}\left(R_h^7\right),\\
\mathrm{Re}\left(\omega_{1,2,2}\right) =& 6-\frac{384}{35\,\pi}\,R_h+\left(\frac{675}{28}-\frac{12\,163\,072}{42\,875\,\pi^2}\right)R_h^2-\left(\frac{1152\,\pi}{35}-\frac{49\,433\,312}{42\,875\,\pi}\right.\\
&\left.+\frac{13\,824}{49\,\pi}\,\log\left(2\right)+\frac{1\,544\,254\,324\,736}{157\,565\,625\,\pi^3}-\frac{442\,368}{175\,\pi^3}\,\zeta\left(3\right)\right)R_h^3+\mathcal{O}\left(R_h^4\right),\\
\mathrm{Im}\left(\omega_{1,2,2}\right) =& -\frac{1792}{5\,\pi}\,R_h^6-\left(\frac{5376}{5}+\frac{385\,024}{75\,\pi^2}\right)R_h^7+\mathcal{O}\left(R_h^8\right).
\end{aligned}
\tag{2.2.93}
$$

Some of the results presented above were shortened for the reader's convenience. The full expressions and more expansions of frequencies for other choices of n, ℓ, and s can be found in the `Mathematica` files on `https://github.com/GlebAminov/BH_PolyLog`. From these, one can see that the irrational numbers entering these QNM frequencies are $\log(2)$, π, and Euler sums.

Analytically computing $f_1^L\left(z\right)$ from (2.2.79), we can also determine the subleading term in the QNM frequency expansion with $n=0$ and $\ell \geq 1$:

$$
\omega_{0,\ell,s}=\ell+2-\frac{2^{2\ell+2}}{\pi}\,\frac{2\,\ell+s^2}{\ell\left(\ell+1\right)}\,\frac{\left(\left(\ell+1\right)!\right)^2}{\left(2\,\ell+2\right)!}\,R_h+\mathcal{O}\left(R_h^2\right). \tag{2.2.94}
$$

For small enough values of R_h, our results agree with the numerical ones obtained earlier in [159]. Since the frequency expansions in higher orders in R_h include multiple zeta values (B.1.14), we use different identities of the form (B.1.15)–(B.1.18) to compute the corresponding numerical values. Tables 2.1–2.3 present the numerical results from the frequency expansions truncated at R_h^7 (in the scalar case with $n=l=0$, the expansion was computed up to order R_h^6 and truncated at the same order). In these tables, bold digits are the ones that are stable and agree with the numerical results obtained directly from the Heun function and the continuity condition (2.1.59). The digit is considered stable if it does not change when higher orders of R_h are added to the expansion of the frequency. For example, below are the numerical results from electromagnetic frequency expansion with $n=0$, $\ell=1$ truncated at different powers of $R_h=1/20$:

$$
\begin{aligned}
R_h: &\quad \omega_{0,1,1}=2.936338022763,\\
R_h^2: &\quad \omega_{0,1,1}=2.932954718005,\\
R_h^3: &\quad \omega_{0,1,1}=2.932365431000,\\
R_h^4: &\quad \omega_{0,1,1}=2.932257833944-0.000031830989\,i,\\
R_h^5: &\quad \omega_{0,1,1}=2.932232789345-0.000042370624\,i,\\
R_h^6: &\quad \omega_{0,1,1}=2.932227305824-0.000051050731\,i,\\
R_h^7: &\quad \omega_{0,1,1}=\mathbf{2.93222}\mathit{6938543}-\mathbf{0.00005}\mathit{3055262}\,i.
\end{aligned}
\tag{2.2.95}
$$

R_h	$\mathrm{Re}\,(\omega_{0,0,0})$	$-\mathrm{Im}\,(\omega_{0,0,0})$
$1/16$	$\mathbf{1.9095}\mathit{9612832}$	$\mathbf{0.0136}\mathit{6850348}$
$1/18$	$\mathbf{1.92054}\mathit{810947}$	$\mathbf{0.0104}\mathit{3093333}$
$1/20$	$\mathbf{1.92919}\mathit{836511}$	$\mathbf{0.00820}\mathit{901816}$
$1/50$	$\mathbf{1.9733862}\mathit{8700}$	$\mathbf{0.0011184}\mathit{9414}$
$1/100$	$\mathbf{1.986986250}\mathit{43}$	$\mathbf{0.000265980}\mathit{52}$

Table 2.1: Numerical results from conformally coupled scalar QNM frequency expansion with $n = 0$, $\ell = 0$.

R_h	$\mathrm{Re}\,(\omega_{0,1,1})$	$-\mathrm{Im}\,(\omega_{0,1,1})$
$1/16$	$\mathbf{2.9136}\mathit{28697405}$	$\mathbf{0.0001}\mathit{51017506}$
$1/18$	$\mathbf{2.92406}\mathit{3021823}$	$\mathbf{0.00008}\mathit{6542953}$
$1/20$	$\mathbf{2.93222}\mathit{6938543}$	$\mathbf{0.00005}\mathit{3055262}$
$1/50$	$\mathbf{2.973953080}\mathit{307}$	$\mathbf{0.000000967}\mathit{146}$
$1/100$	$\mathbf{2.9871273749}\mathit{10}$	$\mathbf{0.0000000550}\mathit{27}$

Table 2.2: Numerical results from electromagnetic QNM frequency expansion with $n = 0$, $\ell = 1$.

R_h	$\mathrm{Re}\,(\omega_{0,2,2})$	$-\mathrm{Im}\,(\omega_{0,2,2})$
$1/15$	$\mathbf{3.90327}\mathit{7526809}$	$\mathbf{0.00000}\mathit{1160789}$
$1/18$	$\mathbf{3.920419}\mathit{438200}$	$\mathbf{0.000000}\mathit{363885}$
$1/20$	$\mathbf{3.928811}\mathit{737917}$	$\mathbf{0.000000}\mathit{186778}$
$1/50$	$\mathbf{3.972361286}\mathit{120}$	$\mathbf{0.000000000}\mathit{619}$
$1/100$	$\mathbf{3.9863033746}\mathit{08}$	$\mathbf{0.0000000000}\mathit{09}$

Table 2.3: Numerical results from odd gravitational QNM frequency expansion with $n = 0$, $\ell = 2$.

2.3 Scalar Sector of Gravitational Perturbations of $SAdS_4$ black holes

Following [160], one can consider a subdivision of gravitational perturbations in different sectors (scalar, vector, or tensor), whose distinction comes from the expansions in scalar, vector, or tensor spherical harmonics on the S^2 component of AdS_4. In Sec. 2.2 we considered the vector sector of gravitational perturbations ($s = 2$). We now focus on the scalar sector and impose a different boundary condition at the AdS boundary, namely a Robin boundary condition [161, 162, 163, 164, 165, 166], see also [167] for very recent developments. This choice of boundary condition is motivated by the AdS/CFT correspondence, and it ensures that the perturbations do not deform the metric on the boundary of AdS.

From the point of view of the dual CFT, these boundary conditions are related to double-trace deformations, see for instance [168, 169, 170] and references therein. In particular, we analyze the so-called *low-lying quasinormal frequencies*, which, according to AdS/CFT duality, are related to hydrodynamic modes of the $3d$ thermal CFT on the boundary [171, 172, 173, 174, 175, 166, 176, 177, 178, 179]. We will therefore expand our quasinormal frequencies for large values of R_h, $R_h \gg 1$, differently from the previous sections. Defining

$$m = \ell(\ell+1) - 2 \quad \text{with } \ell \geq 2, \tag{2.3.96}$$

the equation describing the scalar sector of gravitational perturbations in AdS_4 can be written as (see [160, eq. (3.1)] for the definition of the master variable Φ)

$$\left(\partial_r^2 + \frac{f'(r)}{f(r)}\partial_r + \frac{\omega^2 - V_S(r)}{f(r)^2}\right)\Phi(r) = 0, \tag{2.3.97}$$

where

$$\begin{aligned} f(r) &= 1 - \frac{2M}{r} + r^2, \\ V_S(r) &= \frac{f(r)}{(mr+6M)^2}\left[m^3 + \left(2 + \frac{6M}{r}\right)m^2 + \frac{36M^2}{r^2}\left(m + 2r^2 + \frac{2M}{r}\right)\right]. \end{aligned} \tag{2.3.98}$$

This equation has five regular singularities, located at $r = 0, R_h, R_\pm, R_5$, where

$$R_\pm = \frac{-R_h \pm i\sqrt{4+3R_h^2}}{2}, \quad R_5 = -\frac{3R_h\left(1+R_h^2\right)}{m}. \tag{2.3.99}$$

The new singularity R_5, coming from the potential $V_S(r)$, is in the unphysical region $r < 0$. Similarly to the previous cases, we introduce the change of variables

$$z\left(r\right) = \frac{R_h}{r} \tag{2.3.100}$$

and the new wave function

$$\psi\left(z\right) = r^{-1}e^{i\omega r_*}\Phi\left(r\right). \tag{2.3.101}$$

The master equation (2.3.97) then becomes

$$\psi''(z) + \frac{\mathfrak{f}'(z) - 2z^{-1}\mathfrak{f}(z) + 2i\,\omega R_h^{-1}}{\mathfrak{f}(z)}\psi'(z) - \left(\frac{\mathfrak{f}'(z) - 2z^{-1}\mathfrak{f}(z) + 2i\,\omega R_h^{-1}}{z\,\mathfrak{f}(z)} + \frac{\mathfrak{V}(z)}{\mathfrak{f}(z)^2}\right)\psi(z) = 0, \tag{2.3.102}$$

where

$$\mathfrak{f}(z) = (1-z)\left(1 + z + z^2 + \frac{z^2}{R_h^2}\right),$$
$$\mathfrak{V}(z) = \frac{\mathfrak{f}(z)}{(mR_h + 6Mz)^2}\left[m^3 + \left(2 + \frac{6Mz}{R_h}\right)m^2 + \frac{36M^2z^2}{R_h^2}\left(m + \frac{2Mz}{R_h} + \frac{2R_h^2}{z^2}\right)\right], \tag{2.3.103}$$

and M is related to R_h via

$$2M = R_h\left(1 + R_h^2\right). \tag{2.3.104}$$

The boundary conditions in terms of the ψ function are given by

$$\psi(z) \sim 1 \quad \text{for } z \sim 1,$$
$$\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\psi(z)}{z}\right) + \left[\frac{3(1+R_h^2)}{m} + \frac{i\omega}{R_h}\right]\frac{\psi(z)}{z}\right\}\Bigg|_{z=0} = 0. \tag{2.3.105}$$

The five regular singularities of the equation (2.3.97) have three different scalings with $R_h \to \infty$. The singularity at $r = 0$ doesn't scale, the singularities $R_\pm$ and R_h scale linearly, and R_5 scales as R_h^3. Hence, we will divide the space into three different regions and apply the perturbative method.

The three local variables are

$$x = R_h^3/(mr) + 1/3$$

for the left region (near the AdS boundary),

$$y = R_h^2/r$$

for the middle region, and

$$z = R_h/r$$

for the right one (near the BH horizon).[3] Here the regions are labeled left and right as they appear on the complex z plane (see Figure 2.3). From the point of view of the

[3]We choose to add an intermediate region with local variable y to increase the efficiency of the computation. According to our estimations (2.3.116), without the middle region, one would need to compute at least 48 orders in the expansion of the wave function in the left region (2.3.115) to get the frequency expansion up to ω_5 (assuming we do not increase the number of corrections computed in the right region). Adding the middle region allows us to get the same result by computing $\psi^L(x)$ up to order 15.

complex z plane, the left and middle regions represent two zoomings close to the origin, with different scalings. Considering the normal form of the differential equation (2.3.102),

$$\psi''(z) + V_z(z)\psi(z) = 0, \tag{2.3.106}$$

the potential $V_z(z)$ has the following expansion in $1/R_h$

$$V_z(z) = \frac{z^6 + 16z^3 - 8}{4z^2\left(z^3 - 1\right)^2} + \mathcal{O}\left(\frac{1}{R_h^2}\right). \tag{2.3.107}$$

The two rescalings $x \sim \frac{R_h^2\, z}{m}$ and $y = R_h\, z$ are such that, in both variables, the differential equation in normal form has a potential, $V_x(x)$ and $V_y(y)$, respectively, with non-vanishing leading order in $1/R_h$,

$$\begin{aligned} V_x(x) &= -\frac{2}{x^2} + \mathcal{O}\left(\frac{1}{R_h^2}\right), \\ V_y(y) &= -\frac{2}{y^2} + \mathcal{O}\left(\frac{1}{R_h^2}\right). \end{aligned} \tag{2.3.108}$$

Out of the three, the right region is the one in which it is more challenging to expand the solution of the differential equation. In particular, the solution involves multiple polylogarithms in several variables, which we analyze in Appendix B.3.

Since we work with $R_h \gg 1$, the small parameter is $\alpha = 1/R_h$, and the frequency expansion can be written as

$$\omega = \sum_{k\geq 0} \omega_k \alpha^k. \tag{2.3.109}$$

The intersections of the three regions and the boundary points $r = R_h, \infty$ determine three intervals in which the wave function should be continuous:

$$x \in \left[\frac{1}{3}, \frac{1}{3} + \frac{1}{\alpha\, m}\right], \qquad y \in \left[1, \alpha^{-3/4}\right], \qquad z \in \left[\alpha^{1/4}, 1\right]. \tag{2.3.110}$$

From the point of view of x and y, the first two intervals have infinite lengths, their left endpoints are at finite values and their right endpoints are chosen to meet the next region (and so they become infinite because of the different scalings of the local variables in powers of R_h). Finally, we will derive the low-lying QNM frequencies by requiring that the wave function and its first derivative are continuous at the intersection points $y = 1$ and $z = \alpha^{1/4}$. As we explain later, the second intersection point $z = \alpha^{1/4}$ is chosen to avoid the reshuffling of terms in the wave function expansion (2.3.127).

2.3.1 Left Region

The left region represents the region close to the AdS boundary, where we impose the Robin boundary condition. The local variable in this region is

$$x = \frac{R_h^3}{m\, r} + \frac{1}{3} = \frac{\alpha^{-3}}{m\, r} + \frac{1}{3}, \tag{2.3.111}$$

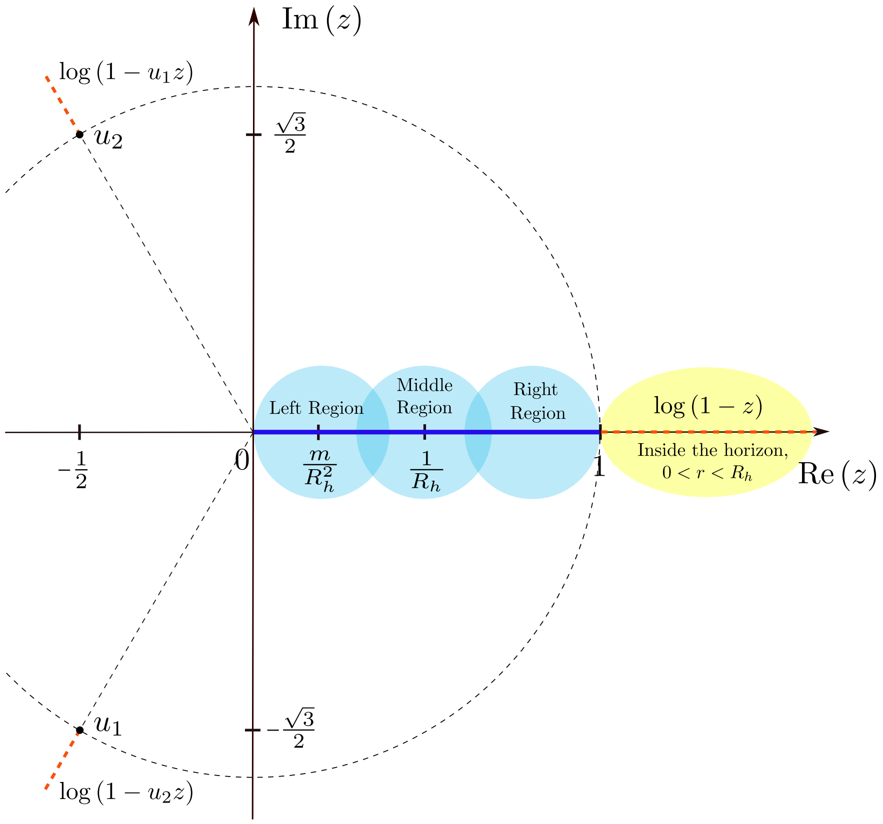

Figure 2.3: Complex z plane for scalar sector of gravitational perturbations in SAdS_4.

and the AdS boundary is at $x = 1/3$. The master equation in the left region is obtained by applying the coordinate transformation $z = \alpha^2 m\,(x - 1/3)$ to (2.3.102) and substituting $\psi\,(z)$ with $\psi^L\,(x)$. In the leading order in α, we get

$$\partial_x^2\,\psi^L\,(x) + \frac{6}{1-3\,x}\,\partial_x\,\psi^L\,(x) - \frac{2\,(1-6\,x)}{x^2\,(1-3\,x)^2}\,\psi^L\,(x) + \mathcal{O}\,(\alpha) = 0. \tag{2.3.112}$$

The two leading order solutions are

$$\begin{aligned} f_0^L\,(x) &= 1 - \frac{1}{3\,x}, \\ g_0^L\,(x) &= x^2\left(x - \frac{1}{3}\right). \end{aligned} \tag{2.3.113}$$

Since f_0^L satisfies the Robin boundary condition

$$\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi^L(x)}{x-\frac{1}{3}}\right) + \left[3(1+\alpha^2) + i\,\alpha^3\,m\,\omega\right]\frac{\psi^L(x)}{x-\frac{1}{3}}\right\}\Bigg|_{x=\frac{1}{3}} = 0, \tag{2.3.114}$$

the following perturbative expansion for the wave function in the left region can be written:

$$\psi^L\,(x) = f_0^L\,(x) + \sum_{k\geq 1} f_k^L\,(x)\,\alpha^k. \tag{2.3.115}$$

We do not use (1.1.26) to compute $f_k^L\,(x)$ as they are simple Laurent polynomials in x. The form of these polynomials depends on whether k is even or odd. The following general result holds for the first 30 computed orders:

$$\begin{aligned} f_{2k}^L\,(x) &= \left(x - \frac{1}{3}\right)\sum_{s=-k-1}^{k-1-\frac{4}{3}\sin\left(k\frac{\pi}{3}\right)^2} \mathfrak{a}_{2k,s}\,x^s, \\ f_{2k-1}^L\,(x) &= \left(x - \frac{1}{3}\right)\sum_{s=0}^{k-3+\frac{4}{3}\sin\left(k\frac{\pi}{3}\right)^2} \mathfrak{a}_{2k-1,s}\,x^s, \end{aligned} \tag{2.3.116}$$

where the coefficients $\mathfrak{a}_{k,s}$ depend on the parameters m and ω_i. For example, we have for $k = 1, 2, 3, 4$:

$$\begin{aligned} f_1^L\,(x) &= 0, & f_3^L\,(x) &= -im\,\omega_0\left(x - \frac{1}{3}\right), \\ f_2^L\,(x) &= \left(x - \frac{1}{3}\right)\frac{1}{3\,x^2}, & f_4^L\,(x) &= \left(x - \frac{1}{3}\right)\left(\frac{1}{9\,x^3} - \frac{1}{3\,x^2} - im\,\omega_1\right). \end{aligned} \tag{2.3.117}$$

In each order in α, the contribution of g_0^L is fixed by the Robin boundary condition. The contribution of f_0^L is arbitrary and can be absorbed into a normalization of the wave function $\psi^L\,(x)$. We choose the normalization so that f_0^L is only present in the leading order.

2.3.2 Middle Region

To match the wave function expansions in the left and right regions, we introduce an intermediate region with the local variable

$$y = \frac{R_h^2}{r} = \frac{\alpha^{-2}}{r}. \tag{2.3.118}$$

The master equation in the middle region is obtained by applying the coordinate transformation $z = \alpha\, y$ to (2.3.102) and substituting $\psi(z)$ with $\psi^M(y)$. In the leading order in α, we get

$$\partial_y^2\, \psi^M(y) - \frac{2}{y}\, \partial_y\, \psi^M(y) + \mathcal{O}(\alpha) = 0. \tag{2.3.119}$$

The two leading order solutions are

$$\begin{aligned} f_0^M(y) &= 1, \\ g_0^M(y) &= y^3. \end{aligned} \tag{2.3.120}$$

Strictly speaking, there is no boundary condition in the middle region. However, there is a way to use the expansion of the wave function in this region and apply the boundary condition near the horizon $y \sim \alpha^{-1}$. This requires a resummation of infinitely many terms, and the results agree with the ones obtained using three regions instead of just two. Here we focus on the procedure with three regions as it allows us to get more orders in the QNM frequency expansion. To justify our choice of functions f_0^M and g_0^M, we can either use the gluing procedure or look at the behavior near the horizon. In the first couple of orders in α, there is no resummation of terms in the wave function $\psi^M(y)$ when we take $y \sim \alpha^{-1}$. Since near the horizon $g_0^M(y) \sim \alpha^{-3}$, it can only appear in orders α^3 and higher. This leads to the following perturbative expansion of the wave function:

$$\psi_M(y) = f_0^M(y) + \sum_{k\geq 1} f_k^M(y)\alpha^k. \tag{2.3.121}$$

Similarly to the left region, the corrections $f_k^M(y)$ are Laurent polynomials of the form

$$f_k^M(y) = \sum_{s=-k}^{k-\frac{4}{3}\sin\left(k\frac{\pi}{3}\right)^2} \mathfrak{b}_{k,s}\, y^s, \tag{2.3.122}$$

where coefficients $\mathfrak{b}_{k,s}$ also depend on the parameters m and ω_i. Starting from order α^3, the gluing procedure fixes the contribution of g_0^M, so we keep the corresponding integration constants c_k^M in the expressions for f_k^M, $k \geq 3$. Out of the 27 computed orders, we present the first 4:

$$
\begin{aligned}
f_1^M(y) &= -\frac{m}{3y}, \\
f_2^M(y) &= \frac{m^2}{9\,y^2} - i\,\omega_0\,y, \\
f_3^M(y) &= -\frac{m^3}{27\,y^3} + \frac{m}{3\,y} - i\,\omega_1\,y + c_3^M y^3, \\
f_4^M(y) &= \frac{m^4}{81\,y^4} - \frac{2\,m^2}{9\,y^2} - i\,\omega_2\,y + \frac{2\,m}{3}\,c_3^M y^2 + c_4^M\,y^3.
\end{aligned}
\tag{2.3.123}
$$

2.3.3 Right Region

The local variable in the right region is z, and the event horizon is at $z = 1$. The leading order in α of (2.3.102) is

$$\partial_z^2\,\psi\,(z) + \frac{\left(z^3+2\right)}{z\,\left(z^3-1\right)}\,\partial_z\,\psi\,(z) + \mathcal{O}\,(\alpha) = 0. \tag{2.3.124}$$

The two leading order solutions are

$$
\begin{aligned}
f_0^R\,(z) &= 1, \\
g_0^R\,(z) &= \log\left(1-z^3\right).
\end{aligned}
\tag{2.3.125}
$$

The Wronskian between these solutions is

$$W_0\,(z) = \frac{3\,z^2}{z^3-1}. \tag{2.3.126}$$

According to the boundary conditions (2.3.105), the wave function in the right region is regular at $z = 1$. The corresponding perturbative expansion of the wave function is then

$$\psi^R(z) = f_0^R\,(z) + \sum_{k\geq 1} f_K^R\,(z)\,\alpha^k. \tag{2.3.127}$$

The corrections $f_k^R\,(z)$ are computed with the help of (1.1.26), where the constants c_k are fixed by the regularity condition at $z = 1$. The integrals in (1.1.26) can be described in terms of the multiple polylogarithms in several variables:

$$\mathrm{Li}_{s_1,\ldots,s_n}\,(z_1,\ldots,z_n) = \sum_{k_1>k_2>\cdots>k_n\geq 1}^{\infty} \frac{z_1^{k_1}\ldots z_n^{k_n}}{k_1^{s_1}\ldots k_n^{s_n}}. \tag{2.3.128}$$

For $s_1 \geq 2$, these functions satisfy

$$z_1\,\partial_{z_1}\mathrm{Li}_{s_1,\ldots,s_n}\,(z_1,\ldots,z_n) = \mathrm{Li}_{s_1-1,\ldots,s_n}\,(z_1,\ldots,z_n)\,, \tag{2.3.129}$$

and for $s_1 = 1$, $k \geq 2$,

$$(1-z_1)\,\partial_{z_1}\mathrm{Li}_{1,s_2,\ldots,s_n}(z_1,\ldots,z_n) = \mathrm{Li}_{s_2,\ldots,s_n}(z_1 z_2, z_3,\ldots,z_n)\,. \tag{2.3.130}$$

The weight and level of $\mathrm{Li}_{s_1,\ldots,s_n}(z_1,\ldots,z_n)$ are $s_1+\cdots+s_n$ and n. When taking the integrals in (1.1.26) with the input from this section, we will only encounter multiple polylogarithms with $s_1 = s_2 = \cdots = s_n = 1$ (see Appendix B.3 for more details). In this case, the weight and level are the same. Moreover, all arguments z_i with $i \geq 2$ are constants and can take one of the three possible values: 1, u_1, and u_2. These constants are the third roots of unity

$$u_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \qquad u_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \tag{2.3.131}$$

that arise in the following decomposition of $g_0^R(z)$:

$$g_0^R(z) = \log(1-z) + \log(1-u_1 z) + \log(1-u_2 z)\,. \tag{2.3.132}$$

Similarly to the previous cases with multiple polylogarithms, the corrections $f_k^R(z)$ at order α^k are described in terms of functions $\mathrm{Li}_{s_1,\ldots,s_n}(z_1,\ldots,z_n)$ of weight k and lower. This allows us to construct a linear basis of functions, in which $f_k^R(z)$ can be expanded:

$$\begin{gathered}
\frac{\sum_{m=-k_1}^{l_1} \zeta_m^R\, z^m}{(1-u_1 z)^{i_1}(1-u_2 z)^{j_1}} \log(1-z)^{p_1}\log(1-u_1 z)^{p_2}\log(1-u_2 z)^{p_3}\,,\\
\frac{\sum_{m=-k_2}^{l_2} \xi_m^R\, z^m}{(1-u_1 z)^{i_2}(1-u_2 z)^{j_2}} \log(1-z)^{p_4}\log(1-u_1 z)^{p_5}\log(1-u_2 z)^{p_6}\,\mathrm{Li}_{\{1\}_n}(z_1,z_2,\ldots,z_n)\,,
\end{gathered} \tag{2.3.133}$$

where $i_{1,2},\ j_{1,2},\ k_{1,2},\ l_{1,2},\ p_j$ are non-negative integers, and $0 \leq p_1+p_2+p_3 \leq k$, $0 \leq p_4+p_5+p_6+n \leq k$. Since the first argument in $\mathrm{Li}_{\{1\}_n}(z_1,z_2,\ldots,z_n)$ can take one of the three possible forms

$$z_1 = z, \quad z_1 = u_1 z, \quad \text{or} \quad z_1 = u_2 z, \tag{2.3.134}$$

we have 3^k functions that can enter the basis at level $k \geq 2$. However, this number is reduced due to the identities that involve multiplication by ordinary logarithm functions $\log(1-z)$, $\log(1-u_1 z)$, and $\log(1-u_2 z)$ (see Appendix B.3). These identities allow us to use only two forms of the first argument $z_1 = u_1 z$ and $z_1 = u_2 z$. The reduced number of multiple polylogarithms that enter the basis is $8 \times 3^{k-3}$ for $k \geq 3$, and just 3 for $k = 2$:

$$\mathrm{Li}_{1,1}(u_1 z, u_1)\,, \quad \mathrm{Li}_{1,1}(u_1 z, u_2)\,, \quad \mathrm{Li}_{1,1}(u_2 z, u_1)\,. \tag{2.3.135}$$

Using `Mathematica`, we compute 7 corrections $f_k^R(z)$; the first two are

$$
\begin{aligned}
f_1^R(z) =& \frac{\omega_0}{\sqrt{3}} \left(u_1 \log\left(1-u_1 z\right) - u_2 \log\left(1-u_2 z\right)\right), \\
f_2^R(z) =& -\frac{m}{3\,z} - \frac{i\,\omega_0^2}{3\sqrt{3}} \left[\, \mathrm{Li}_{1,1}\left(u_1 z, u_1\right) + u_1\, \mathrm{Li}_{1,1}\left(u_2 z, u_1\right) - u_2\, \mathrm{Li}_{1,1}\left(u_1 z, u_2\right) \right] \\
&+ \frac{i\,\omega_0^2}{3\sqrt{3}} \left[\log\left(1-u_1 z\right)^2 - \log\left(1-u_2 z\right)^2 - u_1 \log\left(1-u_1 z\right) \log\left(1-u_2 z\right)\right] \\
&+ \frac{i\,\omega_0^2}{3\sqrt{3}} \log\left(1-z\right) \left[\, u_1 \log\left(1-u_2 z\right) - u_2 \log\left(1-u_1 z\right) \right] + \frac{\omega_1 - i\,\omega_0^2}{\sqrt{3}} \log\left(1-u_2 z\right) \\
&+ \frac{i\,u_2\,\omega_0^2 - u_1\,\omega_1}{\sqrt{3}} \log\left(1-z\right) + b_2^R\, g_0^R\left(z\right),
\end{aligned}
$$

where

$$
b_2^R = \frac{u_1\,\omega_1}{\sqrt{3}} + \frac{i\,\omega_0^2}{3\sqrt{3}} \left[\, u_2 \log\left(1-u_1\right) - u_1 \log\left(1-u_2\right) - 3\,u_2 \,\right]. \tag{2.3.136}
$$

We estimate the following behavior of $f_k^R(z)$ as $z \to 0$ based on the obtained results:

$$
k \geq 1: \quad \begin{aligned} f_{2k-1}^R\left(z\right) &\sim z^{2-k}, \\ f_{2k}^R\left(z\right) &\sim z^{-k}. \end{aligned} \tag{2.3.137}
$$

Thus, to avoid the reshuffling of terms, we choose the gluing point between the middle and the right region to be $z = \alpha^{1/4}$.

2.3.4 Results for QNM frequencies

We need two continuity conditions to determine the QNM frequencies, at $z = \alpha^{1/4}$ and $z = \alpha$:

$$
\begin{aligned}
\psi^M\left(\alpha^{-3/4}\right) &= C_R^M\left(\alpha\right) \psi^R\left(\alpha^{1/4}\right), \\
\partial_z \psi^M\left(z/\alpha\right)\Big|_{z=\alpha^{1/4}} &= C_R^M\left(\alpha\right) \partial_z \psi^R\left(z\right)\Big|_{z=\alpha^{1/4}},
\end{aligned} \tag{2.3.138}
$$

$$
\begin{aligned}
\psi^L\left(1/3 + \left(\alpha\, m\right)^{-1}\right) &= C_M^L\left(\alpha\right) \psi^M\left(1\right), \\
\partial_z \psi^L\left(1/3 + z\left(\alpha^2 m\right)^{-1}\right)\Big|_{z=\alpha} &= C_M^L\left(\alpha\right) \partial_z \psi^M\left(z/\alpha\right)\Big|_{z=\alpha}.
\end{aligned} \tag{2.3.139}
$$

The first condition in (2.3.138) is used to fix the integration constants c_k^M, and the second one gives the coefficients ω_k in the QNM frequency expansion (2.3.109). The first seven computed orders of the wave function expansion in the right region allow us to determine

ω_k up to $k=6$:

$$
\begin{aligned}
\omega_0 =& \sqrt{\frac{m+2}{2}}, \\
\omega_1 =& -\frac{im}{6}, \\
\omega_2 =& \frac{\sqrt{2}\,m}{36\sqrt{m+2}} + \frac{m\sqrt{m+2}}{108\sqrt{2}}\left[15+\sqrt{3}\,\pi - 9\,\log(3)\right], \\
\omega_3 =& -\frac{m\,(m+2)}{18\sqrt{3}}\left[\mathrm{Li}_{1,1}(u_1,u_1) + u_1\mathrm{Li}_{1,1}(u_2,u_1) - u_2\mathrm{Li}_{1,1}(u_1,u_2)\right] \\
&+ \frac{m\,(m+2)}{1296\sqrt{3}}\left[\pi^2 - 6i\,\pi\log(3) + 9\,(u_2 - 3\,u_1)\log(3)^2\right] \\
&+ \frac{im\,(m+3)}{162}\left[9+\sqrt{3}\,\pi - 9\log(3)\right], \\
\omega_4 =& -\frac{i\,m\,(m+2)^{3/2}}{54\sqrt{6}}\Big[\mathrm{Li}_{\{1\}_3}(u_1,u_1,u_1) - u_1\,\mathrm{Li}_{\{1\}_3}(u_1,u_1,1) - u_1\,\mathrm{Li}_{\{1\}_3}(u_1,1,u_2) \\
&-2\,u_2\,\mathrm{Li}_{\{1\}_3}(u_1,u_2,1) - (\mathbf{u_1}\leftrightarrow\mathbf{u_2})\Big] + \dots\,, \\
\omega_5 =& \frac{m\,(m+2)^2}{162\sqrt{3}}\Big[\mathrm{Li}_{\{1\}_4}(u_1,u_1,u_1,1) + u_2\,\mathrm{Li}_{\{1\}_4}(u_1,1,u_1,u_1) - 2\,\mathrm{Li}_{\{1\}_4}(u_1,1,1,u_2) \\
&- u_1\,\mathrm{Li}_{\{1\}_4}(u_1,1,u_2,1) - 2\,u_2\,\mathrm{Li}_{\{1\}_4}(u_1,u_2,1,1) - u_1\,\mathrm{Li}_{\{1\}_4}(u_1,u_2,u_2,1) \\
&- (\mathbf{u_1}\leftrightarrow\mathbf{u_2})\Big] + \frac{m\,(m+2)^2}{486\sqrt{3}}\Big[3\,\mathrm{Li}_{\{1\}_4}(u_1,1,u_1,1) + 6\,u_1\,\mathrm{Li}_{\{1\}_4}(u_1,1,1,u_1) \\
&-2\,u_2\,\mathrm{Li}_{\{1\}_4}(u_2,u_2,u_1,1)\Big] + \dots\,,
\end{aligned}
\tag{2.3.140}
$$

where we shortened the results for ω_4 and ω_5 for readers convenience. The full results, including the result for ω_6, can be found in the `Mathematica` files on `https://github.com/GlebAminov/BH_PolyLog`. Notice that, as compared to the QNM frequencies computed in Sec. 2.1 and Sec. 2.2, here the frequencies involve different irrational numbers, for instance, $\log 3$, $\sqrt{3}$, as well as colored multiple zeta values of level 3.

Upon taking the scaling limit

$$
R_h \to \infty, \quad \ell \to \infty, \quad \frac{2\,\ell}{3\,R_h} \to \mathfrak{q}, \tag{2.3.141}
$$

where $\mathfrak{q}$ stays constant, we reproduce the results for the QNM frequencies of the M2-brane in the AdS_4 background (see Table IV in [175]) which are directly linked to hydrodynamics [171, 173, 174, 172]. Also, the following rescaling of the frequency is needed:

$$
\mathfrak{w} = \frac{2\,\omega}{3\,R_h}. \tag{2.3.142}
$$

Applying this limit to (2.3.140), we obtain an expansion of $\mathfrak{w}$ in $\mathfrak{q}$:

$$\mathfrak{w} = \sum_{k \geq 1} \mathfrak{w}_k \, \mathfrak{q}^k, \tag{2.3.143}$$

where $\mathfrak{w}_1$, $\mathfrak{w}_2$, and $\mathfrak{w}_3$ agree with the results from [175], and the new results are

$$\begin{aligned}
\mathfrak{w}_4 =& -\frac{\sqrt{3}}{16}\left[\mathrm{Li}_{1,1}\left(u_1,u_1\right)+u_1\mathrm{Li}_{1,1}\left(u_2,u_1\right)-u_2\mathrm{Li}_{1,1}\left(u_1,u_2\right)\right]+\frac{72\,i\sqrt{3}+24\,i\,\pi+\pi^2}{384\sqrt{3}}\\
&-\frac{12\,i\sqrt{3}+i\,\pi}{64\sqrt{3}}\log\left(3\right)+\frac{\sqrt{3}}{128}\left(u_2-3\,u_1\right)\log\left(3\right)^2,\\
\mathfrak{w}_5 =& -\frac{i\sqrt{3}}{32\sqrt{2}}\Big[\mathrm{Li}_{\{1\}_3}\left(u_1,u_1,u_1\right)-u_1\,\mathrm{Li}_{\{1\}_3}\left(u_1,u_1,1\right)-u_1\,\mathrm{Li}_{\{1\}_3}\left(u_1,1,u_2\right)\\
&-2\,u_2\,\mathrm{Li}_{\{1\}_3}\left(u_1,u_2,1\right)-\left(\mathbf{u_1}\leftrightarrow\mathbf{u_2}\right)\Big]+\dots,\\
\mathfrak{w}_6 =& \frac{\sqrt{3}}{64}\Big[\mathrm{Li}_{\{1\}_4}\left(u_1,u_1,u_1,1\right)+u_2\,\mathrm{Li}_{\{1\}_4}\left(u_1,1,u_1,u_1\right)-2\,\mathrm{Li}_{\{1\}_4}\left(u_1,1,1,u_2\right)\\
&-u_1\,\mathrm{Li}_{\{1\}_4}\left(u_1,1,u_2,1\right)-2\,u_2\,\mathrm{Li}_{\{1\}_4}\left(u_1,u_2,1,1\right)-u_1\,\mathrm{Li}_{\{1\}_4}\left(u_1,u_2,u_2,1\right)\\
&-\left(\mathbf{u_1}\leftrightarrow\mathbf{u_2}\right)\Big]+\frac{1}{64\sqrt{3}}\Big[3\,\mathrm{Li}_{\{1\}_4}\left(u_1,1,u_1,1\right)+6\,u_1\,\mathrm{Li}_{\{1\}_4}\left(u_1,1,1,u_1\right)\\
&-2\,u_2\,\mathrm{Li}_{\{1\}_4}\left(u_2,u_2,u_1,1\right)\Big]+\dots,
\end{aligned} \tag{2.3.144}$$

where we shortened the results for $\mathfrak{w}_5$ and $\mathfrak{w}_6$ for readers convenience. The full results, including the result for $\mathfrak{w}_7$, can be found in the `Mathematica` files on `https://github.com/GlebAminov/BH_PolyLog`. The numerical values of these coefficients are

$$\begin{aligned}
\mathfrak{w}_1 &= \frac{1}{\sqrt{2}},\\
\mathfrak{w}_2 &= -\frac{i}{4}\,,\\
\mathfrak{w}_3 &= 0.155473446153645...,\\
\mathfrak{w}_4 &= 0.067690388847266...\cdot i,\\
\mathfrak{w}_5 &= -0.010733416957692...,\\
\mathfrak{w}_6 &= 0.013959543659902...\cdot i,\\
\mathfrak{w}_7 &= -0.016615814626711....
\end{aligned} \tag{2.3.145}$$

These alternate between real and imaginary parts, precisely as predicted in [166, 176].

2.4 Perturbations of asymptotically flat Schwarzschild black holes in four dimensions

In this section, we study the same class of linear perturbations around the asymptotically flat Schwarzschild black hole of mass M that are described by the Regge-Wheeler equation (2.1.5) with

$$f(r) = 1 - \frac{2M}{r}. \tag{2.4.146}$$

This can be conveniently rewritten as a $SU(2)$ Seiberg-Witten quantum curve with three fundamental hypermultiplets ($N_f = 3$) [18, 19] or, equivalently, as a confluent Heun equation [6].

The Regge-Wheeler equation (2.1.5) can be rewritten in the form of a quantum spectral curve

$$\widehat{H}\,\Psi\,(x) = E\,\Psi\,(x) \tag{2.4.147}$$

with the following Hamiltonian:

$$\widehat{H} = \hat{p}^2 + t^{1/2}\mathrm{e}^{x}\prod_{j=1}^{2}\left(i\,\hat{p} + M_j + \frac{\hbar}{2}\right) + t^{1/2}\mathrm{e}^{-x}\left(i\,\hat{p} + M_3 - \frac{\hbar}{2}\right). \tag{2.4.148}$$

After fixing $\hbar = 1$, the spectral curve parameters such as energy, masses, and the coupling constant can be written in terms of the black hole parameters ℓ, s, and $M\omega$:

$$E = -\left(\ell + \frac{1}{2}\right)^2, \quad M_1 = -s, \quad M_2 = s, \quad M_3 = 0, \quad t = -4\,i\,M\omega. \tag{2.4.149}$$

The main goal of this section is to find an analytic description of scattering amplitudes for each type of perturbation field. To do so, we utilize three complementary approaches: an analytic small-frequency expansion, the study of the monodromy properties, and the correspondence with the Seiberg-Witten theory [43].

The small parameter, which we denote with t, is proportional to the frequency ω of the perturbation field, leading to a small-frequency expansion of the spectral problem. Around the black hole horizon, we select the local solution representing the purely incoming wave. Near spatial infinity, the local solution is represented as a linear combination of the purely outgoing and purely incoming waves:

$$\psi_{(\infty)} = \mathcal{A}\,\psi^{\text{out}}_{(\infty)} + \mathcal{B}\,\psi^{\text{in}}_{(\infty)}, \tag{2.4.150}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the reflection and incidence coefficients, respectively. The elements of the scattering matrix, which we call *scattering elements* for brevity, $S \equiv S_{\ell,s}$ are then defined as

$$S = \frac{\mathcal{A}}{\mathcal{B}}. \tag{2.4.151}$$

These have been studied semi-analytically [180, 181, 107, 109, 108, 182, 183, 184] and, most recently, via gauge theory methods [185]. Closely related are the studies of Green functions in Schwarzschild spacetime [186, 187, 188, 189]. The monodromy of the two solutions $\psi^{\text{out}}_{(\infty)}$ and $\psi^{\text{in}}_{(\infty)}$ around spatial infinity leads to a partial differential equation on the scattering elements. Solving this equation, it is possible to find the exact $\log t$ dependence of S. The remaining integration constant $\phi(t)$ is given by its Taylor expansion in t and can be further related to the Seiberg Witten B-period [190, 191, 24, 25, 26, 27, 28]. We perform explicit checks of this relation and specify the resummations that occur to express the instanton expansions as small-frequency expansions. As a result, an alternative expression of the scattering amplitudes in terms of the Nekrasov-Shatashvilli free energy [24] is obtained, which aligns with the recent studies of the Kerr-Compton amplitudes carried out in [185]. The poles of the scattering elements will provide results for the quasinormal mode (QNM) frequencies $\omega_{n,\ell,s}$.

2.4.1 Regge-Wheeler equation in different local regions

The first goal is to find local solutions around the Regge-Wheeler equation's regular and irregular singular points in the small-frequency regime. For that purpose, it is necessary to write down the corresponding differential equations in local variables z_R, z_M, and z_L. A wave function transformation is also performed to rewrite the Regge-Wheeler equation as a quantum spectral curve, simplifying the solution in the near-horizon region.

Introducing the local variable

$$z_R = \frac{2\,M}{r} \tag{2.4.152}$$

and redefining the radial part of the perturbation $R\,(r)$ in the Regge-Wheeler equation

$$R(r) = z_R^{-(t+1)/2}\,(1-z_R)^{\frac{t}{2}}\,\mathrm{e}^{t/(2z_R)}\psi(z_R), \tag{2.4.153}$$

we arrive at the differential equation in the near-horizon region:

$$\psi''_{\text{hor}} + \frac{z_R(2\,z_R-1)+t}{z_R^2(z_R-1)}\,\psi'_{\text{hor}} + \frac{(2\ell+1)^2\,z_R + \left(1-4\,s^2\right)z_R^2 - 2\,t}{4\,z_R^3(z_R-1)}\,\psi_{\text{hor}} = 0, \tag{2.4.154}$$

where $\psi_{\text{hor}} = \psi(z_R)$ is the local wave function. The boundary condition at the horizon becomes

$$\psi_{\text{hor}} = \psi^{\text{in}}_{\text{hor}} \sim 1, \quad z_R \to 1. \tag{2.4.155}$$

To set up the perturbative approach, one should be able to write the leading-order solutions of (2.4.154) and their Wronskian in terms of elementary functions. This is indeed possible for integer values of ℓ and s such that $\ell \geq s \geq 0$:

$$\begin{aligned} f_0^{\text{hor}}(z_R) &= \sum_{k=0}^{\ell-s} \frac{(s-\ell)_k\,(-\ell-s)_k}{(-2\ell)_k\,k!}\,z_R^{k-\ell-1/2}, \\ g_0^{\text{hor}}(z_R) &= \sum_{m=-s}^{\ell-1} a_{s\ell m}\,z_R^{-m-1/2} + \log{(1-z_R)}\sum_{m=s}^{\ell} b_{s\ell m}\,z_R^{-m-1/2}, \\ W_0^{\text{hor}}(z^R) &= \frac{2\,\ell+1}{z_R(1-z_R)}, \end{aligned} \tag{2.4.156}$$

where the constants $a_{s\ell m}$, $b_{s\ell m}$ are

$$\begin{aligned} a_{s\ell m} &= -\, b_{s\ell m}\left(H_{\ell+s}+H_{\ell-s}-H_{m+s}-H_{m-s}\right), \\ b_{s\ell m} &= \frac{(-1)^{\ell+m+1}}{(m+s)!\,(m-s)!}\,\frac{(2\,\ell+1)!}{(\ell+s)!\,(\ell-s)!}\,\frac{(\ell+m)!}{(\ell-m)!}. \end{aligned} \tag{2.4.157}$$

The local variable in the near-spatial infinity region is

$$z_L = \frac{t}{z_R}. \tag{2.4.158}$$

The corresponding local wave function $\psi_{(\infty)} = \psi\,(z_L)$ satisfies the following differential equation:

$$\psi''_{(\infty)} + \frac{z_L+1}{z_L-t}\,\psi'_{(\infty)} + \frac{(2\ell+1)^2 z_L - 2\,z_L^2 + t\left(1-4s^2\right)}{4z_L^2(t-z_L)}\,\psi_{(\infty)} = 0, \tag{2.4.159}$$

and the boundary condition at the spatial infinity is

$$\psi_{(\infty)} = \psi^{\text{out}}_{(\infty)} \sim \mathrm{e}^{-z_L}\,z_L^{-t-1/2}, \quad z_L \to \infty. \tag{2.4.160}$$

The behavior of the incoming wave at spatial infinity is needed too:

$$\psi^{\text{in}}_{(\infty)} \sim z_L^{-1/2}, \quad z_L \to \infty. \tag{2.4.161}$$

The leading order solutions and their Wronskian in the near-spatial infinity region are

$$\begin{aligned} f_0^{(\infty)}(z_L) &= \mathrm{e}^{-z_L}\,z_L^{-\ell-1/2}\,p_\ell(z_L), \\ g_0^{(\infty)}(z_L) &= z_L^{-\ell-1/2}\,q_\ell(z_L), \\ W_0^{(\infty)}(z_L) &= (-1)^\ell\,\frac{\mathrm{e}^{-z_L}}{z_L}, \end{aligned} \tag{2.4.162}$$

where $p_\ell(z_L)$ and $q_\ell(z_L)$ are the following polynomials of degree ℓ:

$$\begin{aligned} p_\ell(z_L) &= \sum_{m=0}^{\ell} \binom{\ell-\left\lfloor \frac{m+1}{2}\right\rfloor}{\left\lfloor \frac{m}{2}\right\rfloor} \frac{2^{\ell-m}\left(-2\left\lfloor \frac{m}{2}\right\rfloor+2\ell-1\right)!!}{\left(2\left\lfloor \frac{m+1}{2}\right\rfloor-1\right)!!}\,z_L^m, \\ q_\ell(z_L) &= \sum_{m=0}^{\ell} \binom{\ell-\left\lfloor \frac{m+1}{2}\right\rfloor}{\left\lfloor \frac{m}{2}\right\rfloor} \frac{2^{\ell-m}\left(-2\left\lfloor \frac{m}{2}\right\rfloor+2\ell-1\right)!!}{\left(2\left\lfloor \frac{m+1}{2}\right\rfloor-1\right)!!}\,(-z_L)^m. \end{aligned} \tag{2.4.163}$$

In the intermediate region, the local variable is

$$z_M = \frac{t^{1/2}}{z_R}, \tag{2.4.164}$$

and the corresponding wave function $\psi_{\text{mid}} = \psi(z_M)$ satisfies the differential equation

$$\psi''_{\text{mid}} + \frac{t^{1/2}\,z_M+1}{z_M-t^{1/2}}\,\psi'_{\text{mid}} + \frac{(2\ell+1)^2 z_M + t^{1/2}\left(1-4s^2-2z_M^2\right)}{4z_M^2\left(t^{1/2}-z_M\right)}\,\psi_{\text{mid}} = 0. \tag{2.4.165}$$

The corresponding leading order solutions and their Wronskian are given by

$$\begin{aligned} f_0^{\text{mid}}(z_M) &= z_M^{\ell+\frac{1}{2}}, \\ g_0^{\text{mid}}(z_M) &= z_M^{-\ell-\frac{1}{2}}, \\ W_0^{\text{mid}}(z_M) &= -\frac{2\ell+1}{z_M}. \end{aligned} \tag{2.4.166}$$

2.4.2 Local wave functions

The polylog approach allows us to compute the wave function ψ_{hor} perturbatively in the parameter t up to any given order (the constraint being the exponential growth of the number of multiple polylogarithm functions entering each order). The particular functions appearing in the near-horizon region are multiple polylogarithms in a single variable

$$\text{Li}_{s_1,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1}^{\infty} \frac{z^{k_1}}{k_1^{s_1}\ldots k_n^{s_n}}, \tag{2.4.167}$$

which have been extensively studied and have many known properties [112, 113, 192, 193, 194] . To give a taste of the results, we present a few computed orders of the wave function in the case $\ell = s = 0$:

$$\begin{aligned} \psi_{\text{hor}}^{\text{in}}(z_R) &= z_R^{-1/2} + \frac{t}{2}\, z_R^{-1/2}\left(\log(z_R) - z_R^{-1}\right) + t^2\, \psi_2^{\text{hor}}(z_R) + \mathcal{O}\left(t^3\right), \\ \psi_2^{\text{hor}}(z_R) &= -\frac{\text{Li}_2(z_R)}{4\, z_R^{1/2}} + \frac{\log(z_R)\,\text{Li}_1(z_R)}{4\, z_R^{1/2}} + \frac{5\, z_R + 4}{24\, z_R^{5/2}} - \frac{(11\, z_R + 6)\log(z_R)}{24\, z_R^{3/2}} + \frac{\log^2(z_R)}{8\, z_R^{1/2}}. \end{aligned} \tag{2.4.168}$$

The integration constants c_k^R are fixed in each order in t by requiring regularity as $z_R \to 1$ and expressed in terms of multiple zeta values of weight less or equal to k. Depending on the quantum numbers, the near-horizon wave function was computed up to orders $t_{\text{max}}^R = 13, 14, 15$.

The wave function in the intermediate region is the simplest one since this region includes two regular singularities at $z_M = 0$ and $z_M = \infty$ in the leading order in t. Each order of ψ_{mid} is a Laurent polynomial in $z_M^{1/2}$ and a polynomial in $\log z_M$. The first few orders of the t-expansion in the case $\ell = s = 0$ are

$$\psi_{\text{mid}}(z_M) = z_M^{1/2} - \frac{t^{1/2}}{2}\, z_M^{3/2} + \frac{t}{6}\, z_M^{1/2}\left(z_M^2 - 3\log z_M\right) + \mathcal{O}\left(t^{3/2}\right). \tag{2.4.169}$$

The integration constants c_k^M are determined by the following continuity condition between the two local wave functions $\psi_{\text{hor}}^{\text{in}}$ and ψ_{mid}:

$$\partial_{z_R} \log\left(\frac{\psi_{\text{hor}}^{\text{in}}}{\psi_{\text{mid}}}\right)\Bigg|_{z_R=\lambda} = 0, \tag{2.4.170}$$

where the gluing point λ can be placed anywhere in the interval $\left(t^{1/2}, t^0\right)$, and where t is considered to be real for simplicity (until the computation of the QNMs). Putting λ closer to $t^{1/2}$ would increase the maximum order to which the near-horizon wave function needs to be computed. Thus, we choose $\lambda = t^{1/6}$, which is far enough from $t^{1/2}$ and not too close to t^0. With this choice of λ, the maximum required order in t for the intermediate wave function is at least $t^M_{\max} \sim 5\, t^R_{\max}/2$.

Since radial infinity is an irregular singular point of the differential equation, we must modify the polylog approach to find the local wave function in the near-spatial infinity region. Instead of multiple polylogarithms, new special functions appear in the perturbative expansion - multiple polyexponential integrals $\mathrm{ELi}_{s_1,\ldots,s_n}(z)$. These functions can be defined as iterated integrals of exponential integral $\mathrm{Ei}(z)$ and are much less studied (for the study of related functions see [114, 115, 116, 117, 118, 119, 120]). As with multiple polylogarithms, the level of $\mathrm{ELi}_{s_1,\ldots,s_n}(z)$ is n and the weight is $s_1 + \cdots + s_n$. The first special function of this kind is the exponential integral itself:

$$\mathrm{ELi}_1(z) \equiv \mathrm{Ei}(z) = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k!\, k}, \quad |\mathrm{Arg}(-z)| < \pi. \tag{2.4.171}$$

The functions with higher weight are defined recursively as

$$\begin{aligned} s_1 = 1\colon \quad \mathrm{ELi}_{1,s_2,\ldots,s_n}(z) &= -\int_{-\infty}^{z} \frac{\mathrm{e}^t}{t}\, \mathrm{ELi}_{s_2,\ldots,s_n}(-t)\, \mathrm{d}t, \\ s_1 > 1\colon \quad \mathrm{ELi}_{s_1,s_2,\ldots,s_n}(z) &= \int_{-\infty}^{z} \frac{1}{t}\, \mathrm{ELi}_{s_1-1,s_2,\ldots,s_n}(t)\, \mathrm{d}t. \end{aligned} \tag{2.4.172}$$

All the main properties of these functions can be found in Appendix C.

The solution in the near-spatial infinity region can be written as a linear combination of the purely outgoing wave $\psi^{\text{out}}_{(\infty)}$ and the purely incoming wave $\psi^{\text{in}}_{(\infty)}$:

$$\psi_{(\infty)} = \mathcal{A}\, \psi^{\text{out}}_{(\infty)} + \mathcal{B}\, \psi^{\text{in}}_{(\infty)}. \tag{2.4.173}$$

The first few orders of this solution in the case of $\ell = s = 0$ are

$$\psi^{\text{out}}_{(\infty)}(z_L) = \mathrm{e}^{-z_L} z_L^{-1/2} + t\, \psi^{(\infty)}_{1,\text{out}}(z_L) + t^2\, \psi^{(\infty)}_{2,\text{out}}(z_L) + \mathcal{O}\left(t^3\right), \tag{2.4.174}$$

$$\psi^{(\infty)}_{1,\text{out}}(z_L) = \frac{1}{2}\, z_L^{-1/2}\, \mathrm{ELi}_1(-z_L) + \frac{1}{2}\, \mathrm{e}^{-z_L} z_L^{-3/2} - \mathrm{e}^{-z_L} z_L^{-1/2} \log z_L,$$

$$\begin{aligned} \psi^{(\infty)}_{2,\text{out}}(z_L) = &\frac{1}{4\, z_L^{1/2}} \left[\mathrm{e}^{-z_L}\, \mathrm{ELi}_{1,1}(z_L) + 2\, \mathrm{ELi}_2(-z_L)\right] - \frac{17\, z_L - 6}{24\, z_L^{3/2}}\, \mathrm{ELi}_1(-z_L) + \\ &+ \frac{\mathrm{e}^{-z_L}}{2\, z_L^{1/2}} \log(z_L) \left[\log(z_L) - \mathrm{e}^{z_L}\, \mathrm{ELi}_1(-z_L) - z_L^{-1}\right] + \mathrm{e}^{-z_L}\, \frac{13\, z_L + 8}{24\, z_L^{5/2}}, \end{aligned}$$

$$\psi^{\text{in}}_{(\infty)}(z_L) = z_L^{-1/2} + t\, \psi^{(\infty)}_{1,\text{in}}(z_L) + t^2\, \psi^{(\infty)}_{2,\text{in}}(z_L) + \mathcal{O}\left(t^3\right), \tag{2.4.175}$$

$$\psi^{(\infty)}_{1,\text{in}}(z_L) = \frac{1}{2}\, z_L^{-3/2} - \frac{1}{2}\, \mathrm{e}^{-z_L} z_L^{-1/2}\, \mathrm{ELi}_1(z_L),$$

$$\psi^{(\infty)}_{2,\text{in}}(z_L) = \frac{\mathrm{ELi}_{1,1}(-z_L)}{4\, z_L^{1/2}} + \frac{\mathrm{e}^{-z_L}}{24\, z_L^{1/2}} \left[\mathrm{e}^{z_L}\, \frac{11\, z_L + 8}{z_L^2} - \frac{17\, z_L + 6}{z_L}\, \mathrm{ELi}_1(z_L) + 12\, \mathrm{ELi}_2(z_L)\right].$$

To fix the integration constants c_k^L and determine the coefficients $\mathcal{A}$ and $\mathcal{B}$, we impose the continuity condition:

$$\partial_{z_L} \log\left(\frac{\psi_{(\infty)}}{\psi_{\text{mid}}}\right)\Bigg|_{z_L=\lambda} = 0, \tag{2.4.176}$$

where $\lambda = t^{1/6}$. In the above equation, one needs to know how polyexponential integrals (1.1.38) behave as $z \to 0$. As we show in Appendix C, $\text{ELi}_{s_1,\ldots,s_n}(z)$ admit relations similar to (1.1.35) with another set of functions we call undressed multiple polyexponential functions $el_{s_1,\ldots,s_n}(z)$ defined as

$$el_{s_1,s_2,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1} \frac{z^{k_1}}{k_1!\, k_1^{s_1}\, k_2^{s_2}\ldots k_n^{s_n}}. \tag{2.4.177}$$

As a result of (2.4.176), we obtain $\mathcal{A}$ and $\mathcal{B}$ in the form of series expansions in the parameter t, which also depend on the logarithm of t. Computing $\mathcal{B}/\mathcal{A}$ gives us the perturbative expansion of the inverse scattering elements S^{-1}. The first few orders in the case of $\ell = s = 0$ are

$$\begin{aligned} S^{-1}_{0,0} =& -1 + \frac{1}{2}(1 - i\,\pi - 2\gamma)\left(1 + \gamma\, t + \frac{\gamma^2}{2}\, t^2\right) t + \frac{1}{24}\left(9 - 5\, i\,\pi + 3\,\pi^2 + 12\,\gamma^2\right)(1 + \gamma\, t)\, t^2 \\ &+ \frac{1}{144}\left(-15 + 78\, i\,\pi + 35\,\pi^2 + 3\, i\,\pi^3 - 36\log t - 12\,\zeta(3) - 36\,\gamma - 24\,\gamma^3\right) t^3 + \mathcal{O}\left(t^4\right), \end{aligned} \tag{2.4.178}$$

where γ is the Euler–Mascheroni constant. The maximum order to which we were able to compute the inverse scattering elements is 14 for $\ell = s = 0$ and 13 for $\ell = s = 1, 2$. At the same time, the near-spatial infinity wave function was computed up to order $t^L_{\max} = 16$ for $\ell = s = 0$ and up to $t^L_{\max} = 15$ for higher values of ℓ and s. The first few orders of the inverse scattering elements in the cases $\ell = s = 1, 2$ are

$$\begin{aligned} S^{-1}_{1,1} =& 1 + \frac{1}{4}\left(2\, i\,\pi - 5 + 4\,\gamma\right)\left(1 + \gamma\, t + \frac{\gamma^2}{2}\, t^2\right) t \\ &+ \frac{1}{32}\left(25 - 16\gamma^2 - \frac{206\, i\,\pi}{15} - 4\,\pi^2\right)(1 + \gamma\, t)\, t^2 \\ &+ \frac{1}{96}\left(-33 + 16\,\gamma^3 + 14\, i\,\pi + \frac{37\,\pi^2}{15} - 2\, i\,\pi^3 + 8\,\zeta(3)\right) t^3 + \mathcal{O}\left(t^4\right), \end{aligned} \tag{2.4.179}$$

$$\begin{aligned} S^{-1}_{2,2} =& -1 + \frac{1}{6}\left(10 - 6\,\gamma - 3\, i\,\pi\right)\left(1 + \gamma\, t + \frac{\gamma^2}{2}\, t^2\right) t \\ &+ \frac{1}{72}\left(-100 + 36\,\gamma^2 + \frac{1779\, i\,\pi}{35} + 9\,\pi^2\right)(1 + \gamma\, t)\, t^2 \\ &+ \frac{1}{96}\left(73 - 16\,\gamma^3 - \frac{324\, i\,\pi}{7} - \frac{1244\,\pi^2}{105} + 2\, i\,\pi^3 - 8\,\zeta(3)\right) t^3 + \mathcal{O}\left(t^4\right). \end{aligned} \tag{2.4.180}$$

2.4.3 Monodromy in the near-spatial infinity region

Having computed the perturbative expansion of the near-spatial infinity wave function, one can study the monodromy of the two solutions $\psi^{\text{out}}_{(\infty)}$ and $\psi^{\text{in}}_{(\infty)}$. In the near-spatial infinity region, when $t \to 0$, the two regular singularities at $r = 0$ ($z_L = 0$) and $r = 2\,M$ ($z_L = t$) are very close. A small oriented loop around $z_L = 0$ incircling both singularities is equivalent to a loop around spatial infinity with opposite orientation. Thus, the only monodromy matrix available to us in the near-spatial infinity region is $\mathcal{M} \equiv \mathcal{M}_{(\infty)}$. Changing the branch of the logarithm by

$$\log z_L \to \log z_L + y, \tag{2.4.181}$$

induces the following monodromy transformation of local solutions:

$$\begin{pmatrix} \psi^{\text{out}}_{(\infty)} \\ \psi^{\text{in}}_{(\infty)} \end{pmatrix} \to \mathcal{M}\,(y) \begin{pmatrix} \psi^{\text{out}}_{(\infty)} \\ \psi^{\text{in}}_{(\infty)} \end{pmatrix}, \tag{2.4.182}$$

where the monodromy matrix is a 2×2 matrix:

$$\mathcal{M}(y) = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}. \tag{2.4.183}$$

Here, we treat the parameter y as a continuous variable, not necessarily equal to $2\,\pi i\,n$, $n \in \mathbb{Z}$. Thus, $\mathcal{M}$ is an element of a one-parameter Lie group with an identity element at $y = 0$. The corresponding Lie algebra generator is

$$\mu = \mathcal{M}'(0). \tag{2.4.184}$$

The determinant of $\mathcal{M}$ is

$$\det \mathcal{M} = \mathrm{e}^{-y\,t}, \tag{2.4.185}$$

which results in

$$\operatorname{Tr} \mu = -t. \tag{2.4.186}$$

Similar monodromy matrices were studied earlier using other methods in the context of the confluent Heun equation or Painlevé V equation [73, 74, 77, 195, 78]. As compared to the results in [195], one of the simpler differences is an additional factor of $\mathrm{e}^{-y\,t/2}$ in front of the monodromy matrix, which is due to the wave function transformation.

Each element of the monodromy matrix is obtained as a series expansion in t, which we computed up to order t^L_{max}. These perturbative expansions permit to determine the exact dependence of $\mathcal{M}$ on the parameter y:

$$\begin{aligned} \mathcal{M}_{11} &= \mathrm{e}^{-y\,t/2} \left(\cosh(y\,\beta) + \frac{(\mu_{11} - \mu_{22})}{2\beta} \sinh(y\,\beta) \right), \quad \mathcal{M}_{12} = \mathrm{e}^{-y\,t/2}\, \frac{\mu_{12}}{\beta} \sinh(y\,\beta), \\ \mathcal{M}_{22} &= \mathrm{e}^{-y\,t/2} \left(\cosh(y\,\beta) - \frac{(\mu_{11} - \mu_{22})}{2\beta} \sinh(y\,\beta) \right), \quad \mathcal{M}_{21} = \mathrm{e}^{-y\,t/2}\, \frac{\mu_{21}}{\beta} \sinh(y\,\beta), \end{aligned} \tag{2.4.187}$$

where μ_{ij} are the elements of the generator μ, and β is related to the determinant of μ:

$$\beta = \sqrt{\frac{t^2}{4} - \det \mu}. \tag{2.4.188}$$

In (2.4.187), the parameters that depend on quantum numbers ℓ and s are μ_{ij} and $\beta \equiv \beta_{\ell,s}$.

The perturbative expansions of the two local solutions allow us to determine the elements of μ up to the same order $t_{\max}^L$. For example, the first few orders in the t-expansion of μ_{11} in the case $\ell = s = 0$ are

$$\mu_{11} = -t + \frac{i\,\pi}{4}\,t^2 + \frac{\pi^2}{12}\,t^3 + \mathcal{O}\left(t^4\right). \tag{2.4.189}$$

The corresponding Taylor expansion for $\beta_{\ell,s}$ is

$$\beta_{0,0} = \frac{7}{24}\,t^2 - \frac{9449}{120\,960}\,t^4 + \frac{102\,270\,817}{2\,133\,734\,400}\,t^6 + \mathcal{O}\left(t^8\right). \tag{2.4.190}$$

It is possible to find the exact expressions for μ_{11} and μ_{12} in terms of $\beta_{\ell,s}$ and a new function $\xi_{\ell,s} = \xi_{\ell,s}(t)$:

$$\mu_{11}(t) = -\frac{t}{2} - i\,\beta\,\frac{\mathrm{e}^{-i\,\pi\,t} - \cos(2\,\pi\,\beta)}{\sin(2\,\pi\,\beta)}, \tag{2.4.191}$$

$$\mu_{12}(t) = \frac{2\,\pi\,\beta}{\sin\left(2\,\pi\,\beta\right)}\frac{\left(\frac{t}{2} - \beta\right)\xi_{\ell,s}}{\Gamma\left(1 + \frac{t}{2} + \beta\right)\Gamma\left(1 + \frac{t}{2} - \beta\right)}, \tag{2.4.192}$$

where $\xi_{\ell,s}$ satisfies

$$\xi_{\ell,s}(t)\,\xi_{\ell,s}(-t) = 1. \tag{2.4.193}$$

In the case $\ell = s = 0$, $\xi_{\ell,s}$ admits the following Taylor expansion:

$$\xi_{0,0}(t) = 1 - \frac{5}{6}\,t + \frac{25}{72}\,t^2 - \frac{23}{315}\,t^3 + \frac{41}{72\,576}\,t^4 + \mathcal{O}\left(t^5\right). \tag{2.4.194}$$

The remaining elements μ_{ij} can be obtained using the trace and the determinant of μ:

$$\mu_{11} + \mu_{22} = -t, \quad \mu_{12}\,\mu_{21} = \mu_{11}\,\mu_{22} + \beta^2 - \frac{t^2}{4}. \tag{2.4.195}$$

Following [195], we look at the eigenvalues of the monodromy matrix $\mathcal{M}$ rescaled by $\mathrm{e}^{y\,t/2}$:

$$\mathrm{e}^{y\,t/2}\mathcal{M} = U\begin{pmatrix} \mathrm{e}^{-\beta\,y} & 0 \\ 0 & \mathrm{e}^{\beta\,y} \end{pmatrix} U^{-1}, \tag{2.4.196}$$

which tells us that β is related to the flat modulus a of the corresponding Seiberg-Witten curve. However, β is obtained by resumming instanton corrections using the dictionary between the gauge theory quantities and the black hole parameters. To be more precise, a can be computed by inverting the Matone relation [196] perturbatively in the instanton parameter $\Lambda \equiv t$. The resulting instanton expansion depends on both Λ and the frequency ω (also proportional to t via (2.4.149)). In the $\ell = s = 0$ case, the following expansion for a can be derived:

$$a = \frac{1}{2}\sqrt{8\,\omega^2 - 1} + \frac{i\,\omega\,\Lambda}{4\sqrt{8\,\omega^2 - 1}} + \frac{\left(272\,\omega^4 + 70\,\omega^2 - 11\right)\Lambda^2}{64\,(8\,\omega^2 - 1)^{3/2}\,(8\,\omega^2 + 3)} + \mathcal{O}\left(\Lambda^3\right), \tag{2.4.197}$$

where we fixed $M = 1/2$ for simplicity. When rewriting this expansion using the single parameter t, *infinitely* many orders in Λ contribute to the same order in t. For example, one can check the following:

$$i\,a = -\ell - \frac{1}{2} + \mathcal{O}\left(t^2\right). \tag{2.4.198}$$

This leads us to the relation between a and $\beta_{\ell,s}$:

$$i\,a = -\ell - \frac{1}{2} - \beta_{\ell,s}. \tag{2.4.199}$$

To check this claim, we can approximately compute coefficients in front of t^2 and t^4 for $\ell = s = 0$ using 15 instanton orders in (2.4.197):

$$i\,a \simeq -\frac{1}{2} - 0.29170\,t^2 + 0.07816\,t^4 + \mathcal{O}\left(t^6\right), \tag{2.4.200}$$

which are indeed close to $-7/24 \simeq -0.29167$ and $9449/120\,960 \simeq 0.07812$. The resummation of infinitely many instantons comes from the poles of the Nekrasov partition function in a, and it starts at order $t^{2\ell+2}$ for higher values of ℓ. This means, for example, that for $\ell = 1$, the first two instantons provide an exact expression for a up to order t^2. Indeed, we have for $\ell = s = 1$

$$i\,a = -\frac{3}{2} - \frac{47}{240}\,t^2 + \mathcal{O}\left(t^4\right) \tag{2.4.201}$$

and

$$\beta_{1,1} = \frac{47}{240}\,t^2 - \frac{43\,908\,007}{1\,137\,024\,000}\,t^4 + \mathcal{O}\left(t^6\right). \tag{2.4.202}$$

In Appendix D, generic formulas are presented for the coefficients in front of $t^{2\,k}$, $k = 1, 2, 3$ in the expansion of $\beta_{\ell,s}$, which are valid for $\ell \geq k$ and can be computed via the instanton expansion.

2.4.4 Scattering elements

Applying the monodromy transformation (2.4.182) to the full solution in the near-spatial infinity region

$$\psi_{(\infty)} = \mathcal{A}\,\psi^{\text{out}}_{(\infty)} + \mathcal{B}\,\psi^{\text{in}}_{(\infty)}, \tag{2.4.203}$$

allows us to understand how the scattering elements $S = \mathcal{A}/\mathcal{B}$ behave under the shift of the logarithm $\log t \to \log t + y$:

$$S\left(t, \log t\right) \to \frac{\mathcal{M}_{11}\, S\left(t, \log t + y\right) + \mathcal{M}_{21}}{\mathcal{M}_{12}\, S\left(t, \log t + y\right) + \mathcal{M}_{22}}. \tag{2.4.204}$$

Since changing the branch of the logarithm should not affect any physical quantities, the following functional relation should hold:

$$\frac{\mathcal{M}_{11}\, S\left(t, \log t + y\right) + \mathcal{M}_{21}}{\mathcal{M}_{12}\, S\left(t, \log t + y\right) + \mathcal{M}_{22}} = S\left(t, \log t\right), \tag{2.4.205}$$

which can be verified in the small t expansion. It follows that the y-dependence is only apparent:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\mathcal{M}_{11}\, S\left(t, \log t + y\right) + \mathcal{M}_{21}}{\mathcal{M}_{12}\, S\left(t, \log t + y\right) + \mathcal{M}_{22}}\right]\Bigg|_{y=0} = 0. \tag{2.4.206}$$

At $y = 0$, the monodromy matrix is an identity matrix

$$\mathcal{M}_{11}|_{y=0} = \mathcal{M}_{22}|_{y=0} = 1, \quad \mathcal{M}_{12}|_{y=0} = \mathcal{M}_{21}|_{y=0} = 0, \tag{2.4.207}$$

which leads to the following first-order differential equation on the inverse scattering elements:

$$\frac{\partial}{\partial \log t}\, \frac{1}{S\left(t, \log t\right)} = \frac{\mu_{21}}{S\left(t, \log t\right)^2} + \frac{\mu_{11} - \mu_{22}}{S\left(t, \log t\right)} - \mu_{12}. \tag{2.4.208}$$

The exact dependence of the scattering elements on $\log t$ can thus be derived:

$$\frac{1}{S\left(t, \log t\right)} = -\frac{t + 2\,\mu_{11}}{2\,\mu_{21}} - \frac{\beta}{\mu_{21}} \tanh\left(\phi + \beta \log t\right), \tag{2.4.209}$$

where the integration constant is a function of t: $\phi = \phi(t)$. Computing $\phi(t)$ up to a certain order in t, we verify that it is related to the B-period of the corresponding Seiberg-Witten curve with some additional contributions. In terms of β, we have:

$$\begin{aligned}\phi(t) = i\,\pi\,\beta + \left(\ell + \frac{1}{2}\right)\log t + \log\frac{\Gamma\left(-2\,\ell - 2\,\beta\right)}{\Gamma\left(2\,\ell + 2\,\beta + 2\right)} - \frac{1}{2}\sum_{j=1}^{3}\log\frac{\Gamma\left(m_j - \ell - \beta\right)}{\Gamma\left(m_j + \ell + \beta + 1\right)} \\ -\frac{1}{2}\log\frac{\sin\left[\pi\left(\beta - t/2\right)\right]}{\sin\left[\pi\left(\beta + t/2\right)\right]} + \sum_{k\geq 0}\varphi_{\ell,s,k}\, t^{2\,k},\end{aligned} \tag{2.4.210}$$

where the hypermultiplet masses m_j are

$$m_1 = \frac{t}{2} + s, \quad m_2 = \frac{t}{2} - s, \quad m_3 = \frac{t}{2}, \tag{2.4.211}$$

and the coefficients φ_k can be computed perturbatively. The first few coefficients in the case of $\ell = s = 0$ are

$$\sum_{k\geq 0}\varphi_{0,0,k}\, t^{2\,k} = \log\left(\frac{7}{9}\right) - \frac{8587}{70\,560}\, t^2 + \frac{59\,423\,233}{995\,742\,720}\, t^4 + \mathcal{O}\left(t^6\right). \tag{2.4.212}$$

One might notice that $\phi\,(t)$ depends on $(\ell + 1/2)\log t$, which would seemingly violate the differential equation (2.4.208). However, $1/2\,\log t$ is canceled by the expansion of the log-Gamma functions, and the contribution of $\ell \log t$ can be rewritten as

$$\tanh\left(\ell \log t + x\right) = \frac{t^{2\,\ell}\,\mathrm{e}^{2\,x} - 1}{t^{2\,\ell}\,\mathrm{e}^{2\,x} + 1}, \tag{2.4.213}$$

which makes it invisible for the partial derivative with respect to $\log t$ in the small t expansion. Another consequence of (2.4.213) is that the contribution of $\log(t)$ to the expansion of the inverse scattering amplitudes is delayed to order $t^{2\,\ell+3}$, in agreement with (2.4.178).

Let us comment on the relation with the Seiberg-Witten B-period in more detail. We can see that $\phi\,(t)$ contains the perturbative part of the B-period, which is given by the a-derivative of the Nekrasov-Shatashvili (NS) free energy with $N_f = 3$ and $\hbar = 1$:

$$\frac{\partial\, F_{\text{pert}}}{\partial\, a} = -2\,a\log t - 2\,i\log\frac{\Gamma\,(1+2\,i\,a)}{\Gamma\,(1-2\,i\,a)} + i\sum_{j=1}^{3}\log\frac{\Gamma\,(m_j + i\,a + 1/2)}{\Gamma\,(m_j - i\,a + 1/2)}. \tag{2.4.214}$$

Taking into account relation (2.4.199) between a and β, we can write for $\phi\,(t)$:

$$\phi\,(t) + \beta\log t = i\,\pi\,\beta + \frac{i}{2}\frac{\partial\, F_{\text{pert}}}{\partial\, a} - \frac{1}{2}\log\frac{\sin\left[\pi\,(\beta - t/2)\right]}{\sin\left[\pi\,(\beta + t/2)\right]} + \sum_{k\geq 0}\varphi_{\ell,s,k}\,t^{2\,k}. \tag{2.4.215}$$

This leads to think that the sum

$$\varphi \equiv \varphi_{\ell,s} = \sum_{k\geq 0}\varphi_{\ell,s,k}\,t^{2\,k} \tag{2.4.216}$$

is related to the instanton part of the NS free energy. To confirm this relation, we compute the a-derivative of the instanton expansion and substitute (2.4.197) to get for $\ell = s = 0$:

$$\frac{\partial\, F_{\text{inst}}}{\partial\, a} = \frac{\sqrt{8\,\omega^2 - 1}}{8\,i\,\omega}\,\Lambda + \frac{\left(-1920\,\omega^6 - 912\,\omega^4 + 40\,\omega^2 + 45\right)\Lambda^2}{256\,\omega^2\sqrt{8\,\omega^2-1}\,\left(8\,\omega^2+3\right)^2} + \mathcal{O}\left(\Lambda^3\right). \tag{2.4.217}$$

Rewriting this as an expansion in t will result in resumming infinitely many orders in Λ. Using 15 instanton corrections, we get approximately:

$$\frac{i}{2}\frac{\partial\, F_{\text{inst}}}{\partial\, a} \simeq -0.25095 - 0.12213\,t^2 + 0.05985\,t^4 + \mathcal{O}\left(t^6\right), \tag{2.4.218}$$

where only even powers of t are present consistently with (2.4.212). All the numeric coefficients in the above instanton resummation also agree with (2.4.212), where

$$\varphi_{0,0,0} \simeq -0.25131, \quad \varphi_{0,0,1} \simeq -0.12170, \quad \varphi_{0,0,2} \simeq 0.05968. \tag{2.4.219}$$

In this case, the resummation of instantons starts at order $t^{2\ell}$, which is due to the a-derivative that increases the order of the poles in each instanton contribution. Thus, for

$\ell = 2$, the first two instantons should be enough to match the coefficient $\varphi_{2,s,1}$ exactly. Taking $\ell = s = 2$, we get

$$\frac{i}{2}\frac{\partial F_{\text{inst}}}{\partial a} = \frac{125}{1568}\, t^2 + \mathcal{O}\left(t^4\right), \tag{2.4.220}$$

which agrees with the expansion for $\varphi_{2,2}$:

$$\varphi_{2,2} = \frac{125}{1568}\, t^2 - \frac{53\,950\,959\,337}{2\,280\,051\,680\,256} t^4 + \mathcal{O}\left(t^6\right). \tag{2.4.221}$$

Plugging the relation between $\varphi_{\ell,s}$ and the instanton expansion back into (2.4.215), gives

$$\phi\,(t) + \beta \log t = i\,\pi\,\beta + \frac{i}{2}\frac{\partial F_{\text{pert}}}{\partial a} + \frac{i}{2}\frac{\partial F_{\text{inst}}}{\partial a} - \frac{1}{2}\log\frac{\sin\left[\pi\left(\beta - t/2\right)\right]}{\sin\left[\pi\left(\beta + t/2\right)\right]}, \tag{2.4.222}$$

which provides an alternative description for the scattering elements in terms of the NS free energy and the flat modulus a.

2.4.5 QNM frequencies

The poles of the scattering elements obtained in the previous section determine the quasinormal mode frequencies $\omega_n \equiv \omega_{n,\ell,s}$, provided the relation between t and ω given in (2.4.149). The corresponding condition for the inverse scattering elements is

$$\frac{1}{S\left(-4\,i\,M\omega_n\right)} = 0, \tag{2.4.223}$$

which we call the quantization condition for short. The latter can be rewritten with the help of (2.4.209) as

$$\frac{\mathrm{e}^{2\,\phi + 2\,\beta\,\log(t)} - 1}{\mathrm{e}^{2\,\phi + 2\,\beta\,\log(t)} + 1} = -\frac{t + 2\,\mu_{11}}{2\,\beta}. \tag{2.4.224}$$

Substituting the element of the monodromy generator (2.4.191), we get

$$\mathrm{e}^{2\,\phi + 2\,\beta\,\log(t)} = \frac{e^{i\,\pi(t+2\,\beta)} - 1}{1 - e^{i\,\pi(t-2\,\beta)}} = \mathrm{e}^{2\,i\,\pi\,\beta}\,\frac{\sin\left[\pi\left(\beta + \frac{t}{2}\right)\right]}{\sin\left[\pi\left(\beta - \frac{t}{2}\right)\right]}. \tag{2.4.225}$$

Now, we can use the expression (2.4.210) for $\phi\,(t)$ in terms of β, where $i\,\pi\,\beta$ and the logarithm of sines conveniently cancel with the right-hand side of (2.4.225):

$$\mathrm{e}^{2\,\varphi} t^{2\,\ell + 2\,\beta + 1}\,\frac{\Gamma\left(-2\,\ell - 2\,\beta\right)^2}{\Gamma\left(2\,\ell + 2\,\beta + 2\right)^2}\prod_{j=1}^{3}\frac{\Gamma\left(m_j + \ell + \beta + 1\right)}{\Gamma\left(m_j - \ell - \beta\right)} = 1. \tag{2.4.226}$$

By taking the logarithm, we can also rewrite (2.4.226) in a form equivalent to the quantization of the Seiberg-Witten B-period:

$$\frac{\partial F_{\text{pert}}}{\partial a} + \frac{\partial F_{\text{inst}}}{\partial a} = 2\,\pi\left(n + 1\right), \quad n \in \mathbb{Z}_{\geq 0}, \tag{2.4.227}$$

where a is related to β via (2.4.199).

Quantization condition (2.4.226) requires a simple input in the form of two Taylor expansions in t, where only even powers of t are present:

$$\beta_{\ell,s} = \sum_{k\geq 1} \beta_{\ell,s,k}\, t^{2k}, \quad \varphi_{\ell,s} = \sum_{k\geq 0} \varphi_{\ell,s,k}\, t^{2\,k}. \tag{2.4.228}$$

For given quantum numbers ℓ and s, the coefficients $\beta_{\ell,s,k}$ and $\varphi_{\ell,s,k}$ are rational numbers, except for $\varphi_{0,0,0} = \log(7/9)$. Moreover, it is possible to derive generic formulas for $\beta_{\ell,s,k}$, $\ell \geq k$ and $\varphi_{\ell,s,k}$, $\ell > k$ from the instanton expansion (see Appendix D). Since β is entirely determined by the monodromy in the near-spatial infinity region, the coefficients $\beta_{\ell,s,k}$ can be computed solely from the perturbative expansions of $\psi^{\text{out}}_{(\infty)}$ and $\psi^{\text{in}}_{(\infty)}$. The second Taylor expansion $\varphi_{\ell,s}$ is related to the Seiberg-Witten B-period and thus also requires the knowledge of the near-horizon wave function $\psi^{\text{in}}_{\text{hor}}$.

The radius of convergence r of the t-expansion of $\beta_{\ell,s}$ seems to depend on the angular quantum number ℓ: $r \equiv r_\ell$. For $\ell = 0$, we have approximately $|t| < r_0 \sim 1$. This radius might be related to the poles in the instanton expansion for a (2.4.197). The first pole appears at order Λ^2 and is of the form $8\,\omega^2 + 3$, which is equivalent to $3 - 2\,t^2$, provided $M = 1/2$. If this is true, then r_0 is indeed close to one: $r_0 = \sqrt{3/2}$. However, we don't know if the poles in the instanton expansion are meaningful for the black hole perturbation theory or if they are remnants of the inversion of the Matone relation. Acknowledging its speculative nature, one can continue the above argument for higher values of ℓ. The poles in each order of the instanton expansion are of the form $2\,i\,a \pm k$, $k \in \mathbb{N}$. We have for a in the leading order:

$$a^2 = 2\,\omega^2 - \left(\ell + \frac{1}{2}\right)^2 + \mathcal{O}(\Lambda)\,, \tag{2.4.229}$$

which corresponds to the poles in t at

$$t^2 = \frac{k^2 - (2\,\ell + 1)^2}{2}. \tag{2.4.230}$$

As we mentioned earlier, the pole at $t = 0$ leads to the resummation of the instanton expansion into (2.4.228). The next pole closest to $t = 0$ is given by $k = 2\,\ell$ with $\ell > 0$. Thus, we get the following estimation for the radius of convergence of $\beta_{\ell,s}$ in (2.4.228):

$$\ell > 0: \quad r_\ell = \sqrt{\frac{4\,\ell + 1}{2}}. \tag{2.4.231}$$

By taking the square root of the ratios $|\beta_{\ell,s,k}/\beta_{\ell,s,k+1}|$, we can also estimate the radius of absolute convergence r^{abs}_ℓ for $\ell = 0, 1, 2$:

$$r^{\text{abs}}_0 \simeq 1, \quad r^{\text{abs}}_1 \simeq 1.3, \quad r^{\text{abs}}_2 \simeq 1.6. \tag{2.4.232}$$

For comparison, we have

$$r_0 \simeq 1.2, \quad r_1 \simeq 1.6, \quad r_2 \simeq 2.1. \tag{2.4.233}$$

The above r_ℓ values allow us to compute the fundamental frequencies $\omega_{0,\ell,s}$ via (2.4.226) or (2.4.227). Using the t-expansions of $\beta_{\ell,s}$ and $\varphi_{\ell,s}$ from Appendix D, we get:

$$\begin{aligned} \omega_{0,0,0} &= \pm\mathbf{0.220911} - \mathbf{0.209792}i, \\ \omega_{0,1,1} &= \pm\mathbf{0.497}254 - \mathbf{0.185}939i, \\ \omega_{0,2,2} &= \pm\mathbf{0.747}289 - \mathbf{0.177}543i, \end{aligned} \tag{2.4.234}$$

where the digits in bold agree with the reference numerical results from [197] with the rounding in the last bold digit.

2.5 Summary

In this chapter, we considered different methods to analytically compute QNMs of different spacetime geometries. The gauge theory method, or instanton approach, is not new, as it was first introduced in the context of black hole perturbations in [43]. The polylog method, involving an expansion of the wave solutions in terms of multiple polylogarithms and multiple polyexponential integrals is, instead, a novelty. We remark that, however, other perturbative methods have been developed to study QNMs and scattering amplitudes, such as, for example, the method of matched asymptotic expansions. We investigated in detail the relation between our method and the instanton approach. In the same way, it would be important to relate it also with the MST method [107, 108, 109] in the case of asymptotically flat spacetime. The latter involves a matching of asymptotic expansions in which the local solutions are expressed as series of hypergeometric functions depending on an additional parameter ν - the *renormalized angular momentum*. The coefficients in such series expansions obey a three-term recurrence relation, which permits to express the parameter ν and the outgoing and incoming amplitudes as expansions in powers of $\epsilon = 2\,M\,\omega$ (which equals the instanton parameter that we denoted with t up to a constant factor of $-2\,i$).

The expansions provided by MST method were used in [184, 198, 199] to study the dynamical tidal response of black holes by matching them with the predictions of the point particle effective field theory. This tidal response of black holes is characterized by the so-called dynamical Love numbers. In particular, it was shown that the wave amplitude ratio factorizes into two parts: the near-zone, which carries information about finite-size effects due to the black hole, and the far-zone, which contains the relativistic post-Minkowskian corrections. This factorization of the scattering elements was then analyzed in [185], where it was rewritten in the gauge theory language. In this way, the connection between the instanton approach and the MST method was established, and the parameter ν of the MST method was identified with the gauge modulus a. By the same logic, our parameter β is also related to the renormalized angular momentum ν through (2.4.199). It would be essential to establish the exact form of this relation.

Chapter 3

Determinants of Klein-Gordon operators in black hole geometries

In this chapter, we apply the Gelfand-Yaglom theorem (see Appendix E) to compute determinants of the second-order separable differential operators which compute the one-loop effective actions in the BH backgrounds. More precisely, we consider the conformally coupled Klein-Gordon differential operator in four-dimensional Kerr-de Sitter black holes and the Klein-Gordon differential operator in Schwarzschild anti-de Sitter black holes in five dimensions. In both cases, the action of the scalar field can be written as

$$S[\Phi, g_{\mu\nu}] = \int \mathrm{d}^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \, \nabla_\nu \Phi - \frac{1}{2} \mu^2 \Phi^2 \right), \tag{3.0.1}$$

where $g_{\mu\nu}$ is the metric of the spacetime, and the resulting differential operator can be written as

$$\left[\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \right) - \mu^2 \right] \Phi \equiv \left[\Box - \mu^2 \right] \Phi = 0, \tag{3.0.2}$$

where, for Kerr-de Sitter BH in four dimensions, we fix $\mu^2 = 2$ so that it realizes the conformal coupling if the de Sitter radius has unit norm, whereas, for Schwarzschild anti-de Sitter black hole in five dimensions, we consider μ to be generic. In the latter case, it is convenient to reparametrize μ as

$$\mu^2 = \Delta(\Delta - 4), \tag{3.0.3}$$

where Δ corresponds to the conformal dimension of the scalar field living in the holographic dual $4d$ CFT. We require $\Delta \notin \mathbb{Z}$ in order to avoid logarithmic solutions for the radial function around the AdS boundary.

3.1 One-loop black hole effective actions and Gelfand-Yaglom theorem

The Gelfand-Yaglom theorem provides a way to compute the logarithm of the inverse of the partition functions associated with the above Klein-Gordon differential operators.

The computation of the full determinant $\det\left(\Box-\mu^2\right)$ can be reduced to the computation of the determinant of a radial 1-dimensional operator which depends on the eigenvalues of the other separated problems and their degeneracies. We use the following decomposition in Fourier modes of the wave function Φ

$$\Phi(t,r,\Omega)=\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{\ell,\vec{m}}e^{-i\omega t}S_{\omega,\ell,\vec{m}}(\Omega)R_{\omega,\ell,\vec{m}}(r). \tag{3.1.4}$$

In the spherically symmetric cases, the angular functions $S_{\omega,\ell,\vec{m}}(\Omega)$ coincide with the spherical harmonics $Y_{\ell,\vec{m}}(\Omega)$. Starting from the problem

$$\left(\Box-\mu^2\right)\Phi=\lambda\Phi, \tag{3.1.5}$$

and using (1.1.1), we obtain a system of coupled second-order differential equations for $S_{\omega,\ell,\vec{m}}(\Omega)$ and $R_{\omega,\ell,\vec{m}}(r)$, of the form

$$\begin{aligned}\mathcal{D}_{\mathrm{rad}}R_{\omega,\ell,\vec{m}}(r)&=\left(A_{\ell\vec{m}}+\lambda\right)R_{\omega,\ell,\vec{m}}(r),\\ \mathcal{D}_{\mathrm{ang}}S_{\omega,\ell,\vec{m}}(\Omega)&=-A_{\ell\vec{m}}\,S_{\omega,\ell,\vec{m}}(\Omega),\end{aligned} \tag{3.1.6}$$

for some second-order differential operators $\mathcal{D}_{\mathrm{rad}}$ and $\mathcal{D}_{\mathrm{ang}}$, and where $A_{\ell\vec{m}}$ denotes the separation constant at fixed values of the quantum numbers.

The expression of the separation constant is obtained from the angular equation and then, when substituted into the radial equation, gives the determinant in terms of ω and the quantum numbers. In the Kerr-de Sitter case, the separation constant is expressed as an instanton expansion in terms of NS functions. In the Schwarzschild-(anti-)de Sitter cases, the angular eigenfunctions reduce to the spherical harmonics and the separation constant has an exact expression in terms of the quantum number ℓ.

In the asymptotically de Sitter black hole problems, around the points in which the boundary conditions are imposed – which are horizons of the BH geometry – a basis of independent solutions of the radial equation behaves like

$$R_{\omega,\ell,\vec{m}}(r)\sim\exp(\pm i\omega r_*), \tag{3.1.7}$$

where r_* is the tortoise coordinate. When ω is analytically continued to assume values in the complex plane, the boundary conditions select the correct local solutions according to the sign of the imaginary part of ω. In the asymptotically anti-de Sitter black hole problem, this still holds for the boundary condition imposed around the black hole horizon, but the second boundary condition is imposed at the AdS boundary which is a regular point, and the selected solution depends on the value of the mass of the scalar perturbation.

The full determinant has an expression of the form

$$\log(\det\left(\Box-\mu^2\right))\equiv\int_{-\infty}^{\infty}\mathrm{d}\omega\sum_{\ell,\vec{m}}\log\left(\det\left(\mathcal{D}_{\mathrm{rad}}-A_{\ell\vec{m}}\right)[\omega]\right). \tag{3.1.8}$$

When applying the Gelfand-Yaglom theorem to the 1-dimensional radial operator, we introduce a new variable z such that the radial differential equation can be brought in Heun's form

$$\mathcal{D}_{\text{Heun}}\,\psi(z)=0,$$

$$\mathcal{D}_{\text{Heun}}=\frac{\mathrm{d}^2}{\mathrm{d}\,z^2}+\frac{\frac{1}{4}-a_0^2}{z^2}+\frac{\frac{1}{4}-a_1^2}{(z-1)^2}+\frac{\frac{1}{4}-a_t^2}{(z-t)^2}-\frac{\frac{1}{2}-a_1^2-a_t^2-a_0^2+a_\infty^2+u}{z(z-1)}+\frac{u}{z(z-t)} \tag{3.1.9}$$

and $z=0$ and $z=1$ become the two *singular* points in which we impose the boundary conditions. Let

$$\psi_{i,\lambda}^{(\hat{z})}(z)=(z-\hat{z})^{\frac{1}{2}\pm a_{\hat{z}}}\left[1+\mathcal{O}(z-\hat{z})\right],\quad i=1,2 \tag{3.1.10}$$

be the fundamental system of local solutions around $z=\hat{z}$. The solution selected by the boundary condition at $z=\hat{z}$ is the one having in front of the exponent $a_{\hat{z}}$ the same sign of $\mathrm{Re}(a_{\hat{z}})$. For the problems we consider, this condition changes according to the values of the gravitational quantities. In particular, around the singularities corresponding to horizons of the geometry, the condition depends on the sign of $\mathrm{Im}(\omega)$, when this is analytically continued to take values on the complex plane.

Let us denote with $\psi_{1,\lambda}^{(\hat{z})}(z)$ the solution selected by the boundary condition at $z=\hat{z}$. Using the connection formulae, we can write

$$\psi_{1,\lambda}^{(0)}(z)=\mathcal{C}_{11,\lambda}\psi_{1,\lambda}^{(1)}(z)+\mathcal{C}_{12,\lambda}\psi_{2,\lambda}^{(1)}(z), \tag{3.1.11}$$

where we denote with $\mathcal{C}_{11,\lambda},\mathcal{C}_{12,\lambda}$ the connection coefficients, which depend on λ (but are independent of z).

In order to apply the Gelfand-Yaglom theorem, we introduce a reference problem whose differential operator $\tilde{\mathcal{D}}_{\text{rad}}$ is a Hypergeometric one, obtained by simplifying the Heun differential equation keeping the indices of the singular points at $z=0$ and $z=1$ fixed. When computing ratios of the determinants of the two radial operators (the one of the original problem and the one of the reference problem), we have

$$\frac{\det\left(\mathcal{D}_{\text{rad}}-A_{\ell m}-\lambda\right)}{\det\left(\tilde{\mathcal{D}}_{\text{rad}}-\lambda\right)}\propto\frac{\mathcal{C}_{12,\lambda}}{\tilde{\mathcal{C}}_{12,\lambda}}, \tag{3.1.12}$$

where $\tilde{\mathcal{C}}$ denotes the connection matrix of the reference problem. The above statement holds since both the left-hand side and the right-hand side (as functions of λ) have zeros in the eigenvalues of $\mathcal{D}_{\text{rad}}-A_{\ell m}$ and poles in the eigenvalues of $\tilde{\mathcal{D}}_{\text{rad}}$. The fact that the connection coefficient $\mathcal{C}_{12,\lambda}$ has zeroes in the eigenvalues is due to the fact that, if $\hat{\lambda}$ is an eigenvalue, then $\psi_{1,\lambda=\hat{\lambda}}^{(1)}(1)=0$ because of the boundary condition, and $\psi_{1,\hat{\lambda}}^{(0)}(1)=0$ if and only if $\mathcal{C}_{12,\hat{\lambda}}=0$. Moreover, in the limit $\lambda\to\infty$ the ratio (3.1.12) tends to 1. We thus conclude that

$$\frac{\det\left(\mathcal{D}_{\text{rad}}-A_{\ell m}\right)}{\det\left(\tilde{\mathcal{D}}_{\text{rad}}\right)}=\frac{\mathcal{C}_{12,\lambda=0}}{\tilde{\mathcal{C}}_{12,\lambda=0}}. \tag{3.1.13}$$

Finally, as described in Appendix G, we can compute the regularized determinant for the reference Hypergeometric potential. This provides a solution for the determinant of the radial Heun differential operator, which is of the form

$$\det\left(\mathcal{D}_{\text{rad}} - A_{\ell m}\right) = 2\pi \frac{\mathcal{C}_{12,\lambda=0}}{\Gamma(1+2\theta_0 a_0)\Gamma(2\theta_1 a_1)}, \tag{3.1.14}$$

as obtained in Appendix G.3, where a_0, a_1 denote the indices of the singularities at $z = 0$ and $z = 1$ of the Heun differential operator, and where $\theta_0, \theta_1 = \pm$, the signs being the same of the ones of the real parts of the indices a_0, a_1, respectively.

In the following sections, we concretely compute the determinants for the gravitational problems, and in the last section (see Sec. 3.4) we discuss the results and rewrite the previous formulae more explicitly.

3.2 Kerr-de Sitter spacetime in four dimensions

The four-dimensional Kerr-de Sitter metric in Chambers-Moss coordinates can be written as

$$\begin{aligned} \mathrm{d}s^2 = {} & \frac{r^2+x^2}{\Delta_r}\mathrm{d}r^2 + \frac{r^2+x^2}{(a_{\text{BH}}^2 - x^2)\left(1+\frac{\Lambda}{3}x^2\right)}\mathrm{d}x^2 \\ & - \frac{\Delta_r - (a_{\text{BH}}^2 - x^2)\left(1+\frac{\Lambda}{3}x^2\right)}{(r^2+x^2)\left(1+\frac{\Lambda a_{\text{BH}}^2}{3}\right)^2}\mathrm{d}t^2 \\ & + \frac{(a_{\text{BH}}^2 - x^2)}{a_{\text{BH}}^2(r^2+x^2)\left(1+\frac{\Lambda}{3}a_{\text{BH}}^2\right)^2}\left[(r^2+a_{\text{BH}}^2)^2\left(1+\frac{\Lambda}{3}x^2\right) - (a_{\text{BH}}^2 - x^2)\Delta_r\right]\mathrm{d}\phi^2 \\ & + 2\frac{(a_{\text{BH}}^2 - x^2)}{a_{\text{BH}}(r^2+x^2)\left(1+\frac{\Lambda}{3}a_{\text{BH}}^2\right)^2}\left[\Delta_r - (r^2+a_{\text{BH}}^2)\left(1+\frac{\Lambda}{3}x^2\right)\right]\mathrm{d}t\,\mathrm{d}\phi, \end{aligned} \tag{3.2.15}$$

where

$$\begin{aligned} x &= a_{\text{BH}}\cos\theta, \\ \Delta_r(r) &= r^2 - 2Mr + a_{\text{BH}}^2 - \frac{\Lambda}{3}r^2(r^2+a_{\text{BH}}^2) = -\frac{\Lambda}{3}(r-R_+)(r-R_-)(r-R_h)(r-R_i). \end{aligned} \tag{3.2.16}$$

In the previous equations, M is the mass parameter of the black hole, a_{BH} is the parameter characterizing its angular momentum, $\Lambda > 0$ is the cosmological constant, and we have factorized $\Delta_r(r)$ in linear terms, where R_h is the event horizon, R_i is the inner horizon, and $R_\pm$ represent cosmological horizons, one of which is negative, $R_- \in \mathbb{R}_{<0}$, and the other one is positive and bigger than the event horizon, $R_+ > R_h$. In the following discussion, we fix $\Lambda = 3$ and we work in the *small black hole regime*, which corresponds to taking the black hole radius small compared to the norm of the de Sitter radius, $R_h \ll 1$.

Using the decomposition (1.1.1), the conformally coupled Klein-Gordon equation can be separated into an angular equation and a radial equation which read

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r(r)\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right)+\left[\frac{[\omega(r^2+a_{\mathrm{BH}}^2)-a_{\mathrm{BH}}\,m]^2\left(1+a_{\mathrm{BH}}^2\right)^2}{\Delta_r(r)}-2r^2-A_{\ell m}\right]R(r)=0, \tag{3.2.17}$$

$$\begin{split}&\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(a_{\mathrm{BH}}^2-x^2\right)\left(1+x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]\right.\\ &\left.-\frac{\left\{\left(1+a_{\mathrm{BH}}^2\right)\left[\omega(a_{\mathrm{BH}}^2-x^2)-a_{\mathrm{BH}}m\right]\right\}^2}{(a_{\mathrm{BH}}^2-x^2)(1+x^2)}-2x^2+A_{\ell m}\right\}S(x)=0,\end{split} \tag{3.2.18}$$

where with $A_{\ell m}$ we denote the separation constant. Both equations can be written in Heun's form [146, 200, 82]. We first address the problem of quantization of the separation constant.

3.2.1 Angular Problem

The singularities of the angular equation are

$$\pm a_{\mathrm{BH}}, \pm i. \tag{3.2.19}$$

The Kerr-de Sitter black hole solution is well defined if the a_{BH} parameter lies in the range $0 < a_{\mathrm{BH}} < 2-\sqrt{3}$. Indeed, the extreme cases in which two or more singularities coincide can be obtained by solving the system

$$\begin{cases}\Delta_r(r)=0,\\ \Delta_r'(r)=0.\end{cases} \tag{3.2.20}$$

Solving the system in r and M, gives the solutions

$$M = r - a_{\mathrm{BH}}^2 r - 2r^3, \tag{3.2.21}$$

and

$$r=\begin{cases}\pm\frac{1}{\sqrt{6}}\sqrt{1-a_{\mathrm{BH}}^2-\sqrt{a_{\mathrm{BH}}^4-14a_{\mathrm{BH}}^2+1}},\\ \pm\frac{1}{\sqrt{6}}\sqrt{1-a_{\mathrm{BH}}^2+\sqrt{a_{\mathrm{BH}}^4-14a_{\mathrm{BH}}^2+1}}.\end{cases} \tag{3.2.22}$$

These are consistent with the physical requirements $M > 0$ and $0 < a_{\mathrm{BH}} < 1$ if and only if $0 < a_{\mathrm{BH}} < 2-\sqrt{3}$.

Let us perform the following change of variables:

$$z=\frac{2i(x+a_{\mathrm{BH}})}{(a_{\mathrm{BH}}+i)(x+i)}. \tag{3.2.23}$$

This change of variables maps

$$\begin{split}&(x_4=-i, x_1=-a_{\mathrm{BH}}, x_2=i, x_3=a_{\mathrm{BH}}, \infty)\mapsto\\ &\left(\infty, z_1=0, z_2=1, z_3\equiv\frac{4\,i\,a_{\mathrm{BH}}}{(a_{\mathrm{BH}}+i)^2}, z_\infty\equiv\frac{2i}{a_{\mathrm{BH}}+i}\right).\end{split} \tag{3.2.24}$$

We note that, for $0 < a_{\rm BH} < 2-\sqrt{3}$, one has $|t| < 1$. Let us define

$$\begin{aligned}\Delta_x(x) &= (a_{\rm BH}^2 - x^2)\left(1+x^2\right),\\ \theta_k^{(a)} &= -\frac{(1+a_{\rm BH}^2)[\omega(a_{\rm BH}^2 - x_k^2) - a_{\rm BH}\, m]}{\Delta_x'(x_k)}, \quad k=1,2,3.\end{aligned} \tag{3.2.25}$$

If we transform the angular wave function as

$$S(x) = (z - z_\infty)\prod_{k=1}^{3}(z-z_i)^{-\theta_k^{(a)}}\, w(z), \tag{3.2.26}$$

we can remove the apparent singularity in z_∞ and the angular equation becomes a Heun equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}\right] w(z) = 0, \tag{3.2.27}$$

with

$$\begin{aligned} t &= \frac{4ia_{\rm BH}}{(a_{\rm BH}+i)^2},\\ \alpha &= 1 - i\omega + ia_{\rm BH}(m - a_{\rm BH}\omega), \qquad \beta = 1,\\ \gamma &= 1-m, \qquad \delta = 1 - i\omega + ia_{\rm BH}(m - a_{\rm BH}\omega), \qquad \epsilon = 1+m,\\ q &= \frac{A_{\ell m} + 2a_{\rm BH}\left[\left(1-a_{\rm BH}^2\right)\omega + (a_{\rm BH}-i)m + i\right]}{(a_{\rm BH}+i)^2}.\end{aligned} \tag{3.2.28}$$

The dictionary that gives the parameter of the Heun's operator in normal form (3.1.9), is given by

$$\begin{aligned} a_0 &= \frac{m}{2}, \qquad a_t = -\frac{m}{2},\\ a_1 &= \frac{i}{2}\left[\omega(1+a_{\rm BH}^2) - a_{\rm BH}m\right], \qquad a_\infty = -\frac{i}{2}\left[\omega(1+a_{\rm BH}^2) - a_{\rm BH}m\right],\\ u &= \frac{1 + 2A_{\ell m} - m^2 + 2a_{\rm BH}(i - im^2 + 2m\omega) + a_{\rm BH}^2(-1+8m-3m^2) + 4a_{\rm BH}^3(m-2)\omega}{2(a_{\rm BH}-i)^2}.\end{aligned} \tag{3.2.29}$$

We impose as boundary conditions the regularity of the solutions at $\theta = 0, \pi$, which correspond to $x = \pm a_{\rm BH}$, and so to $z = t$ and $z = 0$.

For $x \sim a_{\rm BH}$ the original angular function has the following two behaviors:

$$\begin{aligned} S_-^{(t)}(x) &\sim (x - a_{\rm BH})^{\frac{m}{2}}\\ S_+^{(t)}(x) &\sim (x - a_{\rm BH})^{-\frac{m}{2}}.\end{aligned} \tag{3.2.30}$$

Therefore, we take $S \sim S_-^{(t)}$ if $m \geq 0$, and we take $S \sim S_+^{(t)}$ if $m < 0$.

For $x \sim -a_{\mathrm{BH}}$ the original angular function has the following two behaviors:

$$
\begin{aligned}
S_-^{(0)}(x) &\sim (x+a_{\mathrm{BH}})^{-\frac{m}{2}} \\
S_+^{(0)}(x) &\sim (x+a_{\mathrm{BH}})^{\frac{m}{2}},
\end{aligned}
\tag{3.2.31}
$$

Therefore, we take $S \sim S_-^{(0)}$ if $m \leq 0$, and we take $S \sim S_+^{(0)}$ if $m > 0$.

The boundary conditions are satisfied if the following requirement is imposed on the v.e.v. parameter a (see Appendix A for the relevant definitions and conventions) which parameterizes the composite monodromy around $x = 0$ and $x = t$:

$$
a = \ell + \frac{1}{2}, \quad \text{with} \quad \ell \geq |m| \quad \text{and } \ell \in \mathbb{N}. \tag{3.2.32}
$$

The v.e.v. parameter a is related to the parameter u of the Heun differential equation through the *Matone relation*

$$
u = -\frac{1}{4} + a_t^2 + a_0^2 - a^2 + t\frac{\partial F(t)}{\partial t}, \tag{3.2.33}
$$

where $F(t)$ is the instanton partition function with four fundamental multiplets in the NS limit (see Appendix A for the relevant definitions and conventions). Using the gravitational dictionary for u and the quantization condition $a = \ell + \frac{1}{2}$, we obtain the following expansion of the separation constant:

$$
A_{\ell ms} = (a_{\mathrm{BH}} - 1)^2 \ell(\ell+1) + 2a_{\mathrm{BH}} m (a_{\mathrm{BH}}^2 \omega - a_{\mathrm{BH}} m - \omega) - (a_{\mathrm{BH}} - 1)^2 t \frac{\partial F(t)}{\partial t}. \tag{3.2.34}
$$

As expected, expanding this expression around $a_{\mathrm{BH}} = 0$ gives

$$
A_{\ell ms} = \ell(\ell+1) - 2\, m\, a_{\mathrm{BH}}\, \omega + \mathcal{O}(a_{\mathrm{BH}}^2). \tag{3.2.35}
$$

3.2.2 Radial Problem and expression for the determinant

For the radial equation, let us perform the following change of variables:

$$
z = \frac{R_+ - R_-}{R_+ - R_h} \cdot \frac{r - R_h}{r - R_-}. \tag{3.2.36}
$$

This sends

$$
\begin{aligned}
&(r_4 = R_-, r_1 = R_h, r_2 = R_+, r_3 = R_i, \infty) \mapsto \\
&\left(\infty, z_1 = 0, z_2 = 1, z_3 := t = \frac{R_+ - R_-}{R_+ - R_h} \cdot \frac{R_i - R_h}{R_i - R_-}, z_\infty := \frac{R_+ - R_-}{R_+ - R_h}\right).
\end{aligned}
\tag{3.2.37}
$$

We remark that $t < 0$, so that in the interval $z \in]0,1[$ there are no singularities. Let us define

$$
\theta_k^{(r)} = \frac{i}{\Delta_r'(r_k)} [\omega(r_k^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \quad k = 1, \ldots, 4. \tag{3.2.38}
$$

If we transform the radial function as

$$R(r) = (z - z_\infty) \prod_{k=1}^{3} (z - z_i)^{-\theta_k^{(r)}} w(z), \tag{3.2.39}$$

we can remove the singularity in z_∞ and the radial equation becomes a Heun equation (3.2.27) with

$$\begin{aligned}
t &= \frac{R_+ - R_-}{R_+ - R_h} \cdot \frac{R_i - R_h}{R_i - R_-}, \\
\alpha &= 1 + \frac{2i}{\Delta_r'(R_-)}[\omega(R_-^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
\beta &= 1, \\
\gamma &= 1 - \frac{2i}{\Delta_r'(R_h)}[\omega(R_h^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
\delta &= 1 - \frac{2i}{\Delta_r'(R_+)}[\omega(R_+^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
\epsilon &= 1 - \frac{2i}{\Delta_r'(R_i)}[\omega(R_i^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
q &= (t-1)(a_0 + a_t) + t(a_1 + a_t) - \frac{t(t-1)}{t - z_\infty} + t\,\alpha + \frac{2R_i^2 + A_{\ell m}}{(R_h - R_+)(R_i - R_-)},
\end{aligned} \tag{3.2.40}$$

where the indices of the singular points are

$$\begin{aligned}
a_0 &= \frac{i}{\Delta_r'(R_h)}[\omega(R_h^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
a_t &= \frac{i}{\Delta_r'(R_i)}[\omega(R_i^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
a_1 &= \frac{i}{\Delta_r'(R_+)}[\omega(R_+^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2), \\
a_\infty &= \frac{i}{\Delta_r'(R_-)}[\omega(R_-^2 + a_{\mathrm{BH}}^2) - a_{\mathrm{BH}} m](1 + a_{\mathrm{BH}}^2),
\end{aligned} \tag{3.2.41}$$

and the parameter u in the Heun equation (3.1.9) is given by

$$u = \frac{-2\,q + 2\,t\,\alpha\,\beta + \gamma\,\epsilon - t\,(\gamma + \delta)\,\epsilon}{2(t-1)}. \tag{3.2.42}$$

We distinguish two cases according to the sign of $\mathrm{Im}(\omega)$.

Let us start from the case $\mathrm{Im}(\omega) > 0$. In this case $\mathrm{Re}(a_0) < 0$ and $\mathrm{Re}(a_1) > 0$. Then, the local solutions of the normal form of the Heun equation (and normalized as in

(E.1.16)) selected by the boundary conditions are

$$\begin{aligned}
\psi_-^{(0)}(z) =& t^{-\epsilon/2} z^{\gamma/2}(z-1)^{\delta/2}(z-t)^{\epsilon/2}\,\text{Heun}\,(t,q,\alpha,\beta,\gamma,\delta,z)\,,\\
\psi_+^{(1)}(z) =& (1-t)^{-\epsilon/2} z^{\gamma/2}(z-1)^{1-\delta/2}(z-t)^{\epsilon/2}\left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta}\times\\
&\text{Heun}\Big(t, q-\alpha(\beta+\delta-2)+(\delta-1)(\alpha+\beta-1-t\gamma),\\
&\alpha+1-\delta, 1+\gamma-\beta, 2-\delta,\gamma, t\frac{1-z}{t-z}\Big).
\end{aligned} \tag{3.2.43}$$

The connection formula between the two local solutions changes according to the position of the singularity t in the z-space. The small black hole regime corresponds to the regime $|t| < 1$[1]. The connection coefficient in terms of which we can express the determinant is the one in front of $\psi_-^{(1)}(z)$ starting from the solution $\psi_-^{(0)}(z)$ in the connection formula (F.2.19):

$$\sum_{\theta'=\pm}\mathcal{M}_{-\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')-}(a,a_1;a_\infty)t^{-a_0+\theta' a}\exp\left(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right). \tag{3.2.44}$$

In the case $\text{Im}(\omega) < 0$, the local (normalized) solutions selected by the boundary conditions are

$$\begin{aligned}
\psi_+^{(0)}(z) =& e^{i\pi(-\delta/2-\epsilon/2)} t^{-\epsilon/2} z^{1-\gamma/2}(z-1)^{\delta/2}(z-t)^{\epsilon/2}\times\\
&\times\text{Heun}\,(t, q-(\gamma-1)(t\,\delta+\epsilon), \alpha+1-\gamma, \beta+1-\gamma, 2-\gamma,\delta,z)\,,\\
\psi_-^{(1)}(z) =& (1-t)^{-\epsilon/2} z^{\gamma/2}(z-1)^{\delta/2}(z-t)^{\epsilon/2}\left(\frac{z-t}{1-t}\right)^{-\alpha}\times\\
&\times\text{Heun}\left(t, q+\alpha(\delta-\beta),\alpha,\delta+\gamma-\beta,\delta,\gamma,t\frac{1-z}{t-z}\right).
\end{aligned} \tag{3.2.45}$$

The connection coefficient in terms of which we can express the determinant is the one in front of $\psi_+^{(1)}(z)$ starting from the solution $\psi_+^{(0)}(z)$ in the connection formula (F.2.19):

$$\sum_{\theta'=\pm}\mathcal{M}_{+\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')+}(a,a_1;a_\infty)t^{a_0+\theta' a}\exp\left(\frac{1}{2}\partial_{a_0}F(t)-\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right). \tag{3.2.46}$$

3.2.3 Determinant of radial operator

We can finally write the result for the determinant of the radial differential operator, following the procedure explained in Appendix G.1 and G.2. The reference problem we

[1]The other regime $|t| > 1$, would lead to a simpler connection formula, more similar to a Hypergeometric-like connection problem, but still involving the presence of the NS functions (see Appendix F.3).

consider for the radial operator is a Hypergeometric problem having the same indices at the singular points $z = 0$ and $z = 1$.

For $\mathrm{Im}(\omega) > 0$, we have $\mathrm{Re}(a_0) < 0$ and $\mathrm{Re}(a_1) > 0$. The formula for the (regularized) determinant reads

$$\begin{aligned}&\sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0-\theta' a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a+a_1+\sigma\, a_\infty\right)} t^{-a_0+\theta' a}\times \\ &\times \exp\left(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right).\end{aligned} \tag{3.2.47}$$

For $\mathrm{Im}(\omega) < 0$, we have $\mathrm{Re}(a_0) > 0$ and $\mathrm{Re}(a_1) < 0$. The formula for the (regularized) determinant reads

$$\begin{aligned}&\sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}+a_0-\theta' a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a-a_1+\sigma\, a_\infty\right)} t^{a_0+\theta' a}\times \\ &\times \exp\left(\frac{1}{2}\partial_{a_0}F(t)-\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right).\end{aligned} \tag{3.2.48}$$

We can summarize the two formulae together introducing $\eta = \mathrm{Im}(\omega)/|\mathrm{Im}(\omega)|$ as

$$\begin{aligned}\det(\mathcal{D}_{\mathrm{rad}} - A_{\ell m}) = &\sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-\eta a_0-\theta' a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a+\eta a_1+\sigma\, a_\infty\right)}\times \\ &\times t^{-\eta a_0+\theta' a}\exp\left(-\frac{\eta}{2}\partial_{a_0}F(t)+\frac{\eta}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right).\end{aligned} \tag{3.2.49}$$

The (anti-)quasinormal modes are directly given by the zeroes of the above expression.

3.2.4 Schwarzschild-de Sitter spacetime in four dimensions

In this subsection, we want to briefly comment on how the previous formula also gives the solution for the determinant of the same operator around the four-dimensional Schwarzschild-de Sitter black hole, which is a spherically symmetric spacetime. In particular, the angular problem, in this case, is solved by the spherical harmonics, and the only nontrivial problem is the radial one, which can be solved precisely as in the previous discussion, but with a simplified dictionary.

The metric describing the Schwarzschild-de Sitter black hole in four dimensions (SdS_4) is given in (2.1.1).

The conformally coupled Klein-Gordon equation in the SdS geometry can be obtained from the Kerr-dS one by sending the rotation parameter $a_{\mathrm{BH}} \to 0$. This also sends the singularity $R_i \to 0$ and the angular equation becomes trivial, giving an exact result for the separation constant

$$A_{\ell m} = \ell(\ell+1). \tag{3.2.50}$$

The radial equation, instead, remains a Heun equation (3.1.9), whose parameters can be deduced from the ones in (3.2.41) and (3.2.42):

$$t = \frac{R_h}{R_-} \cdot \frac{R_+ - R_-}{R_+ - R_h}, \qquad u = -\frac{2\ell(\ell+1) + (R_h + R_+)^2}{2R_+(2R_h + R_+)}, \qquad a_0 = \frac{i\omega R_h}{(R_h - R_-)(R_+ - R_h)},$$
$$a_1 = \frac{i\omega R_+}{(R_+ - R_-)(R_h - R_+)}, \qquad a_t = 0, \qquad a_\infty = \frac{i\omega R_-}{(R_h - R_-)(R_- - R_+)}. \tag{3.2.51}$$

With this new dictionary, the expression of the determinant is given by (3.2.49), as in the Kerr case.

3.2.5 Pure de Sitter spacetime in four dimensions

An additional simplification can be obtained from the previous problem in the limit in which $R_h \to 0$. This leads to the determinant of the same operator in the pure de Sitter spacetime in four dimensions. In this case, the radial problem reduces to a Hypergeometric differential equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{4\ell(\ell+1)(z-1) + \left(\omega^2+1\right) z^2}{4(z-1)^2 z^2}\right]\psi(z) = 0. \tag{3.2.52}$$

The indices of the singularities $z = 0$ and $z = 1$ are

$$a_0 = -\ell - \frac{1}{2}, \quad a_1 = \frac{i\,\omega}{2}. \tag{3.2.53}$$

The sign of $\mathrm{Re}(a_0)$ is always negative, whereas the sign of $\mathrm{Re}(a_1)$ depends on the sign of the imaginary part of the frequency. Therefore, the local solution selected around $z \sim 0$ is the one behaving like

$$\psi_-^{(0)}(z) = z^{\frac{1}{2}+\left(\ell+\frac{1}{2}\right)}\left(1 + \mathcal{O}(z)\right). \tag{3.2.54}$$

The selected solution around $z \sim 1$ is

$$\begin{aligned} \psi_-^{(1)}(z) &= (z-1)^{\frac{1}{2}-\frac{i\omega}{2}}\left(1 + \mathcal{O}(z-1)\right), \quad \text{if } \mathrm{Im}(\omega) > 0, \\ \psi_+^{(1)}(z) &= (z-1)^{\frac{1}{2}+\frac{i\omega}{2}}\left(1 + \mathcal{O}(z-1)\right), \quad \text{if } \mathrm{Im}(\omega) < 0. \end{aligned} \tag{3.2.55}$$

Redefining the wave function as

$$\psi(z) = z^{\ell+1}(z-1)^{\frac{1}{2}-\frac{i\,\omega}{2}} w(z) \tag{3.2.56}$$

we can rewrite the differential equation as in (F.1.1) with

$$a = \ell + 1, \quad b = \ell + 1 - i\omega, \quad c = 2\ell + 2. \tag{3.2.57}$$

Using the connection formulae (F.1.4) and the results in Appendix G.2, the determinant can be written as

$$\frac{2\pi}{\Gamma(\ell+1)\Gamma(\ell+1-i\,\eta\,\omega)}, \tag{3.2.58}$$

where $\eta = \mathrm{Im}(\omega)/|\mathrm{Im}(\omega)|$. The zeros in ω of the previous functions are given by the quasinormal mode frequencies $\omega = -i(\ell + n + 1)$ and by the anti-quasinormal mode frequencies $\omega = i(\ell + n + 1)$.

3.2.6 Reduction of the determinant from Schwarzschild-de Sitter to pure de Sitter

In this subsection, we want to comment on how the result of the determinant in the pure de Sitter geometry can be obtained from the Schwarzschild-de Sitter case in the limit $R_h \to 0$ (or, equivalently, sending to zero the mass of the black hole $M \to 0$). We already stressed that starting from the determinant in the Kerr-de Sitter case and sending the rotation parameter $a_{\mathrm{BH}} \to 0$, one obtains the determinant of the Schwarzschild-de Sitter case, which has the same expression but with the reduced dictionary. This is a smooth limit, in the sense that the result can be obtained simply by looking at the limit of the parameters for $a_{\mathrm{BH}} \to 0$. In the reduction to the pure de Sitter case, the equation becomes a Hypergeometric equation in a non-trivial way, namely by a collision of singularities.

Let us start by rewriting the determinant of the Schwarzschild-de Sitter case written in the following form:

$$\begin{aligned}&\frac{2\pi\sum_{\theta'=\pm}\mathcal{M}_{-\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')-}(a,a_1;a_\infty)}{\Gamma(1-2a_0)\Gamma(2a_1)}\times\\&t^{-a_0+\theta' a}\exp\left(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right),\end{aligned}\tag{3.2.59}$$

where we took the $\mathrm{Im}(\omega) < 0$ case (the $\mathrm{Im}(\omega) > 0$ case is analogous).

By considering the limit $R_h \to 0$, we have to implement the limit $t \to 0$ in the Heun differential operator (G.1.1). Comparing the reduced operator with (3.2.52), we can see that the new index $\tilde{a}_0$ at $z = 0$ is given by

$$\tilde{a}_0^2 = -\frac{1}{4} - u + a_0^2 + a_t^2\bigg|_{R_h\to 0} = \frac{(2\ell+1)^2}{4}.\tag{3.2.60}$$

Note that this is not obtained smoothly from a_0 by sending $R_h \to 0$ because of the collision of singularities. Moreover,

$$a^2\bigg|_{R_h\to 0} = -\frac{1}{4} - u + a_0^2 + a_t^2 = \tilde{a}_0^2.\tag{3.2.61}$$

Indeed, when 0 and t collide, the monodromy parametrized by a becomes simply the monodromy around $z = 0$. Therefore, in (3.2.59), the first connection matrix $\mathcal{M}_{-\theta'}(a_0, a; a_t)$ trivializes and reduces to the identity matrix[2].

Let us now fix, consistently with the previous subsections, the signs $\tilde{a}_0 = -\ell - \frac{1}{2}$ and

[2]This can also be seen from the Liouville three-point functions by considering one of the three insertions to reduce to the identity insertion, see Appendix A in [4] for the detailed definitions and conventions.

$a \to \ell + \frac{1}{2}$. Then, the determinant (3.2.59), in the limit $R_h \to 0$, reduces to

$$\frac{2\pi}{\Gamma(2\ell+2)\Gamma(i\,\omega)}\left[\sum_{\theta'=\pm}\frac{\Gamma\left(1-2\theta'\left(\ell+\frac{1}{2}\right)+\mathcal{O}\left(R_h\right)\right)\Gamma\left(i\,\omega\right)}{\Gamma\left(\frac{1}{2}-\theta'\left(\ell+\frac{1}{2}\right)+\mathcal{O}\left(R_h\right)\right)\Gamma\left(\frac{1}{2}-\theta'\left(\ell+\frac{1}{2}\right)+i\,\omega\right)}\times\right.$$
$$\left.\times\left(-2R_h\right)^{\ell+\frac{1}{2}+\theta'\left(\ell+\frac{1}{2}\right)}\right]\cdot\left[1+\mathcal{O}\left(R_h\log(R_h)\right)\right]= \tag{3.2.62}$$
$$=\frac{2\pi}{\Gamma\left(\ell+1\right)\Gamma\left(\ell+1+i\,\omega\right)}\left[1+\mathcal{O}\left(R_h\log(R_h)\right)\right],$$

which is the result obtained in the previous subsection, (3.2.58). Passing to the final result, we used the fact that the choice of the sign $\theta' = +$ forces the corresponding channel to go to zero, as can be seen from the dependence on $R_h^{2\ell+1}$ and noticing that the ratio of Gamma functions $\Gamma\left(-2\ell+\mathcal{O}\left(R_h\right)\right)/\Gamma\left(-\ell+\mathcal{O}\left(R_h\right)\right)$ gives a finite quantity in the $R_h \to 0$ limit.

3.3 Schwarzschild anti-de Sitter spacetime in five dimensions

The metric of the five-dimensional Schwarzschild-anti-de Sitter black hole ($SAdS_5$) is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2, \tag{3.3.63}$$

where $\mathrm{d}\Omega_3^2$ is the volume element of the 3-sphere and, normalizing the AdS radius to 1,

$$f(r) = \left(1 - \frac{R_h^2}{r^2}\right)\left(r^2 + R_h^2 + 1\right), \tag{3.3.64}$$

where R_h is the radius of the black hole horizon. We again work in the small black hole regime, $0 < R_h \ll 1$.

The wave equation satisfied by (the Fourier modes $R_{\ell,\omega}$ of) a massive scalar field Φ in this black hole background is given by

$$\left[\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3 f(r)\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{\omega^2}{f(r)} - \frac{\ell(\ell+2)}{r^2} - \Delta(\Delta-4)\right]R_{\ell,\omega}(r) = 0, \tag{3.3.65}$$

where Δ is the dimension of the scalar-operator dual to the scalar field in the bulk, related to the mass μ of the field by $\mu = \sqrt{\Delta(\Delta-4)}$. The problem is symmetric under $\Delta \mapsto 4-\Delta$. We assume in what follows $\Delta > 2$ and $\Delta \notin \mathbb{N}$ in order not to be in a log case.

Defining a new variable

$$z = \frac{r^2 - R_h^2}{r^2 + R_h^2 + 1}, \tag{3.3.66}$$

and redefining the wave function as

$$R_{\ell,\omega}(r) = (z-1)^{2-\frac{\Delta}{2}} z^{-\frac{i\omega R_h}{4R_h^2+2}} w_{\ell,\omega}(z), \tag{3.3.67}$$

where

$$t = -\frac{R_h^2}{R_h^2+1}, \quad \gamma = 1 - \frac{i\omega R_h}{2R_h^2+1}, \quad \delta = 3-\Delta, \quad \epsilon = 1, \tag{3.3.68}$$

the differential equation becomes a Heun equation (3.2.27), where the complete dictionary is given by

$$\begin{aligned}
t =& -\frac{R_h^2}{R_h^2+1},\\
\alpha =& \frac{(4-\Delta)\left(2R_h^2+1\right)+\omega\sqrt{R_h^2+1}-iR_h\omega}{4R_h^2+2},\\
\beta =& \frac{(4-\Delta)(1+2R_h^2)-\omega\sqrt{R_h^2+1}-iR_h\omega}{4R_h^2+2},\\
\gamma =& 1-\frac{i\omega R_h}{2R_h^2+1}, \quad \delta = 3-\Delta, \quad \epsilon = 1,\\
q =& -\frac{\left(2R_h^2+1\right)\left(\ell(\ell+2)+(\Delta-4)(\Delta-2)R_h^2\right)+2iR_h\omega\left((\Delta-2)R_h^2+1\right)-R_h^2\omega^2}{8R_h^4+12R_h^2+4}.
\end{aligned} \tag{3.3.69}$$

We remark that again t is real and negative. The parameters of the equation in normal form (3.1.9) are

$$\begin{aligned}
a_0 =& \frac{i\omega R_h}{2(2R_h^2+1)}, \quad a_t = 0, \quad a_1 = \frac{\Delta-2}{2}, \quad a_\infty = \frac{\omega\sqrt{R_h^2+1}}{2(2R_h^2+1)},\\
u =& -\frac{\ell(\ell+2)+2R_h^2+2}{8R_h^2+4}.
\end{aligned} \tag{3.3.70}$$

In the z variable, the black hole horizon is located at $z=0$ and the AdS boundary at $z=1$. The main important difference in this case with respect to the problems in an asymptotically de Sitter spacetime is the fact that the choice of the local solution near the AdS boundary does not depend on the sign of the imaginary part of ω, but just on the parameter Δ. This is because the corresponding boundary condition is imposed at the AdS boundary and not in a horizon of the geometry. With the assumptions we made on Δ, the local solution of the Heun equation in normal form selected at $z=1$ is given by

$$\begin{aligned}
&\psi_+^{(1)}(z) = (1-t)^{-\epsilon/2}z^{\gamma/2}(z-1)^{1-\frac{\delta}{2}}(z-t)^{\epsilon/2}\left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta} \times\\
&\text{Heun}\left(t, q-(\delta-1)\gamma t-(\beta-1)(\alpha-\delta+1), -\beta+\gamma+1, \alpha-\delta+1, 2-\delta, \gamma, t\frac{1-z}{t-z}\right).
\end{aligned} \tag{3.3.71}$$

For the choice of the local solution around $r=R_h$, we again divide the cases according to the sign of $\text{Im}(\omega)$. In the case $\text{Im}(\omega)>0$ we have $\text{Re}(a_0)<0$ and the local solution of the normal form of the Heun equation (and normalized as in (E.1.16)) selected by the boundary condition is

$$\psi_-^{(0)}(z) = t^{-\epsilon/2} z^{\gamma/2} (z-1)^{\delta/2} (z-t)^{\epsilon/2} \mathrm{Heun}\,(t, q, \alpha, \beta, \gamma, \delta, z)\,. \tag{3.3.72}$$

Considering again the regime in which $|t| < 1$, the connection coefficient in terms of which we can express the determinant is the one in front of $\psi_-^{(1)}(z)$ starting from the solution $\psi_-^{(0)}(z)$ in the connection formula (F.2.19), namely

$$\sum_{\theta'=\pm} \mathcal{M}_{-\theta'}(a_0, a; a_t) \mathcal{M}_{(-\theta')-}(a, a_1; a_\infty) t^{-a_0+\theta' a} \exp\left(-\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) - \frac{\theta'}{2}\partial_a F(t)\right). \tag{3.3.73}$$

In the case $\mathrm{Im}(\omega) < 0$, the local (normalized) solution around $z = 0$ selected by the boundary condition is

$$\begin{aligned} \psi_+^{(0)}(z) = {} & e^{i\pi(-\delta/2-\epsilon/2)} t^{-\epsilon/2} z^{1-\gamma/2} (z-1)^{\delta/2} (z-t)^{\epsilon/2} \times \\ & \times \mathrm{Heun}\,(t, q-(\gamma-1)(t\,\delta+\epsilon), \alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, \delta, z)\,. \end{aligned} \tag{3.3.74}$$

The connection coefficient in terms of which we can express the determinant is the one in front of $\psi_-^{(1)}(z)$ starting from the solution $\psi_+^{(0)}(z)$ in the connection formula (F.2.19), that is

$$\sum_{\theta'=\pm} \mathcal{M}_{+\theta'}(a_0, a; a_t) \mathcal{M}_{(-\theta')-}(a, a_1; a_\infty) t^{a_0+\theta' a} \exp\left(\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) - \frac{\theta'}{2}\partial_a F(t)\right). \tag{3.3.75}$$

3.3.1 Determinant of radial operator

We can again write the expression of the (regularized) determinant according to the sign of $\mathrm{Im}(\omega)$, remembering that we always have $\mathrm{Re}(a_1) > 0$.

For $\mathrm{Im}(\omega) > 0$, we have $\mathrm{Re}(a_0) < 0$ and the formula for the (regularized) determinant reads

$$\sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)\, t^{-a_0+\theta' a} \exp\left(-\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) - \frac{\theta'}{2}\partial_a F(t)\right)}{\prod_{\sigma=\pm} \Gamma\left(\frac{1}{2} - a_0 - \theta' a + \sigma\, a_t\right) \Gamma\left(\frac{1}{2} - \theta' a + a_1 + \sigma\, a_\infty\right)}. \tag{3.3.76}$$

For $\mathrm{Im}(\omega) < 0$, we have $\mathrm{Re}(a_0) > 0$ and the formula for the (regularized) determinant reads

$$\sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)\, t^{a_0+\theta' a} \exp\left(\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) - \frac{\theta'}{2}\partial_a F(t)\right)}{\prod_{\sigma=\pm} \Gamma\left(\frac{1}{2} + a_0 - \theta' a + \sigma\, a_t\right) \Gamma\left(\frac{1}{2} - \theta' a + a_1 + \sigma\, a_\infty\right)}. \tag{3.3.77}$$

We can unify the two formulae above by introducing $\eta = \text{Im}(\omega)/|\text{Im}(\omega)|$ and get

$$\det(\mathcal{D}_{\text{rad}} - A_{\ell m}) = \sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-\eta a_0 - \theta' a + \sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a + a_1 + \sigma\, a_\infty\right)} \times \\ \times\, t^{-\eta a_0 + \theta' a} \exp(-\frac{\eta}{2}\partial_{a_0}F(t) + \frac{1}{2}\partial_{a_1}F(t) - \frac{\theta'}{2}\partial_a F(t)). \tag{3.3.78}$$

3.3.2 Pure anti-de Sitter spacetime in five dimensions

As in the asymptotically de Sitter case, sending $R_h \to 0$ simplifies the problem, reducing it to the pure AdS_5 case, whose relevant differential equation is again of Hypergeometric type. Following the same procedure of the SAdS_5 problem, or reducing the dictionary in the limit $R_h = 0$, we can write the radial differential equation in normal form as

$$\psi''(z) + \frac{\ell(\ell+2)(z-1) + z\left[\omega^2(1-z) + z - (\Delta-2)^2\right]}{4(z-1)^2 z^2}\psi(z) = 0, \tag{3.3.79}$$

which is a Hypergeometric differential equation.

The boundary conditions are imposed at the origin $z = 0$ and at the AdS boundary $z = 1$. We notice that both points do not represent horizons of the geometry, and the indices in these singular points do not depend on ω:

$$a_0 = -\frac{\ell+1}{2}, \quad a_1 = 1 - \frac{\Delta}{2}. \tag{3.3.80}$$

Assuming again $\Delta > 2$ and $\Delta \notin \mathbb{N}$, the determinant can be written as

$$\frac{2\pi}{\Gamma\left(\frac{\Delta+\ell-\omega}{2}\right)\Gamma\left(\frac{\Delta+\ell+\omega}{2}\right)}. \tag{3.3.81}$$

We notice that the zeros in ω of the determinant are real and given by the normal modes of AdS_5

$$\omega = -\ell - \Delta - 2n \quad \text{and} \quad \omega = \ell + \Delta + 2n, \quad \text{with } n \in \mathbb{Z}_{\geq 0}. \tag{3.3.82}$$

3.4 Detailed analysis of the effective actions

In the previous sections, we computed the determinant of the radial differential operator in Heun's form, which can be written as

$$\det\left(\mathcal{D}_{\text{rad}} - A_{\ell m}\right) = 2\pi \frac{\mathcal{C}_{12}}{\Gamma(1+2\theta_0 a_0)\Gamma(2\theta_1 a_1)}, \tag{3.4.83}$$

where a_0, a_1 denote the indices of the singularities at $z = 0$ and $z = 1$ of the Heun differential operator, $\mathcal{C}_{12}$ denotes the Heun connection coefficient between the local solution at

$z = 0$ satisfying the boundary condition and the discarded local solution at $z = 1$, and where $\theta_0, \theta_1 = \pm$, according to the sign of $\mathrm{Im}(\omega)$.

However, it is important to notice that the problems we considered are parity-time(PT)-symmetric, and the full determinants (3.1.8) are symmetric for the transformation $\omega \mapsto -\omega$. In particular, the contribution coming from the analytic continuation for $\mathrm{Im}(\omega) < 0$ gives the same result obtained for the analytic continuation for $\mathrm{Im}(\omega) > 0$. More precisely, our determinant for the radial part has the following property:

$$\det\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[\omega] = \begin{cases} \det^{(+)}\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[\omega], & \text{for} \quad \mathrm{Im}(\omega) > 0, \\ \det^{(-)}\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[\omega], & \text{for} \quad \mathrm{Im}(\omega) < 0, \end{cases} \tag{3.4.84}$$

with

$$\det^{(-)}\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[\omega] = \det^{(+)}\left(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}}\right)[-\omega]. \tag{3.4.85}$$

We conclude that our final result for the one-loop effective action of a real scalar field is given by

$$\log(\det\left(\Box - \mu^2\right)) = \int_{-\infty}^{\infty} \mathrm{d}\omega \sum_{\ell,\vec{m}} \log\left(\det^{(+)}(\mathcal{D}_{\mathrm{rad}} - A_{\ell\vec{m}})[\omega]\right). \tag{3.4.86}$$

In the above formula, one has to substitute

$$\begin{aligned} \det^{(+)}(\mathcal{D}_{\mathrm{rad}} - A_{\ell m})[\omega] = \sum_{\theta'=\pm} \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2} - a_0 - \theta' a + \sigma\, a_t\right)\Gamma\left(\frac{1}{2} - \theta' a + a_1 + \sigma\, a_\infty\right)} \times \\ \times\, t^{-a_0 + \theta' a} \exp(-\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) - \frac{\theta'}{2}\partial_a F(t)), \end{aligned} \tag{3.4.87}$$

where, for the analyzed problems, the dictionaries of the quantities are given in (3.2.41) for the Kerr-de Sitter case, in (3.2.51) for the Schwarzschild-de Sitter case, and in (3.3.70) for the Schwarzschild-anti-de Sitter case.

One can see that the two summands in (3.4.87) have different behaviors, which are determined by the exponential factor $t^{\theta' a}$.

Since we always took a to be positive, in the limit in which t is small (that in our problems corresponded to the small black hole regime) we can argue that the term proportional to t^a is subleading compared to the one proportional to t^{-a}. In particular, we can write

$$\begin{aligned} &\log\left(\det^{(+)}(\mathcal{D}_{\mathrm{rad}} - A_{\ell m})[\omega]\right) = \\ &= \log\left(\frac{2\pi\Gamma(2a)\Gamma(1+2a)\, t^{-a_0-a} \exp\left(-\frac{1}{2}\partial_{a_0}F(t) + \frac{1}{2}\partial_{a_1}F(t) + \frac{1}{2}\partial_a F(t)\right)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2} - a_0 + a + \sigma\, a_t\right)\Gamma\left(\frac{1}{2} + a + a_1 + \sigma\, a_\infty\right)}\right) + \\ &\log\left(1 - \frac{\Gamma(-2a)^2}{\Gamma(2a)^2}\prod_{\sigma=\pm}\frac{\Gamma\left(\frac{1}{2} - a_0 + a + \sigma\, a_t\right)\Gamma\left(\frac{1}{2} + a + a_1 + \sigma\, a_\infty\right)}{\Gamma\left(\frac{1}{2} - a_0 - a + \sigma\, a_t\right)\Gamma\left(\frac{1}{2} - a + a_1 + \sigma\, a_\infty\right)} t^{2a} \exp\left(-\partial_a F(t)\right)\right), \end{aligned} \tag{3.4.88}$$

where the second line encodes the correction terms to the leading result in the first line.

This suggests that, in the decomposition in (3.4.87), the effects due to the presence of the black hole are subleading compared to the ones due to the asymptotic geometry. This is equivalent to saying that the contribution to the near-horizon zone is subleading compared to the far-zone (for a discussion on the distinction of these regions see [185]).

We add that, since the leading order of a in the small black hole regime is determined by the angular quantum number ℓ, the previous decomposition is also significant in the limit $\ell \gg 1$. Indeed, for large values of ℓ, the term t^{2a} is exponentially suppressed, and the first line of the previous decomposition already gives a good estimate for the (logarithm of the) radial determinant.

We finally remark that, in the Schwarzschild-de Sitter case, this small R_h expansion gives purely imaginary QNMs (see [157] and the results in Sec. 2.1), as it happens for pure de Sitter spacetime. Analogously, in the Schwarzschild-anti-de Sitter case, neglecting the second channel in (3.4.87), produces purely real QNMs (see [59]), as it happens for pure anti-de Sitter spacetime.

More precisely, in the small R_h expansion of the QNMs,

$$\omega = \sum_{k \geq 0} \omega_k \, R_h^k, \tag{3.4.89}$$

where the dependence on the quantum numbers is implied, the first orders ω_k can be found by looking at the zeros of

$$\begin{aligned} &\frac{\Gamma(2a)\Gamma(1+2a)}{\prod_{\sigma=\pm} \Gamma\left(\frac{1}{2} - a_0 + a + \sigma\, a_t\right) \Gamma\left(\frac{1}{2} + a + a_1 + \sigma\, a_\infty\right)} \times \\ &t^{-a_0-a} \exp\left(-\frac{1}{2}\partial_{a_0} F(t) + \frac{1}{2}\partial_{a_1} F(t) + \frac{1}{2}\partial_a F(t)\right), \end{aligned} \tag{3.4.90}$$

which is equivalent to looking at the poles in the Gamma functions in the denominator[3]. For the four-dimensional Schwarzschild-de Sitter case this gives the correct coefficients ω_k for $0 \leq k \leq 2\ell + 1$ (see the results in Sec. 2.1), while for the five-dimensional Schwarzschild-anti-de Sitter case this gives the correct coefficients ω_k for $0 \leq k \leq 2\ell + 2$ (see [59][4]).

For the higher-order coefficients ω_k, the quantization condition involves both channels

[3]This is justified by gauge theory considerations, since $F(t)$ can be expressed as a series expansion in t (see Appendix A), and, therefore, there are no zeroes in the exponential functions.

[4]In [59] the small expansion parameter is μ which in the small black hole regime behaves like $\mu \sim R_h^2$. The near-horizon zone starts contributing when the QNMs develop an imaginary part, which behaves like $\mu^{\ell+\frac{3}{2}}$.

of the connection coefficient:

$$\begin{aligned}
&\frac{\Gamma(2a)\Gamma(1+2a)\, t^{-a_0-a}}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)}\times\\
&\exp\left(\frac{1}{2}\partial_{a_1}F(t)-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_a F(t)\right)+\\
&\frac{\Gamma(-2a)\Gamma(1-2a)\, t^{-a_0+a}}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0-a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-a+a_1+\sigma\, a_\infty\right)}\times\\
&\exp\left(\frac{1}{2}\partial_{a_1}F(t)-\frac{1}{2}\partial_{a_0}F(t)-\frac{1}{2}\partial_a F(t)\right)=0,
\end{aligned} \tag{3.4.91}$$

that is

$$\frac{\Gamma(-2a)^2\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)}{\Gamma(2a)^2\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0-a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-a+a_1+\sigma\, a_\infty\right)}t^{2a}\exp(-\partial_a F(t))=1. \tag{3.4.92}$$

Again, this is a manifestation of the fact that in the small R_h regime the contribution of the near-horizon zone is delayed compared to the far-zone, and the order of delay is determined by the angular quantum number ℓ.

3.4.1 Wick rotation and the thermal version

In this final part, we analyze the thermal version of the one-loop quantum effective actions, show how our results generalize the ones already present in the literature [139, 140] and reduce to the latter when the relevant differential equation reduces to the Hypergeometric one.

Let us Wick rotate the spacetime metric to real-time by defining $t = i\tau$, where τ has periodicity equal to the inverse of the temperature T of the spacetime. We can introduce the thermal frequencies by setting

$$i\,\omega_k = 2\,\pi\,T\,k, \quad k \in \mathbb{Z}. \tag{3.4.93}$$

With these redefinitions, it is possible to connect our results with the one in [139]. In particular, the results for ω with a positive imaginary part correspond to computation with $k < 0$, whereas the results for ω with a negative imaginary part correspond to computation with $k > 0$.

Let us see the match in the pure de Sitter and anti-de Sitter cases, where the radial differential equations are of Hypergeometric type.

In the four-dimensional de Sitter case, our result, using also the PT symmetry, can be rewritten as[5]

$$\log\left(\det\left(\Box-\mu^2\right)\right)=\sum_{k\in\mathbb{Z}}\sum_{\ell,m}\log\left(\frac{2\pi}{\Gamma\left(\ell+1\right)\Gamma\left(\ell+1+2\pi T\,|k|\right)}\right)= \\ =\sum_{k\in\mathbb{Z}}\sum_{\ell=0}^{\infty}(2\ell+1)\log\left(\frac{2\pi}{\ell!\,\Gamma\left(\ell+1+2\pi T\,|k|\right)}\right), \tag{3.4.94}$$

where we used that the degeneracy for each $\ell \geq 0$ is equal to $2\ell+1$ due to spherical symmetry.

Using Weierstrass's definition of the Gamma function

$$\Gamma\left(\ell+1+2\pi T\,|k|\right)= \\ \frac{\exp(-\gamma\left(\ell+1+2\pi T\,|k|\right))}{\ell+1+2\pi T\,|k|}\prod_{n=1}^{\infty}\left[\left(1+\frac{\ell+1+2\pi T\,|k|}{n}\right)^{-1}\exp\left(\frac{\ell+1+2\pi T\,|k|}{n}\right)\right], \tag{3.4.95}$$

we can write the full determinant as (for the equality in the formula see the comment in footnote 5)

$$\det\left(\Box-\mu^2\right)=\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\left[\frac{2\pi(\ell+1+2\,\pi\,T\,|k|)\prod_{n=1}^{\infty}\left(1+\frac{\ell+1+2\pi T\,|k|}{n}\right)}{\ell!\,\exp(-\gamma\left(\ell+1+2\pi T\,|k|\right))\prod_{n=1}^{\infty}\exp(\frac{\ell+1+2\pi T\,|k|}{n})}\right]^{2\ell+1}= \\ =\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\left[\frac{2\pi(2\,\pi\,T)\prod_{n=1}^{\infty}\frac{2\,\pi\,T}{n}\prod_{n=0}^{\infty}\left(|k|+\frac{\ell+n+1}{2\,\pi\,T}\right)}{\ell!\,\exp(-\gamma\left(\ell+1+2\pi T\,|k|\right))\prod_{n=1}^{\infty}\exp(\frac{\ell+1+2\pi T\,|k|}{n})}\right]^{2\ell+1}= \\ =\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\left[\frac{2\pi}{\ell!\,\exp(-\gamma\left(\ell+1+2\pi T\,|k|\right))}\frac{\prod_{n=0}^{\infty}(2\,\pi\,T)}{\prod_{n=1}^{\infty}n\,\exp(\frac{\ell+1+2\pi T\,|k|}{n})}\right]^{2\ell+1}\times \\ \times\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\prod_{n=0}^{\infty}\left(|k|+\frac{\ell+n+1}{2\,\pi\,T}\right)^{2\ell+1}, \tag{3.4.96}$$

where, in the final result, the first line is just an overall entire function (without any poles and zeroes in k, which corresponds to ω). Notice that since the QNMs of pure de Sitter spacetime are given by $-i(\ell+n+1)$, this result is consistent with formula (2.10) in [140].

In the pure AdS_5 case, the reasoning is analogous and, up to the overall factor, the structure of zeros can be seen from the infinite product arising from the Gamma functions

$$\frac{1}{\Gamma\left(\frac{\Delta+\ell-\omega_k}{2}\right)\Gamma\left(\frac{\Delta+\ell+\omega_k}{2}\right)}=\frac{1}{\Gamma\left(\frac{\Delta+\ell+2\,\pi\,i\,T\,|k|}{2}\right)\Gamma\left(\frac{\Delta+\ell-2\,\pi\,i\,T\,|k|}{2}\right)}. \tag{3.4.97}$$

[5] In writing the equality, we neglect UV divergencies due to the infinite products over the quantum numbers in the right-hand side. These should be cured by subtracting local counterterms, which can be analyzed, for example, with heat kernel methods or WKB-type approximations. In [140], the authors also comment on the possibility of absorbing these divergences into the cosmological constant, Newton's constant, and local couplings to higher curvature terms in the gravity sector.

The infinite product contribution gives

$$\begin{aligned}
&\det\left(\Box-\mu^2\right)\sim\\
&\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\Bigg[\frac{\Delta+\ell+2\,\pi\,i\,T\,|k|}{2}\,\frac{\Delta+\ell-2\,\pi\,i\,T\,|k|}{2}\times\\
&\prod_{n=1}^{\infty}\left(1+\frac{\Delta+\ell+2\,\pi\,i\,T\,|k|}{2n}\right)\left(1+\frac{\Delta+\ell-2\,\pi\,i\,T\,|k|}{2n}\right)\Bigg]^{(\ell+1)^2}=\\
&\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\Bigg[\frac{2\,\pi\,i\,T}{2}\left(|k|+\frac{\Delta+\ell}{2\,\pi\,i\,T}\right)\frac{-2\,\pi\,i\,T}{2}\left(|k|-\frac{\Delta+\ell}{2\,\pi\,i\,T}\right)\times\\
&\prod_{n=1}^{\infty}\frac{2\,\pi\,i\,T}{2}\left(|k|+\frac{2n+\Delta+\ell}{2\,\pi\,i\,T}\right)\frac{-2\,\pi\,i\,T}{2}\left(|k|-\frac{2n+\Delta+\ell}{2\,\pi\,i\,T}\right)\Bigg]^{(\ell+1)^2}=\\
&\sim\prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\left[\prod_{n=0}^{\infty}\left(|k|-i\,\frac{2n+\Delta+\ell}{2\,\pi\,T}\right)\left(|k|+i\,\frac{2n+\Delta+\ell}{2\,\pi\,T}\right)\right]^{(\ell+1)^2}
\end{aligned} \tag{3.4.98}$$

where we used that the degeneration for each $\ell\geq 0$ is given by $(\ell+1)^2$. This again coincides with formula (2.10) in [140] using (3.3.82).

For the cases considered in (3.4.96) and (3.4.98) we can also give the explicit formula for the ζ-function regularized one-loop action. We have

$$\zeta_{\mathrm{dS}_4}(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{dt}{t}\,t^s\,\frac{1+e^{-\beta t}}{1-e^{-\beta t}}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}(2\ell+1)\,e^{i(\ell+n+1)t} \tag{3.4.99}$$

in the four-dimensional de Sitter case, and

$$\zeta_{\mathrm{AdS}_5}(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{dt}{t}\,t^s\,\frac{1+e^{-\beta t}}{1-e^{-\beta t}}\sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}(\ell+1)^2\left[e^{(2\ell+n+\Delta)t}+e^{-(2\ell+n+\Delta)t}\right] \tag{3.4.100}$$

in the five-dimensional anti-de Sitter case.

In the black hole problems, extracting explicitly the relevant factors as in (3.4.98) from the Heun connection coefficients is complicated, since the zeros come from requiring the sum of the two channels in (3.4.87) to vanish, and it is no longer possible to look only in the infinite product structure of the Gamma functions.

However, if we consider the small R_h limit and the decomposition (3.4.88), the leading contribution of the determinant is given by

$$\begin{aligned}
\det\left(\Box-\mu^2\right)\sim\prod_{k\in\mathbb{Z}}\prod_{\ell,\vec{m}}&\frac{2\pi\Gamma(2a)\Gamma(1+2a)t^{-a_0-a}}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\,a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+\sigma\,a_\infty\right)}\times\\
&\times\exp\left(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)+\frac{1}{2}\partial_{a}F(t)\right),
\end{aligned} \tag{3.4.101}$$

where again the substitution $i\,\omega=2\,\pi\,T\,k$ is implied.

In all the considered cases, the leading order of a in the expansion in R_h (and also in the small a_{BH} expansion for the Kerr-de Sitter case) depends only on the angular quantum number ℓ, and both indices a_0 and a_t start with higher orders in R_h, therefore, neglecting the corrections in the last line in (3.4.88), the factors

$$\frac{2\pi\Gamma(2a)\Gamma(1+2a)t^{-a_0-a}}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\, a_t\right)}\exp\left(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)+\frac{1}{2}\partial_a F(t)\right) \tag{3.4.102}$$

only contribute as entire functions and do not give any contributions to zeros or poles in ω, and all the analytic structure can be written by the infinite products in the Gamma functions

$$\Gamma\left(\frac{1}{2}+a+a_1\pm a_\infty\right)\propto\frac{1}{\frac{1}{2}+a+a_1\pm a_\infty}\prod_{n=1}^{\infty}\left(1+\frac{\frac{1}{2}+a+a_1\pm a_\infty}{n}\right)^{-1}. \tag{3.4.103}$$

Moreover, since $t\sim R_h$, only the leading order of a contributes, and the reasoning proceeds as in the Hypergeometric cases.

In the final analytic structure, there is also one important difference between the Kerr-de Sitter case and the spherically symmetric cases, which is given by the degeneracies coming from the angular problem. In the spherically symmetric problems in four dimensions, these are given by $N^{(4)}(\ell)=2\ell+1$, and in five dimensions by $N^{(5)}(\ell)=(\ell+1)^2$, for each $\ell\geq 0$. In this approximation, the analytic structure in the leading order can therefore be read from

$$\det\left(\Box-\mu^2\right)\sim \prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\left[\prod_{\sigma=\pm}\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)\prod_{n=1}^{\infty}\prod_{\sigma=\pm}\left(1+\frac{\frac{1}{2}+a+a_1+\sigma\, a_\infty}{n}\right)\right]^{N^{(d)}(\ell)}, \tag{3.4.104}$$

with $d=4,5$. In the Kerr-de Sitter case, instead, the formula reads

$$\det\left(\Box-\mu^2\right)\sim \prod_{k\in\mathbb{Z}}\prod_{\ell=0}^{\infty}\prod_{m=-\ell}^{\ell}\left[\prod_{\sigma=\pm}\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)\prod_{n=1}^{\infty}\prod_{\sigma=\pm}\left(1+\frac{\frac{1}{2}+a+a_1+\sigma\, a_\infty}{n}\right)\right], \tag{3.4.105}$$

and each pair of values (ℓ,m) gives a different contribution.

Although there are no closed expressions for the QNMs of the generic BH, we still can write approximate formulae by expanding in the BH radius R_h by using the explicit power expansion of the QNMs (3.4.89). As these, to the first order can be found from the zeros of (3.4.104) and (3.4.105), we get the following approximated expressions.

For the four-dimensional Kerr-de Sitter case in the small-rotating regime, one gets

$$\zeta_{\mathrm{KdS}_4}(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{dt}{t}\,t^s\,\frac{1+e^{-\beta t}}{1-e^{-\beta t}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{n=0}^{\infty}e^{\left[i(\ell+n+1)-a_{\mathrm{BH}}\,m+\mathcal{O}(R_h^2,a_{\mathrm{BH}}^2)\right]t}, \tag{3.4.106}$$

where we used that the first order correction in R_h of the QNMs vanishes for any value of the quantum numbers. This reduces in the $a_{\rm BH} \to 0$ Schwarzschild limit to (see Sec. 2.1)

$$\zeta_{\mathrm{SdS}_4}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t}\, t^s\, \frac{1+e^{-\beta t}}{1-e^{-\beta t}} \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} (2\ell+1)\, e^{\left[i(\ell+n+1)-\omega_2\, R_h^2+\mathcal{O}(R_h^3)\right]t}. \qquad (3.4.107)$$

where, for $\ell \geq 0$[6] and $n \geq 0$,

$$\begin{aligned}\omega_2 = &-\frac{i}{8(\ell+1)(2\ell+1)(2\ell-1)(2\ell+3)} \times \Big[\ell^3\left(60n^2+60n+22\right)+ \\ &\ell^2\left(120n^2+122n+45\right)+\ell\left(16n^2+19n+8\right)-\left(44n^2+43n+15\right)\Big].\end{aligned} \qquad (3.4.108)$$

For the five-dimensional Schwarzschild-anti-de Sitter case one has, instead,

$$\begin{aligned}&\zeta_{\mathrm{SAdS}_5}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t}\, t^s \times \\ &\frac{1+e^{-\beta t}}{1-e^{-\beta t}} \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} (\ell+1)^2 \left\{e^{\left[2\ell+n+\Delta-\hat{\omega}_2\, R_h^2+\mathcal{O}(R_h^3)\right]t} + e^{-\left[2\ell+n+\Delta+\hat{\omega}_2\, R_h^2+\mathcal{O}(R_h^3)\right]t}\right\},\end{aligned} \qquad (3.4.109)$$

where (see eq.(47) in [59])

$$\hat{\omega}_2 = -\frac{\Delta^2+\Delta(6n-1)+6n(n-1)}{2(\ell+1)}. \qquad (3.4.110)$$

The full result, rewritten in a form that makes explicit the analytic structure depending on the QNMs (see the quantization condition (3.4.92)), is (for the equality in the formula see the comment in footnote 5)

$$\begin{aligned}&\det\left(\Box-\mu^2\right) = \\ &\prod_{k\in\mathbb{Z}}\prod_{\ell,\vec{m}}\left\{\frac{2\pi\Gamma(2a)\Gamma(1+2a)t^{-a_0-a}\exp(-\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)+\frac{1}{2}\partial_a F(t))}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)}\times\right. \\ &\left.\left[1-\frac{\Gamma(-2a)^2\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0+a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+\sigma\, a_\infty\right)}{\Gamma(2a)^2\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}-a_0-a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-a+a_1+\sigma\, a_\infty\right)}t^{2a}\exp(-\partial_a F(t))\right]\right\},\end{aligned} \qquad (3.4.111)$$

where the substitution (3.4.93) is implied, and the structure depending on QNMs can be read from the second line.

We remark again that, given a fixed ℓ_0, the coefficients of the QNMs expansion (3.4.89) up to order $2\ell_0+1$ (for the four-dimensional cases) or $2\ell_0+2$ (for the five-dimensional case) in R_h can be determined by the poles in (3.4.103), where the additional

[6] For $\ell > 0$ the correction ω_2 can be found from the zeros of (3.4.104). The case $\ell = 0$ is more subtle. The full quantization condition (3.4.92) must be used in this case. However, in the expansion in R_h, the leading order of the v.e.v. parameter a equals 1/2, and the NS function $F(t)$ has a pole for this value of the parameter. To find the analytic expression for ω_2, we first assume ℓ to be generic in (3.4.92), and only in the final expansion in R_h we send $\ell \to 0$.

complication compared to the Hypergeometric cases comes from the fact that a is expressed as an instanton expansion and ω (or, equivalently, k) appears in the coefficients of such expansion.

3.5 Summary

In this chapter, we proved a singular version of the Gelfand-Yaglom theorem and applied it to compute determinants of differential operators relevant in the context of black hole perturbation problems. The final results are written in terms of the connection formulae for Heun's equation. In the final part, we analyzed the thermal version of the one-loop effective actions, showing how our results reduce to the ones already present in the literature [139, 140] when the relevant differential equation simplifies to the Hypergeometric one.

The main novelty of our approach is the use of the techniques from Liouville CFT, and more precisely of the connection formulae for Fuchsian differential equations, in the computation of spectral determinants. It would be interesting to extend this analysis to confluent forms of the differential equations, which arise, as seen in the previous chapter, in asymptotically flat spacetime.

Chapter 4

Convergence of Nekrasov's functions

In this chapter, we study the convergence properties of Nekrasov's instanton partition functions with matter in the adjoint and fundamental representations. We refer to Appendix A for the notations used for the Young diagrams and the building blocks of the Nekrasov functions.

4.1 Convergence of $U(N)$ Instanton Partition Function with adjoint matter

We begin our analysis with the study of the convergence properties of the instanton partition function of $\mathcal{N} = 2^* \ U(N)$ gauge theory

$$Z_{\text{inst}}^{\mathcal{N}=2^*,U(N)} = \sum_{k\geq 0} t^k \sum_{|\vec{Y}|=k} \prod_{i=1}^{N} \prod_{s\in Y_i} \left(1 - \frac{m}{-\epsilon_1 L_{Y_i}(s) + \epsilon_2(A_{Y_i}(s)+1)}\right)\left(1 - \frac{m}{\epsilon_1(L_{Y_i}(t)+1) - \epsilon_2 A_{Y_i}(s)}\right) \times \prod_{1\leq i\neq j\leq N} \prod_{s\in Y_i} \left(1 - \frac{m}{a_i - a_j - \epsilon_1 L_{Y_j}(s) + \epsilon_2(A_{Y_i}(s)+1)}\right) \times \prod_{r\in Y_j} \left(1 - \frac{m}{-a_j + a_i + \epsilon_1(L_{Y_i}(r)+1) - \epsilon_2 A_{Y_j}(r)}\right). \tag{4.1.1}$$

In the products above we collected first the pairs with $i = j$ (in what follows we will call these contributions *diagonal*), and then the pairs (i, j) with $i \neq j$ (in what follows we will call these contributions *nondiagonal*). From a direct inspection of (4.1.1), one can see that the coefficients of the series are well defined under the assumptions

$$\text{Arg}\left(\frac{\epsilon_2}{\epsilon_1}\right) \neq 0 \quad \text{and} \quad \pm(a_i - a_j) \notin \Lambda(\epsilon_1, \epsilon_2) \quad \forall\ 1 \leq i < j \leq N\,, \tag{4.1.2}$$

where $\Lambda(\epsilon_1, \epsilon_2)$ is the 2-dimensional lattice

$$\Lambda(\epsilon_1, \epsilon_2) = \{z \in \mathbb{C} \mid z \in \epsilon_1\mathbb{Z} + \epsilon_2\mathbb{Z}\}, \tag{4.1.3}$$

which we will use in the proof of the

Theorem 4.1.1. *The instanton partition function of the $\mathcal{N} = 2^*$ $U(N)$ gauge theory, as a power series in the complex parameter t, is absolutely convergent at least for*

$$|t| < \left(1 + \frac{|m|}{D(\vec{a}, \epsilon_1, \epsilon_2)}\right)^{-2(N-1)}, \tag{4.1.4}$$

where m is the mass of the adjoint multiplet, and

$$D(\vec{a}, \epsilon_1, \epsilon_2) = \min_{1 \leq i \neq j \leq N} \{ \min_{p \in \Lambda(\epsilon_1, \epsilon_2)} \{|a_i - a_j - p|\}\}. \tag{4.1.5}$$

From this result, two corollaries can be proved. The first comes from the fact that the $\mathcal{N} = 2^*$ instanton partition function reduces to the $\mathcal{N} = 2$ SYM instanton partition function in the double scaling limit $t \to 0$ and $m \to \infty$ with $\Lambda = t\, m^{2N}$ kept finite.

Corollary 4.1.2. *The instanton partition function of the $U(N)$ pure gauge theory, as a power series in the complex parameter Λ, is convergent over the whole complex plane.*

The second corollary comes from the fact that if the mass of the adjoint multiplet goes to zero, $m \to 0$, the $\mathcal{N} = 2^*$ instanton partition function reduces to the $\mathcal{N} = 4$ instanton partition function.

Corollary 4.1.3. *The instanton partition function of the $\mathcal{N} = 4$ $U(N)$ gauge theory, as a power series in the complex parameter t, is convergent in the region $|t| < 1$.*

Remark 4.1.1. *By using known analytic properties of the partition function* (4.1.1)*, one can lift* (4.1.2) *to milder conditions for the values of the a-parameters. Indeed, the second condition, which we imposed to a priori get rid of the possible poles in the non-diagonal part, can be reduced to the set of actual poles as classified in [201, 202, 203].*

Remark 4.1.2. *The content of Corollary 4.1.2 is a higher rank generalisation of an observation about the $SU(2)$ SYM $\mathcal{N} = 2$ instanton partition function given in [42].*

Remark 4.1.3. *Corollary 4.1.3 is trivial. Indeed, it is well known that the $\mathcal{N} = 4$ partition function is equal to $\phi(t)^{-N}$, $\phi(t)$ being the Euler function.*

4.1.1 Proof of Theorem 4.1.1

It is useful to divide the assumption

$$\mathrm{Arg}\left(\frac{\epsilon_2}{\epsilon_1}\right) \neq 0 \tag{4.1.6}$$

in the two subcases:

1. $\mathrm{Im}\left(\frac{\epsilon_2}{\epsilon_1}\right) \neq 0$;
2. $\mathrm{Re}\left(\frac{\epsilon_2}{\epsilon_1}\right) < 0$.

First Subcase

Suppose

$$\mathrm{Im}\left(\frac{\epsilon_2}{\epsilon_1}\right) \neq 0.$$

Let $\delta > 0$ be a real number such that

$$\min\left\{\left|\mathrm{Im}\left(\frac{\epsilon_2}{\epsilon_1}\right)\right|, \left|\mathrm{Im}\left(\frac{\epsilon_1}{\epsilon_2}\right)\right|\right\} > \delta.$$

Notice that $\delta \leq 1$.

We first analyze the products over the boxes of one of the diagrams, say Y_1, whose contributions come from the diagonal factors, namely we look for a bound on

$$\left|\prod_{s\in Y_1}\left(1-\frac{m}{-\epsilon_1 L_{Y_1}(s)+\epsilon_2(A_{Y_1}(s)+1)}\right)\left(1-\frac{m}{\epsilon_1(L_{Y_1}(s)+1)-\epsilon_2 A_{Y_1}(s)}\right)\right|. \tag{4.1.7}$$

An analogous reasoning also holds for the diagonal contributions of the other diagrams $Y_2, \ldots, Y_N$.

We begin by estimating the denominators in the previous product. Let us fix a box $s \in Y_1$, and let us consider the term

$$\frac{1}{|-\epsilon_1 L_{Y_1}(s)+\epsilon_2(A_{Y_1}(s)+1)|}. \tag{4.1.8}$$

By recalling the definition of *hook length*, $h_{Y_1}(s) = L_{Y_1}(s) + A_{Y_1}(s) + 1$, we can without loss of generality suppose

$$A_{Y_1}(s) \geq \frac{h_{Y_1}(s)-1}{2}.$$

Then, if we collect a factor of ϵ_1, we have

$$\begin{aligned} |-\epsilon_1 L_{Y_1}(s)+\epsilon_2(A_{Y_1}(s)+1)| &= |\epsilon_1|\cdot|L_{Y_1}(s)-\frac{\epsilon_2}{\epsilon_1}(A_{Y_1}(s)+1)| \\ &\geq |\epsilon_1|\cdot\left|\mathrm{Im}\left(\frac{\epsilon_2}{\epsilon_1}\right)\right|\cdot(A_{Y_1}(s)+1) \\ &\geq |\epsilon_1|\cdot\delta\cdot\frac{h_{Y_1}(s)+1}{2} \geq |\epsilon_1|\cdot\delta\cdot\frac{h_{Y_1}(s)}{4}. \end{aligned} \tag{4.1.9}$$

Analogously, for the term

$$\frac{1}{|\epsilon_1(L_{Y_1}(s)+1)-\epsilon_2 A_{Y_1}(s)|}, \tag{4.1.10}$$

we have

$$\begin{aligned} |\epsilon_1(L_{Y_1}(s)+1)-\epsilon_2 A_{Y_1}(s)| = |\epsilon_1|\cdot|L_{Y_1}(s)+1-\frac{\epsilon_2}{\epsilon_1}A_{Y_1}(s)| &\geq |\epsilon_1|\cdot\left|\mathrm{Im}\left(\frac{\epsilon_2}{\epsilon_1}\right)\right|\cdot A_{Y_1}(s) \\ &\geq |\epsilon_1|\cdot\delta\cdot\frac{h_{Y_1}(s)-1}{2}. \end{aligned} \tag{4.1.11}$$

Notice that, if $h_{Y_1}(s) = 1$, then both $L_{Y_1}(s) = A_{Y_1}(s) = 0$, and the previous term is simply $|\epsilon_1| = |\epsilon_1| h_{Y_1}(s) \geq |\epsilon_1| \cdot \delta \cdot \frac{h_{Y_1}(s)}{4}$, and, if $h_{Y_1}(s) \geq 2$, then $(h_{Y_1}(s) - 1)/2 \geq h_{Y_1}(s)/4$. Therefore, also this term is always bounded by $|\epsilon_1| \cdot \delta \cdot \frac{h_{Y_1}(s)}{4}$.

If we instead considered a box s for which

$$A_{Y_1}(s) < \frac{h_{Y_1}(s) - 1}{2},$$

we would have

$$L_{Y_1}(s) \geq \frac{h_{Y_1}(s) - 1}{2}.$$

In this case, we collect factors of ϵ_2 from both terms to obtain

$$\begin{aligned} |-\epsilon_1 L_{Y_1}(s) + \epsilon_2 (A_{Y_1}(s) + 1)| &\geq |\epsilon_2| \cdot \delta \cdot \frac{h_{Y_1}(s)}{4}, \\ |\epsilon_1 (L_{Y_1}(s) + 1) - \epsilon_2 A_{Y_1}(s)| &\geq |\epsilon_2| \cdot \delta \cdot \frac{h_{Y_1}(s)}{4}. \end{aligned} \tag{4.1.12}$$

Now, fix

$$|\epsilon| = \min\{|\epsilon_1|, |\epsilon_2|\}. \tag{4.1.13}$$

Then,

$$\begin{aligned} \frac{1}{|-\epsilon_1 L_{Y_1}(s) + \epsilon_2 (A_{Y_1}(s) + 1)|} &\leq \frac{4}{|\epsilon| \cdot \delta \cdot h_{Y_1}(s)}, \\ \frac{1}{|\epsilon_1 (L_{Y_1}(s) + 1) - \epsilon_2 A_{Y_1}(s)|} &\leq \frac{4}{|\epsilon| \cdot \delta \cdot h_{Y_1}(s)}. \end{aligned} \tag{4.1.14}$$

Therefore, we have that

$$\begin{aligned} &\left| \prod_{s \in Y_1} \left(1 - \frac{m}{-\epsilon_1 L_{Y_1}(s) + \epsilon_2 (A_{Y_1}(s) + 1)}\right) \left(1 - \frac{m}{\epsilon_1 (L_{Y_1}(s) + 1) - \epsilon_2 A_{Y_1}(s)}\right) \right| = \\ &\left| \prod_{s \in Y_1} \left(1 + \frac{m^2 - m(\epsilon_1 + \epsilon_2)}{[-\epsilon_1 L_{Y_1}(s) + \epsilon_2 (A_{Y_1}(s) + 1)][\epsilon_1 (L_{Y_1}(s) + 1) - \epsilon_2 A_{Y_1}(s)]}\right) \right| \leq \\ &\prod_{s \in Y_1} \left(1 + \frac{16 |m^2 - m(\epsilon_1 + \epsilon_2)|}{\delta^2 |\epsilon|^2 h_{Y_1}(s)^2}\right) = \prod_{s \in Y_1} \left(1 + \frac{\frac{16|m^2 - m(\epsilon_1 + \epsilon_2)|}{\delta^2 |\epsilon|^2}}{h_{Y_1}(s)^2}\right). \end{aligned} \tag{4.1.15}$$

We now consider the remaining terms, which come from the nondiagonal contributions. We analyze the products over the boxes of Y_1, coming from the pairs $(1, 2)$ and $(2, 1)$. The products over the boxes of the diagram Y_2 in the same pairs will be analogous, and the same holds for any other couple of pairs $(i, j), (j, i)$.

The terms we consider are then

$$\prod_{s \in Y_1} \left(1 - \frac{m}{a_1 - a_2 - \epsilon_1 L_{Y_2}(s) + \epsilon_2 (A_{Y_1}(s) + 1)}\right) \left(1 - \frac{m}{a_2 - a_1 + \epsilon_1 (L_{Y_2}(s) + 1) - \epsilon_2 A_{Y_1}(s)}\right). \tag{4.1.16}$$

With our assumptions on the vev parameters a_i we have that the denominators are never zero. Let us define

$$D_{ij}(\vec{a},\epsilon_1,\epsilon_2) = \min_{p\in\Lambda(\epsilon_1,\epsilon_2)}\{|a_i - a_j - p|\}. \tag{4.1.17}$$

We have that

$$\begin{aligned}
&\Big|\prod_{s\in Y_1}\left(1-\frac{m}{a_1-a_2-\epsilon_1 L_{Y_2}(s)+\epsilon_2(A_{Y_1}(s)+1)}\right)\left(1-\frac{m}{a_2-a_1+\epsilon_1(L_{Y_2}(s)+1)-\epsilon_2 A_{Y_1}(s)}\right)\Big| \leq \\
&\prod_{s\in Y_1}\left(1+\frac{|m|}{|a_1-a_2-\epsilon_1 L_{Y_2}(s)+\epsilon_2(A_{Y_1}(s)+1)|}\right)\left(1+\frac{|m|}{|a_2-a_1+\epsilon_1(L_{Y_2}(s)+1)-\epsilon_2 A_{Y_1}(s)|}\right) \\
&\leq \left(1+\frac{|m|}{D_{12}(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2|Y_1|}.
\end{aligned} \tag{4.1.18}$$

Putting the bounds (4.1.15) and (4.1.18) together, we can conclude

$$\begin{aligned}
&\Big|\sum_{k\geq 0} t^k \sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1-\frac{m}{-\epsilon_1 L_{Y_i}(s)+\epsilon_2(A_{Y_i}(s)+1)}\right)\left(1-\frac{m}{\epsilon_1(L_{Y_i}(s)+1)-\epsilon_2 A_{Y_i}(s)}\right)\times \\
&\quad \prod_{1\leq i\neq j\leq N}\prod_{s\in Y_i}\left(1-\frac{m}{a_i-a_j-\epsilon_1 L_{Y_j}(s)+\epsilon_2(A_{Y_i}(s)+1)}\right)\times \\
&\qquad \prod_{r\in Y_j}\left(1-\frac{m}{-a_j+a_i+\epsilon_1(L_{Y_i}(r)+1)-\epsilon_2 A_{Y_j}(r)}\right)\Big| \leq \\
&\leq \sum_{k\geq 0}|t|^k \sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}}{h_{Y_i}(s)^2}\right)\prod_{i=1}^{N}\prod_{j\neq i}\left(1+\frac{|m|}{D_{ij}(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2|Y_i|}.
\end{aligned} \tag{4.1.19}$$

Now, let us define

$$D(\vec{a},\epsilon_1,\epsilon_2) = \min_{1\leq i\neq j\leq N}\{D_{ij}(\vec{a},\epsilon_1,\epsilon_2)\}. \tag{4.1.20}$$

The following result (which is Theorem 1.2 in [204]) is useful:

Proposition 4.1.4. *For any complex number z the following holds:*

$$\sum_{Y\in\mathbb{Y}} x^{|Y|}\prod_{s\in Y}\left(1-\frac{z}{(h_Y(s))^2}\right) = \prod_{j=1}^{\infty}(1-x^j)^{z-1} = \phi(x)^{z-1}, \tag{4.1.21}$$

where ϕ is the Euler function.

We remind that $\phi(x)$ is convergent for $|x|<1$. Then,

$$\sum_{k\geq 0}|t|^k\sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}}{h_{Y_i}(s)^2}\right)\prod_{i=1}^{N}\prod_{j\neq i}\left(1+\frac{|m|}{D_{ij}(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2|Y_i|}\leq$$

$$\sum_{k\geq 0}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^k\sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}}{h_{Y_i}(s)^2}\right)=$$

$$\sum_{Y_1,\dots,Y_N\in\mathbb{Y}}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^{\sum_{i=1}^{N}|Y_i|}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}}{h_{Y_i}(s)^2}\right)=$$

$$\left\{\sum_{Y\in\mathbb{Y}}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^{|Y|}\prod_{s\in Y}\left(1+\frac{\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}}{h_{Y}(s)^2}\right)\right\}^N=$$

$$\phi\left(|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right)^{N\left(-\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}-1\right)},\tag{4.1.22}$$

where in the last line we used (4.1.21) with

$$x=|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\quad\text{and}\quad z=-\frac{16|m^2-m(\epsilon_1+\epsilon_2)|}{\delta^2|\epsilon|^2}.\tag{4.1.23}$$

Hence, we can conclude that the instanton partition function is convergent in the region defined by

$$|t|<\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{-2(N-1)}.\tag{4.1.24}$$

Second Subcase

Suppose otherwise that

$$\mathrm{Re}\left(\frac{\epsilon_2}{\epsilon_1}\right)<0,$$

and let

$$\beta=-\mathrm{Re}\left(\frac{\epsilon_2}{\epsilon_1}\right)>0.$$

Again, we start by analyzing the products over the boxes of one of the diagrams, say Y_1, coming from the diagonal contributions, that is, we look for a bound on

$$\left|\prod_{s\in Y_1}\left(1-\frac{m}{-\epsilon_1 L_{Y_1}(s)+\epsilon_2(A_{Y_1}(s)+1)}\right)\left(1-\frac{m}{\epsilon_1(L_{Y_1}(s)+1)-\epsilon_2 A_{Y_1}(s)}\right)\right|.\tag{4.1.25}$$

We begin by estimating the denominators. For every box $s \in Y_1$, we have

$$
\begin{aligned}
& |-\epsilon_1 L_{Y_1}(s) + \epsilon_2(A_{Y_1}(s)+1)| = |\epsilon_1| \cdot \left| L_{Y_1}(s) - \frac{\epsilon_2}{\epsilon_1}(A_{Y_1}(s)+1) \right| \geq \\
& |\epsilon_1| \cdot \left| \mathrm{Re}\left[L_{Y_1}(s) - \frac{\epsilon_2}{\epsilon_1}(A_{Y_1}(s)+1) \right] \right| = \\
& |\epsilon_1| \cdot \left| L_{Y_1}(s) - (A_{Y_1}(s)+1)\mathrm{Re}\left(\frac{\epsilon_2}{\epsilon_1}\right) \right| = \\
& |\epsilon_1| \cdot \left| L_{Y_1}(s) + \beta(A_{Y_1}(s)+1) \right|.
\end{aligned}
\tag{4.1.26}
$$

Fix $\gamma = \min\{\beta, 1\}$. Then,

$$
\frac{1}{|-\epsilon_1 L_{Y_1}(s) + \epsilon_2(A_{Y_1}(s)+1)|} \leq \frac{1}{|\epsilon_1|} \frac{1}{\gamma h_{Y_1}(s)}. \tag{4.1.27}
$$

Analogously, for the other term in the product, we have

$$
\frac{1}{|\epsilon_1(L_{Y_1}(s)+1) - \epsilon_2 A_{Y_1}(s)|} \leq \frac{1}{|\epsilon_1|} \frac{1}{\gamma h_{Y_1}(s)}. \tag{4.1.28}
$$

Then,

$$
\begin{aligned}
& \left| \prod_{s \in Y_1} \left(1 - \frac{m}{-\epsilon_1 L_{Y_1}(s) + \epsilon_2(A_{Y_1}(s)+1)}\right)\left(1 - \frac{m}{\epsilon_1(L_{Y_1}(s)+1) - \epsilon_2 A_{Y_1}(s)}\right) \right| = \\
& \left| \prod_{s \in Y_1} \left(1 + \frac{m^2 - m(\epsilon_1+\epsilon_2)}{[-\epsilon_1 L_{Y_1}(s) + \epsilon_2(A_{Y_1}(s)+1)][\epsilon_1(L_{Y_1}(s)+1) - \epsilon_2 A_{Y_1}(s)]}\right) \right| \leq \\
& \prod_{s \in Y_1} \left(1 + \frac{|m^2 - m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2 \gamma^2 h_{Y_1}(s)^2}\right) = \prod_{s \in Y_1} \left(1 + \frac{\frac{|m^2 - m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}}{h_{Y_1}(s)^2}\right).
\end{aligned}
\tag{4.1.29}
$$

An analogous bound holds for all diagrams $Y_2, \ldots, Y_N$. For the nondiagonal contributions we use the same bound (4.1.18) of the previous subsection. Therefore, in this case we have

$$
\begin{aligned}
&\Big|\sum_{k\geq 0} t^k \sum_{|\vec{Y}|=k} \prod_{i=1}^{N}\prod_{s\in Y_i}\left(1-\frac{m}{-\epsilon_1 L_{Y_i}(s)+\epsilon_2(A_{Y_i}(s)+1)}\right)\\
&\left(1-\frac{m}{\epsilon_1(L_{Y_i}(s)+1)-\epsilon_2 A_{Y_i}(s)}\right)\times\\
&\prod_{1\leq i\neq j\leq N}\prod_{s\in Y_i}\left(1-\frac{m}{a_i-a_j-\epsilon_1 L_{Y_j}(s)+\epsilon_2(A_{Y_i}(s)+1)}\right)\\
&\prod_{r\in Y_j}\left(1-\frac{m}{-a_j+a_i+\epsilon_1(L_{Y_i}(r)+1)-\epsilon_2 A_{Y_j}(r)}\right)\Big| \leq\\
&\sum_{k\geq 0}|t|^k \sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}}{h_{Y_1}(s)^2}\right)\prod_{i=1}^{N}\prod_{j\neq i}\left(1+\frac{|m|}{D_{ij}(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2|Y_i|}\leq\\
&\sum_{k\geq 0}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^k \sum_{|\vec{Y}|=k}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}}{h_{Y_1}(s)^2}\right)=\\
&\sum_{Y_1,\dots,Y_N\in\mathbb{Y}}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^{\sum_{i=1}^N|Y_i|}\prod_{i=1}^{N}\prod_{s\in Y_i}\left(1+\frac{\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}}{h_{Y_1}(s)^2}\right)=\\
&\left\{\sum_{Y\in\mathbb{Y}}\left[|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right]^{|Y|}\prod_{s\in Y}\left(1+\frac{\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}}{h_Y(s)^2}\right)\right\}^N=\\
&\phi\left(|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)}\right)^{N\left(-\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}-1\right)},
\end{aligned}
\tag{4.1.30}
$$

where in the last line we used (4.1.21) with

$$
x=|t|\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{2(N-1)} \quad \text{and} \quad z=-\frac{|m^2-m(\epsilon_1+\epsilon_2)|}{|\epsilon_1|^2\gamma^2}. \tag{4.1.31}
$$

Hence, as in the previous case, the instanton partition function is convergent in the region defined by

$$
|t|<\left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{-2(N-1)}. \tag{4.1.32}
$$

Remark 4.1.4. *Let us note that in the case $\epsilon_2/\epsilon_1\in\mathbb{R}_{<0}$ the 2-dimensional lattice $\Lambda(\epsilon_1,\epsilon_2)$ degenerates into a 1-dimensional lattice. Therefore, if we move sufficiently away from the line spanned by ϵ_1 in the complex plane, that is, if, for every $i\neq j$, a_i-a_j has a big enough distance from the set $\{z\in\mathbb{C}\mid z=r\epsilon_1, r\in\mathbb{R}\}$, the constant $D(\vec{a},\epsilon_1,\epsilon_2)$ can become very large and the radius of convergence tends to 1.*

4.1.2 Corollary 4.1.2: from $\mathcal{N}=2^*$ to $\mathcal{N}=2$ SYM

The results on the convergence of the $\mathcal{N}=2$ instanton partition function can be deduced from the ones on the $\mathcal{N}=2^*$ instanton partition function. Indeed, if one considers the double scaling limit in which the mass of the adjoint multiplet m becomes large $m\to\infty$ and the instanton parameter t becomes small, $t\to 0$, in such a way that $\Lambda := t\,m^{2N}$ remains finite, the instanton partition function of the $\mathcal{N}=2^*$ $U(N)$ theory (A.1.8) reduces to (A.1.9) in the expansion parameter Λ instead of t.

From (4.1.32), we find

$$|m|^{2N}|t| \le \left(1+\frac{|m|}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^2\left(\frac{1}{|m|}+\frac{1}{D(\vec{a},\epsilon_1,\epsilon_2)}\right)^{-2N} \qquad (4.1.33)$$

which in the above limit reduces to $|\Lambda|<\infty$.

4.1.3 Corollary 4.1.3: from $\mathcal{N}=2^*$ to $\mathcal{N}=4$

The instanton partition function of the $\mathcal{N}=4$ $U(N)$ gauge theory can be written as

$$Z_{\text{inst}}^{\mathcal{N}=4,U(N)} = \sum_{k\ge 0} t^k \sum_{|Y|=k} 1 = \sum_{k\ge 0} t^k p_N(k) = \prod_{j=1}^{\infty}\frac{1}{(1-t^j)^N} = \phi(t)^{-N}, \qquad (4.1.34)$$

which is convergent in the region $|t|<1$.

This result can also be obtained from the analysis of the $\mathcal{N}=2^*$ $U(N)$ theory setting to zero the mass of the adjoint multiplet, as it is obvious from (4.1.32).

4.2 $U(N)$ Instanton Partition Functions with Fundamental Matter

Also in this case, we work under the assumptions (4.1.2). Moreover, we assume $\epsilon_1+\epsilon_2=0$ and set

$$\epsilon := \epsilon_1 = -\epsilon_2, \quad \alpha_i := a_i/\epsilon, \quad \mu_r := m_r/\epsilon. \qquad (4.2.35)$$

In this notation, the instanton partition function reads

$$Z_{\text{inst}}^{U(N)\ ,N_f} = \sum_{k\ge 0}\left(t\,\epsilon^{N_f-2N}\right)^k \sum_{|\vec{Y}|=k}\prod_{i,j=1}^{N}\prod_{(m,n)\in Y_i}\frac{1}{\alpha_i-\alpha_j-h_{Y_i}((m,n))+(Y_i')_m-(Y_j')_m}$$
$$\prod_{(m,n)\in Y_j}\frac{1}{\alpha_i-\alpha_j+h_{Y_j}((m,n))-(Y_j')_m+(Y_i')_m}$$
$$\prod_{i=1}^{N}\prod_{(m,n)\in Y_i}\prod_{r=1}^{N_f}\left[\alpha_i+\mu_r+m-n\right]. \qquad (4.2.36)$$

We observe that in this case (4.1.2) reduces to $\alpha_i-\alpha_j\notin\mathbb{Z}$ for every $1\le i<j\le N$.

The main result we find is

Theorem 4.2.1. *The instanton partition function of the $U(N)$ gauge theory with $N_f = 2N$ fundamental multiplets has at least a finite radius of convergence.*

The instanton partition function of the $U(N)$ gauge theory with $N_f < 2N$ fundamental multiplets is absolutely convergent over the whole complex plane.

We consider in (4.2.36) the sum starting from $k \geq 1$, as it does not change the convergence properties of the series.

Many steps will be necessary to arrive at our final result, so it is useful to divide the coefficient functions into simpler factors and analyze them separately.

We start by considering the products over the boxes of one specific Young diagram, let us take Y_1, which are

$$\prod_{(m,n)\in Y_1} \frac{\prod_{r=1}^{N_f}[\alpha_1+\mu_r+m-n]}{h_{Y_1}((m,n))^2} \prod_{j\neq 1}\prod_{(m,n)\in Y_1} \frac{1}{(\alpha_1-\alpha_j-h_{Y_1}((m,n))+(Y_1')_m-(Y_j')_m)^2}. \tag{4.2.37}$$

We first analyze the $N_f = 2N$ case of the theorem, in which we have the same number of factors in the numerator and denominator of (4.2.37). In particular, we can factor (4.2.37) in two types of products:

$$\prod_{(m,n)\in Y_1} \frac{\alpha_1+\mu_r+m-n}{h_{Y_1}((m,n))} \quad \text{with} \quad r \in \{1,\ldots,N_f\}, \tag{4.2.38}$$

and

$$\prod_{(m,n)\in Y_1} \frac{\alpha_1+\mu_r+m-n}{\alpha_1-\alpha_j-h_{Y_1}((m,n))+(Y_1')_m-(Y_j')_m} \text{ with } r\in\{1,\ldots,N_f\} \text{ and } j\in\{2,\ldots,N\}. \tag{4.2.39}$$

The key result on the first kind of product is the following

Lemma 4.2.2. *For every Young diagram Y with $k \geq 1$ boxes and for every fixed complex number z, the following inequality holds*

$$\begin{aligned}
&\prod_{(i,j)\in Y}\left|\frac{z+i-j}{h_Y((i,j))}\right| < \\
&\sqrt{\frac{k+2\max\{1,|z|\}\sqrt{k}-1}{2\pi k(2\max\{1,|z|\}\sqrt{k}-1)}}\left(1+\frac{2\sqrt{k}\max\{1,|z|\}-1}{k}\right)^{k}\times \\
&\left(1+\frac{k}{2\sqrt{k}\max\{1,|z|\}-1}\right)^{2\max\{1,|z|\}\sqrt{k}-1}\times \\
&\exp\left(\frac{1}{12(k+2\sqrt{k}\max\{1,|z|\}-1)}-\frac{1}{12k+1}-\frac{1}{12(2\sqrt{k}\max\{1,|z|\}-1)+1}\right).
\end{aligned} \tag{4.2.40}$$

We will denote $f(z,k)$ the function on the right hand side of (4.2.40).

The key result on the second kind of product is the following

Lemma 4.2.3. *For every pair of diagrams* (Y_1, Y_2) *with* $|Y_1|+|Y_2| = k \geq 1$ *and for every pair of fixed complex numbers* z_1, z_2*, the following inequality holds*

$$\prod_{(i,j)\in Y_1} \left| \frac{z_1+i-j}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} \right| \prod_{(i,j)\in Y_2} \left| \frac{z_2+i-j}{\alpha_1-\alpha_2+h_{Y_2}((i,j))-(Y_2')_i+(Y_1')_i} \right|$$
$$\leq \left(\frac{16}{\min\{1, |\alpha_1-\alpha_2|\}} \left(1+\frac{|\alpha_1-\alpha_2|}{C_{12}(\vec{\alpha})} \right) \right)^k f(z_1,k) f(z_2,k), \tag{4.2.41}$$

where

$$C_{ij}(\vec{\alpha}) = \min_{n\in\mathbb{Z}} |\alpha_i-\alpha_j-n| > 0. \tag{4.2.42}$$

We will use the notation

$$g_{ij}(\vec{\alpha}) = \frac{16}{\min\{1, |\alpha_i-\alpha_j|\}} \left(1+\frac{|\alpha_i-\alpha_j|}{C_{ij}(\vec{\alpha})} \right). \tag{4.2.43}$$

4.2.1 Proofs of Lemma 4.2.2 and Lemma 4.2.3

We start by proving Lemma 4.2.2. The following results will be important in the proof. First, the following formula, that appears in equation 7.207 of exercise 7.50 of [205], is crucial:

Proposition 4.2.4. *For a Young diagram* Y *with* k *boxes, if* $c(\sigma)$ *denotes the number of cycles in the permutation* $\sigma \in S_k$*, and* $\chi^Y(\sigma)$ *is the character of the irreducible representation of* S_k *associated to the partition* Y *of* k *and evaluated in the element* $\sigma \in S_k$*, then*

$$\prod_{(i,j)\in Y} \frac{z+i-j}{h_Y((i,j))} = \frac{1}{k!} \sum_{\sigma\in S_k} \chi^Y(\sigma) z^{c(\sigma)}. \tag{4.2.44}$$

This holds as a polynomial identity for all $z \in \mathbb{C}$.

Moreover, we will use also the following result, that is Lemma 5 of [206]:

Proposition 4.2.5. *Let* Y *be a partition of* $k \geq 1$*. Let* $\mathrm{sq}(Y)$ *be the side length of the largest square contained in* Y*; that is, the largest* j *such that* $Y_j \geq j$*. Let* $\sigma \in S_k$ *be a permutation with* $c(\sigma)$ *cycles. Then*

$$|\chi^Y(\sigma)| \leq (2\,\mathrm{sq}(Y))^{c(\sigma)}. \tag{4.2.45}$$

Finally, from [207, 208] and references therein, the following holds

Proposition 4.2.6. *For every natural number* m*, the expectation value of* $m^{c(\sigma)}$*, over all the permutations of* S_k *weighted uniformly, is equal to*

$$E(m^c) = \binom{k+m-1}{k}. \tag{4.2.46}$$

This result can be extended to noninteger m by considering the right hand side of the equation as the generalized binomial coefficient.

Proof of Lemma 4.2.2. From the identity (4.2.44), it follows that

$$\prod_{(i,j)\in Y}\left|\frac{z+i-j}{h_Y((i,j))}\right| = \left|\frac{1}{k!}\sum_{\sigma\in S_k}\chi^Y(\sigma)z^{c(\sigma)}\right| \leq \frac{1}{k!}\sum_{\sigma\in S_k}|\chi^Y(\sigma)|\cdot|z|^{c(\sigma)}. \tag{4.2.47}$$

Moreover, using (4.2.45) and the fact that

$$\mathrm{sq}(Y) \leq \sqrt{k}, \tag{4.2.48}$$

we can conclude

$$|\chi^Y(\sigma)| \leq (2\sqrt{k})^{c(\sigma)}, \tag{4.2.49}$$

so that

$$\left|\prod_{(i,j)\in Y}\frac{z+i-j}{h_Y((i,j))}\right| \leq \frac{1}{k!}\sum_{\sigma\in S_k}[2\sqrt{k}|z|]^{c(\sigma)} \leq \frac{1}{k!}\sum_{\sigma\in S_k}[2\sqrt{k}\max\{1,|z|\}]^{c(\sigma)}. \tag{4.2.50}$$

Now, the expression

$$\frac{1}{k!}\sum_{\sigma\in S_k}[2\sqrt{k}\max\{1,|z|\}]^{c(\sigma)} \tag{4.2.51}$$

is the expectation value of $[2\sqrt{k}\max\{1,|z|\}]^{c(\sigma)}$ with the uniform measure, where all permutations have the same probability, given by $1/k!$. From Proposition 4.2.6, we can use the generalized binomial coefficient in order to obtain

$$\left|\prod_{(i,j)\in Y}\frac{z+i-j}{h_Y((i,j))}\right| \leq \binom{k+2\sqrt{k}\max\{1,|z|\}-1}{k}. \tag{4.2.52}$$

We can write

$$\binom{k+2\sqrt{k}\max\{1,|z|\}-1}{k} = \frac{\Gamma(k+2\sqrt{k}\max\{1,|z|\})}{\Gamma(k+1)\Gamma(2\sqrt{k}\max\{1,|z|\})}. \tag{4.2.53}$$

Using Stirling approximation in the form

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < \Gamma(n+1) < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}, \tag{4.2.54}$$

we have

$$\binom{k+2\sqrt{k}\max\{1,|z|\}-1}{k} <$$

$$\frac{\sqrt{2\pi(k+2\max\{1,|z|\}\sqrt{k}-1)}}{\sqrt{2\pi k}\sqrt{2\pi(2\max\{1,|z|\}\sqrt{k}-1)}} \frac{(k+2\max\{1,|z|\}\sqrt{k}-1)^{k+2\max\{1,|z|\}\sqrt{k}-1}e^k e^{2\max\{1,|z|\}\sqrt{k}-1}}{k^k(2\max\{1,|z|\}\sqrt{k}-1)^{2\max\{1,|z|\}\sqrt{k}-1}e^{k+2\max\{1,|z|\}\sqrt{k}-1}}$$

$$\times \exp\left(\frac{1}{12(k+2\sqrt{k}\max\{1,|z|\}-1)} - \frac{1}{12k+1} - \frac{1}{12(2\sqrt{k}\max\{1,|z|\}-1)+1}\right) =$$

$$\sqrt{\frac{k+2\max\{1,|z|\}\sqrt{k}-1}{2\pi k(2\max\{1,|z|\}\sqrt{k}-1)}}\left(1+\frac{2\sqrt{k}\max\{1,|z|\}-1}{k}\right)^k$$

$$\times\left(1+\frac{k}{2\sqrt{k}\max\{1,|z|\}-1}\right)^{2\max\{1,|z|\}\sqrt{k}-1}$$

$$\times \exp\left(\frac{1}{12(k+2\sqrt{k}\max\{1,|z|\}-1)} - \frac{1}{12k+1} - \frac{1}{12(2\sqrt{k}\max\{1,|z|\}-1)+1}\right). \tag{4.2.55}$$

□

Remark 4.2.1. *Let us remark that the binomial coefficient* (4.2.53) *is increasing in* k*. Indeed, considering the ratio of the binomial coefficient with* $k = r+1$ *and* $k = r$*, we have that*

$$\frac{\binom{r+1+2\sqrt{r+1}\max\{1,|z_l|\}-1}{r+1}}{\binom{r+2\sqrt{r}\max\{1,|z_l|\}-1}{r}} \geq \frac{\binom{r+2\sqrt{r}\max\{1,|z_l|\}}{r+1}}{\binom{r+2\sqrt{r}\max\{1,|z_l|\}-1}{r}} = \frac{r+2\sqrt{r}\max\{1,|z_l|\}}{r+1} \geq 1. \tag{4.2.56}$$

Therefore, when we consider a N*-tuple of Young diagrams* $Y_1,\dots,Y_N$ *with* $|Y_i| = k_i$ *and* $\sum_{i=1}^N k_i = k \geq 1$*, we can bound the quantity* $\prod_{(m,n)\in Y_i}\frac{z+m-n}{h_{Y_i}((m,n))}$ *with* $f(z,k)$*. This bound holds also if the diagram* Y_i *is empty, since* $f(z,k) > 1$ *if* $k \geq 1$*.*

In order to prove Lemma 4.2.3, we need some further preliminary results which we now discuss. Since we have found a sharp bound for the products of the form

$$\prod_{(i,j)\in Y_1}\frac{z+i-j}{h_{Y_1}((i,j))}, \tag{4.2.57}$$

we can write the second type of product (3.4.105) as (we fix $j=2$ in (3.4.105) for simplicity)

$$\begin{aligned}&\prod_{(i,j)\in Y_1}\frac{z+i-j}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\ &\prod_{(i,j)\in Y_1}\frac{z+i-j}{h_{Y_1}((i,j))}\frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i},\end{aligned} \tag{4.2.58}$$

and so we can reduce to estimate the products of the form

$$\prod_{(i,j)\in Y_1} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}. \tag{4.2.59}$$

Let us fix a pair of Young diagrams Y_1, Y_2 with $|Y_1|+|Y_2|=k\geq 1$. Let us consider first the product over the boxes of Y_1. We suppose Y_1 to be nonempty, otherwise the product would clearly be bounded with 1, and the final estimate would also include that case. Let us divide the set of boxes of Y_1 in two subsets: we call $B_1(Y_1)$ the set of boxes of Y_1 for which $h_{Y_1}((i,j))=(Y_1')_i-(Y_2')_i$, and $B_2(Y_1)$ the set of boxes of Y_1 for which $h_{Y_1}((i,j))\neq(Y_1')_i-(Y_2')_i$. We have then

$$\begin{aligned}
&\prod_{(i,j)\in Y_1} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\
&\prod_{(i,j)\in B_1(Y_1)} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\times \\
&\prod_{(i,j)\in B_2(Y_1)} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\
&\prod_{(i,j)\in B_1(Y_1)} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2} \prod_{(i,j)\in B_2(Y_1)} \frac{h_{Y_1}((i,j))}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\times \\
&\prod_{(i,j)\in B_2(Y_1)} \frac{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\
&\prod_{(i,j)\in B_1(Y_1)} \frac{(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2} \prod_{(i,j)\in B_2(Y_1)} \frac{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\times \\
&\prod_{(i,j)\in B_2(Y_1)} \frac{h_{Y_1}((i,j))}{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}.
\end{aligned} \tag{4.2.60}$$

We consider the three products in the last line one by one.

Lemma 4.2.7. *The first product in the last line of* (4.2.60) *can be bounded as follows*

$$\prod_{(i,j)\in B_1(Y_1)} \left|\frac{(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2}\right| \leq \frac{2^k}{\min\{1,|\alpha_1-\alpha_2|\}^k}. \tag{4.2.61}$$

Proof. See Sec. 4.3. □

Remark 4.2.2. *Since for every fundamental hypermultiplet there is an identical product over the boxes in $B_1(Y_2)$, we notice that, for a given index i of the box, only one of the two equalities $h_{Y_1}((i,j))=(Y_1')_i-(Y_2')_i$ and $h_{Y_2}((i,j))=(Y_2')_i-(Y_1')_i$ can be satisfied,* *since the left hand sides are always positive, but the right hand sides are one the opposite of the other. Therefore, for a fixed index i, only one of the factors*

$$\frac{(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2} \quad \text{and} \quad \frac{(Y_2')_i-(Y_1')_i}{\alpha_1-\alpha_2}$$

appears in the product over the boxes in $B_1(Y_1)$ and over the boxes in $B_1(Y_2)$, so the previous estimate actually bounds the product of the two products of the first kind (the one for Y_1 and the one for Y_2).

We now pass to the second product in (4.2.60).

Lemma 4.2.8. *The second product in the last line of* (4.2.60) *can be bounded as follows*

$$\prod_{(i,j)\in B_2(Y_1)}\left|\frac{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\right| \le \left(1+\frac{|\alpha_1-\alpha_2|}{C_{12}(\vec{\alpha})}\right)^{|Y_1|}, \tag{4.2.62}$$

where

$$C_{12}(\vec{\alpha}) = \min_{n\in\mathbb{Z}}|\alpha_1-\alpha_2-n|.$$

Proof. We have

$$\begin{aligned}
&\prod_{(i,j)\in B_2(Y_1)}\frac{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\
&\prod_{(i,j)\in B_2(Y_1)}\frac{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i-(\alpha_1-\alpha_2)}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i} = \\
&\prod_{(i,j)\in B_2(Y_1)}\left(1-\frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\right),
\end{aligned} \tag{4.2.63}$$

and so

$$\prod_{(i,j)\in B_2(Y_1)}\left|\frac{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}{\alpha_1-\alpha_2-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\right| \le \left(1+\frac{|\alpha_1-\alpha_2|}{C_{12}(\vec{\alpha})}\right)^{|Y_1|}, \tag{4.2.64}$$

where

$$C_{12}(\vec{\alpha}) = \min_{n\in\mathbb{Z}}|\alpha_1-\alpha_2-n|.$$

□

We finally bound the third product.

Lemma 4.2.9. *The third product in the last line of* (4.2.60) *can be bounded as follows*

$$\prod_{(i,j)\in B_2(Y_1)}\left|\frac{h_{Y_1}((i,j))}{-h_{Y_1}((i,j))+(Y_1')_i-(Y_2')_i}\right| \le 8^{|Y_1|}. \tag{4.2.65}$$

Proof. See Sec. 4.4. □

Remark 4.2.3. *Referring to the proof in Sec. 4.4 and considering the analogous product over the boxes in Y_2, we would have to bound the product*

$$\prod_{(i,j)\in B_2(Y_2)\cap ith\ row\ of\ Y_2} \left| \frac{h_{Y_2}((i,j))}{h_{Y_2}((i,j)) - [(Y_2')_i - (Y_1')_i]} \right|. \tag{4.2.66}$$

But then, for a fixed i, we have either $(Y_1')_i - (Y_2')_i = 0$, $(Y_1')_i - (Y_2')_i > 0$ or $(Y_2')_i - (Y_1')_i > 0$. If we are in the first case, both products over the boxes in the ith row of Y_1 and over the ith row of Y_2 are bounded by 1. If we are in the second case, the product over the boxes in the ith row of Y_2 is bounded by 1, and, if we are in the third case, the product over the boxes in the ith row of Y_1 is bounded by 1. Therefore, for every i, only one product has to be considered to give an upper bound. Hence, the previous bound, with $|Y_1|$ replaced by k, is a bound for the product of the two products of the third kind (the one for Y_1 and the one for Y_2).

We are now finally ready to prove Lemma 4.2.3.

Proof of Lemma 4.2.3 : Putting together (4.2.61), (4.2.62) and (4.2.65), and using the remarks after the previous lemmas, we conclude that

$$\prod_{(i,j)\in Y_1} \left| \frac{h_{Y_1}((i,j))}{\alpha_1 - \alpha_2 - h_{Y_1}((i,j)) + (Y_1')_i - (Y_2')_i} \right| \prod_{(i,j)\in Y_2} \left| \frac{h_{Y_2}((i,j))}{\alpha_1 - \alpha_2 + h_{Y_2}((i,j)) - (Y_2')_i + (Y_1')_i} \right|$$
$$\leq \left(\frac{16}{\min\{1, |\alpha_1 - \alpha_2|\}} \left(1 + \frac{|\alpha_1 - \alpha_2|}{C_{12}(\vec{\alpha})} \right) \right)^k . \tag{4.2.67}$$

To conclude the proof of lemma 4.2.3, it only remains to include the bounds of the products of the form analyzed in lemma 4.2.2, both for Y_1 and Y_2. Since the inequality

$$\prod_{(i,j)\in Y_l} \frac{z_l + i - j}{h_{Y_l}((i,j))} \leq f(z_l, k) \tag{4.2.68}$$

holds for both $l = 1, 2$, we can write the following estimate

$$\prod_{(i,j)\in Y_1} \left| \frac{z_1 + i - j}{\alpha_1 - \alpha_2 - h_{Y_1}((i,j)) + (Y_1')_i - (Y_2')_i} \right| \prod_{(i,j)\in Y_2} \left| \frac{z_2 + i - j}{\alpha_1 - \alpha_2 + h_{Y_2}((i,j)) - (Y_2')_i + (Y_1')_i} \right|$$
$$\leq \left(\frac{16}{\min\{1, |\alpha_1 - \alpha_2|\}} \left(1 + \frac{|\alpha_1 - \alpha_2|}{C_{12}(\vec{\alpha})} \right) \right)^k f(z_1, k) f(z_2, k). \tag{4.2.69}$$

□

4.2.2 Proof of Theorem 4.2.1

In the $N_f = 2N$ case, using Lemma 4.2.2 and Lemma 4.2.3, we can arrange the products in the numerator and denominator of the coefficients of the instanton partition function (4.2.36) to conclude that

$$|Z_{\text{inst}}^{U(N)\ N_f=2N}| \leq \sum_{k\geq 1} |t|^k p_N(k) \left[\prod_{i=1}^{N}\prod_{r=1}^{N_f=2N} f(\alpha_i+\mu_r,k)\right] \prod_{\{(i,j)\in\{1,\dots,N\}^2\ \mid\ i\neq j\}} g_{ij}(\vec{\alpha})^k, \tag{4.2.70}$$

where $p_N(k)$ denotes the number of N-coloured partitions of the integer k. If $p(k)$ denotes the number of partitions of k, we can bound $p_N(k)$ with $p(k)^{N+1}$, since the former can be seen as the number of partitions of N integers whose sum equals k, and so any of these N numbers has to be smaller than k. Moreover, we can use the following estimate, known as Ramanujan-Hardy formula [209]:

Proposition 4.2.10. *If $p(k)$ is the number of partitions of the natural number k, the following holds:*

$$p(k) \sim \frac{1}{4\sqrt{3}k}\exp\left(\pi\sqrt{\frac{2k}{3}}\right), \quad \textit{for } k\to\infty. \tag{4.2.71}$$

Therefore, applying the root test to (4.2.70), and using that

$$\begin{aligned} &\lim_{k\to\infty} p(k)^{1/k} = \lim_{k\to\infty}\left(\frac{1}{4\sqrt{3}k}\exp\left(\pi\sqrt{\frac{2k}{3}}\right)\right)^{1/k} = 1 \\ &\lim_{k\to\infty}\left(f(\alpha_i+\mu_r,k)\right)^{1/k} = 1 \quad \forall i=1,\dots,N \ \ \forall r=1,\dots,N_f, \end{aligned} \tag{4.2.72}$$

we conclude that the radius of convergence of the right-hand side is given by

$$\prod_{\{(i,j)\in\{1,\dots,N\}^2\ \mid\ i\neq j\}} [g_{ij}(\vec{\alpha})]^{-1} = \prod_{\{(i,j)\in\{1,\dots,N\}^2\ \mid\ i\neq j\}} \left[\frac{16}{\min\{1,|\alpha_i-\alpha_j|\}}\left(1+\frac{|\alpha_i-\alpha_j|}{C_{ij}(\vec{\alpha})}\right)\right]^{-1} \tag{4.2.73}$$

Hence, we can conclude the first part of the theorem, that is the fact that the instanton partition function of the $U(N)$ gauge theory with $N_f = 2N$ fundamental multiplets with the Omega background $\epsilon_1+\epsilon_2 = 0$ is absolutely convergent at least for

$$|t| < \prod_{\{(i,j)\in\{1,\dots,N\}^2\ \mid\ i\neq j\}} \left[\frac{16}{\min\{1,|\alpha_i-\alpha_j|\}}\left(1+\frac{|\alpha_i-\alpha_j|}{C_{ij}(\vec{\alpha})}\right)\right]^{-1}. \tag{4.2.74}$$

The case $N_f < 2N$ can now be easily proved by noticing that the decoupling limit of fundamental hypermultiplets is achieved with the double scaling limit in which $t\to 0$ and one of the masses, say m_1, goes to infinity $m_1\to\infty$, in such a way that $\tilde{\Lambda} = t\,m_1$

remains finite. Indeed, from the expression (A.1.10), one can see that in this limit the function becomes

$$
\begin{aligned}
Z_{\text{inst}}^{U(N),\,N_f} = \sum_{k\geq 0}(t\,m_1)^k \sum_{|\vec{Y}|=k}\prod_{i,j=1}^{N}\prod_{(m,n)\in Y_i} &\frac{1}{a_i - a_j - \epsilon_1 L_{Y_j}((m,n)) + \epsilon_2\left(A_{Y_i}((m,n))+1\right)} \\
\prod_{(m,n)\in Y_j} &\frac{1}{a_i - a_j + \epsilon_1\left(L_{Y_i}((m,n))+1\right) - \epsilon_2 A_{Y_j}((m,n))} \\
\prod_{i=1}^{N}\prod_{(m,n)\in Y_i} &\left[1 + \frac{a_i + \epsilon_1(m-1) + \epsilon_2(n-1)}{m_1}\right] \\
\prod_{i=1}^{N}\prod_{(m,n)\in Y_i}\prod_{r=2}^{N_f} &\left[a_i + \epsilon_1(m-1) + \epsilon_2(n-1) + m_r\right] \to \\
\sum_{k\geq 0}\tilde{\Lambda}^k \sum_{|\vec{Y}|=k}\prod_{i,j=1}^{N}\prod_{(m,n)\in Y_i} &\frac{1}{a_i - a_j - \epsilon_1 L_{Y_j}((m,n)) + \epsilon_2\left(A_{Y_i}((m,n))+1\right)} \\
\prod_{(m,n)\in Y_j} &\frac{1}{a_i - a_j + \epsilon_1\left(L_{Y_i}((m,n))+1\right) - \epsilon_2 A_{Y_j}((m,n))} \\
\prod_{i=1}^{N}\prod_{(m,n)\in Y_i}\prod_{r=2}^{N_f} &\left[a_i + \epsilon_1(m-1) + \epsilon_2(n-1) + m_r\right],
\end{aligned}
\tag{4.2.75}
$$

which is the instanton partition function with one fundamental hypermultiplet less. The radius of convergence of this latter series in $\tilde{\Lambda}$ can be obtained by multiplying (4.2.74) by m_1 and letting $m_1 \to \infty$, which means that the series is absolutely convergent for any $\tilde{\Lambda}$. The proof for lower N_f is obtained by repeated application of the above argument. □

4.3 Proof of Lemma 4.2.7

We know that the boxes in $B_1(Y_1)$ satisfy $h_{Y_1}((i,j)) = (Y_1')_i - (Y_2')_i$. This can happen at most for one box in each row of Y_1, since the left hand side strictly decreases moving on the right on a fixed row of the diagram, while the right hand side remains constant. Therefore, we can bound the product as follows:

$$
\begin{aligned}
\prod_{(i,j)\in B_1(Y_1)}\left|\frac{(Y_1')_i - (Y_2')_i}{\alpha_1 - \alpha_2}\right| = \frac{\prod_{(i,j)\in B_1(Y_1)}|(Y_1')_i - (Y_2')_i|}{|\alpha_1 - \alpha_2|^{|B_1(Y_1)|}} \leq \\
\frac{\max\{1, |(Y_1')_1 - (Y_2')_1|\}\cdots\max\{1, |(Y_1')_{(Y_1)_1} - (Y_2')_{(Y_1)_1}|\}}{|\alpha_1 - \alpha_2|^{|B_1(Y_1)|}},
\end{aligned}
\tag{4.3.76}
$$

where we bounded the product in the numerator with the product of all the differences between rows' lenghts ($(Y_1)_1$ is the height of the first column of Y_1, that is the number of rows of Y_1), and we modified the factors taking the maximum with 1, because it could happen that, in a fixed row i of Y_1, there is not a box which is in $B_1(Y_1)$ and $(Y_1')_i = (Y_2')_i$ holds, and we want to avoid that the right hand side vanishes for this reason.

From the last term in (4.3.76), we can bound the numerator using the geometric-arithmetic mean inequality:

$$\begin{aligned}
&\max\{1, |(Y_1')_1 - (Y_2')_1|\} \cdots \max\{1, |(Y_1')_{(Y_1)_1} - (Y_2')_{(Y_1)_1}|\} \leq \\
&\left(\frac{\max\{1, |(Y_1')_1 - (Y_2')_1|\} + \cdots + \max\{1, |(Y_1')_{(Y_1)_1} - (Y_2')_{(Y_1)_1}|\}}{(Y_1)_1}\right)^{(Y_1)_1} \\
&\leq \left(\frac{k}{(Y_1)_1}\right)^{(Y_1)_1} \leq \binom{k}{(Y_1)_1} \leq 2^k,
\end{aligned} \tag{4.3.77}$$

where we used that for the binomial coefficient, for every $1 \leq k \leq n$, the following bounds always hold

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(\frac{n \cdot e}{k}\right)^k.$$

For the denominator, we have to distinguish the cases in which $|\alpha_1 - \alpha_2| \geq 1$ and $|\alpha_1 - \alpha_2| < 1$. In the first case, we simply bound the fraction with the bound of the numerator; in the second case, we have that, since $(Y_1)_1 \leq k$, $|\alpha_1 - \alpha_2|^{|B_1(Y_1)|} \geq |\alpha_1 - \alpha_2|^k$. Therefore,

$$\prod_{(i,j) \in B_1(Y_1)} \left| \frac{(Y_1')_i - (Y_2')_i}{\alpha_1 - \alpha_2} \right| \leq \frac{2^k}{\min\{1, |\alpha_1 - \alpha_2|\}^k}. \tag{4.3.78}$$

4.4 Proof of Lemma 4.2.9

Let us first find a bound on the product over the boxes in $B_2(Y_1)$ in one fixed row of Y_1. After that, we will multiply the bounds on all the rows of Y_1. We can write

$$\begin{aligned}
&\prod_{(i,j) \in B_2(Y_1) \cap i\text{th row of } Y_1} \left| \frac{h_{Y_1}((i,j))}{-h_{Y_1}((i,j)) + (Y_1')_i - (Y_2')_i} \right| = \\
&\prod_{(i,j) \in B_2(Y_1) \cap i\text{th row of } Y_1} \left| \frac{h_{Y_1}((i,j))}{h_{Y_1}((i,j)) - [(Y_1')_i - (Y_2')_i]} \right|.
\end{aligned} \tag{4.4.79}$$

Note that the denominator is different from 0 for all the factors, since we are only multiplying over the boxes in $B_2(Y_1)$.

We suppose $(Y_1')_i > (Y_2')_i$ for every i, since otherwise the previous product would be clearly bounded by 1 in the ith row.

Then, for a given row i, the product over the boxes in the ith row of Y_1 can be splitted in two parts: the product over the boxes for which $h_{Y_1}((i,j)) - [(Y_1')_i - (Y_2')_i]$ is positive, and the product over the boxes for which the same quantity is negative. Note that, since we are assuming $(Y_1')_i - (Y_2')_i > 0$, the latter product is present if and only if in the ith row there is a box, let us denote it with (i, j^*), such that $h_{Y_1}((i, j^*)) = [(Y_1')_i - (Y_2')_i]$, since the quantity $h_{Y_1}((i,j)) - [(Y_1')_i - (Y_2')_i]$ is strictly decreasing moving to the right on a fixed row.

Therefore, we first consider the product over the boxes for which that quantity is positive (that correspond to the boxes at the left of (i,j^*) if this box is present in the ith row). We can rewrite the factors of this first part of the product as

$$\frac{h_{Y_1}((i,j))}{h_{Y_1}((i,j)) - [(Y_1')_i - (Y_2')_i]} = \frac{[(Y_1')_i - (Y_2')_i] + (Y_2')_i - j + A_{Y_1}((i,j)) + 1}{(Y_2')_i - j + A_{Y_1}((i,j)) + 1}, \tag{4.4.80}$$

which is of the form

$$\prod_{j=1}^{n} \frac{a+b_j}{b_j},$$

with $b_j \in \mathbb{N}$ and $b_{j+1} < b_j$ for all j, and $a > 0$ constant (since moving to the right $A_{Y_1}((i,j))$ decreases). But then a product of this form is bounded by

$$\frac{a+1}{1} \cdot \frac{a+2}{2} \cdots \frac{a+n}{n}. \tag{4.4.81}$$

Indeed, if $a > 0$ and $b > c > 0$, it is always true that

$$\frac{a+b}{b} \le \frac{a+c}{c},$$

since, under those hypothesis,

$$\frac{a+b}{b} \le \frac{a+c}{c} \iff (a+b)c \le (a+c)b \iff ac \le ab \iff c \le b.$$

But then, $b_n \ge 1$ (since it is an integer number and for hypothesis it is positive), and, since $b_{j-1} > b_j$ for all $j = 2, \dots, n$, we have that $b_j \ge n - j + 1$ for all $j = 1, \dots n-1$; so the previous bound holds.

In our case, n is at most $j^* - 1$, so we can bound this first part of the product with

$$\prod_{r=1}^{j^*-1} \frac{[(Y_1')_i - (Y_2')_i] + r}{r} = \binom{(Y_1')_i - (Y_2')_i + j^* - 1}{j^* - 1}. \tag{4.4.82}$$

We can bound the second part of the product (if there are boxes on the right of (i,j^*)) as follows. First, from $h_{Y_1}((i,j^*)) = (Y_1')_i - (Y_2')_i$, it follows that

$$A_{Y_1}((i,j^*)) + (Y_2')_i + 1 = j^*.$$

Then, we rewrite

$$\begin{aligned} h_{Y_1}((i,j^*+r)) &= (Y_1')_i - j^* - r + A_{Y_1}((i,j^*+r)) + 1 = \\ &= (Y_1')_i - (Y_2')_i + (Y_2')_i - j^* - r + A_{Y_1}((i,j^*)) - [A_{Y_1}((i,j^*)) \\ &\quad - A_{Y_1}((i,j^*+r))] + 1 \\ &= (Y_1')_i - (Y_2')_i + (Y_2')_i - [A_{Y_1}((i,j^*)) + (Y_2')_i + 1] - r + A_{Y_1}((i,j^*)) + \\ &\quad - [A_{Y_1}((i,j^*)) - A_{Y_1}((i,j^*+r))] + 1 = \\ &= (Y_1')_i - (Y_2')_i - r - [A_{Y_1}((i,j^*)) - A_{Y_1}((i,j^*+r))], \end{aligned} \tag{4.4.83}$$

for every $0 < r \leq (Y_1')_i - j^*$. Moreover,

$$[(Y_1')_i - (Y_2')_i] - h_{Y_1}((i, j^* + r)) = r + [A_{Y_1}((i, j^*)) - A_{Y_1}((i, j^* + r))]. \tag{4.4.84}$$

Since the quantity $[A_{Y_1}((i, j^*)) - A_{Y_1}((i, j^* + r))]$ is positive, we have that the product over the boxes on the right of (i, j^*) is bounded by

$$\prod_{j=j^*+r} \frac{(Y_1')_i - (Y_2')_i - r}{r} = \prod_{r=1}^{(Y_1')_i - j^*} \frac{(Y_1')_i - (Y_2')_i - r}{r} = \binom{(Y_1')_i - (Y_2')_i - 1}{(Y_1')_i - j^*}. \tag{4.4.85}$$

Putting together (4.4.82) and (4.4.85), the product over the boxes of the ith row of Y_1 is bounded by

$$\binom{(Y_1')_i - (Y_2')_i + j^* - 1}{j^* - 1} \binom{(Y_1')_i - (Y_2')_i - 1}{(Y_1')_i - j^*}. \tag{4.4.86}$$

Since $j^* \leq (Y_1')_i$ and $j^* > (Y_2')_i$, we have that

$$\begin{aligned} &\binom{(Y_1')_i - (Y_2')_i + j^* - 1}{j^* - 1} \leq \binom{2(Y_1')_i - (Y_2')_i - 1}{j^* - 1} \leq 2^{2(Y_1')_i - (Y_2')_i}, \\ &\binom{(Y_1')_i - (Y_2')_i - 1}{(Y_1')_i - j^*} \leq 2^{(Y_1')_i - (Y_2')_i}. \end{aligned} \tag{4.4.87}$$

We conclude that

$$\binom{(Y_1')_i - (Y_2')_i + j^* - 1}{j^* - 1} \binom{(Y_1')_i - (Y_2')_i - 1}{(Y_1')_i - j^*} \leq 2^{3(Y_1')_i}. \tag{4.4.88}$$

Considering the product of this bound for all the rows of Y_1, we can conclude

$$\prod_{(i,j) \in B_2(Y_1)} \left| \frac{h_{Y_1}((i,j))}{-h_{Y_1}((i,j)) + (Y_1')_i - (Y_2')_i} \right| \leq 8^{|Y_1|}. \tag{4.4.89}$$

4.5 On the convergence of Painlevé τ-functions

The Kyiv formula conjectured in [210] states that Painlevé τ-functions can be expressed as discrete Fourier transforms of suitable full Nekrasov partition functions. This is the core issue of Painlevé/gauge theory correspondence [211]. Concretely, according to the Kyiv formula, the PVI τ-function is related to the Nekrasov function as follows

$$\tau^{\mathrm{VI}}(t; \alpha, s) = t^{-\theta_0^2 - \theta_t^2} (1-t)^{\theta_1 \theta_t} \sum_{n \in \mathbb{Z}} s^n t^{(\alpha+n)^2} Z_{\mathrm{1loop}}^{U(2)\ N_f=4}(\alpha + n) Z_{\mathrm{inst}}^{U(2)\ N_f=4}(t, \alpha + n), \tag{4.5.90}$$

where

$$Z_{\mathrm{1loop}}^{U(2)\ N_f=4}(\alpha) = \frac{\prod_{\sigma, \sigma' = \pm} G(1 + \theta_t + \sigma\theta_0 + \sigma'(\alpha + n)) G(1 + \theta_1 + \sigma\theta_\infty + \sigma'(\alpha + n))}{G(1 + 2(\alpha + n)) G(1 - 2(\alpha + n))} \tag{4.5.91}$$

is the one loop contribution to the full partition function written in terms of Barnes G functions, and the re-scaled masses (4.2.35) are related to the θ-parameters by

$$\mu_1 = \theta_1 - \theta_\infty, \quad \mu_2 = \theta_0 - \theta_t, \quad \mu_3 = \theta_0 + \theta_t, \quad \mu_4 = \theta_1 + \theta_\infty.$$

The τ-function (4.5.90) is the one associated to the isomonodromic deformation problem for the Riemann sphere with four regular singularities, with θs parameterizing the associated monodromies.

In order to study the convergence properties of the series (4.5.90), we can make use of the results obtained in the previous section together with the asymptotic behavior of the one-loop coefficients. The latter can be determined from the reflection formula:

$$G(1-z) = \frac{G(1+z)}{(2\pi)^z} \exp\left(\int_0^z \pi z' \cot(\pi z') \mathrm{d}z'\right) \tag{4.5.92}$$

and the asymptotic formula for $z \to \infty$ [212]

$$\begin{aligned} &\log(G(1+a+z)) = \\ &\frac{z+a}{2}\log(2\pi) + \zeta'(-1) - \frac{3z^2}{4} - az + \left(\frac{z^2}{2} - \frac{1}{12} + \frac{a^2}{2} + az\right)\log(z) + \mathcal{O}\left(\frac{1}{z}\right), \end{aligned} \tag{4.5.93}$$

which holds for all $a \in \mathbb{C}$ and where $\zeta'(-1)$ is a known ζ-constant. From this, we have that, for $a \in \mathbb{C}$ and $\mathbb{Z} \ni n \to \infty$,

$$\log(G(1+a+n)) = \frac{n^2}{2}\log(n) - \frac{3n^2}{4} + \mathcal{O}(n\log(n)). \tag{4.5.94}$$

To evaluate the $n \to \infty$ limit of the other set of Barnes functions, we note that the integral in the reflection formula is given by

$$\int_0^z \pi z' \cot(\pi z') \mathrm{d}z' = \frac{\pi z \log(1-\exp(2\pi i z)) - \frac{i}{2}\left(\pi^2 z^2 + \mathrm{Li}_2(\exp(2\pi i z))\right)}{\pi}. \tag{4.5.95}$$

Since the asymptotic of the above integral is given by $-\frac{i}{2}\pi n^2 + \mathcal{O}(n)$, we have that, for every $b \in \mathbb{C}$ and for $\mathbb{Z} \ni n \to \infty$,

$$\log(G(1-b-n)) = \frac{n^2}{2}\log(n) - \frac{3n^2}{4} - \frac{i}{2}\pi n^2 + \mathcal{O}(n\log(n)). \tag{4.5.96}$$

Therefore, neglecting terms of order $n\log(n)$, which are subleading, the one-loop coefficient in the limit $\mathbb{Z} \ni n \to \infty$ reads

$$\begin{aligned} &\frac{\prod_{\sigma,\sigma'=\pm} G(1+\theta_t+\sigma\theta_0+\sigma'(\alpha+n))G(1+\theta_1+\sigma\theta_\infty+\sigma'(\alpha+n))}{G(1+2(\alpha+n))G(1-2(\alpha+n))} \to \\ &\to \frac{\left(n^{\frac{n^2}{2}}\exp(-\frac{3n^2}{4})\right)^8 \left(\exp(-\frac{i\pi n^2}{2})\right)^4}{\left((2n)^{\frac{(2n)^2}{2}}\exp(-\frac{3(2n)^2}{4})\right)^2 \left(\exp(-\frac{i\pi(2n)^2}{2})\right)} = \frac{1}{2^{4n^2}}. \end{aligned} \tag{4.5.97}$$

This immediately implies that the convergence radius of the τ^{VI}-function series is driven by the one of the Z_{inst} coefficient, for which we derived the lower bound (4.2.74) in Theorem 4.2.1. Actually, from modularity, one expects the true region of convergence to be $|t| < 1$.

The τ-functions for Painlevé V and III$_i$ $i = 1, 2, 3$ equations are obtained by implementing in the gauge theory the suitable coalescence limits. These correspond to the holomorphic decoupling of fundamental masses. As far as the one-loop coefficient is concerned, the holomorphic decoupling lowers the number of factors in the numerator of (4.5.91), which implies even stronger convergence properties driven by the denominator, as one can see from (4.5.97). We therefore conclude that the corresponding Painlevé τ-functions have an infinite radius of convergence. Actually, this was already shown to hold for the PIII$_3$ equation in [42].

The above, together with Theorem 4.2.1, provide a proof of the following

Theorem 4.5.1. *Let $2\alpha \notin \mathbb{Z}$. The τ-function for PVI equation has at least a finite radius of absolute and uniform convergence, while those of PV and PIII$_i$ $i = 1, 2, 3$ equations have an infinite radius of absolute and uniform convergence.*

Let us also mention that an extension of Kyiv formula for the isomonodromic deformation problem on the torus was introduced in [213, 214]. For the one-punctured torus the corresponding equations are given by Manin's elliptic form of PVI equation with specific values of the monodromy parameters, and the related τ-function is obtained in terms of the partition function of the $U(2)$ $\mathcal{N} = 2^*$ theory

$$\tau^{U(2)\ \mathcal{N}=2^*}(t; \alpha, s) = Z_D / Z_{\text{twist}}, \tag{4.5.98}$$

where

$$Z_{\text{twist}} = t^{\alpha^2} \eta(t)^{-2} \theta_1(\alpha\tau + \rho + Q(\tau)) \theta_1(\alpha\tau + \rho - Q(\tau))$$

is given in terms of the solution of the corresponding Painlevé equation $Q(\tau)$ and

$$Z_D = \sum_{n\in\mathbb{Z}} s^n t^{(\alpha+n)^2} Z_{\text{1loop}}^{U(2)\ \mathcal{N}=2^*}(\alpha + n) Z_{\text{inst}}^{U(2)\ \mathcal{N}=2^*}(t, \alpha + n) \tag{4.5.99}$$

with $s = e^{2\pi i \rho}$ and $t = e^{2\pi i \tau}$. The one-loop coefficient is given by

$$Z_{\text{1loop}}^{U(2)\ \mathcal{N}=2^*} = \frac{G(1 - \mu - 2(\alpha + n)) G(1 - \mu + 2(\alpha + n))}{G(1 + 2(\alpha + n)) G(1 - 2(\alpha + n))}, \tag{4.5.100}$$

where $\mu = m/\epsilon$ is the re-scaled adjoint mass.

Theorem 4.5.2. *Let $2\alpha \notin \mathbb{Z}$. The τ-function* (4.5.98) *has at least a finite radius of absolute and uniform convergence.*

Proof. With the same asymptotic formulas used before, see Sec. H, we have that, as $n \to \infty$,

$$\frac{G(1 - \mu - 2(\alpha + n)) G(1 - \mu + 2(\alpha + n))}{G(1 + 2(\alpha + n)) G(1 - 2(\alpha + n))} \propto (2n)^{\mu^2} \left(\frac{\sin(\pi(\mu + 2\alpha))}{\sin(2\pi\alpha)} \right)^{2n} \tag{4.5.101}$$

up to $1/n$ corrections, where the proportionality constant is independent of n. This does not spoil the convergence radius of the instanton sector and the proof follows from Theorem 4.1.1. □

4.6 Summary

In this chapter, we proved some results on the convergence of Nekrasov functions in four-dimensional $\mathcal{N} = 2$ gauge theories as power series in the complexified gauge coupling. We proved that, under some genericity assumption of the gauge parameters, if the theory is asymptotically free, then the multi-instanton series has an infinite radius of convergence, whereas, if the theory is conformal, the multi-instanton series has at least a finite radius of convergence. In the final part, we applied our results to analyze the convergence properties of Painlevé τ-functions.

The convergence properties of Nekrasov functions or, thanks to the AGT duality [20], of conformal blocks have been studied with several different methods [37, 38, 215, 40, 41, 42]. The novelty of our approach is the study of these properties with the use of the explicit combinatorial formulae parametrized by Young diagrams. The advantage of our results compared to other methods is the independence of the estimates of the radius of convergence on some parameters of the theory, like the masses of fundamental matter. It would be interesting to see if combining our technique with other methods could help improve these estimates of the radii of convergence in the conformal theories.

Chapter 5

Conclusions and Outlook

In the thesis, we studied black hole linear perturbations from several complementary perspectives. We began Chapter 2 by using the approach based on the NS functions in the context of four-dimensional de Sitter Schwarzschild black holes. In this setup, the NS functions allow us to compute the large ℓ expansion of QNMs systematically. We find that, up to non-perturbative effects in ℓ, the QNMs are (negative) imaginary numbers that are even functions of R_h. To include non-perturbative effects, switching to the polylog approach is convenient. Once non-perturbative effects are included, QNMs are no longer even in R_h. Nonetheless, we still find a branch of purely imaginary modes, thereby providing analytical confirmation of the results obtained through numerical studies in [156, 216, 157, 158]. Exploring the interplay between the NS and polylog approaches would be interesting. In particular, the appearance of multiple polylogarithms and multiple zeta values may be related to the behavior of the NS functions close to their singular points, see e.g. [217, 218, 219].

In Sec. 2.2, we used the polylog method to study conformally coupled scalar, electromagnetic, and vector-type gravitational perturbations in Schwarzschild AdS_4 black holes. The NS functions are less effective for these perturbations because the point at spatial infinity is not a singular point of the equation. If we considered massive scalar perturbation instead, the underlying equation would have five regular singular points and spatial infinity would be mapped into one of them. Hence, we switch to the polylog method for Dirichlet and Robin boundary conditions. As an application, in Sec. 2.3, we use this technique to study the low-lying modes of the scalar sector of gravitational perturbations and compute several orders in the $1/R_h$ expansion. Even in the hydrodynamic expansion, this allowed us to go beyond the results presently available in the literature. From the point of view of holography, the polylog method presents finite spin predictions for the dual 3d CFT.

In Sec. 2.4, we extended the polylog method to study linear perturbations around the asymptotically flat Schwarzschild spacetime. In this case, spatial infinity is an irregular singular point, and we needed to introduce new sets of special functions, multiple polyexponential functions and multiple polyexponential integrals. Considering the properties of the local solutions around spatial infinity under the monodromy transformations, we obtained a partial differential equation on the scattering amplitudes, which fixes their exact dependence on $\log t$. The remaining constant of integration was related to the Seiberg-Witten quantum period via the resummation of infinitely many instantons. We found agreement with the previous results obtained via the Seiberg-Witten theory [43].

In Chapter 3, we analyzed the determinants of differential operators describing scalar field perturbations in Kerr-de Sitter black hole in four dimensions, in Schwarzschild-de Sitter black hole in four dimensions, and in Schwarzschild-anti-de Sitter in five dimensions, where the radial problem is encoded in a Heun differential equation. Applying the techniques from supersymmetric gauge theory, and the connection formulae for Heun equations, we focused on the resulting analytic structure described by QNMs contributions. In particular, we could see that the effects due to the presence of the black hole are subleading compared to the ones due to the asymptotic geometry. This is equivalent to saying that the contribution to the near-horizon zone is subleading compared to the far-zone (for a discussion on the distinction of these regions see [185]).

The Heun connection formulae have an explicit dependence on Nekrasov's functions. This is why, in Chapter 4, we proved some results on the convergence of Nekrasov functions in four-dimensional $\mathcal{N} = 2$ gauge theories as power series in the complexified gauge coupling. In short, we proved that, under some genericity assumption of the gauge parameters, if the theory under scrutiny is *asymptotically free*, then the multi-instanton series has an *infinite* radius of convergence, whereas, if the theory is *conformal*, then the multi-instanton series has at least a *finite* radius of convergence. We also applied our results to analyze the convergence properties of some Painlevé τ-functions, using the Kyiv formula conjectured in [210], which states that Painlevé τ-functions can be expressed as discrete Fourier transforms of suitable full $SU(2)$ Nekrasov partition functions.

About the convergence properties of Nekrasov's functions, one obvious extension of our analysis would be to linear and circular quiver gauge theories in general Ω-background, which, on the two-dimensional CFT counterpart, correspond to conformal blocks with several insertions on the sphere and torus, respectively. It would also be interesting to extend the approach and results of our work to the corresponding five-dimensional gauge theories on a circle. A crucial improvement would be to extend the results to different Ω-backgrounds, especially in the presence of fundamental matter. In particular, this could provide some insights about the convergence properties of the NS functions.

There are many further interesting questions that arise from our analysis and many further directions that deserve to be explored.

First, the same techniques we presented can be applied in different and more complicated black hole geometries, for example, in higher dimensions or in the presence of electric/magnetic charges. Also, different types of perturbation fields, and possibly boundary conditions, can be considered. This typically gives rise to ODEs with more regular and/or irregular singularities. In asymptotically anti-de Sitter spacetime, this could allow us to make contact with past and recent developments in the study of holographic CFTs [220, 221, 222, 223, 174, 224, 168, 169, 170, 148, 147, 225, 149, 150, 151, 152, 153, 154, 155]. When considering rotating and/or charged black holes, instead, our techniques could make it possible to detect phase transitions and/or (in)stabilities of the black hole.

One of the most challenging questions would be to go beyond linear perturbation theory. The analyzed methods allow for the computation of the eigenfunctions and the Green functions, which are essential inputs to go beyond the linear theory.

Also, more technical questions can be addressed within our formalism. The eigenfunctions corresponding to the QNMs are usually not normalizable, and, in general, they do not form a complete set. It would be interesting to study these completeness properties and the *pseudo-spectrum* of the associated differential operators. The latter is relevant to studying the spectral stability of black hole QNM frequencies, namely evaluating how they move in the complex plane under small perturbations of the differential operator [226, 227]. For self-adjoint operators in non-dissipative systems, the spectrum is stable under perturbations [228], meaning that small perturbations of the operator (in some fixed norm) lead to small movements of the operator's eigenvalues on the complex plane. In the case of dissipative systems such as black holes, instead, instabilities could arise because of the non-completeness of the set of eigenfunctions [229, 230, 231, 232, 233, 234, 235, 236, 237, 238].

Appendix A

Gauge theory notations and conventions

In this appendix, we fix our conventions on Young diagrams and gauge theory quantities used throughout the work. For the latter, we mostly follow the notations of [239].

Definition A.1.1. A *partition* of a positive integer k is a finite non-increasing sequence of positive integers $Y_1 \geq \cdots \geq Y_r > 0$ such that $\sum_{i=1}^{r} Y_i = k$.

We denote the number of partitions of k as $p(k)$. The Y_is that appear in a given partition are called *parts* of the partition.

Definition A.1.2. We say that a partition is N*-coloured* if each part of the partition can have N possible colours.

We denote the number of N-coloured partitions of k as $p_N(k)$.

We introduce some important functions related to the partitions of integers. Let τ be a complex number with $\text{Im}\tau > 0$, and let $t = e^{2\pi i \tau}$.

Definition A.1.3. The *Dedekind* η *function* is defined as

$$\eta(t) = t^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - t^n).$$

The requests on τ and t are justified by the following:

Proposition A.1.4. *The infinite product*

$$\prod_{n=1}^{\infty} (1 - t^n)$$

converges absolutely if $|t| < 1$.

Definition A.1.5. The *Euler function* is defined as

$$\phi(t) = \prod_{n=1}^{\infty}(1 - t^n).$$

Note that the Euler function coincides with the Dedekind η function up to a factor $t^{\frac{1}{24}}$.

Proposition A.1.6. *For every $N \geq 1$, the generating function for $p_N(k)$ is given by*

$$\sum_{k=0}^{\infty} p_N(k) t^k = \prod_{j=1}^{\infty} \frac{1}{(1 - t^j)^N}. \tag{A.1.1}$$

We always identify a partition of a natural number k with a Young diagram Y with k boxes, arranged in left-justified rows, with the row lengths in non-increasing order, such that the parts $Y_1 \geq Y_2 \geq \cdots \geq Y_r > 0$ of Y (such that $Y_1 + \cdots + Y_r = k$) denote the heights of the columns of the diagram. Moreover, we denote with $Y'_1 \geq Y'_2 \geq \cdots \geq Y'_s > 0$ the lengths of the rows of Y. We denote with $\mathbb{Y}$ the set of all Young diagrams.

If every box s is labeled with a pair of indices (i, j), with $1 \leq i \leq Y_j$ and $1 \leq j \leq Y'_i$, that denotes its position in the diagram, we define the *arm length* and the *leg length* of s as

$$\begin{aligned} A_Y(s) &= Y_j - i, \\ L_Y(s) &= Y'_i - j, \end{aligned} \tag{A.1.2}$$

respectively.

Moreover, we use the following

Definition A.1.7. If Y is a Young diagram and $s = (i, j)$ is one of its boxes, we call *hook* of s the set of boxes with indices (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$.

We denote with $h_Y((i, j))$ or $h_Y(s)$ the number of boxes in the hook of s in Y. It is easy to see that, if $s \in Y$, then

$$h_Y(s) = A_Y(s) + L_Y(s) + 1. \tag{A.1.3}$$

For a box $s = (i, j)$, we define the following quantities, crucial for the definitions of the instanton partition functions:

$$\begin{aligned} E(a, Y_1, Y_2, s) &= a - \epsilon_1 L_{Y_2}(s) + \epsilon_2 (A_{Y_1}(s) + 1) \\ \varphi\left(a, s = (i, j)\right) &= a + \epsilon_1 (i - 1) + \epsilon_2 (j - 1). \end{aligned} \tag{A.1.4}$$

We are now ready to define the useful contributions for the $U(N)$ instanton partition functions [240, 241]. We begin with the contribution of a bifundamental hypermultiplet of mass m:

$$\begin{aligned} z_{\text{bifund}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; m) = \prod_{i,j=1}^{N} & \prod_{s \in Y_i} (E(a_i - b_j, Y_i, W_j, s) - m) \\ & \prod_{r \in W_j} (\epsilon_1 + \epsilon_2 - E(b_j - a_i, W_j, Y_i, r) - m), \end{aligned} \tag{A.1.5}$$

where with $\vec{Y}$ we denote an N-tuple $\vec{Y} = (Y_1, \dots, Y_N)$ of Young diagrams, and the same for $\vec{W}$, while $\vec{a} = (a_1, \dots, a_N)$ and $\vec{b} = (b_1, \dots, b_N)$ denote the vacuum expectation values (v.e.v.) of the scalar component of the vector multiplets on the Coulomb branch.

From this, the contributions of an adjoint hypermultiplet of mass m and of a vector multiplet can be written as

$$\begin{aligned} z_{\text{adj}}(\vec{a}, \vec{Y}, m) &= z_{\text{bifund}}(\vec{a}, \vec{Y}, \vec{a}, \vec{Y}, m), \\ z_{\text{vec}}(\vec{a}, \vec{Y}) &= [z_{\text{adj}}(\vec{a}, \vec{Y}, 0)]^{-1}. \end{aligned} \tag{A.1.6}$$

Finally, the contributions for fundamental and antifundamental hypermultiplets read as follows:

$$\begin{aligned} z_{\text{fund}}(\vec{a}, \vec{Y}, m) &= \prod_{i=1}^{2} \prod_{s \in Y_i} (\varphi(a_i, s) - m + \epsilon_1 + \epsilon_2), \\ z_{\text{antifund}}(\vec{a}, \vec{Y}, m) &= z_{\text{fund}}(\vec{a}, \vec{Y}, \epsilon_1 + \epsilon_2 - m). \end{aligned} \tag{A.1.7}$$

We finally recall the expressions of Nekrasov's instanton partition functions.

The instanton partition function of the $\mathcal{N} = 2^*$ gauge theory with gauge group $U(N)$ can be written as

$$\begin{aligned} Z_{\text{inst}}^{\mathcal{N}=2^*, U(N)} = \sum_{k \geq 0} t^k \sum_{|\vec{Y}|=k} \prod_{i,j=1}^{N} \prod_{s \in Y_i} & \frac{a_i - a_j - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1) - m}{a_i - a_j - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1)} \\ \prod_{r \in Y_j} & \frac{-a_j + a_i + \epsilon_1 (L_{Y_i}(r) + 1) - \epsilon_2 A_{Y_j}(r) - m}{-a_j + a_i + \epsilon_1 (L_{Y_i}(r) + 1) - \epsilon_2 A_{Y_j}(r)}, \end{aligned} \tag{A.1.8}$$

where the sum over $|\vec{Y}| = k$ means that we are summing over N-tuples of Young diagrams $(Y_1, \dots, Y_N)$ such that the sum of the number of the boxes in all the diagram is equal to k.

The instanton partition function of the $\mathcal{N} = 2$ super Yang–Mills gauge theory with gauge group $U(N)$ can be written as

$$\begin{aligned} Z_{\text{inst}}^{\mathcal{N}=2, U(N)} = \sum_{k \geq 0} t^k \sum_{|\vec{Y}|=k} \prod_{i,j=1}^{N} \prod_{s \in Y_i} & \frac{1}{a_i - a_j - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1)} \\ \prod_{r \in Y_j} & \frac{1}{-a_j + a_i + \epsilon_1 (L_{Y_i}(r) + 1) - \epsilon_2 A_{Y_j}(r)}. \end{aligned} \tag{A.1.9}$$

For what concerns the instanton partition function of the $U(N)$ gauge theory with N_f (anti)fundamental hypermultiplets, our analysis does not depend on whether the matter is in the fundamental or antifundamental representation, and, to simplify the notation, we restrict to consider only the antifundamental matter. Hence, we can write

$$
\begin{aligned}
Z_{\text{inst}}^{\mathcal{N}=2\ U(N),\ N_f} = \sum_{k\geq 0} t^k \sum_{|\vec{Y}|=k} \prod_{i,j=1}^{N} \prod_{(m,n)\in Y_i} & \frac{1}{a_i - a_j - \epsilon_1 L_{Y_j}((m,n)) + \epsilon_2 \left(A_{Y_i}((m,n)) + 1\right)} \\
\prod_{(m,n)\in Y_j} & \frac{1}{a_i - a_j + \epsilon_1 \left(L_{Y_i}((m,n)) + 1\right) - \epsilon_2 A_{Y_j}((m,n))} \\
\prod_{i=1}^{N} \prod_{(m,n)\in Y_i} \prod_{r=1}^{N_f} & \left[a_i + \epsilon_1(m-1) + \epsilon_2(n-1) + m_r\right],
\end{aligned}
\tag{A.1.10}
$$

where m_r, $r = 1, \ldots, N_f$, are the masses of the antifundamental hypermultiplets.

In Sec. 2.1, we used the instanton part of the NS free energy to express the quantization condition for the QNMs. To take the NS limit of the instanton partition function of the $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 4$ fundamental hypermultiplets, it is convenient to redefine the hypermultiplet contribution as

$$
z_{\text{hyp}}\left(\vec{a}, \vec{Y}, m\right) = \prod_{k=1,2} \prod_{(i,j)\in Y_k} \left[a_k + m + \epsilon_1 \left(i - \frac{1}{2}\right) + \epsilon_2 \left(j - \frac{1}{2}\right)\right]. \tag{A.1.11}
$$

We take $\epsilon_1 = 1$ and $\vec{a} = (a, -a)$. We denote with m_1, m_2, m_3, m_4 the masses of the four hypermultiplets and we introduce the gauge parameters a_0, a_t, a_1, a_∞ satisfying

$$
\begin{aligned}
m_1 &= -a_t - a_0, \\
m_2 &= -a_t + a_0, \\
m_3 &= a_\infty + a_1, \\
m_4 &= -a_\infty + a_1.
\end{aligned}
\tag{A.1.12}
$$

Moreover, we denote with t the instanton counting parameter $t = e^{2\pi i \tau}$, where τ is related to the gauge coupling by

$$
\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{\text{YM}}^2}. \tag{A.1.13}
$$

The instanton part of the NS free energy is then given as a power series in t by

$$
F(t) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log\left[(1-t)^{-2\epsilon_2^{-1}\left(\frac{1}{2}+a_1\right)\left(\frac{1}{2}+a_t\right)} \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{a}, \vec{Y}\right) \prod_{i=1}^{4} z_{\text{hyp}}\left(\vec{a}, \vec{Y}, m_i\right)\right]. \tag{A.1.14}
$$

In the text, we also refer to the *full NS free energy*, which contains not only the instanton part but also the classical and one-loop contributions. This is explicitly given by

$$
\begin{aligned}
F_{\text{full}}(t) = F(t) - a^2 \log(t) - \sum_{i=1}^{4} \psi^{(-2)}\left(\frac{1}{2} - a - m_i\right) - \sum_{i=1}^{4} \psi^{(-2)}\left(\frac{1}{2} + a - m_i\right) + \\
+ \psi^{(-2)}(1 + 2a) + \psi^{(-2)}(1 - 2a),
\end{aligned}
\tag{A.1.15}
$$

where

$$\psi^{(-2)}(z) = \int_0^z \mathrm{d}t \log\left[\Gamma(t)\right]. \tag{A.1.16}$$

The gauge parameter a is expressed in a series expansion in the instanton counting parameter t, obtained by inverting the *Matone relation* [196, 242]

$$u = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F(t), \tag{A.1.17}$$

where the parameter u is the complex moduli parametrizing the corresponding SW curve. Explicitly, the expansion reads as follows

$$a = \pm\left\{\sqrt{-\frac{1}{4} - u + a_t^2 + a_0^2} + \frac{\left(\frac{1}{2} + u - a_t^2 - a_0^2 - a_1^2 + a_\infty^2\right)\left(\frac{1}{2} + u - 2a_t^2\right)}{2(1 + 2u - 2a_t^2 - 2a_0^2)\sqrt{-\frac{1}{4} - u + a_t^2 + a_0^2}} t + \mathcal{O}(t^2)\right\}. \tag{A.1.18}$$

Appendix B

Multiple Polylogarithms

B.1 Useful facts about multiple polylogarithms in a single variable

We start by recalling the definition of multiple polylogarithms in a single variable:

$$\mathrm{Li}_{s_1,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1}^{\infty} \frac{z^{k_1}}{k_1^{s_1}\ldots k_n^{s_n}}. \tag{B.1.1}$$

These satisfy

$$z\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{s_1,\ldots,s_n}(z) = \mathrm{Li}_{s_1-1,\ldots,s_n}(z) \tag{B.1.2}$$

for $s_1 \geq 2$, and

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{1,s_2,\ldots,s_n}(z) = \mathrm{Li}_{s_2,\ldots,s_n}(z). \tag{B.1.3}$$

for $s_1 = 1$, $n \geq 2$.

There are many identities between polylogarithms and multiple polylogarithms. Below is the list of identities that are relevant in our case. First, for multiple polylogarithms of the form $\mathrm{Li}_{1,s_2,\ldots,s_n}(z)$, we have:

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{(-1)^n}{n!}\log(1-z)^n. \tag{B.1.4}$$

Taking derivatives and using (2.1.42) and (2.1.43), it is easy to show by induction that

$$n\geq 1: \quad \sum_{k=1}^{n-1}\mathrm{Li}_{k,n-k+1}(z) + 2\,\mathrm{Li}_{n,1}(z) + \log(1-z)\,\mathrm{Li}_n(z) = 0, \tag{B.1.5}$$

$$\begin{cases} m\geq 1, \\ n\geq 1 \end{cases} : \quad \sum_{k=1}^{m-1}\mathrm{Li}_{k,m-k+1,n}(z) + \sum_{k=1}^{n-1}\mathrm{Li}_{m,k,n-k+1}(z) + \mathrm{Li}_{m,1,n}(z) + 2\,\mathrm{Li}_{m,n,1}(z) + \\ + \log(1-z)\,\mathrm{Li}_{m,n}(z) = 0. \tag{B.1.6}$$

Generalizing the last two identities to an arbitrary level, one gets the following identity, which we use to express $\mathrm{Li}_{1,s_1,\ldots,s_n}(z)$ in terms of multiple polylogarithms $\mathrm{Li}_{r_1,\ldots,r_{n+1}}(z)$ with $r_1 \geq 2$:

$$\sum_{i=1}^{n}\sum_{k=1}^{s_i-1}\mathrm{Li}_{s_1,\ldots,s_{i-1},k,s_i',s_{i+1},\ldots,s_n}(z)+\sum_{i=1}^{n-1}\mathrm{Li}_{s_1,\ldots,s_i,1,s_{i+1},\ldots,s_n}(z)+2\,\mathrm{Li}_{s_1,\ldots,s_n,1}(z)+$$
$$+\log(1-z)\,\mathrm{Li}_{s_1,\ldots,s_n}(z)=0, \tag{B.1.7}$$

where in the first double sum, we insert index k in the position of s_i and then move s_i to the next position while modifying it as

$$s_i' = s_i - k + 1. \tag{B.1.8}$$

Up to weight 4, all multiple polylogarithms in a single variable can be expressed as ordinary polylogarithms by combining the above identities and the following ones [192, 243]:

$$\mathrm{Li}_{2,1}(z)+\mathrm{Li}_3(1-z)-\log(1-z)\,\mathrm{Li}_2(1-z)-\frac{1}{2}\log(z)\log(1-z)^2-\zeta(3)=0, \tag{B.1.9}$$

$$\mathrm{Li}_{3,1}(z)-\mathrm{Li}_4(z)+\mathrm{Li}_4(1-z)-\mathrm{Li}_4\left(\frac{z}{z-1}\right)+\log(1-z)\,\mathrm{Li}_3(z)=\frac{1}{24}\log(1-z)^4$$
$$-\frac{1}{6}\log(z)\log(1-z)^3+\frac{\pi^2}{12}\log(1-z)^2+\zeta(3)\log(1-z)+\frac{\pi^4}{90}, \tag{B.1.10}$$

$$\mathrm{Li}_{2,1,1}(z)+\mathrm{Li}_4(1-z)-\log(1-z)\,\mathrm{Li}_3(1-z)+\frac{1}{2}\log(1-z)^2\,\mathrm{Li}_2(1-z)=$$
$$=\frac{\pi^4}{90}-\frac{1}{6}\log(z)\log(1-z)^3, \tag{B.1.11}$$

$$4\,\mathrm{Li}_{3,1}(z)+2\,\mathrm{Li}_{2,2}(z)-\mathrm{Li}_2(z)^2=0. \tag{B.1.12}$$

There are identities for weight higher than 4, but not enough to express all multiple polylogarithms as ordinary polylogarithms. For example, we have for weight 5:

$$\mathrm{Li}_{2,1,1,1}(z)+\mathrm{Li}_5(1-z)-\log(1-z)\,\mathrm{Li}_4(1-z)+\frac{1}{2}\log(1-z)^2\,\mathrm{Li}_3(1-z)=$$
$$\frac{1}{6}\log(1-z)^3\,\mathrm{Li}_2(1-z)+\frac{1}{24}\log(z)\log(1-z)^4+\zeta(5). \tag{B.1.13}$$

The latter can be checked by taking a derivative and using identity B.1.4. For the applications to BH problems, we choose not to use the powers of polylogarithms in any basis, which reduces the number of relevant identities.

Multiple zeta values (MZVs) and Euler sums arise when evaluating the quasinormal mode frequencies:

$$\mathrm{Li}_{s_1,\ldots,s_n}(1)\equiv\zeta(s_1,\ldots,s_n),\quad \mathrm{Li}_{s_1,\ldots,s_n}(-1)\equiv\zeta(-s_1,s_2,\ldots,s_n). \tag{B.1.14}$$

Some of these values can be computed using the known relations [244, 245, 246, 247] of the form:

$$a,b>1: \quad \zeta(a,b)+\zeta(b,a)=\zeta(a)\,\zeta(b)-\zeta(a+b)\,, \tag{B.1.15}$$

$$\zeta(-2\,n,1)=\frac{1}{2}\zeta(2\,n+1)-\frac{2\,n-1}{2}\,\eta(2\,n+1)+\sum_{k=1}^{n-1}\eta(2\,k)\,\zeta(2\,n+1-2\,k)\,, \tag{B.1.16}$$

where

$$\eta(x)=\left(1-2^{1-x}\right)\zeta(x)\,. \tag{B.1.17}$$

In particular, the following MZVs and Euler sums of weight 5 can be written in terms of Riemann ζ-functions [247]:

$$\begin{aligned}
&\zeta(2,3)=\frac{9}{2}\,\zeta(5)-\frac{\pi^2}{3}\,\zeta(3)\,, \quad \zeta(3,2)=\frac{\pi^2}{2}\,\zeta(3)-\frac{11}{2}\,\zeta(5)\,,\\
&\zeta(4,1)=2\,\zeta(5)-\frac{\pi^2}{6}\,\zeta(3)\,, \quad \zeta(-2,3)=\frac{51}{32}\,\zeta(5)-\frac{\pi^2}{8}\,\zeta(3)\,,\\
&\zeta(-3,2)=\frac{41}{32}\,\zeta(5)-\frac{5\,\pi^2}{48}\,\zeta(3)\,, \quad \zeta(-4,1)=\frac{\pi^2}{12}\,\zeta(3)-\frac{29}{32}\,\zeta(5)\,.
\end{aligned} \tag{B.1.18}$$

Lastly, we need expansions of multiple polylogarithms around $z=1$. Such an expansion for the polylogarithm $\mathrm{Li}_n(z)$ with $n\geq 1$ is given by [248, 249]

$$\mathrm{Li}_n(\mathrm{e}^{\mu})=\frac{\mu^{n-1}}{(n-1)!}\left[H_{n-1}-\log(-\mu)\right]+\sum_{\substack{k=0\\k\neq n-1}}^{\infty}\zeta(n-k)\,\frac{\mu^k}{k!}, \tag{B.1.19}$$

where H_n is the n-th harmonic number and $|\mu|<2\pi$. To derive the same for $\mathrm{Li}_{1,n}(z)$, we integrate both sides of the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\,\mathrm{Li}_{1,n}(\mathrm{e}^{\mu})=\frac{\mathrm{e}^{\mu}}{1-\mathrm{e}^{\mu}}\,\mathrm{Li}_n(\mathrm{e}^{\mu})\,, \tag{B.1.20}$$

where

$$\frac{\mathrm{e}^{\mu}}{1-\mathrm{e}^{\mu}}=-\frac{1}{2}-\frac{1}{\mu}-\sum_{j=1}^{\infty}B_{2j}\,\frac{\mu^{2j-1}}{(2j)!}. \tag{B.1.21}$$

Up to a constant of integration $c_{1,n}$ we get:

$$\begin{aligned}
n\geq 2: \quad \mathrm{Li}_{1,n}(\mathrm{e}^{\mu})=&\,c_{1,n}-\zeta(n)\log(-\mu)-\frac{1}{2}\mathrm{Li}_{n+1}(\mathrm{e}^{\mu})-\sum_{\substack{k=1\\k\neq n-1}}^{\infty}\zeta(n-k)\,\frac{\mu^k}{k!\,k}\\
&-\frac{1}{(n-1)!}\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}\,\frac{\mu^{2j+n-1}}{2j+n-1}\left[H_{n-1}+\frac{1}{2j+n-1}-\log(-\mu)\right]\\
&-\sum_{j=1}^{\infty}\sum_{\substack{k=2j\\k\neq 2j+n-1}}^{\infty}\frac{B_{2j}}{(2j)!}\,\frac{\zeta(2j+n-k)}{(k-2j)!}\,\frac{\mu^k}{k}.
\end{aligned} \tag{B.1.22}$$

Using (B.1.14) and the above polylogarithm identities, one obtains the first few coefficients $c_{1,n}$. For example, from (B.1.5) and (B.1.9)–(B.1.12), we get

$$c_{1,2} = -\frac{3}{2}\zeta(3)\,, \quad c_{1,3} = -\frac{\pi^4}{120}. \tag{B.1.23}$$

Now, we can get the expansion for $\mathrm{Li}_{m,n}(\mathrm{e}^{\mu})$ by consecutively integrating (B.1.22):

$$\begin{aligned}\mathrm{Li}_{m,n}(\mathrm{e}^{\mu}) &= \sum_{k=0}^{m-1} c_{m-k,n}\frac{\mu^k}{k!} + \zeta(n)\frac{\mu^{m-1}}{(m-1)!}\left[H_{m-1} - \log(-\mu)\right] - \frac{1}{2}\mathrm{Li}_{m+n}(\mathrm{e}^{\mu}) \\ &- \sum_{\substack{k=1\\ k\neq n-1}}^{\infty}\zeta(n-k)\frac{\mu^{k+m-1}}{k\,(k+m-1)!} - \sum_{j=1}^{\infty}\sum_{\substack{k=2j\\ k\neq 2j+n-1}}^{\infty}\frac{B_{2j}}{(2j)!}\frac{\zeta(2j+n-k)}{(k-2j)!}\frac{(k-1)!}{(k+m-1)!}\mu^{k+m-1} \\ &- \frac{\mu^{n+m}}{(n-1)!}\sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}\frac{(2j+n-2)!\,\mu^{2j-2}}{(2j+n+m-2)!}\left[H_{2j+n+m-2} + H_{n-1} - H_{2j+n-2} - \log(-\mu)\right],\end{aligned} \tag{B.1.24}$$

for every $m \geq 1$, $n \geq 2$. Again, the integration constants $c_{m,n}$ can be computed with the help of the known identities:

$$c_{2,2} = \frac{\pi^4}{72}, \quad c_{1,4} = \frac{\pi^2}{6}\zeta(3) - \frac{5}{2}\zeta(5)\,, \quad c_{2,3} = -\frac{\pi^2}{3}\zeta(3) + 5\zeta(5)\,, \quad c_{3,2} = \frac{\pi^2}{2}\zeta(3) - 5\zeta(5)\,.$$

In the same way, one can derive the expansion for $\mathrm{Li}_{m,1}$ by consecutively integrating $\mathrm{Li}_{1,1}$:

$$\begin{aligned}\mathrm{Li}_{m,1}(\mathrm{e}^{\mu}) &= \sum_{k=0}^{m-2}\zeta(m-k,1)\frac{\mu^k}{k!} - \sum_{k=1}^{\infty}\zeta(1-k)\frac{\mu^{k+m-1}}{(k+m-1)!}\left[\log(-\mu) + H_k - H_{k+m-1}\right] \\ &+ \frac{\mu^{m-1}}{(m-1)!}\left[\frac{1}{2}\log(-\mu)^2 - H_{m-1}\log(-\mu) + H_{m-1,2} + \sum_{k=1}^{m-1}\frac{H_{k-1}}{k}\right] \\ &+ \frac{1}{2}\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\zeta(1-j)\,\zeta(j-k+1)\frac{k!}{(k+m-1)!}\frac{\mu^{k+m-1}}{j!\,(k-j)!},\end{aligned} \tag{B.1.25}$$

for every $m \geq 1$, and where $H_{m,2}$ is the generalized harmonic number of the form

$$H_{m,2} = \sum_{k=1}^{m}\frac{1}{k^2}. \tag{B.1.26}$$

B.2 Solving integral recurrence relations

In sections 2.1.2 and 2.2.2, we claimed that the wave functions $\psi^L(z)$ at order t^k (or, equivalently, R_h^k) are described in terms of multiple polylogarithms of weight k and lower.

Here, we prove this claim, but first, let us clarify the terminology. The notion of weight is related to the power of a logarithm function, as seen in the following identity:

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{(-1)^n}{n!} \log(1-z)^n \,. \tag{B.2.27}$$

Thus, we will ascribe weight to the ordinary logarithm functions as follows. For any product of two logarithms

$$m, n \geq 0: \quad \log(z)^m \log(z-1)^n \,, \tag{B.2.28}$$

the weight equals $n+m \geq 0$. For the product of a logarithm and a multiple polylogarithm

$$n \geq 1,\, m \geq 0: \quad \log(z-1)^m \,\mathrm{Li}_{s_1,\ldots,s_n}(1-z) \,, \tag{B.2.29}$$

the weight is $m + s_1 + \cdots + s_n > 0$. Here we do not consider the other possible product $\log(z)^m \,\mathrm{Li}_{s_1,\ldots,s_n}(1-z)$ because, due to the identities of the form (B.1.7), this product can always be rewritten as a linear combination of multiple polylogarithms. Some simple examples are:

$$\log(z)\, \mathrm{Li}_2(1-z) = -\mathrm{Li}_{1,2}(1-z) - 2\,\mathrm{Li}_{2,1}(1-z) \,, \tag{B.2.30}$$

$$\frac{1}{2}\log(z)^2\, \mathrm{Li}_2(1-z) = \mathrm{Li}_{1,1,2}(1-z) + 2\,\mathrm{Li}_{1,2,1}(1-z) + 3\,\mathrm{Li}_{2,1,1}(1-z) \,. \tag{B.2.31}$$

In general, multiple polylogarithm functions can not be rewritten as powers of ordinary logarithm functions. We will use both logarithms and multiple polylogarithms of a certain weight to build a linear basis in which the wave function can be expanded at a certain order in t. In what follows, all powers of logarithms are non-negative integers.

We are going to prove our claim by induction. In the first order in t, the integrands in the recurrence relations are just rational functions of the form

$$\frac{\sum_{m=0}^{r_0} \alpha_m \, z^m}{z^{i_0}\,(z-1)^{j_0}} \tag{B.2.32}$$

with non-negative integers r_0, i_0, j_0 that depend on the quantum numbers of the scalar, electromagnetic, or gravitational perturbations. These rational functions can be broken up into a sum of monomials in z and poles at $z = 0, 1$ with the help of the identities

$$n, m \geq 0: \quad \frac{z^n}{(z-1)^m} = \sum_{l=0}^{n} \binom{n}{l} (z-1)^{l-m} \,, \tag{B.2.33}$$

$$\frac{1}{z^n\,(1-z)^m} = \sum_{l=1}^{n} \binom{n+m-l-1}{m-1} \frac{1}{z^l} + \sum_{j=1}^{m} \binom{n+m-j-1}{n-1} \frac{1}{(1-z)^j}, \tag{B.2.34}$$

where in the last identity $n, m \geq 1$. Thus, the wave function $\psi^L(z)$ ar order t is described in terms of rational functions and logarithms of weight 1: $\log(z)$ and $\log(z-1)$.

Next, we assume that the integrands in the recurrence relations at order t^{k+1} are linear combinations of functions with maximum weight k :

$$\frac{\sum_{m=0}^{r_1} \alpha_m \, z^m}{z^{i_1} \, (z-1)^{j_1}} \, \log{(z-1)}^{p_1} \log{(z)}^{p_2} \, ,$$

$$\frac{\sum_{m=0}^{r_2} \beta_m \, z^m}{z^{i_2} \, (z-1)^{j_2}} \, \log{(z-1)}^{p_3} \, \mathrm{Li}_{s_1,\ldots,s_n} \, (1-z) \, . \tag{B.2.35}$$

After breaking up rational functions with the help of (B.2.34), we will consider all possible integrals case by case and show that the maximum weight after the integration is $k+1$. Splitting this last part of the proof into three steps is helpful. In each step, we will deal with the following integrals:

1. Integrals that increase the maximum weight by one.
2. Integrals that do not increase the maximum weight and involve only one logarithm or multiple polylogarithm: $\log{(z)}^m$, $\log{(z-1)}^n$, or $\mathrm{Li}_{s_1,\ldots,s_k} \, (1-z)$.
3. Integrals that do not increase the maximum weight and involve the following products of logarithms: $\log{(z)}^m \log{(z-1)}^n$ and $\log{(z-1)}^m \, \mathrm{Li}_{s_1,\ldots,s_n} \, (1-z)$.

Step 1. Four types of integrals increase the maximum weight. In each case the integrand has a factor of z^{-1} or $(z-1)^{-1}$. For the product of two logarithms of weight $n+m$, $n, m \geq 0$ we have:

$$\int \frac{\log{(z)}^m \log{(z-1)}^n}{z} \, \mathrm{d}z = (-1)^{m+n+1} \; m! \, n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \log{(z-1)}^j \, \mathrm{Li}_{n-j+1,\{1\}_m} \, (1-z) \, , \tag{B.2.36}$$

$$\int \frac{\log{(z)}^m \log{(z-1)}^n}{z-1} \, \mathrm{d}z = (-1)^{m+n} \; m! \, n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \log{(z-1)}^j \, \mathrm{Li}_{n-j+2,\{1\}_{m-1}} \, (1-z) \, , \tag{B.2.37}$$

where in the last integral $m \geq 1$ and $\mathrm{Li}_{n,\{1\}_0} \equiv \mathrm{Li}_n$. The resulting weight after the integration is $1+n+m$. In the more general case of integrals involving multiple polylogarithms, we have ($m \geq 0$):

$$\begin{aligned} &\int \frac{\log{(z-1)}^m}{z} \, \mathrm{Li}_{s_1,\ldots,s_n} \, (1-z) \; \mathrm{d}z = \\ &(-1)^{m+1} \; m! \sum_{j=0}^{m} \frac{(-1)^j}{j!} \log{(z-1)}^j \, \mathrm{Li}_{m-j+1,s_1,\ldots,s_n} \, (1-z) \, , \\ &\int \frac{\log{(z-1)}^m}{z-1} \, \mathrm{Li}_{s_1,\ldots,s_n} \, (1-z) \; \mathrm{d}z = \\ &(-1)^m \; m! \sum_{j=0}^{m} \frac{(-1)^j}{j!} \log{(z-1)}^j \, \mathrm{Li}_{s_1+m-j+1,s_2,\ldots,s_n} \, (1-z) \, . \end{aligned} \tag{B.2.38}$$

Again, after the integration, the weight was increased by 1 from $m + s_1 + \cdots + s_n$ to $1 + m + s_1 + \cdots + s_n$. The above identities were obtained by repeated integrations by parts.

Step 2. The integrands in this step are products of one logarithm or multiple polylogarithm with z^n or $(z-1)^n$, with $n \neq -1$. Moreover, it is enough to consider only the negative powers of $(z-1)$ since all positive powers can be reduced to monomials in z. We start with integrals involving the $\log(z)$ function:

$$n \neq -1: \quad \int z^n \log(z)^m \, \mathrm{d}z = (-1)^m \, m! \, \frac{z^{n+1}}{(n+1)^{m+1}} \sum_{j=0}^{m} \frac{(-1)^j}{j!} (n+1)^j \log(z)^j \,, \tag{B.2.39}$$

$$\begin{aligned} n \geq 2: \quad \int \frac{\log(z)}{(z-1)^n} \, \mathrm{d}z = \frac{1}{1-n} \left((-1)^n + \frac{1}{(z-1)^{n-1}} \right) \log(z) - \\ - \frac{(-1)^n}{1-n} \log(z-1) + \frac{(-1)^n}{1-n} \sum_{j=1}^{n-2} \frac{1}{j\,(z-1)^j}, \end{aligned} \tag{B.2.40}$$

$$m \geq 2: \quad \int \frac{\log(z)^m}{(z-1)^2} \, \mathrm{d}z = \frac{z}{1-z} \log(z)^m - (-1)^m \, m! \, \mathrm{Li}_{2,\{1\}_{m-2}}(1-z) \,,$$

$$\begin{aligned} n, m \geq 2: \quad \int \frac{\log(z)^m}{(z-1)^n} \, \mathrm{d}z = \frac{1}{1-n} \left((-1)^n + \frac{1}{(z-1)^{n-1}} \right) \log(z)^m + \\ + (-1)^{m+n} \frac{m!}{1-n} \mathrm{Li}_{2,\{1\}_{m-2}}(1-z) + \frac{m}{1-n} \sum_{l=2}^{n-1} (-1)^{l+n} \int \frac{\log(z)^{m-1}}{(z-1)^l} \, \mathrm{d}z, \end{aligned} \tag{B.2.41}$$

where the last equation allows us to take the corresponding integral recursively. In principle, the integrals with the other logarithm $\log(z-1)$ can be obtained from (B.2.40)–(B.2.41) by shifting the variable $z \to 1-z$. This, however, would change the argument of multiple polylogarithms from $(1-z)$ to z. Since we want our multiple polylogarithms to converge in the disk $|1-z| < 1$ (or $|1-z| \leq 1$ when $s_1 \geq 2$), we rewrite (B.2.40)–(B.2.41) using the function $\log(z-1)$:

$$n \geq 2: \quad \int \frac{\log(z-1)}{z^n} \, \mathrm{d}z = \frac{z^{1-n} - 1}{1-n} \log(z-1) + \frac{\log(z)}{1-n} + \frac{1}{n-1} \sum_{j=1}^{n-2} \frac{1}{j\, z^j},$$

$$\begin{aligned} m \geq 0: \quad \int \frac{\log(z-1)^m}{z^2} \, \mathrm{d}z = \frac{z-1}{z} \log(z-1)^m \\ - (-1)^m \, m! \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} \log(z-1)^j \, \mathrm{Li}_{m-j}(1-z) \,, \end{aligned}$$

$$\begin{aligned} n \geq 2, \\ m \geq 0 \end{aligned} : \quad \int \frac{\log(z-1)^m}{z^n}\,\mathrm{d}z = \frac{z^{1-n}-1}{1-n}\log(z-1)^m + \frac{m}{1-n}\sum_{l=2}^{n-1}\int \frac{\log(z-1)^{m-1}}{z^l}\,\mathrm{d}z$$
$$+ (-1)^m \frac{m!}{1-n}\sum_{j=0}^{m-1}\frac{(-1)^j}{j!}\log(z-1)^j\,\mathrm{Li}_{m-j}(1-z)\,. \tag{B.2.42}$$

In all the integrals taken so far in step 2, we can explicitly see that the maximum weights before and after integration are the same.

For the integrals that involve multiple polylogarithms $\mathrm{Li}_{s_1,\ldots,s_n}(1-z)$, we consider the two cases $s_1 = 1$ and $s_1 \geq 2$ separately. First, we look at the integrals with non-negative powers of z:

$$\begin{cases} n \geq 2, \\ m \geq 0 \end{cases} : \quad \int z^m\,\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{z^{m+1}}{m+1}\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)$$
$$+\frac{1}{m+1}\int z^m\,\mathrm{Li}_{s_2,\ldots,s_n}(1-z)\,\mathrm{d}z, \tag{B.2.43}$$

$$\begin{cases} s_1 \geq 2, \\ m \geq 0 \end{cases} : \quad \int z^m\,\mathrm{Li}_{s_1,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{z^{m+1}-1}{m+1}\mathrm{Li}_{s_1,\ldots,s_n}(1-z)$$
$$-\frac{1}{m+1}\sum_{l=0}^{m}\int z^l\,\mathrm{Li}_{s_1-1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z. \tag{B.2.44}$$

The above integrals can be taken recursively until one gets integrals of the form (B.2.39) and the maximum weight of the final result is equal to k. Similarly, we have for the integrals with negative powers of z (except for $1/z$):

$$\begin{cases} n \geq 2, \\ m \geq 2 \end{cases} : \quad \int \frac{1}{z^m}\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{z^{1-m}}{1-m}\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)$$
$$+\frac{1}{1-m}\int \frac{1}{z^m}\mathrm{Li}_{s_2,\ldots,s_n}(1-z)\,\mathrm{d}z, \tag{B.2.45}$$

$$\begin{cases} s_1 \geq 2, \\ m \geq 2 \end{cases} : \quad \int \frac{1}{z^m}\mathrm{Li}_{s_1,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{1}{1-m}\sum_{l=2}^{m-1}\int \frac{1}{z^l}\mathrm{Li}_{s_1-1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z$$
$$+\frac{\mathrm{Li}_{1,s_1-1,s_2,\ldots,s_n}(1-z)}{m-1} + \frac{z^{1-m}-1}{1-m}\mathrm{Li}_{s_1,\ldots,s_n}(1-z)\,. \tag{B.2.46}$$

Finally, we consider the integrals with multiple polylogarithms divided by $(z-1)^m$, $m \geq 2$:

$$\begin{cases} n \geq 2, \\ m \geq 2 \end{cases} : \quad \int \frac{1}{(z-1)^m}\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{(-1)^m + (z-1)^{1-m}}{1-m}\mathrm{Li}_{1,s_2,\ldots,s_n}(1-z)$$
$$+\frac{(-1)^m}{1-m}\mathrm{Li}_{s_2+1,\ldots,s_n}(1-z) + \frac{(-1)^m}{m-1}\sum_{l=2}^{m-1}\int \frac{(-1)^l}{(z-1)^l}\mathrm{Li}_{s_2,\ldots,s_n}(1-z)\,\mathrm{d}z, \tag{B.2.47}$$

$$\begin{cases} s_1 \geq 2, \\ m \geq 2 \end{cases} : \quad \int \frac{1}{(z-1)^m} \operatorname{Li}_{s_1,\ldots,s_n}(1-z)\,\mathrm{d}z = \frac{(z-1)^{1-m}}{1-m} \operatorname{Li}_{s_1,\ldots,s_n}(1-z)$$

$$+\frac{1}{m-1} \int \frac{1}{(z-1)^m} \operatorname{Li}_{s_1-1,s_2,\ldots,s_n}(1-z)\,\mathrm{d}z. \tag{B.2.48}$$

Step 3. Here we have to deal with four types of integrals:

$$\begin{cases} j \neq -1 \\ n, m \geq 1 \end{cases} : \quad \int z^j \log(z)^m \log(z-1)^n \,\mathrm{d}z, \tag{B.2.49}$$

$$\begin{cases} j \geq 2 \\ n, m \geq 1 \end{cases} : \quad \int \frac{\log(z)^m \log(z-1)^n}{(z-1)^j} \,\mathrm{d}z, \tag{B.2.50}$$

$$\begin{cases} m \neq -1 \\ j \geq 1 \end{cases} : \quad \int z^m \log(z-1)^j \operatorname{Li}_{s_1,\ldots,s_n}(1-z)\,\mathrm{d}z, \tag{B.2.51}$$

$$\begin{cases} m \geq 2 \\ j \geq 1 \end{cases} : \quad \int \frac{\log(z-1)^j}{(z-1)^m} \operatorname{Li}_{s_1,\ldots,s_n}(1-z)\,\mathrm{d}z. \tag{B.2.52}$$

In each case, we can use integration by parts to reduce the weight of one of the logarithms by 1. Applying integration by parts recursively allows us to reduce all integrals of the form (B.2.49)–(B.2.52) to one of the integrals from step 2 or 1. For example, in the case of (B.2.49), we have

$$\int z^j \log(z)^m \log(z-1)^n \,\mathrm{d}z = (-1)^m \; m! \, \frac{z^{j+1}}{(j+1)^{m+1}} \sum_{l=0}^{m} \frac{(-1)^l}{l!} (j+1)^l \log(z)^l \log(z-1)^n$$

$$-(-1)^m \; \frac{m!\, n}{(j+1)^{m+1}} \sum_{l=0}^{m} \frac{(-1)^l}{l!} (j+1)^l \int \frac{z^{j+1}}{z-1} \; \log(z)^l \log(z-1)^{n-1} \,\mathrm{d}z. \tag{B.2.53}$$

We simplify the integral in the *right-hand side* of (B.2.53) by breaking up the rational function $z^{j+1}/(z-1)$ into a sum of monomials in z and poles at $z = 0, 1$:

$$j \geq 0: \quad \frac{z^{j+1}}{z-1} = \frac{1}{z-1} + \sum_{l=0}^{j} z^l, \tag{B.2.54}$$

$$j \leq -2: \quad \frac{z^{j+1}}{z-1} = \frac{1}{z-1} - \sum_{l=1}^{-j-1} z^{-l}. \tag{B.2.55}$$

Almost all the resulting integrals are of the first type (B.2.49) with the power of $\log(z-1)$ reduced by 1. The remaining two integrals

$$\int \frac{\log(z)^j \log(z-1)^{n-1}}{z} \,\mathrm{d}z \quad \text{and} \quad \int \frac{\log(z)^j \log(z-1)^{n-1}}{z-1} \,\mathrm{d}z \tag{B.2.56}$$

were already taken in step 1, and the maximum weight is $n+m = k$. Recursively applying this procedure, one can either express (B.2.49) in terms of integrals like (B.2.56) or reduce it to (B.2.39). In the same way, (B.2.50) can be essentially reduced to (B.2.39) with z replaced by $(z-1)$. Finally, the last two types of integrals, (B.2.51) and (B.2.52), are reducible to a combination of integrals from (B.2.43)–(B.2.48).

To summarize, we have shown by induction that the wave function $\psi^L(z)$ at any order t^k is a linear combination of certain functions of weight k or lower. The only special functions needed are multiple polylogarithms with argument $(1-z)$ (another possible argument would be z, but it would not be consistent with the boundary condition at the AdS boundary in the SAdS case). The same can be done for the wave function in the right region, $\psi^R(z)$.

B.3 Multiple polylogarithms for hydrodynamic QNMs

For the computations of gravitational QNMs in the scalar sector (Sec. 2.3), we introduced an expansion of the solution using the multiple polylogarithms in several variables (2.3.128). An alternative definition could be given in terms of one-forms[1] [250]

$$\mathrm{Li}_{s_1,\dots,s_n}(z_1,\dots,z_n) = \int_0^1 \omega_0^{s_1-1}\omega_{z_1}\omega_0^{s_2-1}\omega_{z_1 z_2}\dots\omega_0^{s_n-1}\omega_{z_1\dots z_n}, \tag{B.3.57}$$

where

$$\omega_z = \begin{cases} \dfrac{z\mathrm{d}t}{1-zt}, & z\neq 0, \\ \dfrac{\mathrm{d}t}{t}, & z=0. \end{cases} \tag{B.3.58}$$

All the integrals in Sec. 2.3 do not include ω_0 after the simplification, which means that $s_1 = s_2 = \dots = s_n = 1$. We define the relevant multiple polylogarithms as

$$\mathrm{Li}_{\{1\}_n}(z_1 z, z_2,\dots,z_n) = \int_0^z \omega_{z_1}\omega_{z_1 z_2}\dots\omega_{z_1\dots z_n}, \tag{B.3.59}$$

where $z_i \in \{1, u_1, u_2\}$ for every $i = 1,\dots,n$ and u_1, u_2 are the third roots of unity (2.3.131). We consider the following products of ordinary logarithm functions and multiple polylogarithms to describe the wave function:

$$\begin{aligned} &\log(1-z)^{p_1}\log(1-u_1\,z)^{p_2}\log(1-u_2\,z)^{p_3} \\ &\log(1-z)^{p_4}\log(1-u_1\,z)^{p_5}\log(1-u_2\,z)^{p_6}\,\mathrm{Li}_{\{1\}_n}(z_1 z, z_2,\dots,z_n)\,. \end{aligned} \tag{B.3.60}$$

At order α^k, only functions with maximum weight k appear, so that $0\le p_1+p_2+p_3\le k$ and $0\le p_4+p_5+p_6+n\le k$. However, at a fixed weight, some identities make the functions listed in (B.3.60) linearly dependent. For example, at level $n=2$, we have the following identities:

$$\begin{aligned} \mathrm{Li}_{1,1}(z,u_1) &= \log(1-u_1\,z)\log(1-z) - \mathrm{Li}_{1,1}(u_1\,z,u_2), \\ \mathrm{Li}_{1,1}(z,u_2) &= \log(1-u_2\,z)\log(1-z) - \mathrm{Li}_{1,1}(u_2\,z,u_1), \\ \mathrm{Li}_{1,1}(u_2\,z,u_2) &= \log(1-u_1\,z)\log(1-u_2\,z) - \mathrm{Li}_{1,1}(u_1\,z,u_1), \end{aligned} \tag{B.3.61}$$

[1] For $\omega_{z_1},\dots,\omega_{z_p}$ differential one-forms, with $\omega_{z_i} = f_{z_i}(t)\mathrm{d}t$ for some function f_{z_i}, we define inductively $\int_0^x \omega_{z_1}\dots\omega_{z_p} = \int_0^x f_{z_1}(t)\mathrm{d}t\int_0^t \omega_{z_2}\dots\omega_{z_p}$.

including the ones that reduce to the single variable case:

$$\mathrm{Li}_{1,1}(z,1) = \frac{1}{2}\,\log\left(1-z\right)^2\ . \tag{B.3.62}$$

Thus, out of 9 possible functions $\mathrm{Li}_{1,1}(z_1 z, z_2)$ at level $k=2$, we only need 3:

$$\mathrm{Li}_{1,1}\left(u_1 z, u_2\right), \quad \mathrm{Li}_{1,1}\left(u_1 z, u_1\right), \quad \mathrm{Li}_{1,1}\left(u_2 z, u_1\right). \tag{B.3.63}$$

In the rest of the appendix, we will try to classify the identities arising at a given level n and find what multiple polylogarithms are needed to form a linear basis in (B.3.60).

According to (B.3.59), there is a one-to-one correspondence between multiple polylogarithms and ordered multisets of one-forms $\{\omega_{z_1}, \omega_{z_1 z_2}, \ldots, \omega_{z_1\ldots z_n}\}$. If two multiple polylogarithms are related by the permutation of the one-forms in the corresponding ordered multisets, then an identity exists between these two. However, this identity could be reducible in the sense that it can be split into smaller ones. To show this, we integrate by parts the right-hand side of (B.3.59):

$$\begin{aligned}&\int_0^z \omega_{z_1}\omega_{z_1z_2}\ldots\omega_{z_1\ldots z_n} = \int_0^z \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Li}_1(z_1 t)\mathrm{d}t\,\omega_{z_1z_2}\ldots\omega_{z_1\ldots z_n} = \\ &= \mathrm{Li}_1(z_1 z)\int_0^z \omega_{z_1z_2}\ldots\omega_{z_1\ldots z_n} - \int_0^z \mathrm{d}t\,\mathrm{Li}_1(z_1 t)\frac{z_1 z_2}{1-z_1z_2t}\omega_{z_1z_2z_3}\ldots\omega_{z_1\ldots z_n},\end{aligned} \tag{B.3.64}$$

where $\mathrm{Li}_1\left(z_1 t\right)$ is the ordinary logarithm function:

$$\mathrm{Li}_1\left(z_1 t\right) = -\log\left(1-z_1 t\right) \tag{B.3.65}$$

and

$$\mathrm{Li}_1(z_1 z)\int_0^z \omega_{z_1z_2}\ldots\omega_{z_1\ldots z_n} = -\log\left(1-z_1 z\right)\mathrm{Li}_{\{1\}_{n-1}}(z_1z_2z, z_3,\ldots,z_n). \tag{B.3.66}$$

Continuing the integration by parts, we obtain

$$\begin{aligned}&\int_0^z \omega_{z_1}\omega_{z_1z_2}\ldots\omega_{z_1\ldots z_n} \supset \int_0^z \mathrm{d}t\mathrm{Li}_1(z_1t)\mathrm{Li}_1(z_1z_2t)\ldots\mathrm{Li}_1(z_1\ldots z_{n-1}t)\frac{z_1\ldots z_n}{1-z_1\ldots z_n t} = \\ &= \int_0^z \mathrm{d}t\mathrm{Li}_1(y_1t)\mathrm{Li}_1(y_2t)\ldots\mathrm{Li}_1(y_{n-1}t)\frac{y_n}{1-y_nt},\end{aligned} \tag{B.3.67}$$

where $y_j = z_1\cdots z_j$ for $j=1,\ldots,n$. From this last integral, one can reconstruct by the reverse process any other multiple polylogarithm for which the representation in (B.3.59) involves the integrals of the same one-forms in a different order. In the intermediate steps of this procedure, there appear products of the form

$$\mathrm{Li}_{\{1\}_{m_1}}(z_1^{(1)}\,z,\ldots,z_{m_1}^{(1)})\cdot\ldots\cdot\mathrm{Li}_{\{1\}_{m_r}}(z_1^{(r)}\,z,\ldots,z_{m_r}^{(r)}), \text{ with } m_1+\cdots+m_r = n. \tag{B.3.68}$$

It is possible to rewrite these in terms of products in (B.3.60) using *shuffle relations* (see for example Eq. (5.4) in [251]). The result is an identity involving two multiple polylogarithms that are related by the permutation of the corresponding one-forms.

Let us describe with a concrete identity at level 4 how this works. We prove that

$$\begin{aligned}\mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2) = & -2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) - \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2) \\ & - \mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\log(1-z).\end{aligned} \tag{B.3.69}$$

By definition, the left-hand side is

$$\begin{aligned}\mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2) = \int_0^z \omega_1\omega_{u_1}\omega_{u_1}\omega_1 = & -\mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\log(1-z) \\ & - \int_0^z \mathrm{Li}_1(t)\frac{u_1\,\mathrm{d}t}{1-u_1 z}\omega_{u_1}\omega_1,\end{aligned} \tag{B.3.70}$$

where in the last equality we integrated by parts. Therefore, we reduce to proving that

$$\int_0^z \mathrm{Li}_1(t)\frac{u_1\,\mathrm{d}t}{1-u_1 z}\omega_{u_1}\omega_1 = 2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2). \tag{B.3.71}$$

We have

$$\begin{aligned}&\int_0^z \mathrm{Li}_1(t)\frac{u_1\,\mathrm{d}t}{1-u_1 z}\omega_{u_1}\omega_1 = \int_0^z \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Li}_{1,1}(u_1 t,u_2)\omega_{u_1}\omega_1 = \\ &\mathrm{Li}_{1,1}(u_1 z,u_2)\mathrm{Li}_{1,1}(u_1 z,u_2) - \int_0^z \mathrm{Li}_{1,1}(u_1 t,u_2)\frac{u_1\,\mathrm{d}t}{1-u_1 t}\omega_1.\end{aligned} \tag{B.3.72}$$

Moreover,

$$\begin{aligned}&\int_0^z \mathrm{Li}_{1,1}(u_1 t,u_2)\frac{u_1\,\mathrm{d}t}{1-u_1 t}\omega_1 = \int_0^z \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Li}_{1,1,1}(u_1 t,1,u_2)\omega_1 = \\ &\mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\mathrm{Li}_1(z) - \int_0^z \mathrm{Li}_{1,1,1}(u_1 t,1,u_2)\frac{\mathrm{d}t}{1-t} = \\ &\mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\mathrm{Li}_1(z) - \mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2).\end{aligned} \tag{B.3.73}$$

Putting together (B.3.71)-(B.3.72)-(B.3.73), it remains to prove that

$$\begin{aligned}&\mathrm{Li}_{1,1}(u_1 z,u_2)\mathrm{Li}_{1,1}(u_1 z,u_2) - \mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\mathrm{Li}_1(z) + \mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2) = \\ &2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2).\end{aligned} \tag{B.3.74}$$

Applying the shuffle relation to the first two terms in left-hand side, we have

$$\begin{aligned}\mathrm{Li}_{1,1}(u_1 z,u_2)\mathrm{Li}_{1,1}(u_1 z,u_2) &= 4\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + 2\mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2), \\ \mathrm{Li}_{1,1,1}(u_1 z,1,u_2)\mathrm{Li}_1(z) &= 2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2) \\ &\quad + \mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2).\end{aligned} \tag{B.3.75}$$

Therefore, as we wanted, the left-hand side of (B.3.74) becomes

$$\begin{aligned}&4\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + 2\mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2) - [2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) \\ &+ \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2) + \mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2)] + \mathrm{Li}_{1,1,1,1}(z,u_1,1,u_2) = \\ &2\mathrm{Li}_{1,1,1,1}(u_1 z,1,u_2,1) + \mathrm{Li}_{1,1,1,1}(u_1 z,u_2,u_1,u_2).\end{aligned} \tag{B.3.76}$$

Let us remark that with the previous procedure, one can find several identities at a fixed level involving the same multiple polylogarithm. To choose which elements to add to the basis, we followed the criterium that we omit the multiple polylogarithms with the first argument z. This criterium comes from the regularity condition on the wave function at $z = 1$. Moreover, when possible, we tried to include the same number of multiple polylogarithms with the first argument $u_1 z$ and with the first argument $u_2 z$ (for example, it is not possible at level $n = 2$).

For completeness, let us write the elements of level 3 that we add to our basis:

$$\begin{aligned}&\mathrm{Li}_{1,1,1}(u_1 z, 1, u_2), \quad \mathrm{Li}_{1,1,1}(u_1 z, 1, u_1), \quad \mathrm{Li}_{1,1,1}(u_1 z, u_2, 1), \quad \mathrm{Li}_{1,1,1}(u_1 z, u_1, u_1), \\ &\mathrm{Li}_{1,1,1}(u_1 z, u_1, 1), \quad \mathrm{Li}_{1,1,1}(u_2 z, 1, u_1), \quad \mathrm{Li}_{1,1,1}(u_2 z, u_1, 1), \quad \mathrm{Li}_{1,1,1}(u_2 z, u_2, u_2),\end{aligned} \tag{B.3.77}$$

and the nontrivial identities with the other functions of the same level (other identities are obtained by exchanging u_1 with u_2):

$$\begin{aligned}
&\mathrm{Li}_{1,1,1}(u_2 z, u_2, 1) = \\
&\quad \mathrm{Li}_{1,1,1}(u_1 z, 1, u_1) + \mathrm{Li}_{1,1}(u_1 z, u_1)\log(1-u_1 z) - \frac{\log(1-u_1 z)^2\log(1-u_2 z)}{2}, \\
&\mathrm{Li}_{1,1,1}(z, u_2, 1) = \\
&\quad \mathrm{Li}_{1,1,1}(u_2 z, 1, u_1) + \mathrm{Li}_{1,1}(u_2 z, u_1)\log(1-u_2 z) - \frac{\log(1-u_2 z)^2\log(1-z)}{2}, \\
&\mathrm{Li}_{1,1,1}(z, u_1, 1) = \\
&\quad \mathrm{Li}_{1,1,1}(u_1 z, 1, u_2) + \mathrm{Li}_{1,1}(u_1 z, u_2)\log(1-u_1 z) - \frac{\log(1-u_1 z)^2\log(1-z)}{2}, \\
&\mathrm{Li}_{1,1,1}(u_2 z, u_2, u_1) = -2\mathrm{Li}_{1,1,1}(u_1 z, u_1, 1) - \mathrm{Li}_{1,1}(u_1 z, u_1)\log(1-u_2 z), \\
&\mathrm{Li}_{1,1,1}(z, u_1, u_1) = \\
&\quad \mathrm{Li}_{1,1,1}(u_2 z, u_2, u_2) - \mathrm{Li}_{1,1}(u_1 z, u_1)\log(1-z) + \mathrm{Li}_{1,1}(u_1 z, u_2)\log(1-u_2 z), \\
&\mathrm{Li}_{1,1,1}(z, u_2, u_1) = -2\mathrm{Li}_{1,1,1}(u_2 z, u_1, 1) - \mathrm{Li}_{1,1}(u_2 z, u_1)\log(1-z), \\
&\mathrm{Li}_{1,1,1}(z, u_1, u_2) = -2\mathrm{Li}_{1,1,1}(u_1 z, u_2, 1) - \mathrm{Li}_{1,1}(u_1 z, u_2)\log(1-z).
\end{aligned} \tag{B.3.78}$$

Computing all the identities up to level $n = 7$, we arrive at the following conclusion: the number of multiple polylogarithms needed to form a basis in (B.3.60) at level $n \geq 3$ is $8 \times 3^{n-3}$. Even though this significantly reduces the number of functions used at a certain level n, we still need to compute the identities for all 3^n functions to go to the next level $n + 1$.

Appendix C

Multiple polyexponential integrals

C.1 Multiple polyexponential functions: definitions

First, we define a set of functions we call undressed multiple polyexponential functions:

$$el_n\left(z\right)=\sum_{k=1}^{\infty}\frac{z^k}{k^n\,k!},\quad el_{s_1,\dots,s_n}\left(z\right)=\sum_{k_1>k_2>\dots>k_n\geq1}\frac{1}{k_1^{s_1}\dots k_n^{s_n}}\frac{z^{k_1}}{k_1!}. \tag{C.1.1}$$

The latter functions are straightforward in terms of their series expansion around $z=0$, but taking the derivative proves more challenging. If the first index s_1 is greater than one, we have a simple polylogarithm-like derivative

$$s_1>1:\quad z\frac{\mathrm{d}}{\mathrm{d}z}el_{s_1,\dots,s_n}\left(z\right)=el_{s_1-1,s_2,\dots,s_n}\left(z\right). \tag{C.1.2}$$

However, when $s_1=1$, the derivative rule becomes harder:

$$z\frac{\mathrm{d}}{\mathrm{d}z}el_{1,s_2,\dots,s_n}\left(z\right)=-el_{s_2,\dots,s_n}\left(z\right)-\left(-1\right)^n\mathrm{e}^z\sum_{\mathrm{op}(s_2)}\cdots\sum_{\mathrm{op}(s_n)}el_{\mathrm{op}(s_2),\dots,\mathrm{op}(s_n)}\left(-z\right), \tag{C.1.3}$$

where we sum over all ordered partitions of $s_i\in\mathbb{N}$, $i\geq2$:

$$\mathrm{op}\left(1\right)=\{1\},\quad\mathrm{op}\left(2\right)=\{2,(1,1)\},\quad\mathrm{op}\left(3\right)=\{3,(2,1),(1,2),(1,1,1)\} \tag{C.1.4}$$

and so on.

Observing that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{\mathrm{op}(n)}el_{1,\mathrm{op}(n)}\left(z\right)=-\sum_{\mathrm{op}(n)}el_{\mathrm{op}(n)}\left(z\right)-\mathrm{e}^z\,el_n\left(-z\right), \tag{C.1.5}$$

one is led to define the following dressed multiple polyexponential function:

$$\mathrm{EL}_{1,n}\left(z\right)\equiv\sum_{\mathrm{op}(n+1)}el_{\mathrm{op}(n+1)}\left(z\right) \tag{C.1.6}$$

which satisfies the following derivative rule:

$$z\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{EL}_{1,n}\left(z\right)=-\mathrm{e}^{z}\,\mathrm{EL}_{n}\left(-z\right), \tag{C.1.7}$$

where

$$\mathrm{EL}_{n}\left(z\right)\equiv el_{n}\left(z\right). \tag{C.1.8}$$

Motivated by the definition of $\mathrm{EL}_{1,n}\left(z\right)$, we define the complete set of dressed multiple polyexponential functions via the following recursive derivative rules:

$$\begin{aligned} &z\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{EL}_{1,s_2,\dots,s_n}\left(z\right)=-\mathrm{e}^{z}\,\mathrm{EL}_{s_2,\dots,s_n}\left(-z\right),\\ s_1>1:\quad &z\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{EL}_{s_1,\dots,s_n}\left(z\right)=\mathrm{EL}_{s_1-1,s_2,\dots,s_n}\left(z\right),\end{aligned} \tag{C.1.9}$$

where the integration constants are fixed by

$$\mathrm{EL}_{s_1,\dots,s_n}\left(0\right)=0. \tag{C.1.10}$$

Compared with undressed functions $el_{s_1,\dots,s_n}\left(z\right)$, the new set of functions has simple derivative rules, but more involved series expansions around $z=0$. In our first example (C.1.6), the Taylor expansion is given by

$$\mathrm{EL}_{1,n}\left(z\right)=\sum_{k_1\geq k_2\geq\cdots\geq k_{n+1}\geq 1}\frac{1}{k_1k_2\dots k_{n+1}}\,\frac{z^{k_1}}{k_1!}. \tag{C.1.11}$$

To generalize the relations and Taylor expansions to an arbitrary number of indices, it is convenient to introduce an operator $\oplus$ between two vectors $\mathbf{v}=(v_1,\dots,v_i)$ and $\mathbf{u}=(u_1,\dots,u_j)$:

$$\mathbf{v}\oplus\mathbf{u}=\left(v_1,\dots,v_{i-1},v_i+u_1,u_2,\dots,u_j\right). \tag{C.1.12}$$

If the second vector is a scalar, we have:

$$\mathbf{v}\oplus u=\left(v_1,\dots,v_{i-1},v_i+u\right). \tag{C.1.13}$$

For an even number of indices $n=2\,k$, the relation between the dressed and undressed functions is

$$\mathrm{EL}_{r_1,s_1,\dots,r_k,s_k}\left(z\right)=\sum_{\mathrm{op}(s_1+1)}\cdots\sum_{\mathrm{op}(s_k+1)}el_{(r_1-1)\oplus\mathrm{op}(s_1+1)\oplus\cdots\oplus(r_k-1)\oplus\mathrm{op}(s_k+1)}\left(z\right). \tag{C.1.14}$$

If, instead, the number of indices n is odd, $n=2\,k+1$ with $k\geq 1$, we have

$$\mathrm{EL}_{r_1,s_1,\dots,r_k,s_k,r_{k+1}}\left(z\right)=\sum_{\mathrm{op}(s_1+1)}\cdots\sum_{\mathrm{op}(s_k+1)}el_{(r_1-1)\oplus\mathrm{op}(s_1+1)\oplus\cdots\oplus(r_k-1)\oplus\mathrm{op}(s_k+1)\oplus r_{k+1}}\left(z\right). \tag{C.1.15}$$

Introducing the notation

$$1 \le j \le k: \quad w_j = \sum_{i=1}^{j} s_i, \tag{C.1.16}$$

we can write down the Taylor expansions for the dressed functions:

$$\mathrm{EL}_{r_1,s_1,\dots,r_k,s_k}(z) = \sum_{k_1 \ge \dots \ge k_{w_k+1} \ge 1} \left(\prod_{j=2}^{w_k+1} \frac{1}{k_j} \right) \left(\prod_{j=1}^{k-1} \frac{1}{k_{w_j+1}^{r_{j+1}}} \right) \frac{z^{k_1}}{k_1^{r_1} k_1!}, \tag{C.1.17}$$

$$\mathrm{EL}_{r_1,s_1,\dots,r_k,s_k,r_{k+1}}(z) = \sum_{k_1 \ge \dots \ge k_{w_k+1} \ge 1} \left(\prod_{j=2}^{w_k+1} \frac{1}{k_j} \right) \left(\prod_{j=1}^{k} \frac{1}{k_{w_j+1}^{r_{j+1}}} \right) \frac{z^{k_1}}{k_1^{r_1} k_1!}. \tag{C.1.18}$$

C.2 Multiple polyexponential integrals

The asymptotic behavior of polyexponential functions $\mathrm{EL}_n(z)$ at $z \to \pm\infty$ is determined by their relations with a set of functions called polyexponential integrals $\mathrm{ELi}_n(z)$, defined as

$$n \ge 2: \quad \mathrm{ELi}_n(z) = \int_{-\infty}^{z} \frac{\mathrm{ELi}_{n-1}(t)}{t} \mathrm{d}t, \quad \mathrm{ELi}_1(z) \equiv \mathrm{Ei}(z). \tag{C.2.19}$$

Using L'Hôpital's rule, the leading asymptotic behavior of $\mathrm{ELi}_n(z)$ at $z \to \pm\infty$ can be derived:

$$\mathrm{ELi}_n(z) = \frac{\mathrm{e}^z}{z^n} \left(1 + O\left(|z|^{-1}\right)\right). \tag{C.2.20}$$

Starting with the known relation for the exponential integral

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \mathrm{EL}_1(z) \tag{C.2.21}$$

and applying recursively the definition (C.2.19), we get the following general result for $n \ge 1$:

$$\mathrm{ELi}_n(z) = \mathrm{EL}_n(z) + \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!} \Gamma^{(n-k)}(1) \log(-z)^k. \tag{C.2.22}$$

The previously defined dressed multiple polyexponential functions have the following property:

$$\mathrm{EL}_{s_1,\dots,s_n}(0) = 0. \tag{C.2.23}$$

Thus, they are naturally described in terms of their Taylor expansions. The corresponding multiple polyexponential integrals tend to zero as $z \to -\infty$ and can be defined using definite integrals:

$$\begin{aligned} \mathrm{ELi}_{1,s_2,\dots,s_n}(z) &= -\int_{-\infty}^{z} \frac{\mathrm{e}^t}{t} \mathrm{ELi}_{s_2,\dots,s_n}(-t)\,\mathrm{d}t, \\ s_1 > 1: \quad \mathrm{ELi}_{s_1,\dots,s_n}(z) &= \int_{-\infty}^{z} \frac{1}{t} \mathrm{ELi}_{s_1-1,s_2,\dots,s_n}(t)\,\mathrm{d}t. \end{aligned} \tag{C.2.24}$$

The general relation between multiple polyexponential integrals and dressed multiple polyexponential functions is

$$\mathrm{ELi}_{s_1,\ldots,s_n}(z) = \mathrm{EL}_{s_1,\ldots,s_n}(z) + \sum_{k_1=1}^{s_1} \frac{(-1)^{k_1-1}}{(k_1-1)!\,(s_1-k_1)!}\,\mathrm{cLi}_{k_1,s_2,\ldots,s_n} \log(-z)^{s_1-k_1}$$

$$+\sum_{i=2}^{n-1} \sum_{s_i\geq k_1\geq\cdots\geq k_i\geq 1} \frac{(-1)^{k_1}}{(k_i-1)!\,(s_i-k_1)!} \prod_{j=1}^{i-1} \binom{s_j-1+k_j-k_{j+1}}{s_j-1} \mathrm{cLi}_{k_i,s_{i+1},\ldots,s_n} \log((-1)^i\, z)^{s_i-k_1}\times \tag{C.2.25}$$

$$\times\mathrm{ELi}_{s_1+k_1-k_2,\ldots,s_{i-1}+k_{i-1}-k_i}(z)$$

$$+\sum_{s_n\geq k_1\geq\cdots\geq k_n\geq 0} \frac{(-1)^{k_1+1}}{k_n!\,(s_n-k_1)!} \prod_{j=1}^{n-1} \binom{s_j-1+k_j-k_{j+1}}{s_j-1} \Gamma^{(k_n)}(1) \log((-1)^n\, z)^{s_n-k_1}\times$$

$$\times\mathrm{ELi}_{s_1+k_1-k_2,\ldots,s_{n-1}+k_{n-1}-k_n}(z),$$

where the constants $\mathrm{cLi}_{k_1,s_2,\ldots,s_n}$ are defined as

$$k_1\geq 1: \quad \mathrm{cLi}_{k_1,s_2,\ldots,s_n} = \int_0^\infty \mathrm{e}^{-t}\log(t)^{k_1-1}\,\mathrm{EL}_{s_2,\ldots,s_n}(t)\,\frac{\mathrm{d}t}{t}. \tag{C.2.26}$$

As for the asymptotic behavior, one can prove that all multiple polyexponential integrals $\mathrm{ELi}_{s_1,\ldots,s_n}(z)$ behave at most like $1/z$ when $z\to-\infty$. The starting point is the asymptotic series for the exponential integral:

$$\mathrm{ELi}_1(z) = \frac{\mathrm{e}^z}{z}\left(\sum_{k=0}^{n-1}\frac{k!}{z^k} + R_n(z)\right), \tag{C.2.27}$$

where $R_n(z)$ is a remainder, which is explicitly given by

$$R_n(z) = n!\,z\,\mathrm{e}^{-z}\int_{-\infty}^{z}\frac{\mathrm{e}^t}{t^{n+1}}\,\mathrm{d}t. \tag{C.2.28}$$

Introducing the *multiple harmonic numbers*, defined as

$$H_m^{(s_1,\ldots,s_n)} \equiv \sum_{j_1=1}^{m}\frac{1}{j_1^{s_1}}\sum_{j_2=1}^{j_1-1}\frac{1}{j_2^{s_2}}\cdots\sum_{j_n=1}^{j_{n-1}-1}\frac{1}{j_n^{s_n}}, \quad m,s_1,\ldots,s_n\in\mathbb{Z}_{\geq 1}, \tag{C.2.29}$$

the asymptotic expansions for multiple polyexponential integrals are given by

$$\begin{aligned}
&\mathrm{ELi}_{s_1,\ldots,s_{2n+1}}(z) = \\
&\mathrm{e}^z \sum_{j=\sum_{m=0}^{n} s_{2m+1}}^{N} \frac{(-1)^{\sum_{m=1}^{n} s_{2m}}}{z^j}\,\Gamma(j)\,H_{j-1}^{(\overbrace{1,\ldots,1}^{s_1-1},s_2+1,\ldots,\overbrace{1,\ldots,1}^{s_{2n+1}-1})} + \mathcal{O}\left(\frac{\mathrm{e}^z}{z^{N+1}}\right), \\
&\mathrm{ELi}_{s_1,\ldots,s_{2n}}(z) = \\
&\sum_{j=\sum_{m=1}^{n} s_{2m}}^{N} \frac{(-1)^{j-1+\sum_{m=1}^{n} s_{2m-1}}}{j^{s_1}\,z^j}\,\Gamma(j)\,H_{j-1}^{(\overbrace{1,\ldots,1}^{s_2-1},s_3+1,\ldots,\overbrace{1,\ldots,1}^{s_{2n}-1})} + \mathcal{O}\left(\frac{1}{z^{N+1}}\right).
\end{aligned} \tag{C.2.30}$$

Appendix D

Results for β, ξ, and φ expansions

The small-frequency approach described in sections 2.4.1 and 2.4.2 provides us with the following perturbative results for $\beta_{\ell,s}$, $\xi_{\ell,s}$, and $\varphi_{\ell,s}$ with $\ell = s = 0,1,2$. When $\beta_{\ell,s}$ and $\xi_{\ell,s}$ are determined up to order t^K, the corresponding expansions of $\varphi_{\ell,s}$ can be computed up to order $t^{K-2\,\ell-2}$. In case $\ell = s = 0$, the wave functions in the near-spatial infinity region were computed up to order t^{16}, which fixes the first 14 orders of $\beta_{0,0}$ and $\xi_{0,0}$ and 12 orders of $\varphi_{0,0}$:

$$\begin{aligned}\beta_{0,0} = &\frac{7}{24}\,t^2 - \frac{9449}{120\,960}\,t^4 + \frac{102\,270\,817}{2\,133\,734\,400}\,t^6 - \frac{4\,988\,909\,608\,861}{150\,556\,299\,264\,000}\,t^8 \\ &+ \frac{72\,237\,319\,625\,071\,987}{2\,655\,813\,119\,016\,960\,000}\,t^{10} - \frac{2\,008\,359\,560\,158\,182\,591\,511}{86\,646\,394\,825\,172\,582\,400\,000}\,t^{12} \\ &+ \frac{202\,956\,264\,764\,788\,667\,222\,561\,313\,859}{9\,656\,699\,112\,995\,968\,225\,640\,448\,000\,000}\,t^{14} + \mathcal{O}\left(t^{16}\right),\end{aligned} \tag{D.1.1}$$

$$\begin{aligned}\xi_{0,0}(t) = &1 - \frac{5}{6}\,t + \frac{25}{72}\,t^2 - \frac{23}{315}\,t^3 + \frac{41}{72\,576}\,t^4 + \frac{225\,487}{17\,781\,120}\,t^5 - \frac{51\,838\,301}{6\,401\,203\,200}\,t^6 \\ &+ \frac{9\,481\,089\,257}{1\,568\,294\,784\,000}\,t^7 - \frac{284\,742\,073\,739}{90\,333\,779\,558\,400}\,t^8 + \frac{722\,761\,773\,679\,487}{304\,311\,919\,887\,360\,000}\,t^9 \\ &- \frac{108\,778\,841\,322\,632\,893}{87\,641\,832\,927\,559\,680\,000}\,t^{10} + \frac{1\,711\,007\,301\,901\,126\,757\,149}{1\,754\,589\,495\,209\,744\,793\,600\,000}\,t^{11} \\ &- \frac{10\,726\,170\,068\,189\,907\,710\,593}{21\,055\,073\,942\,516\,937\,523\,200\,000}\,t^{12} \\ &+ \frac{4\,383\,394\,423\,080\,102\,339\,340\,164\,317}{10\,622\,369\,024\,295\,565\,048\,204\,492\,800\,000}\,t^{13} \\ &- \frac{68\,395\,977\,167\,815\,587\,016\,740\,406\,979}{318\,671\,070\,728\,866\,951\,446\,134\,784\,000\,000}\,t^{14} + \mathcal{O}\left(t^{15}\right),\end{aligned} \tag{D.1.2}$$

$$\varphi_{0,0} = \log\left(\frac{7}{9}\right) - \frac{8587}{70\,560}\,t^2 + \frac{59\,423\,233}{995\,742\,720}\,t^4 - \frac{3\,034\,619\,927\,027}{131\,736\,761\,856\,000}\,t^6 + \frac{2\,636\,632\,572\,686\,279\,887}{136\,331\,740\,109\,537\,280\,000}\,t^8 - \frac{6\,348\,255\,806\,061\,211\,753\,575\,743}{429\,874\,426\,326\,387\,474\,432\,000\,000}\,t^{10} + \frac{120\,127\,293\,342\,168\,835\,533\,078\,304\,741}{9\,463\,565\,130\,736\,048\,861\,127\,639\,040\,000}\,t^{12} + \mathcal{O}\left(t^{14}\right). \tag{D.1.3}$$

For $\ell = s = 1$, we computed the near-infinity local wave functions up to order t^{14}, which gives

$$\beta_{1,1} = \frac{47}{240}\,t^2 - \frac{43\,908\,007}{1\,137\,024\,000}\,t^4 + \frac{1\,897\,955\,762\,232\,049}{126\,588\,975\,206\,400\,000}\,t^6 - \frac{916\,976\,100\,036\,495\,015\,111\,773}{124\,023\,735\,704\,345\,444\,352\,000\,000}\,t^8 + \frac{5\,706\,721\,543\,769\,515\,470\,350\,083\,430\,384\,773}{1\,410\,389\,488\,815\,788\,497\,680\,728\,064\,000\,000\,000}\,t^{10} - \frac{106\,995\,634\,360\,703\,511\,437\,460\,300\,615\,557\,539\,248\,229\,319}{44\,908\,789\,408\,918\,140\,501\,620\,712\,312\,039\,014\,400\,000\,000\,000}\,t^{12} + \mathcal{O}\left(t^{14}\right), \tag{D.1.4}$$

$$\xi_{1,1}(t) = -1 + \frac{5}{4}\,t - \frac{25}{32}\,t^2 + \frac{1471}{4512}\,t^3 - \frac{29\,555}{288\,768}\,t^4 + \frac{26\,305\,804\,327}{1\,004\,674\,406\,400}\,t^5 - \frac{38\,025\,119\,711}{6\,429\,916\,200\,960}\,t^6 + \frac{152\,296\,572\,400\,211\,831}{111\,854\,018\,492\,375\,040\,000}\,t^7 - \frac{1\,089\,548\,738\,109\,409\,027}{2\,863\,462\,873\,404\,801\,024\,000}\,t^8 + \frac{2\,259\,394\,074\,262\,863\,197\,636\,203}{15\,655\,338\,981\,194\,233\,518\,489\,600\,000}\,t^9 - \frac{1\,293\,517\,008\,380\,421\,728\,073\,421}{20\,038\,833\,895\,928\,618\,903\,666\,688\,000}\,t^{10} + \frac{1\,549\,762\,537\,865\,582\,971\,981\,233\,948\,038\,333}{48\,554\,031\,908\,479\,118\,826\,650\,311\,065\,600\,000\,000}\,t^{11} - \frac{6\,224\,927\,205\,898\,953\,592\,055\,383\,850\,794\,531}{395\,494\,659\,909\,066\,276\,987\,987\,988\,316\,160\,000\,000}\,t^{12} + \mathcal{O}\left(t^{13}\right), \tag{D.1.5}$$

$$\varphi_{1,1} = \frac{895\,597}{7\,068\,800}\,t^2 - \frac{455\,691\,732\,736\,543}{4\,407\,171\,729\,408\,000}\,t^4 + \frac{74\,327\,495\,711\,146\,205\,449\,777}{1\,226\,665\,736\,133\,046\,272\,000\,000}\,t^6 - \frac{29\,832\,726\,638\,753\,503\,996\,423\,316\,486\,442\,073}{793\,193\,404\,821\,187\,035\,447\,794\,073\,600\,000\,000}\,t^8 + \mathcal{O}\left(t^{10}\right). \tag{D.1.6}$$

For $\ell = s = 2$, we also computed the near-infinity local wave functions up to order t^{14} and got the following results for $\beta_{2,2}$, $\xi_{2,2}$, and $\varphi_{2,2}$:

$$\begin{aligned}\beta_{2,2} =& \frac{107}{840}\,t^2 - \frac{1\,695\,233}{148\,176\,000}\,t^4 + \frac{76\,720\,109\,901\,233}{30\,764\,716\,012\,800\,000}\,t^6 \\ &- \frac{71\,638\,806\,585\,865\,707\,261\,481}{99\,644\,321\,084\,605\,000\,704\,000\,000}\,t^8 \\ &+ \frac{270\,360\,664\,939\,833\,821\,554\,899\,493\,653\,643}{1\,152\,641\,264\,228\,149\,083\,523\,559\,424\,000\,000\,000}\,t^{10} \\ &- \frac{25\,911\,378\,819\,560\,727\,799\,984\,792\,720\,318\,253\,742\,297\,427}{317\,331\,168\,450\,503\,888\,940\,177\,332\,763\,456\,307\,200\,000\,000\,000}\,t^{12} + \mathcal{O}\left(t^{14}\right),\end{aligned} \tag{D.1.7}$$

$$\begin{aligned}\xi_{2,2}(t) =& 1 - \frac{5}{3}\,t + \frac{25}{18}\,t^2 - \frac{73}{96}\,t^3 + \frac{785}{2592}\,t^4 - \frac{1\,007\,354\,009}{10\,871\,884\,800}\,t^5 + \frac{896\,782\,589}{39\,138\,785\,280}\,t^6 \\ &- \frac{6\,050\,546\,248\,023\,481}{1\,231\,227\,907\,338\,240\,000}\,t^7 + \frac{39\,569\,841\,898\,687}{41\,040\,930\,244\,608\,000}\,t^8 \\ &- \frac{49\,266\,134\,378\,785\,551\,042\,377}{306\,757\,182\,671\,647\,123\,046\,400\,000}\,t^9 \frac{6\,918\,422\,754\,413\,573\,300\,251}{368\,108\,619\,205\,976\,547\,655\,680\,000}\,t^{10} \\ &- \frac{8\,229\,146\,383\,510\,495\,333\,773\,316\,804\,837}{3\,548\,430\,888\,956\,507\,708\,078\,122\,598\,400\,000\,000}\,t^{11} \\ &+ \frac{1\,543\,339\,979\,694\,704\,523\,189\,885\,578\,017}{2\,129\,058\,533\,373\,904\,624\,846\,873\,559\,040\,000\,000}\,t^{12} + \mathcal{O}\left(t^{13}\right),\end{aligned} \tag{D.1.8}$$

$$\varphi_{2,2} = \frac{125}{1568}\,t^2 - \frac{53\,950\,959\,337}{2\,280\,051\,680\,256}\,t^4 + \frac{29\,227\,746\,558\,029\,477\,261}{3\,905\,473\,390\,495\,507\,353\,600}\,t^6 + \mathcal{O}\left(t^8\right). \tag{D.1.9}$$

It is possible to determine the generic expressions for $\beta_{\ell,s,k}$, $\ell \geq k$ and $\varphi_{\ell,s,k}$, $\ell > k$ using the instanton part of the NS free energy F_{inst} with $N_f = 3$. We start by inverting the Matone relation to determine the modulus a as a function of ω, ℓ, s, and the instanton parameter Λ:

$$a^2 = 2\,\omega^2 - \left(\ell + \frac{1}{2}\right)^2 + \Lambda\,\frac{\partial F_{\text{inst}}}{\partial \Lambda}. \tag{D.1.10}$$

In general, when getting the small-frequency expansion of a, infinitely many orders in Λ can contribute to the same order in t or ω. However, when $\ell \geq k \geq 1$, the first $2\,k$ instantons are enough to compute the t-expansions of a and $\beta_{\ell,s}$ up to order t^{2k}. The perturbative expansion of $\varphi_{\ell,s}$ can be determined by taking the a-derivative of F_{inst} and substituting the instanton expansion of a obtained earlier by inverting the Matone relation. Unless $\ell > k$, infinitely many orders in Λ contribute to the same order t^k of $\varphi_{\ell,s}$. For $\ell > k$, we use the first $2\,k$ instantons to obtain the generic expressions of $\varphi_{\ell,s,k}$.

Below are the results for $\beta_{\ell,s,1}$, $\ell \geq 1$ and $\beta_{\ell,s,2}$, $\ell \geq 2$. To simplify the expressions, we introduce the notation $L \equiv \ell\,(\ell+1)$:

$$\ell \geq 1: \quad \beta_{\ell,s,1} = \frac{1}{2\,(2\,\ell-1)_5}\left(15\,L^2 - 11\,L + 6\,s^2\,(L-1) + 3\,s^4\right), \tag{D.1.11}$$

$$\ell \geq 2: \quad \beta_{\ell,s,2} =$$

$$- \frac{2}{((2\,\ell-1)_5)^2\,(2\,\ell-3)_9}\Big\{9\,s^8\,(L-2)\,\big(560\,L^3-840\,L^2+63\,L+45\big)$$
$$+\,4\,s^2\,L\,\big(5040\,L^6-39\,480\,L^5+106\,015\,L^4-124\,514\,L^3+58\,737\,L^2-3528\,L-2970\big)$$
$$+\,2\,s^4\,\big(8400\,L^6-65\,800\,L^5+171\,689\,L^4-152\,670\,L^3+40\,113\,L^2+4752\,L-1620\big)$$
$$+\,L^2\,(L-2)\,\big(18\,480\,L^5-105\,000\,L^4+155\,295\,L^3-82\,625\,L^2+8733\,L+3240\big)$$
$$+\,4\,s^6\,\big(2800\,L^5-19\,880\,L^4+32\,907\,L^3-16\,731\,L^2-81\,L+810\big)\Big\}, \tag{D.1.12}$$

where $(q)_n$ is the rising Pochhammer symbol.

Similarly, we have for $\varphi_{\ell,s,1}$ and $\varphi_{\ell,s,2}$:

$$\ell > 1: \quad \varphi_{\ell,s,1} = \frac{(2\ell+1)^3}{2\,((2\,\ell-1)_5)^2}\left[L^2+4\,s^2\,\big(14\,L^2-24\,L+9\big)+3\,s^4\,(8\,L-3)\right], \tag{D.1.13}$$

$$\ell > 2: \quad \varphi_{\ell,s,2} =$$

$$- \frac{4(2\,\ell+1)^3}{3\,((2\,\ell-3)_9)^2\,((2\,\ell-1)_5)^2}\Big\{3\,(L-2)^2\,L^2\,\big(622\,848\,L^7-6\,212\,544\,L^6+21\,712\,272\,L^5$$
$$-\,32\,554\,372\,L^4+22\,795\,266\,L^3-5\,708\,997\,L^2-1\,088\,640\,L+583\,200\big)$$
$$+\,4\,L\,s^2\,\big(1\,061\,376\,L^9-15\,044\,160\,L^8+86\,481\,808\,L^7-260\,933\,804\,L^6+444\,628\,113\,L^5$$
$$-\,420\,000\,264\,L^4+187\,434\,000\,L^3-5\,375\,808\,L^2-25\,048\,440\,L+6\,415\,200\big)$$
$$+\,4\,s^4\,\big(1\,018\,752\,L^9-14\,267\,744\,L^8+79\,516\,264\,L^7-221\,705\,538\,L^6+322\,081\,092\,L^5$$
$$-\,232\,063\,569\,L^4+61\,460\,370\,L^3+15\,646\,770\,L^2-12\,830\,400\,L+2\,187\,000\big)$$
$$+\,4\,s^6\,\big(735\,232\,L^8-9\,344\,576\,L^7+43\,334\,736\,L^6-93\,552\,300\,L^5+98\,731\,791\,L^4$$
$$-\,43\,634\,700\,L^3-1\,363\,716\,L^2+6\,765\,120\,L-1\,749\,600\big)+135\,(L-2)^2\,s^8\,\big(8448\,L^5$$
$$-\,41\,152\,L^4+49\,488\,L^3-5076\,L^2-5130\,L+2025\big)\Big\}. \tag{D.1.14}$$

As mentioned earlier, the distinction between the cases in which a finite or an infinite number of instantons contribute to the same order in t arises from the presence of the poles at integer values of ℓ in the NS function. We remark that the resummation of infinitely many instantons in the t-expansion differs from the apparent resummation in the pure instanton approach. There, this resummation may be avoided by analytically continuing ℓ to be a generic complex number, and taking the limit $\ell \to \mathbb{Z}_{\geq 0}$ only in the final step of the computation. This procedure is discussed in [185], where the authors show that the *fixed-ℓ prescription* and the *generic-ℓ prescription* lead to the same answer for the computation of the scattering phase-shift in Kerr spacetime. The same reasoning also applies when computing the coefficients of the QNM frequencies of Schwarzschild-de Sitter spacetime in 4 dimensions in the Taylor series expansion around $R_h = 0$ (R_h being the radius of the event horizon) as discussed in footnote 6.

Appendix E

Gelfand-Yaglom theorem

E.1 Gelfand-Yaglom theorem for regular differential operators

Let us introduce the setting in which the standard Gelfand-Yaglom theorem applies. Let

$$\mathcal{D} = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z) \tag{E.1.1}$$

be a second-order differential operator defined on the interval $z = [0, 1]$. Let us consider the eigenvalue problem

$$\mathcal{D}\psi_n = \lambda_n \psi_n, \tag{E.1.2}$$

with ψ_n satisfying the Dirichlet boundary conditions

$$\psi_n(0) = \psi_n(1) = 0, \tag{E.1.3}$$

where $\{\lambda_n\}_n$ is the set of eigenvalues of $\mathcal{D}$, which is required to be discrete, non-degenerate, and bounded from below. Suppose we can solve the associated problem

$$\mathcal{D}\, u_\lambda = \lambda\, u_\lambda, \tag{E.1.4}$$

with u_λ satisfying the boundary conditions

$$u_\lambda(0) = 0, \quad u'_\lambda(0) = 1. \tag{E.1.5}$$

We call this $u_\lambda(z)$ the *normalized solution* of $(\mathcal{D} - \lambda)\, u = 0$ at $z = 0$.

Then, one has that $u_\lambda(1)$ is equal to zero if and only if λ is an eigenvalue of the operator $\mathcal{D}$. Indeed, $u_\lambda(1) = 0$ if and only if u_λ satisfies both the Dirichlet boundary conditions at $z = 0$ and at $z = 1$, but then u_λ coincides with one of the eigenfunctions of $\mathcal{D}$, that is $\lambda = \lambda_n$ for some n.

Let $\tilde{\mathcal{D}}$ be some reference differential operator and $\tilde{u}_\lambda$ the corresponding eigenfunction satisfying (E.1.5). $\tilde{\mathcal{D}}$ is obtained by $\mathcal{D}$ by considering a deformation of the potential $V(z)$. It holds

$$\frac{\det(\mathcal{D} - \lambda)}{\det\left(\tilde{\mathcal{D}} - \lambda\right)} = \frac{u_\lambda(1)}{\tilde{u}_\lambda(1)}. \tag{E.1.6}$$

Indeed, seen as functions of λ, both the left-hand side and the right-hand side have zeros in the eigenvalues of $\mathcal{D}$ and poles in the eigenvalues of $\tilde{\mathcal{D}}$. Therefore, the two must coincide up to a constant. Moreover,

$$\lim_{\lambda\to\infty} \frac{\det(\mathcal{D}-\lambda)}{\det\left(\tilde{\mathcal{D}}-\lambda\right)} = 1, \tag{E.1.7}$$

assuming the deformation of the potential $V(z)$ to be bounded and not modifying the asymptotic of the spectrum. Hence, we can conclude that

$$\frac{\det(\mathcal{D})}{\det\left(\tilde{\mathcal{D}}\right)} = \left.\frac{\det(\mathcal{D}-\lambda)}{\det\left(\tilde{\mathcal{D}}-\lambda\right)}\right|_{\lambda=0} = \frac{u_{\lambda=0}(1)}{\tilde{u}_{\lambda=0}(1)}. \tag{E.1.8}$$

Remark E.1.1. *As a consequence of the theorem, we conclude that the ratio of determinants of two differential operators only depends on the normalized solutions of the corresponding differential equation.*

We also comment on the fact that it is not restrictive to consider differential operators in the normal form (E.1.1). Indeed, let us consider a second-order differential equation of the form

$$\left[a(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + b(y)\frac{\mathrm{d}}{\mathrm{d}y} + c(y)\right]\phi(y) = 0, \tag{E.1.9}$$

with the properties that $b(y)$ is differentiable and $a(y)$ is twice differentiable. We can first redefine the variable as

$$z = \frac{\int_0^y \mathrm{d}\bar{y}\frac{1}{\sqrt{a(\bar{y})}}}{C}, \quad \text{with} \quad C = \int_0^1 \mathrm{d}\bar{y}\frac{1}{\sqrt{a(\bar{y})}}, \tag{E.1.10}$$

so that the interval $y = [0,1]$ is mapped onto the interval $z = [0,1]$ and the differential equation becomes of the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + C\beta(z)\frac{\mathrm{d}}{\mathrm{d}z} + C^2\gamma(z)\right]\phi(z) = 0, \tag{E.1.11}$$

where

$$\beta(z) = \frac{b(y) - \frac{1}{2}a'(y)}{\sqrt{a(y)}}, \quad \gamma(z) = c(y). \tag{E.1.12}$$

Then, redefining the wave function ϕ as

$$\phi(z) = \exp(-\frac{1}{2}\int \mathrm{d}z\, C\beta(z))\psi(z), \tag{E.1.13}$$

the differential equation becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right]\psi(z) = 0, \tag{E.1.14}$$

with

$$V(z) = C^2\gamma(z) - \frac{C^2}{4}\beta(z)^2 - \frac{C}{2}\beta'(z). \tag{E.1.15}$$

E.1.1 Gelfand-Yaglom version for regular singular points

Suppose now that the potential $V(z)$ has regular singular points at $z = 0$ and $z = 1$. We denote with $\frac{1}{2} \pm a_0$ the roots of the indicial equation at $z = 0$. Then, supposing not to be in a log case, there exists a fundamental system of solutions of the differential equation $\mathcal{D}\psi(z) = 0$ around $z = 0$ given by

$$\begin{aligned} \psi_1^{(0)} &= z^{\frac{1}{2}+a_0}\left[1 + \mathcal{O}(z)\right], \\ \psi_2^{(0)} &= z^{\frac{1}{2}-a_0}\left[1 + \mathcal{O}(z)\right], \end{aligned} \tag{E.1.16}$$

and the Wronskian between the two solutions is (constant in z) equal to $2a_0$.

Let us suppose $\mathrm{Re}(a_0) > 0$. In order to apply the Gelfand-Yaglom theorem in the regular singular case, the standard vanishing Dirichlet boundary condition at $z = 0$ is reformulated by asking the function ψ to satisfy

$$\lim_{z\to 0} \left(z^{\frac{1}{2}-a_0}\right)^{-1} \psi(z) = 0. \tag{E.1.17}$$

Analogous formulae hold at $z = 1$.

Suppose that the points $z = 0$ and $z = 1$ are regular singular points of both the equations $\mathcal{D}\psi(z) = 0$ and $\tilde{\mathcal{D}}\tilde{\psi}(z) = 0$ with equal indices $a_0 = \tilde{a}_0$ and $a_1 = \tilde{a}_1$. Suppose, moreover, $\mathrm{Re}(a_0) > 0$ and $\mathrm{Re}(a_1) > 0$. Then,

$$\frac{\det\left(\mathcal{D}\right)}{\det\left(\tilde{\mathcal{D}}\right)} = \frac{\mathcal{C}_{12}}{\tilde{\mathcal{C}}_{12}} \tag{E.1.18}$$

where $\mathcal{C}_{12}$ is the connection coefficient relating the local solutions around the two regular singular points:

$$\psi_1^{(0)}(z) = \mathcal{C}_{11}\psi_1^{(1)}(z) + \mathcal{C}_{12}\psi_2^{(1)}(z), \tag{E.1.19}$$

and the same for $\tilde{\mathcal{C}}_{11}$ and $\tilde{\mathcal{C}}_{12}$.

Let us prove (E.1.18). The strategy is to consider the associated problem as in the standard Gelfand-Yaglom theorem, but taking the normalized solution at a point close to $z = 0$ in terms of the corresponding local solutions (E.1.16). Then, using the connection matrix, it is possible to analytically continue the solution close to the point $z = 1$ and evaluate it there. Removing the cut-off, we get (E.1.18).

The normalized solution at $z = 0$ satisfying $u_{[\delta]}(\delta) = 0$ and $u'_{[\delta]}(\delta) = 1$ is given by

$$u_{[\delta]}(z) = \frac{\psi_i^{(0)}(\delta)}{W(\psi_1^{(0)}, \psi_2^{(0)})(\delta)}\epsilon_{ij}\psi_j^{(0)}(z) = \frac{\psi_i^{(0)}(\delta)}{2a_0}\epsilon_{ij}\psi_j^{(0)}(z), \tag{E.1.20}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{E.1.21}$$

Let us now analytically continue this solution to the neighborhood of $z = 1$, as

$$\psi_i^{(0)}(z) = \mathcal{C}_{ij}\psi_j^{(1)}(z), \tag{E.1.22}$$

and evaluate

$$u_{[\delta]}(1+\delta') = \frac{\psi_i^{(0)}(\delta)}{2a_0}\epsilon_{ij}\mathcal{C}_{jk}\psi_k^{(1)}(1+\delta')\,. \tag{E.1.23}$$

By using the expansion of the local solutions around $z = 0$ and $z = 1$, and denoting

$$\begin{aligned} \rho_1^{(0)} &= \frac{1}{2} + a_0, \quad \rho_2^{(0)} = \frac{1}{2} - a_0, \\ \rho_1^{(1)} &= \frac{1}{2} + a_1, \quad \rho_2^{(1)} = \frac{1}{2} - a_1, \end{aligned} \tag{E.1.24}$$

one finds

$$u_{[\delta]}(1+\delta') = \frac{1}{2a_0}\left(\delta^{\rho_i^{(0)}}\right)\epsilon_{ij}\mathcal{C}_{jk}\left(\delta'^{\rho_k^{(1)}}\right)\left[1+\mathcal{O}\left(\delta,\delta'\right)\right]. \tag{E.1.25}$$

Consider now the ratio

$$\frac{u_{[\delta]}(1+\delta')}{\tilde{u}_{[\delta]}(1+\delta')}, \tag{E.1.26}$$

where the denominator is given by the above procedure for the reference operator $\tilde{\mathcal{D}}$ and with the same cut-off assignment. As we remove the cut-off, in the limit $\delta, \delta' \to 0$ and using the assumptions $a_0, a_1 > 0$, the leading order term is given by

$$\frac{\det(\mathcal{D})}{\det(\tilde{\mathcal{D}})} = \lim_{\delta,\delta'\to 0^+}\frac{u_{[\delta]}(1+\delta')}{\tilde{u}_{[\delta]}(1+\delta')} = \lim_{\delta,\delta'\to 0^+}\frac{-\delta^{\frac{1}{2}-a_0}\mathcal{C}_{12}(\delta')^{\frac{1}{2}-a_1}}{-\delta^{\frac{1}{2}-a_0}\tilde{\mathcal{C}}_{12}(\delta')^{\frac{1}{2}-a_1}} = \frac{\mathcal{C}_{12}}{\tilde{\mathcal{C}}_{12}}. \tag{E.1.27}$$

Similar results were obtained in [144].

Appendix F

Connection problems for Hypergeometric and Heun equations

In this appendix, we recall the connection coefficients that analytically continue the local solutions around the singularity at $z = 0$ in the region around the singularity at $z = 1$ for the Heun and Hypergeometric differential operators.

F.1 Hypergeometric connection formulae

For the Hypergeometric equation

$$\left[z(1-z)\frac{d^2}{dz^2} + (c - (a+b+1)z)\frac{d}{dz} - ab\right] w(z) = 0, \tag{F.1.1}$$

a basis of local solutions around $z = 0$ is given by

$$\begin{aligned} w_-^{(0)}(z) &= {}_2F_1(a, b; c; z), \\ w_+^{(0)}(z) &= z^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z), \end{aligned} \tag{F.1.2}$$

and a basis of local solutions around $z = 1$ is given by

$$\begin{aligned} w_-^{(1)}(z) &= {}_2F_1(a, b; a+b+1-c; 1-z), \\ w_+^{(1)}(z) &= (1-z)^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1; 1-z). \end{aligned} \tag{F.1.3}$$

The connection formulae between these solutions are

$$\begin{aligned} w_-^{(0)}(z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} w_-^{(1)}(z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} w_+^{(1)}(z), \\ w_+^{(0)}(z) &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} w_-^{(1)}(z) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} w_+^{(1)}(z). \end{aligned} \tag{F.1.4}$$

Let us consider the normal form of the equation but supposing the index of the singularity at $z=\infty$ to satisfy $a_\infty^2 = 1/4$:

$$\psi''(z) + \left[\frac{\frac{1}{4}-a_0^2}{z^2} + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} - \frac{\frac{1}{2}-a_0^2-a_1^2}{z(z-1)}\right]\psi(z) = 0, \tag{F.1.5}$$

where the dictionary with the a, b, c parameters is

$$a_0 = \frac{1-c}{2}, \quad a_1 = \frac{c-a-b}{2}, \tag{F.1.6}$$

and with inverse

$$a = 1 - a_0 - a_1, \quad b = -a_0 - a_1, \quad c = 1 - 2a_0. \tag{F.1.7}$$

A basis of local solutions around $z = 0$ is given by

$$\begin{aligned}
\psi_-^{(0)}(z) &= (z-1)^{\frac{1}{2}-a_1} z^{\frac{1}{2}-a_0} w_-^{(0)}(z) = \\
&= (z-1)^{\frac{1}{2}+a_1} z^{\frac{1}{2}-a_0} \, {}_2F_1\left(-a_0+a_1, -a_0+a_1+1; 1-2a_0; z\right), \\
\psi_+^{(0)}(z) &= (z-1)^{\frac{1}{2}-a_1} z^{\frac{1}{2}-a_0} w_+^{(0)}(z) = \\
&= (z-1)^{\frac{1}{2}+a_1} z^{\frac{1}{2}+a_0} \, {}_2F_1\left(a_0+a_1, a_0+a_1+1; 1+2a_0; z\right),
\end{aligned} \tag{F.1.8}$$

whose Wronskian is equal to $2a_0$, and a basis of local solutions around $z = 1$ is given by

$$\begin{aligned}
\psi_-^{(1)}(z) &= (z-1)^{\frac{1}{2}-a_1} z^{\frac{1}{2}-a_0} w_-^{(1)}(z) = \\
&= (z-1)^{\frac{1}{2}-a_1} z^{\frac{1}{2}+a_0} \, {}_2F_1\left(a_0-a_1, 1+a_0-a_1; 1-2a_1; 1-z\right), \\
\psi_+^{(1)}(z) &= (z-1)^{\frac{1}{2}-a_1} z^{\frac{1}{2}-a_0} w_+^{(1)}(z) = \\
&= (z-1)^{\frac{1}{2}+a_1} z^{\frac{1}{2}+a_0} \, {}_2F_1\left(a_0+a_1, 1+a_0+a_1; 1+2a_1; 1-z\right),
\end{aligned} \tag{F.1.9}$$

whose Wronskian is equal to $2a_1$. The corresponding connection formulae read

$$\begin{aligned}
\psi_-^{(0)}(z) &= \frac{\Gamma(1-2a_0)\Gamma(2a_1)}{\Gamma(-a_0+a_1)\Gamma(1-a_0+a_1)}\psi_-^{(1)}(z) + \frac{\Gamma(1-2a_0)\Gamma(-2a_1)}{\Gamma(1-a_0-a_1)\Gamma(-a_0-a_1)}\psi_+^{(1)}(z), \\
\psi_+^{(0)}(z) &= \frac{\Gamma(1+2a_0)\Gamma(2a_1)}{\Gamma(a_0+a_1)\Gamma(1+a_0+a_1)}\psi_-^{(1)}(z) + \frac{\Gamma(1+2a_0)\Gamma(-2a_1)}{\Gamma(1+a_0-a_1)\Gamma(a_0-a_1)}\psi_+^{(1)}(z).
\end{aligned} \tag{F.1.10}$$

F.2 Heun connection formula for $|t| < 1$

The analogous problem was solved for the Heun equation in [4], where connection formulae between semiclassical Liouville conformal blocks were studied. We are interested in the regime in which $|t| < 1$, but, in the next section, we also show the analogous formulae for the regime $|t| > 1$.

The conformal block for small t around $z=0$ (see (1.1.16) and (1.1.19)) reads

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\alpha^{\alpha_t}\alpha_{0\theta}\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\frac{z}{t}\right). \tag{F.2.11}$$

The conformal block for small t around $z=1$ reads

$$t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_t-\Delta_0}(1-t)^{\Delta_\infty+\Delta_0+\Delta_{2,1}-\Delta_t-\Delta_1}(z-t)^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_0 \\ \alpha_t\end{matrix}\alpha^{\alpha_\infty}\alpha_{1\theta}\begin{matrix}\alpha_{2,1} \\ \alpha_1\end{matrix};t,\frac{z-1}{z-t}\right). \tag{F.2.12}$$

In the semiclassical limit, these read

$$\begin{aligned}&\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}a^{a_t}a_{0\theta}\begin{matrix}a_{2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right),\\ &(t(1-t))^{-\frac{1}{2}}(z-t)\mathcal{F}\left(\begin{matrix}a_0 \\ a_t\end{matrix}a^{a_\infty}a_{1\theta}\begin{matrix}a_{2,1} \\ a_1\end{matrix};t,\frac{z-1}{z-t}\right).\end{aligned} \tag{F.2.13}$$

The connection formula between the two semiclassical blocks, written in terms of the connection matrices of the Hypergeometric functions

$$\mathcal{M}_{\theta\theta'}(a_1,a_2;a_3)=\frac{\Gamma(-2\theta'a_2)\Gamma(1+2\theta a_1)}{\Gamma\left(\frac{1}{2}+\theta a_1-\theta'a_2+a_3\right)\Gamma\left(\frac{1}{2}+\theta a_1-\theta'a_2-a_3\right)},\quad\text{where}\quad\theta,\theta'=\pm \tag{F.2.14}$$

reads

$$\begin{aligned}\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}a^{a_t}a_{0\theta}\begin{matrix}a_{2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right)=&\sum_{\theta',\theta''=\pm}\mathcal{M}_{\theta\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')\theta''}(a,a_1;a_\infty)\exp\left(-\frac{\theta'}{2}\partial_aF(t)\right)\\ &\times t^{\theta'a}(t(1-t))^{-\frac{1}{2}}(z-t)\mathcal{F}\left(\begin{matrix}a_0 \\ a_t\end{matrix}a^{a_\infty}a_{1\theta''}\begin{matrix}a_{2,1} \\ a_1\end{matrix};t,\frac{z-1}{z-t}\right),\end{aligned} \tag{F.2.15}$$

$F(t)$ being the classical 4-point conformal block (see Appendix A)

$$F(t)=F\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}a\begin{matrix}a_t \\ a_0\end{matrix};t\right). \tag{F.2.16}$$

We also need to relate the expansions of these semiclassical blocks to the local solutions of the Heun equation in normal form

$$\begin{aligned}&\psi_\theta^{(0)}(z)\sim z^{\frac{1}{2}+\theta a_0}\left[1+\mathcal{O}(z)\right],\\ &\psi_\theta^{(1)}(z)\sim(z-1)^{\frac{1}{2}+\theta a_1}\left[1+\mathcal{O}(z-1)\right].\end{aligned} \tag{F.2.17}$$

The semiclassical blocks' expansions read

$$\begin{aligned}&\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}a^{a_t}a_{0\theta}\begin{matrix}a_{2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right)\sim t^{-\theta a_0}\exp\left(-\frac{\theta}{2}\partial_{a_0}F(t)\right)z^{\frac{1}{2}+\theta a_0}\left[1+\mathcal{O}\left(t,\frac{z}{t}\right)\right],\\ &\mathcal{F}\left(\begin{matrix}a_0 \\ a_t\end{matrix}a^{a_\infty}a_{1\theta''}\begin{matrix}a_{2,1} \\ a_1\end{matrix};t,\frac{z-1}{z-t}\right)\sim\left(\frac{t}{1-t}\right)^{1/2}(z-1)^{\frac{1}{2}+\theta''a_1}\times\\ &\qquad\exp\left(-\frac{\theta''}{2}\partial_{a_1}F(t)\right)\left[1+\mathcal{O}\left(t,\frac{z-1}{z-t}\right)\right],\end{aligned} \tag{F.2.18}$$

where $F(t)$ is the conformal block (F.2.16).

It follows that the connection formula between the solutions of the Heun equation reads

$$\begin{aligned}\psi_\theta^{(0)}(z) &= \sum_{\theta''=\pm} \mathcal{C}_{\theta\theta''}\,\psi_{\theta''}^{(1)}(z), \quad \text{with}\\ \mathcal{C}_{\theta\theta''} &= \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')\theta''}(a,a_1;a_\infty)t^{\theta a_0+\theta' a}\times\\ &\qquad \exp\left(\frac{\theta}{2}\partial_{a_0}F(t)-\frac{\theta''}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right).\end{aligned} \tag{F.2.19}$$

F.3 Heun connection formula for $|t|>1$

The connection formula between the points $z=0$ and $z=1$ in the regime $|t|>1$ is simpler since the two singular points are on the same side of the pants decomposition of the four-punctured sphere.

In the semiclassical limit, the conformal blocks around $z=0$ and $z=1$ read

$$\begin{aligned}&t^{1/2}\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_1\\ \end{matrix}\,a_{0\theta}\,\begin{matrix}a_{2,1}\\a_0\end{matrix};\frac{1}{t},z\right),\\ &(t-1)^{1/2}e^{\theta i\pi a}\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_0\\ \end{matrix}\,a_{1\theta}\,\begin{matrix}a_{2,1}\\a_1\end{matrix};\frac{1}{t-1},1-z\right),\end{aligned} \tag{F.3.20}$$

respectively. The connection formula between them reads

$$\begin{aligned}&t^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_1\\ \end{matrix}\,a_{0\theta}\,\begin{matrix}a_{2,1}\\a_0\end{matrix};\frac{1}{t},z\right)=\\ &\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}\left(a_0,a_1;a\right)(t-1)^{\frac{1}{2}}e^{\theta' i\pi a}\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_1\\ \end{matrix}\,a_{0\theta'}\,\begin{matrix}a_{2,1}\\a_0\end{matrix};\frac{1}{t-1},1-z\right).\end{aligned} \tag{F.3.21}$$

The semiclassical blocks' expansions read

$$\begin{aligned}&\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_1\\ \end{matrix}\,a_{0\theta}\,\begin{matrix}a_{2,1}\\a_0\end{matrix};\frac{1}{t},z\right)\sim t^{-\frac{1}{2}}\exp\left(-\frac{\theta}{2}\partial_{a_0}F(t)\right)z^{\frac{1}{2}+\theta a_0}\left[1+\mathcal{O}\left(\frac{1}{t},z\right)\right],\\ &\mathcal{F}\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_0\\ \end{matrix}\,a_{1\theta'}\,\begin{matrix}a_{2,1}\\a_1\end{matrix};\frac{1}{t-1},1-z\right)\sim(t-1)^{-\frac{1}{2}}\,e^{\theta'\,i\,\pi\,a}\,(z-1)^{\frac{1}{2}+\theta' a_1}\\ &\qquad\qquad\times\exp\left(-\frac{\theta'}{2}\partial_{a_1}F(t)\right)\left[1+\mathcal{O}\left(\frac{1}{t-1},1-z\right)\right],\end{aligned} \tag{F.3.22}$$

where $F(t)$ is the conformal block

$$F=F\left(\begin{matrix}a_t\\a_\infty\end{matrix}\,a\,\begin{matrix}a_1\\a_0\end{matrix};\frac{1}{t}\right). \tag{F.3.23}$$

It follows that the connection formula between the solutions of the Heun equation with the expansion (F.2.17) reads

$$\begin{aligned} &\psi_\theta^{(0)}(z) = \sum_{\theta'=\pm} \mathcal{C}_{\theta\theta'}\, \psi_{\theta'}^{(1)}(z), \quad \text{with} \\ &\mathcal{C}_{\theta\theta'} = \mathcal{M}_{\theta\theta'}(a_0, a_1; a) \exp\left(\frac{\theta}{2}\partial_{a_0} F(t) - \frac{\theta'}{2}\partial_{a_1} F(t)\right). \end{aligned} \tag{F.3.24}$$

Appendix G

Determinant of Heun differential operators and Gelfand-Yaglom theorem

G.1 Heun normalized with Hypergeometric

For applying the Gelfand-Yaglom theorem to the BH problems, we need to compute (ratio of) determinants of differential operators of Heun's and Hypergeometric's type.

Indeed, the spectral problems in which we are interested are encoded by Heun differential equations with both boundary conditions imposed at singular points. If we consider the normal form of the Heun operator

$$\mathcal{D}: \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{\frac{1}{4}-a_0^2}{z^2} + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} + \frac{\frac{1}{4}-a_t^2}{(z-t)^2} - \frac{\frac{1}{2}-a_1^2-a_t^2-a_0^2+a_\infty^2+u}{z(z-1)} + \frac{u}{z(z-t)}\right], \tag{G.1.1}$$

the simpler problem can be taken to be a Hypergeometric operator, which can be obtained by modifying the potential setting $u=0$, $a_t^2=\frac{1}{4}$. For simplicity, we also set $a_\infty^2=\frac{1}{4}$. This gives

$$\tilde{\mathcal{D}}: \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{\frac{1}{4}-a_0^2}{z^2} + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} - \frac{\frac{1}{2}-a_1^2-a_0^2}{z(z-1)}\right]. \tag{G.1.2}$$

This simplification is such that the indices of the singularities at $z=0$ and $z=1$ are kept fixed. Using the connection coefficients for the Heun equation and Hypergeometric equation (see the previous Appendix), we can take the ratio of determinants as the ratio of connection coefficients, distinguishing the cases according to the signs of $\mathrm{Re}(a_0)$ and $\mathrm{Re}(a_1)$.

For example, in the case $\mathrm{Re}(a_0) > 0$ and $\mathrm{Re}(a_1) > 0$, we have

$$\frac{\det(\mathcal{D})}{\det\left(\tilde{\mathcal{D}}\right)} = \frac{\sum_{\theta'=\pm} \mathcal{M}_{+\theta'}(a_0, a; a_t)\mathcal{M}_{(-\theta')-}(a, a_1; a_\infty)t^{a_0+\theta' a} \exp(\frac{1}{2}\partial_{a_0}F(t) + \frac{1}{2}\partial_{a_1}F(t) - \frac{\theta'}{2}\partial_a F(t))}{\frac{\Gamma(1+2a_0)\Gamma(2a_1)}{\Gamma(1+a_0+a_1)\Gamma(a_0+a_1)}}. \tag{G.1.3}$$

G.2 Computation of determinant for Hypergeometric operators

In this Appendix, we use a different method to compute the determinant of generic Hypergeometric differential operators. Using the result of the ratio of determinants (G.1.3), this gives a prescription on how to compute the (regularized) determinant for Heun differential operators.

Let $\mathcal{D}_1$ be the Hypergeometric differential operator in normal form with generic indices of the singularities, parametrized by the parameters a, b, c:

$$\mathcal{D}_1 : \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{2c[z(a+b-1)+1] + z\left[-z(a-b)^2 - 4ab + z\right] - c^2}{4(z-1)^2 z^2}. \tag{G.2.4}$$

Let $\mathcal{D}_2$ be the Hypergeometric differential operator in the form

$$\mathcal{D}_2 : \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{[c-(a+b+1)z]}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{a\,b}{z(1-z)}. \tag{G.2.5}$$

We have that if $\psi_{1,\lambda}(z)$ is an eigenfunction for $\mathcal{D}_1$ with corresponding eigenvalue λ, then

$$\psi_{2,\lambda}(z) = z^{-c/2}(z-1)^{-\frac{a+b+1-c}{2}}\psi_{1,\lambda}(z) \tag{G.2.6}$$

is an eigenfunction for $\mathcal{D}_2$ with the same eigenvalue λ. Indeed,

$$\begin{aligned}&\mathcal{D}_2\psi_{2,\lambda}(z) = \\ &\left[z^{-c/2}(z-1)^{-(a+b+1-c)/2}\mathcal{D}_1\, z^{c/2}(z-1)^{(a+b+1-c)/2}\right]\left[z^{-c/2}(z-1)^{-(a+b+1-c)/2}\psi_{1,\lambda}(z)\right] = \\ &z^{-c/2}(z-1)^{-(a+b+1-c)/2}\mathcal{D}_1\psi_{1,\lambda}(z) = z^{-c/2}(z-1)^{-(a+b+1-c)/2}\lambda\psi_{1,\lambda}(z) = \lambda\psi_{2,\lambda}(z).\end{aligned} \tag{G.2.7}$$

Therefore, the determinant of the two differential operators is the same, since the two have the same eigenvalues.

Now, thanks to the Gelfand-Yaglom theorem and the remark E.1.1, the determinant of $\mathcal{D}_2$ is equal to the determinant of the operator

$$\mathcal{D}_3 : \quad z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}}{\mathrm{d}z} - a\,b, \tag{G.2.8}$$

since the differential equations $\mathcal{D}_2\psi(z) = 0$ and $\mathcal{D}_3\psi(z) = 0$ have the same solutions. Indeed, to apply the Gelfand-Yaglom theorem and compute the determinants of $\mathcal{D}_2$ and $\mathcal{D}_3$, one can transform them in the normal form using the procedure outlined in Appendix E, and finally, normalizing with respect to the same reference operator, one finds the same result for the two computations.

Finally, in order to compute the determinant of $\mathcal{D}_3$, we can look into the eigenvalue problem

$$(\mathcal{D}_3 - \lambda)\, w(z) = 0. \tag{G.2.9}$$

A basis of independent solutions of this differential equation around $z = 0$ is given by

$$\begin{aligned}
w_-^{(0)}(z) &= {}_2F_1\left(-\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}+\frac{a}{2}+\frac{b}{2}, \frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}+\frac{a}{2}+\frac{b}{2}; c; z\right),\\
w_+^{(0)}(z) &= z^{1-c}\,{}_2F_1\Bigg(-\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}+\frac{a}{2}+\frac{b}{2}-c+1,\\
&\qquad \frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}+\frac{a}{2}+\frac{b}{2}-c+1; 2-c; z\Bigg).
\end{aligned} \tag{G.2.10}$$

The selected solution around $z = 0$ is $w_+^{(0)}(z)$. The connection coefficient in front of the discarded solution around $z = 1$ is given by

$$\frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma\left(1+\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}-\frac{a}{2}-\frac{b}{2}\right)\Gamma\left(1-\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda}-\frac{a}{2}-\frac{b}{2}\right)}. \tag{G.2.11}$$

Therefore, the λ_n that ensure the correct boundary conditions for the solution are obtained by the quantization condition

$$1+\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda_n}-\frac{a}{2}-\frac{b}{2} = -n \quad \text{or} \quad 1-\frac{1}{2}\sqrt{a^2-2ab+b^2-4\lambda_n}-\frac{a}{2}-\frac{b}{2} = -n, \tag{G.2.12}$$

with $n \in \mathbb{Z}_{\geq 0}$, that is,

$$\lambda_n = (1-b+n)(-1+a-n). \tag{G.2.13}$$

Hence, denoting with a tilde the regularization of the previous infinite product, the determinant is given by

$$\det\mathcal{D}_3 = \widetilde{\prod}_{n\geq 0}(1-b+n)(-1+a-n) = \frac{2\pi}{\Gamma(1-b)\Gamma(1-a)}, \tag{G.2.14}$$

where we used the Zeta regularization and the *Lerch's formula* [252].

G.3 Regularized determinant for Heun differential operators

Comparing the differential operator $\mathcal{D}_1$ in (G.2.4) with the operator $\tilde{\mathcal{D}}$ in (G.1.2), we have that the two are related by the dictionary

$$a = 1 - a_0 - a_1, \quad \text{and} \quad b = -a_0 - a_1. \tag{G.3.15}$$

With the result of the previous subsection, we have that

$$\det\tilde{\mathcal{D}} = \widetilde{\prod}_{n\geq 0}(1+a_0+a_1+n)(-a_0-a_1-n) = \frac{2\pi}{\Gamma(1+a_0+a_1)\Gamma(a_0+a_1)}. \quad \text{(G.3.16)}$$

We conclude that we can give a formula for the (regularized) determinant of the Heun differential operator (G.1.1), under the assumption $\mathrm{Re}(a_0) > 0$ and $\mathrm{Re}(a_1) > 0$:

$$\begin{aligned}\det\mathcal{D} = \sum_{\theta'=\pm} & \frac{2\pi\Gamma(-2\theta' a)\Gamma(1-2\theta' a)}{\prod_{\sigma=\pm}\Gamma\left(\frac{1}{2}+a_0-\theta' a+\sigma\, a_t\right)\Gamma\left(\frac{1}{2}-\theta' a+a_1+\sigma\, a_\infty\right)}\times \\ & t^{a_0+\theta' a}\exp\left(\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)-\frac{\theta'}{2}\partial_a F(t)\right).\end{aligned} \quad \text{(G.3.17)}$$

Let us remark that this result is equal to the Heun connection coefficient in front of the discarded solution at $z=1$, divided by the two Gamma functions whose arguments depend on the indices of the singularities where the two boundary conditions are imposed. The 2π factor comes from the Zeta function regularization. This normalization gives analogous results as the ones obtained in the work [140], where the subtraction of the Gamma functions is motivated by physical arguments, introducing the *Rindler-like region.*

G.4 Reduction of connection coefficient

Here, we prove that in the limit in which the Heun differential operator $\mathcal{D}$ reduces to the Hypergeometric one, the ratio in (G.1.3) becomes equal to 1. Notice that we are not taking a collision limit, but we are just fitting the parameters so that the singularity at $z=t$ becomes an apparent one.

The differential operator $\mathcal{D}$ in (G.1.1) reduces to $\tilde{\mathcal{D}}$ in (G.1.2) by setting

$$a_t = a_\infty = \frac{1}{2}, \quad u = 0. \quad \text{(G.4.18)}$$

Using the instanton expansion

$$u = -\frac{1}{4}+a_t^2+a_0^2-a^2+t\frac{\partial F(t)}{\partial t}, \quad \text{(G.4.19)}$$

we get

$$a = \pm a_0, \quad F(t) = 0. \quad \text{(G.4.20)}$$

Let us choose the plus sign. In the Hypergeometric connection matrices appearing in the determinant of $\mathcal{D}$

$$\begin{aligned}&\mathcal{M}_{+\theta'}(a_0,a;a_t)\mathcal{M}_{(-\theta')-}(a,a_1;a_\infty) = \\ &\frac{\Gamma(1+2a_0)\Gamma(-2\theta' a)\Gamma(1-2\theta' a)\Gamma(2a_1)}{\Gamma\left(\frac{1}{2}+a_0-\theta' a+a_t\right)\Gamma\left(\frac{1}{2}+a_0-\theta' a-a_t\right)\Gamma\left(\frac{1}{2}-\theta' a+a_1+a_\infty\right)\Gamma\left(\frac{1}{2}-\theta' a+a_1-a_\infty\right)},\end{aligned} \quad \text{(G.4.21)}$$

one can see that the choice $\theta' = +$ makes one of the arguments of the Gamma functions in the denominator equal to 0 under the dictionary (G.4.18) (if we had chosen the different sign $a_0 = -a$, then the reasoning would have been the same with $\theta' = -$). Therefore, the determinant of $\mathcal{D}$ reduces to channel corresponding to $\theta' = -$:

$$\begin{aligned}
&\frac{\Gamma\left(1+2a_0\right)\Gamma\left(2a\right)\Gamma\left(1+2a\right)\Gamma\left(2a_1\right)}{\Gamma\left(\frac{1}{2}+a_0+a+a_t\right)\Gamma\left(\frac{1}{2}+a_0+a-a_t\right)\Gamma\left(\frac{1}{2}+a+a_1+a_\infty\right)\Gamma\left(\frac{1}{2}+a+a_1-a_\infty\right)}\times \\
&t^{a_0-a}\exp\left(\frac{1}{2}\partial_{a_0}F(t)+\frac{1}{2}\partial_{a_1}F(t)+\frac{1}{2}\partial_{a}F(t)\right)\to \\
&\to\frac{\Gamma\left(1+2a_0\right)^2\Gamma\left(2a_0\right)\Gamma\left(2a_1\right)}{\Gamma\left(1+2a_0\right)\Gamma\left(2a_0\right)\Gamma\left(1+a_0+a_1\right)\Gamma\left(a_0+a_1\right)}=\frac{\Gamma\left(1+2a_0\right)\Gamma\left(2a_1\right)}{\Gamma\left(1+a_0+a_1\right)\Gamma\left(a_0+a_1\right)}
\end{aligned} \tag{G.4.22}$$

where we used (G.4.18) to pass to the second line. The last result is precisely the determinant of $\tilde{\mathcal{D}}$ appearing in the denominator of (G.1.3) (equivalently, one of the Hypergeometric connection coefficients).

Appendix H

Useful asymptotics for Barnes G-function

Here we collect some useful asymptotic formulae used in chapter 4.

$$\begin{aligned}
&\log(G(1-\mu+2(\alpha+n))) = \\
&\frac{2n+2\alpha-\mu}{2}\log(2\pi)-\log(A)+\frac{1}{12}-\frac{3(2n)^2}{4}-(2\alpha-\mu)(2n)+ \\
&+\left(\frac{(2n)^2}{2}-\frac{1}{12}+\frac{(2\alpha-\mu)^2}{2}+(2\alpha-\mu)(2n)\right)\log(2n)+\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned} \tag{H.0.1}$$

$$\begin{aligned}
&\log(G(1-\mu-2(\alpha+n))) = \\
&\log(G(1+\mu+2(\alpha+n)))-(\mu+2(\alpha+n))\log(2\pi)+\int_0^{\mu+2(\alpha+n)}\pi z'\cot(\pi z')\mathrm{d}z' = \\
&=\frac{2n+2\alpha+\mu}{2}\log(2\pi)-\log(A)+\frac{1}{12}-\frac{3(2n)^2}{4}-(2\alpha+\mu)(2n)+ \\
&+\left(\frac{(2n)^2}{2}-\frac{1}{12}+\frac{(2\alpha+\mu)^2}{2}+(2\alpha+\mu)(2n)\right)\log(2n)-(\mu+2(a+n))\log(2\pi)+ \\
&+(\mu+2(\alpha+n))\log(1-\exp(2\pi i(\mu+2\alpha)))+ \\
&-\frac{i\left(\pi^2(\mu+2(\alpha+n))^2+\mathrm{Li}_2(\exp(2\pi i(\mu+2\alpha)))\right)}{2\pi}+\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned} \tag{H.0.2}$$

$$\begin{aligned}
&\log(G(1+2(\alpha+n))) = \\
&\frac{2n+2\alpha}{2}\log(2\pi)-\log(A)+\frac{1}{12}-\frac{3(2n)^2}{4}-(2\alpha)(2n)+ \\
&+\left(\frac{(2n)^2}{2}-\frac{1}{12}+\frac{(2\alpha)^2}{2}+(2\alpha)(2n)\right)\log(2n)+\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned} \tag{H.0.3}$$

$$
\begin{aligned}
&\log(G(1-2(\alpha+n))) = \\
&\log(G(1+2(\alpha+n))) - (2(\alpha+n))\log(2\pi) + \int_0^{2(\alpha+n)} \pi z' \cot(\pi z') \mathrm{d}z' = \\
&= \frac{2n+2\alpha}{2}\log(2\pi) - \log(A) + \frac{1}{12} - \frac{3(2n)^2}{4} - (2\alpha)(2n) + \\
&+ \left(\frac{(2n)^2}{2} - \frac{1}{12} + \frac{(2\alpha)^2}{2} + (2\alpha)(2n)\right)\log(2n) - (2(\alpha+n))\log(2\pi) + \\
&+ (2(\alpha+n))\log(1-\exp(4\pi i\alpha)) + \\
&- \frac{i\left(\pi^2(2(\alpha+n))^2 + \mathrm{Li}_2(\exp(4\pi i\alpha))\right)}{2\pi} + \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}
\tag{H.0.4}
$$

Hence, the following holds

$$
\begin{aligned}
&\log\left(\frac{G(1-\mu-2(\alpha+n))G(1-\mu+2(\alpha+n))}{G(1+2(\alpha+n))G(1-2(\alpha+n))}\right) = \\
&2n\log\left(\frac{1-\exp(2\pi i(\mu+2\alpha))}{1-\exp(4\pi i\alpha)}\right) - 2\pi i\mu n + \mu^2\log(2n) - \mu\log(2\pi) \\
&+ 2\alpha\log\left(\frac{1-\exp(2\pi i(\mu+2\alpha))}{1-\exp(4\pi i\alpha)}\right) + \mu\log(1-\exp(2\pi i(\mu+2\alpha))) \\
&- \frac{\mathrm{Li}_2(\exp(2\pi i(\mu+2\alpha))) - \mathrm{Li}_2(\exp(4\pi i\alpha))}{2\pi} - \frac{i\pi}{2}(\mu^2+4\alpha\mu) + \mathcal{O}\left(\frac{1}{n}\right).
\end{aligned}
\tag{H.0.5}
$$

Therefore, up to $1/n$ corrections,

$$
\frac{G(1-\mu-2(\alpha+n))G(1-\mu+2(\alpha+n))}{G(1+2(\alpha+n))G(1-2(\alpha+n))} \propto (2n)^{\mu^2}\left(\frac{\sin(\pi(\mu+2\alpha))}{\sin(2\pi\alpha)}\right)^{2n}, \tag{H.0.6}
$$

where the proportionality constant is independent of n.

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