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Isoperimetric inequalities in non-compact spaces

Candidate:
Davide Manini

Advisor:
Fabio Cavalletti

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Declaration

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Abstract

The sharp isoperimetric inequality for Riemannian manifolds having non-negative Ricci curvature and Euclidean volume growth has been obtained in increasing generality in a number of contributions [1, 46, 19, 51], culminated by Balogh and Kristály [14]. This last results covers also the case of non-smooth metric-measure spaces verifying the non-negative Ricci curvature condition in the synthetic sense of Sturm and Lott–Villani (the so-called CD condition) and the proof exploits the Brunn–Minkowski inequality.

In the first part of the present thesis, we generalize the isoperimetric inequality to the wider class of $\mathsf{MCP}(0, N)$ spaces having Euclidean volume growth. The inequality we prove is sharp and the constant in the lower bound is smaller than the constant found in [14]. Since the Brunn–Minkowski is not available in MCP spaces, our proof is a scaling limit of the localization approach.

In the second part, we present a characterization of the isoperimetric sets in the same generality of [14], i.e., the synthetic CD setting. Namely, we prove that the equality in the isoperimetric inequality can be attained only by metric balls and, whenever this happens, the space is forced, in a measure-theoretic sense, to be a cone. As a corollary, in the setting of general RCD spaces, we derive rigidity for the metric structure, i.e., the space is a cone also in the metric sense, generalizing a result of Antonelli et al. [10]. The proof consists in a careful refinement of the scaling argument presented in first part.

In the third part, we generalize the isoperimetric inequality to the family of irreversible Finsler manifold with non-negative Ricci curvature and Euclidean volume growth. Irreversible Finsler manifolds are not covered by the theory of metric-measure spaces, for the distance induced by the Finsler structure is not symmetric. We also prove a rigidity result analogous to the result we obtained in the synthetic CD setting, namely a rigidity for the isoperimetric set and for the space, in the measure-theoretic sense. The proof of the inequality is based on the Brunn–Minkowski inequality; the rigidity is proved using the argument we developed in the second part.

As a by-product of the rigidity results in both the CD and Finsler setting, we deduce that optimizers in the anisotropic and weighted isoperimetric inequality for Euclidean cones are necessarily the Wulff shapes.

Contents

Chapter 1

Introduction

The isoperimetric problem is perhaps the most ancient problem in Calculus of Variation, as its first mention can be found in the legendary foundation of the city of Carthage by Queen Dido. According to the legend, Queen Dido was allowed to take possession of a portion of land that she could enclose with a leather rope, so she built Carthage on a half-circle whose diameter was given by the shore. More generally, the isoperimetric problem consists in finding the shape of a fixed given volume having least perimeter; this problem can take place in different settings, e.g., in the Euclidean space, on a sphere, on more general class of spaces, etc.

A possible modern mathematical approach to the isoperimetric problem is two-step. First, one establishes an isoperimetric inequality, that is, a lower bound on the perimeter of a shape given in terms of the volume of said shape; one also checks the *sharpness* of this lower bound, that is, whether the lower bound is effectively attained in a certain case. Then, one checks the *rigidity* of the isoperimetric inequality, by giving a complete characterization of the equality case. (One can explore further more sophisticated properties of the isoperimetric inequality; in the present thesis we confine our-self to these two steps).

In order to clarify, let's state the classical isoperimetric inequality in the Euclidean space: if E is a (sufficiently regular) subset of $\mathbb{R}^d$, then it holds that

$$|\partial E| \geq d\omega_d^{\frac{1}{d}}|E|^{1-\frac{1}{d}}, \tag{1.1}$$

where $|\partial E|$ and $|E|$ are the $(d-1)$-dimensional and d-dimensional measure of the boundary of E and E itself (respectively), and ω_d is the measure of the unitary ball of $\mathbb{R}^d$; moreover, the equality is attained if and only if E coincides with a ball of radius $(\frac{|E|}{\omega_d})^{\frac{1}{d}}$. The present inequality has been successfully extended in more general settings where the ambient space is not the Euclidean one. Indeed, it turns out that two are the relevant properties of the Euclidean space needed for such generalizations: 1) the fact that $\mathbb{R}^d$ has non-negative Ricci curvature; 2) its Euclidean volume growth, i.e., a constraint on the growth of the measure of large balls.

We shortly detail what we mean by "Euclidean volume growth". If a complete Riemannian d-manifold (X, g) has non-negative Ricci curvature, then it satisfies the so-called Bishop–Gromov inequality, which states that the function

$$r \mapsto \frac{\mathrm{Vol}_g(B_r(o))}{r^d} \qquad \text{is non-increasing;}$$

here $B_r(o) = \{y : \mathsf{d}_g(o, y) < r\}$ denotes the metric ball of radius r centered in the point o. The asymptotic volume ratio (a.v.r.) of (X, g) is then naturally defined as

$$\mathsf{AVR}_X := \lim_{r\to\infty} \frac{\mathrm{Vol}_g(B_r(o))}{\omega_d r^d} \in [0, 1],$$

where ω_d is the measure of the unitary ball in $\mathbb{R}^d$; it is immediate to verify that the definition is not depending on the point $o \in X$. (To be pedantic, one should also specify the dependence of the a.v.r. on the metric g; however, the context always makes clear what metric we are considering). We say that a d-manifold has Euclidean volume growth (e.v.g.), if its a.v.r. is positive.

Having in mind this definition, we state a theorem by Brendle [19] which generalizes inequality (1.1).

Theorem 1.1 ([19, Theorems 1.1 and 1.2])**.** *Let (X, g) be a d-dimensional Riemannian manifold with non-negative Ricci curvature and e.v.g. If E is a compact domain, then it holds that*

$$\mathcal{H}^{d-1}(\partial E) \geq d\omega_d^{\frac{1}{d}} \mathsf{AVR}_X^{\frac{1}{d}} \mathrm{Vol}_g(E)^{1-\frac{1}{d}}. \tag{1.2}$$

Moreover, if the inequality is saturated by a compact domain E with positive volume, then X is isometric to the Euclidean space $\mathbb{R}^d$ and E coincides with a metric ball.

Some comments are in order. In the statement, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. The r.h.s. of the inequality depends only on the a.v.r. and the measure on the set E. The rigidity is two-fold: one side the equality, like in the classical isoperimetric inequality, implies that the isoperimetric set is a ball; on the other side, also a rigidity of the manifold itself holds true. Finally, we point out that if we specialize to the case $X = \mathbb{R}^d$, we recover the classical isoperimetric inequality.

A natural question is: can we widen the class of spaces admitting a (sharp, rigid) isoperimetric inequality mimicking (1.2)? In particular, does any (sharp, rigid) isoperimetric result hold in the non-smooth setting? The present thesis will answer these questions by providing: 1) sharp isoperimetric inequalities in the non-smooth MCP setting and in the irreversible Finsler setting; 2) rigidity results for the CD setting, irreversible Finsler setting, and for cones in the Euclidean space.

In order to better present the previous literature and the novel results, we give an informal description of the involved geometrical structures, leaving the formal definition to the next chapters.

Description of the geometric structures. If (X, g) is a d-dimensional Riemannian manifold, it naturally comes with a distance d_g and volume measure Vol_g; however, it is natural to consider more general measures other than Vol_g, by multiplying the volume by a weight $h = e^{-\varphi}$, with $\varphi \in C^2(X)$. Since the Ricci curvature quantifies the distortion of the volume measure, when dealing with weighted measures, it loses its meaning. Therefore, the relevant object to control is the N-Ricci tensor introduced by Bakry [12] (also known as *generalized Ricci curvature* or *Bakry–Émery tensor*): if $\varphi \in C^2(M)$, the generalised N-Ricci tensor (with $N \in (0, \infty)$) is defined by

$$\mathrm{Ric}_{g,h,N} := \begin{cases} \mathrm{Ric}_g + \nabla_g^2 \varphi - \frac{\nabla_g \varphi \otimes \nabla_g \varphi}{N-d}, & \text{if } d < N, \\ \mathrm{Ric}_g + \nabla_g^2 \varphi, & \text{if } d = N \text{ and } d\varphi = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

The weighted Riemannian manifold $(X, g, h\,\mathrm{Vol}_g)$ is said to verify the Bakry-Emery Curvature-Dimension condition $\mathsf{CD}(K, N)$ [13], if $\mathrm{Ric}_{g,h,N} \geq Kg$. The $\mathsf{CD}(K, N)$ condition incorporates information on curvature and dimension from both the geometry of (X, g) and the measure $h\,\mathrm{Vol}_g$.

In their seminal works, Lott–Villani [58] and Sturm [79, 80] introduced a synthetic definition of $\mathsf{CD}(K, N)$ for complete and separable metric spaces (X, d) endowed with a (locally-finite Borel) reference measure $\mathfrak{m}$ ("metric-measure space", or m.m.s.). The synthetic $\mathsf{CD}(K, N)$ condition is formulated using the language of Optimal Transport and it is given by the displacement convexity of a certain entropy functional. It was shown that two definitions of $\mathsf{CD}(K, N)$ given by Bakry–Émery and Lott–Sturm–Villani coincide in the smooth Riemannian setting (in both weighted and non-weighted cases). The $\mathsf{CD}(K, N)$ condition enjoys stablility under measured Gromov–Hausdorff convergence of m.m.s.'s and Alexandrov spaces satisfy it. The definition of $\mathsf{CD}(K, N)$ condition is contained in Section 2.1.1.

The stronger $\mathsf{RCD}(K, N)$ (Riemannian Curvature–Dimension) condition, was later introduced [7] in order to exclude the possible non-linearity of the Laplacian and the heat-flow. It can be defined as the $\mathsf{CD}(K, N)$ condition with the additional requirement that the Sobolev space $W^{1,2}(X)$ is an Hilbert space [47]. $\mathsf{RCD}(K, N)$ features many properties, not available for $\mathsf{CD}(K, N)$ spaces, such as a splitting theorem and the Bochner inequality.

A weaker notion of "lower bound on the curvature" is given by the *measure contraction property* $\mathsf{MCP}(K,N)$, introduced in [80, 67]. Roughly speaking, while the $\mathsf{CD}(K,N)$ condition consists in a convexity constraint on a certain entropy functional on the space of probability measure, the $\mathsf{MCP}(K,N)$ condition restricts said convexity constraint only to the case when one of the two measures of the convexity condition is a Dirac's delta. For a d-dimensional unweighted Riemannian manifold, the $\mathsf{MCP}(K,N)$ condition coincides with the $\mathsf{CD}(K,d)$ condition, i.e., $\mathrm{Ric} \geq K$; for weighted manifolds, the MCP condition is weaker than CD, in general. The MCP condition is interesting also because it encompasses certain spaces which are known not to satisfy the $\mathsf{CD}(K,N)$; for example, the three-dimensional Heisenberg group is $\mathsf{MCP}(0,5)$ but not $\mathsf{CD}(K,N)$ for all $K,N \in \mathbb{R}$ [52] (see also [17]). See Section 2.1.2 for the definition of the $\mathsf{MCP}(K,N)$ condition.

Spaces satisfying the $\mathsf{MCP}(0,N)$ condition (and a fortiori the $\mathsf{CD}(0,N)$ condition) admit a Bishop–Gromov inequality; therefore, for an $\mathsf{MCP}(0,N)$ space $(X,\mathsf{d},\mathfrak{m})$, we can define its a.v.r. as

$$\mathsf{AVR}_X := \lim_{r\to\infty} \frac{\mathfrak{m}(B_r(o))}{\omega_N r^N} \in [0,\infty),$$

and ω_N is naturally extended also for $N \notin \mathbb{N}$, using the Γ-function.

A Finsler manifold is a triple $(X,F,\mathfrak{m})$, such that X is a differential manifold (possibly with boundary), $\mathfrak{m}$ a Borel measure, and F a Finsler structure, that is a real-valued function $F : TX \to [0,\infty)$, which is convex, positively homogeneous, and $F(v) = 0$ if and only if $v = 0$ (see Section 5.1 for the precise definition). A Finsler manifold $(X,F,\mathfrak{m})$ is called irreversible if $F(v) \neq F(-v)$; otherwise, the manifold is called reversible. When a Finsler manifold is reversible, then it naturally comes with a (symmetric) distance, therefore the well-established theory of m.m.s.'s apply; by extension, with the expression "reversible setting" we will include m.m.s.'s. Our interest in irreversible Finsler structures lies in the fact that irreversible Finsler manifolds arise in natural phenomena, e.g., the navigation with wind [16] or walking on a steep mountain under the effect of gravity [62]. Recently, Ohta successfully extended the theory of the Curvature-Dimension condition for possibly irreversible Finsler manifolds (see [68, 71, 74]). Namely, a notion of N-Ricci curvature (compatible with the Riemannian one) was introduced, the synthetic definition of $\mathsf{CD}(K,N)$ was adapted to the irreversible setting, and it was proven that a Finsler manifold satisfies the $\mathsf{CD}(K,N)$ condition if and only if $\mathrm{Ric}_N \geq K$. $\mathsf{CD}(K,N)$ Finsler manifolds enjoy several of the properties already known in the reversible setting, for instance, the Bishop–Gromov inequality (see Section 5.1.2). For this reason, the definition of a.v.r. of a $\mathsf{CD}(0,N)$ Finsler manifold is well-given (see Section 5.1.2). More recently, the notion of irreversible metric measure space has been introduced [56], attempting to unify the theory of metric-measure spaces and Finsler manifolds; in the present thesis we confine our-self to the setting of Finsler manifolds (see Remark 1.9).

Review of the literature. The isoperimetric problem in the reversible setting (for both smooth and non-smooth spaces) has been extensively investigated. Perhaps one of the most ancient isoperimetric inequality in the non-Euclidean setting is the Lévy–Gromov inequality [60] (see also [48, Appendix C]): if E is a subset of a Riemannian manifold X, whose Ricci curvature is bounded below by $K > 0$, then $\frac{|\partial E|}{|X|} \geq \frac{|\partial B|}{|S|}$, where S is a sphere with Ricci curvature equal to K and $B \subset S$ is a metric ball with volume $|B| = |E|\frac{|S|}{|X|}$. The equality is attained if and only if X has maximal diameter in the sense of Myers's Theorem and therefore X is isometric to the sphere S and E is a metric ball. More recently, E. Milman [63] gave a sharp isoperimetric inequality for weighted Riemannian manifolds satisfying the $\mathsf{CD}(K, N)$ condition (for any $K \in \mathbb{R}$, $N > 1$), with an additional constraint on the diameter and with unitary measure. Namely, given $K \in \mathbb{R}$, $N > 1$, $D \in (0, \infty]$, he gave a rather explicit description of the so-called *isoperimetric profile* function $\mathcal{I}^{\mathsf{CD}}_{K,N,D} : [0, 1] \to \mathbb{R}$. The isoperimetric profile function $\mathcal{I}^{\mathsf{CD}}_{K,N,D}(v)$ is defined as the infimum of the perimeter among all possible sets of measure v contained in all possible $\mathsf{CD}(K, N)$ manifolds with diameter at most D with unitary volume. Milman's result is sometimes referred as generalized Lévy–Gromov inequality.

In his celebrated monograph [82], Villani asked whether one can find an alternative proof of Lévy–Gromov inequality, using only techniques of Optimal Transport. He motivated his question pointing out that Optimal Transport techniques are often suitable for treating non-smooth spaces. Cavalletti and Mondino [28] first answered Villani's question. They extended Milman's result to the non-smooth setting finding the same lower bound, i.e., if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K, N)$ space with unitary measure, then $\mathsf{P}(E) \geq \mathcal{I}^{\mathsf{CD}}_{K,N,D}(\mathfrak{m}(E))$, for any Borel subset $E \subset X$. Their proof makes use of the localisation method (also known as needle decomposition), a powerful dimensional reduction tool, initially developed by Klartag [54] for Riemannian manifolds and later extended to $\mathsf{CD}(K, N)$ spaces [28]. This method has proven to be very adequate in solving isoperimetric problems and it can be considered an answer to Villani's question. Indeed, the localization method is the core tool used in this thesis. Cavalletti and Santarcangelo [32], still using the localization approach, substituted the CD hypothesis with the MCP condition; in their paper, they described the isoperimetric profile $\mathcal{I}^{\mathsf{MCP}}_{K,N,D} : [0, 1] \to \mathbb{R}$, which turns out to be strictly smaller than $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$.

In the setting of Finsler manifold, less is known. Following the line traced in [28], Ohta [72] extended the localization method to Finsler manifolds and he obtained a lower bound for the perimeter for Finsler manifolds with finite reversibility constant. The reversibility constant (introduced in [76]) of a Finsler structure F on the manifold X is defined as the least constant (possibly infinite) $\Lambda_F \geq 1$ such that $F(-v) \leq \Lambda_F F(v)$ for all vectors $v \in TX$. Ohta proved [72] that given a Finsler manifold $(X, F, \mathfrak{m})$ having finite reversibility constant and unitary measure satisfying the $\mathsf{CD}(K,N)$ condition, with diameter bounded from above by D, it holds that

$$\mathsf{P}(E) \geq \Lambda_F^{-1}\, \mathcal{I}^{\mathsf{CD}}_{K,N,D}(\mathfrak{m}(E)), \qquad \forall E \subset X,$$

where $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$ is the isoperimetric profile function described by E. Milman. The presence of the factor Λ_F^{-1} suggests that the inequality above is not sharp. Indeed, in the case $N = D = \infty$, using a different approach, this factor can be eliminated obtaining a Barky–Ledoux isoperimetric inequality for Finsler manifolds [73].

The rigidity, in the setting of $\mathsf{CD}/\mathsf{MCP}(K,N)$ spaces with diameter bounded by $D \in (0,\infty]$, distinguishes two regimes: $K > 0$ and $K \leq 0$. The regime $K > 0$ has been addressed in [28] and later in [24] with a quantitative analysis. To be precise, it has been proven that if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{CD}(K,N)$ space such that the isoperimetric inequality is saturated, then the isoperimetric set is a ball and the diameter of X is $\sqrt{\frac{K}{N-1}}$, i.e., it is maximal in the sense of Myers theorem. If in addition $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{RCD}(K,N)$ condition, then X is isomorphic, as m.m.s., to a spherical suspensions [28, Theorem 1.4]. For $\mathsf{MCP}(K,N)$ spaces with $K > 0$, the rigidity problem is still open.

In the regime $K \leq 0$, it has been proven [32] that if a $\mathsf{MCP}(K,N)$ space with $K \leq 0$ and diameter at most $D \in (0,\infty)$ saturates the isoperimetric inequality, then the diameter of the space is saturated to D. No analogous result for $\mathsf{CD}(K,N)$ with $K \leq 0$ seems to be present in the literature; however, if one inspects the isoperimetric profiles computed in [63], one recognizes that the map $D \mapsto \mathcal{I}^{\mathsf{CD}}_{K,N,D}(v)$ is strictly decreasing and therefore diameter saturation of the diameter happens in $\mathsf{CD}(K,N)$ spaces, as well.

For irreversible Finsler manifold satisfying the $\mathsf{CD}(K,N)$ condition, the rigidity problem is still open, as well.

Concerning the regime $K = 0$, as we have already pointed out, it is natural to couple the lower bound on the Ricci curvature, with the e.v.g. constraint. Beside Theorem 1.1 [19], the sharp isoperimetric inequality for Riemannian manifolds with Euclidean volume growth has been obtained in increasing generality with different approaches in a number of contributions [1, 46, 51]. The most general version (including as subclasses the previous contributions) is the one obtained by Balogh and Kristály [14] and it is valid for m.m.s.'s verifying the $\mathsf{CD}(0,N)$ condition. Their argument follows from a refined application of the Brunn–Minkowski inequality given by Optimal Transport, and it can be considered a different possible answer to Villani's question.

Theorem 1.2 ([14, Theorem 1.1]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. satisfying the $\mathsf{CD}(0,N)$ condition for some $N > 1$, and having Euclidean volume growth. Then for every bounded Borel subset $E \subset X$ it holds*

$$\mathfrak{m}^+(E) \geq N\omega_N^{\frac{1}{N}}\, \mathsf{AVR}_X^{\frac{1}{N}}\, \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{1.3}$$

Moreover, inequality (1.3) *is sharp.*

Here we denote by $\mathfrak{m}^+$ the Minkowski content. See Section 2.2 and Appendix B for the relation between the Minkowski content and the perimeter.

More challenging to prove are the rigidity properties of (1.3). So far it has been obtained only under special assumptions on the space without matching the generality of Theorem 1.2. Antonelli, Pasqualetto, Pozzetta and Semola [10] generalised the rigidity of 1.1 to the non-smooth setting by considering $\mathsf{RCD}(0, N)$-spaces and removes the regularity assumptions on the boundary of E. However, the assumptions of infinitesimal hilbertianity is kept, and only the unweighted case (i.e., the reference measure is the Hausdorff measure) is considered, as well.

Theorem 1.3 ([10, Theorem 1.3])**.** *Let* $(X, \mathsf{d}, \mathcal{H}^N)$ *be an* $\mathsf{RCD}(0, N)$ *m.m.s. having Euclidean volume growth. Then the equality* (1.3) *holds for some* $E \subset X$ *with* $\mathcal{H}^N(E) \in (0, \infty)$ *if and only if* X *is isometric to a Euclidean metric measure cone over an* $\mathsf{RCD}(N-2, N-1)$ *space and* E *is isometric to a ball centered at one of the tips of* X*.*

Regarding the family of $\mathsf{MCP}(0, N)$ spaces having e.v.g., no isoperimetric inequality is known; the same can be said for irreversible Finsler manifolds satisfying $\mathsf{CD}(0, N)$ having e.v.g.

The table below summarizes the literature, novel contributions, and open problems.

	Inequality	Rigidity
$\mathsf{CD}(K, N)$ spaces with diameter bounded by D	[63, 28, 30]	If $K > 0$, $D = \infty$: [28, 30, 24]. If $K \leq 0$, $D < \infty$: it seems that there is diameter saturation (see p. 6).
$\mathsf{CD}(0, N)$ spaces having e.v.g.	[19, 1, 46, 51, 14]	[19, 10, 9, 11] Theorem 1.5
$\mathsf{MCP}(K, N)$ spaces with diameter bounded by D	[32]	If $K > 0$, $D = \infty$: open. If $K \leq 0$, $D < \infty$: diameter saturation [32].
$\mathsf{MCP}(0, N)$ spaces having e.v.g.	Theorem 1.4	Open
$\mathsf{CD}(K, N)$ Finsler manifolds with diameter bounded by D	[72] (the result is not sharp, except for $N = D = \infty$, $K > 0$ [73])	Open
$\mathsf{CD}(0, N)$ Finsler manifolds having e.v.g.	Theorem 1.7	Theorem 1.8

1.1 Contributions in the present thesis

Sharp isoperimetric inequality in $\mathsf{MCP}(0, N)$ spaces. The first result I present is taken by a paper [25], written in collaboration with my advisor, Fabio Cavalletti.

The next theorem generalizes Theorem 1.2 to the wider class of $\mathsf{MCP}(0, N)$ spaces.

Theorem 1.4 ([25, Theorem 1.2])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching m.m.s. satisfying the* $\mathsf{MCP}(0, N)$ *condition for some* $N > 1$ *and having Euclidean volume growth. Then* *for every Borel subset* $E \subset X$ *with* $\mathfrak{m}(E) < \infty$*, it holds that*

$$\mathfrak{m}^+(E) \geq (N\omega_N \mathsf{AVR}_X)^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{1.4}$$

Moreover, inequality (1.4) *is sharp.*

Some comments on Theorem 1.4 are in order. The constant of (1.4) is slightly smaller than the constant of (1.3). Indeed, the relation between the isoperimetric profile function of $\mathsf{MCP}(0, N)$ and $\mathsf{CD}(0, N)$ space with bounded diameter (i.e., $\mathcal{I}^{\mathsf{MCP}}_{0,N,D} < \mathcal{I}^{\mathsf{CD}}_{0,N,D}$ [32]), is reflected in different constant when considering the isoperimetric inequality for spaces with e.v.g.

The assumption of the space to be essentially non-branching prevents pathological phenomena within the theory of synthetic lower bounds on the Ricci curvature (see for instance the local-to-global property [27]). It is verified both by the class of reversible Finsler manifolds and $\mathsf{RCD}(0, N)$ spaces; in particular it holds for weighted Riemannian manifolds. All the results proved in this thesis requires this assumption.

Finally, let us mention that Theorem 1.4 will not imply a non-trivial isoperimetric inequality in the Heisenberg group of any dimension. Indeed while for instance the first Heisenberg group satisfies $\mathsf{MCP}(0, 5)$, the volume of its geodesic balls grows with the fourth power, giving zero AVR. Therefore, the isoperimetric inequality in the Heisenberg group is still open. Please, refer to [17, 64] for further details about the isoperimetric problem in the Heisenberg group and the wider class of sub-Riemannian manifolds.

The proof of Theorem 1.2 was carried out by using the Brunn–Minkowski inequality, a consequence of the $\mathsf{CD}(0, N)$ condition. In $\mathsf{MCP}(0, N)$ spaces, the Brunn–Minkowski inequality is not available, therefore we will adopt a more sophisticated approach that relies on the localization paradigm. Before sketching the proof, let us mention that recently a *modified* Brunn-Minkowski inequality has been established [15, 17] for a large family of sub-Riemannian manifolds also verifying the $\mathsf{MCP}(0, N)$ condition, for an appropriate choice of $N > 1$. Due to the nonlinearity of the concavity interpolation coefficients, this modified version of Brunn-Minkowski inequality seems not to imply a non-trivial isoperimetric inequality directly. Also the weaker versions of the Brunn-Minkowski inequality obtained in [64], verified again by a large family of sub-Riemannian spaces, are just not tailored to obtain an expansion of the volume of a tubular neighbourhood of a given set.

We sketch the ideas used for proving Theorem 1.4. We can assume the set E to have positive measure and to be bounded (if it is not bounded one proceeds by approximation). We embed the set E in a large ball of radius R and we consider the L^1-Optimal Transport of the reference measure restricted to E and the complementary of E in the ball, respectively. The localization technique, which takes into account the properties of Optimal Transport, produces a family of one-dimensional $\mathsf{MCP}(0, N)$ spaces (called transport rays), with diameter bounded by R. On each transport ray one applies the isoperimetric inequality to the trace of E on the ray, and combine them together. Finally, using a Taylor expansion of the isoperimetric profile $\mathcal{I}^{\mathsf{MCP}}_{0,N,R}$, one takes the limit as $R \to \infty$, and taking into account the a.v.r., one obtain the desired inequality.

The proof is contained in Chapter 3, whereas Chapter 2 presents the notation and technical tools employed in the proof.

Rigidity of the isoperimetric inequality in $\mathsf{CD}(0, N)$ spaces. The second result comes from a preprint [26], written in collaboration with my advisor, Fabio Cavalletti.

The next theorem characterize the equality case of inequality (1.3).

Theorem 1.5 ([26, Theorem 1.4]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying the $\mathsf{CD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Let $E \subset X$ be a bounded Borel set that saturates* (1.3)*.*

Then there exists (a unique) $o \in X$ such that, up to a negligible set, $E = B_\rho(o)$, with $\rho = (\frac{\mathfrak{m}(E)}{\mathsf{AVR}_X \omega_N})^{\frac{1}{N}}$. Moreover, considering the disintegration of $\mathfrak{m}$ with respect to $\mathsf{d}(\cdot, o)$, the measure $\mathfrak{m}$ has the following representation

$$\mathfrak{m} = \int_{\partial B_\rho(o)} \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha), \qquad \mathfrak{q} \in \mathcal{P}(\partial B_\rho(o)), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(X), \tag{1.5}$$

with $\mathfrak{m}_\alpha$ concentrated on the geodesic ray from o through α and $\mathfrak{m}_\alpha$ can be identified (via the unitary speed parametrisation of the ray) with $N\omega_N \mathsf{AVR}_X t^{N-1} \mathcal{L}^1 \llcorner_{[0,\infty)}$.

In the more regular setting of $\mathsf{RCD}(0, N)$ spaces one can invoke [37, Theorem 1.1], the so-called "volume cone implies metric cone", so to improve the measure rigidity of Theorem 1.5 valid in the $\mathsf{CD}(0, N)$ setting to the stronger metric rigidity.

Theorem 1.6 ([26, Theorem 1.5]). *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. verifying the $\mathsf{RCD}(0, N)$ condition for some $N > 1$, and having Euclidean volume growth. Then the equality* (1.3) *holds for some bounded set $E \subset X$ if and only if X is isometric to a Euclidean metric measure cone over an $\mathsf{RCD}(N-2, N-1)$ space and E is isometric to the ball centered at one of the tips of X.*

Thus Theorem 1.6 recovers and extends Theorem 1.3 by allowing more general measures (other than the volume measure or the Hausdorff measure). It should be noticed that the hypothesis on the boundedness of E can be dropped. Indeed, it was proven [11] (see also [9, Theorem 1.3] for the unweighted case) that minimizers of the perimeter are bounded for $\mathsf{RCD}(K, N)$ spaces.

We briefly sketch the line of the proof. Our approach starts from the argument used for proving Theorem 1.4, and refines it. We consider the localisation given by the optimal transport between the given set E and its complement inside a large ball of radius R containing E, producing a disjoint family of $\mathsf{CD}(0,N)$ transport rays. Then one can apply the isoperimetric inequality to the traces of E along the transport rays and conclude the proof of (1.3) by taking $R \to \infty$. This alternative proof is contained in Section 4.2.2.

In order to capture the equality case following this proof it is therefore necessary to deal with this limit procedure. The intuition suggests that whenever a region E attains the equality in (1.3) then for large values of R the one-dimensional traces have to be almost optimal. The almost optimality has to be intended in many respects: along each geodesic ray, the diameter has to be almost optimal, the one-dimensional conditional measures has to be almost t^{N-1} and the set has to be almost an interval starting from the starting point of the ray. The main difficulty here is to perform a quantitative analysis of the right order that will not vanish when $R \to \infty$. This is done in Section 4.3 that culminates with Theorem 4.19 where we summarise the crucial stability estimates for the one-dimensional densities and the geometry of the traces of E.

Then the natural prosecution is to take the limit as $R \to \infty$ and hopefully obtain a disintegration of $\mathfrak{m}$ on the whole space having conditional measures verifying the limit estimates. Disintegration formulas are however typically hard to threat under a limit procedure. For instance, the maximality of the transport rays is likely not preserved preventing any chances to get limit estimates. Nonetheless, the almost maximality of E, and all the almost optimal information deduced from it in Section 4.3 permits to bypass this intricate issue and obtain a well behaved limit disintegration. The limit is analysed in Section 4.4 and summarised by Corollary 4.33. The final part of Section 4.4 is then dedicated to the proof that the optimal set E is a ball and the disintegration formula (1.5) (see Theorems 4.38 and 4.45).

The proof is contained in Chapter 4 (see Chapter 2 for the notation).

Isoperimetric inequality in irreversible Finsler manifolds. The third result is taken from a preprint [61].

If a Finsler manifold $(X, F, \mathfrak{m})$ is reversible, then the induced distance is symmetric and we fall into the well-established theory of m.m.s.'s, and in particular the isoperimetric inequality (1.3) holds true. The next theorem extends Theorem 1.2 to the realm of irreversible Finsler manifolds.

Theorem 1.7 ([61, Theorem 1.3])**.** *Let $(X, F, \mathfrak{m})$ be a Finsler manifold (possibly with convex boundary) satisfying the $\mathsf{CD}(0,N)$ condition for some $N > 1$, such that all closed forward balls are compact and having Euclidean volume growth. Then for every Borel subset $E \subset X$ it holds*

$$\mathsf{P}(E) \geq N \omega_N^{\frac{1}{N}} \mathsf{AVR}_X^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}} \tag{1.6}$$

Moreover, inequality (1.6) *is sharp.*

Let us briefly comments Theorem 1.7. By "forward ball" we denotes sets of the form $B^+(x,r) = \{y : \mathsf{d}_F(x,y) < r\}$; similarly, backward balls are defined with the same condition, swapping x and y. For irreversible Finsler manifold it is essential to specify whether a ball is forward or backward, as the two notions do not coincide. The asymptotic volume ratio is then defined using forward balls (see Section 5.1.2). The hypothesis that forward balls are compact, in the light of Hopf–Rinow theorem, is needed for guaranteeing that for every couple of point there exists a curve of minimal length connecting them.

By convex boundary of a Finsler manifold we mean that given two points in the interior, every curve of minimal length connecting them does not touch the boundary. The possible presence of the boundary might harm the equivalence of the definition of $\mathsf{CD}(0,N)$ in the synthetic sense and in the sense of lower bound on the N-Ricci curvature. In this thesis we take as definition of $\mathsf{CD}(0,N)$ for Finsler manifolds the synthetic one (see Section 5.1.2). Indeed, the main technical tool used in this thesis, the localization or needle decomposition, was developed using the synthetic definition of $\mathsf{CD}(0,N)$ [72]. We also specify that the localization theorem was proved for Finsler manifolds without boundary; however, by inspecting the proof [72], one notices that the presence of a convex boundary is not harmful.

Finally, we remark that, to the best of the author knowledge, besides the Barky–Ledoux inequality [74], there is no other isoperimetric inequality for Finsler manifolds that does not involve the reversibility constant.

The proof strategy of Theorem 1.7 is based on the Brunn–Minkowski inequality and follows closely the proof by Balogh and Kristály [14] of Theorem 1.2. Indeed, in the light of [56], it seems that this inequality holds true also for irreversible metric measure spaces; here we confine our-self to the setting of Finsler manifolds.

One can try to prove the inequality using the localization paradigm. If one follows the proof presented in Chapter 3 or in Section 4.2.2, a factor $\Lambda_F^{-1} \leq 1$ would appear (notice that in this level of generality, we do not assume Λ_F to be finite), and thus one would not obtain a sharp inequality. This lack of sharpness is expected, as the isoperimetric inequality for Finsler manifold is not sharp. However, quite surprisingly, using the localization paradigm, following the ideas we developed for proving Theorem 1.5 we can prove the following rigidity result.

Theorem 1.8 ([61, Theorem 1.4])**.** *Let $(X, F, \mathfrak{m})$ be a Finsler manifold (possibly with convex boundary) satisfying the $\mathsf{CD}(0,N)$ condition for some $N > 1$, such that all closed forward balls are compact and it has Euclidean volume growth and finite reversibility constant $\Lambda_F < \infty$. Assume that for all $x, y \notin \partial X$ and for all geodesics γ connecting x to y, it holds that $\gamma_t \notin \partial X$, for all $t \in [0,1]$. Let $E \subset X$ be a bounded Borel set that saturates* (1.6).

Then there exists (a unique) $o \in X$ such that, up to a negligible set, $E = B^+(o,\rho)$, with

$\rho = (\frac{\mathfrak{m}(E)}{\mathsf{AVR}_X\omega_N})^{\frac{1}{N}}$. *Moreover, considering the disintegration of* $\mathfrak{m}$ *with respect to the* 1*-Lipschitz function* $-\mathsf{d}(o,\cdot)$*, the measure* $\mathfrak{m}$ *has the following representation*

$$\mathfrak{m} = \int_{\partial B^+(o,\rho)} \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha), \qquad \mathfrak{q} \in \mathcal{P}(\partial B^+(o,\rho)), \qquad \mathfrak{m}_\alpha \in \mathcal{M}_+(X),$$

with $\mathfrak{m}_\alpha$ *concentrated on the geodesic ray from* o *through* α *and* $\mathfrak{m}_\alpha$ *can be identified (via the unitary speed parametrisation of the ray) with* $N\omega_N \mathsf{AVR}_X t^{N-1}\mathcal{L}^1\llcorner_{[0,\infty)}$.

The hypothesis of the Theorem above contains the requirement that the reversibility constant Λ_F is finite. This assumption is quite expected since in Finsler manifolds with infinite reversibility certain pathological behaviors may arise (e.g., the Sobolev spaces may not be vector spaces [55, 43]).

Remark 1.9. We point out that the proof we present is not immediately applicable to the family of irreversible metric-measure spaces [56], for two reasons. First, there is no localization theorem for irreversible metric-measure spaces. Second, we used the differential structure of Finsler manifolds in a few points of the proof, in particular by using Rademacher Theorem for Lipschitz functions.

The proof is contained in Chapter 5; Appendixes A and B present a few facts about the perimeter in Finsler manifolds which seem to be novel in this setting. The proof follows more or less the line of the proof of the rigidity theorem for $\mathsf{CD}(0,N)$ spaces; therefore, we will not presents all the details and we will focus more on the issues arising from the possible irreversibility.

1.2 Applications in the Euclidean setting

Theorems 1.5 and 1.8 imply new results also in the Euclidean setting, namely the characterisation of optimal regions for the anisotropic isoperimetric inequality for weighted cones.

The setting is the following one: let $\Sigma \subset \mathbb{R}^d$ be an open convex cone with vertex at the origin, and $H : \mathbb{R}^d \to [0,\infty)$ be a *gauge*, that is a nonnegative, convex and positively homogeneous of degree one function. In other words, H would be a (possibly non-smooth) Finsler structure for $\mathbb{R}^d$, which does not depend on the base-point. Moreover, we denote by w a weight that is supposed to be continuous on $\overline{\Sigma}$ and positive and locally Lipschitz in Σ.

For a smooth set $E \subset \mathbb{R}^d$, the *weighted anisotropic perimeter* relative to the cone Σ is given by

$$\mathsf{P}_{w,H}(E;\Sigma) = \int_{\partial E} H(\nu(x))w(x)\, dS,$$

where $\nu(x)$ is the unit outward normal at $x \in \partial E$, and dS the surface measure. The main result of [20] is the sharp isoperimetric inequality for the weighted anisotropic perimeter: if in addition w is positively homogeneous of degree $\alpha > 0$ and $w^{1/\alpha}$ is concave in Σ, then

$$\frac{\mathsf{P}_{w,H}(E;\Sigma)}{w(E\cap\Sigma)^{\frac{N-1}{N}}} \geq \frac{\mathsf{P}_{w,H}(W;\Sigma)}{w(W\cap\Sigma)^{\frac{N-1}{N}}}, \tag{1.7}$$

where $N = d + \alpha$ and W is the Wulff shape associated to H, for the details see [20, Theorem 1.3]. The expression $w(A)$ with $A \subset \mathbb{R}^d$ is a short-hand notation for the integral of w over A.

The inequality (1.7), taking $w = 1$, $\Sigma = \mathbb{R}^d$, and $H = \|\cdot\|_2$, recovers the classical sharp isoperimetric inequality. Taking $w = 1$ and $H = \|\cdot\|_2$, (1.7) gives back the isoperimetric inequality in convex cones originally obtained by Lions and Pacella [57]. Finally, if $w = 1$ $\Sigma = \mathbb{R}^d$ and H be some other gauge, (1.7) is the Wulff inequality.

As observed in [20], Wulff balls W centered at the origin intersected with Σ are always minimizers (1.7). However in [20] a characterization of the equality case (or a proof of uniqueness of those minimizers), is not carried over (see also [50] for a different approach). Despite the many recent contributions (and an announcement in [20] of a the forthcoming work solving the problem), this characterisation, in its full generality, seems to be still not present in the literature.

We now briefly recall the known results. The characterization of the optimal sets has been obtained in the unweighted and isotropic case ($w = 1$ and $H = \|\cdot\|_2$) for smooth cones in [57] and for general cones in [45] via a quantitative analysis. The same approach of [45] has been recently used in [38] to characterize optimal sets in the unweighted and anisotropic case with the gauge H assumed additionally to be a norm with strictly convex unitary ball. Finally [35] extended [38] to the case of H being a positive gauge still uniformly elliptic. The characterisation in weighted setting has been solved in [34] but only in the isotropic case ($H = \|\cdot\|_2$); in [65], the anisotropic case is treated, but the minimizer is assumed to be convex.

It has been already observed [20, 74] that the assumption that $w^{1/\alpha}$ is concave has a natural interpretation as the $\mathsf{CD}(0, N)$ condition, where $N = d + \alpha$; when H is a norm, it has been observed [14] that (1.7) can be obtained as a particular case of (1.3)

To be precise, let $H_0(v) := \sup_{w\neq 0} \frac{w\cdot v}{H(w)}$ be the dual gauge. If H is a norm, then H_0 is a norm as well and one can associate to the triple Σ, H and w the metric measure space $(\overline{\Sigma}, d_{H_0}, w\mathcal{L}^d)$, where $d_{H_0}(x, y) := H_0(x - y)$. On the other hand, if H is not a norm, but it is smooth and strictly convex, then the dual gauge H_0 is a Finsler structure and the triple $(\overline{\Sigma}, H_0, w\mathcal{L}^d)$ is a Finsler manifold (with boundary when Σ is not the full space). In both cases, the perimeter in the sense either of metric measure spaces or of Finsler manifolds, will indeed coincide with $\mathsf{P}_{w,H}$ [33].

Moreover, in both cases, the spaces we consider satisfies the $\mathsf{CD}(0, N)$ condition (see for instance [82] and [41, Proposition 3.3] for the m.m.s.'s setting and [74, Section 10.1] for the setting of Finsler manifolds). The homogeneity properties of H_0 and w, imply that we can compute the a.v.r., just by computing the volume of the unitary ball:

$$\mathsf{AVR}_\Sigma = \lim_{R\to\infty} \frac{\int_{B^+(R)\cap\Sigma} w\, d\mathcal{L}^d}{\omega_N R^N} = \frac{\int_{B^+(1)\cap\Sigma} w\, d\mathcal{L}^d}{\omega_N} > 0.$$

Indeed, the Lebesgue measure scales with power d, whereas w is α homogeneous, thus the measure of a ball scales with power $N = d + \alpha$. Conversely, the perimeter of the rescaled ball is the derivative w.r.t. the scaling factor of the measure, hence the perimeter of the ball with radius 1 is N times its measure. Recalling that the Wulff shape W of H is the unitary ball of the dual gauge H_0, it is therefore clear that (1.7) can be seen as a particular case of (1.3) or (1.6), when H is a norm or H is a strictly convex, smooth gauge, respectively.

We can therefore apply Theorem 1.5 and Theorem 1.8 to the triples $(\overline{\Sigma}, d_{H_0}, w\mathcal{L}^d)$ and $(\overline{\Sigma}, H_0, w\mathcal{L}^d)$, respectively.

Theorem 1.10 ([26, Theorem 1.7] and [61, Theorem 1.5])**.** *Let $\Sigma \subset \mathbb{R}^d$ be an open convex cone with vertex at the origin, and $H : \mathbb{R}^d \to [0,\infty)$ be a gauge with strictly convex balls. Assume that either*

1. *H is a norm, that is $H(v) = H(-v)$, or*

2. *H is smooth.*

Consider moreover the α-homogeneous weight $w : \bar{\Sigma} \to [0,\infty)$ such that $w^{1/\alpha}$ is concave.

Then the equality in (1.7) is attained if and only if $E = W \cap \Sigma$, where W is a rescaled Wulff shape.

Indeed, hypothesis 1. together with the strict convexity of the balls implies that the metric space (Σ, d_{H_0}) is non-branching. On the other hand, hypothesis 2. implies that H_0 is a smooth Finsler manifold. Therefore, the first part of Theorem 1.5·and Theorem 1.8 says that the isoperimetric set is a ball in the dual gauge of H, i.e., it is a rescaled Wulff shape.

To conclude we stress that assumption on w being α-homogeneous can actually be removed and obtained as a consequence of the measure rigidity part of Theorems 1.5 and 1.8 if we consider the modified version of (1.7) with the asymptotic volume ratio, i.e., we assume the r.h.s. of (1.7) to be equal to $(\omega_{n+\alpha}\mathsf{AVR}_\Sigma)^{1/(d+\alpha)} > 0$. In this case, the second part of Theorems 1.5·and 1.8, regarding the disintegration of the measures along the rays, provides an integration formula in polar coordinates, where the Jacobian determinant grows with exponent $N - 1 = d + \alpha - 1$. Since the density of Lebesgue measure in polar coordinates is $(d-1)$-homogeneous, we deduce that w is α-homogeneous.

1.3 Further developments

The approach developed in this thesis does not seem to have run out its potentiality and further developments are possible in many directions. For example, one can try to refine the analysis in order to obtain a quantitative version of the isoperimetric inequality, i.e., quantify the distance of a given set to a candidate isoperimetric set in terms of the deficit of the inequality. One can also investigate other functional or geometric inequalities (isocapacitive inequality, Sobolev inequality, etc.). The hyperbolic setting is interesting, as well: one can investigate the isoperimetric problem for $\mathsf{CD}(K,N)$ manifolds, with $K<0$; in this case the a.v.r. should be substituted with the hyperbolic volume growth.

Since in the special cases of $\mathsf{RCD}(0,N)$ spaces isoperimetric sets are bounded [11], is it possible to prove the same for general $\mathsf{CD}(0,N)$ spaces, and, therefore, get rid of the boundedness hypothesis in Theorem 1.5? The same question can be asked for the irreversible setting.

Finally, it would be nice to get rid of the essentially non-branching-ness hypothesis. This is of particular interest in the setting of Euclidean cones as it would permit to consider crystalline norms.

Chapter 2

Preliminaries

This chapter contains the principal objects referred in the present thesis and in particular in Chapters 3 and 4. In Section 2.1 we review geodesics in the Wasserstein distance and the CD and MCP conditions; in Section 2.2 the perimeter and in Section 2.3 BV functions in the metric setting. The reader familiar with Optimal Transport and the basics of metric-measure spaces can skip these three sections and needs only to check Section 2.4 for the needle decomposition (localization) which is going to be used throughout the thesis.

2.1 Wasserstein distance and lower bounds on the Ricci curvature

We recall a few fact about Optimal Transport that the reader can find with all the details in any book on the topic (e.g. [81, 82, 78]). We take the opportunity to introduce a few notation that will be used throughout this thesis.

Let X be a Polish space. By $\mathcal{M}^+(X)$ and $\mathcal{P}(X)$ we denote the space of non-negative Borel measures on X and the space of probability measures, respectively. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$, and $c : X \times X \to [0, \infty)$ a continuous function. The Kantorovich Optimal Transport problem between μ_0 and μ_1 with cost c is the optimization problem

$$\inf_{\pi} \int_{X\times X} c(x, y)\, \pi(dxdy), \tag{2.1}$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal, i.e., $(P_1)_\sharp\pi = \mu_0, (P_2)_\sharp\pi = \mu_1$. Of course $P_i, i = 1, 2$ is the projection on the first (resp. second) factor and $(P_i)_\sharp$ denotes the corresponding push-forward map on measures. The infimum is always attained, provided that it is finite.

The so-called dual problem (also known as Rubenstein problem) is the following

$$\sup_{\phi,\psi}\left\{\int_X \phi(x)\,\mu_0(dx)+\int_X \psi(x)\,\mu_1(dx)\right\},$$

where the supremum is taken among all functions $\phi,\psi : X\to\mathbb{R}$, such that $\phi(x)+\psi(y)\le c(x,y)$, for all $x,y\in X$. By contrast, the Kantorovich problem is also called primal problem. A fundamental fact in Optimal Transport is that the minimum of the primal problem and the supremum of the dual problem coincide, provided that the minimum in (2.1) is finite. Under certain general hypotheses (see, e.g., [82, Theorem 5.10]) the supremum of the dual problem is attained by a non-unique couple (ϕ,ψ) and ϕ is called Kantorovich potential for μ_0 and μ_1. For our purposes, these hypotheses will always be satisfied.

We specialize to setting of (X,d) being a complete, separable metric space and the cost being $c=\mathsf{d}^p$, $p\in[1,\infty)$. In this case, it is useful to restrict to the set of probability measures of p-th finite moment, i.e., the family of measures $\mu\in\mathcal{P}(X)$, such that $\int_X \mathsf{d}^p(x,o)\,\mu(dx)<\infty$, for some (hence any) $o\in X$. This family will be denoted by $\mathcal{P}_p(X)$. This set can be naturally endowed with the Wasserstein distance

$$W_p(\mu_0,\mu_1)=\left(\inf_{\pi}\int_{X\times X}\mathsf{d}^p(x,y)\,\pi(dxdy)\right)^{\frac{1}{p}}, \tag{2.2}$$

making $(\mathcal{P}_p(X),W_p)$ a complete separable metric spaces.

Denote the space of geodesics of (X,d) by

$$\mathrm{Geo}(X):=\big\{\gamma\in C([0,1],X):\mathsf{d}(\gamma_s,\gamma_t)=|s-t|\mathsf{d}(\gamma_0,\gamma_1),\ \text{for every } s,t\in[0,1]\big\}.$$

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X),W_2)$ can be lifted to a measure $\nu\in\mathcal{P}(\mathrm{Geo}(X))$, so that $(e_t)_\sharp\,\nu=\mu_t$ for all $t\in[0,1]$, where, for each $t\in[0,1]$, e_t is the evaluation map:

$$e_t:\mathrm{Geo}(X)\to X,\qquad e_t(\gamma):=\gamma_t.$$

Given $\mu_0,\mu_1\in\mathcal{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0,\mu_1)$ the space of all $\nu\in\mathcal{P}(\mathrm{Geo}(X))$ for which $(e_0\otimes e_1)_\sharp\,\nu$ realizes the minimum in (2.2). If (X,d) is geodesic, then the set $\mathrm{OptGeo}(\mu_0,\mu_1)$ is non-empty for any $\mu_0,\mu_1\in\mathcal{P}_2(X)$ and, in particular $(\mathcal{P}_2(X),W_2)$ is geodesics, as well.

A m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is a triple with (X,d) a complete and separable metric space and $\mathfrak{m}$ a Borel non negative measure over X. With no loss in generality for our purposes, we will assume that $X=\mathrm{supp}(\mathfrak{m})$.

A set $F\subset\mathrm{Geo}(X)$ is a set of non-branching geodesics if and only if for any $\gamma^1,\gamma^2\in F$, it holds:

$$\exists\, \bar t \in (0,1) \text{ such that } \forall t \in [0,\bar t] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

With this terminology, we recall from [77] the following definition.

Definition 2.1. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially non-branching* if and only if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, with μ_0, μ_1 absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

2.1.1 Curvature-Dimension condition

The $\mathsf{CD}(K,N)$ for condition for m.m.s.'s has been introduced in the seminal works of Sturm [79, 80] and Lott–Villani [58]; here we briefly recall only the basics in the case $K = 0$, $1 < N < \infty$ (the setting of the present thesis). For the general definition of $\mathsf{CD}(K,N)$ see [58, 79, 80].

To state the definition of $\mathsf{CD}(0,N)$ one needs to the define the N-Rényi entropy. If $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s., and $\mu \in \mathcal{P}(X)$, consider the Lebesgue decomposition of μ w.r.t. the reference measure: $\mu = \rho\mathfrak{m} + \mu_s$, with $\mu_s \perp \mathfrak{m}$. The N-Rényi entropy ($N \in [1,\infty)$) is then defined as

$$S_N(\mu|\mathfrak{m}) := -\int_X \rho^{1-\frac{1}{N}}\, d\mathfrak{m}.$$

Definition 2.2 ($\mathsf{CD}(0,N)$)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. We say that $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}(0,N)$ condition if and only if the N'-Rényi entropy is convex along the geodesics of the Wasserstein space $\forall N' \geq N$, that is, for any couple of absolutely continuous curves $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there exists a geodesics $(\mu_t)_{t\in[0,1]} \in \mathrm{Geo}(\mathcal{P}_2(X))$ connecting μ_0 to μ_1, such that

$$S_{N'}(\mu_t|\mathfrak{m}) \leq (1-t)S_{N'}(\mu_0|\mathfrak{m}) + tS_{N'}(\mu_1|\mathfrak{m}), \qquad \forall N' \geq N.$$

For essentially non-branching spaces, a different definition was given, which was proven to be equivalent to the previous one [29].

Definition 2.3 ($\mathsf{CD}(0,N)$ for essentially non-branching spaces)**.** An essentially non-branching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\mathsf{CD}(0,N)$ if and only if for all $\mu_0, \mu_1 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$, there exists a unique $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$, ν is induced by a map (i.e. $\nu = S_\sharp(\mu_0)$, for some map $S : X \to \mathrm{Geo}(X)$), $\mu_t := (e_t)_\#\nu \ll \mathfrak{m}$ for all $t \in [0,1]$, and writing $\mu_t = \rho_t\mathfrak{m}$, we have for all $t \in [0,1]$:

$$\rho_t^{-1/N}(\gamma_t) \geq (1-t)\,\rho_0^{-1/N}(\gamma_0) + t\,\rho_1^{-1/N}(\gamma_1) \quad \text{for } \nu\text{-a.e. } \gamma \in \mathrm{Geo}(X).$$

If (M,g) is a Riemannian manifold of dimension n and $h \in C^2(M)$, with $h > 0$, then the m.m.s. $(M, \mathsf{d}_g, h\ \mathrm{Vol}_g)$ verifies $\mathsf{CD}(0, N)$ with $N \geq n$ if and only if (see Theorem 1.7 of [80])

$$\mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g - (N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}} \geq 0,$$

in other words if and only if the weighted Riemannian manifold $(M, g, h\ \mathrm{Vol}_g)$ has non-negative generalized N-Ricci tensor. If $N = n$ the generalized N-Ricci tensor $\mathrm{Ric}_{g,h,N} = \mathrm{Ric}_g$ requires h to be constant. In particular, in the case of one-dimensional manifolds, the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$ ($I \subset \mathbb{R}$ is an interval) is $\mathsf{CD}(0, N)$, if and only if

$$\left(h^{\frac{1}{N-1}}\right)'' \leq 0, \qquad \text{in the sense of distributions.}$$

2.1.2 Measure Contraction Property

We now briefly describe the Measure–Contraction Property (MCP). Similarly to the $\mathsf{CD}(K, N)$ condition, we confine our-self to the presentation of the case $K = 0$. For the details, the reader should refer to the paper [67] where this condition was introduced and investigated (see also [80]).

Definition 2.4 ($\mathsf{MCP}(0, N)$)**.** A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is said to satisfy $\mathsf{MCP}(0, N)$ if for any $o \in \mathrm{supp}(\mathfrak{m})$ and $\mu_0 \in \mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ of the form $\mu_0 = \frac{1}{\mathfrak{m}(A)}\mathfrak{m}\llcorner_A$ for some Borel set $A \subset X$ with $0 < \mathfrak{m}(A) < \infty$, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \delta_o)$ such that:

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (\mathrm{e}_t)_\sharp\big((1-t)^N \nu(d\gamma)\big), \qquad \forall t \in [0, 1]. \tag{2.3}$$

The MCP condition can be see as the limiting case of the $\mathsf{CD}(0, N)$ condition when $\mu_0 = \frac{\mathfrak{m}}{\mathfrak{m}(A)}$ and $\mu_1 = \delta_0$. For this reason, the MCP condition is strictly weaker than the CD condition.

If $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s. verifying $\mathsf{MCP}(K, N)$, no matter for which $K \in \mathbb{R}$, then $(\mathrm{supp}(\mathfrak{m}), \mathsf{d})$ is Polish, proper and it is a geodesic space. It should also be noticed that the MCP condition is not local.

The MCP condition justifies its nature as lower bound on the Ricci curvature by the following important fact [67, Theorem 3.2]: if (M, g) is n-dimensional Riemannian manifold with $n \geq 2$, the m.m.s. (M, d_g, Vol_g) verifies $\mathsf{MCP}(K, n)$ if and only if $\mathrm{Ric}_g \geq Kg$, where d_g is the geodesic distance induced by g and Vol_g the volume measure.

If $(M, g, h\,\mathrm{Vol}_g)$ is a weighted manifold and we substitute Ric with the N-Ricci tensor, the just-stated fact is not true anymore. For our purposes, it is interesting to investigate the $\mathsf{MCP}(0, N)$ condition for one-dimensional manifolds, i.e., for the m.m.s. of the form $(I, |\cdot|, h\mathcal{L}^1)$, where $I \subset \mathbb{R}$ is an interval and $h : I \to [0, \infty)$ is a locally integrable function. It is immediate to see that for the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$ the inequality (2.3) for the measure $h\mathcal{L}^1$ is equivalent to the following inequality for the density h:

$$h(tx_1 + (1-t)x_0) \geq (1-t)^{N-1} h(x_0), \qquad \forall x_0, x_1 \in I, \, \forall t \in [0,1], \tag{2.4}$$

see for instance [18, Theorem 9.5] where also the case $K \neq 0$ is discussed. We will call h an $\mathsf{MCP}(0,N)$-density.

Inequality (2.4) implies several known properties that we recall for readers' convenience. If we confine ourselves to the case $I = (a,b)$ with $a, b \in \mathbb{R}$ (2.4) implies (actually is equivalent to)

$$\left(\frac{b - x_1}{b - x_0} \right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{x_1 - a}{x_0 - a} \right)^{N-1},$$

for $x_0 \leq x_1$. If we consider the unbounded case $I = [0, +\infty)$, (2.4) is equivalent to

$$1 \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{x_1}{x_0} \right)^{N-1}. \tag{2.5}$$

In both cases, h is locally Lipschitz in the interior of I and continuous up to the boundary.

We also point out that if $(I, |\cdot|, \mathfrak{m})$ is a $\mathsf{MCP}(0,N)$ space, then $\mathfrak{m}$ is absolutely continuous w.r.t. the Lebesgue measure and therefore the Radon-Nikodym derivative h enjoys all the useful properties we described in the paragraphs above.

Unless otherwise stated, we shall always assume that the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is essentially non-branching and satisfies $\mathsf{CD}(0,N)$, for some $N > 2$ with $\mathrm{supp}(\mathfrak{m}) = X$. This implies directly that (X, d) is a geodesic, complete, and locally compact metric space.

2.2 Perimeter in metric measure spaces

Having in mind classical [2, 3, 66] and more recent [5] literature, we present the definition of perimeter and the basilar related facts. The result presented in this section work for a generic m.m.s., non-necessarily verifying a curvature condition.

Given $u \in \mathrm{Lip}(X)$, the space of real-valued Lipschitz functions over X, its local Lipschitz constant (also known as slope) $|Du|(x)$ at $x \in X$ is defined by

$$|Du|(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{\mathsf{d}(x,y)}.$$

Given a Borel subset $E \subset X$ and Ω open, the perimeter of E relative to Ω is denoted by $\mathsf{P}(E;\Omega)$ and is defined as follows

$$\mathsf{P}(E;\Omega) := \inf \left\{ \liminf_{n \to \infty} \int_\Omega |Du_n| \, d\mathfrak{m} \, : \, u_n \in \mathrm{Lip}(\Omega), \, u_n \to \mathbf{1}_E \text{ in } L^1(\Omega, \mathfrak{m}) \right\}.$$

We say that $E \subset X$ has finite perimeter in X if $\mathsf{P}(E; X) < \infty$. We recall also few properties of the perimeter functions:

(a) (locality) $\mathsf{P}(E; \Omega) = \mathsf{P}(F; \Omega)$, whenever $\mathfrak{m}((E\Delta F) \cap \Omega) = 0$;

(b) (l.s.c.) the map $E \mapsto \mathsf{P}(E; \Omega)$ is lower-semicontinuous with respect to the $L^1_{loc}(\Omega)$ convergence;

(c) (complementation) $\mathsf{P}(E; \Omega) = \mathsf{P}(X \backslash E; \Omega)$.

Moreover, if E is a set of finite perimeter, then the set function $\Omega \to \mathsf{P}(E; A)$ is the restriction to open sets of a finite Borel measure $\mathsf{P}(E; \cdot)$ in X (see Lemma 5.2 of [5]), defined by

$$\mathsf{P}(E; A) := \inf\{\mathsf{P}(E; \Omega) \colon \Omega \supset A, \ \Omega \text{ open}\}.$$

In order to simplify the notation, we will write $\mathsf{P}(E)$ instead of $\mathsf{P}(E; X)$. Finally, we recall that the perimeter can be seen [6] as the l.s.c. envelope (in the L^1 topology) of the Minkowski content

$$\mathfrak{m}^+(E) := \liminf_{\epsilon \to 0} \frac{\mathfrak{m}(E^\epsilon) - \mathfrak{m}(E)}{\epsilon},$$

where $E^\epsilon = \{x \in X : \text{dist}(x, E) < \epsilon\}$. In other words, given $E \subset X$ there exists a sequence E_n such that $\mathfrak{m}(E \bigtriangleup E_n) \to 0$ and $\mathfrak{m}^+(E_n) \to \mathsf{P}(E)$ [6, Theorem 3.6]. We point out that, inspecting the proof contained in [6], one can assume the sets E_n to be bounded.

The *isoperimetric profile function* of $(X, \mathsf{d}, \mathfrak{m})$, denoted by $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}$, is defined as the point-wise maximal function so that $\mathsf{P}(A) \geq \mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mathfrak{m}(A))$ for every Borel set $A \subset X$, that is

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf \big\{\mathsf{P}(A) \colon A \subset X \ \text{ Borel}, \ \mathfrak{m}(A) = v\big\}.$$

Given $\mathcal{X}$ a family of m.m.s. we can consider $\mathcal{I}_{\mathcal{X}}$ the isoperimetric profile of family $\mathcal{X}$, as the point-wise maximal function so that $\mathsf{P}(A) \geq \mathcal{I}_{\mathcal{X}}(\mathfrak{m}(A))$, for $A \subset X$ Borel and $X \in \mathcal{X}$, that is

$$\mathcal{I}_{\mathcal{X}}(v) = \inf\{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) : (X, \mathsf{d}, \mathfrak{m}) \in \mathcal{X}\}.$$

For example, $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$ (resp. $\mathcal{I}^{\mathsf{MCP}}_{K,N,D}$) denote the isoperimetric profile for the family of normalized (i.e., of unitary mass) $\mathsf{CD}(K, N)$ (resp. $\mathsf{MCP}(K, N)$) spaces having diameter not larger than D. Milman [63] (see also [28] for the non-smooth setting) and Cavalletti–Santarcangelo [32] gave a rather explicit description of the functions $\mathcal{I}^{\mathsf{CD}}_{K,N,D}$ and $\mathcal{I}^{\mathsf{MCP}}_{K,N,D}$, respectively.

2.3 BV functions in metric measure spaces

The classical theory of the perimeter in $\mathbb{R}^n$ makes extensive use of BV function. The notion of BV functions in the setting of m.m.s. has been first introduced in [66] and then more deeply studied in [5]. In particular, three different definitions of BV functions has been proven to be equivalent.

One of these three notions is given by relaxation of the energy functional. We say that a function $f \in L^1(X)$ is in $BV_*((X, \mathsf{d}, \mathfrak{m}))$, if there exists a sequence $f_n \in \mathrm{Lip}(X) \cap L^1(X)$ converging to f in L^1, such that $\sup_n \int_X |\nabla f_n| \, d\mathfrak{m} < \infty$. In this case one can define the relaxed total variation

$$|Df|_*(\Omega) := \inf \left\{ \liminf_{n \to \to \infty} \int_X |Df_n| \, d\mathfrak{m} : f_n \in \mathrm{Lip}_{loc}(\Omega), \ f_n \to f \text{ in } L^1(\Omega) \right\},$$

where $\Omega \subset X$ is an open set. It has been shown [66] that the total variation extends uniquely to a finite Borel measure.

Another definition of BV functions is given using test plans. We say that a probability measure $\pi \in \mathcal{P}(C([0,1]; X))$ is a ∞-test plan if: 1) π is concentrated on Lipschitz-continuous curves; 2) there exists a constant $C = C(\pi) > 0$ (named *compression* of the test plan) such that $(e_t)_\# \pi \leq C\mathfrak{m}$. A Borel subset $\Gamma \subset C([0,1]; X)$ is said to be 1-negligible if $\pi(\Gamma) = 0$, for every ∞-test plan π. We say that a function $f \in L^1(X)$ is of weak-bounded variation ($f \in$ w-$BV((X, \mathsf{d}, \mathfrak{m}))$), if the following two conditions holds

1. there exists a 1-negligible subset Γ such that $f \circ \gamma \in BV((0,1))$ for all $\gamma \in C([0,1]; X) \backslash \Gamma$ and
$$|f(\gamma_0) - f(\gamma_1)| \leq |D(f \circ \gamma)|((0,1));$$

2. there exists a measure $\mu \in \mathcal{M}^+(X)$ such that for every ∞-test plan π, for every Borel set $B \subset X$ we have that
$$\int \gamma_\# |D(f \circ \gamma)|(B) \, \pi(d\gamma) \leq C(\pi) \Big\| \sup_{t \in [0,1]} |\dot{\gamma}_t| \Big\|_{L^\infty(\pi)} \mu(B). \tag{2.7}$$

Moreover, one can prove that there exists a least measure satisfying (2.7). Such measure is named weak total variation and it is denoted by $|Df|_w$.

Theorem 2.5 ([5, Theorem 1.1])**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric measure space, with $\mathfrak{m}$ a locally finite Borel measure (i.e. for all $x \in X$ there exists $r > 0$ such that $\mathfrak{m}(B_r(x)) < \infty$). Then the spaces $BV_*((X, \mathsf{d}\, \mathfrak{m}))$ and w-$BV((X, \mathsf{d}, \mathfrak{m}))$ coincide and for every* *function* $f \in BV_*((X,\mathsf{d},\mathfrak{m})) = w\text{-}BV((X,\mathsf{d},\mathfrak{m}))$ *it holds*

$$|Df|_*(B) = |Df|_w(B), \quad \textit{for every Borel set } B.$$

It is clear that a set $E \subset X$ has finite perimeter whenever $\mathbf{1}_E \in BV_*((X,\mathsf{d},\mathfrak{m}))$ and in this case it holds

$$\mathsf{P}(E;\Omega) = |D\mathbf{1}_E|_*(\Omega) = |D\mathbf{1}_E|_w(\Omega), \quad \forall \Omega \subset X \text{ open}.$$

2.4 Localization

The localization method reduces the task of establishing various analytic and geometric inequalities on a full dimensional space to the one-dimensional setting.

In the Euclidean setting goes back to Payne and Weinberger [75], it has been developed and popularised by Gromov and V. Milman [49], Lovász–Simonovits [59], and Kannan–Lovasz–Simonovits [53]. In 2015, Klartag [54] reinterpreted the localization method as a measure disintegration adapted to L^1-Optimal-Transport, and extended it to weighted Riemannian manifolds satisfying $\mathsf{CD}(K,N)$. Cavalletti and Mondino [28] have succeeded to generalise this technique to essentially non-branching m.m.s.'s verifying the $\mathsf{CD}(K,N)$, condition with $N \in (1,\infty)$.

Localization for $\mathsf{MCP}(K,N)$ was, partially and in a different form, already known in 2009, see [18, Theorem 9.5], for non-branching m.m.s.'s. The case of essentially non-branching m.m.s.'s and the effective reformulation after the work of Klartag [54] has been recently discussed in [31, Section 3] to which we refer for all the missing details (see in particular [31, Theorem 3.5]).

Here we only report the case $K = 0$.

Theorem 2.6 (Localization on $\mathsf{MCP}(0,N)$ spaces [31, Theorem 3.5])**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an essentially non-branching m.m.s. with* $\mathrm{supp}(\mathfrak{m}) = X$ *and satisfying* $\mathsf{MCP}(0,N)$*, for some* $N \in (1,\infty)$.

Let $f : X \to R$ *be* $\mathfrak{m}$*-integrable with* $\int_X f\,\mathfrak{m} = 0$ *and* $\int_X |f(x)|\mathsf{d}(x,x_0)\,\mathfrak{m}(dx) < \infty$ *for some (hence for all)* $x_0 \in X$*. Then there exists an* $\mathfrak{m}$*-measurable subset* $\mathcal{T} \subset X$ *(named transport set) and a family* $\{X_\alpha\}_{\alpha \in Q}$ *of subsets of* X*, such that there exists a disintegration of* $\mathfrak{m}\llcorner_{\mathcal{T}}$ *on* $\{X_\alpha\}_{\alpha \in Q}$*:*

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha),$$

and for $\mathfrak{q}$*-a.e.* $\alpha \in Q$*:*

1. X_α *is a closed geodesic in* (X,d).

2. *$\mathfrak{m}_\alpha$ is a Radon measure supported on X_α with $\mathfrak{m}_\alpha \ll \mathcal{H}^1 \llcorner_{X_\alpha}$.*

3. *$(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{MCP}(0, N)$.*

   If the space $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(0, N)$, then also $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{CD}(0, N)$.

4. *$\int f \, d\mathfrak{m}_\alpha = 0$, and $f = 0$ $\mathfrak{m}$-a.e. on $X \setminus \mathcal{T}$.*

Moreover, the X_α are called transport rays and two distinct transport rays can only meet at their extremal points (having measure zero for $\mathfrak{m}_\alpha$).

Few comments are in order.
By $\mathcal{H}^1$ we denote the one-dimensional Hausdorff measure on the underlying metric space.

Given $\{X_\alpha\}_{\alpha \in Q}$ a partition of X, a disintegration of $\mathfrak{m}$ on $\{X_\alpha\}_{\alpha \in Q}$ is a measure space structure $(Q, \mathcal{Q}, \mathfrak{q})$ and a map

$$Q \ni \alpha \mapsto \mathfrak{m}_\alpha \in \mathcal{M}(X, \mathcal{X})$$

such that

1. For $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on X_α.

2. For all $B \in \mathcal{X}$, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable.

3. For all $B \in \mathcal{X}$, $\mathfrak{m}(B) = \int_Q \mathfrak{m}_\alpha(B) \, \mathfrak{q}(d\alpha)$; this is abbreviated by $\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha)$.

We point out that the disintegration is unique for fixed $\mathfrak{q}$. That means that, if there is a family $(\tilde{\mathfrak{m}}_\alpha)_\alpha$ satisfying the conditions above, then for $\mathfrak{q}$-a.e. α, $\mathfrak{m}_\alpha = \tilde{\mathfrak{m}}_\alpha$. If we change $\mathfrak{q}$ with a different measure $\widehat{\mathfrak{q}}$, such that $\widehat{\mathfrak{q}} = \rho \mathfrak{q}$, then the map $\alpha \mapsto \rho(\alpha)\mathfrak{m}_\alpha$ still satisfies the conditions above, with $\widehat{\mathfrak{q}}$ in place of $\mathfrak{q}$.

Concerning the fact that $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{CD}(0, N)$, since (X_α, d) is a geodesic, it is isometric to a real interval and therefore the $\mathsf{CD}(0, N)$ condition is equivalent to have $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner_{X_\alpha}$ and $h_\alpha^{\frac{1}{N-1}}$ being concave (here we are identifying X_α with a real interval).

2.4.1 L^1-optimal transportation

In this section we recall only some facts from the theory of L^1 optimal transportation which are of some interest for this thesis; we refer to [4, 8, 18, 23, 27, 42, 44, 54, 81] and references therein for more details on the theory of L^1 optimal transportation.

Theorem 2.6 has been proven studying the optimal transportation problem between $\mu_0 := f^+\mathfrak{m}$ and $\mu_1 := f^-\mathfrak{m}$, where $f^\pm$ denote the positive and the negative part of f, with the distance as cost function.

By the summability properties of f (see the hypothesis of Theorem 2.6) one deduces the existence of an L^1-Kantorovich potential φ, solution of the dual problem. Using φ we can construct the set

$$\Gamma := \{(x,y) \in X \times X : \varphi(x) - \varphi(y) = \mathsf{d}(x,y)\},$$

inducing a partial order relation whose maximal chains produce a partition made of one dimensional sets of a certain subset of the space, provided the ambient space X verifies some mild regulartiy properties.

This procedure has been already presented and used in several contributions ([8, 18, 44, 54, 81]) when the ambient space is the euclidean space, a manifold or a non-branching metric space (see [18, 21] for extended metric spaces). The analysis in our framework started with [23] and has been refined and extended in [27]; we will follow the notation of [27] to which we refer for more details.

The *transport relation* $\mathcal{R}^e$ and the *transport set with end-points* $\mathcal{T}^e$ are defined as:

$$\mathcal{R}^e := \Gamma \cup \Gamma^{-1} = \{|\varphi(x) - \varphi(y)| = \mathsf{d}(x,y)\}, \quad \mathcal{T}^e := P_1(\mathcal{R}^e \setminus \{x = y\}),$$

where $\{x = y\}$ denotes the diagonal $\{(x,y) \in X^2 : x = y\}$ and $\Gamma^{-1} = \{(x,y) \in X \times X : (y,x) \in \Gamma\}$. Since φ is 1-Lipschitz, Γ, Γ^{-1} and $\mathcal{R}^e$ are closed sets and therefore, from the local compactness of (X, d), σ-compact; consequently $\mathcal{T}^e$ is σ-compact.

We restrict $\mathcal{T}^e$ to a smaller set where $\mathcal{R}^e$ is an equivalent relation. To exclude possible branching we need to consider the following sets, introduced in [23]:

$$\begin{aligned} A^+ &:= \{x \in \mathcal{T}^e : \exists z, w \in \Gamma(x), (z,w) \notin \mathcal{R}^e\}, \\ A^- &:= \{x \in \mathcal{T}^e : \exists z, w \in \Gamma^{-1}(x), (z,w) \notin \mathcal{R}^e\}; \end{aligned} \tag{2.8}$$

where $\Gamma(x) = \{y \in X : (x,y) \in \Gamma\}$ denotes the section of Γ through x in the first coordinate; $\Gamma^{-1}(x)$ and $\mathcal{R}^e(x)$ are defined in the same way. $A^{\pm}$ are called the sets of forward and backward branching points, respectively. Note that both $A^{\pm}$ are σ-compact sets. Then the non-branched transport set has been defined as

$$\mathcal{T} := \mathcal{T}^e \setminus (A^+ \cup A^-),$$

and it is a Borel set; in the same way define the non-branched relation as $\mathcal{R} = \mathcal{R}^e \cap (\mathcal{T} \times \mathcal{T})$. It was shown in [23] (cf. [18]) that $\mathcal{R}$ is an equivalence relation over $\mathcal{T}$ and that for any $x \in \mathcal{T}$, $\mathcal{R}(x) \subset (X, \mathsf{d})$ is isometric to a closed interval in $(\mathbb{R}, |\cdot|)$.

A priori the non-branched transport set $\mathcal{T}$ can be much smaller than $\mathcal{T}^e$. However, under fairly general assumptions one can prove that the sets $A^{\pm}$ of forward and backward branching are both $\mathfrak{m}$-negligible. In [23, Proposition 4.5] this was shown for a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ verifying

$\mathsf{RCD}^*(K,N)$ and $\text{supp}(\mathfrak{m}) = X$. The same proof works for an essentially non-branching m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfying $\mathsf{CD}(0,N)$ and $\text{supp}(\mathfrak{m}) = X$ (see [29]).

One can chose $Q \subset \mathcal{T}$ a Borel section of the equivalence relation $\mathcal{R}$ (this choice is possible as it was shown in [18, Proposition 4.4]). Define the quotient map $\mathfrak{Q} : \mathcal{T} \to Q$ as $\mathfrak{Q}(x) = \alpha$, where α is the unique element of $\mathcal{R}(x) \cap Q$. Given a finite measure $\mathfrak{q} \in \mathcal{M}^+(Q)$, such that $\mathfrak{q} \ll \mathfrak{Q}_\#(\mathfrak{m}\llcorner_{\mathcal{T}})$, the Disintegration Theorem applied to $(\mathcal{T}, \mathcal{B}(\mathcal{T}), \mathfrak{m}\llcorner_{\mathcal{T}})$, gives an essentially unique disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}}$ consistent with the partition of $\mathcal{T}$ given by the equivalence classes $\{\mathcal{R}(\alpha)\}_{\alpha\in Q}$ of $\mathcal{R}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha).$$

In the sequel, we will use also the notation X_α to denote the equivalence class $\mathcal{R}(\alpha)$. Note that such measure $\mathfrak{q}$ can always be build, by taking the push-forward via $\mathfrak{Q}$ of a suitable finite measure absolutely continuous w.r.t. $\mathfrak{m}_{\mathcal{T}}$.

The existence of a measurable section also permits to construct a measurable parametrization of the transport rays. First define the (possibly infinite) length of a transport ray $|X_\alpha| := \sup_{x,y\in X_\alpha} \mathsf{d}(x,y)$. Then, we can define

$$g : \text{Dom}(g) \subset Q \times [0,+\infty) \to \mathcal{T}$$

that associates to (α,t) the unique $x \in \mathcal{R}(\alpha)$ in such a way $\varphi(g(\alpha,t)) - \varphi(g(\alpha,s)) = s - t$, provided $t, s \in (0, |X_\alpha|)$. In other words, $g(\alpha, \cdot\,)$ is the unit-speed, maximal parametrization of X_α such that $\frac{d}{dt}\varphi(g(\alpha,t)) = -1$. We specify that this parametrization ensures that $f(g(\alpha,0)) \geq 0$. By continuity of g w.r.t. the variable t, we extend g, in order to map also the end-points of the rays X_α; the restriction of g to the set $\{(\alpha,t) : t \in (0,|X_\alpha|)\}$ is injective.

Finally to prove that the disintegration is $\mathsf{CD}(0,N)$, i.e. that for $\mathfrak{q}$-a.e. $\alpha \in Q$ the space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is $\mathsf{CD}(0,N)$, one uses the presence of the L^2-Wasserstein geodesics inside the transport set $\mathcal{T}$ (see [22, Lemma 4.6]). We refer to [28, Theorem 4.2] for all the details.

The measure $\mathfrak{m}_\alpha$ will be absolutely continuous w.r.t. $\mathcal{H}^1\llcorner_{X_\alpha}$ as a consequence of the $\mathsf{CD}(0,N)$ condition in one-dimensional spaces: there exists a map $h_\alpha : (0,|X_\alpha|) \to \mathbb{R}$ such that

$$\mathfrak{m}_\alpha = (g(\alpha, \cdot\,))_\#(h_\alpha \mathcal{L}^1\llcorner_{(0,|X_\alpha|)}).$$

The construction does not depend on the function f but only on the L^1-Kantorovich potential φ.

Theorem 2.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. with $\text{supp}(\mathfrak{m}) = X$ and satisfying $\mathsf{CD}(0,N)$, for some $N \in (1,\infty)$. Assume that $\varphi : X \to R$ is a 1-Lipschitz function, and let $\mathcal{T}$ and $(X_\alpha)_{\alpha\in Q}$ be respectively the transport set and the transport rays as they were* *defined in the previous paragraphs. Let Q and $\mathfrak{Q} : \mathcal{T} \to Q$ be the quotient set and the quotient map, respectively, and assume that there exists a measure $\mathfrak{q} \ll \mathfrak{Q}_\#(\mathfrak{m}_\mathcal{T})$. Then there exists a disintegration of $\mathfrak{m}\llcorner_\mathcal{T}$ on $\{X_\alpha\}_{\alpha\in Q}$*

$$\mathfrak{m}\llcorner_\mathcal{T} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

and for $\mathfrak{q}$-a.e. $\alpha \in Q$:

1. *X_α is a closed geodesic in (X, d).*
2. *$\mathfrak{m}_\alpha$ is a Radon measure supported on X_α with $\mathfrak{m}_\alpha \ll \mathcal{H}^1\llcorner_{X_\alpha}$.*
3. *The metric measure space $(X_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ verifies $\mathsf{CD}(0, N)$.*

Theorem 2.6 follows from the previous theorem, provided that we are able to localize constraint $\int_X f \, d\mathfrak{m} = 0$. The localization is a consequence of the properties of the L^1-optimal transport problem (see [28, Theorem 5.1]).

Chapter 3

Sharp isoperimetric inequality in $\mathsf{MCP}(0, N)$ spaces

We present the proof of Theorem 1.4.

3.1 One dimensional reduction

To prove Theorem 1.4 we will need to consider the isoperimetric problem inside a family of large subsets of X with diameter approaching ∞. In order to apply the classical dimension reduction argument furnished by localization theorem (Theorem 2.6), one needs in principle these subsets to also be convex. As the existence of an increasing family of convex subsets recovering at the limit the whole space X is in general false, we will overcome this issue in the following way.

Given any bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then consider $R > 0$ such that $E \subset B_R$ (hereinafter we will adopt the following notation $B_R := B_R(x_0)$). Consider then the following family of zero mean functions:

$$g_R(x) = \left(\chi_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \right) \chi_{B_R}.$$

Clearly g_R satisfies the hypothesis of Theorem 2.6 so we obtain an $\mathfrak{m}$-measurable subset $\mathcal{T}_R \subset X$ and a family $\{X_{\alpha,R}\}_{\alpha \in Q_R}$ of transport rays, such that there exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_R}$ on $\{X_{\alpha,R}\}_{\alpha \in Q_R}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}_R} = \int_{Q_R} \mathfrak{m}_{\alpha,R}\, \mathfrak{q}_R(d\alpha), \qquad \mathfrak{q}_R(Q_R) = 1, \tag{3.1}$$

with the Radon measures $\mathfrak{m}_{\alpha,R}$ having an $\mathsf{MCP}(0, N)$ density with respect to $\mathcal{H}^1\llcorner_{X_{\alpha,R}}$. The localization of the zero mean implies that

$$\mathfrak{m}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\mathfrak{m}_{\alpha,R}(B_R), \qquad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R.$$

By using a unit speed parametrisation of the geodesic $X_{\alpha,R}$, without loss of generality we can assume that $\mathfrak{m}_{\alpha,R} = h_{\alpha,R}\mathcal{L}^1 \llcorner_{[0,\ell_{\alpha,R}]}$, where $\ell_{\alpha,R}$ denotes the (possibly infinite) length of the $X_{\alpha,R}$. Also we specify that the direction of the parametrisation of $X_{\alpha,R}$ is chosen such that $0 \in E$. Equivalently, if u_R denotes a Kantorovich potential associated to the localization of g_R, then the parametrisation is chosen in such a way that u_R is decreasing along $X_{\alpha,R}$ with slope -1.

Now we can define $T_{\alpha,R}$ to be the unique element of $[0,\ell_{\alpha,R}]$ such that $\mathfrak{m}_{\alpha,R}([0,T_{\alpha,R}]) = \mathfrak{m}_{\alpha,R}(B_R)$. Notice that $\mathrm{diam}(B_R \cap X_{\alpha,R}) \leq R + \mathrm{diam}(E)$: if γ is a unit speed parametrization of $X_{\alpha,R}$, then $\mathsf{d}(\gamma_0,\gamma_t) \leq \mathsf{d}(\gamma_0,x_0) + \mathsf{d}(\gamma_t,x_0) \leq \mathrm{diam}(E) + R$, provided $\gamma_t \in B_R \cap X_{\alpha,R}$. Hence the same upper bound is valid for $T_{\alpha,R}$, i.e. $T_{\alpha,R} \leq R + \mathrm{diam}(E)$.

The plan will be to restrict $\mathfrak{m}_{\alpha,R}$ to $[0,T_{\alpha,R}]$ so to have the following disintegration:

$$\mathfrak{m}\llcorner_{\bar{\mathcal{T}}_R} = \int_{Q_R} \bar{\mathfrak{m}}_{\alpha,R}\, \bar{\mathfrak{q}}_R(d\alpha), \tag{3.2}$$

where $\bar{\mathfrak{m}}_{\alpha,R} := \mathfrak{m}_{\alpha,R}\llcorner_{[0,T_{\alpha,R}]}/\mathfrak{m}_{\alpha,R}(B_R)$ are probability measures, $\bar{\mathfrak{q}}_R = \mathfrak{m}_{\cdot,R}(B_R)\mathfrak{q}_R$ (in particular $\bar{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$, using (3.1) and the fact that $B_R \subset \mathcal{T}_R$) and $\bar{\mathcal{T}}_R = \cup_{\alpha\in Q_R}[0,T_{\alpha,R}]$, where we are identifying $[0,T_{\alpha,R}]$ with the geodesic segment of length $T_{\alpha,R}$ of $X_{\alpha,R}$, that will be denoted by $\bar{X}_{\alpha,R}$

The disintegration (3.2) will have applications only if $(E \cap X_{\alpha,R}) \subset [0,T_{\alpha,R}]$, implying that

$$\bar{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \qquad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R.$$

To prove this inclusion we will impose that $E \subset B_{R/4}$. If we denote by $\gamma^{\alpha,R} : [0,\ell_{\alpha,R}] \to X_{\alpha,R}$ the unit speed parametrisation, we notice that

$$\mathsf{d}(\gamma_t^{\alpha,R},x_0) \leq \mathsf{d}(\gamma_0^{\alpha,R},x_0) + t \leq \mathrm{diam}(E) + t \leq \frac{R}{2} + t,$$

where in the second inequality we have used that each starting point of the transport ray has to be inside E, being precisely where $g_R > 0$. Hence $\gamma_t^{\alpha,R} \in B_R$ for all $t < R/2$. This implies that $(\gamma^{\alpha,R})^{-1}(B_R) \supset [0,\min\{R/2,\ell_{\alpha,R}\}]$, hence "no holes" inside $(\gamma^{\alpha,R})^{-1}(B_R)$ before $\min\{R/2,\ell_{\alpha,R}\}$, implying that $T_{\alpha,R} \geq \min\{R/2,\ell_{\alpha,R}\}$.

Since $\mathrm{diam}(E) \leq R/2$, necessarily $(\gamma^{\alpha,R})^{-1}(E) \subset [0,\min\{R/2,\ell_{\alpha,R}\}]$ implying that $(E \cap X_{\alpha,R}) \subset [0,T_{\alpha,R}]$. We summarise this construction in the following

Proposition 3.1. *Given any bounded $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then fix $R > 0$ such that $E \subset B_{R/4}(x_0)$.*

Then there exists a Borel set $\bar{\mathcal{T}}_R \subset X$, with $E \subset \bar{\mathcal{T}}_R$ and a disintegration formula

$$\mathfrak{m}\llcorner_{\bar{\mathcal{T}}_R} = \int_{Q_R} \bar{\mathfrak{m}}_{\alpha,R}\, \bar{\mathfrak{q}}_R(d\alpha), \qquad \bar{\mathfrak{m}}_{\alpha,R}(\bar{X}_{\alpha,R}) = 1, \qquad \bar{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R),$$

such that $\bar{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}$, $\bar{\mathfrak{q}}_R$-a.e. and the one-dimensional m.m.s. $(\bar{X}_{\alpha,R}, \mathsf{d}, \bar{\mathfrak{m}}_{\alpha,R})$ verifies $\mathsf{MCP}(0,N)$ and has diameter bounded by $R + \mathrm{diam}(E)$.

3.2 One dimensional analysis

Proposition 3.1 produces a family of normalized (i.e., of mass one) one-dimensional $\mathsf{MCP}(0,N)$ space, whose diameter is uniformly bounded by $D = R + \mathrm{diam}\, E$. For such a family of spaces the isoperimetric profile function $\mathcal{I}^{\mathsf{MCP}}_{0.N,D}$ has been investigated [32]. The following theorem summerizes the properties of $\mathcal{I}^{\mathsf{MCP}}_{0,N,D}$ that we need (it is stated for $K \in \mathbb{R}$, although we are interested only in the case $K = 0$).

Theorem 3.2 ([32])**.** *Let $K, N, D \in \mathbb{R}$ with $N > 1$ and $D > 0$. Then there exists an explicit non-negative function $\mathcal{I}^{\mathsf{MCP}}_{K,N,D} : [0,1] \to \mathbb{R}$ such that the following holds.*

If $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$ with $\mathfrak{m}(X) = 1$ and having diameter less than D and $A \subset X$, then

$$\mathfrak{m}^+(A) \geq \mathcal{I}^{\mathsf{MCP}}_{K,N,D}(\mathfrak{m}(A)). \tag{3.3}$$

Moreover (3.3) is sharp, i.e. for each $v \in [0,1]$, K, N, D there exists a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m}(X) = 1$ and $A \subset X$ with $\mathfrak{m}(A) = v$ such that (3.3) is an equality.

Finally, if $K = 0$, $D, D' > 0$, then

$$\frac{D'}{D}\, \mathcal{I}^{\mathsf{MCP}}_{0,N,D'} = \mathcal{I}^{\mathsf{MCP}}_{0,N,D}. \tag{3.4}$$

The sharp lower bound on the isoperimetric profile function (3.3) has an explicit expression. We report only the case $K = 0$: for each $v \in [0,1]$

$$\mathcal{I}^{\mathsf{MCP}}_{0,N,D}(v) = f_{0,N,D}(a_{0,N,D}(v)), \tag{3.5}$$

where the function $f_{0,N,D}$ is defined in the following way

$$f_{0,N,D}(x) := \left(\int_{(0,x)} \left(\frac{D-y}{D-x} \right)^{N-1} dy + \int_{(x,D)} \left(\frac{y}{x} \right)^{N-1} dy \right)^{-1},$$

and the function $a_{0,N,D}$ is obtained as follows: define the function

$$v_{0,N,D}(a) = \frac{f_{0,N,D}(a)}{(D-a)^{N-1}} \int_{(0,a)} (D-x)^{N-1}\,dx = f_{0,N,D}(a)\frac{D^N-(D-a)^N}{N(D-a)^{N-1}};$$

as proved in [32], for each N, D it is possible to define the inverse map of v:

$$[0,1] \ni v \longmapsto a_{0,N,D}(v) \in (0,D);$$

hence we have recalled the definition of each function used in (3.5) to construct the lower bound $\mathcal{I}^{\mathsf{MCP}}_{0,N,D}$.

We now look for a simple expansion of $\mathcal{I}^{\mathsf{MCP}}_{0,N,D}(v)$ for v close to 0.

Lemma 3.3. *Fix $N>1$. Then the following estimate for $\mathcal{I}^{\mathsf{MCP}}_{0,N,D}$ hold true for all $D>0$*

$$\mathcal{I}^{\mathsf{MCP}}_{0,N,D}(v) = \frac{N^{\frac{1}{N}}}{D}\left(v^{\frac{N-1}{N}} + o(v^{\frac{N-1}{N}})\right). \tag{3.6}$$

Proof. We start with the case $D=1$ and we recall that

$$f_{0,N,1}(x) = \left(\int_{(0,x)} \left(\frac{1-y}{1-x}\right)^{N-1} dy + \int_{(x,1)} \left(\frac{y}{x}\right)^{N-1} dy\right)^{-1},$$

obtaining

$$\begin{aligned} f_{0,N,1}(x) &= \left(\frac{1-(1-x)^N}{N(1-x)^{N-1}} + \frac{1-x^N}{Nx^{N-1}}\right)^{-1} = N\left(\frac{1}{(1-x)^{N-1}} - 1 + \frac{1}{x^{N-1}}\right)^{-1} \\ &= Nx^{N-1}\left(\left(\frac{x}{1-x}\right)^{N-1} - x^{N-1} + 1\right)^{-1} = Nx^{N-1} + o(x^{N-1}). \end{aligned}$$

Then looking at $v_{0,N,1}(a)$

$$\begin{aligned} v_{0,N,1}(a) &= f_{0,N,1}(a)\frac{1-(1-a)^N}{N(1-a)^{N-1}} \\ &= f_{0,N,1}(a)(a+o(a)) = Na^N + o(a^N), \end{aligned}$$

giving that $a_{0,N,1}(v) = N^{-\frac{1}{N}}v^{\frac{1}{N}} + o(v^{\frac{1}{N}})$ and implying that

$$\mathcal{I}^{\mathsf{MCP}}_{0,N,1}(v) = Na_{0,N,1}(v)^{N-1} + o(a_{0,N,1}(v)^{N-1}) = N^{\frac{1}{N}}v^{\frac{N-1}{N}} + o(v^{\frac{N-1}{N}}). \tag{3.7}$$

To obtain the general case when D is arbitrary, we use (3.4). yielding $\mathcal{I}^{\mathsf{MCP}}_{0,N,1} = D\mathcal{I}^{\mathsf{MCP}}_{0,N,D}$. This means that (3.7) implies (3.6). □

We now deduce Theorem 1.4 from the expansion of (3.6) and Theorem 3.2 applied to a family of one-dimensional spaces. The dimensional reduction argument is a classical application of the localization paradigm.

Theorem 3.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(0,N)$ and having $\mathsf{AVR}_X > 0$. Let $E \subset X$ be any Borel set with $\mathfrak{m}(E) < \infty$, then*

$$\mathsf{P}(E) \geq (N\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}\, \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{3.8}$$

Proof. Assume first E to be a bounded set. Let $x_0 \in E$ be any point and consider $R > 0$ such that $E \subset B_{R/4}(x_0)$; Proposition 3.1 implies we have the following disintegration

$$\mathfrak{m}\llcorner_{\bar{\mathcal{T}}_R} = \int_{Q_R} \bar{\mathfrak{m}}_{\alpha,R}\, \bar{\mathfrak{q}}_R(d\alpha), \qquad \bar{\mathfrak{m}}_{\alpha,R}(\bar{X}_{\alpha,R}) = 1, \qquad \bar{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R),$$

with $E \subset \bar{\mathcal{T}}_R$ and for $\bar{\mathfrak{q}}_R$-a.e. $\alpha \in Q_R$, $\bar{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}$. At this point we can compute the outer Minkowski content of E:

$$\begin{aligned}
\mathfrak{m}^+(E) &= \liminf_{\varepsilon \to 0} \frac{\mathfrak{m}(E^\varepsilon) - \mathfrak{m}(E)}{\varepsilon} \geq \liminf_{\varepsilon \to 0} \frac{\mathfrak{m}(E^\varepsilon \cap \bar{\mathcal{T}}_R) - \mathfrak{m}(E)}{\varepsilon} \\
&\geq \int_{Q_R} \liminf_{\varepsilon \to 0} \frac{\bar{\mathfrak{m}}_{\alpha,R}(E^\varepsilon) - \bar{\mathfrak{m}}_{\alpha,R}(E)}{\varepsilon}\, \bar{\mathfrak{q}}_R(d\alpha) \geq \int_{Q_R} \bar{\mathfrak{m}}^+_{\alpha,R}(E)\, \bar{\mathfrak{q}}_R(d\alpha) \\
&\geq \int_{Q_R} \mathcal{I}^{\mathsf{MCP}}_{0,N,T_{\alpha,R}}\left(\mathfrak{m}(E)/\mathfrak{m}(B_R)\right)\, \bar{\mathfrak{q}}_R(d\alpha),
\end{aligned}$$

where the last inequality follows from $(X_{\alpha,R}, \mathsf{d}, \bar{\mathfrak{m}}_{\alpha,R})$ being an $\mathsf{MCP}(0,N)$ one-dimensional space, and $T_{\alpha,R} \leq R + \operatorname{diam}(E)$ was introduced to obtain (3.2).

Equation (3.4) yields

$$\mathcal{I}^{\mathsf{MCP}}_{0,N,T_{\alpha,R}}(v) = \frac{R + \operatorname{diam}(E)}{T_{\alpha,R}}\, \mathcal{I}^{\mathsf{MCP}}_{0,N,R+\operatorname{diam}(E)}(v) \geq \mathcal{I}^{\mathsf{MCP}}_{0,N,R+\operatorname{diam}(E)}(v),$$

implying the following inequality:

$$\mathfrak{m}^+(E) \geq \mathfrak{m}(B_R)\mathcal{I}^{\mathsf{MCP}}_{0,N,R+\operatorname{diam}(E)}\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right).$$

We continue the chain of inequalities by (3.6):

$$\mathfrak{m}^+(E) \geq \frac{\mathfrak{m}(B_R) N^{\frac{1}{N}}}{R + \operatorname{diam}(E)} \left(\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)^{\frac{N-1}{N}} + o\left(\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)^{\frac{N-1}{N}} \right) \right).$$

From the hypothesis of Euclidean volume growth ($\mathsf{AVR}_X > 0$), one infers that $R \sim \mathfrak{m}(B_R)^{\frac{1}{N}}$

for large values of R, hence we can then take the limit as $R \to \infty$ to obtain

$$\mathfrak{m}^+(E) \geq (N\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}\, \mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{3.9}$$

We now drop the assumption that E is bounded. If E is a possibly unbounded set, then there exists a sequence E_n of bounded sets such that $\mathfrak{m}(E \triangle E_n) \to 0$ and $\mathfrak{m}^+(E_n) \to \mathsf{P}(E)$ (see Section 2.2). We pass to the limit in (3.9) and, since $\mathfrak{m}^+(E) \geq \mathsf{P}(E)$, we conclude. □

3.3 Sharp Inequality

As one can expect from the sharpness of the isoperimetric inequality for compact $\mathsf{MCP}(0,N)$ spaces obtained in [32], also inequality (3.8) is sharp. In particular, if we fix $a, v > 0$, $N > 1$, we can find a $\mathsf{MCP}(0,N)$ space $(X, \mathsf{d}, \mathfrak{m})$, with $\mathsf{AVR}_X = a$, and a subset $E \subset X$ such that $\mathfrak{m}(E) = v$ and $\mathfrak{m}^+(E) = (N\omega_N \mathsf{AVR}_X)^{\frac{1}{N}} \mathfrak{m}(E)^{\frac{N-1}{N}}$. Indeed, consider the one-dimensional space $([0,\infty), |\cdot|, h\mathcal{L})$, with

$$h(x) = \begin{cases} (N\omega_N a)^{\frac{1}{N}} v^{\frac{N-1}{N}} & \text{if } x \leq \left(\dfrac{v}{N\omega_N a}\right)^{\frac{1}{N}}, \\ N\omega_N a x^{N-1} & \text{if } x \geq \left(\dfrac{v}{N\omega_N a}\right)^{\frac{1}{N}}. \end{cases}$$

It is easy to check that h satisfies (2.5) with $K = 0$ and that $\mathsf{AVR}_{([0,\infty),|\cdot|,h\mathcal{L})} = a$. We take $E = [0, (\frac{v}{N\omega_N a})^{\frac{1}{N}}]$, and we trivially have $(h\mathcal{L})(E) = v$ and

$$(h\mathcal{L})^+(E) = h\left(\left(\frac{v}{N\omega_N a}\right)^{\frac{1}{N}}\right) = (N\omega_N a)^{\frac{1}{N}} v^{\frac{N-1}{N}},$$

which corresponds to equality in inequality (3.8). This easy observation concludes, together with Theorem 3.4, the proof of Theorem 1.4.

Chapter 4

Rigidity of the isoperimetric inequality in $\mathsf{CD}(0, N)$ spaces

In this chapter we prove Theorem 1.5.

4.1 One dimensional reduction

In this section we proceed in a way similar to what we have done in Section 3.1, that is, we consider the the isoperimetric problem inside a family of large subsets of X with diameter approaching ∞. Then we will apply the localization theorem obtaining a family of one-dimensional spaces and we will disintegrate the reference measure consistently with the partition. Differently from Section 3.1, we will also provide a sort of "disintegration formula" for the perimeter, i.e., we prove that the perimeter of E in X (considered as a measure) controls the perimeters of E in the rays.

We proceed in the following way.

Given any bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then consider $R > 0$ such that $E \subset B_R$ (hereinafter we will adopt the following notation $B_R := B_R(x_0)$). Consider then the following family of zero mean functions:

$$f_R(x) = \left(\chi_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \right) \chi_{B_R}.$$

Clearly f_R satisfies the hypothesis of Theorem 2.6 so we obtain an $\mathfrak{m}$-measurable subset $\mathcal{T}_R \subset X$ and a family $\{X_{\alpha,R}\}_{\alpha \in Q_R}$ of transport rays, such that there exists a disintegration of $\mathfrak{m}\llcorner_{\mathcal{T}_R}$ on $\{X_{\alpha,R}\}_{\alpha \in Q_R}$:

$$\mathfrak{m}\llcorner_{\mathcal{T}_R} = \int_{Q_R} \mathfrak{m}_{\alpha,R}\, \mathfrak{q}_R(d\alpha), \qquad \mathfrak{q}_R(Q_R) = \mathfrak{m}(\mathcal{T}_R), \tag{4.1}$$

with the probability measures $\mathfrak{m}_{\alpha,R}$ having an $\mathsf{CD}(0,N)$ density with respect to $\mathcal{H}^1 \llcorner_{X_{\alpha,R}}$. The localization of the zero mean implies that

$$\mathfrak{m}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \mathfrak{m}_{\alpha,R}(B_R), \qquad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R. \tag{4.2}$$

We denote by $g_R(\alpha, \cdot\,) : [0, |X_{\alpha,R}|]$ the unit speed parametrisation of the geodesic $X_{\alpha,R}$. For this reason, it holds

$$\mathfrak{m}_{\alpha,R} = (g_R(\alpha, \cdot\,))_\#(h_{\alpha,R} \mathcal{L}^1 \llcorner_{[0,|X_{\alpha,R}|]}),$$

for some $\mathsf{CD}(0,N)$ density $h_{\alpha,R}$.

Also we specify that the direction of the parametrisation of $X_{\alpha,R}$ is chosen such that $g_R(\alpha, 0) \in E$. Equivalently, if φ_R denotes a Kantorovich potential associated to the localization of g_R, then the parametrisation is chosen in such a way that φ_R is decreasing along $X_{\alpha,R}$ with slope -1.

We then define $T_{\alpha,R}$ to be the unique element of $[0, |X_{\alpha,R}|]$ such that

$$\mathfrak{m}_{\alpha,R}(g_R(\alpha, [0, T_{\alpha,R}])) = \mathfrak{m}_{\alpha,R}(B_R) :$$

since $\mathfrak{m}_{\alpha,R}$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner_{X_{\alpha,R}}$ the existence of a unique $T_{\alpha,R}$ follows. Moreover from the measurability in α of $\mathfrak{m}_{\alpha,R}$ we deduce the same measurability for $T_{\alpha,R}$.

Notice that $\operatorname{diam}(B_R \cap X_{\alpha,R}) \leq R + \operatorname{diam}(E)$: since $g_R(\alpha, \cdot\,)$ is a unit speed parametrization of $X_{\alpha,R}$, then $\mathsf{d}(g_R(\alpha,0), g_R(\alpha,t)) \leq \mathsf{d}(g_R(\alpha,0), x_0) + \mathsf{d}(g_R(\alpha,t), x_0) \leq \operatorname{diam}(E) + R$, provided $g_R(\alpha,t) \in B_R \cap X_{\alpha,R}$. Hence the same upper bound is valid for $T_{\alpha,R}$, i.e. $T_{\alpha,R} \leq R + \operatorname{diam}(E)$.

We restrict $\mathfrak{m}_{\alpha,R}$ to $\widehat{X}_{\alpha,R} := g_R(\alpha, [0, T_{\alpha,R}])$ so to have the following disintegration:

$$\mathfrak{m} \llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R} \, \widehat{\mathfrak{q}}_R(d\alpha), \qquad \widehat{\mathfrak{m}}_{\alpha,R} := \frac{\mathfrak{m}_{\alpha,R} \llcorner_{\widehat{X}_{\alpha,R}}}{\mathfrak{m}_{\alpha,R}(B_R)} \in \mathcal{P}(X), \qquad \widehat{\mathfrak{q}}_R = \mathfrak{m}_{\cdot,R}(B_R) \mathfrak{q}_R; \tag{4.3}$$

where $\widehat{\mathcal{T}}_R := \cup_{\alpha \in Q_R} \widehat{X}_{\alpha,R}$; in particular $\widehat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$, using (4.1) and the fact that $B_R \subset \mathcal{T}_R$.

The disintegration (4.3) will be a localisation like (4.2) only if $(E \cap X_{\alpha,R}) \subset \widehat{X}_{\alpha,R}$, implying that

$$\widehat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \qquad \widehat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R.$$

To prove this inclusion we will impose that $E \subset B_{R/4}$. Since $g_R(\alpha, \cdot\,) : [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ has unit speed, we notice that

$$\mathsf{d}(g_R(\alpha,t), x_0) \leq \mathsf{d}(g_R(\alpha,0), x_0) + t \leq \operatorname{diam}(E) + t \leq \frac{R}{2} + t,$$

where in the second inequality we have used that each starting point of the transport ray has to be inside E, being precisely where $f_R > 0$. Hence $g_R(\alpha, t) \in B_R$ for all $t < R/2$. This implies that $((g_R(\alpha, \cdot))^{-1}(B_R) \supset [0, \min\{R/2, |X_{\alpha,R}|\}]$, hence "no holes" inside $(g_R(\alpha, \cdot))^{-1}(B_R)$ before $\min\{R/2, |X_{\alpha,R}|\}$, implying that $|\widehat{X}_{\alpha,R}| \geq \min\{R/2, |X_{\alpha,R}|\}$. Since $\mathrm{diam}(E) \leq R/2$, we deduce that $(g_R(\alpha, \cdot))^{-1}(E) \subset [0, \min\{R/2, |X_{\alpha,R}|\}]$ implying that $(E \cap X_{\alpha,R}) \subset \widehat{X}_{\alpha,R}$.

We can give an explicit description of the measure $\widehat{\mathfrak{q}}_R$ in term of a push-forward via the quotient map $\mathfrak{Q}_R$ of the measure $\mathfrak{m}\llcorner_E$

$$
\begin{aligned}
\widehat{\mathfrak{q}}_R(A) &= \int_{Q_R} \mathbf{1}_A(\alpha) \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \widehat{\mathfrak{m}}_{\alpha,R}(E)\, \widehat{\mathfrak{q}}_R(d\alpha) \\
&= \int_{Q_R} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \widehat{\mathfrak{m}}_{\alpha,R}(E \cap \mathfrak{Q}_R^{-1}(A))\, \widehat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \mathfrak{m}(E \cap \mathfrak{Q}_R^{-1}(A)),
\end{aligned}
$$

hence $\widehat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)$.

We need to study the relation between the perimeter and the disintegration of the measure (4.3). Fix $\Omega \subset X$ an open set and consider the relative perimeter $\mathsf{P}(E;\Omega)$. Let $u_n \in \mathrm{Lip}_{loc}(\Omega)$ be a sequence such that $u_n \to \mathbf{1}_E$ in $L^1_{loc}(\Omega)$ and $\lim_{n\to\infty} \int_\Omega |Du_n|\, d\mathfrak{m} = \mathsf{P}(E;\Omega)$. Using the Fatou Lemma, we can compute

$$
\begin{aligned}
\mathsf{P}(E;\Omega) &= \lim_{n\to\infty} \int_\Omega |Du_n|\, d\mathfrak{m} \geq \liminf_{n\to\infty} \int_{\Omega \cap \widehat{\mathcal{T}}_R} |Du_n|\, d\mathfrak{m} \\
&= \liminf_{n\to\infty} \int_{Q_R} \int_\Omega |Du_n|\, \widehat{\mathfrak{m}}_{\alpha,R}(dx)\, \widehat{\mathfrak{q}}_R(d\alpha) \\
&\geq \int_{Q_R} \liminf_{n\to\infty} \int_\Omega |Du_n|\, \widehat{\mathfrak{m}}_{\alpha,R}(dx)\, \widehat{\mathfrak{q}}_R(d\alpha) \\
&\geq \int_{Q_R} \liminf_{n\to\infty} \int_{X_{\alpha,R} \cap \Omega} |u_n'|\, \widehat{\mathfrak{m}}_{\alpha,R}(dx)\, \widehat{\mathfrak{q}}_R(d\alpha) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\Omega)\, \widehat{\mathfrak{q}}_R(d\alpha),
\end{aligned}
$$

where u_n' denotes the derivative along the curve $g_R(\alpha, \cdot)$ and $\mathsf{P}_{\widehat{X}_{\alpha,R}}$ the perimeter m.m.s. $(\widehat{X}_{\alpha,R}, \mathsf{d}, \widehat{\mathfrak{m}}_{\alpha,R})$.

By arbitrariness of Ω, we deduce the following disintegration inequality

$$
\mathsf{P}(E;\cdot) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\cdot)\, \widehat{\mathfrak{q}}_R(d\alpha).
$$

Moreover, the fact that the geodesic $g_R(\alpha, \cdot) : [0, |\widehat{X}_{\alpha,R}|] \to \widehat{X}_{\alpha,R}$ has unit speed, implies that

$$
\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\cdot) = (g_R(\alpha,\cdot))_\#(\mathsf{P}_{h_{\alpha,R}}((g_R(\alpha,\cdot))^{-1}(E);\cdot)).
$$

We summarise this construction in the following

Proposition 4.1. *Given any bounded $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then fix $R > 0$ such that $E \subset B_{R/4}(x_0)$.*

Then there exists a Borel set $\widehat{\mathcal{T}}_R \subset X$, with $E \subset \widehat{\mathcal{T}}_R$ and a disintegration formula

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha), \qquad \widehat{\mathfrak{m}}_{\alpha,R}(\widehat{X}_{\alpha,R}) = 1, \qquad \widehat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R), \tag{4.4}$$

such that

$$\widehat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \quad \text{for } \widehat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R \quad \text{and} \quad \widehat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E), \tag{4.5}$$

and the one-dimensional m.m.s. $(\widehat{X}_{\alpha,R}, \mathsf{d}, \widehat{\mathfrak{m}}_{\alpha,R})$ verifies $\mathsf{CD}(0,N)$ and has diameter bounded by $R + \mathrm{diam}(E)$. Furthermore, the following formula holds true

$$\mathsf{P}(E;\,\cdot\,) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\,\cdot\,)\, \widehat{\mathfrak{q}}_R(d\alpha). \tag{4.6}$$

The rescaling introduced in Proposition 4.1 will be crucially used to obtain non-trivial limit estimates as $R \to \infty$.

4.2 One dimensional analysis

Proposition 4.1 is the first step to obtain from the optimality of a bounded set E an almost optimality of $E \cap \widehat{X}_{\alpha,R}$. We now have to analyse in details the one-dimensional isoperimetric profile function. This analysis has the same flavor of the analysis in Section 3.2, but in this case the investigation will be carried out in much more details. We also fix few notation and conventions.

We will be considering the m.m.s. $(I, |\cdot|, h\mathcal{L}^1)$, with $I \subset \mathbb{R}$ an interval and verifying the $\mathsf{CD}(0,N)$ condition; when the interval has finite diameter, we will always assume that $I = [0,D]$. We will assume also that $\int_0^D h = 1$, unless otherwise specified. For consistency with the conditional measures from Disintegration theorem, we will use the notation $\mathfrak{m}_h = h\,\mathcal{L}^1$.

We also introduce the functions $v_h : [0,D] \to [0,1]$ and $r_h : [0,1] \to [0,D]$ as

$$v_h(r) := \int_0^r h(s)\, ds, \qquad r_h(v) := (v_h)^{-1}(v);$$

notice that from the $\mathsf{CD}(0,N)$ condition, $h > 0$ over I making v_h invertible and in turn the definition of r_h well-posed.

We will denote by P_h the perimeter in the space $([0,D], |\cdot|, h\mathcal{L}^1_{[0,D]})$. If $E \subset [0,D]$ is a set of finite perimeter, then it can be decomposed (up to a negligible set) in a family of disjoint intervals

$$E = \bigcup_i (a_i, b_i),$$

and the union is at most countable. In this case we have that the perimeter is given by the formula

$$\mathsf{P}_h(E) = \sum_{i: a_i \neq 0} h(a_i) + \sum_{i: b_i \neq D} h(b_i).$$

We shall denote by $\mathcal{I}_h$ the isoperimetric profile $\mathcal{I}_h(v) := \inf_{E: \mathfrak{m}_h(E)=v} \mathsf{P}_h(E)$.

4.2.1 Properties of the isoperimetric profile function

If $(I, |\cdot|, h\mathcal{L}^1)$ is a $\mathsf{CD}(0,N)$ space with diameter at most D, then $\mathcal{I}_h(v) \geq \mathcal{I}^{\mathsf{CD}}_{0,N,D}(v)$

E. Milman [63] gave an explicit description of the isoperimetric profile function $\mathcal{I}^{\mathsf{CD}}_{0,N,D}$, in the smooth setting, whereas Cavalletti–Mondino [28] generalized the result to the non-smooth setting (also in the case $K \neq 0$). The isoperimetric profile computed by Milman [63, Corollary 1.4, Case 4] (see [28, Section 6.1] for the non-smooth analog) is indeed given by the formula

$$\mathcal{I}^{\mathsf{CD}}_{0,N,D}(v) := \frac{N}{D} \inf_{\xi \geq 0} \frac{\left(\min\{v, 1-v\}(\xi+1)^N + \max\{v, 1-v\}\xi^N\right)^{\frac{N-1}{N}}}{(\xi+1)^N - \xi^N},$$

and it is obtained by optimising among a family of one-dimensional spaces. In order to keep the notation short, we will write $\mathcal{I}_{N,D}$ in place of $\mathcal{I}^{\mathsf{CD}}_{0,N,D}$.

We shortly describe the optimization procedure carried out by Milman, and meanwhile we introduce some notation that will be used in the sequel. For our purpose, we consider the model spaces $([0,D], |\cdot|, h_{N,D}(\xi, \cdot)\mathcal{L}^1\llcorner_{[0,D]})$, for $N > 1$, $D > 0$, and, $\xi \geq 0$, where

$$h_{N,D}(\xi, x) := \frac{N}{D^N} \frac{(x + \xi D)^{N-1}}{(\xi+1)^N - \xi^N}. \tag{4.7}$$

For the model spaces, we can easily compute the two functions $v_{N,D}(\xi, \cdot) := v_{h_{N,D}(\xi,\cdot)}$ and $r_{N,D}(\xi, \cdot) := r_{h_{N,D}(\xi,\cdot)}$

$$\begin{aligned} v_{N,D}(\xi, r) &= \frac{(r + \xi D)^N - (\xi D)^N}{D^N((1+\xi)^N - \xi^N)}, \\ r_{N,D}(\xi, v) &= D\left(\left(v(1+\xi)^N + (1-v)\xi^N\right)^{\frac{1}{N}} - \xi\right). \end{aligned} \tag{4.8}$$

We can easily deduce that if E is an isoperimetric set of measure $v \in (0,1)$ for a model space, then (up to a negligible set)

$$E = \begin{cases} [0, r_{N,D}(\xi, v)], & \text{if } v \leq \frac{1}{2}, \\ [r_{N,D}(\xi, 1-v), D], & \text{if } v \geq \frac{1}{2}, \end{cases}$$

with the convention that if $v = \frac{1}{2}$, both cases are possible. Indeed, if $v \leq \frac{1}{2}$ we can "push" all the mass to left obtaining a new set $E' = [0, r_{N,D}(\xi, v)]$; the monotonicity of $h_{N,D}(\xi, \cdot)$ ensures that E' has smaller perimeter than E'. If, on the contrary, $v \geq \frac{1}{2}$, then we have that the complementary $[0, D]\backslash E$ is an isoperimetric set, then $[0, D]\backslash E = [0, r_{N,D}(1-v)]$. This allows us to explicitly compute the isoperimetric profile of the model spaces

$$\begin{aligned}\mathcal{I}_{N,D}(\xi, v) &= h_{N,D}(\xi, r_{N,D}(\min\{v, 1-v\})) \\ &= \frac{N}{D} \frac{\left(\min\{v, 1-v\}(\xi+1)^N + \max\{v, 1-v\}\xi^N\right)^{\frac{N-1}{N}}}{(\xi+1)^N - \xi^N}.\end{aligned}$$

It is therefore clear that

$$\mathcal{I}_{N,D}(v) = \inf_{\xi \geq 0} \mathcal{I}_{N,D}(\xi, v),$$

We also define an auxiliary function $\mathcal{G}_N$ as

$$\mathcal{G}_N(\xi, v) := \frac{\left((\xi+1)^N + (\frac{1}{v} - 1)\xi^N\right)^{\frac{N-1}{N}}}{(\xi+1)^N - \xi^N}. \tag{4.9}$$

Notice that, if $v \leq \frac{1}{2}$, then

$$\mathcal{G}_N(\xi, v) = \frac{D}{N} \frac{\mathcal{I}_{N,D}(\xi, v)}{v^{1-\frac{1}{N}}}.$$

One advantage of this function is that it is not depending on D. This is indeed quite natural, as the isoperimetric profile scales with D.

We obtain the following lower bound, which is the analogous for CD spaces of Lemma 3.3, proved for MCP spaces.

Lemma 4.2. *Fix $N > 1$. Then, we have the following estimate for $\mathcal{I}_{N,D}$*

$$\mathcal{I}_{N,D}(w) \geq \frac{N}{D} w^{1-\frac{1}{N}} (1 - O(w^{\frac{1}{N}})) = \frac{N}{D}(w^{1-\frac{1}{N}} - O(w)), \qquad \textit{as } w \to 0.$$

Proof. Recalling the definition of $\mathcal{G}_N$, what we have to prove becomes

$$\inf_{\xi \geq 0} \mathcal{G}_N(\xi, w) \geq 1 - O(w^{\frac{1}{N}}), \qquad \text{as } w \to 0.$$

The minimum in the infimum in the expression above is attained, at least for all w small enough. Indeed, we have that

$$\mathcal{G}_N(\xi,v)=\frac{\Big(\big(1+\xi^{-1}\big)^N+\big(\frac{1}{w}-1\big)\Big)^{\frac{N-1}{N}}}{\xi\Big(\big(1+\xi^{-1}\big)^N-1\Big)}=\frac{\Big(\big(1+\xi^{-1}\big)^N+\big(\frac{1}{w}-1\big)\Big)^{\frac{N-1}{N}}}{\xi\left(1+N\xi^{-1}-o\left(\xi^{-1}\right)-1\right)}, \tag{4.10}$$

thus the limit $\lim_{\xi\to\infty}\mathcal{G}_N(\xi,w)=(\frac{1}{w}-1)^{(N-1)/N}/N\geq 1=\mathcal{G}_N(0,w)$ implies the coerciveness of $\xi\mapsto\mathcal{G}_N(\xi,w)$. Define $\xi_w\in\arg\min_{\xi\in[0,\infty]}\mathcal{G}_N(\xi,w)$; we trivially have that $\mathcal{G}_N(\xi_w,w)\leq 1$.

First we prove that $\limsup_{w\to0}\xi_w<\infty$ (we soon will improve this estimate). Suppose the contrary, i.e., there exists some sequence $w_n\to0$ such that $\xi_{w_n}\to\infty$. Then we have

$$\begin{aligned}1\geq\limsup_{n\to\infty}\mathcal{G}_N(\xi_{w_n},w_n)&\geq\limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}\xi_{w_n}^{N-1}}{(\xi_{w_n}+1)^N-\xi_{w_n}^N}\\&=\limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}}{\xi_{w_N}(1+N\xi_{w_n}^{-1}+o(\xi_{w_n}^{-1})-1)}=\infty,\end{aligned} \tag{4.11}$$

which is a contradiction. Since $\limsup_{w\to0}\xi_w<\infty$, then we have $(\xi_w+1)^{N-1}-\xi_w^{N-1}\leq C$, for all w small enough, for some constant $C>0$. We improve the estimate above

$$1\geq\limsup_{w\to0}\mathcal{G}_N(\xi_w,w)\geq\limsup_{w\to0}\frac{((\frac{1}{w}-1)\xi_w^N)^{\frac{N-1}{N}}}{(\xi_w+1)^{N-1}-\xi_w^N}\geq\limsup_{w\to0}\frac{((\frac{1}{w}-1)\xi_w^N)^{\frac{N-1}{N}}}{C}, \tag{4.12}$$

which implies $\limsup_{w\to0}\xi_w\leq0$, i.e., $\xi_w\to0$ as $w\to0$. We can improve the estimate again

$$1\geq\limsup_{w\to0}\mathcal{G}_N(\xi_w,w)=\limsup_{w\to0}\frac{\Big((1+\xi_w)^N+\frac{\xi_w^N}{w}-\xi_w^N\Big)^{\frac{N-1}{N}}}{(\xi_w+1)^{N-1}-\xi_w^N}=\left(1+\limsup_{w\to0}\frac{\xi_w^N}{w}\right)^{\frac{N-1}{N}}, \tag{4.13}$$

yielding $\limsup_{w\to0}\xi_w/w^{\frac{1}{N}}\leq0$, i.e., $\xi_w=o(w^{\frac{1}{N}})$ as $w\to0$. Finally we can conclude noticing that

$$\inf_{\xi\geq0}\mathcal{G}_N(\xi,w)=\mathcal{G}_N(\xi_w,w)=\frac{\big((\xi_w+1)^N+\big(\frac{1}{w}-1\big)\,\xi_w^N\big)^{\frac{N-1}{N}}}{(\xi_w+1)^N-\xi_w^N}=\frac{(1+o(1))^{\frac{N-1}{N}}}{1+O(\xi_w)}=1-O(w^{\frac{1}{N}}).\ \square$$

Corollary 4.3. *Fix $N>1$. Then for all $D\geq D'>0$ and for all $h:[0,D']\to\mathbb{R}$ satisfying the $\mathsf{CD}(0,N)$ condition it holds that*

$$\mathsf{P}_h(E)\geq\mathcal{I}_h(\mathfrak{m}_h(E))\geq\frac{N}{D'}\mathfrak{m}_h(E)^{1-\frac{1}{N}}(1-O(\mathfrak{m}_h(E)^{\frac{1}{N}})\geq\frac{N}{D}\mathfrak{m}_h(E)^{1-\frac{1}{N}}(1-O(\mathfrak{m}_h(E)^{\frac{1}{N}}),$$

for any Borel set $E\subset[0,D']$.

4.2.2 Alternative proof of Theorem 1.2

We re-obtain Theorem 1.2 via localization, following the line of the proof of Theorem 3.4.

Theorem 4.4. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an essentially non-branching* $\mathsf{CD}(0,N)$ *space having* $\mathsf{AVR}_X>0$. *Let* $E\subset X$ *be any bounded Borel set then*

$$\mathsf{P}(E)\geq N\omega_N^{\frac{1}{N}}\mathsf{AVR}_X^{\frac{1}{N}}\mathfrak{m}(E)^{\frac{N-1}{N}}. \tag{4.14}$$

Proof. Let $x_0\in E$ be any point. We then consider $R>0$ such that $E\subset B_{R(x)}$. For shortness we will write $B_R=B_R(x_0)$. We use Propsition 4.1 and in particular (4.6), obtaining

$$\mathsf{P}(E)\geq\int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E)\,\widehat{\mathfrak{q}}_R(d\alpha). \tag{4.15}$$

Using Corollary 4.3, and the fact that each ray $\widehat{X}_{\alpha,R}$ has length at most $\operatorname{diam}E+R$, we deduce

$$\begin{aligned}\mathsf{P}(E)&\geq\int_{Q_R}\mathcal{I}_{N,\operatorname{diam}E+R}(\widehat{\mathfrak{m}}_{\alpha,R}(E))\,\widehat{\mathfrak{q}}_R(d\alpha)\geq\mathfrak{m}(B_R)\,\mathcal{I}_{N,\operatorname{diam}E+R}\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)\\&\geq\mathfrak{m}(B_R)\,\frac{N}{\operatorname{diam}E+R}\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)^{1-\frac{1}{N}}\left(1-O\left(\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)^{\frac{1}{N}}\right)\right)\\&=N\left(\frac{\mathfrak{m}(B_R)}{R^N}\right)^{\frac{1}{N}}\mathfrak{m}(E)^{1-\frac{1}{N}}-\frac{O(1)}{\operatorname{diam}E+R}.\end{aligned}$$

We conclude by taking the limit as $R\to\infty$ in the equation above. □

4.2.3 One dimensional reduction for the optimal region

Assuming $E\subset X$ to turn inequality (4.14) into an identity and following the proof of Theorem 4.4, a natural guess is that the r.h.s. of (4.15) converges to the l.h.s. as $R\to\infty$. The measure $\widehat{\mathfrak{q}}_R(Q_R)=\mathfrak{m}(B_R)$ is converging to infinity with order $O(R^N)$, so the integrand should converge to 0 with order $O(R^{-N})$. We now confirm this heuristic.

Definition 4.5. Let $D\geq D'>0$ and let $h:[0,D']\to\mathbb{R}$ be a $\mathsf{CD}(0,N)$ density. If $E\subset[0,D']$ is Borel subset, we define the D-residual of E as

$$\mathrm{Res}_h^D(E):=\frac{D\mathsf{P}_h(E)}{N(\mathfrak{m}_h(E))^{1-\frac{1}{N}}}-1. \tag{4.16}$$

If $v\in(0,1/2)$, we define the D-residual of v as

$$\mathrm{Res}_h^D(v):=\mathrm{Res}_h^D([0,r_h(v)])=\frac{Dh(r_h(v))}{Nv^{1-\frac{1}{N}}}-1. \tag{4.17}$$

Corollary 4.3 can be restated as

$$\mathrm{Res}_h^D(E) \geq -O(\mathfrak{m}_h(E)^{\frac{1}{N}}). \tag{4.18}$$

We now apply the definition of residual to the disintegration rays.

In order to simplify the notation, we denote by $\mathsf{P}_{\alpha,R}$ the perimeter measure of the one-dimensional m.m.s. $(\widehat{X}_{\alpha,R}, \mathsf{d}, \widehat{\mathfrak{m}}_{\alpha,R})$. The measure $\widehat{\mathfrak{m}}_{\alpha,R}$ will be identified with the ray map g to $h_{\alpha,R}\mathcal{L}^1$. Then

$$\begin{aligned}
\mathrm{Res}_{\alpha,R} &:= \mathrm{Res}_{h_{\alpha,R}}^{R+\mathrm{diam}(E)}(g(\alpha,\cdot)^{-1}(E \cap \widehat{X}_{\alpha,R})), \quad \text{for } \alpha \in Q_R,\\
\mathrm{Res}_{x,R} &:= \mathrm{Res}_{\mathfrak{Q}_R(x),R}, \quad \text{for } x \in E.
\end{aligned}$$

The good rays are those rays having small residual. We quantify their abundance.

Proposition 4.6. *Assume that* $(X, \mathsf{d}, \mathfrak{m})$ *is an essentially non-branching* $\mathsf{CD}(0, N)$ *space such that* $\mathsf{AVR}_X > 0$. *If* $E \subset X$ *is a bounded set attaining the identity in the inequality* (4.14), *then*

$$\lim_{R\to\infty} \frac{\|\mathrm{Res}_{\alpha,R}\|_{L^1(Q_R)}}{\mathfrak{m}(B_R)} = 0, \tag{4.19}$$

where the reference measure for the Lebesgue space $L^1(Q_R)$ *is* $\mathfrak{q}_R$.

Proof. We first check that the function $\alpha \to \mathrm{Res}_{\alpha,R}$ is integrable. To this extent, it is enough to check that $(\mathrm{Res}_{\alpha,R})^-$, is integrable; indeed, this last fact derives from the isoperimetric inequality $\mathrm{Res}_{\alpha,R} \geq -O((\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)})^{\frac{1}{N}})$, as stated in (4.18). We can now compute the integral in (4.19)

$$\begin{aligned}
\frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} |\mathrm{Res}_{\alpha,R}|\, \widehat{\mathfrak{q}}_R(d\alpha) &= \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} (2(\mathrm{Res}_{\alpha,R})^- + \mathrm{Res}_{\alpha,R})\, \widehat{\mathfrak{q}}_R(d\alpha)\\
&\leq O\left(\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\right)^{\frac{1}{N}}\right) + \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} \mathrm{Res}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha).
\end{aligned}$$

The first term is infinitesimal, so we focus on the second one

$$\begin{aligned}
\int_{Q_R} \mathrm{Res}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha) &= \int_{Q_R} \left(\frac{(R+\mathrm{diam}(E))\mathsf{P}_{\alpha,R}(E)}{N} \left(\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\right)^{1-\frac{1}{N}} - 1 \right) \widehat{\mathfrak{q}}_R(d\alpha)\\
&= \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1}\, N\mathfrak{m}(E)^{1-\frac{1}{N}}} \int_{Q_R} \mathsf{P}_{\alpha,R}(E)\, \widehat{\mathfrak{q}}_R(d\alpha) - \mathfrak{m}(B_R)\\
&\leq \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1}\, N\mathfrak{m}(E)^{1-\frac{1}{N}}}\, \mathsf{P}(E) - \mathfrak{m}(B_R)
\end{aligned}$$

$$\leq \mathfrak{m}(B_R)\frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}}(\mathsf{AVR}_X\omega_N)^{\frac{1}{N}} - \mathfrak{m}(B_R),$$

yielding

$$\frac{1}{\mathfrak{m}(B_R)}\int_{Q_R}\mathrm{Res}_{\alpha,R}\,\mathfrak{q}_R(d\alpha) \leq \tfrac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}}(\mathsf{AVR}_X\omega_N)^{\frac{1}{N}} - 1,$$

and the r.h.s. goes to 0, as $R \to \infty$. □

Corollary 4.7. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an essentially non-branching* $\mathsf{CD}(0,N)$ *space having* $\mathsf{AVR}_X > 0$*. Let* $E \subset X$ *be a set saturating the isoperimetric inequality* (4.14)*, then it holds true:*

$$\lim_{R\to\infty}\left\|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\right\|_{L^1(E)} = 0.$$

Proof. A direct computation gives

$$\begin{aligned}\left\|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\right\|_{L^1(E)} &= \int_{Q_R}\int_E |\mathrm{Res}_{\mathfrak{Q}_R(x),R}|\,\widehat{\mathfrak{m}}_{\alpha,R}(dx)\,\widehat{\mathfrak{q}}_R(d\alpha)\\ &= \int_{Q_R}|\mathrm{Res}_{\alpha,R}|\,\widehat{\mathfrak{m}}_{\alpha,R}(E)\,\widehat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\left\|\mathrm{Res}_{\alpha,R}\right\|_{L^1(Q_R)} \to 0.\end{aligned}$$ □

4.3 Analysis along the good rays

We now use the residual to control how distant is the density $h:[0,D']\to\mathbb{R}$ from the model density $x\in[0,D]\mapsto Nx^{N-1}/D$ as well as the one-dimensional traces of E from the optimal ones.

The results in this section go in the direction of proving that, given $D \geq D' > 0$, $h : [0,D'] \to \mathbb{R}$ a $\mathsf{CD}(0,N)$ density, a subset $E \subset [0,D']$, if the measure $\mathfrak{m}_h(E)$ and the residual $\mathrm{Res}_h^D(E)$ are small, then the set E is closed to the interval $[0, D\mathfrak{m}_h(E)^{\frac{1}{N}}]$ and the density h is closed to the model density Nx^{N-1}/D.

Remark 4.8. We will make an extensive use of Landau's "big-O" and "small-o" notation. If we are in a situation where several variables appear, but only a few of them are converging, either the "big-O" or "small-o" could depend on the non-converging variables.

In our setting, the converging variables will be $w \to 0$ and $\delta \to 0$. The free variables will be: 1) D, a bound from above on the diameter of the space; 2) $D' \in (0,D]$, the diameter of the space; 3) $h:[0,D']$ a $\mathsf{CD}(0,N)$ density; 4) $E\subset[0,D']$ a set with measure $\mathfrak{m}_h(E)=w$ and residual $\mathrm{Res}_h^D(E)\leq\delta$.

The following estimates are infinitesimal expansions as $w\to 0$ and $\delta\to 0$ and whenever a "big-O" or "small-o" appears, it has to be understood that this expression can be substituted with a function going to 0 with the same order uniformly w.r.t. the other free variables.

Remark 4.9. Another point to remark is the fact that we focus only on the case when E is on the left. We will sometimes assume that E is of the form $[0,r] \subset [0,D']$ and sometimes that $E \subset [0,L]$, with the tacit understanding that $r \ll D'$ or $L \ll D'$. This is possible because the rays come from the L^1-optimal transport problem from the measure $\frac{\mathfrak{m}\llcorner_E}{\mathfrak{m}(E)}$ to the measure $\frac{\mathfrak{m}\llcorner_{B_R}}{\mathfrak{m}(B_R)}$, where E is our original set. Hence the rays are lines starting from E and going away, thus the intersection of E with any ray lays at the beginning of the ray.

4.3.1 Almost rigidity of the diameter

We start our analysis focusing on the diameter of the space: the inequality $D \geq D'$ tends to be saturated if $\mathfrak{m}_h(E) = w \to 0$ and $\mathrm{Res}_h^D(E) \leq \delta \to 0$. It follows from the fact that the isoperimetric profile $\mathcal{I}_{N,D}$ scales according to D.

Proposition 4.10. *Fix $N > 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$D' \geq D(1 - o(1)), \tag{4.20}$$

where $D \geq D' > 0$ and $h : [0,D'] \to \mathbb{R}$ is a $\mathsf{CD}(0,N)$ density such that $E \subset [0,D']$ is a subset satisfying $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \leq \delta$.

Proof. The definition of residual (4.16) gives

$$\frac{N}{D'} w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^{D'}(E)) = \mathsf{P}_h(E) = \frac{N}{D} w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^{D}(E)).$$

Since $\mathrm{Res}_h^D(E) \geq O(w^{\frac{1}{N}})$ by (4.18), if w is small enough, we can multiply by $D' w^{\frac{1}{N}-1}/(N(1 + \mathrm{Res}_h^D(E))$, obtaining

$$\frac{D'}{D} = \frac{1 + \mathrm{Res}_h^{D'}(E)}{1 + \mathrm{Res}_h^{D}(E)} \geq \frac{1 - O(w^{\frac{1}{N}})}{1 + \mathrm{Res}_h^{D}(E)} \geq \frac{1 - O(w^{\frac{1}{N}})}{1 + \delta} = 1 - o(1). \qquad \square$$

4.3.2 Almost rigidity of the set E: the convex case

We now prove that the set E has to be close to $[0, D\mathfrak{m}_h(E)^{\frac{1}{N}}]$. We start considering the special case when the set E of the form $E = [0,r]$.

Proposition 4.11. *Fix $N > 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$r_h(w) \leq D(w^{\frac{1}{N}}(1 + o(1))), \tag{4.21}$$

$$r_h(w) \geq D(w^{\frac{1}{N}}(1 + o(1))), \tag{4.22}$$

where $D \geq D' > 0$ and $h : [0,D'] \to \mathbb{R}$ is a $\mathsf{CD}(0,N)$ density such that the residual $\mathrm{Res}_h^D(w) = \mathrm{Res}_h^D([0, r_h(w)]) \leq \delta$.

Proof. In order to simplify the notation, we write $r = r_h(w)$.

Part 1 Inequality (4.21)**.**
By the $\mathsf{CD}(0,N)$ of the function h, we have that $h(x) \leq \frac{h(r)}{r^{N-1}}x^{N-1}$, for $r \leq x \leq D'$. If we integrate in $[r, D']$ we obtain

$$1 - w \leq \int_r^{D'} \frac{h(r)}{r^{N-1}}x^{N-1}dx = \frac{h(r)(D'^N - r^N)}{Nr^{N-1}} \leq \frac{h(r)D'^N}{Nr^{N-1}} \leq \frac{h(r)D^N}{Nr^{N-1}},$$

yielding to

$$r^{N-1} \leq \frac{D^N}{N(1-w)}h(r) = \frac{D^N}{N(1-w)}\frac{N}{D}w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(w)) \leq (Dw^{\frac{1}{N}})^{N-1}\frac{1+\delta}{1-w}.$$

Part 2 Inequality (4.22)**.**
This second part is a bit more difficult. The first step is to show that we can lead back ourselves to the case of model spaces, namely that we can assume $h = h_{N,D'}(\xi, \cdot)$ for some $\xi \geq 0$ (cfr. (4.7)). That is, we want to show that given h, we find ξ, such that $\mathrm{Res}^D_{h_{N,D'}(\xi,\cdot)}(w) \leq \mathrm{Res}_h^D(w) \leq \delta$ and $r_{h_{N,D'}(\xi,\cdot)}(w) \leq r$.

To this extent, consider the function $s : [0, \infty) \to \mathbb{R}$ given by

$$s(a) := \int_r^{D'} \Big(h(r)^{\frac{1}{N-1}} + a(x-r)\Big)^{N-1} dx.$$

Clearly this function is strictly increasing and it holds

$$s\left(\frac{h(r)^{\frac{1}{N-1}}}{r}\right) = \int_r^{D'} \frac{h(r)}{r^{N-1}}x^{N-1}dx \geq \int_r^{D'} h(x)dx = 1 - w,$$
$$s(0) = (D'-r)h(r) = (D'-r)\frac{N}{D}w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(w)) \leq 2Nw_N^{1-\frac{1}{N}} < 1 - w_N \leq 1 - w,$$

where in the second line we assumed that $\mathrm{Res}_h^D(w) \leq \delta \leq 1$ and $w \leq w_N$ (for some $w_N > 0$ depending only on N), which is possible since $w \to 0$ and $\delta \to 0$. From the two inequalities above, it follows that there exist a unique $a \in (0, h(r)^{\frac{1}{N-1}}/r]$, such that $s(a) = 1 - w$. We can define the $\mathsf{CD}(0,N)$ density $\bar{h}(x) := (h(r)^{\frac{1}{N-1}} + a(x-r))^{N-1}$, which satisfies

$$\int_r^{D'} \bar{h}(x)dx = \int_r^{D'} h(x)dx = 1 - w.$$

By mean-value theorem, there exists $y \in (r, D')$ such that $h(y) = \bar{h}(y)$, thus, by convexity of

$h^{\frac{1}{N-1}}$, $\bar{h}(x) \geq h(x)$ for all $x \in [0, r]$. This implies that

$$\int_0^r \bar{h}(x)\, dx \geq \int_0^r h(x)\, dx = w.$$

Define

$$\begin{aligned} V &:= \int_0^{D'} \bar{h}(x)dx = \frac{(h(r)^{\frac{1}{N-1}} + a(D'-r))^N - (h(r)^{\frac{1}{N-1}} - ar)^N}{Na} \\ &= \int_r^{D'} \bar{h}(x)dx + \int_0^r \bar{h}(x)dx \geq 1 - w + \int_0^r h(x)dx = 1, \\ \bar{r} &:= r_{\bar{h}}(wV) \leq r_{\bar{h}}(V - (1-w)) = r_{\bar{h}}\left(\int_0^r \bar{h}(x)dx\right) = r, \end{aligned}$$

where $wV \leq V - (1-w)$ follows from $1 - w \in [0,1]$ and $V \geq 1$. Finally, we renormalize $\bar{h}$, defining

$$\widehat{h}(x) := \frac{\bar{h}(x)}{V} = Na \frac{(h(r)^{\frac{1}{N-1}} + a(x-r))^{N-1}}{(h(r)^{\frac{1}{N-1}} + a(D'-r))^N - (h(r)^{\frac{1}{N-1}} - ar)^N}.$$

If we set $\xi = \frac{h(r)^{\frac{1}{N-1}} - ar}{aD'} \geq 0$, then it turns out that (cfr. (4.7))

$$\widehat{h}(x) = h_{N,D'}(\xi, x) = \frac{N(x + D'\xi)^{N-1}}{D'^N((1+\xi)^N - \xi^N)}.$$

This function satisfies

$$r_{N,D'}(\xi, w) = \bar{r} \leq r_h(w) \quad \text{ and } \quad h_{N,D'}(\xi, \bar{r}) \leq \bar{h}(\bar{r}) \leq \bar{h}(r) = h(r)$$

(the inequality $\bar{h}(\bar{r}) \leq \bar{h}(r)$ follows from the fact that $a \geq 0$, hence $\bar{h}$ is non increasing). This latter inequality can be restated as

$$\mathrm{Res}^D_{h_{N,D'}(\xi,\cdot)}(w) \leq \mathrm{Res}^D_h(w) \leq \delta. \tag{4.23}$$

For this reason we can assume that h is of the type $h_{N,D'}(\cdot, \xi)$ for some $\xi \geq 0$.

Recalling Equation (4.8), we notice that

$$r_{N,D'}(\xi, w) = D'\left((w(1+\xi)^N + (1-w)\xi^N)^{\frac{1}{N}} - \xi\right) \geq D'(w^{\frac{1}{N}} - \xi). \tag{4.24}$$

What we are going to prove is that ξ is “small” in a sense that we will soon specify. Using the definition of residual, inequality (4.23) can be restated as (we already defined $\mathcal{G}_N$ in

Equation (4.9))

$$\mathcal{G}_N(\xi,w)=\frac{((1+\xi)^N+(\frac{1}{w}-1)\xi^N)^{\frac{N-1}{N}}}{(1+\xi)^N-\xi^N}\le\frac{D'}{D}(1+\delta)\le 1+\delta.$$

Define the set

$$L_\delta(w):=\{\xi:\mathcal{G}_N(\eta,w)>1+\delta:\forall\eta>\xi\}.$$

We have already proved in (4.10) that $\lim_{\xi\to\infty}\mathcal{G}_N(\xi,w)=N^{-1}w^{\frac{1-N}{N}}$, hence the set $L_\delta(w)$ is non-empty.

At this point define the function $\xi_\delta(w):=\inf L_\delta(w)$. By the definition of ξ_δ and the continuity of $\mathcal{G}_N$, it clearly holds that

$$\begin{aligned}&\mathcal{G}_N(\xi,w)\le 1+\delta \quad\Longrightarrow\quad \xi\le\xi_\delta(w),\\&\mathcal{G}_N(\xi_\delta(w),w)=1+\delta.\end{aligned}$$

Now we follow the line of the proof of Proposition 4.2. First, like in (4.11), we can see that $\xi_\delta(w)$ is bounded as $w\to 0$ and $\delta\to 0$. Indeed, suppose the contrary, i.e., that there exists two sequences $w_n\to 0$ and $\delta_n\to 0$ such that $\xi_{\delta_n}(w_n)\to\infty$. Then we have

$$\begin{aligned}1&\ge\limsup_{n\to\infty}\mathcal{G}_N(\xi_{\delta_n}(w_n),w_n)\ge\limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}\xi_{\delta_n}(w_n)^{N-1}}{(\xi_{\delta_n}(w_n)+1)^N-\xi_{\delta_n}(w_n)^N}\\&\ge\limsup_{n\to\infty}\frac{(\frac{1}{w_n}-1)^{\frac{N-1}{N}}}{\xi_{\delta_n}(w_n)\left(\left(\frac{1}{\xi_{\delta_n}(w_n)}+1\right)^N-1\right)}=\infty,\end{aligned}$$

which is a contradiction. Like in (4.12) we can prove that $\xi_\delta(w)\to 0$, as $w\to 0$ and $\delta\to 0$:

$$1\ge\limsup_{\substack{w\to0\\ \delta\to0}}\mathcal{G}_N(\xi_\delta(w),w)\ge\limsup_{\substack{w\to0\\ \delta\to0}}\frac{((\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{(\xi_\delta(w)+1)^N-\xi_\delta(w)^N}\ge\limsup_{\substack{w\to0\\ \delta\to0}}\frac{((\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{C}.$$

Finally, like in (4.13), we have that

$$\begin{aligned}1=\lim_{\substack{w\to0\\ \delta\to0}}\mathcal{G}_N(\xi_\delta,(w),w)&=\lim_{\substack{w\to0\\ \delta\to0}}\frac{((1+\xi_\delta(w))^N+(\frac{1}{w}-1)\xi_\delta(w)^N)^{\frac{N-1}{N}}}{(1+\xi_\delta(w))^N-\xi_\delta(w)^N}\\&=\lim_{\substack{w\to0\\ \delta\to0}}\left(1+\frac{\xi_\delta(w)^N}{w}\right)^{\frac{N-1}{N}},\end{aligned}$$

yielding

$$\lim_{\substack{w \to 0 \\ \delta \to 0}} \frac{\xi_\delta(w)^N}{w} = 1.$$

Using Landau's notation, the above becomes $\xi_\delta(w) = o(w^{\frac{1}{N}})$, as $w \to 0$ and $\delta \to 0$.

At this point we can recall (4.24), obtaining

$$r_{N,D'}(\xi_\delta(w), w) \geq D'(w^{\frac{1}{N}} - \xi_\delta(w)) \geq D'(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})).$$

If we use the estimate (4.20), we can continue the chain on inequalities and conclude:

$$\frac{r_{N,D'}(\xi_\delta(w), w)}{D} \geq \frac{D'}{D}(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) \geq (1 - o(1))(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) = w^{\frac{1}{N}}(1 - o(1)). \qquad \square$$

4.3.3 Almost rigidity of the set E: the general case

We now drop the assumption $E = [0, r]$. Up to a negligible set, $E = \bigcup_{i \in \mathbb{N}}(a_i, b_i)$ where the intervals (a_i, b_i) are far away from each other (i.e. $b_i < a_j$ or $b_j < a_i$, for $i \neq j$). By boundedness of the original set of our isoperimetric problem, we can also assume that E is included in the interval $[0, L]$, for some $L > 0$. Define $b(E) := \operatorname{ess\,sup} E \leq L$.

In the next proposition we exclude the existence of a sequence such that (a_{i_n}, b_{i_n}) goes to $b(E)$.

Lemma 4.12. *Fix $N > 1$ and $L > 0$. Then there exists two constants $\bar{w} > 0$ and $\bar{\delta} > 0$ (depending only on N and L) such that the following happens. For all $D \geq D' > 0$ with $D \geq 3L$, for all $h : [0, D'] \to \mathbb{R}$ satisfying the $\mathsf{CD}(0, N)$ condition, and for all $E \subset [0, L]$, such that $\mathfrak{m}_h(E) \leq \bar{w}$ and $\operatorname{Res}_h^D(E) \leq \bar{\delta}$, there exists $a \in [0, b(E))$ and an at most countable family of intervals $((a_i, b_i))_i$ such that, up to a negligible set,*

$$E = \bigcup_i (a_i, b_i) \cup (a, b(E)),$$

with $a_i, b_i < a$, $\forall i$.

Moreover, h is strictly increasing on $[0, b(E)]$.

Proof. By Proposition 4.10, we have that, if $\mathfrak{m}_h(E)$ and $\operatorname{Res}_h^D(E)$ are small enough, then D' is closed to $D \geq 3L$ and in particular $D' \geq 2L$. We already know that the set E is of the form $E = \bigcup_i (a_i, b_i)$ (up to a negligible set); our aim is to prove that there exists j such that $a_i, b_i < a_j$, for all $i \neq j$. In this case $a = a_j$. Suppose the contrary, i.e., $\forall j$, $\exists i \neq j$ such that $a_i > a_j$. With this assumption, we can build a sequence $(j_n)_n$, so that $(a_{j_n})_n$ is increasing, thus converging to some $y \in (0, L]$. By continuity of h, we have that $h(a_{j_n}) \to h(y) > 0$. We can compute the perimeter

$$\infty = \sum_{n\in\mathbb{N}} h(a_{j_n}) \le \mathsf{P}_h(E) = \frac{N}{D}(\mathfrak{m}_h(E))^{1-\frac{1}{N}}(1+\mathrm{Res}_h^D(E)) < \infty,$$

which is a contradiction.

It remains to prove that h is increasing on $[0,b(E)]$. In order to simplify the notation, let $b := b(E)$. Denote by $t := \lim_{z\searrow 0}(h(b+z)^{\frac{1}{N-1}} - h(b)^{\frac{1}{N-1}})/z$ the right-derivative of $h^{\frac{1}{N-1}}$ in b, which must exists because $h^{\frac{1}{N-1}}$ is concave. We want to prove that $t > 0$; from this and the fact that $h^{\frac{1}{N-1}}$ is concave it will follow that h is strictly increasing in $[0,b]$. Suppose the contrary, i.e., $t \le 0$. Then, by concavity of $h^{\frac{1}{N-1}}$, we have that

$$h(x) \le h(b)\left(\frac{D'-x}{D'-b}\right)^{N-1}, \quad \forall x\in[0,b], \quad \text{and} \quad h(x)\le h(b), \quad \forall x \in [b,D'].$$

If we integrate we obtain

$$\begin{aligned} 1 &\le \int_0^b h(b)\left(\frac{D'-x}{D'-b}\right)^{N-1} dx + \int_b^{D'} h(b)\,dx \\ &= \frac{h(b)}{N}\left(\frac{D'^N-(D'-b)^N}{(D'-b)^{N-1}} + N(D'-b)\right) \le \frac{\mathsf{P}_h(E)}{N}\left(\frac{D'^N}{(D'-b)^{N-1}} + ND'\right) \\ &= \frac{\mathsf{P}_h(E)D'}{N}\left(\left(1-\frac{b}{D'}\right)^{1-N} + N\right) = \frac{\mathsf{P}_h(E)D'}{N}\left(1+(N-1)\frac{b}{D'} + o\left(\frac{b}{D'}\right) + N\right). \end{aligned} \tag{4.25}$$

Consider the two factors in the r.h.s. of the estimate above. The former is controlled just using the definition of residual

$$\frac{\mathsf{P}_h(E)D'}{N} \le \frac{\mathsf{P}_h(E)D}{N} = \mathfrak{m}_h(E)^{1-\frac{1}{N}}(1+\mathrm{Res}_h^D(E)),$$

and, if $\mathfrak{m}_h(E)\to 0$ and $\mathrm{Res}_h^D(E)$ is bounded, then the term above goes to 0. Regarding the latter factor, we just need to prove that $\frac{b}{D'}$ is bounded:

$$\frac{b}{D'} \le \frac{L}{D'} \le \frac{L}{2L} = \frac{1}{2}.$$

If we put together this last two estimates, we obtain that the r.h.s. of (4.25) converges to 0 as $\mathfrak{m}_h(E)\to 0$ and $\mathrm{Res}_h^D(E)\to 0$, whereas the l.h.s. is equal to 1, obtaining a contradiction. □

Remark 4.13. What we have just proven is that there exists a right-extremal connected component for the set E and this component is precisely the interval $(a, b(E))$. We will denote by $a(E)$ the number a given by the just-proven proposition. Since our estimates are infinitesimal expansions in the limit as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E) \to 0$, we will always assume that $\mathfrak{m}_h(E) \le \bar{w}$ and $\mathrm{Res}_h^D(E) \le \bar{\delta}$, so that the expression $a(E)$ makes sense. We will make an extensive use of the fact that h is increasing in the interval $[0, b(E)]$: since, again, our estimates are in the limit as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}_h^D(E) \to 0$, the fact that h is increasing in $[0, b(E)]$ will be taken into account, without explicitly referring to the previous Lemma.

We now prove that the component $(a(E), b(E))$ tends to fill the set E and that $b(E)$ converges as expected to $D\mathfrak{m}_h(E)^{\frac{1}{N}}$.

Proposition 4.14. *Fix $N > 1$ and $L > 0$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$b(E) \le Dw^{\frac{1}{N}} + Do(w^{\frac{1}{N}}) \tag{4.26}$$

$$b(E) \ge Dw^{\frac{1}{N}} - Do(w^{\frac{1}{N}}) \tag{4.27}$$

$$a(E) \le Do(w^{\frac{1}{N}}), \tag{4.28}$$

where $D \ge 3L$, $D' \in (0, D]$, $h : [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \le \delta$.

Proof.

Part 1 Inequality (4.27)**.**
Since the density h is strictly increasing on $[0, b(E)]$ and $E \subset [0, b(E)]$ (up to a null measure set), we have that $r_h(w) \le b(E)$ and

$$\mathrm{Res}_h^D(w) = \frac{Dh(r_h(w))}{Nw^{1-\frac{1}{N}}} - 1 \le \frac{Dh(b(E))}{Nw^{1-\frac{1}{N}}} - 1 \le \frac{D\mathsf{P}_h(E)}{Nw^{1-\frac{1}{N}}} - 1 = \mathrm{Res}_h^D(E) \le \delta.$$

We now exploit Proposition 4.11 (in particular the estimate (4.22)), yielding

$$D(w^{\frac{1}{N}} - o(w^{\frac{1}{N}})) \le r_h(w) \le b(E),$$

and we have concluded the proof of (4.27).

Part 2 Inequality (4.28)**.**
First we prove that $a(E) < r_h(w)$ for w and δ small enough. Suppose the contrary, i.e., that $a(E) \ge r_h(w)$. This implies that $h(a(E)) \ge h(r_h(w))$, hence $\mathsf{P}_h(E) \ge 2h(r_h(w))$. We deduce that

$$-O(w^{\frac{1}{N}}) \le \mathrm{Res}_h^D(w) = \frac{Dh(r_h(w))}{Nw^{1-\frac{1}{N}}} - 1 \le \frac{D\mathsf{P}_h(E)}{2Nw^{1-\frac{1}{N}}} - 1 = \frac{1}{2}(\mathrm{Res}_h^D(E) - 1) \ge \frac{\delta - 1}{2}.$$

If we take the limit as $w \to 0$ and $\delta \to 0$ we obtain a contradiction.

We exploit the Bishop–Gromov inequality and the isoperimetric inequality (respectively) to obtain

$$h(a(E)) \geq h(r_h(w)) \left(\frac{a(E)}{r_h(w)}\right)^{N-1}$$
$$h(b(E)) \geq h(r_h(w)) \geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})),$$

Putting together the inequalities above and using the definition of residual we obtain

$$\begin{aligned}\frac{N}{D} w^{1-\frac{1}{N}}(1 + \mathrm{Res}_h^D(E)) = \mathsf{P}_h(E) &\geq h(b(E)) + h(a(E)) \geq h(r_h(w)) + h(a(E)) \\ &\geq h(r_h(w)) \left(1 + \left(\frac{a(E)}{r_h(w)}\right)^{N-1}\right) \\ &\geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})) \left(1 + \left(\frac{a(E)}{r_h(w)}\right)^{N-1}\right),\end{aligned}$$

yielding

$$\begin{aligned}a(E) &\leq r_h(w) \left(\frac{1 + \mathrm{Res}_h^D(E)}{1 + O(w^{\frac{1}{N}})} - 1\right)^{\frac{1}{N-1}} \leq r_h(w) \left((1+\delta)(1 - O(w^{\frac{1}{N}})) - 1\right)^{\frac{1}{N-1}} \leq r_h(w)\, o(1) \\ &\leq D w^{\frac{1}{N}}(1 + o(1)) o(1) = D o(w^{\frac{1}{N}}),\end{aligned}$$

where the estimate (4.21) was taken into account. This concludes the proof of (4.28).

Part 3 Inequality (4.26).
Since

$$\int_E h = \int_0^{r_h(w)} h,$$

we can deduce (together with the fact that $a(E) \leq r_h(w) \leq b(E)$)

$$\int_{E\cap[0,r_h(w)]} h + \int_{r_h(w)}^{b(E)} h = \int_{E\cap[0,r_h(w)]} h + \int_{[0,r_h(w)]\setminus E} h = \int_{E\cap[0,r_h(w)]} h + \int_{[0,a(E)]\setminus E} h,$$

hence

$$(b(E) - r_h(w))\, h(r_h(w)) \leq \int_{r_h(w)}^{b(E)} h = \int_{[0,a(E)]\setminus E} h \leq \int_0^{a(E)} h \leq a(E)\, h(a(E)),$$

yielding

$$b(E) - r_h(w) \leq a(E)\, \frac{h(a(E))}{h(r_h(w))} \leq a(E).$$

We conclude by combining the inequality above with the already-proven estimate (4.27) and the estimate (4.21) from Proposition 4.11. □

4.3.4 Almost rigidity of the density h

We now prove that the density h converges to the density of the model space Nx^{N-1}/D^N. Relying on the Bishop–Gromov inequality, we obtain an estimate of h from below.

Proposition 4.15. *Fix $N > 1$ and $L > 0$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$h(x) \geq \frac{N}{D^N} x^{N-1}(1 - o(1)), \qquad \text{uniformly w.r.t. } x \in [0, b(E)], \tag{4.29}$$

where $D \geq 3L$, $D' \in (0, D]$, $h : [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0, N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\operatorname{Res}_h^D(E) \leq \delta$.

Proof. Fix $x \in [0, b(E)]$. We can compute, using the Bishop–Gromov inequality

$$h(x) \geq h(b(E)) \frac{x^{N-1}}{b(E)^{N-1}} \geq h(r_h(w)) \frac{x^{N-1}}{b(E)^{N-1}}.$$

The first factor is controlled using the isoperimetric inequality

$$h(r_h(w)) \geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})) = \frac{N}{D} w^{1-\frac{1}{N}}(1 - o(1)).$$

For the term $b(E)$ we use the estimate (4.26)

$$b(E) \leq D w^{\frac{1}{N}}(1 + o(1)).$$

The thesis follows from the combination of these last two inequalities. □

Before going on we prove the following, purely technical lemma.

Lemma 4.16. *Fix $N > 1$ and consider the function $f : [0, 1) \times [0, \infty] \to \mathbb{R}$ given by*

$$f(t, \eta) = \frac{1 + \eta - t^N}{1 - t}.$$

Define the function g by

$$g(\eta) = \sup\{t - s : f(t, 0) \leq f(s, \eta)\}. \tag{4.30}$$

Then $\lim_{\eta \to 0} g(\eta) = 0$.

Proof. The proof is by contradiction. Suppose that there exists $\epsilon > 0$ and three sequences in $(\eta_n)_n$, $(t_n)_n$, and $(s_n)_n$, such that $\eta_n \to 0$, $f(t_n, 0) \leq f(s_n, \eta_n)$, and $t_n - s_n > \epsilon$. Up to a taking a sub-sequence, we can assume that $t_n \to t$ and $s_n \to s$, hence $1 \geq t \geq s+\epsilon$. The functions $f(\cdot,\eta_n)$ converge to $f(\cdot,0)$, uniformly in the interval $[0,1-\frac{\epsilon}{2}]$. This implies $f(s_n,\eta_n) \to f(s,0)$, yielding $f(t,0) \leq f(s,0)$. Since $t \mapsto f(t,0)$ is strictly increasing, we obtain $t \leq s \leq t-\epsilon$, which is a contradiction. $\square$

We now obtain an estimate of h from above in the interval $[a(E),b(E)]$ going in the opposite direction of the Bishop–Gromov inequality.

Proposition 4.17. *Fix $N>1$, $L>0$. Fix $N>1$ and $L>0$. The following estimates hold for $w\to 0$ and $\delta \to 0$*

$$h(x) \leq h(b(E))\left(\frac{x}{b(E)}+o(1)\right)^{N-1}, \qquad \textit{uniformly w.r.t. } x \in [a(E),b(E)], \tag{4.31}$$

where $D \geq 3L$, $D' \in (0,D]$, $h:[0,D'] \to \mathbb{R}$ is a $\mathsf{CD}(0,N)$ density, and the set $E \subset [0,L]$ satisfies $\mathfrak{m}_h(E)=w$ and $\mathrm{Res}_h^D(E) \leq \delta$.

Proof. Fix $x \in [a(E),b(E)]$ and, in order to simplify the notation, define

$$a := a(E), \quad b := b(E), \quad k := h(x)^{\frac{1}{N-1}}, \quad l := h(b(E))^{\frac{1}{N-1}}.$$

By concavity of $h^{\frac{1}{N-1}}$, it holds true that

$$\begin{aligned} h(y) &\geq \left(\frac{y}{x}\right)^{N-1} k^{N-1}, \quad \forall y \in [a,x], \\ h(y) &\geq \left(l+(k-l)\frac{b-y}{b-x}\right)^{N-1}, \quad \forall y \in [x,b]. \end{aligned}$$

We can integrate these two inequalities, obtaining

$$\begin{aligned} w &\geq \int_a^x \frac{y^{N-1}}{x^{N-1}} k^{N-1}\, dy + \int_x^b \left(l+(k-l)\frac{b-y}{b-x}\right)^{N-1} dy \\ &= \frac{k^{N-1}\left(x^N-a^N\right)}{N x^{N-1}} + \frac{b-x}{N}\,\frac{l^N-k^N}{l-k}, \end{aligned}$$

yielding

$$\begin{aligned} \frac{1-\left(\frac{k}{l}\right)^N}{1-\frac{k}{l}} &\leq \frac{Nw - \frac{k^{N-1}(x^N-a^N)}{x^{N-1}}}{l^{N-1}(b-x)} = \frac{\frac{Nw}{bl^{N-1}} - \frac{k^{N-1}(x^N-a^N)}{b(lx)^{N-1}}}{1-\frac{x}{b}} \\ &\leq \frac{\frac{Nw}{bl^{N-1}} - \frac{x^N-a^N}{b^N}}{1-\frac{x}{b}} = \frac{\frac{Nw}{bl^{N-1}} + \frac{a^N}{b^N} - \frac{x^N}{b^N}}{1-\frac{x}{b}}, \end{aligned}$$

where in the last inequality we used the Bishop–Gromov inequality written in the form $\frac{k^{N-1}}{l^{N-1}} \geq \frac{x^{N-1}}{b^{N-1}}$. At this point, we estimate the terms $\frac{Nw}{bl^{N-1}}$ and $\frac{a^N}{b^N}$. Regarding the former, taking into account (4.27) and the isoperimetric inequality, we notice

$$\frac{Nw}{bl^{N-1}} = \frac{Nw}{b(E)\,h(b(E))} \le \frac{Nw}{b(E)\,h(r_h(w))} \le \frac{Nw}{Dw^{\frac{1}{N}}(1-o(1))\,\frac{N}{D}w^{1-\frac{1}{N}}(1-O(w^{\frac{1}{N}})} = 1+o(1).$$

The latter term is even more simple (recall (4.26) and (4.28))

$$\frac{a^N}{b^N} = \frac{a(E)^N}{b(E)^N} \le \frac{D^N o(w)}{D^N w(1-o(1))^N} = o(1).$$

We can put all the pieces together obtaining

$$f\left(\frac{k}{l},0\right) = \frac{1-\left(\frac{k}{l}\right)^N}{1-\frac{k}{l}} \le \frac{\frac{Nw}{bl^{N-1}}+\frac{a^N}{b^N}-\frac{x^N}{b^N}}{1-\frac{x}{b}} \le \frac{1+o(1)-\frac{x^N}{b^N}}{1-\frac{x}{b}} = f\left(\frac{x}{b},o(1)\right),$$

where f is the function of Lemma 4.16. We can apply said Lemma (and in particular (4.30)) and we get

$$\frac{k}{l}-\frac{x}{b} \le g(o(1)) = o(1).$$

If we explicit the definitions of k, l, and b, it turns out that the inequality above is precisely the thesis. □

4.3.5 Rescaling the diameter and renormalizing the measure

We now obtain a first limit estimate of the densities h. The presence of factor $\frac{1}{D^N}$ in the estimate (4.29) suggests the need of a rescaling to get a non-trivial limit estimate. We will rescale by $\frac{1}{b(E)}$ and renormalise the measure by $\mathfrak{m}_h(E)$.

Fix $k>0$ and define the rescaling transformation $S_k(x) = x/k$. If $h : [0,D'] \to \mathbb{R}$ is a density and $E \subset [0,L]$, we can define

$$\nu_{h,E} = (S_{b(E)})_\# \left(\frac{\mathfrak{m}_h \llcorner E}{\mathfrak{m}_h(E)}\right) \in \mathcal{P}([0,1]).$$

The probability measure $\nu_{h,E}$ is absolutely continuous w.r.t. $\mathcal{L}^1$. Denote by $\tilde{h}_E : [0,1] \to \mathbb{R}$ the Radon–Nikodym derivative $\frac{d\nu_{h,e}}{d\mathcal{L}^1}$. The density $\tilde{h}_E$ can be computed explicitly

$$\tilde{h}_E(t) = \mathbf{1}_E(b(E)t)\frac{b(E)}{\mathfrak{m}_h(E)}\,h(b(E)t). \tag{4.32}$$

Since E could be disconnected, the indicator function in (4.32) prevents $\tilde{h}_E^{\frac{1}{N-1}}$ from being concave and therefore $([0,1],|\cdot|,\nu_{h,E})$ from satisfing the $\mathsf{CD}(0,N)$ condition.

Proposition 4.18. *Fix $N > 1$ and $L > 0$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$\left\| \tilde{h}_E - N t^{N-1} \right\|_{L^\infty(0,1)} \le o(1)$$

where $D \ge 3L$, $D' \in (0, D]$, $h : [0, D'] \to \mathbb{R}$ is a $\mathsf{CD}(0,N)$ density, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}_h^D(E) \le \delta$.

Proof. Fix $t \in [0,1]$. The proof is divided in four parts.

Part 1 Estimate from below and $t > \frac{a(E)}{b(E)}$.
Since $t > \frac{a(E)}{b(E)}$, then $t\, b(E) \in E$ (for a.e. t). By a direct computation, we have

$$\begin{aligned} \tilde{h}_E(t) &= \frac{b(E)}{w}\, h(tb(E)) \ge \frac{N b(E)^N}{D^N w}\, t^{N-1}(1 - o(1)) \\ &\ge \frac{N D^N w (1 + o(1))^N}{D^N w}\, t^{N-1}(1 - o(1)) = N t^{N-1} - N t^{N-1} o(1), \end{aligned}$$

where we have used the estimate (4.29), with $x = tb(E)$, in the first inequality and (4.27) in the first and second inequalities, respectively. Since $t \in [0,1]$, then $-t^{N-1} o(1) \ge o(1)$ and we conclude this first part.

Part 2 Estimate from below and $t \le \frac{a(E)}{b(E)}$.
In this case it may happen that $t\, b(E) \notin E$, so the best we can say about $\tilde{h}_E$ is that it is non-negative in t. The point here is to exploit the fact that the interval $[0, \frac{a(E)}{b(E)}]$ is "short" and that $t \le \frac{a(E)}{b(E)}$. By a direct computation (we recall (4.27) and (4.28)) we have

$$\begin{aligned} \tilde{h}_E(t) &\ge 0 \ge N t^{N-1} - N t^{N-1} \ge N t^{N-1} - N \frac{a(E)^{N-1}}{b(E)^{N-1}} \\ &\ge N t^{N-1} - N \frac{D^{N-1} o(w^{1-\frac{1}{N}})}{D^{N-1} w^{1-\frac{1}{N}} (1 + o(1))^{N-1}} \ge N t^{N-1} - o(1). \end{aligned}$$

Part 3 Estimate from above and $t > \frac{a(E)}{b(E)}$.
We take into account the estimate (4.31), with $x = tb(E)$ and compute

$$\begin{aligned} \tilde{h}_E(t) &= \frac{b(E)}{w}\, h(tb(E)) \le \frac{b(E)}{w}\, h(b(E))(t + o(1))^{N-1} \le \frac{b(E)}{w}\, h(b(E))(t^{N-1} + o(1)) \\ &\le \frac{D w^{\frac{1}{N}} (1 + o(1))}{w}\, \mathsf{P}_h(E)(t^{N-1} + o(1)) \\ &= \frac{D w^{\frac{1}{N}} (1 + o(1))}{w}\, \frac{N}{D} w^{1-\frac{1}{N}} (1 + \mathrm{Res}_h^D(E))(t^{N-1} + o(1)) \\ &\le N(1 + o(1))(1 + \delta)(t^{N-1} + o(1)) = N t^{N-1} + o(1) \end{aligned}$$

(in the second inequality we exploited the uniform continuity of $t \in [0,1] \mapsto t^{N-1}$; in the third one, estimate (4.26)).

Part 4 Estimate from above and $t \le \frac{a(E)}{b(E)}$.
Fix $\epsilon > 0$ and compute

$$\begin{aligned}\tilde{h}_E(t) &= b(E)\frac{\mathbf{1}_E(tb(E))}{\mathfrak{m}_h(E)}h(b(E)t) \le \frac{b(E)}{\mathfrak{m}_h(E)}h(b(E)t)\\ &\le \frac{b(E)}{\mathfrak{m}_h(E)}h\left(b(E)\left(\frac{a(E)}{b(E)}+\epsilon\right)\right) = \tilde{h}_E\left(\frac{a(E)}{b(E)}+\epsilon\right),\end{aligned}$$

and the last equality holds true for a.e. ϵ small enough. At this point we can take into account the previous part and continue

$$\tilde{h}_E(t) \le \tilde{h}_E\left(\frac{a(E)}{b(E)}+\epsilon\right) \le N\left(\frac{a(E)}{b(E)}+\epsilon\right)^{N-1} + o(1).$$

If we take the limit as $\epsilon \to 0$ we can conclude

$$\tilde{h}_E(t) \le N\left(\frac{a(E)}{b(E)}\right)^{N-1} + o(1) \le o(1) \le Nt^{N-1} + o(1). \qquad \square$$

The following theorem summerizes the content of this section.

Theorem 4.19. *Fix $N > 1$ and $L > 0$. Then there exists a function $\omega : \mathrm{Dom}(\omega) \subset (0,\infty) \times \mathbb{R} \to \mathbb{R}$, infinitesimal in 0, such that the following holds. For all $D \ge 3L$, $D' \in (0, D)$, for all $h : [0, D'] \to \mathbb{R}$ a $\mathsf{CD}(0,N)$ density, and for all $E \subset [0, L]$, it holds*

$$\left|b(E) - D\mathfrak{m}_h(E)^{\frac{1}{N}}\right| \le D\mathfrak{m}_h(E)^{\frac{1}{N}}\omega(\mathfrak{m}_h(E), \mathrm{Res}_h^D(E)), \tag{4.33}$$

$$\left\|\tilde{h}_E - Nt^{N-1}\right\|_{L^\infty} \le \omega(\mathfrak{m}_h(E), \mathrm{Res}_h^D(E)), \tag{4.34}$$

where $b(E) = \operatorname{ess\,sup} E$ and $\tilde{h}_E$ is the Radon-Nikodym derivative of $\mathfrak{m}_h(E)^{-1}(S_{b(E)})_\#\mathfrak{m}_h\llcorner_E$, with $S_{b(E)}(x) = x/b(E)$.

4.4 Passage to the limit as $R \to \infty$

We now go back to the studying the identity case of the isoperimetric inequality: E is a bounded Borel such that

$$\mathsf{P}(E) = N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}\mathfrak{m}(E)^{1-\frac{1}{N}},$$

where $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching $\mathsf{CD}(0,N)$ space having $\mathsf{AVR}_X > 0$ and

We make use of the notation of Section 4.1; denote by φ_R the Kantorovich potential associated to f_R and (4.1). Since the construction does not change if we add a constant to φ_R, we can assume that φ_R are equibounded on every bounded set. Using the Ascoli–Arzelà theorem and a diagonal argument we deduce that, up to subsequences, φ_R converges to a certain 1-Lipschitz function φ_∞, uniformly on every bounded set.

We recall the disintegration given by Proposition 4.1,

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}\,\widehat{\mathfrak{q}}_R(d\alpha), \quad \text{and} \quad \mathsf{P}(E;\,\cdot\,) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\,\cdot\,)\,\widehat{\mathfrak{q}}_R(d\alpha). \tag{4.35}$$

We would like to take the limit in the disintegration formula (4.35). To the knowledge of the author there is no easy way to take such limit. For this reason, the effort of this section goes in the direction to understand how the properties of the disintegration behave at the limit.

4.4.1 Passage to the limit of the radius

We start by defining the *radius* function $r_R : \bar{E} \to [0, \operatorname{diam} E]$. Fix $x \in E \cap \widehat{\mathcal{T}}_R$ and let $E_{x,R} := (g_R(\mathfrak{Q}_R(x), \cdot))^{-1}(E) \subset [0, |\widehat{X}_{\mathfrak{Q}_R(x),R}|]$. Define

$$r_R(x) := \begin{cases} \operatorname{ess\,sup} E_{x,R}, & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ 0, & \text{otherwise.} \end{cases} \tag{4.36}$$

Notice that $r_R(x) = b(E_{x,E})$, where the notation $b(E)$ was introduced in 4.3.3.

The function r_R is defined on $\bar{E}$ for two motivations: we require a common domain not depending on R and the domain must be a compact metric spaces.

Remark 4.20. The set $E \cap \widehat{\mathcal{T}}_R$ has full $\mathfrak{m}\llcorner_E$-measure in $\bar{E}$. This means that it does not really matter how r_R is defined outside $E \cap \widehat{\mathcal{T}}_R$. This fact is particularly relevant, because we will only take limits in the $\mathfrak{m}\llcorner_E$-a.e. sense or in senses which are weaker than the pointwise convergence.

The next proposition ensures that, in limit as $R \to \infty$, the function r_R converges to the constant $(\frac{\mathfrak{m}_h(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$, which is precisely the radius that we expect.

Proposition 4.21. *Up to subsequences it holds true*

$$\lim_{R\to\infty} r_R = \left(\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\right)^{\frac{1}{N}}, \qquad \mathfrak{m}\llcorner_E - a.e..$$

Proof. By Corollary 4.7 we have that $\left\|\mathrm{Res}_{R,\mathfrak{Q}_R(x)}\right\|_{L^1(\bar{E};\mathfrak{m}\llcorner_E)} \to 0$, as $R \to \infty$, hence there exists a negligible subset $N \subset E$ and a sequence $R_n \to \infty$, such that $\lim_{n\to\infty} \mathrm{Res}_{x,R_n} = 0$, for all $x \in E \setminus N$.

Define $G := \bigcap_n \widehat{\mathcal{T}}_{R_n} \setminus N$ and notice that $\mathfrak{m}(E \setminus G) = 0$. Now fix $n \in \mathbb{N}$ and $x \in G$ and let $\alpha := \mathfrak{Q}_{R_n}(x) \in Q_{R_n}$. By triangular inequality, it holds

$$\left| r_{R_n}(x) - \left(\tfrac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\right)^{\frac{1}{N}} \right| \le \left| r_{R_n}(x) - (R_n + \operatorname{diam} E)\left(\tfrac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{\frac{1}{N}} \right| + \left| (R_n + \operatorname{diam} E)\left(\tfrac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{\frac{1}{N}} - \left(\tfrac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X}\right)^{\frac{1}{N}} \right|,$$

and the second term goes to 0 by definition of AVR.

Let's focus on the first term. Consider the ray $(\widehat{X}_{\alpha,R_n}, \mathsf{d}, \widehat{\mathfrak{m}}_{\alpha,R_n})$. By definition, we have that

$$\mathrm{Res}_{h_{\alpha,R_n}}^{R_n + \operatorname{diam} E}(E_{x,R_n}) = \mathrm{Res}_{\alpha,R_n}$$

We are in position to use Theorem 4.19 and, in particular, estimate (4.33) implies

$$\begin{aligned}
\left| r_{R_n}(x) - (R_n + \operatorname{diam} E)\left(\tfrac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{\frac{1}{N}} \right| &= \left| r_{R_n}(x) - (R_n + \operatorname{diam} E)(\mathfrak{m}_{h_{\alpha,R_n}}(E_{x,R_n}))^{\frac{1}{N}} \right| \\
&\le (R_n + \operatorname{diam} E)\mathfrak{m}_{h_{\alpha,R_n}}(E)^{\frac{1}{N}} \omega(\mathfrak{m}_{h_{\alpha,R_n}}(E), \mathrm{Res}_{h_{\alpha,R_n}}^{R_n + \operatorname{diam} E}(E_{x,R_n})) \\
&= (R_n + \operatorname{diam} E)\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}\right)^{\frac{1}{N}} \omega\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}, \mathrm{Res}_{\mathfrak{Q}_R(x),R_n}\right).
\end{aligned}$$

Since the r.h.s. in the inequality above is infinitesimal, we can take the limit as $n \to \infty$ and conclude. □

Hence in the limit the length of the rays converge to a well defined constant; this will turn out to be the radius of E. From now on we will write $\rho := (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$.

4.4.2 Passage to the limit of the rays

Consider now a constant-speed parametrization of the rays inside E:

$$\gamma_s^{x,R} := \begin{cases} g_R(\mathfrak{Q}_R(x), s\, r_R(x)), & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ x, & \text{otherwise,} \end{cases}$$

where $x \in \bar{E}$ and $s \in [0,1]$. Remark 4.20 applies also to the map $x \mapsto \gamma^{x,R}$. A direct consequence of the definition of $\gamma^{x,R}$ is

$$\mathsf{d}(\gamma_t^{x,R}, \gamma_s^{x,R}) = \varphi_R(\gamma_t^{x,R}) - \varphi_R(\gamma_s^{x,R}), \quad \forall\, 0 \le t \le s \le 1, \text{ for } \mathfrak{m}\text{-a.e. } x \in E, \tag{4.37}$$

$$\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R}) = r_R(x), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E, \tag{4.38}$$

$$x \in \gamma^{x,R}, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{4.39}$$

We stress out the order of the quantifiers in (4.37): said equation has to be understood in the sense that $\exists N \subset E$ such that $\mathfrak{m}(N)=0$ and $\forall t \le s$, $\forall x \in E \setminus N$, (4.37) holds true. Regarding (4.39), we point out that the expression $x \in \gamma^{x,R}$ means that $\exists t \in [0,1]$ such that $x = \gamma_t^{x,R}$, or, equivalently, $\min_{t\in[0,1]} \mathsf{d}(x,\gamma_t^{x,R}) = 0$.

In order to capture the limit behaviour of $\gamma^{x,R}$ as $R \to \infty$ we proceed as follows. First define $K := \{\gamma \in \mathrm{Geo}(X) : \gamma_0, \gamma_1 \in \bar{E}\}$. Since a $\mathsf{CD}(K,N)$ space is locally compact and E is bounded, $\bar{E}$ is compact and so is K. Then define the measure

$$\tau_R := (\mathrm{Id} \times \gamma^{\cdot,R})_\# \mathfrak{m}\llcorner_E \in \mathcal{M}(\bar{E} \times K).$$

The measures τ_R have mass $\mathfrak{m}(E)$ and enjoy the following immediate properties

$$(P_1)_\# \tau_R = \mathfrak{m}\llcorner_E, \qquad \text{and} \qquad \gamma = \gamma^{x,R}, \qquad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \bar{E} \times K.$$

We can restate the properties (4.37)–(4.39) using a more measure-theoretic language

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)) = 0, \quad \forall\, 0 \le t \le s \le 1, \text{ for } \tau_R\text{-a.e. } (x,\gamma) \in \bar{E} \times K, \tag{4.40}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x) = 0, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \bar{E} \times K, \tag{4.41}$$

$$x \in \gamma, \quad \text{for } \tau_R\text{-a.e. } (x,\gamma) \in \bar{E} \times K \tag{4.42}$$

Since the measures τ_R have the same mass and $\bar{E} \times K$ is compact, the family of measures $(\tau_R)_{R>0}$ is tight, thus we can extract a sub-sequence (which we do not relabel) such that $\tau_R \rightharpoonup \tau$ weakly, i.e., $\int_{\bar{E}\times K} \psi \, d\tau_R \to \int_{\bar{E}\times K} \psi \, d\tau$, for all $\psi \in C_b(\bar{E} \times K)$.

The next proposition affirms that the properties (4.40)–(4.42) pass to the limit as $R \to \infty$.

Proposition 4.22. *For τ-a.e. $(x,\gamma) \in \bar{E} \times K$, it holds thas*

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)), \quad \forall\, 0 \le t \le s \le 1, \tag{4.43}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{4.44}$$

$$x \in \gamma. \tag{4.45}$$

Proof. Fix $t \le s$ and integrate (4.40) in $\bar{E} \times K$, obtaining

$$0 = \int_{\bar{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_R(e_t(\gamma)) + \varphi_R(e_s(\gamma)))\, \tau_R(dx\, d\gamma) = \int_{\bar{E}\times K} L^{t,s}_{\varphi_R}(\gamma)\, \tau_R(dx\, d\gamma),$$

where we have set $L^{t,s}_\psi(\gamma) := \mathsf{d}(e_t(\gamma), e_s(\gamma)) - \psi(e_t(\gamma)) + \psi(e_s(\gamma))$. The map $L^{t,s}_{\varphi_R} : K \to \mathbb{R}$ is clearly continuous and converges uniformly (recall that $\varphi_R \to \varphi_\infty$ uniformly on every compact) to $L^{t,s}_{\varphi_\infty}$. For this reason we can take the limit in the equation above obtaining

$$0 = \int_{\bar{E}\times K} L^{t,s}_{\varphi_\infty}(\gamma)\,\tau(dx\,d\gamma) = \int_{\bar{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \varphi_\infty(e_t(\gamma)) + \varphi_\infty(e_s(\gamma)))\,\tau(dx\,d\gamma).$$

The 1-lipschitzianity of φ_∞, yields $L^{t,s}_{\varphi_\infty}(\gamma) \geq 0$, $\forall \gamma \in K$, hence

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)) \quad \text{for } \tau\text{-a.e. } (x,\gamma) \in \bar{E} \times K.$$

In order to conclude, fix $P \subset [0,1]$ a countable dense subset, and find a τ-negligible set $N \subset \bar{E} \times K$ such that

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)), \quad \forall t,s \in P, \text{ with } t \leq s, \forall (x,\gamma) \in (\bar{E} \times K)\backslash N.$$

If we have $0 \leq t \leq s \leq 1$, we approximate t and s with two sequences in P and we can pass to the limit in the equation above concluding the proof of (4.43).

Now we prove (4.44). The idea is similar, but in this case we need to be more careful, because the function r_R fails to be continuous. Like before, we can integrate Equation (4.41) obtaining

$$0 = \int_{\bar{E}\times X} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\,\tau_R(dx\,d\gamma).$$

If the functions r_R were continuous and converged uniformly to ρ, then we could pass to the limit and conclude. Unfortunately Proposition 4.21, provides a limit only the a.e. sense. We overcome this issue using Lusin's and Egorov's theorems. Fix $\epsilon > 0$ and find a compact set $L \subset E$, such that: 1) the restrictions $r_R|_L$ are continuous; 2) the restricted maps $r_R|_L$ converge uniformly to ρ; 3) $\mathfrak{m}(E\backslash L) \leq \epsilon$. We can now compute the limit

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{\bar{E}\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\,\tau_R(dx\,d\gamma) \\ &\geq \liminf_{R\to\infty} \int_{L\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\,\tau_R(dx\,d\gamma) \\ &\geq \int_{L\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - \rho|\,\tau(dx\,d\gamma) \geq 0, \end{aligned}$$

hence

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \qquad \text{for } \tau\text{-a.e. } (x,\gamma) \in L \times K.$$

This means that the equation above holds true except for a set of measure at most ϵ. By arbitrariness of ϵ, we conclude the proof of (4.44). Finally we prove (4.45). Consider the continuous, non-negative function $L(x,\gamma) := \inf_{t\in[0,1]} \mathsf{d}(x, e_t(\gamma))$. Equation (4.42) implies

$$0 = \int_{\bar{E}\times K} L(x,\gamma)\, \tau_R(dx\, d\gamma).$$

The equation above passes to the limit as $R \to \infty$, hence we deduce $L(x,\gamma) = 0$ for τ-a.e. $(x,\gamma) \in \bar{E} \times K$, which is precisely (4.42). □

4.4.3 Disintegration of the measure and the perimeter

Recalling the disintegration formula (4.35), we define the map $\bar{E} \ni x \mapsto \mu_{x,R} \in \mathcal{P}(\bar{E})$ as

$$\mu_{x,R} := \begin{cases} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (\widehat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R}) \llcorner_E, & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ \delta_x, & \text{otherwise.} \end{cases}$$

This new family of measures satisfies the disintegration formula

$$\mathfrak{m}\llcorner_E = \int_{\bar{E}} \mu_{x,R}\, \mathfrak{m}\llcorner_E(dx). \tag{4.46}$$

Indeed, by a direct computation (recall (4.4)–(4.5))

$$\begin{aligned} \mathfrak{m}(A\cap E) &= \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}(A\cap E)\, \widehat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}(A\cap E)\, (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)(d\alpha) \\ &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_X \widehat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R}(A\cap E)\, \mathfrak{m}\llcorner_E(dx) = \int_X \mu_{x,R}(A)\, \mathfrak{m}\llcorner_E(dx). \end{aligned}$$

Remark 4.23. We give a few details regarding the measurablity of the integrand function in Equation (4.46). Said equation should be interpreted in the following sense: the map $x \mapsto \mu_{x,R}(A)$ is measurable and the formula (4.46) holds. Indeed, the map $x \mapsto \mu_{x,R}(A)$ is (up to excluding the negligible set $\bar{E}\backslash(E \cap \widehat{\mathcal{T}}_R)$) the composition of $Q_R \ni \alpha \mapsto \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \widehat{\mathfrak{m}}_{\alpha,R}(A\cap E)$ and the projection $\mathfrak{Q}_R$. The former map is $\widehat{\mathfrak{q}}_R$-measurable, while the map $\mathfrak{Q}_R$ is $\mathfrak{m}$-measurable, with respect to the σ-algebra of Q_R, thus the composition is measurable.

Since $\widehat{\mathfrak{m}}_{\alpha,R} = (g_R(\alpha,\cdot))_\#(h_{\alpha,R}\mathcal{L}^1\llcorner_{[0,|\widehat{X}_{\alpha,R}|]})$, we can explicitly compute the measure $\mu_{x,R}$ (recall that by (4.36) $r_R(x) = \operatorname{ess\,sup} E_{x,R}$, for $\mathfrak{m}\llcorner_E$-a.e. x)

$$\begin{aligned} \mu_{x,R} &= \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (g_R(\mathfrak{Q}_R(x),\cdot))_\# \left((g_R(\mathfrak{Q}_R(x),\cdot))^{-1}(E) h_{\mathfrak{Q}_R(x),R} \mathcal{L}^1\llcorner_{[0,r_R(r)]} \right) \\ &= (g_R(\mathfrak{Q}_R(x),\cdot))_\# \left(\mathbf{1}_{E_{x,R}} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{\mathfrak{Q}_R(x),R} \mathcal{L}^1\llcorner_{[0,r_R(x)]} \right) \\ &= (\gamma^{x,R})_\# (\tilde{h}_E^{x,R} \mathcal{L}^1\llcorner_{[0,1]}), \qquad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \bar{E} \end{aligned}$$

where

$$\tilde{h}_E^{x,R}(t) = \mathbf{1}_{E_{x,R}}(r_R(x)t)\, r_R(x) \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\, h_{\mathfrak{Q}_R(x),R}(r_R(x)t).$$

Thanks to (4.6), we can perform a similar operation for the perimeter. Having in mind that $h_{R,\mathfrak{Q}_R(x)}(r_R(x))\delta_{r_R(x)} \le \mathsf{P}_{h_{R,\mathfrak{Q}_R(x)}}(E_{x,R};\,\cdot)$, we define the map

$$p_{x,R} := \begin{cases} \min\left\{\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{R,\mathfrak{Q}_R(x)}(r_R(x)),\, \frac{N}{\rho}\right\} \delta_{g_R(\mathfrak{Q}_R(x), r_R(x))}, & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ \frac{N}{\rho}\delta_x, & \text{if } x \in \bar{E} \backslash (E \cap \mathcal{T}_R). \end{cases}$$

Using the maps $\gamma^{x,R}$ and $\tilde{h}_{x,R}$, we can rewrite $p_{x,R}$ as

$$p_{x,R} = \begin{cases} \min\left\{\dfrac{\tilde{h}_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R})},\, \dfrac{N}{\rho}\right\} \delta_{\gamma_1^{x,R}}., & \text{if } x \in E \cap \widehat{\mathcal{T}}_R, \\ \dfrac{N}{\rho}\delta_x, & \text{if } x \in \bar{E} \backslash (E \cap \mathcal{T}_R). \end{cases}$$

The definition of $p_{x,R}$ immediately yields

$$p_{x,R} \le \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;\,\cdot), \qquad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \bar{E},$$

hence we deduce the following "disintegration" formula (equations (4.6) and (4.5) are taken into account)

$$\begin{aligned} \mathsf{P}(E;A) &\ge \int_{Q_R} \mathsf{P}_{X_{\alpha,R}}(E;A)\, \widehat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{\bar{E}} \mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;A)\, \mathfrak{m}\llcorner_E(dx) \\ &\ge \int_{\bar{E}} p_{x,R}(A)\, \mathfrak{m}(dx), \quad \forall A \subset \bar{E} \text{ Borel}. \end{aligned} \tag{4.47}$$

Let $F := e_{(0,1)}(K) = \{\gamma_t : \gamma \in K, t \in [0,1]\}$ and let $S \subset \mathcal{M}^+(F)$ be the subset of the non-negative measures on F with mass at most N/ρ. We endow the sets $\mathcal{P}(F)$ and S with the weak topology of measures. Since K and F are compact Hausdorff spaces, by Riesz–Markov Representation Theorem, the weak topology on $\mathcal{P}(F)$ and S, coincides with the weak* topology induced by the duality against continuous functions $C(F)$. It is well-known that the weak* convergence can be metrized on bounded sets, if the primal space is separable. For instance, a possible suitable metric is given by

$$d(\mu,\nu) = \sum_{k=1}^{\infty} \frac{1}{2^k \|f_k\|_\infty} \left| \int_X f_k\, d\mu - \int_X f_k\, d\nu \right|, \tag{4.48}$$

where $\{f_k\}_k$ is dense set in $C(X)$. We endow the spaces $\mathcal{P}(F)$ and S with the distance defined in (4.48).

Define the map $G_R : \bar{E} \times K \to \mathcal{P}(F) \times S$, as

$$G_R(x,\gamma) := \left(\gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}), \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)} \right).$$

The function G_R is measurable w.r.t. the variable x and continuous w.r.t. the variable γ. At this point we can define the measure

$$\sigma_R := (\mathrm{Id} \times G_R)_\# \tau_R \in \mathcal{M}^+(\bar{E} \times K \times \mathcal{P}(F) \times S).$$

Notice that the mass of σ_R is $\mathfrak{m}(E)$ for all $R > 0$. In order to simplify the notation, set $Z = \bar{E} \times K \times \mathcal{P}(F) \times S$.

Proposition 4.24. *The measure σ_R satisfies the following properties*

$$\int_E \psi \, d\mathfrak{m} = \int_Z \int_E \psi(y) \, \mu(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp), \quad \forall \psi \in C_b^0(\bar{E}), \tag{4.49}$$

$$\int_{\bar{E}} \psi(y) \, \mathsf{P}(E, dy) \geq \int_Z \int_{\bar{E}} \psi(y) \, p(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp), \quad \forall \psi \in C_b^0(\bar{E}), \psi \geq 0. \tag{4.50}$$

Proof. Fix a test function $\psi \in C_b^0(\bar{E})$. First we notice that for σ_R-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $\mu = \mu_{x,R}$. Indeed, it holds that

$$\mu = \gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}) = (\gamma^{x,R})_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]}) = \mu_{x,R}, \qquad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p) \in Z,$$

and we used the fact that $\gamma = \gamma_{x,R}$ for τ_R-a.e. $(x,\gamma) \in \bar{E} \times K$. We conclude this first part by a direct computation

$$\begin{aligned} \int_E \psi \, d\mathfrak{m} &= \int_E \int_E \psi(y) \, \mu_{x,R} \, \mathfrak{m}(dx) = \int_Z \int_E \psi(y) \, \mu_{x,R}(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp) \\ &= \int_Z \int_E \psi(y) \, \mu(dy) \, \sigma_R(dx \, d\gamma \, d\mu \, dp), \end{aligned}$$

we conclude the proof of inequality (4.49).

Now fix an open set $\Omega \subset X$ and compute using (4.47)

$$\begin{aligned} \mathsf{P}(E;\Omega) &\geq \int_E \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R})}, \frac{N}{\rho} \right\} \delta_{\gamma_1^{x,R}}(\Omega) \, d\mathfrak{m}(dx) \\ &= \int_Z \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma^{x,R}), e_1(\gamma^{x,R}))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma^{x,R})}(\Omega) \, d\sigma_R(dx \, d\gamma \, d\mu \, dp) \end{aligned}$$

$$= \int_Z \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)}(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp).$$

If we use the fact that

$$p = \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}}{\mathfrak{m}(E)^{\frac{1}{N}}} \right\} \delta_{e_1(\gamma)}(\Omega), \qquad \text{for } \sigma_R\text{-a.e. } (x, \gamma, \mu, p) \in Z,$$

we continue the chain of inequalities obtaining

$$\begin{aligned} \mathsf{P}(E; \Omega) &\geq \int_Z \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)}(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp) \\ &= \int_Z p(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp). \end{aligned}$$

Since the perimeter is outer-regular, i.e., $\mathsf{P}(E; A) = \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A \text{ is open}\}$, we can conclude. □

At this point we are in position to take the limit as $R \to \infty$, as the properties we have proven pass to the limit, but before proceeding we prove the following technical Lemma.

Lemma 4.25. *Let X be a Polish space, let Y, Z be two compact Polish spaces, and let $\mathfrak{m}$ be a finite Radon measure on X. Consider a sequence of functions $f_n : X \times Y \to Z$ and $f : X \times Y \to Z$, such that f and f_n are Borel-measurable in the first variable and continuous in the second. Assume that for $\mathfrak{m}$-a.e. $x \in X$ the sequence $f_n(x, \cdot)$ converges uniformly to $f(x, \cdot)$. Consider a sequence of measures $\mu_n \in \mathcal{M}^+(X \times Y)$ such that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{M}^+(X \times Y)$ and $(\pi_X)_\# \mu_n = \mathfrak{m}$.*

Then it holds

$$(\mathrm{Id} \times f_n)_\# \mu_n \rightharpoonup (\mathrm{Id} \times f)_\# \mu, \qquad \textit{weakly in } \mathcal{M}(X \times Y \times Z).$$

Proof. In order to simplify the notation, set $\nu_n = (\mathrm{Id} \times f_n)_\# \mu_n$ and $\nu = (\mathrm{Id} \times f)_\# \mu$. Fix $\epsilon > 0$.

We would like to use an extension of the Egorov's and Lusin's Theorems for functions taking values in separable metric spaces. The reader can find a proof these theorems in [40, Theorem 7.5.1] (for the Egorov's Theorem) and [39, Appendix D] (for the Lusin's Theorem). In this setting, we deal with maps taking value in $C(Y, Z)$, the space of continuous functions between the compact spaces Y and Z, which turns out to be separable.

Using said Theorems, we can find a compact $K \subset X$ such that: 1) the maps $x \in K \mapsto f_n(x, \cdot) \in C(Y, Z)$ are continuous (and the same holds for f in place of f_n); 2) the restricted maps $x \in K \mapsto f_n(x, \cdot)$ converge to $x \in K \mapsto f(x, \cdot)$, uniformly in the space $C(K, C(Y, Z))$;

3) $\mathfrak{m}(X\backslash K) \leq \epsilon$. Regarding point 2), this immediately implies that the restriction $f_n|_{K\times Y} \to f|_{K\times Y}$ converges uniformly in $K \times Y$.

We test the convergence of ν_n against $\varphi \in C_b^0(X \times Y \times Z)$

$$\begin{aligned}\left|\int_{X\times Y\times Z} \varphi\, d\nu_n - \int_{X\times Y\times Z} \varphi\, d\nu\right| &\leq \|\varphi\|_{C^0} \left(\nu_n((X\backslash K)\times Y\times Z)\right.\\ &\quad \left.+ \nu((X\backslash K)\times Y\times Z)\right)\\ &\quad + \left|\int_{K\times Y\times Z} \varphi\, d\nu_n - \int_{K\times Y\times Z} \varphi\, d\nu\right|\\ &= \|\varphi\|_{C^0} \left(\mathfrak{m}(X\backslash K) + \mathfrak{m}(X\backslash K)\right)\\ &\quad + \left|\int_{K\times Y\times Z} \varphi\, d\nu_n - \int_{K\times Y\times Z} \varphi\, d\nu\right|\\ &\leq 2\epsilon\, \|\varphi\|_{C^0} + \left|\int_{K\times Y\times Z} \varphi\, d\nu_n - \int_{K\times Y\times Z} \varphi\, d\nu\right|.\end{aligned}$$

We focus on the second term and we compute the integral

$$\int_{K\times Y\times Z} \varphi\, d\nu_n = \int_{K\times Y} \varphi(x,y,f_n(x,y))\, \mu_n(dx\, dy).$$

The function $\varphi|_{K\times Y\times Z}$ is uniformly continuous (because it is continuous and defined on a compact space), hence $\varphi(x,y,f_n(x,y))$ converges to $\varphi(x,y,f(x,y))$ uniformly in $K\times Y$. For this reason, together with the fact that $\mu_n \rightharpoonup \mu$ weakly we can take the limit in the equation above obtaining

$$\lim_{n\to\infty} \int_{K\times Y} \varphi(x,y,f_n(x,y))\, \mu_n(dx\, dy) = \int_{K\times Y} \varphi(x,y,f(x,y))\, \mu(dx\, dy) = \int_{K\times Y\times Z} \varphi\, d\nu,$$

and this concludes the proof. □

Corollary 4.26. *Consider the function $G : \bar{E} \times K \to \mathcal{P}(F) \times S$ defined as*

$$G(x,\gamma) = \left(\gamma_\#(Nt^{N-1}\mathcal{L}^1\llcorner_{[0,1]}), \max\left\{\frac{N}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho}\right\}\delta_{e_1(\gamma)}\right),$$

and let $\sigma := (\mathrm{Id} \times G)_\#\tau$. Then we have that $\sigma_R \rightharpoonup \sigma$ in the weak topology of measures.

Proof. We just need to check the hypotheses of the previous Lemma. The set $\bar{E}$ is compact, hence complete and separable. The set K is compact and so is $\mathcal{P}(F) \times S$ (w.r.t. the weak topology). As we have already pointed out, the maps G_R are measurable in the first variable and continuous in the second variable. Finally, we need to see that for a.e. x, the limit

$G_R(x,\gamma) \to G(x,\gamma)$ holds uniformly w.r.t. γ. Fix x and γ and pick $\psi \in C_b(F)$ a test function. Compute

$$\left| \int_F \psi(y)\, \gamma_\#(\tilde{h}_E^{x,R} \mathcal{L}^1 \llcorner_{[0,1]})(dy) - \int_F \psi(y)\, \gamma_\#(Nt^{N-1} \mathcal{L}^1 \llcorner_{[0,1]})(dy) \right|$$
$$= \left| \int_0^1 \psi(\gamma_t)(\tilde{h}_E^{x,R} - Nt^{N-1})\, dt \right| \le \|\psi\|_{C(F)} \left\| \tilde{h}_E^{x,R} - Nt^{N-1} \right\|_{L^\infty}.$$

The r.h.s. of the inequality above does not depend on γ (but only on x and ψ) and converges to 0 by Theorem 4.19, in particular (4.34). This means that the first component of $G_R(x,\gamma)$ converges (in the weak topology of $\mathcal{P}(F)$), uniformly w.r.t. γ (see (4.48)). For the other component the proof is analogous, so we omit it. □

The next proposition reports all the relevant properties of the limit measure σ.

Proposition 4.27. *The measure σ satisfies the following disintegration formulae*

$$\int_E \psi(y)\, \mathsf{m}(dy) = \int_Z \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp), \quad \forall \psi \in L^1(E; \mathsf{m}\llcorner_E), \tag{4.51}$$
$$\int_{\bar{E}} \psi(y) \mathsf{P}(E; dy) = \frac{N}{\rho} \int_Z \psi(e_1(\gamma))\psi\, \sigma(dx\, d\gamma\, d\mu\, dp), \quad \forall \psi \in L^1(\bar{E}; \mathsf{P}(E;\,\cdot\,)). \tag{4.52}$$

Furthermore, for σ-a.e. $(x,\gamma,\mu,p) \in Z$ it holds

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \varphi_\infty(e_t(\gamma)) - \varphi_\infty(e_s(\gamma)), \quad \forall 0 \le t \le s \le 1, \tag{4.53}$$
$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{4.54}$$
$$x \in \gamma, \tag{4.55}$$
$$\mu = \gamma_\#(Nt^{N-1}\mathcal{L}^1 \llcorner_{[0,1]}), \tag{4.56}$$
$$p = \frac{N}{\rho} \delta_{e_1(\gamma)}. \tag{4.57}$$

Proof. Equations (4.53)–(4.55) are just a restatement of (4.43)–(4.45), respectively. Equation (4.56) is an immediate consequence of the definition of G. Similarly, taking into account (4.54), we can deduce (4.57)

$$p = \min\left\{ \frac{N}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)} = \frac{N}{\rho} \delta_{e_1(\gamma)}.$$

In order to prove (4.51), fix $\psi \in C_b^0(F) = C_b^0(e_{(0,1)}(K))$ and define the function $L_\psi : \mathcal{P}(F) \to \mathbb{R}$ as $L_\psi(\mu) = \int_F \psi\, d\mu$. This function is bounded and continuous w.r.t. the weak topology of $\mathcal{P}(F)$. Hence, we take into account the definition of weak convergence of measures and we compute the limit using (4.49) and (4.56)

$$\begin{aligned}\int_E \psi\, d\mathfrak{m} &= \lim_{R\to\infty}\int_Z\int_F \psi(y)\,\mu(dy)\,\sigma_R(dx\,d\gamma\,d\mu\,dp) = \lim_{R\to\infty}\int_Z L_\psi(\mu)\,\sigma_R(dx\,d\gamma\,d\mu\,dp)\\ &= \int_Z\int_F \psi(y)\,\mu(dy)\,\sigma(dx\,d\gamma\,d\mu\,dp) = \int_Z\int_0^1 \psi(e_t(\gamma))Nt^{N-1}\,dt\,\sigma(dx\,d\gamma\,d\mu\,dp).\end{aligned}$$

Using standard approximation arguments, we see that the equation above holds true also for any $\psi \in L^1(E;\mathfrak{m}\llcorner_E)$.

Regarding (4.52), using the same argument we can deduce that

$$\int_{\bar{E}} \psi(y)\,\mathsf{P}(E;dy) \geq \frac{N}{\rho}\int_Z \psi(e_1(\gamma))\,\sigma(dx\,d\gamma\,d\mu\,dp), \quad \forall\psi \in L^1(\bar{E};\mathsf{P}(E;\,\cdot\,)),\ \psi\geq 0.$$

If we test the inequality above with $\psi = 1$, the inequality is saturated meaning that the two measures have the same mass, so the inequality improves to an equality. □

4.4.4 Back to the classical localization notation

We are now in position to re-obtain a "classical" disintegration formula for both the measure $\mathfrak{m}$ and the perimeter of E.

We recall the definition of some of the objects that were introduced in Subsection 2.4.1. For instance, let $\Gamma_\infty = \{(x,y) : \varphi_\infty(x) - \varphi_\infty(y) = \mathsf{d}(x,y)\}$ and $\mathcal{R}^e_\infty = \Gamma_\infty \cup \Gamma^{-1}_\infty$ be the transport relation. The transport set with endpoints is $\mathcal{T}^e_\infty := P_1(\mathcal{R}^e_\infty\setminus\{x=y\})$; clearly $E \subset \mathcal{T}^e_\infty$, up to a negligible set. The sets of forward and backward branching points are defined as

$$\begin{aligned}A^+_\infty &:= \{x \in \mathcal{T}^e_\infty : \exists z,w \in \Gamma_\infty(x), (z,w)\notin \mathcal{R}^e_\infty\},\\ A^-_\infty &:= \{x \in \mathcal{T}^e_\infty : \exists z,w \in \Gamma^{-1}_\infty(x), (z,w)\notin \mathcal{R}^e_\infty\}.\end{aligned}$$

The transport set is defined as $\mathcal{T}_\infty := \mathcal{T}^e_\infty\setminus(A^+_\infty \cup A^-_\infty)$; since the sets A^+_∞ and A^-_∞ are negligible, then $\mathcal{T}_\infty$ has full measure in $\mathcal{T}^e_\infty$. Let Q_∞ be the quotient set and let $\mathfrak{Q}_\infty : \mathcal{T}_\infty \to Q_\infty$ be the quotient map; denote by $X_{\alpha,\infty} := \mathfrak{Q}^{-1}(\alpha)$ the disintegration rays and let $g_\infty : \mathrm{Dom}(g_\infty) \subset \mathbb{R}\times Q_\infty \to X$ be the parametrization of the rays such that $\frac{d}{dt}\varphi_\infty(g_\infty(t,\alpha)) = -1$. For every $\alpha \in Q_\infty$, let $t_\alpha : \overline{X_{\alpha,\infty}} \to [0,\infty)$ be the function $t_\alpha(x) := (g(\alpha,\cdot))^{-1} = \mathsf{d}(x, g_\infty(\mathfrak{Q}_\infty(x),0))$; the function t_α measures how much a point is translates from the starting point of the ray $X_{\alpha,\infty}$.

The following proposition guarantees that the geodesics on which the measure σ is supported lay on the transport set $\mathcal{T}_\infty$.

Proposition 4.28. *For σ-a.e. $(x,\gamma,\mu,p) \in Z$, it holds that $e_t(\gamma) \notin A^+_\infty \cup A^-_\infty$, for all $t \in (0,1)$.*

Proof. Fix $\epsilon > 0$ and let

$$P := \{(x, \gamma, \mu, p) \in Z : e_\epsilon(\gamma) \in A^+_\infty \text{ and conditions (4.51)–(4.57) holds}\}$$

Notice that by definition of A^+_∞, if $(x, \gamma, \mu, p) \in P$, then $\gamma_t \in A^+_\infty$, for all $t \in [0, \epsilon]$, thus we can compute

$$\begin{aligned} 0 = \mathfrak{m}(A^+_\infty) &= \int_Z \int_0^1 \mathbf{1}_{A^+_\infty}(e_t(\gamma)) N t^{N-1} \, dt \, \sigma(dx \, d\gamma \, d\mu \, dp) \\ &\geq \int_P \int_0^\epsilon \mathbf{1}_{A^+_\infty}(e_t(\gamma)) N t^{N-1} \, dt \, \sigma(dx \, d\gamma \, d\mu \, dp) \geq \epsilon^N \sigma(P), \end{aligned}$$

so P is negligible. Fix now $(x, \gamma, \mu, p) \notin P$. By definition of A^+_∞ and P, we have that $\gamma_t \notin A^+_\infty$, for all $t \in [\epsilon, 1]$. By arbitrariness of ϵ, we deduce that for σ-a.e $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \notin A^+_\infty$, for all $t \in (0, 1]$. The proof for the set A^-_∞ is analogous. □

Corollary 4.29. *For σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \in \overline{X_{\mathfrak{Q}(x),\infty}}$ and*

$$e_t(\gamma) = g(\mathfrak{Q}(x), t_{\mathfrak{Q}(x)}(e_0(\gamma)) + \rho t). \tag{4.58}$$

Define $\hat{\mathfrak{q}} := \frac{1}{\mathfrak{m}(E)} (\mathfrak{Q}_\infty)_\#(\mathfrak{m} \llcorner_E) \ll (\mathfrak{Q}_\infty)_\# \mathfrak{m} \llcorner_{\mathcal{T}_\infty}$ and let $\tilde{\mathfrak{q}}$ be a probability measure such that $(\mathfrak{Q}_\infty)_\# \mathfrak{m} \llcorner_{\mathcal{T}_\infty} \ll \tilde{\mathfrak{q}}$. The disintegration theorem gives the following two formulae

$$\mathfrak{m} \llcorner_E = \int_{Q_\infty} \hat{\mathfrak{m}}_{\alpha,\infty} \, \hat{\mathfrak{q}}(d\alpha), \quad \text{and} \quad \mathfrak{m} \llcorner_{\mathcal{T}_\infty} = \int_{Q_\infty} \tilde{\mathfrak{m}}_{\alpha,\infty} \, \tilde{\mathfrak{q}}(d\alpha), \tag{4.59}$$

where the measures $\hat{\mathfrak{m}}_{\alpha,\infty}$ and $\tilde{\mathfrak{m}}_{\alpha,\infty}$ are supported on $X_{\alpha,\infty}$. By comparing the two expressions above, it turns out that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha) \, \hat{\mathfrak{m}}_{\alpha,\infty} = \mathbf{1}_E \tilde{\mathfrak{m}}_{\alpha,\infty}$. Theorem 2.7, ensures that the space $(X_{\alpha,\infty}, \mathsf{d}, \tilde{\mathfrak{m}}_{\alpha,\infty})$ satisfies the $\mathsf{CD}(0, N)$ condition. Note that the disintegration $\alpha \mapsto \hat{\mathfrak{m}}_{\alpha,\infty}$ does not fall under the hypothesis of Theorem 2.7: indeed, in this case we are disintegrating a measure concentrated on E and not on the transport set $\mathcal{T}_\infty$. Define the functions $\hat{h}_\alpha$ and $\tilde{h}_\alpha$ as the functions such that

$$\hat{\mathfrak{m}}_{\alpha,\infty} = (g(\alpha, \cdot))_\#(\hat{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}), \quad \text{and} \quad \tilde{\mathfrak{m}}_{\alpha,\infty} = (g(\alpha, \cdot))_\#(\tilde{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}).$$

Clearly, it holds that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha) \hat{h}_\alpha(t) = \mathbf{1}_E(g(\alpha, t)) \tilde{h}_\alpha(t)$, thus we can derive a somehow weaker concavity condition for the function $\hat{h}_\alpha^{\frac{1}{N-1}}$: for all $x_0, x_1 \in (0, |X_{\alpha,\infty}|)$ and for all $t \in [0, 1]$, it holds that

$$\hat{h}_\alpha((1-t)x_0 + t x_1)^{\frac{1}{N-1}} \geq (1-t)\hat{h}_\alpha(x_0)^{\frac{1}{N-1}} + t \hat{h}_\alpha(x_1)^{\frac{1}{N-1}}, \quad \text{if } \hat{h}_\alpha((1-t)x_0 + t x_1) > 0.$$

The inequality above implies that

$$\text{the map } r \mapsto \frac{\hat{h}_\alpha(r)}{r^{N-1}} \text{ is decreasing on the set } \{r \in (0, |X_{\alpha,\infty}|) : \hat{h}_\alpha(r) > 0\}. \tag{4.60}$$

Define the set $\hat{Z} \subset Z$ as

$$\hat{Z} := \{(x,\gamma,\mu,p) \in Z : x \in E \cap \mathcal{T}_\infty, \text{ and the properties given by Equations (4.51)–(4.52) and (4.58) holds}\}.$$

Clearly $\hat{Z}$ has full σ-measure in Z. We give a partition for $\hat{Z}$

$$\hat{Z}_\alpha := \{(x,\gamma,\mu,p) \in \hat{Z} : \mathfrak{Q}_\infty(x) = \alpha\},$$

and we disintegrate the measure σ according to the partition $(\hat{Z}_\alpha)_{\alpha \in Q_\infty}$

$$\sigma = \int_{Q_\infty} \sigma_\alpha \, \mathfrak{q}(d\alpha), \tag{4.61}$$

where the measures σ_α are supported on $\hat{Z}_\alpha$. Moreover, let $\nu_\alpha \in \mathcal{P}([0,\infty))$ be the measure given by

$$\nu_\alpha := \frac{1}{\mathfrak{m}(E)} (t_\alpha \circ e_0 \circ \pi_K)_\#(\sigma_\alpha)$$

(we recall that $t_\alpha = (g(\alpha, \cdot))^{-1}$ and $\pi_K(x,\gamma,\mu,p) = \gamma$).

The following proposition states that the density $\hat{h}_\alpha$ is given by the convolution of the model density and the measure ν_α.

Proposition 4.30. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r) = N\omega_N \mathsf{AVR}_X \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r) \, \nu_\alpha(dt), \quad \forall r \in (0, |X_{\alpha,\infty}|).$$

Proof. Fix $\psi \in L^1(\mathfrak{m}\llcorner_E)$ and compute its integral using Equations (4.51) and (4.61)

$$\begin{aligned}\int_E \psi(x)\,\mathfrak{m}(dx) &= \int_{\hat{Z}} \int_0^1 \psi(e_t(\gamma)) N t^{N-1} \, dt \, \sigma(dx\,d\gamma\,d\mu\,dp) \\ &= \int_{Q_\infty} \int_{\hat{Z}_\alpha} \int_0^1 \psi(e_t(\gamma)) N t^{N-1} \, dt \, \sigma_\alpha(dx\,d\gamma\,d\mu\,dp) \, \mathfrak{q}(d\alpha).\end{aligned}$$

Fix now $\alpha \in Q_\infty$ and compute (recall (4.58) and the definition of $\hat{Z}$)

$$\int_{\hat{Z}_\alpha} \int_0^1 \psi(e_t(\gamma)) N t^{N-1} \, dt \, \sigma_\alpha(dx\,d\gamma\,d\mu\,dp) = \int_{\hat{Z}_\alpha} \int_0^\rho \psi(e_{s/\rho}(\gamma)) N \frac{s^{N-1}}{\rho^N} \, ds \, \sigma_\alpha(dx\,d\gamma\,d\mu\,dp)$$

$$
\begin{aligned}
&= \int_{\hat{Z}_\alpha} \int_0^\rho \psi(g_\infty(\mathfrak{Q}(x), t(\alpha,\gamma_0)+s)) N \frac{s^{N-1}}{\rho^N}\, ds\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\
&= \int_{\hat{Z}_\alpha} \int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha, r)) N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0), t(\alpha,\gamma_0)+\rho)}(r)\, dr\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\
&= \int_0^{|X_{\alpha,\infty}|} \psi(g_\infty(\alpha, r)) \int_{\hat{Z}_\alpha} N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0), t(\alpha,\gamma_0)+\rho)}(r)\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp)\, dr,
\end{aligned}
$$

hence, by the uniqueness of the disintegration, we deduce that

$$
\begin{aligned}
\hat{h}_\alpha(r) &= \int_{\hat{Z}_\alpha} N \frac{(r-t(\alpha,\gamma_0))^{N-1}}{\rho^N} \mathbf{1}_{(t(\alpha,\gamma_0), t(\alpha,\gamma_0)+\rho)}(r)\, \sigma_\alpha(dx\, d\gamma\, d\mu\, dp) \\
&= N\omega_N \mathsf{AVR}_X \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r)\, \nu_\alpha(dt).
\end{aligned}
$$
□

Proposition 4.31. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that $\nu_\alpha = \delta_0$.*

Proof. Let $T := \inf \operatorname{supp} \nu_\alpha$. If we set $r \in (T, T+\rho)$, we can compute

$$
\begin{aligned}
\frac{\hat{h}_{\alpha,\infty}(r)}{N\omega_N \mathsf{AVR}_X} &= \int_{[0,\infty)} (r-t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r)\, \nu_\alpha(dt) = \int_{[T,r)} (r-t)^{N-1}\, \nu_\alpha(dt) \\
&\geq \int_{[T,r)} \left(\frac{r-T}{2} \mathbf{1}_{[T,(r+T)/2]}(t) \right)^{N-1} \nu_\alpha(dt) = \frac{(r-T)^{N-1}}{2^{N-1}} \nu_\alpha([T, \tfrac{r+T}{2}]).
\end{aligned}
\tag{4.62}
$$

By definition of T, we have that $\nu_\alpha([T, \frac{r+T}{2}]) > 0$, hence $\hat{h}_\alpha(r) > 0$, for all $r \in (T, T+\rho)$. On the other hand

$$
\begin{aligned}
\hat{h}_{\alpha,\infty}(r) &= N\omega_N \mathsf{AVR}_X \int_{[T,r)} (r-t)^{N-1}\, \nu_\alpha(dt) \\
&\leq N\omega_N \mathsf{AVR}_X (r-T)^{N-1}\, \nu_\alpha([T,r)) \to 0. \quad \text{as } r \to T^+.
\end{aligned}
\tag{4.63}
$$

We claim that $T = 0$. Indeed, if $T > 0$, then $\lim_{r \to T^+} \hat{h}_\alpha(r)/r^{N-1} = 0$ contradicting (4.60).

We now derive the non-increasing function

$$
(0,\rho) \ni r \mapsto \frac{\hat{h}_\alpha(r)}{r^{N-1}} = \frac{N\omega_N \mathsf{AVR}_X}{r^{N-1}} \int_{[0,r)} (r-t)^{N-1}\, \nu_\alpha(dt),
$$

obtaining

$$
0 \geq N\omega_N \mathsf{AVR}_X \left(\frac{1-N}{r^N} \int_{[0,r)} (r-t)^{N-1}\, \nu_\alpha(dt) + \frac{1}{r^{N-1}} \frac{d}{dr} \int_{[0,r)} (r-t)^{N-1}\, \nu_\alpha(dt) \right).
$$

The second term can be computed as

$$\begin{aligned}
&\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\
&\qquad=\lim_{h\to 0}\int_{[r,r+h)}\frac{(r+h-t)^{N-1}}{h}\,\nu_\alpha(dt)+\lim_{h\to 0}\int_{[0,r)}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\,\nu_\alpha(dt)\\
&\qquad\geq 0+\int_{[0,r)}\lim_{h\to 0}\frac{(r+h-t)^{N-1}-(r-t)^{N-1}}{h}\,\nu_\alpha(dt)=(N-1)\int_{[0,r)}(r-t)^{N-2}\,\nu_\alpha(dt),
\end{aligned}$$

yielding

$$\begin{aligned}
0&\geq(1-N)\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)+r\frac{d}{dr}\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)\\
&\geq(N-1)\int_{[0,r)}\left(r(r-t)^{N-2}-(r-t)^{N-1}\right)\nu_\alpha(dt)=(N-1)\int_{[0,r)}t(r-t)^{N-2}\,\nu_\alpha(dt).
\end{aligned}$$

The inequality above implies that $\nu_\alpha((0,r))=0$, for all $r\in(0,\rho)$, hence $\nu_\alpha(0,\rho)=0$. We deduce that

$$\hat{h}_\alpha(r)=N\omega_N\mathsf{AVR}_X\int_{[0,r)}(r-t)^{N-1}\,\nu_\alpha(dt)=N\omega_N\mathsf{AVR}_X\,r^{N-1}\nu_\alpha(\{0\}),\quad\forall r\in(0,\rho).$$

If $\nu_\alpha([\rho,\infty))=0$, then $\nu_\alpha=\delta_0$ (because ν_α has mass 1) completing the proof. Assume on the contrary that $\nu_\alpha([\rho,\infty))>0$, and let $S:=\inf\operatorname{supp}(\nu_\alpha\llcorner_{[\rho,\infty)})\geq\rho$. In this case we follow the computations (4.62) and (4.63), with S in place of T, deducing $\lim_{r\to S^+}\hat{h}_\alpha(r)=0$, contradicting (4.60). □

Corollary 4.32. *For $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, for σ_α-a.e. $(x,\gamma,\mu,p)\in Z_\alpha$, it holds that $e_t(\gamma)=g(\alpha,\rho t)$, $\forall t\in[0,1]$.*

Proof. The fact that $\nu_\alpha=\delta_0$, implies $t_\alpha(\gamma_0)=0$ for σ_α-a.e. $(x,\gamma\,\mu,p)\in\hat{Z}_\alpha$, hence, recalling (4.58) and the definition of $\hat{Z}$, we have that $e_t(\gamma)=g(\alpha,t_\alpha(e_0)+\rho t)=g(\alpha,\rho t)$. □

The next corollary concludes the discussion of the limiting procedures of the localization.

Corollary 4.33. *For $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r)=N\omega_N\mathsf{AVR}_X\mathbf{1}_{(0,\rho)}(r)r^{N-1}.$$

Moreover, the following disintegration formulae hold

$$\mathfrak{m} = N\omega_N \mathsf{AVR}_X \int_{Q_\infty} (g(\alpha,\,\cdot\,))_\#(r^{N-1}\,\mathcal{L}^1\llcorner_{(0,\rho)})\,\hat{\mathfrak{q}}(d\alpha), \tag{4.64}$$

$$\mathsf{P}(E;\,\cdot\,) = \mathsf{P}(E)\int_{Q_\infty} \delta_{g(\alpha,\rho)}\,\hat{\mathfrak{q}}(d\alpha). \tag{4.65}$$

Proof. The only non-trivial part is Equation (4.65). Using (4.52) and Corollary 4.32, we can deduce that $\forall \psi \in L^1(\bar{E};\mathsf{P}(E;\,\cdot\,))$

$$\begin{aligned}\int_{\bar{E}} \psi(x)\,\mathsf{P}(E;dx) &= \frac{N}{\rho}\int_{\hat{Z}} \psi(e_1(\gamma))\psi\,\sigma(dx\,d\gamma\,d\mu\,dp)\\ &= \frac{N}{\rho}\int_{Q_\infty}\int_{\hat{Z}_\alpha} \psi(e_1(\gamma))\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,\hat{\mathfrak{q}}(d\alpha)\\ &= \frac{N}{\rho}\int_{Q_\infty} \psi(g(\alpha,\rho))\int_{\hat{Z}_\alpha} \sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,\hat{\mathfrak{q}}(d\alpha). \end{aligned}$$

□

4.5 E is a ball

The aim of this section is to prove that E coincides with a ball of radius ρ. Before starting the proof, we give a few technical lemmas. The first Lemma states that a BV function with null differential on an open connected set is constant. This fact is already known for Sobolev functions and it follows from either the Sobolev-to-Lipschitz property or the local Poincaré inequality.

Lemma 4.34. *Let $(X,\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}(K,N)$ space with $X = \operatorname{supp}\mathfrak{m}$ and let $\Omega \subset X$ be an open connected set. If $v \in w\text{-}\mathrm{BV}((\Omega,\mathsf{d},\mathfrak{m}))$ and $|Du| = 0$, then u is constant in Ω (i.e., there exists $C \in \mathbb{R}$ such that $v(x) = C$ for $\mathfrak{m}$-a.e. $x \in \Omega$).*

Proof. The proof in given only for the case $K = 0$. We refer to Section 2.3 for the notation. Fix $x \in \Omega$ and let $r > 0$ such that $B_{3r}(x) \subset \Omega$. Assume by contradiction that there are two constants $a < b$ such that the sets

$$A := \{y \in B_r(x) : v(y) \le a\} \quad \text{and} \quad B := \{y \in B_r(x) : v(y) \ge b\}$$

have strictly positive measure. Consider the probability measures $\mu_0 = \frac{\mathfrak{m}\llcorner A}{\mathfrak{m}(A)}$ and $\mu_1 = \frac{\mathfrak{m}\llcorner B}{\mathfrak{m}(B)}$. Let $\pi \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and $\mu_t = (e_t)_\#\pi$. The $\mathsf{CD}(K,N)$ condition (as stated in Definition 2.3) reads

$$\begin{aligned}\rho_t(\gamma_t) &\le \Big((1-t)\rho_0^{-1/N}(\gamma_0) + t\rho_1^{-1/N}(\gamma_1)\Big)^{-N} \le (1-t)\rho_0(\gamma_0) + t\rho_1(\gamma_1)\\ &= \mathfrak{m}(A)^{-1} + \mathfrak{m}(B)^{-1}, \quad \text{for } \pi\text{-a.e. } \gamma,\end{aligned}$$

and this proves that there exists a constant $C > 0$ such that $(e_t)_{\#}\pi \le C\mathfrak{m}$. What we have proven and the fact that $\mathrm{Lip}(\gamma) = \mathsf{d}(\gamma_0, \gamma_1) \le 2r$, for π-a.e. γ, implies that π is an ∞-test plan. For π-a.e. γ, we have that $|D(v \circ \gamma)|([0,1]) \ge b - a$, because γ is a curve from A to B, thus

$$b - a \le \int \gamma_{\#}|D(v \circ \gamma)|(X)\, \pi(d\gamma) \le C\, \|\mathrm{Lip}(\gamma)\|_{L^\infty(\pi)}\, \mu(X) \le 2rC\mu(X),$$

where μ is any weak upper gradient for v. Since we can chose the null measure as weak upper gradient we obtain a contradiction. Thus there exists a constant c_x such that $v = c_x$ a.e. in $B_r(x)$. Taking into account the connectedness of Ω, we deduce that v is globally constant. □

The following Lemma is topological. It can be seen as a weak formulation of the following statement: let Ω be an open connected subset of a topological space X and let $E \subset X$ be any set; if $\Omega \cap E \neq \emptyset$ and $\Omega \backslash E \neq \emptyset$, then $\partial E \cap \Omega \neq \emptyset$.

Lemma 4.35. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching $\mathsf{CD}(K,N)$ space with $X = \operatorname{supp}\mathfrak{m}$. Let $E \subset X$ be a Borel set and let $\Omega \subset X$ be an open connected set. If $\mathfrak{m}(E \cap \Omega) > 0$ and $\mathfrak{m}(\Omega \backslash E) > 0$, then $\mathsf{P}(E;\Omega) > 0$.*

Proof. The proof is by contradiction. Assume that $\mathsf{P}(E;\Omega) = 0$. Then there exists a sequence $u_n \in \mathrm{Lip}_{loc}(\Omega)$ such that $u_n \to \mathbf{1}_E$ in L^1_{loc} and $\int_\Omega |\operatorname{lip} u_n|\, d\mathfrak{m} \to 0$. This immediately implies that $u_n \to v$ in the space $\mathrm{BV}_*((\Omega, \mathsf{d}, \mathfrak{m}))$ for some $v \in \mathrm{BV}_*((\Omega, \mathsf{d}, \mathfrak{m}))$ such that $|Dv| = 0$. By uniqueness of the limit, $\mathbf{1}_E = v$ a.e. in Ω, whereas Lemma 4.34 implies that v is constant, which is a contradiction. □

The next Lemma ensures that if two balls coincide, then they must share their center.

Lemma 4.36. *Assume that $(X, \mathsf{d}, \mathfrak{m})$ is an essentially non-branching, $\mathsf{CD}(K,N)$ space with $X = \operatorname{supp}\mathfrak{m}$ and let $x, y \in X$ and $r > 0$. If $B_r(x) = B_r(y)$ and $\mathfrak{m}(X \backslash B_r(x)) > 0$, then $x = y$.*

Proof. Assume by contradiction $x \neq y$. Since (X, d) is a geodesic space $(B_r(x))^t = B_{r+t}(x) = B_{r+t}(y)$, hence if $z \in X$ is such that $\mathsf{d}(z, x) = r + t$, then $z \in B_{r+t+\epsilon}(x) = B_{r+t+\epsilon}(y)$, for all $\epsilon > 0$, thus $\mathsf{d}(z, y) \le r + t = \mathsf{d}(z, x)$. We deduce $\mathsf{d}(z, y) = \mathsf{d}(z, x)$, for all $z \in X \backslash B_r(x)$. Consider now two disjoint sets $A, B \subset X \backslash B_r(x)$, such that $\mathfrak{m}(A) = \mathfrak{m}(B)$. Consider the maps

$$T(z) = \begin{cases} x, & \text{if } z \in A, \\ y, & \text{if } z \in B, \end{cases} \qquad S(z) = \begin{cases} y, & \text{if } z \in A, \\ x, & \text{if } z \in B. \end{cases}$$

Since $\mathsf{d}(S(z), x) = \mathsf{d}(S(z), y) = \mathsf{d}(T(z), x) = \mathsf{d}(T, x), \forall z \in A \cup B$, these maps are two different solutions of the Monge problem $\inf_R \int_{A\cup B} \mathsf{d}^2(z, R(z))\, \mathfrak{m}(dz)$, among all possible maps $R : X \to X$ such that $R_{\#}(\mathfrak{m}\llcorner_{A\cup B}) = \mathfrak{m}(A)(\delta_x + \delta_y)$. Since said problem admits a unique solution [29, Theorem 5.1], we have found a contradiction. □

Proposition 4.37. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\varphi_\infty(g_\infty(\alpha,0)) \leq \operatorname*{ess\,sup}_E \varphi_\infty, \qquad \textit{and} \qquad \varphi_\infty(g_\infty(\alpha,\rho)) \geq \operatorname*{ess\,inf}_E \varphi_\infty.$$

Proof. We prove only the former inequality, for the latter has the same proof. In order to simplify the notation define $M := \operatorname{ess\,sup}_E \varphi_\infty$. Let $H := \{\alpha \in Q_\infty : \varphi_\infty(g_\infty(\alpha,0)) \geq M + 2\epsilon\}$. Define the following measure on E

$$\mathfrak{n}(T) = N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon \mathbf{1}_T(g_\infty(\alpha,r)) r^{N-1}\, dr\, \hat{\mathfrak{q}}(d\alpha), \quad \forall T \subset E \text{ Borel}.$$

Clearly, $\mathfrak{n} \ll \mathfrak{m}$ (compare with (4.64)), so $\varphi_\infty(x) \leq M$, for $\mathfrak{n}$-a.e. $x \in E$. We can compute the integral

$$\begin{aligned} 0 &\geq \int_E (\varphi_\infty(x) - M)\, \mathfrak{n}(dx) = N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon (\varphi_\infty(g_\infty(\alpha,t)) - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon (\varphi_\infty(g_\infty(\alpha,0)) - t - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &\geq N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon \epsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) = \epsilon^N \hat{\mathfrak{q}}(H). \end{aligned}$$

We deduce that $\hat{\mathfrak{q}}(H) = 0$ and, by arbitrariness of ϵ, we can conclude. □

Theorem 4.38. *There exists a unique point $o \in X$, such that, up to a negligible set, $E = B_\rho(o)$, where $\rho = (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$. Moreover, it holds that*

$$\varphi_\infty(o) = \operatorname*{ess\,sup}_E \varphi_\infty = \max_{B_\rho(o)} \varphi_\infty. \tag{4.66}$$

Proof. Define $\tilde{E} := \operatorname{supp} \mathbf{1}_E$. Recall that by definition of support, $\tilde{E} = \bigcup_C C$, where the intersection is taken among all closed sets C such that $\mathfrak{m}(E \backslash C) = 0$; and in particular $\mathfrak{m}(E \backslash \tilde{E}) = 0$. Let $o \in \arg\max_{\tilde{E}} \varphi_\infty$. The uniqueness will follow from Lemma 4.36.

First we prove the first equality of (4.66). Let $N := \{x \in E : \varphi_\infty(x) > \varphi_\infty(o) = \max_{\tilde{E}} \varphi_\infty\}$. By definition of maximum, $N \cap \tilde{E} = \emptyset$, so $N \subset E \backslash \tilde{E}$, hence $\mathfrak{m}(N) = 0$, thus $M = \operatorname{ess\,sup}_E \varphi_\infty \leq \varphi_\infty(o)$. On the other side, consider the open set $P := \{x : \varphi(x) > M\}$. By definition of essential supremum, we have that $\mathfrak{m}(E \cap P) = 0$, hence $\tilde{E} \subset X \backslash P$, thus $\varphi_\infty(o) \leq M$. the other eqiality in (4.66) will follow from the fact $E = B_\rho(o)$ (up to a negligible set).

It is sufficient to prove only that $B_\rho(o) \subset E$, for the other inclusion is a consequence. Indeed, the Bishop–Gromov inequality, together with the definition of a.v.r. yields

$$\mathfrak{m}(E) \geq \mathfrak{m}(B_\rho(o)) \geq \omega_N \mathsf{AVR}_X \rho^N = \mathfrak{m}(E),$$

and the equality of measures improves to an equality of sets.

Fix now $\epsilon > 0$ and define $A = B_{\rho-\epsilon}(o)$. If $\mathfrak{m}(E\backslash A) = 0$, then we deduce that $B_{\rho-\epsilon}(o) \subset E$ and, by arbitrariness of ϵ, we can conclude.

Suppose the contrary, i.e., that $\mathfrak{m}(E\backslash A) > 0$. Clearly A is connected and $\mathfrak{m}(A \cap E) > 0$ (otherwise $o \notin \tilde{E}$), so we exploit Lemma 4.35 obtaining $\mathsf{P}(E; A) > 0$. Define $H = \{\alpha \in Q_\infty : g_\infty(\alpha, \rho) \in A\}$. A simple computation shows that the set H is non-negligible (recall (4.65))

$$0 < \frac{\mathsf{P}(E; A)}{\mathsf{P}(E)} = \int_{Q_\infty} \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \int_H \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \hat{\mathfrak{q}}(H).$$

The lipschitz-continuity of φ_∞ yields

$$\varphi_\infty(x) \geq \varphi_\infty(o) - \rho + \epsilon \geq M - \rho + \epsilon, \quad \forall x \in A = B_{\rho-\epsilon}(o)$$

hence

$$\varphi_\infty(g_\infty(\alpha, \rho)) \geq M - \rho + \epsilon, \quad \forall \alpha \in H.$$

We continue the chain of inequalities, obtaining

$$\varphi_\infty(g_\infty(\alpha, 0)) \geq \varphi_\infty(g_\infty(\alpha, \rho)) + \rho \geq M + \epsilon, \quad \forall \alpha \in H.$$

The line above, together with the fact that $\hat{\mathfrak{q}}(H) > 0$, contradicts Proposition 4.37. □

4.5.1 $\varphi_\infty(x)$ coincides with $-\mathsf{d}(x, o)$

The present section is devoted in proving that, $\varphi_\infty(x) = -\mathsf{d}(x, o) + \varphi_\infty(o)$.

Proposition 4.39. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\mathsf{d}(o, g(\alpha, t)) = t, \quad \forall t \in [0, \rho]. \tag{4.67}$$

Proof. The 1-lipschitzianity of φ_∞, and the fact that $E = B_\rho(o)$ (up to a negligible set) implies that, $\varphi_\infty(x) \geq \varphi_\infty(o) - \rho$, for $\mathfrak{m}$-a.e. $x \in E$. Thus we deduce, using Proposition 4.37 and Equation (4.66), that

$$\varphi_\infty(g_\infty(\alpha, 0)) \leq \varphi_\infty(o), \quad \text{and} \quad \varphi_\infty(g_\infty(\alpha, \rho)) \geq \varphi_\infty(o) - \rho.$$

Since $\frac{d}{dt}\varphi_\infty(g_\infty(\alpha, t)) = -1$, $t \in (o, \rho)$, the inequalities above are saturated and

$$\varphi_\infty(g_\infty(\alpha, t)) = \varphi_\infty(o) - t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty.$$

The 1-lipschitzianity of φ_∞, together with the Equation above, yields

$$\mathsf{d}(o, g_\infty(\alpha, t)) \geq \varphi_\infty(o) - \varphi_\infty(g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty. \tag{4.68}$$

Fix $\epsilon > 0$ and let $C = \{\alpha \in Q_\infty : \mathsf{d}(o, g_\infty(\alpha, 0)) > 2\epsilon\}$. The function $f(t) := \inf\{\mathsf{d}(o, g_\infty(\alpha, t)) : \alpha \in C\}$ is 1-Lipschitz and satisfies $f(0) \geq 2\epsilon$, hence $f(t) \geq 2\epsilon - t$, yielding (cfr. (4.68))

$$f(t) \geq \max\{(2\epsilon - t), t\} \geq \epsilon.$$

The inequality above implies that $g_\infty(\alpha, t) \notin B_\epsilon(o)$ for all $t \in [0, 1]$, for all $\alpha \in C$. We compute the measure of $B_\epsilon(o)$ using the disintegration formula (4.64)

$$\begin{aligned}\frac{\mathfrak{m}(B_\epsilon(o))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty} \int_0^\rho \mathbf{1}_{B_\epsilon(o)}(g_\infty(\alpha, t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= \int_{Q_\infty \setminus C} \int_0^\rho \mathbf{1}_{B_\epsilon(o)}(g_\infty(\alpha, t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha).\end{aligned}$$

If $\mathbf{1}_{B_\epsilon(o)}(g_\infty(\alpha, t)) = 1$, then inequality (4.68) yields $t \leq \epsilon$, so we continue the computation

$$\begin{aligned}\frac{\mathfrak{m}(B_\epsilon(o))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty \setminus C} \int_0^\rho \mathbf{1}_{B_\epsilon(o)}(g_\infty(\alpha, t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= \int_{Q_\infty \setminus C} \int_0^\epsilon \mathbf{1}_{B_\epsilon(o)}(g_\infty(\alpha, t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &\leq \int_{Q_\infty \setminus C} \int_0^\epsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) = (\hat{\mathfrak{q}}(Q_\infty) - \hat{\mathfrak{q}}(C))\frac{\epsilon^N}{N}.\end{aligned}$$

On the other hand, the Bishop–Gromov inequality states that

$$\mathfrak{m}(B_\epsilon(o)) \geq \frac{\epsilon^N}{\rho^N}\mathfrak{m}(B_\rho(o)) = \frac{\epsilon^N}{\rho^N}\mathfrak{m}(E) = \epsilon^N \omega_N \mathsf{AVR}_X,$$

thus, comparing with the previous inequality, we obtain $\hat{\mathfrak{q}}(C) = 0$. By arbitrariness of ϵ, we deduce that $g_\infty(\alpha, 0) = o$ for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$.

Finally, using again (4.68), we can conclude

$$t \leq \mathsf{d}(o, g_\infty(\alpha, t)) \leq \mathsf{d}(o, g_\infty(\alpha, 0)) + \mathsf{d}(g_\infty(\alpha, 0), g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e } \alpha \in Q_\infty.$$

□

Corollary 4.40. *It holds that for all $x \in B_\rho(o)$, $\varphi_\infty(x) = \varphi_\infty(o) = -\mathsf{d}(x, o)$.*

Proof. If $x \in E \cap \mathcal{T}_\infty$, then $x = g(\alpha, t)$ for some t, with $\alpha = \mathfrak{Q}_\infty(x)$. By the previous proposition we may assume that $g_\infty(\alpha,0)=o$, hence we have that

$$\varphi_\infty(x)-\varphi_\infty(o)=\varphi_\infty(g_\infty(\alpha,t))-\varphi_\infty(g_\infty(\alpha,0))=-\mathsf{d}(g_\infty(\alpha,t),g_\infty(\alpha,0))=-\mathsf{d}(x,o).$$

Since $\mathcal{T}_\infty\cap E$ has full measure in $B_\rho(o)$ and $\operatorname{supp}\mathfrak{m}=X$, we conclude. □

4.5.2 Localization of the whole space

At this point, we are in position to extend the localization given in Section 4.4.4 to the whole space X. Since we do not know the behaviour of φ_∞ outside $B_\rho(o)$, we take as reference function $-\mathsf{d}(o,\cdot)$, which coincides with φ_∞ on $B_\rho(o)$.

In this section we will use some of the concept introduced in Subsection 2.4.1. In particular we will refer to transport relation $\mathcal{R}^e$; the transport set $\mathcal{T}$ turns out to have full $\mathfrak{m}$-measure. We will denote by Q the quotient set and $\mathfrak{Q}:\mathcal{T}\to Q$ be the quotient map; let $X_\alpha:=\mathfrak{Q}^{-1}(\alpha)$ be the disintegration rays and let $g:\mathrm{Dom}(g)\subset\mathbb{R}\times Q\to X$ be the standard parametrization. Define $\mathfrak{q}:=\frac{1}{\mathfrak{m}(E)}\mathfrak{Q}_\#(\mathfrak{m}\llcorner_E)$ (note that for the moment we still do not know if $\mathfrak{Q}_\#(\mathfrak{m}\llcorner_E)\ll\mathfrak{q}$).

Proposition 4.41. *For $\mathfrak{q}$-a.e. $\alpha\in Q$, it holds that $\mathsf{d}(o,g(\alpha,t))=t$, for all $t\in[0,|X_\alpha|]$.*

Proof. Let $\tilde{\mathfrak{q}}\in\mathcal{P}(Q)$ be a measure such that $\tilde{\mathfrak{q}}\ll\mathfrak{Q}_\#(\mathfrak{m})\ll\tilde{\mathfrak{q}}$. The maximality of the rays (see [27, Theorem 7.10]) guarantees that $\mathring{\mathcal{R}}^e(\alpha)\subset X_\alpha$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha\in Q$, where $\mathring{\mathcal{R}}^e(\alpha)$ denotes the relative interior of $\mathcal{R}^e(\alpha)$. By definition of distance $o\in\mathcal{R}^b(\alpha)$, for all $\alpha\in Q$, thus $g(\alpha,0)=o$ for $\tilde{\mathfrak{q}}$-a.e. $\alpha\in Q$. Since $\mathfrak{q}\ll\mathfrak{Q}_\#\mathfrak{m}\ll\tilde{\mathfrak{q}}$, the thesis follows. □

Proposition 4.42. *It holds true that $\mathfrak{Q}_\#\mathfrak{m}\ll\mathfrak{q}$.*

Proof. Let $\tilde{\mathfrak{q}}\in\mathcal{P}(Q)$ be a measure such that $\mathfrak{Q}_\#(\mathfrak{m})\ll\tilde{\mathfrak{q}}$. Using the Localization Theorem, we get that $\mathfrak{m}=\int_Q\tilde{\mathfrak{m}}_\alpha\,\tilde{\mathfrak{q}}(d\alpha)$, where the measures $\tilde{\mathfrak{m}}_\alpha$ are supported on X_α and satisfy the $\mathsf{CD}(0,N)$ condition. Let $A\subset Q$ be a set such that $\mathfrak{q}(A)=0$, that is $0=\mathfrak{m}(B_\rho(0)\cap\mathfrak{Q}^{-1}(A))=\int_A\mathfrak{m}(B_\rho(o))\,\tilde{\mathfrak{q}}(d\alpha)$, thus $\tilde{\mathfrak{m}}_\alpha(B_\rho)=0$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha\in A$ Since $\mathsf{d}(o,g(\alpha,t))=t$, for $\tilde{\mathfrak{q}}$-a.e. $\alpha\in A$ (compare with the previous proof), the $\mathsf{CD}(0,N)$ condition applied to every $\tilde{\mathfrak{m}}_\alpha$ yields $\tilde{\mathfrak{m}}_\alpha=0$ for $\tilde{\mathfrak{q}}$-a.e. $\alpha\in Q$. It follows that $\mathfrak{m}(\mathfrak{Q}^{-1}(A))=0$. □

The previous proposition allows us to use the Theorem 2.7, hence there exists a unique disintegration for the measure $\mathfrak{m}$

$$\mathfrak{m}=\int_Q\mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha),\tag{4.69}$$

such that: 1) the measures $\mathfrak{m}_\alpha$ are supported on X_α; 2) the space $(X_\alpha,\mathsf{d},\mathfrak{m}_\alpha)$ satisfy the $\mathsf{CD}(0,N)$ condition. We denote by $h_\alpha:(0,|X_\alpha|)\to\mathbb{R}$ the density function such that $\mathfrak{m}_\alpha=(g(\alpha,\cdot))_\#(h_\alpha\mathcal{L}\llcorner_{(0,|X_\alpha|)})$.

The next two propositions bound together the localization obtained in section 4.4.4 (in particular Corollary (4.33)) with the localization using $-\mathsf{d}(o,\cdot)$ as 1-Lipschitz reference function.

Proposition 4.43. *There exists a unique measurable map $L : \mathrm{Dom}(L) \subset Q_\infty \to Q$ such that the domain of L has full $\hat{\mathfrak{q}}$ in Q_∞ and it holds*

$$L(\mathfrak{Q}_\infty(x)) = \mathfrak{Q}(x), \quad \forall x \in B_\rho(o) \cap \mathcal{T}_\infty \cap \mathcal{T}, \qquad \text{and} \qquad \mathfrak{q} = L_\# \hat{\mathfrak{q}}.$$

Proof. Since $\varphi_\infty = \varphi_\infty(o) - \mathsf{d}(o,\cdot)$ on $B_\rho(o)$, the partitions $(X_{\alpha,\infty})_{\alpha \in Q_\infty}$ and $(X_\alpha)_{\alpha\in Q}$ agree on the set $B_\rho(o) \cap \mathcal{T}_\infty \cap \mathcal{T}$, that is, given $x, y \in B_\rho(o) \cap \mathcal{T}_\infty \cap \mathcal{T}$, we have that $(x, y) \in \mathcal{R}_\infty$ if and only if $(x, y) \in \mathcal{R}$. Consider the set

$$H := \{(x, \alpha, \beta) \in (B_\rho(o) \cap \mathcal{T}_\infty \cap \mathcal{T}) \times Q_\infty \times Q : \mathfrak{Q}_\infty(x) = \alpha \text{ and } \mathfrak{Q}(x) = \beta\},$$

and let $G := \pi_{Q_\infty \times Q}(H)$ be the projection of H on the second and third variable. For what we have said G is the graph of a map $L : \mathrm{Dom}(L) \subset Q_\infty \to Q$. The other properties easily follow. □

Proposition 4.44. *For q-a.e. $\alpha \in Q$, it holds that $|X_\alpha| \geq \rho$ and*

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1}, \quad \forall r \in [0, \rho].$$

Proof. Comparing Equation (4.67) with Proposition 4.41 we deduce that for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that

$$g_\infty(\alpha, t) = g_\infty(L(\alpha), t), \quad \forall t \in (0, \min\{\rho, |X_\alpha|\}).$$

Comparing the disintegration formulas (4.59) and (4.69), we deduce

$$\mathfrak{m}\llcorner_E = \int_Q \hat{\mathfrak{m}}_{\alpha,\infty}\, \hat{\mathfrak{q}}(d\alpha) = \int_Q \mathfrak{m}_\alpha \llcorner_E\, \mathfrak{q}(d\alpha) = \int_{Q_\infty} (\mathfrak{m}_{L(\alpha)})\llcorner_E\, \hat{\mathfrak{q}}(d\alpha),$$

hence $\hat{\mathfrak{m}}_{\alpha,\infty} = (\mathfrak{m}_{L(\alpha)})\llcorner_E$, thus, recalling (4.64), we deduce that

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1}, \quad \forall r \in (0, \min\{\rho, |X_\alpha|\}).$$

The fact that $|X_\alpha| \geq \rho$ follows from the expression above. □

Theorem 4.45. *For $\mathfrak{q}$-a.e. $\alpha \in Q$, it holds that $|X_\alpha| = \infty$ and*

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1}, \quad \forall r > 0.$$

Proof. Fix $\epsilon > 0$ and let

$$C := \left\{ \alpha \in Q : \lim_{R\to\infty} \int_0^R h_\alpha / R^N < \omega_N \mathsf{AVR}_X (1-\epsilon) \right\},$$

with the convention that the limit above is 0 if $|X_\alpha| < \infty$ (notice that the limit always exists and it is not larger than $\omega_N \mathsf{AVR}_X$ by the Bishop–Gromov inequality applied to each density h_α). We compute the a.v.r. using the disintegration

$$\begin{aligned}
\mathsf{AVR}_X \omega_N &= \lim_{R\to\infty} \frac{\mathfrak{m}(B_R)}{R^N} = \lim_{R\to\infty} \int_Q \int_0^R \frac{h_\alpha(t)}{R^N}\, dt\, \mathfrak{q}(d\alpha) = \int_Q \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\, dt\, \mathfrak{q}(d\alpha) \\
&= \int_C \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\, dt\, \mathfrak{q}(d\alpha) + \int_{Q\setminus C} \lim_{R\to\infty} \int_0^R \frac{h_\alpha(t)}{R^N}\, dt\, \mathfrak{q}(d\alpha) \\
&\le \int_C \omega_N \mathsf{AVR}_X (1-\epsilon)\, \mathfrak{q}(d\alpha) + \int_{Q\setminus C} \omega_N \mathsf{AVR}_X\, \mathfrak{q}(d\alpha) = \omega_N \mathsf{AVR}_X (1 - \epsilon \mathfrak{q}(C)),
\end{aligned}$$

thus $\mathfrak{q}(C) = 0$. By arbitrariness of ϵ we deduce that $\lim_{R\to\infty} \int_0^R h_\alpha / R^N = \omega_N \mathsf{AVR}_X$, hence $h_\alpha(t) = N\omega_N \mathsf{AVR}_X t^{N-1}$, for $\mathfrak{q}$-a.e. $\alpha \in \tilde{Q}$. □

The proof of Theorem 1.5 is therefore concluded.

Chapter 5

Isoperimetric inequality in irreversible Finsler manifolds

This chapter contains the proofs of the isoperimetric inequality for Finsler manifolds and its rigidity. Section 5.1 contains a few facts on Finsler geometry. Section 5.2 is devoted to the proof of Theorem 1.7, whereas the remaining sections contain the proof of Theorem 1.8

5.1 Finsler geometry

We quickly recall the basic notions regarding Finsler manifolds. The reader should refer to the monograph [74] for more details. We adopt the convention that a manifold may have a boundary, unless otherwise stated. We require the boundary to be Lipschitz.

Definition 5.1. Let X be a connected n-dimensional manifold. We say that a function $F : TX \to [0, \infty)$ is a Finsler structure on X if

1. (Regularity) F is C^∞ on $TX\backslash 0$, where 0 denotes the null section;
2. (Positive 1-homogeneity) For all $c > 0$, $v \in TX$, it holds that $F(cv) = cF(v)$;
3. (Strong convexity) On each tangent space T_xX, the function F^2 is strictly convex.

The reader should notice that in general $F(v) \neq F(-v)$. This feature, known as irreversibility, is what precludes us from applying the theory of m.m.s.'s. We define the reversibility constant of a Finsler structure as

$$\Lambda_{X,F} := \sup_{v \in TX : v \neq 0} \frac{F(v)}{F(-v)} \in [1, \infty],$$

or, in other words, $\Lambda_{X,F} \in [1,\infty]$ is the least constant such that $F(v) \leq \Lambda_{X,F} F(-v)$, for all $v \in TX$. Later we will restrict ourself to the family of Finsler structures with finite reversibility. If no confusion arises, we shall write $\Lambda_F = \Lambda_{X,F}$. If X is compact, then $\Lambda_{X,F} < \infty$.

We define the speed of a C^1 curve η as $F(\dot{\eta})$. The notion of speed naturally induces a length functional

$$\mathrm{Length}(\eta) := \int_0^1 F(\dot{\eta})\, dt,$$

and thus we have a natural notion of distance between two points given by

$$\mathsf{d}_{X,F}(x,y) := \inf_{\eta} \{\mathrm{Length}(\eta) : \eta_0 = x, \text{ and } \eta_1 = y\}.$$

Whenever no confusion arises, we shall write $\mathsf{d} = \mathsf{d}_{X,F}$. The distance d satisfies the usual properties of a distance, with the exception of the symmetry

$$\mathsf{d}(x,y) \leq \mathsf{d}(x,z) + \mathsf{d}(z,y), \quad \forall x,y,z \in M, \qquad \text{and} \qquad \mathsf{d}(x,y) = 0 \quad \Leftrightarrow \quad x = y.$$

Remark 5.2. We reassure the reader on the fact that the lack of symmetry of the distance does not harm most of the classical theory of metric spaces. Indeed, one can build g_1 and g_2, two Riemannian metric on TX, such that

$$\sqrt{g_1(v,v)} \leq F(v) \leq \sqrt{g_2(v,v)}, \qquad \forall v \in TX.$$

Such metrics can be built on local charts and then glued together using a partition of the unity. Furthermore, such metrics can be chosen so that $g_2 \leq f g_1$, for some continuous function $f : X \to [1,\infty)$.

Using these metrics one can reobtain many classical results for free. In particular, we will make use of the Ascoli–Arzelà theorem, the fact that locally Lipschitz functions (as will be later introduced) are differentiable almost everywhere, and that locally Lipschitz functions with compact support are globally Lipschitz.

We define the forward and backward balls, respectively, as

$$B^+(x,r) := \{y \in M : \mathsf{d}(x,y) < r\}, \qquad B^-(x,r) := \{y \in M : \mathsf{d}(y,x) < r\}.$$

If $A \subset M$, we define its (forward) ϵ-enlargement as $B^+(A,\epsilon) := \bigcup_{x \in A} B^+(x,\epsilon)$. The topology induced by forward and backward balls coincides an it is indeed the topology of the manifold; therefore the distance function is continuous in the product topology. We say that a set A is forward (resp. backward) bounded, if for some (hence any) $x_0 \in X$, there exists $r > 0$ such that $A \subset B^+(x_0,r)$ (resp. $A \subset B^-(x_0,r)$). We say that a set is bounded if it both backward and forward bounded. We denote by $\operatorname{diam} A := \sup_{x,y\in A} \mathsf{d}(x,y)$ the diameter of a set; a set has finite diameter if and only if it is bounded.

Later we will impose some compactness on the manifold we deal with. Namely, we will require that closed, forward bounded sets are compact, or, equivalently that closed forward balls are compact. This condition implies that given two points there exists a geodesics (as defined in the next paragraph) joining these two points.

A curve $\gamma : [0,l] \to M$ is called geodesics if it minimizes the length and its speed is constant. We point out that, if $\gamma : [0,l] \to M$ is a geodesics, in general the reverse curve $t \mapsto \gamma_{l-t}$ may fail to be a geodesics due to the possible irreversibility of the manifold. We will denote by $\mathrm{Geo}(X)$ the set of geodesics with domain the interval $[0,1]$. Like in the reversible setting, if $\gamma \in \mathrm{Geo}(X)$ is a geodesics, it holds that

$$\mathsf{d}(\gamma_t,\gamma_s) = (s-t)\,\mathsf{d}(\gamma_0,\gamma_1), \qquad \forall 0 \le t \le s \le 1;$$

in this case the condition $t \le s$ cannot be lifted. In the case when the boundary is not empty, we will require some convexity, namely, that for all points $x, y \in X\backslash\partial X$, and for all geodesics γ connecting x to y, we have that γ does not touch the boundary.

Given a submanifold $Y \subset X$, we can identify the tangent bundle TY as a subset of TX via the standard immersion. With this notation, we can restrict the Finsler structure F to TY; clearly, $F|_{TY}$ satisfies Definition 5.1. Regarding the reversibility constant and the distance, one immediately sees that

$$\Lambda_{Y,F} \le \Lambda_{X,F}, \qquad \text{and} \qquad \mathsf{d}_{X,F}(x,y) \le \mathsf{d}_{Y,F}(x,y), \quad \forall x,y \in Y.$$

We define the dual function $F^* : T^*X \to [0,\infty)$ as

$$F^*(\omega) := \sup\{\omega(v) : v \in T_xX, \text{ and } F(v) \le 1\}, \qquad \text{if } \omega \in T_x^*M.$$

Notice that, while we have that $\omega(v) \le F^*(\omega)F(v)$, the "reverse" inequality may not hold: $\omega(v) \not\ge -F^*(\omega)F(v)$. We define the Legendre transform $\mathcal{L} : T_x^*X \to T_xX$ as $\mathcal{L}(\omega) = v$, where $v \in T_xM$ is the unique vector such that $F(v) = F^*(\omega)$ and $\omega(v) = F(v)^2$ (the uniqueness follows from the fact that F^2 is smooth and strictly convex). Given a differentiable function $f : M \to \mathbb{R}$, we define its gradient as $\nabla f(x) := \mathcal{L}(df(x))$. Please note that in general $\nabla(-f) \ne -\nabla f$.

We say that a function $f : X \to \mathbb{R}$ is L-Lipschitz (with $L \geq 0$), if [1]

$$-L\mathsf{d}(y,x) \leq f(x) - f(y) \leq L\mathsf{d}(x,y), \quad \forall x, y \in M. \tag{5.1}$$

We point out that the first inequality in (5.1) follows from the second by swapping x with y. The family of L-Lipschitz functions is stable by pointwise limits; the infimum or the supremum of L-Lipschitz functions is still L-Lipschitz. Moreover, by Ascoli–Arzelà theorem, the family of L-Lipschitz functions forms a compact set in the topology of local uniform convergence. If f is L-Lipschitz, then $-f$ is $(\Lambda_F L)$-Lipschitz. Two examples of 1-Lipschitz functions are given by $f(x) = -\mathsf{d}(o,x)$ and $g(x) = \mathsf{d}(x,o)$, for some fixed point o.

We define the (descending) slope of a locally Lipschitz function f as

$$|\partial f|(x) := \limsup_{y \to x} \frac{(f(x) - f(y))^+}{\mathsf{d}(x,y)}.$$

Obviously, if f is L-Lipschitz, then $|\partial f| \leq L$. If $Y \subset X$ is a submanifold, and $f : X \to \mathbb{R}$, then $|\partial_Y f| \leq |\partial_X f|$ in Y, where these two expressions have the meaning of the slope of f as seen as a function defined in Y and X, respectively. If f is differentiable at $x \in X$, the slope can be computed as $|\partial f|(x) = F^*(-df(x)) = F(\nabla(-f)(x))$, hence for locally Lipschitz functions $|\partial f| = F(\nabla(-f))$ almost everywhere.

Finally, we would like to endow a manifold with a measure. Differently from the Riemannian case, there is no canonical measure induced from the Finsler structure. On the other hand the theory of m.m.s.'s does not require any strong assumption on the reference measure and, a priori, this measure might have nothing to do with the Hausdorff measure. For this reason we will only require for the reference measure to have a smooth density when expressed in coordinates. We conclude this section with the definition of Finsler manifold.

Definition 5.3. A triple $(X, F, \mathfrak{m})$ is called Finsler manifold, provided that X is a connected differential manifold (possibly with boundary) F is a Finsler structure on X and $\mathfrak{m}$ is a positive smooth measure, i.e., given $x_1, \ldots, x_n$ local coordinates, we have that

$$\mathfrak{m} = f\, dx_1 \ldots dx_n, \qquad \text{with } f > 0 \text{ and } f \in C^\infty.$$

5.1.1 Perimeter

We summarize the topic of the perimeter in Finsler manifolds; the reader check Section 2.2 for more details.

[1]Please note that we have chosen a sign convention different from [72, 74] (a function f is L-lipschitz in our sense, if $-f$ is L-Lipschitz in the sense of the cited papers). However, this sign convention is consistent with the Kantorovich potential decreases along the transport rays.

The definition of perimeter is given analogously to how is given for m.m.s., this time using the descending slope. Given a Borel subset $E \subset X$ and Ω open, the perimeter of E relative to Ω is denoted by $\mathsf{P}(E;\Omega)$ and is defined as follows

$$\mathsf{P}(E;\Omega) := \inf \left\{ \liminf_{n\to\infty} \int_\Omega |\partial u_n| \, d\mathfrak{m}; u_n \in \mathrm{Lip}_{loc}(\Omega), u_n \to \mathbf{1}_E \text{ in } L^1_{loc}(\Omega) \right\}.$$

The perimeter for Finsler manifolds enjoys the properties reported in Section 2.2, such as the locality and the lower semicontinuity, but, due to the possible irreversibility of the Finsler structure, the complementation property does not hold in general.

Like for the metric-measure setting, if E is a set of finite perimeter, then the set function $A \to \mathsf{P}(E;A)$ is the restriction to open sets of a finite Borel measure $\mathsf{P}(E;\cdot)$ in X. This fact seems to be novel and it is proven in details in Appendix A.

Finally, given a subset $E \subset M$, we define its (forward) Minkowski content as

$$\mathfrak{m}^+(E) := \liminf_{\epsilon\to 0^+} \frac{\mathfrak{m}(B^+(E,\epsilon)) - \mathfrak{m}(E)}{\epsilon}.$$

It can be shown that the perimeter is the l.s.c. relaxation of the Minkowski content w.r.t. the L^1 distance of sets. The proof of this fact can be found in Appendix B.

5.1.2 Wasserstein distance and the Curvature-Dimension condition

In this section, first we shortly present the Wasserstein (irreversible) distance over Finsler manifolds, then the CD condition is presented. The ideas are more or less the same of Section 2.1 (which we refer for more details).

Let $(X, F, \mathfrak{m})$ be a Finsler manifold, such that all closed forward balls are compact. We say that a measure $\mu \in \mathcal{P}(X)$ has finite p-moment if

$$\int_X (\mathsf{d}(o,x) + \mathsf{d}(x,o))^p \, \mu(dx), \qquad \text{for some (hence any) } o \in X.$$

The set of measures having finite p-th moment is denoted by $\mathcal{P}_p(X)$. The p-Wasserstein (irreversible) distance $W_p(\mu_0,\mu_1)$, for $\mu_0,\mu_1 \in \mathcal{P}_p(X)$, is then defined as

$$W_p(\mu_0,\mu_1) := \left(\min_\pi \int_{X\times X} \mathsf{d}^p(x,y) \, \pi(dxdy) \right)^{\frac{1}{p}} < \infty.$$

A geodesic in the Wasserstein space $(\mathcal{P}_p(X), W_p)$ is a curve $\mu : [0,1] \to \mathcal{P}_p$ such that

$$W_p(\mu_t,\mu_s) = (s-t) W_p(\mu_0,\mu_1), \qquad \forall 0 \le t \le s \le 1.$$

It can be shown that if μ_0 and μ_1 are absolutely continuous, there exists a unique geodesics connecting μ_0 to μ_1.

The $\mathsf{CD}(K,N)$ for condition for Finsler manifolds has been introduced in [69] (see also the survey [70]). Here we report only the case $K=0$.

Definition 5.4 ($\mathsf{CD}(0,N)$ for Finsler manifolds)**.** Let $(X,F,\mathfrak{m})$ be a Finsler manifold, such that closed forward balls are compact and let $N\in[\dim X,\infty)$. We say that $(X,F,\mathfrak{m})$ satisfies the $\mathsf{CD}(0,N)$ condition if and only if the N'-Rényi entropy is convex along the geodesics of the Wasserstein space $\forall N'\geq N$, that is, for any couple of absolutely continuous curves $\mu_0,\mu_1\in\mathcal{P}_2(X)$, it holds that

$$S_{N'}(\mu_t|\mathfrak{m})\leq(1-t)S_{N'}(\mu_0|\mathfrak{m})+tS_{N'}(\mu_1|\mathfrak{m}),\qquad\forall N'\geq N,$$

where $(\mu_t)_{t\in[0,1]}$ is the unique geodesic connecting μ_0 to μ_1.

Similarly to the Riemannian case, a notion of weighted N-Ricci curvature, still denoted by Ric_N, has been introduced. Here we do not give the definition of Ric_N, for it is quite lenghty and useless for our purposes. Ohta [68] proved that a Finsler manifold without boundary satisfies the $\mathsf{CD}(0,N)$ condition if and only if $\mathrm{Ric}_N\geq 0$. The possible presence of the boundary in the manifolds does not harm the results of this thesis; indeed, we rely only on the curvature dimension condition given by Definition 5.4 and never on Ric_N.

Among many consequences of the $\mathsf{CD}(0,N)$ condition, two are of our interest. One is the Brunn–Minkowski inequality (see, e.g., [74, Theorem 18.8]). Given two measurable subsets A and B of a $\mathsf{CD}(0,N)$ Finsler manifold $(X,F,\mathfrak{m})$, we define

$$\begin{aligned}Z_t(A,B):=&\{\gamma_t:\gamma\text{ is a geodesics such that }\gamma_0\in A\text{ and }\gamma_1\in B\}\\=&\{z:\exists x\in A,y\in B:\mathsf{d}(x,z)=t\mathsf{d}(x,y)\text{ and }\mathsf{d}(z,y)=(1-t)\mathsf{d}(x,y)\}.\end{aligned}$$

With this notation, we have the Brunn–Minkowski inequality

$$\mathfrak{m}(Z_t(A,B))^{\frac{1}{N}}\geq(1-t)\mathfrak{m}(A)^{\frac{1}{N}}+t\mathfrak{m}(B)^{\frac{1}{N}},\qquad t\in[0,1].$$

The other property we are interested in is the Bishop–Gromov inequality that states

$$\frac{\mathfrak{m}(B^+(x,r))}{r^N}\geq\frac{\mathfrak{m}(B^+(x,R))}{R^N},\qquad\forall 0<r\leq R,$$

for any fixed point $x\in X$. This inequality guarantees that the definition of asymptotic volume ratio given below is well posed

$$\mathsf{AVR}_X := \lim_{R\to\infty} \frac{\mathfrak{m}(B^+(x,R))}{\omega_N R^N}.$$

5.1.3 Localization

This section presents the localization technique as developed by Ohta [72]. The localization paradigm for Finsler manifolds is quite similar with the localization for m.m.s.'s. In this section, we only recall the main points of the localization in Finsler manifolds, with more emphasis to the differences with the metric-measure setting.

In his work, Ohta considered manifolds without boundary. However, his proof also work for manifolds with boundary satisfying the following convexity assumption: for all points x, y in the interior and for all geodesics γ connecting x to y, it holds that γ does not touch the boundary.

Consider a $\mathsf{CD}(0,N)$ Finsler manifold $(X, F, \mathfrak{m})$ and a function $f \in L^1(\mathfrak{m})$ with finite first moment such that

$$\int_X f\, d\mathfrak{m} = 0.$$

The function f induces two absolutely continuous measures $\mu_0 = f^+\mathfrak{m}$ and $\mu_1 = f^-\mathfrak{m}$. Let ϕ be a 1-Lipschitz Kantorovich potential for μ_0 and μ_1. We can construct the set

$$\Gamma := \{(x,y) \in X \times X \colon \phi(x) - \phi(y) = \mathsf{d}(x,y)\},$$ [2]

inducing a partial order relation.

We now slightly depart from the approach of Section 2.4.1. The maximal chains of this order relation turns out to be the image of curves of maximal slope for ϕ with unitary speed. To be more precise, we say that a unitary speed geodesics $\gamma : \mathrm{Dom}(\gamma) \subset \mathbb{R} \to X$ is a non-degenerate transport curve, if its domain has at least two points, $\frac{d}{dt}\phi(\gamma(t)) = -1$, and γ cannot be extended to a larger domain.

Given a point $x \in X$, three possible cases are possible.

- There is no non-degenerate transport curve passing through x. We denote by $\mathcal{D}$ the set of such points. The set $\mathcal{D}$ is generally large.
- There is exactly 1 non-degenerate transport curve passing through x. Such points form the transport set that will be denoted by $\mathcal{T}$. A fundamental property of $\mathcal{T}$ is that the f is constantly 0 a.e. in $X \backslash \mathcal{T}$.

[2] Plese, notice that we use a different sign convention from [72, 74]. However, this convention is consistent with Section 2.4.1.

- There are 2 or more non-degenerate transport curves passing through x. Such points are called branching points and the set that they form will be denoted by $\mathcal{A}$. The set $\mathcal{A}$ turns out to be negligible.

We will also refer to the sets of forward (resp. backward) branching points $\mathcal{A}^+$ (resp. $\mathcal{A}^-$), defined in the same way as in (2.8).

On the transport set, we define the equivalence relation $\mathcal{R}$ as $\mathcal{R} = (\Gamma \cup \Gamma^{-1}) \cap (\mathcal{T} \times \mathcal{T})$, and the equivalence classes turns out to be precisely the images of the transport curves. Similarly to the metric-measure setting, one chooses $Q \subset \mathcal{T}$ a measurable section of the equivalence relation $\mathcal{R}$ and define the quotient map $\mathfrak{Q} : \mathcal{T} \to Q$. The transport rays are the equivalence classes and are denoted by $(X_\alpha)_{\alpha \in Q}$.

We still denote by $g : \mathrm{Dom}(g) \subset Q \times [0, +\infty) \to \mathcal{T}$, the measurable parametrization of the transport rays, such that $\inf \mathrm{Dom}(g(\alpha, \cdot)) = 0$. The map g enjoys all the properties described in Section 2.4.1. Define also $|X_\alpha| := \sup \mathrm{Dom}(g(\alpha, \cdot)) = 0$. We stress out that due to the possible irreversibility of the manifold, $|X_\alpha|$ is not the diameter of X_α, in general. Indeed, on one hand we have that $|X_\alpha| = \mathsf{d}(g(\alpha, 0), g(\alpha, |X_\alpha|))$; on the other, the irreversibility of d might cause the diameter be larger than $|X_\alpha|$.

The transport rays naturally come with the structure of one-dimensional oriented manifold, with the orientation given by $\partial_t g(\alpha, t)$, the velocity of the parametrization. We endow X_α with the Finsler structure given by the restriction of F to X_α; notice that $F(\partial_t g(\alpha, t)) = 1$. As we already pointed out, it holds that

$$\mathsf{d}_{X,F}(x, y) \leq \mathsf{d}_{X_\alpha,F}(x, y), \qquad \forall x, y \in X_\alpha;$$

if $(x, y) \in \Gamma$, the inequality above is saturated, hence

$$\mathsf{d}(g(\alpha, t), g(\alpha, s)) = s - t, \qquad \forall 0 \leq t \leq s \leq |X_\alpha|.$$

We are in position to apply the Disintegration Theorem to $(\mathcal{T}, \mathcal{B}(\mathcal{T}), \mathfrak{m}\llcorner_{\mathcal{T}})$, obtaining a disintegration

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha \, \mathfrak{q}(d\alpha),$$

for some suitable probability measure $\mathfrak{q}$ and for a family of measures $(\mathfrak{m}_\alpha)_{\alpha \in Q}$, each of them is supported on X_α. The transport ray X_α is endowed with the measure $\mathfrak{m}_\alpha$, making $(X_\alpha, F, \mathfrak{m}_\alpha)$ a one-dimensional oriented Finsler manifold.

Differently from the reversible case, it might happen that the transport rays fail to satisfy the $\mathsf{CD}(0, N)$ condition. However, a bound from below on the Ricci curvature can be given in a certain sense than now we specify. It can be proved that $\mathfrak{m}_\alpha = (g(\alpha, \cdot))_\#(h_\alpha \mathcal{L}^1_{(0,|X_\alpha|)})$, for a certain non-negative function h_α. The function h_α satisfies $(h_\alpha^{\frac{1}{N-1}})'' \le 0$ in the distributional sense, i.e., the function $h_\alpha^{\frac{1}{N-1}}$ is concave. Here we can recognize the $\mathsf{CD}(0,N)$ for weighted Riemannian manifolds, namely that the space $([0,|X_\alpha|], |\cdot|, h_\alpha \mathcal{L}^1_{[0,|X_\alpha|]})$ satisfies the $\mathsf{CD}(0,N)$ condition. This fact leads us to the following definition.

Definition 5.5. Let $(X, F, \mathfrak{m})$ be a Finsler manifold diffeomorphic to an interval, endowed with an orientation given by a vector field v, such that $F(v) = 1$. We say that $(X, F, \mathfrak{m})$ satisfies the oriented $\mathsf{CD}(0,N)$ condition ($N > 1$), if the following happens. There exists $g : \mathrm{Dom}(g) \subset \mathbb{R} \to X$ a parametrization of X such that $\partial_t g(t) = v(g(t))$ and $h : \mathrm{Dom}(g) \to [0,\infty)$, a function such that $g_\#(h\mathcal{L}^1) = \mathfrak{m}$ and $h^{\frac{1}{N-1}}$ is concave.

With this definition, clearly holds that the transport rays satisfy the oriented $\mathsf{CD}(0,N)$ condition. For the reader used with the notion of N-Ricci curvature, we point out that the oriented $\mathsf{CD}(0,N)$ condition is equivalent to the fact $\mathrm{Ric}_N(\partial_t g(\alpha,t)) \ge 0$.

Finally, we point out that, as a consequence of the properties of the optimal transport, we can localize the constraint $\int_X f\, d\mathfrak{m} = 0$, i.e. it holds that $\int_X f\, d\mathfrak{m}_\alpha = 0$, for $\mathfrak{q}$-a.e. $\alpha \in Q$.

We summarize this section in the following theorem.

Theorem 5.6. *Let $(X, F, \mathfrak{m})$ be a Finsler manifold satisfying $\mathsf{CD}(0,N)$, for some $N \in (1,\infty)$.*

Let $f \in L^1(\mathfrak{m})$ with $\int_X f\, \mathfrak{m} = 0$ and

$$\int_X (\mathsf{d}(o,x) + \mathsf{d}(x,o))|f(x)|\, \mathfrak{m}(dx) < \infty, \qquad \text{for some (hence any) } o \in X.$$

Then there exists a measurable subset $\mathcal{T} \subset X$ (named transport set), a family $\{(X_\alpha, F, \mathfrak{m}_\alpha)\}_{\alpha \in Q}$ of oriented one-dimensional submanifolds of X (named transport rays), and a measurable function $g : \mathrm{Dom}(g) \subset Q \times [0,\infty)$ such that the following happens.

The function f is zero $\mathfrak{m}$-a.e. in $X \backslash \mathcal{T}$ and $\mathfrak{m}\llcorner_{\mathcal{T}}$ can be disintegrated in the following way

$$\mathfrak{m}\llcorner_{\mathcal{T}} = \int_Q \mathfrak{m}_\alpha\, \mathfrak{q}(d\alpha).$$

Moreover, for $\mathfrak{q}$-a.e. $\alpha \in Q$, the transport ray $(X_\alpha, F, \mathfrak{m}_\alpha)$ is parametrized by the unitary speed geodesics $g(\alpha,\cdot)$, it satisfies the oriented $\mathsf{CD}(0,N)$ condition, and it holds that

$$\int f\, d\mathfrak{m}_\alpha = 0, \tag{5.2}$$

Furthermore, two distinct transport rays can only meet at their extremal points (having measure zero for $\mathfrak{m}_\alpha$).

5.2 Proof of Theorem 1.7

We devout this section in proving Theorem 1.7.

Proof of Theorem 1.7. We will first prove that

$$\mathfrak{m}^+(E) \geq N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}} \mathfrak{m}(E)^{1-\frac{1}{N}}, \qquad \forall E \subset X \text{ bounded.}$$

From the inequality above the thesis will immediately follow by Theorem B.5.

Fix $E \subset X$ bounded and $x_0 \in E$; set $d = \operatorname{diam} E$. Fix $R > 0$ so that $E \subset B^+(x_0, R)$. We claim that $Z_t(E, B^+(x_0, R)) \subset B^+(E, t(d+R))$. Indeed, let $z \in Z_t(E, B^+(x_0, R))$, hence there exist $x \in E$ and $y \in B^+(x_0, R)$ so that $\mathsf{d}(x, z) = t\mathsf{d}(x, y)$. By triangular inequality we deduce that

$$\mathsf{d}(x, z) = t\mathsf{d}(x, y) \leq t(\mathsf{d}(x, x_0) + \mathsf{d}(x_0, y)) \leq t(d + R),$$

thus $z \in B^+(E, t(d + R))$, proving the claim. We are in position to compute the Minkowski content, using the Brunn–Minkowski inequality

$$\begin{aligned}
\mathfrak{m}^+(E) &= \liminf_{\epsilon \to 0} \frac{\mathfrak{m}(B^+(E, \epsilon)) - \mathfrak{m}(E)}{\epsilon} = \liminf_{t \to 0} \frac{\mathfrak{m}(B^+(E, t(d + R))) - \mathfrak{m}(E)}{t(d + R)} \\
&\geq \liminf_{t \to 0} \frac{\mathfrak{m}(Z_t(E, B^+(x_0, R)) - \mathfrak{m}(E)}{t(d + R)} \\
&\geq \liminf_{t \to 0} \frac{((1 - t)\mathfrak{m}(E)^{\frac{1}{N}} + t\mathfrak{m}(B^+(x_0, R))^{\frac{1}{N}})^N - \mathfrak{m}(E)}{t(d + R)} \\
&\geq \liminf_{t \to 0} \frac{\mathfrak{m}(E) + N\mathfrak{m}(E)^{1-\frac{1}{N}} t(\mathfrak{m}(B^+(x_0, R))^{\frac{1}{N}} - \mathfrak{m}(E)^{\frac{1}{N}}) + o(t) - \mathfrak{m}(E)}{t(d + R)} \\
&= N\mathfrak{m}(E)^{1-\frac{1}{N}} \frac{\mathfrak{m}(B^+(x_0, R))^{\frac{1}{N}} - \mathfrak{m}(E)^{\frac{1}{N}}}{d + R}.
\end{aligned}$$

By taking the limit as $R \to \infty$, recalling the definition of AVR_X, we conclude. □

5.3 Localization of the measure and the perimeter

From now on we assume that every Finsler manifold has finite reversibility constant, that every closed forward ball is compact, and the following convexity hypothesis: for all x, y in the interior of a manifold and for all geodesics γ connecting them, we have that γ does not touch the boundary. To prove Theorem 1.8 we consider the isoperimetric problem inside a ball with larger and larger radius. In order to apply the needle decomposition given by the Localization Theorem 2.6, one also needs in principle the balls to be convex. As in general balls fail to be convex, we will overcome this issue in the following way.

Given a bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix a point $x_0 \in E$ and then consider $R > 0$ such that $E \subset B_R$ (hereinafter we will adopt the following notation $B_R := B^+(x_0, R)$). Consider then the following family of null-average functions:

$$f_R(x) = \chi_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \chi_{B_R}.$$

Clearly, f_R falls in the hypothesis of Theorem 2.6. Thus we obtain a measurable subset $\mathcal{T}_R \subset X$ (the transport set) and a family $\{(X_{\alpha,R}, F, \mathfrak{m}_{\alpha,R})\}_{\alpha \in Q_R}$ of transport rays, so that the measure $\mathfrak{m} \llcorner_{\mathcal{T}_R}$ can be disintegrated:

$$\mathfrak{m} \llcorner_{\mathcal{T}_R} = \int_{Q_R} \mathfrak{m}_{\alpha,R} \, \mathfrak{q}_R(d\alpha), \qquad \mathfrak{q}_R(Q_R) = \mathfrak{m}(\mathcal{T}_R), \tag{5.3}$$

where $\mathfrak{m}_{\alpha,R}$ are probability densities supported on $X_{\alpha,R}$. We denote by $g_R(\alpha, \cdot) : [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ the unit speed parametrisation of the transport ray $X_{\alpha,R}$, in the direction given by the natural orientation of the disintegration ray $X_{\alpha,R}$. With this notation, it holds

$$\mathfrak{m}_{\alpha,R} = (g_R(\alpha, \cdot))_\#(h_{\alpha,R} \mathcal{L}^1 \llcorner_{[0,|X_{\alpha,R}|]}),$$

for some $\mathsf{CD}(0, N)$ density $h_{\alpha,R}$. The localization of the zero mean implies that (see (5.2))

$$\mathfrak{m}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)} \mathfrak{m}_{\alpha,R}(B_R), \qquad \mathfrak{q}_R\text{-a.e. } \alpha \in Q_R.$$

Unfortunately, the presence of the factor $\mathfrak{m}_{\alpha,R}(B_R)$ in the r.h.s. of the equation does make the quantity $\mathfrak{m}_{\alpha,R}$ independent of α, harming the localization approach. To get rid of this factor we proceed as follows.

We define $T_{\alpha,R}$ to be the unique element of $[0, |X_{\alpha,R}|]$ such that

$$\mathfrak{m}_{\alpha,R}(g_R(\alpha, [0, T_{\alpha,R}])) = \int_0^{T_{\alpha,R}} h_{\alpha,R}(x) \, dx = \mathfrak{m}_{\alpha,R}(B_R)$$

The measurability in α of $\mathfrak{m}_{\alpha,R}$ implies the same measurability for $T_{\alpha,R}$.

Notice that $|X_{\alpha,R}| \leq R + \operatorname{diam}(E)$: since $g_R(\alpha, \cdot)$ is the unit speed parametrization of $X_{\alpha,R}$, then

$$\mathsf{d}(g_R(\alpha, 0), g_R(\alpha, |X_{\alpha,R}|)) \leq \mathsf{d}(g_R(\alpha, 0), x_0) + \mathsf{d}(x_0, g_R(\alpha, |X_{\alpha,R}|)) \leq \operatorname{diam}(E) + R,$$

and consequently we deduce $T_{\alpha,R} \leq R + \operatorname{diam}(E)$. We restrict $\mathfrak{m}_{\alpha,R}$ to $\widehat{X}_{\alpha,R} := g_R(\alpha, [0, T_{\alpha,R}])$, having the following disintegration formula:

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha), \quad \widehat{\mathfrak{m}}_{\alpha,R} := \frac{\mathfrak{m}_{\alpha,R}\llcorner_{\widehat{X}_{\alpha,R}}}{\mathfrak{m}_{\alpha,R}(B_R)} \in \mathcal{P}(X), \quad \widehat{\mathfrak{q}}_R = \mathfrak{m}_{\cdot,R}(B_R)\, \mathfrak{q}_R; \tag{5.4}$$

where $\widehat{\mathcal{T}}_R := \cup_{\alpha \in Q_R} \widehat{X}_{\alpha,R}$. Using (5.3) and the fact that $B_R \subset \mathcal{T}_R$, we get $\widehat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R)$,

The disintegration (5.4) will be a useful localisation only if $(E \cap X_{\alpha,R}) \subset \widehat{X}_{\alpha,R}$; in this case we have

$$\widehat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \qquad \widehat{\mathfrak{q}}_R\text{-a.e. } \alpha \in Q_R,$$

obtaining a localization constraint independent of α. To prove this inclusion we will impose that $E \subset B_{R/(4\Lambda_F)}$. Since $g_R(\alpha, \cdot) : [0, |X_{\alpha,R}|] \to X_{\alpha,R}$ has unitary speed, we notice that

$$\mathsf{d}(x_0, g_R(\alpha,t)) \le \mathsf{d}(x_0, g_R(\alpha,0)) + \mathsf{d}(g_R(\alpha,0), g_R(\alpha,t)) \le \frac{R}{4\Lambda_F} + t \le \frac{R}{2} + t,$$

where in the second inequality we have used that each starting point of the transport ray has to be inside $E \subset B_{R/(4\Lambda_F)}$, being precisely where $f_R > 0$. The inequality above yields $g_R(\alpha,t) \in B_R$ for all $t < R/2$, hence $((g_R(\alpha, \cdot))^{-1}(B_R) \supset [0, \min\{R/2, |X_{\alpha,R}|\}]$, thus there are no "holes" inside $(g_R(\alpha, \cdot))^{-1}(B_R)$ before $\min\{R/2, |X_{\alpha,R}|\}$, implying that $|\widehat{X}_{\alpha,R}| \ge \min\{R/2, |X_{\alpha,R}|\}$. Fix $x \in E \cap X_{\alpha,R}$ and let $t \in [0, |X_{\alpha,R}|]$ be such that $x = g_R(\alpha,t)$. It holds that

$$t = \mathsf{d}(g_R(\alpha,0), x) \le \mathsf{d}(g_R(\alpha,0), x_0) + \mathsf{d}(x_0, x) \le (\Lambda + 1)\frac{R}{4\Lambda} \le \frac{R}{2},$$

where in the second inequality we used that $g_R(\alpha,0), x \in E \subset B_{R/(4\Lambda)}$. The inequality immediately implies $(g_R(\alpha, \cdot))^{-1}(E) \subset [0, \min\{R/2, |X_{\alpha,R}|\}]$, hence $E \cap X_{R,\alpha} \subset \widehat{X}_{\alpha,R}$, as we desired.

We describe explicitly the measure $\widehat{\mathfrak{q}}_R$ in term of a push-forward via the quotient map $\mathfrak{Q}_R$ of the measure $\mathfrak{m}\llcorner_E$

$$\begin{aligned}\widehat{\mathfrak{q}}_R(A) &= \int_{Q_R} \mathbf{1}_A(\alpha) \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \widehat{\mathfrak{m}}_{\alpha,R}(E)\, \widehat{\mathfrak{q}}_R(d\alpha) \\ &= \int_{Q_R} \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \widehat{\mathfrak{m}}_{\alpha,R}(E \cap \mathfrak{Q}_R^{-1}(A))\, \widehat{\mathfrak{q}}_R(d\alpha) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \mathfrak{m}(E \cap \mathfrak{Q}_R^{-1}(A)),\end{aligned}$$

hence $\widehat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)$.

We study now the relation between the perimeter and the disintegration of the measure (5.4). Let $\Omega \subset X$ be an open set and consider the relative perimeter $\mathsf{P}(E;\Omega)$. Let $u_n \in \mathrm{Lip}_{loc}(\Omega)$ be a sequence such that $u_n \to \mathbf{1}_E$ in $L^1_{loc}(\Omega)$ and $\lim_{n\to\infty} \int_\Omega |\partial u_n|\, d\mathfrak{m} = \mathsf{P}(E;\Omega)$.

Using the Fatou Lemma, we can compute

$$\begin{aligned}
\mathsf{P}(E;\Omega) &= \lim_{n\to\infty}\int_\Omega |\partial u_n|\,d\mathfrak{m} \ge \liminf_{n\to\infty}\int_{\Omega\cap\widehat{\mathcal{T}}_R} |\partial u_n|\,d\mathfrak{m} \\
&= \liminf_{n\to\infty}\int_{Q_R}\int_\Omega |\partial u_n|\,\widehat{\mathfrak{m}}_{\alpha,R}(dx)\,\widehat{\mathfrak{q}}_R(d\alpha) \ge \int_{Q_R}\liminf_{n\to\infty}\int_\Omega |\partial u_n|\,\widehat{\mathfrak{m}}_{\alpha,R}(dx)\,\widehat{\mathfrak{q}}_R(d\alpha) \\
&\ge \int_{Q_R}\liminf_{n\to\infty}\int_{X_{\alpha,R}\cap\Omega} |\partial_{X_{R,\alpha}} u_n|\,\widehat{\mathfrak{m}}_{\alpha,R}(dx)\,\widehat{\mathfrak{q}}_R(d\alpha) \ge \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\Omega)\,\widehat{\mathfrak{q}}_R(d\alpha),
\end{aligned}$$

where $|\partial_{X_{\alpha,R}}u|$ denotes the slope of the restriction of u to the transport ray $\widehat{X}_{\alpha,R}$ and $\mathsf{P}_{\widehat{X}_{\alpha,R}}$ the perimeter in the submanifold $(\widehat{X}_{\alpha,R}, F, \widehat{\mathfrak{m}}_{\alpha,R})$.

By arbitrariness of Ω, we deduce the following disintegration inequality

$$\mathsf{P}(E;\,\cdot\,) \ge \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\,\cdot\,)\,\widehat{\mathfrak{q}}_R(d\alpha).$$

We summarise this construction in the following

Proposition 5.7. *Let $(X,F,\mathfrak{m})$ be a $\mathsf{CD}(0,N)$ Finsler manifold with finite reversibility $\Lambda_F < \infty$. Given any bounded set $E \subset X$ with $0 < \mathfrak{m}(E) < \infty$, fix any point $x_0 \in E$ and then fix $R > 0$ such that $E \subset B_{R/(4\Lambda_F)}(x_0)$.*

Then there exists a Borel set $\widehat{\mathcal{T}}_R \subset X$, with $E \subset \widehat{\mathcal{T}}_R$ and a disintegration formula

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R}\widehat{\mathfrak{m}}_{\alpha,R}\,\widehat{\mathfrak{q}}_R(d\alpha), \qquad \widehat{\mathfrak{m}}_{\alpha,R}(\widehat{X}_{\alpha,R}) = 1, \qquad \widehat{\mathfrak{q}}_R(Q_R) = \mathfrak{m}(B_R), \tag{5.5}$$

such that

$$\widehat{\mathfrak{m}}_{\alpha,R}(E) = \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, \quad \text{for } \widehat{\mathfrak{q}}_R\text{-a.e. } \alpha\in Q_R \quad \text{and} \quad \widehat{\mathfrak{q}}_R = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}(\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E), \tag{5.6}$$

Moreover, the transport ray $(\widehat{X}_{\alpha,R}, F, \widehat{\mathfrak{m}}_{\alpha,R})$ satisfies the oriented $\mathsf{CD}(0,N)$ condition and it holds that $|X_\alpha| \le R + \operatorname{diam}(E)$. Furthermore, the following formula holds true

$$\mathsf{P}(E;\,\cdot\,) \ge \int_{Q_R}\mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\,\cdot\,)\,\widehat{\mathfrak{q}}_R(d\alpha). \tag{5.7}$$

The rescaling introduced in Proposition 5.7 will be crucially used to obtain non-trivial limit estimates as $R \to \infty$.

5.4 One dimensional analysis

Proposition 5.7 is the first step to obtain from the optimality of a bounded set E an almost optimality of $E \cap \widehat{X}_{\alpha,R}$. We now have to analyse in details the behaviour of the perimeter in one-dimensional oriented Finsler manifolds.

We fix few notation and conventions. A one dimensional oriented Finsler manifold can be identified with the manifold $(I, F, \mathfrak{m})$, where $I \subset \mathbb{R}$ is an interval. Without loss of generality we assume that the orientation is given by the coordinated vector field ∂_t on I. Since we are studying manifolds arising from the localization, we shall consider only Finsler structures that satisfy $F(\partial_t) = 1$. Thus, it is clear that the Finsler structure is completely determined by $F(-\partial_t)$; for this reason, with a slight abuse of notation, we will denote by F, the real-valued function given by $F(-\partial_t)$. With this convention, the reversibility constant turns out to be

$$\Lambda_{I,F} = \sup_{x \in I} \left\{ \max \left\{ F(x), \frac{1}{F(x)} \right\} \right\}.$$

When the interval has finite diameter, we will always assume that $I = [0, D]$. Notice that D in general is not the diameter, for it may happen that $\mathsf{d}(D, 0) > \mathsf{d}(0, D) = D$; however, it holds that $\operatorname{diam}(I, F) \leq \Lambda_{I,F} D$.

If $(I, F, \mathfrak{m})$ satisfies the oriented $\mathsf{CD}(0, N)$ condition, then it happens that $\mathfrak{m}$ is absolutely continuous w.r.t. the Lebesgue measure $\mathcal{L}^1$ and

$$(h^{\frac{1}{N-1}})'' \leq 0, \qquad \text{in the sense of distributions, where } h = \frac{d\mathfrak{m}}{d\mathcal{L}^1}.$$

We stress out that if $(I, F, h\mathcal{L}^1 \llcorner_I)$ satisfies the oriented $\mathsf{CD}(0, N)$ condition, then the reversible manifold $(I, |\cdot|, h\mathcal{L}^1 \llcorner_I)$ satisfies the $\mathsf{CD}(0, N)$ condition.

We make use of the notation introduced in Section 4.2; in particular, we recall a few definitions:

$$\mathfrak{m}_h := h\mathcal{L}^1 \llcorner_I, \qquad v_h(r) := \int_0^r h(s)\, ds, \qquad r_h(v) := (v_h)^{-1}(v),$$

We will denote by $\mathsf{P}_{F,h}$ the perimeter in the Finsler manifold $(I, F, h\mathcal{L}^1 \llcorner_I)$. If the Finsler structure is the Euclidean one, in order to simplify the notation, we shall write $\mathsf{P}_h = \mathsf{P}_{|\cdot|,h}$; this notation is compatible with the notation we used in the Finsler setting. As we have already pointed out, if $E \subset [0, D]$ is a set of finite perimeter, then it can be decomposed as $E = \bigcup_i (a_i, b_i)$. In this case, we have that the perimeter is given by the formula

$$\mathsf{P}_{F,h}(E) = \sum_{i: a_i \neq 0} F(a_i) h(a_i) + \sum_{i: b_i \neq D} h(b_i),$$

which takes into account the Finsler structure F. It should be noticed, that the Finsler structure is seen only by the left-pointing part of the boundary. Therefore, if E is of the form $[0,b]$, then $\mathsf{P}_{F,h}(E) = \mathsf{P}_h(E)$. From the equation above, we immediately deduce a lower bound on the irreversible perimeter in terms of the reversible perimeter

$$\mathsf{P}_{F,h}(E) \geq \Lambda_{I,F}^{-1}\mathsf{P}_h(E).$$

We can continue this chain on inequalities with the isoperimetric inequality for the reversible setting (see Section 4.2.1), obtaining

$$\mathsf{P}_{F,h}(E) \geq \Lambda_{I,F}^{-1}\mathcal{I}_{N,D}(\mathfrak{m}_h(E)),$$

provided that h is defined on an interval $[0, D']$, with $D' \leq D$. Once again, we notice that D is not an upper bound on the diameter of the space, in general, due to the irreversibility. We can now obtain an expansion of the inequality above as a corollary of Lemma 4.2.

Corollary 5.8. *Fix $N > 1$. Then for all $D \geq D' > 0$ and for all one-dimensional oriented Finsler manifolds $([0,D'], F, h\mathcal{L}^1)$ satisfying the oriented $\mathsf{CD}(0,N)$ condition, it holds that*

$$\begin{aligned} \mathsf{P}_{F,h}(E) &\geq \frac{\mathcal{I}_{N,D}(\mathfrak{m}_h(E))}{\Lambda_F} \geq \frac{N}{\Lambda_F D'}\mathfrak{m}_h(E)^{1-\frac{1}{N}}(1 - O(\mathfrak{m}_h(E)^{\frac{1}{N}}) \\ &\geq \frac{N}{\Lambda_F D}\mathfrak{m}_h(E)^{1-\frac{1}{N}}(1 - O(\mathfrak{m}_h(E)^{\frac{1}{N}}), \end{aligned} \tag{5.8}$$

for any Borel set $E \subset [0, D']$.

Remark 5.9. The lower bound in (5.8) is very rough for our purposes. If one attempted to prove the isoperimetric inequality (1.6), by adapting the proof contained in Section 4.2.2, the inverse of the reversibility constant would appear in the lower bound.

The only reason why the factor Λ_F^{-1} appears in (5.8), is that the part of the boundary where the external normal vector "points to the left" might be non-empty. Indeed, if E is of the form $[0,b]$, then $\mathsf{P}_{F,h}(E) = \mathsf{P}_{|\cdot|,h}(E)$. We will see that the part of the boundary "pointing to the left" contributes little to the perimeter.

5.4.1 One dimensional reduction for the optimal region

We give the definition of the residual of a set in a irreversible setting.

Definition 5.10. Let $D \geq D' > 0$ and let $([0,D'], F, h\mathcal{L}^1)$ be a one-dimensional Finsler manifold satisfying the oriented $\mathsf{CD}(0,N)$ condition. If $E \subset [0,D']$ is Borel set, we define its D-residual as

$$\mathrm{Res}_{F,h}^D(E) := \frac{D\mathsf{P}_{F,h}(E)}{N(\mathfrak{m}_h(E))^{1-\frac{1}{N}}} - 1.$$

If $v \in (0,1/2)$, then we easily see that

$$\mathrm{Res}_h^D(v) = \mathrm{Res}_{F,h}^D([0, r_h(v)]) = \frac{Dh(r_h(v))}{Nv^{1-\frac{1}{N}}} - 1,$$

where $\mathrm{Res}_h^D(v)$ is the residual defined in (4.17).

Using the residual, Inequality 5.8 can be restated as

$$\mathrm{Res}_{F,h}^D(E) \geq \Lambda_F^{-1} - 1 - O(\mathfrak{m}_h(E)^{\frac{1}{N}}). \tag{5.9}$$

We are ready to apply the definition of residual to the disintegration rays. In order to ease the notation, we let $\mathsf{P}_{\alpha,R} = \mathsf{P}_{(\widehat{X}_{\alpha,R}, F, \widehat{\mathfrak{m}}_{\alpha,R})}$. The measure $\widehat{\mathfrak{m}}_{\alpha,R}$ will be identified with the ray map $g_R(\alpha,\cdot)$ to $h_{\alpha,R}\mathcal{L}^1$, thus we define

$$\begin{aligned}
&\mathrm{Res}_{\alpha,R} := \mathrm{Res}_{F,h_{\alpha,R}}^{R+\mathrm{diam}(E)}(g(\alpha,\cdot)^{-1}(E \cap \widehat{X}_{\alpha,R})), \quad \text{for } \alpha \in Q_R,\\
&\mathrm{Res}_{x,R} := \mathrm{Res}_{\mathfrak{Q}_R(x),R}, \quad \text{for } x \in E.
\end{aligned}$$

The good rays are those rays having small residual. We quantify their abundance in the following proposition, which is the irreversible analogous of Proposition 4.6.

Proposition 5.11. *Assume that* $(X, F, \mathfrak{m})$ *is a* $\mathsf{CD}(0,N)$ *Finsler manifold, such that* $\mathsf{AVR}_X > 0$*. If* $E \subset X$ *is a bounded set attaining the identity in the inequality* (1.6)*, then*

$$\limsup_{R\to\infty} \frac{1}{\mathfrak{m}(B_R)} \int_{Q_R} \mathrm{Res}_{\alpha,R}\, \mathfrak{q}_R(d\alpha) \leq 0. \tag{5.10}$$

Proof. In order to check that the function $\alpha \to \mathrm{Res}_{\alpha,R}$ is integrable, it is enough to check that $(\mathrm{Res}_{\alpha,R})^-$, is integrable. This last fact derives from the isoperimetric inequality $\mathrm{Res}_{\alpha,R} \geq \Lambda_F^{-1} - 1 - O((\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)})^{\frac{1}{N}})$, as stated in (5.9). We can now compute the integral in (5.10)

$$\begin{aligned}
\int_{Q_R} \mathrm{Res}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha) &= \int_{Q_R} \left(\frac{(R+\mathrm{diam}(E))\mathsf{P}_{\alpha,R}(E)}{N} \left(\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)}\right)^{1-\frac{1}{N}} - 1 \right) \widehat{\mathfrak{q}}_R(d\alpha)\\
&= \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1} N \mathfrak{m}(E)^{1-\frac{1}{N}}} \int_{Q_R} \mathsf{P}_{\alpha,R}(E)\, \widehat{\mathfrak{q}}_R(d\alpha) - \mathfrak{m}(B_R)\\
&\leq \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}-1} N \mathfrak{m}(E)^{1-\frac{1}{N}}} \mathsf{P}(E) - \mathfrak{m}(B_R)\\
&\leq \mathfrak{m}(B_R) \frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}} (\mathsf{AVR}_X \omega_N)^{\frac{1}{N}} - \mathfrak{m}(B_R),
\end{aligned}$$

yielding

$$\frac{1}{\mathfrak{m}(B_R)}\int_{Q_R}\mathrm{Res}_{\alpha,R}\,\mathfrak{q}_R(d\alpha)\le\frac{R+\mathrm{diam}(E)}{\mathfrak{m}(B_R)^{\frac{1}{N}}}(\mathsf{AVR}_X\omega_N)^{\frac{1}{N}}-1,$$

and the r.h.s. goes to 0, as $R\to\infty$. □

Corollary 5.12. *Let* $(X,F,\mathfrak{m})$ *be a* $\mathsf{CD}(0,N)$ *Finsler manifold, having* $\mathsf{AVR}_X>0$*. Let* $E\subset X$ *be a set saturating the isoperimetric inequality* (1.6)*, then it holds true that*

$$\limsup_{R\to\infty}\int_E\mathrm{Res}_{\mathfrak{Q}_R(x),R}\,\mathfrak{m}(dx)\le 0.$$

Proof. A direct computation gives

$$\begin{aligned}\int_E\mathrm{Res}_{\mathfrak{Q}_R(x),R}\,\mathfrak{m}(dx)&=\int_{Q_R}\int_E\mathrm{Res}_{\mathfrak{Q}_R(x),R}\,\widehat{\mathfrak{m}}_{\alpha,R}(dx)\,\widehat{\mathfrak{q}}_R(d\alpha)\\&=\int_{Q_R}\mathrm{Res}_{\alpha,R}\,\widehat{\mathfrak{m}}_{\alpha,R}(E)\,\widehat{\mathfrak{q}}_R(d\alpha)\\&=\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\int_{Q_R}\mathrm{Res}_{\alpha,R}\,\mathfrak{q}_R(d\alpha),\end{aligned}$$

and one concludes by taking the superior limit. □

Remark 5.13. At first sight, it looks like that Proposition 5.11 is much weaker than its reversible correspondent Proposition 4.6 (an similarly Corollary 5.12 seems weaker to Corollary 4.7. To be precise, it seems that an infinitesimal lower bound on the residual seems missing. Nonetheless, we will see that the missing lower bound will appear as a by-product of the almost rigidity of the rays.

5.5 Analysis along the good rays

Like in the reversible setting, we now deduce a few one-dimensional almost-rigidity estimates. Most of these estimates are carried out in Section 4.3

5.5.1 Almost rigidity of the set E and of the length of the ray

Like in the reversible setting, up to a negligible set, it holds that $E=\bigcup_{i\in\mathbb{N}}(a_i,b_i)$. The boundedness of the original set of our isoperimetric problem, implies that $E\subset[0,L]$, for some $L>0$. Define $b(E):=\operatorname{ess\,sup}E\le L$.

In the next lemma we prove that $b(E)$ is in the essential boundary of E; this lemma is the non-reversible equivalent of Lemma 4.12.

Lemma 5.14. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. Then there exists two constants $\bar{w} > 0$ and $\bar{\delta} > 0$ (depending only on N, L, and Λ) such that the following happens. For all $D \geq D' > 0$ with $D \geq 4L\Lambda$, for all $([0, D'], F, h\mathcal{L}^1)$ a one-dimensional Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \leq \Lambda$, and for all $E \subset [0, L]$, such that $\mathfrak{m}_h(E) \leq \bar{w}$ and $\mathrm{Res}^D_{F,h}(E) \leq \bar{\delta}$, there exists $a \in [0, b(E))$ and an at-most-countable family of intervals $((a_i, b_i))_i$ such that, up to a negligible set,*

$$E = \bigcup_i (a_i, b_i) \cup (a, b(E)), \tag{5.11}$$

with $a_i, b_i < a$, $\forall i$.

Moreover, h is strictly increasing on $[0, b(E)]$.

Proof. Taking into account the definition of residual and the isoperimetric inequality (5.9), choosing $\bar{\delta} \leq 1$, we can deduce that

$$\frac{D'}{D} \geq \frac{1 + \mathrm{Res}^{D'}_{F,h}(E)}{1 + \mathrm{Res}^{D}_{F,h}(E)} \geq \frac{1 + \Lambda_F^{-1} - 1 - O(w^{\frac{1}{N}})}{1 + \bar{\delta}} \geq \frac{\Lambda^{-1}}{2} - O(w^{\frac{1}{N}})).$$

If we choose $\bar{w}$ small enough, taking into account the hypothesis $D \geq 4L\Lambda$, we deduce $D' \geq 2L$. At this point one can follow the proof of Lemma 4.12 to conclude that E takes the form of equation (5.11).

Finally, we prove that h increases on $[0, b(E)]$. Let $b := b(E)$. Following the reasoning as in the proof of Lemma 4.12, one obtain an inequality similar to (4.25) (using the fact that $h(b) \leq \mathsf{P}_{F,h}(E)$)

$$1 \leq \frac{\mathsf{P}_{F,h}(E) D'}{N} \left(1 + (N-1)\frac{b}{D'} + o\left(\frac{b}{D'}\right) + N \right). \tag{5.12}$$

The first factor in the r.h.s. of the estimate above is controlled just using the definition of residual

$$\frac{\mathsf{P}_{F,h}(E) D'}{N} \leq \frac{\mathsf{P}_{F,h}(E) D}{N} = \mathfrak{m}_h(E)^{1-\frac{1}{N}}(1 + \mathrm{Res}^D_{F,h}(E)),$$

and, if $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}^D_{F,h}(E)$ is bounded, then the term above goes to 0. The second factor, is bounded (see the proof of Lemma 4.12). If we put together this last two estimates, we deduce that the r.h.s. of (5.12) is infinitesimal as $\mathfrak{m}_h(E) \to 0$ and $\mathrm{Res}^D_{F,h}(E) \to 0$, obtaining a contradiction. □

Like in the reversible case, we will denote by $a(E)$ the number a given by Proposition 5.14; Remark 4.13 still hold.

Remark 5.15. Since $r_h(\mathfrak{m}_h(E)) \leq b(E)$ and h is increasing, we have that $\mathrm{Res}^D_h(\mathfrak{m}_h(E)) \leq \mathrm{Res}^D_{F,h}(E)$. Therefore, the estimates contained in Proposition 4.11 still hold true for the Finsler setting.

Proposition 5.16. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$D' \geq D(1 - o(1)) \tag{5.13}$$
$$b(E) \leq Dw^{\frac{1}{N}} + Do(w^{\frac{1}{N}}) \tag{5.14}$$
$$b(E) \geq Dw^{\frac{1}{N}} - Do(w^{\frac{1}{N}}) \tag{5.15}$$
$$a(E) \leq Do(w^{\frac{1}{N}}), \tag{5.16}$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathcal{L}^1)$ is a one-dimensional Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof.

Part 1 Inequalities (5.13) **and** (5.15)**.**
Since h is decreasing on $[0, b(E)]$, we have that $r_h(w) \leq b(E)$, thus $h(r_h(w)) \leq h(b(E)) \leq \mathsf{P}_{F,h}(E)$, hence $\mathrm{Res}^D_h(w) \leq \mathrm{Res}^D_{F,h}(E)$. Estimate (5.13) follows from estimate (4.20), whereas estimate (5.15) is a consequence of (4.22).

Part 2 Inequality (5.16)**.**
First we prove that $a(E) < r_h(w)$, for w and δ small enough. Suppose on the contrary that $a(E) \geq r_h(w)$, implying that $h(a(E)) \geq h(r_h(w))$, hence $\mathsf{P}_{F,h}(E) \geq \Lambda^{-1}h(a(E)) + h(b(E)) \geq (1 + \Lambda^{-1})h(r_h(w))$. We deduce that (compare with (5.9))

$$\begin{aligned} -O(w^{\frac{1}{N}}) \leq \mathrm{Res}^D_h(w) &= \frac{Dh(r_h(w))}{Nw^{1-\frac{1}{N}}} - 1 \leq \frac{D\mathsf{P}_{F,h}(E)}{(1+\Lambda^{-1})Nw^{1-\frac{1}{N}}} - 1 \\ &= \frac{1}{1+\Lambda^{-1}}(\mathrm{Res}^D_{F,h}(E) - \Lambda^{-1}) \leq \frac{\delta - \Lambda^{-1}}{1+\Lambda^{-1}}. \end{aligned}$$

If we take the limit as $w \to 0$ and $\delta \to 0$ we obtain a contradiction.

Using the Bishop–Gromov inequality and the isoperimetric inequality (respectively), we get

$$\begin{aligned} h(a(E)) &\geq h(r_h(w)) \left(\frac{a(E)}{r_h(w)}\right)^{N-1} \\ h(b(E)) &\geq h(r_h(w)) \geq \frac{N}{D} w^{1-\frac{1}{N}}(1 - O(w^{\frac{1}{N}})). \end{aligned}$$

We put together the inequalities above obtaining

$$\begin{aligned}\frac{N}{D}w^{1-\frac{1}{N}}(1+\mathrm{Res}^D_{F,h}(E)) = \mathsf{P}_{F,h}(E) &\geq h(b(E)) + \Lambda^{-1}h(a(E)) \geq h(r_h(w)) + \Lambda^{-1}h(a(E)) \\ &\geq h(r_h(w))\left(1+\Lambda^{-1}\left(\frac{a(E)}{r_h(w)}\right)^{N-1}\right) \\ &\geq \frac{N}{D}w^{1-\frac{1}{N}}(1-O(w^{\frac{1}{N}}))\left(1+\Lambda^{-1}\left(\frac{a(E)}{r_h(w)}\right)^{N-1}\right),\end{aligned}$$

hence

$$\begin{aligned}a(E) &\leq r_h(w)\Lambda^{\frac{1}{N-1}}\left(\frac{1+\mathrm{Res}^D_{F,h}(E)}{1+O(w^{\frac{1}{N}})}-1\right)^{\frac{1}{N-1}} \\ &\leq r_h(w)\Lambda^{\frac{1}{N-1}}\left((1+\delta)(1-O(w^{\frac{1}{N}}))-1\right)^{\frac{1}{N-1}} \leq r_h(w)\,o(1) \\ &\leq Dw^{\frac{1}{N}}(1+o(1))o(1) = Do(w^{\frac{1}{N}}),\end{aligned}$$

where the estimate (4.21) was taken into account.

Part 3 Inequality (5.14)**.**
Following exactly the same steps of Part 3 of the proof of Proposition 4.14, we arrive at

$$b(E) - r_h(w) \leq a(E)\,\frac{h(a(E))}{h(r_h(w))} \leq a(E).$$

Combining the inequality above, the already-proven estimate (5.15), and the estimate (4.21), we reach the conclusion. □

5.5.2 Almost rigidity of the density h

In this section we prove that the density h converges to the model density Nx^{N-1}/D^N. The bound from below is easy and the proof is not different form the proof Proposition 4.15.

Proposition 5.17. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$h(x) \geq \frac{N}{D^N}x^{N-1}(1-o(1)), \qquad \textit{uniformly w.r.t. } x \in [0, b(E)], \tag{5.17}$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0,D'], F, h\mathcal{L}^1)$ is a one dimensional Finsler manifold satisfying the $\mathsf{CD}(0,N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0,L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof. Fix $x \in [0, b(E)]$. The Bishop–Gromov inequality yields

$$h(x) \geq h(b(E)) \frac{x^{N-1}}{b(E)^{N-1}} \geq h(r_h(w)) \frac{x^{N-1}}{b(E)^{N-1}}.$$

The first factor $h(r_h(w))$ is controlled using the isoperimetric inequality (compare with Corollary 4.3), whereas the term $b(E)$ is controlled using estimate (5.14). □

The following corollary gives a lower boundary for the residual, under the hypothesis that the (positive part of the) residual is bounded from above, improving inequality (5.9). This surprising self-improving estimate, states that if the residual is small, i.e., it is bounded by above by a positive small constant, than it cannot be too small, i.e., it is bounded by below by a small negative constant.

Corollary 5.18. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$\mathrm{Res}^D_{F,h}(E) \geq -o(1)$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathcal{L}^1)$ is a one-dimensional Finsler manifold satisfying the $\mathsf{CD}(0, N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof. By a direct computation, recalling estimates (5.17) and (5.15), we obtain

$$\mathrm{Res}^D_{F,h}(E) \geq \frac{Dh(b(E))}{Nw^{1-\frac{1}{N}}} - 1 \geq \frac{b(E)^{N-1}(1 - o(1))}{D^{N-1}w^{1-\frac{1}{N}}} - 1 \geq \frac{(w^{\frac{1}{N}}(1 - o(1)))^{N-1}}{w^{1-\frac{1}{N}}} - 1 \geq o(1). \quad \square$$

We now obtain an upper bound for h in the interval $[a(E), b(E)]$ going in the opposite direction of the Bishop–Gromov inequality.

Proposition 5.19. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. The following estimates hold for $w \to 0$ and $\delta \to 0$*

$$h(x) \leq h(b(E)) \left(\frac{x}{b(E)} + o(1)\right)^{N-1}, \quad \textit{uniformly w.r.t. } x \in [0, b(E)], \tag{5.18}$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathcal{L}^1)$ is a one-dimensional Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition, with $\Lambda_F \leq \Lambda$, and the set $E \subset [0, L]$ satisfies $\mathfrak{m}_h(E) = w$ and $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof. Fix $x \in [a(E), b(E)]$.

Case 1 $x \in [0, r_h(w)]$**.**
Since $\mathrm{Res}_h^D(w) \leq \mathrm{Res}_{F,h}^D(E)$, Estimate 4.31 yields

$$h(x) \leq h(r_h(w)) \left(\frac{x}{r_h(w)} + o(1) \right)^{N-1} \leq h(b(E)) \left(\frac{x}{r_h(w)} + o(1) \right)^{N-1}. \tag{5.19}$$

Estimates (5.14) and (4.22) yield

$$\frac{b(E)}{r_h(w)} \leq \frac{Dw^{\frac{1}{N}} + Do(w^{\frac{1}{N}})}{Dw^{\frac{1}{N}} - Do(w^{\frac{1}{N}})} = 1 + o(1). \tag{5.20}$$

Plugging the inequality above in (5.19) gives

$$h(x) \leq h(b(E)) \left(\frac{x}{b(E)}(1 + o(1)) + o(1) \right)^{N-1} \leq h(b(E)) \left(\frac{x}{b(E)} + o(1) \right)^{N-1},$$

which is precisely the thesis.

Case 2 $x \in [r_h(w), b(E)]$**.**
The Bishop–Gromov inequality yields

$$h(x) \leq h(r_h(w)) \left(\frac{x}{r_h(w)} \right)^{N-1} \leq h(b(E)) \left(\frac{x}{b(E)} + o(1) \right)^{N-1},$$

where in the last inequality we used (5.20). □

5.5.3 Rescaling the diameter and renormalizing the measure

In this last section, as we did in Section 4.3.5, we rescale the density h and the set E by a factor $\frac{1}{b(E)}$ and renormalize the measure by a factor $\mathfrak{m}_h(E)$.

Given a density $h : [0, D'] \to \mathbb{R}$ and $E \subset [0, L]$, we define

$$\nu_{h,E} = (S_{b(E)})_\# \left(\frac{\mathfrak{m}_h \llcorner E}{\mathfrak{m}_h(E)} \right) \in \mathcal{P}([0,1]).$$

Let $\tilde{h}_E : [0,1] \to \mathbb{R}$ be the density of $\nu_{h,e}$, that is

$$\tilde{h}_E(t) = \mathbf{1}_E(b(E)t) \frac{b(E)}{\mathfrak{m}_h(E)} h(b(E)t).$$

We now estimate the density $\tilde{h}_E$.

Proposition 5.20. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. The following estimates hold for $w \to 0$*

and $\delta \to 0$

$$\left\| \tilde{h}_E - Nt^{N-1} \right\|_{L^\infty(0,1)} \leq o(1)$$

where $D \geq 4L\Lambda$, $D' \in (0, D]$, $([0, D'], F, h\mathcal{L}^1)$ *is a one-dimensional Finsler manifold satisfying the* $\mathsf{CD}(0, N)$ *condition, with* $\Lambda_F \leq \Lambda$, *and the set* $E \subset [0, L]$ *satisfies* $\mathfrak{m}_h(E) = w$ *and* $\mathrm{Res}^D_{F,h}(E) \leq \delta$.

Proof. Fix $t \in [0, 1]$. The proof is divided in three parts.

Part 1 Estimate from below and $t > \frac{a(E)}{b(E)}$**.**

Since $t > \frac{a(E)}{b(E)}$, then $t\, b(E) \in E$ (for a.e. t). A direct computation, gives

$$\begin{aligned} \tilde{h}_E(t) &= \frac{b(E)}{w}\, h(tb(E)) \geq \frac{Nb(E)^N}{D^N w}\, t^{N-1}(1 - o(1)) \\ &\geq \frac{ND^N w(1 + o(1))^N}{D^N w}\, t^{N-1}(1 - o(1)) = Nt^{N-1} - o(1), \end{aligned}$$

having used the estimate (5.17), with $x = tb(E)$, in the first inequality and (5.15) in the second inequality.

Part 2 Estimate from below and $t \leq \frac{a(E)}{b(E)}$**.**

In this case it may happen that $t\, b(E) \notin E$, so the best estimate from below is the non-negativity. For this reason, here we exploit the fact that the interval $[0, \frac{a(E)}{b(E)}]$ is “short” and that $t \leq \frac{a(E)}{b(E)}$. A direct computation gives (recall (5.15) and (5.16))

$$\begin{aligned} \tilde{h}_E(t) &\geq 0 \geq Nt^{N-1} - Nt^{N-1} \geq Nt^{N-1} - N\frac{a(E)^{N-1}}{b(E)^{N-1}} \\ &\geq Nt^{N-1} - N\frac{D^{N-1} o(w^{1-\frac{1}{N}})}{D^{N-1} w^{1-\frac{1}{N}}(1 + o(1))^{N-1}} \geq Nt^{N-1} - o(1). \end{aligned}$$

Part 3 Estimate from above.

We use estimate (5.18), with $x = tb(E)$, deducing

$$\begin{aligned} \tilde{h}_E(t) &= \frac{b(E)}{w}\, h(tb(E)) \leq \frac{b(E)}{w}\, h(b(E))(t + o(1))^{N-1} \leq \frac{b(E)}{w}\, h(b(E))(t^{N-1} + o(1)) \\ &\leq \frac{Dw^{\frac{1}{N}}(1 + o(1))}{w}\, \mathsf{P}_{F,h}(E)(t^{N-1} + o(1)) \\ &= \frac{Dw^{\frac{1}{N}}(1 + o(1))}{w}\, \frac{N}{D} w^{1-\frac{1}{N}}(1 + \mathrm{Res}^D_{F,h}(E))(t^{N-1} + o(1)) \\ &\leq N(1 + o(1))(1 + \delta)(t^{N-1} + o(1)) = Nt^{N-1} + o(1) \end{aligned}$$

(in the second inequality we used the uniform continuity of $t \in [0, 1] \mapsto t^{N-1}$; in the third one, estimate (5.14)). □

The following theorem summerizes the contents of this section. This theorem is the irreversible version of Theorem 4.19. Notice that the function ω takes as argument the positive part of the residual and not the residual itself.

Theorem 5.21. *Fix $N > 1$, $L > 0$, and $\Lambda \geq 1$. Then there exists a function $\omega : (0,\infty) \times [0,\infty) \to \mathbb{R}$, infinitesimal in 0, such that the following holds. For all $D \geq 4L\Lambda$, $D' \in (0, D)$, for all $([0, D'], F, h\mathcal{L}^1)$ one-dimensional Finsler manifold satisfying the oriented $\mathsf{CD}(0, N)$ condition with $\Lambda_F \leq \Lambda$, and for all $E \subset [0, L]$, it holds that*

$$\left| b(E) - D\mathfrak{m}_h(E)^{\frac{1}{N}} \right| \leq D\mathfrak{m}_h(E)^{\frac{1}{N}} \omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+), \tag{5.21}$$

$$\left\| \tilde{h}_E - Nt^{N-1} \right\|_{L^\infty} \leq \omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+),$$

$$\mathrm{Res}^D_{F,h}(E) \geq -\omega(\mathfrak{m}_h(E), (\mathrm{Res}^D_{F,h}(E))^+), \tag{5.22}$$

where $b(E) = \operatorname{ess\,sup} E$ and $\tilde{h}_E$ is the Radon–Nikodym derivative of $\mathfrak{m}_h(E)^{-1}(S_{b(E)})_\#\mathfrak{m}_h\llcorner_E$, with $S_{b(E)}(x) = x/b(E)$.

5.6 Passage to the limit as $R \to \infty$

We now go back to the studying the identity case of the isoperimetric inequality. Fix $E \subset X$ a bounded isoperimetric Borel set with positive measure. We quickly recall the notation introduced in Section 5.3. Denote by ϕ_R the 1-Lipschitz Kantorovich potential associated to $f_R = \mathbf{1}_E - \frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}\mathbf{1}_{B_R}$. Without loss of generality we assume that ϕ_R is equibounded on every bounded set. The Ascoli–Arzelà theorem implies that that, up to subsequences, ϕ_R converges to a certain 1-Lipschitz function ϕ_∞, uniformly on every compact set.

We recall the disintegration given by Proposition 5.7

$$\mathfrak{m}\llcorner_{\widehat{\mathcal{T}}_R} = \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}\, \widehat{\mathfrak{q}}_R(d\alpha), \quad \text{and} \quad \mathsf{P}(E;\,\cdot\,) \geq \int_{Q_R} \mathsf{P}_{\widehat{X}_{\alpha,R}}(E;\,\cdot\,)\, \widehat{\mathfrak{q}}_R(d\alpha). \tag{5.23}$$

We are now in position to improve Corollary 5.12, recovering the same strength of Corollary 4.7.

Proposition 5.22. *Up to taking subsequences, it holds that*

$$\lim_{R\to\infty} \mathrm{Res}_{\mathfrak{Q}_R(x),R} = 0, \qquad \mathfrak{m}\llcorner_E\text{-}a.e.. \tag{5.24}$$

Proof. Corollary 5.12 guarantees that

$$\limsup_{R\to\infty} \int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\, \mathfrak{m}(dx) \leq 0,$$

Using estimate (5.22), we estimate the negative part of the residual

$$(\mathrm{Res}_{\mathfrak{Q}_R(x),R})^- \leq \omega\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, (\mathrm{Res}_{\mathfrak{Q}_R(x),R})^+\right) = \omega\left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_R)}, 0\right),$$

where ω is a function, infinitesimal in $(0,0)$. The L^1-norm of the residual is given by

$$\left\|\mathrm{Res}_{\mathfrak{Q}_R(x),R}\right\|_{L^1(E;\mathfrak{m})} = 2\int_E (\mathrm{Res}_{\mathfrak{Q}_R(x),R})^-\, d\mathfrak{m} + \int_E \mathrm{Res}_{\mathfrak{Q}_R(x),R}\, d\mathfrak{m}.$$

Taking into account the previous inequality and, again, Corollary 5.12, we deduce that the residual $\mathrm{Res}_{\mathfrak{Q}_R(x),R}$, converges to 0 in L^1. By taking a subsequence, we obtain (5.24). □

At this point, the proof proceeds more or less in the same way as in the reversible setting. For this reason, we will be a bit sloppy, letting the reader to check Chapter 4 for the details.

Throughout this section, we set $\rho = (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$.

5.6.1 Passage to the limit of the radius

The radius function $r_R : \overline{E} \to [0, \operatorname{diam} E]$ is defined as follows. Fix $x \in E \cap \widehat{\mathcal{T}}_R$ and let $E_{x,R} := (g_R(\mathfrak{Q}_R(x),\cdot))^{-1}(E) \subset [0, |\widehat{X}_{\mathfrak{Q}_R(x),R}|]$. Define

$$r_R(x) := \operatorname{ess\,sup} E_{x,R}, \qquad \text{if } x \in E \cap \widehat{\mathcal{T}}_R.$$

Notice that $r_R(x) = b(E_{x,E})$, where the notation $b(E)$ was introduced in Section 5.5.1.

The next proposition ensures that, in limit as $R \to \infty$, the function r_R converges to ρ, which is precisely the radius that we expect.

Proposition 5.23. *Up to subsequences it holds true*

$$\lim_{R\to\infty} r_R = \rho, \qquad \mathfrak{m}\llcorner_E\text{–}a.e..$$

Proof. By Proposition 5.22, there exists a sequence R_n and a negligible subset $N \subset E$, such that $\lim_{n\to\infty} \mathrm{Res}_{\mathfrak{Q}_{R_n}(x),R_n} = 0$, for all $x \in E\setminus N$.

Define $G := \bigcap_n \widehat{\mathcal{T}}_{R_n}\setminus N$ and notice that $\mathfrak{m}(E\setminus G) = 0$. Fix $n \in \mathbb{N}$ and $x \in G$ and let

$\alpha := \mathfrak{Q}_{R_n}(x) \in Q_{R_n}$. Clearly, it holds

$$|r_{R_n}(x) - \rho| \le \left| r_{R_n}(x) - (R_n + \operatorname{diam} E) \left(\tfrac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})} \right)^{\frac{1}{N}} \right| + \left| (R_n + \operatorname{diam} E) \left(\tfrac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})} \right)^{\frac{1}{N}} - \rho \right|.$$

The second term goes to 0 by definition of AVR, so we focus on the first term. Consider the ray $(\widehat{X}_{\alpha,R_n}, F, \widehat{\mathfrak{m}}_{\alpha,R_n})$. By definition, we have that

$$\operatorname{Res}^{R_n + \operatorname{diam} E}_{h_{\alpha,R_n}}(E_{x,R_n}) = \operatorname{Res}_{\alpha,R_n}$$

We can now use Theorem 5.21 (in particular estimate (5.21)), obtaining

$$\begin{aligned} \left| r_{R_n}(x) - (R_n + \operatorname{diam} E) \left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})} \right)^{\frac{1}{N}} \right| &= \left| r_{R_n}(x) - (R_n + \operatorname{diam} E)(\mathfrak{m}_{h_{\alpha,R_n}}(E_{x,R_n}))^{\frac{1}{N}} \right| \\ &\le (R_n + \operatorname{diam} E) \left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})} \right)^{\frac{1}{N}} \omega \left(\frac{\mathfrak{m}(E)}{\mathfrak{m}(B_{R_n})}, (\operatorname{Res}_{\mathfrak{Q}_R(x),R_n})^+ \right). \end{aligned}$$

□

5.6.2 Passage to the limit of the rays

Consider now a constant-speed parametrization of the rays inside the set E:

$$\gamma^{x,R}_s := g_R(\mathfrak{Q}_R(x), s\, r_R(x)), \qquad \text{if } x \in E \cap \widehat{\mathcal{T}}_R.$$

A direct consequence of the definition of $\gamma^{x,R}$ and the properties of the disintegration are

$$\mathsf{d}(\gamma^{x,R}_t, \gamma^{x,R}_s) = \phi_R(\gamma^{x,R}_t) - \phi_R(\gamma^{x,R}_s), \quad \forall\, 0 \le t \le s \le 1, \text{ for } \mathfrak{m}\text{-a.e. } x \in E, \tag{5.25}$$

$$\mathsf{d}(\gamma^{x,R}_0, \gamma^{x,R}_1) = r_R(x), \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in E, \tag{5.26}$$

$$x \in \gamma^{x,R}, \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{5.27}$$

In order to compute the limit as $R \to \infty$ we proceed as follows. Define the compact set $K := \{\gamma \in \operatorname{Geo}(X) : \gamma_0, \gamma_1 \in \overline{E}\}$. Define the measure (having mass $\mathfrak{m}(E)$)

$$\tau_R := (\operatorname{Id} \times \gamma^{\cdot,R})_\# \mathfrak{m}\llcorner_E \in \mathcal{M}(\overline{E} \times K),$$

with the property that $(P_1)_\# \tau_R = \mathfrak{m}\llcorner_E$ and $\gamma = \gamma^{x,R}$, for τ_R-a.e. $(x,\gamma) \in \overline{E} \times K$. Properties (5.25)–(5.27) can be restated using a more measure-theoretic language

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \phi_R(e_t(\gamma)) + \phi_R(e_s(\gamma)) = 0, \tag{5.28}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x) = 0, , \tag{5.29}$$

$$x \in \gamma, \tag{5.30}$$

for τ_R-a.e. $(x, \gamma) \in \overline{E} \times K$. By tightness, we can extract a sub-sequence such that $\tau_R \rightharpoonup \tau$ weakly. The next proposition guarantees that the properties (5.28)–(5.30) pass to the limit as $R \to \infty$.

Proposition 5.24. *For τ-a.e. $(x, \gamma) \in \overline{E} \times K$, it holds that*

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \phi_\infty(e_t(\gamma)) - \phi_\infty(e_s(\gamma)), \quad \forall\, 0 \le t \le s \le 1, \tag{5.31}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{5.32}$$

$$x \in \gamma. \tag{5.33}$$

Proof. Fix $t \le s$ and integrate (5.28) in $\overline{E} \times K$, obtaining

$$0 = \int_{\overline{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \phi_R(e_t(\gamma)) + \phi_R(e_s(\gamma)))\, \tau_R(dx\, d\gamma) = \int_{\overline{E}\times K} L^{t,s}_{\phi_R}(\gamma)\, \tau_R(dx\, d\gamma),$$

having set $L^{t,s}_\psi(\gamma) := \mathsf{d}(e_t(\gamma), e_s(\gamma)) - \psi(e_t(\gamma)) + \psi(e_s(\gamma))$. The map $L^{t,s}_{\phi_R} : K \to \mathbb{R}$ is clearly continuous and converges uniformly to $L^{t,s}_{\phi_\infty}$. Therefore, we can take the limit in the equation above obtaining

$$0 = \int_{\overline{E}\times K} L^{t,s}_{\phi_\infty}(\gamma)\, \tau(dx\, d\gamma) = \int_{\overline{E}\times K} (\mathsf{d}(e_t(\gamma), e_s(\gamma)) - \phi_\infty(e_t(\gamma)) + \phi_\infty(e_s(\gamma)))\, \tau(dx\, d\gamma).$$

The 1-lipschitzianity of ϕ_∞, yields $L^{t,s}_{\phi_\infty}(\gamma) \ge 0$, $\forall \gamma \in K$, hence

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \phi_\infty(e_t(\gamma)) - \phi_\infty(e_s(\gamma)) \quad \text{for } \tau\text{-a.e. } (x, \gamma) \in \overline{E} \times K.$$

We conclude by an approximation argument for t and s in obtaining (5.31).

Now we prove (5.32). We integrate Equation (5.29) obtaining

$$0 = \int_{\overline{E}\times X} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma).$$

By Lusin's and Egorov's theorems there exists a compact $M \subset E$, such that: 1) the restrictions $r_R|_M$ are continuous; 2) the restricted maps $r_R|_M$ converge uniformly to ρ; 3) $\mathfrak{m}(E \backslash M) \le \epsilon$, for $\epsilon > 0$ sufficiently small. We now compute the limit

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{\overline{E}\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma) \\ &\ge \liminf_{R\to\infty} \int_{M\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - r_R(x)|\, \tau_R(dx\, d\gamma) \\ &\ge \int_{M\times K} |\mathsf{d}(e_0(\gamma), e_1(\gamma)) - \rho|\, \tau(dx\, d\gamma) \ge 0, \end{aligned}$$

$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho$, for τ-a.e. $(x, \gamma) \in M \times K$. By arbitrariness of $\epsilon > 0$ one concludes.

Finally we prove (5.33). This is done by testing the weak convergence against the function $L(x, \gamma) := \inf_{t\in[0,1]} \mathsf{d}(x, e_t(\gamma))$. □

5.6.3 Disintegration of the measure and the perimeter

Having in mind the disintegration formula (5.23), we define the map $\overline{E} \ni x \mapsto \mu_{x,R} \in \mathcal{P}(\overline{E})$ as

$$\mu_{x,R} := \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \, (\widehat{\mathfrak{m}}_{\mathfrak{Q}_R(x),R})\llcorner_E, \quad \text{if } x \in E \cap \widehat{\mathcal{T}}_R,$$

A direct computation (recall (5.5)–(5.6)) gives

$$\mathfrak{m}(A \cap E) = \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \int_{Q_R} \widehat{\mathfrak{m}}_{\alpha,R}(A \cap E) \, (\mathfrak{Q}_R)_\#(\mathfrak{m}\llcorner_E)(d\alpha) = \int_X \mu_{x,R}(A) \, \mathfrak{m}\llcorner_E(dx),$$

therefore

$$\mathfrak{m}\llcorner_E = \int_E \mu_{x,R} \, \mathfrak{m}\llcorner_E(dx).$$

The measure $\mu_{x,R}$ is given by

$$\mu_{x,R} = (\gamma^{x,R})_\#(\tilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]}), \quad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \overline{E}$$

where

$$\tilde{h}_E^{x,R}(t) = \mathbf{1}_{E_{x,R}}(r_R(x)t) \, r_R(x) \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \, h_{\mathfrak{Q}_R(x),R}(r_R(x)t).$$

Having in mind (5.7), we can perform a similar operation for the perimeter. Indeed, in the natural parametrization of the rays, if we consider only the "right extremal" of $E_{x,R}$ and the fact that $F(\partial_t) = 1$, it holds that

$$h_{R,\mathfrak{Q}_R(x)}(r_R(x))\delta_{r_R(x)} \leq \mathsf{P}_{F,h_{R,\mathfrak{Q}_R(x)}}(E_{x,R}; \cdot).$$

This observation, naturally leads to the definition

$$p_{x,R} := \min\left\{\frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} h_{R,\mathfrak{Q}_R(x)}(r_R(x)), \frac{N}{\rho}\right\} \delta_{g_R(\mathfrak{Q}_R(x), r_R(x))}, \qquad \text{if } x \in E \cap \widehat{\mathcal{T}}_R,$$

Using the maps $\gamma^{x,R}$ and $\tilde{h}_{x,R}$, we rewrite $p_{x,R}$

$$p_{x,R} = \min\left\{\frac{\tilde{h}_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R})}, \frac{N}{\rho}\right\} \delta_{\gamma_1^{x,R}}, \quad \text{if } x \in E \cap \widehat{\mathcal{T}}_R.$$

By definition of $p_{x,R}$ we have that

$$p_{x,R} \le \frac{\mathfrak{m}(B_R)}{\mathfrak{m}(E)} \mathsf{P}_{X_{R,\mathfrak{Q}_R(x)}}(E;\,\cdot\,), \qquad \text{for } \mathfrak{m}\llcorner_E\text{-a.e. } x \in \overline{E},$$

deducing the following "disintegration" formula (equations (5.7) and (5.6) are taken into account)

$$\mathsf{P}(E;A) \ge \int_{Q_R} \mathsf{P}_{X_{\alpha,R}}(E;A)\,\widehat{\mathfrak{q}}_R(d\alpha) \ge \int_{\overline{E}} p_{x,R}(A)\,\mathfrak{m}\llcorner_E(dx), \quad \forall A \subset \overline{E} \text{ Borel.} \tag{5.34}$$

Define now the compact set $F := e_{(0,1)}(K) = \{\gamma_t : \gamma \in K, t \in [0,1]\}$ and let $S \subset \mathcal{M}^+(F)$ be the subset of the non-negative measures on F with mass at most N/ρ. The sets $\mathcal{P}(F)$ and S are naturally endowed with the weak topology of measures. Like in the reversible setting, we metrize these two compact spaces with a metric constructed like in Equation (4.48). Define now the map $G_R : \overline{E} \times K \to \mathcal{P}(F) \times S$, as

$$G_R(x,\gamma) := \left(\gamma_\#(\tilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]}), \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)} \right).$$

Clearly, the function G_R is measurable and continuous w.r.t. the variable x and γ, respectively. Define the measure (having mass $\mathfrak{m}(E)$)

$$\sigma_R := (\mathrm{Id} \times G_R)_\# \tau_R \in \mathcal{M}^+(\overline{E} \times K \times \mathcal{P}(F) \times S).$$

In order to ease the notation, we set $Z = \overline{E} \times K \times \mathcal{P}(F) \times S$.

Proposition 5.25. *The measure σ_R enjoys the following properties*

$$\int_E \psi\, d\mathfrak{m} = \int_Z \int_E \psi(y)\,\mu(dy)\,\sigma_R(dx\,d\gamma\,d\mu\,dp), \quad \forall \psi \in C_b^0(\overline{E}), \tag{5.35}$$

$$\int_{\overline{E}} \psi(y)\,\mathsf{P}(E,dy) \ge \int_Z \int_{\overline{E}} \psi(y)\,p(dy)\,\sigma_R(dx\,d\gamma\,d\mu\,dp), \quad \forall \psi \in C_b^0(\overline{E}), \psi \ge 0. \tag{5.36}$$

Proof. Fix a test function $\psi \in C_b^0(\overline{E})$. Notice that for σ_R-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $\mu = \mu_{x,R}$, because

$$\mu = \gamma_\#(\tilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]}) = (\gamma^{x,R})_\#(\tilde{h}_E^{x,R}\mathcal{L}^1\llcorner_{[0,1]}) = \mu_{x,R}, \qquad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p) \in Z.$$

A direct computation gives

$$\int_E \psi\, d\mathfrak{m} = \int_E \int_E \psi(y)\,\mu_{x,R}\,\mathfrak{m}(dx) = \int_Z \int_E \psi(y)\,\mu_{x,R}(dy)\,\sigma_R(dx\,d\gamma\,d\mu\,dp)$$

$$= \int_Z \int_E \psi(y)\, \mu(dy)\, \sigma_R(dx\, d\gamma\, d\mu\, dp).$$

Now fix an open set $\Omega \subset X$. Since

$$p = \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N(\omega_N \mathsf{AVR}_X)^{\frac{1}{N}}}{\mathfrak{m}(E)^{\frac{1}{N}}} \right\} \delta_{e_1(\gamma)}(\Omega), \qquad \text{for } \sigma_R\text{-a.e. } (x,\gamma,\mu,p) \in Z,$$

we can compute (recall (5.34))

$$\begin{aligned}
\mathsf{P}(E;\Omega) &\geq \int_E \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(\gamma_0^{x,R}, \gamma_1^{x,R})}, \frac{N}{\rho} \right\} \delta_{\gamma_1^{x,R}}(\Omega)\, d\mathfrak{m}(dx) \\
&= \int_Z \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma^{x,R}), e_1(\gamma^{x,R}))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma^{x,R})}(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp) \\
&= \int_Z \min\left\{ \frac{\tilde{h}^E_{x,R}(1)}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)}(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp). \\
&= \int_Z p(\Omega)\, d\sigma_R(dx\, d\gamma\, d\mu\, dp).
\end{aligned}$$

Since $\mathsf{P}(E;A) = \inf\{\mathsf{P}(E;\Omega) : \Omega \supset A \text{ is open}\}$, for any Borel set A, we can conclude. □

Proposition 5.26. *Consider the function $G : \overline{E} \times K \to \mathcal{P}(F) \times S$ defined as*

$$G(x,\gamma) = \left(\gamma_\#(Nt^{N-1}\mathcal{L}^1 \llcorner_{[0,1]}), \max\left\{ \frac{N}{\mathsf{d}(e_0(\gamma), e_1(\gamma))}, \frac{N}{\rho} \right\} \delta_{e_1(\gamma)} \right),$$

and let $\sigma := (\mathrm{Id} \times G)_\# \tau$. Then it holds that $\sigma_R \rightharpoonup \sigma$ in the weak topology of measures.

Proof. The thesis is an immediate consequence of Lemma 4.25. Therefore, one only needs to check the hypotheses of said Lemma, and this is done like in the reversible setting (see the proof of Corollary 4.26). □

We conclude this section with a proposition reporting all the relevant properties of the limit measure σ, whose proof is carried out like in the reversible setting (see Proposition 4.27.

Proposition 5.27. *The measure σ satisfies the following disintegration formulae*

$$\int_E \psi(y)\, \mathfrak{m}(dy) = \int_Z \int_0^1 \psi(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp), \quad \forall \psi \in L^1(E; \mathfrak{m}\llcorner_E), \tag{5.37}$$

$$\int_{\overline{E}} \psi(y) \mathsf{P}(E; dy) = \frac{N}{\rho} \int_Z \psi(e_1(\gamma)) \psi\, \sigma(dx\, d\gamma\, d\mu\, dp), \quad \forall \psi \in L^1(\overline{E}; \mathsf{P}(E;\cdot)). \tag{5.38}$$

Furthermore, for σ-a.e. $(x, \gamma, \mu, p) \in Z$ it holds

$$\mathsf{d}(e_t(\gamma), e_s(\gamma)) = \phi_\infty(e_t(\gamma)) - \phi_\infty(e_s(\gamma)), \quad \forall 0 \le t \le s \le 1, \tag{5.39}$$

$$\mathsf{d}(e_0(\gamma), e_1(\gamma)) = \rho, \tag{5.40}$$

$$x \in \gamma, \tag{5.41}$$

$$\mu = \gamma_\#(N t^{N-1} \mathcal{L}^1 \llcorner_{[0,1]}), \tag{5.42}$$

$$p = \frac{N}{\rho} \delta_{e_1(\gamma)}. \tag{5.43}$$

5.6.4 Back to the classical localization notation

We are now in position to re-obtain a "classical" disintegration formula for the measure $\mathfrak{m}$, as well as for the relative perimeter of E.

We recall the definition of some of the objects that were introduced in Section 5.1.3. For instance, let $\Gamma_\infty := \{(x, y) : \phi_\infty(x) - \phi_\infty(y) = \mathsf{d}(x, y)\}$ and let $\mathcal{T}_\infty$ be the transport set, i.e., the family of points passing through only one non-degenerate transport curve. Let $\mathcal{A}_\infty$ the set of branching points (i.e. points where two of more non-degenerate transport curves pass). The sets of forward and backward branching points are defined as

$$\mathcal{A}^+_\infty := \{x \in \mathcal{A}_\infty : \exists y \neq x \text{ such that } (x, y) \in \Gamma_\infty\},$$
$$\mathcal{A}^-_\infty := \{x \in \mathcal{A}_\infty : \exists y \neq x \text{ such that } (y, x) \in \Gamma_\infty\}.$$

We recall that $\mathcal{A}_\infty = \mathcal{A}^+_\infty \cup \mathcal{A}^-_\infty$ and that $\mathcal{A}_\infty$ is negligible. Let Q_∞ be the quotient set and let $\mathfrak{Q}_\infty : \mathcal{T}_\infty \to Q_\infty$ be the quotient map; denote by $X_{\alpha,\infty} := \mathfrak{Q}^{-1}(\alpha)$ the disintegration rays and let $g_\infty : \mathrm{Dom}(g_\infty) \subset Q_\infty \times [0, \infty) \to X$ be the standard parametrization of the rays.

We introduce the function $t_\alpha : \overline{X_{\alpha,\infty}} \to [0, \infty)$ defined as

$$t_\alpha(x) := (g_\infty(\alpha, \cdot))^{-1} = \mathsf{d}(g_\infty(\mathfrak{Q}_\infty(x), 0), x);$$

the function t_α measures how much a point is translates from the starting point of the ray $X_{\alpha,\infty}$.

The following proposition guarantees that the geodesics on which the measure σ is supported lay on the transport set $\mathcal{T}_\infty$.

Proposition 5.28. *For σ-a.e. $(x, \gamma, \mu, p) \in Z$, it holds that $e_t(\gamma) \in \mathcal{T}_\infty$, for all $t \in (0, 1)$.*

Proof. Clearly, for σ-a.e. $(x, \gamma, \mu, p) \in Z$, γ is non-degenerate, hence $e_t(\gamma) \notin \mathcal{D}$, where $\mathcal{D}$ is the set where no non-degenerate transport curve pass. Therefore we need only to check that $e_t(\gamma) \notin \mathcal{A}^\infty$. We will prove only that $e_t(\gamma) \neq \mathcal{A}^+_\infty$, for the case $e_t(\gamma) \neq \mathcal{A}^-_\infty$ is analogous. Fix

$\epsilon > 0$ and let

$$P := \{(x,\gamma,\mu,p) \in Z : e_\epsilon(\gamma) \in \mathcal{A}^+_\infty \text{ and conditions (5.37)–(5.43) holds}\}$$

Notice that by definition of $\mathcal{A}^+_\infty$, if $(x,\gamma,\mu,p) \in P$, then $\gamma_t \in \mathcal{A}^+_\infty$, for all $t \in [0,\epsilon]$, thus we can compute

$$\begin{aligned} 0 = \mathfrak{m}(\mathcal{A}^+_\infty) &= \int_Z \int_0^1 \mathbf{1}_{\mathcal{A}^+_\infty}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \\ &\geq \int_P \int_0^\epsilon \mathbf{1}_{\mathcal{A}^+_\infty}(e_t(\gamma)) N t^{N-1}\, dt\, \sigma(dx\, d\gamma\, d\mu\, dp) \geq \epsilon^N \sigma(P), \end{aligned}$$

thus P is negligible. Fix now $(x,\gamma,\mu,p) \notin P$. By definition of $\mathcal{A}^+_\infty$ and P, we have that $\gamma_t \notin \mathcal{A}^+_\infty$, for all $t \in [\epsilon,1]$. By arbitrariness of ϵ, we deduce that for σ-a.e $(x,\gamma,\mu,p) \in Z$, it holds that $e_t(\gamma) \notin \mathcal{A}^+_\infty$, for all $t \in (0,1]$. □

Corollary 5.29. *It holds that $E \subset \mathcal{T}_\infty$ and for σ-a.e. $(x,\gamma,\mu,p) \in Z$, we have that $e_t(\gamma) \in \overline{X_{\mathfrak{Q}(x),\infty}}$ and*

$$e_t(\gamma) = g_\infty(\mathfrak{Q}(x), t_{\mathfrak{Q}(x)}(e_0(\gamma)) + \rho t). \tag{5.44}$$

Define $\hat{\mathfrak{q}} := \frac{1}{\mathfrak{m}(E)}(\mathfrak{Q}_\infty)_\#(\mathfrak{m}\llcorner_E) \ll (\mathfrak{Q}_\infty)_\# \mathfrak{m}\llcorner_{\mathcal{T}_\infty}$ and let $\tilde{\mathfrak{q}}$ be a probability measure such that $(\mathfrak{Q}_\infty)_\#\mathfrak{m}\llcorner_{\mathcal{T}_\infty} \ll \tilde{\mathfrak{q}}$. The disintegration theorem gives the following two formulae

$$\mathfrak{m}\llcorner_E = \int_{Q_\infty} \hat{\mathfrak{m}}_{\alpha,\infty}\, \hat{\mathfrak{q}}(d\alpha), \qquad \text{and} \qquad \mathfrak{m}\llcorner_{\mathcal{T}_\infty} = \int_{Q_\infty} \tilde{\mathfrak{m}}_{\alpha,\infty}\, \tilde{\mathfrak{q}}(d\alpha),$$

where the measures $\hat{\mathfrak{m}}_{\alpha,\infty}$ and $\tilde{\mathfrak{m}}_{\alpha,\infty}$ are supported on $X_{\alpha,\infty}$. By comparing the two expressions above, it turns out that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\, \hat{\mathfrak{m}}_{\alpha,\infty} = \mathbf{1}_E \tilde{\mathfrak{m}}_{\alpha,\infty}$. The Localization Theorem 2.6, ensures that the transport rays $(X_{\alpha,\infty}, F, \tilde{\mathfrak{m}}_{\alpha,\infty})$ satisfies the oriented $\mathsf{CD}(0,N)$ condition. On the contrary, we cannot deduce the same condition for the other disintegration, because the reference measure is restricted to the set E and not the transport set. Consider the densities $\hat{h}_\alpha$ and $\tilde{h}_\alpha$ given by

$$\hat{\mathfrak{m}}_{\alpha,\infty} = (g_\infty(\alpha,\cdot))_\#(\hat{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}), \qquad \text{and} \qquad \tilde{\mathfrak{m}}_{\alpha,\infty} = (g_\infty(\alpha,\cdot))_\#(\tilde{h}_\alpha \mathcal{L}^1_{(0,|X_{\alpha,\infty}|)}).$$

Clearly, it holds that $\frac{d\hat{\mathfrak{q}}}{d\tilde{\mathfrak{q}}}(\alpha)\hat{h}_\alpha(t) = \mathbf{1}_E(g(\alpha,t))\tilde{h}_\alpha(t)$, thus we can derive a somehow weaker concavity condition for the function $\hat{h}_\alpha^{\frac{1}{N-1}}$: for all $x_0, x_1 \in (0,|X_{\alpha,\infty}|)$ and for all $t \in [0,1]$, it holds that

$$\hat{h}_\alpha((1-t)x_0 + tx_1)^{\frac{1}{N-1}} \geq (1-t)\hat{h}_\alpha(x_0)^{\frac{1}{N-1}} + t\hat{h}_\alpha(x_1)^{\frac{1}{N-1}}, \qquad \text{if } \hat{h}_\alpha((1-t)x_0 + tx_1) > 0.$$

A natural consequence is the following "Bishop–Gromov inequality"

$$\text{the map } r \mapsto \frac{\hat{h}_\alpha(r)}{r^{N-1}} \text{ is decreasing on the set } \{r \in (0, |X_{\alpha,\infty}|) : \hat{h}_\alpha(r) > 0\}.$$

Define the full-measure set $\hat{Z} \subset Z$ as

$$\hat{Z} := \{(x, \gamma, \mu, p) \in Z : x \in E \cap \mathcal{T}_\infty, \text{ and the properties given by Equations (5.37)–(5.38) and (5.44) holds}\}.$$

We partitionate $\hat{Z}$ in the following way

$$\hat{Z}_\alpha := \{(x, \gamma, \mu, p) \in \hat{Z} : \mathfrak{Q}_\infty(x) = \alpha\},$$

and we disintegrate the measure σ according to the partition $(\hat{Z}_\alpha)_{\alpha \in Q_\infty}$

$$\sigma = \int_{Q_\infty} \sigma_\alpha \, \mathfrak{q}(d\alpha),$$

where the probability measures σ_α are supported on $\hat{Z}_\alpha$. Moreover, let $\nu_\alpha \in \mathcal{P}([0, \infty))$ be the measure given by

$$\nu_\alpha := \frac{1}{\mathfrak{m}(E)} (t_\alpha \circ e_0 \circ \pi_K)_\#(\sigma_\alpha)$$

(we recall that $t_\alpha = (g_\infty(\alpha, \cdot))^{-1}$ and $\pi_K(x, \gamma, \mu, p) = \gamma$).

The following two propositions have exactly the same proof of Proposition 4.30 and 4.31, therefore the proofs are omitted.

Proposition 5.30. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r) = N\omega_N \mathsf{AVR}_X \int_{[0,\infty)} (r - t)^{N-1} \mathbf{1}_{(t,t+\rho)}(r) \, \nu_\alpha(dt), \quad \forall r \in (0, |X_{\alpha,\infty}|).$$

Proposition 5.31. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that $\nu_\alpha = \delta_0$.*

Corollary 5.32. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, for σ_α-a.e. $(x, \gamma, \mu, p) \in Z_\alpha$, it holds that $e_t(\gamma) = g(\alpha, \rho t)$, $\forall t \in [0, 1]$.*

Proof. The fact that $\nu_\alpha = \delta_0$, implies $t_\alpha(\gamma_0) = 0$ for σ_α-a.e. $(x, \gamma\, \mu, p) \in \hat{Z}_\alpha$, hence, recalling the disintegration formula (5.44) and the definition of $\hat{Z}$, we deduce that $e_t(\gamma) = g(\alpha, t_\alpha(e_0) + \rho t) = g(\alpha, \rho t)$. □

The next corollary concludes the discussion of the limiting procedures of the disintegration.

Corollary 5.33. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\hat{h}_\alpha(r) = N\omega_N \mathsf{AVR}_X \mathbf{1}_{(0,\rho)}(r) r^{N-1}.$$

Moreover, the following disintegration formulae hold true

$$\mathfrak{m}\llcorner_E = N\omega_N \mathsf{AVR}_X \int_{Q_\infty} (g_\infty(\alpha,\,\cdot\,))_\#(r^{N-1}\,\mathcal{L}^1\llcorner_{(0,\rho)})\,\hat{\mathfrak{q}}(d\alpha), \tag{5.45}$$

$$\mathsf{P}(E;\,\cdot\,) = \mathsf{P}(E)\int_{Q_\infty} \delta_{g_\infty(\alpha,\rho)}\,\hat{\mathfrak{q}}(d\alpha). \tag{5.46}$$

Proof. We need only to prove Equation (5.46). Equation (5.38) and Corollary 5.32 yield

$$\begin{aligned}\int_{\overline{E}} \psi(x)\,\mathsf{P}(E;dx) &= \frac{N}{\rho}\int_{\hat{Z}} \psi(e_1(\gamma))\psi\,\sigma(dx\,d\gamma\,d\mu\,dp)\\ &= \frac{N}{\rho}\int_{Q_\infty}\int_{\hat{Z}_\alpha} \psi(e_1(\gamma))\,\sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,\hat{\mathfrak{q}}(d\alpha)\\ &= \frac{N}{\rho}\int_{Q_\infty} \psi(g_\infty(\alpha,\rho))\int_{\hat{Z}_\alpha} \sigma_\alpha(dx\,d\gamma\,d\mu\,dp)\,\hat{\mathfrak{q}}(d\alpha), \quad \forall \psi \in L^1(\overline{E};\mathsf{P}(E;\,\cdot\,)).\end{aligned}$$ □

5.7 E is a ball

Next lemma is the Finsler version of Lemma 4.35.

Lemma 5.34. *Let $(X, F, \mathfrak{m})$ be Finsler manifold (with possible infinite reversibility). Let $E \subset X$ be a Borel set and let $\Omega \subset X$ be an open connected set with finite measure. If $\mathfrak{m}(E\cap\Omega) > 0$ and $\mathfrak{m}(\Omega\backslash E) > 0$, then $\mathsf{P}(E;\Omega) > 0$.*

Proof. If the manifold is reversible, then one can just apply Lemma 4.35, and therefore we can assume that the manifold is irreversible. As we stressed out (see Remark 5.2), there exists a Riemannian metric g, such that its dual metric g^{-1} in T^*X satisfies $\sqrt{g^{-1}(\omega,\omega)} \le F^*(\omega)$, for all $\omega \in T^*X$. By definition of perimeter, there exists a sequence $u_n \in \mathrm{Lip}_{loc}(\Omega)$ such that $u_n \to \mathbf{1}_E$ in L^1_{loc} and $\int_\Omega |\partial u_n|\,d\mathfrak{m} \to \mathsf{P}_{(X,F,\mathfrak{m})}(E;\Omega)$. Since $g^{-1}(du_n, du_n) \le F^*(-du_n) = |\partial u_n|$ a.e. in Ω, we conclude that $\mathsf{P}_{(X,g,\mathfrak{m})}(E;\Omega) \le \mathsf{P}_{(X,F,\mathfrak{m})}(E;\Omega)$. □

Proposition 5.35. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\phi_\infty(g_\infty(\alpha,0)) \le \operatorname*{ess\,sup}_E \phi_\infty, \quad \textit{and} \quad \phi_\infty(g_\infty(\alpha,\rho)) \ge \operatorname*{ess\,inf}_E \phi_\infty.$$

Proof. We prove only the former inequality; the latter has the same proof. Define $M := \operatorname{ess\,sup}_E \phi_\infty$ and $H := \{\alpha \in Q_\infty : \phi_\infty(g_\infty(\alpha,0)) \ge M + 2\epsilon\}$. Consider the following measure on E

$$\mathfrak{n}(T) := N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon \mathbf{1}_T(g_\infty(\alpha, r)) r^{N-1}\, dr\, \hat{\mathfrak{q}}(d\alpha), \quad \forall T \subset E \text{ Borel}.$$

Clearly, $\mathfrak{n} \ll \mathfrak{m}$, thus $\phi_\infty(x) \leq M$, for $\mathfrak{n}$-a.e. $x \in E$. If we compute the integral

$$\begin{aligned}
0 &\geq \int_E (\phi_\infty(x) - M)\, \mathfrak{n}(dx) = N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon (\phi_\infty(g_\infty(\alpha, t)) - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\
&= N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon (\phi_\infty(g_\infty(\alpha, 0)) - t - M)\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\
&\geq N\omega_N \mathsf{AVR}_X \int_H \int_0^\epsilon \epsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) = \epsilon^N \hat{\mathfrak{q}}(H).
\end{aligned}$$

we can deduce that $\hat{\mathfrak{q}}(H) = 0$ and, by arbitrariness of ϵ, we conclude. □

Theorem 5.36. *There exists a (unique) point $o \in X$, such that, up to a negligible set, $E = B^+(o, \rho)$, where $\rho = (\frac{\mathfrak{m}(E)}{\omega_N \mathsf{AVR}_X})^{\frac{1}{N}}$. Moreover, it holds that*

$$\phi_\infty(o) = \operatorname*{ess\,sup}_E \phi_\infty = \max_{B^+(o,\rho)} \phi_\infty. \tag{5.47}$$

Proof. Define $\tilde{E} := \operatorname{supp} \mathbf{1}_E$. Recall that by definition of support, $\tilde{E} = \bigcup_C C$, where the intersection is taken among all closed sets C such that $\mathfrak{m}(E \backslash C) = 0$; and in particular $\mathfrak{m}(E \backslash \tilde{E}) = 0$. Let $o \in \arg\max_{\tilde{E}} \phi_\infty$. By definition of $\tilde{E}$, we have that $\max_{\tilde{E}} \phi_\infty = \operatorname{ess\,sup}_E \phi_\infty$, deducing the first equality of (5.47). The other equality in (5.47) will follow from the fact $E = B^+(o, \rho)$ (up to a negligible set).

It is sufficient to prove only that $B^+(o, \rho) \subset E$, for the other inclusion is automatic Indeed, the Bishop–Gromov inequality, together with the definition of a.v.r. yields

$$\mathfrak{m}(E) \geq \mathfrak{m}(B^+(o, \rho)) \geq \omega_N \mathsf{AVR}_X \rho^N = \mathfrak{m}(E),$$

and the equality of measures improves to an equality of sets.

Fix now $\epsilon > 0$ and define $A = B^+(o, \rho - \epsilon)$. If $\mathfrak{m}(A \backslash E) = 0$, then we deduce that $B^+(o, \rho - \epsilon) \subset E$ and, by arbitrariness of ϵ, we can conclude.

Suppose on the contrary that $\mathfrak{m}(A \backslash E) > 0$. Clearly A is connected and $\mathfrak{m}(A \cap E) > 0$ (otherwise $o \notin \tilde{E}$), so we can apply Lemma 5.34 obtaining $\mathsf{P}(E; A) > 0$. Define $H = \{\alpha \in Q_\infty : g_\infty(\alpha, \rho) \in A\}$. The set H is non-negligible because (recall (5.46))

$$0 < \frac{\mathsf{P}(E; A)}{\mathsf{P}(E)} = \int_{Q_\infty} \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \int_H \mathbf{1}_A(g_\infty(\alpha, \rho))\, \hat{\mathfrak{q}}(d\alpha) = \hat{\mathfrak{q}}(H).$$

By lipschitz-continuity of ϕ_∞ we deduce

$$\phi_\infty(x) \geq \phi_\infty(o) - \rho + \epsilon \geq M - \rho + \epsilon, \quad \forall x \in A = B^+(o, \rho - \epsilon)$$

hence

$$\phi_\infty(g_\infty(\alpha, \rho)) \geq M - \rho + \epsilon, \quad \forall \alpha \in H.$$

Continuing the chain of inequalities, we arrive at

$$\phi_\infty(g_\infty(\alpha, 0)) = \phi_\infty(g_\infty(\alpha, \rho)) + \rho \geq M + \epsilon, \quad \forall \alpha \in H.$$

The line above, together with the fact that $\hat{\mathfrak{q}}(H) > 0$, contradicts Proposition 5.35. □

5.7.1 $\phi_\infty(x)$ coincides with $-\mathsf{d}(o, x)$

The present section is devoted in proving that, $\phi_\infty(x) = -\mathsf{d}(o, x) + \phi_\infty(o)$. Please notice that here there are some differences with the reversible setting and in particular the fact that $\Lambda_F < \infty$ is taken into account.

Proposition 5.37. *For $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$, it holds that*

$$\mathsf{d}(o, g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho]. \tag{5.48}$$

Proof. By the 1-lipschitzianity of ϕ_∞ and the fact that $E = B^+(o, \rho)$ (up to a negligible set) we deduce that $\phi_\infty(x) \geq \phi_\infty(o) - \rho$, for $\mathfrak{m}$-a.e. $x \in E$. Henceforth, Proposition 5.35 and Equation (5.47) yield

$$\phi_\infty(g_\infty(\alpha, 0)) \leq \phi_\infty(o), \quad \text{and} \quad \phi_\infty(g_\infty(\alpha, \rho)) \geq \phi_\infty(o) - \rho.$$

Since $\frac{d}{dt}\phi_\infty(g_\infty(\alpha, t)) = -1$, $t \in (o, \rho)$, the inequalities above are saturated, i.e., it holds that

$$\phi_\infty(g_\infty(\alpha, t)) = \phi_\infty(o) - t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty.$$

Using again the 1-lipschitzianity of ϕ_∞, we arrive at

$$\mathsf{d}(o, g_\infty(\alpha, t)) \geq \phi_\infty(o) - \phi_\infty(g_\infty(\alpha, t)) = t, \quad \forall t \in [0, \rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e. } \alpha \in Q_\infty. \tag{5.49}$$

Now fix $\epsilon > 0$ and let $C = \{\alpha \in Q_\infty : \mathsf{d}(o, g_\infty(\alpha, 0)) > (1 + \Lambda_F)\epsilon\}$, where Λ_F is the reversibility constant. Define the function $f(t) := \inf\{\mathsf{d}(o, g_\infty(\alpha, t)) : \alpha \in C\}$. Clearly, f is Λ_F-Lipschitz and satisfies $f(0) \geq (1+\Lambda_F)\epsilon$, hence $f(t) \geq (1+\Lambda_F)\epsilon - \Lambda_F t$, yielding (cfr. (5.49))

$$f(t) \geq \max\{((1+\Lambda_F)\epsilon - \Lambda_F t), t\} \geq \epsilon.$$

The inequality above implies that $g_\infty(\alpha,t) \notin B^+(o,\epsilon)$ for all $t \in [0,1]$, for all $\alpha \in C$. We compute $\mathfrak{m}(B^+(0,\epsilon))$ using the disintegration formula (5.45)

$$\begin{aligned}\frac{\mathfrak{m}(B^+(o,\epsilon))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty}\int_0^\rho \mathbf{1}_{B^+(o,\epsilon)}(g_\infty(\alpha,t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B^+(o,\epsilon)}(g_\infty(\alpha,t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha).\end{aligned}$$

If $\mathbf{1}_{B^+(o,\epsilon)}(g_\infty(\alpha,t)) = 1$, then inequality (5.49) yields $t \leq \epsilon$, so we continue the computation

$$\begin{aligned}\frac{\mathfrak{m}(B^+(o,\epsilon))}{N\omega_N \mathsf{AVR}_X} &= \int_{Q_\infty\setminus C}\int_0^\rho \mathbf{1}_{B^+(o,\epsilon)}(g_\infty(\alpha,t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= \int_{Q_\infty\setminus C}\int_0^\epsilon \mathbf{1}_{B^+(o,\epsilon)}(g_\infty(\alpha,t))\, t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &\leq \int_{Q_\infty\setminus C}\int_0^\epsilon t^{N-1}\, dt\, \hat{\mathfrak{q}}(d\alpha) \\ &= (\hat{\mathfrak{q}}(Q_\infty) - \hat{\mathfrak{q}}(C))\frac{\epsilon^N}{N}.\end{aligned}$$

On the other hand, the Bishop–Gromov inequality yields

$$\mathfrak{m}(B^+(o,\epsilon)) \geq \frac{\epsilon^N}{\rho^N}\mathfrak{m}(B^+(o,\rho)) = \frac{\epsilon^N}{\rho^N}\mathfrak{m}(E) = \epsilon^N \omega_N \mathsf{AVR}_X.$$

The comparison of the two previous inequality gives $\hat{\mathfrak{q}}(C) = 0$. By arbitrariness of ϵ, we deduce that $g_\infty(\alpha,0) = o$ for $\hat{\mathfrak{q}}$-a.e. $\alpha \in Q_\infty$.

Finally, using again (5.49), we conclude

$$t \leq \mathsf{d}(o, g_\infty(\alpha,t)) \leq \mathsf{d}(o, g_\infty(\alpha,0)) + \mathsf{d}(g_\infty(\alpha,0), g_\infty(\alpha,t)) = t, \quad \forall t \in [0,\rho], \text{ for } \hat{\mathfrak{q}}\text{-a.e } \alpha \in Q_\infty.$$

□

Corollary 5.38. *It holds that for all $x \in B^+(o,\rho)$, $\phi_\infty(x) = \phi_\infty(o) - \mathsf{d}(o,x)$.*

Proof. If $x \in E \cap \mathcal{T}_\infty$, then $x = g(\alpha,t)$, for some t, with $\alpha = \mathfrak{Q}_\infty(x)$. By the previous proposition we may assume that $g_\infty(\alpha,0) = o$, hence we have that

$$\phi_\infty(x) - \phi_\infty(o) = \phi_\infty(g_\infty(\alpha,t)) - \phi_\infty(g_\infty(\alpha,0)) = -\mathsf{d}(g_\infty(\alpha,0), g_\infty(\alpha,t)) = -\mathsf{d}(o,x).$$

Since $\mathcal{T}_\infty \cap E$ has full measure in $B^+(o,\rho)$, we conclude. □

5.7.2 Localization of the whole space

We can now extend the localization given in Section 5.6.4 to the whole space X. Since we do not know the behaviour of ϕ_∞ outside $B^+(o,\rho)$, we take as reference 1-Lipschitz function $-\mathsf{d}(o,\cdot)$, which coincides with ϕ_∞ on $B^+(o,\rho)$: we disintegrate using $-\mathsf{d}(o,\cdot)$ and we see that this disintegration coincides with the one given Section 5.6.4 in the set E. From this fact, and the geometric properties of the space, we will conclude.

We recall some of the concepts introduced in Subsection 5.1.3, applied to the 1-Lipschitz function $-\mathsf{d}(o,\cdot)$. The set $\mathcal{D}$ where no non degenerate transport curve pass is empty, for we can connect o to any point with a geodesic. The set of branching points, $\mathcal{A}$, contains only o and elements of the boundary of the manifolds; this follows from the uniqueness of the geodesics. For this reason, the transport set $\mathcal{T}$ coincides with $X\backslash\{o\}$. Let $Q\subset\mathcal{T}$ be a measurable section and let $\mathfrak{Q}:\mathcal{T}\to Q$ be the quotient map; let $X_\alpha:=\mathfrak{Q}^{-1}(\alpha)$ be the disintegration rays and let $g:\mathrm{Dom}(g)\subset Q\times\mathbb{R}\to X$ be the standard parametrization. The map $t\mapsto g(\alpha,t)$ is the unitary speed parametrization of the geodesic connecting o to α and then maximally extended. Define $\mathfrak{q}:=\frac{1}{\mathfrak{m}(E)}\mathfrak{Q}_\#(\mathfrak{m}\llcorner_E)$. Using the $\mathsf{CD}(0,N)$ condition, one immediately sees that $\mathfrak{Q}_\#(\mathfrak{m})\ll\mathfrak{q}$.

We are in position to use Theorem 2.6, hence there exists a unique disintegration for the measure $\mathfrak{m}$

$$\mathfrak{m}=\int_Q \mathfrak{m}_\alpha\,\mathfrak{q}(d\alpha), \tag{5.50}$$

where the measures $\mathfrak{m}_\alpha$ are supported on X_α and the transport ray $(X_\alpha,F,\mathfrak{m}_\alpha)$ satisfy the oriented $\mathsf{CD}(0,N)$ condition. We denote by $h_\alpha:(0,|X_\alpha|)\to\mathbb{R}$ the density function satisfying $\mathfrak{m}_\alpha=(g(\alpha,\cdot))_\#(h_\alpha\mathcal{L}^1\llcorner_{(0,|X_\alpha|)})$.

The disintegration obtained in Section 5.6.4 (in particular Corollary (5.33)) and the disintegration given by (5.50) are bounded by the fact that there exists a (unique) measurable map $L:\mathrm{Dom}(L)\subset Q_\infty\to Q$ such that $\mathcal{D}(L)$ has full $\hat{\mathfrak{q}}$-measure in Q_∞ and it holds

$$L(\mathfrak{Q}_\infty(x))=\mathfrak{Q}(x),\quad \forall x\in B^+(o,\rho)\cap\mathcal{T}_\infty\cap\mathcal{T},\qquad\text{and}\qquad \mathfrak{q}=L_\#\hat{\mathfrak{q}}.$$

The presence of this maps permits to prove the following Proposition.

Proposition 5.39. *For q-a.e. $\alpha\in Q$, it holds that $|X_\alpha|\geq\rho$ and*

$$h_\alpha(r)=N\omega_N\mathsf{AVR}_X r^{N-1},\quad \forall r\in[0,\rho].$$

Proof. Using Equation (5.48), we deduce that for $\hat{\mathfrak{q}}$-a.e. $\alpha\in Q_\infty$, it holds that $g_\infty(\alpha,t)=g(L(\alpha),t)$, $\forall t\in(0,\min\{\rho,|X_\alpha|\})$. Since in the disintegration (5.45), all rays have length ρ, we deduce that $|X_\alpha|\geq\rho$. Moreover, we obtain $\hat{\mathfrak{m}}_{\alpha,\infty}=(\mathfrak{m}_{L(\alpha)})\llcorner_E$, concluding. □

Theorem 5.40. *For* q*-a.e.* $\alpha \in Q$*, it holds that* $|X_\alpha| = \infty$ *and*

$$h_\alpha(r) = N\omega_N \mathsf{AVR}_X r^{N-1}, \quad \forall r > 0.$$

The proof of the theorem above is the same as for Theorem 4.45, and thus it is omitted. The proof of Theorem 1.8 is therefore concluded.

Appendix A

The relative perimeter as a Borel measure

This appendix is devoted in proving that the relative perimeter can be extended uniquely to a Borel measure. Notice that in the result that follow, it is not needed the fact that $\Lambda_F < \infty$, the compactness of closed balls, and any convexity hypothesis. We follow the line traced in [66].

We recall the definition of relative perimeter: fixed a Borel set $E \subset \Omega$ of a Finsler manifold $(X, F, \mathfrak{m})$, and fixed $\Omega \subset X$, we define the perimeter of E relative to Ω as

$$\mathsf{P}(E; \Omega) := \inf\{\liminf_{n\to\infty} \int_\Omega |\partial u_n| \, d\mathfrak{m} : u_n \in \mathrm{Lip}_{loc}(\Omega) \text{ and } u_n \to \mathbf{1}_E \text{ in } L^1_{loc}(\Omega)\}.$$

The infimum is clearly realized by a certain sequence u_n. Using a truncation argument we may assume that u_n takes values in $[0, 1]$; moreover, by passing to subsequences, we may also assume that u_n converges also in the $\mathfrak{m}$-a.e. sense. If, in addition, Ω has finite measure, we may also assume (by dominated convergence theorem) that $u_n \to \mathbf{1}_E$ in $L^1(\Omega)$. These assumptions will always be assumed tacitly, when dealing with a sequence realizing the minimum in the definition of perimeter.

The slope satisfies the following calculus rules, in the $\mathfrak{m}$-a.e. sense

$$\begin{aligned}
&|\partial(f+g)| \le |\partial f| + |\partial g|, \qquad |\partial(-f)| \le \Lambda_F |\partial f|, \\
&|\partial(fg)| \le f|\partial g| + g|\partial f|, \qquad \text{if } f, g \ge 0, \\
&|\partial(fg)| \le \Lambda_F(|f||\partial g| + |g||\partial f|).
\end{aligned}$$

The proof is straightforward, once we know that $|\partial f|(x) = F^*(-df(x))$, for $\mathfrak{m}$-a.e. $x \in X$.

The next lemma permits us to join two Lipschitz functions defined on overlapping domains.

Lemma A.1. *Let $(X, F, \mathfrak{m})$ be a Finsler manifold. Let $N, M \subset X$ be two open sets such that*

$\partial M \cap \partial N = \emptyset$ and $\Lambda_{F,M\cap N} < \infty$. Then there exist an open set H such that $\overline{H} \subset N \cap M$ and a constant $c = c(M,N)$ such that the following happen. For all $u \in \mathrm{Lip}_{loc}(M), v \in \mathrm{Lip}_{loc}(N)$, for all $\epsilon > 0$, there exists a function $w \in \mathrm{Lip}_{loc}(M \cup N)$, such that

$$w = u \text{ in } M\backslash N, \qquad w = v \text{ in } N\backslash M, \qquad \min\{u,v\} \le w \le \max\{u,v\} \text{ in } M \cap N,$$

and it holds that

$$\int_{M\cup N} |\partial w| \, d\mathfrak{m} \le \int_M |\partial u| \, d\mathfrak{m} + \int_N |\partial v| \, d\mathfrak{m} + c \int_H |v - w| \, d\mathfrak{m} + \epsilon. \tag{A.1}$$

Proof. The hypothesis $\partial M \cap \partial N = \emptyset$ yields $\overline{M\backslash N} \cap \overline{N\backslash M} = \emptyset$. Define $d := \inf\{\mathsf{d}(x,y), x \in M\backslash N, y \in N\backslash M\}$. and consider the function $\phi : M \cup N \to \mathbb{R}$ defined as

$$\phi(x) := \max\left\{1 - \frac{3}{d} \sup_{y \in B^+(M\backslash N, d/3)} \mathsf{d}(y,x), 0\right\}.$$

The function ϕ is $(3/d)$-Lipschitz and attains the values 1 and 0 in a neighborhood of $M\backslash N$ and $N\backslash M$, respectively. Define $H = \phi^{-1}((0,1))$. Clearly it holds $\overline{H} \subset M \cap N$. Fix now $\epsilon > 0$ and find $k \in \mathbb{N}$ such that

$$\int_H (|\partial u| + |\partial v|) \, d\mathfrak{m} \le \Lambda_{F,M\cap N}^{-2} \epsilon k.$$

Define H_i and ψ_i $(i = 1 \dots k)$ as

$$H_i = \phi^{-1}\left(\left(\frac{i-1}{k}, \frac{i}{k}\right)\right), \qquad \psi_i = \min\left\{3\left(k\phi - i + \frac{2}{3}\right)^+, 1\right\}.$$

Clearly, ψ_i is $(9k/d)$-Lipschitz and it is locally constant outside H_i. Define $w_i = \psi_i u + (1-\psi_i)v$. We compute the slope of w_i in H_i using the calculus rules for the slope

$$\begin{aligned}
|\partial w_i| &= |\partial(v + \psi_i(u-v))| \le |\partial v| + |\partial(\psi_i(u-v))| \\
&\le |\partial v| + \Lambda_{F,M\cap N}|\partial \psi_i||u-v| + \Lambda_{F,M\cap N}|\partial(u-v)|\psi_i \\
&\le |\partial v| + \frac{9k}{d}\Lambda_{F,M\cap N}|u-v| + \Lambda_{F,M\cap N}^2(|\partial u| + |\partial v|).
\end{aligned}$$

Outside H_i the slope of w_i is either $|\partial u|$ or $|\partial v|$. Integrating over $M \cup N$, we obtain

$$\begin{aligned}
\int_{M\cup N} |\partial w_i| \, d\mathfrak{m} \le \int_M |\partial u| \, d\mathfrak{m} + \int_N |\partial v| \, d\mathfrak{m} + \frac{9k\Lambda_{F,M\cap N}}{d} \int_{H_i} |u-v| \, d\mathfrak{m} \\
+ \Lambda_{F,M\cap N}^2 \int_{H_i} (|\partial u| + |\partial v|) \, d\mathfrak{m}.
\end{aligned}$$

Summing over i and dividing by k, we deduce that

$$\frac{1}{k}\sum_{i=1}^{k}\int_{M\cup N}|\partial w_i|\,d\mathfrak{m} \le \int_M |\partial u|\,d\mathfrak{m} + \int_N |\partial v|\,d\mathfrak{m} + \frac{9\Lambda_{F,M\cap N}}{d}\int_H |u-v|\,d\mathfrak{m} + \epsilon,$$

hence there exists an index i_0 such that $w = w_{i_0}$ satisfies (A.1), with $c = 9\Lambda_{F,M\cap N}/d$. □

Theorem A.2. *Let* $(X, F, \mathfrak{m})$ *be a Finsler manifold, and let* $E \subset X$ *be a Borel set. Then it holds that*

1. *(Monotonicity)* $\mathsf{P}(E; A) \le \mathsf{P}(E; B)$*, if* $A \subset B$*,*
2. *(Superadditivity)* $\mathsf{P}(E; A \cup B) \ge \mathsf{P}(E; A) + \mathsf{P}(E; B)$*, if* $A \cap B = \emptyset$*,*
3. *(Inner regularity)* $\mathsf{P}(E; A) = \sup\{\mathsf{P}(E; B) : B \subset A$ *is open with compact closure in* $A\}$*,*
4. *(Subadditivity)* $\mathsf{P}(E; A \cup B) \le \mathsf{P}(E; A) + \mathsf{P}(E; B)$*,*

for all open sets A*,* B*.*

Moreover, if for any Borel set A *we define* $\mathsf{P}(E; A) := \inf\{\mathsf{P}(E; B) : B \supset A$ *is open*$\}$*, then the map* $A \mapsto \mathsf{P}(E; A)$ *is a Borel measure.*

Proof. The monotonicity and superadditivity are immediate consequences of the definition of perimeter. Let's consider the inner regularity. Fix an open set A, such that $\sup\{\mathsf{P}(E; B) : B \subset A\} < \infty$ (otherwise the proof is trivial). Find $(A_j)_j$ a sequence of open sets with compact closure such that $\overline{A_j} \subset A_{j+1}$, and $\bigcup_j A_j = A$, and define $C_j = A_{2j} \setminus \overline{A_{2j-3}}$. Since $C_{2j} \cap C_{2k} = \emptyset$, if $j \ne k$, by superadditivity, we have that $\sum_j \mathsf{P}(E; C_{2j}) < \infty$, and analogously $\sum_j \mathsf{P}(E; C_{2j+1}) < \infty$. Fix $\epsilon > 0$; there exists J, such that

$$\sum_{j=J}^{\infty} \mathsf{P}(E; C_j) \le \epsilon 2^{-4}.$$

Let $A := C_{J+2}$, $B' := A_{J+1}$, $F_h := C_{J+h-1}$, and $G_h := \bigcup_{i=1}^{h} F_i$; all these sets have compact closure, thus the irreversibility constant is finite on these sets.

By definition of perimeter, there exists a sequence $\psi_{m,h} \in \mathrm{Lip}_{loc}(F_h)$ such that $\psi_{m,h} \to \mathbf{1}_E$ in $L^1(F_h)$ and

$$\int_{F_h} |\partial \psi_{m,h}|\,d\mathfrak{m} \le \mathsf{P}(E; F_h) + 2^{-2-m-h}.$$

Notice that G_h has compact closure, hence $\Lambda_{F,G_n\cap F_{h+1}} < \infty$, thus we are in position to use Lemma A.1 applied to the sets G_h and F_{h+1}. Said Lemma gives a set $H_h \subset G_h \cap F_{h+1}$ and a constant c_h, that will be used soon. Clearly, up to passing to subsequences, we can assume that

$$c_h \int_{H_h} |\psi_{m,h+1} - \psi_{m,h}| \, d\mathfrak{m} \leq \epsilon 2^{-10-h}.$$

We define inductively on h a sequence of functions $u_{m,h} : G_h \to \mathbb{R}$ as follows. For the initial step, take $u_{m,1} = \psi_{m,1}$. For the inductive step, apply Lemma A.1 to the functions $u_{m,h}$ and $\psi_{m,h+1}$ obtaining a function $u_{m,h+1}$ such that

$$\begin{aligned} \int_{G_{h+1}} |\partial u_{m,h+1}| \, d\mathfrak{m} \leq \int_{G_h} |\partial u_{m,h}| \, d\mathfrak{m} + \int_{F_{h+1}} |\partial \psi_{m,h+1}| \, d\mathfrak{m} \\ + c_h \int_{H_h} |u_{m,h} - \psi_{m,h+1}| \, d\mathfrak{m} + \epsilon 2^{-10-h}. \end{aligned}$$

Since $u_{m,h+1} = \psi_{m,h+1}$ on $F_{h+1} \backslash G_h$ and $u_{m,h+1} = u_{m,h}$ on $G_h \backslash F_{h+1}$, we can deduce by induction that

$$\begin{aligned} \int_{G_{h+1}} |\partial u_{m,h+1}| \, d\mathfrak{m} &\leq \sum_{i=1}^{h+1} \int_{F_i} |\partial \psi_{m,i}| \, d\mathfrak{m} + \sum_{i=1}^{h} \left(c_i \int_{H_i} |\psi_{m,i} - \psi_{m,i+1}| \, d\mathfrak{m} + \epsilon 2^{-10-i} \right) \\ &\leq \sum_{i=1}^{h+1} \int_{F_i} |\partial \psi_{m,i}| \, d\mathfrak{m} + \epsilon 2^{-8} \leq \sum_{i=1}^{h+1} \mathsf{P}(E; F_i) + 2^{-m} + \epsilon 2^{-8}. \end{aligned}$$

We define $u_m(x) = u_{m,h}(x)$, whenever $x \in G_{h-1}$ (the definition is well-posed) and we integrate its slope

$$\begin{aligned} \int_{A \backslash \overline{B'}} |\partial u_m| \, d\mathfrak{m} &\leq \lim_{h \to \infty} \int_{G_h} |\partial u_{m,h-1}| \, d\mathfrak{m} \leq \sum_{h=1}^{\infty} \mathsf{P}(E; F_h) + 2^{-m} + \epsilon 2^{-8} \\ &= \sum_{h=1}^{\infty} \mathsf{P}(E; C_{J+h-1}) + 2^{-m} + \epsilon 2^{-8} \leq \epsilon 2^{-3} + 2^{-m}. \end{aligned}$$

The sequence u_m converges to $\mathbf{1}_E$ in $L^1(G_h)$ for all h, hence it converges in $L^1_{loc}(A \backslash \overline{B'})$.

We take now $v_m \in \mathrm{Lip}_{loc}(B)$ converging to $\mathbf{1}_E$ in $L^1(B)$ such that

$$\mathsf{P}(E; B) \leq \int_B |\partial v_m| \, d\mathfrak{m} + 2^{-m}.$$

We are in position to use Lemma A.1 again with the sets $A \backslash \overline{B'}$ and B and find an open set H and a constant c, such that for all m there exists a function $w_m : A \to \mathbb{R}$ such that

$$\int_A |\partial w_m| \, d\mathfrak{m} \leq \int_{A \backslash \overline{B'}} |\partial u_m| \, d\mathfrak{m} + \int_B |\partial v_m| \, d\mathfrak{m} + \int_H |u_m - v_m| \, d\mathfrak{m} + \epsilon 2^{-3}$$

$$
\begin{aligned}
&\leq 2^{-m} + \epsilon 2^{-3} + 2^{-m} + \int_H |u_m - \mathbf{1}_E| \, d\mathfrak{m} + \int_H |\mathbf{1}_E - v_m| \, d\mathfrak{m} + \epsilon 2^{-3} \\
&\leq 2^{1-m} + \epsilon + \int_{G_3} |u_m - \mathbf{1}_E| \, d\mathfrak{m} + \int_B |\mathbf{1}_E - v_m| \, d\mathfrak{m}.
\end{aligned}
$$

By taking the limit as $m \to \infty$, we deduce that $\mathsf{P}(E; A) \leq \mathsf{P}(E; B) + \epsilon$, concluding the proof of the inner regularity.

We prove now the subadditivity. Fix A and B two open sets and let A' and B' compactly included in A and B, respectively. We will prove that $\mathsf{P}(E; A' \cup B') \leq \mathsf{P}(E; A') + \mathsf{P}(E; B')$. From this fact and the inner regularity, the subadditivity will follow. Consider $u_n \in \mathrm{Lip}_{loc}(A')$ and $v_n \in \mathrm{Lip}_{loc}(B')$ converging in L^1 to $\mathbf{1}_E$, such that

$$
\int_{A'} |\partial u_n| \, d\mathfrak{m} \leq \mathsf{P}(E; A') + \frac{1}{n}, \qquad \text{and} \qquad \int_{B'} |\partial v_n| \, d\mathfrak{m} \leq \mathsf{P}(E; B') + \frac{1}{n}.
$$

Apply Lemma A.1 to the sets A' and B', and find $H \subset A' \cap B'$ and $c > 0$ such that, for all $n > 0$, there exists a function w_n satisfying

$$
\int_{A' \cup B'} |\partial_n w_n| \, d\mathfrak{m} \leq \int_{A'} |\partial_n u_n| \, d\mathfrak{m} + \int_{B'} |\partial_n v_n| \, d\mathfrak{m} + c \int_H |u_n - v_n| \, d\mathfrak{m} + \frac{1}{n}.
$$

We conclude by taking the limit as $n \to \infty$.

The fact that the relative perimeter can be extended to a Borel measure, is a consequence of a well-known Theorem of De Giorgi and Letta [36], that states that the conditions we have just proven are sufficient to obtain such a measure. □

Appendix B

Relaxation of the Minkowski content

In this appendix we give a proof of the fact that the perimeter can be seen as the l.s.c. relaxation on the Minkowski content. The proof follows the line of [6], with some extra attention to the irreversibility of the space. In the case $X = \mathbb{R}^d$, this was already proven in [33], with a different technique.

Proposition B.1. *Let $(X, F, \mathfrak{m})$ be a Finsler manifold and $E \subset X$ be a Borel set. Then it holds that*

$$\mathfrak{m}^+(E) \geq \mathsf{P}(E).$$

Proof. We consider the case $\mathfrak{m}^+(E) < \infty$ (the other is trivial). This implies that $\mathfrak{m}(\overline{E} \backslash E) = 0$, hence, without loss of generality, we may assume that E is closed. Consider the ϵ^{-1}-Lipschitz function

$$f_\epsilon(x) := \max \left\{ 1 - \frac{1}{\epsilon} \sup_{y \in B^+(E, \epsilon^2)} \mathsf{d}(y, x), 0 \right\}.$$

Clearly $f_\epsilon \to \mathbf{1}_E$ in $L^1(\mathfrak{m})$. In $B^+(E, \epsilon^2)$ it is equal to 1, hence $|\partial f_\epsilon|(x) = 0$, for all $x \in E$. Conversely, in $X \backslash B^+(E, \epsilon + \epsilon^2)$ it attains its minimum, hence $|\partial f_\epsilon|(x) = 0$ for all $x \in X \backslash B^+(E, \epsilon + \epsilon^2)$. We compute the integral

$$\begin{aligned} \int_X |\partial f_\epsilon|(x) \, \mathfrak{m}(dx) &= \int_{B^+(E, \epsilon + \epsilon^2) \backslash E} |\partial f_\epsilon|(x) \, \mathfrak{m}(dx) \leq \int_{B^+(E, \epsilon + \epsilon^2) \backslash E} \frac{1}{\epsilon} \, \mathfrak{m}(dx) \\ &= \frac{\mathfrak{m}(B^+(E, \epsilon + \epsilon^2) \backslash E)}{\epsilon} = (1 + \epsilon) \frac{\mathfrak{m}(B^+(E, \epsilon + \epsilon^2)) - \mathfrak{m}(E)}{\epsilon + \epsilon^2}. \end{aligned}$$

By taking the inferior limit as $\epsilon \to 0$, we conclude. □

The previous proposition guarantees that the l.s.c. envelope of the Minkowski content is not smaller than the perimeter. The reverse is a bit more difficult and, at a certain point, we will require that closed forward balls are compact.

We consider the "semigroup" $(T_t)_{t\geq 0}$ given by the formula

$$T_t f(x) := \sup_{y\in B^-(x,t)} f(y), \qquad T_0 f = f.$$

Note that the ball in the supremum is backward. The semigroup T_t enjoys the following immediate property.

Lemma B.2. *It holds that $T_{t+s}f \geq T_t(T_s f)$ and, if f is locally Lipschitz*

$$\limsup_{t\to 0^+} \frac{T_t f - f}{t} \leq |\partial f|, \qquad \mathfrak{m}\text{-a.e. in } X.$$

Proof. Regarding the first part, fix $x \in X$, and $\epsilon > 0$. By definition there exists y such that $\mathsf{d}(y,x) < t$ and $(T_t(T_s f))(x) \leq (T_s f)(y) + \epsilon$. Similarly, there exists z such that $\mathsf{d}(z,y) < s$ and $(T_s f)(y) \leq f(z) + \epsilon$. By triangular inequality, we have that $\mathsf{d}(z,x) < t+s$, thus

$$(T_{t+s}f)(x) \geq f(z) \geq (T_s f)(y) - \epsilon \geq (T_t(T_s f))(x) - 2\epsilon.$$

By arbitrariness of ϵ, we conclude the first part.

Regarding the second part, fix $x \in X$. By a direct computation we deduce

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{(T_t f)(x) - f(x)}{t} &= \inf_{r>0} \sup_{t\in(0,r)} \frac{\sup_{y\in B^-(x,t)} f(y) - f(x)}{t} \\
&= \inf_{r>0} \sup_{t\in(0,r)} \sup_{y\in B^-(x,t)} \frac{(f(y) - f(x))^+}{t} \\
&\leq \inf_{r>0} \sup_{t\in(0,r)} \sup_{y\in B^-(x,t)} \frac{(f(y) - f(x))^+}{\mathsf{d}(y,x)} \\
&= \limsup_{y\to x} \frac{(f(y) - f(x))^+}{\mathsf{d}(y,x)}.
\end{aligned}$$

If x is a point where f is differentiable, then the last term of the inequality above is equal to $F^*(-df) = |\partial f|(x)$, concluding the proof. □

We prove now a sort of coarea formula.

Lemma B.3. *Consider $(X, F, \mathfrak{m})$ a Finsler manifold. If $f : X \to [0, \mathbb{R})$ is a Lipschitz function with compact support, it holds that*

$$\int_0^\infty \mathfrak{m}^+(\{f \geq t\})\, dt \leq \int_X |\partial f|(x)\, \mathfrak{m}(dx).$$

Proof. In the first place, we notice that $\int_0^\infty \mathbf{1}_{\{f\geq t\}}(x)\, dt = f(x)$. Fix $t \geq 0$ and $h > 0$. If

$x \in B^+(\{f \geq t\}, h)$, then $(T_h f)(x) \geq t$, or in other words $\mathbf{1}_{B^+(\{f\geq t\},h)} \leq \mathbf{1}_{\{(T_h f)\geq t\}}$. By integrating over t we obtain

$$\int_0^\infty \mathbf{1}_{B^+(\{f\geq t\},h)}(x)\,dt \leq \int_0^\infty \mathbf{1}_{\{(T_h f)\geq t\}}(x)\,dt \leq (T_h f)(x).$$

By subtracting the first equation to the inequality above, integrating over x and using Fubini's theorem, we obtain

$$\int_0^\infty \frac{\mathfrak{m}(B^+(\{f \geq t\}, h)) - \mathfrak{m}(\{f \geq t\})}{h}\,dt \leq \int_X \frac{(T_h f)(x) - f(x)}{h}\,\mathfrak{m}(dx).$$

The set $\{f \geq 0\}$ is compact, hence for $h > 0$ sufficiently small $B^+(\{f \geq 0\}, h)$ is compact. Moreover, $\frac{T_h f - f}{h}$ is smaller than the Lipschitz constant of f, hence the integrand in the r.h.s. is dominated by an L^1 function. We take the inferior and superior limit in the l.h.s. and r.h.s. (respectively) of the inequality above; the Fatou's Lemma brings us to the conclusion. □

We now prove that we can, without loss of generality, assume that the functions of a sequence attaining the minimum in the definition of the perimeter have compact support.

Proposition B.4. *Let $(X, F, \mathfrak{m})$ be a Finsler manifold, such that all closed forward balls are compact and let $E \subset X$ be a Borel set with finite measure. Then there exists a sequence of Lipschitz functions with compact support, $(w_n)_n$, such that $w_n \to \mathbf{1}_E$ in L^1 and $\mathsf{P}(E) = \lim_{n\to\infty} \int_X |\partial w_n|\,d\mathfrak{m}$.*

Proof. Fix $E \subset X$ with finite measure, such that $\mathsf{P}(E) < \infty$ (otherwise the proof is trivial). Let $A_n := B^+(o, n)$ for some o, fixed once and for all. Up to taking subsequences, we can assume that $\mathfrak{m}(E \backslash A_n) \leq 2^{-n}$. Let ϕ_n be the 3-Lipschitz function given by

$$\phi_n(x) := \left(1 - 3 \inf_{y \in B^+(A_n, \frac{1}{3})} \mathsf{d}(y, x)\right)^+.$$

This function takes value 1 and 0 in a neighborhood of $\overline{A_n}$ and $X \backslash A_{n+1}$, respectively. By definition of perimeter, there exist a sequence $u_n : A_n \to [0, 1]$ of locally Lipschitz function such that

$$\mathsf{P}(A_n) \geq \int_{A_n} |\partial u_n|\,d\mathfrak{m} - 2^{-n}, \qquad \text{and} \qquad \|u_n - \mathbf{1}_E\|_{L^1(A_n)} \leq 2^{-n}.$$

Define the function $w_n := \phi_n u_{n+1}$. This function is Lipschitz with compact support. We compute its distance to $\mathbf{1}_E$

$$\int_X |w_n - \mathbf{1}_E|\, d\mathfrak{m} \leq \int_{A_n} |u_{n+1} - \mathbf{1}_E|\, d\mathfrak{m} + 2\, \mathfrak{m}(E \cap A_{n+1} \backslash A_n) + \mathfrak{m}(E \backslash A_{n+1}) \leq 2^{3-n},$$

thus $w_n \to \mathbf{1}_E$ in $L^1(X)$. Using the fact $|\partial w_n| \leq \phi_n |\partial u_{n+1}| + |\partial \phi_n| u_{n+1}$, we deduce

$$\begin{aligned}\int_X |\partial w_n|\, d\mathfrak{m} &\leq \int_{A_n} |\partial u_{n+1}|\, d\mathfrak{m} + \int_{A_{n+1}\backslash A_n} \phi_n |\partial u_{n+1}|\, d\mathfrak{m} + \int_{A_{n+1}\backslash A_n} u_{n+1}\, d\mathfrak{m} \\ &\leq \int_{A_{n+1}} |\partial u_{n+1}|\, d\mathfrak{m} + 2^{-1-n} \leq \mathsf{P}(E; A_n) + 2^{-n} \leq \mathsf{P}(E) + 2^{-n}. \end{aligned}$$

□

Theorem B.5. *Consider $(X, F, \mathfrak{m})$ a Finsler manifold, such that all closed forward balls are compact. Let $E \subset X$ be a Borel set with finite measure. Then there exists $(E_n)_n$, a sequence of compact sets, such that $\mathfrak{m}(E_n \triangle E) \to 0$ and*

$$\mathsf{P}(E) \geq \limsup_{n\to\infty} \mathfrak{m}^+(E_n).$$

Proof. Proposition B.4 guarantees the existence of a sequence $(f_n)_n$ of Lipschitz functions with compact support, such that $f_n \to \mathbf{1}_E$ in $L^1(\mathfrak{m})$ and

$$\mathsf{P}(E) = \lim_{n\to\infty} \int_X |\partial f_n|(x)\, \mathfrak{m}(dx).$$

Clearly we may assume that $0 \leq f_n \leq 1$. Fix $\epsilon \in (0, \frac{1}{2})$. By Lemma B.3, there exists $t_n^\epsilon \in (\epsilon, 1-\epsilon)$ such that

$$\mathfrak{m}^+(\{f_n \geq t_n^\epsilon\}) \leq \frac{1}{1-2\epsilon} \int_X |\partial f_n|(x)\, \mathfrak{m}(dx).$$

Define $E_n^\epsilon := \{f_n \geq t_n^\epsilon\}$. Since $\mathfrak{m}(E_n^\epsilon \triangle E) \to 0$, by taking an appropriate choice of $\epsilon = \epsilon_n$, we conclude. □

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