Research Article

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A new proof of compactness in G(S)BD

https://doi.org/10.1515/acv-2021-0041
Received April 26, 2021; revised November 17, 2021; accepted November 25, 2021

Abstract: We prove a compactness result in GBD which also provides a new proof of the compactness theorem in GSBD, due to Chambolle and Crismale. Our proof is based on a Fréchet–Kolmogorov compactness criterion and does not rely on Korn or Poincaré–Korn inequalities.

Keywords: Generalized functions of bounded deformation, compactness, brittle fracture

MSC 2010: 49J45, 74R10

Communicated by: Jan Kristensen

1 Introduction

In this paper, we prove a compactness result in GBD, which in particular provides an alternative proof of the compactness theorem in GSBD obtained by Chambolle and Crismale in [5, Theorem 1.1]. Referring to Section 2 for the notation used below, the theorem reads as follows.

Theorem 1.1. *Let $U \subseteq \mathbb{R}^n$ be an open bounded subset of $\mathbb{R}^n$ and let $u_k \in \mathrm{GBD}(U)$ be such that*

$$\sup_{k\in\mathbb{N}} \hat{\mu}_{u_k}(U) < +\infty. \tag{1.1}$$

Then there exists a subsequence, still denoted by u_k, such that the set

$$A := \{x \in U : |u_k(x)| \to +\infty \text{ as } k \to \infty\}$$

has finite perimeter, i.e. $u_k \to u$ a.e. in $U \setminus A$ for some function $u \in \mathrm{GBD}(U)$ with $u = 0$ in A. Furthermore,

$$\mathcal{H}^{n-1}(\partial^* A) \le \lim_{\sigma\to\infty} \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k}), \tag{1.2}$$

where $J^\sigma_{u_k} := \{x \in J_{u_k} : |[u_k(x)]| \ge \sigma\}$.

We notice that the main difference to [5] is that we do not request equi-integrability of the approximate symmetric gradient $e(u_k)$ and boundedness of the measure of the jump sets J_{u_k}, but only boundedness of $\hat{\mu}_{u_k}(U)$, which is the natural assumption for sequences in $\mathrm{GBD}(U)$. Hence, when passing to the limit, the absolutely continuous and the singular parts of $\hat{\mu}_{u_k}$ could interact. For this reason, it is not possible to get weak L^1-convergence of the approximate symmetric gradients or lower-semicontinuity of the measure of the jump.

Nevertheless, we are able to recover the lower-semicontinuity (1.2) for the set A where $|u_k| \to +\infty$. In particular, formula (1.2) highlights that the emergence of the singular set A results from an uncontrolled jump discontinuity along the sequence u_k. Hence, an equi-boundedness of the measure of the super-level

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sets $J^\sigma_{u_k}$, i.e.

$$\text{for every } \varepsilon > 0 \text{ there exists } \sigma_\varepsilon \in \mathbb{N} \text{ such that } \mathcal{H}^{n-1}(J^\sigma_{u_k}) < \varepsilon \text{ for } \sigma \geq \sigma_\varepsilon \text{ and } k \in \mathbb{N},$$

guarantees $\partial^* A = \emptyset$.

The GSBD-result [5, Theorem 1.1] is recovered by replacing (1.1) with

$$\sup_{k\in\mathbb{N}} \int_U \phi(|e(u_k)|)\, \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k}) < +\infty \tag{1.3}$$

for a positive function ϕ with superlinear growth at infinity. The novelty of our proof, presented in Section 3, concerns the compactness part of Theorem 1.1. It is based on the Fréchet–Kolmogorov criterion and makes no use of Korn or Poincaré–Korn-type of inequalities [3] (see also [2, 7, 8]), which are instead the key tools of [5]. The remaining lower-semicontinuity results of [5, Theorem 1.1] can be obtained by standard arguments.

2 Preliminaries and notation

We briefly recall here the notation used throughout the paper. For $d, k \in \mathbb{N}$, we denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the Lebesgue and the k-dimensional Hausdorff measure in $\mathbb{R}^d$, respectively. Given $F \subseteq \mathbb{R}^d$, we indicate with $\dim_{\mathcal{H}}(F)$ the Hausdorff dimension of F. For all compact subsets F_1 and F_2 of $\mathbb{R}^d$, $\mathrm{dist}_{\mathcal{H}}(F_1, F_2)$ stands for the Hausdorff distance between F_1 and F_2. We denote by $\mathbf{1}_E$ the characteristic function of a set $E \subseteq \mathbb{R}^d$. For every measurable set $\Omega \subseteq \mathbb{R}^d$ and every measurable function $u\colon \Omega \to \mathbb{R}^d$, we further set J_u to be the set of approximate discontinuity points of u and

$$J^\sigma_u := \{x \in J_u : |[u](x)| \geq \sigma\}, \quad \sigma > 0,$$

where $[u](x) := u^+(x) - u^-(x)$ with $u^\pm(x)$ being the unilateral approximate limit of u at x.

For $m, \ell \in \mathbb{N}$ we denote by $\mathbb{M}^{m\times\ell}$ the space of $m \times \ell$ matrices with real coefficients, and set $\mathbb{M}^m := \mathbb{M}^{m\times m}$. The symbol $\mathbb{M}^m_{\mathrm{sym}}$ (resp. $\mathbb{M}^m_{\mathrm{skw}}$) indicates the subspace of $\mathbb{M}^m$ of squared symmetric (resp. skew-symmetric) matrices of order m. We further denote by $\mathrm{SO}(m)$ the set of rotation matrices.

Let us now fix $n \in \mathbb{N} \setminus \{0\}$. For every $\xi \in \mathbb{S}^{n-1}$, π_ξ stands for the projection over the subspace $\xi^\perp$ orthogonal to ξ. For every measurable set $V \subseteq \mathbb{R}^n$, every $\xi \in \mathbb{S}^{n-1}$, and every $y \in \mathbb{R}^n$, we set

$$\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\}, \quad V^\xi_y := \{t \in \mathbb{R} : y + t\xi \in V\}.$$

For $V \subseteq \mathbb{R}^n$ measurable, $\xi \in \mathbb{S}^{n-1}$, and $y \in \mathbb{R}^n$, we define

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi \quad \text{for every } t \in V^\xi_y.$$

For every open bounded subset U of $\mathbb{R}^n$, the space $\mathrm{GBD}(U)$ of generalized functions of bounded deformation [6] is defined as the set of measurable functions $u\colon U \to \mathbb{R}^n$ which admit a positive Radon measure $\lambda \in \mathcal{M}^+_b(U)$ such that for every $\xi \in \mathbb{S}^{n-1}$ one of the following two equivalent conditions is satisfied [6, Theorem 3.5]:

- For every $\theta \in C^1(\mathbb{R}; [-\frac{1}{2}; \frac{1}{2}])$ such that $0 \leq \theta' \leq 1$, the partial derivative $D_\xi(\theta(u\cdot\xi))$ is a Radon measure in U and
$$|D_\xi(\theta(u \cdot \xi))|(B) \leq \lambda(B)$$
for every Borel subset B of U.
- For $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$, the function $\hat{u}^\xi_y$ belongs to $\mathrm{BV}_{\mathrm{loc}}(U^\xi_y)$ and
$$\int_{\Pi^\xi} |(D\hat{u}^\xi_y)|(B^\xi_y \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\hat{u}^\xi_y})\, \mathrm{d}\mathcal{H}^{n-1}(y) \leq \lambda(B) \tag{2.1}$$
for every Borel subset B of U.

A function u belongs to GSBD(U) if $\hat{u}^\xi_y \in SBV_{loc}(U^\xi_y)$ and (2.1) holds. Every function $u \in GBD(U)$ admits an approximate symmetric gradient $e(u) \in L^1(U;\mathbb{M}^n_{sym})$. The jump set J_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable with approximate unit normal vector ν_u. We will also use measures $\hat{\mu}^\xi, \hat{\mu}_u \in \mathcal{M}^+_b(U)$ defined in [6, Definitions 4.10 and 4.16] for $u \in GBD(U)$ and $\xi \in \mathbb{S}^{n-1}$. We further refer to [6] for an exhaustive discussion on the fine properties of functions in GBD(U).

3 Proof of Theorem 1.1

This section is devoted to the presentation of an alternative proof of Theorem 1.1, based on the Fréchet–Kolmogorov compactness criterion. We start by giving two definitions.

Definition 3.1. Let $\Xi = \{\xi_1, \dots, \xi_n\}$ denote an orthonormal basis of $\mathbb{R}^n$. We define

$$S_{\Xi,0} := \bigcup_{\xi\in\Xi}\{x \in \mathbb{R}^n : |x| = 1,\ x \in \Pi^\xi\}.$$

Given $\delta > 0$, we define the δ-neighborhood of $S_{\Xi,0}$ by

$$S_{\Xi,\delta} := \{x \in \mathbb{R}^n : |x| = 1,\ \mathrm{dist}(x, S_{\Xi,0}) < \delta\}.$$

Definition 3.2. In order to simplify the notation, given a family $\mathcal{K}$ and a positive natural number m, we denote by $\mathcal{K}_m$ the set consisting of all subsets of $\mathcal{K}$ containing exactly m-elements of $\mathcal{K}$, i.e.

$$\mathcal{K}_m := \{\mathcal{Z} \in \mathrm{P}(\mathcal{K}) : \#\mathcal{Z} = m\}.$$

In order to prove Theorem 1.1, we need the following two lemmas, which allow us to construct a suitable orthonormal basis of $\mathbb{R}^n$ that will be used to test the Fréchet–Kolmogorov compactness criterium.

Lemma 3.3. *Let $M \in \mathbb{N}$ be such that $M \geq n$ and consider a family $\mathcal{K} := \{\Xi_1, \dots, \Xi_M\}$ of orthonormal bases of $\mathbb{R}^n$ such that for every $\mathcal{Z} \in \mathcal{K}_n$,*

$$\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset. \tag{3.1}$$

Then there exists a further orthonormal basis $\Sigma = \{\xi_1, \dots, \xi_n\}$ such that for every $\mathcal{Z} \in \mathcal{K}_{n-1}$,

$$S_{\Sigma,0} \cap \bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset. \tag{3.2}$$

Proof. First of all, notice that whenever $\mathcal{Z} \in \mathcal{K}_n$ is such that

$$\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset,$$

then we have

$$\mathcal{H}^0\Big(\bigcap_{\Xi\in\mathcal{X}} S_{\Xi,0}\Big) < +\infty \quad \text{for every } \mathcal{X} \in \mathcal{Z}_{n-1}. \tag{3.3}$$

Indeed, let us suppose by contradiction that (3.3) does not hold for some $\mathcal{X} \in \mathcal{Z}_{n-1}$. Since for $\Xi \in \mathcal{X}$ we have that each $S_{\Xi,0}$ is a finite union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ intersected with $\mathbb{S}^{n-1}$, the equality

$$\mathcal{H}^0\Big(\bigcap_{\Xi\in\mathcal{X}} S_{\Xi,0}\Big) = +\infty$$

implies that

$$\dim_{\mathcal{H}}\Big(\bigcap_{\Xi\in\mathcal{X}} S_{\Xi,0}\Big) \geq 1.$$

As a consequence, we get

$$\dim_{\mathcal{H}}\Big(\bigcap_{\Xi\in\mathcal{X}} \bigcup_{\xi\in\Xi}\{\xi^\perp\}\Big) \geq 2.$$

Hence, if we denote by $\overline{\Xi}$ the basis contained in $\mathcal{Z} \setminus \mathcal{X}$, then, by using Grassmann's formula,

$$\dim(V) + \dim(W) - \dim(V \cap W) = \dim(V + W) \le n,$$

which is valid for each couple V, W of vector subspaces of $\mathbb{R}^n$, we deduce

$$\dim_{\mathcal{H}}\Big(\bigcup_{\xi\in\overline{\Xi}}\{\xi^\perp\} \cap \bigcap_{\Xi\in\mathcal{X}}\bigcup_{\xi\in\Xi}\{\xi^\perp\}\Big) \ge 1.$$

Hence,

$$\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} \ne \emptyset,$$

which is a contradiction to the assumption (3.1).

Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and let $\mathrm{SO}(n)$ be the group of special orthogonal matrices. It can be endowed with the structure of an $(\frac{n^2-n}{2})$-dimensional submanifold of $\mathbb{R}^{n^2}$. We can identify an element $O \in \mathrm{SO}(n)$ with an $(n \times n)$-matrix whose columns are the vectors of an orthonormal basis Ξ written with respect to $\{e_1, \dots, e_n\}$ and vice versa.

In order to show the existence of Σ satisfying (3.2), we prove the following stronger condition: given $\mathcal{Z} \in \mathcal{K}_{n-1}$, for $\mathcal{H}^{(n^2-n)/2}$-a.e. choice of Σ we have that

$$S_{\Sigma,0} \cap \bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset. \tag{3.4}$$

This easily implies the existence of an orthonormal basis Σ satisfying (3.2), as the choice of $\mathcal{Z} \in \mathcal{K}_{n-1}$ is finite. To show (3.4), for every $i \in \{1, \dots, n\}$ let us define the smooth map

$$\Lambda_i\colon\ \mathrm{SO}(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\} \to \mathbb{S}^{n-1}$$

by

$$\Lambda_i(\Sigma, y) := \sum_{j<i} y_j \xi_j + \sum_{j>i} y_{j-1}\xi_j,$$

where ξ_j denotes the j-th column vector of the matrix representing Σ. In order to show (3.4), we claim that it is enough to prove that for every $x \in \mathbb{S}^{n-1}$ we have

$$\mathcal{H}^{(n^2-n)/2}(\pi_{\mathrm{SO}(n)}(\{\Lambda_i^{-1}(x)\})) = 0 \quad \text{for } i \in \{1, \dots, n\}, \tag{3.5}$$

where

$$\pi_{\mathrm{SO}(n)}\colon\ \mathrm{SO}(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\} \to \mathrm{SO}(n)$$

is the canonical projection map. Indeed, if Σ does not belong to $\pi_{\mathrm{SO}(n)}(\{\Lambda_i^{-1}(x)\})$ for every $x \in \bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0}$ and for every $i \in \{1, \dots, n\}$, then, by using the definition of the map Λ_i, we deduce immediately that Σ satisfies

$$S_{\Sigma,0} \cap \bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset.$$

Therefore, if (3.5) holds, then the set (remember that $\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0}$ is a discrete set)

$$\bigcup_{i=1}^{n} \bigcup_{x\in\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0}} \pi_{\mathrm{SO}(n)}(\{\Lambda_i^{-1}(x)\})$$

is of $\mathcal{H}^{(n^2-n)/2}$-measure zero and (3.4) holds true. Thus, $\mathcal{H}^{(n^2-n)/2}$-a.e. Σ satisfies (3.2).

To prove (3.5) it is enough to show that the differential of Λ_i has full rank at every point

$$z \in \mathrm{SO}(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\}.$$

Indeed, this implies that $\Lambda_i^{-1}(x)$ is an $(\frac{n^2-n-2}{2})$-dimensional submanifold for every $x \in \mathbb{S}^{n-1}$, which ensures the validity of (3.5) since

$$\begin{aligned}
&\#(\{\pi_{\mathrm{SO}(n)}^{-1}(\Xi)\} \cap \{\Lambda_i^{-1}(x)\}) = 1, && x \in \mathbb{S}^{n-1},\\
&\frac{n^2-n-2}{2} < \frac{n^2-n}{2} = \dim_{\mathcal{H}}(\mathrm{SO}(n)), && n \ge 2.
\end{aligned}$$

Notice that Λ_i is the restriction to $SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\}$ of the map $\tilde{\Lambda}_i \colon \mathbb{M}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by

$$\tilde{\Lambda}_i(\Theta, y) := \sum_{j<i} y_j \theta_j + \sum_{j>i} y_{j-1} \theta_j,$$

where θ_j is the j-th column vector of the matrix $\Theta \in \mathbb{M}^n$. To show that the differential of Λ_i has full rank everywhere, it is enough to check that for every $z \in SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\}$ the differential of $\tilde{\Lambda}_i$ restricted to $\mathrm{Tan}(SO(n) \times \{y \in \mathbb{R}^{n-1} : |y| = 1\}, z)$ has rank equal to $n-1$. By using the relation

$$\tilde{\Lambda}_i(M\Theta, y) = M\tilde{\Lambda}_i(\Theta, y),$$

valid for every $M \in \mathbb{M}^n$, we can reduce ourselves to the case $z = (\mathrm{I}, \overline{y})$, where I denotes the identity matrix and $\overline{y} \in \mathbb{R}^{n-1}$ is such that $|\overline{y}| = 1$. It is well known that

$$\mathrm{Tan}(SO(n) \times \{\zeta \in \mathbb{R}^{n-1} : |\zeta| = 1\}, z) \cong \mathbb{M}^n_{\mathrm{skw}} \times \mathrm{Tan}(\{\zeta \in \mathbb{R}^{n-1} : |\zeta| = 1\}, \overline{y}),$$

where $\mathbb{M}^n_{\mathrm{skw}}$ denotes the space of skew symmetric matrices. Using that $\mathbb{R}^{n^2+n-1} \cong \mathbb{M}^n \times \mathbb{R}^{n-1}$, we identify a point $Z \in \mathbb{R}^{n^2+n-1}$ as

$$Z = ((x^i_j)^n_{i,j=1}, y_1, \dots, y_{n-1}).$$

A direct computation shows that the differential of Λ_i at the point $(\mathrm{I}, \overline{y})$ acting on the vector Z is given by

$$d\tilde{\Lambda}_i(\mathrm{I}, \overline{y})[Z] = \sum_{l=1}^{n} \sum_{j<i} (x^j_l \overline{y}_j + \delta_{jl} y_j) e_l + \sum_{j>i} (x^j_l \overline{y}_{j-1} + \delta_{jl} y_{j-1}) e_l.$$

It is better to introduce the matrix $P_i \in \mathbb{M}^{n \times (n-1)}$ defined by

$$(P_i)^m_k := \begin{cases} \delta_{km} & \text{if } 1 \le m < i, \\ \delta_{k-1m} & \text{if } i \le m \le n-1. \end{cases}$$

Roughly speaking, given $X \in \mathbb{M}^{l \times n}$, the product XP_i is the matrix in $\mathbb{M}^{l \times (n-1)}$ obtained by removing from X the i-th column, while given $Y \in \mathbb{M}^{(n-1) \times l}$, the product $P_i Y$ is the matrix in $\mathbb{M}^{n \times l}$ obtained by adding a new row made of zero entries at the i-th position. With this definition, the linear map $d\Lambda_i(\mathrm{I}, \overline{y})(\cdot)$ can be rewritten more compactly as

$$d\Lambda_i(\mathrm{I}, \overline{y})[(X, y)] = XP_i\overline{y} + P_i y, \quad X \in \mathbb{M}^n_{\mathrm{skw}},\ y \in \mathrm{Tan}(\{\zeta \in \mathbb{R}^{n-1} : |\zeta| = 1\}, \overline{y}).$$

Given $O \in SO(n-1)$ such that $O\tilde{e}_1 = \overline{y}$ (where $\{\tilde{e}_1, \dots, \tilde{e}_{n-1}\}$ denotes the reference orthonormal basis of $\mathbb{R}^{n-1}$), we can rewrite the system as

$$d\Lambda_i(\mathrm{I}, \overline{y})[(X, y)] = XP_i O\tilde{e}_1 + P_i y, \quad X \in \mathbb{M}^n_{\mathrm{skw}},\ y \in \mathrm{Tan}(\{\zeta \in \mathbb{R}^{n-1} : |\zeta| = 1\}, \overline{y}).$$

Hence, by the well-known relation

$$\dim(V) - \dim(\mathrm{Im}[\alpha]) = \dim(\ker[\alpha]), \tag{3.6}$$

valid for every linear map $\alpha \colon V \to W$ and all finite-dimensional vector spaces V and W, if we want to prove that $d\Lambda_i(\mathrm{I}, \overline{y})$ has full rank, i.e.

$$\dim(\mathrm{Im}[(\cdot)P_i O\tilde{e}_1 + P_i(\cdot)]) = n-1,$$

since

$$n-1 \ge \dim(\mathrm{Im}[(\cdot)P_i O\tilde{e}_1 + P_i(\cdot)]) \ge \dim(\mathrm{Im}[(\cdot)P_i O\tilde{e}_1])$$

(where the first inequality comes from $\mathrm{Im}[d\Lambda_i(\mathrm{I}, \overline{y})] \subset \mathrm{Tan}(\mathbb{S}^{n-1}, \Lambda_i(\mathrm{I}, \overline{y}))$), it is enough to show that

$$\dim(\mathrm{Im}[(\cdot)P_i O\tilde{e}_1]) = n-1. \tag{3.7}$$

Again by relation (3.6), we can reduce ourselves to find the dimension of the kernel of the map

$$\mathbb{M}^n_{\mathrm{skw}} \ni X \mapsto XP_i O\tilde{e}_1.$$

But this dimension can easily be computed to be

$$\dim(\ker[(\cdot)P_i O\tilde{e}_1]) = \sum_{k=1}^{n-2} k = \frac{(n-2)(n-1)}{2},$$

which immediately implies (3.7). □

Remark 3.4. By a standard argument from linear algebra, it is possible to construct n orthonormal bases of $\mathbb{R}^n$, say $\mathcal{K} = \{\Xi_1, \dots, \Xi_n\}$, satisfying

$$\bigcap_{\Xi\in\mathcal{K}} S_{\Xi,0} = \emptyset.$$

Moreover, given $U \subset SO(n)$ open, Ξ_i can be chosen in such a way that

$$\Xi_i \in U, \quad i \in \{1, \dots, n\}.$$

Therefore, Lemma 3.3, and in particular condition (3.4), tells us that for every $M \in \mathbb{N}$ ($M \geq n$) we can always find a family of orthonormal bases of $\mathbb{R}^n$, say $\mathcal{K} = \{\Xi_1, \dots, \Xi_M\}$, satisfying (3.1) and

$$\Xi_i \in U, \quad i \in \{1, \dots, M\}.$$

Lemma 3.5. *Let $A \subset \mathbb{R}^n$ be a measurable set with $\mathcal{L}^n(A) < \infty$, let $(B_k)_{k=1}^{\infty}$ be measurable subsets of A, and let $(v_k)_{k=1}^{\infty}$ be measurable functions $v_k : B_k \to \mathbb{S}^{n-1}$. Then, given a sequence $\epsilon_h \searrow 0$, there exist a sequence $\delta_h \searrow 0$ with $\delta_h > 0$, a map $\phi : \mathbb{N} \to \mathbb{N}$, and an orthonormal basis Ξ of $\mathbb{R}^n$ such that, up to passing through a subsequence on k,*

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi,\delta_h})) \leq \epsilon_h \quad \textit{for every } k \geq \phi(h).$$

Proof. We claim that for every natural number $N \geq n$, for every $j \in \{0, 1, \dots, n-1\}$, for every $\varepsilon > 0$, and for every open set $U \subset SO(n)$ there exist $\delta > 0$ and a family of orthonormal bases $\mathcal{K} := \{\Xi_1, \dots, \Xi_N\} \subseteq U$ such that, up to subsequences on k, we have

$$\mathcal{L}^n\Big(v_k^{-1}\Big(\Big\{x \in \bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,\delta} : \mathcal{Z} \in \mathcal{K}_{n-j}\Big\}\Big)\Big) \leq \varepsilon, \quad k = 1, 2, \dots, \tag{3.8}$$

$$\Xi \in U, \quad \Xi \in \mathcal{K}. \tag{3.9}$$

Clearly, the pair $(\delta, \mathcal{K})$ depends on (N, j, ε), but we do not emphasize this fact. We proceed by induction on j. The case $j = 0$: given $N \in \mathbb{N}$, $\varepsilon > 0$, and any open set $U \subset y(n)$, we can make use of Lemma 3.3 and Remark 3.4 to find N orthonormal bases $\mathcal{K} = \{\Xi_1, \dots, \Xi_N\} \subseteq U$ such that

$$\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,0} = \emptyset \quad \text{for } \mathcal{Z} \in \mathcal{K}_n.$$

Since the $S_{\Xi,0}$ are closed sets, there exists $\delta > 0$ such that

$$\bigcap_{\Xi\in\mathcal{Z}} S_{\Xi,\delta} = \emptyset \quad \text{for } \mathcal{Z} \in \mathcal{K}_n.$$

Hence, (3.8) is satisfied with $j = 0$ and (3.9) holds true.

We want to prove the same for $0 < j \leq n-1$. For this purpose, we fix a natural number $M \geq n$, a parameter $\varepsilon > 0$, and an open set $U \subset SO(n)$. By using the induction hypothesis, we may suppose that (3.8) and (3.9) hold true for $j-1$. This means that, given $N \geq n$ and $\tilde{\varepsilon} > 0$ (to be chosen later), we find $\delta > 0$ and orthonormal bases $\mathcal{K} = \{\Xi_1, \dots, \Xi_N\}$ such that (3.8) and (3.9) hold true for $j-1$. Choose $\mathcal{Z} \in \mathcal{K}_M$ and consider the set

$$S^{n-j}_{\mathcal{Z},\delta} := \bigcup_{q\in\mathcal{Z}_{n-j}} \bigcap_{\Xi\in q} S_{\Xi,\delta}, \tag{3.10}$$

which is the union of all possible $(n-j)$-intersections of sets of the form $S_{\Xi,\delta}$ for $\Xi \in \mathcal{Z}$.

We recall the following identity valid for any finite family of subsets of A, say $(B)_{l=1}^{L}$:

$$\mathcal{L}^n\Big(\bigcup_{l=1}^{L} B_l\Big) = \sum_{l=1}^{L} \mathcal{L}^n(B_l) - \sum_{l_1<l_2}^{L} \mathcal{L}^n(B_{l_1} \cap B_{l_2}) + \dots + (-1)^{L-1}\mathcal{L}^n\Big(\bigcap_{l=1}^{L} B_l\Big). \tag{3.11}$$

Now we partition $\mathcal{K}$ into $\frac{N}{M}$ disjoint subsets (without loss of generality, we may choose N to be an integer multiple of M) each of which belongs to $\mathcal{K}_M$. We call this partition $\mathcal{P}$. By construction, any l-intersection of sets of the form $S^{n-j}_{\mathcal{Z},\delta}$ with $\mathcal{Z} \in \mathcal{P}$ can be written as the union of $\binom{M}{n-j}^l$ sets each of which, thanks to the fact that (we use that $\mathcal{P}$ is a partition)

$$Z_1, Z_2 \in \mathcal{P} \quad \text{implies} \quad Z_1 \cap Z_2 = \emptyset,$$

is the intersection of at least $n-(j-1)$ different sets of the form $S_{\Xi,\delta}$ with $\Xi \in \mathcal{K}$. Taking this last fact into account, if we replace the sets B_j by $v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})$ and $L = \frac{N}{M}$ in identity (3.11), we obtain

$$\mathcal{L}^n\Big(\bigcup_{\mathcal{Z}\in\mathcal{P}} v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})\Big) \geq \sum_{\mathcal{Z}\in\mathcal{P}} \mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) - \sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon}, \quad k = 1, 2, \ldots, \tag{3.12}$$

where we have used the inductive hypothesis (3.8) for $j-1$ to estimate the remaining terms in the right-hand side of (3.11).

Now suppose that for every $\mathcal{Z} \in \mathcal{K}_M$ it holds true for some k that

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) > \varepsilon. \tag{3.13}$$

Then inequality (3.12) implies

$$\mathcal{L}^n\Big(\bigcup_{\mathcal{Z}\in\mathcal{P}} v_k^{-1}(S_{\mathcal{Z},\delta})\Big) > \frac{N}{M}\varepsilon - \sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon}. \tag{3.14}$$

Therefore, if we choose N sufficiently large in such a way that

$$\frac{N}{M}\varepsilon \geq 2\mathcal{L}^n(A),$$

and $\tilde{\varepsilon} > 0$ such that

$$\sum_{l=2}^{N/M} \binom{M}{n-j}^l \tilde{\varepsilon} < \mathcal{L}^n(A),$$

then (3.14) implies that for every k there exists $\mathcal{Z}^k \in \mathcal{P}$ for which (3.13) does not hold, i.e.

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z}^k,\delta})) \leq \varepsilon, \quad k = 1, 2, \ldots,$$

where we have used that B_k, the domain of v_k, is contained in A. Since $\mathcal{P}$ is a finite family, we may suppose that, up to subsequences on k, we find a common $\mathcal{Z} \in \mathcal{P}$ for which

$$\mathcal{L}^n(v_k^{-1}(S^{n-j}_{\mathcal{Z},\delta})) \leq \varepsilon, \quad k = 1, 2, \ldots. \tag{3.15}$$

Taking into account the definition of $S^{n-j}_{\mathcal{Z},\delta}$ (see (3.10)), formula (3.15) gives our claim for j. Finally, by induction, this implies the validity of our claim for every $j \in \{0, \ldots, n\}$.

Now we prove the lemma. For $j = n-1$, the claim says in particular that we find an orthonormal basis Ξ_0 and $\delta_0 > 0$ such that, up to passing to a subsequence on k, we have

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_0,\delta_0})) \leq \epsilon_0, \quad k = 1, 2, \ldots.$$

Notice that, by using a continuity argument, we find a neighborhood U_0 of Ξ_0 in $\mathrm{SO}(n)$ such that

$$S_{\Xi,\delta_0/2} \Subset S_{\Xi_0,\delta}, \quad \Xi \in U_0.$$

By applying again the claim, we find an orthonormal basis $\Xi_1 \in U_0$ and $\tilde{\delta}_1 > 0$ such that, up to passing to a further subsequence on k, we have

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_1,\tilde{\delta}_1})) \leq \epsilon_1, \quad k = 1, 2, \ldots.$$

Hence if we set $\delta_1 := \min\{\tilde{\delta}_1, \delta_0/2\}$, we obtain as well

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi_1,\delta_1})) \leq \epsilon_1, \quad k = 1, 2, \ldots,$$
$$S_{\Xi_1,\delta_1} \Subset S_{\Xi_0,\delta_0}.$$

Proceeding again by induction, we find for every $h = 1, 2, \dots$ an orthonormal basis Ξ_h, $\delta_h > 0$, and a subsequence $(k^h_\ell)_\ell$ such that

$$\mathcal{L}^n(v^{-1}_{k^h_\ell}(S_{\Xi_h,\delta_h})) \le \epsilon_h, \quad \ell = 1, 2, \dots,$$
$$S_{\Xi_h,\delta_h} \Subset S_{\Xi_{h-1},\delta_{h-1}},$$
$$(k^h_\ell)_\ell \subset (k^{h-1}_\ell)_\ell.$$

If we denote with abuse of notation the diagonal sequence $(k^h_h)_h$ simply as k, then we can find a map $\phi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathcal{L}^n(v^{-1}_k(S_{\Xi_h,\delta_h})) \le \epsilon_h, \quad k \ge \phi(h) \tag{3.16}$$
$$S_{\Xi_h,\delta_h} \Subset S_{\Xi_{h-1},\delta_{h-1}}. \tag{3.17}$$

Since the family $(S_{\Xi_h,0})_h$ is made of compact subsets of $\mathbb{S}^{n-1}$, then it is relatively compact with respect to the Hausdorff distance. This means that, up to a subsequence on h, we find an orthonormal basis Ξ such that

$$\lim_{h\to\infty} \mathrm{dist}_{\mathcal{H}}(S_{\Xi_h,0}, S_{\Xi,0}) = 0.$$

By using (3.17) and the fact that S_{Ξ_h,δ_h} are relatively open subsets of $\mathbb{S}^{n-1}$, this last convergence tells us that for every h the compact inclusion $S_{\Xi,0} \Subset S_{\Xi_h,\delta_h}$ holds true. But this implies that, up to defining suitable $\delta'_h > 0$ with $\delta'_h \le \delta_h$, we can write

$$S_{\Xi,\delta'_h} \Subset S_{\Xi_h,\delta_h}, \quad h \in \mathbb{N}.$$

Finally, with abuse of notation, we set $\delta_h := \delta'_h$ for every h. Then (3.16) implies

$$\mathcal{L}^n(v^{-1}_k(S_{\Xi,\delta_h})) \le \epsilon_h, \quad k \ge \phi(h),\ h \in \mathbb{N}.$$

This gives the desired result. □

Remark 3.6. Given $U \subset \mathbb{R}^n$, $u \in GBD(U)$, and $\sigma \ge 1$, we have that

$$\mathcal{H}^{n-1}(J^\sigma_u) \le 4n\hat{\mu}_u(U).$$

Indeed, given $\epsilon > 0$, one can consider a partition of $\mathbb{S}^{n-1}$ into a finite family of measurable sets $\{S_1, \dots, S_M\}$ such that for every $m = 1, \dots, M$ there exists an orthonormal basis $\Xi_m = \{\xi^m_1, \dots, \xi^m_n\}$ with $\xi \cdot \xi^m_i \ge \frac{1}{4}$ for every $\xi \in S_m$ and for every $i, j \in \{1, \dots, n\}$ and $m \in \{1, \dots, M\}$. Consider then the partition of J^σ_u given by $\{B_1, \dots, B_M\}$, where

$$B_m := \{x \in J^\sigma_u : [u(x)]/|[u(x)]| \in S_m\}.$$

We then have

$$\begin{aligned}
\mathcal{H}^{n-1}(J^\sigma_u) &\le \sum_{m=1}^{M} \sum_{\xi\in\Xi_m} \int_{B_m} |v_u \cdot \xi| \,d\mathcal{H}^{n-1} \\
&= \sum_{m=1}^{M} \sum_{\xi\in\Xi_m} \int_{\Pi_\xi} \mathcal{H}^0((B_m)^\xi_y) \,d\mathcal{H}^{n-1}(y) \\
&= \sum_{m=1}^{M} \sum_{\xi\in\Xi_m} \int_{\Pi_\xi} \mathcal{H}^0(J^1_{4\hat{u}^\xi_y} \cap (B_m)^\xi_y) \,d\mathcal{H}^{n-1}(y) \\
&= \sum_{m=1}^{M} \sum_{\xi\in\Xi_m} \hat{\mu}^\xi_{4u}(B_m) \\
&\le n \sum_{m=1}^{M} \hat{\mu}_{4u}(B_m) \\
&\le n\hat{\mu}_{4u}(U) \\
&\le 4n\mu_u(U),
\end{aligned}$$

where we have used that $|[4\hat{u}^\xi_y](t)| \ge 1$ for every $t \in J_{4\hat{u}^\xi_y} \cap (B_m)^\xi_y$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ with $\xi \in \Xi_m$.

Remark 3.7. Let $U \subset \mathbb{R}^n$ and $u \in \mathrm{GBD}(U)$. Given $\xi \in \mathbb{S}^{n-1}$ and $\sigma > 1$, if we introduce the map $\hat{\mu}^\xi_\sigma \colon \mathcal{B}(U) \to \overline{\mathbb{R}}$ as

$$\hat{\mu}^\xi_\sigma(B) := \int_{\Pi^\xi} |D\hat{u}^\xi_y|(B^\xi_y \setminus J^\sigma_{\hat{u}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^\sigma_{\hat{u}^\xi_y}) \,\mathrm{d}\mathcal{H}^{n-1}(y), \quad B \in \mathcal{B}(U),$$

then we have $\hat{\mu}^\xi_\sigma \in \mathcal{M}^+_b(U)$. More precisely, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$|D\hat{u}^\xi_y|(B \setminus J^\sigma_{\hat{u}^\xi_y}) + \mathcal{H}^0(B \cap J^\sigma_{\hat{u}^\xi_y}) \le |D\hat{u}^\xi_y|(B \setminus J^1_{\hat{u}^\xi_y}) + \mathcal{H}^0(B \cap J^1_{\hat{u}^\xi_y}) + (\sigma - 1)\mathcal{H}^0(B \cap (J^1_{\hat{u}^\xi_y} \setminus J^\sigma_{\hat{u}^\xi_y})), \quad B \in \mathcal{B}(U^\xi_y)$$

(notice that for $\mathcal{H}^{n-1}$-a.e. y the right-hand side is a finite measure thanks to Remark 3.6). By using the inclusion $J^1_{\hat{v}^\xi_y} \subset (J^1_v)^\xi_y$, valid for every $v \in \mathrm{GBD}(U)$ for every $\xi \in \mathbb{S}^{n-1}$, and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, we deduce

$$\hat{\mu}^\xi_\sigma(B) \le \hat{\mu}^\xi(B) + (\sigma - 1) \int_{B \cap J^1_u} |\nu_u \cdot \xi| \,\mathrm{d}\mathcal{H}^{n-1}, \quad B \in \mathcal{B}(U). \tag{3.18}$$

Finally, Remark 3.6 and the definition of $\hat{\mu}^\xi$ (see [6, Definition 4.10]) imply that the right-hand side of (3.18) is a finite measure, and so is $\hat{\mu}^\xi_\sigma$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\tau(t) := \arctan(t)$. We claim that for every $i \in \{1, \dots, n\}$ the family $(\tau(u_k \cdot e_i))_k$ is relatively compact in $L^1(U)$, where $\{e_i\}_{i=1}^n$ denotes a suitable orthonormal basis of $\mathbb{R}^n$. Now given $\epsilon_h \searrow 0$, by using Lemma 3.5, there exists $\delta_h \searrow 0$ such that if we define $B_k := \{|u_k| \neq 0\}$ and $v_k \colon B_k \to \mathbb{S}^{n-1}$ by $v_k := u_k/|u_k|$, then

$$\mathcal{L}^n(v_k^{-1}(S_{\Xi,\delta_h})) \le \epsilon_h \quad \text{for every } k \ge \phi(h),$$

for a suitable orthonormal basis Ξ and a suitable map $\phi \colon \mathbb{N} \to \mathbb{N}$.

In order to simplify the notation, let us denote $\Xi = \{e_1, \dots, e_n\}$. Fix $i \in \{1, \dots, n\}$ and set

$$\xi^t_j := \frac{\sqrt{t}}{\sqrt{t + t^2}} e_i + \frac{t}{\sqrt{t + t^2}} e_j \in \mathbb{S}^{n-1}$$

for every $j \neq i$ and $t > 0$. Notice that

$$|\xi^t_j - e_i| \le \sqrt{2t} \quad \text{and} \quad \left| \frac{\xi^t_j - e_i}{|\xi^t_j - e_i|} - e_j \right| \le \sqrt{2t}. \tag{3.19}$$

We define $U_t := \{x \in U : \mathrm{dist}(\partial U, x) > t\}$. Since we want to apply the Fréchet–Kolmogorov theorem, we have to estimate for $x \in U_t$,

$$\begin{aligned} &|\tau(u_k(x + te_j) \cdot e_i) - \tau(u_k(x) \cdot e_i)| \\ &\quad \le |\tau(u_k(x + te_j) \cdot e_i) - \tau(u_k(x + te_j) \cdot \xi^t_j)| + |\tau(u_k(x + te_j) \cdot \xi^t_j) - \tau(u_k(x - \sqrt{t}e_i) \cdot \xi^t_j)| \\ &\qquad + |\tau(u_k(x - \sqrt{t}e_i) \cdot \xi^t_j) - \tau(u_k(x - \sqrt{t}e_i) \cdot e_i)| + |\tau(u_k(x - \sqrt{t}e_i) \cdot e_i) - \tau(u_k(x) \cdot e_i)|. \end{aligned}$$

Now notice that, by the definition of S_{Ξ,δ_h} (see Definition 3.1), there exists a positive constant $c = c(\delta_h)$ such that for every $x \in U \setminus v_k^{-1}(S_{\Xi,\delta_h/2})$ and every $i, j \in \{1, \dots, n\}$,

$$|u_k(x) \cdot e_i| \ge c(\delta_h)|u_k(x) \cdot e_j| \quad \text{for every } k \text{ and } h. \tag{3.20}$$

Moreover, by taking into account (3.19), we deduce the existence of a dimensional parameter $\bar{t} > 0$ such that

$$|z \cdot \xi^t_j|^2 \ge 2^{-1}|z \cdot e_i|^2, \quad t \le \bar{t},\ z \in \mathbb{R}^n,\ i, j \in \{1, \dots, n\}, \tag{3.21}$$

$$\left| z \cdot \frac{\xi^t_j - e_i}{|\xi^t_j - e_i|} \right| \le 2|z \cdot e_j|, \quad t \le \bar{t},\ z \in \mathbb{R}^n,\ i, j \in \{1, \dots, n\}. \tag{3.22}$$

For every $t \le \bar{t}$, if $x \in U_t$ and $x \notin v_k^{-1}(S_{\Xi,\delta_h/2}) - te_j$, by using (3.19) and (3.20)–(3.22), we can write

$$
\begin{aligned}
&|\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x+te_j)\cdot \xi_j^t)| \\
&\quad = \Bigg| \int_{u_k(x+te_j)\cdot e_i}^{u_k(x+te_j)\cdot \xi_j^t} \frac{\mathrm{d}s}{1+s^2} \Bigg| \\
&\quad \le \max\Big\{ \frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot e_i|^2}, \frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot \xi_j^t|^2} \Big\} \Big| u_k(x+te_j)\cdot \frac{\xi_j^t - e_i}{|\xi_j^t - e_i|} \Big| \\
&\quad \le \max\Big\{ \frac{\sqrt{2t}}{1+|u_k(x+te_j)\cdot e_i|^2}, \frac{\sqrt{2t}}{1+2^{-1}|u_k(x+te_j)\cdot e_i|^2} \Big\} \Big| u_k(x+te_j)\cdot \frac{\xi_j^t - e_i}{|\xi_j^t - e_i|} \Big| \\
&\quad \le \frac{2\sqrt{2t}}{1+2^{-1}|u_k(x+te_j)\cdot e_i|^2} |u_k(x+te_j)\cdot e_j| \le \frac{2\sqrt{t}}{c(\delta_h)}.
\end{aligned} \tag{3.23}
$$

Analogously, if $x \in U_t$ and $x \notin v_k^{-1}(S_{\Xi,\delta_h/2}) + \sqrt{t}e_i$, we have

$$
|\tau(u_k(x-\sqrt{t}e_i)\cdot \xi_j^t) - \tau(u_k(x-\sqrt{t}e_i)\cdot e_i)| \le \frac{2\sqrt{t}}{c(\delta_h)}. \tag{3.24}
$$

Hence, from (3.23) and (3.24) we infer that for every $t \le \bar{t}$,

$$
\int_{U_t} |\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x+te_j)\cdot \xi_j^t)| \,\mathrm{d}x \le |U| \frac{2\sqrt{t}}{c(\delta_h)} + \pi\epsilon_h
$$

and

$$
\int_{U_t} |\tau(u_k(x-\sqrt{t}e_i)\cdot e_i) - \tau(u_k(x-\sqrt{t}e_i)\cdot \xi_j^t)| \,\mathrm{d}x \le |U| \frac{2\sqrt{t}}{c(\delta_h)} + \pi\epsilon_h.
$$

Moreover, setting $s_t := \sqrt{t+t^2}$, we can write

$$
\begin{aligned}
&\int_{U_t} |\tau(u_k(x+te_j)\cdot \xi_j^t) - \tau(u_k(x-\sqrt{t}e_i)\cdot \xi_j^t)| \,\mathrm{d}x \\
&\quad = \int_{U_t} |\tau(u_k(x-\sqrt{t}e_i + s_t\xi_j^t)\cdot \xi_j^t) - \tau(u_k(x-\sqrt{t}e_i)\cdot \xi_j^t)| \,\mathrm{d}x \\
&\quad = \int_{U_t+\sqrt{t}e_i} |\tau(u_k(x+s_t\xi_j^t)\cdot \xi_j^t) - \tau(u_k(x)\cdot \xi_j^t)| \,\mathrm{d}x \\
&\quad \le \int_{\Pi_{\xi_j^t}} \Big(\int_{(U_t+\sqrt{t}e_i)_y^{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|((s,s+s_t)) \,\mathrm{d}s \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y).
\end{aligned} \tag{3.25}
$$

By a mollification argument, we have that

$$
\int_{\Pi_{\xi_j^t}} \Big(\int_{(U_t+\sqrt{t}e_i)_y^{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|((s,s+s_t)) \,\mathrm{d}s \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\Pi_{\xi_j^t}} \Big(\int_0^{s_t} |D\tau(\hat{u}_y^{\xi_j^t})|((U_t+\sqrt{t}e_i)_y^{\xi_j^t} + \lambda) \,\mathrm{d}\lambda \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y),
$$

so that we obtain from (3.25) that

$$
\begin{aligned}
\int_{U_t} |\tau(u_k(x+te_j)\cdot \xi_j^t) - \tau(u_k(x-\sqrt{t}e_i)\cdot \xi_j^t)| \,\mathrm{d}x &\le \int_{\Pi_{\xi_j^t}} \Big(\int_0^{s_t} |D\tau(\hat{u}_y^{\xi_j^t})|((U_t+\sqrt{t}e_i)_y^{\xi_j^t} + \lambda) \,\mathrm{d}\lambda \Big) \,\mathrm{d}\mathcal{H}^{n-1}(y) \\
&\le \int_0^{s_t} \Big(\int_{\Pi_{\xi_j^t}} |D\tau(\hat{u}_y^{\xi_j^t})|(U_y^{\xi_j^t}) \,\mathrm{d}\mathcal{H}^{n-1}(y) \Big) \,\mathrm{d}\lambda \\
&\le \pi s_t \hat{\mu}_{u_k}(U).
\end{aligned}
$$

Analogously,

$$\int_{U_t} |\tau(u_k(x-\sqrt{t}e_i)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le \pi\sqrt{t}\hat{\mu}_{u_k}(U).$$

Summarizing, we have shown that if t_h is such that $t_h \in (0, \bar{t}]$ and

$$|U|\frac{2\sqrt{t_h}}{c(\delta_h)} \le \epsilon_h \quad \text{and} \quad \pi s_{t_h}\hat{\mu}_{u_k}(U) \le \epsilon_h,$$

then for every $t \le t_h$ we have for every $e_j \in \Xi$,

$$\int_{U_t} |\tau(u_k(x+te_j)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le 10\epsilon_h \quad \text{for every } k \ge \phi(h).$$

As a consequence, there exists a positive constant $L = L(n)$ such that

$$\int_{U_t} |\tau(u_k(x+t\xi)\cdot e_i) - \tau(u_k(x)\cdot e_i)|\,\mathrm{d}x \le L(n)\epsilon_h \quad \xi \in \mathbb{S}^{n-1},\ k \ge \phi(h),\ t \le t_h.$$

Since the index i chosen at the beginning was arbitrary, this means also that if we consider the diffeomorphism $\psi\colon \mathbb{R}^n \to (-\pi/2, \pi/2)^n$ defined by $\psi(x) := (\tau(x_1), \dots, \tau(x_n))$, then

$$\int_{U_t} |\psi(u_k(x+t\xi)) - \psi(u_k(x))|\,\mathrm{d}x \le L'(n)\epsilon_h, \quad \xi \in \mathbb{S}^{n-1},\ k \ge \phi(h),\ t \le t_h.$$

By the Fréchet–Kolmogorov theorem, this last inequality implies that the sequence $\psi(u_k)$ is relatively compact in $L^1(U;\mathbb{R}^n)$. Hence, we can pass to another subsequence, still denoted by $\psi(u_k)$, such that $\psi(u_k) \to v$ as $k \to \infty$ strongly in $L^1(U;\mathbb{R}^n)$. By eventually passing through another subsequence, we may suppose $\psi(u_k(x)) \to v(x)$ a.e. in U as $k \to \infty$. As a consequence, there exists a measurable $u\colon U \to \overline{\mathbb{R}}$ such that $u_k(x) \to u(x)$ as $k \to \infty$ a.e. in

$$U \setminus \Big\{x \in U : v(x) \in \partial\Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)^n\Big\}.$$

Moreover, $|u_k(x)| \to +\infty$ if and only if, for at least one index i, $u_k(x)\cdot e_i \to \pm\infty$ (clearly, $\tau(u\cdot e_i) = v_i$) or equivalently if and only if $x \in \{x \in U : v(x) \in \partial(-\frac{\pi}{2}, \frac{\pi}{2})^n\}$. Thus, we obtain that $u_k \to u$ a.e. in $U \setminus A$ as $k \to \infty$.

To show that $A := \{x \in U : |u_k(x)| \to +\infty\}$ has finite perimeter, the argument follows that in [4]. We give a sketch of the proof.

It is easy to check that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ it holds true that

$$x \in A \quad \text{if and only if} \quad \lim_{k\to\infty} \tau(u_k(x)\cdot\xi) = \pm\frac{\pi}{2} \qquad \text{for a.e. } x \in U. \tag{3.26}$$

Now fix $\sigma \ge 1$. First of all, using also (3.26), we can follow a standard measure theoretic argument which shows that we can extract a subsequence, still denoted as $(u_k)_k$, such that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ it holds true that

$$\tau((\hat{u}_k)^\xi_y) \to v^\xi_y := \begin{cases} \tau(\hat{u}^\xi_y) & \text{on } U^\xi_y \setminus A^\xi_y, \\ \pm\dfrac{\pi}{2} & \text{on } A^\xi_y, \end{cases} \qquad \text{in } L^1(U^\xi_y). \tag{3.27}$$

Fix $\epsilon > 0$. By the Fatou lemma and Remarks 3.6 and 3.7, we estimate

$$\begin{aligned}
&\int_{\Pi_\xi} \liminf_{k\to\infty}[\epsilon|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat{u}_k)^\xi_y})]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\quad\le \int_{\Pi_\xi} \liminf_{k\to\infty}[\epsilon|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap (J^\sigma_{u_k})^\xi_y)]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\
&\quad\le \limsup_{k\to\infty}\Big(\epsilon\hat{\mu}^\xi_{u_k}(U) + \epsilon(\sigma-1)\int_{U\cap J^1_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}\Big) + \liminf_{k\to\infty}\int_{U\cap J^\sigma_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad\le \epsilon\sup_{k\in\mathbb{N}}(1+4n(\sigma-1))\hat{\mu}_{u_k}(U) + \liminf_{k\to\infty}\int_{U\cap J^\sigma_{u_k}} |\nu_{u_k}\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1} < +\infty.
\end{aligned} \tag{3.28}$$

For $\mathcal{H}^{n-1}$-a.e. y, we can thus consider a subsequence depending on y but still denoted by $(u_k)_k$ such that

$$\sup_{k\in\mathbb{N}} \epsilon|D(\hat{u}_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat{u}_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat{u}_k)^\xi_y}) < +\infty. \tag{3.29}$$

Now we study the behavior of a sequence of one-dimensional functions satisfying (3.29). Let $(a, b) \subset \mathbb{R}$ be a non-empty open interval and suppose that $(f_k)_k$ is a sequence in $\mathrm{BV}_{\mathrm{loc}}((a, b))$ satisfying

$$\sup_{k\in\mathbb{N}} |Df_k|((a, b) \setminus J^\sigma_{f_k}) + \mathcal{H}^0(J^\sigma_{f_k}) < \infty. \tag{3.30}$$

We write $f_k = f^1_k + f^2_k$ for $f^1_k, f^2_k : (a, b) \to \mathbb{R}$ defined by

$$f^1_k(t) := Df_k((a, t) \setminus J^\sigma_{f_k}) \quad \text{and} \quad f^2_k(t) := f_k(a) + Df_k((a, t) \cap J^\sigma_{f_k}).$$

We study the convergence of f^1_k and f^2_k separately.

Inequality (3.30) tells us that, up to extracting a further not relabelled subsequence,

$$f^1_k \to f^1 \quad \text{pointwise a.e. for some } f^1 \in \mathrm{BV}((a, b)) \text{ as } k \to \infty. \tag{3.31}$$

As for $(f^2_k)_k$, by inequality (3.30) we may suppose that, up to extracting a further not relabelled subsequence, there exists a finite set $J \subset [a, b]$ such that

$$\mathcal{H}^0(J) \le \sup_{k\in\mathbb{N}} \mathcal{H}^0(J^\sigma_{f_k}), \tag{3.32}$$

$$J^\sigma_{f_k} \to J \quad \text{in Hausdorff distance as } k \to \infty. \tag{3.33}$$

Then (3.32), (3.33) together with the fact that, by construction, f^2_k is a piecewise constant function allow us to deduce that any pointwise limit function f^2 for $(f^2_k)_k$ must be of the form

$$f^2(t) = \sum_{l=1}^{M} \alpha_l \mathbf{1}_{(a_l, a_{l+1})}(t) \quad \text{for } t \in (a, b),$$

for a suitable $M \le \mathcal{H}^0(J \cap (a, b)) + 1$, for suitable $\alpha_l \in \mathbb{R} \cup \{\pm\infty\}$ with $\alpha_l \ne \alpha_{l+1}$, and for suitable $a_l \in J$ with $a_l < a_{l+1}$ and $a_1 = a$, $a_{\mathcal{H}^0(J\cap(a,b))+2} = b$. Up to extracting a further not relabelled subsequence, we may suppose $f^2_k \to f^2$ pointwise a.e. Now if $\alpha_l \in \{\pm\infty\}$, $l \ne 1$ and $l \ne \mathcal{H}^0(J \cap (a, b)) + 1$, we set

$$T_{l,k} := \Big\{t \in J^\sigma_{f^2_k} : |t - a_l| \le \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|\Big\},$$

$$T_{l+1,k} := \Big\{t \in J^\sigma_{f^2_k} : |t - a_{l+1}| \le \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|\Big\},$$

while if $l = 1$ we set

$$T_{l,k} := \Big\{t \in J^\sigma_{f^2_k} : |t - a_{l+1}| \le \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|\Big\},$$

and if $l = M$ we set

$$T_{l,k} := \Big\{t \in J^\sigma_{f^2_k} : |t - a_l| \le \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|\Big\}.$$

By (3.33), we have $T_{l,k} \ne \emptyset$ for every but sufficiently large k, and thanks to the definition of $T_{l,k}$ any sequence $(t_{l,k})_k$ with $t_{l,k} \in T_{l,k}$ is such that $t_{l,k} \to \alpha_l$ as $k \to \infty$. We claim that for every $l \in \{1, \dots, M\}$ there exists one of such sequences $(t_{l,k})_k$ such that

$$\lim_{k\to\infty} |[f^2_k(t_{l,k})]| = +\infty. \tag{3.34}$$

Suppose by contradiction that there exists l and a subsequence k_j such that

$$\sup_{j\in\mathbb{N}} \max_{t\in T_{l,k_j}} |[f^2_{k_j}(t)]| < +\infty.$$

Then we are in the following situation: we choose one of the endpoints a_l or a_{l+1}, for example a_l, (in the case $l = 1$ we choose a_{l+1}, and in the case $l = M$ we choose a_l) and the sequence

$$v_j := f^2_{k_j} \llcorner \Big(a_l - \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|, a_l + \frac{1}{2} \min_{t_1,t_2\in J} |t_1 - t_2|\Big)$$

satisfies

$$\begin{aligned}&v_j \text{ is piecewise constant},\\&J_{v_j} = T_{l,k_j} \quad \text{and} \quad J_{v_j} \to a_l \text{ in Hausdorff distance as } j \to \infty,\\&\sup_{j\in\mathbb{N}} \mathcal{H}^0(T_{l,k_j}) < +\infty, \quad \sup_{j\in\mathbb{N}} \max_{t\in J_{v_j}} |[v_j](t)| < +\infty.\end{aligned}$$

It is easy to see that the previous conditions are in contradiction to the fact that

$$f^2 \llcorner \Big(a_l - \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|, a_l + \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|\Big),$$

i.e. the pointwise limit of v_j is such that f^2 has a non-finite jump point at a_l. This proves our claim. Our claim implies in particular that, since $(f_k^1)_k$ is equibounded, the sequence $t_{l,k}$ satisfying (3.34) is actually contained for every but sufficiently large k in $J^\sigma_{f_k}$ (roughly speaking, the jumps of f_k^1 cannot compensate a non-bounded sequence of jumps of f_k^2). Clearly, since the intervals

$$\Big\{t : |t - a_l| < \frac{1}{2}\min_{t_1,t_2\in J} |t_1 - t_2|\Big\}$$

are pairwise disjoint for $l \in \{2, \dots, M\}$ (we are avoiding the end points a and b), we have actually proved the following lower semi-continuity property

$$\mathcal{H}^0(\partial^*\{f = \pm\infty\}) = \mathcal{H}^0(\{t \in (a, b) \cap J_f : |[f(t)]| = \infty\}) \le \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{f_k}), \tag{3.35}$$

where $f := f_1 + f_2$. Notice that the set J_f is well defined since f is the sum of a (bounded) BV function and a piecewise constant function, which might assume values $\pm\infty$, but jumps only at finitely many points.

Having this in mind, we can come back to our original problem. Fix $\xi \in \mathbb{S}^{n-1}$ satisfying (3.27). Given $y \in \Pi^\xi$ for which (3.27) and (3.29) hold true, we can pass through a not relabelled subsequence (depending on y) for which

$$\liminf_{k\to\infty} [\epsilon|D(\hat u_k)^\xi_y|(U^\xi_y \setminus J^\sigma_{(\hat u_k)^\xi_y}) + \mathcal{H}^0(U^\xi_y \cap J^\sigma_{(\hat u_k)^\xi_y})]$$

is actually a limit. Passing through a further not relabelled subsequence, we may also suppose that (3.35) holds true in each connected component of U^ξ_y, i.e.

$$\mathcal{H}^0\Big(\partial^*\Big\{v^\xi_y = \frac{\pm\pi}{2}\Big\}\Big) \le \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{(\hat u_k)^\xi_y}).$$

Notice that $|v^\xi_y| < \pi/2$ a.e. on $U^\xi_y \setminus A^\xi_y$, and hence $\{v^\xi_y = \pm\pi/2\} = A^\xi_y$ a.e., and so $\partial^*\{v^\xi_y = \pm\pi/2\} = \partial^* A^\xi_y$. In particular,

$$\mathcal{H}^0(\partial^* A^\xi_y) \le \liminf_{k\to\infty} \mathcal{H}^0(J^\sigma_{(\hat u_k)^\xi_y}) \tag{3.36}$$

Therefore, by passing through suitable subsequences, each depending on y, when computing the liminf inside the left-hand side integral of (3.28) and by using (3.36), we infer

$$\int_{\Pi^\xi} \mathcal{H}^0(\partial^* A^\xi_y)\, d\mathcal{H}^{n-1}(y) \le \epsilon \sup_{k\in\mathbb{N}}(1 + 4n(\sigma - 1))\hat\mu_{u_k}(U) + \liminf_{k\to\infty} \int_{U\cap J^\sigma_{u_k}} |\nu_{u_k} \cdot \xi|\, d\mathcal{H}^{n-1}. \tag{3.37}$$

The arbitrariness of ξ implies that (3.37) holds for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. Hence, we deduce that A has finite perimeter in U. In addition, by taking the integral on $\mathbb{S}^{n-1}$ on both sides of (3.37), we infer

$$\alpha_n\mathcal{H}^{n-1}(\partial^* A) \le \epsilon n\omega_n(1 + 4n(\sigma - 1)) \sup_{k\in\mathbb{N}} \hat\mu_{u_k}(U) + \alpha_n \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k}),$$

where $\alpha_n := \int_{\mathbb{S}^{n-1}} |\nu \cdot \xi|$. Moreover, the arbitrariness of $\epsilon > 0$ tells us

$$\mathcal{H}^{n-1}(\partial^* A) \le \liminf_{k\to\infty} \mathcal{H}^{n-1}(J^\sigma_{u_k}).$$

Finally, by the arbitrariness of $\sigma \ge 1$ and by the fact that $J^{\sigma_1} \subset J^{\sigma_2}$ for $\sigma_1 \ge \sigma_2$, we conclude (1.2).

In order to show that u can be extended to the whole of U as a function in $\mathrm{GBD}(U)$, we define the sequence of $\mathrm{GBD}(U)$ functions by

$$\tilde{u}_k(x) := \begin{cases} u_k(x) & \text{if } x \in U \setminus A, \\ 0 & \text{if } x \in A. \end{cases}$$

Clearly, if we define v by

$$v(x) := \begin{cases} u(x) & \text{if } x \in U \setminus A, \\ 0 & \text{if } x \in A, \end{cases} \tag{3.38}$$

then we have $\tilde{u}_k \to v$ a.e. in U and

$$\sup_{k \in \mathbb{N}} \hat{\mu}_{\tilde{u}_k}(U) \le \sup_{k \in \mathbb{N}} \hat{\mu}_{u_k}(U) + \mathcal{H}^{n-1}(\partial^* A) < +\infty.$$

Therefore, by using the technique developed in [1, 6], we can conclude $v \in \mathrm{GBD}(U)$. □

Remark 3.8. Under the additional assumption (1.3) with $u_k \in \mathrm{GSBD}(U)$, we can obtain the further information $e(u_k)\mathbf{1}_{U \setminus A} \rightharpoonup e(u)$ in $L^1(U; \mathbb{M}^n_{\mathrm{sym}})$ thanks to $e(\tilde{u}_k) \rightharpoonup e(u)$ in $L^1(U; \mathbb{M}^n_{\mathrm{sym}})$ together with the fact $e(u_k)\mathbf{1}_{U \setminus A} = e(\tilde{u}_k)$ for every $k \in \mathbb{N}$. Moreover, (3.35) can be modified in the following way:

$$\mathcal{H}^0(J_f \cup \partial^* \{f = \pm\infty\}) \le \liminf_{k \to \infty} \mathcal{H}^0(J_{f_k}),$$

from which it is possible to deduce that

$$\mathcal{H}^{n-1}(J_u \cup \partial^* A) \le \liminf_{k \to \infty} \mathcal{H}^{n-1}(J_{u_k}).$$

Condition (1.3) would also imply that in (3.28) we actually control

$$\int_{\Pi_\xi} \liminf_{k \to \infty} \Big[\int_{U^\xi_y} \epsilon \phi(|(\dot{u}_k)^\xi_y(t)|)\, dt + \mathcal{H}^0(U^\xi_y \cap J_{(\hat{u}_k)^\xi_y}) \Big]\, d\mathcal{H}^{n-1}(y) < +\infty,$$

where $(\dot{u}_k)^\xi_y$ denotes the absolutely continuous part of $D(\hat{u}_k)^\xi_y$. This in turns allows us to use the well-known compactness result for SBV functions in one variable to deduce that the pointwise limit function f^1 in (3.31) belongs to $\mathrm{SBV}((a,b))$. For this reason, the techniques of [1, 6] can be adapted to deduce $v \in \mathrm{GSBD}(U)$ (see (3.38) for the definition of v). The convergence of $e(u_k)$ to $e(u)$ in $L^2(\Omega \setminus A; \mathbb{M}^n_{\mathrm{sym}})$ follows instead by the arguments of [5, pp. 10–11].

Funding: S. Almi acknowledges the support of the OeAD-WTZ project CZ 01/2021 and of the FWF through the project I 5149. E. Tasso was partially supported by the Austrian Science Fund (FWF) through the project F65.

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