



PhD Thesis

Physics Area

Astroparticle and Gravitational Physics

Conformal Bootstrap for AdS Loops

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To my parents.

غرض نقشی ست کز ما باز ماند
که هستی را نمی بینم بقایی

— سعدی، دیباچه گلستان

*The aim is to leave behind a trace of us,
for I see no permanence in existence.*

— Saadi, Preface to the *Golestan*

Abstract

This thesis studies loop corrections to four-point functions of identical scalar operators in four-dimensional conformal field theories with weakly coupled AdS duals. The main goal is to determine how much of the quantum dynamics in AdS, at all orders in the large- N perturbation, is encoded in lower-order CFT data. We address this question using analytic bootstrap methods.

We first consider a scalar theory in AdS with a quartic interaction and isolate a contribution to the correlator that, at any order in $1/N$, is completely determined by tree-level data. This contribution controls the leading logarithmic behavior of double-trace anomalous dimensions in the large-spin expansion. We resum these effects to all orders in $1/N$ and show that they admit an effective description in AdS. In Mellin space, the flat-space limit of the same contribution reproduces consecutive unitarity cuts of the corresponding bubble diagrams [1].

We then study the appearance of higher-trace operators in theories dual to bulk ϕ^3 and ϕ^4 interactions. Infinite spin sums of double-trace data generate contributions which, after crossing, have the structure of higher-trace exchanges. This shows how multi-trace sectors arise as a necessary consequence of crossing symmetry and allows us to determine part of their OPE data. Finally, we introduce a diagrammatic framework to organize the different terms in the conformal block expansion. This framework makes the relations between lower- and higher-trace data more transparent and allows crossing symmetry to be applied to individual diagrammatic topologies [2].

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Chapter 1

Introduction

Most of our present understanding of nature rests on two remarkably successful theoretical frameworks. One is quantum field theory, which describes elementary particles and their interactions, and the other is general relativity, which describes gravity as the geometry of spacetime. Each of them has been tested with great precision within their own domain, but we still lack a complete description of situations in which quantum effects and gravity are simultaneously important. Anti-de Sitter (AdS) space provides a controlled setting in which this problem can be investigated. Although it does not describe the geometry of our universe directly, its high degree of symmetry and its special role in holography make it a useful laboratory for studying quantum field theory in curved spacetime and aspects of quantum gravity.

An important question is then how the interactions of quantum fields should be described in AdS. A principal way to study the nature of interactions between particles is through scattering. The idea is to prepare particles in the far past, let them interact with each other, and then observe them after they become separated again. In flat spacetime, this picture provides a well-defined description of the scattering problem and allows us to compare theoretical calculations with measurable outcomes. However, in generic curved spacetimes, this picture may not apply. In particular, in AdS, particles cannot remain freely separated forever. The geometry effectively confines them and brings them back into interaction. As a result, the usual freely separated incoming and outgoing states cannot be defined. Therefore, the usual notion of scattering should be replaced by another description in AdS that still contains the physical information about how states interact and how more complicated intermediate states are produced.

A complete description of these interactions should also include quantum corrections. In perturbative quantum field theory, the different contributions are organized by Feynman diagrams. Namely, we have tree-level diagrams, which represent the leading interaction, and loop diagrams, which describe the quantum corrections. In flat spacetime, the principles of quantum theory, in particular

unitarity, imposes relations between loop amplitudes and simpler, lower-order amplitudes [3, 4]. More broadly, decades of work in this setting have revealed a rich physical and mathematical structure for scattering amplitudes in flat space [5–7]. This naturally raises the question of whether analogous structures can be formulated in AdS [8]. Studying loop effects in AdS is therefore important for clarifying how quantum consistency and multi-particle dynamics are encoded in curved spacetimes. However, as we mentioned above, the ordinary scattering framework is not available in AdS. So to address this question, we first need a framework that describes AdS interactions in terms of well-defined observables.

The AdS/CFT correspondence provides such a framework by relating quantum field theory in AdS to correlation functions in a conformal field theory (CFT) defined on its boundary [9–11]. For CFTs with a large- N expansion and a large gap in their spectrum, the dual bulk theory is expected to admit a weakly coupled effective field theory description [12]. In this regime, the $1/N$ expansion of boundary correlators is related to perturbation theory in AdS [13–16]. This perturbative expansion is represented by Witten diagrams, which are the AdS analogue of Feynman diagrams. In these diagrams, bulk interaction vertices are connected to operator insertion points on the boundary.

Conformal field theories are also important in their own right. Many physical systems near continuous phase transitions exhibit scale-invariant behavior and can be described by CFTs. Although these systems may be physically very different, certain properties near the transition can be universal. Therefore, CFTs can provide a common framework for understanding a broad class of phenomena and studying their universal properties.

Coming back to the AdS/CFT context, the strong constraints imposed by conformal symmetry make the boundary CFT a powerful tool for studying AdS loop physics, which is otherwise very hard to access through direct computations. A CFT four-point function can be decomposed into conformal blocks, each representing the exchange of a family of operators in a chosen operator product expansion (OPE) channel. The dimensions of the exchanged operators and their OPE coefficients constitute the CFT data. Crossing symmetry, which requires the OPE decompositions in different channels to describe the same correlation function, imposes non-trivial consistency conditions on the CFT data. The conformal bootstrap combines these conditions with unitarity to constrain and, in some cases, determine the CFT data. [17–19].

At one loop, in a scalar theory dual to a bulk ϕ^4 interaction, crossing symmetry allows the corrections to the double-trace data to be reconstructed from tree-level information [8]¹. At higher loops, tree-level data are no longer sufficient to

¹Related one-loop double-trace data in ϕ^4 theory have also been computed directly in AdS₄ [20] and AdS₃ [21]. See also [22–29] for other works on loop physics using direct AdS computations.

determine the complete correlator. New corrections from previous orders enter, and higher-trace operators also begin to contribute. Nevertheless, a particular part of the answer remains fixed by tree-level data at every order.

This thesis does not attempt to determine complete correlators at arbitrary loop order. Instead, it addresses three related problems. The first is which part of the higher-loop data continues to be fixed by tree-level information and how this part can be understood in the bulk scattering picture. The second is how the higher-trace operators required by crossing symmetry arise from lower-trace CFT data. The third is how to organize the different products of OPE data and the corresponding terms in the conformal block expansion in a diagrammatic way that keeps track of their channel dependence. We study these questions for four-point functions of identical scalar operators $\mathcal{O}$ in four-dimensional CFTs with weakly coupled AdS duals in large- N expansion.

The first problem is considered in a theory with a non-derivative quartic interaction in the bulk and hence a $\mathbb{Z}_2$ symmetry in the CFT side. At every order $N^{-2\kappa}$, we isolate a contribution to the correlator that depends only on the leading OPE coefficients and the tree-level anomalous dimensions of double-trace operators. At one loop, this contribution determines the complete correction to the double-trace data. At higher loops, it gives only part of the answer, but we will argue that this is a physically meaningful part.

The anomalous dimensions obtained from this contribution have a characteristic large-spin behavior. Denoting the conformal spin by J , at order $N^{-2\kappa}$ the result takes the form

$$\gamma_\ell^{(\kappa)} \Big|_{\text{leading log}} = \log^{\kappa-2} J \left(\frac{c_0^{(\kappa)}}{J^{2\Delta_\mathcal{O}}} + \frac{c_1^{(\kappa)}}{J^{2\Delta_\mathcal{O}+2}} + \frac{c_2^{(\kappa)}}{J^{2\Delta_\mathcal{O}+4}} + \cdots \right), \quad \kappa \geq 2. \quad (1.1)$$

So, at every order in $1/N$, tree-level data determine the coefficient of the highest power of $\log J$ to all orders in the $1/J$ expansion.

The part of the OPE data described by (1.1) has also a interpretation in terms of bulk unitarity cuts.² When expressed in Mellin space, its flat-space limit at order $N^{-2\kappa}$ reproduces the consecutive cuts of a chain of $\kappa - 1$ bubbles. The correspondence therefore relates the part of the CFT data fixed by tree-level information to the maximally cut part of the bulk loop amplitude.

The second problem studied in this thesis concerns higher-trace operators. Their appearance can be understood by combining crossing symmetry with the infinite spin sums appearing in the conformal block expansion of the double-trace sector. The sum over a double-trace family can generate terms with new behavior that is not visible in the individual conformal blocks. After crossing, some of these terms have precisely the form required by higher-trace exchanges.

²Related analyses in the case of $\mathcal{N} = 4$ SYM were carried out in [30, 31].

We apply this idea to theories dual to bulk ϕ^4 and ϕ^3 interactions. In the quartic theory, infinite sums over double-trace operators produce the contributions associated with quadruple-trace operators at order N^{-8} . We use these terms to determine part of the OPE data of the quadruple-trace family. The result is obtained as a large-spin expansion for a general external dimension and in closed form in a special case.

In the theory with a cubic interaction, the absence of a $\mathbb{Z}_2$ symmetry allows operators containing both even and odd numbers of $\mathcal{O}$'s to appear in the OPE. k -trace operators enter at order N^{-2k} , and part of their large-spin OPE data can be extracted from spin sums constructed from double-trace information. This connects to the problem of understanding multi-particle states in AdS from CFT data [32, 33].

The third problem is motivated by the relation between the contribution fixed by tree-level data and the maximal cuts of bubble diagrams [1]. We introduce a more general diagrammatic representation for organizing products of OPE data and the corresponding terms in the conformal block expansion. This representation keeps track of their channel dependence and makes it possible to apply crossing symmetry to subsets of terms associated with a given diagrammatic topology. In particular, it identifies the crossed-channel spin sums responsible for the higher-trace contributions mentioned above.

1.1 Structure of the thesis

The thesis is organized into three parts. Part I introduces the necessary background. Chapter 2 reviews the main elements of conformal field theory, including correlation functions, the operator product expansion, conformal blocks, crossing symmetry, and generalized free fields. Chapter 3 reviews the AdS/CFT correspondence, the relation between the large- N expansion and bulk perturbation theory, the interpretation of multi-trace operators, and Witten diagrams.

Part II introduces the analytic methods used in the later chapters. Chapter 4 presents the Lorentzian inversion formula and explains how CFT data can be extracted at finite spin and in a large-spin expansion. It also introduces the peeling method used in the explicit calculations. Chapter 5 discusses infinite spin sums and finite-spin corrections to the large-spin solution. Chapter 6 introduces the Mellin space representation of CFT correlators and explains how flat-space scattering amplitudes can be extracted from it.

Part III contains the main results of the thesis. Chapter 7 studies the universal contribution fixed by tree-level data, and their relation to maximal cuts of bubble diagrams. Chapter 8 studies how higher-trace operators arise from infinite spin sums in theories dual to ϕ^4 and ϕ^3 interactions. Chapter 9 introduces a diagrammatic representation for organizing the terms in the conformal block ex-

pansion and their associated OPE data. The final chapter summarizes the results and discusses possible directions for future works.

For clarity, the origin of material adapted from the author's publications is as follows. Chapter 7 largely reproduces, with adaptations to notation and presentation, material from *Bubbles in AdS* [1], which is joint work with Agnese Bissi and Giulia Fardelli. Chapters 8 and 9 largely reproduce material from the author's paper *Higher-Trace Operators and Cut Diagrammatics in the Conformal Block Expansion* [2]. Some material in other chapters is also adapted from the latter paper, particularly the discussions in Section 4.3.2 and Section 5.1, as well as some points in Section 10.1.

Part I

Foundations

Chapter 2

Basics of Conformal Field Theory

A conformal field theory is a quantum field theory invariant under a symmetry group known as the conformal group. Conformal field theory provides the basic framework for the discussion in this thesis. Conformal symmetry strongly constrains correlation functions of local operators. In this theory, the position dependence of two-point and three-point functions is completely fixed up to a set of constants. Four-point functions are the first correlators whose position dependence is not completely fixed by conformal symmetry and therefore contains dynamical information. The four-point function of identical scalar operators will be the main object of study throughout this thesis.

In this chapter, we review the basic ingredients of conformal field theory that will be needed later. We begin with the definition of conformal transformations and derive the conformal algebra. We then discuss local operators, distinguishing between primary operators and their descendants. Next, we study correlation functions of primary operators and organize four-point functions using the conformal block expansion. Finally, we introduce generalized free field theory as a simple example of a conformal field theory.

2.1 Conformal symmetry and local operators

2.1.1 Conformal transformations

A conformal group can be placed on a d -dimensional manifold $\mathcal{M}$ with a background metric $g_{\mu\nu}(x)$. A conformal transformation is a coordinate transformation

$$x^\mu \mapsto x'^\mu(x) \tag{2.1}$$

such that,

$$\frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x') = \Omega(x)^2 g_{\mu\nu}(x). \tag{2.2}$$

Namely, such transformation preserves angles, but allowing a local change of scale.

Restricting to the infinitesimal transformations,

$$x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x). \quad (2.3)$$

Expanding (2.2) to first order in ϵ^μ gives

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d} (\nabla_\rho \epsilon^\rho) g_{\mu\nu}. \quad (2.4)$$

This is the conformal Killing equation, whose solutions generate infinitesimal conformal transformations.

In the particular case of the flat space, which is the relevant case for this thesis, the conformal Killing equation becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial_\rho \epsilon^\rho) \delta_{\mu\nu}. \quad (2.5)$$

For $d > 2$, its most general solution is

$$\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - b^\mu x^2, \quad (2.6)$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The four terms in (2.6) generate translations, rotations, dilatations and special conformal transformations, respectively. The case $d = 2$ is special because the local conformal symmetry becomes infinite-dimensional. We will not consider that case in this thesis.

We denote the corresponding generators of these transformations by P_μ , $M_{\mu\nu}$, D and K_μ . In our conventions, their action on functions of the coordinates may be represented as

$$P_\mu = -\partial_\mu, \quad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad (2.7)$$

$$D = -x^\rho \partial_\rho, \quad K_\mu = x^2 \partial_\mu - 2x_\mu x^\rho \partial_\rho. \quad (2.8)$$

For $d > 2$, the conformal group of flat Euclidean space is locally isomorphic to $SO(d+1, 1)$. In Lorentzian signature, the corresponding group is locally isomorphic to $SO(d, 2)$.

The conformal algebra

The generators of conformal transformations satisfy the commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho}, \quad (2.9)$$

$$[M_{\mu\nu}, P_\rho] = \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \quad (2.10)$$

$$[M_{\mu\nu}, K_\rho] = \delta_{\nu\rho}K_\mu - \delta_{\mu\rho}K_\nu, \quad (2.11)$$

$$[D, P_\mu] = P_\mu, \quad (2.12)$$

$$[D, K_\mu] = -K_\mu, \quad (2.13)$$

$$[D, M_{\mu\nu}] = 0, \quad (2.14)$$

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}, \quad (2.15)$$

$$[P_\mu, P_\nu] = 0, \quad (2.16)$$

$$[K_\mu, K_\nu] = 0. \quad (2.17)$$

The conformal algebra in Euclidean signature is $\mathfrak{so}(d+1, 1)$, and in Lorentzian signature it is $\mathfrak{so}(d, 2)$.

The commutators with D show that P_μ and K_μ raise and lower, respectively, the eigenvalue of the dilatation operator. This property will be important in the construction of conformal representations of local operators, as we will now discuss.

2.1.2 Local operators

Local operators can be defined at the origin and transferred elsewhere by

$$\mathcal{O}(x) = e^{x \cdot P} \mathcal{O}(0) e^{-x \cdot P}. \quad (2.18)$$

A local operator at the origin is characterized by its “scaling dimension” and by its transformation law under rotation. For a scalar operator, the rotational representation is trivial. More generally, local operators may carry Lorentz indices. In this thesis, we are mostly working in scenarios in which we can restrict ourselves into the symmetric traceless representation of the conformal group, namely

$$\mathcal{O}_{\mu_1 \dots \mu_\ell} = \mathcal{O}_{(\mu_1 \dots \mu_\ell)}(x), \quad \delta^{\mu_i \mu_j} \mathcal{O}_{\mu_1 \dots \mu_\ell}(x) = 0. \quad (2.19)$$

The integer ℓ is the rank of the tensor and is referred to as the spin of the operator.

A convenient way to write such operators is to contract them with a null polarization vector z^μ , with $z^2 = 0$ [34],

$$\mathcal{O}_\ell(x, z) = z^{\mu_1} \dots z^{\mu_\ell} \mathcal{O}_{\mu_1 \dots \mu_\ell}(x). \quad (2.20)$$

Having set the representation with respect to rotation, we now move to the remaining generators. As we mentioned earlier the generators K_μ and P_μ act as lowering and raising operators on D eigenvalues. So we can build a representation using the condition:

$$[K_\nu, \mathcal{O}_{\mu_1 \dots \mu_\ell}(0)] = 0 \quad (2.21)$$

together with

$$[D, \mathcal{O}_{\mu_1 \dots \mu_\ell}(0)] = \Delta \mathcal{O}_{\mu_1 \dots \mu_\ell}(0). \quad (2.22)$$

These two conditions define a Primary operator. We can furthermore define a descendant operator by acting with powers of translation generator P_μ on the primary.

A primary together with all of its descendant forms a conformal multiplet.

In a Unitary CFT, conformal dimensions are bounded from below. For scalar primary in d dimensions we have the Unitarity bound [35]:

$$\Delta \geq \frac{d-2}{2}, \quad (2.23)$$

and for traceless symmetric tensor of spin $\ell \geq 1$:

$$\Delta \geq \ell + d - 2. \quad (2.24)$$

State–Operator Correspondence

To relate local operators to states, we should quantize the theory in a way adapted to conformal symmetry. In ordinary canonical quantization one choose a time coordinate and define states on constant-time hypersurfaces. Then the Hamiltonian operator generates the evolution between these time surfaces. In Euclidean CFT, Dilatation provides a natural notion of evolution. This suggests using the radial coordinate as Euclidean time. So we foliate the Euclidean space by spheres centered at the origin, and states are defined on such spheres at radius r . Radial evolution describes how this state changes when the sphere is expanded or contracted.

Writing the flat Euclidean coordinates as

$$x^\mu = r n^\mu, \quad n^2 = 1, \quad (2.25)$$

the metric takes the form:

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 = e^{2\tau} (d\tau^2 + d\Omega_{d-1}^2), \quad (2.26)$$

with $\tau = \log r$.

So the flat Euclidean space with origin removed is conformally related to the cylinder $\mathbb{R}_\tau \times S^{d-1}$. Therefore Dilatation operator D plays the role of Hamiltonian on the cylinder. Therefore we can write

$$|\Psi(\tau_2)\rangle = e^{-(\tau_2 - \tau_1)D} |\Psi(\tau_1)\rangle. \quad (2.27)$$

Now consider the Euclidean path integral over a ball centred at the origin. If the fields are fixed on the boundary sphere, this path integral defines a wavefunctional and hence a state on S^{d-1} . Inserting a local operator at the origin changes this state. We therefore associate to every local operator a state:

$$|\mathcal{O}\rangle \equiv \mathcal{O}(0) |0\rangle. \quad (2.28)$$

Conversely, a state on S^{d-1} may be represented by an operator insertion at the origin. This identification is known as state-operator correspondence.

For an operator with definite scaling dimension

$$[D, \mathcal{O}(0)] = \Delta_{\mathcal{O}} \mathcal{O}(0), \quad (2.29)$$

using the invariance of the vacuum under dilatations,

$$D |0\rangle = 0, \quad (2.30)$$

we obtain

$$D |\mathcal{O}\rangle = \Delta_{\mathcal{O}} |\mathcal{O}\rangle. \quad (2.31)$$

Therefore, the scaling dimension of an operator is equal to the energy of the corresponding state on the cylinder.

This also gives a state interpretation of conformal multiplet. A primary operator satisfies

$$[K_{\mu}, \mathcal{O}(0)] = 0. \quad (2.32)$$

Since the vacuum is conformally invariant, it follows that

$$K_{\mu} |0\rangle = 0. \quad (2.33)$$

Moreover, the conformal algebra gives

$$[D, P_{\mu}] = P_{\mu}, \quad [D, K_{\mu}] = -K_{\mu}. \quad (2.34)$$

Therefore the primary state is a lowest energy state within a conformal multiplet and descendants are obtained by acting with translations

$$P_{\mu_1} \cdots P_{\mu_n} |\mathcal{O}\rangle, \quad (2.35)$$

and they have cylinder energy $\Delta_{\mathcal{O}} + n$.

The dual bra state is defined by radial reflection which maps the origin to infinity

$$\langle \mathcal{O} | \equiv \lim_{x \rightarrow \infty} x^{2\Delta_{\mathcal{O}}} \langle 0 | \mathcal{O}^{\dagger}(x). \quad (2.36)$$

Therefore we can write the inner product

$$\langle \mathcal{O} | \mathcal{O} \rangle = \lim_{x \rightarrow \infty} x^{2\Delta_{\mathcal{O}}} \langle \mathcal{O}^{\dagger}(x) \mathcal{O}(0) \rangle. \quad (2.37)$$

In a unitary theory, this should be positive definite. Enforcing this condition and requiring it also for the descendant states leads to the unitarity conditions stated earlier in (2.23) and (2.24).

2.2 Correlation functions

We now want to discuss the correlation functions of scalar primary operators. Conformal symmetry fixes the form of 2- and 3-point functions up to a constant.

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \frac{C_{12} \delta_{\Delta_1, \Delta_2}}{(x_{12}^2)^{\Delta_1}}, \quad (2.38)$$

where $x_{12} = x_1 - x_2$ and Δ_i is the scaling dimension of $\mathcal{O}_i$. We normalize the 2-point function such that

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{(x_{12}^2)^{\Delta_{\mathcal{O}}}}. \quad (2.39)$$

For three-point functions:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{f_{123}}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}} (x_{23}^2)^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} (x_{13}^2)^{\frac{\Delta_1 + \Delta_3 - \Delta_2}{2}}}. \quad (2.40)$$

The coefficient f_{123} is called a three-point structure constant, and it is theory-dependent. In a unitary theory, these coefficients are real.

The first correlator whose spacetime dependence is not completely fixed by conformal symmetry is the 4-point function. For identical scalar primaries $\mathcal{O}$, conformal symmetry constrains its form to

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{\mathcal{G}(u, v)}{(x_{12}^2)^{\Delta_{\mathcal{O}}} (x_{34}^2)^{\Delta_{\mathcal{O}}}}, \quad (2.41)$$

where

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (2.42)$$

are called conformal cross ratios. It is often useful to parameterize them by

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}). \quad (2.43)$$

In Euclidean signature, $\bar{z}$ is the complex conjugate of z . In Lorentzian signature z and $\bar{z}$ are independent real variables [36].

Crossing symmetry

Permutation symmetry of the operators in the correlation function implies crossing symmetry.

Exchanging $x_1 \leftrightarrow x_3$ gives

$$\mathcal{G}(u, v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}(v, u). \quad (2.44)$$

Similarly, exchanging $x_1 \leftrightarrow x_2$ gives:

$$\mathcal{G}(u, v) = \mathcal{G}\left(\frac{u}{v}, \frac{1}{v}\right). \quad (2.45)$$

2.2.1 OPE and conformal blocks

To extract the dynamical content of the 4-point function, we use the operator product expansion (OPE). For two local operators inserted sufficiently close to one another, their product may be expanded as a sum over local operators at one point. For scalar primaries:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_p} f_{12\mathcal{O}_p} C(x, \partial) \mathcal{O}_p(0), \quad (2.46)$$

where the sum runs over conformal primaries, and the descendants are generated by the differential operator $C(x, \partial)$. The coefficient $f_{12\mathcal{O}_p}$ is called the OPE coefficient and with the standard normalization of two-point functions, is equal to the three-point structure constant that appears in $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_p \rangle$ [36].

We refer to the set of operators in the spectrum together with their OPE coefficients as OPE data. All n -point functions can be built by repeatedly inserting the OPE. Therefore, we can say that a CFT is specified by the OPE data.

In the case of identical scalar operators, one can easily see that only even-spin operators can appear in the $\mathcal{O} \times \mathcal{O}$ OPE.

Conformal block expansion

Applying the OPE inside a 4-point function, we get a decomposition into contributions of conformal multiplets. Choosing a basis in which two-point functions are diagonal, the expansion in the $12 \rightarrow 34$ channel for identical scalars will be:

$$\begin{aligned} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle &= \sum_{\mathcal{O}_p, \mathcal{O}_q} f_{\mathcal{O}\mathcal{O}\mathcal{O}_p} f_{\mathcal{O}\mathcal{O}\mathcal{O}_q} C(x_{12}, \partial_2) C(x_{34}, \partial_4) \langle \mathcal{O}_p(x_2)\mathcal{O}_q(x_4) \rangle \\ &= \sum_{\mathcal{O}_p} f_{\mathcal{O}\mathcal{O}\mathcal{O}_p}^2 \mathcal{G}_{\Delta_p, \ell_p}^{\Delta_{\mathcal{O}}}(x_i), \end{aligned} \quad (2.47)$$

where

$$\mathcal{G}_{\Delta_p, \ell_p}^{\Delta_{\mathcal{O}}}(x_i) = C(x_{12}, \partial_2) C(x_{34}, \partial_4) \langle \mathcal{O}_p(x_2)\mathcal{O}_p(x_4) \rangle. \quad (2.48)$$

We can rewrite this as

$$\mathcal{G}_{\Delta_p, \ell_p}^{\Delta_{\mathcal{O}}}(x_i) = \frac{g_{\Delta_p, \ell_p}(u, v)}{(x_{12}^2)^{\Delta_{\mathcal{O}}}(x_{34}^2)^{\Delta_{\mathcal{O}}}}. \quad (2.49)$$

Therefore using (2.41) we have

$$\mathcal{G}(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell} g_{\Delta, \ell}(u, v), \quad a_{\Delta, \ell} = f_{\mathcal{O}\mathcal{O}\mathcal{O}_{\Delta, \ell}}^2. \quad (2.50)$$

$g_{\Delta,\ell}(u, v)$ is the conformal block associated with the exchange of the corresponding conformal multiplet. The conformal blocks are fixed completely by symmetry, and the dynamical CFT data are encoded in the coefficients $a_{\Delta,\ell}$.

Conformal blocks are eigenfunctions of the quadratic and quartic Casimir operators of the conformal group. In four dimensions, they satisfy

$$\mathcal{D}_2(g_{\Delta,\ell}(u, v)) = \frac{1}{2}((\Delta - 4)\Delta + \ell(\ell + 2))(g_{\Delta,\ell}(u, v)), \quad (2.51)$$

$$\mathcal{D}_4(g_{\Delta,\ell}(u, v)) = \ell(\ell + 2)(\Delta - 3)(\Delta - 1)(g_{\Delta,\ell}(u, v)), \quad (2.52)$$

where

$$\mathcal{D}_2 = \mathcal{D} + \bar{\mathcal{D}} + 2\frac{z\bar{z}}{z - \bar{z}}((1 - z)\partial - (1 - \bar{z})\bar{\partial}), \quad (2.53)$$

$$\mathcal{D}_4 = \left(\frac{z\bar{z}}{z - \bar{z}}\right)^2(\mathcal{D} - \bar{\mathcal{D}})\left(\frac{z - \bar{z}}{z\bar{z}}\right)^2(\mathcal{D} - \bar{\mathcal{D}}). \quad (2.54)$$

Here

$$\mathcal{D} = (1 - z)z^2\partial^2 - z^2\partial, \quad (2.55)$$

and $\bar{\mathcal{D}}$ is obtained by replacing z with $\bar{z}$ and $\partial = \partial_z$ with $\bar{\partial} = \partial_{\bar{z}}$.

In general dimensions, the quadratic Casimir eigenvalue is

$$C_{\Delta,\ell} = \frac{1}{2}[\Delta(\Delta - d) + \ell(\ell + d - 2)]. \quad (2.56)$$

This characterization leads to the explicit closed form expression for conformal blocks at $d = 4$. Namely, [37]

$$g_{\Delta,\ell}(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left[k_{\frac{\Delta+\ell}{2}}(z) k_{\frac{\Delta-\ell-2}{2}}(\bar{z}) - k_{\frac{\Delta+\ell}{2}}(\bar{z}) k_{\frac{\Delta-\ell-2}{2}}(z) \right]. \quad (2.57)$$

where

$$k_\beta(x) = x^\beta F_\beta(x), \quad F_\beta(x) = {}_2F_1(\beta, \beta, 2\beta; x). \quad (2.58)$$

At small u , the conformal blocks behave as

$$g_{\Delta,\ell}(u, v) \xrightarrow{u \ll 1} u^{\frac{\tau}{2}} \times (\text{function of } v), \quad (2.59)$$

where $\tau = \Delta - \ell$ is the twist of the exchanged operator. We factor out this leading power and define the reduced block

$$g_{\Delta,\ell}(u, v) = u^{\frac{\Delta-\ell}{2}} \tilde{g}_{\Delta,\ell}(u, v). \quad (2.60)$$

The function $\tilde{g}_{\Delta,\ell}(u, v)$ is regular as $u \rightarrow 0$. Its behavior at small v is given by

$$\tilde{g}_{\Delta,\ell}(u, v) \xrightarrow{v \ll 1} a(u, v) \log v + b(u, v), \quad (2.61)$$

where $a(u, v)$ and $b(u, v)$ admit a regular series expansion at small u and v .

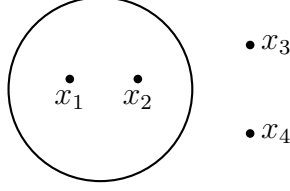


Figure 2.1: A sphere separating the pair of operators to which the OPE is applied from the remaining insertions.

Radius of Convergence

The convergence of the conformal block expansion follows from the convergence of the OPE, which we now discuss.

In a generic local quantum field theory, the OPE is typically understood as an asymptotic expansion. However, in a unitary Euclidean CFT, the OPE has a genuine finite radius of convergence. This is an important structural advantage of CFT over generic QFT.

Geometrically, the OPE $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ converges when there exists a sphere separating the points x_1, x_2 from all other insertions in the correlator, as illustrated in Figure 2.1.

The two operator insertions inside the sphere prepare a state on the sphere. In radial quantization, this state can be expanded in a complete basis of cylinder energy eigenstates. The OPE therefore converges whenever the sphere separates the two chosen insertions from all other operators.

The conformal block expansion may be truncated by retaining only primary operators with $\Delta < \Delta_*$. At fixed cross ratios inside the region of convergence, the resulting error decreases exponentially with Δ_* [38].

2.2.2 Conformal bootstrap

Writing the crossing equation for a four-point function in terms of its conformal block expansion is often called the bootstrap equation. For identical scalar operators, it takes the form

$$F_{0,0}(u, v) + \sum_{\substack{\Delta, \ell \text{ even} \\ (\Delta, \ell) \neq (0, 0)}} a_{\Delta, \ell} F_{\Delta, \ell}(u, v) = 0, \quad (2.62)$$

where

$$F_{\Delta, \ell}(u, v) = v^{\Delta_{\mathcal{O}}} g_{\Delta, \ell}(u, v) - u^{\Delta_{\mathcal{O}}} g_{\Delta, \ell}(v, u). \quad (2.63)$$

Here the first term is the contribution of the identity operator, for which

$$g_{0,0}(u, v) = 1, \quad a_{0,0} = 1. \quad (2.64)$$

The bootstrap equation constrains the spectrum and OPE coefficients of the theory. In a unitary CFT, the coefficients $a_{\Delta,\ell}$ are non-negative. Therefore, we may regard each $F_{\Delta,\ell}(u, v)$ as a vector in the vector space of functions of u and v , with non-negative coefficients in the crossing equation.

The basic idea of the numerical bootstrap [17] is to assume a possible spectrum and test whether it is compatible with crossing symmetry and unitarity. For example, suppose that there is no scalar primary with dimension in the interval (Δ_1, Δ_2) . If one can find a linear functional α such that

$$\alpha[F_{0,0}] > 0, \quad (2.65)$$

and

$$\alpha[F_{\Delta,0}] \geq 0, \quad \Delta \geq \frac{d-2}{2}, \quad \Delta \notin (\Delta_1, \Delta_2), \quad (2.66)$$

together with

$$\alpha[F_{\Delta,\ell}] \geq 0, \quad \Delta \geq \ell + d - 2, \quad \ell \geq 2, \quad \ell \text{ even}, \quad (2.67)$$

then applying α to the crossing equation gives

$$0 = \alpha[F_{0,0}] + \sum_{\substack{\Delta, \ell \text{ even} \\ (\Delta,\ell) \neq (0,0)}} a_{\Delta,\ell} \alpha[F_{\Delta,\ell}] > 0, \quad (2.68)$$

which is a contradiction. The assumed spectral gap is therefore excluded.

In practice, the functional is approximated by a finite linear combination of derivatives evaluated at the crossing-symmetric point [17, 18, 39, 40],

$$\alpha[f] = \sum_{m+n \leq \Lambda} \alpha_{m,n} \partial_z^m \partial_{\bar{z}}^n f(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}}. \quad (2.69)$$

The search for a functional satisfying the required positivity conditions then becomes a numerical optimization problem. By increasing the number of derivatives, we can obtain stronger bounds on the allowed CFT data.

A different approach is the analytic bootstrap [8, 18, 19, 41–52]. In this approach, one often considers controlled limits of the cross ratios, such as the double-lightcone limit

$$z \rightarrow 0, \quad \bar{z} \rightarrow 1. \quad (2.70)$$

In such limits, the singular behavior generated by operators in one channel must be reproduced by an infinite sum of conformal blocks in the crossed channel. This relation often determines asymptotic properties of the OPE data, in particular the large-spin behavior of anomalous dimensions and OPE coefficients [42, 43].

The results in this thesis are obtained using analytic bootstrap methods. In particular, later chapters will use the Lorentzian inversion formula [53] to extract OPE data from the crossed-channel double discontinuity. We now discuss generalized free field theory, which gives a simple exact solution of crossing symmetry and provides the starting point for the large- N expansions considered later.

2.3 Generalized free fields

A generalized free field is a scalar primary whose correlators are built from two-point functions by Wick contractions. Namely,

$$\begin{aligned} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle_{\text{GFF}} &= \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle \langle \mathcal{O}(x_3)\mathcal{O}(x_4) \rangle \\ &\quad + \langle \mathcal{O}(x_1)\mathcal{O}(x_3) \rangle \langle \mathcal{O}(x_2)\mathcal{O}(x_4) \rangle \\ &\quad + \langle \mathcal{O}(x_1)\mathcal{O}(x_4) \rangle \langle \mathcal{O}(x_2)\mathcal{O}(x_3) \rangle \\ &= \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} + \frac{1}{(x_{13}^2 x_{24}^2)^{\Delta_{\mathcal{O}}}} + \frac{1}{(x_{14}^2 x_{23}^2)^{\Delta_{\mathcal{O}}}}. \end{aligned} \quad (2.71)$$

Therefore,

$$\mathcal{G}_{\text{GFF}}(u, v) = 1 + u^{\Delta_{\mathcal{O}}} + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}. \quad (2.72)$$

We refer to this as the disconnected contribution.

In the $12 \rightarrow 34$ OPE channel, the first term is the contribution of the identity operator. Crossing symmetry requires the remaining two terms to be reproduced by an infinite tower of double-trace operators,

$$[\mathcal{O}\mathcal{O}]_{n,\ell} \sim \mathcal{O} \square^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}, \quad n = 0, 1, \dots, \quad \ell = 0, 2, 4, \dots, \quad (2.73)$$

with their scaling dimensions given by

$$\Delta_{n,\ell}^{(0)} = 2\Delta_{\mathcal{O}} + 2n + \ell. \quad (2.74)$$

The conformal block decomposition of the generalized free field correlator is therefore

$$\mathcal{G}_{\text{GFF}}(u, v) = 1 + \sum_{n=0}^{\infty} \sum_{\ell=0,2,\dots} a_{n,\ell}^{(0)} u^{\Delta_{\mathcal{O}}+n} \tilde{g}_{2\Delta_{\mathcal{O}}+2n+\ell,\ell}(u, v). \quad (2.75)$$

Equivalently,

$$u^{\Delta_{\mathcal{O}}} + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} = \sum_{n=0}^{\infty} \sum_{\ell=0,2,\dots} a_{n,\ell}^{(0)} g_{2\Delta_{\mathcal{O}}+2n+\ell,\ell}(u, v). \quad (2.76)$$

To see how the coefficients are fixed, consider the limit $z \rightarrow 0$ with $\bar{z}$ fixed. The leading contribution of a double-trace block is

$$g_{2\Delta_{\mathcal{O}}+2n+\ell,\ell}(z, \bar{z}) = z^{\Delta_{\mathcal{O}}+n} k_{\Delta_{\mathcal{O}}+n+\ell}(\bar{z}) + O(z^{\Delta_{\mathcal{O}}+n+1}). \quad (2.77)$$

At leading order in z , the crossing equation becomes

$$\bar{z}^{\Delta_{\mathcal{O}}} \left[1 + \frac{1}{(1-\bar{z})^{\Delta_{\mathcal{O}}}} \right] = \sum_{\ell=0,2,\dots} a_{0,\ell}^{(0)} k_{\Delta_{\mathcal{O}}+\ell}(\bar{z}). \quad (2.78)$$

Expanding both sides around $\bar{z} = 0$ determines the coefficients $a_{0,\ell}^{(0)}$ recursively. The leading term gives $a_{0,0}^{(0)} = 2$, and at higher orders the contributions of the lower-spin blocks are subtracted to determine the coefficients at higher spin. Matching the higher powers of z and repeating the same procedure determines the full tower of OPE coefficients.

In four dimensions, the result is [12]

$$a_{n,\ell}^{(0)} = \frac{2(\ell+1)(2\Delta_{\mathcal{O}}+2n+\ell-2)}{(\Delta_{\mathcal{O}}-1)^2} C_n^{\Delta_{\mathcal{O}}-1} C_{n+\ell+1}^{\Delta_{\mathcal{O}}-1}, \quad (2.79)$$

where

$$C_n^{\alpha} = \frac{\Gamma^2(\alpha+n)\Gamma(2\alpha+n-1)}{\Gamma(n+1)\Gamma^2(\alpha)\Gamma(2\alpha+2n-1)}. \quad (2.80)$$

Therefore, generalized free field theory provides an exact crossing-symmetric solution whose $\mathcal{O} \times \mathcal{O}$ OPE contains only the identity and double-trace operators. In later chapters, the dimensions and OPE coefficients of these double-trace operators will receive corrections in the large- N expansion.

Chapter 3

AdS/CFT and holographic correlators

AdS/CFT Correspondence [9–11] is a conjectured equivalence between a conformal field theory in d spacetime dimensions and a quantum gravity theory in the $d+1$ dimensional Anti-de Sitter space. The CFT lives on the conformal boundary of AdS. This correspondence provides a concrete realization of the holographic principle [54, 55].

A CFT with a perturbative bulk description have a special structure. It possess a parameter N which control the perturbation in the bulk [56]. N is the measure of number of degrees of freedom in the CFT. In the gauge theory examples, N can be regarded as the rank of the gauge group. Then the CFT should admit an expansion around $N \rightarrow \infty$ limit in which correlation functions become generalized free. In addition, for the bulk theory to admit a local low-energy description, the spectrum of low-dimension operators must be sufficiently sparse [12].

The purpose of this chapter is to introduce the ingredient of this holographic description that will be used throughout the thesis. We begin by reviewing the AdS space and the basic AdS/CFT dictionary between CFT operators and bulk fields. We then discuss the large N limit and discuss how the $\frac{1}{N}$ expansion of CFT correlation function is related to perturbation theory in the bulk. Next we will introduce the distinction between single-trace and multi-trace operators and explore their interpretation as one-particle and multi-particle states in AdS. Finally we review Witten diagrams, which provide the bulk representation of CFT correlation functions and make the relation between the large- N expansion and the bulk loop expansion explicit.

3.1 Basics of AdS/CFT

AdS space

Anti-de Sitter space is a maximally symmetric spacetime of constant negative curvature. In the Poincare patch, the Euclidean metric of AdS_{d+1} can be written as

$$ds^2 = \frac{R_{\text{AdS}}^2}{z^2} (dz^2 + dx^\mu dx_\mu), \quad z > 0. \quad (3.1)$$

where R_{AdS} is the AdS radius. x^μ with $\mu = 1, \dots, d$ are coordinates along the boundary directions, and z is the radial coordinate. The surface $z = 0$ defines the conformal boundary of AdS, and lies at infinite proper distance. A rescaling of the coordinates

$$x^\mu \rightarrow \lambda x^\mu, \quad z \rightarrow \lambda z \quad (3.2)$$

leaves the AdS metric invariant. More generally AdS isometries reproduce the conformal transformations on the boundary theory. The isometry group of Euclidean AdS_{d+1} is $SO(d+1, 1)$ which is the conformal group of a Euclidean CFT in d dimensions. Similarly, the isometry group of Lorentzian AdS_{d+1} is $SO(d, 2)$, which is the conformal group of a Lorentzian CFT in d dimensions.

AdS/CFT Dictionary

The AdS/CFT correspondence associates every local primary operator in the CFT with a field in the AdS bulk. Schematically

$$\mathcal{O}_{\Delta, \ell}(x) \longleftrightarrow \phi_{\Delta, \ell}(X) \quad (3.3)$$

where $\mathcal{O}_{\Delta, \ell}$ is a primary operator of scaling dimension Δ and spin ℓ , and $\phi_{\Delta, \ell}$ is a bulk field with the corresponding spin. In particular, a scalar primary is dual to a bulk scalar field, a conserved current is dual to a bulk gauge field, and the CFT stress tensor is dual to the graviton in the bulk.

For a scalar field of mass m , the relation between its bulk mass and the scaling dimension of the dual operator is given by

$$m^2 R_{\text{AdS}}^2 = \Delta(\Delta - d). \quad (3.4)$$

More generally the quantum numbers of bulk fields are determined by conformal representation of the dual operators.

3.1.1 Correlation functions from the bulk

To make the relation between bulk fields and CFT operators more precise, consider a scalar field ϕ in Euclidean AdS_{d+1} with free action

$$S_{\text{bulk}}^{\text{free}}[\phi] = \frac{1}{2} \int d^{d+1}X \sqrt{g} (g^{MN} \partial_M \phi \partial_N \phi + m^2 \phi^2). \quad (3.5)$$

Varying this action gives the Klein–Gordon equation

$$(\nabla^2 - m^2) \phi = 0. \quad (3.6)$$

Using the metric (3.1), this equation becomes

$$[z^2 \partial_z^2 - (d-1)z \partial_z + z^2 \partial_\mu \partial^\mu - m^2 R_{\text{AdS}}^2] \phi(z, x) = 0. \quad (3.7)$$

Near the boundary, looking for a solution of the form

$$\phi(z, x) = z^\alpha f(x) \quad (3.8)$$

gives

$$\alpha(\alpha - d) - m^2 R_{\text{AdS}}^2 = 0. \quad (3.9)$$

Therefore, near the boundary we can write

$$\phi(z, x) = z^{\Delta_-} \phi_{(0)}(x) + z^{\Delta_+} \phi_{(1)}(x) + \cdots, \quad (3.10)$$

where

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R_{\text{AdS}}^2}. \quad (3.11)$$

Near the boundary the solution with $\Delta_- = d - \Delta_+$ dominates and we can write

$$\phi(z, x) \Big|_{z \rightarrow 0} = z^{d-\Delta_+} J(x). \quad (3.12)$$

where we used $\phi_{(0)}(x) = J(x)$, which we identify it with the source of the dual scalar operator. This source defines the CFT partition function

$$Z_{\text{CFT}}[J] = \int [D\varphi] \exp \left[-S_{\text{CFT}}[\varphi] + \int d^d x J(x) \mathcal{O}(x) \right]. \quad (3.13)$$

Therefore $\Delta_+ = \Delta$ is the scaling dimension of the dual CFT operator. Taking functional derivatives of the partition function gives the connected correlation functions,

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle_{\text{conn}} = \frac{\delta^n \log Z_{\text{CFT}}[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0}. \quad (3.14)$$

The AdS/CFT prescription identifies

$$Z_{\text{CFT}}[J] = Z_{\text{AdS}}[J] \quad (3.15)$$

with the boundary condition

$$\lim_{z \rightarrow 0} z^{\Delta-d} \phi(z, x) = J(x). \quad (3.16)$$

In the semiclassical limit, the bulk path integral is dominated by the classical solution ϕ_J satisfying this boundary condition. Namely

$$Z_{\text{AdS}}[J] \simeq \exp \left[-S_{\text{bulk}}^{\text{on-shell}}[\phi_J] \right]. \quad (3.17)$$

Therefore, in the semiclassical limit, the on-shell action is a functional of the boundary source J , and its functional derivatives generate connected CFT correlation functions. These contributions are represented diagrammatically by tree-level Witten diagrams, which we discuss their construction in section 3.4. Loop-level Witten diagrams also arise from quantum fluctuations around the classical solution.

3.2 Large N CFTs

The AdS/CFT Correspondence can in principle relate a CFT to a quantum gravity theory in AdS, without requiring a perturbative bulk description. However, in this thesis we consider a more restricted regime of CFTs that admit an expansion controlled by a large parameter N . In gauge theory examples, N is the rank of the gauge group. More generally it denotes a parameter with respect to which CFT correlation function and OPE data can be organized in an expansion around $N \rightarrow \infty$. Now when a bulk dual exist, this gives a perturbative description as bulk interactions and loop corrections are suppressed by inverse powers of N .

In the case of gauge theory, local operators can be organized according to the number of traces over color indices, schematically, operators of the form

$$\text{Tr}(\Phi^k) \quad (3.18)$$

are called single-trace operators, and products such as

$$\text{Tr}(\Phi^{k_1}) \text{Tr}(\Phi^{k_2}) \quad (3.19)$$

are double-trace operators. In a general large N CFT, we retain this terminology. Operationally, single-trace operators are basic operators dual to the fundamental fields in the bulk, and multi-trace operators are composite operators built from products of single traces. The bulk interpretation of these operators will be discussed in the next section.

We choose the parameter N such that the stress tensor two point function, known as central charge C_T scale as $C_T \sim N^2$. Then we can normalize the single-trace operators in the theory such that their connected n -point function scale as

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_{\text{conn}} = \mathcal{O}(N^{2-n}). \quad (3.20)$$

For example for a scalar single trace operator $\mathcal{O}$:

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{1}{x_{12}^{2\Delta_{\mathcal{O}}}}. \quad (3.21)$$

And its 4-point function has the following large N expansion:

$$\begin{aligned} \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle &= \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle \langle \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle \\ &+ \langle \mathcal{O}(x_1) \mathcal{O}(x_3) \rangle \langle \mathcal{O}(x_2) \mathcal{O}(x_4) \rangle \\ &+ \langle \mathcal{O}(x_1) \mathcal{O}(x_4) \rangle \langle \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle + \mathcal{O}\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.22)$$

At strict $N = \infty$, the correlator then has the same factorized form as a generalized free field correlator. Interactions first become visible in the subleading terms of the $\frac{1}{N}$ expansion.

Bulk couplings. The bulk interpretation of large- N factorization follows directly from the relation between C_T and the bulk Newton constant. The Einstein-Hilbert term in the bulk action takes the form

$$S_{\text{grav}} = \frac{1}{16\pi G_N} \int d^{d+1}X \sqrt{g} \left(R + \frac{d(d-1)}{R_{\text{AdS}}^2} + \dots \right). \quad (3.23)$$

where R is the Ricci scalar.

The AdS/CFT dictionary relate the overall coefficient of this action to the CFT central charge,

$$\frac{R_{\text{AdS}}^{d-1}}{G_N} \sim C_T \sim N^2. \quad (3.24)$$

Hence, the dimensionless bulk gravitational coupling behaves as

$$g_{\text{bulk}} \sim \sqrt{\frac{G_N}{R_{\text{AdS}}^{d-1}}} \sim \frac{1}{N}. \quad (3.25)$$

So in this sense, large N corresponds to weak bulk interactions.

3.2.1 Large Gap

To write the bulk theory as a local effective field theory description, there is another condition that the dual CFT should have. Namely, the CFT should have a parametrically large gap in the single-trace spectrum.

Suppose that, besides the stress tensor and a set of few light operators, the first additional single-trace operators appears only above a large gap

$$\Delta_{\text{gap}} \gg 1. \quad (3.26)$$

For a sufficiently heavy bulk field, its mass and dimension are related by

$$mR_{\text{AdS}} \approx \Delta. \quad (3.27)$$

Therefore, the gap implies that the corresponding bulk degrees of freedom have masses

$$M_{\text{heavy}}R_{\text{AdS}} \sim \Delta_{\text{gap}} \gg 1. \quad (3.28)$$

At energies well below M_{heavy} , these fields can be integrated out. The large gap is therefore the condition that allows the bulk theory to be approximated by a local low energy effective field theory.

The distinct role of large N and large gap is particularly visible in the example of type IIB string theory on $\text{AdS}_5 \times S^5$. In this example the 't Hooft coupling is given by

$$\lambda_{\text{th}} = g_{\text{YM}}^2 N = 4\pi g_s N, \quad (3.29)$$

where g_{YM} denotes the coupling of the boundary $\mathcal{N} = 4$ super-Yang–Mills theory, and g_s is the string coupling.

The AdS radius is related to the string length $\ell_s = \sqrt{\alpha'}$ by

$$\frac{R_{\text{AdS}}^2}{\alpha'} = \sqrt{\lambda_{\text{th}}}. \quad (3.30)$$

At fixed λ_{th} the limit $N \rightarrow \infty$ gives

$$g_s \sim \frac{\lambda_{\text{th}}}{N} \rightarrow 0. \quad (3.31)$$

So string-loop corrections are suppressed.

On the other hand,

$$\frac{\ell_s}{R_{\text{AdS}}} \sim \lambda_{\text{th}}^{-\frac{1}{4}}. \quad (3.32)$$

Therefore when $\lambda_{\text{th}} \gg 1$, the massive string excitations have

$$m_{\text{string}}R_{\text{AdS}} \sim \frac{R_{\text{AdS}}}{\ell_s} \sim \lambda_{\text{th}}^{\frac{1}{4}} \gg 1. \quad (3.33)$$

So N suppresses the quantum bulk corrections, and λ_{th} suppresses finite-string-length corrections.

So when both conditions hold, the low energy dynamics of a light scalar field can be written schematically as

$$S_{\text{bulk}} = \int d^{d+1}X \sqrt{g} \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g_3}{N}\phi^3 + \frac{g_4}{N^2}\phi^4 + \dots \right]. \quad (3.34)$$

Thus cubic interactions are of order $\frac{1}{N}$, and quartic interactions are of order $\frac{1}{N^2}$. Moreover, each additional bulk loop introduce a further suppression by $\frac{1}{N^2}$. This reproduces the large N expansion of connected CFT correlation functions.

3.3 Multi-particle states in the bulk

To discuss the bulk interpretation of CFT operators, let us consider Lorentzian AdS in global coordinates,

$$ds^2 = R_{\text{AdS}}^2 \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{d-1}^2 \right). \quad (3.35)$$

Near the boundary, $\rho \rightarrow \infty$, the metric becomes, up to an overall Weyl factor,

$$ds_{\partial\text{AdS}}^2 = -dt^2 + d\Omega_{d-1}^2. \quad (3.36)$$

So, the conformal boundary of global AdS is the cylinder $\mathbb{R} \times S^{d-1}$, which is the spacetime naturally associated with radial quantization in the CFT. Translations in global AdS time are therefore identified with dilatations in the CFT [10, 57]. So we can write

$$H_{\text{AdS}} = \frac{D}{R_{\text{AdS}}}. \quad (3.37)$$

For a scalar primary operator, the state-operator correspondence gives

$$|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle, \quad D|\mathcal{O}\rangle = \Delta_{\mathcal{O}}|\mathcal{O}\rangle. \quad (3.38)$$

Therefore, the corresponding state has bulk energy

$$E_{\text{AdS}} = \frac{\Delta_{\mathcal{O}}}{R_{\text{AdS}}}. \quad (3.39)$$

Consider now a scalar single-trace operator $\mathcal{O}$ dual to a bulk scalar field ϕ . The lowest mode of ϕ has energy $\Delta_{\mathcal{O}}/R_{\text{AdS}}$. So, the state created by the single-trace primary is identified with the one-particle state associated with the lowest normal mode of the bulk field,

$$\mathcal{O}(0)|0\rangle \quad \longleftrightarrow \quad a_{\mathcal{O}}^\dagger|0\rangle. \quad (3.40)$$

More generally, the descendants of $\mathcal{O}$ correspond to excited one-particle states of the same bulk field.

The interpretation of multi-trace operators follows from large- N factorization. Consider two single-trace operators $\mathcal{O}_1$ and $\mathcal{O}_2$. A double-trace primary has the schematic form

$$[\mathcal{O}_1\mathcal{O}_2]_{n,\ell} \sim \mathcal{O}_1 \square^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}_2 + \cdots, \quad (3.41)$$

At leading order in the large- N expansion, its dimension is

$$\Delta_{[\mathcal{O}_1\mathcal{O}_2]_{n,\ell}}^{(0)} = \Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2} + 2n + \ell. \quad (3.42)$$

These are exactly the lowest energies of the irreducible two-particle sectors in global AdS [57]. At large N , inner products of states built from two single-trace operators factorize schematically as

$$\begin{aligned} \langle \mathcal{O}_3 \mathcal{O}_4 | \mathcal{O}_1 \mathcal{O}_2 \rangle &= \langle \mathcal{O}_3 | \mathcal{O}_1 \rangle \langle \mathcal{O}_4 | \mathcal{O}_2 \rangle \\ &+ \langle \mathcal{O}_3 | \mathcal{O}_2 \rangle \langle \mathcal{O}_4 | \mathcal{O}_1 \rangle + \mathcal{O} \left(\frac{1}{N^2} \right). \end{aligned} \quad (3.43)$$

Therefore, the double-trace primary creates a two-particle state,

$$[\mathcal{O}_1 \mathcal{O}_2]_{n,\ell}(0)|0\rangle \longleftrightarrow |\phi_1 \phi_2; n, \ell\rangle. \quad (3.44)$$

Bulk interactions shift the energy of this state. Equivalently, the double-trace operator acquires an anomalous dimension,

$$\Delta_{[\mathcal{O}_1 \mathcal{O}_2]_{n,\ell}} = \Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2} + 2n + \ell + \gamma_{n,\ell}, \quad (3.45)$$

so that

$$E_{\phi_1 \phi_2; n, \ell} = \frac{\Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2} + 2n + \ell + \gamma_{n,\ell}}{R_{\text{AdS}}}. \quad (3.46)$$

Therefore, $\gamma_{n,\ell}/R_{\text{AdS}}$ is interpreted as the interaction energy of the two-particle state.

The same argument extends to higher-trace operators [58]. Namely, an operator built from k single-trace operators creates a k -particle state in AdS.

3.4 Witten Diagrams

Witten diagrams provide a bulk representation of CFT correlation functions [10, 11]. They are the AdS analogues of Feynman diagrams. External operator insertions are connected to bulk interaction vertices by bulk-to-boundary propagators, and internal lines are represented by bulk-to-bulk propagators.

Bulk-to-Boundary Propagator. Consider a scalar field ϕ dual to a scalar primary operator $\mathcal{O}$ of dimension Δ . The free bulk field obeys

$$(\nabla^2 - m^2) \phi(z, x) = 0 \quad (3.47)$$

with the boundary condition

$$\phi(z, x) \underset{z \rightarrow 0}{\sim} z^{d-\Delta} J(x), \quad \Delta > \frac{d}{2}. \quad (3.48)$$

Then requiring regularity in the interior of Euclidean AdS selects a unique solution. So we can write

$$\phi_J(z, x) = \int d^d x' K_\Delta(z, x, x') J(x'), \quad (3.49)$$

where $K_\Delta(z, x, x')$ is the bulk-to-boundary propagator. Here (z, x) denotes a bulk point and x' denotes a boundary point. In the Poincaré patch, the scalar bulk-to-boundary propagator is given by

$$K_\Delta(z, x, x') = C_\Delta \left(\frac{z}{z^2 + |x - x'|^2} \right)^\Delta, \quad (3.50)$$

where

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})}. \quad (3.51)$$

It obeys

$$(\nabla^2 - m^2) K_\Delta(z, x; x') = 0 \quad (3.52)$$

with

$$m^2 R_{\text{AdS}}^2 = \Delta(\Delta - d). \quad (3.53)$$

The bulk-to-boundary propagator describes how a source inserted at the boundary point x' produces a field profile in AdS.

Bulk-to-Bulk Propagator. To describe internal propagation within the bulk, we can introduce the bulk-to-bulk propagator. Let X and Y be two bulk points. The scalar bulk-to-bulk propagator $G_\Delta(X, Y)$ is defined as the Green function satisfying

$$(\nabla_X^2 - m^2) G_\Delta(X, Y) = -\frac{\delta^{(d+1)}(X - Y)}{\sqrt{g(X)}}. \quad (3.54)$$

Therefore, $G_\Delta(X, Y)$ describes the field produced at the point X by a point-like source placed at the bulk point Y . When $X = (z, x)$ approaches the boundary with Y fixed, we have the behavior

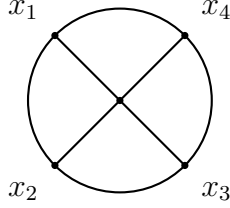
$$G_\Delta(X, Y) \sim z^\Delta, \quad z \longrightarrow 0. \quad (3.55)$$

3.4.1 Examples

Witten diagrams are constructed by combining bulk propagators with interaction vertices and integrating over the positions of the bulk vertices.

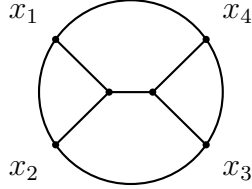
For example, in the case of a quartic interaction, the corresponding four-point contact Witten diagram is

$$\mathcal{W}_{\text{contact}}(x_i) = \lambda \int_{\text{AdS}} d^{d+1}X \sqrt{g(X)} \prod_{i=1}^4 K_{\Delta_i}(X; x_i). \quad (3.56)$$



As another example, consider the tree-level exchange diagram. For the exchange of a scalar field of dimension Δ_p , it takes the form

$$\begin{aligned} \mathcal{W}_{\text{exchange}}(x_i) = & g_{12p} g_{34p} \int_{\text{AdS}} d^{d+1}X \sqrt{g(X)} \int_{\text{AdS}} d^{d+1}Y \sqrt{g(Y)} \\ & \times K_{\Delta_1}(X; x_1) K_{\Delta_2}(X; x_2) G_{\Delta_p}(X, Y) K_{\Delta_3}(Y; x_3) K_{\Delta_4}(Y; x_4). \end{aligned} \quad (3.57)$$



Beyond tree level, Witten diagrams are constructed by the same rules but contain closed loops of bulk-to-bulk propagators, with the positions of all bulk vertices integrated over AdS. These diagrams describe quantum corrections in the bulk. In a large- N holographic CFT, each additional bulk loop is suppressed by a factor of $1/N^2$. For example, a one-loop bubble diagram contributes to the four-point function at order $\mathcal{O}(1/N^4)$, compared with the $\mathcal{O}(1/N^2)$ tree-level contact diagram.

Later in this thesis, we will study such diagrams in more details, but rather than evaluating them directly in the bulk using bulk-to-bulk and bulk-to-boundary propagators, we will study them from a boundary perspective using conformal bootstrap techniques. However, before turning to this analysis, we now devote the next few chapters to develop the necessary tools.

Part II

Analytic Methods for Holographic Correlators

Chapter 4

The Lorentzian Inversion Formula

In the previous chapters, we introduced the conformal block expansion and the crossing equation for four-point functions of identical scalar operators. In this chapter, we introduce the Lorentzian inversion formula, originally derived in [53]. This formula is a powerful tool for extracting CFT data from the singular part of the correlator. It is analogous in spirit to a dispersion relation, but instead of reconstructing a function from its imaginary part, it reconstructs the OPE data from the double discontinuity of the correlator. Moreover, the Lorentzian inversion formula gives an analytic continuation of the OPE data in the spin variable.

In this chapter, we give a sketch of the proof following the light-ray derivation of [59]. After stating the formula, we explain how its poles encode the CFT data and how it can be used in large-spin perturbation theory. We will also provide specific examples.

4.1 Proof sketch

4.1.1 Partial wave expansion

In Chapter 2 we introduced the conformal block expansion of a conformal four-point function. The four-point function may equivalently be expanded in conformal partial waves [59–61],

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \sum_{\ell=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \rho(\Delta, \ell) \Psi_{\Delta, \ell}(x_i). \quad (4.1)$$

The integral runs over the upper half of the principal series. The conformal partial wave is given by

$$\Psi_{\Delta,\ell}(x_i) = \int d^d x \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}_{\Delta,\ell}(x) \rangle \langle \tilde{\mathcal{O}}_{\Delta,\ell}(x) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle. \quad (4.2)$$

Here $\tilde{\mathcal{O}}_{\Delta,\ell}$ denotes the shadow of $\mathcal{O}_{\Delta,\ell}$ and has scaling dimension $d - \Delta$. We have suppressed the spin indices in this schematic expression. A conformal partial wave is a single-valued eigenfunction of the quadratic conformal Casimir. Since the Casimir eigenvalue is invariant under $\Delta \rightarrow d - \Delta$, it can be written as a linear combination of the block of an operator and the block of its shadow:

$$\Psi_{\Delta,\ell}(x_i) = \frac{1}{(x_{12}^2)^{\Delta_{\mathcal{O}}}(x_{34}^2)^{\Delta_{\mathcal{O}}}} [K_{\Delta,\ell} g_{\Delta,\ell}(u, v) + K_{d-\Delta,\ell} g_{d-\Delta,\ell}(u, v)], \quad (4.3)$$

where $K_{\Delta,\ell}$ and $K_{d-\Delta,\ell}$ are kinematic factors whose explicit form is not important for this discussion.

Using the shadow symmetry of the partial waves and their coefficients, we can unfold the contour in (4.1) and rewrite the expansion as

$$\mathcal{G}(u, v) = \sum_{\ell=0}^{\infty} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} c(\Delta, \ell) g_{\Delta,\ell}(u, v), \quad (4.4)$$

where $c(\Delta, \ell) = \rho(\Delta, \ell) K_{\Delta,\ell}$.

By comparing (4.4) with the ordinary conformal block expansion, we see that near a physical pole associated with an operator of dimension Δ_* and spin ℓ , the function $c(\Delta, \ell)$ behaves as

$$c(\Delta, \ell) \underset{\Delta \rightarrow \Delta_*}{\sim} \frac{a_{\Delta_*,\ell}}{\Delta_* - \Delta}, \quad (4.5)$$

where $a_{\Delta_*,\ell} = f_{\tilde{\mathcal{O}}\mathcal{O}\mathcal{O}_*}^2$ is the squared OPE coefficient. Thus, the function $c(\Delta, \ell)$ contains the OPE data. This is the same object that the inversion formula reconstructs from an integral over the singular part of the correlator.

Now consider the following object:

$$P_{\Delta,\ell}(x, w; x_3, x_4) = \mathcal{N}_{\Delta,\ell} \int d^d x_1 d^d x_2 \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle \\ \times \langle \tilde{\mathcal{O}}(x_1) \tilde{\mathcal{O}}(x_2) \mathcal{O}_{\Delta,\ell}(x, w) \rangle, \quad (4.6)$$

where $\mathcal{N}_{\Delta,\ell}$ is a normalization factor chosen consistently with the normalization of $c(\Delta, \ell)$, and we used index-free notation for the spin- ℓ operator. Here $\tilde{\mathcal{O}}$ denotes the shadow of the external scalar operator $\mathcal{O}$.

By conformal covariance, this object is proportional to the unique three-point structure involving $\mathcal{O}_{\Delta,\ell}$ and two external scalars [59],

$$P_{\Delta,\ell}(x, w; x_3, x_4) = c(\Delta, \ell) \langle \mathcal{O}_{\Delta,\ell}(x, w) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle. \quad (4.7)$$

With the normalization specified above, the proportionality coefficient is the same function $c(\Delta, \ell)$ that appears in (4.4) and stores the OPE data.

To express $c(\Delta, \ell)$ as a Lorentzian integral, we analytically continue (4.6) to Lorentzian signature. We now discuss this continuation.

4.1.2 Lorentzian continuation

We recall that the Euclidean correlator is single-valued whenever no two operator insertions coincide. After continuation to Lorentzian signature, singularities can also appear when two insertions become null separated. Therefore, Lorentzian correlators depend on the ordering of the operators. A Lorentzian correlator with a specified ordering is called a Wightman correlator, and it can be obtained from the Euclidean correlator by the prescription

$$t_i \longrightarrow t_i - i\epsilon_i, \quad (4.8)$$

where the relative ordering of the infinitesimal parameters ϵ_i determines the ordering of the Lorentzian operators.

Different Lorentzian orderings are associated with different analytic continuations around branch points of the correlator in the complex z and $\bar{z}$ planes. For the Lorentzian inversion formula, the relevant branch point is $\bar{z} = 1$. We therefore define the double discontinuity around $\bar{z} = 1$ by

$$\text{dDisc} [\mathcal{G}(z, \bar{z})] = \mathcal{G}(z, \bar{z}) - \frac{1}{2} \mathcal{G}^\circ(z, \bar{z}) - \frac{1}{2} \mathcal{G}^\circ(z, \bar{z}), \quad (4.9)$$

where $\mathcal{G}^\circ$ and $\mathcal{G}^\circ$ are the two analytic continuations obtained by taking $\bar{z}$ around $\bar{z} = 1$ in opposite directions. In a Lorentzian configuration in which x_4 lies in the future lightcone of x_1 , and x_2 lies in the future lightcone of x_3 , this double discontinuity is related to the following double commutator:

$$(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}} \langle [\mathcal{O}(x_4), \mathcal{O}(x_1)] [\mathcal{O}(x_2), \mathcal{O}(x_3)] \rangle = 2 \text{dDisc} [\mathcal{G}(z, \bar{z})]. \quad (4.10)$$

Some examples of the action of the double discontinuity that will be used in the rest of the thesis are

$$\text{dDisc} [(1 - \bar{z})^\alpha] = 2 \sin^2(\pi\alpha) (1 - \bar{z})^\alpha, \quad (4.11)$$

and

$$\text{dDisc} [\log(1 - \bar{z})] = 0, \quad \text{dDisc} [\log^2(1 - \bar{z})] = 4\pi^2. \quad (4.12)$$

More generally,

$$\text{dDisc} [\log^k(1 - \bar{z})] = 2\pi^2 k(k-1) \log^{k-2}(1 - \bar{z}) + \text{lower powers of } \log(1 - \bar{z}). \quad (4.13)$$

We now return to the object $P_{\Delta,\ell}$ in (4.6). To obtain the Lorentzian inversion formula, one analytically continues this object to Lorentzian signature and applies a light transform to the exchanged operator [59].

The light transform is defined by

$$\mathbf{L}[\mathcal{O}_{\Delta,\ell}](X, Z) = \int_{-\infty}^{\infty} d\alpha \mathcal{O}_{\Delta,\ell}(Z - \alpha X, -X), \quad (4.14)$$

where (X, Z) denote embedding-space coordinates. Although the light-transformed operator is non-local in physical spacetime, it transforms covariantly under the conformal group. Its quantum numbers are

$$\mathbf{L} : (\Delta, \ell) \mapsto (1 - \ell, 1 - \Delta). \quad (4.15)$$

The important property for us is that, when $\text{Re}(\Delta + \ell) > 1$, the light transform annihilates the vacuum [59]:

$$\mathbf{L}[\mathcal{O}_{\Delta,\ell}] |0\rangle = 0. \quad (4.16)$$

The proof of this statement uses the analyticity properties of Lorentzian correlators and a contour argument in the auxiliary α integral.

Applying the light transform to both sides of (4.7), we obtain

$$\mathbf{L}[P_{\Delta,\ell}](x, w; x_3, x_4) = c(\Delta, \ell) \langle \mathcal{O}(x_4) \mathbf{L}[\mathcal{O}_{\Delta,\ell}](x, w) \mathcal{O}(x_3) \rangle. \quad (4.17)$$

The left-hand side can also be computed by applying the light transform directly to the integral definition (4.6). After the Lorentzian continuation and the use of the property (4.16), the light transform of (4.6) can be written as

$$\begin{aligned} \mathbf{L}[P_{\Delta,\ell}](x, w; x_3, x_4) &= \mathcal{N}_{\Delta,\ell} \int_{\mathcal{R}_t} d^d x_1 d^d x_2 \langle [\mathcal{O}(x_4), \mathcal{O}(x_1)] [\mathcal{O}(x_2), \mathcal{O}(x_3)] \rangle \\ &\quad \times \langle \tilde{\mathcal{O}}(x_1) \mathbf{L}[\mathcal{O}_{\Delta,\ell}](x, w) \tilde{\mathcal{O}}(x_2) \rangle \\ &\quad + (1 \leftrightarrow 2). \end{aligned} \quad (4.18)$$

Here $\mathcal{R}_t$ is the Lorentzian region selected by the light transform in which the two commutators have support. In particular, x_1 is in the past of x_4 , and x_2 is in the future of x_3 . The first term is the t -channel contribution, and the term $(1 \leftrightarrow 2)$ gives the u -channel contribution.

To isolate $c(\Delta, \ell)$, one pairs the Lorentzian expression obtained above with the three-point structure dual to the light-transformed operator. This removes the

remaining dependence on the position and polarization of the light-ray operator. Conformal covariance then fixes the resulting kernel. In terms of the usual scalar conformal blocks, it is

$$g_{\ell+d-1, \Delta-d+1}(z, \bar{z}), \quad (4.19)$$

in which the shifted labels are a consequence of the light transform and the Lorentzian pairing. All in all, the t -channel contribution will be given by

$$c^{(t)}(\Delta, \ell) = \frac{1}{4} \kappa_{\frac{\Delta+\ell}{2}} \int_0^1 dz d\bar{z} \frac{|z - \bar{z}|^{d-2}}{(z\bar{z})^d} g_{\ell+d-1, \Delta-d+1}(z, \bar{z}) d\text{Disc}[\mathcal{G}(z, \bar{z})]. \quad (4.20)$$

where $\kappa_\beta = \frac{\Gamma(\beta)^4}{2\pi^2 \Gamma(2\beta-1) \Gamma(2\beta)}$, and we used (4.10) to write the double commutator as a double discontinuity. The u -channel gives a similar contribution $c^{(u)}(\Delta, \ell)$, and the complete answer is

$$c(\Delta, \ell) = c^{(t)}(\Delta, \ell) + (-1)^\ell c^{(u)}(\Delta, \ell). \quad (4.21)$$

For identical scalar operators, the two crossed-channel contributions are equal. In this case, specializing further to four spacetime dimensions, the inversion formula takes the form

$$c(\Delta, \ell) = \frac{1 + (-1)^\ell}{4} \kappa_{\frac{\Delta+\ell}{2}} \int_0^1 dz d\bar{z} \frac{(z - \bar{z})^2}{z^4 \bar{z}^4} g_{\ell+3, \Delta-3}(z, \bar{z}) d\text{Disc}[\mathcal{G}(z, \bar{z})]. \quad (4.22)$$

In a unitary CFT, boundedness of the correlator in the Regge limit guarantees the validity of the formula for $\ell > 1$ [53]. This completes the proof sketch of the Lorentzian inversion formula. Finally, it is also worth mentioning that this formula gives separate analytic continuations for even and odd spins. For identical external scalars, only the even-spin continuation is nonzero.

In the next sections, we explain how the poles of $c(\Delta, \ell)$ encode the CFT data in a large- N theory and how the formula can be used in the large-spin expansion.

4.2 Extracting CFT data

We can expand a CFT holographic correlator in large- N . Namely:

$$\mathcal{G}(u, v) = \mathcal{G}^{(0)}(u, v) + \frac{1}{N^2} \mathcal{G}^{(1)}(u, v) + \frac{1}{N^4} \mathcal{G}^{(2)}(u, v) + \dots \quad (4.23)$$

In a large- N CFT, the positions and residues of the poles in the OPE density function $c(\Delta, \ell)$ also admit a perturbative expansion. Consider a family of double-trace operators $[\mathcal{OO}]_{n, \ell}$, whose leading dimensions are $\Delta_{n, \ell}^{(0)} = 2\Delta_{\mathcal{O}} + 2n + \ell$. In

large- N expansion their dimensions and their squared OPE coefficients take the form

$$\Delta_{n,\ell} = \Delta_{n,\ell}^{(0)} + \frac{\gamma_{n,\ell}^{(1)}}{N^2} + \frac{\gamma_{n,\ell}^{(2)}}{N^4} + \dots, \quad (4.24)$$

$$a_{n,\ell} = a_{n,\ell}^{(0)} + \frac{a_{n,\ell}^{(1)}}{N^2} + \frac{a_{n,\ell}^{(2)}}{N^4} + \dots. \quad (4.25)$$

These data enter in the contribution of double trace operators into $c(\Delta, \ell)$. Namely

$$c(\Delta, \ell) \supset \frac{a_{n,\ell}}{\Delta_{n,\ell} - \Delta}. \quad (4.26)$$

Expanding this pole in powers of $1/N^2$, we find

$$\begin{aligned} \frac{a_{n,\ell}}{\Delta_{n,\ell} - \Delta} &= \frac{a_{n,\ell}^{(0)}}{\Delta_{n,\ell}^{(0)} - \Delta} \\ &+ \frac{1}{N^2} \left[\frac{a_{n,\ell}^{(1)}}{\Delta_{n,\ell}^{(0)} - \Delta} - \frac{a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)}}{\left(\Delta_{n,\ell}^{(0)} - \Delta\right)^2} \right] \\ &+ \frac{1}{N^4} \left[\frac{a_{n,\ell}^{(2)}}{\Delta_{n,\ell}^{(0)} - \Delta} - \frac{a_{n,\ell}^{(1)} \gamma_{n,\ell}^{(1)} + a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(2)}}{\left(\Delta_{n,\ell}^{(0)} - \Delta\right)^2} \right. \\ &\quad \left. + \frac{a_{n,\ell}^{(0)} \left(\gamma_{n,\ell}^{(1)}\right)^2}{\left(\Delta_{n,\ell}^{(0)} - \Delta\right)^3} \right] + \dots. \end{aligned} \quad (4.27)$$

Therefore, the OPE data can be read off order by order in the large- N expansion from the coefficients of the corresponding poles.

4.2.1 Generalized free field data

At order N^0 , we have the generalized free field correlator,

$$\mathcal{G}^{(0)}(u, v) = 1 + u^{\Delta_{\mathcal{O}}} + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}. \quad (4.28)$$

The first two terms have vanishing double discontinuity around $\bar{z} = 1$. Therefore, the only contribution to the inversion formula comes from the third term. Its double discontinuity is

$$\text{dDisc} \left[\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \right] = 2 \sin^2(\pi \Delta_{\mathcal{O}}) \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}. \quad (4.29)$$

Substituting (4.29) into the four-dimensional inversion formula gives

$$c^{(0)}(\Delta, \ell) = \frac{1 + (-1)^\ell}{2} \kappa_{\frac{\Delta+\ell}{2}} \sin^2(\pi \Delta_{\mathcal{O}}) \int_0^1 dz \int_0^1 d\bar{z} \frac{(z - \bar{z})^2}{z^4 \bar{z}^4} \times g_{\ell+3, \Delta-3}(z, \bar{z}) \left[\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right]^{\Delta_{\mathcal{O}}}. \quad (4.30)$$

This integral can be evaluated exactly. Using the explicit four-dimensional conformal blocks (2.57), one finds [19]

$$c^{(0)}(\Delta, \ell) = \frac{2^{2-\Delta}(\Delta-2)\Delta_{\mathcal{O}} [1 + (-1)^\ell] (\ell+1)\Gamma^2(2-\Delta_{\mathcal{O}})\Gamma(-\Delta_{\mathcal{O}})\Gamma(-\Delta+\ell+4)}{\Gamma(1-\Delta_{\mathcal{O}})\Gamma^2(\Delta_{\mathcal{O}})\Gamma^2\left(\frac{-\Delta+\ell+4}{2}\right)\Gamma(\Delta+\ell-1)\Gamma\left(\frac{-\Delta-2\Delta_{\mathcal{O}}+\ell+8}{2}\right)} \times \frac{\Gamma^2\left(\frac{\Delta+\ell}{2}\right)\Gamma\left(-\frac{\Delta}{2}+\Delta_{\mathcal{O}}+\frac{\ell}{2}\right)\Gamma\left(\frac{\Delta+2\Delta_{\mathcal{O}}+\ell-4}{2}\right)}{\Gamma\left(\frac{\Delta-2\Delta_{\mathcal{O}}+\ell+4}{2}\right)}. \quad (4.31)$$

The physical poles of this expression are located at

$$\Delta = 2\Delta_{\mathcal{O}} + 2n + \ell, \quad n = 0, 1, 2, \dots, \quad (4.32)$$

which reproduces the generalized free field double-trace spectrum.

Expanding (4.31) around these poles gives

$$c^{(0)}(\Delta, \ell) \underset{\Delta \rightarrow 2\Delta_{\mathcal{O}} + 2n + \ell}{\sim} \frac{a_{n,\ell}^{(0)}}{2\Delta_{\mathcal{O}} + 2n + \ell - \Delta}. \quad (4.33)$$

We can therefore read off the coefficients of the poles $a_{n,\ell}^{(0)}$, recovering (2.79).

4.2.2 Scalar exchange

Now let us consider another example in which the inversion integral can be evaluated without taking any limit of the cross-ratios. Suppose that the $\mathcal{O} \times \mathcal{O}$ OPE contains a scalar operator U with $\Delta_U = 2$. and squared OPE coefficient $a_{2,0}$.

The contribution of this operator to the crossed-channel conformal block expansion of $\mathcal{G}^{(1)}$ is given by

$$\mathcal{G}^{(1)}(z, \bar{z}) \supset a_{2,0} \frac{(z\bar{z})^{\Delta_{\mathcal{O}}} [(1-z)(1-\bar{z})]^{1-\Delta_{\mathcal{O}}}}{z - \bar{z}} \log\left(\frac{z}{\bar{z}}\right). \quad (4.34)$$

Taking the double discontinuity of it around $\bar{z} = 1$ gives

$$\text{dDisc} [\mathcal{G}^{(1)}(z, \bar{z})] = 2a_{2,0} \sin^2(\pi \Delta_{\mathcal{O}}) \frac{(z\bar{z})^{\Delta_{\mathcal{O}}} [(1-z)(1-\bar{z})]^{1-\Delta_{\mathcal{O}}}}{z - \bar{z}} \log\left(\frac{z}{\bar{z}}\right). \quad (4.35)$$

Inserting (4.35) into the inversion formula and evaluating the integral gives a function $c^{(1)}(\Delta, \ell)$ with double and simple poles at

$$\Delta = 2\Delta_{\mathcal{O}} + 2n + \ell. \quad (4.36)$$

The double-pole contribution is

$$c^{(1)}(\Delta, \ell) \supset -\frac{a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\text{as}}}{(2\Delta_{\mathcal{O}} + 2n + \ell - \Delta)^2}. \quad (4.37)$$

Extracting the coefficient of the double pole gives

$$\gamma_{n,\ell}^{(1),\text{as}} = -\frac{2a_{2,0}(\Delta_{\mathcal{O}} - 1)^2}{(\ell + 1)(\ell + 2\Delta_{\mathcal{O}} + 2n - 2)}, \quad (4.38)$$

where the superscript “as” denotes the asymptotic large-spin solution. This agrees with the resummed asymptotic solution obtained from crossing symmetry [50].

Note that the Lorentzian inversion formula determines the contribution in the OPE data that is analytic in spin. The result $\gamma_{n,\ell}^{(1),\text{as}}$ has infinite support in spin. The complete crossing-symmetric solution also contains a contribution with finite support. For this scalar exchange, that additional contribution is confined to $\ell = 0$ [50] and will be discussed in the next chapter.

4.3 Large-spin perturbation theory

The inversion formula can be evaluated over the complete Lorentzian integration region, as in the examples above. However, in many example, the full double integral cannot be performed in closed form. In such cases, the inversion formula can instead be used to construct a systematic large-spin expansion of the CFT data [19, 49, 53].

For identical scalar operators, the integrand of the inversion formula is symmetric under $z \leftrightarrow \bar{z}$. We may therefore restrict the integration region to $0 < z < \bar{z} < 1$ and multiply the result by two. We can then expand the integrand around

$$z \rightarrow 0, \quad \bar{z} \rightarrow 1. \quad (4.39)$$

The former limit isolates a particular double-trace family, and the latter determines its large-spin behavior. This regime is often referred to as the double-lightcone limit.

In the limit $z \rightarrow 0$, the four-dimensional conformal block appearing in the inversion kernel behaves as

$$g_{\ell+3,\Delta-3}(z, \bar{z}) = z^{\frac{\ell-\Delta+6}{2}} k_{\frac{\Delta+\ell}{2}}(\bar{z}) + \dots \quad (4.40)$$

Now at a given order in the large- N expansion, suppose that the double discontinuity of the correlator has an expansion of the form

$$\text{dDisc} [\mathcal{G}(z, \bar{z})] = \sum_{n=0}^{\infty} z^{\Delta_{\mathcal{O}}+n} \sum_{p=0}^{\infty} f_{n,p}(\bar{z}) \log^p z. \quad (4.41)$$

Combining this expansion with the measure and the small- z expansion of the inversion kernel reorganizes the integrand in powers of z . At order n , the resulting coefficient contains the contribution from $f_{n,p}(\bar{z})$, together with contributions from $f_{m,p}(\bar{z})$, with $m < n$, multiplied by subleading terms in the block expansion. So the z -integral takes the form

$$\int_0^1 dz z^{\Delta_{\mathcal{O}}+n+\frac{\ell-\Delta}{2}-1} \log^p z = \frac{(-1)^p 2^{p+1} p!}{\left(\Delta_{n,\ell}^{(0)} - \Delta\right)^{p+1}}. \quad (4.42)$$

The total power of z in the inversion integrand therefore selects the double-trace family labelled by n , and the power of $\log z$ determines the order of the pole. In particular,

$$z^{\frac{\Delta_{n,\ell}^{(0)} - \Delta}{2} - 1} \longrightarrow \frac{2}{\Delta_{n,\ell}^{(0)} - \Delta}, \quad (4.43)$$

$$z^{\frac{\Delta_{n,\ell}^{(0)} - \Delta}{2} - 1} \log z \longrightarrow -\frac{4}{\left(\Delta_{n,\ell}^{(0)} - \Delta\right)^2}. \quad (4.44)$$

So according to (4.27), the non-logarithmic terms determine corrections to the OPE coefficients, and terms proportional to $\log z$ determine anomalous dimensions.

After the pole associated with a fixed n has been isolated, its coefficient is determined by the remaining $\bar{z}$ integral which has the form

$$\mathcal{J}_\beta[D] = \int_0^1 d\bar{z} \bar{z}^{\beta-2} {}_2F_1(\beta, \beta; 2\beta; \bar{z}) D(\bar{z}), \quad (4.45)$$

where $\beta = \frac{\Delta+\ell}{2}$. At the position of the pole associated with $[\mathcal{OO}]_{n,\ell}$, we have

$$\beta = \ell + \Delta_{\mathcal{O}} + n. \quad (4.46)$$

The integral (4.45) contains the full dependence on the spin and can sometimes be evaluated exactly. However, at large ℓ , it localizes near $\bar{z} = 1$, so the expansion of $D(\bar{z})$ around this region determines the asymptotic expansion of the CFT data.

It is natural to organize this expansion in terms of the conformal spin

$$J^2 = \beta(\beta - 1) = (\ell + \Delta_{\mathcal{O}} + n)(\ell + \Delta_{\mathcal{O}} + n - 1). \quad (4.47)$$

The expansion obtained from the inversion formula then takes the form of a series in inverse powers of J , possibly accompanied by powers of $\log J$. The practical procedure for extracting successive terms in this series will be described next.

4.3.1 The peeling method

We now develop a systematic large- J expansion of $\mathcal{J}_\beta[D]$. The basic ingredient that we use is the integral

$$\begin{aligned} I(q_1, q_2) &:= \int_0^1 d\bar{z} \bar{z}^{q_1} (1 - \bar{z})^{q_2} {}_2F_1(q_3, q_3; 2q_3; \bar{z}) \\ &= \frac{\Gamma(q_1 + 1)\Gamma^2(q_2 + 1)\Gamma(2q_1 + 2q_2 + 4)}{\Gamma^2(q_1 + q_2 + 2)\Gamma(q_1 + 2q_2 + 3)}, \end{aligned} \quad (4.48)$$

where

$$q_3 = q_1 + q_2 + 2. \quad (4.49)$$

To use this identity we expand $D(\bar{z})$ near $\bar{z} = 1$ in the form

$$D(\bar{z}) = \sum_{m=0}^{\infty} d_m \frac{(1 - \bar{z})^{\alpha+m}}{\bar{z}^{\alpha+m}}, \quad (4.50)$$

where α is the leading power of $1 - \bar{z}$.

Substituting the m -th term of this expansion into $\mathcal{J}_\beta[D]$ gives

$$\begin{aligned} &\int_0^1 d\bar{z} \bar{z}^{\beta-\alpha-m-2} (1 - \bar{z})^{\alpha+m} {}_2F_1(\beta, \beta; 2\beta; \bar{z}) \\ &= I(\beta - \alpha - m - 2, \alpha + m), \end{aligned} \quad (4.51)$$

for which the condition (4.49) is satisfied. Using (4.48), we obtain

$$I(\beta - \alpha - m - 2, \alpha + m) = \frac{\Gamma(\beta - \alpha - m - 1)\Gamma^2(\alpha + m + 1)\Gamma(2\beta)}{\Gamma^2(\beta)\Gamma(\beta + \alpha + m + 1)}. \quad (4.52)$$

The coefficients d_m can be found recursively without constructing the full expansion (4.50). The first one is

$$d_0 = \lim_{\bar{z} \rightarrow 1} \frac{\bar{z}^\alpha}{(1 - \bar{z})^\alpha} D(\bar{z}). \quad (4.53)$$

After subtracting the terms already determined, the next coefficient is obtained from

$$d_m = \lim_{\bar{z} \rightarrow 1} \frac{\bar{z}^{\alpha+m}}{(1 - \bar{z})^{\alpha+m}} \left[D(\bar{z}) - \sum_{j=0}^{m-1} d_j \frac{(1 - \bar{z})^{\alpha+j}}{\bar{z}^{\alpha+j}} \right]. \quad (4.54)$$

So, at each step, the leading term near $\bar{z} = 1$ is removed before the next one is extracted. This successive subtraction is the reason for the name peeling method.

Having found the coefficients, the integral can then be written as

$$\mathcal{J}_\beta[D] = \sum_{m=0}^{\infty} d_m I(\beta - \alpha - m - 2, \alpha + m). \quad (4.55)$$

This is a series expansion in $1/J^2$. To see that, note that

$$\begin{aligned} & \frac{I(\beta - \alpha - m - 2, \alpha + m)}{I(\beta - \alpha - 2, \alpha)} \\ &= \frac{\Gamma^2(\alpha + m + 1)}{\Gamma^2(\alpha + 1)} \prod_{r=1}^m \frac{1}{J^2 - (\alpha + r)(\alpha + r + 1)} \sim \frac{\Gamma^2(\alpha + m + 1)}{\Gamma^2(\alpha + 1)} \frac{1}{J^{2m}}. \end{aligned} \quad (4.56)$$

Therefore, each subleading term is suppressed by two additional inverse powers of the conformal spin. The expansion is consequently organized in even inverse powers of J , in agreement with large-spin reciprocity [48].

Logarithms of $1 - \bar{z}$ can be treated using the representation

$$\log^q(1 - \bar{z}) = \lim_{w \rightarrow \infty} w^q [(1 - \bar{z})^{1/w} - 1]^q. \quad (4.57)$$

After expanding the power in brackets, the peeling procedure is applied to each term with α replaced by $\alpha + r/w$, where $r = 0, \dots, q$. We then take the limit $w \rightarrow \infty$. This produces powers of $\log J$ in the large-spin expansion.

4.3.2 Generic exchange

We now illustrate the large-spin expansion with the exchange of a generic primary operator. Let U denote a generic primary operator in the $\mathcal{O} \times \mathcal{O}$ OPE of dimension Δ_U , spin ℓ_U , twist $\tau_U = \Delta_U - \ell_U$, and squared OPE coefficient a_{Δ_U, ℓ_U} . Its direct-channel contribution to the correlator can be written as

$$\mathcal{G}(u, v) \supset a_{\Delta_U, \ell_U} u^{\tau_U/2} \tilde{g}_U(u, v), \quad (4.58)$$

where $\tilde{g}_U$ denotes the corresponding reduced block. By crossing symmetry, this contribution implies a term in the crossed channel

$$\mathcal{G}(u, v) \supset a_{\Delta_U, \ell_U} u^{\Delta_{\mathcal{O}}} v^{\tau_U/2 - \Delta_{\mathcal{O}}} \tilde{g}_U(v, u). \quad (4.59)$$

Inserting (4.59) into the Lorentzian inversion formula and taking the mentioned double lightcone limit gives the contribution of U to the anomalous dimensions of double trace operators at large spin expansion. If U first enters the OPE with

coefficient of order N^{-p} , then its contribution to the correlator starts at order N^{-2p} . For leading twist, the large-spin anomalous dimension takes the form¹

$$N^{2p}\gamma_{0,\ell} = \frac{\mathcal{A}(\Delta_{\mathcal{O}}, \ell_U, \tau_U)}{J^{\tau_U}} + \frac{\mathcal{B}(\Delta_{\mathcal{O}}, \ell_U, \tau_U)}{J^{\tau_U+2}} + \mathcal{O}\left(\frac{1}{J^{\tau_U+4}}\right), \quad (4.60)$$

where $J^2 = (\ell + \Delta_{\mathcal{O}})(\ell + \Delta_{\mathcal{O}} - 1)$ is the conformal spin for the leading twist double trace operator.

The coefficients $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} = -2a_{\Delta_U, \ell_U} \frac{\Gamma(\Delta_{\mathcal{O}})^2}{\Gamma(\Delta_{\mathcal{O}} - \frac{\tau_U}{2})^2} \frac{\Gamma(2\ell_U + \tau_U)}{\Gamma(\ell_U + \frac{\tau_U}{2})^2} \left[1 - \frac{\Gamma(\tau_U - 2)}{\Gamma(-\ell_U)\Gamma(\ell_U + \tau_U - 1)} \right], \quad (4.61)$$

$$\mathcal{B} = \frac{a_{\Delta_U, \ell_U}}{24} \frac{\Gamma(\Delta_{\mathcal{O}})^2}{\Gamma(\Delta_{\mathcal{O}} - \frac{\tau_U}{2})^2} \frac{\Gamma(2\ell_U + \tau_U)}{\Gamma(\ell_U + \frac{\tau_U}{2})^2} \left[\mathcal{P}(\Delta_{\mathcal{O}}, \tau_U) + \frac{\mathcal{R}(\Delta_{\mathcal{O}}, \ell_U, \tau_U)\Gamma(\tau_U - 2)}{\ell_U\Gamma(-\ell_U)\Gamma(\ell_U + \tau_U)} \right], \quad (4.62)$$

where

$$\mathcal{P}(\Delta_{\mathcal{O}}, \tau_U) = \tau_U^3 - 4\tau_U \left(1 + 3(\Delta_{\mathcal{O}} - 2)\Delta_{\mathcal{O}} \right) - 24 \left(1 - \Delta_{\mathcal{O}} + \frac{\tau_U}{2} \right)^2, \quad (4.63)$$

and

$$\begin{aligned} \mathcal{R}(\Delta_{\mathcal{O}}, \ell_U, \tau_U) = & 12\Delta_{\mathcal{O}}^2 [4\ell_U^2 - (\tau_U - 2)^2 + 4\ell_U(\tau_U - 1)] - 3(\tau_U^2 - 4)^2 \\ & - 4\ell_U^2(-12 - 10\tau_U + \tau_U^3) - 4\ell_U(\tau_U - 1)(-12 - 10\tau_U + \tau_U^3) \\ & + 12\Delta_{\mathcal{O}} [(\tau_U - 2)^2(\tau_U + 2) + \ell_U^2(\tau_U^2 - 4\tau_U - 8) \\ & + \ell_U(\tau_U - 1)(\tau_U^2 - 4\tau_U - 8)]. \end{aligned} \quad (4.64)$$

For the case of single trace exchange, as a simple example, suppose that $U = \mathcal{O}$. Then $p = 1$, $\tau_U = \Delta_{\mathcal{O}}$, and $\ell_U = 0$. Inserting these data in (4.60) gives

$$\begin{aligned} \gamma_{0,\ell}^{(1),\text{as}} = & -\frac{2a_{\Delta_{\mathcal{O}},0}\Gamma(\Delta_{\mathcal{O}})^3}{\Gamma(\frac{\Delta_{\mathcal{O}}}{2})^4} \frac{1}{J^{\Delta_{\mathcal{O}}}} \\ & - \frac{a_{\Delta_{\mathcal{O}},0}\Gamma(\Delta_{\mathcal{O}})^3}{24\Gamma(\frac{\Delta_{\mathcal{O}}}{2})^4} \frac{\Delta_{\mathcal{O}}(\Delta_{\mathcal{O}} - 2)(11\Delta_{\mathcal{O}}^2 - 10\Delta_{\mathcal{O}} - 4)}{\Delta_{\mathcal{O}} - 1} \frac{1}{J^{\Delta_{\mathcal{O}}+2}} + \mathcal{O}\left(\frac{1}{J^{\Delta_{\mathcal{O}}+4}}\right). \end{aligned} \quad (4.65)$$

Here $a_{\Delta_{\mathcal{O}},0} = f_{\mathcal{O}\mathcal{O}\mathcal{O}}^2$. The result obtained in this way is the asymptotic solution with infinite support in spin. In general, as we already mentioned, crossing symmetry also requires a contribution with finite support in spin. We will discuss this finite-spin contributions for some special example in the next chapter. In chapter 9, we will distinguish these asymptotic and finite pieces diagrammatically.

¹Using the peeling method, this can be continued systematically to higher orders in $1/J^2$.

Chapter 5

Spin sums and finite-spin reconstruction

In the previous chapters we introduced the Lorentzian inversion formula and we discussed how it can be used to extract the CFT data from the singular behavior of the correlator in the crossed channel. We also provided some examples.

However, in the conformal block expansion, observing the singular behavior of the terms is not always trivial. For a finite sum of exchanged operators one may freely interchange the double discontinuity in the inversion formula with the sum. For an infinite tower this is not automatic. Namely:

$$\text{dDisc}\left[\sum_{n,\ell} h_{n,\ell}\right] \neq \sum_{n,\ell} \text{dDisc}[h_{n,\ell}], \quad \sum_{n,\ell} h_{n,\ell} = \mathcal{G}(u, v). \quad (5.1)$$

Infinite spin sums can generate singularities which are invisible term by term. This mechanism will be central in chapter 8.

A sufficient condition to make the inequality of (5.1) into equality is that the function $h_{n,\ell}$ has finite support in spin.

A convenient way to organize such sums is through twist conformal blocks. These functions collect all operators belonging to a fixed twist family. We use the Casimir recursion relations to find the behavior of such infinite sums around $\bar{z} = 1$.

Another reason to study such sums is that the large spin solution obtained from the inversion formula does not always give the complete solution to crossing. In the presence of tree-level exchange, crossing requires additional contributions with finite support in spin. We will review the construction of a finite-dimensional projection of the crossing equation which determines these terms. We will first apply it to a scalar exchange and then to a contact interaction, for which the asymptotic contribution vanishes and the entire anomalous dimension has finite spin support.

5.1 Spin sums

Consider the double-trace family $[\mathcal{OO}]_{n,\ell}$, with $\tau_n = 2\Delta_{\mathcal{O}} + 2n$. We define the family of spin sums

$$H_n^{(m)}(z, \bar{z}) = \sum_{\ell} a_{n,\ell}^{(0)} \frac{(z\bar{z})^{\tau_n/2}}{J^{2m}} \tilde{g}_{n,\ell}(z, \bar{z}), \quad (5.2)$$

where J is the conformal spin and

$$g_{\tau_n+\ell,\ell}(z, \bar{z}) = (z\bar{z})^{\tau_n/2} \tilde{g}_{n,\ell}(z, \bar{z}). \quad (5.3)$$

(5.2) represents the general structure of spin sums that appear in the conformal block expansion of double-trace operators. In the literature, these objects are often referred to as twist conformal blocks [8, 49, 50, 62].

Inserting the explicit form of the four-dimensional conformal blocks (2.57) into (5.2), we obtain

$$H_n^{(m)}(z, \bar{z}) = \frac{1}{z - \bar{z}} \left(\bar{z}^{\tau_n/2} F_{\frac{\tau_n-2}{2}}(\bar{z}) \tilde{H}_n^{(m)}(z) - z^{\tau_n/2} F_{\frac{\tau_n-2}{2}}(z) \tilde{H}_n^{(m)}(\bar{z}) \right), \quad (5.4)$$

where

$$\tilde{H}_n^{(m)}(z) = \sum_{\ell} \frac{a_{n,\ell}^{(0)}}{J^{2m}} z^{\tau_n/2+\ell+1} F_{\frac{\tau_n}{2}+\ell}(z), \quad \tau_n = 2\Delta_{\mathcal{O}} + 2n \quad (5.5)$$

Our goal is to determine these functions in the limit $\bar{z} \rightarrow 1$. The functions $\tilde{H}_n^{(m)}(z)$ satisfy a recursive differential equation of the form [50]

$$D_{4d} \tilde{H}_n^{(m+1)}(z) = \tilde{H}_n^{(m)}(z), \quad D_{4d} = z D z^{-1}, \quad (5.6)$$

where the differential operator D is given by

$$D = (1 - z) z^2 \partial_z^2 - z^2 \partial_z. \quad (5.7)$$

A similar relation holds for $\tilde{H}_n^{(m)}(\bar{z})$.

Given $\tilde{H}_n^{(0)}(\bar{z})$, the recursive relation allows us to determine $\tilde{H}_n^{(m)}(\bar{z})$ for all m . We now review the steps of [50] to extract $\tilde{H}_n^{(0)}(\bar{z})$.

$\tilde{H}_n^{(0)}(\bar{z})$ appears in the conformal block expansion of the generalized free field correlator,

$$\mathcal{G}^{(0)}(z, \bar{z}) = 1 + \frac{1}{z - \bar{z}} \left(\sum_{n,\ell} a_{n,\ell}^{(0)} z \bar{z} k_{\frac{\tau_n}{2}+\ell}(z) k_{\frac{\tau_n}{2}-1}(\bar{z}) - \sum_n z k_{\frac{\tau_n}{2}-1}(z) \tilde{H}_n^{(0)}(\bar{z}) \right). \quad (5.8)$$

To isolate $\tilde{H}_n^{(0)}(\bar{z})$, we use the orthogonality relation

$$\oint_{z=0} \frac{dz}{2\pi i} \frac{1}{z^2} k_{\frac{\tau_m}{2}-1}(z) k_{2-\frac{\tau_m}{2}}(z) = \delta_{n,m}. \quad (5.9)$$

This yields

$$\tilde{H}_n^{(0)}(\bar{z}) = B_n^{(0)}(\bar{z}) + P_n^{(0)}(\bar{z}), \quad (5.10)$$

where

$$B_n^{(0)}(\bar{z}) = \sum_{m,\ell} \delta_{m+\ell,n-1} a_{m,\ell}^{(0)} \bar{z} k_{\frac{\tau_m}{2}-1}(\bar{z}), \quad (5.11)$$

and

$$P_n^{(0)}(\bar{z}) = - \oint_{z=0} \frac{dz}{2\pi i} \frac{1}{z^3} k_{2-\frac{\tau_n}{2}}(z) (z - \bar{z}) \left(\mathcal{G}^{(0)}(z, \bar{z}) - 1 \right). \quad (5.12)$$

Note that

$$\bar{D}_{4d} \bar{z} k_\alpha(\bar{z}) = \alpha(\alpha - 1) \bar{z} k_\alpha(\bar{z}). \quad (5.13)$$

This implies that the functions $B_n^{(m)}(\bar{z})$ can be constructed iteratively such that

$$\bar{D}_{4d} B_n^{(m+1)}(\bar{z}) = B_n^{(m)}(\bar{z}). \quad (5.14)$$

So we can also define $P_n^{(m)}(\bar{z})$ to satisfy

$$\bar{D}_{4d} P_n^{(m+1)}(\bar{z}) = P_n^{(m)}(\bar{z}). \quad (5.15)$$

The functions $B_n^{(m)}(\bar{z})$ exhibit the same behavior as a single conformal block in the limit $\bar{z} \rightarrow 1$. In contrast, $P_n^{(m)}(\bar{z})$ can display new behavior, arising from the infinite sum over spins.

From the definition of $P_n^{(0)}$, we find that near $\bar{z} = 1$

$$P_n^{(0)}(\bar{z}) = \frac{Y_n}{(1 - \bar{z})^{\Delta_{\mathcal{O}}}} + \mathcal{O}((1 - \bar{z})^{-\Delta_{\mathcal{O}}+1}). \quad (5.16)$$

where

$$Y_n = \frac{4^{-n} (\Delta_{\mathcal{O}} - 1)_n (2\Delta_{\mathcal{O}} - 3)_n}{(1)_n (\Delta_{\mathcal{O}} - \frac{3}{2})_n}. \quad (5.17)$$

Using the recursion relation (5.15), we can determine the behavior of $P_n^{(m)}(\bar{z})$ near $\bar{z} = 1$. For $\Delta_{\mathcal{O}} - m \notin \mathbb{Z}_{\leq 0}$, we find

$$P_n^{(m)}(\bar{z}) = \frac{Y_n}{((1 - \Delta_{\mathcal{O}})_m)^2} (1 - \bar{z})^{-\Delta_{\mathcal{O}}+m} + \mathcal{O}((1 - \bar{z})^{-\Delta_{\mathcal{O}}+m+1}). \quad (5.18)$$

In the case $\Delta_{\mathcal{O}} - m \in \mathbb{Z}_{\leq 0}$, we parametrize $m = \Delta_{\mathcal{O}} + p$ with $p \in \mathbb{Z}_{\geq 0}$. In such case, near $\bar{z} = 1$ we find

$$P_n^{(m=\Delta_{\mathcal{O}}+p)}(\bar{z}) = \frac{1}{2(p!)^2 \Gamma^2(\Delta_{\mathcal{O}})} Y_n (1 - \bar{z})^p \log^2(1 - \bar{z}) + \mathcal{O}((1 - \bar{z})^p \log(1 - \bar{z})). \quad (5.19)$$

Finally, we note that in some cases we encounter a more general class of spin sums of the form

$$\tilde{H}_n^{(m)(q)}(z) = \sum_{\ell} a_{n,\ell} \frac{\log^q J}{J^{2m}} z^{\tau_n/2+\ell+1} F_{\frac{\tau_n}{2}+\ell}(z), \quad (5.20)$$

such that the previously defined functions correspond to the case $q = 0$, i.e.

$$\tilde{H}_n^{(m)}(z) = \tilde{H}_n^{(m)(0)}(z). \quad (5.21)$$

For $q = 1$, these functions can be written as

$$\tilde{H}_n^{(m)(1)}(z) = -\frac{1}{2} \frac{\partial}{\partial a} \sum_{\ell} a_{n,\ell} \frac{1}{J^{2(m+a)}} z^{\tau_n/2+\ell+1} F_{\frac{\tau_n}{2}+\ell}(z) \Big|_{a=0} \quad (5.22)$$

$$= -\frac{1}{2} \frac{\partial}{\partial a} \tilde{H}_n^{(m+a)}(z) \Big|_{a=0} \quad (5.23)$$

$$= -\frac{1}{2} \frac{\partial}{\partial a} \left(B_n^{(m+a)}(z) + P_n^{(m+a)}(z) \right) \Big|_{a=0}. \quad (5.24)$$

This allows us to define the corresponding decompositions $B_n^{(m)(1)}$ and $P_n^{(m)(1)}$. In the case $\Delta_{\mathcal{O}} - m \notin \mathbb{Z}_{\leq 0}$, we find for the leading behavior near $\bar{z} = 1$

$$\begin{aligned} P_n^{(m)(1)}(\bar{z}) &= -\frac{Y_n}{2((1-\Delta_{\mathcal{O}})_m)^2} (1-\bar{z})^{-\Delta_{\mathcal{O}}+m} [\log(1-\bar{z}) - 2\psi(1-\Delta_{\mathcal{O}}+m)] \\ &\quad + \mathcal{O}((1-\bar{z})^{-\Delta_{\mathcal{O}}+m+1} \log(1-\bar{z})). \end{aligned} \quad (5.25)$$

where $\psi(x)$ denotes the digamma function.

5.2 Finite-spin reconstruction

As we discussed in the previous chapter, the Lorentzian inversion formula determines the part of the OPE data which is analytic in spin. However, for a tree-level exchange, this analytic part does not fully satisfy the crossing equation by itself. The full double-trace anomalous dimension induced by such exchange at tree-level can be written as

$$\gamma_{n,\ell}^{(1)} = \gamma_{n,\ell}^{(1),\text{as}} + \gamma_{n,\ell}^{(1),\text{fin}}. \quad (5.26)$$

Here $\gamma_{n,\ell}^{(1),\text{as}}$ is the analytic piece coming from the inversion formula and the term $\gamma_{n,\ell}^{(1),\text{fin}}$ is required in order to complete the exchange contribution into a crossing-symmetric solution.

We now derive a set of finite-dimensional equations which determines $\gamma_{n,\ell}^{(1),\text{fin}}$. This part is basically a quick review of the construction of [50].

We consider the exchange of a scalar operator of dimension two. Its asymptotic contribution was derived in the previous chapter using the inversion formula. We recall that it is given by

$$\gamma_{n,\ell}^{(1),\text{as}} = -\frac{2a_{2,0}(\Delta_{\mathcal{O}} - 1)^2}{(\ell + 1)(\ell + 2\Delta_{\mathcal{O}} + 2n - 2)}, \quad (5.27)$$

The correlator at order N^{-2} take the following analytic structure around $z = 0$ and $\bar{z} = 1$.

$$\begin{aligned} \mathcal{G}^{(1)}(z, \bar{z}) = & \eta_0(z, \bar{z}) \log z \log(1 - \bar{z}) + \eta_1(z, \bar{z}) \log z \\ & + \eta_2(z, \bar{z}) \log(1 - \bar{z}) + \eta_3(z, \bar{z}), \end{aligned} \quad (5.28)$$

where the function $\eta_i(z, \bar{z})$ contain no logarithms in this expansion. Now let us focus on the function $\eta_0(z, \bar{z})$. This function receive contributions only from the anomalous dimensions, since the corrections to the OPE coefficients do not generate $\log z$ at this order. Moreover, η_0 satisfy the crossing equation by itself. Namely,

$$\left(\frac{1 - \bar{z}}{\bar{z}}\right)^{\Delta_{\mathcal{O}}} \eta_0(z, \bar{z}) = \left(\frac{z}{1 - z}\right)^{\Delta_{\mathcal{O}}} \eta_0(1 - \bar{z}, 1 - z). \quad (5.29)$$

For later convenience, we define

$$\tilde{k}_{\beta}(\bar{z}) = -\frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \bar{z}^{\beta} {}_2F_1(\beta, \beta; 1; 1 - \bar{z}), \quad (5.30)$$

so that

$$k_{\beta}(\bar{z}) = \tilde{k}_{\beta}(\bar{z}) \log(1 - \bar{z}) + \text{terms without } \log(1 - \bar{z}). \quad (5.31)$$

The coefficient η_0 can then be written as

$$\begin{aligned} \eta_0(z, \bar{z}) = & \sum_{n,\ell} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\text{fin}} z \bar{z} \left[k_{\frac{\tau_n}{2}+\ell}(z) \tilde{k}_{\frac{\tau_n}{2}-1}(\bar{z}) - k_{\frac{\tau_n}{2}-1}(z) \tilde{k}_{\frac{\tau_n}{2}+\ell}(\bar{z}) \right] \\ & + \sum_{n,\ell} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\text{as}} z \bar{z} k_{\frac{\tau_n}{2}+\ell}(z) \tilde{k}_{\frac{\tau_n}{2}-1}(\bar{z}) \\ & - \sum_{m=0}^{\infty} z k_{\Delta_{\mathcal{O}}+m-1}(z) \tilde{S}_m(\bar{z}). \end{aligned} \quad (5.32)$$

Here $\tilde{S}_m(\bar{z})$ denotes the coefficient of $\log(1 - \bar{z})$ in the spin sum $S_m(\bar{z})$, which is defined as

$$S_n(\bar{z}) = \sum_{\ell} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\text{as}} \bar{z}^{\frac{\tau_n}{2}+\ell+1} F_{\frac{\tau_n}{2}+\ell}(\bar{z}). \quad (5.33)$$

The infinite spin sum in this expression can affect the behavior around $\bar{z} = 1$. Hence, in this case, we should first do the spin sum and then expand around $\bar{z} = 1$.

Using the form of $\gamma_{n,\ell}^{(1),\text{as}}$ given in (5.27) we can write the corresponding recursion relation for $S_n(\bar{z})$.

$$\left[\bar{D}_{4d} - (\Delta_{\mathcal{O}} + n - 1)(\Delta_{\mathcal{O}} + n - 2) \right] S_n(\bar{z}) = -2a_{2,0}(\Delta_{\mathcal{O}} - 1)^2 \tilde{H}_n^{(0)}(\bar{z}). \quad (5.34)$$

We remind that $\tilde{H}_n^{(0)}(\bar{z}) = B_n^{(0)}(\bar{z}) + P_n^{(0)}(\bar{z})$, where the $P_n^{(0)}$ part does not contain $\log(1 - \bar{z})$, so only $B_n^{(0)}$ is relevant here. We recall that

$$B_n^{(0)}(\bar{z}) = \sum_{m,\ell} \delta_{m+\ell,n-1} a_{m,\ell}^{(0)} \bar{z} k_{\frac{\tau_m}{2}-1}(\bar{z}), \quad (5.35)$$

So now using (5.13) in (5.34) we find

$$\tilde{S}_m(\bar{z}) = -2a_{2,0}(\Delta_{\mathcal{O}} - 1)^2 \sum_{n,\ell} \frac{\delta_{n+\ell,m-1}}{\lambda_{m,n}} a_{n,\ell}^{(0)} \bar{z} k_{\frac{\tau_n}{2}-1}(\bar{z}), \quad (5.36)$$

where

$$\lambda_{m,n} = (n - m)(2\Delta_{\mathcal{O}} + m + n - 3). \quad (5.37)$$

Although $S_m(\bar{z})$ was defined by an infinite spin sum, the Kronecker delta restricts $\tilde{S}_m(\bar{z})$ to finitely many terms.

We now turn the crossing equation into a system of algebraic equations. For this purpose, we use the orthogonality relation (5.9) and we define

$$I_{m,m'}^{(\Delta_{\mathcal{O}})} = \oint_{z=0} \frac{dz}{2\pi i} \frac{z^{\Delta_{\mathcal{O}}-3}}{(1-z)^{\Delta_{\mathcal{O}}-1}} \tilde{k}_{\frac{\tau_m}{2}}(1-z) k_{1-\frac{\tau_{m'}}{2}}(z). \quad (5.38)$$

Substituting (5.32) into (5.29) and applying these projectors gives

$$\begin{aligned} & \sum_{n,\ell} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\text{fin}} \left[\delta_{n+\ell,p} I_{n-1,q}^{(\Delta_{\mathcal{O}})} - \delta_{n-1,p} I_{n+\ell,q}^{(\Delta_{\mathcal{O}})} \right] \\ & + \sum_{n,\ell} a_{n,\ell}^{(0)} \left[\gamma_{n,\ell}^{(1),\text{as}} + \frac{2a_{2,0}(\Delta_{\mathcal{O}} - 1)^2}{\lambda_{p+1,n}} \right] \delta_{n+\ell,p} I_{n-1,q}^{(\Delta_{\mathcal{O}})} = (p \leftrightarrow q). \end{aligned} \quad (5.39)$$

This equation holds for all non-negative integers p and q . The first sum contains the unknown finite-spin contribution, and the second is fixed by the asymptotic solution. For each p and q , the Kronecker deltas reduce the equation to finitely many terms.

Now we can solve (5.39) for $\gamma_{n,\ell}^{(1),\text{fin}}$ by inserting $\gamma_{n,\ell}^{(1),\text{as}}$ as given in (5.27). This yields:

$$\gamma_{n,\ell}^{(1),\text{fin}} = \delta_{\ell,0} \frac{a_{2,0}(\Delta_{\mathcal{O}} - 1)^2(n+1)(2\Delta_{\mathcal{O}} + n - 3)}{2(\Delta_{\mathcal{O}} + n - 1)(2\Delta_{\mathcal{O}} + 2n - 3)(2\Delta_{\mathcal{O}} + 2n - 1)}. \quad (5.40)$$

Therefore, the finite contribution has support only at $\ell = 0$. In Chapter 9, we distinguish diagrammatically between the finite and asymptotic contributions.

Finally, we also mention that for a quartic contact interaction, no single-trace operator is exchanged. So in this case we have to solve:

$$\sum_{n,\ell} a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1),\phi^4} \left[\delta_{n+\ell,p} I_{n-1,q}^{(\Delta_{\mathcal{O}})} - \delta_{n-1,p} I_{n+\ell,q}^{(\Delta_{\mathcal{O}})} \right] = (p \leftrightarrow q). \quad (5.41)$$

Solving this equation for a quartic interaction without derivatives yields [12]:

$$\gamma_{n,\ell}^{(1),\phi^4} = \delta_{\ell,0} \gamma_{0,0}^{(1),\phi^4} \frac{(2\Delta_{\mathcal{O}} - 1)(n+1)(\Delta_{\mathcal{O}} + n - 1)(2\Delta_{\mathcal{O}} + n - 3)}{(\Delta_{\mathcal{O}} - 1)(2\Delta_{\mathcal{O}} + 2n - 3)(2\Delta_{\mathcal{O}} + 2n - 1)}, \quad (5.42)$$

which has support only at spin zero.

The overall normalization of this solution is not fixed by crossing symmetry. However, in this thesis we fix this normalization by setting $\gamma_{0,0}^{(1),\phi^4} = 1$.

Finally, we mention that here we used the superscript ϕ^4 to distinguish the contact contribution from the scalar-exchange contribution. In later chapters, we might omit this superscript whenever the distinction is clear from the context, or we might use alternative notation when both contributions are present.

Chapter 6

Mellin space

The Mellin transform provides a representation of correlation functions that is well adapted to conformal field theory. Similar to the Fourier transform, which is naturally associated with translations, the Mellin transform is adapted to scale transformations. For holographic correlation functions, it provides a natural framework for describing amplitudes in AdS [13, 14, 63–66]. Mellin space has also been used in Polyakov–Mellin bootstrap methods and in dispersion relations for CFT correlators [67–75].

The Mellin transformed function of holographic correlation functions is called the Mellin amplitude. The Mellin amplitude has an analytic structure that closely resembles that of a flat-space scattering amplitude. Local contact interactions in AdS give polynomial Mellin amplitudes, and the exchange of a bulk field gives a sequences of poles in the Mellin amplitude whose positions are determined by the dimension of the dual CFT operator and their residues encode the spin dependence of the exchanged operator.

However, this analogy should be understood with some care. A boundary correlator in AdS does not define an ordinary S-matrix, since AdS does not possess the standard asymptotic states of flat spacetime. The Mellin amplitude is a representation of the boundary correlation function, or equivalently of the Witten diagrams that compute it. Nevertheless, it provides the natural AdS analogue of a scattering amplitude, so we will regard Mellin amplitudes as the appropriate notion of amplitudes in AdS. Moreover, in the limit in which the AdS radius becomes large, the Mellin variables can be related directly to flat-space kinematic invariants, and the corresponding flat-space scattering amplitude can be recovered [13].

In this chapter, we first introduce the ordinary Mellin transform of a function of one variable. This simple example makes the relation between the singularities of the Mellin transform and the power-law expansion of the original function manifest. We then introduce the Mellin representation of four-point functions of identical scalar operators. Finally, we review the flat-space limit that will be used

in the following chapters.

6.1 Mellin transform

Consider a function $f(x)$ defined for $x > 0$. Its Mellin transform is defined by

$$\tilde{f}(s) = \int_0^\infty dx x^{s-1} f(x), \quad (6.1)$$

whenever the integral converges. The inverse transform is

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} x^{-s} \tilde{f}(s), \quad (6.2)$$

where the real number c is chosen inside a vertical strip in the complex s -plane in which (6.1) converges. This region is usually referred to as the fundamental strip.

Writing $x = e^y$ and $s = c + i\nu$, and defining $g_c(y) = e^{cy} f(e^y)$, we obtain

$$\tilde{f}(c + i\nu) = \int_{-\infty}^{\infty} dy g_c(y) e^{i\nu y}. \quad (6.3)$$

So, along a fixed vertical contour, the Mellin transform is the Fourier transform of $g_c(y)$ in the logarithmic variable $y = \log x$, and rescaling of x become translation of y .

The main property that will be useful for CFT correlators is the relation between poles in Mellin space and power-law expansions of a function in position space. Suppose that, as $x \rightarrow 0$, the function contains a term

$$f(x) \sim x^\alpha \log^p x. \quad (6.4)$$

Its contribution to the Mellin transform near $s = -\alpha$ follows from

$$\int_0^1 dx x^{s+\alpha-1} \log^p x = \frac{(-1)^p p!}{(s + \alpha)^{p+1}}. \quad (6.5)$$

Therefore, a power x^α produces a pole at $s = -\alpha$, and a term $x^\alpha \log^p x$ produces a pole of order $p + 1$.

Similarly, if the large- x expansion contains

$$f(x) \sim x^{-\beta} \log^p x, \quad x \rightarrow \infty, \quad (6.6)$$

then

$$\int_1^\infty dx x^{s-\beta-1} \log^p x = \frac{p!}{(\beta - s)^{p+1}}. \quad (6.7)$$

Therefore, the poles to the left of the fundamental strip encode the small- x expansion, and the poles to its right encode the large- x expansion. These expansions can be recovered from (6.2) by deforming the integration contour and summing the enclosed residues.

As a simple example, consider

$$f(x) = \frac{1}{1+x}. \quad (6.8)$$

Its Mellin transform is

$$\begin{aligned} \tilde{f}(s) &= \int_0^\infty dx \frac{x^{s-1}}{1+x} \\ &= \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad 0 < \text{Re } s < 1. \end{aligned} \quad (6.9)$$

The poles at $s = 0, -1, -2, \dots$ determine the expansion around $x = 0$. Closing the inverse-Mellin contour to the left gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1. \quad (6.10)$$

Instead, closing the contour to the right and picking up the poles at $s = 1, 2, 3, \dots$ gives

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} (-1)^{n-1} x^{-n}, \quad |x| > 1. \quad (6.11)$$

The two asymptotic regions of the same function are therefore encoded by poles on opposite sides of the fundamental strip.

6.2 Mellin amplitudes for CFT correlators

We now apply the preceding discussion to the four-point function of identical scalar primary operators. As in the previous chapters, we write

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{\mathcal{G}(u, v)}{x_{12}^{2\Delta_{\mathcal{O}}} x_{34}^{2\Delta_{\mathcal{O}}}}, \quad (6.12)$$

where

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (6.13)$$

The standard Mellin amplitude is associated with the connected part of the correlator. In the large- N theories considered in this thesis, we separate

$$\mathcal{G}(u, v) = \mathcal{G}^{(0)}(u, v) + \mathcal{G}_{\text{conn}}(u, v), \quad (6.14)$$

where

$$\mathcal{G}^{(0)}(u, v) = 1 + u^{\Delta_{\mathcal{O}}} + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \quad (6.15)$$

is the GFF contribution. It is given by disconnected Wick contractions and, in the bulk description, by disconnected Witten diagrams. These disconnected terms can be represented in Mellin space only distributionally and are not included in the ordinary meromorphic Mellin amplitude. We therefore define the Mellin representation for $\mathcal{G}_{\text{conn}}$.

A direct double Mellin transform with respect to u and v would already provide a valid representation of the connected correlator. However, in CFT it is useful to extract a universal product of Gamma functions from the transform. For identical external operators, the Mellin amplitude $M(s, t)$ is defined by

$$\begin{aligned} \mathcal{G}_{\text{conn}}(u, v) = & \int_{-i\infty}^{i\infty} \frac{ds dt}{(4\pi i)^2} u^{\frac{t}{2}} v^{\frac{\hat{u}-2\Delta_{\mathcal{O}}}{2}} \\ & \times \Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-t}{2}\right) \Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-s}{2}\right) \Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-\hat{u}}{2}\right) M(s, t), \end{aligned} \quad (6.16)$$

where the three Mellin variables satisfy

$$s + t + \hat{u} = 4\Delta_{\mathcal{O}}. \quad (6.17)$$

The variables $s, t, \hat{u}$ are Mandelstam-like variables. For identical external operators, crossing symmetry becomes

$$M(s, t) = M(s, \hat{u}) = M(t, s). \quad (6.18)$$

The Gamma functions in (6.16) are factored out because they contain universal poles associated with the double-trace operators built from the external field. The remaining function $M(s, t)$ contains the interaction-dependent information and is what we call the Mellin amplitude.

Large- N expansion

The connected correlator inherits the large- N expansion introduced in the previous chapters,

$$\mathcal{G}_{\text{conn}}(u, v) = \frac{1}{N^2} \mathcal{G}^{(1)}(u, v) + \frac{1}{N^4} \mathcal{G}^{(2)}(u, v) + \dots \quad (6.19)$$

Correspondingly, the Mellin amplitude admits the expansion

$$M(s, t) = \frac{1}{N^2} M_{\text{tree}}(s, t) + \frac{1}{N^4} M_{1\text{-loop}}(s, t) + \frac{1}{N^6} M_{2\text{-loop}}(s, t) + \dots \quad (6.20)$$

Here M_{tree} is the Mellin transform of $\mathcal{G}^{(1)}$, and more generally the Mellin amplitude associated with $\mathcal{G}^{(\kappa)}$ corresponds to the $(\kappa - 1)$ -loop contribution in the bulk perturbative expansion.

6.2.1 Mellin poles

The one-variable discussion of Section 6.1 can be applied directly to (6.16). For example, consider the dependence on the variable t , which is conjugate to the cross ratio u . The corresponding Gamma function has double poles at

$$t = t_m, \quad t_m = 2\Delta_{\mathcal{O}} + 2m, \quad m = 0, 1, 2, \dots \quad (6.21)$$

Namely, near $t = t_m$,

$$\Gamma^2\left(\frac{2\Delta_{\mathcal{O}} - t}{2}\right) \sim \frac{4}{(m!)^2} \frac{1}{(t - t_m)^2} + \dots \quad (6.22)$$

To determine the corresponding position-space behavior, it is enough to use

$$\text{Res}_{t=t_0} \left[\frac{u^{t/2}}{(t - t_0)^q} \right] = \frac{u^{t_0/2}}{(q - 1)!} \left(\frac{\log u}{2} \right)^{q-1}. \quad (6.23)$$

A pole of order q in the full Mellin integrand therefore produces a term proportional to $\log^{q-1} u$.

In particular, if the Mellin amplitude is regular at $t = t_m$, the double pole of the Mellin measure produces a term of the form

$$u^{\Delta_{\mathcal{O}}+m} \log u. \quad (6.24)$$

If instead the Mellin amplitude itself contains a pole of order r ,

$$M(s, t) \sim \frac{R_m^{(r)}(s)}{(t - t_m)^r}, \quad (6.25)$$

then the full integrand has a pole of order $r + 2$ and gives

$$\mathcal{G}_{\text{conn}}(u, v) \supset u^{\Delta_{\mathcal{O}}+m} \log^{r+1} u. \quad (6.26)$$

Therefore, a simple pole in the Mellin amplitude produces a $\log^2 u$ term, a double pole produces a $\log^3 u$ term, and so on.

More generally, at order $N^{-2\kappa}$, the contribution proportional to $\log^\kappa u$ is encoded by poles of order $\kappa - 1$ in the Mellin amplitude. This relation will be important in chapter 7.

The pole structure of the Mellin amplitude provides a counterpart of the conformal block decomposition. Consider a primary operator $\mathcal{O}_p$ of dimension Δ_p , spin ℓ_p , and twist $\tau_p = \Delta_p - \ell_p$ appearing in the $\mathcal{O} \times \mathcal{O}$ OPE. Its contribution to the polar part of the t -channel Mellin amplitude takes the form

$$M(s, t) = \sum_p a_{\Delta_p, \ell_p} \sum_{r=0}^{\infty} \frac{\mathcal{Q}_{\ell_p, r}(s, \tau_p)}{t - (\tau_p + 2r)} \quad (6.27)$$

The sum over p runs over the primary operators exchanged in the chosen OPE channel, and the integer r labels their descendants.

The residues $\mathcal{Q}_{\ell_p, r}(s, \tau_p)$ are Mack polynomials [64, 76]. It is useful to separate a normalization factor and write

$$\mathcal{Q}_{\ell_p, m}(s, \tau_p) = \frac{-2\Gamma(\Delta_p + \ell_p)(\Delta_p - 1)_{\ell_p}}{4^{\ell_p} m! (\Delta_p - 1)_m \Gamma^4\left(\frac{\Delta_p + \ell_p}{2}\right) \Gamma^2\left(\Delta_{\mathcal{O}} - m - \frac{\Delta_p - \ell_p}{2}\right)} Q_{\ell_p, m}(s, \tau_p), \quad (6.28)$$

where $Q_{\ell_p, r}(s, \tau_p)$ is a polynomial of degree ℓ_p in s , and $Q_{0, r}(s, \tau_p) = 1$. So for scalar exchanges the residues are independent of the second Mellin variable. For spinning exchange, the polynomial dependence of the residues plays the role of the angular dependence of a flat-space partial wave.

The relation (6.27) is written up to some contributions that are regular at the exchange poles. In a local bulk theory, these ambiguities are polynomials in the Mellin variables and correspond to contact Witten diagrams. At loop level, they are related to the freedom of adding local counterterms and hence to the choice of renormalization scheme.

We now want to specialize the pole decomposition to the double-trace operators that appear in the large- N expansion. Their twists take the form

$$\tau_{n, \ell} = 2\Delta_{\mathcal{O}} + 2n + \frac{\gamma_{n, \ell}^{(1)}}{N^2} + \frac{\gamma_{n, \ell}^{(2)}}{N^4} + \dots \quad (6.29)$$

An important feature of (6.28) is that the normalization of the Mack polynomial vanishes when the twist is set equal to its generalized free field value. Namely,

$$\begin{aligned} \frac{1}{\Gamma^2\left(\Delta_{\mathcal{O}} - r - \frac{\tau_{n, \ell}}{2}\right)} &= \frac{1}{\Gamma^2\left(-n - r - \frac{\gamma_{n, \ell}^{(1)}}{2N^2} + \dots\right)} \\ &= \frac{[(n+r)!]^2}{4N^4} \left(\gamma_{n, \ell}^{(1)}\right)^2 + O(N^{-6}). \end{aligned} \quad (6.30)$$

Thus, the contribution of double-trace operators to the polar part of the Mellin amplitude begins at order N^{-4} and is proportional to the square of their tree-level anomalous dimensions.

Combining the primary label n with the descendant level r , the poles occur at

$$t = 2\Delta_{\mathcal{O}} + 2n + 2r. \quad (6.31)$$

Defining $m = n + r$, all pairs (n, r) with the same value of m contribute to the same Mellin pole. The one-loop contribution can therefore be organized as

$$M_{1\text{-loop}}(s, t) = \sum_{m=0}^{\infty} \frac{R_m^{(1)}(s)}{t - (2\Delta_{\mathcal{O}} + 2m)} + \text{crossed}. \quad (6.32)$$

Here “crossed” denotes the analogous pole sequences in the s - and $\hat{u}$ -channels. The residue $R_m^{(1)}(s)$ receives contributions from all double-trace primaries with $0 \leq n \leq m$.

When the tree-level anomalous dimensions have support only at $\ell = 0$, the Mack polynomials become independent of s , and the residues become constants. In that case and specializing further to $\Delta_{\mathcal{O}} = 2$, we find

$$R_m^{(1)} = \sum_{n=0}^m \frac{-4(2n+1)(2n+3)\Gamma(m+1)^2}{\Gamma(m-n+1)\Gamma(m+n+3)} \left(\gamma_{n,0}^{(1)}\right)^2. \quad (6.33)$$

This formula shows explicitly how the residues of the one-loop Mellin amplitude are determined by tree-level double-trace data.

When there are several tree-level contributions, we can decompose

$$\gamma_{n,\ell}^{(1)} = \sum_i \gamma_{n,\ell}^{(1)(i)}. \quad (6.34)$$

Then the square entering (6.30) becomes

$$\left(\gamma_{n,\ell}^{(1)}\right)^2 = \sum_{i,j} \gamma_{n,\ell}^{(1)(i)} \gamma_{n,\ell}^{(1)(j)}. \quad (6.35)$$

Consequently, the residues of the one-loop Mellin amplitude can be decomposed as

$$R_m^{(1)} = \sum_{i,j} [R_m^{(1)}]_{(i)(j)}, \quad (6.36)$$

where

$$[R_m^{(1)}]_{(i)(j)} = \sum_{n=0}^m \frac{-4(2n+1)(2n+3)\Gamma(m+1)^2}{\Gamma(m-n+1)\Gamma(m+n+3)} \gamma_{n,0}^{(1)(i)} \gamma_{n,0}^{(1)(j)}. \quad (6.37)$$

This decomposition will be used in chapter 9 when we introduce the diagrammatic representation of products of OPE data.

6.3 Flat-space limit

A holographic description of the flat-space S-matrix can be obtained by taking the AdS radius to infinity. In this limit, the bulk dynamics effectively reduce to those of flat spacetime, and correlators in the boundary CFT are expected to reproduce flat-space scattering amplitudes. As first emphasized in [13] and later further developed in [14], this relation is most transparently realized at the

level of the Mellin amplitude. Indeed the Mellin variables s_{ij} ¹ admit a direct interpretation as the kinematic invariants in a scattering amplitude, $s_{ij} \sim p_i \cdot p_j$, and the flat-space amplitude can be obtained by taking them to be large. More concretely

$$M(s_{ij}) \approx \frac{R_{\text{AdS}}^{-d+3}}{\mathcal{N} \Gamma(\frac{1}{2} \sum_i \Delta_i - \frac{d}{2})} \int_0^\infty d\beta \beta^{\frac{1}{2} \sum_i \Delta_i - \frac{d}{2} - 1} e^{-\beta} A\left(S_{ij} = \frac{2\beta}{R_{\text{AdS}}^2} s_{ij}\right), \quad s_{ij} \gg 1, \quad (6.38a)$$

$$A(S_{ij}) = \frac{\mathcal{N} \Gamma(\frac{1}{2} \sum_i \Delta_i - \frac{d}{2})}{R_{\text{AdS}}^{-d+3}} \lim_{R_{\text{AdS}} \rightarrow \infty} \int_{-i\infty}^{+i\infty} \frac{d\alpha}{2\pi i} \alpha^{\frac{d}{2} - \frac{1}{2} \sum_i \Delta_i} e^\alpha M\left(s_{ij} = \frac{R_{\text{AdS}}^2}{2\alpha} S_{ij}\right), \quad (6.38b)$$

where R_{AdS} is the radius of AdS_{d+1} , and $\mathcal{N}$ is a normalization constant which, in our normalization, we take $\mathcal{N} = 8\pi^2$.

¹With respect to our previous conventions: $s_{12} = t$, $s_{13} = s$ and $s_{14} = \hat{u}$.

Part III

Loop Dynamics in AdS

Chapter 7

Bubbles in AdS

In the previous chapters, we introduced the large- N expansion of holographic four-point functions and the analytic tools used to extract CFT data from crossing symmetry. We now apply these tools to loop corrections in a four-dimensional large- N CFT dual to a scalar theory in AdS with a ϕ^4 interaction.

At order N^{-4} , the full double discontinuity of the correlator is completely fixed by tree-level OPE data. Therefore, at this order, using the Lorentzian inversion formula we can extract the full correction to the double-trace OPE data in large-spin expansion. This provides an alternative derivation of the one-loop results previously obtained in [8].

At higher orders, the full double discontinuity is no longer determined by tree-level data alone. Nevertheless, at every order $N^{-2\kappa}$, the conformal block expansion of $\mathcal{G}^{(\kappa)}(u, v)$ contains a contribution proportional to $a_{n,0}^{(0)}(\gamma_{n,0}^{(1)})^\kappa$. This is the term containing the highest power $\log^\kappa u$ in the direct channel, which becomes the term containing the highest power $\log^\kappa v$ after crossing. Inserting this contribution in the crossed-channel into the inversion formula fixes the coefficient of $\log^{\kappa-2} J$ in the large-spin expansion of the double-trace anomalous dimensions. So, although tree-level data do not determine the complete higher-order result, they determine its leading logarithmic dependence on the conformal spin. We then resum these leading-logarithmic contributions to all orders in $1/N$ and organize them in terms of effective double-trace exchanges with shifted twists.

By reformulating the same contribution in Mellin space, we establish a three-way correspondence at every order $N^{-2\kappa}$ between the term proportional to $\log^\kappa u$ in the conformal block expansion, the coefficient of $\log^{\kappa-2} J$ in the large-spin expansion of the double-trace anomalous dimensions, and the residues of the poles of order $\kappa - 1$ in the Mellin amplitude. All three quantities are completely fixed by tree-level OPE data and encode the same contribution to the correlator. In the flat-space limit, this part of the Mellin amplitude reproduces the maximal unitarity cut of the corresponding $(\kappa - 1)$ -loop bubble diagram.

The chapter is organized as follows. Section 7.1 reviews the tree-level data and the one-loop correction. Section 7.2 extends the analysis to higher orders and develops the effective scalar-exchange description. Section 7.3 constructs the corresponding restricted Mellin amplitude, and Section 7.4 studies its flat-space limit and its relation to cuts of bubble diagrams.

7.1 Tree-level data and one-loop correction

We consider the four-point function of an identical scalar single-trace operator $\mathcal{O}$ in a four-dimensional large- N CFT. We impose a $\mathbb{Z}_2$ symmetry under $\mathcal{O} \rightarrow -\mathcal{O}$ and focus on the sector dual to a scalar theory in AdS with a ϕ^4 interaction. Gravitational interactions and additional single-trace exchanges will be neglected. The $\mathcal{O} \times \mathcal{O}$ OPE therefore contains the identity and the double-trace operators $[\mathcal{OO}]_{n,\ell}$, together with higher even-trace operators that can enter at subleading orders.

We use the large- N expansions introduced in (4.23)–(4.25). For the non-derivative quartic interaction, the tree-level corrections have support only at spin zero [12]. Fixing the normalization by requiring $\gamma_{0,0}^{(1)} = 1$, they are

$$\gamma_{n,0}^{(1)} = \frac{(2\Delta_{\mathcal{O}} - 1)(n + 1)(2\Delta_{\mathcal{O}} + n - 3)(\Delta_{\mathcal{O}} + n - 1)}{(\Delta_{\mathcal{O}} - 1)(2\Delta_{\mathcal{O}} + 2n - 3)(2\Delta_{\mathcal{O}} + 2n - 1)}, \quad (7.1)$$

which is what we derived in (5.42), and

$$a_{n,0}^{(1)} = \frac{1}{2} \partial_n \left(a_{n,0}^{(0)} \gamma_{n,0}^{(1)} \right). \quad (7.2)$$

All corrections with $\ell > 0$ vanish at this order. For $\Delta_{\mathcal{O}} = 2$, the anomalous dimensions reduce to

$$\gamma_{n,0}^{(1)} = \frac{3(n + 1)^3}{(2n + 1)(2n + 3)}. \quad (7.3)$$

At order N^{-4} , the conformal block expansion contains the term

$$\mathcal{G}^{(2)}(u, v) \supset \frac{1}{8} \sum_n a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^2 u^{\Delta_{\mathcal{O}}+n} (\log u + \partial_n)^2 \tilde{g}_{2\Delta_{\mathcal{O}}+2n,0}(u, v). \quad (7.4)$$

After crossing, its leading logarithmic part is

$$\mathcal{G}^{(2)}(u, v) \supset \frac{1}{8} u^{\Delta_{\mathcal{O}}} \log^2 v \sum_n a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^2 v^n \tilde{g}_{2\Delta_{\mathcal{O}}+2n,0}(v, u). \quad (7.5)$$

The other terms contain at most one power of $\log v$ and have vanishing double discontinuity. Therefore, using (4.12), we obtain

$$\text{dDisc} [\mathcal{G}^{(2)}(z, \bar{z})] = \frac{\pi^2}{2} u^{\Delta_{\mathcal{O}}} \sum_n a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^2 v^n \tilde{g}_{2\Delta_{\mathcal{O}}+2n,0}(v, u). \quad (7.6)$$

So, at this order, the full double discontinuity is determined entirely by the generalized-free-field OPE coefficients and the tree-level anomalous dimensions.

We now set $\Delta_{\mathcal{O}} = 2$ and project onto the leading-twist double-trace operators by retaining the leading term in the small- z expansion. We define the functions $\mathcal{D}^{(2)}(\bar{z})$ and $\mathcal{R}^{(2)}(\bar{z})$ by

$$\text{dDisc} [\mathcal{G}^{(2)}(z, \bar{z})] = z^2 [\mathcal{D}^{(2)}(\bar{z}) \log z + \mathcal{R}^{(2)}(\bar{z})] + O(z^3). \quad (7.7)$$

The function $\mathcal{D}^{(2)}(\bar{z})$ determines the anomalous dimension, and $\mathcal{R}^{(2)}(\bar{z})$ determines the correction to the OPE coefficient. Expanding the crossed-channel scalar blocks and performing the sum over n , we find near $\bar{z} = 1$

$$\mathcal{D}^{(2)}(\bar{z}) = -6\pi^2 - \frac{108\pi^2}{5} \left(\frac{1-\bar{z}}{\bar{z}} \right) - \frac{342\pi^2}{35} \left(\frac{1-\bar{z}}{\bar{z}} \right)^2 + \dots. \quad (7.8)$$

Using the small- z form of the inversion formula kernel, the z -integral of the logarithmic term will be

$$\int_0^1 dz z^{1+\frac{\ell-\Delta}{2}} \log z = -\frac{4}{(4+\ell-\Delta)^2}. \quad (7.9)$$

It therefore produces a double pole at $\Delta = 4 + \ell$. The coefficient of this double pole is determined by the remaining $\bar{z}$ -integral,

$$\int_0^1 d\bar{z} \bar{z}^{\frac{\Delta+\ell-4}{2}} {}_2F_1 \left(\frac{\Delta+\ell}{2}, \frac{\Delta+\ell}{2}; \Delta+\ell; \bar{z} \right) \mathcal{D}^{(2)}(\bar{z}). \quad (7.10)$$

This integral is evaluated at large spin using the peeling method described in Section 4.3.1. Each successive term in (7.8) contributes two additional inverse powers of the conformal spin $J^2 = (\ell+2)(\ell+1)$. The resulting anomalous dimension is

$$\gamma_{0,\ell}^{(2)} = -\frac{6}{J^4} \left(1 + \frac{18}{5J^2} + \frac{96}{7J^4} + \frac{360}{7J^6} + \frac{74304}{385J^8} + \dots \right). \quad (7.11)$$

This agrees with the one-loop result obtained in [8].

Applying the same procedure to $\mathcal{R}^{(2)}(\bar{z})$ gives

$$\frac{a_{0,\ell}^{(2)}}{a_{0,\ell}^{(0)}} = \frac{6}{J^4} \left(1 + \frac{59}{10J^2} + \frac{1048}{35J^4} + \dots \right). \quad (7.12)$$

7.2 Higher orders

7.2.1 Anomalous dimensions

At order $1/N^4$, the contribution in (7.5) captures the full double discontinuity of the correlator and therefore determines both $\gamma_{0,\ell}^{(2)}$ and $a_{0,\ell}^{(2)}$ completely. At higher orders this is no longer the case: dDisc receives contributions not only from tree-level data, and in general fixing the corrections at order $1/N^{2\kappa}$ requires knowledge of the full set of OPE data at order $1/N^{2(\kappa-1)}$. Nevertheless, there is always a universal part of the correlator that depends only on the leading OPE coefficients $a_{n,0}^{(0)}$ and tree-level anomalous dimensions $\gamma_{n,0}^{(1)}$, which produces a non-vanishing dDisc. This contribution is the higher-order analogue of (7.5), namely

$$\mathcal{G}^{(\kappa)} \supset \frac{1}{2^\kappa \kappa!} u^2 \log^\kappa v \sum_n a_{n,0}^{(0)} (\gamma_{n,0}^{(1)})^\kappa v^n \tilde{g}_{4+2n,0}(v, u), \quad (7.13)$$

where we have set $\Delta_{\mathcal{O}} = 2$. While this does not reproduce the full dDisc, we will see that these “leading-log” contributions still control a specific part of the OPE data. To distinguish the data extracted by inverting only this partial contribution from the full result, we will denote them with a $\hat{\cdot}$.

To see how this mechanism works, we first project onto the leading-twist sector and evaluate the double discontinuity using (4.13). As in the one-loop analysis, the term proportional to $\log z$ produces a double pole at $\Delta = 4 + \ell$. At general κ , its residue contains the combination

$$\frac{1}{(\Delta - (4 + \ell))^2} \left[a_{0,\ell}^{(0)} \hat{\gamma}_{0,\ell}^{(\kappa)} + \sum_{i=2}^{\kappa-2} \hat{a}_{0,\ell}^{(i)} \hat{\gamma}_{0,\ell}^{(\kappa-i)} \right]. \quad (7.14)$$

For $\kappa = 3$, the sum in this expression is absent, and the double pole directly determines $\hat{\gamma}_{0,\ell}^{(3)}$. The corresponding contribution in (7.13) is

$$\mathcal{G}^{(3)} \supset \frac{1}{48} u^2 \log^3 v \sum_n a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^3 v^n \tilde{g}_{4+2n,0}(v, u). \quad (7.15)$$

The only new feature compared with the one-loop calculation is the additional logarithm in the $\bar{z}$ -integral, since

$$\text{dDisc} [\log^3(1 - \bar{z})] = 12\pi^2 \log(1 - \bar{z}). \quad (7.16)$$

We treat this logarithm using

$$\log(1 - \bar{z}) = \lim_{w \rightarrow \infty} w [(1 - \bar{z})^{1/w} - 1], \quad (7.17)$$

which is the $q = 1$ case of (4.57). So applying the peeling procedure of Section 4.3.1 we find:

$$\hat{\gamma}_{0,\ell}^{(3)} = \frac{6}{J^4} (\log J + \gamma_E) + \frac{924 (\log J + \gamma_E) - 899}{25J^6} + \dots, \quad (7.18)$$

where γ_E is the Euler–Mascheroni constant. The simple pole similarly gives

$$\frac{\hat{a}_{0,\ell}^{(3)}}{a_{0,\ell}^{(0)}} = -\frac{6}{J^4} (\log J + \gamma_E) + \dots. \quad (7.19)$$

A remarkable feature of these results is the appearance of $\log J$, unlike the results at previous order. This is significant for two reasons. First, from the inversion formula, terms of the form $\log^{\kappa-2} J$ can only arise from the integration of $\log^{\kappa-2}(1 - \bar{z})$. As follows from (4.13), this is the highest logarithm generated in the double discontinuity by the $\log^\kappa v$ contribution in (7.13). So it follows that the $\log J$ part of $\hat{\gamma}^{(3)}$ exactly matches the corresponding piece of the full anomalous dimensions, and more generally

$$\hat{\gamma}_{0,\ell}^{(\kappa)} \Big|_{\log^{\kappa-2} J} = \gamma_{0,\ell}^{(\kappa)} \Big|_{\log^{\kappa-2} J} \quad (7.20)$$

with an analogous statement for the OPE coefficients. Second, this behavior could have been anticipated from the structure of the correlator and in particular its behavior under crossing symmetry. In fact, when expanded in conformal blocks, $\mathcal{G}^{(3)}(u, v)$ contains a contribution of the form

$$\mathcal{G}^{(3)}(u, v) \supset \frac{1}{2} u^2 \log u \sum_{\ell} a_{0,\ell}^{(0)} \gamma_{0,\ell}^{(3)} \tilde{g}_{4+\ell,\ell}(u, v). \quad (7.21)$$

At large ℓ , or rather large J , this contribution leads to [49]

$$\log u \sum_J a_{0,J}^{(0)} \frac{\log J}{J^4} \tilde{g}_{4+J,J}(u, v) \approx \log u \log^3 v. \quad (7.22)$$

This $\log v$ originates exclusively from the $\log J$ term and, upon crossing, provides precisely the contribution required to reproduce the $a^{(0)}(\gamma^{(1)})^3 \log^3 u \log v$ structure in (7.13). For the same reason at higher order we expect $\log^{\kappa-2} J$. Starting from $\kappa = 4$, however, the situation becomes more subtle because the residue of the double pole contains not only the new correction to the anomalous dimension but also contributions proportional to lower-order data. Concretely, at $\kappa = 4$, as follows from (7.14), the relevant structure is

$$\hat{\gamma}_{0,\ell}^{(4)} + \frac{\hat{a}_{0,\ell}^{(2)} \hat{\gamma}_{0,\ell}^{(2)}}{a_{0,\ell}^{(0)}}. \quad (7.23)$$

Nevertheless, using the explicit results of (7.11) and (7.12), one sees that the second term cannot generate $\log^2 J$. Hence any $\log^2 J$ produced by the inversion formula at this order must come entirely from $\hat{\gamma}_{0,\ell}^{(4)}$. And by the same logic as above, this contribution coincides with the corresponding piece of the full anomalous dimension. In general, we can therefore state that the leading-logarithmic part of the double-pole residue in (7.14) determines the coefficient of $\log^{\kappa-2} J$ in the large- J expansion of $\gamma_{0,\ell}^{(\kappa)}$,

$$\gamma_{0,\ell}^{(\kappa)} \approx \log^{\kappa-2} J \left(\frac{\delta_1^{(\kappa)}}{J^4} + \frac{\delta_2^{(\kappa)}}{J^6} + \frac{\delta_3^{(\kappa)}}{J^8} + \dots \right) + \dots, \quad (7.24)$$

where explicit evaluation for several values of κ yields

$$\begin{aligned} \delta_1^{(\kappa)} &= \frac{6(-1)^{\kappa+1}}{\Gamma(\kappa-1)}, \\ \delta_2^{(\kappa)} &= \frac{4(-1)^\kappa}{\Gamma(\kappa-1)} \left(1 - 4 \left(\frac{8}{5} \right)^{\kappa-1} \right), \\ \delta_3^{(\kappa)} &= \frac{4(-1)^{\kappa+1}}{\Gamma(\kappa-1)} \left(1 - 7 \left(\frac{8}{5} \right)^\kappa + 7 \left(\frac{81}{35} \right)^\kappa \right). \end{aligned} \quad (7.25)$$

Reinstating the factors of $1/N$, this series can actually be resummed in κ , giving the leading large J contribution to the leading twist anomalous dimension $\gamma_{0,\ell} \equiv \gamma_\ell$

$$\begin{aligned} \gamma_\ell(N) \approx \frac{1}{N^4} \left[-\frac{6J^{-\frac{1}{N^2}}}{J^4} + \frac{4}{J^6} \left(J^{-\frac{1}{N^2}} - \frac{32}{5} J^{-\frac{8}{5N^2}} \right) \right. \\ \left. - \frac{4}{J^8} \left(J^{-\frac{1}{N^2}} - \frac{448}{25} J^{-\frac{8}{5N^2}} + \frac{6561}{175} J^{-\frac{81}{35N^2}} \right) \right] + \dots. \end{aligned} \quad (7.26)$$

Here the powers of J can be recognized as precisely the tree level anomalous dimensions $\gamma_{n,0}^{(1)}$ defined in (7.3),

$$\gamma_{0,0}^{(1)} = 1, \quad \gamma_{1,0}^{(1)} = \frac{8}{5}, \quad \gamma_{2,0}^{(1)} = \frac{81}{35}. \quad (7.27)$$

This observation leads to a compact expression for the resummation of the leading logarithms,

$$\hat{\gamma}_{0,\ell}(N)|_{\text{leading log}} \approx \frac{1}{N^4} \sum_{n,m \geq 0} \frac{f_{nm}^{(0)}}{J^{4+2n+2m+\epsilon_n}}, \quad \epsilon_n = \frac{\gamma_{n,0}^{(1)}}{N^2}, \quad (7.28)$$

where the coefficients $f_{nm}^{(0)}$ are independent of N . In the next section, we will interpret this result from an effective field theory point of view.

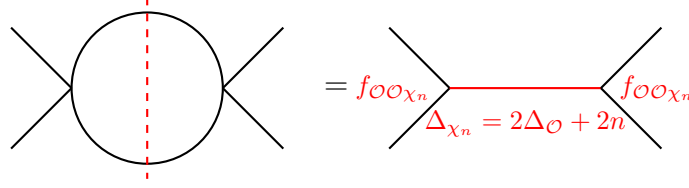


Figure 7.1: Spectral decomposition of bubble diagrams as a sum over tree-level scalar exchanges.

7.2.2 Interpretation as a scalar exchange in AdS

To understand how an effective description emerges, let us return to the case $\kappa = 2$. In this situation, the one-loop bubble diagram can be decomposed, via the spectral representation of [16], into a sum over tree-level scalar exchanges χ_n , with dimension $2\Delta_{\mathcal{O}} + 2n$, as illustrated in Fig. 7.1. At this order, using the appropriate OPE coefficients and setting $\Delta_{\chi_n} = 4 + 2n$, one recovers the result for $\gamma_{0,\ell}^{(2)}$ in (7.11), where the exchange of the scalar of dimension $4 + 2n$ contributes starting from $1/J^{4+2n}$. At higher orders in $1/N$, we would like to push this analogy further and formulate an effective field theory at large spin. The idea is to not consider explicit loop computations and instead promote each of the scalars χ_n to dynamical fields in AdS with shifted dimensions $\Delta_{\chi_n} = 4 + 2n + \frac{1}{N^2}\gamma_{n,0}^{(1)}$. At large spin, the anomalous dimension of leading-twist double-trace operators induced by the exchange of an AdS scalar takes the form [42]

$$\gamma_{\ell} \approx -\frac{1}{J^{\Delta_{\chi_n}}} \frac{2f_{\mathcal{O}\mathcal{O}\chi_n}^2 \Gamma(\Delta_{\mathcal{O}})^2 \Gamma(\Delta_{\chi_n})}{\Gamma\left(\Delta_{\mathcal{O}} - \frac{\Delta_{\chi_n}}{2}\right)^2 \Gamma\left(\frac{\Delta_{\chi_n}}{2}\right)^2}. \quad (7.29)$$

Focusing for simplicity on the case $n = 0$, and substituting the large- N expansion of the OPE coefficient and the shifted dimension, one immediately reproduces the leading $\log J$ term multiplying $1/J^4$

$$\gamma_{\ell} \approx -\frac{3}{J^4} \sum_{\kappa=2} \frac{1}{N^{2\kappa}} a_{0,0}^{(0)} \left(\gamma_{0,0}^{(1)}\right)^{\kappa} \frac{(-1)^{\kappa} \log^{\kappa-2} J}{\Gamma(\kappa-1)}, \quad (7.30)$$

where we have used $\gamma_{0,0}^{(1)} = 1$ and $a_{0,0}^{(0)} = 2$. Similar logic applies for $n \geq 1$. In this case, the expression in (7.29) does not fully determine the coefficient of the $1/J^{4+2n}$ term, but it does fix the universal contribution corresponding to

$$-\frac{1}{N^4} \frac{36(n+1)^8 \Gamma(n+1)^4}{(2n+1)\Gamma(2n+4)} \frac{1}{J^{4+2n+\gamma_{n,0}^{(1)}/N^2}}. \quad (7.31)$$

7.2.3 Subleading twist

So far our analysis has focused on the leading-twist operators, but the same procedure can be applied to double-trace operators with higher twists. As an illustration, consider the case $n = 1$. By extracting the z^3 contribution from dDisc, one finds at large spin, for the first few orders in $1/N$,

$$\begin{aligned}\gamma_{1,\ell}^{(2)} &\approx -\frac{30}{J^4} \left(1 + \frac{314}{25J^2} + \cdots \right), \\ \hat{\gamma}_{1,\ell}^{(3)} &\approx \frac{30}{J^4} \log J \left(1 + \frac{2402}{125J^2} + \cdots \right) + \cdots, \\ \hat{\gamma}_{1,\ell}^{(4)} &\approx -\frac{15}{J^4} \log^2 J \left(1 + \frac{18666}{625J^2} + \cdots \right) + \cdots.\end{aligned}\tag{7.32}$$

where we have omitted terms suppressed by lower powers of $\log J$ and in this case $J^2 = (\ell + 3)(\ell + 2)$.

It is important to mention that starting from twist eight, the anomalous dimensions extracted from the double poles of the inversion formula may not coincide directly with those of the pure double-trace operators. In fact, at $1/N^4$ order and beyond, quadruple and higher-trace operators also appear in the OPE and mix with double traces of the same classical dimensions.¹ As a result, what is obtained from the double-pole structure is an average anomalous dimension over all such operators, and a proper diagonalization of the mixing problem would be required to isolate the individual contributions.

7.3 Mellin space

The Mellin representation of the four-point function, its crossing properties, and the pole structure of loop amplitudes were reviewed in Chapter 6. For higher loops, we will instead focus on constructing a Mellin amplitude that reproduces only the leading-log contribution in (7.13), which takes the form

$$M_{(\kappa-1)\text{-loop}}^R(s, t) = \sum_{m=0}^{\infty} \frac{R_m^{(\kappa-1)}(s)}{(t - (2\Delta_{\mathcal{O}} + 2m))^{\kappa-1}} + \text{crossed}.\tag{7.33}$$

Although this is only part of the full solution, it already captures significant physical information. In what follows, we show how the coefficients $R_m^{(\kappa-1)}$ can be derived from tree-level data and how they reproduce the anomalous dimensions $\gamma_{0,\ell}^{(\kappa)}$ obtained previously.

¹Recall that we define quadruple-trace operators and in general Q -trace operators as operators with $\tau = Q\Delta_{\mathcal{O}} + 2n$.

7.3.1 Extracting the $R_m^{(\kappa-1)}$ residues

The analogue of the conformal block decomposition for the Mellin amplitude and the normalization of the corresponding Mack polynomials were reviewed in Section 6.2.1. Specializing to the case where $\mathcal{O}_p$ are the double-trace operators with scaling dimensions and OPE coefficients given in (4.24) and (4.25), we get

$$M(s, t) = \sum_{m,n} \frac{\left(a_{n,\ell}^{(0)} + \frac{a_{n,\ell}^{(1)}}{N^2} + \dots \right) \mathcal{Q}_{\ell,m} \left(s, 2\Delta_{\mathcal{O}} + 2n + \frac{\gamma_{n,\ell}^{(1)}}{N^2} + \frac{\gamma_{n,\ell}^{(2)}}{N^4} + \dots \right)}{t - \left(2\Delta_{\mathcal{O}} + 2n + 2m + \frac{\gamma_{n,\ell}^{(1)}}{N^2} + \frac{\gamma_{n,\ell}^{(2)}}{N^4} + \dots \right)} + \dots \quad (7.34)$$

We first expand the Mack polynomial and by using (6.28) it is straightforward to verify that

$$\mathcal{Q}_{\ell,m} \left(s, 2\Delta_{\mathcal{O}} + 2n + \frac{\gamma_{n,\ell}^{(1)}}{N^2} + \frac{\gamma_{n,\ell}^{(2)}}{N^4} + \dots \right) \propto \left(\frac{\gamma_{n,\ell}^{(1)}}{N^2} \right)^2 \mathcal{Q}_{\ell,m}(s, 2\Delta_{\mathcal{O}} + 2n) + O(N^{-6}). \quad (7.35)$$

In our setup, only the spin-zero tree-level anomalous dimension are non vanishing, we can therefore set $\ell = 0$, and upon further specifying $\Delta_{\mathcal{O}} = 2$, we obtain

$$\mathcal{Q}_{0,m} = -\frac{(\gamma_{n,\ell}^{(1)})^2}{N^4} \frac{(2n+3)\Gamma(2n+3)^2\Gamma(m+n+1)^2}{2\Gamma(m+1)\Gamma(n+2)^4\Gamma(m+2n+3)} + \dots \quad (7.36)$$

Plugging this into (7.34), we find that at order $N^{-2\kappa}$ the Mellin amplitude contains a contribution of the form

$$M_{(\kappa-1)\text{-loop}}(s, t) \supset \frac{-a_{n,0}^{(0)}(\gamma_{n,0}^{(1)})^\kappa}{(t - (4 + 2n + 2m))^{\kappa-1}} \frac{(2n+3)\Gamma(2n+3)^2\Gamma(m+n+1)^2}{2\Gamma(m+1)\Gamma(n+2)^4\Gamma(m+2n+3)}. \quad (7.37)$$

This is precisely the term to be compared with $M_{(\kappa-1)\text{-loop}}^R$ in (7.33) in order to extract the residues $R_m^{(\kappa-1)}$. After substituting the explicit OPE data and changing variables $m \rightarrow m - n$, we finally obtain

$$R_m^{(\kappa-1)} = -\sum_{n=0}^m \frac{4 \cdot 3^\kappa (n+1)^{3\kappa} \Gamma(m+1)^2}{(2n+1)^{\kappa-1} (2n+3)^{\kappa-1} \Gamma(m-n+1) \Gamma(m+n+3)}. \quad (7.38)$$

Apart from the case $\kappa = 2$ this expression cannot be resummed in an explicit closed form ². However, in the form given above it becomes particularly convenient to extract the large- m behavior, which will be relevant in the following.

²It is however possible to get a recursive sequence.

First observe that for m, n large

$$\begin{aligned} \frac{\Gamma(m+1)^2}{\Gamma(m-n+1)\Gamma(m+n+3)} &= \frac{1}{(m+1)(m+2)} \frac{\binom{2m+2}{m-n}}{\binom{2m+2}{m}} \\ &\approx \frac{1}{(m+1)(m+2)} e^{-\frac{n(n+2)}{m+1}}, \end{aligned} \quad (7.39)$$

where we have used the expansion of the binomial coefficient $\binom{x}{y}$ for large argument. Expanding then all the other terms, we obtain

$$\begin{aligned} R_m^{(\kappa-1)} &\approx \sum_{n=0}^m -\frac{3^\kappa n^{\kappa+2}}{4^{\kappa-2} m^2} e^{-\frac{n^2}{m}} \approx -\frac{3^\kappa m^{\frac{\kappa-1}{2}}}{4^{\kappa-2}} \int_0^\infty dx x^{\kappa+2} e^{-x^2} \\ &= -\frac{3^\kappa}{2^{2\kappa-3}} \Gamma\left(\frac{\kappa+3}{2}\right) m^{\frac{\kappa-1}{2}}, \end{aligned} \quad (7.40)$$

where we have used $x = n/\sqrt{m}$. Remarkably, this expression makes manifest the connection between the residues at order $N^{-2\kappa}$ and the one-loop result

$$R_m^{(\kappa-1)} \approx -6\Gamma\left(\frac{\kappa+3}{2}\right) \left(\frac{-2}{9\sqrt{\pi}}\right)^{\kappa-1} (R_m^{(1)})^{\kappa-1}. \quad (7.41)$$

This factorized structure closely mirrors what happens in flat space, where higher-loop bubble diagrams can be understood as repeated insertions of the one-loop one. We will return to this analogy in more detail in Sec. 7.4.

7.3.2 Connection to $\hat{\gamma}_{0,\ell}^{(\kappa)}$

So far, we have showed that the coefficients $R_m^{(\kappa-1)}$ are uniquely determined from the tree-level OPE data. At the same time, in Sec. 7.2, we have seen that the same information fixes the leading $\log J$ contributions to the anomalous dimensions at each loop order. This strongly suggests a direct correspondence between the $R_m^{(\kappa-1)}$ terms in (7.33) and the leading large- J behavior of the anomalous dimensions in (7.23).

As derived in detail in [8], by matching powers of $\log u$ and performing a conformal block decomposition in Mellin space (analogous to (7.4)), one can express the correction to the anomalous dimension at order $N^{-2\kappa}$ in terms of the corresponding Mellin amplitude

$$\gamma_{0,\ell>0}^{(\kappa)} = -\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds M'_{(\kappa-1)\text{-loop}}(s, 2\Delta_{\mathcal{O}}) \Gamma\left(\frac{s}{2}\right)^2 \Gamma\left(\frac{2\Delta_{\mathcal{O}}-s}{2}\right)^2 {}_3F_2\left(\begin{matrix} -\ell, & \ell+2\Delta_{\mathcal{O}}-1, & \frac{s}{2} \\ \Delta_{\mathcal{O}}, & \Delta_{\mathcal{O}} \end{matrix}; 1\right) \quad (7.42)$$

where $M'_{(\kappa-1)\text{-loop}}(s, 2\Delta_{\mathcal{O}})$ denotes the Mellin amplitude with the $t = 2\Delta_{\mathcal{O}}$ pole subtracted. For large ℓ , or equivalently large conformal spin J , the hypergeometric function admits the approximation

$${}_3F_2\left(\begin{matrix} -\ell, \ell + 2\Delta_{\mathcal{O}} - 1, \frac{s}{2}; 1 \end{matrix}\right) \approx \frac{\Gamma(\Delta_{\mathcal{O}})^2}{\Gamma(\frac{2\Delta_{\mathcal{O}}-s}{2})^2} \beta_J(s) + (-1)^\ell \frac{\Gamma(\Delta_{\mathcal{O}})^2}{\Gamma(\frac{s}{2})^2} \beta_J(2\Delta_{\mathcal{O}} - s),$$

$$\beta_J(s) = \frac{1}{J^s} \left(1 + \frac{3\Delta_{\mathcal{O}}s^2 - s^3 - 3s^2 - 2s}{12J^2} + \dots \right). \quad (7.43)$$

At leading order this gives

$$\gamma_{0,\ell \gg 1}^{(\kappa)} \approx -\frac{\Gamma(\Delta_{\mathcal{O}})^2}{2\pi i} \int_{-i\infty}^{i\infty} ds M'_{(\kappa-1)\text{-loop}}(s, 2\Delta_{\mathcal{O}}) \left(\frac{\Gamma(\frac{s}{2})^2}{J^s} + \frac{\Gamma(\frac{2\Delta_{\mathcal{O}}-s}{2})^2}{J^{2\Delta_{\mathcal{O}}-s}} \right), \quad (7.44)$$

where we have set $(-1)^\ell = 1$, since only even spins are exchanged. From this expression, it is straightforward to see that the leading $\log J$ term can only arise from the part of the Mellin amplitude we have denoted $M_{(\kappa-1)\text{-loop}}^R$ in (7.33).

$$\gamma_{0,\ell \gg 1}^{(\kappa)} \approx -\frac{\Gamma(\Delta_{\mathcal{O}})^2}{2\pi i} \int_{-i\infty}^{i\infty} ds \sum_{m=0}^{\infty} \frac{R_m^{(\kappa-1)}}{(s - (2\Delta_{\mathcal{O}} + 2m))^{\kappa-1}} \left(\frac{\Gamma(\frac{s}{2})^2}{J^s} + \frac{\Gamma(\frac{2\Delta_{\mathcal{O}}-s}{2})^2}{J^{2\Delta_{\mathcal{O}}-s}} \right), \quad (7.45)$$

The first term can be evaluated by closing the contour to the right and picking up the pole at $s = 2\Delta_{\mathcal{O}} + 2m$. Since each of them contributes a factor $J^{-(2\Delta_{\mathcal{O}}+2m)}$, for the leading contribution at large spin we can set $m = 0$. To evaluate the second term, we can close the contour to the left and since there are no poles, it simply vanishes. We therefore obtain

$$\begin{aligned} \gamma_{0,\ell \gg 1}^{(\kappa)} &\approx \frac{(-1)^\kappa}{\Gamma(\kappa-1)} \frac{\log^{\kappa-2} J}{J^{2\Delta_{\mathcal{O}}}} R_0^{(\kappa-1)} \Gamma(\Delta_{\mathcal{O}})^4 + \dots \\ &= -a_{0,0}^{(0)} (\gamma_{0,0}^{(1)})^\kappa \frac{(-1)^\kappa \log^{\kappa-2} J}{2\Gamma(\kappa-1)} \Gamma(2\Delta_{\mathcal{O}}), \end{aligned} \quad (7.46)$$

where in the second line we have used

$$R_0^{(\kappa-1)} = -a_{0,0}^{(0)} (\gamma_{0,0}^{(1)})^\kappa \frac{\Gamma(2\Delta_{\mathcal{O}})}{2\Gamma(\Delta_{\mathcal{O}})^4}. \quad (7.47)$$

Notice that setting $\Delta_{\mathcal{O}} = 2$, this expression exactly reproduces our previous result in (7.30) and more generally, it makes manifest the direct correspondence between the leading $\log J$ behavior of the anomalous dimension at $(\kappa-1)$ -loops and the restricted Mellin amplitude $M_{(\kappa-1)\text{-loop}}^R$.

7.4 Flat-space limit

The flat-space limit and the integral transforms relating Mellin amplitudes to flat-space scattering amplitudes were reviewed in Section 6.3. We now apply this relation to the Mellin amplitudes obtained above.

Let us illustrate this by analyzing the one-loop result. For $\Delta_i = 2$, the Mellin amplitude takes the form

$$M_{1\text{-loop}}(s, t) = \sum_m \frac{R_m^{(1)}}{t - (4 + 2m)} + \text{crossed}. \quad (7.48)$$

To apply the flat-space limit, we must take $s, t \gg 1$ large. In this regime, the m sum is dominated by terms with $m \sim s, t \gg 1$, so it can be approximated by an integral. Focusing on a single channel, one finds

$$M_{1\text{-loop}}(t) \approx -\frac{27}{8} \sqrt{\pi} \int_0^\infty dm \frac{m^{1/2}}{t - 2m}, \quad (7.49)$$

where we have used the large m approximation of $R_m^{(1)}$ in (7.41). The integral is divergent, but it can be regularized as

$$M_{1\text{-loop}}(t) = -\frac{27}{8} \sqrt{\pi} \int_0^\infty dm \frac{m^j e^{-w^2/m}}{t - 2m}, \quad (7.50)$$

so that one can first evaluate the integral and then set $j = \frac{1}{2}$ and $w = 0$. This yields

$$M_{1\text{-loop}}(t) \stackrel{t \gg 1}{\approx} -\frac{27\pi^{\frac{3}{2}} \sqrt{-t}}{16\sqrt{2}}. \quad (7.51)$$

Plugging this result into (6.38b) we get

$$A_{1\text{-loop}} = \frac{9\pi R_{\text{AdS}}^2}{8N^4} \mathcal{N} \sqrt{-T}. \quad (7.52)$$

This should be compared to the corresponding flat-space amplitude, namely the bubble diagram in five dimensions. A straightforward evaluation of the integral, including its symmetry factor, yields

$$A_{1\text{-bubble}} = \lambda_{\text{flat}}^2 \frac{\sqrt{-T}}{256\pi}. \quad (7.53)$$

The two results have exactly the same $\sqrt{-T}$ behavior, and this comparison fixes the flat-space coupling to be

$$|\lambda_{\text{flat}}| = \frac{12\sqrt{2}\pi R_{\text{AdS}}}{N^2} \sqrt{\mathcal{N}} = \frac{48\pi^2 R_{\text{AdS}}}{N^2}, \quad (7.54)$$

where in the last equality we used $\mathcal{N} = 8\pi^2$.

From two loops onward, the flat space amplitude receives contributions from Feynman diagrams of different topologies, but at each order there is always a contribution from a chain of $(\kappa - 1)$ bubbles at $(\kappa - 1)$ -loop order. In flat space such diagrams are simple to compute, as they factorize:

$$A_{(\kappa-1)\text{-bubble}} = \frac{(A_{1\text{-bubble}})^{\kappa-1}}{\lambda_{\text{flat}}^{\kappa-2}}. \quad (7.55)$$

Substituting this into the flat-space formula (6.38a) predicts a Mellin amplitude of the form

$$\begin{aligned} M_{(\kappa-1)\text{-bubble}}(t) &\approx -\frac{1}{N^{2\kappa}} (-t)^{\frac{\kappa-1}{2}} \mathcal{N}^{\frac{\kappa-2}{2}} \pi 3^\kappa 2^{\frac{15}{2}-5\kappa} \Gamma\left(\frac{\kappa+3}{2}\right) \\ &= \frac{1}{N^{2\kappa}} (M_{1\text{-loop}}(t))^{\kappa-1} \mathcal{N}^{\frac{\kappa-2}{2}} 2^{3-\frac{\kappa}{2}} (-1)^\kappa 3^{3-2\kappa} \pi^{\frac{5}{2}-\frac{3\kappa}{2}} \Gamma\left(\frac{\kappa+3}{2}\right), \end{aligned} \quad (7.56)$$

with $M_{1\text{-loop}}(t)$ in (7.51). It is now interesting to interpret this result in light of our expression for the R_m coefficients. Focusing again on a single channel, we recall that

$$M_{(\kappa-1)\text{-loop}}^R = \frac{1}{N^{2\kappa}} \sum_m \frac{R_m^{(\kappa-1)}}{(t - (4 + 2m))^{\kappa-1}}. \quad (7.57)$$

Using the large- m expansion of the residues given in (7.41), this can be rewritten as

$$M_{(\kappa-1)\text{-loop}}^R = -6\Gamma\left(\frac{\kappa+3}{2}\right) \left(\frac{-2}{9\sqrt{\pi}}\right)^{\kappa-1} \frac{1}{N^{2\kappa}} \sum_m \frac{(R_m^{(1)})^{\kappa-1}}{(t - t_m)^{\kappa-1}}, \quad (7.58)$$

where we have defined $t_m \equiv 4 + 2m$. This result should be compared to

$$(M_{1\text{-loop}}(t))^{\kappa-1} = \left(\sum_m \frac{R_m^{(1)}}{t - t_m} \right)^{\kappa-1}. \quad (7.59)$$

The two structures are evidently not equivalent: expanding the latter produces, in addition to the diagonal terms $\frac{(R_m^{(1)})^{\kappa-1}}{(t-t_m)^{\kappa-1}}$, also cross terms with $m_i \neq m_j$. However these cross-terms generate lower-order poles³, thus contributing to subleading

³As an example, consider $\kappa = 3$

$$\begin{aligned} (M_{1\text{-loop}}(t))^2 &= \sum_m \frac{(R_m^{(1)})^2}{(t - t_m)^2} + 2 \sum_{m < n} \frac{R_m^{(1)} R_n^{(1)}}{(t - t_m)(t - t_n)}, \\ \frac{1}{(t - t_m)(t - t_n)} &= \frac{1}{(t_m - t_n)} \left(\frac{1}{t - t_m} - \frac{1}{t - t_n} \right) = \frac{1}{2(m - n)} \left(\frac{1}{t - t_m} - \frac{1}{t - t_n} \right). \end{aligned} \quad (7.60)$$

logarithms in position space. So as far as the leading log results is concerned we can approximate

$$M_{(\kappa-1)\text{-loop}}^R \Big|_{\text{leading log}} = -6\Gamma\left(\frac{\kappa+3}{2}\right) \left(\frac{-2}{9\sqrt{\pi}}\right)^{\kappa-1} \frac{1}{N^{2\kappa}} (M_{1\text{-loop}}(t))^{\kappa-1}, \quad (7.61)$$

and fixing $\mathcal{N} = 8\pi^2$ this exactly matches the expectation from $(\kappa-1)$ -bubbles in (7.56). In this way, we have shown how M^R precisely captures the contribution of bubble-like diagrams, providing the Mellin-space counterpart of the flat-space factorization property of bubble diagrams.

This correspondence with flat-space bubble diagrams also clarifies the role of the R_m coefficients. In flat space, the $(\kappa-1)$ -loop bubble diagrams are the only diagrams that admit a maximal cut in one channel, obtained by putting all propagators on shell. Concretely, each propagator is replaced by a delta function,

$$\frac{1}{p^2} \rightarrow 2\pi i \delta(p^2), \quad (7.62)$$

so that the diagram reduces to a product of on-shell phase-space measures. For a chain of $(\kappa-1)$ bubbles, this procedure yields

$$\max \text{ cut}_{t\text{-channel}} [A_{(\kappa-1)\text{-loop}}] \propto (-T)^{\frac{\kappa-1}{2}}. \quad (7.63)$$

Remarkably, the same scaling is already encoded in the Mellin amplitude M^R , and in particular in the residues of its $(\kappa-1)$ poles. In the large- m regime, it is enough to identify $m \simeq T/2$ in the expansion of R_m , obtaining

$$R^{(\kappa-1)}(m = T/2) \propto T^{\frac{\kappa-1}{2}}, \quad (7.64)$$

which exactly reproduces the dependence of the flat-space maximal cut. This observation, first emphasized in [31], shows that the Mellin-space residues provide a direct counterpart of flat-space maximal cuts. More conceptually, it highlights that information contained purely in tree-level data already fixes universal aspects of loop amplitudes, as illustrated in Fig. 7.2.

$$\frac{1}{2^\kappa \kappa!} \log^\kappa u \sum_n u^{\Delta_{\mathcal{O}}+n} a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^\kappa \tilde{g}_{2\Delta_{\mathcal{O}}+2n}(u, v) \longleftrightarrow R_m^{(\kappa-1)} \longleftrightarrow \begin{array}{c} \text{bubble diagram} \\ \vdots \\ \text{bubble diagram} \end{array}$$

Figure 7.2: Schematic relation between the order- $(\kappa - 1)$ residues of the Mellin amplitude, fully determined by tree-level anomalous dimensions, and the maximal cut of the $(\kappa - 1)$ -loop bubble diagrams.

Chapter 8

Higher-trace operators

In the previous chapter, we studied a universal part of the double-trace OPE data that is completely fixed by tree-level data. We showed that this contribution is related to maximal cuts of bubble Witten diagrams and that, at order $N^{-2\kappa}$, it determines the leading $\log^{\kappa-2} J$ behavior of the double-trace anomalous dimensions.

One aim of this chapter is to understand what additional structures are generated in the conformal block expansion once infinite spin sums of double-trace data are treated carefully. A key point, discussed in Chapter 5, is that such sums can produce behavior in the small cross-ratio limit which is not visible at the level of individual blocks. After applying crossing symmetry, these terms can have the power-law structure expected from the exchange of higher-trace operators. In this way, higher-trace operators appear as a necessary part of the crossing-symmetric completion of the lower-trace data.

We analyze this mechanism in theories dual to bulk ϕ^4 and ϕ^3 interactions. In the ϕ^4 theory, the crossed image of double-trace spin sums at order N^{-8} gives rise to quadruple-trace operators. We compute the leading-twist OPE coefficients of these operators, as well as their first anomalous dimensions. In the ϕ^3 theory, k -trace operators appear at order N^{-2k} , and we determine the leading logarithmic part of their large-spin OPE coefficients. The diagrammatic origin of these relations will be discussed in the next chapter.

Before turning to the higher-trace sector, we collect the double-trace data that will be needed in the calculation.

8.1 Double-trace data

8.1.1 Bubble data

In section 4.3.2 we studied the effect of generic exchange of an operator U to the double-trace spectrum. Here we want to specify to the case in which the operator U itself belongs to the double-trace tower. One should first observe that the generic-exchange result in (4.60) vanishes whenever

$$\tau_U - 2\Delta_{\mathcal{O}} \in 2\mathbb{Z}_{\geq 0}, \quad (8.1)$$

which precisely corresponds to the bare twists of double-trace operators. The non-trivial contribution appears only after including anomalous dimensions and, in general, after summing over the relevant tower of exchanged operators.

As discussed in Chapter 5, an infinite spin sum can generate singularities that are invisible term by term. However, in the case of a ϕ^4 interaction, the tree-level OPE data have support only at spin zero. The corresponding anomalous dimensions and OPE coefficients were given in (7.1) and (7.2). Because of this finite support in spin, one may treat each exchanged operator independently and afterwards sum over the level n .

We introduce the shorthand

$$\epsilon_n \equiv \frac{\gamma_{n,0}^{(1)}}{N^2}, \quad \alpha_n \equiv a_{n,0}^{(0)} + \frac{a_{n,0}^{(1)}}{N^2}. \quad (8.2)$$

We then define an effective scalar exchange with

$$\tau_U = 2\Delta_{\mathcal{O}} + 2n + \epsilon_n, \quad \ell_U = 0, \quad a_{\Delta_U,0} = \alpha_n. \quad (8.3)$$

This leads to

$$\hat{\gamma}_{0,\ell} = \sum_{n,m \geq 0} \frac{f_{nm}}{J^{2\Delta_{\mathcal{O}}+2n+2m+\epsilon_n}}. \quad (8.4)$$

The hat indicates that we keep only the contribution determined by tree-level OPE data. The coefficients f_{nm} depend implicitly on N and they can be expanded as:

$$f_{nm} = \frac{f_{nm}^{(0)}}{N^4} + O(N^{-6}). \quad (8.5)$$

This effective description was discussed in the previous chapter for $\Delta_{\mathcal{O}} = 2$. Here we extend it to generic $\Delta_{\mathcal{O}}$ by reporting the first few coefficients. Using (4.61) and (4.62), we find

$$f_{00} = -2\alpha_0 \frac{\Gamma(\Delta_{\mathcal{O}})^2 \Gamma(2\Delta_{\mathcal{O}} + \epsilon_0)}{\Gamma(-\frac{\epsilon_0}{2})^2 \Gamma(\Delta_{\mathcal{O}} + \frac{\epsilon_0}{2})^2}, \quad (8.6)$$

$$f_{10} = -2\alpha_1 \frac{\Gamma(\Delta_{\mathcal{O}})^2 \Gamma(2\Delta_{\mathcal{O}} + 2 + \epsilon_1)}{\Gamma(-1 - \frac{\epsilon_1}{2})^2 \Gamma(\Delta_{\mathcal{O}} + 1 + \frac{\epsilon_1}{2})^2}, \quad (8.7)$$

and

$$f_{01} = -\frac{S_0}{48} f_{00}, \quad (8.8)$$

where

$$S_0 = \frac{x_0 [x_0^3 - 4x_0^2 - 12\Delta_{\mathcal{O}}^2 x_0 + 36\Delta_{\mathcal{O}} x_0 - 16x_0 - 8]}{x_0 - 1}, \quad x_0 = 2\Delta_{\mathcal{O}} + \epsilon_0. \quad (8.9)$$

The same method can be used to obtain arbitrarily high orders in the large-spin expansion.

Expanding these expressions at large N gives loop-level double-trace data. At one loop, namely at order N^{-4} , we find

$$\begin{aligned} \gamma_{0,\ell}^{(2)} = & -\frac{\Gamma(2\Delta_{\mathcal{O}})}{J^{2\Delta_{\mathcal{O}}}} \\ & + \frac{\Delta_{\mathcal{O}} [1 + \Delta_{\mathcal{O}} (8 - \Delta_{\mathcal{O}} (7 + 2\Delta_{\mathcal{O}}))] \Gamma(2\Delta_{\mathcal{O}})}{3(1 + 2\Delta_{\mathcal{O}}) J^{2\Delta_{\mathcal{O}}+2}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+4}}\right). \end{aligned} \quad (8.10)$$

At two loops, tree-level data fix the coefficient of $\log J$ in the anomalous dimension:

$$\begin{aligned} \gamma_{0,\ell}^{(3)} \Big|_{\log J} = & \frac{\Gamma(2\Delta_{\mathcal{O}})}{J^{2\Delta_{\mathcal{O}}}} \\ & + \frac{(\Delta_{\mathcal{O}} - 1) \Delta_{\mathcal{O}} [1 + \Delta_{\mathcal{O}} (11 + 4\Delta_{\mathcal{O}} (11 + \Delta_{\mathcal{O}}))] \Gamma(2\Delta_{\mathcal{O}})}{3(1 + 2\Delta_{\mathcal{O}})^2 J^{2\Delta_{\mathcal{O}}+2}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+4}}\right). \end{aligned} \quad (8.11)$$

More generally, at order $N^{-2\kappa}$, the part of the higher-loop OPE data fixed by tree-level input is the leading logarithmic term

$$\gamma_{n,\ell}^{(\kappa)} \Big|_{\log^{\kappa-2} J}. \quad (8.12)$$

Subleading twist

So far we have focused mostly on the leading-twist double-trace family $[\mathcal{OO}]_{0,\ell}$. In the next sections we will also need the large-spin behavior of the full tower $[\mathcal{OO}]_{n,\ell}$. The analysis of the previous subsection can be systematically extended to subleading twists, but for our purposes we only need the leading large-spin behavior at fixed n . We therefore briefly recall the standard large-spin result of [46].

For identical external scalar operators of dimension $\Delta_{\mathcal{O}}$, the anomalous dimension of the double-trace operator $[\mathcal{OO}]_{n,\ell}$ induced by the exchange of an operator with twist τ_U and spin ℓ_U has the leading large-spin form

$$\gamma_{n,\ell} = \frac{\gamma_n}{J^{\tau_U}} + \mathcal{O}\left(\frac{1}{J^{\tau_U+2}}\right), \quad (8.13)$$

where for the family $[\mathcal{OO}]_{n,\ell}$ we use the conformal spin defined as $J^2 = (\ell + \Delta_{\mathcal{O}} + n)(\ell + \Delta_{\mathcal{O}} + n - 1)$. The coefficient γ_n is determined by a finite sum

$$\gamma_n = \sum_{m=0}^n c_{n,m} L_m, \quad (8.14)$$

where

$$c_{n,m} = \frac{1}{8} \left[\frac{\Gamma(\Delta_{\mathcal{O}})}{\Gamma(\Delta_{\mathcal{O}} + m - 1)} \right]^2 (2\Delta_{\mathcal{O}} + n - 3)_m (-1)^{n+m} \frac{n!}{(n-m)!} \left[\frac{\Gamma(\Delta_{\mathcal{O}} - 1)}{\Gamma(\Delta_{\mathcal{O}} - \frac{\tau_U}{2})} \right]^2, \quad (8.15)$$

and

$$L_m = -16 a_{\Delta_U, \ell_U} \frac{\Gamma(2\ell_U + \tau_U) \Gamma(m + \ell_U + \frac{\tau_U}{2})^2}{\Gamma(m+1)^2 \Gamma(\ell_U + \frac{\tau_U}{2})^4} {}_3F_2 \left[\begin{matrix} -m, -m, -1 - \ell_U + \Delta_{\mathcal{O}} - \frac{\tau_U}{2} \\ 1 - m - \ell_U - \frac{\tau_U}{2}, 1 - m - \ell_U - \frac{\tau_U}{2} \end{matrix}; 1 \right]. \quad (8.16)$$

Here we remind that a_{Δ_U, ℓ_U} denotes the squared OPE coefficient of the exchanged operator.

As an example, at $n = 0$ only the $m = 0$ term contributes in (8.14), and one finds

$$\gamma_0 = -2a_{\Delta_U, \ell_U} \frac{\Gamma(\Delta_{\mathcal{O}})^2}{\Gamma(\Delta_{\mathcal{O}} - \frac{\tau_U}{2})^2} \frac{\Gamma(2\ell_U + \tau_U)}{\Gamma(\ell_U + \frac{\tau_U}{2})^2}. \quad (8.17)$$

Let us now specialize this formula to the case of ϕ^4 interaction in the bulk. As discussed earlier, in order to obtain the effect of tree-level data at all loops, we can consider an effective exchange with scaling dimension and OPE coefficient given in (8.3). Inserting this into (8.14) and expanding in $1/N$, one finds at order N^{-4} :

$$\gamma_{n,\ell}^{(2)} = \frac{\zeta_n}{J^{2\Delta_{\mathcal{O}}}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+2}}\right), \quad (8.18)$$

where the coefficient ζ_n can be find in closed form

$$\zeta_n = 2(1 - 2\Delta_{\mathcal{O}}) [2\Delta_{\mathcal{O}}^2 + (4n - 5)\Delta_{\mathcal{O}} + 2n^2 - 6n + 3] \frac{\Gamma(2\Delta_{\mathcal{O}} + n - 3)}{\Gamma(n+1)}. \quad (8.19)$$

At order N^{-4} , the result is completely fixed by this effective tree-level exchange. At order N^{-6} , the tree-level data fix only the coefficient of $\log J$ in the anomalous dimension. Therefore we find

$$\gamma_{n,\ell}^{(3)} \Big|_{\log J} = -\frac{\zeta_n}{J^{2\Delta_{\mathcal{O}}}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+2}}\right). \quad (8.20)$$

In the following section, these large-spin data will be inserted into spin sums.

8.2 Higher-trace operators

At higher orders in the $1/N$ expansion multi-trace operators appear in the $\mathcal{O} \times \mathcal{O}$ OPE. Thus, beyond the double-trace sector, one must also keep track of the possible exchange of higher-trace operators.

A simplification can be obtained by imposing a $\mathbb{Z}_2$ symmetry under $\mathcal{O} \rightarrow -\mathcal{O}$. This forbids operators built from an odd number of $\mathcal{O}$'s in the $\mathcal{O} \times \mathcal{O}$ OPE. This is the situation relevant for a bulk theory with a ϕ^4 interaction, and is the setup used in the previous chapter. Even in this restricted setup, higher-trace operators with an even number of single trace constituents appear at subleading orders in the large- N expansion. However, these operators do not affect the coefficient of the leading $\log J$ in the large-spin expansion of the double-trace OPE data studied in the previous chapter.

In this section, we are interested in the OPE data of the higher-trace operators themselves. We first analyze the appearance of quadruple-trace operators in the ϕ^4 theory. We then discuss the ϕ^3 theory, where the $\mathbb{Z}_2$ symmetry is absent and k -trace operators appear at order N^{-2k} for integer $k \geq 3$.

8.2.1 ϕ^4 theory

In the $\mathbb{Z}_2$ -symmetric case, the OPE takes the schematic form

$$\mathcal{O} \times \mathcal{O} = \mathbf{1} + [\mathcal{O}\mathcal{O}]_{n,\ell} + \frac{1}{N^4}[\mathcal{O}^4]_{q,\ell,I} + \cdots. \quad (8.21)$$

Here $[\mathcal{O}^4]_{q,\ell,I}$ denotes a quadruple-trace primary built from four copies of $\mathcal{O}$. The label q denotes its twist level, and I denotes a possible degeneracy label. Unlike in the double-trace sector, the labels q and ℓ need not specify a unique quadruple-trace primary. At fixed total twist and spin there can be several independent quadruple-trace primaries, corresponding to different ways of distributing derivatives among the four single-trace constituents.

The coupling of quadruple-trace operators to the external pair is suppressed by N^{-4} , provided the quadruple-trace operators are orthogonal to lower-trace families. Therefore their squared OPE coefficients first contribute to the connected four-point function at order N^{-8} .

We expand the scaling dimension of quadruple-trace operators in large N as

$$\Delta_{[\mathcal{O}^4]_{q,\ell,I}} = 4\Delta_{\mathcal{O}} + 2q + \ell + \frac{\delta_{q,\ell,I}^{(1)}}{N^2} + \cdots, \quad (8.22)$$

and the corresponding squared OPE coefficients as

$$a_{[\mathcal{O}^4]_{q,\ell,I}} = \frac{b_{q,\ell,I}^{(0)}}{N^8} + \frac{b_{q,\ell,I}^{(1)}}{N^{10}} + \cdots. \quad (8.23)$$

After summing over the degeneracy label, we define

$$b_{q,\ell}^{(0)} \equiv \sum_I b_{q,\ell,I}^{(0)}, \quad (8.24)$$

$$b_{q,\ell}^{(0)} \delta_{q,\ell}^{(1)} \equiv \sum_I b_{q,\ell,I}^{(0)} \delta_{q,\ell,I}^{(1)}. \quad (8.25)$$

So the contribution of quadruple-trace operators to the correlator at order N^{-8} takes the form

$$\mathcal{G}^{(4)}(u, v) \supset \sum_{q,\ell} b_{q,\ell}^{(0)} u^{2\Delta_{\mathcal{O}}+q} \tilde{g}_{q,\ell}^{(4)}(u, v), \quad (8.26)$$

where

$$g_{4\Delta_{\mathcal{O}}+2q+\ell,\ell}(u, v) = u^{2\Delta_{\mathcal{O}}+q} \tilde{g}_{q,\ell}^{(4)}(u, v). \quad (8.27)$$

We now consider the contribution to the same order coming from the double-trace sector. Among the various terms, there is in particular

$$\mathcal{G}^{(4)}(u, v) \supset \sum_{n,\ell} \frac{1}{8} a_{n,\ell}^{(0)} u^{\Delta_{\mathcal{O}}+n} \left(\gamma_{n,\ell}^{(2)} \right)^2 \log^2 u \tilde{g}_{n,\ell}(u, v). \quad (8.28)$$

Using the large-spin result in (8.18), we can write

$$\left(\gamma_{n,\ell}^{(2)} \right)^2 = \sum_{\rho \geq 0} \frac{c_{n,\rho}}{J^{4\Delta_{\mathcal{O}}+2\rho}}. \quad (8.29)$$

Inserting this into (8.28) and using the spin sums discussed in chapter 5, this produces, in the small- u and small- v limit,

$$\mathcal{G}^{(4)}(u, v) \sim u^{\Delta_{\mathcal{O}}+n} v^{\Delta_{\mathcal{O}}+\rho} \log^2 u. \quad (8.30)$$

Crossing symmetry then maps this behavior to

$$\mathcal{G}^{(4)}(u, v) \sim u^{2\Delta_{\mathcal{O}}+\rho} v^n \log^2 v. \quad (8.31)$$

The power of u in (8.31) cannot come from the direct-channel double-trace sector. Instead, it matches the power expected from a quadruple-trace block (8.26). Thus we see that crossing symmetry forces the exchange of quadruple-trace operators in the $\mathcal{O} \times \mathcal{O}$ OPE. This will become even more clear when we discuss the diagrammatic representation in the next chapter.

OPE coefficient

We now want to find the leading OPE coefficient $b_{q,\ell}^{(0)}$ for $q = 0$. Our strategy is to apply the Lorentzian inversion formula to $\mathcal{G}^{(4)}(u, v)$ in the crossed channel. We then identify the poles of $c_{\Delta,\ell}$ at the positions of the quadruple-trace operators and read off their residues. For this purpose, the only relevant contribution is the term displayed in (8.28). Using the functions $H_n^{(m)}(z, \bar{z})$ defined in (5.2) the contribution in (8.28) can be written as

$$\frac{1}{8} \log^2(z\bar{z}) \sum_{n,\rho} c_{n,\rho} H_n^{(2\Delta_{\mathcal{O}}+\rho)}(z, \bar{z}). \quad (8.32)$$

We are interested in the behavior of this expression in the $\bar{z} \rightarrow 1$ limit. In the decomposition of $H_n^{(m)}$ in (5.4), only the second term is relevant for extracting the crossed-channel singularity that gives rise to the leading quadruple-trace pole. The first term has the behavior of an individual conformal block near $\bar{z} = 1$, and therefore does not generate the behavior of interest. Thus we focus on

$$-\frac{1}{8} \frac{\log^2(z\bar{z})}{z - \bar{z}} \sum_{n,\rho} c_{n,\rho} z^{\tau_n/2} F_{\frac{\tau_n-2}{2}}(z) \tilde{H}_n^{(2\Delta_{\mathcal{O}}+\rho)}(\bar{z}). \quad (8.33)$$

Near $\bar{z} = 1$, we may replace $\tilde{H}_n$ by P_n , which is defined in (5.18). Since the other terms again have the behavior of individual conformal blocks in this limit. Doing so we find

$$\mathcal{G}^{(4)}(z, \bar{z}) \supset -\frac{\log^2 z (1 - \bar{z})^{\Delta_{\mathcal{O}}}}{8(z - 1) [(1 - \Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \sum_n c_{n,0} Y_n z^{\tau_n/2} F_{\frac{\tau_n-2}{2}}(z) + \mathcal{O}((1 - \bar{z})^{\Delta_{\mathcal{O}}+1}). \quad (8.34)$$

Here the coefficients Y_n are defined in (5.17). Note that at leading order in $(1 - \bar{z})$ expansion, only the $\rho = 0$ term contributes. In other words, only the leading term in the large-spin expansion of $\gamma_{n,\ell}^{(2)}$ is needed to extract $b_{0,\ell}^{(0)}$.

Before proceeding, let us comment on the range of validity of this expression. As explained in the chapter 5, the relation used above holds provided

$$\Delta_{\mathcal{O}} - m \notin \mathbb{Z}_{\leq 0}. \quad (8.35)$$

In the present case $m = 2\Delta_{\mathcal{O}} + \rho$, and therefore this condition excludes integer values of $\Delta_{\mathcal{O}}$. This is precisely the case in which the double-trace and quadruple-trace towers can mix with each other. In what follows then we assume that $\Delta_{\mathcal{O}}$ is non-integer.

We now use crossing symmetry to find the crossed image of (8.34). It is convenient to define the crossing operator S by

$$S[f(z, \bar{z})] = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_{\mathcal{O}}} f(1 - \bar{z}, 1 - z). \quad (8.36)$$

Under this transformation, the limit $\bar{z} \rightarrow 1$ is mapped to the limit $z \rightarrow 0$. Acting with $\mathbf{S}$ on (8.34), and keeping only the leading term in z we obtain

$$\mathcal{G}^{(4)}(z, \bar{z}) \supset \frac{1}{8} \frac{\log^2(1 - \bar{z})}{\bar{z}^{1-\Delta_{\mathcal{O}}}} \frac{z^{2\Delta_{\mathcal{O}}}}{[(1 - \Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \sum_n c_{n,0} (1 - \bar{z})^n F_{\frac{n-2}{2}}(1 - \bar{z}) Y_n. \quad (8.37)$$

We may now insert (8.37) into the inversion integral (4.22). In the limit that we are taking, the z and $\bar{z}$ integrals decouple. The z -integral takes the form

$$\int_0^1 dz z^{\frac{\ell-\Delta}{2}-1+2\Delta_{\mathcal{O}}} = -\frac{2}{\Delta - (4\Delta_{\mathcal{O}} + \ell)}. \quad (8.38)$$

Therefore we see that the pole is located at the position of the leading-twist quadruple-trace operator. The residue of this pole is determined by the remaining $\bar{z}$ -integral.

We evaluate the $\bar{z}$ -integral by expanding the summand in (8.37) around $\bar{z} = 1$ and integrating term by term, following the peeling method introduced in section 4.3.1. This expansion gives the large-spin expansion of the final result in powers of $1/\ell$. Therefore, to obtain the first few terms in the large-spin expansion of $b_{0,\ell}^{(0)}$, it is sufficient to know the coefficients $c_{n,0}$ for the corresponding finite range of n .

The coefficients $c_{n,0}$ can be extracted from (8.18). Namely we have

$$c_{n,0} = \zeta_n^2 = 4(1 - 2\Delta_{\mathcal{O}})^2 [2\Delta_{\mathcal{O}}^2 + (4n - 5)\Delta_{\mathcal{O}} + 2n^2 - 6n + 3]^2 \frac{\Gamma(2\Delta_{\mathcal{O}} + n - 3)^2}{\Gamma(n + 1)^2}. \quad (8.39)$$

For half-integer values of $\Delta_{\mathcal{O}}$, the sum over n in (8.37) can be performed exactly. For example, for $\Delta_{\mathcal{O}} = \frac{5}{2}$, we find

$$\sum_n c_{n,0} (1 - \bar{z})^n F_{n+\frac{3}{2}}(1 - \bar{z}) Y_n = -\frac{36 [3\bar{z}(\bar{z}(43\bar{z} - 365) + 725) - 1225]}{\bar{z}^{9/2}}. \quad (8.40)$$

In this case the remaining $\bar{z}$ -integral in the inversion formula can be performed exactly, without expanding around $\bar{z} = 1$. So the resulting leading quadruple-trace OPE coefficient for $\Delta_{\mathcal{O}} = \frac{5}{2}$ is

$$b_{0,\ell}^{(0)} = \frac{\sqrt{\pi} 2^{5-2\ell} (2\ell + 9)^2 (\ell(\ell + 9) + 23)^2 \Gamma(\ell + 1)}{225(\ell + 5)(\ell + 6)(\ell + 7)(\ell + 8) \Gamma(\ell + \frac{9}{2})}. \quad (8.41)$$

For generic non-integer $\Delta_{\mathcal{O}}$, the leading large-spin behavior is

$$b_{0,\ell}^{(0)} = \frac{\sqrt{\pi} 2^{-4\Delta_{\mathcal{O}}-2\ell+1} \Gamma(2\Delta_{\mathcal{O}})^2}{\ell^{3/2} [(1 - \Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \left[1 + \mathcal{O}\left(\frac{1}{\ell}\right) \right]. \quad (8.42)$$

Higher-order terms in the $1/\ell$ expansion can be obtained systematically by keeping more terms in the expansion of the summand in (8.37) around $\bar{z} = 1$. Thus the leading large-spin double-trace data are sufficient to fix the leading-twist quadruple-trace OPE coefficient to all orders in the large-spin expansion. For special half-integer values of $\Delta_{\mathcal{O}}$, the sum over n can even be performed in closed form, giving an exact expression for $b_{0,\ell}^{(0)}$.

Anomalous dimension

For quadruple-trace operators, the function $c_{\Delta,\ell}$ in the inversion formula (4.22) has the following expansion

$$c_{\Delta,\ell} = -\frac{1}{N^8} \frac{b_{q,\ell}^{(0)}}{\Delta - (4\Delta_{\mathcal{O}} + 2q + \ell)} - \frac{1}{N^{10}} \left[\frac{b_{q,\ell}^{(1)}}{\Delta - (4\Delta_{\mathcal{O}} + 2q + \ell)} + \frac{b_{q,\ell}^{(0)} \delta_{q,\ell}^{(1)}}{(\Delta - (4\Delta_{\mathcal{O}} + 2q + \ell))^2} \right] + \dots \quad (8.43)$$

So the anomalous dimension $\delta_{q,\ell}^{(1)}$ should be extracted from the double pole at the quadruple-trace position.

The relevant source of this double pole is the following term in the double-trace sector at order N^{-10} :

$$\mathcal{G}^{(5)}(u, v) \supset \sum_{n,\ell} \frac{1}{4} a_{n,\ell}^{(0)} u^{\Delta_{\mathcal{O}}+n} \gamma_{n,\ell}^{(2)} \gamma_{n,\ell}^{(3)} \Big|_{\log J} \left(\log u + \frac{\partial}{\partial n} \right)^2 \tilde{g}_{n,\ell}(u, v). \quad (8.44)$$

Note that the double pole is controlled only by the part of $\gamma_{n,\ell}^{(3)}$ which is proportional to $\log J$ (See (5.25)). Therefore, the knowledge of this piece is sufficient to determine the quadruple-trace anomalous dimension $\delta_{q,\ell}^{(1)}$.

As in the analysis of the OPE coefficient, we restrict our attention to the lowest-twist quadruple-trace operators. Therefore, in the large-spin expansion

$$\gamma_{n,\ell}^{(2)} \gamma_{n,\ell}^{(3)} \Big|_{\log J} = \sum_{\rho \geq 0} \frac{w_{n,\rho}}{J^{4\Delta_{\mathcal{O}}+2\rho}}, \quad (8.45)$$

only the term with $\rho = 0$ is needed. From (8.18) and (8.20), we then have

$$w_{n,0} = -\zeta_n^2 = -c_{n,0}, \quad (8.46)$$

where $c_{n,0}$ is given in (8.39).

Using the discussion in Chapter 5, in particular (5.25), the relevant part of (8.44) in the limit $\bar{z} \rightarrow 1$ becomes

$$\mathcal{G}^{(5)}(u, v) \supset \frac{1}{8[(1 - \Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \frac{\log^2(z\bar{z})}{z - 1} \sum_n w_{n,0} Y_n z^{\tau_n/2} F_{\frac{\tau_n-2}{2}}(z) (1-\bar{z})^{\Delta_{\mathcal{O}}} \log(1-\bar{z}). \quad (8.47)$$

We now use crossing symmetry in the form prescribed in (8.36). In the crossed channel, and in the limit $z \rightarrow 0$, this gives

$$\mathcal{G}^{(5)}(u, v) \supset \frac{\log^2(1 - \bar{z})}{8[(1 - \Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \frac{z^{2\Delta_{\mathcal{O}}} \log z}{\bar{z}^{1-\Delta_{\mathcal{O}}}} \sum_n c_{n,0} Y_n(1 - \bar{z})^n F_{\frac{\tau_n-2}{2}}(1 - \bar{z}). \quad (8.48)$$

In this limit, the z and $\bar{z}$ integrals in the inversion formula decouple. The z -integral gives

$$\int_0^1 dz z^{\frac{\ell-\Delta}{2}-1+2\Delta_{\mathcal{O}}} \log z = -\frac{4}{(\Delta - (4\Delta_{\mathcal{O}} + \ell))^2}. \quad (8.49)$$

This produces the expected double pole at the position of the lowest-twist quadruple-trace operator. Now it remains to evaluate the $\bar{z}$ -integral, but comparing (8.48) with (8.37) we see that the $\bar{z}$ integral is identical to the one for the OPE coefficient. Therefore we find

$$\delta_{0,\ell}^{(1)} = 2. \quad (8.50)$$

8.2.2 ϕ^3 theory

We now turn to a bulk theory with a cubic interaction. In the CFT this means that $\mathcal{O}$ itself appears in the $\mathcal{O} \times \mathcal{O}$ OPE, so the $\mathbb{Z}_2$ symmetry of the previous subsection is absent. Schematically,

$$\mathcal{O} \times \mathcal{O} = \mathbf{1} + [\mathcal{O}\mathcal{O}]_{n,\ell} + \frac{1}{N} \mathcal{O} + \sum_{k=3}^{\sigma} \frac{1}{N^k} [\mathcal{O}^k]_{r,\ell,I} + \dots. \quad (8.51)$$

Here $[\mathcal{O}^k]_{r,\ell,I}$ denotes a k -trace family built out of $\mathcal{O}$, r labels the twist level, and I is a possible degeneracy label within that family. Moreover, we assume that, up to order $1/N^\sigma$ in the OPE, different trace families do not mix with each other.

The scaling dimension and squared OPE coefficient of a k -trace operator can be expanded in large N as

$$\Delta_{[\mathcal{O}^k]_{r,\ell,I}} = k\Delta_{\mathcal{O}} + 2r + \ell + \frac{\eta_{k,r,\ell,I}^{(1)}}{N^2} + \dots, \quad (8.52)$$

$$a_{[\mathcal{O}^k]_{r,\ell,I}} = \frac{\mathfrak{a}_{k,r,\ell,I}^{(0)}}{N^{2k}} + \dots. \quad (8.53)$$

As before, we can sum over the degeneracy label and define

$$\mathfrak{a}_{k,r,\ell}^{(0)} \equiv \sum_I \mathfrak{a}_{k,r,\ell,I}^{(0)}. \quad (8.54)$$

Our goal is to find this coefficient for the leading twist $r = 0$. The strategy is the same as in the ϕ^4 analysis. We start from the double-trace part of the conformal block expansion and isolate the terms which, after the infinite spin sum and crossing, have the correct small- z behavior to be interpreted as the exchange of k -trace operators.

At order N^{-2k} in the correlator, several double-trace terms have this property. Among them, the term with the highest power of $\log u$ is

$$\mathcal{G}^{(k)}(u, v) \supset \frac{1}{2^k k!} \sum_{n, \ell} a_{n, \ell}^{(0)} u^{\Delta_{\mathcal{O}} + n} \left(\gamma_{n, \ell}^{(1), \text{as}} \right)^k \log^k u \tilde{g}_{n, \ell}(u, v). \quad (8.55)$$

Here $\gamma_{n, \ell}^{(1), \text{as}}$ denotes the asymptotic part of the tree-level anomalous dimension, given in (4.65) for $n = 0$.

We now take the crossed channel of (8.55), perform the infinite spin sum using the method of chapter 5, and insert the result into the Lorentzian inversion formula to read off the OPE coefficients. We will not repeat the steps of the calculation, since they are very similar to the ϕ^4 case. Let us only note that, although (8.55) is not the only term that contributes to the k -trace OPE coefficient, it is the term with the highest power of $\log u$. It therefore uniquely fixes the coefficient of the leading $\log^{k-2} \ell$ behavior of $\mathfrak{a}_{k, 0, \ell}^{(0)}$ at large spin. Thus we find

$$\mathfrak{a}_{k, 0, \ell \gg 1}^{(0)} = \frac{(-1)^k 2^{1-k\Delta_{\mathcal{O}}-2\ell} \sqrt{\pi} (\tilde{\gamma}_0)^k \log^{k-2} \ell}{(k-2)! \left[(1 - \Delta_{\mathcal{O}}) \frac{k\Delta_{\mathcal{O}}}{2} \right]^2} \frac{1}{\ell^{3/2}} + \dots \quad (8.56)$$

where $\tilde{\gamma}_0$ can be read off from (4.65) and is given by

$$\tilde{\gamma}_0 = -\frac{2a_{\Delta_{\mathcal{O}}, 0} \Gamma(\Delta_{\mathcal{O}})^3}{\Gamma\left(\frac{\Delta_{\mathcal{O}}}{2}\right)^4}. \quad (8.57)$$

We remind that this result is written under the no-mixing assumption stated earlier. Namely for $k \leq \sigma$.

Chapter 9

Cut Diagrammatics

In this chapter we introduce a diagrammatic notation for organizing the terms that appear in the conformal block expansion. These diagrams keep track of the combinations of OPE data multiplying a given conformal block.

The motivation for this construction comes from the analysis of bubble diagrams carried out in Chapter 7, originally presented in [1]. In that case, it was observed that a part of the conformal block expansion that is completely fixed by tree-level data is related to the AdS Bubble diagrams with maximal unitarity cuts. By “maximal” we mean that the bubble diagram is cut iteratively through all its internal propagators. Here we extend this observation into a more systematic dictionary. Namely, terms in the conformal block expansion are represented by cut diagrams. The cuts need not be maximal. We use vertical cuts to denote direct-channel expansion and horizontal cuts to denote the crossed-channel expansion.

This construction is closely related to existing holographic unitarity methods [77]. In these methods, loop-level CFT data are reconstructed from products of lower-order OPE data. The bulk formulation of [77] gives this factorization a direct diagrammatic meaning. Namely, a cut puts internal bulk propagators on shell and factorizes a Witten diagram into lower-loop subdiagrams.

Our construction starts instead directly from the conformal block expansion. When an internal cut separates a diagram into subdiagrams, the associated OPE data factorize into lower-loop data according to a product rule that we will introduce. In this sense, our internal cuts, whether drawn vertically or horizontally, have the same algebraic behavior as unitarity cuts. Since the two orientations correspond to different OPE channels, this suggests that internal vertical and horizontal cuts may be related to bulk unitarity cuts in the corresponding channels. However, in this work, we do not define the cuts by an on-shell operation in the bulk, nor do we establish that the internal cuts introduced here are AdS unitarity cuts.

Regardless of this possible bulk interpretation, this diagrammatic notation

has two useful purposes on the CFT side. First, it provides a more detailed bookkeeping of the CFT data contributing at a given order in $1/N$. Second, it makes it possible to identify crossing-closed classes of diagrams, namely subsets of terms that are mapped into themselves under crossing.

Let us recall the conformal-block expansion of the four-point function,

$$\mathcal{G}(u, v) = \sum_{\Delta, \ell} a_{\Delta, \ell} u^{\frac{\Delta-\ell}{2}} \tilde{g}_{\Delta, \ell}(u, v). \quad (9.1)$$

In the large- N expansion we write

$$\mathcal{G}(u, v) = \mathcal{G}^{(0)}(u, v) + \frac{1}{N^2} \mathcal{G}^{(1)}(u, v) + \frac{1}{N^4} \mathcal{G}^{(2)}(u, v) + \dots \quad (9.2)$$

This is the CFT counterpart of the bulk loop expansion. At each order in $1/N$, our goal is to associate the different terms in the conformal-block expansion with diagrams carrying specified cuts.

9.1 Disconnected contribution

At leading order the correlator is the generalized-free-field correlator,

$$\mathcal{G}^{(0)}(u, v) = 1 + u^{\Delta_{\mathcal{O}}} + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} = 1 + \sum_{n, \ell} a_{n, \ell}^{(0)} u^{\Delta_{\mathcal{O}}+n} \tilde{g}_{n, \ell}(u, v), \quad (9.3)$$

where the sum runs over the double-trace operators $[\mathcal{OO}]_{n, \ell}$. In this decomposition, the first term is the identity contribution, and the remaining disconnected pieces are reproduced by the exchange of the double-trace tower in the direct channel.

In terms of Witten diagrams, we can represent the three disconnected contributions by

$$\mathcal{G}^{(0)}(u, v) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}. \quad (9.4)$$

Note that the first diagram admits a vertical cut which does not intersect any propagator. We interpret this cut as the exchange of the identity operator in the direct-channel OPE, and we make the following diagrammatic identification

$$\left(\begin{array}{c} | \\ \vdots \\ | \end{array} \right) = 1. \quad (9.5)$$

The other two diagrams admit a two-particle cut. Such a cut represents the exchange of two-particle states, dual to the double-trace operators $[\mathcal{OO}]_{n, \ell}$. Thus,

for a fixed exchanged double-trace operator, we make the identification

$$\left[\left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) + \left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) \right]_{n,\ell} = a_{n,\ell}^{(0)}. \quad (9.6)$$

This motivates the following shorthand. A dot-cut keeps track of a fixed pairing of the two particles across the cut. For identical external operators, we want the cut to include the two possible pairings of the external legs on one side of the cut. So we define the dash-cut in the following way,

$$\left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) = \frac{1}{\mathfrak{s}} \left[\left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) + \left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) \right]. \quad (9.7)$$

where

$$\mathfrak{s} = \begin{cases} 2, & \text{if the two dot-cut diagrams on the right-hand side coincide,} \\ 1, & \text{otherwise.} \end{cases} \quad (9.8)$$

So the factor $\mathfrak{s}$ removes the double counting when the two outgoing pairings give the same diagram.

Using the dash-cut definition (9.7), the relation (9.6) can be written more simply as

$$\left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right)_{n,\ell} = a_{n,\ell}^{(0)}. \quad (9.9)$$

From now on, unless stated otherwise, by a cut we mean a dash-cut. When we need to keep track of a fixed pairing across the cut, we will explicitly refer to it as a dot-cut.

Two-fold notation. We use two related diagrammatic notations. A diagram inside parentheses denotes the OPE data factor multiplying a conformal block. The same diagram enclosed by a circle denotes the corresponding contribution to the correlator, after summing over n and ℓ . Thus we write

$$\left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right) = \sum_{n,\ell} u^{\Delta_{\mathcal{O}+n}} \left(\begin{array}{c} \vdots \\ \text{---} \\ \vdots \\ \text{---} \\ \vdots \end{array} \right)_{n,\ell} \tilde{g}_{n,\ell}(u, v). \quad (9.10)$$

For the identity contribution, the corresponding OPE coefficient is normalized to one, as in (9.5). Therefore the disconnected correlator can be written diagram-

matically as

$$\mathcal{G}^{(0)}(u, v) = \text{circle with two vertical dashed lines} + \text{circle with two horizontal dashed lines}. \quad (9.11)$$

The first term represents the identity exchange, and the second term represents the contribution of the double-trace tower in the direct channel.

Crossing. The crossing operator $\mathbf{S}$ defined in (8.36) acts on the circled diagrams, as these diagrams denote contributions to the correlator. Diagrammatically, we represent this action by a rotation by $\pi/2$, which also exchanges vertical and horizontal cuts.

At the level of OPE data, such rotation does not change the identification. For instance,

$$\left(\text{circle with two horizontal dashed lines} \right)_{n,\ell} = \left(\text{circle with two vertical dashed lines} \right)_{n,\ell} = a_{n,\ell}^{(0)}. \quad (9.12)$$

But for circled diagrams, we have for instance,

$$\left(\text{circle with two vertical dashed lines} \right) = \mathbf{S} \left[\text{circle with two horizontal dashed lines} \right]. \quad (9.13)$$

With this notation, crossing symmetry equation for the disconnected correlator can be written diagrammatically as

$$\left(\text{circle with two vertical dashed lines} \right) + \left(\text{circle with two horizontal dashed lines} \right) = \left(\text{circle with two vertical dashed lines} \right) + \left(\text{circle with two horizontal dashed lines} \right). \quad (9.14)$$

9.2 Higher orders

Let us now move beyond the disconnected contribution. We want to formulate more general rules for cutting and merging two-particle states, corresponding to double-trace operators in the CFT.

Before doing so, we recall that multi-particle states in AdS are dual to multi-trace operators in the boundary CFT [58, 78]. A diagram may admit cuts through one, two, three, four, or more internal propagators. These cuts therefore signal the exchange of single-trace, double-trace, triple-trace, quadruple-trace, or higher-trace operators in the OPE. We call a cut which intersects p -particle propagators a p -particle cut.

In this work we focus mainly on general rules for decomposing two-particle cuts. Other types of cuts may also appear, and in those cases we treat the corresponding cut diagrams as building blocks, together with their associated OPE data.

For example, a tree-level single-trace exchange in ϕ^3 theory contains a one-particle cut which can be identified as

$$\left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right) = a_{\Delta_{\mathcal{O},0}}. \quad (9.15)$$

Now let us return to the two-particle cuts. We first consider the case where a two-particle cut goes through the external legs. In that case we make the following identifications:

$$\left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(0)} \equiv a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(k)(i)}, \quad \left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(1)} \equiv a_{n,\ell}^{(k)(i)}. \quad (9.16)$$

The superscripts (0) and (1) in the diagrams distinguish two different identifications. The label (i) distinguishes inequivalent diagrams contributing to the same OPE data, so we have

$$a_{n,\ell}^{(k)} = \sum_i a_{n,\ell}^{(k)(i)}, \quad \gamma_{n,\ell}^{(k)} = \sum_i \gamma_{n,\ell}^{(k)(i)}. \quad (9.17)$$

At tree level, this notation gives the following external two-particle cut identifications:

$$\left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(0)} \equiv a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)}, \quad \left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(1)} \equiv a_{n,\ell}^{(1)(1)}, \quad (9.18)$$

$$\left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(0)} \equiv a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(2)}, \quad \left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(1)} \equiv a_{n,\ell}^{(1)(2)}, \quad (9.19)$$

$$\left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(0)} \equiv a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(3)}, \quad \left(\begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \\ \diagdown \quad \text{---} \quad \diagup \end{array} \right)_{n,\ell}^{(1)} \equiv a_{n,\ell}^{(1)(3)}. \quad (9.20)$$

The index $i = 1, 2, 3$ labels the three inequivalent tree-level diagrams shown above.¹

¹The separation of the tree-level anomalous dimension into different bulk channels was also considered in [77].

At one loop, the same rule gives contributions proportional to $a_{n,\ell}^{(0)}\gamma_{n,\ell}^{(2)}$ and $a_{n,\ell}^{(2)}$. For example,

$$\left(\text{diagram} \right)_{n,\ell}^{(0)} \equiv a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(2)(1)}, \quad \left(\text{diagram} \right)_{n,\ell}^{(1)} \equiv a_{n,\ell}^{(2)(1)}. \quad (9.21)$$

Product rule. Now we want to consider two-particle cuts which go through internal propagators. Such a cut splits the diagram into two subdiagrams, which are glued along the same two-particle state. We denote this gluing operation by $\otimes_2$.

We define the product rule as:

$$X \otimes_2 Y \equiv \frac{XY}{a_{n,\ell}^{(0)}}. \quad (9.22)$$

The division by $a_{n,\ell}^{(0)}$ removes the extra free OPE coefficient which is counted twice when the two cut subdiagrams are multiplied.

As a simple check, applying this rule to the disconnected two-particle cut gives $a_{n,\ell}^{(0)} \otimes_2 a_{n,\ell}^{(0)} = a_{n,\ell}^{(0)}$, as expected.

A diagram split into f subdiagrams carries a label $\vec{\nu} = (\nu_1, \dots, \nu_f)$, where each entry specifies which OPE data identification is used for the corresponding subdiagram. Here $\nu_i \in \{0, 1\}$. We keep only the choices satisfying $\sum_i \nu_i \leq 1$. Contributions with $\sum_i \nu_i > 1$ can be absorbed into higher-order OPE data associated with diagrams with fewer cuts.

As a simple example, consider the case of one-loop bubble diagram, in which its internal two-particle cut factorizes into two tree-level cut diagrams:

$$\left(\text{diagram} \right)_{n,\ell}^{(\nu_1, \nu_2)} = \left(\text{diagram} \right)_{n,\ell}^{(\nu_1)} \otimes_2 \left(\text{diagram} \right)_{n,\ell}^{(\nu_2)}. \quad (9.23)$$

For $\vec{\nu} = (0, 0)$, this gives

$$\left(\text{diagram} \right)_{n,\ell}^{(0,0)} = \left(a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \right) \otimes_2 \left(a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \right) = a_{n,\ell}^{(0)} \left(\gamma_{n,\ell}^{(1)(1)} \right)^2. \quad (9.24)$$

Similarly, for a $(r-1)$ -loop bubble diagram with maximal internal cuts we have

$$\begin{aligned} \left(\text{diagram} \right)_{n,\ell}^{(0, \dots, 0)} &= \underbrace{\left(a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \right) \otimes_2 \cdots \otimes_2 \left(a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \right)}_{r \text{ factors}} \\ &= a_{n,\ell}^{(0)} \left(\gamma_{n,\ell}^{(1)(1)} \right)^r. \end{aligned} \quad (9.25)$$

This is exactly the maximal-cut contribution studied in [1]. This term, through crossing and inversion formula, is related to an external two-particle cut in the crossed channel. Namely,

$$\left(\text{Diagram with two circles and a dashed line} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(r)(1)}. \quad (9.26)$$

Circled diagrams and crossing. As we mentioned earlier, in our construction, a diagram inside parentheses denotes the OPE data factor multiplying a conformal block, while the corresponding circled diagram denotes its contribution to the correlator after summing over the exchanged double-trace operators.

Let Γ_c be a diagram with a given two-particle cut denoted by c . We denote the corresponding OPE data factor by

$$(\Gamma_c)_{n,\ell}^{\vec{\nu}}. \quad (9.27)$$

Here $\vec{\nu}$ specifies the OPE data assignment associated with the subdiagrams produced by the cuts. Schematically, this factor is a product of OPE coefficients and anomalous dimensions of the form

$$(\Gamma_c)_{n,\ell}^{\vec{\nu}} \sim a_{n,\ell}^{(s)} \prod_{m \geq 1} \left(\gamma_{n,\ell}^{(m)} \right)^{\omega_m}. \quad (9.28)$$

The integer ω_m counts how many factors of the order $1/N^{2m}$ anomalous dimension appear in this term, and s denotes the order of the OPE coefficient. Both s and the ω_m are fixed by the diagram and by the choice of $\vec{\nu}$.

The corresponding contribution to the correlator is obtained by inserting this OPE data into the conformal-block expansion. Therefore, the associated circled diagram is defined by

$$\left(\Gamma_c \right)^{\vec{\nu}} = \sum_{n,\ell} u^{\Delta_{\mathcal{O}}+n} (\Gamma_c)_{n,\ell}^{\vec{\nu}} \frac{(\log u + \partial_n)^\Omega}{\prod_{m \geq 1} 2^{\omega_m} \omega_m!} \tilde{g}_{n,\ell}(u, v), \quad (9.29)$$

where $\Omega = \sum_{m \geq 1} \omega_m$.

We can also sum over different $\vec{\nu}$ assignments and define

$$\left(\Gamma_c \right) \equiv \sum_{\vec{\nu} \in \mathcal{N}(\Gamma_c)} \left(\Gamma_c \right)^{\vec{\nu}}. \quad (9.30)$$

Here $\mathcal{N}(\Gamma_c)$ denotes the set of inequivalent allowed assignments. assignments which give exactly the same OPE data factor, are identified and should not counted separately.

The full correlator at order $1/N^{2k}$ can then be organized as a sum over cut diagrams,

$$\mathcal{G}^{(k)}(u, v) = \sum_{\Gamma \in \mathcal{D}^{(k)}} \sum_{c \in \mathcal{C}_v(\Gamma)} \frac{\mathfrak{s}_v(\Gamma)}{2} \left(\Gamma_c \right). \quad (9.31)$$

Here $\mathcal{D}^{(k)}$ denotes the set of connected diagrams contributing at order $1/N^{2k}$, and $\mathcal{C}_v(\Gamma)$ denotes the set of inequivalent allowed vertical cuttings of a fixed diagram Γ . If two cuts give the same contribution to the conformal-block expansion, we keep only one representative. The factor $\mathfrak{s}_v(\Gamma)$ is the dash-cut symmetry factor defined in (9.8), evaluated for the vertical pairing. Therefore $\mathfrak{s}_v(\Gamma)/2$ removes the overcounting from diagrams related only by this pairing exchange.

Crossing maps vertical cuts to horizontal cuts. Therefore the usual crossing equation for the full correlator can be written diagrammatically as

$$\sum_{\Gamma \in \mathcal{D}^{(k)}} \sum_{c \in \mathcal{C}_v(\Gamma)} \frac{\mathfrak{s}_v(\Gamma)}{2} \left(\Gamma_c \right) = \sum_{\Gamma \in \mathcal{D}^{(k)}} \sum_{c \in \mathcal{C}_h(\Gamma)} \frac{\mathfrak{s}_h(\Gamma)}{2} \left(\Gamma_c \right). \quad (9.32)$$

Here $\mathcal{C}_h(\Gamma)$ is defined analogously as the set of inequivalent allowed horizontal cuttings, and $\mathfrak{s}_h(\Gamma)$ is the same dash-cut symmetry factor evaluated for the horizontal pairing.

(9.32) is simply the standard crossing relation rewritten in terms of cut diagrams. The advantage of the notation is that it allows us to ask a sharper question, whether crossing can be imposed on smaller classes of diagrams, rather than only on the full correlator.

With the diagrammatic representation, we can write a subset of correlator corresponding to a subset of diagrams which are closed under crossing. For identical external particles one can take a fixed topology and sum over its channel realizations to obtain a crossing-symmetric combination. Therefore we can write

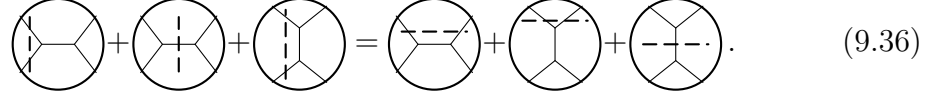
$$\mathcal{G}_{[\Gamma]}^{(k)}(u, v) = \sum_{\Gamma' \in [\Gamma]} \sum_{c \in \mathcal{C}_v(\Gamma')} \frac{\mathfrak{s}_v(\Gamma')}{2} \left(\Gamma'_c \right). \quad (9.33)$$

where $[\Gamma]$ denotes the crossing orbit of a diagram Γ , namely the set of diagrams obtained from Γ by permuting the external points. If this orbit is closed under crossing, then we expect a relation of the form

$$\sum_{\Gamma' \in [\Gamma]} \sum_{c \in \mathcal{C}_v(\Gamma')} \frac{\mathfrak{s}_v(\Gamma')}{2} \left(\Gamma'_c \right) = \sum_{\Gamma' \in [\Gamma]} \sum_{c \in \mathcal{C}_h(\Gamma')} \frac{\mathfrak{s}_h(\Gamma')}{2} \left(\Gamma'_c \right). \quad (9.34)$$

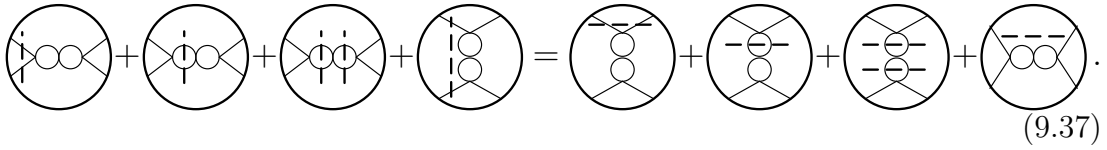
At tree level, this give two independent crossing equations for the contact and exchange topologies, which can be diagrammatically represented as

$$\text{Contact Diagram} = \text{Crossed Diagram}. \quad (9.35)$$



$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} = \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6}. \quad (9.36)$$

We know that (9.35) and (9.36) hold true, for example from [50]. However, (9.34) can generate more interesting crossing relations at higher orders. For example, at two loops in ϕ^4 theory, the 2-bubble topology is not the only topology contributing to the correlator. Nevertheless, its crossing orbit forms a crossing-closed class by itself. We can therefore write



$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8}. \quad (9.37)$$

9.3 Crossing and inversion formula

In the previous section we organized crossing at the level of crossing-closed orbits. For a diagram topology Γ , we wrote the corresponding orbit contribution after summing over all channel realizations $\Gamma' \in [\Gamma]$ as (9.33). We now want to refine this organization by keeping a single channel realization Γ' . This cannot be achieved directly using dash-cuts, since dash-cuts can mix different channel realizations. Therefore, to isolate a fixed Γ' , we use dot-cuts.

We write

$$\mathcal{G}_{\Gamma'}(u, v) \equiv \sum_{\dot{c} \in \dot{\mathcal{C}}_v(\Gamma')} \left(\Gamma'_{\dot{c}} \right), \quad (9.38)$$

where $\dot{\mathcal{C}}_v(\Gamma')$ denotes the set of vertical dot-cuts on Γ' .

Equation (9.38) should be understood as an auxiliary representation. The left-hand side is the ordinary Witten diagram associated with the fixed channel realization Γ' . But the right-hand side is not written in the meaningful dash-cut dictionary of the conformal-block expansion. We remind that individual dot-cuts do not directly correspond to the OPE data. However, this representation is still useful because the dot-cuts keep track of the pairings explicitly and can later be recombined into dash-cuts.

We can now use this auxiliary representation to write a refined crossing relation. Since $\mathcal{G}_{\Gamma'}(u, v)$ is the ordinary Witten diagram in a fixed channel, crossing acts on it in the usual way. In the dot-cut representation this gives

$$\sum_{\dot{c} \in \dot{\mathcal{C}}_v(\Gamma')} \left(\Gamma'_{\dot{c}} \right) = \sum_{\dot{c} \in \dot{\mathcal{C}}_h(\Gamma')} \left(\Gamma'_{\dot{c}} \right). \quad (9.39)$$

Here $\dot{\mathcal{C}}_h(\Gamma')$ denotes the set of horizontal dot-cuts. This equation should still be understood as an auxiliary statement, since the dot-cut terms do not directly translate into the conformal-block expansion.

However, in many cases the dot-cut equation can be rewritten by replacing the dot-cuts with the meaningful dash-cuts, up to external horizontal cuts. For example, for the exchange diagram one obtains relations of the schematic form:

$$\text{Diagram 1} = \text{Diagram 2} + [\text{external horizontal cuts}]. \quad (9.40)$$

$$\text{Diagram 3} + \text{Diagram 4} = [\text{external horizontal cuts}]. \quad (9.41)$$

Diagrams with external horizontal cuts have vanishing double discontinuity in the crossed channel. Hence they can be ignored when applying the inversion formula. Therefore, to read off the OPE data associated with a given vertical cut, we apply the inversion formula to the possible internal horizontal cuts.

If a diagram does not admit any internal horizontal cut, as in (9.41), then the corresponding OPE data can only have finite support in spin. On the other hand, for the case (9.40), we can write

$$\left(\text{Diagram 1} \right)_{n,\ell}^{(0)} = \left(\text{LIF} \left[\text{Diagram 2} \right] \right)_{[\mathcal{O}\mathcal{O}]_{n,\ell}}^{(2)}. \quad (9.42)$$

Where

$$\left(\text{LIF}[\mathcal{X}] \right)_{[\mathcal{O}\mathcal{O}]_{n,\ell}}^{(i)}$$

denotes the coefficient of the i -th order pole of the inversion function obtained from $\mathcal{X} \subset \mathcal{G}(u, v)$, at the double-trace position $\Delta = 2\Delta_{\mathcal{O}} + 2n + \ell$

Using (9.42) we can find the known asymptotic solution. For $\Delta_{\mathcal{O}} = 2$ this gives

$$\gamma_{n,\ell}^{(1)(2)} = -\frac{2a_{2,0}}{(\ell+1)(\ell+2n+2)}. \quad (9.43)$$

This fixes the part of the exchange contribution associated with the internal horizontal cut. To determine the remaining part, we use the full crossing relation for the exchange topology, written in the form (9.36). This gives the finite-spin contribution

$$\gamma_{n,\ell}^{(1)(3)} = \delta_{\ell,0} \frac{1}{2} \frac{(n+1)}{(2n+1)(2n+3)} a_{2,0}. \quad (9.44)$$

The support at $\ell = 0$ is consistent with the fact that this contribution is not associated with an internal horizontal cut.

This split of the tree-level exchange anomalous dimension into an asymptotic piece and a finite-spin piece was already observed in [50] and it was also discussed in Chapter 5. The new point here is that this two pieces admit distinct diagrammatic representations, as in (9.19) and (9.20).

One-loop check

As an application, and also as a check of the diagrammatic interpretation above, we can build one-loop objects by gluing the tree-level building blocks. We have three tree-level pieces,

$$\gamma_{n,\ell}^{(1)(1)}, \quad \gamma_{n,\ell}^{(1)(2)}, \quad \gamma_{n,\ell}^{(1)(3)}, \quad (9.45)$$

with their diagrammatic representation given in (9.18), (9.19), and (9.20).

By merging these tree-level data we can obtain different one-loop diagrams. Namely

$$\Gamma_1 = \text{diagram}, \quad \left(\text{diagram} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \left(\gamma_{n,\ell}^{(1)(1)} \right)^2 \quad (9.46)$$

$$\Gamma_2 = \text{diagram}, \quad \left(\text{diagram} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \gamma_{n,\ell}^{(1)(2)} \quad (9.47)$$

$$\Gamma_3 = \text{diagram}, \quad \left(\text{diagram} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(1)} \gamma_{n,\ell}^{(1)(3)} \quad (9.48)$$

$$\Gamma_4 = \text{diagram}, \quad \left(\text{diagram} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \gamma_{n,\ell}^{(1)(2)} \gamma_{n,\ell}^{(1)(3)} \quad (9.49)$$

$$\Gamma_5 = \text{diagram}, \quad \left(\text{diagram} \right)_{n,\ell}^{(0)} = a_{n,\ell}^{(0)} \left(\gamma_{n,\ell}^{(1)(3)} \right)^2. \quad (9.50)$$

We can use the discussion in the Chapter 6 to obtain the Mellin amplitudes associated to each of these diagrams. The Mellin amplitude has the form (6.32)

and corresponding $R_m^{(1)}$ factors can be read off from (6.37). So we find

$$\begin{aligned}
R_m^{(1)}(\Gamma_1) &= -\frac{9(m+1)(3m+4)(1)_m}{2(2m+3)\left(\frac{3}{2}\right)_m}, & R_m^{(1)}(\Gamma_2) &= \frac{6(1)_m}{\left(\frac{3}{2}\right)_m} a_{2,0}, \\
R_m^{(1)}(\Gamma_3) &= -\frac{3(m+2)(1)_m}{2(2m+3)\left(\frac{3}{2}\right)_m} a_{2,0}, & R_m^{(1)}(\Gamma_4) &= \frac{2(m+1)(1)_m - \left(\frac{3}{2}\right)_m}{(m+1)^2 \left(\frac{3}{2}\right)_m} (a_{2,0})^2, \\
R_m^{(1)}(\Gamma_5) &= -\frac{(1)_m}{2(2m+3)\left(\frac{3}{2}\right)_m} (a_{2,0})^2.
\end{aligned} \tag{9.51}$$

Now inserting these data into the flat-space limit formula (6.38b) we find the corresponding flat-space amplitudes

$$\mathcal{A}(\Gamma_1) = \frac{9}{8}\pi R_{\text{AdS}}^2 \sqrt{-T}, \quad \mathcal{A}(\Gamma_2) = \frac{6\pi}{\sqrt{-T}} a_{2,0}, \tag{9.52}$$

$$\mathcal{A}(\Gamma_3) = -\frac{3\pi}{4\sqrt{-T}} a_{2,0}, \quad \mathcal{A}(\Gamma_4) = -\frac{4\pi}{R_{\text{AdS}}^2 (-T)^{3/2}} (a_{2,0})^2, \tag{9.53}$$

$$\mathcal{A}(\Gamma_5) = \frac{\pi}{2R_{\text{AdS}}^2 (-T)^{3/2}} (a_{2,0})^2. \tag{9.54}$$

This is exactly the corresponding flat-space amplitude in the massless limit and upon fixing

$$|\lambda| = \frac{48R_{\text{AdS}}\pi^2}{N^2}, \quad g^2 = \frac{32\pi^2}{R_{\text{AdS}}N^2} a_{2,0}, \tag{9.55}$$

where λ and g are the quartic and cubic coupling constants.

This shows consistency of associating $\gamma_{n,\ell}^{(1)(2)}$ and $\gamma_{n,\ell}^{(1)(3)}$ given in (9.43) and (9.44) with diagrams (9.19), and (9.20).

9.4 Diagrammatic interpretation of the higher-trace analysis

Finally, we can try to diagrammatically express the calculation performed in Section 8.2. The advantage of the diagrammatic representation is that it makes manifest, already at the level of cuts, which terms in the double-trace sector are related by crossing to the appearance of higher-trace operators.

In particular, for ϕ^4 theory at three loops, the coefficient $b_{q,\ell}^{(0)}$ studied in Section 8.2.1 is associated with the vertical cut of the diagram shown in (9.56). Diagrammatically, what we did was to determine this OPE data by applying the

Lorentzian inversion formula to the corresponding horizontal internal cut in the crossed channel,

$$b_{q,\ell}^{(0)} = \left(\text{diagram}^{(0)} \right)_{q,\ell} = \left(\text{LIF} \left[\text{diagram}^{(1)} \right] \right)_{[\mathcal{O}^4]_{q,\ell}}^{(1)}. \quad (9.56)$$

The important point is that this relation can be read directly from the diagrams, before carrying out the explicit spin sum. The horizontal internal cut on the right hand side is exactly the contribution whose crossed-channel double discontinuity produces the direct-channel data on the left-hand side. In this way, the diagrammatic representation tells us in advance which part of the double-trace sector is responsible, after crossing, for the higher-trace operators that appear in the direct channel.

Similarly, the calculation in section 8.2.1 that obtained the anomalous dimension of this quadruple-trace family can be diagrammatically represented as

$$b_{q,\ell}^{(0)} \delta_{q,\ell}^{(1)} = \left(\text{LIF} \left[\text{diagram}^{(2)} \right] \right)_{[\mathcal{O}^4]_{q,\ell}}^{(2)}. \quad (9.57)$$

For the ϕ^3 theory, the same idea applies to the ladder diagrams studied in Section 8.2.2. In that case the leading large-spin logarithm is obtained from the horizontal cuts of the ladder diagram. Diagrammatically

$$\left[\text{diagram}^{(0)} \right]_{r,\ell}^{(0)} = \left(\text{LIF} \left[\text{diagram}^{(1)} \right] \right)_{[\mathcal{O}^k]_{r,\ell}}^{(1)} \Big|_{\log^{k-2} \ell}. \quad (9.58)$$

Chapter 10

Conclusion

In this thesis we studied loop corrections to four-point functions of identical scalar operators in four-dimensional conformal field theories with weakly coupled AdS duals in the large- N expansion. The central question was how much of the higher-order bulk perturbative expansion is encoded in lower-order CFT data. Our analysis developed along three closely related lines. First, we isolated a part of the double-trace data that is fixed at every order by tree-level input and studied its descriptions in position space and Mellin space. Second, we investigated how infinite sums over lower-trace operators generate the higher-trace sectors by requiring the crossing symmetry. Third, we discussed how the different terms in the conformal block expansion can be organized diagrammatically in a way that keeps track of their associated OPE data. This diagrammatic representation also makes the higher-trace appearance and the discussion on infinite spin-sums more clear.

We now summarize the main results and discuss a few directions for future works.

10.1 Summary of results

1. We isolated a contribution to the double-trace conformal block expansion that is fixed entirely by tree-level OPE data. At order $N^{-2\kappa}$, this is the term containing the highest power of $\log u$,

$$\mathcal{G}^{(\kappa)}(u, v) \supset \frac{\log^\kappa u}{2^\kappa \kappa!} \sum_{n \geq 0} a_{n,0}^{(0)} \left(\gamma_{n,0}^{(1)} \right)^\kappa u^{\Delta_{\mathcal{O}}+n} \tilde{g}_{2\Delta_{\mathcal{O}}+2n,0}(u, v). \quad (10.1)$$

Although this term does not determine the complete higher-order correction for $\kappa > 2$, its crossed image can be inserted into the Lorentzian inversion formula, and at each order, it fixes the coefficient of the leading power $\log^{\kappa-2} J$ in the large-spin expansion of the anomalous dimension of the double-trace family $[\mathcal{OO}]_{n,\ell}$.

In other word, denoting by $\hat{\gamma}_{n,\ell}^{(\kappa)}$ the contribution obtained from (10.1), we find

$$\hat{\gamma}_{n,\ell}^{(\kappa)} \Big|_{\log^{\kappa-2} J} = \gamma_{n,\ell}^{(\kappa)} \Big|_{\log^{\kappa-2} J} = \sum_{i \geq 0} \frac{c_{n,i}^{(\kappa)}}{J^{2\Delta_{\mathcal{O}}+2i}}, \quad (10.2)$$

where the coefficients $c_{n,i}^{(\kappa)}$ are determined by tree-level OPE data.

2. The same contribution was reformulated in Mellin space. At order $N^{-2\kappa}$, it is encoded in the restricted $(\kappa - 1)$ -loop Mellin amplitude

$$M_{(\kappa-1)\text{-loop}}^R(s, t) = \sum_{m=0}^{\infty} \frac{R_m^{(\kappa-1)}}{[t - (2\Delta_{\mathcal{O}} + 2m)]^{\kappa-1}} + \text{crossed}, \quad (10.3)$$

whose residues $R_m^{(\kappa-1)}$ are fixed entirely by tree-level OPE data. The flat-space limit of this restricted amplitude reproduces the contribution obtained from consecutive unitarity cuts, or equivalently the maximal cut, of the corresponding $(\kappa - 1)$ -loop bubble chain. This establishes a relation between the part of the CFT correlator fixed by tree-level data and a specific part of the associated flat-space loop amplitude.

3. We also showed that the leading powers of $\log J$ in the large-spin expansion of the double-trace anomalous dimensions, obtained at successive orders in $1/N$, can be resummed to all orders and organized in terms of effective double-trace exchanges with shifted twists.

$$\tau_n^{\text{eff}} = 2\Delta_{\mathcal{O}} + 2n + \epsilon_n, \quad \epsilon_n = \frac{\gamma_{n,0}^{(1)}}{N^2}. \quad (10.4)$$

For the leading-twist family, this gives

$$\hat{\gamma}_{0,\ell} = \sum_{n,m \geq 0} \frac{f_{nm}}{J^{2\Delta_{\mathcal{O}}+2n+2m+\epsilon_n}}. \quad (10.5)$$

Expanding $J^{-\epsilon_n}$ reproduces the leading $\log^{\kappa-2} J$ contribution at every order. This description was developed for $\Delta_{\mathcal{O}} = 2$ in Chapter 7 and extended to generic $\Delta_{\mathcal{O}}$ in Chapter 8, where the first few coefficients f_{nm} were computed explicitly in (8.6)–(8.8).

Separately, using the large-spin results of [46] we determined the leading large-spin behavior of the full double-trace tower at fixed n :

$$\gamma_{n,\ell}^{(2)} = \frac{\zeta_n}{J^{2\Delta_{\mathcal{O}}}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+2}}\right), \quad \gamma_{n,\ell}^{(3)} \Big|_{\log J} = -\frac{\zeta_n}{J^{2\Delta_{\mathcal{O}}}} + \mathcal{O}\left(\frac{1}{J^{2\Delta_{\mathcal{O}}+2}}\right), \quad (10.6)$$

where ζ_n is given in closed form in (8.19).

4. We showed how higher-trace operators appear as a consequence of crossing symmetry once infinite spin sums over double-trace operators are taken into account. In the ϕ^4 theory, provided that no mixing between double-trace family and quadruple-trace family occurs, the crossed image of double-trace spin sums forces the appearance of quadruple-trace operators at order N^{-8} . For the leading-twist quadruple-trace family, we computed the OPE coefficient $b_{0,\ell}^{(0)}$. For the special case of $\Delta_{\mathcal{O}} = \frac{5}{2}$ we obtained the full spin dependence in closed form.

$$b_{0,\ell}^{(0)} = \frac{\sqrt{\pi} 2^{5-2\ell} (2\ell+9)^2 (\ell(\ell+9)+23)^2 \Gamma(\ell+1)}{225(\ell+5)(\ell+6)(\ell+7)(\ell+8)\Gamma(\ell+\frac{9}{2})}. \quad (10.7)$$

For generic non-integer $\Delta_{\mathcal{O}}$, we found

$$b_{0,\ell}^{(0)} = \frac{\sqrt{\pi} 2^{-4\Delta_{\mathcal{O}}-2\ell+1} \Gamma(2\Delta_{\mathcal{O}})^2}{\ell^{3/2} [(1-\Delta_{\mathcal{O}})_{2\Delta_{\mathcal{O}}}]^2} \left[1 + \mathcal{O}\left(\frac{1}{\ell}\right) \right]. \quad (10.8)$$

We also computed the corresponding leading-twist anomalous dimension and obtained

$$\delta_{0,\ell}^{(1)} = 2. \quad (10.9)$$

5. We also studied theories with a cubic bulk interaction. Under the no-mixing assumption used in this analysis, k -trace operators can appear at order N^{-2k} . We determined the leading large-spin behavior of the leading-twist k -trace OPE coefficient:

$$\mathfrak{a}_{k,0,\ell \gg 1}^{(0)} = \frac{(-1)^k 2^{1-k\Delta_{\mathcal{O}}-2\ell} \sqrt{\pi} (\tilde{\gamma}_0)^k \log^{k-2} \ell}{(k-2)! [(1-\Delta_{\mathcal{O}})_{\frac{k\Delta_{\mathcal{O}}}{2}}]^2} \frac{1}{\ell^{3/2}} + \dots \quad (10.10)$$

6. We introduced a diagrammatic representation for the conformal-block expansion. We took diagrams inside parentheses to denote products of OPE data, and diagrams enclosed in a circle to denote the corresponding contributions to the correlator. This notation gives a refinement of crossing symmetry at two different levels.

The first level is the crossing-closed orbit of a topology. For a crossing orbit $[\Gamma]$, we wrote

$$\sum_{\Gamma' \in [\Gamma]} \sum_{c \in \mathcal{C}_v(\Gamma')} \frac{\mathfrak{s}_v(\Gamma')}{2} \left(\Gamma'_c \right) = \sum_{\Gamma' \in [\Gamma]} \sum_{c \in \mathcal{C}_h(\Gamma')} \frac{\mathfrak{s}_h(\Gamma')}{2} \left(\Gamma'_c \right). \quad (9.34)$$

This expresses crossing not only for the full correlator, but also for the subset of terms associated with a crossing-closed diagrammatic topology.

The second level is an auxiliary fixed-channel relation written in terms of dot-cuts. Since dot-cuts keep track of the pairing across the cut, they allow one to

resolve a fixed channel diagram Γ' before recombining the result into dash-cuts. At this auxiliary level we wrote

$$\sum_{\dot{c} \in \dot{\mathcal{C}}_v(\Gamma')} \bigcirc_{\dot{c}}^{\Gamma'} = \sum_{\dot{c} \in \dot{\mathcal{C}}_h(\Gamma')} \bigcirc_{\dot{c}}^{\Gamma'}. \quad (9.39)$$

Although this relation is auxiliary, it is still useful. In many examples, dot-cut terms can be recombined into dash-cut terms, up to additional external horizontal cuts. The latter have vanishing crossed-channel double-discontinuity and therefore do not contribute to the Lorentzian inversion formula. In this way, the dot-cut relation refines the use of the inversion formula: it gives a diagrammatic way to identify which crossed-channel terms should be probed in order to extract a given piece of OPE data.

7. Using the relation (9.39) for scalar exchange, we showed that the asymptotic and finite-spin parts of the tree-level anomalous dimension, given in (5.27) and (5.40), have different diagrammatic origins. For $\Delta_{\mathcal{O}} = 2$

$$\frac{1}{a_{n,\ell}^{(0)}} \left(\text{diagram} \right)_{n,\ell}^{(0)} = \gamma_{n,\ell}^{(1),\text{as}} = -\frac{2a_{2,0}}{(\ell+1)(\ell+2n+2)}, \quad (10.11)$$

$$\frac{1}{a_{n,\ell}^{(0)}} \left(\text{diagram} \right)_{n,\ell}^{(0)} = \gamma_{n,\ell}^{(1),\text{fin}} = \delta_{\ell,0} \frac{1}{2} \frac{n+1}{(2n+1)(2n+3)} a_{2,0}. \quad (10.12)$$

As a check of this assignment, we glued the tree-level building blocks in scalar $\phi^3 + \phi^4$ theory and computed the corresponding one-loop Mellin amplitudes. We saw that their flat-space limits reproduce the expected massless flat-space amplitudes.

8. Finally, we gave a diagrammatic interpretation of the higher-trace analysis. In the ϕ^4 theory, the diagrammatic representation makes manifest that the leading quadruple-trace OPE coefficient is associated with the infinite double-trace spin sum appearing in (8.28). This is encoded directly in the diagram (9.56). In this sense, the diagrammatic representation helps identify which infinite spin sums can generate higher-trace data, and which crossed cuts should be used to extract that data through the Lorentzian inversion formula. Similarly, in the ϕ^3 theory, the diagrammatic relation (9.58) shows how the higher-trace OPE data are extracted from the crossed-channel terms in (8.55).

10.2 Outlook

Several questions remain open. One is the bulk interpretation of the cut diagrams introduced in Chapter 9. While the maximal cuts of bubble diagrams

reduce in the flat-space limit to ordinary unitarity cuts, it would be interesting to determine whether the more general non-maximal cuts admit a similar flat-space interpretation or can be understood directly as AdS unitarity cuts, perhaps along the lines of [77].

Another important direction is to relax the no-mixing assumption used in the higher-trace analysis. This may require including disconnected contributions and could modify the order in the large- N expansion at which some higher-trace data first appear.

Finally, extending the product rule of Chapter 9 from two-particle to higher-particle cuts, including the additional degeneracies of higher-trace states, would provide a more systematic organization of these contributions.

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