

SISSA

Scuola
Internazionale
Superiore di
Studi Avanzati

Physics Area - PhD course in
Astrophysics and Cosmology

Unveiling the Interaction between Dark Matter and Gravity: from Galaxy Kinematics to Cosmology

Candidate:
Francesco Benetti

Supervisors:
Prof. Andrea Lapi
Prof. Stefano Liberati
Dott. Samuele Silveravalle

Academic Year 2025-26



*Sempre caro mi fu quest'ermo colle,
e questa siepe, che da tanta parte
dell'ultimo orizzonte il guardo esclude.
Ma sedendo e mirando, interminati
spazi di là da quella, e sovrumani
silenzi, e profondissima quiete
io nel pensier mi fingo, ove per poco
il cor non si spaura. E come il vento
odo stormir tra queste piante, io quello
infinito silenzio a questa voce
vo comparando: e mi sovvien l'eterno,
e le morte stagioni, e la presente
e viva, e il suon di lei.
Così tra questa immensità s'annega il
pensier mio:
e il naufragar m'è dolce in questo mare*

Giacomo Leopardi

Acknowledgements

Declaration of Autorship

I, Francesco Benetti, declare that this thesis titled, *Unveiling the Interaction between Dark Matter and Gravity: from Galaxy Kinematics to Cosmology*, and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at SISSA.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at SISSA or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Abstract

The Cold Dark Matter paradigm emerged as the prevailing solution to the cosmic missing mass conundrum. While highly successful on cosmological scales, this model is partially inconsistent with galaxy-scale observations. This thesis addresses such a challenge by pursuing a dual approach.

The first part presents a model featuring a non-local interaction between cold dark matter and gravity. The model, referred to as fractional gravity, in its Newtonian setting is based on a modified Poisson equation featuring the fractional Laplacian and two new parameters: the fractional index s and length scale ℓ . To begin with, I will provide the solution to this equation for the gravitational potential generated by a Navarro-Frenk-White distribution of cold dark matter. Remarkably, the result can be interpreted in terms of an almost cored effective density distribution, similar in shape to a cored isothermal sphere, thus solving the cusp-core problem of cold dark matter. Moreover, the predicted profiles are consistent with the core-surface density relation observed for dwarf galaxies. A Bayesian analysis will then test fractional gravity against stacked rotation curves of a broad sample of spiral galaxies. In terms of reduced chi-squared and BIC, the fits yielded by this model are always superior to the standard, cold dark matter Navarro-Frenk-White profile. On the other hand, I will show how this model can explain the observed interplay between dark matter and baryons in late-type galaxies, embodied by tight dynamical scaling relations such as the Radial Acceleration Relation.

The analysis predicts a scaling between the parameters s and ℓ and the virial mass of dark matter haloes, with non-local effects being stronger in smaller systems. After using the fractional gravity model to fit galaxy clusters' pressure profiles from the X-COP collaboration, I will show that such scaling holds consistently from the dwarf-galaxy regime up to galaxy clusters' virial masses. Overall, fractional gravity shows a rich phenomenology in a comprehensive set of scales capable of addressing long-standing issues of the cold dark matter paradigm.

To go beyond the Newtonian setting, I will extend fractional gravity to a relativistic framework, deriving it from a suitable action. I will show that the resulting field equations can be interpreted as Einstein equations sourced by an effective stress-energy tensor for the dark matter component, with pressure, heat vectors, and anisotropic stresses emerging dynamically from the coupling with gravity. I will then study propagating degrees of freedom in this model, from scalar fields to gravitational waves, showing that non-locality introduces memory effects which extend the propagation inside the past light-cone.

The second part of this thesis is devoted to study a relativistic model, named non-minimally coupled dark matter. Although models of this kind had been already studied in other contexts [1, 2, 3, 4], in the present case this provides a localization of fractional gravity. The greater

simplicity of this model enables to do explicit calculations in the strong gravity regime, giving a taste of the corresponding solutions in the non-local theory.

I will solve the equations of non-minimally coupled dark matter in three relevant scenarios. First, I will show how the dynamically generated pressure is able to balance gravity by supporting spherically-symmetric, static equilibrium configurations of dark matter. The resulting objects are regular and horizonless, but ultra-compact, so that, if formed in the early universe, could have provided the seeds for supermassive black holes observed at high redshifts by the JWST collaboration.

Next, I will turn the attention to cosmology, computing the evolution of the universe in the non-minimally coupled dark matter framework. Remarkably, at early times a phase of accelerated expansion driven by the non-minimal coupling emerges, so that inflation is a natural prediction of the model. At even earlier times the fate of the universe depends on the spatial curvature, with the remarkable possibility of avoiding the initial singularity in favour of a bouncing universe in the case of negative curvature.

Since no inflationary model can call itself complete without discussing the formation and growth of linear perturbations, in the final chapter I will provide a complete analysis. In particular I will compute the linear power spectrum of scalar, vector, and tensor perturbations, and show that in the case of a bouncing cosmology one obtains near scale-invariant power spectra, in agreement with observations of the Cosmic Microwave Background.

All in all, the non-minimally coupled dark matter model presents a rich and interesting phenomenology, providing seeds for supermassive black holes, naturally predicting inflation, and generating a power spectrum of perturbations compatible with observations.

This Ph.D. thesis is based on the following publications and submitted papers:

1. [Dark Matter in Fractional Gravity I: Astrophysical Tests on Galactic Scales](#), by **Benetti, F.**, Lapi, A., Gandolfi, G., Salucci, P., Danese, L., **ApJ** 949 65
2. [Dark Matter in Fractional Gravity II: Tests in Galaxy Clusters](#), by **Benetti, F.**, Lapi, A., Gandolfi, G., Haridasu, S., Danese, L., **Universe**, 9(7), 329.
Recognitions: This work was selected by the Universe editorial board as the cover of the July 2023 issue.
3. [Dark Matter in Fractional Gravity III: Dwarf Galaxies Kinematics](#), by **Benetti, F.**, Lapi, A., Gandolfi, G., Butt, M., Boumetcha, Y., Haridasu, S., **Universe**, 9(11), 478
4. [A Relativistic Scalar Model for Fractional Interaction between Dark Matter and Gravity](#), by **Benetti, F.**, Lapi, A., Gandolfi, G., Liberati, S., **Class. Quantum Grav.**, 41 175010.
5. [A Relativistic Tensorial Model for Fractional Interaction between Dark Matter and Gravity](#), by **Benetti, F.**, Lapi, A., Silveravalle S., Liberati, S., Haridau, S., Boumechta Y., Butt, M., Baccigalupi, C., **Phys. Rev. D** 111, 024026 534.
6. [Ultra-compact Objects of Non-minimally Coupled Dark Matter](#), by **Benetti, F.**, Lapi, A., Silveravalle, S., Liberati, S., **JCAP**, 03 (2025) 029.
7. [Cosmology with a Non-minimally Coupled Dark Matter Fluid I. Background Evolution](#), by Silveravalle, S., Lapi, A., **Benetti, F.**, Liberati, S., **JCAP**, 11 (2025) 067.
8. [Cosmology with a Non-minimally Coupled Dark Matter Fluid II. Cosmological Perturbations](#), by Silveravalle, S., Lapi, A., **Benetti, F.**, Liberati, S., arXiv:2607.00137 , submitted to **JCAP**.
9. [Caustic and hydrostatic mass bias: Implications for modified gravity](#), by Butt, M., Haridasu, S., Boumechta, Y., **Benetti, F.**, Pizzuti, L., Baccigalupi, C., Lapi, A., **Phys. Rev. D** 109, 064006

10. [Mass Modeling and Kinematics of Galaxy Clusters in Modified Gravity](#), by Pizzuti, L., Boumechta, Y., Haridasu, S., Pombo, M., Dossena, S., Butt, M., **Benetti, F.**, Baccigalupi, C., Lapi A., **JCAP** 11 (2024) 014
11. [Modified gravity in galaxy clusters: Joint analysis of Hydrostatics and Caustics](#), by Butt, M., Haridasu, S., Boumechta, Y., **Benetti, F.**, Pizzuti, L., Baccigalupi, C., Lapi A., **JCAP** 11 (2024) 014

Contents

Acknowledgements	iii
Abstract	iii
List of Figures	x
List of Tables	xiii
List of symbols	xvii
1 Chapter 1	1
1.1 The rise of the dark matter paradigm	1
1.2 The small scales problems of cold dark matter	5
1.3 Modified gravity as an alternative to dark matter	11
1.4 Non-standard couplings between Dark Matter and gravity	12
1.5 This thesis	17
I First part	23
2 Chapter 2	25
2.1 Fractional integrals and derivatives	26
2.2 Fractional Laplacian	28
2.2.1 Spectral definition	28
2.2.2 Singular integral representation	29
2.2.3 Elliptic Extension, Heat Semigroup and the Balakrishnan Formula . . .	31
2.3 Fractional d'Alembertian	33
2.4 Summary and conclusions	34

3	Chapter 3	35
3.1	Dark Matter in Fractional Gravity	36
3.2	Bayesian analysis of stacked RCs	39
3.2.1	Results	40
3.3	Recovering empirical relations in late-type galaxies	46
3.3.1	Disk vs halo mass relation	47
3.3.2	Radial Acceleration Relation	47
3.3.3	Universal Core Surface Density and Core Radius vs. Disk Scale-length	49
3.4	An alternative MONDian viewpoint?	51
3.5	Summary and conclusions	54
4	Chapter 4	57
4.1	Galaxy Clusters: Theoretical Background	58
4.1.1	Forward Modeling of the ICM Thermodynamics	59
4.2	Clusters' Bayesian Data Analysis	61
4.2.1	Results	62
4.2.2	Summary	73
4.3	Dwarf Irregulars and Dwarf Ellipticals	73
4.3.1	Dynamical modeling	74
4.3.2	Dwarfs' Data and Bayesian Analysis	75
4.3.3	Results	77
4.4	Summary and conclusions	88
5	Chapter 5	89
5.1	Non-local gravity theories	89
5.2	Relativistic Fractional Gravity	92
5.2.1	Weak field limit	94
5.2.2	Newtonian and Post-Newtonian limit	95
5.2.3	The effective DM stress-energy tensor	97
5.2.4	Scalar–vector–tensor decomposition	99
5.3	Testing RFG with lensing data of galaxy clusters	102
5.4	Summary and conclusions	109
II	Second part	113
6	Chapter 6	115
6.1	Introduction	115

6.2	Non-minimally coupled Dark Matter	116
6.2.1	Field Equations and effective stress-energy tensor	118
6.3	Ultra-compact objects of NMC-DM	120
6.3.1	Properties of the NMC-UCOs	122
6.4	Orbits around NMC-UCOs	126
6.4.1	Orbital portraits	130
6.4.2	UCO's pseudo-shadow	131
6.5	Summary and conclusions	132
7	Chapter 7	135
7.1	Non-minimally coupled cosmology	136
7.1.1	Cosmic evolution	138
7.1.2	NMC-driven inflation and bounce	143
7.2	Summary and conclusions	145
8	Chapter 8	147
8.1	Theory of cosmological perturbations with a non-minimal coupling	148
8.1.1	Scalar perturbations	150
8.1.2	Vector perturbations	154
8.1.3	Tensor perturbations	157
8.2	Generation of perturbations and primordial power spectrum	159
8.2.1	Power spectrum within inflationary scenario	160
8.2.2	Power spectrum within bouncing scenario	162
8.3	Summary and conclusions	169
9	Chapter 9	173
9.1	Summary of the main results	173
9.2	Future prospects	176
9.2.1	From phenomenology to a fundamental origin of fractional gravity	176
9.2.2	Larger and more homogeneous galaxy samples	177
9.2.3	Strong-gravity dynamics and the stability of NMC ultra-compact objects	177
9.2.4	Beyond linear perturbations	178
9.2.5	A more complete treatment of primordial perturbations	178
9.2.6	Towards a unified dark-sector phenomenology	179
9.2.7	Final remarks	180

III	Appendix	183
A	Solutions to the Fractional Poisson Equation	185
B	Variation of the Relativistic Fractional Gravity Action	189
C	Retarded Green Function for the Fractional d'Alembertian	193
D	Cosmology with a Non-Minimally Coupled Radiation fluid	197
E	Evolution of Perturbations around the Bounce	199
F	Solution of the Scalar Sector in the NMC-Epoch	203
	Bibliography	205

List of Figures

3.1	Normalized rotation curve an effective density for the NFW profile for $s \in [1, 3/2]$	38
3.2	Fit to stacked rotation curve for the dwarf galaxy sample and the LSB spirals, with Newtonian and fractional gravity.	42
3.3	Same as Fig. 3.2 for the stacked rotation curves of the HSB spirals sample. . .	43
3.4	MCMC posterior distributions in the fractional gravity framework for the fractional parameter s , the fractional scale-length (in units of the NFW scale radius) ℓ/r_s , the halo mass M_H and the disk mass M_d	44
3.5	Dependence of the parameter s and of the scale-length ℓ in on the halo mass M_H , as derived from rotation curve modeling in fractional gravity.	46
3.6	The disk (stellar) mass vs. halo mass relation from rotation curve modeling in Newtonian and in fractional gravity.	48
3.7	The radial acceleration relation (RAR).	50
3.8	The core surface density $\rho_0 r_0$ vs. the core radius r_0 and the core radius r_0 vs. the disk scale-length r_d	51
3.9	Results for the MONDian viewpoint of fractional gravity for a subset of representative stacked RCs.	55
4.1	Effective DM mass profile in fractional gravity as a function of the radial coordinate for different values of the fractional index s	60
4.2	Fits to the stacked sample profiles of the X-COP clusters in fractional gravity, according to our Bayesian analysis.	65
4.3	Fits to the individual pressure profiles of the X-COP clusters in fractional gravity, according to our Bayesian analysis.	66
4.4	Same as Figure 4.3 for the density profiles.	67
4.5	MCMC posterior distributions of the fractional parameter s , the fractional length scale ℓ , the DM concentration c_{500} , and the DM mass M_{500} , for two representative clusters of the X-COP sample: A2255 and ZW1215	68

4.6	Posterior distributions of fractional index s , fractional length-scale ℓ , concentration c_{500} and DM mass M_{500} , from the fits to the X-COP density and pressure profiles in fractional gravity.	70
4.7	Concentration c_{200} vs. DM mass M_{200} relation for the X-COP sample in fractional gravity.	71
4.8	Scaling relations in fractional gravity from galaxies to clusters: fractional index s and length-scale ℓ vs. DM mass M_{200}	72
4.9	MCMC posterior distributions of the stellar mass-to-light ratio M_*/L , the DM mass M_{200} and the fractional index s , together with the fits to the rotation curve, for the galaxy NGC3741 and DDO64.	78
4.10	Fits to the rotation curves for other 6 dwIrr galaxies.	81
4.11	MCMC posterior distributions of the mass-to-light ratio M_*/L , the symmetrized anisotropy parameter β_{sym} , the mass M_{200} and the fractional index s , together with the fits to the l.o.s. dispersion profile for the dwSph galaxy Sculptor and the ultra-diffuse galaxy DragonFly 44.	85
4.12	Fits to the l.o.s. dispersions profile for other 6 dwSph galaxies.	86
4.13	Fractional index as a function of DM mass from the outcomes of our Bayesian analysis, together with the Radial Acceleration Relation extended to the dwarfs sample.	87
5.1	Fits to the lensing convergence profiles $\kappa_\infty(R)$ of the individual and of the stacked CLASH cluster sample according to the EFG and TFG model.	107
6.1	Radial profiles of UCOs solutions: density $\rho(r)$, cumulative mass $M(< r)$, metric component g^{rr} , metric component $-g_{tt}$, extrinsic curvature component $-\kappa_{tt}$, and effective radial pressure $p_r(r)$	123
6.2	Relevant physical length-scales of UCOs solutions: boundary radius R_b and light rings, measured in units of total mass M of the UCO, and plotted as a function of the UCO central density $\rho(0)$	127
6.3	Relevant examples of orbital trajectories for massless and massive particles around an UCO solution with central density $\hat{\rho}_0 = 0.95$	129
6.4	Orbital portraits diagrams for the UCO solution with central density $\hat{\rho}_0 = 0.95$	130
6.5	Pseudo-shadow of the UCO solution with central density $\hat{\rho}_0 = 0.95$	131
7.1	Evolution of the density parameters as a function of redshift $1 + z \equiv a^{-1}$	140
7.2	The dynamical state parameter w as a function of redshift.	141
7.3	The comoving Hubble horizon as a function of redshift.	142

8.1	Pictorial representation of the evolution of perturbations generated during a an inflationary era, showing their dependence on the wavenumber k and on the scale factor a	161
8.2	Pictorial representation of the evolution of perturbations generated during a contracting phase in a bouncing cosmology, showing their dependence on the wavenumber k and on the scale factor a	164
8.3	Possible shapes of the tensor-to-scalar ratio for different values of $\bar{k}_{\text{MAT}}^{(\text{T})}$	168
8.4	Fit of Planck data with our prediction for the primordial power spectrum. . . .	170

List of Tables

3.1	Marginalized posterior estimates (mean and 1σ confidence intervals) for the fractional and Newtonian gravity. Columns report the values of: fractional parameter s , fractional scale-length ℓ , halo mass M_{H} , disk mass M_{d} , reduced χ_r^2 . The last column refer to the difference in values of the Bayesian inference criterion (BIC) between the two models.	45
4.1	Marginalized posterior estimates (mean and 68% confidence limits are reported) for the parameters from the MCMC analysis of the X-COP sample in fractional gravity.	69
4.2	Properties of the galaxy sample considered in this work: half-light radius, total luminosity, and mass-to-light ratio estimated from stellar population synthesis models, used in building the priors of our Bayesian analysis.	76
4.3	Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the individual rotation-dominated dwIrr in fractional and Newtonian gravity.	79
4.4	Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the individual dispersion-dominated dwarf spheroidal in fractional and Newtonian gravity.	82
5.1	Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the CLASH clusters in EFG, in TFG and in GR.	108

List of Abbreviations

BAO	Baryon Acoustic Oscillations
BBN	Big Bang Nucleosynthesis
BIC	Bayes Information Criterion
BTFR	Baryonic Tully–Fisher Relation
CDM	Cold Dark Matter
CLASH	Cluster Lensing and Supernova survey with Hubble
CMB	Cosmic Microwave Background
COBE	COsmic Background Explorer
DE	Dark Energy
DM	Cold Dark Matter
DwIrr	Dwarf Irregular
DwSph	Dwarf Spheroidal
EFG	Einstein Fractional Gravity
FG	Fractional Gravity
GR	General Relativity
GW	Gravitational Wave
HDM	Hot Dark Matter
HSB	High Surface Brightness
JWST	James Webb Space Telescope
ICM	Intracluster Medium
LSB	Low Surface Brightness
MACHO	Massive Halo Compact Object
MCMC	Monte Carlo Markov Chain
MOND	Modified Newtonian Dynamics
NFW	Navarro–Frenk–White
NMC	Non-Minimal Coupling
RAR	Radial Acceleration Relation
RC	Rotation Curve

RFG	Relativistic Fractional Gravity
SET	Stress-Energy Tensor
SFG	Scalar Fractional Gravity
SIDM	Self-Interacting Dark Matter
SZ	Sunyaev–Zel’dovich
TeV S	Tensor Vector Scalar theories
TFG	Tensorial Fractional Gravity
WDM	Warm Dark Matter
WIMP	Weakly Interacting Massive Particle
WL	Weak Lensing
WMAP	Wilkinson Microwave Anisotropy Probe
X-COP	XMM-Newton - Cluster Outskirts Project

Chapter 1

Introduction

1.1 The rise of the dark matter paradigm

A multitude of astrophysical and cosmological probes have now firmly established that visible matter (or baryons), in the form of stars, galaxies, gas, etc., constitute only some 15% of the total matter content in the Universe, the rest being in the form of ‘dark matter’ (DM). The term dark matter denotes a component of matter that interacts sufficiently weakly with electromagnetic radiation that it cannot be directly observed through its emission or absorption of light, but whose presence is inferred from its gravitational effects. The modern dark matter paradigm did not emerge from a single observation; rather, it developed through the convergence of evidence from galaxy clusters, galactic dynamics, primordial nucleosynthesis, large-scale structure, gravitational lensing and, more recently, precision measurements of the cosmic microwave background (CMB). A comprehensive historical account of this development can be found in [5].

Historically, the first compelling evidence for the existence of dark matter came from the study of galaxy clusters. In 1933, Fritz Zwicky analyzed the velocity dispersion of galaxies in the Coma Cluster using the virial theorem [6]. He found that galaxies were moving far too quickly to be gravitationally bound by the visible matter alone. Zwicky inferred the presence of what he dubbed “dunkle Materie” (dark matter), estimating that the cluster contained far more mass than was observable in stars.

The cluster evidence was subsequently complemented by observations on galactic scales. Measurements of the rotation curves of spiral galaxies demonstrated that the orbital velocity of stars and gas does not decrease with radius as expected for a system in which most of the mass follows the observed luminous distribution. Rubin, Ford and Thonnard measured rotation curves for 21 Sc galaxies over a wide range of radii and luminosities [7]. Their observations showed that the rotation curves generally remained approximately flat at large

radii. Subsequent measurements of different morphological classes reinforced this conclusion [8]. Since in Newtonian gravity the circular velocity at a given radius is linked to the enclosed mass by $v_c^2 = GM(< r)/r$, if gravity was generated only by visible matter, one would expect a decaying of circular velocity $v \propto r^{-1/2}$ at large radii. If one does not want to abandon the Newtonian gravity paradigm in favor of some modified gravity theory to accomodate for the observed phenomenon, the unavoidable implication of an approximately flat rotation curve is that galaxies are embedded in extended, massive halos of unseen matter [9, 10, 11].

During the 1970s and early 1980s, the missing-mass problem became increasingly connected with cosmology and particle physics. An important development was the realisation that the matter responsible for the gravitational effects could not simply consist of ordinary baryonic matter in arbitrary quantities. Big Bang nucleosynthesis (BBN) provides an independent constraint on the cosmic baryon density through the predicted primordial abundances of light nuclei. At the same time, cosmological studies demonstrated that stable relic particles could contribute significantly to the total matter density of the Universe [12]. These developments provided an early indication that the dominant matter component should be non-baryonic.

This motivated the search for a form of matter that is both sufficiently abundant and sufficiently weakly interacting to have escaped direct detection. An important distinction subsequently emerged between *hot*, *warm* and *cold* dark matter, according to the typical velocity of the particles during the epoch when cosmic structures began to form. Relativistic or semi-relativistic dark matter can erase primordial density fluctuations on small scales through free streaming, whereas sufficiently cold dark matter retains fluctuations on galactic and larger scales.

The cold dark matter (CDM) hypothesis was developed into a quantitative model during the early 1980s. Peebles demonstrated that non-relativistic dark matter could provide the basis for the formation of large-scale structure [13], while Bond, Szalay and Turner investigated the evolution of fluctuations in gravitationally dominated cosmologies [14]. The framework was subsequently developed in detail by Blumenthal, Faber, Primack and Rees, who demonstrated that a Universe dominated by cold, collisionless dark matter could produce a hierarchical pattern of structure formation consistent with the observed distribution of galaxies and clusters [15].

The physical motivation for CDM is particularly significant. Since prior to decoupling from the photon fluid at around $z \simeq 1100$, perturbations in baryonic matter were suppressed by photon pressure (Silk damping), there wasn't sufficient time after recombination for these perturbations to grow rapidly enough to form galaxies and clusters. In this context, cold dark matter provides the necessary gravitational scaffolding for structure formation. Because cold dark matter particles possess relatively small thermal velocities, their free-streaming length is sufficiently short that density perturbations over galactic and larger scales survive. Furthermore,

due to their very weak interaction with the photon fluid, they are not subject to Silk damping either. Dark matter could consequently begin to collapse gravitationally before recombination, forming potential wells into which baryonic matter subsequently fell. The resulting hierarchical scenario naturally produces the formation of dark-matter halos, followed by galaxies, groups and clusters. The CDM framework therefore transformed dark matter from an explanation of an observed mass discrepancy into a central dynamical ingredient of cosmic structure formation.

A further important development came in the 1980s with the advent of X-ray astronomy. Galaxy clusters were observed to contain large amounts of hot, diffuse gas emitting via thermal bremsstrahlung. The temperature of this gas—measured through X-ray spectra—is correlated with the depth of the gravitational potential well [16, 17, 18, 19]. These measurements showed that the gas is too hot to be confined by the gravity of visible galaxies alone, and that the total gravitational mass must be significantly larger than the luminous mass, thus providing an independent confirmation of Zwicky’s earlier conclusions, now supported by entirely different observational techniques. In the same years the development of gravitational lensing as an observational tool allowed astronomers to map mass distributions directly, regardless of whether the matter emits light. Strong and weak lensing studies [20, 21] revealed: mass distributions in clusters that exceed luminous matter; smooth, extended dark matter halos around galaxies; direct spatial separation between baryonic matter and gravitational mass in systems like the Bullet Cluster. The latter, in particular, became a critical test case: during the collision of two clusters, the hot gas (baryonic matter) was slowed by interaction, while the dominant mass component—traced via lensing—passed through largely unaffected [22, 23, 24].

At the same time, measurements of the cosmic microwave background provided an increasingly precise and independent probe of the matter content of the Universe. The CMB anisotropies encode information about the physical densities of baryons and dark matter because these densities determine the evolution of the photon–baryon fluid and consequently the acoustic structure of the CMB power spectrum. The development of increasingly precise CMB measurements therefore allowed the dark-matter abundance to be determined without relying on the dynamics of individual galaxies or clusters.

The Wilkinson Microwave Anisotropy Probe (WMAP) established a high-precision cosmological parameter framework in which a spatially flat Λ CDM model provided a successful description of the observed CMB anisotropies [25]. The Planck satellite subsequently improved these measurements substantially. The Planck 2018 analysis found that the base six-parameter, spatially flat Λ CDM model provides a good description of the temperature, polarization and lensing data [26]. In this framework, the physical cold-dark-matter density is measured to be approximately $\Omega_{\text{CDM}} h^2 \simeq 0.120$, while the total matter density is approximately $\Omega_{\text{m}} h^2 \simeq 0.143$. These measurements illustrate the remarkable precision with which the dark-matter abundance

can now be inferred from cosmological observations.

The modern Λ CDM model therefore represents the convergence of several independent observational and theoretical developments. In its simplest form, it assumes a Universe containing radiation, baryonic matter, cold dark matter and a cosmological constant, Λ . The dark-matter component dominates the matter density and provides the gravitational potential responsible for the growth of structure, while the cosmological constant drives the accelerated expansion of the late-time Universe. The success of this model in describing the CMB, galaxy clustering, gravitational lensing and the large-scale matter distribution has made Λ CDM the standard reference framework for modern cosmology [5].

Importantly, however, the success of Λ CDM does not constitute an identification of the microscopic nature of dark matter. The cosmological model specifies the required macroscopic properties of the dark component—most importantly that it is predominantly cold, non-baryonic and effectively collisionless on cosmological scales—but does not identify the underlying particle or field. This distinction has motivated an extensive particle-physics programme aimed at determining the nature of the dark sector.

Among the most extensively studied candidates are weakly interacting massive particles (WIMPs), which fits the model of early Universe relic DM particles. The Standard Model of particle physics does not predict WIMP-like particles. However, various extensions, such as Supersymmetry, predict hypothetical elementary particles that are good WIMP candidates (this coincidence is sometimes dubbed the WIMP miracle [27]). One can constrain WIMPs' cross-section by comparing the present-day relic abundance of DM with theoretical predictions of WIMPs' relic density [28]. At the high temperatures of the early Universe, DM particles and their antiparticles form from and annihilate into lighter Standard Model particles. As the Universe expands and cools, the average thermal energy of these lighter particles decreases, becoming insufficient to form DM particle-antiparticle pairs. Parallely, the annihilation of DM particle-antiparticle pairs still proceeds. Net, the number density of DM particles starts decreasing exponentially after crossing the temperature $T \sim m_{\text{DM}}/20$ [29]. The DM particle number density drops, becoming so low that particle-antiparticle reactions cease entirely. As the Universe expands, the total DM particle number remains roughly constant [30]. Now, the self-annihilation cross-section only weakly depends on the DM particle mass and hence can be approximated as

$$\langle \sigma v \rangle [\text{cm}^3 \text{s}^{-1}] \simeq \frac{3 \times 10^{-27} \text{cm}^3 \text{s}^{-1}}{\Omega_{\text{CDM}} h^2}. \quad (1.1)$$

The measured density $\Omega_{\text{CDM}} h^2 \simeq 0.1$, is then achieved by a self-annihilation cross-section of $\langle \sigma v \rangle \simeq 10^{-26} \text{cm}^3 \text{s}^{-1}$ [31], a value roughly corresponding to the cross-section for the weak interaction. Particles with a larger interaction cross-section would continue to annihilate for a longer time and thus would have a smaller number density when the annihilation interaction

ceases.

Other relevant dark matter candidates include QCD axions, emerging from the Peccei–Quinn mechanism as a solution to the strong-CP problem, massive neutrinos (whether sterile or not) and supersymmetric particles like neutralinos, gravitinos, sneutrinos, axinos [27], and Kaluza-Klein particles [32]. These possibilities illustrate an important change in the status of dark matter: whereas the original problem was primarily astrophysical, it has evolved into a major question at the interface between cosmology and particle physics.

The absence of an experimentally confirmed non-gravitational detection has nevertheless challenged the simplest realisations of the WIMP paradigm and has motivated a much broader exploration of the dark sector. Direct-detection experiments search for the scattering of dark-matter particles off nuclei or electrons, indirect searches look for Standard Model particles produced by dark-matter annihilation or decay, while collider experiments search for missing momentum associated with the production of invisible particles. At the same time, cosmological and astrophysical observations continue to test the assumption that dark matter is cold and collisionless.

Consequently, the current dark-matter framework can be understood as having two distinct but interconnected components. On cosmological and astrophysical scales, there is compelling evidence for a non-luminous matter component whose gravitational effects are observed from galactic scales to the largest observable structures. This component is successfully incorporated into the Λ CDM model (according to which the Universe is composed for 5% by baryons, for 25% by cold dark matter, and for 70% by dark energy) and its cosmic abundance is now measured with percent-level precision. On microscopic scales, however, the particle nature of dark matter remains unknown.

1.2 The small scales problems of cold dark matter

Although on cosmological scales the CDM hypothesis is remarkably consistent with data coming e.g. from CMB and galaxy surveys, on (sub)galactic scales it is known to face some severe challenges.

Cusp-core problem Gravity-only N-body simulations predict the density distribution of virialized halos for collisionless DM to follow the Navarro-Frenk-White (NFW) profile [33]:

$$\rho(r) = \frac{\rho_s r_s^3}{r (r + r_s)^2} . \quad (1.2)$$

Here r_s is a scale radius and ρ_s a characteristic density. The logarithmic density slope $\gamma \equiv d \ln \rho / d \ln r$ takes on values around -3 in the outskirts, -2 at $r \approx r_s$ and -1 in the inner region,

so that the NFW profile exhibits a density cusp at small radii. The size of a halo is usually associated with its virial radius R_v , commonly defined as the radius of a sphere at redshift z containing an overdensity with average density $\Delta_c \rho_{\text{crit}}$, where Δ_c is the critical overdensity for virialisation and $\rho_{\text{crit}} = 3H^2(8\pi G)^{-1}$ is the critical density for closure of the Universe. For an Einstein–de Sitter Universe ($\Omega_m = 1$), $\Delta_c = 178\pi^2$, yet, in general, this quantity depends on redshift. In Λ CDM cosmology, $\Delta_c \sim 101$ at $z = 0$ and $\Delta_c \sim 178$ at high redshifts. However, a wildly adopted and simple choice is the redshift independent value $\Delta_c \sim 200$.

The total mass of a DM halo is traditionally associated with its virial mass

$$M_v = \frac{4\pi}{3} \Delta_c \rho_{\text{crit}} R_v^3, \quad (1.3)$$

so that the virial radius reads

$$R_v = \left(\frac{2GM_v}{\Delta_c(z)H^2(z)} \right)^{1/3}. \quad (1.4)$$

At $z = 0$ and with $\Delta_c = 200$, one obtains

$$R_{200} \simeq 206.3 \left(\frac{M_{200}}{10^{12} M_\odot} \right)^{1/3} \left(\frac{h}{0.7} \right)^{-2/3}, \quad (1.5)$$

where $h \equiv H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$.

The concentration of a NFW profile is defined by the ratio $c = R_v/r_s$. The associated cumulative mass is given by

$$M(< r) = 4\pi \int_0^r dr' r'^2 \rho(r') = M_s \left[\ln \left(1 + \frac{r}{r_s} \right) - \frac{r/r_s}{1 + r/r_s} \right]. \quad (1.6)$$

Here $M_s \equiv 4\pi \rho_s r_s^3 = g(c)M_v$, with $g(c) = [\ln(1+c) - c/(1+c)]^{-1}$. From this the rotational velocity of a test mass follows as a consequence of Birkhoff theorem

$$v^2(r) = \frac{GM(< r)}{r} = \frac{GM_s}{r_s} \left[\frac{\ln(1 + r/r_s)}{r/r_s} - \frac{1}{1 + r/r_s} \right]. \quad (1.7)$$

At small radii, this implies $v(r) \propto r^{1/2}$. However, observations of rotation curves (RCs) in dark-matter-dominated dwarf galaxies show a different behavior: they are consistent with solid-body rotation, $v(r) \propto r$ in the inner regions. Under the assumption of Birkhoff's theorem, this corresponds to an approximately constant central density, i.e., a core, in stark contrast with the cuspy inner profile predicted by CDM N-body simulations [34, 35, 36, 37].

Missing satellites and too-big-to-fail problems The highest-resolution cosmological simulations of Milky Way (MW)-sized halos in the Λ CDM paradigm have demonstrated that

dark matter clumps exist at all resolved mass scales, with no break in the subhalo mass function down to the numerical convergence limit. One would therefore expect that any MW-sized halo hosts hundreds to thousands of subhalos spanning a wide range of masses. However, the number of observed dwarf satellite galaxies is orders of magnitude lower than the number of subhalos predicted by simulations. This discrepancy can be explained by the fact that dark matter halos become increasingly inefficient at forming galaxies at low masses. Processes such as supernova feedback, reionization heating, and tidal stripping suppress galaxy formation within smaller subhalos. When these effects are taken into account, the tension between simulations and observations regarding the number of satellites is significantly alleviated, with some studies even reporting more observed satellites than predicted by simulations.[38]. A natural solution to the missing satellites problem within Λ CDM is therefore to associate the known Milky Way satellites with the largest dark matter subhalos (where “largest” refers to either present-day mass or peak mass), while attributing the absence of observable galaxies in smaller subhalos to galaxy formation physics. As pointed out by Boylan-Kolchin, Bullock and Kaplinghat [39], this solution leads to a testable prediction: the inferred central masses of Milky Way satellites should be consistent with those of the most massive subhalos in Λ CDM simulations of MW-mass halos. However, comparisons between observed central masses and Λ CDM predictions from the Aquarius [40] and Via Lactea II [41] simulations reveal a discrepancy: the most massive Λ CDM subhalos are systematically too centrally dense to host the brightest Milky Way satellites. Although some subhalos have central masses comparable to those of the observed satellites, these are not among the most massive ones. This raises an important question: why would galaxies fail to form in the most massive subhalos, yet successfully form in lower-mass dark matter satellites? This puzzle is known as the “too big to fail” problem.

Relations between dark and luminous matter The emergence of tight empirical relations between properties of the dark and luminous components in disk-dominated galaxies, such as the radial acceleration relation (RAR) [42, 43, 44], the universal core surface density [45] and the scaling between the core radius and the disk scale-length [46] (more on these in Chapter 3) are puzzling and seem to be indicative of a new dark sector and/or of non-gravitational coupling between DM particles and baryons [47]. Some of the above findings can in principle be explained in CDM by invoking physical processes causing transfer of energy and angular momentum from the baryons to DM particles, such as dynamical friction [48, 49] or feedback effects from stars and active galactic nuclei [50, 51], but in recent years it has become progressively clearer that a fine-tuning for these solutions is required to explain in detail the current observations.

This small scales ‘crisis’ has triggered the consideration of non-standard particle candidates [52, 53];

Hot Dark Matter Hot dark matter (HDM) refers to a dark-matter component composed of light particles that were relativistic or semi-relativistic during the epoch when cosmological structures began to form. Thermal relic neutrinos provide the canonical example, while thermal axions with masses of order $\mathcal{O}(1\text{ eV})$ have also been extensively studied as a possible HDM component [54, 55, 56, 57]. More recently, light or massive dark photons, associated with a hidden or dark U(1) gauge symmetry, have been considered as additional candidates for a hot relic component [58]. The large thermal velocities of HDM particles allow them to propagate over cosmological distances and erase density perturbations below their characteristic free-streaming scale. Consequently, free streaming suppresses the growth of matter perturbations on sufficiently small scales, reducing the efficiency with which HDM can cluster within galactic-sized potential wells. This scale-dependent suppression implies that HDM cannot constitute the dominant component of dark matter in the Universe while simultaneously reproducing the observed abundance of small-scale cosmic structure. The same physics is reflected in the cosmic microwave background (CMB) and large-scale structure observations, which place stringent constraints on the abundance and mass of light thermal relics. These considerations have motivated cosmological scenarios in which the hot component constitutes only a subdominant fraction of the total dark matter, with the remainder provided by a colder component capable of efficiently clustering on galactic and smaller scales [59, 60].

Warm Dark Matter Warm dark matter (WDM) provides an intermediate scenario between the cold- and hot-dark-matter paradigms. In contrast to cold dark matter, WDM particles possess a non-negligible primordial velocity dispersion, while their free-streaming scale is substantially smaller than that of HDM. This allows WDM to retain the successful large-scale predictions of CDM while suppressing the formation of structure below a characteristic scale. In particular, keV-scale particles can generate sufficient free streaming to modify the abundance and internal properties of low-mass halos, making WDM an interesting framework for addressing some of the small-scale challenges associated with CDM. A wide variety of WDM candidates and production mechanisms have been proposed, differing in their decoupling histories, interactions, and subsequent decay processes. Among the most extensively studied candidates are sterile neutrinos, which are spin-1/2 particles that can be produced through non-thermal mechanisms and may never have reached thermal equilibrium with the primordial plasma [61, 62]. Their cosmological properties depend strongly on the production mechanism, and therefore no unique sterile-neutrino mass can be assigned to WDM in general. Nevertheless, sterile-neutrino models with masses in the $\mathcal{O}(\text{keV})$ range have historically provided viable WDM scenarios and have been studied extensively using cosmological simulations and observations [63, 64]. Other proposed WDM candidates include axinos, the fermionic superpartners of axions, which can acquire suitable relic abundances through thermal or non-thermal production mechanisms [65, 66, 67, 68].

Gravitinos, spin-3/2 particles arising naturally in supersymmetric extensions of the Standard Model, provide another possible realization of WDM, although their viable parameter space is strongly constrained by cosmological considerations such as structure formation and Big Bang nucleosynthesis [69, 70, 71, 72, 73, 74, 75, 76, 77]. More generally, WDM can also be realized by non-supersymmetric particles with appropriately suppressed primordial velocities. The finite velocity dispersion of WDM particles leads to free streaming that erases density perturbations below a characteristic scale. Consequently, WDM suppresses the formation of low-mass halos relative to CDM, while leaving sufficiently large scales comparatively unaffected. This scale-dependent suppression can alleviate some of the small-scale tensions historically associated with CDM, including the core-cusp problem, the abundance of low-mass satellites, and the “too-big-to-fail” problem [78, 79]. However, whether WDM provides a satisfactory resolution of these tensions depends on the particle-physics model and its associated phase-space distribution, and not simply on the particle mass. The same free-streaming mechanism that makes WDM phenomenologically attractive also provides a powerful means of constraining it. Observations of the abundance of satellite galaxies, strong gravitational lensing, stellar streams, and the Lyman- α forest probe progressively smaller cosmological scales and therefore place stringent lower limits on the WDM particle mass [80, 81, 82]. In particular, analyses combining strong-lensing systems and galaxy-count information have obtained lower bounds in the multi-keV regime, with limits reaching approximately 9.8 keV at the 95% confidence level in some analyses [83]. Combining constraints from stellar streams, galaxy counts, and strong lensing can yield even stronger limits, with lower bounds of order 10 keV reported in the literature [84]. These increasingly stringent constraints indicate that the $\sim 1\text{--}2$ keV mass range, once regarded as particularly interesting for alleviating CDM small-scale tensions, is strongly disfavored for several commonly considered WDM production scenarios. Consequently, current WDM phenomenology must simultaneously satisfy small-scale structure constraints and the requirements imposed by the relic-production mechanism.

Self-Interacting Dark Matter The standard cold dark matter (CDM) paradigm assumes that dark matter (DM) is non-relativistic and effectively collisionless on cosmological and galactic scales, interacting primarily through gravity. An important extension is the possibility that DM particles experience non-gravitational self-interactions mediated by a new dark-sector force. This scenario, known as self-interacting dark matter (SIDM), can modify the internal structure and dynamical evolution of dark matter haloes while leaving the large-scale evolution of density perturbations essentially unchanged in the simplest models [85, 86]. Self-interactions facilitate the transfer of energy and momentum between particles and can therefore substantially alter the density profiles and shapes of galactic and sub-galactic haloes. As a halo evolves, elastic self-scattering transports energy through the system and tends to thermalize

the inner region. This process can produce an approximately isothermal central region with a finite-density core, while the outer halo remains comparatively unaffected and can retain a profile close to the standard NFW form [87, 88]. The formation of cores provides one of the principal motivations for SIDM, as observations of dwarf and low-surface-brightness galaxies have revealed a diversity of inner density profiles that is not straightforwardly reproduced by collisionless, dark-matter-only simulations [89, 90, 91]. SIDM has also been investigated as a possible mechanism for reducing the abundance and central densities of satellite subhaloes. Self-scattering can facilitate the evaporation or tidal disruption of subhaloes, particularly in the inner regions of a host halo, although the extent to which SIDM changes the subhalo abundance depends strongly on the velocity dependence of the self-interaction cross section [85, 92, 93]. The phenomenology of SIDM is particularly relevant to the observed diversity of galaxy rotation curves. Measurements of dwarf and low-mass galaxies indicate a broad range of inner density slopes, including systems consistent with approximately constant-density cores as well as galaxies with substantially steeper central profiles [89, 90, 91]. Such diversity can arise naturally in SIDM because the degree of thermalization depends on the halo mass, concentration, assembly history, and velocity-dependent scattering rate. This provides a possible explanation for variations in inner halo structure without requiring the baryonic processes associated with star formation and feedback to account for the entire observed diversity [94]. Nevertheless, baryonic effects remain important, and the observed rotation curves do not uniquely establish the presence of DM self-interactions. In the simplest SIDM scenario, in which DM interacts only with itself through elastic scattering and remains decoupled from additional relativistic species, the linear evolution of large-scale density perturbations is essentially unchanged from CDM. Because elastic self-scattering conserves energy and momentum, it primarily redistributes momentum within the dark sector rather than directly modifying the background evolution or the linear matter power spectrum on sufficiently large scales [86]. More general dark-sector models, however, introduce interactions between DM and a relativistic or effectively massless species, commonly referred to as dark radiation. In such scenarios, the dark matter and dark radiation can form a tightly coupled fluid in the early Universe, giving rise to dark acoustic oscillations (DAOs) and collisional damping in the matter power spectrum [86, 95]. The resulting suppression and oscillatory features can extend over scales relevant to the formation of low-mass haloes, providing a connection between particle-physics interactions in the dark sector and observable small-scale structure. Before kinetic decoupling, momentum exchange between DM and the relativistic species prevents the DM from freely clustering below the corresponding acoustic and diffusion scales. As the momentum-transfer rate falls below the Hubble expansion rate, DM kinetically decouples from the relativistic component. The temperature of kinetic decoupling therefore determines the characteristic scale at which the matter power spectrum begins to

deviate from the standard CDM prediction [96]. The resulting small-scale suppression can be substantially stronger than that generated by elastic DM self-interactions alone, and its detailed shape depends on the mediator properties, interaction strength, abundance of dark radiation, and thermal history of the dark sector [95, 96]. Consequently, SIDM models with dark radiation provide a broader framework in which halo-scale self-interactions and early-Universe modifications to the matter power spectrum can be studied within a common particle-physics description.

Other captivating possibilities include fuzzy or particle-wave DM [97, 98], or dark photon-dark matter [99, 100]. All these possibilities, despite being still viable to some extent, have been strongly constrained by various astrophysical probes, most noticeably Lyman- α forest [81], high-redshift galaxy counts [101], cosmic reionization [102], gravitational lensing [103], integrated 21 cm data [104], γ -ray emission [105], fossil records of the Local Group [106], dwarf galaxy profiles and scaling relations [107], Milky Way satellite galaxies [108], and cosmic star formation rate density [109], or a combination of these.

1.3 Modified gravity as an alternative to dark matter

Despite all the indirect probes of its existence and the big efforts carried on with colliders [110, 111] or with direct [112, 113] and indirect [114, 115] searches in the sky, still no firm detection of DM particles has been made so far. This lack of detection has motivated, on different grounds, the emergence of modified gravity theories, which aim to explain the observed phenomena not via a dark particle or dark sector, but via a departure of the gravitational interaction from the Newtonian expression at small or large scales. Perhaps the most famous of these alternatives is constituted by modified Newtonian dynamics (MOND), originally proposed by Milgrom in 1983 [116] and further investigated in a rich literature [117, 118, 119]. As the name suggests, MOND aims to explain galactic dynamics through a modification of Newtonian gravity (or more generally of the Newton second law) that comes into action at accelerations well below a definite universal threshold $a_0 \simeq 1.2 \times 10^{-10} \text{ m/s}^2$. More specifically, in the context of MOND is assumed that below a_0 the Poisson equation is modified

$$\nabla \cdot \left(\mu \left(\frac{|\nabla \Phi|}{a_0} \right) \nabla \Phi \right) = 4\pi G \rho; \quad (1.8)$$

here ρ is the baryonic mass density, and $\mu(x)$ is an interpolating function whose asymptotic values are 1 for $x \gg 1$, and x as $x \ll 1$. Thus, one recovers Newtonian gravity and the deep MONDian regime at accelerations well above and below a_0 respectively. In the case of a point-like particle the low-acceleration limit yields a gravitational field $g = \sqrt{GMa_0}/r$. In this way MOND is able to recover the saturation of the RC of spiral galaxies at large radii without including a DM

component, with only baryons sourcing the gravitational field. In particular MOND (or theories reducing to it in the weak-field limit) can properly fit galactic RCs of low-surface-brightness (LSB) spirals, and most dwarf galaxies, even at small radii, where CDM requires baryonic feedback [120, 121]. Furthermore MOND provides a satisfying description of the RAR [122], and the tight empirical relations observed between the galaxy kinematics and the baryonic mass (which, in CDM are interpreted as to link the dark and luminous components, as explained in Section 1.2) emerge naturally within the theory. Remarkably, MOND predicts a relation between the asymptotic circular velocity of a spiral galaxy and its baryonic mass $v_c^4 = GMa_0$, known as the baryonic Tully-Fisher relation, which has been verified with very small scatter [123, 124]. On the other hand, the performances of MOND are highly debated when moving to larger scales. In galaxy clusters the modification of Newtonian gravity at low accelerations does not fully remove the discrepancy between the visible baryonic mass and the required dynamical mass, leaving a factor of 2-5, depending on the cluster. The leftover discrepancy can be inferred from galaxy velocities, X-ray gas temperature (which, assuming hydrostatic equilibrium, is directly linked to the cluster mass), and gravitational lensing of background galaxies. The situation worsens on even larger cosmological scales, where relativistic extensions of MOND, most notably TeVeS, [117] struggle in reproducing the growth of perturbations, the CMB anisotropies' power spectrum, and the amplitude, shape, and correlation statistics of the baryon acoustic oscillations (BAO) [125, 126, 127, 128, 129, 130]. Remarkably, some of these issues can be addressed in the context of MOND with the introduction of additional components, such as the sterile neutrinos or generic extra fields. Since the reason to consider MOND in the first place was to get rid of additional matter beside the luminous one, this weakens MOND's original simplicity advantage.

1.4 Non-standard couplings between Dark Matter and gravity

Given the successes of MOND at galactic scales and of Λ CDM on larger, cosmological scales, one would be tempted to construct a model that strikes an intermediate course between the two paradigms. An intriguing possibility is to envisage that, under particular conditions, realized in virialized structures and at late times, CDM couples non-minimally with gravity. In the presence of this coupling, the gravitational potential generated by a given dark matter distribution gets modified, thereby affecting the dynamics of visible matter. In this way a MONDian behavior could be enforced, and the empirical relationships between dark and visible matter would emerge naturally.

A natural way to implement the non-minimal coupling between dark matter and gravity at the level of the action, is via a disformal transformation between physical and gravitational

metric, the former being the one whose geodesics are followed by free falling matter, while the latter describing the dynamics of the gravitational field. In General Relativity the two are assumed to be the same, but in a generic theory this is not necessarily true. In these cases the physical metric is constructed from the gravitational one with the introduction of additional fields. In [131] Bekenstein showed that the most general transformation between the two metrics which preserves the Weak Equivalence Principle (WEP) and causality is the disformal transformation

$$\tilde{g}_{\mu\nu} = e^{2\phi} (\mathcal{A}(\chi)g_{\mu\nu} + \mathcal{B}(\chi)\nabla_\mu\phi\nabla_\nu\phi) , \quad (1.9)$$

where ϕ is an additional field, $\chi = -\frac{1}{2}g_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$, and $\mathcal{A}, \mathcal{B}$ are two functions on which causality, i.e. the existence of light cones, imposes constraints.

Consider an action involving both a gravitational and a matter Lagrangian. If we express them in term of the physical metric, the so-called (*Jordan frame*), then the extra field ϕ gets non-minimally coupled to gravity, while the matter fields are minimally coupled. On the other hand if we express both parts with respect to the gravitational metric (*Einstein frame*), then the metric action for gravity takes the standard Einstein-Hilbert form, while matter gets coupled to the extra field. This suggest to identify the extra field ϕ with dark matter, so that it gets non-minimally coupled with gravity in the Jordan frame. Viewed in the Einstein frame this then translates to a coupling between dark and luminous matter. To implement such a model, Bruneton et al. [132] started from an action of the form

$$S = S_{\text{EH}}[g] + S_{\text{MAT}}[g, \psi] + S_{\text{DM}}[g, \phi] + S_{\text{INT}}[g, \phi, \psi] , \quad (1.10)$$

where the ψ and ϕ are thought as collective variables encoding baryons and DM particles, respectively. To mimic a MONDian behavior at galactic scales, the coupling should be able to produce an extra-force that applies to baryons only. This can be achieved assuming that the interaction term is such that

$$S_{\text{MAT}}[g, \psi] + S_{\text{INT}}[g, \phi, \psi] \simeq S_{\text{MAT}}[g + h, \psi] , \quad (1.11)$$

where the net effect of the interaction is summarized into a modification of the physical metric felt by the baryons $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + h_{\mu\nu}$. Up to order $O(h^2)$, the full action can be expressed in terms of $\tilde{g}_{\mu\nu}$, and we get that the DM field gets non-minimally coupled to gravity in the following way

$$S = S_{\text{EH}}[\tilde{g}] + S_{\text{MAT}}[\tilde{g}, \psi] + S_{\text{DM}}[\tilde{g}, \phi] - \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} (G^{\mu\nu}(\tilde{g}) + \tilde{g}^{\mu\nu}\Lambda) h_{\mu\nu} . \quad (1.12)$$

The detailed shape of $h_{\mu\nu}$ can then be fixed by the requirement to mimic a MONDian behavior at the scales of interest.

With the form of $h_{\mu\nu}$ tuned in this way, the model is able to reduce to GR+CDM in the early Universe and in the center of galaxy clusters, and to MOND in regions of low accelerations, thus fully reconciling the two frameworks. Nevertheless, it suffers some ambiguities and interpretational shortcomings. First of all, the model has been formulated in terms of scalar fields, while a description in terms of fluid variables seems more appropriate to describe gravitational phenomena at the galactic and cosmological level, provided that the averaging scale necessary to define the fluid variables is small enough, compared to the size of the structures that we want to analyze. Moreover, a non-minimal coupling between dark matter and gravity seems more easily justifiable at the fluid level, since it does not translate directly into a violation of the WEP. Indeed, as long as single particles have standard coupling with gravity, the WEP is safe, while it can be violated at the level of the collective behavior of the fluid. Finally, the needed interaction term must be carefully designed, at the field-theoretic level, in a way that seems very counterintuitive.

At the fluid level there are two main mechanisms that may produce a non-minimal coupling: either it appears through an averaging procedure that brings from particles to fluids, or it can emerge from some collective behavior of the DM particles, as in Bose Einstein Condensates (BEC).

In an attempt to solve these issues, Bettoni and collaborators [1, 2, 133, 134] considered a model in which DM is described as a perfect fluid with a barotropic equation of state which couples non minimally with gravity. Working with perfect fluids, there are no many possibilities to couple DM to gravity, given that we have at our disposal only scalars and the four velocity at each spacetime point. If we add the constraint that we want to keep the gravitational field equations of second order in the metric tensor, so that we do not introduce additional physical modes for the graviton, there are only two possible couplings, namely a conformal one $R\psi(\rho)$ and a disformal one $R_{\mu\nu}\xi(\rho)u^\mu u^\nu$. The action of the non-minimally coupled fluid is thus

$$S = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R (1 + \alpha_{\text{Scal}} \psi(\rho)) + \frac{\alpha_{\text{Ric}} c^3}{16\pi G} \int d^4x \sqrt{-g} R_{\mu\nu} \xi(\rho) u^\mu u^\nu + S_{\text{DM}}. \quad (1.13)$$

Here

$$S_{\text{DM}} = -2c \int d^4x \left[\sqrt{-g} \rho(n, s) + J^\mu \left(\partial_\mu \phi + s \partial_\mu \theta + \beta_A \partial_\mu \alpha^A \right) \right] \quad (1.14)$$

is the DM perfect fluid action, where n is the particle number density, s is the entropy per particle, and the second term implements the constraints for the flow of perfect fluid through the four vector $J^\mu = \sqrt{-g} n u^\mu$. The function $\psi(\rho)$ controls the coupling of the dark fluid to the Ricci scalar, while the function $\xi(\rho)$ mediates the coupling to the Ricci tensor. Both these

functions are dimensionless, and hence they must involve, for dimensional reasons, at least another density parameter ρ_* which sets a characteristic, phenomenological, scale of the model. Finally, the dimensionless constants α_{Scal} , α_{Ric} control the strength of the non-minimal coupling. Performing the variation of the action and taking the static, non-relativistic limit, the modified Poisson equation for the Newtonian gravitational field can be derived

$$\Delta\Phi_N = 4\pi G \left(\rho - \frac{\alpha_{\text{Ric}}}{2} \rho_{\text{fid}} \Delta\tilde{\xi} + \alpha_{\text{Scal}} \rho_{\text{fid}} \Delta\tilde{\psi} \right), \quad (1.15)$$

where $\tilde{\xi}, \tilde{\psi}$ are rescaled quantities, and ρ_{fid} is a fiducial density defined such that $\tilde{\psi}(\rho_{\text{fid}}) = 1$. The main feature of this equation is that the gravitational potential has as sources not only the mass density ρ , but also its variations in space; this means that inhomogeneities in the dark matter distribution can enhance or suppress the gravitational interaction, depending on their sign. Furthermore, in light of the fact that the operative definition of the dark matter mass density is given in terms of the quantity that appears on the right hand side of the equations of motion for the gravitational field (either the full relativistic field equations, or the Poisson equation in the Newtonian limit), one might define an effective mass density and effective stresses, that do not coincide with those that are defined out of the fluid action when the non-minimal coupling is absent. In particular, the right hand side of the Einstein equations in the full relativistic regime define an effective Stress Energy Tensor (SET) for the dark matter fluid depending on higher derivatives of the fluid variables and on curvature. This is indeed what should be expected, given that the basic idea of non-minimal coupling is that the field, or fluid in this case, is able to probe geometry on a given length scale, not only point-wise as in the standard case. Upon contraction of the effective SET, the additional terms can be interpreted as an isotropic and anisotropic pressure, the latter being present only in the case of conformal coupling. Given that we started with the action for a pressureless fluid, this has to be considered to be dynamically generated by the non-minimal coupling between dark matter and gravity.

Following the line of Bettoni, in a series of papers [135, 4, 136] Gandolfi and collaborators developed a model of non-minimal coupling whose action features an interaction term of the form $S_{\text{INT}}[g, \phi, \psi] \propto \int d^4x \sqrt{-g} T_{\text{bar}}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$, where $T_{\text{bar}}^{\mu\nu}$ is the baryons stress-energy tensor, while ϕ represents dark matter. Expressing the full action in terms of the physical metric, the DM field gets non-minimally coupled to gravity via a term of the form

$$S_{\text{NMC}}[\tilde{g}, \phi] = \epsilon L^2 \int d^4x \sqrt{-\tilde{g}} \tilde{G}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi. \quad (1.16)$$

Here L is a coupling length that must be introduced for dimensional consistency, while $\epsilon = \pm 1$ is a polarity parameter of the coupling, whose sign is indeterminated a priori and must be fixed

by requiring consistency of the solution of the field equations. By taking the Newtonian limit of the field equations derived from the full action of the model, and adopting a representation in terms of fluid variables, in the end one obtains a modified Poisson equation for the Newtonian gravitational field

$$\Delta\phi = 4\pi G \left(\rho_{\text{bar}} + \rho_{\text{dm}} - \epsilon L^2 \Delta\rho_{\text{dm}} \right). \quad (1.17)$$

As was the case in (1.15), this modified Poisson equation implies that the source for gravity is not just the total matter density in itself, but also the DM inhomogeneities or spatial variations.

Gandolfi and collaborators studied the self-gravitating equilibria of non-minimally coupled dark matter halos, solving (1.17) together with the Euler and continuity equations, which are not modified by the non-minimal coupling in the Newtonian limit. To close the system, they considered a power law equation of state $p(\rho, r) = \rho_\alpha \sigma_\alpha^2 \left(\frac{\rho}{\rho_\alpha} \right)^\Gamma \left(\frac{r}{r_\alpha} \right)^\alpha$, with $\Gamma = 5/3$, and $\alpha = 35/27$, appropriated for the gravitational assembly of DM halos via accretion and mergers from the cosmic web. Looking for power law density profiles $\rho \propto r^{-\gamma}$, they found that, to have physical solutions, ϵ must be negative, otherwise the inner profile diverges at a finite radius. Then, parametrizing the strength of the NMC with $\eta \equiv \epsilon \frac{L^2}{r_\alpha^2}$, for any negative value of η , there is an optimal value of $\kappa \equiv 4\pi G \rho_\alpha r_\alpha^2 / \sigma_\alpha^2$, such that the inner profile is physical, with limiting central slope $\gamma_0 = 0$, i.e. a core. As for the outer behavior, the profile has a cutoff at a finite radius, setting the effective halo boundary. In particular, for $-0.1 < \eta < -0.01$, the shape of the NMC solution out to several r_{-2} (the radius at which $d \log \rho / d \log r = -2$) follows closely the Burkert profile, commonly exploited to fit observations of dwarf galaxies. At larger distances there is a progressive deviation from the Burkert profile, since the latter has been designed to have a limiting slope close to the NFW one, while the NMC solution gets truncated at a finite radius. However, for an appreciable range of η values this is not a concern, since the truncation occurs at radii much beyond r_{-2} that are scantily probed by observations. To reproduce the Burkert profile over an extended radial range outward of r_{-2} is a remarkable property of the NMC solution, since other models, such as those based on Bose-Einstein condensate DM, predict a cored density profile, but the deviation from the Burkert shape are appreciable, in terms of a prominent cutoff or steep decline, just outside the core.

Even more remarkably, when tested again stacked rotation curves of local spiral galaxies with different average velocities at the optical radius, including dwarfs and low-surface brightness systems, the NMC performs way better than NFW, and comparable to (in some instances even better than) the phenomenological Burkert profile, in terms of χ^2 . Furthermore, to fit the rotation curves, the model predicts a scaling of the NMC length scale with the mass of the halo $L \propto M_v^{0.8}$, suggesting that the NMC may emerge as the result of the collective behavior of the dark matter particles confined in a virialized system.

As two further tests Gandolfi et al. considered the universal core surface density and the

Radial Acceleration Relation. The former, defined as the product of the core density and the core radius $\Sigma_0 \equiv \rho_0 \times r_0$, as stated in Section 1.2, is found to have an almost constant value $\Sigma_0 \simeq 75_{-45}^{+55} M_\odot \text{ pc}^{-2}$ among different dwarf galaxies with halo mass $M \leq 10^{11} M_\odot$. In the NMC model, one finds

$$\Sigma_0 \simeq 50 \left(\frac{\Delta_{vir}}{100} \right) E_z^{0.3} M_\odot \text{ pc}^{-2}, \quad (1.18)$$

where Δ_{vir} is the nonlinear threshold for virialization, with values around 100 at $z \simeq 0$, and $E_z \equiv \Omega_m(1+z)^3 + \Omega_\Lambda$ takes into account the formation redshift of the halo. Thus, the surface density turns out to be independent on the halo mass and core radius, and only weakly dependent on formation redshift. As for the RAR, it is found that the NMC can reproduce the observed relation between baryonic and total accelerations at all scales, when extrapolating down to smaller masses the dependence $L \propto M_v^{0.8}$ found from the RC analysis.

1.5 This thesis

This thesis is devoted to the study of two models involving non-standard interactions between cold dark matter and gravity, and of their phenomenological implications for galaxy kinematics and cosmology.

Since our focus is on phenomenology, we remain agnostic about the particle nature of dark matter and adopt a fluid description. In particular, we model cold dark matter as a pressureless fluid (dust), characterized solely by its energy density ρ and four-velocity u^μ . Deviations from the standard cold dark matter paradigm are then encoded in modifications to these fluid variables, together with the emergence of a non-vanishing pressure.

We also leave open, for the time being, the question of the physical mechanism underlying the emergence of these non-standard interactions. However, our analysis shows that, at least in the first model, they are triggered by the confinement of dark matter within virialized structures, pointing towards a possible collective behaviour of the dark matter fluid.

The first framework is a non-local theory, dubbed fractional gravity (FG), relying on fractional calculus, the branch of mathematics dealing with non-integer derivatives and integrals. In [137] Giusti proposed a theory of modified gravity in which the Laplacian operator appearing in the Poisson equation was promoted to a Fractional Laplacian, resulting in what he named fractional Poisson equation (FPE)

$$(-\Delta)^s \Phi(x) = -4\pi G \ell^{2-2s} \rho(x). \quad (1.19)$$

Here $s \in [1, 3/2)$ is the fractional index, while ℓ is a length scale that must be introduced to recover the right dimensionality.

The peculiar feature of this equation is that the appearing of a non-integer derivative operator introduces non-locality in the theory, meaning that the gravitational potential at a space time point has as a source the density distribution not only at the same point, but in an extended domain. In its original formulation, the fractional interaction was intended to apply only to baryons, as the theory was developed to reproduce MOND-like behavior in galaxies and thereby eliminate the need for dark matter. In this work, however, we adopt a fundamentally different approach by assigning the fractional gravitational coupling to the dark matter component, while baryons obey the standard Poisson equation. Given the stringent constraints on the coupling between visible matter and gravity imposed by Solar System experiments, together with the strong astrophysical and cosmological evidence supporting the existence of dark matter, we argue that this is the more appropriate framework to work in.

To assist readers who may not be fully familiar with fractional calculus, in Chapter 2 we provide an introduction to the subject—by no means exhaustive—with particular emphasis on the fractional Laplacian, its operatorial definitions, and its main properties.

Chapter 3 and 4 are devoted to Newtonian fractional gravity (NFG) and its application in galaxies and clusters. In particular in Chapter 3 we compute the gravitational potential, solution to (1.19) sourced by the NFW density distribution (1.2) for a generic value of the fractional exponent s . We show that the rotation curve derived from this potential would appear, to an observer interpreting the data within the framework of standard Newtonian gravity, as the one produced by an almost cored density profile, thereby naturally resolving the cusp–core problem of cold dark matter. We then fit stacked Rotation Curves of spiral galaxies of different types, from dwarfs, to high and low surface brightness spirals, via a Bayesian MCMC technique, so inferring relevant galaxy properties and at the same time constraining the parameters ruling the strength and length-scale of fractional gravity. As an independent test of NFG in late-type galaxies, we show that the empirical relations observed in spirals between the parameters of the baryonic disk and the DM halo, when the latter is modeled with a NFW or Burkert profile, emerge naturally as a consequence of the non locality of the theory. In Chapter 4 we move to much smaller and larger scales, testing NFG in galaxy clusters and individual dwarfs. First we model the density and pressure distributions of the intracluster medium (ICM), working out the hydrostatic equilibrium equation in fractional gravity. Such theoretical framework is then compared with data from the XMM-Newton Cluster Outskirts Project (X-COP), which consists of 12 clusters with well-observed X-ray and SZ data. We then perform a Bayesian analysis of the thermodynamic profiles of the X-COP sample and infer constraints on the fractional gravity parameters at these scales. We then turn our attention to the smallest scales, where the effect of fractional gravity are expected to be more relevant, focusing on individual dwarf galaxies. Specifically, we probe NFG and infer the values of its parameters both in

irregular dwarf (dwIrr) galaxies by exploiting the rotation velocity profile inferred from stellar and gas kinematical measurements, and in dispersion-dominated dwarf spheroidals (dwSph) by exploiting the velocity dispersion profile inferred from the dynamics of stellar tracers. Our findings prove that NFG is able to reproduce the properties of virialized systems spanning ten orders of magnitude in dark matter mass, producing fits which are always superior (or at least comparable to) the ones obtain with standard cold dark matter.

Chapter 5 is probably the most technical of this work. Motivated by the successes of fractional gravity at the Newtonian level, we develop a full relativistic extension of the theory. This will enable us to study phenomena which are exclusive to the relativistic setting, such as propagation of gravitational waves and gravitational lensing. After a review of non-local gravity theories, we introduce the action of Relativistic Fractional Gravity (RFG) in its most general form, and derive the associated field equations. We then take the weak field and Newtonian limit, making the connection with the previous chapters, and discussing mode propagation via a scalar-vector-tensor decomposition analysis. Finally, we test two particular flavors of RFG to exploit the lensing convergence profiles of the individual and stacked clusters from the CLASH sample, finding a very good agreement with data.

The non-local nature of the interaction makes the full relativistic fractional gravity equations far too complex to be solved in the strong gravity regime. For this reason we devote the rest of this thesis to study a local model of non-minimal coupled dark matter (NMC-DM) whose action is inspired from the RFG one, but without the complications implied by non-locality.

In Chapter 6 we introduce the action of the model, both in the Jordan and in the Einstein frame, and we compute the field equations. We show that they can be interpreted as Einstein equations sourced by the stress-energy tensor of a fluid with both isotropic and anisotropic pressure, being dynamically generated by the coupling of pressureless dark matter with gravity. The appearing of the pressure offers the captivating possibility to form dark matter configurations in hydrostatic equilibrium. We thus solve the NMC-DM field equations taking as an ansatz a static, spherically symmetric, and asymptotically flat metric. We find that physical solutions exist for a limited range of the central density, taking the form of regular, horizonless ultra-compact objects of non-minimally coupled dark matter (NMC-UCOs). We then proceed to analyze the properties of NMC-UCOs, finding that they feature an almost constant density across their radius, a mass proportional to the length scale associated to NMC, and that, for a restricted range of central density, an internal and an external light ring. After studying the orbits of both massless and massive particles around NMC-UCOs, we implement a basic ray-tracing simulation to describe the gravitational lensing effects around these objects in terms of a pseudo-shadow, that makes connection with what is usually done in the context of black holes. If observed, the peculiar light pattern around NMC-UCOs could constitute a clear

signature for the detection of such exotic objects.

The last two chapters are devoted to cosmology in the context of NMC–DM. Since the non-minimal coupling is designed to become effective only when dark matter densities exceed a given threshold, its impact is expected to be most significant in the early Universe. Chapter 7 focuses on background cosmology. By solving the modified Friedmann, Raychaudhuri, and continuity equations for a Universe composed of baryons, non-minimally coupled cold dark matter, photons, and a cosmological constant, we show that cosmic evolution is indeed altered at early times. In particular, during the radiation-dominated era the Universe undergoes a phase of accelerated expansion, with the redshift at which this begins determined by the characteristic length scale of the model. Remarkably, adopting the same length scale that yields NMC-UCO masses compatible with observations of high-redshift supermassive black holes, we find that this accelerated phase starts before nucleosynthesis, thus preserving all established constraints on standard cosmic evolution. At even earlier times, the behavior of the background depends on the spatial curvature. For flat or positively curved Universes, the accelerated phase terminates in the past at a singularity, as in General Relativity, although with modified expansion rates. For negative curvature, however, the model admits the appealing possibility of a cosmological bounce, in which the Universe reaches a finite, non-zero minimum scale factor. Strikingly, using the same NMC length scale and imposing the curvature constraints derived by Planck Collaboration, we find that the accelerated phase lasts long enough to solve both the horizon and flatness problems—two of the central motivations for inflationary models.

For NMC–DM to provide a viable alternative to inflation, however, it must not only address the standard problems of background cosmology, but also correctly reproduce the evolution of cosmological perturbations. Chapter 8 is therefore devoted to a detailed analysis of linear scalar, vector, and tensor perturbations within NMC–DM. For scalar perturbations, we solve the perturbed field and conservation equations in terms of the comoving curvature perturbation, $(\mathcal{R})$, deriving its scaling with wavenumber across the deep NMC, radiation-dominated, and matter-dominated eras. This allows us to compute the scalar power spectrum. We find that only the bouncing cosmology scenario produces a spectrum compatible with observations of the CMB anisotropies by the Planck Collaboration. This compatibility, however, requires a modification of the standard initial conditions: instead of the usual Bunch–Davies vacuum, perturbations must originate from a highly populated quantum state. In the presence of a bounce, where perturbations are generated during a contracting matter- or radiation-dominated phase, this assumption is physically well motivated. We also study the evolution of vector and tensor perturbations (gravitational waves). Vector modes are enhanced only at very small wavenumbers; therefore, if their initial amplitude is small (for instance, of quantum origin), all observationally relevant modes remain negligible today. Tensor perturbations evolve similarly to

the comoving curvature perturbations outside the NMC-dominated phase, differing only in that they propagate at the speed of light. During the NMC-dominated contracting phase, however, the tensor solution develops an exponentially growing mode. To preserve the validity of the linear approximation, this mode must be removed. As a result, tensor perturbations reach the bounce with negligible amplitude, but the growing mode becomes dominant again during the subsequent expanding phase. Consequently, by the end of the NMC-dominated epoch, tensor modes recover approximately the same amplitude they had at its onset. The resulting tensor power spectrum has the same form as the scalar spectrum, but with an additional k^2 dependence. Achieving a scalar-to-tensor ratio consistent with the current observational bound, ($r < 0.036$), therefore requires appropriately tuning the initial conditions, such that tensor perturbations also begin in highly excited states with occupation numbers that decrease over time.

I

First part

Chapter 2

Fractional Calculus

Fractional calculus is a natural extension of classical calculus in which the concepts of differentiation and integration are generalized from integer orders to arbitrary real or complex orders. Although the idea of non-integer differentiation dates back to a correspondence between Leibniz and L'Hôpital in the late seventeenth century, it remained primarily a mathematical curiosity for many years. Over the past few decades, however, fractional calculus has evolved into a mature and rapidly expanding field, driven by both theoretical developments and its remarkable ability to model complex physical phenomena.

Unlike classical differential operators, fractional operators possess a non-local character: the fractional derivative of a function at a given point generally depends on its values over an extended domain rather than solely on its local behavior. This inherent memory effect makes fractional calculus particularly suitable for describing systems with hereditary properties, long-range interactions, and anomalous dynamics. As a result, fractional models have found successful applications in numerous disciplines, including viscoelasticity, diffusion processes, control theory, signal processing, bioengineering, finance, and others.

Since fractional calculus provides the mathematical framework underlying the theory of fractional gravity—the subject of the first part of this work—and because its concepts and techniques are not yet widely familiar within the astrophysics community, this chapter provides a brief introduction to the fundamentals of fractional calculus for the reader's convenience.

We begin by defining the Riemann-Liouville fractional integral and fractional derivative in one dimension, together with an other type of fractional derivative first introduced by Caputo, which proves to be more suitable to physics-related initial value problems. After having analyzed some of the properties of these objects, we move to the most relevant operator to us, that is the Fractional Laplacian $(-\Delta)^s$, defined on $\mathbb{R}^n$.

Several definitions of this operator, such as the singular integral representation, the heat semigroup definition, elliptic extension, the Balakrishnan formula, and the spectral represen-

tation, are presented. These turn out to be all equivalent in the common range of definition, but to differ in the values that can be assumed by the fractional index s . In particular, the spectral representation, which is based on the properties of the Laplace operator under Fourier transform, will turn out to be the most useful to our purposes, since the fractional Poisson equation is most easily solved in the Fourier domain.

To further extend fractional gravity to a relativistic setting we need to be able to treat space and time on an equal setting. This is achieved by the introduction of the fractional d'Alembertian $\square^s$. In Section 2.3 we will provide an introduction to this operator, highlighting the ambiguities in its definition and specifying the one used in this work.

2.1 Fractional integrals and derivatives

In one dimension, the simplest definition of fractional integral can be viewed as a natural extension of the well known formula, usually attributed to Cauchy, valid for any integer n

$$\mathcal{I}^n f(x) = \frac{1}{(n-1)!} \int_0^x dy f(y) (x-y)^{n-1}. \quad (2.1)$$

The above relates the n -fold primitive of a function $f(x)$ to a convolution integral, as it can be easily verified by repeated integration by parts, provided that f satisfies $f(x) = 0$ for $x < 0$. One is immediately led to extend the above formula for any positive real index s by means of the Euler gamma function $\Gamma(s) \equiv \int_0^\infty dx x^{s-1} e^{-x}$. In fact, since for any integer n , $\Gamma(n) = (n-1)!$, one can define the fractional integral of order s via the expression

$$\mathcal{I}^s f(x) \equiv \frac{1}{\Gamma(s)} \int_0^x dy f(y) (x-y)^{s-1}, \quad (2.2)$$

prescribing $\mathcal{I}^0 f(x) = f(x)$. The fractional integral enjoys the semi-group property $\mathcal{I}^s \mathcal{I}^p = \mathcal{I}^{s+p}$, which implies commutativity $\mathcal{I}^s \mathcal{I}^p = \mathcal{I}^p \mathcal{I}^s$, and the straightforward action on power function

$$\mathcal{I}^s x^a = \frac{\Gamma(a+1)}{\Gamma(a+s+1)} x^{a+s}. \quad (2.3)$$

Having defined the fractional integral, it is natural to assume that the fractional derivative could be defined as its inverse operation; however, this idea must be implemented with some care. In particular, note that even for integer n the differentiation $\mathcal{D}^n$ and integration $\mathcal{I}^n$ operators in general do not commute, as $\mathcal{D}^n$ is only a left-inverse of $\mathcal{I}^n$; in fact, $\mathcal{D}^n \mathcal{I}^n f(x) = f(x)$ holds but the converse is not necessarily true due to the appearance of numerous integration constants, related to the value of the function and its derivatives at the boundary points. Therefore, given an integer m one defines the *Riemann-Liouville* fractional derivative $\mathcal{D}_{\text{RL}}^s$ of order $s \in (m-1, m)$

in such a way that $\mathcal{D}_{\text{RL}}^s \equiv \mathcal{D}^m \mathcal{I}^{m-s}$, or more explicitly

$$\mathcal{D}_{\text{RL}}^s f(x) \equiv \frac{d^m}{dx^m} \left[\frac{1}{\Gamma(m-s)} \int_0^x dy f(y) (x-y)^{m-s-1} \right], \quad (2.4)$$

with $\mathcal{D}_{\text{RL}}^0 f(x) = f(x)$. Using this definition and the semi-group property, one has $\mathcal{D}_{\text{RL}}^s \mathcal{I}^s f(x) = \mathcal{D}^m \mathcal{I}^{m-s} \mathcal{I}^s f(x) = \mathcal{D}^m \mathcal{I}^m f(x) = f(x)$ so that $\mathcal{D}_{\text{RL}}^s$ is truly a left inverse of $\mathcal{I}^s$. It is also trivial to verify the action on power functions

$$\mathcal{D}_{\text{RL}}^s x^a = \frac{\Gamma(a+1)}{\Gamma(a-s+1)} x^{a-s}, \quad (2.5)$$

and on constants

$$\mathcal{D}_{\text{RL}}^s y_0 = \frac{y_0}{\Gamma(1-s)} x^{-s}. \quad (2.6)$$

The definition of fractional derivative implies a inherently non-local behavior, since $\mathcal{D}_{\text{RL}}^s f(x)$ depends not only on the values of the argument in an infinitesimal neighborhood around x , as in standard calculus, but in an extended domain. When considering functions of time, such in fractional differential equations $\mathcal{D}_{\text{RL},t}^s x(t) = g(x(t), t)$, this translates into memory effects, meaning that the system stores memory of its past behavior. This is the case for example for the stress in viscoelastic materials, anomalous diffusion in turbulent fluids, or polarization in dielectric materials.

The fact that the Riemann-Liouville derivative does not yield zero when applied on constants motivates to look for an alternative definition of fractional derivative which has this desirable property. The Caputo Fractional Derivative, originally introduced by Caputo in [138], reverses the order of integration and differentiation

$$\mathcal{D}_c^s f(x) \equiv \frac{1}{\Gamma(m-s)} \int_0^x dy (x-y)^{m-s-1} \frac{d^m}{dy^m} f(y), \quad (2.7)$$

for $s \in (m-1, m)$. This definition is more restrictive than (2.4), in that it requires the absolute integrability of the derivative of order m of the function f .

In general, $\mathcal{D}_{\text{RL}}^s f(x) \neq \mathcal{D}_c^s f(x)$: in fact, assuming that the passage of the m -derivative under the integral sign is legitimate, it is easy to see that, for $s \in (m-1, m)$ and $x > 0$:

$$\mathcal{D}_c^s f(x) = \mathcal{D}_{\text{RL}}^s \left(f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!} \right), \quad (2.8)$$

so that the two definitions are equivalent only if the function f , along with its first $m-1$ derivatives, vanish as $x \rightarrow 0^+$. The subtraction of the Taylor polynomial of degree $m-1$ at 0^+ provides a regularization of the fractional derivative. One can deduce that if two functions f

and g admit the same Riemann-Liouville fractional derivative, then the following relation must hold

$$f(x) = g(x) + \sum_{k=1}^m c_k x^{s-k}, \quad (2.9)$$

while to share the same Caputo derivative

$$f(x) = g(x) + \sum_{k=1}^m c_k x^{m-k}. \quad (2.10)$$

It is therefore clear that the Caputo fractional derivative is more suitable for inclusion in fractional differential equations modeling physical phenomena, since the boundary conditions can be specified by assigning values to the physical variables and their integer-order derivatives. In contrast, when the Riemann–Liouville derivative is used, the boundary conditions must instead be expressed in terms of fractional integrals or fractional derivatives.

2.2 Fractional Laplacian

Having acquired familiarity with fractional integrals and derivatives in one dimension, we now move to higher dimensions, focusing on the fractional Laplace operator, thus providing the mathematical framework for fractional gravity.

The standard Laplacian Δ is a linear, second-order, elliptic, semi-positive, self-adjoint, local differential operator, whose applications in physics range from wave mechanics, gravity, electrodynamics, hydrodynamics, biophysics and probability theory. However, some physical phenomena have been experimentally shown to exhibit a non-local behavior, that has motivated the extension of the Laplacian to a fractional operator $(-\Delta)^s$. As it is usually the case for fractional operators, this extension can be performed in different ways, leading to several definitions of the fractional Laplacian. Although these definitions can be shown to be all equivalent on the whole $\mathbb{R}^n$, they differ when restricted to bounded domains $\Omega \subset \mathbb{R}^n$, and they usually cover different subsets of the interval $(0, n/2)$ for the values assumed by the index s , with some definition only well defined for $s \in (0, 1)$. Here we provide a brief presentation of the main definitions, starting with the one which is more suitable for our scopes. For an excellent comparative overview on these topics, treated with much more mathematical rigor, we refer the interested reader to [139].

2.2.1 Spectral definition

One of the simplest routes to attain a non-integer order extension to the Laplacian operator is by making use of its multiplicative action on the Fourier transform of a function. It is well

known that the standard Laplacian is a Fourier multiplier on the space $\mathcal{S}(\mathbb{R}^n)$ of Schwartz functions on $\mathbb{R}^n$, meaning that if $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\mathcal{F}\{-\Delta f\}(\mathbf{k}) = k^2 \mathcal{F}\{f\}(\mathbf{k}) , \quad (2.11)$$

where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform, which in our notation reads

$$\mathcal{F}\{f\}(\mathbf{k}) \equiv \int_{\mathbb{R}^n} d^n \mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) , \quad (2.12)$$

and hereafter $k \equiv |\mathbf{k}|$, and $x \equiv |\mathbf{x}|$. The fractional Laplacian of order $s \in (0, n/2)$ is then defined as the linear operator satisfying

$$\mathcal{F}\{(-\Delta)^s f\}(\mathbf{k}) = k^{2s} \mathcal{F}\{f\}(\mathbf{k}) . \quad (2.13)$$

Taking the inverse Fourier transform $\mathcal{F}^{-1}\{\cdot\}$ on both sides of (2.13) one obtains an explicit form for the fractional Laplacian in real space as

$$(-\Delta)^s f(\mathbf{x}) = \mathcal{F}^{-1}\{k^{2s} \mathcal{F}\{f\}(\mathbf{k})\}(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{d^n \mathbf{k}}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} k^{2s} \int_{\mathbb{R}^n} d^n \mathbf{y} e^{-i\mathbf{k} \cdot \mathbf{y}} f(\mathbf{y}) . \quad (2.14)$$

This approach can be made more rigorous by using the spectral theorem for densely-defined, self-adjoint linear operators on some Hilbert space $\mathcal{H}$, to find the unique projection-valued measure E_λ and the spectrum $\sigma(-\Delta)$, associated with the standard Laplacian, and from there construct the analogue counterparts for its fractional extension. We will not follow this very rigorous approach in this work, since the operative definition presented above is enough for our purposes, but we refer the reader to [140], in which one can also find a discussion on the larger functional spaces on which this definition is well defined.

2.2.2 Singular integral representation

For $s \in (-n/2, 0)$ the multiplier k^{2s} has an inverse Fourier transform in the sense of distributions

$$\mathcal{F}^{-1}\{k^{2s}\}(\mathbf{x}) = \frac{4^s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} |\Gamma(-s)|} x^{-(n+2s)} . \quad (2.15)$$

Using this and the convolution theorem of the inverse Fourier transform, one can find a representation of the fractional Laplacian as a single integral on position space

$$(-\Delta)^s f(\mathbf{x}) = \mathcal{F}^{-1}\{k^{2s}\} * f(\mathbf{x}) = \frac{4^s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} |\Gamma(-s)|} \int_{\mathbb{R}^n} d^n \mathbf{y} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}} . \quad (2.16)$$

For positive values of s , the above derivation fails because k^{2s} does not have an inverse Fourier transform, even in the sense of distributions. Moreover the representation (2.16) fails because the singularity encountered when $\mathbf{x} \rightarrow \mathbf{y}$ is no longer integrable. However, starting from (2.16), an argument based on analytical continuation in s , with f and $\mathbf{x}$ held fixed (see e.g. [141]), yields the following formula for the fractional Laplacian for $s \in (0, 1)$

$$(-\Delta)^s f(\mathbf{x}) = \frac{4^s \Gamma(\frac{n}{2} + s)}{\pi^{n/2} |\Gamma(-s)|} \text{PV} \int_{\mathbb{R}^n} d^n \mathbf{y} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}}, \quad (2.17)$$

where PV denotes the principal value

$$\text{PV} \int_{\mathbb{R}^n} d^n \mathbf{y} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}} \equiv \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n / B_\epsilon(\mathbf{x})} d^n \mathbf{y} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n+2s}}. \quad (2.18)$$

Here the integral is taken over all $\mathbb{R}^n$ minus the ball centered at $\mathbf{x}$ with radius ϵ . When one takes the limit as $\epsilon \rightarrow 0^+$, the difference $f(\mathbf{x}) - f(\mathbf{y})$ in the numerator, which vanishes at the singularity, and takes positive and negative values around it, allows the principal value to exist for smooth functions decaying sufficiently fast.

Note that, although this definition allows a representation of the fractional Laplacian as a single integral in position space, it is restricted to $s \in (0, 1)$. From (2.17), one can appreciate the non-local nature of the fractional Laplace operator: the value of the function $(-\Delta)^s f$ at a particular point $\mathbf{x}$ depends on the value of the f in all of space. Since there is no time propagation, as it is the case, for example, in the solution of the d'Alambert equation in $\mathbb{R}^n$, the dependence is instantaneous and thus $(-\Delta)^s$ has a strong non-local nature.

In analogy with what we have done with the 1D fractional derivative starting from the associated fractional integral, one may wonder whether there is an operator which acts like the inverse of the fractional Laplacian. This is provided by the Riesz potential $\mathcal{R}_s$

$$\mathcal{R}_s f(\mathbf{x}) \equiv \frac{\Gamma(\frac{n-s}{2})}{2^s \pi^{n/2} \Gamma(\frac{s}{2})} \int d^n \mathbf{y} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n-s}}. \quad (2.19)$$

The properties of this operator are discussed in details in [139]; here, we will limit to say that $\mathcal{R}_{2s}$ provides a sort of inverse of the fractional Laplacian, meaning that it can be shown to satisfy

$$(-\Delta)^s \mathcal{R}_{2s} f(\mathbf{x}) = \mathcal{R}_{2s} (-\Delta)^s f(\mathbf{x}) = f(\mathbf{x}) \quad (2.20)$$

so that one can naively set $\mathcal{R}_{2s} = (-\Delta)^{-s}$.

2.2.3 Elliptic Extension, Heat Semigroup and the Balakrishnan Formula

For completeness, we now describe three representations of the Fractional Laplacian in $\mathbb{R}^n$, which require the solution of equations involving the standard Laplacian.

Elliptic extension The extension method, also known as the Dirichlet-to-Neumann map, requires the solution of a degenerate elliptic equation in the $(n + 1)$ -dimensional half-plane, using the function $f(\mathbf{x})$ as the Dirichlet boundary data, followed by a type-of normal derivative of the solution. Let $f(\mathbf{x})$ be a function on $\mathbb{R}^n$, and consider its extension $\tilde{f}(\mathbf{x}, t)$ on $\mathbb{R}^n \times [0, \infty)$, solution of the following Cauchy problem with Dirichlet boundary conditions

$$\begin{cases} \frac{\partial^2 \tilde{f}}{\partial t^2} + \frac{1-2s}{t} \frac{\partial \tilde{f}}{\partial t} + \Delta_{\mathbf{x}} \tilde{f} = 0 \\ \tilde{f}(\mathbf{x}, 0) = f(\mathbf{x}) ; \end{cases} \quad (2.21)$$

then, there is a constant $c(n, s)$ such that

$$(-\Delta)^s f(\mathbf{x}) = c(n, s) \lim_{t \rightarrow 0^+} \frac{\tilde{f}(\mathbf{x}, t) - \tilde{f}(\mathbf{x}, 0)}{t^{2s}}. \quad (2.22)$$

One can interpret this as saying that the extension problem localizes the fractional Laplacian: to compute $(-\Delta)^s f(\mathbf{x})$ it is enough to know the extension $\tilde{f}(\mathbf{x}, t)$ in some upper half ball around the point $(\mathbf{x}, 0)$. This result has stimulated some interesting physical speculations. Indeed, it is possible to reconcile a local description of nature with the fact that some physical phenomena exhibit non-local behaviours, and thus they need to be described via non-local operators, if we assume that we live on a four-dimensional membrane in a larger five-dimensional spacetime. In this way, what appears to us as non-locality, is only the consequence of us not having access to the auxiliary dimension, playing the role of the variable t in the extension problem. Following this line of reasoning, A. Frassino and O. Panella in [142] have shown that the quantum behavior of a non-local scalar quantum field theory built out of the fractional Laplacian in n dimensions, can be described in terms of a local action in $n + 1$ dimensions, which can be quantized using the canonical formalism, though giving up local commutativity.

Heat semigroup representation Given a real number $\lambda \geq 0$ and $s \in (0, 1)$, the integral representation of the Euler Gamma function implies

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-t\lambda} - 1}{t^{s+1}} dt. \quad (2.23)$$

Now, suppose we have a linear, positive semidefinite operator L , with spectral representation $L = \int_0^\infty dE_\lambda \lambda$. Then the spectral representation of its fractional power $L^s = \int_0^\infty dE_\lambda \lambda^s$. Given

a function f which is acted on by L , one then has

$$\begin{aligned} L^s f(\mathbf{x}) &= \int_0^\infty dE_\lambda \lambda^s f(\mathbf{x}) = \frac{1}{\Gamma(-s)} \int_0^\infty dE_\lambda \int_0^\infty \frac{e^{-t\lambda} - 1}{t^{s+1}} dt f(\mathbf{x}) \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-tL} - I}{t^{s+1}} dt f(\mathbf{x}), \end{aligned} \quad (2.24)$$

where we have switched the order of integration and we have used that $e^{-tL} = \int_0^\infty dE_\lambda e^{-t\lambda}$, and the completeness of the spectrum $I = \int_0^\infty dE_\lambda$. In the expression (2.24) $w(\mathbf{x}, t) \equiv \exp(-tL) f(\mathbf{x})$ is the solution to the heat-like equation for the operator L with initial value $f(\mathbf{x})$; namely $w(\mathbf{x}, t)$ solves the following Cauchy problem in $\mathbb{R}^n \times [0, \infty)$

$$\begin{cases} (\partial_t + L)w = 0 \\ w(\mathbf{x}, 0) = f(\mathbf{x}). \end{cases} \quad (2.25)$$

In particular, with $L = -\Delta$, (2.25) is just the heat equation with an initial temperature distribution $f(\mathbf{x})$

$$\begin{cases} (\partial_t - \Delta_{\mathbf{x}})w = 0 \\ w(\mathbf{x}, 0) = f(\mathbf{x}), \end{cases} \quad (2.26)$$

and (2.24) yields another representation of the fractional Laplacian

$$(-\Delta)^s f(\mathbf{x}) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{t\Delta} - I}{t^{s+1}} dt f(\mathbf{x}). \quad (2.27)$$

Starting from an analogue identity, also valid for $\lambda > 0$

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} dt, \quad (2.28)$$

one can obtain a representation of the inverse fractional Laplacian

$$(-\Delta)^{-s} f(\mathbf{x}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{t\Delta} dt f(\mathbf{x}). \quad (2.29)$$

Balakrishnan formula The expression

$$(-\Delta)^s f(\mathbf{x}) = \frac{\sin(\pi s)}{\pi} \int_0^\infty \Delta (tI - \Delta)^{-1} f(\mathbf{x}) t^{s-1} dt, \quad (2.30)$$

known as Balakrishnan formula, first introduced in [143], expresses the fractional Laplacian of a function as an integral involving the classical Laplacian and its resolvent $(tI - \Delta^{-1})$. Naively, this formula can be obtained by using another integral representation of the power of a real

positive number λ

$$\lambda^s = \frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{\lambda}{\lambda + t} t^{s-1} dt, \quad (2.31)$$

which can be easily proven by the substitution $x = \lambda(1 - t)/t$, bringing the integral to a representation of the Beta function, and recalling the Euler reflection formula $\Gamma(s)\Gamma(1 - s) = \pi/\sin(\pi s)$. Then, using that the resolvent $(tI - \Delta)^{-1}$ is a Fourier multiplier with symbol $(k^2 + t)^{-1}$, we can make the formal substitution $\lambda/(\lambda + t) \rightarrow \Delta(tI - \Delta)^{-1} f(\mathbf{x})$, obtaining (2.30).

2.3 Fractional d'Alembertian

The d'Alembertian, or wave operator, is a fundamental differential operator in mathematical physics and is conventionally defined on Minkowski space by $\square = \partial_t^2 - c^2 \Delta$, up to the choice of metric signature and normalization of the temporal coordinate. The operator is hyperbolic and plays a central role in the formulation of relativistic wave equations, including the classical wave equation, the Klein–Gordon equation, and various field equations. Fractional generalizations of the d'Alembertian arise naturally when one seeks nonlocal models of wave propagation, anomalous dispersion, or relativistic equations with non-integer differentiation.

As with the Laplacian, the term fractional d'Alembertian does not refer to a single universally accepted operator. Rather, several inequivalent constructions have been proposed, depending on whether fractionalization is applied to the temporal and spatial derivatives separately, such as in $D_t^{2\alpha} + c^{2\beta}(-\Delta)^\beta$, with different choices of temporal derivative (Caputo, Riemann–Liouville or others) leading to different initial-value structures, domains and solutions, or to the full operator $\square^s$. This distinction is important because the resulting operators can have substantially different analytical and physical properties.

Unlike the fractional Laplacian, whose Fourier symbol is non-negative in the Euclidean setting, the Lorentzian symbol of the d'Alembertian changes sign. Hence, a straightforward replacement of the integer power by a fractional one $(\omega^2 - c^2|\mathbf{k}|^2)^{s/2}$ is generally not sufficient to define an unambiguous real-valued operator in the Fourier domain. A fractional power requires additional structure, such as a complex branch prescription, a causal prescription, a spectral definition on an appropriate function space, or an alternative operator constructed from positive elliptic components.

A conceptually distinct construction is obtained through functional calculus. If a suitable realization of $\square$ is available on a chosen function space, one can attempt to define $\square^s$ for non-integer s by means of the spectrum or an appropriate resolvent representation. In this framework, the fractional d'Alembertian is regarded as a nonlocal operator whose action is determined globally by the spectral properties of the underlying wave operator. This approach

has the advantage of preserving the operator-level structure of the d'Alembertian, but it requires careful treatment of the fact that the Lorentzian operator is hyperbolic rather than positive and elliptic.

Causality introduces a further complication. For relativistic applications, it is often desirable that the operator be compatible with the causal structure of Minkowski space. Different prescriptions for handling the zeros and sign changes of the Fourier symbol can lead to retarded, advanced, Feynman-type, or other Green-function structures. Thus, the specification of a fractional d'Alembertian generally involves more than the exponent α : the underlying function space, operator domain, boundary or initial conditions, and analytic prescription must also be stated.

The distinction between the various definitions becomes particularly important when discussing Lorentz invariance. A construction based directly on the Lorentz-invariant Fourier symbol can retain the formal symmetry of the underlying d'Alembertian, whereas replacing the temporal and spatial derivatives by unrelated fractional orders may introduce preferred directions or scales. Consequently, a fractional model should be classified according to the symmetry properties it is intended to preserve rather than being assumed to inherit all properties of the classical operator.

Throughout this work, in particular in Chapter 5, we will consider a definition of fractional d'Alembertian relying on the wave operator in Minkowski space to preserve Lorentz invariance and avoid superluminal motion. To ensure causality the solution to the corresponding fractional wave equation will be constructed by convolution with the retarded Green function, derived in Appendix C from an appropriate contour in the complex plane.

2.4 Summary and conclusions

In this chapter we have provided the reader with a short primer on fractional calculus and in particular on the fractional Laplacian and d'Alembertian. The material presented here is not by any means complete, nor exhaustive. The interested reader can find more details on the definition and properties of the fractional Laplacian in [140] and in the classic book by Stein [144]. We also refer the reader to [145] for a comprehensive review of the representations of the fractional d'Alembertian.

Chapter 3

Newtonian Fractional Gravity

In this chapter, we introduce a framework in which cold dark matter (CDM) exhibits a non-local coupling to gravity. We demonstrate that this theory, referred to as fractional gravity (FG), naturally accounts for the kinematic properties observed in galactic structures, such as dwarfs, which remain difficult to reconcile within the standard CDM paradigm under Newtonian gravity. In FG, although the standard law of inertia continues to hold, the gravitational potential associated to a given CDM distribution is determined by a modified Poisson equation of the fractional type. The latter, having a inherently non-local nature, can profoundly alter the dynamics of CDM particles with respect to the Newtonian case (based on a standard Poisson equation), flattening the CDM density distribution in the inner region of galactic systems.

In Section 3.1 we first present the fractional Poisson equation (FPE) and its solution in the case of a Navarro-Frenk-White (NFW) profile, showing that the non-local nature of FG can significantly alter the profile of the CDM density distribution in virialized systems. In Section 3.2 we then fit stacked Rotation Curves (RCs) of spiral galaxies of different types, from dwarfs, to high and low surface brightness spirals, via a Bayesian MCMC technique, so inferring relevant galaxy properties and at the same time constraining the parameters ruling the strength and length-scale of FG. In Section 3.3 we then show that several empirical relations observed in spirals between the parameters of the baryonic disk and the DM halo, when the latter is modeled with a NFW or Burkert profile, are correctly reproduced: a fact that itself represents an independent test of FG in late-type galaxies. Finally, in Section 3.4 we repeat our MCMC analysis in the context of a MONDian viewpoint, in which no CDM is assumed and baryons instead originate FG effects. We show that this attempt fails to correctly reproduce the stacked RCs of spiral galaxies, thus implying the need of a CDM component even when FG is taken into account.

3.1 Dark Matter in Fractional Gravity

We assume that CDM, when confined in virialized structures, feels a non-local interaction with gravity. In this framework, the potential associated to a given dark matter distribution $\rho(\mathbf{r})$ is derived from the fractional Poisson equation

$$(-\Delta)^s \Phi(\mathbf{r}) = -4\pi G \ell^{2-2s} \rho(\mathbf{r}), \quad (3.1)$$

where $(-\Delta)^s$ is the spectral definition of the fractional Laplacian introduced in Chapter 2, $s \in [1, 3/2]$ is the fractional index (as stated in Chapter 2, this range of values for s is required to avoid divergences in three dimensional space), and ℓ is a fractional length-scale (more on the meaning of these parameters at the end of this section). To be conservative with respect to the standard CDM paradigm in virialized systems, we have solved the FPE sourced by the NFW density distribution (1.2). In this way the non-local interaction is introduced as a perturbative correction to Newtonian dynamics, based on Equation (3.1): we retain the density distribution emerging from CMD-only N-body simulations, but the resulting gravitational potential is found by solving a fractional differential equation.

As detailed in Appendix A, for $s \in [1, 3/2]$ we find the solution

$$\begin{aligned} \Phi_s(r) = & -\frac{GM_s}{r_s} \frac{1}{2^{2s}\sqrt{\pi}} \left(\frac{\ell}{r_s}\right)^{2-2s} \frac{\Gamma\left(\frac{3}{2}-s\right)}{\Gamma(s+1)} \frac{r_s}{r} \left\{ \frac{2\pi s}{\sin(2\pi s)} \left[\left(1 + \frac{r}{r_s}\right)^{2s-2} - \left(1 - \frac{r}{r_s}\right)^{2s-2} \right] + \right. \\ & + \frac{(r/r_s)^{2s}}{1 - (r/r_s)^2} \left[\left(1 + \frac{r}{r_s}\right) {}_2F_1\left(1, 1, 2s+1, \frac{r}{r_s}\right) + \left(1 - \frac{r}{r_s}\right) {}_2F_1\left(1, 1, 2s+1, -\frac{r}{r_s}\right) + \right. \\ & \left. \left. - \frac{4s}{2s-1} \right] \right\}, \quad (3.2) \end{aligned}$$

in terms of the Euler Gamma function $\Gamma(x)$ and of the ordinary hypergeometric function ${}_2F_1(x)$; using the properties of these functions it is straightforward to verify that $\Phi_{s=1}(r)$ coincides with the Newtonian expression

$$\Phi(r) = -\frac{GM_s}{r} \log\left(1 + \frac{r}{r_s}\right). \quad (3.3)$$

The limiting case $s = 3/2$ must be treated with care, since it requires regularization, as again discussed in detail in Appendix A. We find the solution

$$\begin{aligned} \Phi_{s=3/2}(r) = & -\frac{GM_s}{\ell} \frac{1}{\pi} \frac{r_s}{r} \left\{ 2 \frac{r}{r_s} \left[\log\left(\frac{r}{r_s}\right) - 1 \right] - \left(1 + \frac{r}{r_s}\right) \log\left(\frac{r}{r_s}\right) \log\left(1 + \frac{r}{r_s}\right) + \right. \\ & \left. + \left(\frac{r}{r_s} - 1\right) \text{Li}_2\left(1 - \frac{r}{r_s}\right) - \left(1 + \frac{r}{r_s}\right) \text{Li}_2\left(-\frac{r}{r_s}\right) + \frac{\pi^2}{6} \right\}, \end{aligned} \quad (3.4)$$

in terms of the dilogarithm function $\text{Li}_2(x) \equiv \sum_{k=1}^{\infty} x^k/k^2$. The corresponding RC computed after $v_s^2(r) = r |d\Phi_s/dr|$, which remains true since the standard law of inertia continues to hold in fractional gravity, is illustrated in Figure ?? (left panel) for different values of the fractional index s . For s increasing above one (corresponding to Newtonian gravity), we find that the RC is steeper in the inner region, behaving as $v_s(r) \propto r^{s-1/2}$ for $s < 3/2$ and $v_{s=3/2}(r) \propto r \sqrt{-\log r/r_s}$ for $s = 3/2$; curiously, for the limiting value $s = 3/2$ the RC tends to closely mirror a (Newtonian) solid body. Contrariwise, in the outskirts the RC in fractional gravity is appreciably flatter than the Newtonian case, behaving as $v_s(r) \propto r^{s-3/2}$ for $s < 3/2$ and as $v_{s=3/2}(r) \propto \sqrt{\log r/r_s}$ for $s = 3/2$.

In FG the Birkhoff theorem does not hold, but one can insist in writing $v_s^2(r) = G M_{\text{eff},s}(< r)/r$ in terms of an effective mass $M_{\text{eff},s}(< r)$, and then differentiate this expression to obtain an effective density profile via $\rho_{\text{eff},s}(r) = (1/4\pi r^2) \times dM_{\text{eff},s}/dr$. In general the effective density is the source of the fractional potential $\Phi_s(\mathbf{r})$ in the ordinary Poisson equation, meaning that

$$\Delta \Phi_s(\mathbf{r}) = 4\pi G \rho_{\text{eff},s}(\mathbf{r}). \quad (3.5)$$

By comparing (3.1) and (3.5) we can see that the two density functions are related by

$$\rho_{\text{eff},s}(\mathbf{r}) = (-\ell^2 \Delta)^{1-s} \rho(\mathbf{r}). \quad (3.6)$$

The effective density has a nice interpretation: it is the density profile that one would infer by looking at the RC associated to $\Phi_s(r)$, and interpreting the result in terms of Newtonian gravity. We illustrate such an effective density in the case of the NFW profile, in Fig. ?? (right panel) for different values of the fractional index s . With s increasing from unity (Newtonian case) the effective density profile progressively flattens. Specifically, in the inner region a cored-like behavior tends to be enforced, while in the outskirts the effective profile resembles an isothermal sphere. To have a grasp on the overall effect, consider the $s = 3/2$ case where the effective

density has a particularly simple analytic expression, given by

$$\rho_{\text{eff},s=3/2}(r) = \frac{2\rho_s r_s}{\pi\ell} \frac{\ln(r/r_s)}{(r/r_s)^2 - 1} ; \quad (3.7)$$

interestingly, the logarithmic slope of this profile is -2 in the outskirts, -1 at $r \approx r_s$ and $\lesssim -0.5$ for $r \lesssim r_s/10$. All in all, the effective density is practically indistinguishable from the run of a cored pseudo-isothermal sphere. This slope around $r \sim 0.1 r_s$ is remarkably close to the observational value estimated from the kinematic of individual dwarf galaxies [37]. We thus see that FG naturally solves the cusp-core problem of CDM: the mismatch between observed and predicted galaxy RCs exists only because one compares the data with the velocity derived from the Poisson equation (sourced by the NFW density), while the one computed from the fractional version of the equation has the right behavior at all radii. On the other hand, if one wants to rely on the Poisson equation, then the appropriate density to compare the data with is the effective density, which is indeed more cored in the inner region.

The physical reason behind the development of a core is that the fractional Laplacian is inherently non-local; the effect is markedly evident in Fourier space, where it is easily found that $\mathcal{F}\{\rho_{\text{eff},s}\}/\mathcal{F}\{\rho\} \propto (\lambda/\ell)^{2s-2}$ in terms of the Fourier transform $\mathcal{F}$ and of the wavelength $\lambda = 2\pi/k$ associated to the Fourier mode with wavenumber k . This means that FG causes, with respect to the Newtonian case, a transfer of power from modes with $\lambda \lesssim \ell$, which are suppressed, to modes with $\lambda \gtrsim \ell$, which are instead enhanced. Therefore the parameter s measures the strength of non-locality, while the length-scale ℓ can be interpreted as the typical size below which gravitational effects are somewhat reduced and above which they are instead amplified by non-locality (around $r \approx \ell$ the dynamics is almost unaffected and indistinguishable from the Newtonian case).

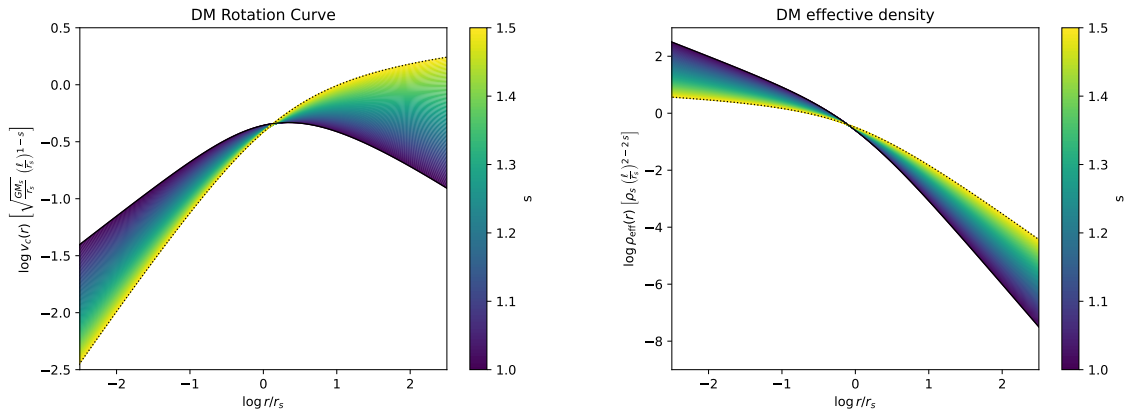


Figure 3.1. Normalized rotation curve and effective density for the NFW profile for $s \in [1, 3/2]$

3.2 Bayesian analysis of stacked RCs

As a first test to the fractional gravity framework we fit observed stacked RCs of local spiral galaxies; these include high surface brightness (HSB) [10, 146] systems divided in 11 optical radial/velocity bins¹, low surface brightness (LSB) [44, 147] systems divided in 5 bins, and 1 bin for dwarf galaxies (Dw) [148]. The stacked RCs are built by co-adding high-quality, high-resolution individual data of thousands of galaxies with similar properties (e.g., Hubble type, magnitude, optical radii and velocity); details on this procedure can be found in [10] and [149].

We mass-model the stacked RCs $v_{\text{model}}^2(r) = v_{\text{d}}^2(r) + v_{\text{DM}}^2(r)$ as the sum of a disk component $v_{\text{d}}^2(r)$ and of a DM component $v_{\text{DM}}^2(r)$. We compute the gravitational potential originated by the baryons in the disk assuming standard Newtonian gravity, since we do not expect and/or require non-local effects for them; thus $v_{\text{d}}^2(r) = G M_{\text{d}}(< r)/r$ applies in terms of the disk cumulative mass $M_{\text{d}}(< r)$. As customary, we assume the disk radial distribution to follow a razor-thin exponential disk [150] with surface density $\Sigma_{\text{d}}(r) = (M_{\text{d}}/2\pi r_{\text{d}}^2) e^{-r/r_{\text{d}}}$ in terms of the disk scale-length $r_{\text{d}} \approx r_{\text{opt}}/3.2$. The related contribution to the total RC is the well-known expression $v_{\text{d}}^2(r) = (G M_{\text{d}}/r_{\text{d}}) \times 2y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)]$, in terms of $y \equiv r/(2r_{\text{d}})$ and of the modified Bessel functions $I_{0,1}$ and $K_{0,1}$. Since the fit is performed in a radial range $r \lesssim r_{\text{opt}}$ we have checked that any contribution from a gaseous disk (typically more important at larger radii) is negligible and largely unconstrained, so we include only the stellar disk in the mass-modeling.

The CDM component of the RC is modeled in the fractional gravity framework according to $v_{\text{DM}}^2(r) = |r d\Phi/dr|$ where $\Phi(r)$ is the potential given by Equations (3.2) and (3.4). Given the halo mass M_{H} , we determine the halo virial radius as $R_{\text{H}} \approx 260 (M_{\text{H}}/10^{12} M_{\odot})^{1/3}$ kpc, set the concentration $c \approx 8 (M_{\text{H}}/10^{12} h^{-1})^{-0.1}$ according to the prescription by Dutton & Maccio [151], and hence determine the reference density $\rho_s = (3 M_{\text{H}}/4\pi R_{\text{H}}^2) \times c^3/[\ln(1+c) - c/(1+c)]$ and the scale radius $r_s = R_{\text{H}}/c$; typical values of the latter are in the range $r_s \approx 5 - 50$ kpc.

All in all, our model RCs depends on four parameters: the fractional index s , the fractional length-scale ℓ , the halo mass M_{H} and the disk mass M_{d} ; in fact, we find it convenient to perform inference on the non-dimensional ratio ℓ/r_s instead of ℓ . To estimate the aforementioned parameters we adopt a Bayesian MCMC technique, numerically implemented via the Python package `emcee` [152]. We use a standard log-likelihood $\mathcal{L}(\theta) \equiv -\chi^2(\theta)/2$ where $\theta = \{s, \log \ell/r_s, \log M_{\text{H}}, \log M_{\text{d}}\}$ is the vector of parameters, and $\chi^2 = \sum_j [v_{\text{model}}(r_j|\theta) - v_{\text{obs}}(r_j)]^2 / \sigma_{v_{\text{obs}}}^2(r_j)$ is obtained by comparing our empirical model expectations $v_{\text{model}}(r_j|\theta)$ to the data $v_{\text{obs}}(r_j)$ with their uncertainties $\sigma_{v_{\text{obs}}}^2(r_j)$, summing over the different radii r_j of each dataset (bin).

¹The optical radius is $r_{\text{opt}} \approx 3.2 r_{\text{d}}$ in terms of the exponential scale-length r_{d} of the (stellar) disk; the optical velocity is the circular velocity at the optical radius.

We adopt flat priors $\pi(\theta)$ on the parameters within the ranges $s \in [1, 3/2]$, $\log \ell/r_s \in [-3, 1]$, $\log M_{\text{H}} [M_{\odot}] \in [7, 14]$, and $\log M_{\text{d}} [M_{\odot}] \in [6, 13]$. We then sample the posterior distribution $\mathcal{P}(\theta) \propto \mathcal{L}(\theta) \pi(\theta)$ by running `emcee` with 10^4 steps and 200 walkers; each walker is initialized with a random position uniformly sampled from the (flat) priors. To speed up convergence, we adopt a mixture of differential evolution [153] and snooker [154] moves of the walkers, in proportion of 0.8 and 0.2, respectively. After checking the auto-correlation time, we remove the first 20% of the flattened chain to ensure the burn-in; the typical acceptance fractions of the various runs are in the range 30 – 40%.

3.2.1 Results

The outcomes of the fitting procedure are illustrated in Fig. 3.2 for the dwarf and LSB spirals samples, and in Fig. 3.3 for HSB spirals. In each panel, the bestfit (solid lines) and the 1σ credible intervals sampled from the posterior (shaded areas) are shown for the halo (green), disk (blue), and total (red) RC models; for reference, the results in Newtonian gravity is also displayed (dashed lines). MCMC posterior distributions in the FG framework for some representative bins are shown in the corner plot of Figure 3.4. Finally, Table 1 reports the marginalized posterior estimates of the fitting parameters, the reduced χ_r^2 of the fits, and the difference in the Bayesian inference criterion (BIC) for model comparison between the Newtonian and fractional gravity fits².

The FG framework performs substantially better than the Newtonian case in several bins, especially those relative to small and intermediate mass systems, where the CDM component dominates or appreciably contributes to the total RC out to r_{opt} . This is manifest both from the quality of the fit in terms of a substantial improvement in the reduced χ_r^2 , and in terms of the strongly negative values of the ΔBIC (meaning that FG is favored, in a Bayesian sense, over the Newtonian case). For example, one can consider the dwarfs RC shown in the top panel of Fig. 3.2, where the contribution of baryons to the dynamics is known to be negligible, hence the overall RC is determined solely by the CDM component. In the Newtonian case, the cuspy behavior of the NFW density distribution (with standard values of the concentration) struggles in reproducing the inner part of the RC; historically, this has been one of the first evidence of the cusp-core problem. However, the dynamics in FG is substantially modified, in such a way that the RC sourced by the NFW density profile can actually reproduce the observed inner RC to a remarkable accuracy; this is because, as discussed in Section 3.1, the gravitational

²The BIC is defined as $\text{BIC} = 2 \ln \mathcal{L}_{\text{max}} + N_{\text{par}} \ln N_{\text{data}}$ in terms of the maximum likelihood estimate $\mathcal{L}_{\text{max}}$, of the number of parameters N_{par} , and the number of data points N_{data} . The BIC comes from approximating the Bayes factor, which gives the posterior odds of one model against another, presuming that the models are equally favored a priori. Note that what matters is the relative value of the BIC among different models; in particular, a difference of around ten or more indicates evidence in favor of the model with the smaller value.

potential sourced by the NFW density in FG mirrors that sourced by a cored density distribution (e.g., a pseudo-isothermal sphere) in Newtonian gravity. In more massive galaxies, where the contribution of baryons to the inner RC out to r_{opt} becomes increasingly relevant and eventually dominates over CDM, the evidence for FG gets weaker; in such systems the fits in fractional and Newtonian gravity turn out to be very similar (though with an appreciable difference in the estimated total mass, see below). However, we caveat that the contribution of the DM to the RC in massive spirals emerges at radii $r \gtrsim r_{\text{opt}}$ where the data are still subject to appreciable uncertainties; more precise observations over an extended radial range would be necessary before drawing a definitive conclusion on the impact of FG in massive spirals.

In Fig. 3.5 we illustrate the behavior of the fractional gravity index s and length-scale ℓ on halo mass. The former features a decreasing behavior as a function of halo mass, passing from values around $s \approx 1.3 - 1.4$ in dwarf galaxies with $M_{\text{H}} \lesssim 10^{10} M_{\odot}$ to $s \approx 1.2 - 1.3$ in intermediate mass galaxies, to $s \lesssim 1.1$ in massive galaxies with $M_{\text{H}} \gtrsim \text{some } 10^{11} M_{\odot}$. This implies a strongly fractional behavior of the DM component in small and intermediate mass systems, whereas deviations from Newtonian gravity in high mass systems seems to be more limited (at least given the present quality of the data in the outer regions where the DM component can be probed in massive galaxies, see above). We fit the $s - M_{\text{H}}$ relation with a linear shape $s = a + b(\log M_{\text{H}}[M_{\odot}] - 11)$ via an ODR algorithm, to obtain bestfit parameters $a = 1.21 \pm 0.02$ and $b = -0.13 \pm 0.01$ and a reduced $\chi_r^2 \approx 0.58$; a nonlinear fit (dotted line) $s = \frac{5}{4} + \frac{1}{4} \tanh[c(\log M_{\text{H}}[M_{\odot}] - d)]$ with asymptotic values $s = 1$ and 1.5 at small and large masses yields bestfit parameters $c = -0.68 \pm 0.09$, $d = 10.71 \pm 0.12$ and a modest improvement in the reduced $\chi_r^2 \approx 0.52$. As to the fractional length-scale, there is a tendency of increasing with halo mass. However, it may be stressed that the determination of ℓ is robust only in small and intermediate mass galaxies with strong fractional gravity behavior, or in other words where the index s is substantially larger than unity. Contrariwise, in massive galaxies, where s gets close to 1, the fractional Poisson equation (3.1) becomes almost independent on ℓ (the quantity ℓ^{2s-2} appears on the RHS), so that any robust inference on this parameter is difficult. We fit the $\ell - M_{\text{H}}$ relation with a linear shape $\ell[\text{kpc}] = a + b(\log M_{\text{H}}[M_{\odot}] - 11)$ via an ODR algorithm, to obtain bestfit parameters $a = 0.27 \pm 0.13$ and $b = 0.57 \pm 0.11$ and a reduced $\chi_r^2 \approx 1.25$; a nonlinear fit (dotted line) in terms of a double powerlaw shape leaves unchanged the reduced $\chi_r^2 \approx 1.32$ and yields large errors on the fitted parameters. Interestingly, the bestfit linear relation implies a scaling of ℓ very similar to the NFW scale radius r_s , in such a way that $\ell/r_s \approx 0.25 \pm 0.15$ is approximately independent of halo mass. The trends of the parameters s and ℓ with halo mass may be suggestive that the effects of fractional gravity on progressively larger scales become weaker; this would retain the successes of the standard CDM paradigm in Newtonian gravity on cosmological scales. However, given the still large uncertainties affecting

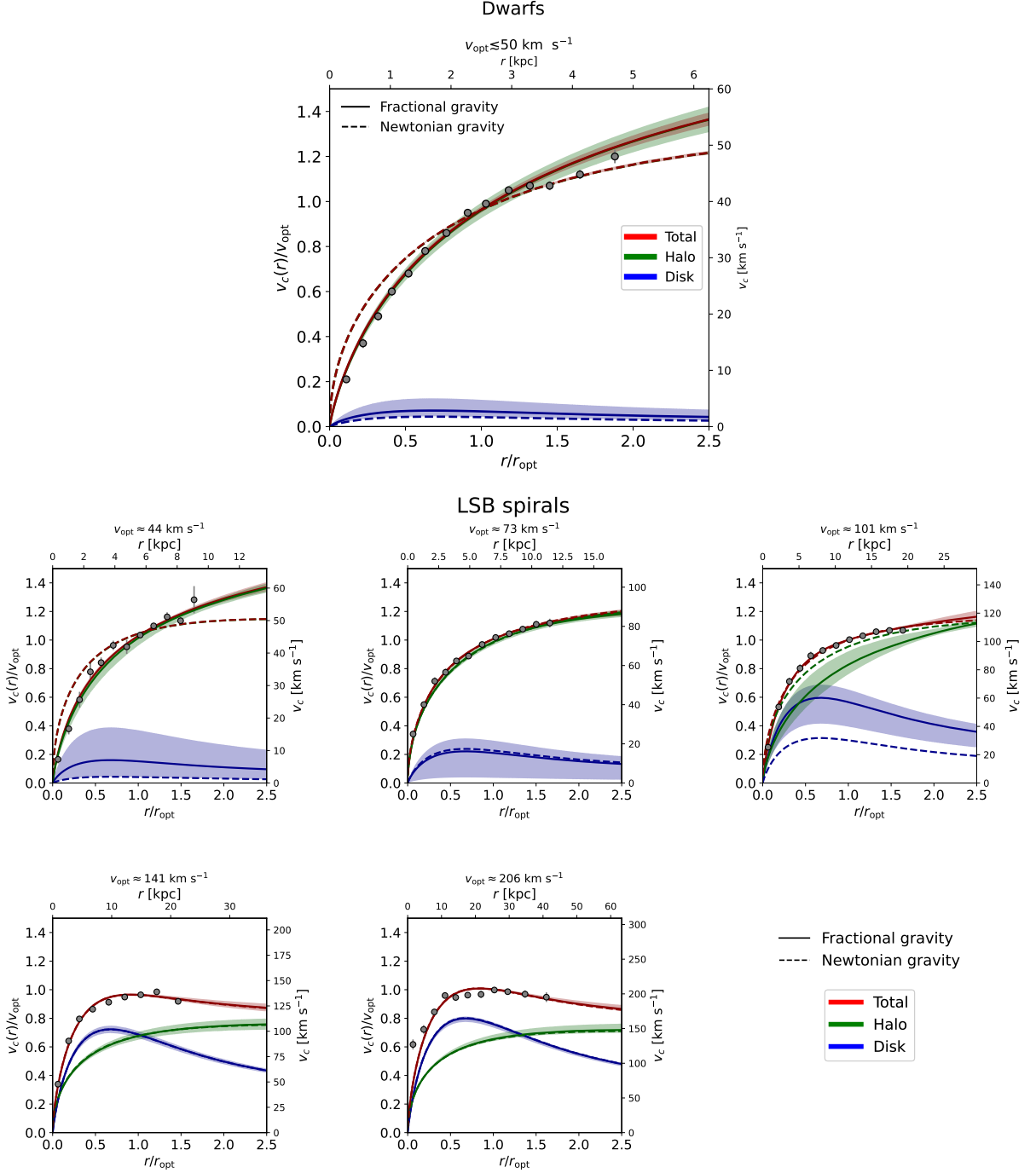


Figure 3.2. Fit to stacked rotation curve (filled circles) for the dwarf galaxy sample (top), and the LSB spirals (bottom), with Newtonian (dashed lines) and fractional (solid lines) gravity. Red lines refer to the total rotation curve, blue line to the disk component, and green lines to the halo component. For the fractional case, the shaded areas illustrate the 1σ credible intervals from sampling the posterior distribution.

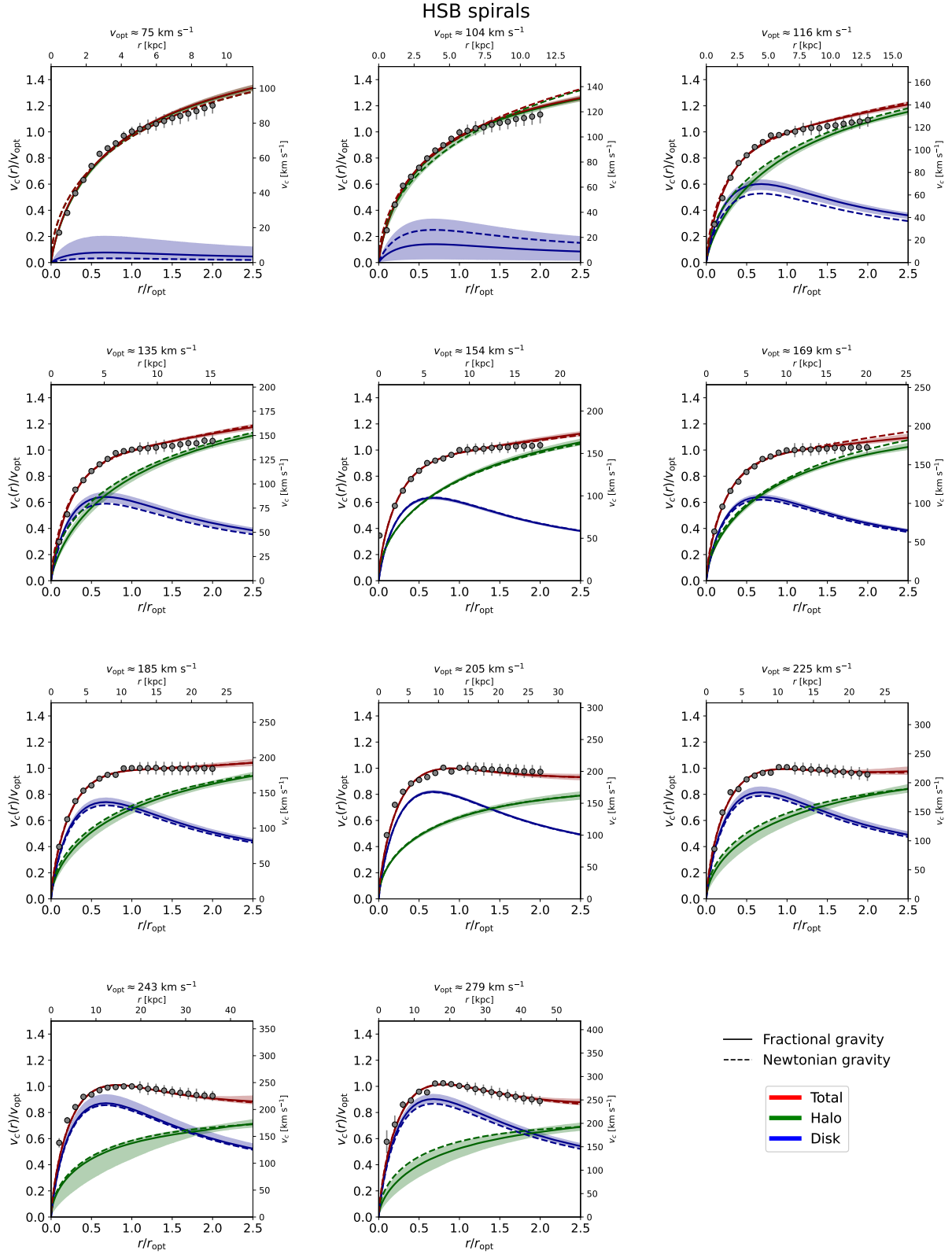


Figure 3.3. Same as Fig. 3.2 for the stacked rotation curves of the HSB spirals sample.

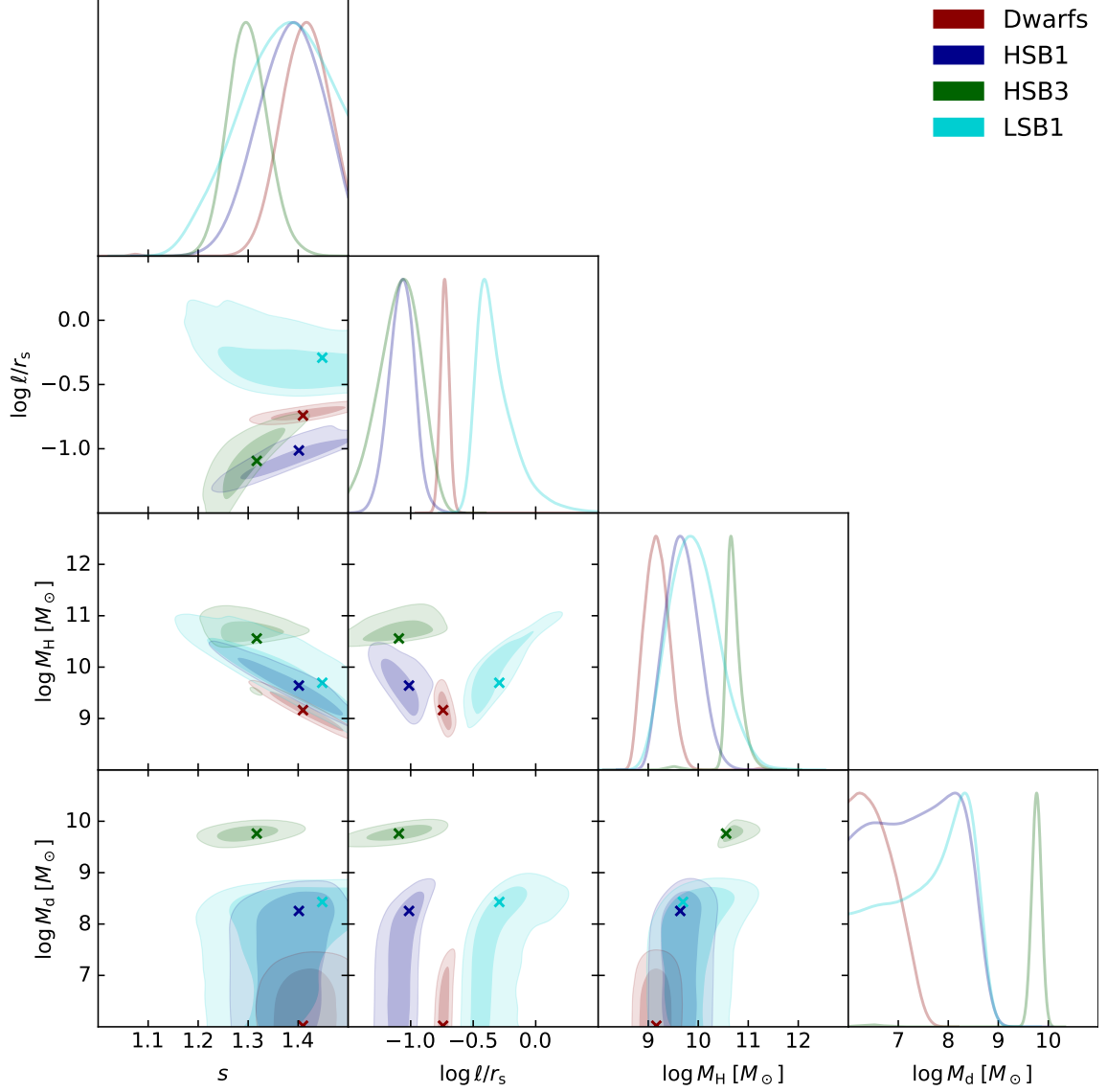


Figure 3.4. MCMC posterior distributions in the fractional gravity framework for the fractional parameter s , the fractional scale-length (in units of the NFW scale radius) ℓ/r_s , the halo mass M_H and the disk mass M_d . Colored contours/lines refer to the most representative cases where there is strong evidence (from χ^2 and BIC) that fractional gravity performs substantially better than the Newtonian one: red is for dwarfs, blue for HSB1 spirals, green for HSB3 spirals and cyan for LSB1 spirals. The contours show 1σ and 2σ confidence intervals, crosses mark the maximum likelihood estimates, and the marginalized distributions are in arbitrary units (normalized to 1 at their maximum value).

Table 3.1. Marginalized posterior estimates (mean and 1σ confidence intervals) for the fractional and Newtonian gravity. Columns report the values of: fractional parameter s , fractional scale-length ℓ , halo mass M_H , disk mass M_d , reduced χ_r^2 . The last column refer to the difference in values of the Bayesian inference criterion (BIC) between the two models.

Bin	Fractional gravity					Newtonian gravity			
	s	$\log \ell / r_s$	$\log M_H [\text{M}_\odot]$	$\log M_d [\text{M}_\odot]$	χ_r^2	$\log M_H [\text{M}_\odot]$	$\log M_d [\text{M}_\odot]$	χ_r^2	ΔBIC
Dwarfs	$1.41^{+0.05}_{-0.04}$	$-0.72^{+0.03}_{-0.04}$	$9.18^{+0.22}_{-0.26}$	$6.59^{+0.38}_{-0.38}$	4.91	$10.52^{+0.03}_{-0.01}$	$6.25^{+0.57}_{-0.57}$	25.5	-250
HSB1	$1.38^{+0.07}_{-0.06}$	$-1.07^{+0.11}_{-0.09}$	$9.68^{+0.31}_{-0.36}$	$7.38^{+0.79}_{-0.89}$	1.37	$11.69^{+0.03}_{-0.03}$	$6.78^{+0.51}_{-0.51}$	4.36	-50
HSB2	$1.35^{+0.06}_{-0.06}$	$-1.39^{+0.17}_{-0.17}$	$9.85^{+0.40}_{-0.36}$	$8.03^{+1.3}_{-0.67}$	0.94	$12.34^{+0.06}_{-0.06}$	$8.81^{+0.20}_{-0.07}$	3.54	-42
HSB3	$1.30^{+0.03}_{-0.04}$	$-1.11^{+0.21}_{-0.13}$	$10.70^{+0.11}_{-0.16}$	$9.71^{+0.10}_{-0.03}$	1.98	$12.24^{+0.04}_{-0.06}$	$9.62^{+0.06}_{-0.01}$	4.87	-51
HSB4	$1.27^{+0.05}_{-0.05}$	$-1.17^{+0.27}_{-0.17}$	$10.98^{+0.14}_{-0.18}$	$9.87^{+0.21}_{-0.11}$	1.70	$12.40^{+0.03}_{-0.08}$	$9.88^{+0.09}_{-0.06}$	3.53	-30
HSB5	$1.03^{+0.10}_{-0.03}$	$-0.72^{+0.89}_{-0.89}$	$12.16^{+0.31}_{-0.20}$	$9.99^{+0.22}_{-0.18}$	17.7	$12.41^{+0.04}_{-0.06}$	$10.17^{+0.05}_{-0.01}$	15.8	+5
HSB6	$1.11^{+0.05}_{-0.04}$	$-1.59^{+0.37}_{-0.37}$	$11.74^{+0.27}_{-0.54}$	$10.34^{+0.10}_{-0.09}$	0.79	$12.60^{+0.04}_{-0.06}$	$10.27^{+0.06}_{-0.03}$	1.13	-2
HSB7	$1.08^{+0.07}_{-0.07}$	$-0.68^{+0.78}_{-0.78}$	$12.28^{+0.37}_{-0.31}$	$10.61^{+0.03}_{+0.04}$	0.52	$12.45^{+0.05}_{-0.07}$	$10.54^{+0.05}_{-0.02}$	0.47	+5
HSB8	$1.02^{+0.05}_{-0.05}$	$-0.64^{+0.86}_{-0.86}$	$12.16^{+0.15}_{-0.07}$	$10.82^{+0.05}_{+0.02}$	3.74	$12.24^{+0.06}_{-0.09}$	$10.79^{+0.06}_{-0.04}$	3.33	+5
HSB9	$1.09^{+0.09}_{-0.09}$	$-0.40^{+0.76}_{-0.76}$	$12.41^{+0.27}_{-0.37}$	$10.97^{+0.06}_{-0.04}$	1.42	$12.48^{+0.05}_{-0.09}$	$10.88^{+0.08}_{-0.06}$	1.26	+5
HSB10	$1.11^{+0.09}_{-0.09}$	$-0.29^{+0.78}_{-0.78}$	$12.45^{+0.03}_{-0.46}$	$11.19^{+0.06}_{-0.04}$	2.25	$12.26^{+0.05}_{-0.08}$	$11.11^{+0.07}_{-0.05}$	1.99	+5
HSB11	$1.12^{+0.09}_{-0.09}$	$-0.02^{+0.65}_{-0.65}$	$12.56^{+0.10}_{-0.34}$	$11.43^{+0.04}_{-0.03}$	0.66	$12.39^{+0.07}_{-0.09}$	$11.35^{+0.06}_{-0.03}$	0.59	+6
LSB1	$1.36^{+0.11}_{-0.06}$	$-0.31^{+0.07}_{-0.19}$	$9.93^{+0.42}_{-0.55}$	$7.57^{+1.1}_{-0.6}$	1.84	$10.46^{+0.04}_{-0.01}$	$6.67^{+0.66}_{-0.66}$	12.3	-103
LSB2	$1.08^{+0.09}_{-0.09}$	$-1.14^{+0.79}_{-0.79}$	$10.83^{+0.67}_{-0.03}$	$8.17^{+0.90}_{-0.15}$	0.65	$11.29^{+0.06}_{-0.02}$	$8.39^{+0.47}_{-0.06}$	0.52	+5
LSB3	$1.23^{+0.06}_{-0.10}$	$-0.17^{+0.54}_{-0.49}$	$11.44^{+0.60}_{-0.69}$	$9.81^{+0.27}_{-0.03}$	0.25	$11.64^{+0.04}_{-0.04}$	$9.29^{+0.17}_{-0.05}$	2.37	-16
LSB4	$1.03^{+0.06}_{-0.06}$	$-0.80^{+0.84}_{-0.84}$	$11.39^{+0.18}_{-0.04}$	$10.35^{+0.13}_{+0.07}$	2.42	$11.48^{+0.12}_{-0.03}$	$10.43^{+0.04}_{-0.02}$	1.74	+4
LSB5	$1.05^{+0.09}_{-0.09}$	$-0.70^{+0.91}_{-0.91}$	$11.64^{+0.36}_{-0.28}$	$10.73^{+0.39}_{-0.36}$	8.9	$11.99^{+0.02}_{-0.06}$	$11.04^{+0.08}_{-0.05}$	8.8	-9

the kinematic analysis of massive galaxies, it would be interesting to validate our results by looking at larger systems like clusters of galaxies. In fact, the overall gravitational potential of the cluster is dominated by the DM component, while most of the baryons are in the form of a hot intracluster medium in hydrostatic equilibrium within the DM potential well. In FG, one may expect the thermodynamic properties of the intracluster medium (e.g., density, temperature, pressure, entropy runs) to be substantially modified, so that comparing with those empirically inferred from X-ray, Sunyaev-Zel'dovich effect and weak lensing data could be extremely informative on fractional gravity parameters at scales much larger than galaxies'. For this reason, the next chapter features a detailed analysis of fractional gravity in galaxy clusters.

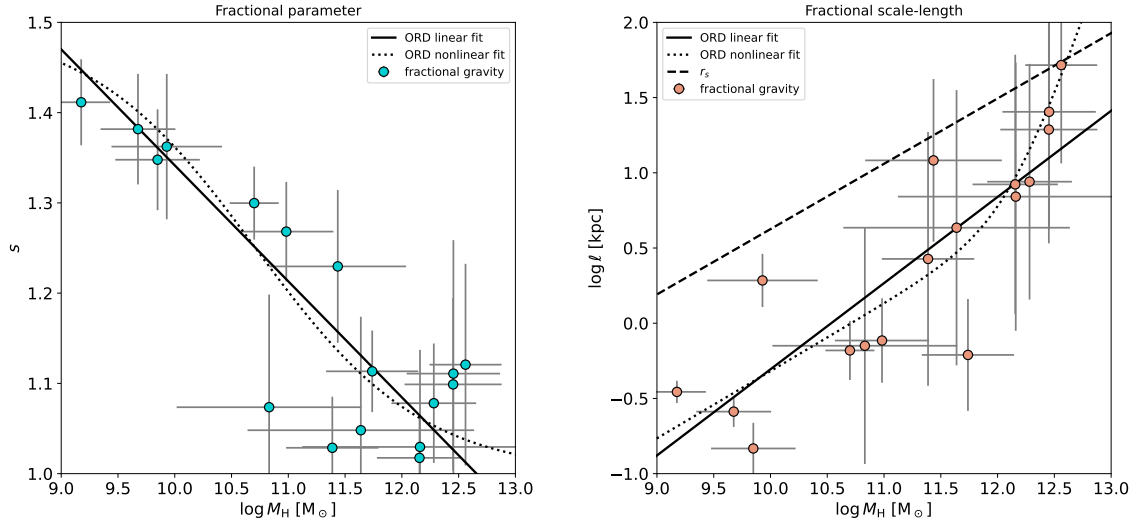


Figure 3.5. Dependence of the parameter s and of the scale-length ℓ in on the halo mass M_H , as derived from rotation curve modeling in fractional gravity. The solid and dotted lines show linear and nonlinear orthogonal regression distance fits to the data (see text for details). In the right panel, the dashed line displays the values of the scale radius r_s .

3.3 Recovering empirical relations in late-type galaxies

When the CDM component in late-type galaxies is modeled with a NFW or Burkert profile, tight empirical relationships have been observed between the halo and the disk parameters. Those include the disk vs halo mass relation, the radial acceleration relation (RAR), the universal core surface density, and the core radius vs disk scale length relation.

In this section, we demonstrate that fractional gravity successfully reproduces these observational findings using the best-fit parameters derived from our MCMC analysis. This result is

non-trivial, as fitting galaxy rotation curves does not, in general, ensure the correct behavior of these relations [155].

3.3.1 Disk vs halo mass relation

In Fig. 3.6 we illustrate the disk (stellar) vs. halo mass relation constructed with the bestfit estimates in fractional (magenta circles) and Newtonian (orange squares) gravity. For reference, the dot-dashed and dashed lines are the relations inferred by Lapi et al. [149] and Karukes & Salucci [148] by fitting galaxy RCs of different masses with the cored Burkert profile, that are also in agreement with abundance matching determinations [156, 149]. The inferred masses in the fractional gravity framework are in excellent agreement with the Lapi et al. and Karukes & Salucci results. Contrariwise, there is instead a clear tendency for the NFW fits in Newtonian gravity to yield higher DM masses at fixed stellar mass (especially so for small galaxies), that are somewhat inconsistent with the aforementioned empirical determinations; in the literature it has been pointed out that more reasonable halo masses can be obtained by letting the concentration parameter free to vary in the fit, but at the price of obtaining concentration values in disagreement with the concentration vs. mass relation measured in cosmological simulations [109, 147]. We fit the stellar mass vs. halo mass relation in fractional gravity via a orthogonal distance regression algorithm (ODR) that takes into account errorbars in both axes. Using a linear (solid line) shape $\log M_d[M_\odot] = a + b(\log M_H[M_\odot] - 11)$ we obtain bestfit parameters $a = 9.31 \pm 0.11$ and $b = 1.34 \pm 0.09$ and a reduced $\chi_r^2 \approx 0.37$; a nonlinear fit (dotted line) inspired by Equation. (2) in [156] yields only a modest improvement in the reduced $\chi_r^2 \approx 0.31$ and large uncertainties on the fitted parameters.

3.3.2 Radial Acceleration Relation

The radial acceleration relation (RAR), namely a relationship between the total g_{tot} vs. the baryonic acceleration g_{bar} has been proposed by McGaugh et al. [157] via the analysis of individual RCs of the SPARC sample [158]; note that the acceleration of any component can be simply computed from the RC as $g \equiv v^2(r)/r$. The RAR is thought to subsume and generalize a plethora of well-known dynamical laws of galaxies [43], although its tightness and physical interpretation are still debated [44, 47]. Here we just aim to show that the RAR emerges from the dynamics of galaxies in the fractional gravity framework, using as inputs the outcomes from the previous analysis of stacked RC data.

Toward this purpose, note that the RAR is a local scaling law that combines data at different radii in galaxies with different masses, which feature different contribution of stellar/gas disk and DM. Therefore we approach the problem via a semi-empirical method: we first build up mock RCs of galaxies with different halo masses, by exploiting well known empirical relationships

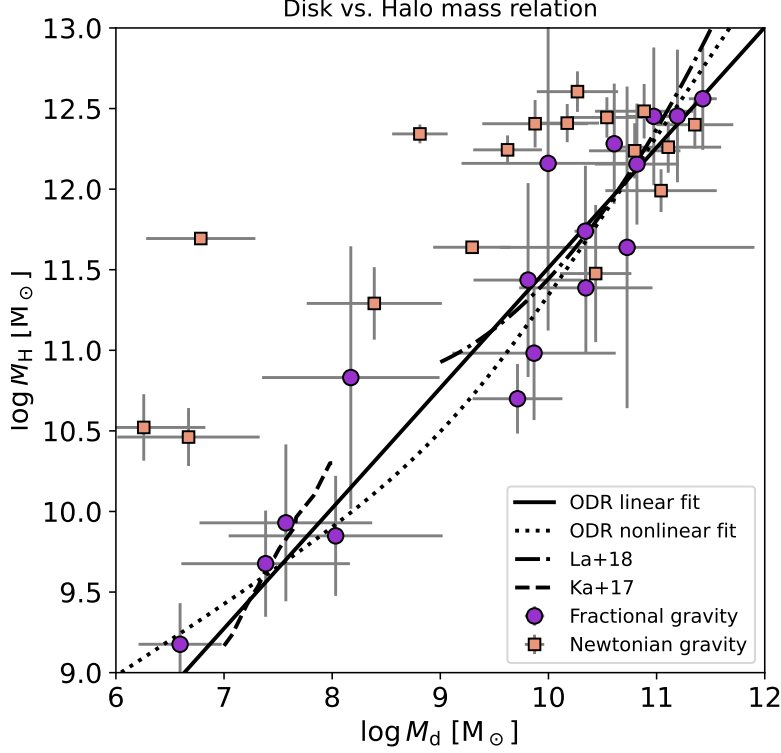


Figure 3.6. The disk (stellar) mass vs. halo mass relation from rotation curve modeling in Newtonian (orange squares) and in fractional gravity (magenta circles). The dotted and solid lines show linear and nonlinear orthogonal regression distance fits to the fractional gravity data (see text for details). For reference, the dot-dashed line is the average relation found by Lapi et al. (2018) and the dashed line is the one by Karukes & Salucci (2017) from the analysis of the galaxy RC with the empirical Burkert profile for the halo component.

between halo mass and baryonic properties; then we sample the RCs to derive the total and baryonic accelerations and construct the RAR. Such a procedure has been described in [4]; see their Section 4, to which we defer the reader for details. The only difference with respect to that work is that the RC of the DM component is modeled in fractional gravity (with a NFW density profile).

The RAR is illustrated in Fig. 3.7. The observational determination are shown in grey: data for spiral galaxies (binned) are from McGaugh et al. ([157]; squares), for Local Group dwarf spheroidal from Lelli et al. ([43]; triangles), for Coma Cluster Ultra Diffuse Galaxies from Freundlich et al. ([159]; reversed triangles), for Centaurus A dwarf spheroidals from Muller et al. ([123]; diamonds), and for dwarf LSB with $g_{\text{bar}}(0.4 \lesssim r/R_{\text{opt}} \lesssim 1) < -11$ from Di Paolo et al. ([44]; stars). For reference, the dotted black line displays the one-to-one relation $g_{\text{tot}} = g_{\text{bar}}$.

The big green circles illustrate the outcome for $s = 1$ (independent of ℓ and corresponding to Newtonian gravity); as pointed out in [160] the observed RAR is reasonably reproduced at large baryonic acceleration g_{bar} , but it is considerably overpredicted at small g_{bar} . The big red and blue circles show the outcomes in fractional gravity for $s = 1.2$ and $s = 1.4$, respectively, and $\ell/r_s = 0.25$ (typical values from the stacked RC analysis); the shaded area displays instead how the predicted RAR changes when ℓ/r_s is varied in the range $0.1 - 0.5$ (extremal values from the stacked RC analysis). Finally, the solid black line is the outcome by assuming the scaling of s and ℓ/r_s with halo mass M_H from the stacked RC analysis (see Fig. 3.5). All in all, the RAR in fractional gravity has a shape and an overall scatter pleasingly consistent with the data.

These results can be understood as follows. The high acceleration regime of the RAR is mainly dominated by the contribution at small radii in high mass galaxies, where the total gravitational acceleration is anyway dominated by baryons, to imply $g_{\text{tot}} \approx g_{\text{bar}}$. Contrariwise, at small g_{bar} the behavior of the RAR is dictated by small/intermediate radii in intermediate and low mass galaxies, where the baryon acceleration is dominated by the stellar disk, while the total gravitational acceleration is contributed by both the disk and the halo; for the NFW density profile in Newtonian gravity the halo typically dominates over the disk and the RAR deviates upward with respect to the high acceleration regime. However, in fractional gravity such an upward deviation is mitigated because the inner RC corresponding to the NFW density cusp is somewhat steepened $v_{\text{DM}} \propto r^{s-1/2} (\ell/r_s)^{1-s}$, hence the total acceleration toward the center is appreciably smaller than in Newtonian dynamics, especially for s approaching the limiting value 1.5 and for higher ℓ/r_s ratios. We conclude by noticing that this behavior is different from intrinsically cored models in Newtonian dynamics, that as demonstrated in [4] (see their Figure 7) tend to enforce a low-acceleration behavior similar to (or only slightly different from) the high-acceleration regime.

3.3.3 Universal Core Surface Density and Core Radius vs. Disk Scale-length

It has been well established [45, 161, 162, 163] that, when fitting galaxy RCs with a cored profile like Burkert or a pseudo-isothermal sphere, the product of the core density ρ_0 and core radius r_0 , which is a kind of core surface density, is an approximately universal constant among different galaxies; typical values of the latter have been estimated around $\rho_0 r_0 \approx 140^{+80}_{-30} M_\odot \text{ pc}^{-2}$ by Donato et al. [164], and around $\rho_0 r_0 \approx 75^{+85}_{-45} M_\odot \text{ pc}^{-2}$ by Burkert [163]. This universality in the core surface density is a somewhat unexpected property that poses a serious challenge to any theoretical model of core formation [107, 165]. We now aim to illustrate that halos in the fractional gravity framework are broadly consistent with such a remarkable scaling law.

To fairly compare with the literature universal core surface density relation, we need to identify an equivalent core density and core radius for a DM halo described by fractional

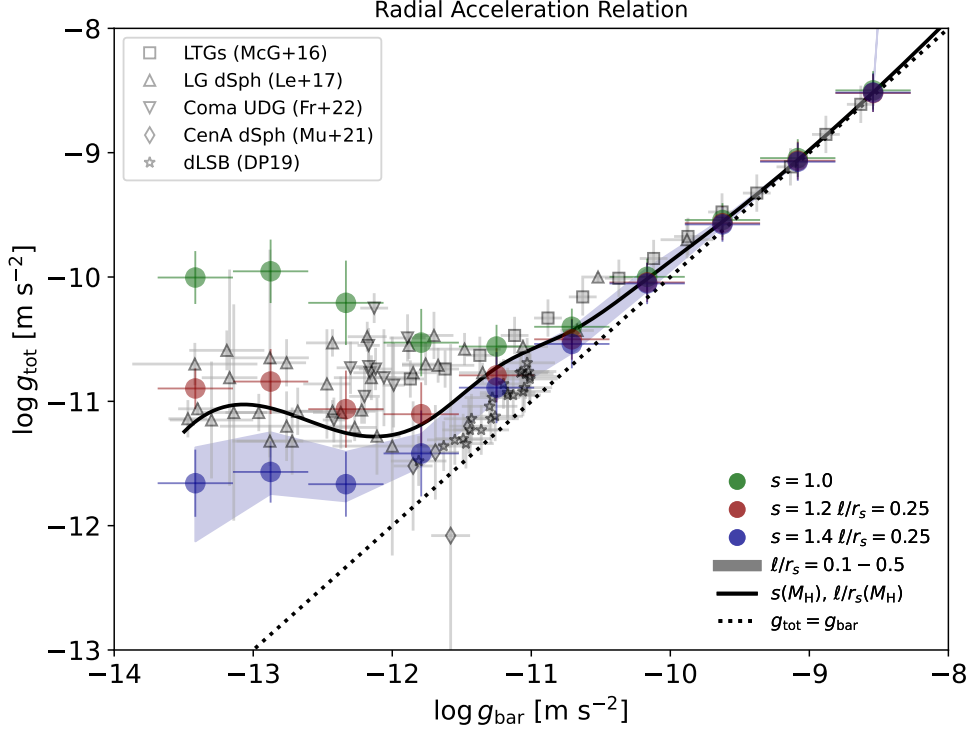


Figure 3.7. The radial acceleration relation (RAR). Green circles illustrate the outcome for $s = 1$ (independent of ℓ and corresponding to Newtonian gravity), red circles for $s = 1.2$ and $\ell/r_s = 0.25$, and blue circles for $s = 1.4$ and $\ell/r_s = 0.25$; the red and blue shaded area show the effect of varying ℓ/r_s in the range $0.1 - 0.5$. Finally the black curve is on adopting a halo mass-dependence in s and ℓ/r_s as emerging from our analysis of stacked rotation curve (see Fig. 3.5). For reference, the dotted black line displays the one-to-one relation $g_{\text{tot}} = g_{\text{bar}}$. Data for spiral galaxies (binned) are from McGaugh et al. (2016; squares), for Local Group dwarf spheroidal from Lelli et al. (2017; triangles), for Coma Cluster Ultra Diffuse Galaxies from Freundlich et al. (2022; reversed triangles), for Centaurus A dwarf spheroidals from Muller et al. (2022; diamonds), and for dwarf LSB with $g_{\text{bar}}(0.4 \lesssim r/R_{\text{opt}} \lesssim 1) < -11$ from Di Paolo et al. (2019; stars).

gravity. To this purpose, we start from the bestfit rendition to the stacked RCs of Section 3.2.1 in the fractional gravity framework, refit each of these RCs with an equivalent Burkert or pseudo-isothermal sphere RC via a least-square minimization algorithm, and finally extract the core density and core radius. The outcome of this procedure is illustrated in Figure 3.8 (left panel), where the core surface density is plotted against the core radius: magenta circles refer to the pseudo-isothermal sphere refitting, and cyan squares to the Burkert profile refitting. All in all, the measured core surface densities in the fractional gravity framework are broadly

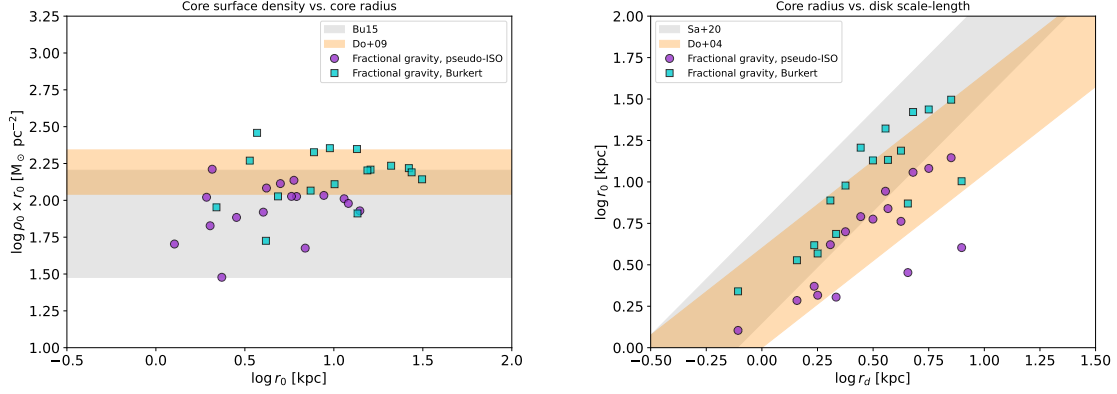


Figure 3.8. The core surface density $\rho_0 r_0$ vs. the core radius r_0 (left) and the core radius r_0 vs. the disk scale-length r_d (right). Purple circles and cyan squares illustrate the results from this work, when the core density and radius are determined by rendering our bestfit rotation curves from fractional gravity with the cored pseudo-isothermal or Burkert shapes (see text for details). In the top panel grey and orange shaded areas display the empirical scaling relations by Burkert (2015) and Donato et al. (2009), while in the bottom panel those by Salucci et al. (2020) and Donato et al. (2004).

consistent with the universal relation and its overall normalization.

Another interesting relation known to hold [46, 148, 149, 47] from dwarf to massive galaxies involves the core radius r_0 and the disk scale-length r_d . Its origin is uncertain, with some of the above authors arguing for an evidence in favor of a non-gravitational interaction between baryons and DM. To check such a relationship in fractional gravity, we extract r_0 as described above for every stacked RC bin, and plot it as a function of the average $r_d = r_{\text{opt}}/3.2$ in the bin. The outcome is illustrated in Figure 3.8 (right panel). We find a pleasing agreement with the determinations by Donato et al. [46] and Salucci et al. [47]. In fractional gravity the r_0 vs. r_d relation can be traced back to non-local effects, that ultimately enforce the scaling with halo mass of the core radius and of the disk mass, the latter being in turn trivially related to the disk-scale-length.

3.4 An alternative MONDian viewpoint?

As recalled in Chapter 1, the evidence for the presence of DM in the Universe goes much beyond galaxy kinematics; thus our fiducial approach in this work has been to explore whether fractional gravity associated to the DM component could possibly alleviate some of the small-scale issues emerging from the analysis of galaxy RCs, especially dwarfs. However, for the sake of completeness and to ease the comparison with other literature studies, it is worth exploring

an alternative MONDian viewpoint, where no DM component is considered, and fractional gravity is associated to the baryons. The underlying idea is that, similarly to the original version of the MOND or of other modified gravity theories, DM phenomenology could be emulated by baryons in fractional gravity.

We have repeated the analysis of Section 3.2 adopting this MONDian viewpoint. Specifically, we have assumed that no DM component is present, and we have fitted the stacked RCs using the circular velocity of a razor-thin exponential disk model in fractional gravity (provided in Appendix A); we recall that for the systems considered here, the contribution to the stacked RCs from any bulge or gas component can be safely neglected. In this setup the free parameters of the model are the disk mass M_d , the fractional index s and the fractional length-scale ℓ . As for ℓ , we have tried either to let it vary as a free fitting parameter, or to set it to the value $\ell \sim (2/\pi) \times \sqrt{G M_d / a_0}$ with $a_0 \approx 1.2 \times 10^{-10} \text{ m s}^{-2}$ that can be naively expected in a MONDian interpretation of the fractional gravity theory [137]; we find that the fits to the stacked RCs in these two cases are practically indistinguishable, so we focus on the latter one. The results of the fitting procedure, on a subset of representative RCs of our sample, are summarized in Fig. 3.9. In each panel, we illustrate the bestfit RC in the MONDian approach with no DM, and report for comparison the one from our fiducial setup with DM; we also indicate the corresponding bestfit estimates of the fractional index s and of the disk mass M_d .

An important point to stress is that the MONDian viewpoint shares with our fiducial setup a similar decreasing trend of s , from values $s \approx 1.5$ in small galaxies toward $s \approx 1$ in the largest systems. This is essentially imposed by the behavior of the measured RCs in the inner region, which behave like a solid body in dwarfs while are steeply rising in massive galaxies. However, it is seen that the MONDian viewpoint struggles somewhat in reproducing the outer behavior of the RCs, especially in small galaxies. This is because for s appreciably larger than one, the RC of a razor-thin disk in fractional gravity tends to saturate to a constant in the outer regions, while the measured RCs in small galaxies are quite steeply rising even in the outermost sampled point. Moreover, the fit of the MONDian viewpoint in the outer regions does not substantially improve in moving toward massive galaxies; this is because s progressively decreases (as required to fit the inner part, see above), so the outer RC in fractional gravity tends to recover the steeply decreasing trend of the Newtonian case, while the measured RCs retain a rather flat or modestly decreasing behavior.

One can speculate that perhaps the MONDian viewpoint can be reconciled with the observed RCs if the fractional index s is allowed to vary with the radius, to originate a sort of variable-order fractional theory (see discussion in [137]). However, some caveats are in order: first, non-trivial numerical techniques for solving the fractional Poisson equation must be developed in such a case, since the analytic treatment becomes impossible; second, it is not clear whether

conceptually the variable order theory is well posed, since fractional gravity is inherently non-local, hence a simple dependence $s(r/\ell)$ on the local radial coordinate does not seem easily justified; third, to describe the radial dependence of s other parameters must be introduced, which plainly make the theory subjects to additional degeneracies, and less elegant or predictive.

Another issue with the MONDian viewpoint is that, lacking the DM component, the correct normalization of the RC (even in the inner region) can be obtained only with disk masses systematically larger than in our fiducial setup. This may constitute a problem, especially in dwarf spheroidals, since the visible mass that can be inferred from photometry (I -band) could be appreciably smaller. In other words, the mass-to-light ratios that can be derived from the bestfit stellar mass in the MONDian viewpoint tend to be substantially larger than the typical values $M_*/L_I \lesssim 1.5$ expected on the basis of stellar population synthesis models [166]. To highlight this point, in each panel of Figure 3.9 we have reported the average I -band luminosity in the bin and the maximum disk mass that can be inferred from such a photometry, according to the relationship between luminosity and disk-to-total mass at the optical radius, estimated by Salucci et al. [167, 10]. There is clearly a tendency for the bestfit stellar masses in the MONDian viewpoint to exceed such values.

We also note that the values of the fractional length-scale derived in the MONDian viewpoint from the disk masses according to the relation $\ell \sim (2/\pi) \times \sqrt{G M_d/a_0}$ range from 0.3 kpc in dwarfs to about 10 kpc in massive galaxies. For comparison, in our fiducial approach with DM ℓ ranges from 0.3 kpc in dwarfs to a few tens kpc in massive galaxies (see right panel in Figure 3.5). Thus the overall dependence of ℓ on the total mass in the two approaches is rather similar within the respective fitting uncertainties.

In the context of the MONDian viewpoint, it is worth mentioning literature studies where fractional gravity is invoked as an alternative to DM in galaxies. A relevant example is the theory of Newtonian fractional-dimensional gravity by Varieschi [168, 169, 170, 171, 172, 173], that introduces a generalized law of Newtonian gravity in a spatial dimension $D < 3$, representing the local effective Hausdorff dimension of the matter distribution. Another approach by Calcagni [174, 175, 176, 177, 178] relies on multi-fractional spacetimes with variable Hausdorff and spectral dimensions, directly inspired from quantum gravity theories. At variance with our fiducial setup, both these theories adopt a viewpoint where DM is not present, and the galaxy kinematics is interpreted as a pure geometrical effect. Another important difference with our approach (even in the MONDian viewpoint with no DM) is that the above theories are formulated in terms of integer-order, local operators (though generalization are possible), whereas the fractional Laplacian exploited in this study is inherently non-local. Recently, both Varieschi's and Calcagni's theories have been tested against a few prototypical examples of individual galaxy RCs. The former decently reproduce the measured RCs, though at the price

of introducing a radially-dependent Hausdorff dimension $D(r)$ different from every galaxy, and thus the fact of introducing an infinite number of degrees of freedom in the fitting procedure. The latter can accommodate the observed rising or flattening behavior of the observed RCs in the intermediate and outer regions, while struggles somewhat in fitting the inner behavior. Furthermore, both theories lack an extended analysis on different mass systems, and especially dwarf galaxies with poor baryonic content, where the gravitational field is dominated by dark matter at any radius.

3.5 Summary and conclusions

In this chapter we have introduced the Fractional Gravity framework and we have tested it against the stacked rotation curves of spiral galaxies of different sizes and types. All in all FG is able to reproduce very efficiently the observed galaxy kinematics. In particular in those systems, such as dwarfs, when CDM in Newtonian gravity yields a cuspy density profile at small radii, FG predicts instead the emergence of a cored-like distribution, in accordance with observations.

The best fit parameters of the MCMC procedure show that the strength of FG, represented by the fractional parameter s , decreases with the mass of the halo, thus implying that the needs of non-locality becomes less and less important in larger systems. Contextually, the fractional length scale ℓ increases approximately as the radius of the halo. The lesson to be learned is that the FG parameters are not universal constants, but should be thought of as emerging from the dynamics of the CDM in virialized systems.

We have then verified that CDM in Fractional Gravity is able to successfully reproduce the tight empirical relations between the halo and disk parameters observed in spiral galaxies: from the disk vs halo mass relation, to the RAR, to the universal core surface density, to the core radius-disk scale length relation.

Finally, to ease the comparison with other literature studies, we have repeated the analysis of galaxy rotation curves adopting a MONDian view point, assuming that no DM component is present, and that baryons originate Fractional Gravity. Our results show that the MONDian viewpoint struggles in reproducing the outer behavior of the RCs, both in small and large galaxies. This is because the value of s is forced to be larger than one from the fitting procedure, in order to reproduce the innermost part of the RC. This fact, in the case of an exponential disk profile, leads to a decreasing of the RC at large radii, thus failing to reproduce the observed rising or saturation of in the outermost points.

The trend found for the fractional parameters s and ℓ suggests that fractional gravity effects are more relevant in smaller systems, while they tend to weaken in larger, more massive systems. In the next chapter we will explore in more detail this trend, extending our analysis to both

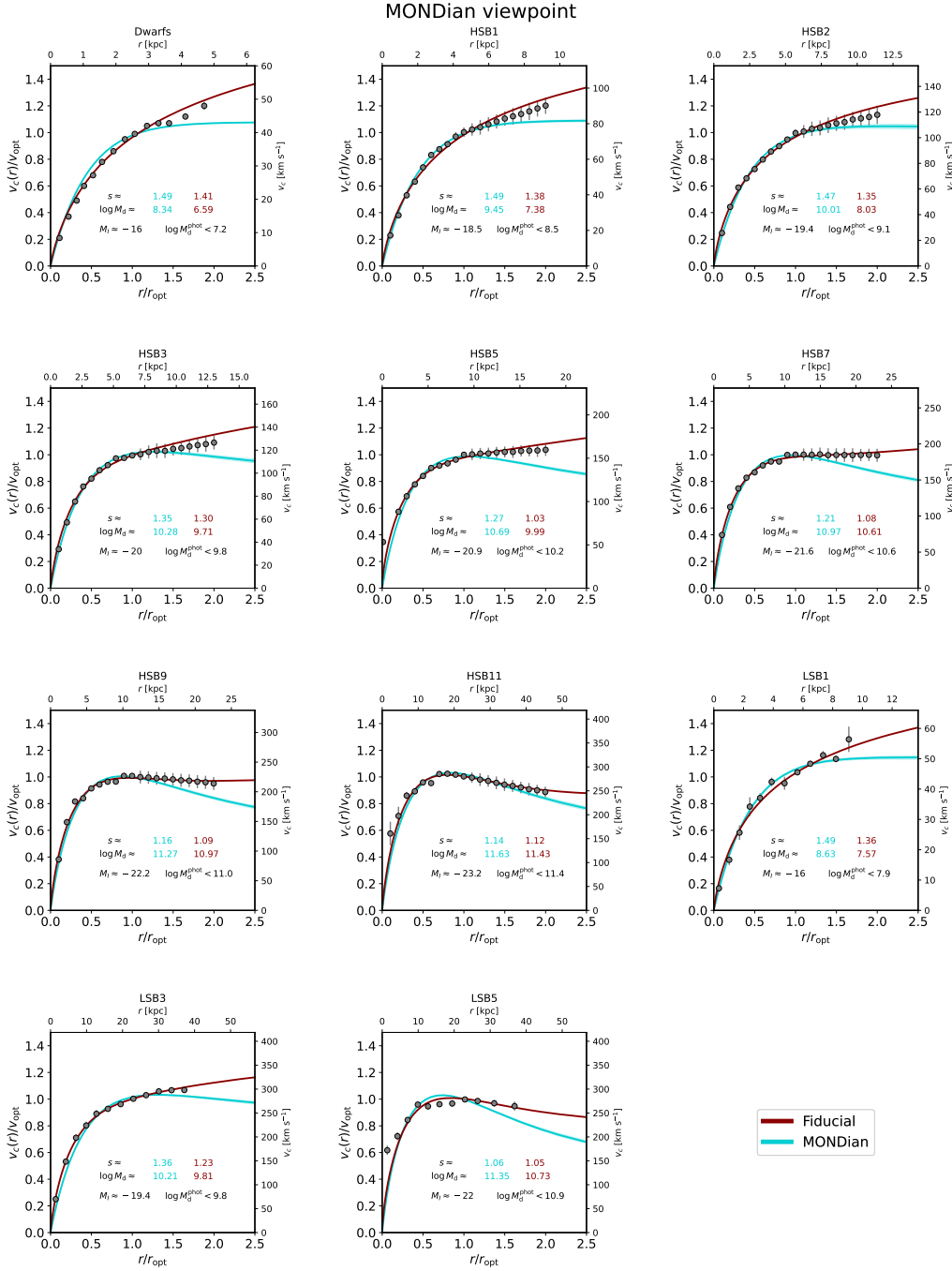


Figure 3.9. Figure showing results for the MONDian viewpoint of fractional gravity (see Section. 3.4). For a subset of representative stacked RCs we illustrate in red the bestfit from our fiducial setup of fractional gravity with DM, and in cyan the bestfit from the MONDian setup with no DM. In each panel we also report in color the corresponding bestfit values of the parameter s and $\log M_d [M_\odot]$, and in black the maximal disk mass expected on the basis of the average I -band magnitude of the bin and typical mass-to-light-ratios, as estimated from the scaling relation by Salucci et al. (2008; see also Persic & Salucci 1990). Note that in the MONDian viewpoint the fractional length-scale is related to the disk mass by $\ell \sim (2/\pi) \times \sqrt{G M_d/a_0}$, and ranges from ≈ 0.3 to ≈ 10 kpc when moving from dwarfs to massive galaxies.

smaller and larger systems, studying individual Milky Way satellites and galaxy clusters. This will help us to have a grasp of fractional gravity effects over a range of ten orders of magnitude in Dark Matter mass, from $10^5 M_\odot$ to $10^{15} M_\odot$.

Chapter 4

Individual tests at small and large scales

In the last chapter we have tested our fractional gravity (FG) framework against the stacked rotation curves (RCs) of galaxies of different sizes and types: from dwarfs to high and low surface brightness spirals. We have found that FG performs very well in fitting the RCs, attaining better results with respect to Newtonian gravity, and that the empirical relations observed between the properties of galaxies and their dark matter (DM) halos emerge as a natural consequence of non-locality.

In this chapter, we aim to extend our investigation on FG to much smaller and larger scales and test its effects in individual systems. First we study galaxy clusters, the largest bound structures in the Universe, with DM masses of $10^{14-15} M_{\odot}$, and size of few Mpc. To this purpose, we forward model the density and pressure distributions of the intracluster medium (ICM), working out the hydrostatic equilibrium equation in FG. Such theoretical framework is then compared with data from the XMM-Newton Cluster Outskirts Project (X-COP) [179, 180, 181, 182], which consists of 12 clusters with well-observed X-ray and Sunyaev–Zel’dovich (SZ) data, providing density and pressure profiles over an extended radial range of ~ 0.2 -2 Mpc. We then perform a Bayesian analysis of the thermodynamic profiles of the X-COP sample and infer constraints on the FG parameters at these scales, for individual clusters and also for clusters stacked together.

We then turn our attention to the smallest scales, where the FG effects are expected to be more relevant, focusing on individual dwarf galaxies. Specifically, we probe FG and infer the values of its parameters both in irregular dwarf (dwIrr) galaxies, by exploiting the rotation velocity profile inferred from stellar and gas kinematical measurements, and in dispersion-dominated dwarf spheroidals (dwSph) by exploiting the velocity dispersion profile inferred from the dynamics of stellar tracers.

The analysis presented in this chapter will help us to derive a clear-cut trend for the strength of FG effects over ten orders of magnitudes in DM mass, from dwarf galaxies to galaxy clusters.

This chapter is structured as follows: in Section 4.1 we give an introduction to galaxy clusters, and we explain how we can probe the density and pressure distribution of the ICM, starting from X-ray and SZ data; in Section 4.2 we present the results of our bayesian data analysis, deriving the density and pressure profiles of the individual and stacked clusters, and the scaling of the fractional parameters with mass of the system. Then, we turn to dwarf galaxies: in Section 4.3 we provide an introduction to dwIrr and dwSph and we describe the relevant observables; in Section 4.3.2 we present the related bayesian analysis, resulting in the fit to the RCs and dispersion velocities, and to an extension of the scaling of ℓ and s to small scales. Finally, in Section 4.2.2 we summarise our findings and highlight future perspectives. Throughout the work, we adopt the standard, flat Λ CDM cosmology [183] with rounded parameter values: a matter density $\Omega_M \approx 0.3$, a baryon density $\Omega_b \approx 0.05$, the Hubble constant $H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$, with $h \approx 0.7$.

4.1 Galaxy Clusters: Theoretical Background

In this section, we give an introduction to galaxy clusters, discussing their characteristic features and physical properties. We then present the fundamental equations necessary to model the ICM in hydrostatic equilibrium and to derive its pressure profile in the context of the FG framework.

Galaxy clusters constitute the largest bound structures in the Universe, with DM masses $M \sim 10^{14-15} M_\odot$ and sizes extending out to a few Mpcs. In these systems, most of the baryons are in the form of a hot diffuse gas, referred to as the intracluster medium (ICM), with a mass ratio over the DM very close to the cosmic fraction $\Omega_b/\Omega_M \approx 0.16$ [183].

The density $n(r)$ and temperature $T(r)$ distributions of the ICM throughout the cluster can be probed thanks to the copious X-ray powers $L_X \propto n^2 \sqrt{T} R^3 \sim 10^{44-46} \text{ erg s}^{-1}$ emitted by the ICM via thermal Bremsstrahlung and high-excitation lines [184, 185]. The inferred high average temperatures $k_B T \sim \text{several keV}$ s and low average number densities $n \sim 10^{-3} \text{ cm}^{-3}$ make the ICM the best plasma in the Universe ever, with thermal to electrostatic energy ratios $k_B T / e^2 n^{1/3} \sim 10^{12}$.

In addition, the pressure distribution $p(r)$ can be probed thanks to the SZ [186, 187]) effect, arising when the hot ICM electrons Compton upscatter the CMB photons crossing the cluster, tilting the latter's black-body spectrum toward high energies. In the microwave band, such a tilt mimics a diminution of the CMB temperature proportional to the Comptonization parameter $y \propto \int d\ell p(r)$, which encompasses the line-of-sight integral of the pressure profile. Combining

X-ray and SZ data allows one to reconstruct the ICM thermodynamic profiles throughout most of the cluster volume, from the center to a few times R_{500} or even beyond the virial boundary ¹.

In massive and sufficiently relaxed clusters, the ICM is expected to settle in hydrostatic equilibrium within the overall gravitational potential well mainly provided by the DM component. Under this assumption, the gas density profile reconstructed from X-rays and the gas pressure profile from SZ data can be combined to probe the shape of the DM gravitational potential and check whether this is consistent with the DM density run extracted from N -body simulations in the Λ CDM cosmology. This is the rationale of many investigations aimed at exploiting galaxy clusters to probe modified gravity scenarios [136, 188, 189, 190, 191, 192, 193] which have been developed to solve cosmological problems such as the origin of dark energy [194, 195, 196], and/or to alleviate small-scales issues of the standard CDM paradigm [160, 197, 198, 199, 200, 3].

4.1.1 Forward Modeling of the ICM Thermodynamics

Assuming hydrostatic equilibrium and spherical symmetry, the distribution of the ICM in the overall gravitational potential well is ruled by the equation

$$\frac{1}{\rho_{\text{gas}}} \frac{dP_{\text{gas}}}{dr} = - \left| \frac{d\Phi}{dr} \right|, \quad (4.1)$$

where $\Phi = \Phi_{\text{DM}} + \Phi_{\text{gas}}$ is the total gravitational potential with main contributions from DM and gas, ρ_{gas} is the gas mass density, and P_{gas} is the gas pressure.

One can conveniently write $\rho_{\text{gas}} = \mu m_p n_{\text{gas}}$ in terms of the mean molecular weight $\mu \approx 0.6$ and of the gas number density n_{gas} , which is in turn easily related to the electron density by the expression $n_{\text{gas}} \approx 1.8 n_e$, applying for a fully ionized plasma at high temperatures and a subsolar chemical composition typical of the ICM. The observed electron density profile $n_e(r)$ of individual clusters inferred from X-ray observations is often empirically rendered by the (simplified version of the) Vikhlinin profile [201]

$$n_e(r) = n_0 \frac{(r/r_c)^{-\alpha/2} [1 + (r/r_t)^{-\epsilon/6}]}{[1 + (r/r_c)^2]^{3\beta/2 - \alpha/4}}; \quad (4.2)$$

where n_0 is the central density, r_c and r_t are a core and a transition radius ($r_c < r_t$), α , β , and $\epsilon < 5$ are three slopes characterizing the inner, intermediate, and outer radial behavior. The gas mass can then be computed as $M_{\text{gas}}(< r) = 4\pi \int_0^r dr' r'^2 \rho_{\text{gas}}(r')$ and the gas contribution to the hydrostatic balance is fully specified by $|d\Phi_{\text{gas}}/dr| = G M_{\text{gas}}(< r)/r^2$.

As to the DM contribution, we can exploit the results of the previous chapter and write

¹Hereafter, R_Δ indicates the radius where the average DM density is Δ times the critical density $\rho_c(z)$ at the redshift z of the cluster.

$|d\Phi_{\text{DM}}/dr| = G M_{\text{eff},s}(< r)/r^2$ in terms of the fractional gravity's effective mass $M_{\text{eff},s}(< r)$ illustrated in Figure 4.1, which depends on the parameters s and ℓ ; in the standard case (corresponding to $s = 1$), this is just the DM mass profile of Equation (1.6). The mass profile is also a function of the NFW scale radius r_s and mass M_s ; for the present analysis, it is convenient to trade off these parameters for the mass M_{500} and the concentration $c_{500} \equiv R_{500}/r_s$ at the reference radius R_{500} . The conversion between these variables can be performed easily using the relations $M_{500} = 4\pi 500 \rho_c(z) R_{500}^3/3$ and $M_{500} = M_s/[\ln(1 + c_{500}) - c_{500}/(1 + c_{500})]$ stemming from the definition of R_{500} and from the adopted NFW mass distribution.

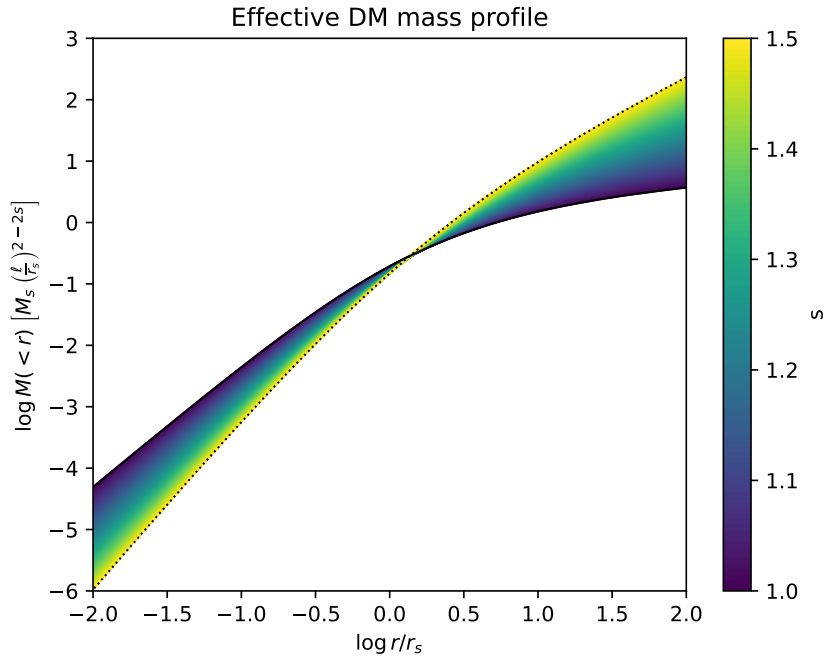


Figure 4.1. Effective DM mass profile (appropriately normalized in units of $M_s (\ell/r_s)^{2-2s}$) in fractional gravity as a function of the radial coordinate (normalized to the NFW scale radius r_s). The profile is shown for different values of the fractional index s (color-coded) from 1 (solid black line, corresponding to Newtonian gravity) to the limiting value $s = 1.5$ (dotted black line).

Then, the solution to the hydrostatic equilibrium equation (4.1) is given by

$$P_{\text{gas}}(r) = -1.8 \mu m_p \int_r^\infty dr' n_e(r') \frac{G [M_{\text{eff},s}(< r') + M_{\text{gas}}(< r')]}{r'^2}, \quad (4.3)$$

where the zero pressure at infinity has been taken as a boundary condition. ²

²Note that in computing the overall gravitational potential Φ , we have neglected the stellar contribution Φ_* , mainly originated by the brightest central galaxy; this would add a term $d\Phi_*/dr = G M_*(< r)/r^2$ to the

Observationally, X-ray surface brightness and spectroscopic data can probe the electron density $n_e(r)$ and the temperature $T_{\text{gas}}(r)$ profiles, whence the pressure profile $P_{\text{gas}}(r) \propto n_{\text{gas}}(r) T_{\text{gas}}(r)$ can be derived, although sensitivity and background issues make such a determination robust only in the region out to R_{500} . In the outskirts, SZ observations can complement X-ray data in probing the pressure profile, though with some caveats about conversion from line-of-sight-integrated to spherically averaged quantities. The rationale of the above forward modeling of the hydrostatic equilibrium is to test the fractional gravity parameters entering in the effective mass profile $M_{\text{eff},s}(< r)$ by simultaneously fitting the observed electron density profile via Equation (4.2) and the observed pressure profile via Equation (4.3).

4.2 Clusters' Bayesian Data Analysis

We tested the FG framework by exploiting the X-COP sample [179, 180, 181, 182] of 12 massive galaxy clusters in the redshift range $0.04 \lesssim z \lesssim 0.1$, and featuring typical sizes $R_{500} \sim 1\text{--}1.5$ Mpc and masses $M_{500} \sim 10^{14}\text{--}10^{15} M_{\odot}$. The X-COP clusters were selected to allow a robust reconstruction of the electron density and gas pressure profiles out to R_{200} via a joint analysis of high-quality X-ray data from *XMM-Newton* and of high signal-to-noise SZ observations from *Planck*. Another important property of the sample is that the hydrostatic equilibrium holds to a high accuracy, with at most mild levels on nonthermal pressure support in the outermost regions, as demonstrated by the analysis of [180, 182]; this is particularly important, since nonthermal effects can appreciably affect the mass estimation in the outer regions [202, 203] and potentially induce spurious effects in constraining modified gravity parameters [204]. All in all, X-COP is currently the largest cluster sample available so far for robust mass-modeling studies over an extended radial range, and as such it has been extensively exploited to probe modified-gravity scenarios [191, 192, 193, 136].

To estimate the parameters $\theta_{\text{M}} \equiv (s, \ell, c_{500}, M_{500})$ describing the effective mass profile $M_{\text{eff},s}(< r)$, alongside with those $\theta_{n_e} \equiv (n_0, \alpha, \beta, \epsilon, r_c, r_t)$ describing the electron density profile $n_e(r)$, we adopted a Bayesian framework and built the joint log-likelihood

$$\log \mathcal{L}(\theta) = \log \mathcal{L}_{P_{\text{gas}}}(\theta_{\text{M}}, \theta_{n_e}) + \log \mathcal{L}_{n_e}(\theta_{n_e}) . \quad (4.4)$$

Each term in the log-likelihood reads $\log \mathcal{L}(\theta) = -\chi^2(\theta)/2$, where the chi-square $\chi^2(\theta) = \sum_i [\mathcal{M}(\theta, r_i) - \mathcal{D}(r_i)]^2 / \sigma_{\mathcal{D}}^2(r_i)$ was obtained by comparing our empirical model expectations $\mathcal{M}(\theta, r_i)$ to the data values $\mathcal{D}(r_i)$ with their uncertainties $\sigma_{\mathcal{D}}(r_i)$, summing over the different

integrand on the right-hand side of Equation (4.3). For the X-COP cluster sample exploited in this work (stellar profiles were available for 5 out of 12 clusters), the related contribution has been shown by [182] to become relevant only for $r \lesssim 0.02 R_{500} \sim 20$ kpc and as such can barely influence the innermost available data point of the pressure profile; as a consequence, our results were negligibly affected, as we also checked numerically.

radial coordinates r_i of the data (approximately 65 points for n_e and 20 points per P_{gas} , with small variations around these numbers from cluster to cluster); note that for the pressure data from SZ observations, we took into account the full covariance matrix.

We adopted flat priors $\pi(\theta)$ on all the parameters; specifically, for those entering the effective mass profile in fractional gravity we took $s \in [1, 3/2]$, $\log \ell \text{ (Mpc)} \in [-3, 3]$, $\log c_{500} \in [-2, 2]$, $\log M_{500}(M_\odot) \in [13, 16]$. We then sampled the parameter posterior distributions $\mathcal{P}(\theta) \propto \mathcal{L}(\theta) \pi(\theta)$ via the MCMC Python package `emcee` [152], running it with 10^4 steps and 200 walkers for every individual cluster; each walker was initialized with a random position uniformly sampled from the (flat) priors. After checking the auto-correlation time, we removed the first 20% of the flattened chain to ensure burn-in; the typical acceptance fractions of the various runs were in the range 30-40%.

4.2.1 Results

We first tested the performance of FG by stacking the X-COP data of all the clusters in the sample. Specifically, we built stacked electron density and pressure profiles by normalizing the individual profiles of the 12 clusters at a reference radius R_{500} , by co-adding them in radial bins of normalized radii r/R_{500} , and by computing the corresponding mean and standard deviation. The outcome of this procedure is illustrated in Figure 4.2: the crosses mark the stacked profiles, and for reference, the gray lines show the individual ones. All in all, the FG frameworks fit the stacked profiles to a remarkable accuracy.

In Figures 4.3 and 4.4, we illustrate the outcome of the fitting procedure on the 12 individual pressure and density profiles of the X-COP sample. In each panel, the best fit (solid lines) and the 2σ credible intervals sampled from the posterior (shaded areas) are shown. The reduced χ_r^2 value of the joint fit to the pressure and density profiles is also reported in Figure 4.3. Overall, the fits in the FG framework are very good. In a few cases (such as A3266 and A2319), the reduced χ_r^2 is somewhat large, but this should not raise any alarm, since the outcome is caused by some peculiar feature in the density profile reconstructed from X-ray data (oscillation in the data points at intermediate radii) or because of some outlier data in the pressure profile reconstructed from SZ (especially in the innermost or outermost radii); note that we retained all data points in our analysis, including them in the reduced χ_r^2 computation.

In Figure 4.5, we illustrate the MCMC posterior distributions for two representative clusters in the sample, namely A2255 and ZW1215; for clarity, we restricted the plot to the subspace of parameters entering the effective mass profile. Magenta/contour lines display the results in our fiducial setup, where no mass prior was imposed; the white cross marks the best-fit value of the parameters.

The corner plots illustrate a clear degeneracy between the fractional length-scale parameter

ℓ and the DM mass M_{500} . This is somewhat expected since the effective mass profile entering the hydrostatic equilibrium equation scales like $M_{500} \ell^{2-2s}$. Therefore, it is possible to obtain the same normalization of the pressure profile, at a given density profile, by changing M_{500} and ℓ in the same direction. Since s does not deviate much from unity, the ℓ dependence is weak, implying that to compensate a rather small change in mass requires a substantial variation in ℓ ; on the other hand, this is also at the origin of the rather loose constraints that can be derived on the parameter ℓ with the present cluster sample.

The situation is expected to improve if a mass prior from other probes such as weak lensing (WL) is introduced in the analysis. However, one must be careful and use WL mass estimates that are independent from assumptions on the shape of the lensing potential; this is because in FG, the lensing potential corresponding to a given mass distribution would be different from the standard case, thus causing an inconsistency. For five X-COP clusters (A85, A1795, A2029, A2142, and ZW1215), such nonparametric WL mass determinations are available in the literature [205].³ The outcome of exploiting the WL mass prior on the marginalized distributions of the parameters is illustrated by the cyan contours/lines in Figure 4.5. The DM mass posterior estimate of ZW1215 is made considerably more precise, and as a consequence of the above degeneracy, the estimate of the fractional length scale ℓ is also appreciably tightened. In any case, the posterior distributions on all the parameters for the analysis without and with the WL mass prior are consistent within 1σ .

Figure 4.6 summarizes the posterior distributions of the fractional index s , fractional length scale ℓ , concentration c_{500} , and DM mass M_{500} . Table 4.1 reports the marginalized posterior estimates (mean and 1σ credible intervals) of these parameters for all the individual X-COP clusters (including the WL mass prior when available), and for the stacked sample. On average, it is seen that the deviations of the fractional index s from unity are modest in clusters, and this originates rather loose constraints on the length scale ℓ . The inferred values of the DM mass M_{500} and concentration c_{500} are reasonable and consistent with that estimated by a variety of other methods in standard gravity [182]; we also checked that the same agreement applied for the gas fraction, as expected given the very good fits to the gas density profiles.

In Figure 4.7, we checked the concentration vs. the DM mass relation for the X-COP sample in FG. To fairly compare with the relation expected from N -body simulations in the Λ CDM framework, we converted our fitting variables c_{500} and M_{500} at a reference radius R_{500} to the corresponding values c_{200} and M_{200} at R_{200} ; this is a trivial rescaling given the adopted NFW density profile. In Figure 4.7, we show as filled magenta circles the outcome for individual

³Actually, in principle, fractional gravity can also alter somewhat the total depth of the gravitational potential, thus biasing the overall WL mass estimates; however, given that the fractional gravity masses estimated without WL prior and the Newtonian ones are consistent with each other within 2σ (see fifth and last column in Table 4.1), we ignored such a small bias and used the Newtonian WL masses as prior, with their uncertainties, in the fractional gravity analysis.

X-COP clusters and with a magenta cross that for the stacked sample. It is seen that the estimates of c_{200} and M_{200} in FG are fairly consistent in shape and scatter with the concentration vs. mass relation extracted from N -body simulations [151].

In passing, we note that the clusters A644, A1644, A2255, and A2319 have been shown not to favor an NFW mass profile, but rather a cored Burkert-like one (e.g., a Burkert, Hernquist, or pseudo-isothermal distribution) [180]. When forward modeling the pressure profiles in standard gravity with the NFW density distribution, this causes inconsistent results (especially in mass and concentration values) and/or a poor fit [191]. Contrariwise, such values and fits in FG stay reasonable and good, since the mass profile entering the hydrostatic equilibrium equation is not the true NFW mass, but the effective mass, which, as mentioned in Section 4.1.1, mirrors that of a cored profile. For these four clusters, we also checked that using a cored Burkert-like density distributions in place of the NFW one as an input in our FG framework did not substantially improve the fits to pressure profiles, and rather forced the fractional index to values $s \approx 1$ compatible with pure Newtonian gravity. In fact, fractional gravity actually reconciles the NFW density distribution from simulations with the observed galactic dynamics, which are empirically described via cored, Burkert-like profiles. Moreover, A2319 have been shown to be characterized by an appreciable nonthermal support in the outskirts [206], which causes some difficulties in forward modeling and fitting the pressure profiles via the usual hydrostatic equilibrium equation in standard gravity. Instead, curiously, in FG, the fits stay good, suggesting that such a nonlocal framework may constitute an effective rendition for the effects of a nonthermal support on the pressure distribution.

In Figure 4.8, we explore the scaling of the FG parameters with the DM mass. For this purpose, we put together the analysis of the X-COP clusters from this chapter, and the constraints coming from the fitting of stacked galaxy rotation curves explored in Chapter 3. These joint datasets covered six orders of magnitude in DM mass from $M_{200} \sim 10^9 M_\odot$ to $10^{15} M_\odot$. As to the fractional index s , we confirmed the decreasing trend with the DM mass, passing from values around $s \approx 1.4$ in dwarf galaxies, to $s \sim 1.2 - 1.3$ in intermediate mass galaxies, to $s \sim 1.1$ in massive galaxies and clusters. We described the s vs. M_{200} relation by a linear fit (dashed line) with shape $s = a + b(\log M_{200}(M_\odot) - 11)$ via an orthogonal distance regression (ODR) algorithm that took into account the error bars on both axis; we obtained the best-fit parameters $a = 1.24 \pm 0.02$ and $b = -0.057 \pm 0.006$ and a reduced $\chi_r^2 \approx 1.87$; a nonlinear fit (solid line) $s = (5/4) + (1/4) \tanh[c(\log M_{200}(M_\odot) - d)]$ interpolating between asymptotic values $s = 1$ and 1.5 at small and large masses yielded the best-fit parameters $c = -0.39 \pm 0.06$, $d = 10.76 \pm 0.25$ and a reduced $\chi_r^2 \approx 1.34$. As to the fractional length scale, there was an increasing trend with the DM mass, extending the results of Chapter 3 at the cluster scales. We fit the ℓ vs. M_{200} relation with a linear shape $\ell(\text{Mpc}) = a + b(\log M_{200}(M_\odot) - 11)$ via an ODR

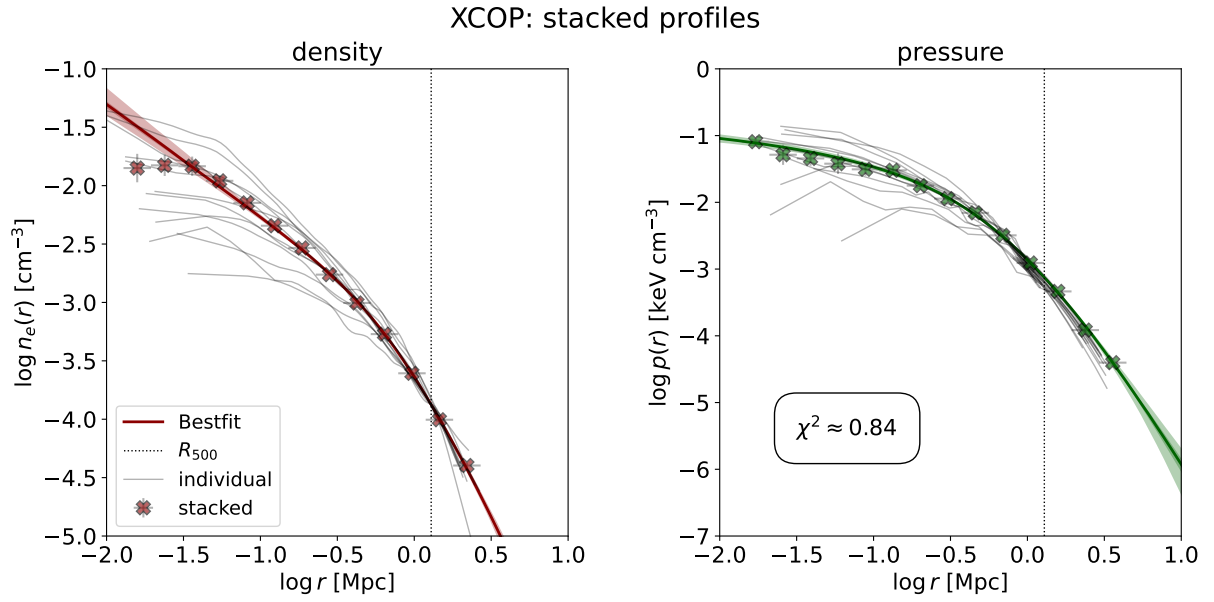


Figure 4.2. Fits to the stacked sample profiles of the X-COP clusters in fractional gravity, according to the Bayesian analysis described in Section ?? . Crosses show the stacked sample profiles, while gray lines show the individual ones. Colored solid lines illustrate the median, and the shaded areas show the 2σ credible interval from sampling the posterior distribution of the fits. The vertical dotted lines marks the average reference radius R_{500} of the sample. The χ_r^2 value of the joint fit to the density and pressure profile is reported in the right panel.

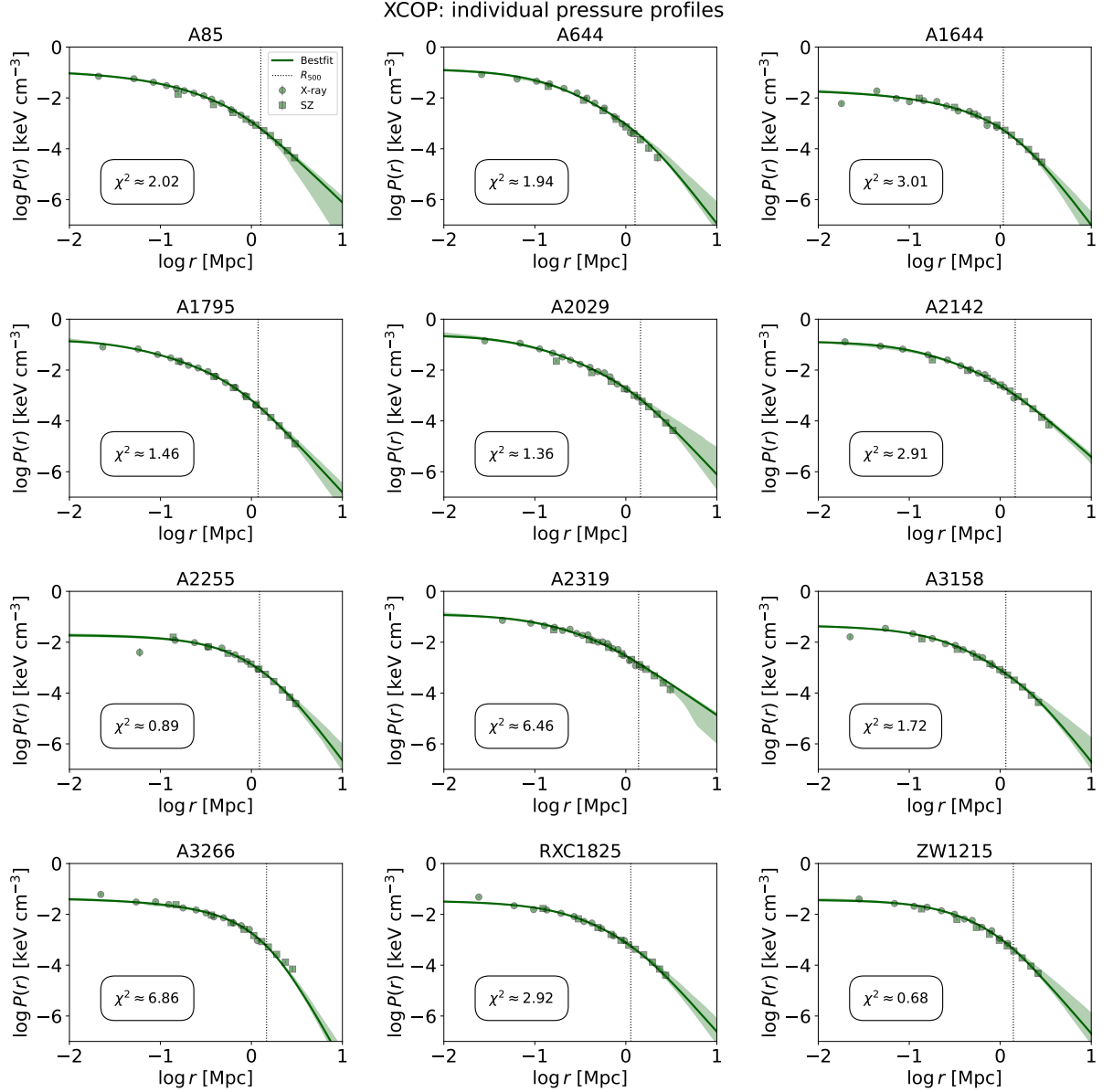


Figure 4.3. Fits to the individual pressure profiles of the X-COP clusters in fractional gravity, according to the Bayesian analysis described in Section ?? . Circles refer to X-ray and squares to SZ data. Solid lines illustrate the median, and the shaded areas show the 2σ credible interval from sampling the posterior distribution of the fits. The vertical dotted lines mark the reference radius R_{500} . The reduced χ_r^2 value of the joint fit to the pressure and density profiles is reported in each panel.

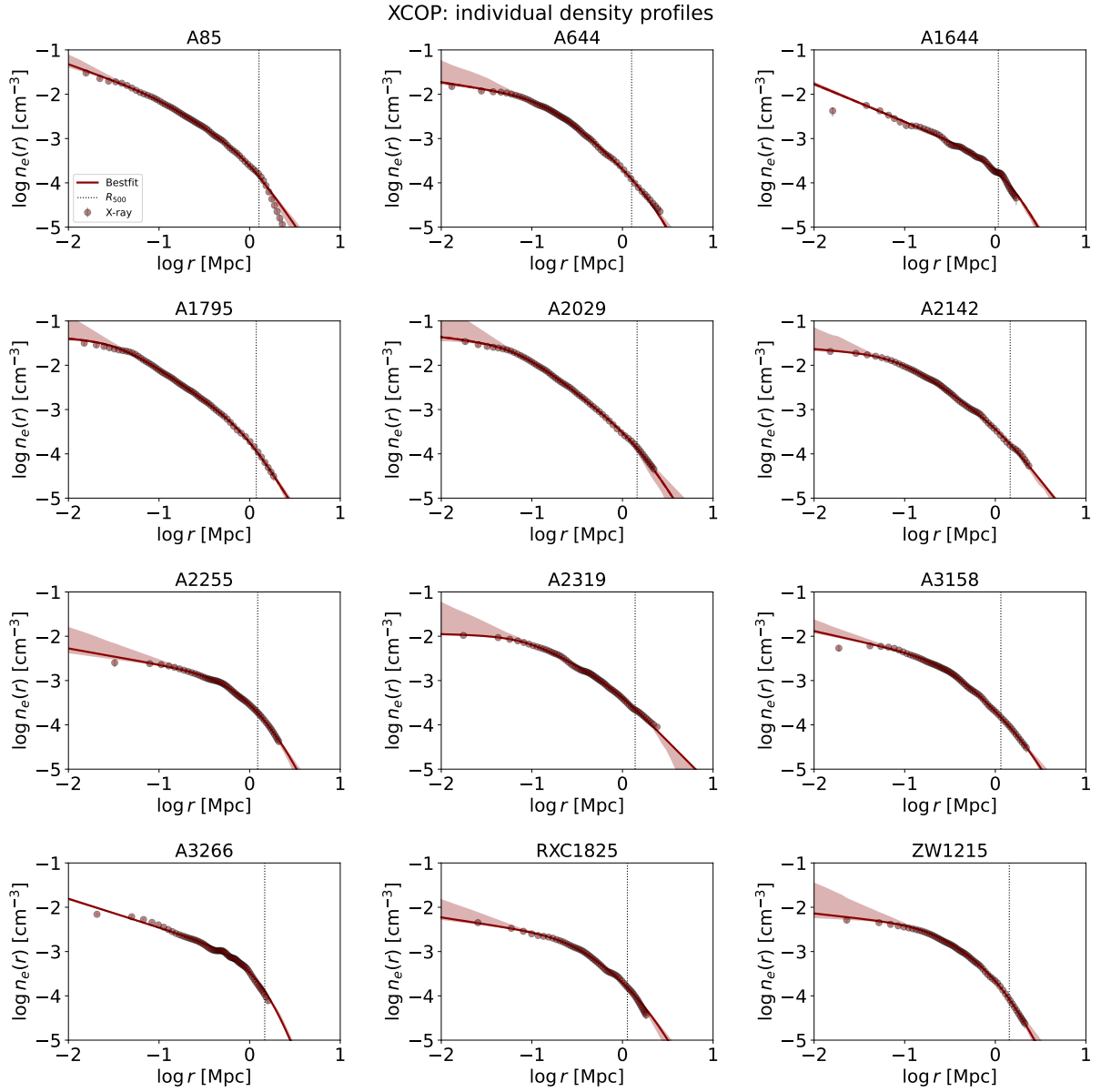


Figure 4.4. Same as Figure 4.3 for the density profiles.

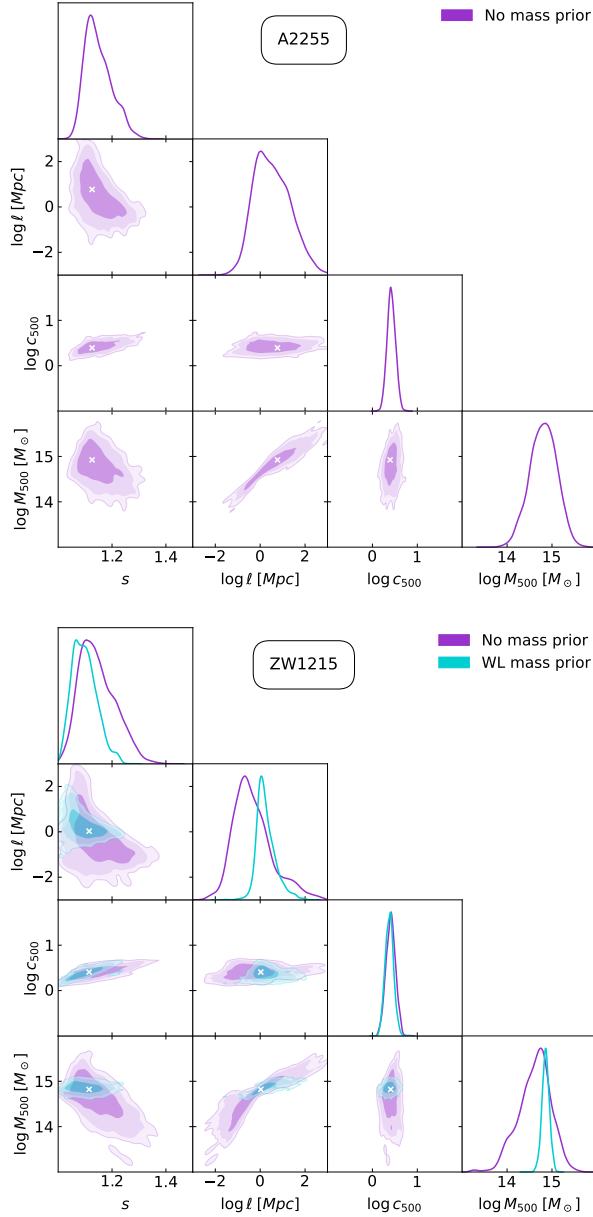


Figure 4.5. MCMC posterior distributions of the fractional parameter s , the fractional length scale ℓ , the DM concentration c_{500} , and the DM mass M_{500} , for two representative clusters of the X-COP sample: A2255 (**top**) and ZW1215 (**bottom**). Magenta contours/lines refer to the analysis with no mass prior, and cyan contours/lines to that with a weak lensing mass prior (only available for ZW1215 in the bottom panel). The contours show 1, 2, and 3 σ confidence intervals, white crosses mark the maximum likelihood estimates, and the marginalized distributions are in arbitrary units (normalized to unity at their maximum value).

Table 4.1. Marginalized posterior estimates (mean and 68% confidence limits are reported) for the parameters from the MCMC analysis of the X-COP sample in fractional gravity. Columns report the values of the fractional parameter s , the fractional length scale ℓ , the DM mass M_{500} , the halo concentration c_{500} , and the reduced χ_r^2 of the joint fit to the density and pressure profiles. The different portions of the table refer to the fit to individual clusters with no mass prior, to individual clusters with weak lensing mass priors (marked WL, and only available for five clusters from [205]), and to the stacked sample. For reference, the last column reports the best-fit DM mass M_{500} from [182].

Cluster	s	$\log \ell$ (Mpc)	$\log c_{500}$	$\log M_{500}$ ($M_\odot$)	χ_r^2	$\log M_{500}^{\text{Eck22}}$ ($M_\odot$)
A85	$1.11^{+0.04}_{-0.08}$	$-1.2^{+0.5}_{-0.8}$	$0.49^{+0.07}_{-0.1}$	$14.27^{+0.37}_{-0.23}$	2.02	14.70
A644	$1.02^{+0.01}_{-0.01}$	$-0.8^{+1.6}_{-1.6}$	$0.66^{+0.03}_{-0.03}$	$14.56^{+0.09}_{-0.04}$	1.94	14.89
A1644	$1.19^{+0.05}_{-0.09}$	$-0.4^{+0.5}_{-0.9}$	$0.49^{+0.09}_{-0.09}$	$14.19^{+0.46}_{-0.35}$	3.01	14.49
A1795	$1.05^{+0.01}_{-0.05}$	$-1.9^{+0.2}_{-1.1}$	$0.54^{+0.02}_{-0.04}$	$14.28^{+0.29}_{-0.12}$	1.46	14.65
A2029	$1.03^{+0.05}_{-0.05}$	$-0.4^{+1.6}_{-1.6}$	$0.55^{+0.02}_{-0.06}$	$14.70^{+0.18}_{-0.07}$	1.36	14.84
A2142	$1.03^{+0.02}_{-0.02}$	$-0.2^{+1.4}_{-2.1}$	$0.42^{+0.04}_{-0.07}$	$14.76^{+0.15}_{-0.11}$	2.91	14.95
A2255	$1.15^{+0.04}_{-0.06}$	$+0.5^{+0.7}_{-0.9}$	$0.42^{+0.09}_{-0.10}$	$14.81^{+0.31}_{-0.31}$	0.89	14.69
A2319	$1.01^{+0.02}_{-0.02}$	$-0.2^{+1.6}_{-1.6}$	$0.59^{+0.02}_{-0.03}$	$14.69^{+0.04}_{-0.02}$	6.46	14.90
A3158	$1.11^{+0.03}_{-0.09}$	$-0.9^{+0.6}_{-1.2}$	$0.48^{+0.07}_{-0.12}$	$14.18^{+0.43}_{-0.25}$	1.72	14.63
A3266	$1.09^{+0.02}_{-0.09}$	$+0.8^{+0.4}_{-0.9}$	$0.36^{+0.09}_{-0.21}$	$14.93^{+0.07}_{-0.07}$	6.86	14.87
RXC1825	$1.03^{+0.01}_{-0.02}$	$+1.4^{+0.6}_{-1.1}$	$0.56^{+0.04}_{-0.06}$	$14.59^{+0.04}_{-0.04}$	2.92	14.59
ZW1215	$1.15^{+0.05}_{-0.08}$	$-0.3^{+0.6}_{-1.0}$	$0.41^{+0.11}_{-0.11}$	$14.61^{+0.40}_{-0.31}$	0.68	14.85
A85+WL	$1.04^{+0.01}_{-0.02}$	$+1.3^{+0.8}_{-0.9}$	$0.49^{+0.06}_{-0.06}$	$14.79^{+0.09}_{-0.09}$	2.02	14.70
A1795+WL	$1.04^{+0.01}_{-0.01}$	$+2.3^{+1.5}_{-1.5}$	$0.68^{+0.05}_{-0.05}$	$14.81^{+0.08}_{-0.08}$	1.47	14.65
A2029+WL	$1.03^{+0.01}_{-0.01}$	$+2.0^{+0.9}_{-0.3}$	$0.61^{+0.05}_{-0.05}$	$14.95^{+0.06}_{-0.06}$	1.36	14.84
A2142+WL	$1.04^{+0.01}_{-0.02}$	$+1.9^{+0.8}_{-0.5}$	$0.51^{+0.05}_{-0.06}$	$15.01^{+0.06}_{-0.06}$	2.91	14.95
ZW1215+WL	$1.10^{+0.04}_{-0.05}$	$+0.3^{+0.3}_{-0.5}$	$0.38^{+0.09}_{-0.09}$	$14.86^{+0.08}_{-0.08}$	0.68	14.85
Stacked	$1.03^{+0.02}_{-0.02}$	$+0.3^{+1.3}_{-1.3}$	$0.43^{+0.04}_{-0.07}$	$14.76^{+0.08}_{-0.08}$	0.84	14.75

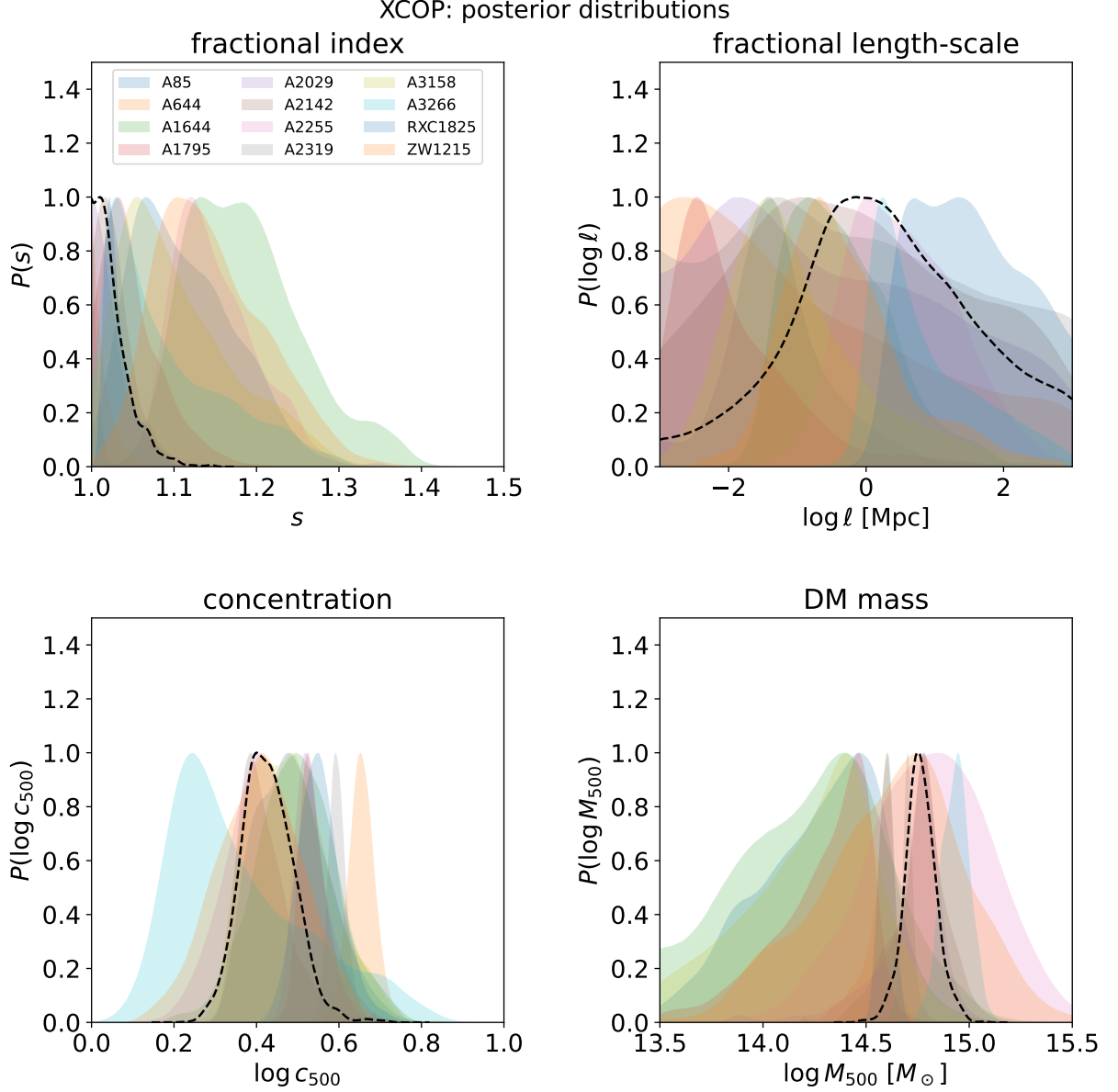


Figure 4.6. Posterior distributions (normalized to unity at their maximum values) of fractional index s (top left), fractional length-scale ℓ (top right), concentration c_{500} (bottom left) and DM mass M_{500} (bottom right), from the fits to the X-COP density and pressure profiles in fractional gravity. Colored areas refer to individual clusters (as detailed in the legend), and the black dashed line to the stacked sample.

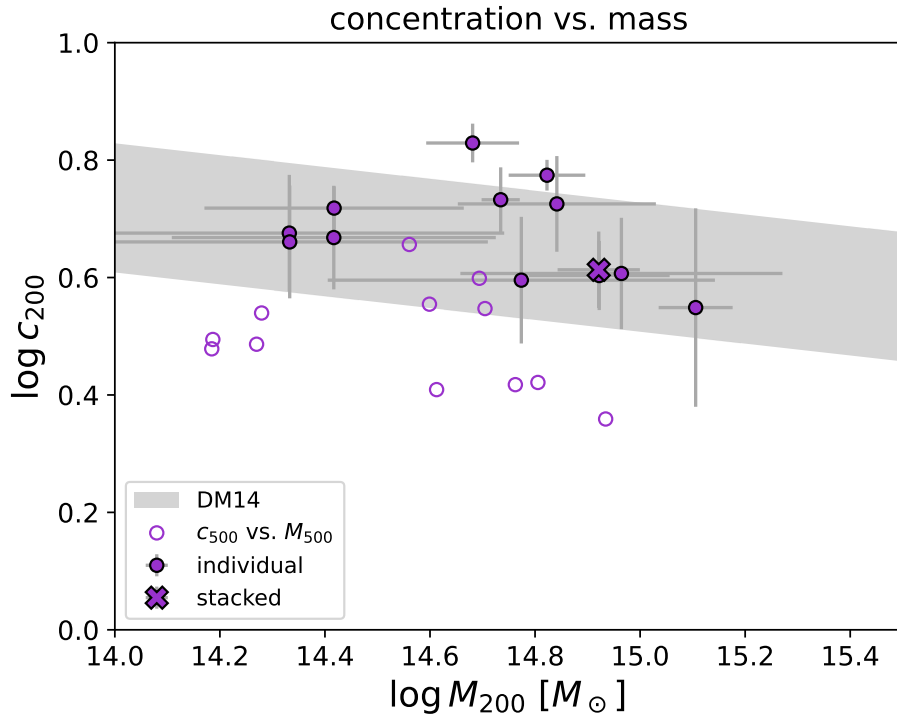


Figure 4.7. Concentration c_{200} vs. DM mass M_{200} relation for the X-COP sample in fractional gravity. Gray shaded area is the relation predicted from N -body simulations in the Λ CDM cosmology [207]. Filled magenta circles refer to individual clusters and the magenta cross to the stacked sample, while open magenta circles show (with no error bars for clarity) the corresponding c_{500} and M_{500} values that are the actual fitting parameters.

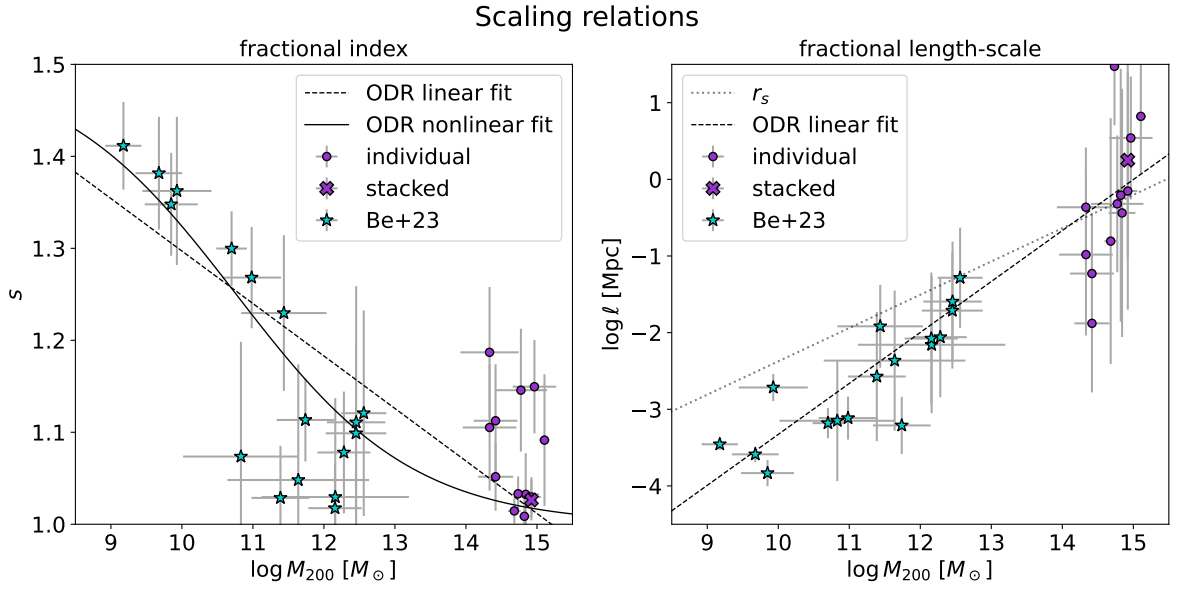


Figure 4.8. Scaling relations in fractional gravity from galaxies to clusters: fractional index s (**left**) and length-scale ℓ (**right**) vs. DM mass M_{200} . Magenta circles refer to individual clusters and magenta crosses to the stacked cluster sample, while cyan stars refer to the stacked rotation curve of galaxies analyzed in Chapter 3. Dashed lines display an ODR linear fit to the overall data, while a solid line in the left panel shows a nonlinear ODR fit with limiting values 1 and 1.5 from large to small masses, and a dotted line in the right panel illustrates the scale radius r_s of the NFW profile (see Section 4.2.1 for further details).

algorithm, to obtain the best-fit parameters $a = -2.66 \pm 0.09$, $b = 0.66 \pm 0.06$ and a reduced $\chi_r^2 \approx 1.09$. This relation was somewhat steeper than the scaling with the DM mass of the NFW scale radius r_s , in such a way that in dwarf galaxies $\ell/r_s \approx 0.25$ but this ratio increased to around one at the cluster scales.

4.2.2 Summary

From the analysis presented in this section we found that FG performed remarkably well in modeling the ICM profiles for the X-COP sample. We also checked that the relationship between the concentration of the DM profile and the DM mass still remained consistent with the expectations of N -body simulations in the Λ CDM framework. Finally, we confirmed the weakening of the fractional gravity effects toward more massive systems and derived the overall scaling of the fractional gravity parameters from dwarf galaxies to massive clusters, over six orders of magnitude in DM mass. Such an overall trend implies that FG can substantially alleviate the small-scale issues of the standard DM paradigm, while remaining successful on large cosmological scales.

The present analysis clearly shows that whatever the ultimate origin of the FG behavior in the DM component is, most of its effects manifest in small DM masses. This motivates us to turn our attention to individual dwarf galaxies, exploring whether FG correctly reproduces their characteristic velocity and dispersion distributions.

4.3 Dwarf Irregulars and Dwarf Ellipticals

In this section we aim at digging deeper into the regime where FG effects are expected to be more relevant, focusing on individual dwarf galaxies. In these objects the cusp-core problem is observationally very pressing, and its solution via baryonic effects is difficult to be envisaged given the paucity of baryons. Specifically, we probe FG both in irregular dwarf (dwIrr) galaxies by exploiting the rotation velocity profile inferred from stellar and gas kinematical measurements, and in dispersion-dominated dwarf spheroidals (dwSph) by exploiting the velocity dispersion profile inferred from the dynamics of stellar tracers.

Dwarf irregulars can be considered extreme late-type spirals; they usually present well defined rotation curves that can be obtained from HI measurements. With respect to standard spirals, dwIrr are considered more ideal to study the properties of DM, because the absence of a central bulge makes the analysis simpler, and also because they present a large gaseous component, which allows the rotation curves to be traced to much larger radii.

Dwarf spheroidals on the other hand are dispersion-dominated, faint objects of very low surface brightness, with very little dust or gas and an old stellar population. Their typical sizes

range from a tens to hundreds of parsecs. They are found in the Local Group as companions to the Milky Way and to the Andromeda Galaxy (M31). Ursa Minor, Carina, and Draco, the known dwarf spheroidal galaxies with the lowest luminosities, have mass-to-light ratios (M/L) greater than that of the Milky Way, implying a large amount of dark matter. The distribution of the latter in these systems can be inferred by looking at the velocity dispersion of tracers (e.g. stars), projected along the line of sight.

4.3.1 Dynamical modeling

For a rotation-dominated system, the crucial quantity to be compared with the data is the total rotation velocity, which is given by

$$v_{\text{rot}}^2(r) = v_{\text{halo}}^2(r) + \frac{M_{\star}}{L} v_{\text{disk}}^2(r) + v_{\text{gas}}^2(r) \quad (4.5)$$

where $v_{\text{halo}}^2 = G M_{\text{eff},s}(< r)/r$ is the contribution from the CDM halo, v_{gas}^2 is the gas contribution from HI measurements, and v_{disk}^2 is the contribution from the disk starlight appropriately converted into the stellar mass one via a global mass-to-light ratio $M_{\star}/L$. The overall rotation velocity depends on three parameters, namely the stellar mass-to-light ratio $M_{\star}/L$, the total mass of the system M_{200} , and the fractional index s .

For a dispersion-dominated system, the crucial observable is the velocity dispersion projected along the line-of-sight (l.o.s.), which is given by [208, 209]

$$\sigma_{\text{los}}^2(R) = \frac{2}{\Sigma_{\star}(R)} \int_R^{\infty} dr \left[1 - \beta \frac{R^2}{r^2} \right] \frac{\rho_{\star}(r) \sigma_r^2(r) r}{\sqrt{r^2 - R^2}} \quad (4.6)$$

where $\beta \equiv 1 - \sigma_{\theta}^2/\sigma_r^2$ is the anisotropy parameter (hereafter assumed constant with radius) and

$$\Sigma_{\star}(R) = 2 \int_R^{\infty} dr \frac{\rho_{\star}(r) r}{\sqrt{r^2 - R^2}} \quad (4.7)$$

is the surface mass density of the tracers (e.g., stars) in terms of the volume one $\rho_{\star}(r)$; in addition, the radial velocity dispersion is obtained by solving the Jeans equation, which yields

$$\sigma_r^2(r) = \frac{1}{r^{2\beta} \rho_{\star}(r)} \int_r^{\infty} dr' r'^{2\beta} \rho_{\star}(r') \frac{G [M_{\text{eff},s}(< r') + M_{\star}(< r')]}{r'^2} \quad (4.8)$$

in terms of the tracer mass $M_{\star}(< r) = 4\pi \int_0^r dr' r'^2 \rho_{\star}(r')$. Typically, for the dispersion-dominated galaxies considered in this work, stellar tracers are exploited, with a density distri-

bution following the Plummer’s model ⁴ [211]. In such a case $\rho_*(r) \propto (1 + r^2/r_{1/2}^2)^{-5/2}$ and $\Sigma_* \propto (1 + r^2/r_{1/2}^2)^{-2}$, with $r_{1/2}$ the 2D half-light radius; the normalization is derived from the observed surface luminosity profile via a global mass-to-light ratio M_*/L . The overall dispersion velocity depends on four parameters, i.e., the mass-to-light ratio M_*/L , the anisotropy parameter β , the mass of the system M_{200} and the fractional index s .

4.3.2 Dwarfs’ Data and Bayesian Analysis

We probe the FG framework by exploiting the rotation velocity profiles of dwIrr galaxies and the l.o.s. dispersion velocity profiles of dwSph.

For rotation-dominated systems, we rely upon the SPARC database [158, 212], which provides the stellar and gas contribution to the rotation velocity as found by numerically solving the standard Poisson equation for the observed surface brightness profile at $3.6\,\mu\text{m}$ (with a reference stellar mass-to-light ratio $M_*/L = 1$, so that this must be rescaled in building the total rotation velocity as in Equation 4.5) and the HI surface density profile. Specifically, we consider the 8 galaxies in SPARC that are classified as dwIrr and are flagged as having high quality data on the rotation curve: D631-7, DDO64, DDO161, UGC731, UGC5005, UGC5414, UGC7608, NGC3741. The latter is the galaxy with the best dataset, and actually constitutes the object with the most extended rotation curve (relative to the half-light radius) measured to date.

For dispersion-dominated systems, we consider 7 Milky-Way dwSph for which high quality determination of the spatially-resolved dispersion velocity has been obtained via stellar tracers ([213, 214]; see also data collection by [215] and references therein) and for which tidal effects are not appreciably influencing the inner kinematics: Carina, Leo I, Leo II, Sculptor, Draco, Sextans, and Fornax. To these we add the dispersion-dominated ultra-diffuse galaxy DragonFly 44 [210]. Note that ultra-diffuse galaxies are a mixed bag of objects with very different properties: some feature large angular momentum, a rich gas reservoir with ongoing active star-formation activity [216, 217, 218, 219]; others show no signs of rotation, a poor gas content and old stellar population in passive evolution [210, 220]. DragonFly 44 belongs to this last category, and being dispersion-dominated has been treated here along with the dwSph sample. In addition, it is a particularly interesting object since it features a very small stellar mass with respect to its large size; in fact, the DM mass is expected to dominate even at small radii. Therefore DragonFly 44 has been exploited as an useful laboratory to test the nature of DM and gravity [210, 221].

In Table 4.2 we report some relevant properties of the galaxies considered in our analysis. Specifically, the first two columns list the circularized half-light radius of the projected surface brightness profile and the total luminosity as determined from photometric observations (uncer-

⁴For DragonFly 44 we actually exploit a Sersic surface density profile, as detailed in [210].

Table 4.2. Properties of the galaxy sample considered in this work: half-light radius, total luminosity, and mass-to-light ratio estimated from stellar population synthesis models (used in building the priors of our Bayesian analysis, see Section ??). Top half of the table includes rotation-dominated dwarfs, for which the listed quantities refer to the luminosity at $3.6\,\mu\text{m}$; bottom half of the Table includes dispersion-dominated dwarfs, for which the listed quantities refer to the V -band luminosity.

Galaxy	$r_{1/2}$ [kpc]	$\log L [L_\odot]$	$M_\star/L [M_\odot/L_\odot]$
D631-7	1.22	8.28	0.5 ± 0.1
DDO64	1.22	8.18	0.5 ± 0.1
DDO161	2.04	8.74	0.5 ± 0.1
UGC731	1.40	8.51	0.5 ± 0.1
UGC5005	5.0	9.61	0.5 ± 0.1
UGC5414	2.33	9.05	0.5 ± 0.1
UGC7608	1.60	8.42	0.5 ± 0.1
NGC3741	0.32	7.45	0.5 ± 0.1
Carina	0.27	5.57	3.4 ± 2.9
Leo I	0.29	6.74	8.8 ± 5.6
Leo II	0.22	5.87	0.4 ± 0.4
Sculptor	0.31	6.36	3.6 ± 2.0
Draco	0.24	5.45	11.1 ± 4.7
Sextans	0.75	5.64	8.5 ± 3.3
Fornax	0.79	7.31	7.1 ± 6.0
DragonFly 44	3.87	8.37	1.5 ± 0.4

tainties are negligible for the purpose of this analysis); these quantities refer to the $3.6\,\mu\text{m}$ band for dwIrr and to the V -band for dwSph. The third column lists the stellar mass-to-light-ratio expected from stellar population synthesis models [222, 166, 223]: for disk-dominated dwIrr values $M_\star/L \approx 0.5$ applies with little uncertainties at $3.6\,\mu\text{m}$; for dwSph the $M_\star/L$ values are estimated from the $V - I$ color index and thus are more dispersed and uncertain [215, 210].

For our Bayesian analysis, we consider the parameter set $\theta_r \equiv (M_\star/L, M_{200}, s)$ for rotation-dominated dwarfs and $\theta_d \equiv (M_\star/L, \beta, M_{200}, s)$ for dispersion-dominated ones. To estimate these parameters, we adopted a Bayesian framework and built the Gaussian log-likelihood

$$\log \mathcal{L}(\theta) = -\chi^2(\theta)/2 . \quad (4.9)$$

where the chi-square $\chi^2(\theta) = \sum_i [\mathcal{M}(\theta, r_i) - \mathcal{D}(r_i)]^2 / \sigma_{\mathcal{D}}^2(r_i)$ is obtained by comparing our empirical model expectations $\mathcal{M}(\theta, r_i)$ to the data values $\mathcal{D}(r_i)$ with their uncertainties $\sigma_{\mathcal{D}}(r_i)$, summing over the different radial coordinates r_i of the data.

We adopt flat priors $\pi(\theta)$ on $s \in [1, 3/2]$ and on $\log M_{200} [M_\odot] \in [6, 13]$. Moreover, we

assume a lognormal prior on $\log M_*/L$ with average and dispersion as expected from stellar population synthesis models (see Table 4.2). As to β , since by definition it varies in the range $(-\infty, 1]$, we actually prefer to perform inference on the symmetrized version $\beta_{\text{sym}} \equiv \beta/(2 - \beta)$ that maps the original quantity in a compact domain $\beta_{\text{sym}} \in (-1, 1]$; a flat prior on β_{sym} within this range is used. Finally, to help robustly break any possible degeneracy between the halo and stellar masses, we follow [212] and add as a Λ CDM prior the stellar mass vs. halo mass relation derived from multi-epoch abundance matching by [156], which is also consistent with independent observational determinations from satellite kinematics [224], rotation curve modeling [149], and weak lensing analysis [225].

We then sample the parameter posterior distributions $\mathcal{P}(\theta) \propto \mathcal{L}(\theta) \pi(\theta)$ via the MCMC Python package `emcee` [152], running it with 10^4 steps and 100 walkers; each walker is initialized with a random position extracted from the priors discussed above. To speed up convergence, we adopt a mixture of differential evolution and snooker moves of the walkers, in proportion of 0.8 and 0.2 respectively, that emulates a parallel tempering algorithm. After checking the auto-correlation time, we remove the first 30% of the flattened chain to ensure burn-in; the typical acceptance fractions of the various runs are around 30%.

4.3.3 Results

The results of our Bayesian analysis for rotation-dominated dwIrr galaxies are displayed in Figures 4.9, 4.10 and in Table 4.3. Specifically, in the top panel of Figure 4.9 we illustrate the MCMC posterior distributions for two representative dwIrr in the sample, namely NGC3741 (the one with the best and most extended data) and DDO64. Red lines/contours refer to the outcomes for FG, and green ones for Newtonian gravity; the white crosses mark the best-fit value of the parameters in FG. In the bottom panel the best fit (solid lines) and the 2σ credible intervals (shaded areas) sampled from the posteriors are shown. The solid line is for the total rotation velocity, while the dashed and dotted lines show the halo and disk contribution, respectively; for comparison, the Newtonian bestfit to the total velocity is reported in green. In Figure 4.10 the bestfits in FG and in the Newtonian case are illustrated for the other 6 dwIrr in the sample. In Table 4.3 we summarize the marginalized posterior estimates of the parameters, both in FG and in the standard Newtonian case (marked with $s = 1$). Columns report the median values and the 1σ credible intervals of the stellar mass-to-light ratio M_*/L , of the DM mass M_{200} , and of the fractional index s ; the reduced χ_r^2 of the fit, and the Bayesian inference criterion (BIC) for model comparison are also reported.

The FG fits are always excellent, and comparable or better than those in Newtonian gravity. In particular, for D631-7, DDO64, DDO161, UGC5005, UGC5414, and NGC3741 there is a clear preference for FG both in terms of χ_r^2 and of the BIC. In such cases, the fractional index

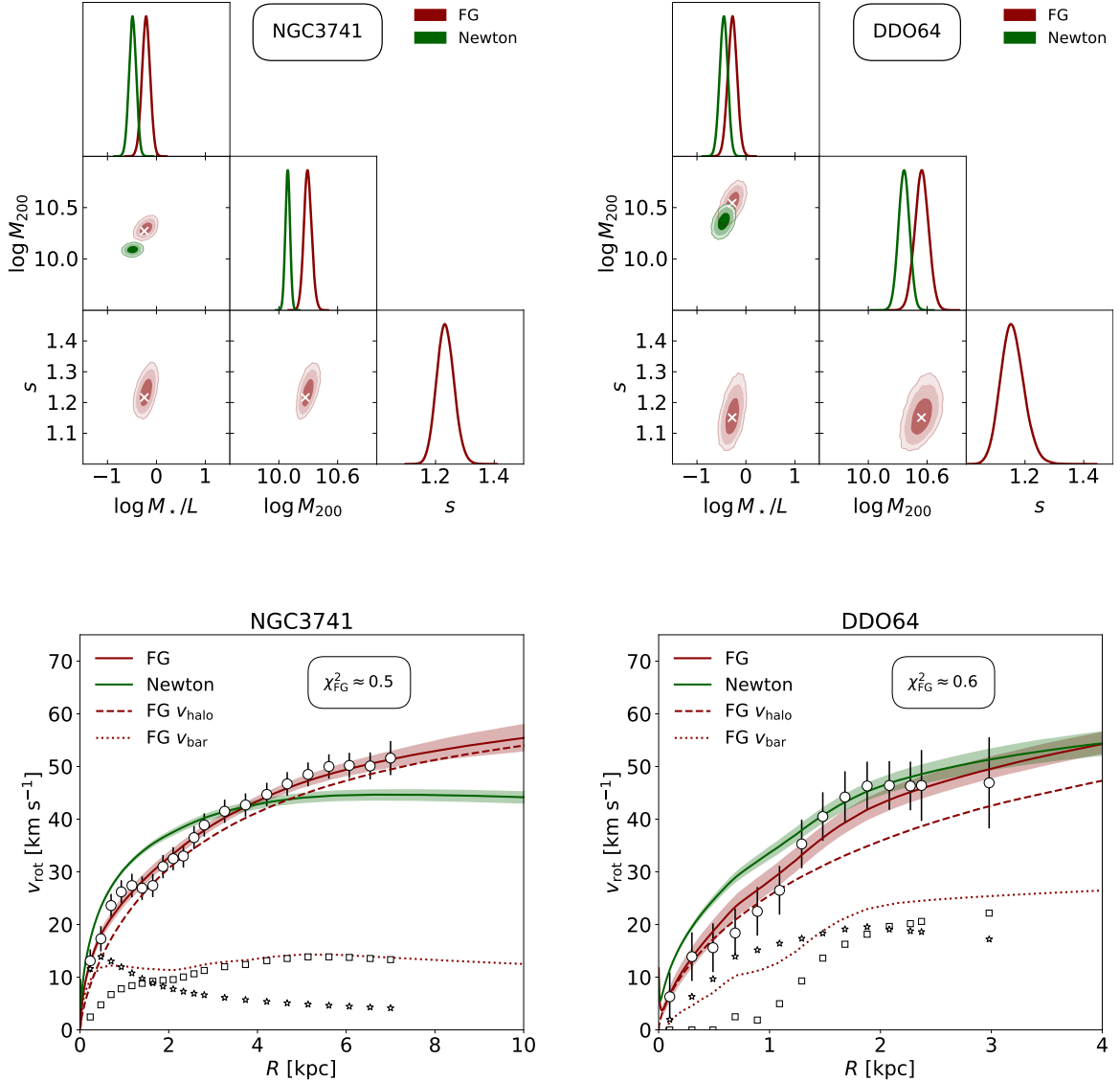


Figure 4.9. Top panel: MCMC posterior distributions of the stellar mass-to-light ratio M_*/L , the DM mass M_{200} and the fractional index s for the galaxy NGC3741 (left) and DDO64 (right). Colored contours/lines refer to the standard Newtonian (green) and to the FG framework (red). The contours show 1 – 2 – 3 σ confidence intervals, with the bestfit values in FG identified by white crosses. The marginalized distributions are in arbitrary units (normalized to 1 at their maximum value). Bottom panel: Fits to the rotation curve with the Newtonian (green) and the FG (red) framework for the galaxy NGC3741 (left) and DDO64 (right). Solid lines refer to the total rotation velocity, while (for clarity only in FG) the dashed line highlights the halo contribution and the dotted line the baryonic one. Solid lines illustrate the median, and the shaded areas show the 2 σ credible interval from sampling the posterior distribution. The value of the reduced χ^2_r of the fit for FG is also reported. Circles represent data from the SPARC database [158] for the total rotation curve, while the contributions from the stellar (for $M_*/L = 1$) and gaseous disk are highlighted by the starred and squared symbols, respectively.

Table 4.3. Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the individual rotation-dominated dwIrr in fractional and Newtonian gravity (marked by $s = 1$). Columns report the values of the stellar mass-to-light ratio $M_\star/L$, of the DM mass M_{200} , of the fractional index s , of the reduced χ_r^2 for the overall fit, and of the Bayesian inference criterion (BIC) for model comparison.

Galaxy	$\log M_\star/L$ [$M_\odot/L_\odot$]	$\log M_{200}$ [$M_\odot$]	s	χ_r^2	BIC
D631-7	$-0.39^{+0.07}_{-0.07}$	$10.59^{+0.06}_{-0.06}$	> 1.47	4.25	63
	$-0.77^{+0.07}_{-0.07}$	$10.15^{+0.03}_{-0.03}$	1	19.50	279
DDO64	$-0.28^{+0.09}_{-0.09}$	$10.55^{+0.07}_{-0.07}$	$1.16^{+0.04}_{-0.04}$	0.56	14
	$-0.45^{+0.08}_{-0.08}$	$10.36^{+0.06}_{-0.06}$	1	1.66	25
DDO161	$-0.76^{+0.07}_{-0.07}$	$10.26^{+0.03}_{-0.02}$	$1.35^{+0.02}_{-0.02}$	0.75	31
	$-0.82^{+0.06}_{-0.06}$	$10.44^{+0.01}_{-0.01}$	1	15.27	450
UGC731	$-0.26^{+0.08}_{-0.08}$	$10.70^{+0.03}_{-0.03}$	$1.05^{+0.02}_{-0.03}$	0.30	12
	$-0.29^{+0.08}_{-0.08}$	$10.67^{+0.02}_{-0.02}$	1	0.66	14
UGC5005	$-0.44^{+0.08}_{-0.08}$	$11.00^{+0.06}_{-0.06}$	$1.29^{+0.06}_{-0.06}$	0.64	15
	$-0.47^{+0.08}_{-0.08}$	$10.98^{+0.05}_{-0.05}$	1	3.13	35
UGC5414	$-0.28^{+0.09}_{-0.09}$	$10.91^{+0.07}_{-0.07}$	$1.28^{+0.04}_{-0.06}$	0.42	8
	$-0.67^{+0.07}_{-0.07}$	$10.54^{+0.04}_{-0.04}$	1	7.37	34
UGC7608	$-0.25^{+0.09}_{-0.09}$	$10.68^{+0.07}_{-0.07}$	$1.08^{+0.03}_{-0.06}$	1.26	14
	$-0.29^{+0.09}_{-0.09}$	$10.63^{+0.06}_{-0.06}$	1	1.35	14
NGC3741	$-0.21^{+0.08}_{-0.08}$	$10.29^{+0.04}_{-0.04}$	$1.23^{+0.03}_{-0.03}$	0.47	20
	$-0.49^{+0.07}_{-0.07}$	$10.09^{+0.02}_{-0.02}$	1	4.65	97

takes on typical values $s \approx 1.2 - 1.3$, the stellar mass-to-light ratios $M_\star/L$ are slightly larger than for the Newtonian case and more in line with the value around 0.5 expected from stellar population synthesis models, and the DM masses M_{200} are appreciably larger of factors a few than in the Newtonian fit. In the other cases, namely UGC731 and UGC7608, s is close to 1, the estimates of the fitting parameters in FG and in the Newtonian setting are consistent to within 1σ , and the overall quality of the fits are comparable. We have looked for some property of these two galaxies that could correlate with their smaller values of s , but were unable to

reach a definite conclusion. Maybe an interesting evidence comes from the kinematics of their HI disks, which appear asymmetric and disturbed by a past or ongoing gravitational interaction [226]; this could possibly alters the shape of the outer rotation curve and originate a variant outcome when modeling it in the FG framework. An extended sample of dwIrr with high quality rotation curve and environmental characterization would be needed to investigate further the issue in a statistically sound manner.

As it can be seen in the top panel of Figure 4.9 for the representative cases of NGC3741 and DDO64, in FG there is no strong degeneracy in the fitting parameters, besides a weak direct dependence between s and both $M_\star/L$ and M_{200} . The bottom panel of the same figure shows that for NGC3741 and DDO64 the halo component largely dominates the rotation curve, with the baryonic contribution being relevant only in the innermost region within a few $r_{1/2}$; this situation is shared by all dwIrr in the analyzed sample. Therefore the shape of the rotation curve strongly constrains the halo mass/density profile; in particular, the rising trend of the rotation velocity out to large radii is difficult to be reproduced with a NFW profile in standard gravity, while the task can be easily achieved in the FG framework. The results for dispersion-dominated galaxies are displayed in Figures 4.11, 4.12 and in Table 4.4. Figure 4.11 focuses on the representative dwSph Sculptor and on the ultra-diffuse galaxy DragonFly 44. In the top panel of Figure 4.11 we illustrate the MCMC posterior distributions for Sculptor and DragonFly 44. As above, red lines/contours refer to the outcomes for FG, and green ones for Newtonian gravity, with the white crosses marking the best-fit value of the parameters in FG. In the bottom panel the best fit (solid lines) and the 2σ credible intervals (shaded areas) sampled from the posteriors are shown, with the reference Newtonian fit in green. In Figure 4.12 the fits in FG and in the Newtonian case are illustrated for the other 6 dwSph in the sample. In Table 4.4 we summarize the marginalized posterior estimates of the parameters for dwSph, both in FG and in the standard Newtonian case (marked with $s = 1$). Columns report the median values and the 1σ credible intervals of the stellar mass-to-light ratio $M_\star/L$, of the symmetrized anisotropy parameter β_{sym} , of the DM mass M_{200} , and of the fractional index s ; the reduced χ_r^2 of the fit, and the BIC are also reported.

The FG fits to dispersion-dominated galaxies are very good, and in several instances appreciably better than in Newtonian gravity. In particular, for Carina, Leo I, Sculptor, and Sextans there is a clear preference for FG both in terms of χ_r^2 and in terms of the BIC. In such cases, the fractional index takes on typical values $s \gtrsim 1.2$, the stellar mass-to-light ratios $M_\star/L$ are appreciably larger than for the Newtonian case and more in line with the prior from stellar population synthesis models, and the DM masses M_{200} are substantially larger of factors several with respect to the Newtonian fits. In other cases, namely Leo II and Draco, the index $s \lesssim 1.1$ is smaller, the estimates of the fitting parameters in FG and in the Newtonian setting

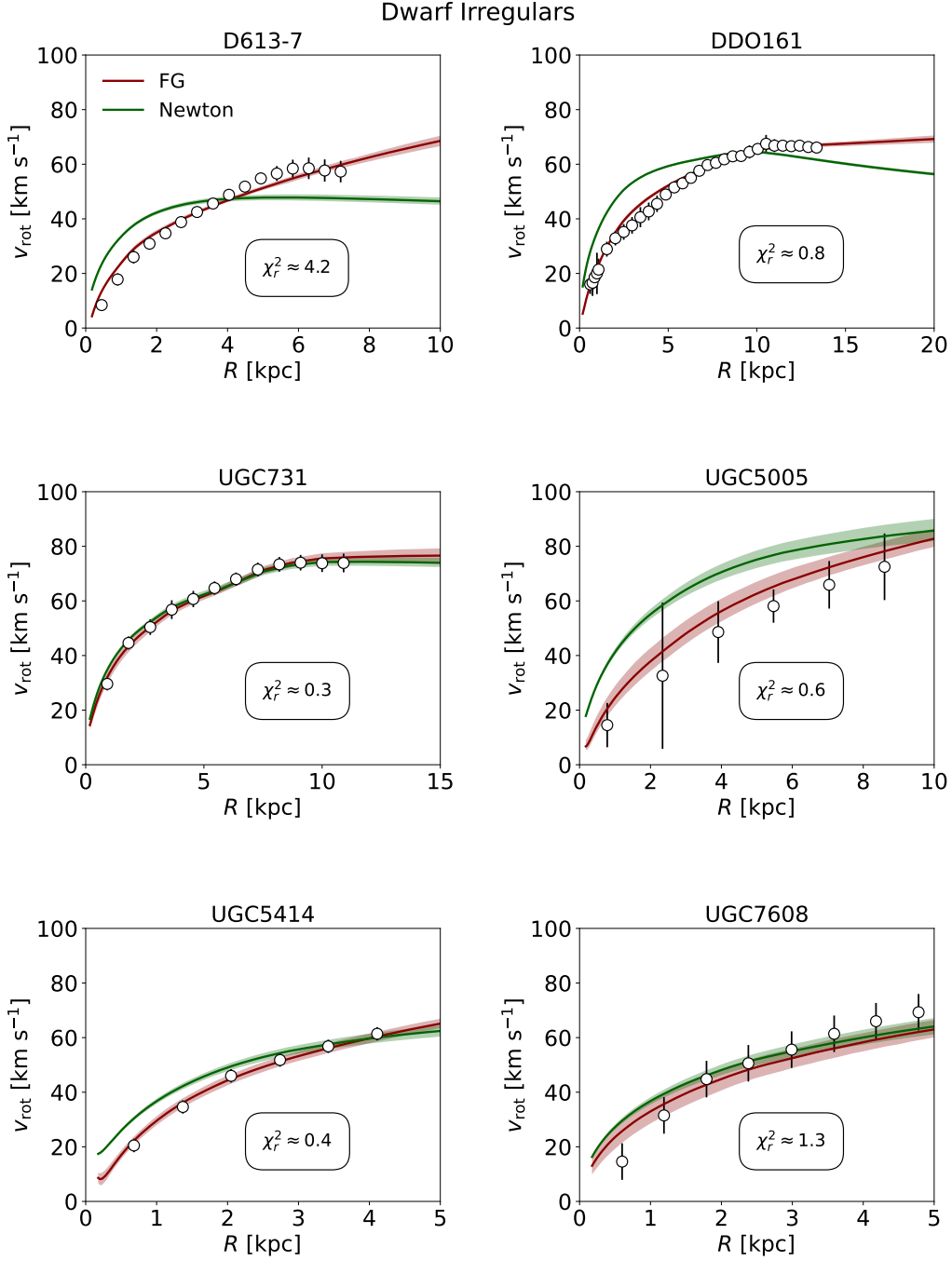


Figure 4.10. Same as bottom panel in the previous figures for other 6 dwIrr galaxies, as labeled. For clarity, data and models only for the total rotation curves are shown.

are consistent to within 3σ , and the overall quality of the fits are comparable. Finally, in the cases of Fornax and DragonFly 44 there is a clear preference for large values of $s \approx 1.5$, but

Table 4.4. Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the individual dispersion-dominated dwarf spheroidal in fractional (first lines) and Newtonian (second lines, with $s = 1$) gravity. Columns report the values of the stellar mass-to-light ratio $M_\star/L$, of the symmetrized anisotropy parameter β_{sym} , of the DM mass M_{200} , of the fractional index s , of the reduced χ_r^2 for the overall fit, and of the Bayesian inference criterion (BIC) for model comparison.

Galaxy	$\log M_\star/L$ [$M_\odot/L_\odot$]	β_{sym}	$\log M_{200}$ [$M_\odot$]	s	χ_r^2	BIC
Carina	$+0.54^{+0.26}_{-0.26}$	$+0.19^{+0.11}_{-0.11}$	$9.77^{+0.12}_{-0.12}$	$1.37^{+0.04}_{-0.05}$	1.67	48
	$-0.75^{+0.28}_{-0.16}$	$-0.81^{+0.013}_{-0.12}$	$9.06^{+0.13}_{-0.06}$	1	3.49	87
Leo I	$+0.74^{+0.12}_{-0.07}$	$-0.08^{+0.16}_{-0.16}$	$10.35^{+0.08}_{-0.07}$	$1.37^{+0.11}_{-0.06}$	0.64	17
	$+0.19^{+0.16}_{-0.12}$	$-0.80^{+0.05}_{-0.07}$	$10.03^{+0.09}_{-0.07}$	1	2.64	39
Leo II	$-0.40^{+0.28}_{-0.28}$	$+0.37^{+0.25}_{-0.25}$	$9.50^{+0.13}_{-0.13}$	$1.11^{+0.04}_{-0.05}$	0.37	11
	$-0.74^{+0.24}_{-0.24}$	$+0.09^{+0.27}_{-0.27}$	$9.32^{+0.11}_{-0.11}$	1	0.42	9
Sculptor	$+0.60^{+0.19}_{-0.19}$	$-0.04^{+0.06}_{-0.06}$	$10.13^{+0.09}_{-0.09}$	$1.23^{+0.03}_{-0.05}$	1.32	54
	$-0.37^{+0.13}_{-0.13}$	$-0.32^{+0.06}_{-0.06}$	$9.51^{+0.06}_{-0.06}$	1	1.65	65
Draco	$+1.05^{+0.14}_{-0.14}$	$+0.01^{+0.17}_{-0.17}$	$9.93^{+0.09}_{-0.09}$	$1.13^{+0.03}_{-0.03}$	0.27	14
	$+0.75^{+0.12}_{-0.12}$	$-0.32^{+0.24}_{-0.17}$	$9.71^{+0.07}_{-0.07}$	1	0.56	15
Sextans	$+0.90^{+0.13}_{-0.13}$	$-0.34^{+0.07}_{-0.05}$	$9.94^{+0.08}_{-0.08}$	> 1.45	1.54	24
	$+0.54^{+0.12}_{-0.12}$	$-0.89^{+0.03}_{-0.07}$	$9.62^{+0.07}_{-0.07}$	1	3.07	38
Fornax	$-0.36^{+0.15}_{-0.12}$	$+0.16^{+0.03}_{-0.03}$	$10.03^{+0.1}_{-0.08}$	> 1.49	1.30	69
	< -1.84	$-0.16^{+0.04}_{-0.04}$	$9.08^{+0.03}_{-0.03}$	1	1.15	62
DragonFly 44	$+0.17^{+0.09}_{-0.09}$	$-0.12^{+0.11}_{-0.07}$	$10.79^{+0.07}_{-0.07}$	> 1.31	0.46	12
	$+0.09^{+0.09}_{-0.09}$	$-0.48^{+0.08}_{-0.08}$	$10.69^{+0.06}_{-0.06}$	1	1.27	16

the improvement in the FG fits with respect to the Newtonian case, albeit clear at a visual inspection, is not statistically significant enough to make definite conclusions.

As it can be seen in the top panel of Figure 4.11 for Sculptor and DragonFly 44, the most relevant degeneracy between the fitting parameters involves s and the β_{sym} , in such a way that FG models with larger s tend to be more isotropic. In fact, for Sculptor and DragonFly 44 the FG fit shows preference for almost isotropic orbits, while the Newtonian fit favors

tangentially-dominated motions.

The bottom panel of the same Figure illustrates visually the quality of the FG fit for Sculptor and DragonFly 44, which is excellent within the scatter of the datapoints. FG performs definitely better than the Newtonian case for Sculptor, while for DragonFly 44 the evidence is made barely significant in terms of reduced χ_r^2 and of the BIC. In this respect, however, it is also interesting to look at the inset where the excess kurtosis $\Delta\kappa$ (the kurtosis is related to the fourth velocity moments of the stellar tracers, and the excess is respect to the value 3 for a reference Gaussian velocity distribution) is illustrated. Although the measured value is largely uncertain, there is clear a tendency for a definite positive $\Delta\kappa$; qualitatively, this is consistent with the FG result at 2σ , while being highly discordant (more than 3σ) with the Newtonian fit. It is worth mentioning that the estimated $M_{200} \lesssim 10^{11} M_\odot$ from our analysis in FG is consistent with the recent determination from the abundance of globular clusters in DragonFly 44 by [227].

By inspecting Tables 4.3 and 4.4 overall one can conclude that the evidence in favor of FG is less compelling in dwSph with respect to dwIrr. This is due to several reasons. First, the main observable for dwIrr is the rotation velocity, which is a direct probe of the mass profile; contrariwise, in dwSph the l.o.s. dispersion profile encases the mass profile in an integrated way, weighted by a kernel that depends on the tracer profiles and on the anisotropy parameter. In addition, the priors on the stellar mass-to-light ratio from population synthesis model are looser for dwSph than for dwIrr (especially when considering $3.6 \mu\text{m}$ luminosities for the latter). Finally, the observed l.o.s. dispersion profiles are more scattered and featureless with respect to the rotation curves. Thus it should not be surprising that the constraints from dwSph are less statistically significant. Nevertheless, these systems may offer an environment where any evidence in favor of FG is more robust, since the lack of baryons even in the innermost regions does not allow us to rely on different interpretations related to baryonic-induced modification of the DM profile.

Finally, in Fig. 4.13 we illustrate two interesting scaling relations, that constitute relevant crosschecks of our results. The diagram on the left panel displays the fractional index s as a function of the DM mass M_{200} . Apart for a few exceptions (objects with $s \lesssim 1.1$), the values from our analysis of individual dwarf galaxies in FG are consistent to within $2 - 3\sigma$ with the expectation from the relationship derived in Chapter 3 from fitting stacked rotation curves of rotation-dominated galaxies.

The diagram on the right reports the Radial Acceleration (RAR) between the total acceleration g_{tot} and the baryonic one g_{bar} . This is an empirical relationship known to hold for different kind of galaxies, whose average and $1 - 2 - 3\sigma$ dispersion is plotted as an orange line surrounded by shaded areas. For rotationally supported systems we compute

$g_{\text{bar}} = [v_{\text{gas}}^2(r_{1/2}) + (M_{\star}/L) \times v_{\text{disk}}^2(r_{1/2})]/r_{1/2}$ and $g_{\text{tot}} = v_{\text{tot}}^2(r_{1/2})/r_{1/2}$ in terms of Equation (4.5). For dispersion-dominated systems we instead estimate $g_{\text{bar}} = G(M_{\star}/L) \times L_{\star}/2r_{1/2}^2$ and $g_{\text{tot}} = 3\sigma_r^2(r_{1/2})/r_{1/2}$ in terms of Equation (4.8). For the sake of simplicity we compute the accelerations at $r_{1/2}$, by using the bestfit values of $M_{\star}/L$, M_{200} and s from our analysis in FG. Reassuringly, almost all our estimated accelerations are consistent within $2 - 3\sigma$ with the RAR by [43], with the dwIrr galaxies clustering around the value of g_{bar} where the relation starts to flatten, and with some dwSph tracing the flat portion of the RAR and its scatter.

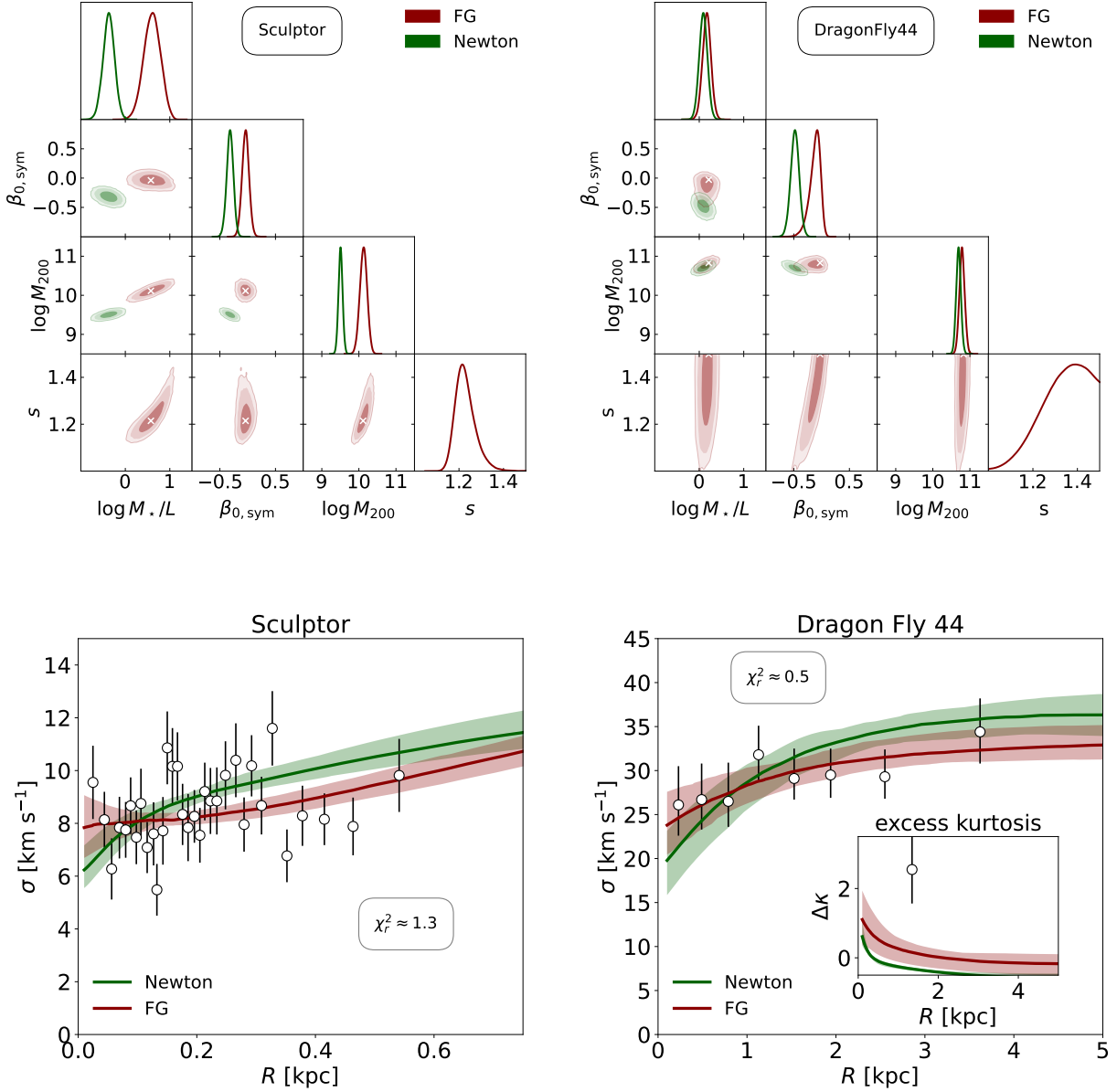


Figure 4.11. Top panel: MCMC posterior distributions of the mass-to-light ratio $M_{\star}/L$, the symmetrized anisotropy parameter β_{sym} , the mass M_{200} and the fractional index s for the dwSph galaxy Sculptor (left) and the ultra-diffuse galaxy DragonFly 44 (right). Colored contours/lines refer to the standard Newtonian (green) and to the FG framework (red). The contours show 1 – 2 – 3 σ confidence intervals, with the bestfit values in FG identified by white crosses. The marginalized distributions are in arbitrary units (normalized to 1 at their maximum value). Bottom panel: Fits to the l.o.s. dispersion profile with the Newtonian (green) and the FG (red) framework for the dwSph galaxy Sculptor (left) and DragonFly 44 (right). The inset on the right bottom panel illustrates the excess kurtosis $\Delta\kappa$ with respect to a Gaussian velocity distribution. Solid lines illustrate the median, and the shaded areas show the 2 σ credible interval from sampling the posterior distribution. The value of the reduced χ_r^2 of the fit for FG is also reported. Circles represent data from [213] for Sculptor and from [210] for DragonFly 44.

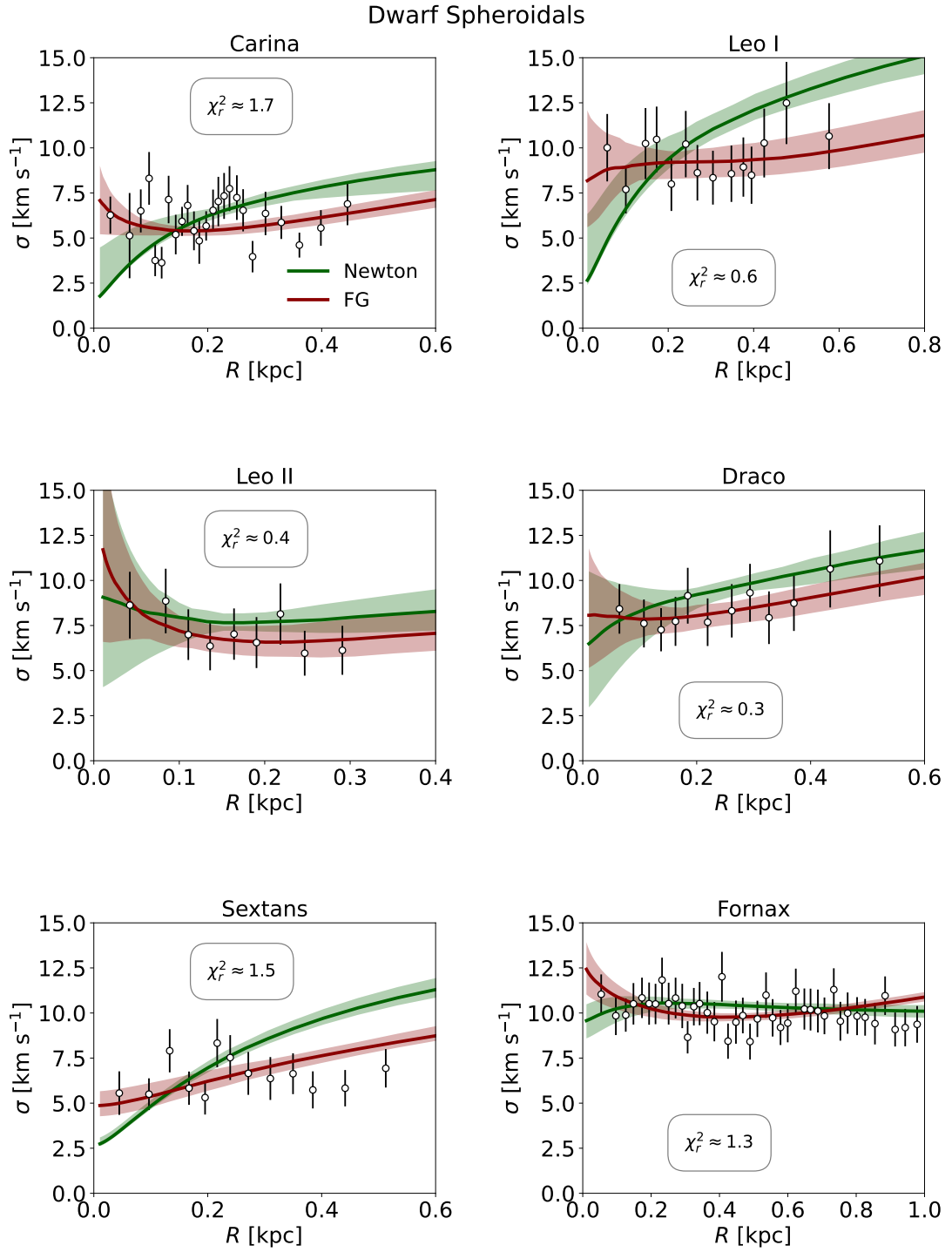


Figure 4.12. Same as bottom panel in the previous figure for 6 dwSph galaxies, as labeled.

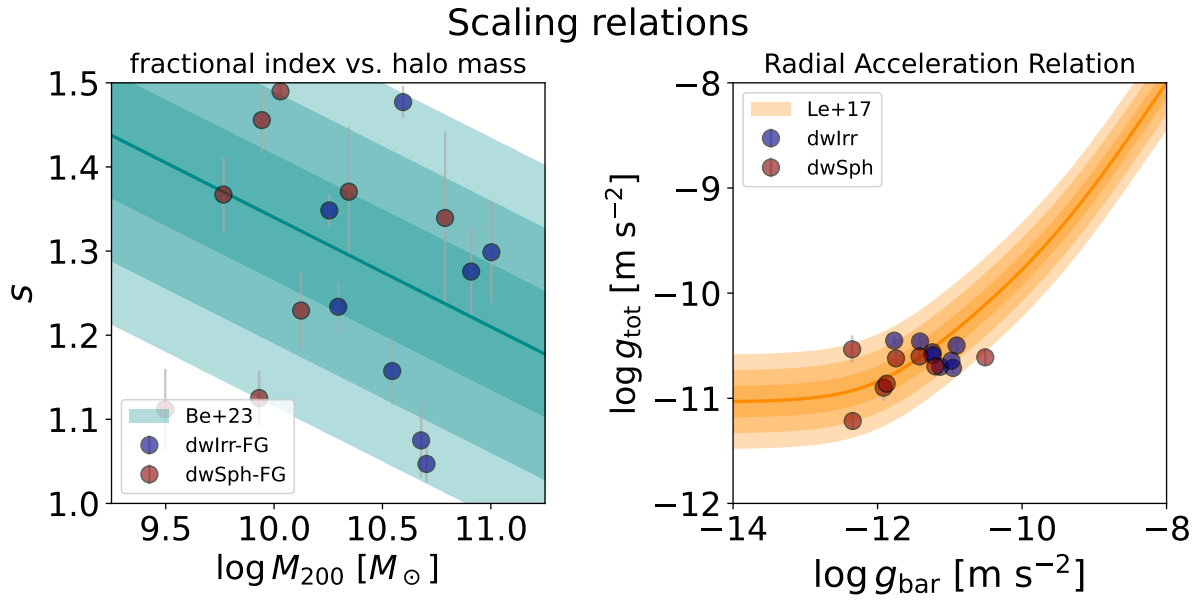


Figure 4.13. Left: fractional index as a function of DM mass from the outcomes of our Bayesian analysis. Blue circles refer to dwIrr and red ones to dwSph; the cyan line and shaded areas illustrate the bestfit relation and $1 - 2 - 3\sigma$ dispersion from the analysis of stacked rotation curves for disc-dominated galaxies from Chapter 3. Right: the Radial Acceleration Relation. Blue symbols refer to dwIrr and red ones to dwSph; the orange line and shaded areas illustrate the bestfit relation and $1 - 2 - 3\sigma$ dispersion from the determination by [43].

4.4 Summary and conclusions

In this chapter we have tested our fractional gravity framework in the smallest and largest bound systems, by fitting the density and pressure profiles of the ICM in 12 galaxy clusters of the XCOP sample, the rotation velocity profiles inferred from stellar and gas kinematic measurements in 8 dwarf irregulars, and the projected velocity dispersion profiles inferred from the observed dynamics of stellar tracers in 7 dwarf spheroidals and in the ultra-diffuse galaxy DragonFly 44. Our results show that FG is able to reproduce the observed properties of systems spanning ten orders in DM mass: from dwarfs of $M \sim 10^5 M_\odot$, to clusters of $M \sim 10^{15} M_\odot$. Furthermore, the fits to the FG parameters s and ℓ inferred from the MCMC procedure confirm the trend observed in the previous chapter: non-local effects associated to FG tend to be stronger at smaller scales, causing a strong departure from standard Newtonian gravity, and to weaken in larger systems. This can be seen in the value of the fractional exponent s , taking values as high as $1.45 - 1.5$ in most dwarfs, decreasing to $1.1 - 1.3$ in intermediate mass spirals, and settling near the Newtonian limit $s = 1$ in massive clusters. This allows FG to solve the small scales issues of the standard Λ CDM, by reconciling with data the DM density distribution expected from N-body simulations, while saving its successes on large cosmological scales. Contextually, the parameter ℓ features a scaling with the mass of the system comparable to the scale radius r_s , when the rotation curve is interpreted in terms of a NFW profile. These findings suggest that, whatever physical mechanisms sits at the origin of FG, its effects are strongly dependent on the size and DM mass of the gravitationally bound, virialized, system considered.

This chapter concludes the part of this work dedicated to the investigation of the effects of fractional gravity in galaxies and clusters. In those systems the velocities involved are very low with respect to the speed of light ($10^{-5} \lesssim v/c \lesssim 10^{-3}$) and the gravitational potentials are weak ($10^{-8} \lesssim \Phi/c^2 \lesssim 10^{-5}$), so that a Newtonian version of fractional gravity based on the fractional Poisson equation (3.1) is enough to describe their dynamics.

In the next chapter we will put the theory to more rigorous theoretical grounds, by developing a fully relativistic extension to fractional gravity, stemming from a suitable action. This will allow us to gain useful insights on the physical nature of the effective non-local density (3.6), and to explore the effects of fractional gravity to phenomena which cannot be described in the Newtonian limit of the theory, such as the propagation of gravitational waves, and gravitational lensing.

Chapter 5

Relativistic Fractional Gravity

Motivated by the successes of fractional gravity (FG) at the Newtonian level, we devote this chapter to develop a full relativistic extension to the theory. This will enable us to explore FG in regimes in which Newtonian approximation cannot be applied, to study phenomena which are exclusive to the relativistic setting (e.g., propagation of gravitational waves), and provide a more straightforward physical interpretation of the non-locality effects associated to the theory.

In Section 5.1 we give a brief review of non-local gravity theories, focusing on their motivations and successes. Section 5.2 is the main one of this chapter. In there we present the action of relativistic fractional gravity (RFG), derive the associated field equations and reinterpret them in terms of an effective dark matter (DM) stress-energy tensor; we investigate the weak field and post-Newtonian limit of the theory, and we perform a scalar-vector-tensor decomposition analysis to discuss the propagation of modes. In Section 5.3 we test RFG against gravitational lensing data of the clusters of the CLASH sample by fitting the lensing convergence profiles. Finally, in Section 5.4 we summarize our findings and outline future perspectives.

5.1 Non-local gravity theories

General Relativity (GR) has been extraordinarily successful in describing gravitational phenomena across a vast range of length scales, from laboratory experiments and Solar-System tests to the dynamics of compact objects and the evolution of the Universe. The direct detection of gravitational waves and the increasing precision of cosmological observations have further confirmed its predictive power. Nevertheless, despite its empirical success, GR is widely regarded as an effective classical theory, expected to break down in regimes where quantum effects become significant or where the spacetime curvature approaches the Planck scale.

The limitations of Einstein's theory emerge from both theoretical and observational considerations. On the one hand, the perturbative quantization of GR leads to a non-renormalizable

quantum field theory, indicating that Einstein gravity cannot provide a complete ultraviolet (UV) description of the gravitational interaction. Moreover, the occurrence of spacetime singularities in black-hole interiors and in the early Universe signals the breakdown of the classical theory in regions of extreme curvature. On the other hand, cosmological observations require the existence of dark energy to trigger an accelerated expansion, whose physical nature remains unknown.

Among the many proposals extending GR, non-local theories of gravity occupy a particularly interesting position. Rather than modifying gravity by introducing additional local degrees of freedom or higher-curvature invariants of finite order, these theories replace the strictly local character of Einstein's equations with interactions involving non-local differential operators. Such operators typically depend on arbitrary functions of the covariant d'Alembertian operator $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, or of its inverse. The resulting gravitational interaction depends not only on the local values of the fields but also on their behavior over finite regions of spacetime, introducing an intrinsic form of gravitational memory.

The appearance of non-local operators is not merely a phenomenological modification of GR but is in fact expected from several independent theoretical considerations. In quantum field theory, integrating out massive or massless degrees of freedom naturally generates non-local terms in the effective action. Covariant perturbation theory developed by Barvinsky, Vilkovisky, and collaborators demonstrated that the quantum effective action in curved spacetime generically contains operators such as $R \log\left(\frac{\square}{\mu^2}\right) R$, $R \square^{-1} R$, together with more general non-local form factors acting on curvature invariants [228]. Similarly, string field theory gives rise to exponential functions of the d'Alembertian, reflecting the extended nature of fundamental strings and the intrinsic non-locality of string interactions [229]. From this perspective, non-locality should be regarded as a generic feature of quantum effective descriptions rather than an ad hoc modification of gravity.

Although non-local gravity encompasses a broad class of theories, it is useful to distinguish two major families according to the physical scale at which the non-local operators become relevant. The first class consists of ultraviolet non-local theories, commonly referred to as infinite-derivative or weakly non-local gravity. The second class comprises infrared non-local theories, typically constructed from inverse powers of the d'Alembertian operator and motivated by cosmological applications.

Ultraviolet non-local gravity was originally developed in the search for a consistent quantum theory of gravity [230, 231, 232]. The starting point was the observation that local higher-derivative theories containing quadratic curvature invariants, such as R^2 or $R_{\mu\nu} R^{\mu\nu}$, improve the renormalization properties of Einstein gravity but inevitably introduce additional propagating degrees of freedom, including massive spin-two ghosts that violate perturbative

unitarity. The central idea behind infinite-derivative gravity is that replacing finite polynomials in the d'Alembertian by entire analytic functions may improve the ultraviolet behavior of the graviton propagator without introducing new poles. Schematically, the Einstein-Hilbert action is generalized as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R + R \mathcal{F}_0(\square) R + R_{\mu\nu} \mathcal{F}_2(\square) R^{\mu\nu} + C_{\mu\nu\rho\sigma} \mathcal{F}_4(\square) C^{\mu\nu\rho\sigma}] , \quad (5.1)$$

where the operators $\mathcal{F}_i(\square) = \sum_{n=0}^{\infty} f_n \left(\frac{\square^n}{M_s^{2(n+1)}} \right)$ are entire functions admitting infinite derivative expansions, with M_s the non-locality scale. Appropriate choices of these functions preserve the massless graviton as the only propagating degree of freedom while exponentially suppressing ultraviolet modes. This approach has motivated extensive investigations of ghost-free quantum gravity, singularity resolution, regular black-hole geometries, non-singular cosmological solutions, and super-renormalizable or finite theories of gravity [233, 234, 235].

A conceptually distinct class of theories modifies gravity in the infrared through inverse powers of the d'Alembertian operator [236, 237, 238]. Rather than addressing the ultraviolet completion of gravity, these models aim to explain the observed late-time acceleration of the Universe without introducing a cosmological constant or additional dynamical scalar fields. Typical examples involve operators such as $R \square^{-1} R$, or $R \square^{-2} R$, which become relevant only on cosmological length scales. These theories preserve general covariance while modifying the effective gravitational dynamics at large distances, leading to cosmological acceleration generated dynamically by the non-local interaction. Their construction is also motivated by the observation that inverse d'Alembertian operators naturally emerge in quantum effective actions after integrating out massless fields, making them plausible infrared corrections to Einstein gravity.

Although ultraviolet and infrared non-local theories were originally motivated by different problems, they share a common mathematical framework based on non-polynomial functions of the d'Alembertian operator. More generally, one may regard the gravitational action as depending on an arbitrary form factor, $\mathcal{F}(\square)$, whose analytic structure determines the physical properties of the theory. Entire analytic functions characterize weakly non-local ultraviolet completions, while non-analytic functions, including inverse powers and logarithms of $\square$, describe long-distance quantum corrections and infrared modifications of gravity. From this perspective, the distinction between ultraviolet and infrared non-locality reflects the different spectral properties of the corresponding form factors rather than two fundamentally unrelated theoretical frameworks.

In a nutshell, that infinite derivatives theories are non-local can be understood with the following basic argument: in order to calculate the first derivative of a standard function, its

value at two infinitesimally close points must be known; similarly, the n -th derivative requires knowledge of the function value at $n + 1$ different points. This means that infinitely many derivatives of a field require knowledge of its value throughout all spacetime. Another way to see this is by looking at the action of the translation operator of a function $e^{\epsilon \frac{d}{dx}} f(x) = f(x + \epsilon)$. Since $e^{\epsilon \frac{d}{dx}} = \sum_n \frac{\epsilon^n}{n!} \left(\frac{d^n}{dx^n} \right)$, this suggests that actions involving infinitely many derivatives can be recast in terms of fields translated at different points in spacetime.

On the other hand, inverse powers of the d'Alembertian lead to manifestly non-local theories, since they are represented as integral kernels acting via the Green function of the d'Alembertian

$$(\Box^{-1} f)(x) = \int d^4 y \sqrt{-g(y)} G(x, y) f(y) . \quad (5.2)$$

To different choices of Green functions correspond alternative definitions of these inverse operators. In particular, requiring causality, meaning that an event at a spacetime point could only be influenced by events in its past but not in its future, amounts to select the retarded green function $G_{\text{ret}}(x - y)$. Note that this choice can be made only a posteriori, by replacing any advanced Green function in the field equations with its retarded counterpart by hand, since both types of Green functions naturally emerge in the variation of the action of a non-local gravity theory [239].

More recently, the landscape of non-local gravity has expanded to include fractional differential operators, generalizing the functions $\mathcal{F}(\Box)$ to fractional powers $(-\Box)^s$, where s is a real parameter. Fractional operators provide a natural mathematical description of anomalous scaling, long-range correlations, and memory effects, and have been investigated in connection with multifractional spacetimes [174, 175, 240], dimensional flow [241, 242] in quantum gravity, and effective descriptions of non-local interactions. Fractional models establish an interesting bridge between non-local gravity, fractal geometry, and quantum effective field theory. Being non analytic in the whole complex plane, fractional operators introduce a stronger departure from local gravity than non-local operators constructed with infinite derivatives of integer order. Once quantized, these kind of theories have been shown to be unitary and finite at one loop for particular values of the fractional exponent s [176].

5.2 Relativistic Fractional Gravity

In this section we present the action of relativistic fractional gravity (RFG), and the associated field equations. In extending CDM in FG to a relativistic setting, we aim to meet a series of requirements:

- The theory must be generally covariant and locally Lorentz invariant. This requirement restricts the class of admissible non-local operators to those depending exclusively on the

fractional d'Alembertian $(-\square)^s \equiv (-\nabla_\mu \nabla^\mu)^s$.

- Baryonic matter should obey standard GR; in particular, it must follow the geodesics of the metric solving the field equations. To satisfy this requirement, we transfer the non-locality from the gravitational sector of the action to the dark matter sector. The ultimate goal is for the field equations to take the form $G^{\mu\nu} = 8\pi G (T_{\text{BAR}}^{\mu\nu} + T_{\text{DM eff}}^{\mu\nu})$, where dark matter sources the gravitational field through an effective non-local stress-energy tensor, representing the covariant generalization of the effective density in NFG.
- In the Newtonian limit of the field equations, valid for slowly moving and nearly static sources in the weak-field regime, the theory should reproduce the fractional Poisson equation (1.19). We anticipate that this requirement imposes a relation among the non-local operators appearing in the action, thereby restricting the space of viable RFG theories, although not determining it uniquely.

Inspired by these guidelines, we propose an action featuring a scalar term, in which the scalar curvature couples with the trace of the DM stress-energy tensor via a non-local operator $\mathcal{F}_0(\square)$, and a tensorial term, in which the Ricci tensor couples with the full DM SET via a second non-local operator $\mathcal{F}_2(\square)$. Here we work with fluid variables, and we consider CDM to be well represented by dust, so that $T_{\text{DM}}^{\mu\nu} = \rho_{\text{DM}} u^\mu u^\nu$.

The RFG action then reads¹

$$S_{\text{RFG}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} R + S_{\text{BAR}} + S_{\text{DM}} + \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R \mathcal{F}_0(\square) T_{\text{DM}} + R_{\alpha\beta} \mathcal{F}_2(\square) T_{\text{DM}}^{\alpha\beta} \right) \quad (5.3)$$

where S_{BAR} and S_{DM} denote the matter actions for baryons and CDM respectively, $T_{\text{DM}}^{\alpha\beta} \equiv -2/\sqrt{-g} \delta S_{\text{DM}} / \delta g_{\alpha\beta}$ is the DM stress-energy tensor and $T_{\text{DM}} \equiv g_{\alpha\beta} T_{\text{DM}}^{\alpha\beta}$ the corresponding trace. The field equations of the theory can then be derived by varying the action (5.3) with respect to the inverse metric (we refer the reader to Appendix B for the full derivation)

¹We remark that the action presented here is the most general one that can be constrained by requiring that it has the right Newtonian limit. Indeed, although in principle one could include also a term of the form $R_{\mu\alpha\nu\beta} \mathcal{F}_4(\square) T_{\text{DM}}^{\mu\nu} T_{\text{DM}}^{\alpha\beta}$ involving the Riemann (or Weyl) tensor and two copies of the DM SET, we stress that this term would be subdominant in the Newtonian limit, and thus couldn't be constrained.

$$\begin{aligned}
 \frac{G_{\mu\nu}}{8\pi G} = & T_{\text{BAR}\mu\nu} + T_{\text{DM}\mu\nu} + (T_{\text{DM}\mu\nu} - g_{\mu\nu} T_{\text{DM}}) \mathcal{F}_0 R - 2(G_{\mu\nu} + g_{\mu\nu} \square - \nabla_{(\mu} \nabla_{\nu)}) \mathcal{F}_0 T_{\text{DM}} + \\
 & - 2R \frac{\delta \mathcal{F}_0}{\delta g^{\mu\nu}} T_{\text{DM}} - \square \mathcal{F}_2 T_{\text{DM}\mu\nu} + 2\nabla^\alpha \nabla_{(\mu} \mathcal{F}_2 T_{\text{DM}\nu)\alpha} - g_{\mu\nu} \nabla_\alpha \nabla_\beta \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \\
 & + g_{\mu\nu} R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} - g_{\mu\nu} T_{\text{DM}}^{\alpha\beta} \mathcal{F}_2 R_{\alpha\beta} - \frac{T_{\text{DM}\mu\nu}}{T_{\text{DM}}} T_{\text{DM}}^{\alpha\beta} \mathcal{F}_2 R_{\alpha\beta} - 2R_{\alpha\beta} \frac{\delta \mathcal{F}_2}{\delta g^{\mu\nu}} T_{\text{DM}}^{\alpha\beta}. \quad (5.4)
 \end{aligned}$$

In obtaining Equation (5.4) we have discarded a boundary term, which pops out when one integrates by parts in the variation of the action. Since, as long as the DM SET vanishes asymptotically so does this boundary term, one can reabsorb it by adding to the action (5.3) a Gibbons-Hawking-York counter-term [243, 244] and no additional degree of freedom is introduced in the theory (see again Appendix B).

As was expected, DM couples with the gravitational field not only via the absolute value of its SET, but also via its variations in space and time, so that the dynamics of an inhomogeneous, or non static dark matter system is far richer than what occurs when the fractional interaction is not taken into account. An immediate consequence of this fact is that the DM SET is no longer conserved on shell, since the contracted Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ do not imply $\nabla_\mu T_{\text{DM}}^{\mu\nu} = 0$. Indeed, by taking the covariant derivative of (5.4), and assuming the baryonic SET to be separately conserved $\nabla_\mu T_{\text{bar}}^{\mu\nu} = 0$, we find

$$\begin{aligned}
 \nabla^\mu T_{\text{DM}\mu\nu} = & -\nabla^\mu \left[(T_{\text{DM}\mu\nu} - g_{\mu\nu} T_{\text{DM}}) \mathcal{F}_0 R - 2(G_{\mu\nu} + g_{\mu\nu} \square - \nabla_{(\mu} \nabla_{\nu)}) \mathcal{F}_0 T_{\text{DM}} + \right. \\
 & - 2R \frac{\delta \mathcal{F}_0}{\delta g^{\mu\nu}} T_{\text{DM}} - \square \mathcal{F}_2 T_{\text{DM}\mu\nu} + 2\nabla^\alpha \nabla_{(\mu} \mathcal{F}_2 T_{\text{DM}\nu)\alpha} + g_{\mu\nu} R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \\
 & \left. - g_{\mu\nu} T_{\text{DM}}^{\alpha\beta} \mathcal{F}_2 R_{\alpha\beta} - \frac{T_{\text{DM}\mu\nu}}{T_{\text{DM}}} T_{\text{DM}}^{\alpha\beta} \mathcal{F}_2 R_{\alpha\beta} - 2R_{\alpha\beta} \frac{\delta \mathcal{F}_2}{\delta g^{\mu\nu}} T_{\text{DM}}^{\alpha\beta} \right]. \quad (5.5)
 \end{aligned}$$

5.2.1 Weak field limit

To connect RFG with the Newtonian framework we have developed in the previous chapters, we need first to take the weak field limit of the field equations (5.4). We thus routinely assume that it is possible to choose a reference frame where the dynamics of the gravitational field is encoded within a small perturbation over the Minkowski background $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \gamma_{\mu\nu}$, with $\varepsilon \ll 1$; this assumption enables to simplify the equations considerably.

First, having the metric as a small perturbation requires the SET to be at least linear in ε , so that all the terms in Equation (5.4) involving the product of the curvature and the SET can be neglected at the linear order. Furthermore, since the non-local operators $\mathcal{F}_{0,2}(\square)$ always act on an object linear in ε , at this order they can be replaced with their expression in Minkowski spacetime $\mathcal{F}_{0,2}(\square_\eta)$, where $\square_\eta = -\partial_0^2 + \Delta$. In the following we will omit the subscript η for simplicity; hereafter anytime a d'Alembertian operator appears, it must be meant as its expression in Minkowski space. Finally, it's not difficult to see that in the weak field limit (5.5) reduces to the usual conservation equation for DM $\partial_\mu T_{\text{DM}}^{\mu\nu} = 0$, since all the additional terms on the RHS are at higher order in ε . This condition can then be reinserted into the field equations to achieve the cancellation of some terms. Taking into account all these simplifications, Equation (5.4), to linear order in ε , turns into the much more manageable form

$$\frac{G_{\mu\nu}^{(\gamma)}}{8\pi G} = T_{\text{BAR}\mu\nu} + T_{\text{DM}\mu\nu} - \square \mathcal{F}_2(\square) T_{\text{DM}\mu\nu} + 2(\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \mathcal{F}_0(\square) T_{\text{DM}}, \quad (5.6)$$

in terms of the linearized Einstein tensor

$$2G_{\mu\nu}^{(\gamma)} = 2\partial_\alpha \partial_{(\mu} \gamma_{\nu)}^\alpha - \partial_\mu \partial_\nu \gamma - \square \gamma_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta \gamma^{\alpha\beta} + \eta_{\mu\nu} \square \gamma. \quad (5.7)$$

5.2.2 Newtonian and Post-Newtonian limit

If we further assume the metric to be quasi static and the source to move slowly with respect to the speed of light c , we can expand Equation (5.6) in powers of $1/c$ to obtain the Newtonian and post-Newtonian approximations. In this setting it is customary to express the metric perturbation in terms of potentials, constructed from the matter density and velocity, with real coefficients known as post-Newtonian parameters. In a gravity theory, these must be computed from the field equations, and then allow a direct comparison with experimental data. We refer the reader to [245] for a discussion on this procedure, known as Parametrized post-Newtonian (PPN) formalism. If we retain only terms up to order c^{-2} , the metric perturbation in the standard PPN coordinate system can be written as

$$\gamma_{\mu\nu} = \begin{pmatrix} -\frac{2\Phi}{c^2} & \\ & -\frac{2\Psi}{c^2} \delta_{ij} \end{pmatrix}. \quad (5.8)$$

Here Φ encodes the time-time curvature and corresponds to the potential satisfying the analogous of the Poisson equation in the Newtonian limit of a given gravity theory. On the other hand, Ψ represents the space-space curvature and is usually written as $\Psi = \gamma \Phi$ in terms of the PPN parameter γ , representing the amount of space curvature per unit rest mass.

Inserting Equation (5.8) into (5.7) we find, up to order c^{-2} , the expansion:

$$G_{00}^{(\gamma)} = \frac{2\Delta\Psi}{c^2}, \quad G_{0i}^{(\gamma)} = 0, \quad G_{ij}^{(\gamma)} = (\partial_i\partial_j - \delta_{ij}\Delta) \frac{\Psi - \Phi}{c^2}. \quad (5.9)$$

By performing the expansion at the same order on the right hand side of Equation (5.6), and taking into account that for a Newtonian source the spatial stresses T_{0i}, T_{ij} are suppressed with respect to the rest mass density $T_{00} = c^2\rho$, we obtain the equations satisfied by the two potentials

$$\Delta\Phi = 4\pi G [\rho_{\text{BAR}} + (1 + (2\mathcal{F}_0 - \mathcal{F}_2)\Delta)\rho_{\text{DM}}], \quad (5.10)$$

$$\Delta\Psi = 4\pi G [\rho_{\text{BAR}} + (1 - (2\mathcal{F}_0 + \mathcal{F}_2)\Delta)\rho_{\text{DM}}]. \quad (5.11)$$

To be consistent with NFG, the RHS of (5.10) concerning dark matter should coincide with the effective density appearing on the RHS in Equation (3.6). This requirement imposes the following relation between the non-local operators $\mathcal{F}_0$ and $\mathcal{F}_2$

$$\mathcal{F}(\square) \equiv \mathcal{F}_2(\square) - 2\mathcal{F}_0(\square) = \frac{1 - (-\ell^2\square)^{1-s}}{\square}. \quad (5.12)$$

Since this constraint does not fix the form of the operators uniquely, one can construct different theories by appropriately choosing the scalar and tensorial operators. For instance, a scalar model, known as Relativistic Scalar Fractional Gravity (RSFG), is obtained setting $\mathcal{F}_0 = -\mathcal{F}/2$, $\mathcal{F}_2 = 0$, while a pure tensorial model (Relativistic Tensorial Fractional Gravity, or RTFG) can be obtained with $\mathcal{F}_0 = 0$, $\mathcal{F}_2 = \mathcal{F}$. Of particular interest is the choice $\mathcal{F}_0 = -\mathcal{F}/4$, $\mathcal{F}_2 = \mathcal{F}/2$, for which the scalar and tensorial part in the action (5.3) combine, leading to a coupling of the type $G_{\mu\nu}\mathcal{F}(\square)T_{\text{DM}}^{\mu\nu}$. Since the Einstein tensor appears already in the action, we will refer to this theory as Einstein Fractional Gravity, or EFG for short.

The PPN parameter γ takes the explicit form

$$\gamma = 1 - 4 \frac{\Delta^{-1}\mathcal{F}_0\Delta\rho_{\text{DM}}}{\Delta^{-1}\rho_{\text{BAR}} - \ell^2(-\ell^2\Delta)^{-s}\rho_{\text{DM}}}, \quad (5.13)$$

so that in TFG retains the GR value $\gamma = 1$, while is modified and space dependent in other fractional gravity theories. We stress that since only the DM component couples non-locally with gravity, while in the presence of only baryonic matter the theory reduces to GR, the present result is not ruled out by the solar system constraints [246] $\gamma - 1 \sim (2.1 \pm 2.3) \times 10^{-5}$ from the *Cassini* experiment.

5.2.3 The effective DM stress-energy tensor

One can insist in writing Equation (5.4) as an ordinary Einstein equation

$$G^{\mu\nu} = 8\pi G (T_{\text{bar}}^{\mu\nu} + T_{\text{DM eff}}^{\mu\nu}) , \quad (5.14)$$

by introducing an effective SET for the CDM component. To gain some insights on the physical interpretation of the latter, we can parametrize it in the most general form as

$$T_{\text{DM eff}}^{\mu\nu} = \rho_{\text{DM eff}} u^\mu u^\nu + q_{\text{DM eff}}^\mu u^\nu + q_{\text{DM eff}}^\nu u^\mu + p_{\text{DM eff}} h^{\mu\nu} + \pi_{\text{DM eff}}^{\mu\nu} . \quad (5.15)$$

Here $\rho_{\text{DM eff}}$ is the energy density measured by an observer comoving with the fluid; $q_{\text{DM eff}}^\mu$ is the spacelike heat flux vector, with $g_{\mu\nu} q_{\text{DM eff}}^\mu q_{\text{DM eff}}^\nu = +1$, $g_{\mu\nu} u^\mu q_{\text{DM eff}}^\nu = 0$; $h^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ is the projection of the metric on the hypersurface orthogonal to u^μ ; $p_{\text{DM eff}}$ is the isotropic pressure, and $\pi_{\text{DM eff}}^{\mu\nu}$ the anisotropic stress, with $u_\mu \pi_{\text{DM eff}}^{\mu\nu} = \pi_{\text{DM eff}}^{\mu\nu} u_\nu = g_{\mu\nu} \pi_{\text{DM eff}}^{\mu\nu} = 0$. The components of the stress-energy tensor can be recovered by contraction

$$\begin{aligned} \rho_{\text{DM eff}} &= T_{\text{DM eff}}^{\mu\nu} u_\mu u_\nu , & p_{\text{DM eff}} &= \frac{1}{3} h_{\mu\nu} T_{\text{DM eff}}^{\mu\nu} , \\ q_{\text{DM eff}}^\mu &= -h_{\alpha}^\mu T_{\text{DM eff}}^{\alpha\nu} u_\nu , & \pi_{\text{DM eff}}^{\mu\nu} &= h_{\alpha}^\mu h_{\beta}^\nu T_{\text{DM eff}}^{\alpha\beta} - \frac{1}{3} h^{\mu\nu} h_{\alpha\beta} T_{\text{DM eff}}^{\alpha\beta} . \end{aligned} \quad (5.16)$$

Thus, at the level of the field equations, the fractional interaction with gravity can be interpreted as a modification of the dark matter stress-energy tensor: even if the action describing dark matter is constructed as that of collisionless dust, the effective SET may still exhibit non-zero heat fluxes, as well as isotropic and anisotropic stresses. Note that the energy density is also modified, with $\rho_{\text{DM eff}}$ accounting for the interaction with gravity, while ρ_{DM} denotes the density of the DM fluid appearing in the action. Interestingly, this is reminiscent of what occurs, for instance, in particle physics, where the measured mass or charge of a particle differs from the corresponding parameters appearing in the Lagrangian, as they become dressed through interactions with other fields. In the same way, the components of the effective SET have a dynamical nature, in the sense that they depend on the gravitational field they themselves generate. As such, they may take different forms and play different roles depending on the system under consideration. Nevertheless, since dark matter is detected only through its gravitational signature, its effective stress-energy tensor may be regarded as providing the physically relevant description of the dark matter fluid. In light of this interpretation, Equation (5.5) expresses the conservation of the effective SET $\nabla_\mu T_{\text{DM eff}}^{\mu\nu} = 0$. On the other end it is clear that the perfect fluid DM SET cannot be conserved, since the fractional interaction with gravity induces heat flows that dissipate energy, together with isotropic and anisotropic stresses which do work on the system.

We can gain insights on the physical interpretation of the effective Newtonian DM density (3.6) appearing in the FPE by looking at the expression for the effective DM SET in the weak field limit, in a frame comoving with the fluid, so that $u^\mu = \delta_0^\mu$

$$T_{\text{DM eff}}^{\mu\nu} = T_{\text{DM}}^{\mu\nu} - \square \mathcal{F}_2(\square) T_{\text{DM}}^{\mu\nu} + 2(\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) \mathcal{F}_0(\square) T_{\text{DM}} , \quad (5.17)$$

with $T_{\text{DM}}^{\mu\nu} = \rho_{\text{DM}} \delta_0^\mu \delta_0^\nu$. The components of the effective DM SET are given by

$$\begin{aligned} \rho_{\text{DM eff}} &= \rho_{\text{DM}} + \partial_0^2 \mathcal{F}_2 \rho_{\text{DM}} - \Delta (2\mathcal{F}_0 + \mathcal{F}_2) \rho_{\text{DM}} , \\ p_{\text{DM eff}} &= 2 \left(\frac{2}{3} \Delta - \partial_0^2 \right) \mathcal{F}_0 \rho_{\text{DM}} , \\ q_{\text{DM eff}}^0 &= 0, \quad q_{\text{DM eff}}^i = -2 \partial^i \partial_0 \mathcal{F}_0 \rho_{\text{DM}} , \\ \pi_{\text{DM eff}}^{ij} &= 2 \left(\frac{1}{3} \delta^{ij} \Delta - \partial^i \partial^j \right) \mathcal{F}_0 \rho_{\text{DM}} . \end{aligned} \quad (5.18)$$

Note that the isotropic pressure, the heat flux, and the anisotropic stress are generated only when the action includes a scalar term, so that $\mathcal{F}_0 \neq 0$. Thus, TFG somewhat occupies a special place in the space of admissible fractional gravity theories, as it is the only case for which the DM effective SET retains the form of pressureless dust, although with a modified energy density $\rho_{\text{DM eff}} = (-\square)^{1-s} \rho_{\text{DM}}$.

In the Newtonian limit, where time derivatives can be neglected with respect to spatial ones, it is instructive to look at the combination $\rho_{\text{DM eff}} + 3 p_{\text{DM eff}}$ (this is the expression producing for instance the acceleration of the expansion factor in a Friedmann Lemaitre Robertson Walker cosmology)

$$\rho_{\text{DM eff}} + 3 p_{\text{DM eff}} = \rho_{\text{DM}} - \Delta \mathcal{F} \rho_{\text{DM}} = (-\Delta)^{1-s} \rho_{\text{DM}} . \quad (5.19)$$

The function on the right is exactly the effective density sourcing the gravitational potential in the Newtonian limit of FG (3.6). This result, which is independent on the particular form of the operators, and thus is valid in any RFG theory, offers a clear interpretation of the results obtained in Chapters 3 and 4 on the galaxies rotation curves. There, we have regarded the effective DM density as the distribution an observer would infer interpreting the data using Newtonian gravity, and we have seen as it can be recovered from the local density via enhancing of the long wavelength modes, and suppression of the short wavelength ones. In light of (5.19) it is clear that this transfer of power between modes originates from the action of the effective pressure, which, unlike in GR, is relevant even in the Newtonian setting, partially counterbalancing gravity and smoothing the inner density cusp in galaxies, turning a NFW profile into an almost cored one.

We also note that the expected relation

$$\Delta (\Psi - \Phi) = 12\pi G \left(\frac{\partial_i \partial_j}{\Delta} - \frac{1}{3} \delta_{ij} \right) \pi^{ij} \quad (5.20)$$

between the difference of the lensing and the Newtonian potentials and the anisotropic stress holds.

5.2.4 Scalar–vector–tensor decomposition

The weak field limit is also the relevant setting for studying the generation and propagation of scalar, vector, and tensor modes. To this aim we perform a Scalar-Vector-Tensor (SVT) decomposition of Equation (5.6) by separating the metric and effective SET into irreducible representations of spatial rotations. For instance γ_{00} , γ_{0i} , and γ_{ij} transform respectively as a scalar, a vector, and a tensor under spatial rotations, the same being true for $T_{\text{DM}00}$, $T_{\text{DM}0i}$, and $T_{\text{DM}ij}$. Moreover, we can perform a Helmholtz decomposition of the vector components into the gradient of a scalar plus a divergenceless vector (i.e. a curl), and similarly decompose the tensor components into two scalars, a divergenceless vector and a transverse traceless tensor. We can thus parametrize the metric and the effective SET as

$$\begin{aligned} \gamma_{00} &= 2\phi, \\ \gamma_{0i} &= \partial_i \alpha + \beta_i, \\ \gamma_{ij} &= \frac{1}{3} H \delta_{ij} + \partial_{(i} \epsilon_{j)} + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \Delta \right) \lambda + \gamma_{ij}^{\text{TT}}, \end{aligned} \quad (5.21)$$

$$\begin{aligned} T_{\text{DM eff } 00} &= (1 - \square \mathcal{F}_2) \rho_{\text{DM}} - 2\Delta \mathcal{F}_0 \rho_{\text{DM}}, \\ T_{\text{DM eff } 0i} &= \partial_i [(1 - \square \mathcal{F}_2) s - 2\partial_0 \rho_{\text{DM}}] + (1 - \square \mathcal{F}_2) s_i, \\ T_{\text{DM eff } ij} &= -2 \left(\partial_0^2 - \frac{2}{3} \Delta \right) \mathcal{F}_0 \rho_{\text{DM}} \delta_{ij} + (1 - \square \mathcal{F}_2) \partial_{(i} \sigma_{j)} + \\ &\quad + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \Delta \right) [(1 - \square \mathcal{F}_2) \sigma - 2\mathcal{F}_0 \rho_{\text{DM}}] + (1 - \square \mathcal{F}_2) \sigma_{ij}. \end{aligned} \quad (5.22)$$

Here $\phi, \alpha, H, \lambda, s, \sigma$ are scalars under rotations; $\beta_i, \epsilon_i, s_i, \sigma_i$ are divergenceless vectors, meaning that $\partial^i \beta_i = \partial^i \epsilon_i = \partial^i s_i = \partial^i \sigma_i = 0$; γ_{ij}^{TT} and σ_{ij} are transverse and traceless tensors, with $\partial^i \gamma_{ij}^{\text{TT}} = \delta^{ij} \gamma_{ij}^{\text{TT}} = \partial^i \sigma_{ij} = \delta^{ij} \sigma_{ij} = 0$. By performing the same SVT decomposition on the generators of gauge transformations, one can check that the following Bardeen's variables are gauge invariant [247]

$$\psi = -\phi + \partial_0 k - \frac{1}{2} \partial_0^2 \lambda,$$

$$\begin{aligned}\theta &= \frac{1}{3} (H - \Delta \lambda) , \\ \Sigma_i &= \beta_i - \frac{1}{2} \partial_0 \epsilon_i .\end{aligned}\tag{5.23}$$

In terms of those the first order Einstein tensor is given by [248]:

$$\begin{aligned}G_{00}^{(\gamma)} &= -\Delta \theta , \\ G_{0i}^{(\gamma)} &= -\partial_0 \partial_i \theta , \\ G_{ij}^{(\gamma)} &= -\frac{1}{2} \partial_i \partial_j (2\psi + \theta) + \delta_{ij} \left[\frac{1}{2} \Delta (2\psi + \theta) - \partial_0^2 \theta \right] .\end{aligned}\tag{5.24}$$

We then substitute this decomposition into Equation (5.6) and use the conservation law $\partial_\mu T_{\text{DM eff}}^{\mu\nu} = 0$, to find that some of the equations are automatically satisfied on shell, while the others can be simplified to

$$\begin{aligned}\Delta \theta &= -8\pi G (1 - \square \mathcal{F}_2 - 2\Delta \mathcal{F}_0) \rho_{\text{DM}} , \\ \Delta \psi &= 4\pi G [(1 - \square \mathcal{F}_2 + 2\Delta \mathcal{F}_0) \rho_{\text{DM}} - 3(1 - \square \mathcal{F}_2) \partial_0 s] , \\ \Delta \Sigma_i &= -16\pi G (1 - \square \mathcal{F}_2) s_i , \\ \square \gamma_{ij}^{\text{TT}} &= -16\pi G (1 - \square \mathcal{F}_2) \sigma_{ij} .\end{aligned}\tag{5.25}$$

In order to understand the physical consequences of Equation (5.25), one needs to write explicitly the fractional operators $\mathcal{F}_{0,2}$, thus selecting a relativistic fractional gravity theory. We will focus on SFG and TFG, as they provide the clearest interpretations.

In SFG, one has $\mathcal{F}_0 = -\mathcal{F}/2$, and $\mathcal{F}_2 = 0$, resulting in

$$\begin{aligned}\Delta\theta &= -8\pi G (1 + \Delta\mathcal{F}) \rho_{\text{DM}} , \\ \Delta\psi &= 4\pi G [(1 - \Delta\mathcal{F}) \rho_{\text{DM}} - 3\partial_0 s] , \\ \Delta\Sigma_i &= -16\pi G s_i , \\ \square\gamma_{ij}^{\text{TT}} &= -16\pi G \sigma_{ij} .\end{aligned}\tag{5.26}$$

Due to the peculiar form of the coupling between DM and gravity in SFG, the vector and tensor modes are not modified with respect to GR. In particular, gravitational waves have two polarizations and propagate at the speed of light. New features appear instead in the scalar modes θ and ϕ . When the non local interaction is neglected, the conservation of the energy momentum tensor implies that these degrees of freedom do not propagate, leaving as the only propagating modes the pure tensorial ones, representing gravitational waves. However, in the current framework, this is not the case, since the presence of $\mathcal{F}(\square)$, containing inverse powers of the d'Alembertian, implies that these scalar modes do propagate. In fact, the solution of (5.26) for θ is

$$\begin{aligned}\theta(t, \mathbf{x}) &= 2G \int_{\mathbb{R}^3} d^3\mathbf{x}' \frac{\rho_{\text{DM}}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + 2G \int_{\mathbb{R}^3} d^3\mathbf{x}' \frac{\rho_{\text{DM}}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \\ &\quad - \frac{4^{2-s}G}{\Gamma(s)\Gamma(s-1)} \int_{\mathbb{R}^3} d^3\mathbf{x}' \int_{-\infty}^{t-|\mathbf{x}-\mathbf{x}'|} \frac{\rho_{\text{DM}}(t', \mathbf{x}')}{[(t-t')^2 - |\mathbf{x} - \mathbf{x}'|^2]^{2-s}} ,\end{aligned}\tag{5.27}$$

with an analogue solution holding for ψ . The above expression comprises three terms: the first one is the instantaneous Newtonian solution, the second one represents the generation of a scalar wave propagating at the speed of light in vacuum, and the last one is of non-local nature. The Heaviside function in this last term restricts the integration in the interior of the past light cone, thus preventing acausal or superluminal motion (we refer to Appendix C for a derivation of the retarded Green function for the fractional d'Alembertian with $s > 1$ in Minkowskian spacetime, whose convolution with the density yields the last term). Note that, if the source is static, the two Bardeen variables are related to the two scalar potentials of the post-Newtonian expansion via $\psi = \Phi$, $\theta = -2\Psi$. The appearance of scalar propagating degrees of freedom alongside gravitational waves is a common feature of many scalar-tensor gravity models which predict a massive or a massless scalar field satisfying a modified Klein-Gordon

equation [249, 250, 251, 252]. In the present model these scalar modes are associated only with the DM component and feature both a static and a propagating term.

In TFG, one instead has $\mathcal{F}_0 = 0$, and $\mathcal{F}_2 = \mathcal{F}$, resulting in

$$\begin{aligned}\Delta\theta &= -8\pi G(-\square)^{1-s} \rho_{\text{DM}}, \\ \Delta\psi &= 4\pi G(-\square)^{1-s} (\rho_{\text{DM}} - 3\partial_0 s), \\ \Delta\Sigma_i &= -16\pi G(-\square)^{1-s} s_i, \\ \square\gamma_{ij}^{\text{TT}} &= -16\pi G(-\square)^{1-s} \sigma_{ij}.\end{aligned}\tag{5.28}$$

In this case all the modes get modified with respect to GR by the action of the same non-local operator, namely the Fractional d'Alembertian of order $1 - s$. It is instructive to look at the solution for the pure tensorial modes $\gamma_{ij}^{\text{TT}} = 16\pi G(-\square)^{-s} \sigma_{ij}$. Referring again to C, the solution is given by

$$\gamma_{ij}^{\text{TT}}(t, \mathbf{x}) = \frac{2^{5-2s} G}{\Gamma(s) \Gamma(s-1)} \int_{\mathbb{R}^3} d^3\mathbf{x}' \int_{-\infty}^{t-|\mathbf{x}-\mathbf{x}'|} dt' \frac{\sigma_{ij}(t', \mathbf{x}')}{[(t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2]^{2-s}}. \tag{5.29}$$

The departing from the standard GR result is transparent. There the integration is along the boundary of the past light cone, so that the solution describes gravitational waves propagating at the speed of light. When the fractional interaction is present however, the memory effects associated with non-locality imply that the integration must be taken over the entire observer's past light cone. Thus, the solution (5.29) describes wavepackets that, although featuring a maximum on the light cone, have power-law tails propagating in its interior. The speed of light still represents the limiting velocity for the propagation of the signal, so that causality is preserved, but it cannot be thought as the characteristic speed of the wavepacket.

5.3 Testing RFG with lensing data of galaxy clusters

We now aim to test the weak field limit of RFG by exploiting gravitational lensing data of galaxy clusters. In this context, it is customary to define the angular distances between the source and the observer D_s , the lens and the observer D_l , and the source and the lens D_{ls} , where in a flat FLRW Universe

$$D_{ij} = \frac{c}{1+z_j} \int_{z_i}^{z_j} \frac{dz}{H(z)} \tag{5.30}$$

holds in terms of the Hubble parameter $H(z) = H_0^2 [\Omega_m (1+z)^3 + 1 - \Omega_m]$. Hereafter the angular distances are computed in the standard Λ CDM cosmology with parameters from [253].

In the thin-lens approximation the lens is considered to lay on a two-dimensional plane, since its dimension along the line of sight is much smaller than the other distances involved. In such a case the lens primary function is to deflect the light beams coming from the source by an angle on the lens plane given by

$$\hat{\alpha} = \int_{-\infty}^{+\infty} dw \vec{\nabla}_{\perp} \left(\frac{\Phi(R, w) + \Psi(R, w)}{c^2} \right), \quad (5.31)$$

where $\vec{\nabla}_{\perp}$ represents the two-dimensional gradient operator, which is perpendicular to the path of the light, R denotes the two-dimensional projected radius on the lens plane, and w indicates the coordinate that specifies the position along the line of sight. The deflection angle $\hat{\alpha}$ can be described using the effective lensing potential $\hat{\alpha} = (D_l/D_{ls} D_s) \vec{\nabla}_{\perp} \Phi_{\text{lens}}$, where

$$\Phi_{\text{lens}}(R) = \frac{D_{ls} D_s}{D_l} \int_{-\infty}^{+\infty} dw \left(\frac{\Phi(R, w) + \Psi(R, w)}{c^2} \right). \quad (5.32)$$

The Laplacian Δ_r of Equation (5.32), taken with respect to the three dimensional radius $r = \sqrt{R^2 + w^2}$, defines the so called ‘convergence’

$$\kappa(R) = \frac{D_l D_{ls}}{D_s} \int_{-\infty}^{+\infty} dw \Delta_r \left(\frac{\Phi(R, w) + \Psi(R, w)}{2 c^2} \right). \quad (5.33)$$

Neglecting the contribution of baryons, in RFG we have from Equations (5.10) and (5.11) the relevant expressions

$$\Delta_r \Phi = 4\pi G (1 - \mathcal{F} \Delta) \rho_{\text{DM}} = 4\pi G \rho_{\text{DM eff}}^{\text{N}}, \quad (5.34)$$

$$\Delta_r \Psi = 4\pi G (1 - (4\mathcal{F}_0 + \mathcal{F}) \Delta) \rho_{\text{DM}}, \quad (5.35)$$

where $\rho_{\text{DM eff}}^{\text{N}}$ denotes the effective density appearing in the Poisson equation in the Newtonian limit of the theory.

The convergence is then

$$\kappa(R) = \frac{1}{\Sigma_{\text{crit}}} \int_{-\infty}^{+\infty} dw (\rho_{\text{DM eff}}^{\text{N}} - 2 \mathcal{F}_0 \Delta \rho_{\text{DM}}), \quad (5.36)$$

in terms of the so called critical surface density $\Sigma_{\text{crit}} \equiv (c^2/4\pi G) (D_s/D_{ls} D_l)$. As we have shown in Chapters 3 and 4, FG effects tend to become negligible on large scales. This implies that background cosmology and thus the angular distances D_{ij} entering the lensing analysis are

not appreciably altered with respect to their standard expressions in Λ CDM. Moreover, the standard lensing formalism recalled above can be applied, since on large scales the metric in the weak field limit over an expanding background can be approximately put in the same form as GR [254], apart from the difference in the gravitational and lensing potentials induced by FG effects within virialized structures.

In SFG $\mathcal{F}_0 = -\mathcal{F}/2$ holds and the convergence from Equation (5.36)

$$\kappa_{\text{SFG}}(R) = \frac{1}{\Sigma_{\text{crit}}} \int_{-\infty}^{+\infty} dw \rho_{\text{DM}} \quad (5.37)$$

coincides with that of GR; in this case the gravitational potential is sourced by the effective density $\rho_{\text{DM eff}}^{\text{N}}$ while the convergence is determined by the local DM density ρ_{DM} . In TFG $\mathcal{F}_0 = 0$ holds and the convergence reads:

$$\kappa_{\text{TFG}}(R) = \frac{1}{\Sigma_{\text{crit}}} \int_{-\infty}^{+\infty} dw \rho_{\text{DM eff}}^{\text{N}} ; \quad (5.38)$$

in this case both the gravitational potential and the lensing convergence are generated by the same effective DM density $\rho_{\text{DM eff}}^{\text{N}}$. In EFG $\mathcal{F}_0 = -\mathcal{F}/4$ holds and the convergence

$$\kappa_{\text{EFG}}(R) = \frac{1}{\Sigma_{\text{crit}}} \int_{-\infty}^{+\infty} dw \left(\frac{\rho_{\text{DM eff}}^{\text{N}} + \rho_{\text{DM}}}{2} \right) , \quad (5.39)$$

involves a sort of arithmetic average between the local and effective density distributions, which retains some difference from the source of the gravitational potential $\rho_{\text{DM eff}}^{\text{N}}$.

The EFG model can be motivated by the fact that it is the most natural way to combine the scalar and tensorial parts of the action in a single term. However, it may suffer of problems at the scales of galaxies, since recent data (e.g., [255, 256]) indicate a remarkable agreement among DM mass estimates based from weak lensing (that depend on lensing convergence) and from kinematical measurements of galaxy rotation curves (that depend on the gravitational potential); the tension can be even stronger for the SFG model and other flavors of the theory. From this point of view, the TFG model may instead be preferred since it is the only FG framework where such an agreement is ensured by construction. We will therefore present the analysis of cluster lensing data for both the EFG and the FG model (recall that as for lensing the FG coincides with GR).

Specifically, we test FG by exploiting the lensing convergence profiles of the individual and stacked clusters from the CLASH sample [257, 258, 259]. This consists of 20 galaxy clusters with redshifts in the range $0.2 \lesssim z \lesssim 0.7$ for which radial convergence profiles have been reconstructed from a joint analysis of weak lensing shear/magnification data and strong lensing

observations. We discard one of the X-ray selected cluster (RXJ1532) from the CLASH sample since no multiple images have been identified in it and therefore the mass reconstruction is based only on wide-field data. For the sake of simplicity, as in [257, 258] we report and analyze the CLASH data in terms of $\kappa_\infty(R)$ profiles, meaning that the Σ_{crit} is computed in the limit of a very distant source (i.e., $z_s \rightarrow \infty$).

We adopt a Bayesian framework, characterized by the parameter set

$$\theta \equiv (s, \log \ell/r_s, \log M_{200}, \log c_{200}), \quad (5.40)$$

where s and ℓ are the FG index and length-scale, while M_{200} and c_{200} are the DM mass and concentration at R_{200} , the reference radius within which the average DM density is 200 times the critical density of the Universe, and $r_s \equiv R_{200}/c_{200}$ is the NFW scale radius. We assume a Gaussian log-likelihood

$$\ln \mathcal{L}(\theta) = -\chi^2(\theta)/2, \quad (5.41)$$

where the chi-square $\chi^2(\theta) = \sum_{ij} [\kappa_\infty^{\mathcal{M}}(\theta, r_i) - \kappa_\infty^{\mathcal{D}}(r_i)] \mathcal{C}_{ij}^{-1} [\kappa_\infty^{\mathcal{M}}(\theta, r_j) - \kappa_\infty^{\mathcal{D}}(r_j)]$ is obtained by comparing our model expectations $\kappa_\infty^{\mathcal{M}}(\theta, r_i)$ to the data values $\kappa_\infty^{\mathcal{D}}(r_i)$, summing over radial coordinates r_i and taking into account the variance-covariance matrix $\mathcal{C}_{ij}$ among radial bins (see [257]). More in detail, our fitting model is constituted by $\kappa_\infty(R)$ profiles computed according to Equation (5.38) for TFG or (5.39) for EFG; note that in our analysis we neglect the contribution to the convergence profile from baryonic matter (e.g., stars, hot gas) since this is expected to be subdominant in the range of radii probed by the CLASH data.

We adopt flat priors $\pi(\theta)$ on the parameters $s \in [1, 3/2]$, $\log \ell/r_s \in [-3, 1]$, $\log c_{200} \in [0, 2]$ and $\log M_{200} [M_\odot] \in [12, 16]$. Moreover, we include a log-normal prior to take into account the relation $c_{200} - M_{200}$ as measured from N -body simulations in the Λ CDM cosmology [151, 260] with dispersion around 0.15 dex. Two caveats are in order. First, we choose to fit the ratio ℓ/r_s rather than ℓ itself. As shown by our previous analysis of galaxy rotation curves, ℓ is not expected to be universal across all scales; instead, it likely depends on halo mass and possibly other environmental properties. By contrast, the ratio ℓ/r_s is expected to be more nearly uniform across different systems. However, again from our analysis in Chapter 4, one can anticipate that even the ratio ℓ/r_s will be poorly constrained when focusing on massive galaxy clusters, since at these scales the index s is expected to deviate mildly from one (FG effects are expected to be weak), and the overall normalization of the effective mass profile solely depends on $(\ell/r_s)^{2-2s}$. The second caveat concerns the adopted prior on the $c_{200} - M_{200}$ relationship. From the thermodynamic analysis of the XCOP sample we have found that in FG the concentration scales with halo mass very similarly to the Λ CDM case; this is again a manifestation of the fact that FG effects in massive clusters tend to be weak. Thus in the present

analysis we impose a standard Λ CDM prior on the halo concentration vs. mass relationship; however, we have found a posteriori that the impact of the adopted prior on the outcomes of our analysis is mild, though it helps somewhat in constraining the fractional index s and hence to highlight the fractional deviations from Newtonian gravity.

We sample the parameter posterior distributions $\mathcal{P}(\theta) \propto \mathcal{L}(\theta) \pi(\theta)$ via the MCMC Python package `emcee` [152], running it with 10^4 steps and 100 walkers; each walker is initialized with a random position extracted from the priors discussed above. To speed up convergence, we adopt a mixture of differential evolution and snooker moves of the walkers, in proportion of 0.8 and 0.2 respectively, that emulates a parallel tempering algorithm. After checking the auto-correlation time, we remove the first 20% of the flattened chain to ensure burn-in; the typical acceptance fractions of the various runs are around 30%.

The results of the fit to the convergence profiles $\kappa_\infty(R)$ are shown in Figure 5.1, where the solid lines with shaded areas illustrate the median and the 2σ credible interval, in magenta for EFG and in cyan for TFG; for reference, dashed lines display the bestfits in Newtonian gravity. All in all, both the EFG and the TFG fits are always very good, both in the inner and outer portions of the convergence profiles. Note, however, that in the outermost radial range the two-halo term (from large-scale clustering) can somewhat alter the profile, and this has not been included in the present analysis.

In Table 5.1 the marginalized posterior estimates (median and 1σ confidence intervals) of the parameters are reported. In about half of the clusters a deviation of the index s from the Newtonian gravity value of 1 is favored. The ℓ/r_s ratio, due to its degeneracy with the s and the mass M_{200} , is marginally constrained, and in many instances only a loose lower limit can be inferred. The estimates of the mass (concentration) tend to be slightly smaller (higher) than in GR, though consistent within the 2σ uncertainties. Unfortunately, the present quality and radial extent of the data does not allow to statistically prefer the FG fits over the general relativity ones, or to discern a clear preference for EFG vs. TFG, in terms of reduced χ^2 (nor Bayesian evidence criterions). However, we can fairly conclude that both the FG models perform decently well on this data sample, at a level comparable with the standard general relativistic setting.

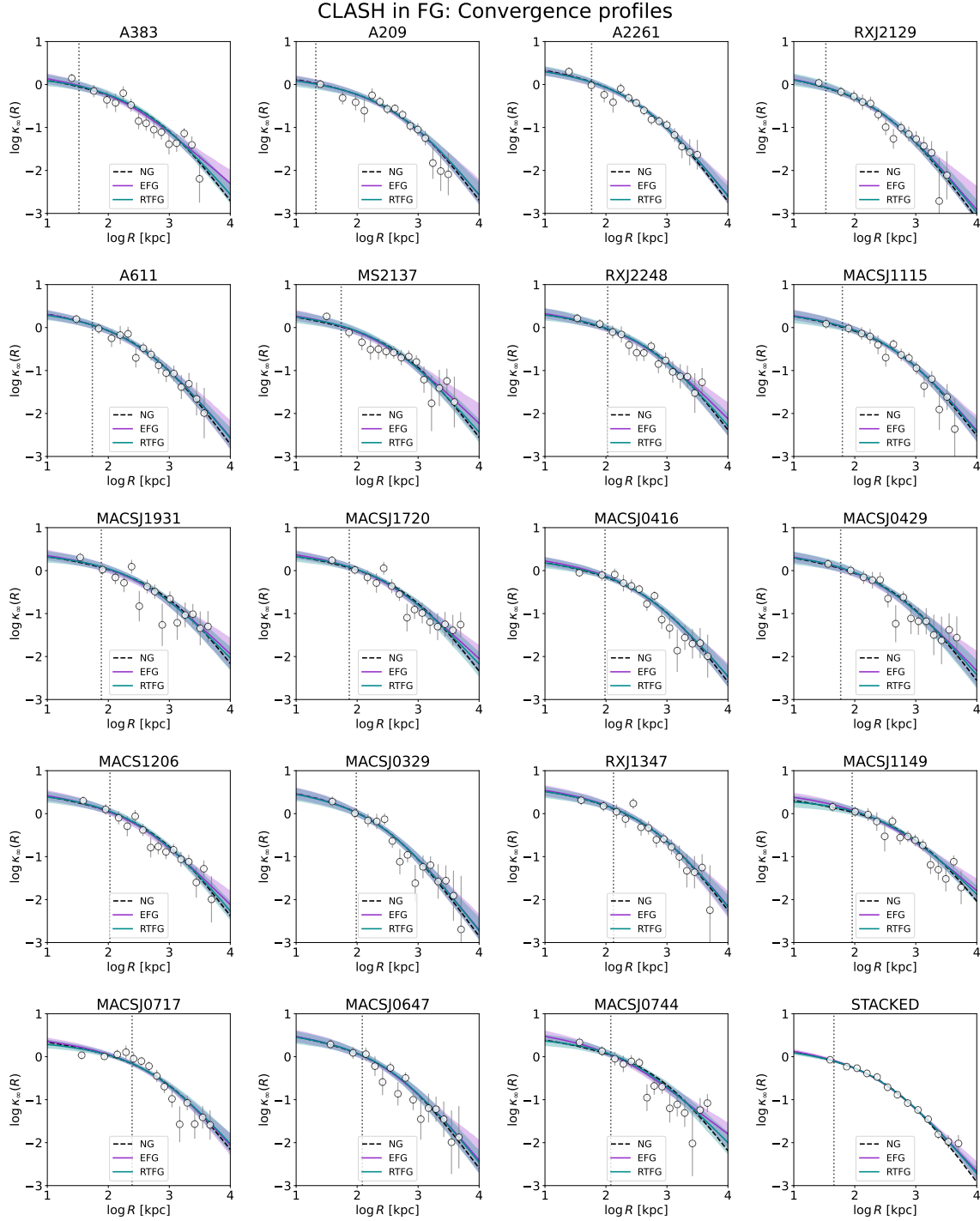


Figure 5.1. Fits to the lensing convergence profiles $\kappa_\infty(R)$ of the individual and of the stacked CLASH cluster sample according to the EFG (magenta) and TFG (cyan) model. Solid lines with shaded areas illustrate the median and the 2σ credible interval from sampling the posterior distribution. Dashed black lines display, for comparison, the bestfits in general relativity. The vertical dotted line marks the Einstein radius in the lens plane, on assuming a reference source redshift $z_s \sim 2$.

Table 5.1. Marginalized posterior estimates (mean and 1σ confidence intervals are reported) for the parameters from the MCMC analysis of the CLASH clusters in EFG (first lines), in TFG (second lines) and in GR (third lines, with $s = 1$). Columns report the values of fractional index s , fractional length-scale $\log \ell/r_s$, concentration $\log c_{200}$, DM mass $\log M_{200}[M_\odot]$ in solar units, and the values of the reduced χ_r^2 for the overall fit.

Cluster	s	$\log \ell/r_s$	$\log c_{200}$	$\log M_{200}$	χ_r^2
A383	$1.28^{+0.15}_{-0.11}$	$-0.08^{+0.93}_{-0.30}$	$0.77^{+0.14}_{-0.13}$	$14.99^{+0.30}_{-0.19}$	1.18
	$1.11^{+0.05}_{-0.09}$	$-0.72^{+1.00}_{-0.60}$	$0.68^{+0.11}_{-0.16}$	$14.93^{+0.32}_{-0.21}$	1.64
	1		$0.44^{+0.10}_{-0.09}$	$15.30^{+0.07}_{-0.07}$	1.49
A209	< 1.14	> -0.99	$0.57^{+0.09}_{-0.10}$	$15.14^{+0.21}_{-0.11}$	0.84
	< 1.10	$-0.61^{+1.20}_{-0.64}$	$0.59^{+0.09}_{-0.12}$	$15.09^{+0.24}_{-0.13}$	0.83
	1		$0.48^{+0.09}_{-0.09}$	$15.32^{+0.08}_{-0.08}$	0.68
A2261	< 1.13	$-0.9^{+1.3}_{-1.0}$	$0.70^{+0.08}_{-0.10}$	$15.12^{+0.27}_{-0.13}$	0.49
	< 1.08	$-1.11^{+0.98}_{-0.98}$	$0.69^{+0.08}_{-0.10}$	$15.08^{+0.27}_{-0.14}$	0.57
	1		$0.63^{+0.08}_{-0.08}$	$15.36^{+0.06}_{-0.06}$	0.49
RXJ2129	< 1.18	> -0.557	$0.72^{+0.09}_{-0.13}$	$14.79^{+0.20}_{-0.11}$	0.56
	< 1.08	$-0.56^{+1.50}_{-0.49}$	$0.70^{+0.09}_{-0.12}$	$14.75^{+0.21}_{-0.11}$	0.63
	1		$0.62^{+0.09}_{-0.09}$	$14.91^{+0.08}_{-0.08}$	0.56
A611	< 1.15	> -0.922	$0.67^{+0.09}_{-0.11}$	$15.08^{+0.22}_{-0.12}$	0.42
	< 1.08	$> -0.74^{+1.30}_{-0.79}$	$0.66^{+0.09}_{-0.11}$	$15.03^{+0.24}_{-0.13}$	0.42
	1		$0.56^{+0.09}_{-0.09}$	$15.24^{+0.08}_{-0.08}$	0.36
MS2137	$1.26^{+0.12}_{-0.12}$	$-0.21^{+0.96}_{-0.36}$	$0.71^{+0.14}_{-0.14}$	$14.94^{+0.30}_{-0.20}$	0.79
	$1.11^{+0.05}_{-0.09}$	$-0.68^{+1.00}_{-0.61}$	$0.65^{+0.11}_{-0.15}$	$14.91^{+0.32}_{-0.20}$	0.99
	1		$0.44^{+0.11}_{-0.10}$	$15.28^{+0.09}_{-0.09}$	0.86
RXJ2248	$1.23^{+0.10}_{-0.19}$	> 0	$0.63^{+0.12}_{-0.12}$	$15.29^{+0.20}_{-0.14}$	0.47
	< 1.11	$-0.21^{+1.20}_{-0.27}$	$0.61^{+0.11}_{-0.14}$	$15.24^{+0.22}_{-0.13}$	0.58
	1		$0.41^{+0.12}_{-0.10}$	$15.39^{+0.09}_{-0.09}$	0.51
MACSJ1115	< 1.17	> -0.72	$0.58^{+0.09}_{-0.11}$	$15.11^{+0.21}_{-0.11}$	0.61
	< 1.12	$> -0.42^{+1.20}_{-0.48}$	$0.59^{+0.09}_{-0.13}$	$15.07^{+0.23}_{-0.13}$	0.58
	1		$0.42^{+0.10}_{-0.10}$	$15.25^{+0.07}_{-0.07}$	0.48
MACSJ1931	$1.25^{+0.11}_{-0.11}$	$-0.55^{+0.91}_{-0.52}$	$0.64^{+0.13}_{-0.13}$	$15.05^{+0.32}_{-0.26}$	0.99
	$1.16^{+0.08}_{-0.09}$	$-0.70^{+0.81}_{-0.43}$	$0.63^{+0.13}_{-0.15}$	$15.00^{+0.28}_{-0.28}$	1.05
	1		$0.34^{+0.12}_{-0.11}$	$15.56^{+0.09}_{-0.09}$	0.91
MACSJ1720	$1.25^{+0.09}_{-0.11}$	$-0.32^{+0.98}_{-0.47}$	$0.68^{+0.13}_{-0.13}$	$15.01^{+0.32}_{-0.23}$	0.98
	$1.15^{+0.06}_{-0.07}$	$-0.57^{+0.77}_{-0.50}$	$0.67^{+0.13}_{-0.15}$	$14.92^{+0.30}_{-0.25}$	1.05
	1		$0.39^{+0.10}_{-0.10}$	$15.40^{+0.09}_{-0.09}$	1.05
MACSJ0416	< 1.18	> -0.36	$0.54^{+0.09}_{-0.09}$	$15.02^{+0.18}_{-0.12}$	1.03
	$1.09^{+0.04}_{-0.08}$	$-0.09^{+1.01}_{-0.22}$	$0.58^{+0.11}_{-0.13}$	$14.97^{+0.20}_{-0.13}$	0.99
	1		$0.36^{+0.09}_{-0.09}$	$15.09^{+0.07}_{-0.08}$	0.84
MACSJ0429	$1.19^{+0.08}_{-0.15}$	$-0.36^{+1.20}_{-0.39}$	$0.66^{+0.12}_{-0.12}$	$14.92^{+0.26}_{-0.18}$	0.55
	$1.11^{+0.04}_{-0.09}$	$-0.51^{+1.00}_{-0.54}$	$0.66^{+0.12}_{-0.14}$	$14.87^{+0.28}_{-0.19}$	0.57
	1		$0.44^{+0.13}_{-0.11}$	$15.19^{+0.09}_{-0.10}$	0.54
MACSJ1206	$1.24^{+0.12}_{-0.14}$	> -0.03	$0.68^{+0.12}_{-0.12}$	$15.19^{+0.23}_{-0.12}$	0.72
	$1.09^{+0.04}_{-0.08}$	$-0.32^{+1.10}_{-0.44}$	$0.64^{+0.11}_{-0.15}$	$15.13^{+0.25}_{-0.14}$	0.91
	1		$0.43^{+0.09}_{-0.09}$	$15.34^{+0.07}_{-0.07}$	0.85

Cluster	s	$\log \ell/r_s$	$\log c_{200}$	$\log M_{200}$	χ_r^2
MACSJ0329	< 1.11	> -0.98	$0.72^{+0.09}_{-0.11}$	$14.92^{+0.18}_{-0.09}$	0.97
	< 1.07	$-0.70^{+1.60}_{-0.58}$	$0.72^{+0.09}_{-0.11}$	$14.88^{+0.20}_{-0.11}$	0.98
	1		$0.66^{+0.11}_{-0.11}$	$15.02^{+0.08}_{-0.08}$	0.83
RXJ1347	< 1.15	> -0.72	$0.60^{+0.10}_{-0.10}$	$15.41^{+0.18}_{-0.11}$	0.89
	< 1.08	$-0.56^{+1.5}_{-0.45}$	$0.61^{+0.09}_{-0.12}$	$15.37^{+0.20}_{-0.12}$	0.89
	1		$0.49^{+0.11}_{-0.11}$	$15.51^{+0.07}_{-0.08}$	0.76
MACSJ1149	$1.31^{+0.10}_{-0.09}$	$0.08^{+0.58}_{-0.35}$	$0.58^{+0.12}_{-0.12}$	$15.14^{+0.28}_{-0.20}$	0.85
	$1.20^{+0.07}_{-0.07}$	$0.01^{+0.45}_{-0.37}$	$0.63^{+0.14}_{-0.14}$	$15.08^{+0.28}_{-0.24}$	0.86
	1		$0.17^{+0.09}_{-0.09}$	$15.41^{+0.07}_{-0.07}$	0.83
MACSJ0717	$1.17^{+0.07}_{-0.11}$	> -0.13	$0.45^{+0.09}_{-0.09}$	$15.29^{+0.18}_{-0.12}$	1.68
	$1.13^{+0.05}_{-0.06}$	$+0.23^{+0.64}_{-0.27}$	$0.56^{+0.12}_{-0.12}$	$15.28^{+0.21}_{-0.14}$	1.49
	1		$0.24^{+0.09}_{-0.08}$	$15.35^{+0.07}_{-0.07}$	1.38
MACSJ0647	< 1.17	> -0.19	$0.62^{+0.11}_{-0.11}$	$15.08^{+0.14}_{-0.11}$	0.68
	< 1.08	> -0.46	$0.63^{+0.10}_{-0.12}$	$15.05^{+0.15}_{-0.11}$	0.68
	1		$0.50^{+0.12}_{-0.11}$	$15.11^{+0.09}_{-0.09}$	0.57
MACSJ0744	$1.36^{+0.13}_{-0.05}$	$0.36^{+0.52}_{-0.21}$	$0.68^{+0.11}_{-0.11}$	$15.05^{+0.25}_{-0.16}$	1.10
	$1.19^{+0.07}_{-0.07}$	$0.09^{+0.47}_{-0.37}$	$0.69^{+0.14}_{-0.14}$	$14.95^{+0.28}_{-0.2}$	1.42
	1		$0.23^{+0.11}_{-0.11}$	$15.27^{+0.08}_{-0.08}$	1.45
STACKED	$1.18^{+0.07}_{-0.08}$	> 0.28	$0.64^{+0.09}_{-0.06}$	$14.88^{+0.14}_{-0.06}$	0.74
	$1.10^{+0.04}_{-0.04}$	> 0.14	$0.69^{+0.11}_{-0.11}$	$14.84^{+0.16}_{-0.07}$	0.83
	1		$0.46^{+0.03}_{-0.03}$	$14.91^{+0.03}_{-0.02}$	0.94

5.4 Summary and conclusions

In this chapter we have extended the fractional gravity framework to a full relativistic theory, deriving the field equations from an action which features the coupling between curvature and the dark matter SET (considered to be the one for dust), mediated by non-local operators. To be more general, we have considered both a scalar $R\mathcal{F}_0(\square)T_{\text{DM}}$, and a tensorial $R_{\mu\nu}\mathcal{F}_2(\square)T_{\text{DM}}^{\mu\nu}$ coupling. The resulting field equations make transparent that, in the context of fractional gravity, the spacetime curvature is sourced not only by the value of the DM density, but also by its space and time variations, leading to a much richer dynamics for non-homogeneous or non-static systems.

We have then derived the weak field and Newtonian limit of the field equations, showing that in the latter case the fractional Poisson equation (3.1) is correctly reproduced, provided a relation between the two non-local operators $\mathcal{F}_0(\square)$ and $\mathcal{F}_2(\square)$ holds. Since this requirements does not completely fix the form of the coupling, we have considered different viable fractional gravity theories, dubbed Scalar (SFG), Tensorial (TFG), and Einstein (EFG) fractional gravity.

Inspired by the form of the field equations, we have noted that they can be written as

Einstein equations sourced by an effective DM SET, which includes the non-local coupling with gravity $G^{\mu\nu} = 8\pi G T_{\text{DM eff}}^{\mu\nu}$. Remarkably the latter can be interpreted as the SET of a fluid with a modified energy density, an emerging isotropic pressure and anisotropic stress, as we have shown explicitly in the weak field limit. This fact clarifies the physical mechanism which turns the density distribution sourcing the classical Poisson equation, to its effective counterpart (3.6). Indeed, the latter emerges dynamically as the result of the action of the non-local isotropic pressure, which keeps gravitating even in the Newtonian limit.

Finally, we have tested TFG and EFG against the lensing convergence profiles $k_\infty(R)$ of the individual and of the stacked CLASH sample, consisting of 20 galaxy clusters with redshifts in the range $0.2 \leq z \leq 0.7$, finding that both the FG models perform decently well on this data sample, at a level comparable with the standard general relativistic setting. This is in accordance to the scaling of the parameters s and ℓ , found in Chapters 3 and 4 in the Newtonian limit of fractional gravity.

To summarize, in the last three chapters we have explored the effects of fractional gravity in gravitationally bound systems, in the Newtonian and weak field relativistic limit, finding that it is able to:

- Fit the rotation curve of galaxies of different shapes and types, from ultra-diffuse dwarfs, to HSB and LSB spirals, at a level better or at least comparable with standard Newtonian gravity. In particular the greater departure from the latter occurs in the smallest systems, when FG always performs better at the level of the reduced χ_r^2 and BIC.
- Solve the cusp-core problem of CDM by modifying the equilibrium density profile through the action of a pressure component, emerging dynamically from the non-local coupling between dark matter and gravity.
- Reproduce and explain naturally the empirical relations observed in spirals between dark and luminous matter, as a consequence of non-locality.
- Fit the density, pressure, and lensing convergence profiles of galaxy clusters of the X-COP sample, at a level comparable with Newtonian gravity or GR.

Despite these remarkable successes in gravitationally bound astrophysical systems, a complete treatment of fractional gravity in the relativistic regime remains out of reach, as the field equations (5.14) are far too complex to admit solutions even under simple metric ansätze. The main difficulty arises from the fact that the nonlocal operators appearing in the equations are functions of the fractional d'Alembertian, whose explicit form depends on the metric itself, which is the unknown to be determined by the same field equations. In the non-local gravity literature, this difficulty is sometimes circumvented through simplifying—albeit rather

speculative—assumptions. In the context of cosmology in higher derivative theories of gravity several authors look for solutions satisfying relations of the type $\square R = \lambda_1 R + \lambda_2$ [234], or $\square R_{\mu\nu} = \lambda_1 R_{\mu\nu} + \lambda_2 g_{\mu\nu}$ [178], so that the action of the n -th power of the d'Alembertian on the curvature can be easily computed. In the analysis of black hole solutions in non-local gravity, it is customary to work in the weak field limit of the theory, justifying this assumption a posteriori by checking that non-local effects remove the central singularity [233]. In our view, assumptions of this kind lack a solid theoretical justification, and the resulting solutions, although leading to interesting scenarios, should therefore not be regarded as physically meaningful.

In the second part of this work, we adopt a different approach to the problem by studying a local model, dubbed non-minimally coupled dark matter (NMCDM), whose action is inspired by that of EFG. In the absence of non-local operators, the field equations of NMCDM can be solved in both cosmological and black hole settings, leading to remarkable results. These findings can, in turn, provide valuable insight into the behavior of fractional gravity in analogous scenarios.

II

Second part

Chapter 6

Non-Minimally Coupled Dark Matter

6.1 Introduction

The main lesson to be learned from the previous chapters is that, although we have successfully formulated a generalization of Fractional Gravity within the framework of General Relativity, the resulting field equations are far too involved to be solved in a fully relativistic setting. Ultimately, this difficulty stems from the presence of the fractional d'Alembertian in the non-local operators $\mathcal{F}_0(\square)$ and $\mathcal{F}_2(\square)$, whose explicit form depends on the geometry of the system—and therefore on the solution of the field equations themselves. This problem can be addressed only in the weak-field regime, since in that case the deviation of the d'Alembertian from its Minkowski-space expression can be regarded as second order and therefore neglected.

Following this line of reasoning, one might be led to explore a model in which the coupling between dark matter and gravity takes a form similar to that of Fractional Gravity, but without involving non-local operators, and to use it to investigate phenomena that require a fully relativistic treatment, such as the formation of black-hole-sized objects or the evolution of the Universe. The results obtained using this model, hereafter referred to as Non-Minimally Coupled Dark Matter (NMC-DM), can then be used to guide our intuition about how the systems under consideration would behave in the context of Fractional Gravity.

The plan of this chapter is as follows: in Section 6.2 we introduce the specific model under analysis and we derive the associated field equations, showing that they can be interpreted in terms of an effective stress-energy tensor (SET) for the dark matter component. In Section 6.3 we solve the equations with a spherically symmetric, static, asymptotically flat ansatz for the metric, and argue that the main properties of the solutions allow us to interpret them as regular, horizonless, ultra compact objects (UCOs); in Section 6.4 we present the geodesics

of UCOs spacetimes, with a particular focus on the appearance of a pseudo-shadow due to extreme lensing effects; finally, in Section 6.5 we summarize the results, and discuss the possible astrophysical implications of these solutions.

6.2 Non-minimally coupled Dark Matter

For DM masses $m_\chi \sim 20 - 100$ GeV, the upper bound on the dark matter-nucleon cross section is of the order $\sigma_{\chi N} < 10^{-47} \text{ cm}^2$, slightly above the irreducible neutrinos background, and much smaller than the cross section $\sigma_{\chi\chi} \sim 10^{-36} \text{ cm}^2$ required by the WIMP miracle. These results seem to indicate that, at least for the range of mass considered, there is little room left for a WIMP miracle, and that either CDM particles have a very different mass, or they interact with themselves and with the Standard Model ones with very different strengths. On the other hand the current constraints on the weak force for CDM particles open the captivating possibility of considering non Standard Model interactions for particles in the WIMP mass range. Nonetheless, we expect these effects to become relevant at sufficiently high densities to avoid contradiction with the standard CDM model on large cosmological scales. An order of magnitude estimation can be found by requiring the mean free path $\lambda_{\text{mfp}} = m_\chi / \rho_\chi \sigma_{\chi\chi}$ of the DM particle to be of the same order of the curvature radius (expressed, e.g., by the inverse of the square root of the Ricci scalar) $R_c = c(8\pi G \rho_\chi)^{-1/2}$. This leads to the definition of a threshold density

$$\rho_{\text{th}} = \left(\frac{8\pi G}{c^2} \right) \left(\frac{\sigma_{\chi\chi}}{m_\chi} \right)^{-2}, \quad (6.1)$$

above which the interaction starts to have macroscopical gravitational effects. Assuming a mass $m_\chi \sim 100$ GeV and a cross section $\sigma_{\chi\chi} \sim 10^{-36} \text{ cm}^2$, leads to a threshold density of $\rho_{\text{th}} \sim 10^2 \text{ g/cm}^3$, much larger than the universal background, and to length scales $\lambda_{\text{mfp}} \sim R_c \sim 10^{12} \text{ cm}$, typical of stars or massive compact objects. Nonetheless, we note that such high densities for dark matter are not pertaining to current virialized halos, but could have been attained in the early Universe. Since almost the only certain thing about dark matter is that it couples with gravity, a natural choice for this interaction is a non-minimal coupling with the gravitational field, i.e. $\sigma_{\chi\chi} \rightarrow \sigma_{\chi g}$. Sufficiently small values of $\sigma_{\chi g}$ suggest the intriguing possibility that the non-minimal coupling with gravity could have a universal nature, being present also for Standard Model particles, but that for the latter it would be non relevant because its strength is outclassed by the Standard Model interactions. On the other hand, lacking any other stronger interactions, the non-minimal coupling would be effective for CDM particles.

As we did in fractional gravity, also in the non-minimally coupled dark matter model we

assume cold dark matter (CDM), characterized as a pressureless fluid, to be non-minimally coupled with the spacetime curvature. Although the coupling can be realized in general with a combination of a scalar and a tensorial term $\alpha_0 R T_{\text{DM}} + \alpha_2 R_{\mu\nu} T_{\text{DM}}^{\mu\nu}$ in the action, we choose the particular combination $\alpha_0 = -\alpha_2/2$, so that the dark matter SET couples directly with the Einstein tensor. As we will show below, at leading order this choice allows the non-minimal coupling to be interpreted as a gravitationally mediated interaction between luminous and dark matter, as well as a gravitationally mediated self-interaction of dark matter, thus being consistent with our interpretation in terms of a cross section $\sigma_{\chi\chi}$.

Neglecting baryons for simplicity, the total action of the theory then reads

$$S_{\text{NMC}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \mathcal{L}_{\text{DM}} + \epsilon L^2 G_{\mu\nu} T_{\text{DM}}^{\mu\nu} \right], \quad (6.2)$$

where the first term is the Einstein-Hilbert Lagrangian and $\mathcal{L}_{\text{DM}}$ denotes the DM Lagrangian. As for the non-minimal coupling term, we consider a perturbative-like construction where $T_{\text{DM}}^{\mu\nu} \equiv (2/\sqrt{-g}) \delta(\sqrt{-g} \mathcal{L}_{\text{DM}}) / \delta g_{\mu\nu}$ is the DM stress-energy tensor of the minimally coupled theory. In [261] the authors have shown that for a barotropic fluid the Lagrangian $\mathcal{L} = -\rho$ leads to the desired energy momentum tensor of a perfect fluid, provided the conservation of particle number $\nabla_\mu(\rho u^\mu) = 0$ applies. As we will see below, this equation fails to be valid once the DM becomes non-minimally coupled with gravity, hence it is not clear how to select a suitable DM Lagrangian from which $T_{\text{DM}}^{\mu\nu} = \rho u^\mu u^\nu$ can be derived. This fact, although being problematic at conceptual level, is irrelevant in the present context since the DM Lagrangian does not appear explicitly in the field equations, which we will consider a valid effective description of a non-minimal coupling.

To complete the interaction Lagrangian we include the characteristic length L required for the correct dimensionality, and $\epsilon = \pm 1$ which is a dimensionless ‘polarity’ parameter. The length L can be seen as the macroscopical manifestation of the cross section to mass ratio $\sigma_{\chi g}/m_\chi$ and, as we will make clear in the next section, it will completely determine the scales of the solutions. The polarity parameter ϵ is instead related with the nature of the non-minimal interaction. It has been shown that in a model with many similarities [135], the choice $\epsilon = -1$ leads to stable cored configurations for DM halos, while the opposite choice leads to very cuspy inner shapes. In the present context of ultra-compact objects we initially considered both possibilities, but found that the case $\epsilon = +1$ always generates singular solutions, with the emergence of wormhole-like effects; as our focus is on regular configurations, we will set $\epsilon = -1$. With the language of Chapter 5, the NMC–DM model can then be seen as a localized version of Einstein Fractional Gravity, in which $-2\mathcal{F}_0 = \mathcal{F}_2 = -L^2$.

Non-minimal coupling between DM and gravity can also be interpreted as a DM self-interaction mediated by gravity. This fact can be seen rigorously through a disformal transfor-

mation into the Einstein frame, which turns the action (6.2) into

$$S_{\text{NMC}} = \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left[\frac{1}{16\pi G} \tilde{R} + \mathcal{F}(\rho, \tilde{g}_{\mu\nu}) + \mathcal{G}(\rho) \mathcal{L}_{\text{DM}} \right], \quad (6.3)$$

that is a minimally coupled fluid with non-trivial self-interactions. A less rigorous, but more intuitive way to understand this self-interaction is by truly employing a perturbative construction with $R = R^{(0)} + \varepsilon R^{(1)}$, $\mathcal{L}_{\text{DM}} = \mathcal{L}_{\text{DM}}^{(0)} + \varepsilon \mathcal{L}_{\text{DM}}^{(1)}$ and $L^2 = \varepsilon L^2$; the action then takes the form

$$\begin{aligned} S_{\text{NMC}}^{(0)} &= \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left[\frac{c^4}{16\pi G} R^{(0)} + \mathcal{L}_{\text{DM}}^{(0)} \right], \\ S_{\text{NMC}}^{(1)} &= \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left[\frac{c^4}{16\pi G} R^{(1)} + \mathcal{L}_{\text{DM}}^{(1)} - L^2 G_{\mu\nu}^{(0)} T^{(0)\mu\nu} \right] = \\ &= \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left[\frac{c^4}{16\pi G} R^{(1)} + \mathcal{L}_{\text{DM}}^{(1)} - 8\pi G L^2 T_{\mu\nu}^{(0)} T^{(0)\mu\nu} \right], \end{aligned} \quad (6.4)$$

where in the last line we used the 0th-order Einstein equations. While this construction effectively shows the self-interacting nature of the non-minimal coupling, having a $T_{\mu\nu} T^{\mu\nu}$ term in the action, we will consider the action (6.2) in its completeness to address non-perturbative effects of this coupling.

6.2.1 Field Equations and effective stress-energy tensor

The field equations of the theory can be derived directly from those of RFG (5.14) with the substitution $\mathcal{F}_0 = -L^2/2$, $\mathcal{F}_2 = -L^2$. We thus obtain

$$\begin{aligned} \frac{1}{8\pi G} G^{\mu\nu} &= T_{\text{DM}}^{\mu\nu} - L^2 \left[(G^{\mu\nu} + g^{\mu\nu} \square - \nabla^{(\mu} \nabla^{\nu)}) T_{\text{DM}} - \square T_{\text{DM}}^{\mu\nu} + 2 \nabla^\alpha \nabla^{(\mu} T_{\text{DM}\alpha}^{\nu)} \right. \\ &\quad \left. - g^{\mu\nu} \nabla_\alpha \nabla_\beta T_{\text{DM}}^{\alpha\beta} - \frac{R}{2} (T_{\text{DM}}^{\mu\nu} - g^{\mu\nu} T_{\text{DM}}) - \frac{T_{\text{DM}}^{\mu\nu}}{T_{\text{DM}}} R_{\alpha\beta} T_{\text{DM}}^{\alpha\beta} \right]. \end{aligned} \quad (6.5)$$

Also in this model the DM stress-energy tensor is not conserved, as can be seen by taking the covariant divergence of (6.5) and using the contracted Bianchi identities

$$\begin{aligned} \nabla_\mu T_{\text{DM}}^{\mu\nu} &= L^2 \left\{ 2 \nabla_\mu (R_{\lambda}^{\nu} T_{\text{DM}}^{\lambda\mu}) + \frac{R}{2} \nabla^\nu T_{\text{DM}} - \nabla^\nu R_{\mu\lambda} T_{\text{DM}}^{\mu\lambda} \right. \\ &\quad \left. - \nabla_\mu \left[\frac{R}{2} (T_{\text{DM}}^{\mu\nu} - g^{\mu\nu} T_{\text{DM}}) + \frac{T_{\text{DM}}^{\mu\nu}}{T_{\text{DM}}} R_{\alpha\beta} T_{\text{DM}}^{\alpha\beta} \right] \right\}. \end{aligned} \quad (6.6)$$

As we did in Chapter 5, we can view Equation (6.2) as the Einstein field equations of General Relativity sourced by an effective energy tensor $G^{\mu\nu} = 8\pi G T_{\text{DM eff}}^{\mu\nu}$. The latter features both a

radial and a tangential pressure, and thus it can be written in the most general form as

$$T_{\text{DM eff}}^{\mu\nu} = \rho_{\text{DM eff}} u^\mu u^\nu + q_{\text{DM eff}}^\mu u^\nu + q_{\text{DM eff}}^\nu u^\mu + p_{\text{DM eff}} h^{\mu\nu} + \pi_{\text{DM eff}}^{\mu\nu}. \quad (6.7)$$

Here $\rho_{\text{DM eff}}$ is the energy density measured by an observer comoving with the fluid; $q_{\text{DM eff}}^\mu$ is the spacelike heat flux vector, with $q_{\text{DM eff}}^\mu q_{\text{DM eff} \mu} = +1$, $q_{\text{DM eff}}^\mu u_\mu = 0$; $h^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ is the projection of the metric on the hypersurface orthogonal to u^μ ; $p_{\text{DM eff}}$ is the isotropic pressure, and $\pi_{\text{DM eff}}^{\mu\nu}$ the traceless anisotropic stress, with $u_\mu \pi_{\text{DM eff}}^{\mu\nu} = \pi_{\text{DM eff}}^{\mu\nu} u_\nu = g_{\mu\nu} \pi_{\text{DM}}^{\mu\nu} = 0$. They can be recovered by contraction

$$\begin{aligned} \rho_{\text{DM eff}} &= T_{\text{DM eff}}^{\mu\nu} u_\mu u_\nu, & p_{\text{DM}} &= \frac{1}{3} h_{\mu\nu} T_{\text{DM eff}}^{\mu\nu}, \\ q_{\text{DM eff}}^\mu &= -h_\alpha^\mu T_{\text{DM eff}}^{\alpha\nu} u_\nu, & \pi_{\text{DM eff}}^{\mu\nu} &= h_\alpha^\mu h_\beta^\nu T_{\text{DM eff}}^{\alpha\beta} - \frac{1}{3} h^{\mu\nu} h_{\alpha\beta} T_{\text{DM eff}}^{\alpha\beta}. \end{aligned} \quad (6.8)$$

In particular, in the weak field limit, when one assumes $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \gamma_{\mu\nu}$, the effective DM SET takes the simple form

$$T_{\text{DM eff}}^{\mu\nu} = T_{\text{DM}}^{\mu\nu} + L^2 [\square T_{\text{DM}}^{\mu\nu} + (\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) T_{\text{DM}}]. \quad (6.9)$$

If we consider an observer comoving with the fluid, $u^\mu = \delta_0^\mu$, $h_{\mu\nu} = \text{diag}(0, \delta_{ij})$, and the components of the effective SET can be written explicitly

$$\begin{aligned} \rho_{\text{DM eff}} &= \rho_{\text{DM}} - L^2 \partial_0^2 \rho_{\text{DM}}, \\ p_{\text{DM eff}} &= L^2 \left(\frac{2}{3} \Delta - \partial_0^2 \right) \rho_{\text{DM}}, \\ q_{\text{DM eff}}^0 &= 0, \quad q_{\text{DM eff}}^i = -L^2 \partial^i \partial_0 \rho_{\text{DM}}, \\ \pi_{\text{DM eff}}^{ij} &= L^2 \left(\frac{1}{3} \delta^{ij} \Delta - \partial^i \partial^j \right) \rho_{\text{DM}}. \end{aligned} \quad (6.10)$$

Equation (6.6) then expresses the conservation of the effective DM SET; in particular, its projection onto the fluid velocity u^μ is a continuity equation for the DM fluid, while the projection on surfaces orthogonal to u^μ yields the relativistic Euler equation, where a dynamically-generated force emerges, implying a deviation of the DM fluid from the geodesic motion. All that suggests the remarkable possibility that the non-minimally coupled DM fluid can settle in a hydrostatic equilibrium configuration where, to use a Newtonian language, the pressure gradient balances the gravitational attraction. This motivates us to look for static solutions of equation (6.5) in a spherically symmetric configuration.

6.3 Ultra-compact objects of NMC–DM

We look for a static, spherically symmetric, asymptotically flat and regular solution of equation (6.5). In terms of the time t measured by a static observer at infinity, the areal coordinate r , and the angular coordinates θ and ϕ , the ansatz for the metric can be written in full generality as

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left(1 - \frac{2GM(<r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (6.11)$$

The metric depends on two unknown functions, namely the Newtonian potential $\Phi(r)$, and the cumulative mass distribution $M(<r)$. The latter can be identified with the Misner-Sharp mass, that is the combined total energy of the fluid and gravitational field inside a sphere of radius r . Another unknown is represented by the density distribution of the DM fluid in the hydrostatic equilibrium configuration $\rho(r)$. Altogether one has to deal with 14 equations: 10 field equations (6.5) and 4 conservation equations (6.6), but the form of the metric and the general covariance of the theory imply that 11 of them are either identically zero or equivalent to the remaining 3, thus leaving just 3 independent equations for 3 unknowns.

In particular, choosing the tt and rr components of (6.5) and the r component of (6.6), we obtain a system of first-order ordinary differential equations in (M, ρ) and a trivial equation for Φ . For the numerical integration will be useful to rescale the variables as

$$r \rightarrow Lx, \quad M(<r) \rightarrow \frac{c^2 L}{8\pi G} \hat{M}(<x), \quad \rho(r) \rightarrow \frac{c^2}{8\pi G L^2} \hat{\rho}(x) \quad (6.12)$$

(where we have written the speed of light explicitly), using which we can write the system of equations in $(\hat{M}, \hat{\rho})$ as

$$\begin{cases} \hat{M}'(<x) = 4\pi x^2 \frac{\hat{\rho}(x)}{1 - \hat{\rho}(x)}, \\ \hat{\rho}'(x) = -\frac{4\pi x^3 \hat{\rho}^2(x) + \hat{M}(<x) [1 - \hat{\rho}(x)]^2}{4x [4\pi x - \hat{M}(<x)] [1 - \hat{\rho}(x)]}. \end{cases} \quad (6.13)$$

The equation for the gravitational potential can be directly integrated in terms of $\hat{M}'$ or of $\hat{\rho}$ as:

$$\Phi(x) = \Phi_0 + \ln \left[\frac{x^2}{4\pi x^2 + \hat{M}'(<x)} \right] = \tilde{\Phi}_0 + \ln [1 - \hat{\rho}(x)] , \quad (6.14)$$

where Φ_0 (or $\tilde{\Phi}_0 = \Phi_0 - \ln 4\pi$) is an integration constant to be set by the boundary conditions (see below). The form of the effective DM SET simplifies in this context, since staticity and spherical symmetry respectively imply that $u_\mu = (u_t, 0, 0, 0)$ and $q_\mu = 0$. Furthermore, the radial and tangential pressure components are given by $p_r = p + \pi_r^r$, $p_t = p + \pi_\theta^\theta$, with the

constraints $\pi_\theta^\theta = \pi_\phi^\phi = -\pi_r^r/2$. The explicit form of the effective density and radial pressure in terms of $\hat{\rho}(x)$ and $\hat{M}(< x)$ are given by

$$\hat{\rho}_{\text{eff}}(x) = \frac{\hat{\rho}(x)}{1 - \hat{\rho}(x)}, \quad \hat{p}_r(x) = \frac{4\pi x^3 \hat{\rho}^2(x) - \hat{M}(< x) [1 - \hat{\rho}(x)]^2}{8\pi x^3 [1 - \hat{\rho}(x)]^2}, \quad (6.15)$$

while the tangential pressure has the more involved expression

$$\begin{aligned} \hat{p}_t(x) = & -\frac{1}{64\pi x^3 [4\pi x - \hat{M}(< x)]} \left\{ 5\hat{M}^2(< x) + \right. \\ & - 8\pi x \hat{M}(< x) \frac{[2 - 7x^2] \hat{\rho}^2(x) - [4 - 2x^2] \hat{\rho}(x) + 2}{[1 - \hat{\rho}(x)]^2} + \\ & \left. + 16\pi^2 x^4 \hat{\rho}(x) \frac{[x^2 - 12] \hat{\rho}^3(x) + 28\hat{\rho}^2(x)^2 - 20\hat{\rho}(x) + 4}{[1 - \hat{\rho}(x)]^4} \right\}. \end{aligned} \quad (6.16)$$

Note that the condition $\hat{\rho}(x) = 1$ defines a critical density $\rho_c = c^2/8\pi G L^2$ for which the radial pressure and the effective energy density diverge. We have solved equations (6.13) numerically with initial conditions $\hat{M}(0) = 0$, $\hat{\rho}(0) = \hat{\rho}_0$. As it is evident from the first equation, in order for $\hat{M}'$ to be well behaved near the origin $\hat{\rho}_0 \in [0, 1]$ must apply. We thus obtain a continuous family of solutions differing by the central value of the density, with $\hat{\rho}_0 = 1$ (that is $\rho(0) = \rho_c$) representing the limiting one.

We identify the boundary of the compact object R_b with the radius where the effective radial pressure drops to zero, so that no net radial force acts there. For $r > R_b$ we have matched our solutions with the Schwarzschild metric, since the theory must reduce to General Relativity where the DM stress energy tensor vanishes. Note that in more realistic situations the UCOs are expected to live in an ambient DM environment, but the idealized solution considered here is a good approximation as long as the density in the interior is orders of magnitude higher than that outside. For the range of densities considered in this work, the latter condition is certainly met if UCOs live inside DM halos, where the average density is $\approx 200 \times \rho_c$, where ρ_c is the critical density of the Universe. The normalization of the potential (6.14) is chosen such that $\Phi(R_b) = \frac{1}{2} \ln \left[1 - \frac{2GM(< R_b)}{c^2 R_b} \right]$, and we have set $M(< r) = M(< R_b)$ and $\rho(r) = 0$ for $r > R_b$. However, note that the density drops to zero from a finite value at R_b ; physically this can be interpreted by stating that, beyond R_b , the DM particles can diffuse freely from the equilibrium configuration, thus reducing drastically the value of the local density.

An alternative choice would be to define the boundary as the minimum radius $R_\star$ at which

the density attains zero, i.e. $\rho(R_\star) = 0$. However, since $R_\star > R_b$, we have $p_r(R_\star) < 0$, so that the DM fluid would feel an attractive pressure at $R_\star$, making its identification as a boundary ill defined. This ambiguity is not present in General Relativity, where the equilibrium configuration of a spherically symmetric star is found by solving the Tolman-Oppenheimer-Volkoff equations: the form of the latter implies that, for any reasonable equation of state of the matter fluid, the density and pressure approach zero at the same radius, making the identification of the boundary straightforward. In our model, since gravity is generated not only by the DM density but also by its radial variations, the equation of state for the effective stress energy tensor is far from trivial and no simultaneous roots of ρ and p_r are present.

It is interesting to notice that the choice of boundary R_b ensures that the potential Φ is of class C^1 , while the derivative of the mass $M'(< r)$ has a discontinuity at the boundary due to the sudden drop of density. We have checked that having instead identified the boundary with $R_\star$ would have resulted in a mass function of class C^1 and a discontinuous derivative of the potential Φ' . In particular, it is not possible to make both components of the metric to be of class C^1 over the entire domain. However, a discontinuity in the derivative of the potential would have much deeper consequences on the physical structure of the spacetime.

The extrinsic curvature $\kappa_{\mu\nu}(r)$ for a surface of constant radius has a discontinuity at the surface $R_\star$, while it remains continuous using the boundary R_b ; this discontinuity is inherited by the expansion and shear tensors for a congruence of geodesics, which at the physical level means that nearby free-falling particles would perceive a jump in their relative velocities, that is an infinite relative acceleration. While the choice of boundary R_b removes such divergences, the relative accelerations of nearby geodesics, as well as invariant contractions of curvature tensors, will have a jump at the boundary. These jumps are unavoidable because they result from the drastic change in gravitational interaction caused by a hard boundary on the extent of the non-minimal coupling. In a realistic setting we expect the non-minimal interaction to diminish smoothly in such a way that the energy density and the curvature tensors get regularized. Nonetheless, we accept and interpret these discontinuities as an effective description for a very sharp falloff in the interaction.

6.3.1 Properties of the NMC-UCOs

In Figure 6.1 we illustrate the density $\rho(r)$, the cumulative mass $M(< r)$, the metric components g_{rr}^{-1} and $-g_{tt}$, the effective radial pressure $p_r(r)$, and the temporal component of the extrinsic curvature $-\kappa_{tt}(r)$ for a surface of constant radius. The colormap refers to the different solutions parametrized by the value of the central density, with the limiting solution with $\hat{\rho}_0 = 1$ highlighted by the dashed black line.

First of all we notice from the top-left panel that there is no solution for $\hat{\rho}_0 < 1/4$; this

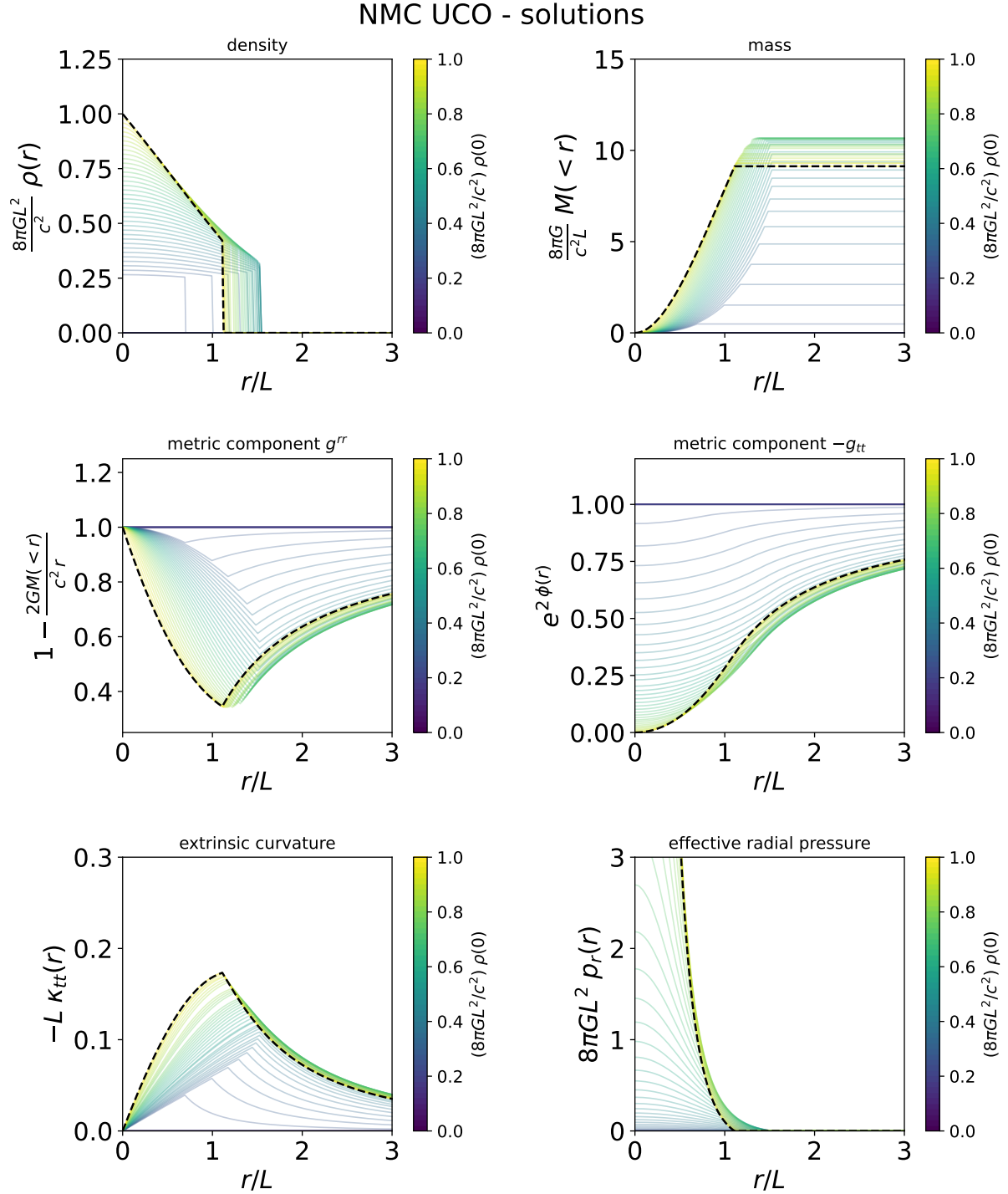


Figure 6.1. Radial profiles of UCOs solutions: density $\rho(r)$ (top left), cumulative mass $M(< r)$ (top right), metric component g^{rr} (middle left), metric component $-g_{tt}$, extrinsic curvature component $-\kappa_{tt}$ (bottom left), and effective radial pressure $p_r(r)$ (bottom right). The various solutions are parameterized by the allowed values of the central density $\rho(0)$, as specified by the color scale. The dashed black line highlights the limiting solutions featuring the maximal value of the central density.

minimum central density is the one for which the central radial pressure

$$\hat{p}_{r,0} = \frac{\hat{\rho}_0 (4\hat{\rho}_0 - 1)}{6(1 - \hat{\rho}_0)^2} \quad (6.17)$$

is zero and the solution becomes trivial. For smaller central densities it is not possible to define the boundary R_b without introducing a negative energy density for the fluid, which is unphysical for a standard DM framework. Being left only with the possibility of the ill-defined boundary R_* , we argue that central densities $\hat{\rho}_0 < 1/4$ are not sufficient to sustain a stable solution. The density range of the solutions is therefore extremely small, with central values $\hat{\rho}_0 \in [0.25, 1]$ and values at the boundary $\hat{\rho}(R_b) \in [0.25, \sim 0.422]$. In fact, these solutions are present only at a precise scale and have a particularly steady density in their interior, suggesting a physical process well-localized in the density domain.

In particular, the small range of the rescaled central density $\hat{\rho}_0(x)$ and mass $\hat{M}(x)$ implies that these objects cannot be relevant for the behavior of entire DM halos. In fact, in principle the value of L could be rescaled by orders of magnitude up to galactic length scales; however, this would lead to a correspondent rescaling of the physical mass and central density of the object. Taking, for instance, $L \approx 3 \times 10^{22} \text{ cm} \approx 30 \text{ kpc}$, corresponding to the radius of the core of the Milky Way halo [262], would result in $M \approx 10^{16} M_\odot$, while the estimated virial mass of the entire halo is only $M \approx 10^{12} M_\odot$. At the same time the density of the object would amount to $\rho \approx 10^{-15} \text{ g/cm}^3$, compared to the estimate of $\rho \approx 10^{-23} \text{ g/cm}^3$ for the Milky-Way core [262].

The radial density profile shows that the solutions grow in size with the central density up to $\hat{\rho}_0 \sim 0.47$, after which they start to shrink until they reach the limiting solution. On the other hand, the top-right panel shows that the total mass $M(< R_b)$ reaches a maximum at $\hat{\rho}_0 \sim 0.66$, after which it starts to decrease; interestingly, the mass of the limiting solution is almost the same as that of the one with the maximum radius. The mass-radius relation for these object will then follow a curve similar to the one of strange stars [263], where at low densities they have small masses and radii increasing concurrently with the central density, while denser objects have larger masses and radii which instead decrease with the central density.

The metric components shown in the middle panels have familiar profiles, with the notable exception for the presence of corners in the radial component g^{rr} due to our choice of the boundary. As discussed before, this behavior at the physical level translates in sudden changes of particle accelerations due to the discontinuous change in density, while the differentiability of the temporal component g_{tt} guarantees that they maintain continuous relative velocities. This is explicitly shown by the continuity of the extrinsic curvature displayed in the bottom-left panel, which in fact indicate that there is no localized stress-energy tensor on the surface. We stress that the UCO solution does not feature any horizon since the g^{rr} component never vanishes.

As the central density approaches the critical value, the temporal component of the metric reduces its central value up to zero for the limiting solution. The vanishing of g_{tt} can be interpreted in a Newtonian sense as an infinite gravitational potential Φ , and in a relativistic sense as an infinite redshift for a photon emitted in the origin and measured at spatial infinity $z = e^{-\Phi(0)} - 1$. In both cases we can state that for the limiting solution the gravitational field in the origin is so intense that an infinite amount of energy is required to leave it. Such a strong gravitational attraction has to be balanced by an equally strong radial pressure, which is in fact shown to diverge for the limiting solution in the bottom-right panel, in agreement with (6.17). The divergence of the effective pressure, which by definition corresponds to the rr component of the Einstein tensor, indicate that the solution becomes singular. However, this singularity does not have to be interpreted as an exotic naked singularity but rather as a limit for static solutions before gravitational collapse, as the Buchdahl limit in General Relativity.

Nonetheless, the presence of extreme curvatures suggests that we are in a very strong gravity regime. In Figure 6.2 we show some relevant length-scales which indicate that the compactness of the solutions is close to the typical size of a black hole mimicker, especially when one approaches the limiting solution. In particular, in red we show the radius in mass units, which approaches the size of the light ring (i.e. the radius where a circular orbit for photons is allowed) of the Schwarzschild solution $R = 3GM/c^2$, and in the inset we show the compactness parameter $\sigma = 1 - 2GM(< R_b)/c^2 R_b$ which goes from 1 at $\hat{\rho}_0 = 1/4$ to $\sim 1/3$ at $\hat{\rho}_0 = 1$. Such a high compactness leads to the formation of two light rings, as usual for very compact regular solutions, of which the larger describes unstable circular photon orbits and the smaller stable ones (we will explain the definition of light ring in the next section). Finally, we can conclude that our choice of naming these solutions ultra-compact objects is thus justified.

As a final remark, we briefly discuss the stability of our UCO solutions. A rigorous treatment of this issue would require a numerical integration of the equations for time-dependent perturbations, which is postponed to future works. Nevertheless, we can gain some insight into linear stability by considering the behavior of test fields. Specifically, the wave equation for a test field of spin $s = 0, 1, 2$ can be reduced to an equation for a scalar function $\psi_s(r)$ with form [264]

$$\frac{d^2 \psi_s(r)}{dr^{*2}} + [\omega^2 - V_s(r)] \psi_s(r) = 0, \quad (6.18)$$

in terms of the effective potential

$$V_s(r) = e^{2\phi(r)} \left\{ \frac{1-s}{r} \left[1 - \frac{2GM(< r)}{c^2 r} \right] \left[\phi'(r) + \frac{(1+2s)GM(< r) - rGM'(< r)}{r(c^2 r - 2GM(< r))} \right] + \frac{\ell(\ell+1)}{r^2} \right\}; \quad (6.19)$$

in the above the test fields have been decomposed as $\Psi_s(t, r, \theta, \varphi) = e^{-i\omega t} Y_{\ell m}(\theta, \varphi) \psi_s(r)/r$, and

r^* in Equation (6.18) is the tortoise coordinate defined by $r^* = \int dr e^{-\phi(r)} \left[1 - \frac{2GM(<r)}{c^2 r}\right]^{-1/2}$. The test fields are unstable if there exists an exponentially growing mode with positive imaginary part. A simple argument for stability [265] can be inferred from the positivity of the potential $V_s(r)$. In a regular spacetime the tortoise coordinate domain can be extended from $[0, \infty)$ to $(-\infty, \infty)$, and the positivity of the potential $V_s(r)$ ensures that the operator $-\frac{d^2}{dr^{*2}} + V_s(r)$ is positively definite in the Hilbert space of square integrable functions $\mathcal{L}^2(r^*, dr^*)$. The positivity of the eigenvalues, in turn, implies the absence of growing modes.

Coming to our UCO solutions, the potential is always positive for a $s = 1$ field, and for a $s = 2$ field when $\ell \geq 1$. However, tensor perturbations with $\ell = 0, 1$ are generally regarded as redefinitions of mass and angular momentum, and are not typically considered in the stability analysis. For a scalar field ($s = 0$) the potential is always positive for $\ell \geq 1$, but for $\ell = 0$ it requires $\hat{\rho}_0 \gtrsim 0.59$. All in all, this preliminary analysis suggests then that the most compact objects are stable against linear perturbations. Note, however, that the presence of a stable light ring could lead to long-lived decaying modes and nonlinear instability effects [264], but these require a detailed analysis which is much beyond the scope of this thesis.

6.4 Orbits around NMC-UCOs

To have insight into the physical nature of NMC-UCOs, we analyze both the timelike and null geodesics of their spacetime. As a matter of fact, geodesics analysis is particularly instructive in this case, as we expect both massive and massless test particles to interact with the DM fluid mainly through the gravitational field rather than through the direct NMC, therefore following its geodesics. As such their motion is described by the Lagrangian for a free falling particle

$$\mathcal{L}_p = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{2} e^{2\Phi(r)} \dot{t}^2 + \frac{1}{2} \left[1 - \frac{2GM(<r)}{r}\right]^{-1} \dot{r}^2 + \frac{1}{2} r^2 \dot{\phi}^2, \quad (6.20)$$

where a dot denotes differentiation with respect to an affine parameter λ along the trajectory, and without loss of generality we have assumed the motion to take place on the plane $\theta = \pi/2$.

Differentiation of (6.20) with respect to λ yields the geodesic equation

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0, \quad (6.21)$$

implying that the Lagrangian itself is constant along the trajectory. Thus we can restrict to consider the simpler first order equation

$$\eta = -e^{-2\Phi(r)} E^2 + \left(1 - \frac{2GM(<r)}{r}\right)^{-1} \dot{r}^2 + \frac{j^2}{r^2}, \quad (6.22)$$

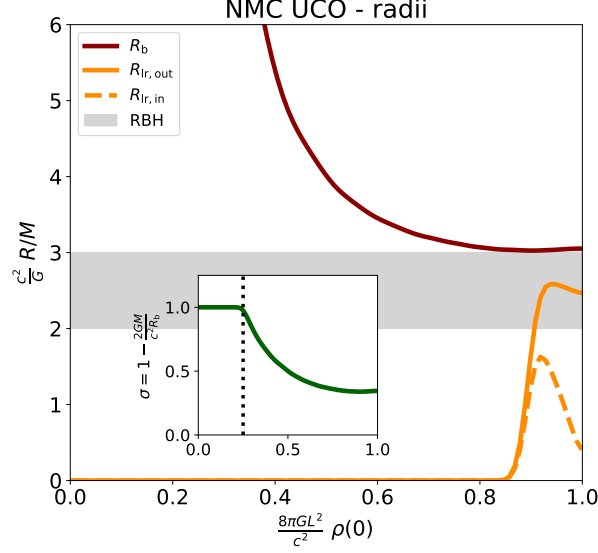


Figure 6.2. Relevant physical length-scales of UCOs solutions: boundary radius R_b (red) and light rings (orange; dashed for the inner and solid for the outer); these are measured in units of total mass M of the UCO, and are plotted as a function of the UCO central density $\rho(0)$ defining the various solutions. The grey shaded area displays the typical size of spherically symmetric regular black hole solutions (e.g., black hole mimickers) in general relativity. The inset displays the compactness parameter $\sigma = 1 - 2GM/c^2 R_b$ (green); the dotted line is the limiting value of the density $\frac{8\pi G L^2}{c^2} \rho(0) = 0.25$ below which no physical solution exists.

where the energy E , the angular momentum j , and the parameter η are the first integrals of motion associated to the invariance of the Lagrangian under time translations, rotations in the considered plane, and reparametrizations of the affine parameter along the particle trajectory, i.e.:

$$\begin{aligned} E &\equiv -\frac{\partial \mathcal{L}_p}{\partial \dot{t}} = e^{2\Phi} \dot{t}, \\ j &\equiv \frac{\partial \mathcal{L}_p}{\partial \dot{\phi}} = r^2 \dot{\phi}, \\ \eta &\equiv g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \end{aligned} \tag{6.23}$$

Specifically, $\eta = 0$ applies to massless particles while, if λ is chosen to be the proper time along the trajectory, $\eta = -1$ applies to massive ones. Equation (6.22) can be recast as

$$\dot{r}^2 = \left(1 - \frac{2GM(< r)}{r}\right) \left(e^{-2\Phi} E^2 - \frac{j^2}{r^2} + \eta\right); \tag{6.24}$$

being horizon-less and regular at the origin, our solution has $1 - \frac{2GM(<r)}{r} > 0$ in all of space, implying that a photon and a massive particle can propagate only in those regions where

$$e^{-2\Phi} E^2 - \frac{j^2}{r^2} \geq -\eta, \quad (6.25)$$

with equality corresponding to a turning point with $\dot{r} = 0$. In particular, a radial geodesic starting at r_0 , having $j = 0$, will fall through the center of the UCO re-emerging on the opposite side, since $e^{2\Phi(r)}$ is a monotone increasing function of the radius and $E = e^{2\Phi(r_0)}$ holds. On the other hand, if $j \neq 0$ a centrifugal barrier is present, preventing the particle to reach the center.

To understand what happens at the turning point, we look at the radial component of the geodesic equation (6.21) with $\dot{r} = 0$:

$$\ddot{r} = \left(1 - \frac{2GM(<r)}{r}\right) \frac{e^{-2\Phi}}{r} \left[(1 - r\Phi')E^2 - \eta e^{2\Phi}\right]. \quad (6.26)$$

This implies that a particle will experience an attractive or a repulsive force at the turning point depending whether $(1 - r\Phi')E^2 - \eta e^{2\Phi}$ is below or above 0 there. This condition is of particular interest in the case of massless particles, where it reduces to whether $r\Phi'$ is above or below 1, with no dependence on the energy. In particular, if there exist one or more radii R_{lr} such that $R_{\text{lr}}\Phi'(R_{\text{lr}}) = 1$, then a photon with energy and angular momentum satisfying $E/|j| = e^{\Phi(R_{\text{lr}})}/R_{\text{lr}}$ will move on a circular orbit there: a radius with this property is known as a ‘light ring’. A detailed numerical analysis of the solution reveals that for a central value of the normalized density less than $\rho_0 \lesssim 0.85$ no light rings exist, while for higher densities two of them appear: an external one which describes unstable circular orbits and an internal one which describes stable orbits. The stability of the orbits is manifest if we recall that $r\Phi' \rightarrow 0$ at large distances, and then circular orbits close to the external light ring will experience a repulsive force on the outside and an attractive one on the inside, while close to the internal light ring they will experience repulsive force on the inside and attractive on the outside. For this reason, close to the internal light ring bounded photon orbits are possible. Although the light ring position $R_{\text{lr}} < R_{\text{b}}$ is internal to the boundary of the UCO, this name is appropriate since the object is made of DM and hence is ‘transparent’ to radiation.

In Figure 6.3 we showcase some relevant examples of orbital trajectories for massless and massive test particles in the gravitational field of the UCO, obtained by solving the orbital equation

$$\left(\frac{dr}{d\phi}\right)^2 = r^2 \left[1 - \frac{2GM(<r)}{r}\right] \left(r^2 e^{-2\Phi} \frac{E^2}{j^2} + \eta \frac{r^2}{j^2} - 1\right) \quad (6.27)$$

with different initial conditions r_0 , and different values of the energy E and angular momentum j . Top diagrams show the trajectories in polar coordinates, magenta for massless and green

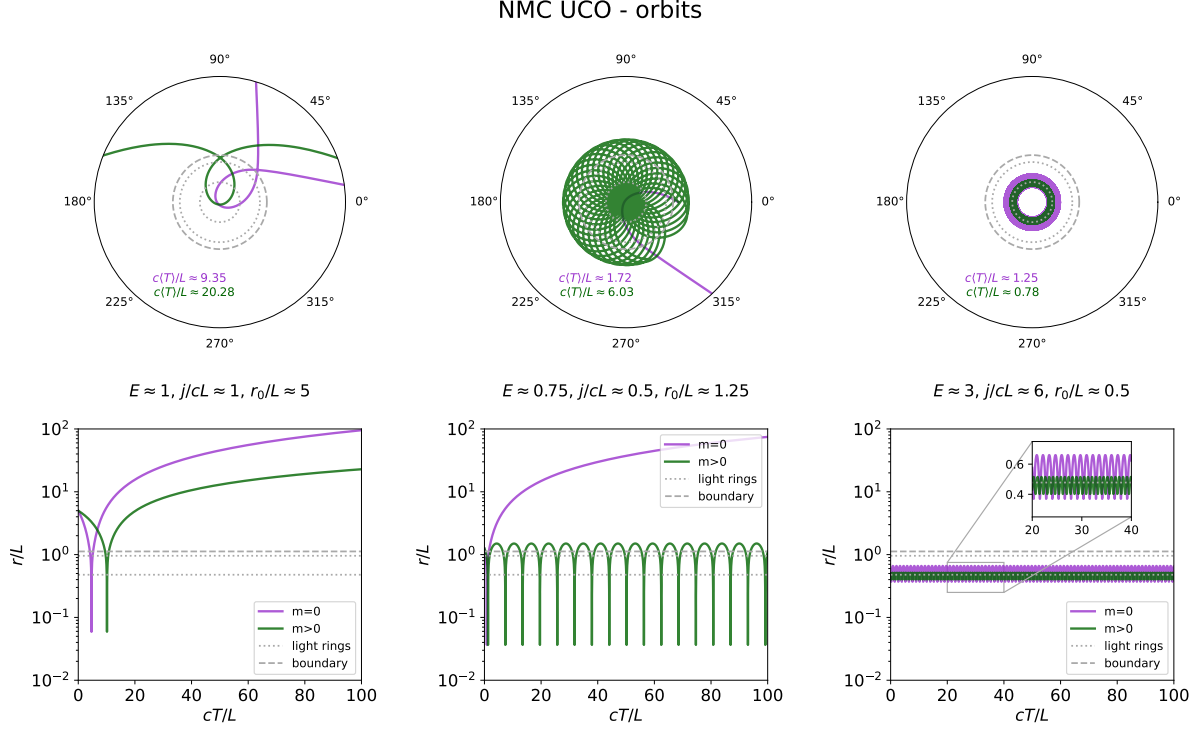


Figure 6.3. Relevant examples of orbital trajectories for massless (magenta) and massive (green) particles around an UCO solution with central density $\hat{\rho}_0 = 0.95$. Top diagram show the trajectories in polar coordinates, bottom diagrams illustrate the radial position of the test particle as a function of proper time. The dashed grey lines displays the position of the boundary, dotted grey lines that of the inner and outer light rings. Left diagrams refer to an unbound orbit, middle diagrams to a loose bound orbit for massive particles, and right diagrams to a tightly bound orbit for both massless and massive particles; the values of the energy parameter E , of the angular momentum j , of the starting radial coordinate r_0 and the average proper period $\langle T \rangle$ of the orbit are reported (polar initial angle is assumed to be null).

for massive particles. The average orbital proper time is also reported. Bottom diagrams show instead the radial position of the test particle as a function of proper time (for massive particles), or of the affine parameter along the trajectory (for massless ones). Left diagrams refer to an unbound orbit, middle diagrams to a loose bound orbit for massive particles (note the pericenter precession), and right diagrams to a tightly bound orbit for both kind of particles. To ease the visualization, the boundary of the UCO and the light rings are also displayed in all diagrams as continuous and dashed grey lines, respectively. Note that proper times are of the order of L/c even for massive particles, typical of ultra-relativistic motion; this copes with the very high compactness of the UCO.

6.4.1 Orbital portraits

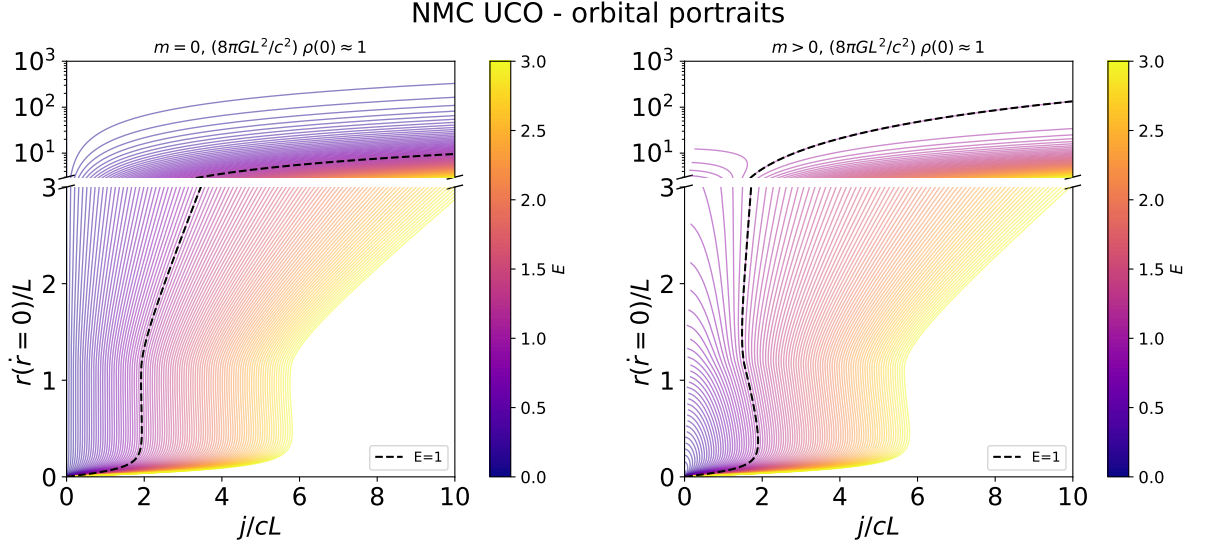


Figure 6.4. Orbital portraits diagrams for the UCO solution with central density $\hat{\rho}_0 = 0.95$, slightly below the maximal value. Left panel refers to a massless and right panel to a massive test particle. The radius r at the radial inversion point(s) with $\dot{r} = 0$ is shown as a function of the angular momentum j (x -axis) and of the energy parameter E (color-scale) of the test particle. The dashed black line highlights the portraits for $E = 1$. Note that in both the panels, the scale of the y -axis starts linear at the bottom and then becomes logarithmic at the top.

While Figure 6.3 makes clear and intuitive what are the possible orbits in UCO spacetimes, it is far from giving a complete picture. Toward this purpose, in Figure 6.4 we show the orbital portraits diagrams for an UCO solution with a central density near the maximum value, high enough to develop light rings. We show the radius r at the radial inversion point(s) with $\dot{r} = 0$, as functions of the angular momentum (x -axis) and of the energy parameter (color-scale) of the test particle. Left panel refer to a massless and right panel to a massive test particle. The dashed black line refers to the case $E = 1$, appropriate e.g. for a particle starting at rest from infinity. In both panels, the scale of the y -axis starts linear at the bottom and then becomes logarithmic at the top.

The number of intersections between a colored curve (corresponding to a given energy E) and a vertical line (corresponding to a specific value of the angular momentum j) hints to the nature of the possible orbits. In fact, such intersections highlight the different radii r at which motion inversion with $\dot{r} = 0$ can occur. When only one intersection is present (e.g., for both massive and massless particles at high j), the particles will reach a minimum radius with zero radial velocity, and then they will be pushed away by the centrifugal barrier and reach spatial

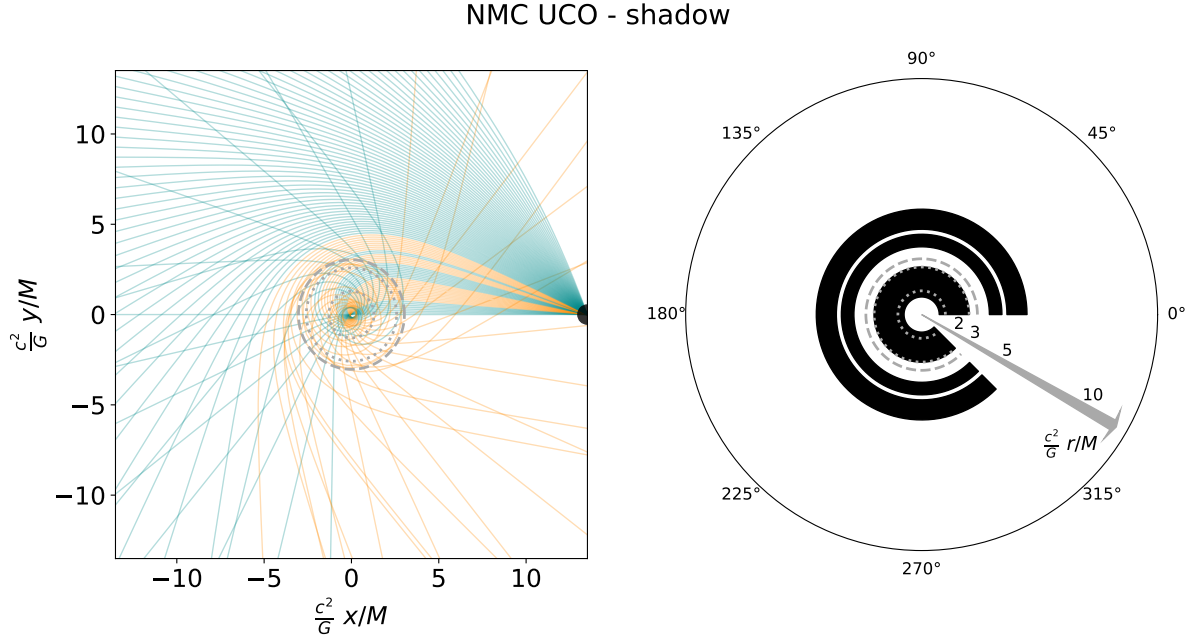


Figure 6.5. Pseudo-shadow of the UCO solution with central density $\hat{\rho}_0 = 0.95$. Left panel illustrates a ray-tracing simulation, with the observer at the black spot on the right and an illuminated screen assumed at $x = -\infty$. Solid lines are light rays traced back in time from the observer with different impact parameters j/E : in the remote past, cyan rays hit the illuminated screen at $x = -\infty$, and hence can be seen by the observer; orange rays are instead deflected back to $x = +\infty$ and are precluded to vision. Dashed grey lines displays the position of the boundary, dotted grey lines that of the inner and outer light rings. Right panel is a polar plot illustrating the face-on view of the UCO from the observer position: as a consequence of gravitational lensing, a multiple photon ring structure is originated around the typical radii $(c^2/G)r \approx 1.8 - 4 - 5 M$.

infinity. When there are two intersections (e.g., for massive particles with $E \lesssim 1$ and $j/cL \lesssim 2$), a bound orbits is possible between a maximum and minimum radius. Finally, when there are three intersections (this can be clearly seen for massive particles with $E \approx 1$ and $j/cL \approx 1.8$), two kind of motion can occur: if the orbit starts from a large radius, then the particle will encounter the external inversion point and will be pushed away to spatial infinity; contrariwise, if the orbit starts between the two internal turning points, it can be bounded.

6.4.2 UCO's pseudo-shadow

A century after the pioneering observation of gravitational light deflection during a solar eclipse that provided the first smoking-gun test of General Relativity, in 2019 the Event Horizon Telescope (EHT) Collaboration published the first ‘image’ of a BH [266]. In fact, the light

rays passing close to a BH can be deflected very strongly and even travel on circular orbits in correspondence of the light rings. This strong deflection, together with the fact that no light comes out of a BH, allows the latter to be imaged as a dark disk in the sky, known as ‘BH shadow’. For a Schwarzschild black hole the size of the shadow is $r_{\text{sh}} = 3\sqrt{3} M \approx 5.2 M$ (e.g., see [267] for a derivation and an extended discussion on black holes’ shadows).

The NMC-UCOs do not feature an horizon and, being composed of DM, are completely transparent to light rays; nevertheless, the notion of a shadow in a wide sense can be introduced also for them. The left panel of Figure 6.5 shows a ray-tracing simulation, with the observer positioned on the black spot at the extreme right of the plot and an illuminated screen assumed at $x = -\infty$. Solid lines represent light rays from the observer traced back in time for different impact parameters j/E . Light rays stemming from the observer are divided into two classes: those that in the remote past hit the illuminated screen at $x = -\infty$ can be seen by the observer and are highlighted in cyan; orange rays are instead deflected back to $x = +\infty$ and not reaching the illuminated screen are precluded to vision. The right panel is a polar plot illustrating the face-on view of the UCO from the observer position. This originates a pseudo-shadow at $\approx 5.2 M$, close to the size of that for a Schwarzschild black hole. Inside it, a multiple photon ring structure around the typical radii $(c^2/G) r \approx 1.8 - 4 - 5 M$ emerges, as generically expected for horizonless compact objects [268].

In principle, if close enough so as to avoid resolution problems, a NMC-UCO could be spotted via the distinctive features of its pseudo-shadow in EHT-like data. However, one problem could be that in a gas rich environment, like our galaxy, these objects could have accreted baryons, subsequently collapsing into a standard singular black hole. Even though a detailed study of the accretion and collapse of NMC-UCOs will be postponed to a future work, a very simple calculation shows that the horizon will exit the boundary once the UCO has accreted half of its original mass. Assuming Eddington accretion this will happen within a few e -folding times. However, if one of these objects had formed in (or migrated to) a region particularly devoid of baryons, such as a globular cluster, the possibility to observe it today would increase significantly. If detected, NMC-UCOs would represent a unique opportunity for the understanding of the interplay between DM and gravity.

6.5 Summary and conclusions

In this chapter we have presented a general relativistic action featuring a non-minimal coupling between the stress-energy tensor of a collisionless DM fluid and gravity. The latter could be described in terms of a scattering cross section between DM particles and gravitons which, if sufficiently small, opens the intriguing possibility that the non-minimal coupling with gravity

could be present also for standard model particles, but that for the latter its strength would be outclassed by the other interactions.

We have shown that the non-minimal coupling originates an anisotropic effective pressure able to counteract gravity and to support the otherwise collisionless DM fluid into static, spherically symmetric configurations. Investigating the properties of such configurations, we have found that they are horizonless, non singular ultra-compact objects (NMC-UCOs), with a very limited range of allowed central densities. We have also analyzed the orbits of massive and massless test particles in the gravitational field of NMC-UCOs, providing some specific examples of both bound and unbound orbits and a general discussion in terms of phase portraits. Finally, we have implemented a basic ray-tracing simulation to study the gravitational lensing effects around NMC-UCOs. We have effectively described these in terms of a pseudo-shadow that makes connection with what is usually done in the context of black holes. If observed, the peculiar light pattern around NMC-UCOs could constitute a clear signature for the detection of such exotic objects.

A caveat concerns the densities required to form NMC-UCOs, that ultimately depend on the magnitude of the lengthscale parameter L appearing in the relativistic theory; in particular for $L \sim 10^{10}$ cm one finds densities $\rho \sim 10^7$ g/cm³ and masses of astrophysical interest $M \sim 10^4 M_\odot$. However, such high densities, necessary to make the non-minimal coupling to kick in, are considerably higher than the average ones occurring within virialized DM halos. Nevertheless, the possibility remains that the necessary conditions could have been met during the early fast and violent collapse of a DM halo or in the early Universe. In the latter instance, the environmental conditions could be prone for a copious production of horizonless NMC-UCOs that, at variance with primordial black holes, will not suffer evaporation-related issues and thus may potentially survive till much later cosmic times. When baryons had cooled sufficiently, they could fall into the potential wells of NMC-UCOs, triggering their collapse. Thus, NMC-UCOs could have provided the primordial gravitational wells that facilitated the early formation of supermassive black hole seeds, potentially offering an explanation for the unexpectedly large population of such objects recently observed at high redshift by JWST [269].

With this motivation in mind, in the next chapter we investigate the background evolution of the Universe within the NMC-DM framework, revealing a radical departure from Λ CDM at early times, when the average DM density of the Universe exceeds the critical density.

Chapter 7

Cosmology with Non-Minimally Coupled Dark Matter

Our current cosmological model is based on a simple framework: a background isotropic and homogeneous spacetime, filled with matter, radiation and vacuum energy, evolving according to the laws of General Relativity. In its standard form, most matter is assumed to be cold dark matter (CDM), and vacuum energy is modeled as a cosmological constant Λ , to originate the Λ CDM model [270]. In this description the Universe underwent radiation and matter-dominated phases with decelerated expansion, followed by late-time acceleration driven by Λ . Despite its success in describing large-scale phenomena, Λ CDM faces challenges across many fronts.

Evolving the Einstein equations backward in time leads to a spacetime singularity, where physical quantities diverge. At such high energies, a quantum theory of gravity is expected to replace classical gravity and resolve the singularity, though no current proposal has provided experimentally falsifiable predictions [271]. Observations indicate that vacuum energy (or dark energy, DE) constitutes roughly 69% of the total energy of the Universe [272, 273], but its theoretical value differs by over 120 orders of magnitude from Quantum Field Theory predictions. About 84% of matter is thought to be CDM, consisting of massive particles that interact gravitationally and possibly very weakly via other forces, but its properties exclude them from being part of the Standard Model [274]. While alternatives to both DE [271, 275] and CDM [276] have been proposed, in the case of CDM the overwhelming evidence strongly supports its existence [17, 9, 277, 24]. The Λ CDM model is also seriously challenged by precision cosmology. In particular, the values of the Hubble parameter H_0 and the matter clustering parameter S_8 inferred from early-Universe data are in tension with those from model-independent late-time observations [278], while the interpretation of DE as a cosmological constant has been questioned by recent Baryon Acoustic Oscillation analyses [279].

Finally, the Λ CDM scenario faces two major fine-tuning issues: the flatness and horizon

problems. The flatness problem refers to the near spatial flatness of the Universe today, which requires the early total energy density to be fine-tuned to within one part in 10^{60} . The horizon problem arises from the observed isotropy of the Cosmic Microwave Background (CMB), despite the regions emitting it not having been in causal contact [280], requiring then extremely uniform initial conditions. To resolve these issues, the model is typically extended with a period of accelerated expansion before radiation domination. This is generally achieved through inflation: a phase of nearly exponential expansion driven by a new, unknown scalar field slowly rolling down a flat potential [281, 282]. Inflation ends when the field reaches the potential minimum and decays into standard particles via reheating, a process still not fully understood [283]. Inflation also helps to dilute exotic relics from beyond the Standard Model theories (e.g., magnetic monopoles), making those theories consistent with their non-observation.

The non-minimally coupled dark matter (NMCDM) model introduced in Chapter 6 is not intended to fully resolve the known issues of Λ CDM, but rather to investigate possible effects that arise when open problems in different sectors, such as dark matter, dark energy and General Relativity, are considered together. As we will show in the following sections, cosmology with a NMCDM fluid actually moves in the direction of addressing some of the problems of the Λ CDM model, naturally providing an inflationary phase and the possibility of avoiding the Big Bang singularity through a cosmological bounce.

This chapter is organized as follows: in Section 7.1 we introduce the modified Friedmann and Raychaudhuri equations of NMCDM cosmology, and we study the consequent evolution of the Universe, finding that it gets substantially altered with respect to Λ CDM at high redshifts, from the point in which the average CDM density exceeds a threshold value. At those early times a phase of accelerated expansion is enforced by the non-minimal coupling of the CDM component with gravity, while at even higher redshifts the evolution of the Universe depends on the coupling between gravity and spatial curvature. In particular, for negative curvature, there is the appealing possibility to remove the initial singularity in favour of a bouncing scenario. Finally, in Section 7.2 we summarize our findings and discuss further developments beyond the background evolution.

7.1 Non-minimally coupled cosmology

To look for implications of the non-minimal coupling in a cosmological context, we model the spacetime with a Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - k r^2} + r^2 d\Omega^2 \right); \quad (7.1)$$

here t is the cosmic time measured by comoving observers, $a(t)$ is the scale factor, and $k = 0, \pm R_0^{-2}$ is the Gaussian curvature of space at present time. The matter content of the model comprises four comoving and homogeneous perfect fluids: baryons, CDM, radiation (photons and neutrinos), and DE (in the form of a cosmological constant Λ). The equations of state for these cosmic fluids have the form $p_i(t) = w_i \rho_i(t)$, with $w_b = w_{\text{DM}} = 0$, $w_r = 1/3$, $w_\Lambda = -1$.

The only two non-vanishing components of the field equations (6.5) with the ansatz (7.1) are the time-time component and a combination of the latter with the trace

$$H^2 = \frac{8\pi G}{3} \frac{\rho_{\text{DM}} + \rho_b + \rho_r + \rho_\Lambda}{1 + 8\pi G L^2 \rho_{\text{DM}}} - \frac{k}{a^2} \frac{1 - 8\pi G L^2 \rho_{\text{DM}}}{1 + 8\pi G L^2 \rho_{\text{DM}}}, \quad (7.2a)$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3} \frac{\left(1 - 9L^2 \left(H^2 - \frac{k}{a^2}\right)\right) \rho_{\text{DM}} + \rho_b + 2\rho_r - 2\rho_\Lambda}{1 + 8\pi G L^2 \rho_{\text{DM}}}. \quad (7.2b)$$

Equation (7.2a) is the Friedmann equation, while (7.2b) is the Raychaudhuri equation. The time component of (6.6) yields instead the continuity equation

$$\left[1 + 3L^2 \left(H^2 + \frac{k}{a^2}\right)\right] (\dot{\rho}_{\text{DM}} + 3H\rho_{\text{DM}}) + \dot{\rho}_b + 3H\rho_b + \dot{\rho}_r + 4H\rho_r + \dot{\rho}_\Lambda = 0. \quad (7.3)$$

If there are no energy transfers between different components of the cosmic fluid, then Equation (7.3) splits into the four conditions $\dot{\rho}_i + 3H(1 + w_i)\rho_i = 0$, one for each component, with solution $\rho_i(t) = \rho_{i,0} a(t)^{-3(1+w_i)}$. Here the subscript “0” refers to a quantity evaluated at the present time, and a_0 has been set equal to one. Note that with the FLRW ansatz for the metric the additional terms due to the non-minimal coupling with gravity factorize, so that CDM behaves like dust.

In the context of cosmology it is customary to express the evolution of quantities in term of redshift $1 + z = a^{-1}$, and to define the dimensionless density parameters

$$\begin{aligned} \Omega_i &\equiv \frac{8\pi G}{3H^2} \rho_i, \\ \Omega_k &\equiv -\frac{k}{a^2 H^2}, \end{aligned} \quad (7.4)$$

related to their present values by

$$\begin{aligned} \Omega_i &= (H_0/H)^2 \Omega_{i,0} (1+z)^{3(1+w_i)}, \\ \Omega_k &= (H_0/H)^2 \Omega_{k,0} (1+z)^2. \end{aligned} \quad (7.5)$$

Note that the evolution of Ω_k suggests the analogy with a fluid with an equation of state parameter $w_k = -1/3$. To keep track of the relevance of the non-minimal coupling during the

evolution of the Universe, we also define the quantity

$$\chi \equiv 8\pi GL^2 \rho_{\text{DM}} = 3L^2 H^2 \Omega_{\text{DM}}, \quad (7.6)$$

evolving as $\chi = \chi_0 (1+z)^3$. We would like to underline that $\rho_{\text{DM}} = (8\pi GL^2)^{-1}$, i.e. $\chi = 1$, corresponds precisely to the critical density at which NMC UCOs, explored in Chapter 6, develop a central singularity. However, in the cosmological context there is no divergence in the curvature invariants at this value, and the evolution is regular at any finite redshift. We also stress that a singularity would indeed appear at $\chi = 1$ if we had chosen the polarity parameter $\epsilon = +1$ in (6.2), which would reverse the sign of all terms multiplied by L^2 . This supports our decision to avoid $\epsilon = +1$, as it leads to singular configurations in both cosmological and astrophysical contexts.

With the above notation, Equation (7.2a) can be written as a constraint, valid at any time, involving the sum of the density parameters

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k (1 - \chi) = 1 + \chi, \quad (7.7)$$

where $\Omega_m \equiv \Omega_{\text{DM}} + \Omega_b$ denotes the total matter density parameter. Taking into account the scaling of the various parameters with redshift (7.5), we can further write the Friedmann equation (7.2a) explicitly as a function of redshift

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 (1 - \chi_0(1+z)^3)}{1 + \chi_0(1+z)^3}. \quad (7.8)$$

In the calculations presented in the next subsections we use values of the density parameters and of the Hubble constant today from the Planck collaboration [277], and $\chi_0 = 5.85 \times 10^{-34}$ given by our fiducial value of L .

7.1.1 Cosmic evolution

A first grasp of the evolution of the Universe within this model can be attained by looking at equation (7.8) at different epochs, starting from the present time and going towards higher redshifts. First of all, one notes that the non-minimal coupling of DM with gravity affects the evolution of the Universe only at early times when $\chi \geq 1$, while it can be completely neglected at later times (at least at the background level) when the expansion of the Universe causes χ to drop significantly below one. Thus, the late time evolution will be indistinguishable from the one in the standard Λ CDM paradigm

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2, \quad (7.9)$$

with a DE dominated phase extending to redshift $\log(1+z) \simeq 0.11$, a matter dominated epoch in the range $0.11 \lesssim \log(1+z) \lesssim 3.54$, and radiation domination at higher redshifts. As the DM density increases with redshift, a point will be reached when $\chi \simeq 1$ and at earlier times, even though radiation is still the dominant component, the NMC of DM with gravity causes the evolution to depart from the Λ CDM scenario. The relevant redshift is given by

$$\log(1+z_{\text{DM}}) \equiv -\frac{1}{3} \log(3\Omega_{\text{DM},0}) - \frac{2}{3} \log(H_0 L) \simeq 11.08. \quad (7.10)$$

We note that the considered value of L ensures that the NMC becomes irrelevant before neutrino decoupling, and thus earlier than any observable phenomenon.

We can get an intuition about the evolution in the NMC-dominated era by looking at Equation (7.8) in the deep limit $\chi \gg 1$, and neglecting matter and DE, since they are subdominant at such high redshifts

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{r,0}}{\chi_0} (1+z) - \Omega_{k,0} (1+z)^2. \quad (7.11)$$

For a spatial curvature parameter close to its limiting value $|\Omega_{k,0}| \sim 10^{-3}$, the first term on the right hand side of (7.11) dominates until a redshift

$$\log(1+z_k) \equiv \log\left(\frac{\Omega_{r,0}}{3|\Omega_{k,0}|\Omega_{\text{DM},0}}\right) - 2\log(H_0 L) \simeq 32.20, \quad (7.12)$$

leading to an almost quadratic evolution of the scale factor during this period

$$a(t) = a(t_k) \left(1 + \frac{H_0}{2} \sqrt{\frac{\Omega_{r,0}}{a(t_k)\chi_0}} (t - t_k)\right)^2. \quad (7.13)$$

At even higher redshifts $z \gg z_k$, the evolution of the Universe strongly depends on the sign of the spatial curvature. In the case of a vanishing curvature $\Omega_k = 0$, the Universe is dominated by radiation and follows the evolution in equation (7.13) at all times. On the other hand, even if a small positive curvature (corresponding to $\Omega_k < 0$) is present, at very early time it would be the dominant component, yielding a linear expansion of the Universe ($a(t) \propto \sqrt{kt}$). One can thus conclude that both a positive or vanishing curvature imply the existence of a singularity in the remote past, although attained at different rates: quadratic in the first case, linear in the second. The most interesting case however, is that of a negative spatial curvature ($\Omega_k > 0$). Looking at Equation (7.11), we see that in this scenario the expansion rate vanishes at z_k signaling the presence of a cosmological bounce. It is interesting to note that, for the chosen value of L and the limiting value of spatial curvature $|\Omega_{k,0}| \sim 10^{-3}$, the bounce would

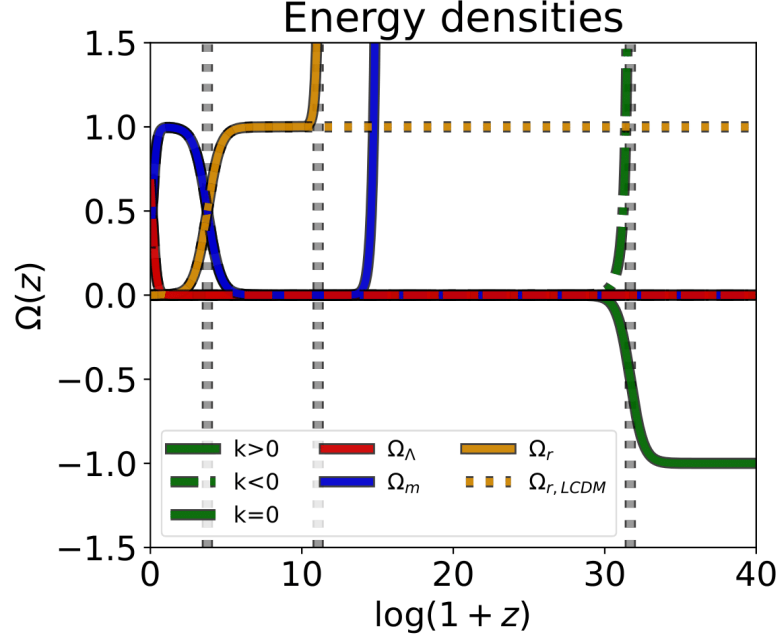


Figure 7.1. Evolution of the density parameters as a function of redshift $1+z \equiv a^{-1}$. Solid lines represent the solution in the present non-minimal coupling model, while the dotted lines show the corresponding ones in the Λ CDM model. The vertical dotted grey lines represent matter-radiation equality at $\log(1+z_{\text{eq}}) \simeq 3.54$, the onset of the NMC at redshift $\log(1+z_{\text{NMC}}) \simeq 11.08$, and the beginning of the curvature era (or the redshift of the bounce), at redshift $\log(1+z_k) \simeq 32.20$.

have occurred when the radiation temperature was extremely close to the Planck temperature $T_{\text{p}} \sim 10^{32} K$.

Figure 7.1 shows the evolution of the density parameters (7.4) as a function of redshift. Solid lines represent the NMC solution, while the dotted lines show the corresponding solution in the Λ CDM model. The vertical dotted grey lines represent matter-radiation equality at $\log(1+z_{\text{eq}}) \simeq 3.54$, the onset of the NMC at redshift $\log(1+z_{\text{NMC}}) \simeq 11.08$, and the beginning of the curvature era (or the redshift of the bounce), at redshift $\log(1+z_k) \simeq 32.20$. The green lines indicate the quantity $\Omega_k \frac{1-\chi}{1+\chi}$ with $k \lesseqgtr 0$. From this plot it is clear that radiation remains the dominant component of the cosmic fluid even during the NMC-dominated phase, while at the same time the dynamics does not follow any longer the standard Friedmann equation, as the constraint equation (7.7) does not longer enforces the density parameters to sum to one.

To keep track of the roles played by the different fluid components during the evolution of the Universe, it is useful to define a dynamical state parameter

$$w \equiv - \left(1 + \frac{2}{3} \frac{\dot{H}}{H^2} \right), \quad (7.14)$$

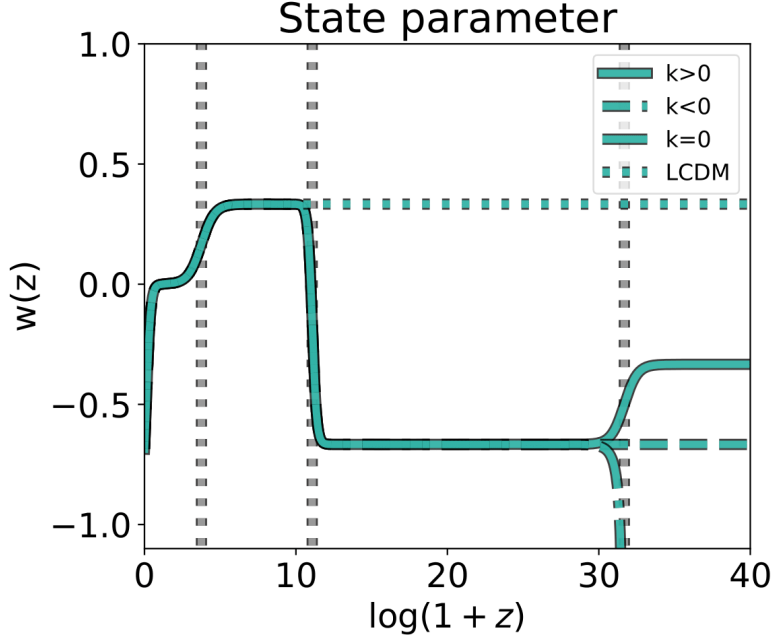


Figure 7.2. The dynamical state parameter w as a function of redshift. The solid, dashed and dotted-dashed lines are the behavior in the presence of NMC with negative, null, or positive curvature respectively, while the dotted line is the one in Λ CDM. The vertical dotted grey lines represent matter-radiation equality at $\log(1+z_{\text{eq}}) \simeq 3.54$, the onset of the NMC at redshift $\log(1+z_{\text{NMC}}) \simeq 11.08$, and the beginning of the curvature era (or the redshift of the bounce), at redshift $\log(1+z_k) \simeq 32.20$.

so that at any time in which w is almost constant, the scale factor evolves as

$$a(t) \simeq \left(\frac{t}{t_0} \right)^{\frac{2}{3(1+w)}}. \quad (7.15)$$

In the absence of NMC, when a fluid component dominates the energy content of the Universe, (7.14) coincides with the state parameter of that component. In the presence of the NMC between DM and gravity this correspondence is more involved. Substituting Equation (7.2) in (7.14), we find

$$w = \frac{\Omega_m + 2\Omega_r - 2\Omega_\Lambda - 3\chi(1 + \Omega_k)}{3(1 + \chi)} - \frac{1}{3}. \quad (7.16)$$

When $\chi \ll 1$ one has the familiar values $w = -1$, $w = 0$, $w = 1/3$ during DE, matter, and radiation domination respectively. In the deep NMC regime $\chi \gg 1$, even if radiation is still the dominant component, the constraint (7.7) requires $\Omega_r \simeq \chi$, leading to $w = -2/3$, and to an accelerated quadratic expansion. At very high redshift, in the case of a positive curvature $\Omega_{k,0} < 0$, the curvature dominates the energy density and the effective parameter increases to

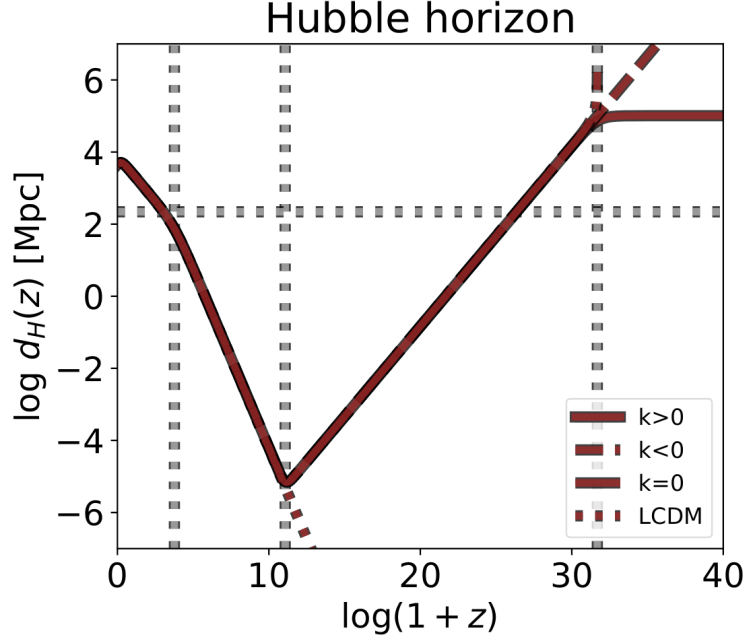


Figure 7.3. The comoving Hubble horizon as a function of redshift. The solid, dashed and dotted-dashed lines are the behavior in the presence of NMC with negative, null, or positive curvature respectively, while the dotted line is the one in Λ CDM. The vertical dotted grey lines represent matter-radiation equality at $\log(1+z_{\text{eq}}) \simeq 3.54$, the onset of the NMC at redshift $\log(1+z_{\text{NMC}}) \simeq 11.08$, and the beginning of the curvature era (or the redshift of the bounce), at redshift $\log(1+z_k) \simeq 32.20$. The horizontal dashed line shows the value of the horizon at the epoch of recombination at $\log(1+z_{\text{CMB}}) \simeq 3.04$. The line further intersects the horizon at $\log(1+z_{\text{end}}) \simeq 26.36$.

$w = -1/3$. The evolution of the dynamical state parameter as a function of redshift is shown in Figure 7.2. The solid, dashed and dot-dashed lines are the behavior in the presence of NMC with negative, null, or positive curvature respectively, while the dotted line is the one in Λ CDM. As mentioned, this plot has two peculiar features. First, one sees a steep decrease of the state parameter at the onset of NMC, when $\chi \simeq 1$, leading to an accelerated phase of expansion at higher redshifts. We want to stress that during this epoch radiation is still the dominant energy component, meaning that $\rho_r \gg \rho_m \gg \rho_\Lambda$, but the dynamical state parameter departs from the value of $w_r = 1/3$, characteristic of radiation, because of the non-trivial coupling between DM and gravity. Furthermore, if a curvature is present, at redshift z_k , it becomes the dominant component, and either the state parameter increases to $w_k = -1/3$, characterizing curvature domination, or the Universe has a bounce and the state parameter is no longer well defined.

7.1.2 NMC-driven inflation and bounce

As opposed to standard inflationary scenarios, which require the introduction of an unknown scalar field, the NMCDM model features an accelerated epoch that emerges naturally from the non-minimal coupling between dark matter and gravity, once $\chi \geq 1$. This accelerated phase, during which the comoving Hubble horizon decreases with time, offers the possibility of solving the horizon and flatness problems without introducing new components to the energy content of the Universe. Figure 7.3 shows the evolution of the comoving Hubble horizon, defined as

$$d_H(z) = \frac{1}{a(z)H(z)}, \quad (7.17)$$

as a function of redshift. In order to solve the horizon problem, this phase of accelerated expansion must last long enough for the comoving Hubble horizon to grow from its minimum value at the onset of radiation domination to the value it has at recombination, around redshift $z_{\text{CMB}} \simeq 1100$. The flatness problem is commonly described in terms of a critical total energy density that would make the Universe spatially flat. In GR, one has that

$$\rho_c = \frac{3H^2}{8\pi G}, \quad \Rightarrow \quad \frac{\rho_m + \rho_r + \rho_\Lambda}{\rho_c} - 1 = \frac{k}{a^2 H^2} = -\Omega_k = d_H^2 k, \quad (7.18)$$

implying that to ensure $|\Omega_k| < 10^{-3}$ today, the total energy density in the early Universe must have satisfied $\rho \sim \rho_c(1 \pm 10^{-60})$, which is extremely fine-tuned. In the NMCDM model, however, equation (7.18) is modified as follows:

$$\rho_c = \frac{3H^2(1+\chi)}{8\pi G}, \quad \Rightarrow \quad \frac{\rho_m + \rho_r + \rho_\Lambda}{\rho_c} - 1 = \frac{k}{a^2 H^2} \frac{1-\chi}{1+\chi} = d_H^2 k \frac{1-\chi}{1+\chi}, \quad (7.19)$$

that is, the deviation from flatness is now proportional to $d_H^2 k \frac{1-\chi}{1+\chi}$ instead of $d_H^2 k$. Nonetheless, it is evident that solving the horizon problem also alleviates the flatness problem, since the deviation from the critical density remains proportional to d_H^2 , except during the short period where $\chi \sim 1$, which, however, does not significantly affect the overall evolution. In Figure 7.3 the horizontal dotted line shows the value of the comoving Hubble horizon at recombination. The line further intersects the horizon at $\log(1+z_{\text{end}}) \simeq 26.54$, and therefore to solve the horizon problem the accelerated expansion had to start before z_{end} , and then to last a minimum of $\log(1+z_{\text{end}}) - \log(1+z_{\text{NMC}}) \simeq 15.46$. Considering z_k as the beginning of the accelerated phase, we thus find

$$\log(1+z_k) - \log(1+z_{\text{NMC}}) = \log\left(\frac{\Omega_{r,0}}{|\Omega_{k,0}|(3\Omega_{\text{DM},0})^{2/3}}\right) - \frac{4}{3}\log(H_0 L) \gtrsim 15.46, \quad (7.20)$$

which implies

$$|\Omega_{k,0}| \lesssim \frac{\Omega_{r,0}}{(3\Omega_{\text{DM},0}H_0^2L^2)^{2/3}} 10^{-15.46} = \Omega_{r,0}\chi_0^{-2/3} 10^{-15.46} \simeq 4.56 \times 10^2, \quad (7.21)$$

that is clearly well guaranteed by the Planck collaboration constraint $|\Omega_{k,0}| < 1.2 \times 10^{-3}$. Thus, as it is evident from Figure 7.3, the epoch of accelerated expansion provided by the NMC between DM and gravity is sufficient to solve the horizon problem. In particular, with the parameters considered we find $\log(1+z_k) - \log(1+z_{\text{NMC}}) \simeq 21.12$, which would correspond to approximately $N = 48.62$ e -folding.

As for the beginning of the Universe, we would like to review the possibility of a cosmological bounce at redshift z_k with negative spatial curvature. It is clear from the structure of Equation (7.11) that, for positive $\Omega_{k,0}$, higher redshift values are not permitted. However, the absence of divergences in the curvature invariants indicates that the spacetime can be extended, implying that the scale factor must begin to increase again. Equation (7.11) can indeed be integrated for positive $\Omega_{k,0}$, and is solved by

$$a(t) = \frac{\chi_0\Omega_{k,0}}{\Omega_{r,0}} + \frac{H_0^2\Omega_{r,0}}{4\chi_0}(t-t_k)^2, \quad (7.22)$$

which clearly shows that the scale factor reached a minimum at t_k , meaning the Universe had undergone a cosmic bounce. It is a peculiar feature of this model, however, that the bouncing solution is found in the case of negative spatial curvature. Bounce models commonly found in the literature generally require positive spatial curvature in order to avoid violations of the energy conditions for matter components [284]. In other words, the negative energy contribution of positive spatial curvature (for which, we recall, $\Omega_k < 0$) is what reverses the attractive nature of gravity. In our case, however, the non-minimal coupling drastically changes the role of spatial curvature: whenever the non-minimal coupling is relevant, it is negative spatial curvature that contributes repulsively to gravity, allowing a contracting Universe to bounce. We note that a bouncing Universe with positive spatial curvature could be found if the non-minimally coupled dark matter were a relativistic fluid. Nonetheless, we leave this case to Appendix D, as it requires unnatural changes to the parameters of the theory.

Furthermore, the extreme energies at which the bounce occurs suggest that the fluid approximation may break down, requiring a more fundamental description of DM. Nonetheless, modified Friedmann equations can arise as effective descriptions of gravity even in purely quantum gravity regimes, leading to bouncing cosmologies [285, 286]. Moreover, one can argue that a fluid description is appropriate when the mean free path is smaller than the characteristic length scale of the system, a condition that holds even more robustly in the high-density

regime. Our results indicate that, in general, non-minimally coupled DM tends to interact with spatial curvature, potentially making it the dominant component in the very early Universe and leading to a cosmological bounce due to the gravitationally repulsive effect it can produce, since curvature is not constrained by energy conditions and can naturally take both positive and negative values.

7.2 Summary and conclusions

In this chapter, we described the impact of the non-minimal coupling between dark matter and gravity, introduced in Chapter 6, on the background cosmological evolution. In this framework the Friedmann and Raychaudhuri equations are modified by the non-minimal coupling, while the conservation equations remain unchanged. It is thus possible to infer the cosmic expansion directly from the evolution of the energy densities. We found that the non-minimal coupling drives a phase of accelerated expansion in the early Universe, which, for the considered value of the length scale, ends just before neutrino decoupling. We verified that this accelerated phase lasts long enough to solve the horizon problem, and could therefore serve as an inflationary epoch that does not require the introduction of additional fields.

Within this model the origin of the Universe depends strongly on spatial curvature. While positive or zero curvature always leads to an initial singularity, although reached with different expansion rates, negative spatial curvature allows for the possibility of a cosmological bounce. The non-minimal coupling to spatial curvature makes it the dominant component in the very early Universe and causes it to reverse its usual effect: negative curvature reduces the expansion rate, while positive curvature increases it, opposite to what happens in standard cosmology. The scale factor at which the bounce occurs depends on the length scale of the coupling and the current value of spatial curvature. For a curvature value close to current observational bounds, the bounce takes place precisely at the Planck epoch.

While the natural emergence of an accelerated expansion phase and a cosmological bounce is appealing, these phenomena come with important caveats. In particular, the presence of derivatives of the energy density in the effective stress–energy tensor suggests that high-frequency perturbations could be amplified relative to low-frequency ones, thus distorting the nearly scale-invariant power spectrum of cosmological perturbations, predicted by inflationary models. Since the scaling of this power spectrum is now severely constrained by precision CMB measurements, a detailed analysis of perturbations is necessary before this accelerated expansion phase can be regarded as a viable inflationary epoch. Moreover, the power-law behavior of the background prevents the use of slow-roll approximations, so that a full analysis of cosmological perturbations is required. The next chapter will be devoted to the generation and evolution of such scalar,

vector and tensor perturbations in the NMC–DM cosmological scenario. A bounce might then be a generic feature of non-minimally coupled models, and could provide an alternative to inflation as a mechanism for generating a nearly scale-invariant power spectrum of cosmological perturbations [\[287\]](#).

Chapter 8

Linear Cosmological Perturbations with a Non-minimally Coupled Dark Matter Fluid

Linear cosmological perturbation theory provides the essential framework for connecting cosmological models with observations. While the homogeneous and isotropic Friedman Lemaître Robertson Walker (FLRW) background successfully describes the global evolution of the Universe, the formation of galaxies, clusters, and the observed anisotropies of the CMB originate from small inhomogeneous perturbations about this background. Since these perturbations remain small over a significant range of cosmic history, they can be treated within linear theory, allowing the Einstein and matter field equations to be consistently expanded to first order.

The modern theory of cosmological perturbations has been developed over several decades and provides a gauge-invariant description of scalar, vector, and tensor fluctuations [288, 289, 290, 291, 292]. Scalar perturbations are responsible for density fluctuations and structure formation, vector perturbations generally decay in an expanding Universe unless sustained by additional sources, while tensor perturbations correspond to primordial gravitational waves. A fundamental aspect of perturbation theory is the treatment of gauge freedom, which arises because the perturbed spacetime can be described using different coordinate systems. The gauge-invariant formalism developed by Bardeen [289] and subsequently refined by many authors, allows physical perturbations to be separated from spurious coordinate artifacts, providing variables whose evolution has an unambiguous physical interpretation.

For a perfect fluid minimally coupled to gravity, the linearized Einstein equations combine with the conservation of the energy–momentum tensor (SET) to produce a closed system governing the evolution of the metric perturbations, density contrast, velocity potential, and pressure

fluctuations. These equations form the basis for the theoretical prediction of cosmological observables and are implemented in modern Boltzmann codes used to analyze observational data.

In this chapter we aim to extend further the study of cosmology in the non-minimally coupled dark matter (NMC–DM) framework by following the evolution of linear perturbations throughout the modified cosmic history. The results are then compared with the relevant observables, such as the matter power spectrum, the CMB anisotropies, and the scalar-to-tensor ratio.

In Section 8.1 we introduce the formalism of linear perturbations over a FLRW background, we present the relevant equations in the NMC–DM framework for the evolution of scalar, vector, and tensor perturbations, and their solutions in the relevant regimes.

In Section 8.2 we then compute the matter power spectrum and the tensor-to-scalar ratio, both assuming an initial big bang singularity followed by an inflationary phase, and a bouncing scenario, in which the singularity is removed and the current expansion of the Universe is preceded by a symmetric contracting phase. We anticipate that the resulting power spectra are very different in the two cases and that only that resulting from a bounce is able to reproduce observations of CMB anisotropies, though at the price of modifying the initial perturbations' quantum state. This fact is very promising, since the same model can accommodate for a resolution of the initial singularity at the background level, and leads to a power spectrum consistent with observations. Finally, in Section 8.3 we summarize our findings.

8.1 Theory of cosmological perturbations with a non-minimal coupling

A consistent cosmological model requires the analysis of perturbations. This analysis becomes even more important when the model is proposed as an alternative to inflation, since it must provide the primordial perturbations that evolve into CMB anisotropies and large-scale structure. In this section, we present the equations governing perturbations in the presence of non-minimally coupled dark matter, and their solutions in the relevant regimes. A complete perturbation analysis is necessary, since the accelerated expansion phase of our model cannot be described within the slow-roll regime. The slow-roll parameters are

$$\epsilon_{\text{H}} := -\frac{\dot{H}}{H^2} = \frac{1}{2}, \quad \eta_{\text{H}} := -\frac{1}{2} \frac{\ddot{H}}{\dot{H}H} = \frac{1}{2}, \quad (8.1)$$

which do not satisfy the condition $\epsilon_{\text{H}}, \eta_{\text{H}} \ll 1$.

In this and the following sections, we switch to conformal time $d\eta = a^{-1}dt$, so that

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + \gamma_{ij} dx^i dx^j \right), \quad f' := \frac{df}{d\eta} = a \frac{df}{dt} = a\dot{f}, \quad \mathcal{H} := \frac{a'}{a} = \dot{a} = aH, \quad (8.2)$$

where $\mathcal{H}$ denotes the conformal Hubble parameter, and its inverse corresponds to the comoving Hubble radius $\mathcal{H}^{-1} = (aH)^{-1} = d_H$.

General perturbations of the metric and stress–energy tensor can be decomposed into scalar, vector, and tensor sectors, according to their transformation properties under spatial rotations. This decomposition is particularly useful at linear order, since the equations for the different sectors decouple, allowing their evolution to be studied independently.

Scalar perturbations represent fluctuations in the gravitational potential and in the energy density and pressure of the cosmic fluid. They are the most relevant sector for cosmological observations, as they generate the temperature anisotropies in the CMB and seed the formation of large-scale structure. The most general scalar perturbations of the metric and stress–energy tensor can be written as [293]

$$\delta g_{\mu\nu}^{(s)} = a^2 \begin{pmatrix} -2\phi & D_i B \\ D_i B & -2(\psi \gamma_{ij} - D_i D_j E) \end{pmatrix}, \quad \delta T_{\nu}^{\mu(s)} = \begin{pmatrix} -\delta\rho & (\rho + p) D_i (V - B) \\ -(\rho + p) D^i V & \delta p \delta^i_j - D^i D_j \sigma \end{pmatrix}, \quad (8.3)$$

where ϕ , ψ , B , E , $\delta\rho$, δp , V , and σ are functions of the spacetime variables, and D_i indicates the covariant spatial derivative.

Vector perturbations induce rotational distortions of spacetime and frame-dragging effects. These modes are not observed at cosmological scales, and therefore a viable model must predict that they decay or remain negligible. The most general vector perturbations of the metric and stress–energy tensor can be written as

$$\delta g_{\mu\nu}^{(v)} = a^2 \begin{pmatrix} 0 & -S_i \\ -S_i & D_j F_i + D_i F_j \end{pmatrix}, \quad \delta T_{\nu}^{\mu(v)} = \begin{pmatrix} 0 & (\rho + p) (U_i - S_i) \\ -(\rho + p) U^i & D_j \nu^i + D^i \nu_j \end{pmatrix}, \quad (8.4)$$

where S_i , F_i , U_i and ν_i are transverse three-vectors, satisfying $D_i S^i = D_i F^i = D_i U^i = D_i \nu^i = 0$.

Tensor perturbations induce transverse and traceless distortions of spacetime, i.e. primordial gravitational waves. They represent the propagating gravitational degrees of freedom in the absence of anisotropic stress, and their detection constrains models of the early Universe. The most general tensor perturbations of the metric and stress–energy tensor can be written as

$$\delta g_{\mu\nu}^{(t)} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \quad \delta T_{\nu}^{\mu(t)} = \begin{pmatrix} 0 & 0 \\ 0 & \pi^i_j \end{pmatrix}, \quad (8.5)$$

where h_{ij} and π_{ij} are symmetric, transverse, and traceless spatial tensors satisfying $h^i_i = D_i h^{ij} = \pi^i_i = D_i \pi^{ij} = 0$.

In this work, we assume the absence of anisotropic stress, setting $\sigma = \nu_i = \pi_{ij} = 0$, and restrict to adiabatic perturbations, so that $\delta p = c_s^2 \delta \rho$.

8.1.1 Scalar perturbations

Most cosmological phenomenology lies in scalar perturbations. Their dynamics can be complex, since the coupling between density fluctuations and the gravitational potential can produce feedback effects and lead to instabilities. They are also the only sector directly constrained by observations, and for this reason our analysis will mainly focus on them.

We first note that the functions defined in equations (8.3) are not gauge invariant. We introduce the gauge-invariant variables

$$\begin{aligned}\Phi &:= \phi + \mathcal{H}(B - E') + (B - E')', & \Psi &:= \psi - \mathcal{H}(B - E'), \\ v_n &:= V_n + B - E', & \delta_n &:= \rho_n^{-1} (\delta \rho_n - 3(1 + w_n) \mathcal{H}(B - E'))\end{aligned}\quad (8.6)$$

where $n = \text{DM}, \text{B}, \text{R}, \Lambda$. As in GR, using these gauge-invariant variables we obtain a system of equations equivalent to that in the longitudinal gauge, i.e. with $B = E = 0$. In particular, the 00, 0i, ii and off-diagonal ij components of the Einstein equations are

$$\begin{aligned}-3(1 + \chi) \mathcal{H}(\Psi' + \mathcal{H}\Phi) + (1 - \chi) (\nabla^2 \Psi + 3\kappa \Psi) - \chi \left(2\mathcal{H} \nabla^2 v_{\text{DM}} - \frac{3}{2} (\mathcal{H}^2 - \kappa) \delta_{\text{DM}} \right) &= \\ &= 4\pi G a^2 \sum_n \rho_n \delta_n, \\ D_i \left((1 - \chi) (\Psi' + \mathcal{H}\Phi) - \chi \left(\frac{1}{2} (4\mathcal{H}' - \mathcal{H}^2 + 3\kappa) v_{\text{DM}} + \mathcal{H} \delta_{\text{DM}} \right) \right) &= \\ &= D_i \left(4\pi G a^2 \sum_n (1 + w_n) \rho_n v_n \right), \\ (1 + \chi) \left(\Psi'' + 2\mathcal{H}\Psi' - \kappa \Psi + \mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2) \Phi - \frac{1}{2} \nabla^2 \Psi + \frac{1}{2} \nabla^2 \Phi \right) + \\ -\chi \left(3\mathcal{H}^2 \Phi + \kappa \Psi + \nabla^2 \Phi + \mathcal{H} \delta'_{\text{DM}} + (\mathcal{H}' - \mathcal{H}^2 + \kappa) \delta_{\text{DM}} - \nabla^2 v'_{\text{DM}} + \mathcal{H} \nabla^2 v'_{\text{DM}} + \frac{1}{2} \nabla^2 \delta_{\text{DM}} \right) &= \\ &= 4\pi G a^2 \sum_n w_n \rho_n \delta_n, \\ D_i D_j \left(\frac{1}{2} (1 - \chi) (\Psi - \Phi) - \chi \left(v'_{\text{DM}} - \mathcal{H} v_{\text{DM}} - \frac{1}{2} \delta_{\text{DM}} \right) \right) &= 0, \end{aligned}\quad (8.7)$$

where the sums are on the indices $n = \text{DM}, \text{B}, \text{R}, \Lambda$, and we used that for barotropic fluids $c_{s,n}^2 = w_n$. The conservation of the minimally coupled stress-energy tensors, namely those of radiation, baryons, and dark energy, leads to

$$\begin{aligned} \delta'_m - (1 + w_m) (3\Psi' + \nabla^2 v_m) &= 0, \\ D_i \left(v'_m + (1 - 3w_m) \mathcal{H} v_m - \Phi - \frac{w_m}{1 + w_m} \delta_m \right) &= 0; \end{aligned} \quad (8.8)$$

where $m = \text{B}, \text{R}, \Lambda$. Let us note that for the cosmological constant component ($w_\Lambda = -1$) the conservation of the stress-energy tensor implies $\delta_\Lambda = \delta'_\Lambda = 0$. For the DM component, we must consider the divergence of the full system of field equations (6.5), or equivalently the conservation of the effective stress-energy tensor (6.7)

$$\begin{aligned} \delta'_{\text{DM}} - 3\Psi' - \nabla^2 v_{\text{DM}} &= 0, \\ D_i \left(v'_{\text{DM}} + \mathcal{H} v_{\text{DM}} - \Phi + \frac{\chi}{2\pi G a^2 \rho_{\text{DM}}} \left(\Psi'' + \frac{1}{2} \mathcal{H} \Psi' + \mathcal{H} \Phi' - \frac{1}{2} \nabla^2 \Psi - \frac{3}{2} \kappa \Psi + \right. \right. \\ \left. \left. + \frac{1}{4} (4\mathcal{H}' - \mathcal{H}^2 - 3\kappa) \Phi + \frac{1}{4} (4\mathcal{H}' - \mathcal{H}^2 - \kappa) v'_{\text{DM}} + \frac{1}{4} (4\mathcal{H}'' - (6\mathcal{H}' - \mathcal{H}^2 - \kappa) \mathcal{H}) v_{\text{DM}} \right) \right) &= 0. \end{aligned} \quad (8.9)$$

Interestingly, the continuity equation for DM perturbations is left unchanged, while the Euler equation is significantly modified. As expected, in the absence of non-minimal coupling ($\chi \rightarrow 0$), equations (8.7–8.9) reduce to those of GR.

For the rest of this chapter, we will consider small spatial curvature $\kappa \sim 0$; this assumption is justified by the fact that the length scales of perturbations are much smaller than the curvature radius of the Universe, with the notable exception of the bounce, which, however, does not change the global evolution of perturbations, as we show in Appendix E. We also move to Fourier space $D_i \rightarrow i k_i$, $\nabla^2 \rightarrow -k^2$ and consider the effective state parameter w and sound speed c_s^2 , defined as

$$w := - \left(1 + \frac{2}{3} \frac{\dot{H}}{H^2} \right) = - \frac{1}{3} \left(1 + 2 \frac{\mathcal{H}'}{\mathcal{H}^2} \right), \quad c_s^2 := \frac{\sum_n c_{s,n}^2 \rho_n \delta_n}{\sum_n \rho_n \delta_n} = \frac{\sum_n w_n \rho_n \delta_n}{\sum_n \rho_n \delta_n}. \quad (8.10)$$

In particular, we approximate the evolution of the Universe as a sequence of epochs where w and c_s^2 are constant (namely, a Λ -dominated epoch with $w \sim c_s^2 \sim -1$, a matter-dominated epoch with $w \sim c_s^2 \sim 0$, a radiation-dominated epoch with $w \sim c_s^2 \sim 1/3$, and a NMC-dominated

epoch with $w \sim -2/3$ and $c_s^2 \sim 1/3$). With these definitions, all the information about the minimally coupled fluids is encoded in w and c_s^2 .

After using the off-diagonal ij component of equations (8.7) to solve for Φ , the dynamics can be described using the two equations (8.9) and the combination $\mathcal{E}_{ii} - c_s^2 \mathcal{E}_{00} = 0$ of equations (8.7), which together form a system at first order in δ_{DM} and at second order in Ψ and v_{DM} . The dynamics of baryon and radiation perturbations follow from equations (8.8), once Φ and Ψ have been determined by this system.

Comoving curvature perturbation

It is standard in cosmology to track the evolution of metric perturbations using the comoving curvature perturbation, that is, the perturbation of the spatial metric ψ (which in GR is equivalent to the Newtonian potential ϕ in the absence of anisotropic stress) as seen by an observer who is comoving with the cosmic fluid. Practically, it is defined as $\mathcal{R} := \psi + \mathcal{H}\xi^0$, where ξ^0 is fixed by the comoving gauge, defined by $T_{0i} = 0$:

$$T_{0i} = \sum_n (1 + w_n) \rho_n (v_n - \xi^0) = 0 \quad \implies \quad \xi^0 = \frac{\sum_n (1 + w_n) \rho_n v_n}{\sum_n (1 + w_n) \rho_n}, \quad (8.11)$$

where we used the gauge transformation for the velocity fields $v_n \rightarrow v_n - \xi^0$. Using the $0i$ and the off-diagonal ij components of equations (8.7), and the background equations (7.2a)- (7.2b) expressed in conformal time, we obtain

$$\mathcal{R} = \Psi + \frac{2}{3} \frac{\Psi + \frac{\Psi'}{\mathcal{H}} - \chi \left(\Psi + \frac{\Psi'}{\mathcal{H}} - \left(\frac{7}{2} + 3w \right) \mathcal{H} v_{\text{DM}} + 2v'_{\text{DM}} \right)}{1 + w + \chi(2 + w)}, \quad (8.12)$$

which reduces to the standard definition in the GR limit $\chi \ll 1$.

The final form of the equations is found by substituting Ψ and Ψ' with $\mathcal{R}$ and $\mathcal{R}'$. However, the full set of equations is very lengthy and not particularly instructive, so we do not report it here¹. In the GR limit, we recover the standard equations

$$\begin{aligned} \mathcal{R}'' + 2\mathcal{H}\mathcal{R}' + c_s^2 k^2 \mathcal{R} &= 0, \\ \delta'_{\text{DM}} + k^2 v_{\text{DM}} - \frac{9}{2} (1 + c_s^2) \mathcal{H} \mathcal{R} - \frac{9}{4} (1 + c_s^2) (5 + 3c_s^2) \mathcal{H}^2 \frac{\mathcal{R}'}{c_s^2 k^2} &= 0, \\ v'_{\text{DM}} + \mathcal{H} v_{\text{DM}} + \frac{3}{2} (1 + c_s^2) \mathcal{H} \frac{\mathcal{R}'}{c_s^2 k^2} &= 0; \end{aligned} \quad (8.13)$$

¹We can provide it in the form of a **Mathematica** notebook on request.

the equation for the comoving curvature perturbation can be integrated directly, yielding

$$\mathcal{R} = C_1 \frac{H_{-\nu}^{(1)}(c_s k \eta)}{\eta^\nu} + C_2 \frac{H_{-\nu}^{(2)}(c_s k \eta)}{\eta^\nu}, \quad (8.14)$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are Hankel functions of the first and second kind, $\nu = \frac{3}{2} \frac{1-w}{1+3w}$, and C_1 and C_2 are integration constants which depend on w and c_s^2 .

To investigate the effect of the non-minimal coupling on perturbations, we consider the NMC-dominated epoch by taking the limit $\chi \gg 1$ and setting $w = -2/3$ and $c_s^2 = 1/3$. The equations of motion in this regime are

$$\begin{aligned} \mathcal{R}'' - 2\mathcal{H} \frac{2k^2 - 3\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \mathcal{R}' + \frac{4}{3} \frac{2k^2 - 3\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \mathcal{R} - \frac{\mathcal{H}}{4} v_{\text{DM}}'' - \frac{3}{4} \frac{\mathcal{H}^4}{4k^2 - 3\mathcal{H}^2} v_{\text{DM}}' - \frac{\mathcal{H}}{24} (4k^2 - 3\mathcal{H}^2) v_{\text{DM}} &= 0, \\ \delta_{\text{DM}}' + 3\mathcal{H} \frac{8k^2 - 3\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} v_{\text{DM}}' + \left(k^2 + \frac{3\mathcal{H}^2}{2} \right) v_{\text{DM}} - \frac{36\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \mathcal{R}' + \frac{24k^2 \mathcal{H}}{4k^2 - 3\mathcal{H}^2} \mathcal{R} &= 0, \\ v_{\text{DM}}' + \frac{5\mathcal{H}}{2} \frac{4k^2 - 3\mathcal{H}^2}{k^2 + 3\mathcal{H}^2} v_{\text{DM}} + \frac{1}{2} \frac{4k^2 - 3\mathcal{H}^2}{k^2 + 3\mathcal{H}^2} \delta_{\text{DM}} + \frac{2}{\mathcal{H}} \frac{2k^2 - 9\mathcal{H}^2}{k^2 + 3\mathcal{H}^2} \mathcal{R}' + \frac{3(2k^2 + \mathcal{H}^2)}{k^2 + 3\mathcal{H}^2} \mathcal{R} &= 0. \end{aligned} \quad (8.15)$$

While this system still appears quite involved, it can be solved analytically after introducing an appropriate variable, as shown in Appendix F. The comoving curvature perturbation in the NMC-dominated epoch then takes the form

$$\mathcal{R} = C_{1,\text{NMC}} k^2 \frac{e^{i\sqrt{\frac{2}{3}}k\eta}}{\eta} + C_{2,\text{NMC}} k^2 \frac{e^{-i\sqrt{\frac{2}{3}}k\eta}}{\eta} + C_{3,\text{NMC}} \frac{e^{k\eta}}{\eta} + C_{4,\text{NMC}} \frac{e^{-k\eta}}{\eta}, \quad (8.16)$$

where the $C_{i,\text{NMC}}$ are the integration constants of the NMC epoch. The real exponential factors are due to non-oscillatory effects in the velocity field, and their associated integration constants can in principle be set to zero; nonetheless, during the NMC-dominated epoch the conformal time η decreases toward $\eta \ll 1/k$, and therefore they contribute to the comoving curvature perturbation as a constant term. We also note that the solution in equation (8.16) has the familiar form of oscillations modulated by the conformal time, but with an additional factor k^2 , which modifies the scale dependence of the perturbations.

To understand the physical behavior of perturbations, it is convenient to distinguish between a super-horizon regime, in which the perturbation length scales are larger than the Hubble radius ($k \ll \mathcal{H}$), and a deep sub-horizon regime, in which the perturbations are much smaller than the horizon and can therefore interact dynamically ($k \gg \mathcal{H}$). In the super-horizon regime,

the solutions in equations (8.14) and (8.16) reduce respectively to

$$\mathcal{R}_{k \ll \mathcal{H}} \sim \frac{i\Gamma(-\nu) \left(\frac{1}{2}c_s k\right)^\nu (C_2 - C_1)}{\pi} + \frac{C_1 + C_2 + i \cot(\pi\nu) (C_2 - C_1)}{\Gamma(1 - \nu) \left(\frac{1}{2}c_s k\right)^\nu} \eta^{-2\nu}, \quad (8.17)$$

where Γ is Euler gamma function, and

$$\begin{aligned} \mathcal{R}_{k \ll \mathcal{H}} \sim & k \left(i\sqrt{\frac{2}{3}} k^2 (C_{1,\text{NMC}} - C_{2,\text{NMC}}) + C_{3,\text{NMC}} - C_{4,\text{NMC}} \right) + \\ & + \left(k^2 (C_{1,\text{NMC}} + C_{2,\text{NMC}}) + C_{3,\text{NMC}} + C_{4,\text{NMC}} \right) \eta^{-1}. \end{aligned} \quad (8.18)$$

In both cases, the solution consists of a constant mode and a time-dependent mode. However, while in the GR case the time-dependent mode decays in an expanding Universe as $\eta^{-2\nu} \propto a^{-\frac{3}{2}(1-w)}$, in the NMC case it instead grows as $\eta^{-1} \propto a^{1/2}$. In the deep sub-horizon regime, equations (8.14) and (8.16) reduce respectively to

$$\mathcal{R}_{k \gg \mathcal{H}} \sim C_1 \frac{e^{-i\frac{\pi}{4}(1-2\nu)}}{\sqrt{\frac{\pi}{2}c_s k}} \frac{e^{ic_s k \eta}}{\eta^{\frac{1}{2}+\nu}} + C_2 \frac{e^{i\frac{\pi}{4}(1-2\nu)}}{\sqrt{\frac{\pi}{2}c_s k}} \frac{e^{-ic_s k \eta}}{\eta^{\frac{1}{2}+\nu}}, \quad (8.19)$$

and

$$\mathcal{R}_{k \gg \mathcal{H}} \sim C_{1,\text{NMC}} k^2 \frac{e^{i\sqrt{\frac{2}{3}}k\eta}}{\eta} + C_{2,\text{NMC}} k^2 \frac{e^{-i\sqrt{\frac{2}{3}}k\eta}}{\eta} + C_{(3,4),\text{NMC}} \frac{e^{\pm k\eta}}{\eta}, \quad (8.20)$$

where the last term depends on whether η is positive or negative. To preserve the validity of the linear approximation in the sub-horizon regime, we set $C_{3,\text{NMC}} = 0$ in an expanding Universe and $C_{4,\text{NMC}} = 0$ in an eventual contracting phase. In both cases, the solutions have oscillations modulated by a time-dependent amplitude. As in the super-horizon regime, in the GR case the amplitude decays in an expanding Universe as $\eta^{-\frac{1+2\nu}{2}} \propto a^{-1}$, while in the NMC case it instead grows as $\eta^{-1} \propto a^{1/2}$.

8.1.2 Vector perturbations

For vector perturbations, gauge-invariant variables are defined as

$$\Psi_i := S_i + F'_i, \quad V_{n,i} := U_{n,i} - F'_i, \quad (8.21)$$

and the equations are equivalent to those obtained in the gauge $F_i = 0$. The non-vanishing components of the Einstein equations, namely the $0i$ and ij ones, are

$$\begin{aligned} \left[(1 + 2\chi) \mathcal{H}^2 - (1 - \chi) \mathcal{H}' \right] \Psi_i - \frac{1}{2} \chi \left(4\mathcal{H}' - \mathcal{H}^2 \right) V_{\text{DM},i} - \frac{1}{4} (1 - 3\chi) \nabla^2 \Psi_i - \frac{1}{2} \chi \nabla^2 V_{\text{DM},i} = \\ = 4\pi G a^2 \sum_n (1 + w_n) \rho_n V_{n,i}, \\ (1 - \chi) \left(D_i \Psi_j' + D_j \Psi_i' \right) + (2 + \chi) \mathcal{H} (D_i \Psi_j + D_j \Psi_i) + 2\chi \left(D_i V_{\text{DM},j}' + D_j V_{\text{DM},i}' \right) + \\ - 2\chi \mathcal{H} (D_i V_{\text{DM},j} + D_j V_{\text{DM},i}) = 0. \end{aligned} \quad (8.22)$$

The ii components take a particularly simple form:

$$D_i \left[(1 - \chi) \Psi_i' + (2 + \chi) \mathcal{H} \Psi_i + 2\chi V_{\text{DM},i}' - 2\chi \mathcal{H} V_{\text{DM},i} \right] = 0. \quad (8.23)$$

The conservation of the minimally coupled stress-energy tensors reads

$$(1 + w_m) \left[V_{m,i}' - \Psi_i' + (1 - 3w_m) \mathcal{H} (V_{m,i} - \Psi_i) \right] = 0, \quad (8.24)$$

where $m = \text{B, R, } \Lambda$, which vanishes identically for a cosmological constant ($w_\Lambda = -1$), since it does not support vector perturbations (see equation (8.4)). Taking the divergence of the equations of motion and using the contracted Bianchi identities, one obtains a modified conservation equation for DM:

$$\begin{aligned} V_{\text{DM},i}' - \Psi_i' + \mathcal{H} (V_{\text{DM},i} - \Psi_i) - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} \left[\left(\mathcal{H}^2 - 4\mathcal{H}' \right) (V_{\text{DM},i}' - \Psi_i') + \right. \\ \left. - \left(4\mathcal{H}'' - 6\mathcal{H}'\mathcal{H} + \mathcal{H}^3 \right) (V_{\text{DM},i} - \Psi_i) + \nabla^2 \Psi_i' - \mathcal{H} \nabla^2 \Psi_i \right] = 0, \end{aligned} \quad (8.25)$$

The conservation equations (8.24) can be readily integrated to give

$$V_{m,i} = \frac{C_{m,i}}{a^{1-3w}} + \Psi_i, \quad (8.26)$$

and the modified conservation equation for dark matter can be integrated in Fourier space as

$$V_{\text{DM},i} = \frac{C_{\text{DM},i}}{a} \frac{1}{1 - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} (\mathcal{H}^2 - 4\mathcal{H}')} + \left(1 - k^2 \frac{\frac{\chi}{8\pi G a^2 \rho_{\text{DM}}}}{1 - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} (\mathcal{H}^2 - 4\mathcal{H}')} \right) \Psi_i. \quad (8.27)$$

Finally, using equation (8.27), the ii components of the Einstein equations (8.23) can be integrated to obtain

$$\Psi_i = -\frac{C_{\text{DM},i}}{a} \frac{\frac{2\chi}{1+\chi}}{1 - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} \left(\mathcal{H}^2 - 4\mathcal{H}' + \frac{2\chi}{1+\chi} k^2 \right)} + \frac{C_{\Psi,i}}{(1+\chi) a^2} \frac{1 - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} (\mathcal{H}^2 - 4\mathcal{H}')}{1 - \frac{\chi}{8\pi G a^2 \rho_{\text{DM}}} \left(\mathcal{H}^2 - 4\mathcal{H}' + \frac{2\chi}{1+\chi} k^2 \right)}, \quad (8.28)$$

while the $0i$ components impose the constraints on the integration constants

$$C_{\Psi,i} = \frac{16\pi G}{k^2} \sum_n (1 + w_n) \rho_{n,0} C_{n,i}. \quad (8.29)$$

These perturbations have a singular behavior when the denominator in equation (8.28) vanishes. This occurs at a scale factor

$$a_s \sim \frac{3\chi_0 k^2}{4\pi G \rho_{\text{R},0}} = 6 \frac{\rho_{\text{DM},0}}{\rho_{\text{R},0}} L^2 k^2. \quad (8.30)$$

While this singularity does not affect the global evolution of the perturbations, we restrict the analysis to modes generated after this point.

Even if a full analytical form is present, to study the evolution of vector perturbations it is instructive to consider separately the GR limit and the NMC-dominated phase. The perturbations of the minimally coupled fluids, described by equation (8.26), follow the same evolution as the gravitational perturbation Ψ_i , up to the addition of a constant or decaying mode; for this reason, we focus only on the evolution of the gravitational and dark matter perturbations. In the GR limit, they have the standard behavior

$$\Psi_i \sim \frac{C_{\Psi,i}}{a^2}, \quad V_{\text{DM},i} \sim \frac{C_{\text{DM},i}}{a} + \frac{C_{\Psi,i}}{a^2}. \quad (8.31)$$

In the NMC-dominated epoch, they instead evolve as

$$\begin{aligned} \Psi_i &\sim \frac{8\pi G \rho_{\text{DM},0} C_{\text{DM},i}}{\chi_0 k^2} a = \frac{C_{\text{DM},i}}{L^2 k^2} a, \\ V_{\text{DM},i} &\sim \frac{1}{2} \left(\frac{C_{\Psi,i}}{\chi_0} + \frac{8\pi G \rho_{\text{DM},0} C_{\text{DM},i}}{\chi_0 k^2} \right) a = \frac{1}{2} \left(\frac{C_{\Psi,i}}{\chi_0} + \frac{C_{\text{DM},i}}{L^2 k^2} \right) a. \end{aligned} \quad (8.32)$$

The key difference with respect to the standard case is therefore a growth proportional to the scale factor in the early Universe.

If we consider gravitational perturbations generated after the singular point in equation (8.28),

their amplitude at recombination scales as

$$\Psi_i(a_{\text{CMB}}) \propto \Psi_i(a_s) \frac{a_{\text{NMC}}}{a_s} \frac{a_{\text{NMC}}^2}{a_{\text{CMB}}^2} \sim \Psi_i(a_s) \frac{4\pi G \rho_{\text{R},0}}{3} \frac{1}{k^2} \frac{1}{a_{\text{CMB}}^2}, \quad (8.33)$$

where we used $a_{\text{NMC}} = (8\pi G L^2 \rho_{\text{DM},0})^{1/3}$ for which $\chi|_{a=a_{\text{NMC}}} = 1$. Perturbations are enhanced with respect to their initial value for wave numbers

$$k < \frac{1}{a_{\text{CMB}}} \sqrt{\frac{4\pi G \rho_{\text{R},0}}{3}} = (1 + z_{\text{CMB}}) \sqrt{\frac{\Omega_{\text{R},0}}{2}} H_0 \sim 1.65 \times 10^{-3} \text{Mpc}^{-1}. \quad (8.34)$$

If the initial perturbations are small (e.g. of quantum origin), all modes that are observationally relevant today remain of negligible amplitude.

8.1.3 Tensor perturbations

To derive the equations for tensor perturbations, it is convenient to express them as

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8.35)$$

The field equations take a particularly simple form by combining the $(1, 1)$ and $(2, 2)$ components. In this way, we obtain

$$(1 + \chi)h_+'' + (2 - \chi)\mathcal{H}h_+' + (1 - \chi)k^2 h_+ = 0, \quad (8.36)$$

and an analogous equation for $h_\times$. In the GR limit, these equations reduce to

$$h_{+/\times}'' + 2\mathcal{H}h_{+/\times}' + k^2 h_{+/\times} = 0. \quad (8.37)$$

Thus, the two polarization modes h_+ and $h_\times$ satisfy the same equation as the comoving curvature perturbation $\mathcal{R}$ in Equation (8.13), the only difference being that tensor perturbations travel at the speed of light $c_s^2 = 1$. The solution is therefore

$$h_{+/\times} = C_1^{+/\times} \frac{H_{-\nu}^{(1)}(k\eta)}{\eta^\nu} + C_2^{+/\times} \frac{H_{-\nu}^{(2)}(k\eta)}{\eta^\nu}, \quad (8.38)$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are Bessel functions of the first and second kind, $\nu = \frac{3}{2} \frac{1-w}{1+3w}$, and $C_1^{+/\times}$ and $C_2^{+/\times}$ are integration constants which depend on w .

On the other hand, in the NMC limit tensor perturbations satisfy

$$h''_{+/\times} - \mathcal{H} h'_{+/\times} - k^2 h_{+/\times} = 0, \quad (8.39)$$

which can be solved immediately, once we recall that in the NMC-dominated epoch $\mathcal{H} = -2/\eta$, to give

$$h_{+/\times} = C_{1,\text{NMC}}^{+/\times} \frac{e^{k\eta}}{\eta} + C_{2,\text{NMC}}^{+/\times} \frac{e^{-k\eta}}{\eta}, \quad (8.40)$$

where $C_{1,\text{NMC}}^{+/\times}$ and $C_{2,\text{NMC}}^{+/\times}$ are the integration constants during the NMC epoch. The absence of oscillatory sub-horizon modes indicates that a standard adiabatic vacuum cannot be defined during the NMC-dominated epoch; consequently, tensor perturbations cannot be generated from vacuum quantum fluctuations, but can only correspond to the classical evolution of pre-existing perturbations.

In the super-horizon regime, the solutions in Equations (8.38) and (8.40) reduce respectively to

$$h_{+/\times}^{k \ll \mathcal{H}} \sim \frac{i\Gamma(-\nu) \left(\frac{1}{2}k\right)^\nu (C_2^{+/\times} - C_1^{+/\times})}{\pi} + \frac{C_1^{+/\times} + C_2^{+/\times} + i \cot(\pi\nu) (C_2^{+/\times} - C_1^{+/\times})}{\Gamma(1-\nu) \left(\frac{1}{2}k\right)^\nu} \eta^{-2\nu}, \quad (8.41)$$

and

$$h_{+/\times}^{k \ll \mathcal{H}} \sim \frac{1}{2} (C_{1,\text{NMC}}^{+/\times} - C_{2,\text{NMC}}^{+/\times}) + (C_{1,\text{NMC}}^{+/\times} + C_{2,\text{NMC}}^{+/\times}) \eta^{-1}, \quad (8.42)$$

which evolve precisely as scalar perturbations with speed $c_s^2 = 1$. In the deep sub-horizon regime, Equations (8.38) and (8.40) reduce to

$$h_{+/\times}^{k \gg \mathcal{H}} \sim C_1^{+/\times} \frac{e^{-i\frac{\pi}{4}(1+2\nu)}}{\sqrt{\frac{\pi}{2}k}} \frac{e^{ik\eta}}{\eta^{\frac{1}{2}-\nu}} + C_2^{+/\times} \frac{e^{i\frac{\pi}{4}(1+2\nu)}}{\sqrt{\frac{\pi}{2}k}} \frac{e^{-ik\eta}}{\eta^{\frac{1}{2}-\nu}}, \quad (8.43)$$

and

$$h_{+/\times}^{k \gg \mathcal{H}} \sim C_{(1,2),\text{NMC}}^{+/\times} \frac{e^{\pm k\eta}}{\eta}. \quad (8.44)$$

Also in this case they evolve as scalar perturbations, with the notable difference that requiring the validity of the linear approximation sets $h_{+/\times}^{k \gg \mathcal{H}} \sim 0$ in the NMC-dominated epoch, and no primordial gravitational waves are generated in the early Universe.

8.2 Generation of perturbations and primordial power spectrum

The solutions derived in the previous section provide the necessary ingredients to study the generation of primordial perturbations. These perturbations set the initial conditions for cosmological anisotropies and can therefore be constrained through observations of the CMB and large-scale structure.

In the standard inflationary scenario, primordial perturbations originate from quantum fluctuations generated in the early Universe. During the accelerated expansion phase, the comoving horizon shrinks, so that fluctuations initially inside the horizon are stretched to super-horizon scales, where they effectively freeze. After the accelerated phase ends, they re-enter the horizon and source the observed anisotropies.

At the practical level, introducing the Mukhanov variable $u = a\mathcal{R}$ [294], the action at second order in perturbation theory reduces to that of a scalar field in flat spacetime with a time-dependent effective mass. The quantization of this scalar field fixes the normalization of the Fourier modes once an initial state is specified, usually taken to be the Bunch–Davies vacuum [295]. On super-horizon scales the comoving curvature perturbation $\mathcal{R}$ remains constant, implying $u \propto a$. Modes with larger wavelengths exit the horizon earlier and therefore undergo a longer amplification. Since horizon crossing is defined by $k \sim \mathcal{H}$, this translates into a characteristic dependence of the perturbation amplitude on the wavenumber k .

In the next subsections we adapt this procedure to our model, first introducing variables suitable for quantization and then following their evolution up to the epoch probed by the CMB. In the first subsection we consider the case in which primordial fluctuations are generated during the accelerated expansion phase. Once suitable variables are identified, the calculation is relatively straightforward, but the resulting predictions are incompatible with observations. In the second subsection we consider the case in which primordial fluctuations are generated during a contracting phase in a bouncing scenario. The evolution is considerably more involved, but we find that it is possible to obtain a power spectrum compatible with current observations. This requires an excited initial state rather than the Bunch–Davies vacuum. While such a choice does not appear unreasonable in a matter or radiation dominated contracting Universe, it substantially complicates the construction of the primordial spectrum.

The quantity directly related to observations is the power spectrum of perturbations. The power spectrum $\mathcal{P}_p(k)$ of a quantity $p(\mathbf{x}, t)$ is defined as

$$\int_0^\infty \frac{dk}{k} \mathcal{P}_p(k) := \langle 0 | p(\mathbf{x}, t) p(\mathbf{x} + \Delta\mathbf{x}, t) | 0 \rangle = \int_0^\infty \frac{d^3k}{(2\pi)^3} |p_{\mathbf{k}}|^2 \quad \Longrightarrow \quad \mathcal{P}_p(k) := \frac{k^3}{(2\pi)^2} |p_{\mathbf{k}}|^2, \quad (8.45)$$

where the $p_{\mathbf{k}}$ are the Fourier modes with wavenumber $\mathbf{k}$. The main cosmological observables

are the scalar spectral index

$$n_s := 1 + \frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k}, \quad (8.46)$$

where $\mathcal{P}_{\mathcal{R}}(k)$ is the power spectrum of comoving curvature perturbations, and the tensor-to-scalar ratio

$$r := \frac{\mathcal{P}_h}{\mathcal{P}_{\mathcal{R}}}, \quad (8.47)$$

where $\mathcal{P}_h$ is the power spectrum of tensor perturbations $h_{+/\times}$. The constraints from Planck collaboration are $n_s = 0.9665 \pm 0.0038$ and $r < 0.036$, together with negligible vector perturbations [277].

8.2.1 Power spectrum within inflationary scenario

We first consider the case in which primordial fluctuations are generated during the accelerated expansion phase, so that the NMC-dominated epoch effectively plays the role of an inflationary era. In standard cosmological perturbation theory, the Mukhanov variable is derived from the action at second order in perturbations, which identifies the canonical degree of freedom. Deriving the full quadratic action for the present non-minimally coupled fluid system is technically involved and lies beyond the scope of this work. We therefore adopt a more pragmatic approach and work directly with the equations of motion, introducing a field redefinition that recasts the perturbation equation into a Mukhanov–Sasaki form. While this procedure does not by itself establish the canonical structure of the theory, it reproduces the correct dynamical equation, and we therefore adopt it as the effective canonical degree of freedom for quantization, in analogy with the standard construction.

Defining the variable

$$u = \frac{1}{6a} \left| 1 - \frac{4}{3} \frac{k^2}{\mathcal{H}^2} \right|^{-1/2} \left(\mathcal{R} - \frac{\mathcal{H}}{4} v_{\text{DM}} \right), \quad (8.48)$$

(8.15) reduces to

$$u'' + \left\{ \frac{2}{3} k^2 - \left[\frac{1}{2} + \left(\frac{\frac{k}{\mathcal{H}}}{\frac{4}{3} \frac{k^2}{\mathcal{H}^2} - 1} \right)^2 \right] \mathcal{H}^2 \right\} u = 0, \quad (8.49)$$

which has the form of a Klein–Gordon equation for modes of wavenumber $\tilde{k} = \sqrt{\frac{2}{3}} k$ with a time-dependent effective mass. Fluctuations are in the quantum regime when they are deeply sub-horizon, where the equation simplifies and admits the solution

$$u'' + \frac{2}{3} k^2 u = 0, \quad \implies \quad u = \alpha_k e^{-i\sqrt{\frac{2}{3}} k \eta} + \beta_k e^{i\sqrt{\frac{2}{3}} k \eta}. \quad (8.50)$$

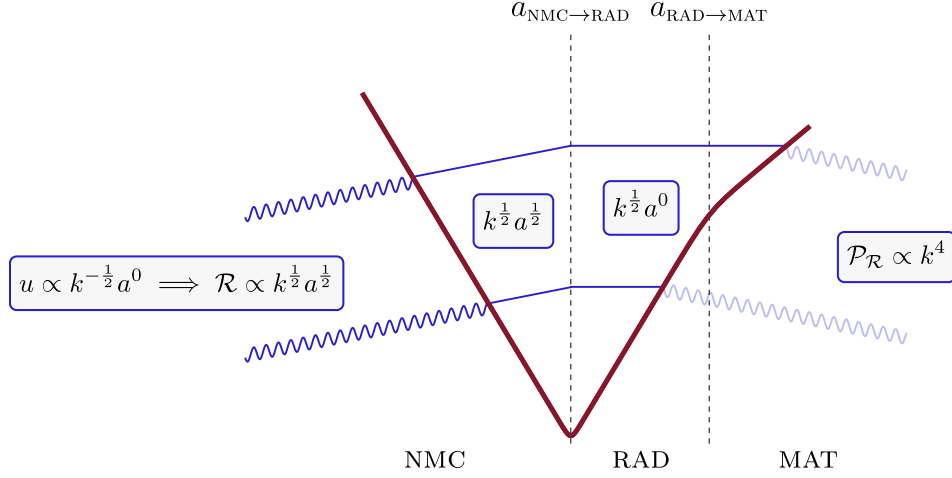


Figure 8.1. Pictorial representation of the evolution of perturbations generated during an inflationary era, showing their dependence on the wavenumber k and on the scale factor a . All modes originate from a quantum state normalized as $k^{-1/2}$, which implies a $k^{1/2}$ -dependence for the comoving curvature perturbation. The right-hand side of the figure shows the resulting k -dependence of the power spectrum.

Imposing the Bunch–Davies vacuum selects only positive frequencies, and thus fixes the normalization as

$$\alpha_k = \left(\sqrt{\frac{2}{3}} k \right)^{-1/2}, \quad \beta_k = 0. \quad (8.51)$$

Using the sub-horizon limit of the comoving curvature perturbation (8.20), the sub-horizon limit of definition (8.48), and the solution for the velocity field (8.27), we obtain the corresponding initial conditions for $\mathcal{R}$:

$$C_{1,\text{NMC}} = \frac{8\sqrt{3}}{5k} \beta_k = 0, \quad C_{2,\text{NMC}} = \frac{8\sqrt{3}}{5k} \alpha_k = \frac{4 \cdot 6^{3/4}}{5} \frac{1}{k^{3/2}}. \quad (8.52)$$

When perturbations exit the horizon, the growing mode in (8.18) rapidly dominates over the constant one, and by the end of the NMC-dominated epoch their amplitude is

$$\mathcal{R} \sim k^2 C_{2,\text{NMC}} \eta_{\text{NMC}}^{-1} = \frac{4 \cdot 6^{3/4}}{5 \eta_{\text{NMC}}} k^{1/2}, \quad (8.53)$$

as illustrated in figure 8.1.

At later times the evolution follows GR, where the constant mode dominates, so that perturbations re-enter the horizon with amplitude (8.53). The corresponding power spectrum

at horizon entry is

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{(2\pi)^2} |\mathcal{R}|^2 \sim \frac{24\sqrt{6}}{(5\pi\eta_{\text{NMC}})^2} k^4. \quad (8.54)$$

The scalar spectral index is therefore $n_s = 5$, in clear tension with observations. Perturbations generated during the accelerated phase cannot account for the observed cosmological fluctuations, and alternative scenarios must be considered.

8.2.2 Power spectrum within bouncing scenario

An alternative to generating primordial fluctuations during the accelerated phase is that they originate during a contracting phase in a bouncing cosmology. In standard models of primordial perturbations in bouncing cosmologies an asymmetric bounce is normally required. Indeed, in order to provide the observed nearly scale-invariant spectrum, the pre-bounce Universe has to be matter dominated while becoming radiation dominated immediately after the bounce. This is because, perturbations generated deep inside the horizon are normalized by imposing the Bunch–Davies vacuum, $u \propto k^{-1/2}$. As they exit the horizon during contraction, the time-dependent mode grows and dominates over the constant one, leading to an amplification by a factor k^{-1} and hence to a scale-invariant power spectrum at horizon re-entry after the bounce [287, 296].

Interestingly, in our case, no change in the matter content of the pre-bounce Universe is required, since the model itself predicts a symmetric cosmological bounce for negative spatial curvature. Nonetheless, as we will show, obtaining an almost scale-invariant spectrum requires a different (albeit we shall argue not unnatural) tuning of initial conditions with respect to the Bunch–Davies vacuum.

In order to make the above discussion quantitative, let us introduce the aforementioned Mukhanov variable $u = a\mathcal{R}$. As anticipated, by doing so the first equation in (8.13) takes the form of a Klein–Gordon equation for modes of wavenumber $\tilde{k} = c_s k$ with a time-dependent effective mass. Explicitly

$$u'' + \left(c_s^2 k^2 - \frac{1}{2} (1 - 3w) \mathcal{H}^2 \right) u = 0. \quad (8.55)$$

To study the quantum regime, we consider the deep sub-horizon limit, in which the equation simplifies and admits the solution

$$u'' + c_s^2 k^2 u = 0 \quad \implies \quad u = \alpha_k e^{-ic_s k \eta} + \beta_k e^{ic_s k \eta}. \quad (8.56)$$

Now, it is worth noticing that, during its pre-bounce contracting phase, the Universe was probably already filled by matter and radiation, so that the Bunch–Davies vacuum may not be its more appropriate description. While it is quite difficult to “guess” an alternative, non-vacuum,

initial condition, we can still “reverse-engineer” checking which ones would have lead in our bouncing scenario to the observed quasi scale-invariant spectrum.

In order to construct suitable initial conditions, we recall that standard commutation relations imply

$$u\bar{u}' - u'\bar{u} = 2i \quad \implies \quad |\alpha_k|^2 - |\beta_k|^2 = \frac{1}{c_s k}. \quad (8.57)$$

Comparing the sub-horizon solution for the comoving curvature perturbation in equation (8.19) with the solution for the Mukhanov variable in equation (8.56), this condition translates into the following constraint on the integration constants of $\mathcal{R}$:

$$|C_2|^2 - |C_1|^2 = \frac{\pi}{2}. \quad (8.58)$$

This relation fixes one combination of the two integration constants, leaving a single free parameter once the coefficients are taken to be real. Rather than working directly with C_1 and C_2 , it is convenient to parametrize the remaining freedom by introducing a characteristic scale in momentum space, $\bar{k}^{(s)}$, defined as the wavenumber for which the amplitudes of the constant and growing modes are equal at $\eta = \bar{\eta}$, which corresponds to

$$\frac{|C_1 + C_2|}{|C_1 - C_2|} = \left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|. \quad (8.59)$$

With this definition, modes with $k < \bar{k}^{(s)}$ reach $\eta = \bar{\eta}$ dominated by the growing mode, whereas modes with $k > \bar{k}^{(s)}$ dominated by the constant mode.

Combining equations (8.58) and (8.59), the integration constants can be written as

$$\begin{aligned} C_1 &= \mp \sqrt{\frac{\pi}{8}} \frac{1 - \left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|}{\left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|^{1/2}}, \\ C_2 &= \pm \sqrt{\frac{\pi}{8}} \frac{1 + \left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|}{\left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|^{1/2}}. \end{aligned} \quad (8.60)$$

These conditions correspond to an initial state of the Mukhanov variable with occupation number

$$n_k = |\beta_k|^2 = \frac{|C_1|^2}{\frac{\pi}{2} c_s k} = \frac{\left(1 - \left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right| \right)^2}{4 \left| \frac{\Gamma(1-\nu)\Gamma(-\nu)}{2^{2\nu}\pi} \left(c_s \bar{k}^{(s)} \bar{\eta} \right)^{2\nu} + \cot(\pi\nu) \right|}. \quad (8.61)$$

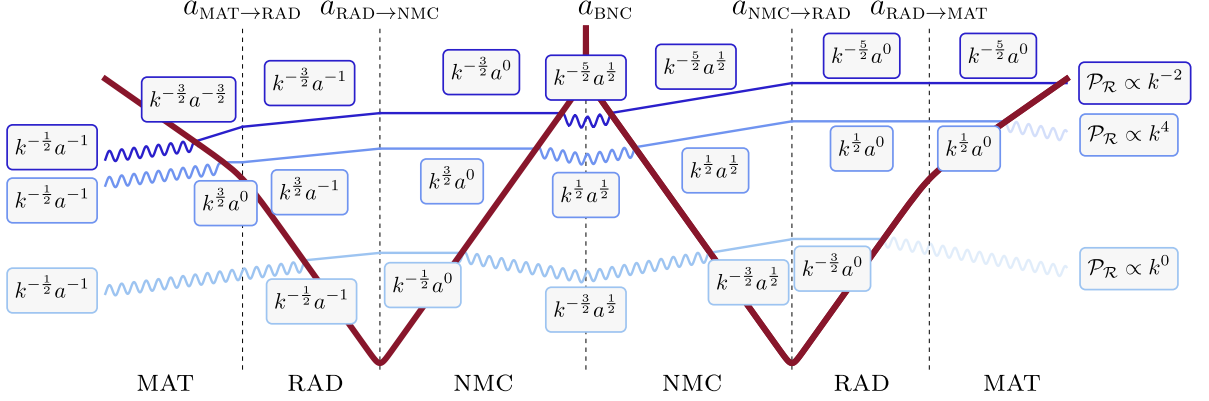


Figure 8.2. Pictorial representation of the evolution of perturbations generated during a contracting phase in a bouncing cosmology, showing their dependence on the wavenumber k and on the scale factor a . All modes originate from a quantum state normalized as $k^{-1/2}$. The upper mode in the figure exits the horizon during matter domination and reaches matter-radiation equality with the growing mode dominant. The central mode also exits the horizon during matter domination, but reaches matter-radiation equality with the constant mode dominant. The lower mode exits the horizon during radiation domination. The right-hand side of the figure shows the resulting k -dependence of the power spectrum.

The Bunch–Davies vacuum is recovered for the particular choice

$$\bar{k}_{\text{BD}}^{(s)} = \frac{2}{c_s \bar{\eta}} \left(\frac{\pi (\pm 1 - \cot(\pi \nu))}{\Gamma(1 - \nu) \Gamma(-\nu)} \right)^{1/2\nu}, \quad (8.62)$$

for which $C_1 = 0$ and $C_2 = \sqrt{\pi/2}$, and therefore $n_k = 0$. Values of $\bar{k}^{(s)}$ either smaller or larger than $\bar{k}_{\text{BD}}^{(s)}$ correspond to excited states.

We note that adopting analogous initial conditions in the inflationary scenario, so as to keep the constant mode of equation (8.18) at horizon exit, would introduce an additional factor of k in the comoving curvature perturbation. The resulting scalar spectral index would then be $n_s = 7$, making the predictions even more incompatible with observations.

With this freedom in the choice of initial conditions, we can now follow the evolution of perturbations up to the CMB and derive a prediction for the primordial power spectrum. However, as shown in figure 8.2, the evolution of perturbations in the bouncing scenario is considerably more involved than in the inflationary case. It is convenient to separate the evolution into different stages and discuss each of them independently.

Contraction phase during matter domination. The evolution of perturbations in a matter-dominated Universe requires some care, since the solution in equation (8.14) becomes singular in the limit $c_s \rightarrow 0$, or identically constant if one sets $C_1 = -C_2$ to avoid the divergence.

A useful approximation is obtained by taking the limit $\omega \rightarrow 0$, while keeping c_s finite. The full solution then reads

$$\mathcal{R} = -C_{1,\text{MAT}} \sqrt{\frac{2}{\pi c_s^3 k^3}} (1 - i c_s k \eta) \frac{e^{i c_s k \eta}}{\eta^3} - C_{2,\text{MAT}} \sqrt{\frac{2}{\pi c_s^3 k^3}} (1 + i c_s k \eta) \frac{e^{-i c_s k \eta}}{\eta^3}, \quad (8.63)$$

where, during matter domination, $a \propto \eta^2$, so that $|\eta| \propto \sqrt{a}$ up to a normalization fixed by the background solution, and we made explicit that the integration constants $C_{1,\text{MAT}}$ and $C_{2,\text{MAT}}$ refer to the matter-dominated epoch. In the super-horizon limit it reduces to

$$\mathcal{R}_{k \ll \mathcal{H}} \sim \frac{i}{3} \sqrt{\frac{2}{\pi}} (c_s k)^{3/2} (C_{2,\text{MAT}} - C_{1,\text{MAT}}) - \sqrt{\frac{2}{\pi}} \frac{C_{1,\text{MAT}} + C_{2,\text{MAT}}}{(c_s k)^{3/2}} a^{-3/2}. \quad (8.64)$$

We choose as initial conditions

$$C_{1,\text{MAT}} = \mp \sqrt{\frac{\pi}{8}} \frac{1 - \left(\frac{c_s}{\sqrt[3]{3}} \bar{k}_{\text{MAT}}^{(s)} \sqrt{a_{\text{MRE}}} \right)^3}{\left(\frac{c_s}{\sqrt[3]{3}} \bar{k}_{\text{MAT}}^{(s)} \sqrt{a_{\text{MRE}}} \right)^{3/2}}, \quad C_{2,\text{MAT}} = \pm \sqrt{\frac{\pi}{8}} \frac{1 + \left(\frac{c_s}{\sqrt[3]{3}} \bar{k}_{\text{MAT}}^{(s)} \sqrt{a_{\text{MRE}}} \right)^3}{\left(\frac{c_s}{\sqrt[3]{3}} \bar{k}_{\text{MAT}}^{(s)} \sqrt{a_{\text{MRE}}} \right)^{3/2}}, \quad (8.65)$$

so that modes with $k < \bar{k}_{\text{MAT}}^{(s)}$ reach matter–radiation equality dominated by the growing mode, while modes with $k > \bar{k}_{\text{MAT}}^{(s)}$ dominated by the constant mode. Allowing for such an intermediate scale $\bar{k}_{\text{MAT}}^{(s)}$, comoving curvature perturbations at the end of this stage scale as

$$\mathcal{R}_{\text{MAT} \rightarrow \text{RAD}} \propto \begin{cases} k^{-3/2} & \text{for } k < \bar{k}_{\text{MAT}}^{(s)} \\ k^{3/2} & \text{for } \bar{k}_{\text{MAT}}^{(s)} < k < k_{\text{MRE}}, \end{cases} \quad (8.66)$$

where $k_{\text{MRE}} \sim \mathcal{H}_{\text{MRE}} \sim 1.04 \times 10^{-2} \text{ Mpc}^{-1}$, is the perturbation’s wavenumber at matter-radiation equality.

Contraction phase during radiation domination. The evolution of perturbations in a radiation-dominated Universe is particularly simple, since the full solution in equation (8.14), with $c_s^2 \sim \omega \sim 1/3$, reduces to

$$\mathcal{R} = \frac{C_{1,\text{RAD}}}{\sqrt{\frac{\pi}{2} \frac{1}{\sqrt{3}} k}} \frac{e^{i \frac{1}{3} k \eta}}{\eta} + \frac{C_{2,\text{RAD}}}{\sqrt{\frac{\pi}{2} \frac{1}{\sqrt{3}} k}} \frac{e^{-i \frac{1}{3} k \eta}}{\eta}, \quad (8.67)$$

where, during radiation domination, $a \sim \eta$ up to a normalization fixed by the background solution. In the super-horizon limit it reduces to

$$\mathcal{R}_{k \ll \mathcal{H}} \sim i \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{\sqrt{3}}} k (C_{2,\text{RAD}} - C_{1,\text{RAD}}) + \sqrt{\frac{2}{\pi}} \frac{C_{1,\text{RAD}} + C_{2,\text{RAD}}}{\sqrt{\frac{1}{\sqrt{3}}}} a^{-1}. \quad (8.68)$$

Modes that are already super-horizon are rescaled by the common factor $\eta_{\text{MRE}}/\eta_{\text{NMC}} = a_{\text{MRE}}/a_{\text{NMC}}$. For modes exiting the horizon during this phase, we fix the integration constants as

$$C_{1,\text{RAD}} \sim \mp \sqrt{\frac{\pi}{8}} \frac{1 - \frac{1}{\sqrt{3}} \bar{k}_{\text{RAD}}^{(s)} a_{\text{NMC}}}{\left(\frac{1}{\sqrt{3}} \bar{k}_{\text{RAD}}^{(s)} a_{\text{NMC}}\right)^{1/2}}, \quad C_{2,\text{RAD}} \sim \pm \sqrt{\frac{\pi}{8}} \frac{1 + \frac{1}{\sqrt{3}} \bar{k}_{\text{RAD}}^{(s)} a_{\text{NMC}}}{\left(\frac{1}{\sqrt{3}} \bar{k}_{\text{RAD}}^{(s)} a_{\text{NMC}}\right)^{1/2}}, \quad (8.69)$$

so that perturbations with $k < \bar{k}_{\text{RAD}}^{(s)}$ reach the onset of the non-minimal coupling dominated by the growing mode, while those with $k > \bar{k}_{\text{RAD}}^{(s)}$ dominated by the constant mode. To remain consistent with observational constraints, however, only the growing modes should be observed, and thus we choose $\bar{k}_{\text{RAD}}^{(s)} > \max(k_{\text{obs}})$. These modes do not acquire additional k -dependence since they scale as $\eta^{-1} = a^{-1}$ both inside and outside the horizon. At the end of this stage, comoving curvature perturbations behave as

$$\mathcal{R}_{\text{RAD} \rightarrow \text{NMC}} \propto \begin{cases} k^{-3/2} & \text{for } k < \bar{k}_{\text{MAT}}^{(s)} \\ k^{3/2} & \text{for } \bar{k}_{\text{MAT}}^{(s)} < k < k_{\text{MRE}} \\ k^{-1/2} & \text{for } k > k_{\text{MRE}}. \end{cases} \quad (8.70)$$

Contraction phase during NMC domination. When the Universe reaches the NMC-dominated phase, all relevant modes are already super-horizon. Their evolution is described by equation (8.18), which contains a constant mode and a time-dependent one scaling as $\eta^{-1} = a^{1/2}$. The latter rapidly decreases and becomes subdominant. Comparing the super-horizon behavior in equation (8.18) with the sub-horizon one in equation (8.20), one finds that perturbations are amplified by a factor k^{-1} . At the bounce, comoving curvature perturbations therefore scale as

$$\mathcal{R}_{\text{BNC}} \propto \begin{cases} k^{-5/2} & \text{for } k < \bar{k}_{\text{MAT}}^{(s)} \\ k^{1/2} & \text{for } \bar{k}_{\text{MAT}}^{(s)} < k < k_{\text{MRE}} \\ k^{-3/2} & \text{for } k > k_{\text{MRE}}. \end{cases} \quad (8.71)$$

Expansion phase during NMC domination. In the sub-horizon regime, perturbations evolve as in equation (8.20), namely as oscillations modulated by a factor $\eta^{-1} = a^{1/2}$. In a symmetric bounce scenario, each mode exits the horizon after the bounce at the same value of

the scale factor at which it entered before the bounce, with a small common enhancement, as shown in Appendix E. No additional k -dependence is generated at horizon exit, since the time-dependent mode dominates due to its scaling as $\eta^{-1} = a^{1/2}$. Comoving curvature perturbations therefore reach the end of the NMC-dominated stage with

$$\mathcal{R}_{\text{NMC} \rightarrow \text{RAD}} \propto \begin{cases} k^{-5/2} & \text{for } k < \bar{k}_{\text{MAT}}^{(s)} \\ k^{1/2} & \text{for } \bar{k}_{\text{MAT}}^{(s)} < k < k_{\text{MRE}} \\ k^{-3/2} & \text{for } k > k_{\text{MRE}}. \end{cases} \quad (8.72)$$

Expansion phase during radiation and matter dominations. Finally, once the NMC becomes negligible, the evolution is again described by GR. In an expanding Universe, the constant mode of super-horizon perturbations dominates during both radiation and matter domination, so comoving curvature perturbations re-enter the horizon with the amplitude shown in equation (8.72). Evaluating the power spectrum at horizon entry then gives

$$\mathcal{P}_{\mathcal{R}}|_{\text{H-entry}} \propto \begin{cases} k^{-2} & \text{for } k < \bar{k}_{\text{MAT}}^{(s)} \\ k^4 & \text{for } \bar{k}_{\text{MAT}}^{(s)} < k < k_{\text{MRE}} \\ k^0 & \text{for } k > k_{\text{MRE}}. \end{cases} \quad (8.73)$$

This clarifies the role of the intermediate scale $\bar{k}_{\text{MAT}}^{(s)}$. If we keep only the growing modes, as is usually done in cosmology, the k^{-2} branch, combined with the integrated Sachs–Wolfe effect, would produce an excessive enhancement of power at large scales, in tension with observations. The k^4 branch, on the other hand, is partially compensated by the same effect.

Before confronting the resulting scalar spectrum with CMB temperature anisotropies, it is useful to consider the tensor sector. In particular, the observational bound on the tensor-to-scalar ratio already provides a constraint on $\bar{k}_{\text{MAT}}^{(s)}$.

Tensor-to-scalar ratio. The evolution of tensor perturbations in the GR limit is nearly identical to that of comoving curvature perturbations, with the only difference that tensor modes propagate at the speed of light, $c_s = 1$. During the NMC-dominated phase of the contracting Universe, however, the solution contains an exponentially growing mode. To preserve the validity of the linear approximation, this mode must be removed by setting $C_{2,\text{NMC}}^{+,\times} = 0$ in equation (8.39). The remaining mode is exponentially suppressed during contraction, so that tensor perturbations reach the bounce with negligible amplitude. After the bounce, the same solution becomes exponentially growing. In a symmetric bounce scenario, this growth compensates for the suppression experienced during the contracting phase, and tensor perturbations therefore

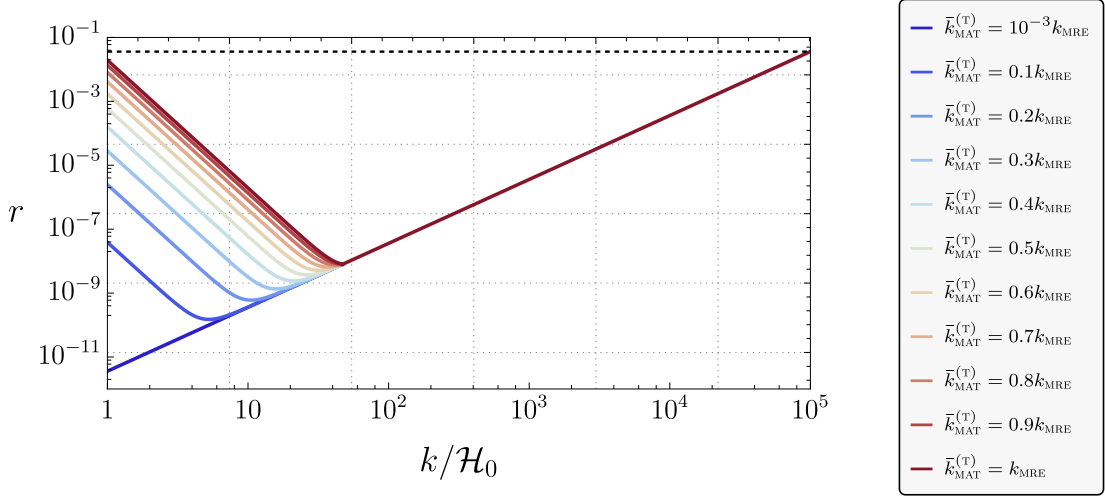


Figure 8.3. Possible shapes of the tensor-to-scalar ratio for different values of $\bar{k}_{\text{MAT}}^{(\text{T})}$, given in the legend as fractions of k_{MRE} . The parameter $\bar{k}_{\text{RAD}}^{(\text{T})}$ is fixed to $\max(k_{\text{obs}}) = 10^5$, while $\bar{k}_{\text{MAT}}^{(\text{s})}$ and $\bar{k}_{\text{RAD}}^{(\text{s})}$ are chosen to satisfy continuity and to make r saturate the Planck bound $r = 0.036$ at $k = \max(k_{\text{obs}})$, under the conditions $\bar{k}_{\text{MAT}}^{(\text{s})} < \min(k_{\text{obs}}) = 1$ and $\bar{k}_{\text{RAD}}^{(\text{s})} > \max(k_{\text{obs}}) = 10^5$.

emerge from the NMC-dominated epoch with approximately the same amplitude they had when entering it. Unlike comoving curvature perturbations, tensor modes do not acquire the additional k^{-1} enhancement generated during the NMC phase. Their power spectrum is therefore obtained from that of $\mathcal{R}$ by setting $c_s = 1$ and removing this extra factor:

$$\mathcal{P}_h \propto k^2 \mathcal{P}_{\mathcal{R}}|_{c_s=1}. \quad (8.74)$$

We parametrize the initial conditions for tensor perturbations in analogy with scalar perturbations in equations (8.65) and (8.69), introducing the corresponding intermediate scales $\bar{k}_{\text{MAT}}^{(\text{T})}$ and $\bar{k}_{\text{RAD}}^{(\text{T})}$. The tensor-to-scalar ratio then takes the form

$$r_{\text{MAT}} = k^2 \left(\frac{\bar{k}_{\text{MAT}}^{(\text{s})}}{\bar{k}_{\text{MAT}}^{(\text{T})}} \right)^3 \frac{k^6 + \left(\bar{k}_{\text{MAT}}^{(\text{T})} \right)^6}{k^6 + \left(\bar{k}_{\text{MAT}}^{(\text{s})} \right)^6}, \quad r_{\text{RAD}} = k^2 \frac{\bar{k}_{\text{RAD}}^{(\text{s})}}{\bar{k}_{\text{RAD}}^{(\text{T})}} \frac{k^2 + \left(\bar{k}_{\text{RAD}}^{(\text{T})} \right)^2}{k^2 + \left(\bar{k}_{\text{RAD}}^{(\text{s})} \right)^2}, \quad (8.75)$$

during matter and radiation domination, respectively, and where k is in units of $\mathcal{H}_0$.

At $k = \bar{k}_{\text{MAT}}^{(\text{s})}$, the tensor-to-scalar ratio satisfies the Planck bound $r < r_{\text{PL}} = 0.036$ only if $\bar{k}_{\text{MAT}}^{(\text{s})} < \sqrt{r_{\text{PL}}} \sim 0.190$, which is smaller than the wavenumber of the largest observable mode, $\min(k_{\text{obs}}) \sim 1$ in these units. Since we also require $\bar{k}_{\text{RAD}}^{(\text{s})} > \max(k_{\text{obs}})$, the tensor-to-scalar ratio

scales as

$$r_{\text{MAT}} \propto \begin{cases} k^{-4}, & \text{for } k < \bar{k}_{\text{MAT}}^{(\text{T})} \\ k^2, & \text{for } \bar{k}_{\text{MAT}}^{(\text{T})} < k < k_{\text{MRE}} \end{cases}, \quad r_{\text{RAD}} \propto \begin{cases} k^2, & \text{for } k_{\text{MRE}} < k < \bar{k}_{\text{RAD}}^{(\text{T})} \\ k^4, & \text{for } k > \bar{k}_{\text{RAD}}^{(\text{T})} \end{cases}. \quad (8.76)$$

Requiring that the tensor-to-scalar ratio satisfy the Planck bound over the whole observable range places only weak constraints on $\bar{k}_{\text{MAT}}^{(\text{T})}$ and $\bar{k}_{\text{RAD}}^{(\text{T})}$. As a result, the precise shape of the tensor-to-scalar ratio is not a robust prediction of the model. Nevertheless, if $\bar{k}_{\text{RAD}}^{(\text{T})}$ is chosen such that r saturates the observational bound at the largest observable scales, and continuity is imposed, the tensor-to-scalar ratio as a function of the wavenumber k resembles the examples shown in figure 8.3: in the radiation-dominated branch, r always increases with k , while in the matter-dominated branch it can either increase—leading to very small values on large scales—or decrease, making it potentially accessible to future observations.

A tensor-to-scalar ratio that could be detected by future experiments, while remaining close to the current upper bound on large scales, requires $\bar{k}_{\text{MAT}}^{(\text{s})} < \sqrt{r_{\text{PL}}}$, a larger $\bar{k}_{\text{MAT}}^{(\text{T})} \sim k_{\text{MRE}}$, and large $\bar{k}_{\text{RAD}}^{(\text{s})}$, $\bar{k}_{\text{RAD}}^{(\text{T})} > \max(k_{\text{obs}})$. This implies that perturbations start in highly excited states, with scalar perturbations having a larger occupation number than tensor perturbations.

Fit of Planck TT-power spectrum. We use the matter power spectrum in equation (8.73) to compute the CMB temperature angular power spectrum using the Weinberg’s [297, 298] semi-analytic approximation for the Fourier coefficients. Note that such a formula is known to have same limitations with respect to the full numerical solutions of the Boltzmann equation for the photon-baryon fluid: it slightly underestimates the first peak, slightly overestimates the third peak, and slightly under-damp the high- ℓ tail; however, these deviations are rather small, limited to $\lesssim 10\%$ for the peak amplitudes and to $\lesssim 1\%$ for the peak positions. Since it is quite complex to implement in the present Boltzmann solvers (e.g., CAMB or CLASS) the peculiarities of the NMC evolution, and since the piece-wise powerlaw description of the input NMC matter power spectrum is anyhow very crude, for a first semi-quantitative assessment we can confidently rely on the Weinberg’s approximation. Choosing a $\bar{k}_{\text{MAT}}^{(\text{s})} < \min(k_{\text{obs}})$, we obtain the spectrum shown in Figure 8.4. Overall, the agreement of our result with the Planck [253] data is encouraging, and could motivate further, more precise analysis.

8.3 Summary and conclusions

In this chapter, we studied the generation of primordial cosmological perturbations in a Universe where dark matter is non-minimally coupled to gravity at the fluid level.

At the background level, the non-minimal coupling drives a phase of accelerated expansion

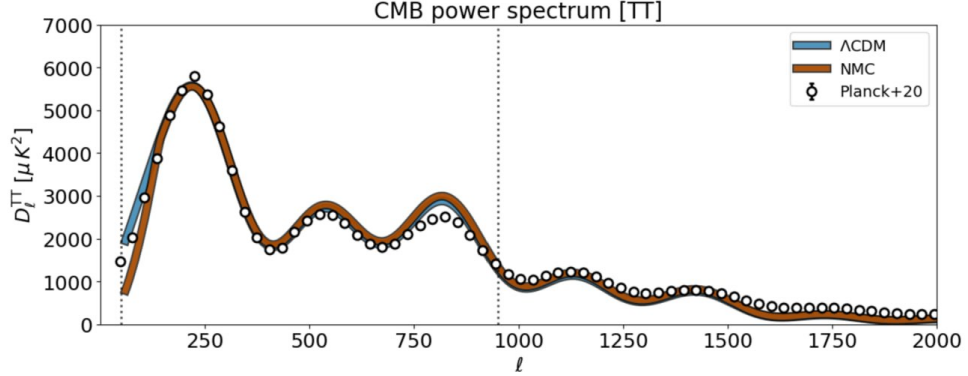


Figure 8.4. Fit of Planck data with our prediction for the primordial power spectrum.

at early times, lasting long enough to solve the horizon and flatness problems. At even earlier times, for negative spatial curvature, the initial singularity can be avoided, since the Universe undergoes a bounce from a previous contracting phase.

In Section 8.1, we presented the full equations for scalar, vector, and tensor perturbations. In the scalar sector, we showed that the equations for comoving curvature perturbations can be solved analytically both in the GR limit and in the regime where the non-minimal interaction dominates, separately; in the latter case, comoving curvature perturbations grow as the Universe expands. In the vector sector, the equations can be solved analytically without any approximation, and we found that vector modes are in general strongly suppressed at late times. In the tensor sector, analytic solutions can again be obtained in both the GR limit and the NMC-dominated epoch; in the latter case, tensor perturbations are described by exponential modes, which either invalidate the linear approximation or remain strongly suppressed.

In section 8.2, we showed how perturbations may be generated from quantum fluctuations and how they evolve up to the present epoch. If primordial fluctuations are generated during the accelerated expansion phase, perturbations receive an additional k^2 enhancement with respect to standard inflationary scenarios. The model therefore predicts a scalar spectral index $n_s = 5$, in clear tension with observations.

If instead perturbations are generated during the contracting phase of a bouncing cosmology, the resulting scalar power spectrum is scale-invariant for multipoles $\ell \gtrsim 150$, while perturbations at small wavenumber are suppressed as k^4 . The integrated Sachs–Wolfe effect partially compensates for this suppression, and the resulting CMB power spectrum shows encouraging agreement with Planck data. The tensor-to-scalar ratio can be expressed in terms of several free parameters, and the model is therefore not highly predictive in this respect. Nevertheless, it is straightforward to obtain values consistent with current constraints while still leaving room for a possible detection in future experiments.

Obtaining observables compatible with current data requires a different prescription for the initial conditions of the perturbations. In particular, while standard inflationary scenarios usually assume that the perturbations begin in a vacuum state, our model requires the corresponding quantum system to start in a highly excited state. This assumption does not appear unnatural, since perturbations are generated when the background Universe is matter — or radiation — dominated. We also require tensor perturbations to begin with a lower occupation number than scalar ones. Since these two sectors are not constrained to share the same initial state, this additional assumption does not seem unreasonable.

These results are encouraging, but several open issues remain. First, the present analysis relies on a piecewise approximation for the background evolution, which should be replaced by a full numerical treatment to obtain precise predictions. Second, the power spectrum obtained in the bouncing scenario requires both specific initial conditions and highly excited perturbation states; the physical origin of these states, and the role of their backreaction on the background evolution, deserve further study. Third, our treatment assumes a symmetric bounce, and it remains to be understood how robust the results are under more general bouncing dynamics. Finally, while the present work focused on the scalar spectral index and tensor-to-scalar ratio, further observational tests should include non-Gaussianity and large-scale structure constraints.

Chapter 9

Conclusions and Future Prospects

The nature of dark matter remains one of the central open questions in modern physics. On cosmological scales, the cold dark matter paradigm provides an extraordinarily successful description of the observed Universe, from the anisotropies of the cosmic microwave background to the formation and distribution of large-scale structure. At galactic scales, however, a number of observations point towards a more complex phenomenology. In particular, the inner structure of dark matter haloes, the dynamics of dwarf galaxies, and the remarkably tight relations between baryonic and dark matter components raise questions about whether the standard assumption of a pressureless and minimally coupled dark matter fluid is sufficient.

This thesis has investigated the possibility that part of the phenomenology usually attributed to the microscopic properties of dark matter may instead originate from a non-standard interaction between dark matter and gravity. Two complementary approaches have been pursued. The first is fractional gravity, in which the gravitational interaction is made non-local through a fractional power of the Laplacian and its relativistic generalization. The second is a local relativistic model in which dark matter is non-minimally coupled to gravity. Although the two frameworks differ substantially in their mathematical structure, they share a common physical motivation: to investigate whether modifying the interaction between dark matter and gravity can provide a unified description of phenomena occurring from galactic scales to the early Universe.

9.1 Summary of the main results

The first part of this thesis has focused on fractional gravity (FG). In its Newtonian formulation, the theory modifies the Poisson equation through a fractional Laplacian characterized by the fractional index s and the length scale ℓ . The analysis of an NFW dark matter distribution showed that the resulting gravitational field can be interpreted in terms of an effective density profile

that is significantly less cuspy than the original distribution. In particular, the effective profile approaches an approximately cored configuration in the regime relevant for low-mass systems, providing a possible explanation of the cusp–core problem without requiring a modification of the underlying NFW dark matter distribution. The resulting phenomenology also reproduces the observed relation between the central surface density and core radius of dwarf galaxies.

The model was then confronted with galaxy rotation curves through Bayesian analyses of stacked systems spanning dwarf, low-surface-brightness and high-surface-brightness galaxies. The fractional-gravity description provided fits that were systematically better than, or at least competitive with, the corresponding standard Newtonian treatment based on an NFW halo. Importantly, the analysis did not merely provide an improved phenomenological fit to individual rotation curves. The posterior distributions revealed a systematic dependence of the fractional parameters on halo mass. The non-local interaction becomes increasingly important towards smaller systems, while the Newtonian limit is approached for more massive haloes. This mass dependence provides a possible explanation for why deviations from the standard cold dark matter predictions are predominantly observed on galactic and sub-galactic scales.

An important consequence of this framework is that several empirical relations involving the baryonic and dark components of late-type galaxies emerge naturally from the non-local modification of gravity. In particular, the Radial Acceleration Relation, the disk–halo mass relation, and the relations involving the core surface density and disk scale-length can be recovered within the same framework. This result is particularly relevant because these correlations represent some of the most striking pieces of evidence for a non-trivial interplay between baryons and dark matter.

The analysis was subsequently extended to galaxy clusters and individual dwarf galaxies. For the X-COP cluster sample, the density and pressure profiles of the intracluster medium were modelled within fractional gravity and confronted with X-ray and Sunyaev–Zel’dovich observations. The model was found to reproduce the thermodynamic profiles of the clusters while preserving a concentration–mass relation compatible with expectations from standard structure formation. The fractional parameters also continued to follow the scaling inferred at galactic scales.

At the opposite end of the mass spectrum, the dynamics of dwarf irregulars, dwarf spheroidals and the ultra-diffuse galaxy DragonFly 44 were investigated. Taken together with the spiral-galaxy and cluster analyses, these results show that fractional gravity can describe virialized systems over approximately ten orders of magnitude in dark matter mass. Most importantly, the same qualitative trend persists across this enormous range: the fractional index approaches the Newtonian value $s = 1$ in massive systems, whereas it can approach $s \simeq 1.5$ in the smallest galaxies. Thus, the phenomenology is not that of a modification with a fixed strength at all

scales, but rather of an interaction whose effective importance depends systematically on the mass and size of the gravitating system.

The relativistic extension of fractional gravity represents the next step towards establishing whether this phenomenology can be embedded consistently into a relativistic theory. An action was constructed whose field equations reduce to the previously studied fractional-gravity framework in the appropriate weak-field limit. The resulting equations can be interpreted as Einstein equations sourced by an effective stress-energy tensor associated with the dark matter component. Interestingly, this effective description contains not only an energy density but also dynamically generated pressure, heat flux and anisotropic stresses.

The relativistic formulation also reveals an intrinsically non-local aspect of the theory. The propagation of perturbations is affected by the fractional d'Alembertian, leading to memory effects and allowing the response at a given spacetime point to depend on an extended region of its past light-cone. This opens the possibility of testing fractional gravity through genuinely relativistic phenomena, including gravitational lensing and gravitational-wave propagation. In fact, the lensing analysis performed using the CLASH galaxy-cluster sample showed good agreement between the relativistic fractional-gravity models considered and the observed convergence profiles.

The second part of this thesis followed a different strategy. The non-minimally coupled dark matter (NMC-DM) model was introduced as a local relativistic realization inspired by the structure of fractional gravity. The main advantage of this formulation is that, by replacing the non-local interaction with a local non-minimal coupling, it becomes possible to study strong-gravity configurations explicitly.

A striking consequence of the coupling is the dynamical generation of an effective pressure from an underlying pressureless dark matter component. This pressure can counteract gravitational collapse and support static, spherically symmetric configurations. The resulting solutions are regular and horizonless ultra-compact objects. Their density profiles are approximately constant over much of their extent, while their compactness can approach that of black-hole mimickers. For a restricted range of central densities, the configurations possess an internal and an external light ring, giving rise to a characteristic optical structure. A ray-tracing analysis consequently showed that these objects can possess a pseudo-shadow, providing a potential observational discriminator between NMC ultra-compact objects and black holes.

The cosmological implications of the same interaction are equally significant. Because the non-minimal coupling becomes important when the dark matter density is sufficiently large, its effects are naturally concentrated in the early Universe. Solving the modified background equations reveals a phase of accelerated expansion driven entirely by the coupling, without the need to introduce an additional inflaton field. For the parameter range investigated in the

thesis, this phase lasts sufficiently long to address the horizon and flatness problems. At still earlier times, the fate of the Universe depends on its spatial curvature. In particular, negative spatial curvature allows the initial singularity to be replaced by a cosmological bounce.

The study of cosmological perturbations provides an essential test of this scenario. The thesis derived the equations governing scalar, vector and tensor perturbations in the NMC–DM framework and followed their evolution through the relevant cosmological epochs. The inflationary and bouncing scenarios lead to qualitatively different primordial spectra. The bouncing scenario is particularly interesting because it can simultaneously remove the initial singularity at the background level and generate a primordial scalar spectrum compatible with the observed CMB anisotropies. This agreement, however, requires a departure from the simplest choice of the initial quantum state of the perturbations.

The resulting picture is therefore a remarkable one. Within a single modification of the dark matter–gravity interaction, one can obtain non-trivial galactic dynamics, regular ultra-compact configurations, an early phase of accelerated expansion, a possible cosmological bounce, and primordial perturbations with an approximately scale-invariant spectrum. These results do not establish that non-standard dark matter–gravity interactions are realized in Nature. Rather, they demonstrate that such interactions constitute a viable and phenomenologically rich direction that deserves further investigation.

9.2 Future prospects

The results obtained in this thesis naturally lead to several directions for future work.

9.2.1 From phenomenology to a fundamental origin of fractional gravity

Perhaps the most important open question concerns the physical origin of the fractional interaction. The analyses presented in this thesis establish a consistent phenomenological picture, including a mass-dependent scaling of the fractional parameters, but do not identify a microscopic mechanism responsible for the appearance of the fractional operator.

A fundamental theory should explain why the interaction is characterized by a fractional index s , why the associated length scale ℓ follows the empirical scaling found across virialized systems, and why the modification becomes negligible in massive systems. One possible direction is to investigate whether fractional operators can emerge as effective descriptions of an underlying dark-sector theory, for example after integrating out additional degrees of freedom or through renormalization-group effects. Fractional operators have already been considered in broader contexts involving non-local gravity, anomalous scaling and effective descriptions of quantum gravitational systems, making this connection particularly interesting.

A related issue concerns the relation between the Newtonian fractional theory and its relativistic completion. The present work provides the relativistic field equations and demonstrates the correct weak-field limit, but a complete understanding of the theory requires a systematic analysis of its degrees of freedom, initial-value formulation, causality and stability. In particular, the role of the choice of Green function in defining the non-local operators deserves further attention, since causal propagation requires the appropriate retarded prescription.

9.2.2 Larger and more homogeneous galaxy samples

The galaxy analyses presented in this thesis demonstrate the potential of fractional gravity, but future observational samples will allow the parameter scaling to be tested much more stringently. In particular, large samples of spatially resolved rotation curves can establish whether the relation between s , ℓ , and halo mass remains valid across a broader range of morphological types, surface brightnesses and environments.

Dwarf galaxies are especially important because the fractional effects are predicted to be strongest in this regime. Improved measurements of stellar and gas kinematics, together with better control of inclination, non-circular motions, stellar mass-to-light ratios and velocity anisotropy, will be crucial for determining whether the inferred departures from standard gravity are robust against astrophysical systematics.

The same strategy should be applied to galaxy clusters using independent mass tracers. Combining X-ray hydrostatics, SZ observations, strong and weak lensing, and galaxy dynamics would provide an especially powerful consistency test because these observables probe the gravitational potential through different physical mechanisms.

9.2.3 Strong-gravity dynamics and the stability of NMC ultra-compact objects

The NMC ultra-compact objects found in this thesis are among the most intriguing predictions of the model. However, establishing their physical relevance requires going beyond the static solutions.

The first essential step is a complete dynamical stability analysis. The preliminary analysis presented in the thesis suggests that the stability properties can be investigated through perturbations of the relevant fields, but a rigorous treatment requires the numerical evolution of time-dependent perturbations. This was explicitly identified as a problem postponed to future work.

Such an analysis should determine whether the static configurations represent dynamically stable compact objects or transient configurations preceding collapse. It should also establish

the allowed range of central densities and determine whether stable branches exist when realistic perturbations and dynamical formation processes are taken into account.

A second important question concerns formation. If NMC-UCOs are to provide seeds for the supermassive black holes observed at high redshift, a viable cosmological mechanism must be identified that can produce sufficiently massive objects with the required compactness. Their abundance and mass function should then be evolved through cosmic history and compared with the observed population of high-redshift quasars.

Finally, the optical properties calculated in this thesis should be extended to more realistic observational predictions. The presence of light rings and a pseudo-shadow suggests that NMC-UCOs may be distinguishable from black holes through high-resolution imaging. Their accretion dynamics, electromagnetic emission, lensing properties and possible gravitational-wave signatures represent promising avenues for future investigation.

9.2.4 Beyond linear perturbations

The cosmological analysis developed here represents an important first step towards understanding the consequences of the non-minimal coupling, but a complete cosmological model must ultimately describe nonlinear structure formation.

A natural development would be the implementation of the NMC-DM equations in a numerical Boltzmann solver and, subsequently, in cosmological N-body or hydrodynamical simulations. This would permit a direct computation of the CMB anisotropies, matter power spectrum and growth of structure without relying on the piecewise or semi-analytic approximations adopted in the present analysis.

Such an implementation is particularly important because the perturbation equations differ substantially from those of minimally coupled CDM. The continuity equation for the dark matter perturbations remains unchanged, whereas the Euler equation acquires significant modifications from the non-minimal coupling. This modification could leave observable signatures in the evolution of density perturbations even if the background cosmology remains close to the standard one at late times.

The next step should therefore be a fully numerical treatment extending continuously from the NMC-dominated epoch through radiation domination, matter domination and the late-time dark-energy era. This would allow the model to be confronted simultaneously with CMB, baryon acoustic oscillations, weak lensing and galaxy-clustering data.

9.2.5 A more complete treatment of primordial perturbations

The primordial perturbation analysis presented in this thesis provides encouraging evidence that the bouncing scenario can reproduce the observed near scale-invariant spectrum. Nevertheless,

several aspects remain to be investigated.

First, the primordial spectrum should be obtained by solving the full perturbation equations numerically across the bounce rather than by matching the asymptotic solutions of the different cosmological epochs. This is especially important because the bounce is precisely the regime where the standard approximations used away from the bounce cease to be reliable.

Second, the quantum state of the perturbations deserves a deeper theoretical justification. The agreement with CMB data in the bouncing scenario requires a modified initial state, and a fundamental explanation of this state would significantly strengthen the predictive power of the model. The thesis also finds that the detailed tensor-to-scalar ratio is not uniquely fixed by the model parameters and can exhibit qualitatively different behaviors depending on the characteristic scales of the initial perturbations.

Third, a complete calculation should include the effects of the modified primordial spectrum on the full CMB temperature and polarization spectra. The semi-analytic comparison with Planck data performed here is encouraging, but the thesis itself emphasizes the limitations of the approximation and the motivation for a more precise numerical treatment.

The development of a dedicated Boltzmann code for NMC—DM would therefore represent one of the most important next steps. Such a code would allow the model to be constrained directly against current observations and would make quantitative predictions for future CMB and large-scale-structure surveys.

9.2.6 Towards a unified dark-sector phenomenology

The results of this thesis suggest a broader perspective on the dark matter problem. The standard paradigm assumes that dark matter is pressureless, minimally coupled and effectively collisionless, with all departures from this behaviour arising from astrophysical processes or from the microscopic particle properties of the dark sector. The models studied here explore the complementary possibility that the interaction between dark matter and gravity itself may possess a richer structure.

At galactic scales, non-locality can modify the effective gravitational response in a way that produces cored density distributions and tight dynamical scaling relations. At strong gravity, a non-minimal coupling can dynamically generate pressure and support ultra-compact configurations. At cosmological densities, the same coupling can modify the background expansion and potentially replace the initial singularity with a bounce. Finally, the perturbation equations show that the modified interaction can affect the primordial fluctuations from which the observed cosmic structure originates.

This hierarchy is particularly suggestive because the characteristic strength of the modifications is naturally associated with the density or mass scale of the system. The phenomenology

found in this thesis therefore points towards a picture in which the dark matter–gravity interaction is not universally described by the same local, pressureless approximation, but instead exhibits scale-dependent effective behaviour.

Whether this picture can be elevated from an effective phenomenological framework to a fundamental theory remains an open question. The answer will require simultaneous progress on the theoretical, numerical and observational fronts. Theoretical consistency must be established through a complete analysis of the degrees of freedom, stability and causality; numerical tools must connect the early Universe to nonlinear structure formation; and observations must test the predicted deviations from standard gravity over the largest possible range of scales.

9.2.7 Final remarks

The central message of this thesis is that the dark matter problem may contain information not only about the nature of the dark matter component itself, but also about the way in which that component interacts with gravity.

Fractional gravity provides a framework in which non-locality modifies the gravitational response of dark matter and naturally becomes more important in low-mass systems. Its success in describing galaxies, dwarf galaxies and clusters, together with the relativistic formulation developed here, demonstrates that this idea can be formulated as a coherent phenomenological programme.

Non-minimally coupled dark matter provides a complementary local realization in which the interaction with gravity dynamically generates pressure and produces qualitatively new relativistic phenomena. The existence of regular ultra-compact objects, an early accelerated phase, a possible cosmological bounce and primordial spectra compatible with CMB observations illustrates the breadth of the phenomenology that can emerge from a seemingly simple modification of the dark matter–gravity coupling.

At the same time, the results presented here should be regarded as the beginning of a research program rather than as a definitive solution to the dark matter problem. Several crucial questions remain open: the fundamental origin of the non-local interaction, the stability and formation of the ultra-compact solutions, the fully nonlinear evolution of cosmological perturbations, and the confrontation of the models with a comprehensive set of cosmological and astrophysical observations.

The most compelling prospect is therefore not necessarily that one of the models studied in this thesis will replace Λ CDM in its entirety, but that the investigation of non-standard dark matter–gravity interactions may reveal which aspects of the dark sector are genuinely fundamental and which may instead emerge from an effective description of gravity. The remarkable fact that a common modification of the dark matter–gravity interaction can manifest

itself in phenomena ranging from the inner structure of dwarf galaxies to the primordial Universe makes this question particularly worthy of further study.

Ultimately, the decisive test will come from connecting the different scales explored in this thesis within one predictive framework. If the scaling relations inferred from galaxies and clusters, the relativistic signatures of non-locality, the strong-gravity solutions of the local theory, and the cosmological predictions can be brought together and confronted simultaneously with future observations, it may become possible to determine whether the apparent tensions of the dark matter paradigm are telling us something new about dark matter itself—or about gravity.

III

Appendix

Appendix A

Solutions to the Fractional Poisson Equation

We derive the solution to the fractional Poisson equation for $s \in [1, 3/2]$ in the case of a NFW and a razor thin exponential disk distribution.

The Green function to the fractional Poisson equation is the solution of

$$(-\Delta)^s \mathcal{G}_s(\mathbf{r}) = -4\pi G \ell^{2-2s} \delta^3(\mathbf{r}) \quad (\text{A.1})$$

Applying the Riesz operator for $s \in (0, 3/2)$ one finds that

$$\mathcal{G}_s(\mathbf{r}) = -4\pi G \ell^{2-2s} \mathcal{R}_{2s} \delta^3(\mathbf{r}) = -\frac{G \ell^{2-2s} \Gamma\left(\frac{3}{2} - s\right)}{4^{s-1} \sqrt{\pi} \Gamma(s)} \frac{1}{r^{3-2s}} ; \quad (\text{A.2})$$

it is straightforward to check that $\mathcal{G}_{s=1}(\mathbf{r})$ collapses to the Newtonian Green function. The case $s = 3/2$ is a bit tricky, and may be solved more conveniently in Fourier space; there the Poisson equation reads $\mathcal{F}\{\mathcal{G}_s\}(\mathbf{k}) = -4\pi G/\ell k^3$ and thus $\mathcal{G}_s = -(4\pi G/\ell) \times \mathcal{F}^{-1}\{1/k^3\}$. Given that $-k^2 \mathcal{F}\{\ln(r/\ell)\} = \mathcal{F}\{\Delta \log(r/\ell)\} = \mathcal{F}\{1/r^2\} = 2\pi^2/k$ (where the last equality is meant to hold in the principal value sense) one finds $\mathcal{F}\{\ln(r/\ell)\} = -2\pi^2/k^3$ and hence

$$\mathcal{G}_{s=3/2}(\mathbf{r}) = \frac{2G}{\pi \ell} \ln\left(\frac{r}{\ell}\right) . \quad (\text{A.3})$$

The potential for a point mass m located at the origin goes like $\Phi_s(\mathbf{r}) \propto G m \ell^{2-2s}/r^{3-2s}$ for $s \in (0, 3/2)$ and like $\Phi_{s=3/2}(\mathbf{r}) \propto (G m/\ell) \times \ln(r/\ell)$ for $s = 3/2$, so decreasing faster than the Newtonian case ($s = 1$) for $s \in (0, 1)$ and slower for $s \in (1, 3/2]$.

Correspondingly, the rotational velocity $v^2(r) = r |d\Phi/dr|$ of a test mass on a circular orbit in such a potential goes like $v_s(r) \propto \sqrt{Gm/\ell} \times (\ell/r)^{3/2-s}$. Curiously for $s = 3/2$ it is found

that $v_{s=3/2}(r) = \sqrt{2Gm/\pi\ell}$ is constant, an occurrence that some authors have interpreted as a connection with MOND theories [137]; in fact, since at low accelerations MOND implies that $v^4 = Gm a_0$ in terms of a universal constant $a_0 \approx 1.2 \times 10^{-10} \text{ m s}^{-2}$, the two expressions are consistent provided that $\ell = (2/\pi) \times \sqrt{Gm/a_0}$ applies.

Consider now a spherically symmetric distribution $\rho(\mathbf{r}) = \rho(r)$, with $r = |\mathbf{r}|$. It is convenient to use coordinates $\mathbf{r}' = (r', \theta, \phi)$, placing the point $\mathbf{r}$ where the potential is calculated at the north pole; using Eq. (A.2) for $s \in [1, 3/2)$ one has

$$\begin{aligned} \Phi_s(r) = \int d^3\mathbf{r}' \mathcal{G}_s(\mathbf{r} - \mathbf{r}') \rho(r') &= -\frac{G \ell^{2-2s} \Gamma\left(\frac{3}{2} - s\right)}{4^{s-1} \sqrt{\pi} \Gamma(s)} \int_0^{2\pi} d\phi \int_0^\infty dr' r'^2 \rho(r') \times \\ &\times \int_0^\pi d\theta \frac{\sin \theta}{(r^2 + r'^2 - 2 r r' \cos \theta)^{3/2-s}}. \end{aligned} \quad (\text{A.4})$$

A straightforward calculation yields the solution

$$\Phi_s(r) = -\frac{\sqrt{\pi} G \ell^{2-2s} \Gamma\left(\frac{3}{2} - s\right)}{4^{s-3/2} (2s-1) \Gamma(s)} \frac{\mathcal{J}_s^\rho(r)}{r}, \quad (\text{A.5})$$

where

$$\mathcal{J}_s^\rho(r) = \int_0^\infty dr' r' \rho(r') [(r+r')^{2s-1} - |r-r'|^{2s-1}]. \quad (\text{A.6})$$

In the case $s = 3/2$ one has instead to use Eq. (A.3), obtaining

$$\begin{aligned} \Phi_{s=3/2}(r) = \int d^3\mathbf{r}' \mathcal{G}_{s=3/2}(\mathbf{r} - \mathbf{r}') \rho(r') &= \frac{2G}{\pi\ell} \int_0^{2\pi} d\phi \int_0^\infty dr' r'^2 \rho(r') \times \\ &\times \int_0^\pi d\theta \sin \theta \ln \left(\frac{\sqrt{r^2 + r'^2 - 2 r r' \cos \theta}}{\ell} \right), \end{aligned} \quad (\text{A.7})$$

$$\times \int_0^\pi d\theta \sin \theta \ln \left(\frac{\sqrt{r^2 + r'^2 - 2 r r' \cos \theta}}{\ell} \right), \quad (\text{A.8})$$

which yields the integral representation

$$\Phi_{s=3/2}(r) = \frac{2G\ell}{r} \int_0^\infty dr' r' \rho(r') \left[\left(\frac{r+r'}{\ell} \right)^2 \ln \left(\frac{r+r'}{\ell} \right) - \left(\frac{r-r'}{\ell} \right)^2 \ln \left(\frac{r-r'}{\ell} \right) - 2 \frac{r r'}{\ell^2} \right]. \quad (\text{A.9})$$

Another method of solving the fractional Poisson equation for spherically-symmetric distribution, especially useful for the $s = 3/2$ case, is to work in the Fourier domain, where the density can be represented as

$$\mathcal{F}\{\rho\}(k) = \frac{4\pi}{k} \int_0^\infty dr r \rho(r) \sin(kr). \quad (\text{A.10})$$

Then by inverse Fourier transform the fractional potential for $s \in [1, 3/2]$ is given by

$$\Phi_s(r) = -\frac{2G \ell^{2-2s}}{\pi r} \int_0^\infty dk \frac{\sin(kr)}{k^{2s-1}} \mathcal{F}\{\rho\}(k) . \quad (\text{A.11})$$

Navarro-Frenk-White profile

As stated in the main text, the density distribuion of the NFW profile is

$$\rho(r) = \frac{\rho_s r_s^3}{r (r + r_s)^2} , \quad (\text{A.12})$$

where ρ_s and r_s the characteristic density and radius respectively.

For $s \in [1, 3/2]$ one has to evaluate the integral

$$\mathcal{J}_s^\rho(r) = \rho_s r_s^3 \int_0^\infty dr' \frac{(r + r')^{2s-1} - |r - r'|^{2s-1}}{(r' + r_s)^2} , \quad (\text{A.13})$$

that can be easily see to converge to

$$\begin{aligned} \Phi_s(r) = & -\sqrt{\pi} G \rho_s r_s^2 \left(\frac{2\ell}{r_s} \right)^{2-2s} \frac{\Gamma\left(\frac{3}{2} - s\right)}{\Gamma(s+1)} \frac{r_s}{r} \left\{ \frac{2\pi s}{\sin(2\pi s)} \left[\left(1 + \frac{r}{r_s}\right)^{2s-2} - \left(1 - \frac{r}{r_s}\right)^{2s-2} \right] + \right. \\ & \left. + \frac{(r/r_s)^{2s}}{1 - (r/r_s)^2} \left[\left(1 + \frac{r}{r_s}\right) {}_2F_1\left(1, 1, 2s+1, \frac{r}{r_s}\right) + \left(1 - \frac{r}{r_s}\right) {}_2F_1\left(1, 1, 2s+1, -\frac{r}{r_s}\right) - \frac{4s}{2s-1} \right] \right\} . \end{aligned} \quad (\text{A.14})$$

In the above ${}_2F_1$ is the ordinary hypergeometric function, defined by the power series ${}_2F_1(a, b, c; x) \equiv \sum_{k=0}^\infty (a)_k (b)_k x^k / (c)_k k!$ in terms of the Pochhammer's symbols $(q)_0 = 1$ and $(q)_k = q(q+1) \dots (q+k-1)$ for any positive integer k .

The case $s = 3/2$ is more easily treated in the Fourier domain, and requires regularizing the integrals appearing in Eq. (A.11); the result reads

$$\begin{aligned} \Phi_{s=3/2}(r) = & -\frac{4G \rho_s r_s^3}{\ell} \frac{r_s}{r} \left\{ 2 \frac{r}{r_s} \left[\log\left(\frac{r}{r_s}\right) - 1 \right] - \left(1 + \frac{r}{r_s}\right) \log\left(\frac{r}{r_s}\right) \log\left(1 + \frac{r}{r_s}\right) + \right. \\ & \left. + \left(\frac{r}{r_s} - 1\right) \text{Li}_2\left(1 - \frac{r}{r_s}\right) - \left(1 + \frac{r}{r_s}\right) \text{Li}_2\left(-\frac{r}{r_s}\right) + \frac{\pi^2}{6} \right\} , \end{aligned} \quad (\text{A.15})$$

in terms of the dilogarithm function $\text{Li}_2(x) \equiv \sum_{k=1}^\infty x^k / k^2$.

Razor-thin exponential disk

The density profile of a razor-thin exponential disk in cylindrical coordinates is defined as

$$\rho(R, z) = \frac{M_\infty}{2\pi r_d^2} e^{-R/r_d} \delta_D(z), \quad (\text{A.16})$$

in terms of the radius r_d . The Newtonian potential on the plane of the disk ($z = 0$) reads

$$\Phi_{s=1}(R, 0) = -\frac{GM_\infty}{r_d} \frac{R}{2r_d} \left[I_0\left(\frac{R}{2r_d}\right) K_1\left(\frac{R}{2r_d}\right) - I_1\left(\frac{R}{2r_d}\right) K_0\left(\frac{R}{2r_d}\right) \right]. \quad (\text{A.17})$$

The solution of the fractional Poisson equation may be most easily found in Fourier space after Eq. (A.11); for an index $s \in [1, 3/2)$ we find

$$\begin{aligned} \Phi_s(R, 0) = & -\frac{GM_\infty}{r_d} \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \left(\frac{\ell}{r_d}\right)^{2-2s} \left[2^{2s} \frac{\Gamma(2-2s)\Gamma(s)}{\Gamma(1-s)} {}_1F_2\left(\frac{3}{2}-s; 1, 1-s, \frac{R^2}{4r_d^2}\right) + \right. \\ & \left. - \frac{\pi}{\sin(\pi s)\Gamma^2(s+1)} \left(\frac{R}{2r_d}\right)^{2s} {}_1F_2\left(\frac{3}{2}; 1+s, 1+s, \frac{R^2}{4r_d^2}\right) \right]. \end{aligned}$$

In the case $s = 3/2$, after regularization of the integrals, we obtain

$$\Phi_{s=3/2}(R, 0) = -\frac{GM_\infty}{\ell} \frac{2}{\pi} \left[\text{Ei}\left(-\frac{R}{r_d}\right) - e^{-R/r_d} - \ln\left(\frac{R}{r_d}\right) \right], \quad (\text{A.18})$$

in terms of the exponential integral function $\text{Ei}(x) \equiv \int_{-\infty}^x dt e^t/t$.

Appendix B

Variation of the Relativistic Fractional Gravity Action

We derive the field equations of relativistic fractional gravity (5.4) by varying the corresponding action (5.3). This comprises the Einstein-Hilbert action, the matter actions for baryons and cold dark matter, a scalar, and a tensorial fractional term

$$\begin{aligned} S_{\text{EH}} &= \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} R, \\ S_{\text{MAT}} &= S_{\text{BAR}} + S_{\text{DM}}, \\ S_{\text{SFG}} &= \int_{\mathcal{M}} d^4x \sqrt{-g} R \mathcal{F}_0(\square) T_{\text{DM}}, \\ S_{\text{TFG}} &= \int_{\mathcal{M}} d^4x \sqrt{-g} R_{\alpha\beta} \mathcal{F}_2(\square) T_{\text{DM}}^{\alpha\beta}. \end{aligned} \tag{B.1}$$

The first two terms yield the Einstein equations

$$\delta S_{\text{EH}} + \delta S_{\text{MAT}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\frac{G_{\mu\nu}}{16\pi G} - \frac{1}{2} T_{\text{BAR}\mu\nu} - \frac{1}{2} T_{\text{DM}\mu\nu} \right) \delta g^{\mu\nu}. \tag{B.2}$$

To deal with the other two terms one needs to recall the variation of the Ricci scalar and tensor

$$\delta R = (R_{\mu\nu} + g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \delta g^{\mu\nu} , \quad (\text{B.3})$$

$$\delta R_{\mu\nu} = \nabla^\alpha \nabla_{(\mu} \delta g_{\nu)\alpha} - \frac{1}{2} \square \delta g_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} \nabla_{(\mu} \nabla_{\nu)} \delta g_{\alpha\beta} , \quad (\text{B.4})$$

together with the variation of the stress-energy tensor for a dust fluid $T_{\text{DM}}^{\mu\nu} = \rho u^\mu u^\nu$ and of its trace

$$\delta T_{\text{DM}}^{\alpha\beta} = \frac{1}{2} T_{\text{DM}}^{\alpha\beta} (g_{\mu\nu} - u_\mu u_\nu) \delta g^{\mu\nu} , \quad \delta T_{\text{DM}} = \frac{1}{2} (T_{\text{DM}} g_{\mu\nu} - T_{\text{DM} \mu\nu}) \delta g^{\mu\nu} . \quad (\text{B.5})$$

Then, the variation of the scalar and tensorial fractional terms yield respectively

$$\begin{aligned} \delta S_{\text{SFG}} &= \int_{\mathcal{M}} d^4x \left(\delta(\sqrt{-g}) R \mathcal{F}_0 T_{\text{DM}} + \sqrt{-g} \delta R \mathcal{F}_0 T_{\text{DM}} + \sqrt{-g} R \delta \mathcal{F}_0 T_{\text{DM}} + \sqrt{-g} \mathcal{F}_0 R \delta T_{\text{DM}} \right) = \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} R \mathcal{F}_0 T_{\text{DM}} + R \frac{\delta \mathcal{F}_0}{\delta g^{\mu\nu}} T_{\text{DM}} + \frac{1}{2} (T_{\text{DM}} g_{\mu\nu} - T_{\text{DM} \mu\nu}) \mathcal{F}_0 R \right) \delta g^{\mu\nu} + \\ &+ \int_{\mathcal{M}} d^4x \sqrt{-g} [(R_{\mu\nu} + g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \delta g^{\mu\nu}] \mathcal{F}_0 T_{\text{DM}} = \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left[(G_{\mu\nu} + g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \mathcal{F}_0 T_{\text{DM}} + R \frac{\delta \mathcal{F}_0}{\delta g^{\mu\nu}} T_{\text{DM}} + \frac{1}{2} (T_{\text{DM}} g_{\mu\nu} - T_{\text{DM} \mu\nu}) \mathcal{F}_0 R \right] \delta g^{\mu\nu} + \\ &+ \epsilon \oint_{\partial \mathcal{M}} d^3y \sqrt{h} N_\alpha \left[g_{\mu\nu} (\nabla^\alpha \delta g^{\mu\nu} - \nabla^\nu \delta g^{\mu\alpha}) \mathcal{F}_0 T_{\text{DM}} - 2 \delta g^{\alpha\beta} \nabla_\beta \mathcal{F}_0 T_{\text{DM}} + 2 \delta g^{\mu\nu} g_{\mu\nu} \nabla^\alpha \mathcal{F}_0 T_{\text{DM}} \right] . \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned}
 \delta S_{\text{TFG}} &= \int_{\mathcal{M}} d^4x \left(\delta(\sqrt{-g}) R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \sqrt{-g} \delta R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \sqrt{-g} R_{\alpha\beta} \delta \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \sqrt{-g} \mathcal{F}_2 R_{\alpha\beta} \delta T_{\text{DM}}^{\alpha\beta} \right) = \\
 &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + R_{\alpha\beta} \frac{\delta \mathcal{F}_2}{\delta g^{\mu\nu}} T_{\text{DM}}^{\alpha\beta} + \frac{1}{2} T_{\text{DM}}^{\alpha\beta} (g_{\mu\nu} - u_\mu u_\nu) \mathcal{F}_2 R_{\alpha\beta} \right) \delta g^{\mu\nu} + \\
 &+ \int_{\mathcal{M}} d^4x \sqrt{-g} \left(\nabla^\mu \nabla_{(\alpha} \delta g_{\beta)\mu} - \frac{1}{2} \square \delta g_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} \nabla_{(\alpha} \nabla_{\beta)} \delta g_{\mu\nu} \right) \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} = \\
 &= \int_{\mathcal{M}} d^4x \sqrt{-g} \left(-\frac{1}{2} g_{\mu\nu} R_{\alpha\beta} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \frac{1}{2} T_{\text{DM}}^{\alpha\beta} (g_{\mu\nu} + \frac{T_{\text{DM}\mu\nu}}{T_{\text{DM}}}) \mathcal{F}_2 R_{\alpha\beta} + \right. \\
 &+ R_{\alpha\beta} \frac{\delta \mathcal{F}_2}{\delta g^{\mu\nu}} T_{\text{DM}}^{\alpha\beta} - \nabla^\alpha \nabla_{(\mu} \mathcal{F}_2 T_{\text{DM},\nu)\alpha} + \frac{1}{2} \square \mathcal{F}_2 T_{\text{DM}\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla_{(\alpha} \nabla_{\beta)} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} \left. \right) \delta g^{\mu\nu} + \\
 &+ \epsilon \oint_{\partial\mathcal{M}} d^3y \sqrt{h} \left[\left(n^\rho \nabla_{(\alpha} \delta g_{\beta)\rho} - \frac{1}{2} n^\rho \nabla_\rho \delta g_{\alpha\beta} - \frac{1}{2} g^{\rho\sigma} n_{(\alpha} \nabla_{\beta)} \delta g_{\rho\sigma} \right) \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \right. \\
 &\left. - n_{(\alpha} \delta g_{\beta)\rho} \nabla^\rho \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \frac{1}{2} n_\rho \delta g_{\alpha\beta} \nabla^\rho \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} + \frac{1}{2} g^{\rho\sigma} \delta g_{\rho\sigma} n_{(\alpha} \nabla_{\beta)} \mathcal{F}_2 T_{\text{DM}}^{\alpha\beta} \right]. \tag{B.7}
 \end{aligned}$$

In both cases, boundary terms arise when the divergence theorem is used to transfer derivatives from the metric variation to the stress-energy tensor. The boundary of the manifold is a three-dimensional hypersurface with volume element $dS_\alpha = \epsilon \sqrt{h} N_\alpha d^3y$. Here, N^α is the unit normal vector to the hypersurface, $\epsilon = N^\alpha N_\alpha = \pm 1$, depending on whether the hypersurface is spacelike or timelike, and $h_{\mu\nu}$ is the induced metric.

Two types of boundary terms appear. The first are proportional to $\delta g^{\mu\nu}$ and involve derivatives of the stress-energy tensor. The second contain derivatives of $\delta g^{\mu\nu}$ and are proportional to $\mathcal{F}_0 T_{\text{DM}}$ or $\mathcal{F}_2 T_{\text{DM}}^{\alpha\beta}$. The former vanish identically because $\delta g^{\mu\nu}$ is held fixed on the boundary. Although the derivatives of $\delta g^{\mu\nu}$ are not fixed there, the corresponding boundary terms also vanish provided the dark matter stress-energy tensor vanishes on the boundary—a standard assumption throughout this work. Under this assumption, both $\mathcal{F}_0 T_{\text{DM}}$ and $\mathcal{F}_2 T_{\text{DM}}^{\alpha\beta}$ vanish on the boundary, allowing all boundary contributions to be consistently discarded.

Collecting (B.2), (B.6), and (B.7) we obtain the field equations (5.4).

Appendix C

Retarded Green Function for the Fractional d'Alembertian

We derive the retarded Green function for the fractional d'Alembertian $(-\square)^s$ with $s > 1$ in the case in Minkowski space, exploited in Chapter 5.

By definition the Green function satisfies the functional equation

$$(-\square_x)^s \mathcal{G}_{\text{ret}}(x - y) = \delta^4(x - y), \quad (\text{C.1})$$

for any two spacetime points $x = (x^0, \mathbf{x})$ and $y = (y^0, \mathbf{y})$. This equation is best solved recalling the action of the Fractional d'Alembertian in Fourier space; specifically, for any function f , the expression $\mathcal{F}\{(-\square)^s f\}(k) = (k_\mu k^\mu)^s \mathcal{F}\{f\}(k)$ applies with $k = (k^0, \mathbf{k})$, so that

$$\mathcal{F}\{\mathcal{G}_{\text{ret}}\}(k) = (k_\mu k^\mu)^{-s}. \quad (\text{C.2})$$

Inverting the Fourier transform, we obtain:

$$\begin{aligned} \mathcal{G}_{\text{ret}}(x - y) &= \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik_\mu (x^\mu - y^\mu)}}{(k_\nu k^\nu)^s} = \\ &= \frac{1}{4\pi^3 |\mathbf{x} - \mathbf{y}|} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| \sin(|\mathbf{k}| |\mathbf{x} - \mathbf{y}|) \int_{-\infty}^\infty dk^0 \frac{e^{-ik^0 (x^0 - y^0)}}{[|\mathbf{k}|^2 - (k^0 + i\epsilon)^2]^s}, \end{aligned} \quad (\text{C.3})$$

We then perform a Wick rotation by setting $q = \epsilon - ik^0$, to get:

$$\begin{aligned} \mathcal{G}_{\text{ret}}(x - y) &= \lim_{\epsilon \rightarrow 0^+} \Theta(x^0 - y^0) \frac{e^{-\epsilon(x^0 - y^0)}}{4\pi^3 i |\mathbf{x} - \mathbf{y}|} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| \sin(|\mathbf{k}| |\mathbf{x} - \mathbf{y}|) \int_{\epsilon - i\infty}^{\epsilon + i\infty} dq \frac{e^{q(x^0 - y^0)}}{(|\mathbf{k}|^2 + q^2)^s} = \\ &= \frac{\Theta(x^0 - y^0) (x^0 - y^0)^{s-1/2}}{2^{s+1/2} \pi^{3/2} \Gamma(s) |\mathbf{x} - \mathbf{y}|} \int_0^\infty dk |\mathbf{k}|^{3/2-s} \sin(|\mathbf{k}| |\mathbf{x} - \mathbf{y}|) J_{s-1/2}(|\mathbf{k}| (x^0 - y^0)) \end{aligned} \quad (\text{C.4})$$

where in the last integral of the first line we have recognized the inverse Laplace transform of the quantity $(k^2 + q^2)^{-s}$, and then we have computed it in terms of the Bessel function $J_\nu(z)$ of the first kind and order ν . To proceed further we can exploit the result that the special integral $I(a, b, \lambda, \nu) \equiv \int_0^\infty dz z^\lambda \sin(bz) J_\nu(az)$ for $-\text{Re}(\nu) - 1 < \text{Re}(\lambda) + 1 < 3/2$ can be written in terms of the Gaussian hypergeometric function ${}_2F_1$ as (e.g., see [299], p.730, eq. 6.699)

$$\begin{aligned} I(a, b, \lambda, \nu) &= \frac{2^{\lambda+1}}{a^{\lambda+2}} \frac{\Gamma\left(\frac{2+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda}{2}\right)} {}_2F_1\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right), \quad \text{if } 0 < b < a \\ &= \left(\frac{a}{2}\right)^\nu \frac{1}{b^{\lambda+\nu+1}} \frac{\Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)} \sin\left[\pi\left(\frac{\lambda+\nu+1}{2}\right)\right] \times \\ &\quad \times {}_2F_1\left(\frac{\lambda+\nu+2}{2}, \frac{\lambda+\nu+1}{2}; \nu+1; \frac{a^2}{b^2}\right), \quad \text{if } 0 < a < b. \end{aligned} \quad (\text{C.5})$$

In the present context $\lambda = 3/2 - s$, $\nu = s - 1/2$, $a = x^0 - y^0$, $b = |\mathbf{x} - \mathbf{y}|$ applies, so that $(\lambda + \nu + 1)/2 = 1$ holds. This implies that the integral in Equation (C.4) vanishes whenever $|\mathbf{x} - \mathbf{y}| > x^0 - y^0$, so preventing superluminal motion. Therefore the support of the Green function coincides with the interior of the past light cone for the point x , where it is explicitly given by:

$$\mathcal{G}_{\text{ret}}(x - y) = \frac{(x^0 - y^0)^{2s-4}}{2^{2s-1} \pi \Gamma(s) \Gamma(s-1)} {}_2F_1\left(\frac{3}{2}, 2-s; \frac{3}{2}; \left[\frac{|\mathbf{x} - \mathbf{y}|}{x^0 - y^0}\right]^2\right) \Theta(x^0 - y^0 - |\mathbf{x} - \mathbf{y}|). \quad (\text{C.6})$$

Finally, since ${}_2F_1\left(\frac{3}{2}, 2-s; \frac{3}{2}; z\right) = (1 - z^2)^{s-2}$, we obtain:

$$\mathcal{G}_{\text{ret}}(x - y) = \frac{[(x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2]^{s-2}}{2^{2s-1} \pi \Gamma(s) \Gamma(s-1)} \Theta(x^0 - y^0 - |\mathbf{x} - \mathbf{y}|), \quad (\text{C.7})$$

which is employed in convolution with the source function to derive the propagation of scalar and tensor modes in Chapter 5.

That (C.7) correctly reduces to the retarded Green function of the standard d'Alembertian operator as $s \rightarrow 1^+$, can be seen using the distributional identity

$$\lim_{\epsilon \rightarrow 0^+} \Theta(\sigma) \frac{\sigma^{\epsilon-1}}{\Gamma(\epsilon)} = \delta(\sigma). \quad (\text{C.8})$$

From this one finds

$$\begin{aligned} \lim_{s \rightarrow 1^+} \mathcal{G}_{\text{ret}}(x - y) &= \frac{\Theta(x^0 - y^0)}{2\pi} \lim_{s \rightarrow 1^+} \frac{[(x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2]^{(s-1)-1}}{\Gamma(s-1)} \Theta((x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2) \\ &= \frac{\Theta(x^0 - y^0)}{2\pi} \delta((x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2) = \frac{\delta(x^0 - y^0 - |\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|}, \end{aligned} \quad (\text{C.9})$$

where in the last step we took advantage of the well known property of the delta function $\delta(f(x)) = \sum_i \frac{\delta(x_i)}{|df/dx(x_i)|}$, the sum running through the zeroes x_i of the function $f(x)$.

Appendix D

Cosmology with a Non-Minimally Coupled Radiation fluid

We study the evolution of the universe in the case in which dark matter is modeled as a relativistic fluid rather than as dust. This approach accounts for the possibility that the non-minimal coupling becomes relevant at sufficiently early times, when dark matter particles have relativistic velocities. The Friedmann equation (7.8) in this case becomes

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{b,0}(1+z)^3 + \Omega_{rel,0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2(1 + \epsilon\bar{\chi}_0(1+z)^4)}{1 + \epsilon\bar{\chi}_0(1+z)^4}, \quad (\text{D.1})$$

where $\bar{\chi}_0 = \frac{16}{3}\pi GL^2\rho_{\text{DM}}$, $\Omega_{rel,0} = \Omega_{r,0} + \Omega_{\text{DM},0}$, and we leave the polarity parameter ϵ unspecified. Three major differences appear from equation (D.1):

- to avoid a curvature singularity at a finite value of z , we must set $\epsilon = +1$;
- the end of the NMC phase is delayed, with $\log(1 + \bar{z}_{NMC}) = \frac{3}{4}\log(1 + z_{NMC})$;
- curvature is non-minimally coupled as $\Omega_k(1 + \bar{\chi})$ rather than $\Omega_k(1 - \chi)$.

The first two points pose serious challenges to the model. The first suggests that the non-minimal coupling term in the action would need to change sign as the equation of state parameter transitions from $w = 0$ to $w = 1/3$, yet there is no natural way to implement such a change. The second indicates that the end of the non-minimal coupling–driven phase occurs much later in cosmic history, potentially spoiling standard cosmological predictions and contradicting its own assumptions, since dark matter is expected to be non-relativistic at later times. The only possible workaround is to assume that the length scale L decreases by several orders of magnitude as w changes from 0 to $1/3$, but again, no natural mechanism exists to justify this transition.

Nonetheless, if we assume that such mechanisms exist, the resulting evolution is quite intriguing. When the non-minimal coupling is significant, i.e. when $\bar{\chi} \gg 1$, equation (D.1) becomes

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{rel,0}}{\bar{\chi}_0} + \Omega_{k,0}(1+z)^2. \quad (\text{D.2})$$

Whenever $\Omega_{k,0}(1+z)^2 \ll 1$, the universe undergoes a phase of exponential expansion, as predicted by standard inflationary models. Moreover, the cosmological bounce occurs with positive spatial curvature $k > 0$ (and $\Omega_k < 0$), like in conventional cosmological bounce scenarios [284, 300].

Appendix E

Evolution of Perturbations around the Bounce

Close to the bounce, the conformal Hubble parameter no longer satisfies the differential equation

$$\mathcal{H}' \sim -\frac{1}{2}(1+3w)\mathcal{H}^2, \quad (\text{E.1})$$

but instead obeys

$$\mathcal{H}' \sim \frac{1}{2}(\mathcal{H}^2 - \kappa). \quad (\text{E.2})$$

For negative spatial curvature, its dependence on conformal time is therefore

$$\mathcal{H} \sim \sqrt{|\kappa|} \tan\left(\frac{1}{2}\sqrt{|\kappa|}\eta\right), \quad (\text{E.3})$$

and the conformal time can be written in terms of the scale factor as

$$\eta \sim \frac{2}{\sqrt{|\kappa|}} \arccos\left(\sqrt{\frac{a_{\text{BNC}}}{a}}\right), \quad \implies \quad \mathcal{H} \sim \sqrt{|\kappa| \left(\frac{a}{a_{\text{BNC}}} - 1\right)}, \quad (\text{E.4})$$

where a_{BNC} is the value of the scale factor at the bounce.

If we use Equation (E.2) instead of Equation (E.1), the expression for the comoving curvature perturbation becomes

$$\mathcal{R} = \Psi + \frac{2}{3} \frac{\Psi + \frac{\Psi'}{\mathcal{H}} - \chi \left(\Psi + \frac{\Psi'}{\mathcal{H}} - \left(\frac{3}{2} + \frac{|\kappa|}{2\mathcal{H}^2} \right) \mathcal{H}v_{\text{DM}} + 2v'_{\text{DM}} \right)}{\frac{1}{3} - \frac{|\kappa|}{2\mathcal{H}^2} + \frac{4}{3}\chi \left(1 + \frac{|\kappa|}{\mathcal{H}^2} \right)}, \quad (\text{E.5})$$

and the equations for δ_{DM} , $\mathcal{R}$, and v_{DM} , in the limit of $\chi \gg 1$ take the form

$$\begin{aligned}
 & \delta'_{\text{DM}} - \frac{18\kappa\mathcal{H}(2\kappa - \mathcal{H}^2)}{18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4} \delta_{\text{DM}} - \frac{36(2\kappa^2 - 3\kappa\mathcal{H}^2 + \mathcal{H}^4)}{18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4} \mathcal{R}' + \\
 & - \frac{12(2k^2 - 3\kappa)(k - \mathcal{H}^2)}{18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4} \mathcal{H}\mathcal{R} + \frac{3(6\kappa^2 + (8k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)}{18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4} \mathcal{H}v'_{\text{DM}} + \\
 & + \frac{18\kappa^2(2k^2 - \kappa) + (2k^2 - 27\kappa)(4k^2 + 3\kappa)\mathcal{H}^2 + 6(k^2 + 18\kappa)\mathcal{H}^4 - 9\mathcal{H}^6}{2(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} v_{\text{DM}} = 0 \\
 & \mathcal{R}'' + \frac{\kappa(27\kappa + 4k^2) - (15\kappa + 4k^2)\mathcal{H}^2 + 6\mathcal{H}^4}{18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4} \mathcal{H}\mathcal{R}' + \\
 & + \frac{9\kappa^2(4k^2 - 15\kappa) + 4(4k^2 - 9\kappa)(k^2 + 3\kappa)\mathcal{H}^2 - 3(8k^2 - 9\kappa)\mathcal{H}^4}{6(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \mathcal{R} + \frac{5\kappa + \mathcal{H}^2}{4(\kappa - \mathcal{H}^2)} \mathcal{H}v''_{\text{DM}} + \\
 & - \frac{3(30\kappa^4 - 12\kappa^3\mathcal{H}^2 + \kappa(16k^2 - 19\kappa)\mathcal{H}^4 - 10\kappa\mathcal{H}^6 + \mathcal{H}^8)}{4(\kappa - \mathcal{H}^2)(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} v'_{\text{DM}} + \\
 & - \left(6\kappa^3(16k^2 - 153\kappa) - \kappa(21k^4 + 900k^2\kappa - 261\kappa^2)\mathcal{H}^2 - (16k^4 - 276k^2\kappa - 927\kappa^2)\mathcal{H}^4 + \right. \\
 & \left. + 3(8k^2 - 87\kappa)\mathcal{H}^6 + 9\mathcal{H}^8 \right) \left(24(\kappa - \mathcal{H}^2)(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4) \right)^{-1} \mathcal{H}v_{\text{DM}} + \\
 & + \frac{9\kappa^3 + \kappa(4k^2 + 9\kappa)\mathcal{H}^2 - (2k^2 + 9\kappa)\mathcal{H}^4 + 3\mathcal{H}^6}{(\kappa - \mathcal{H}^2)(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \kappa\delta_{\text{DM}} = 0, \\
 & \frac{(5\kappa + \mathcal{H}^2)(9\kappa^2 + (k^2 - 15\kappa)\mathcal{H}^2 + 3\mathcal{H}^4)}{2(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} v'_{\text{DM}} + \\
 & + \frac{4\kappa^2(5k^2 - 24\kappa) - \kappa(76k^2 - 69\kappa)\mathcal{H}^2 + 2(10k^2 + 21\kappa)\mathcal{H}^4 + 16\mathcal{H}^6}{4(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \mathcal{H}v_{\text{DM}} + \\
 & - \frac{(2k^2 + 21\kappa - 9\mathcal{H}^2)(\kappa - \mathcal{H}^2)}{2(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \mathcal{H}\mathcal{R}' + \\
 & + \frac{3(\kappa - \mathcal{H}^2)(\kappa(2k^2 + 9\kappa) - (2k^2 - 4\kappa)\mathcal{H}^2 - \mathcal{H}^4)}{2(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \mathcal{R} + \\
 & - \frac{18\kappa^3 + \kappa(8k^2 + 15\kappa)\mathcal{H}^2 - 4(k^2 + 3\kappa)\mathcal{H}^4 + 3\mathcal{H}^6}{4(18\kappa^2 + (4k^2 - 9\kappa)\mathcal{H}^2 - 3\mathcal{H}^4)} \delta_{\text{DM}} = 0.
 \end{aligned} \tag{E.6}$$

Equations (E.6) do not admit a simple analytic solution; however, an approximate solution can be found in the limit $\mathcal{H} \rightarrow 0$. In this regime, δ_{DM} , $\mathcal{R}$ and v_{DM} are linear combinations of exponential functions whose exponents,

$$\pm k\eta \sqrt{\frac{4}{15} \left(-4 + \frac{9|\kappa|}{8k^2} \pm \sqrt{-14 - \frac{351|\kappa|}{4k^2} + \frac{2241\kappa^2}{64k^4}} \right)}, \tag{E.7}$$

always contain non-zero real and imaginary parts. Expanding the solutions around $\eta = 0$, we obtain

$$\begin{aligned}\delta_{\text{DM}} &\sim C_{3,\text{BNC}} + \left(4C_{2,\text{BNC}} - (k^2 + 3|\kappa|) C_{4,\text{BNC}}\right) \eta, \\ \mathcal{R} &\sim C_{1,\text{BNC}} + C_{2,\text{BNC}}\eta, \\ v_{\text{DM}} &\sim C_{4,\text{BNC}} + \left(\frac{C_{3,\text{BNC}}}{5} + \frac{(2k^2 - 9|\kappa|) C_{1,\text{BNC}}}{15|\kappa|}\right) \eta.\end{aligned}\tag{E.8}$$

If the constant mode dominates before the bounce, while the time-dependent mode dominates after the bounce, all modes of the comoving curvature perturbation receive a small enhancement proportional to

$$C_{2,\text{BNC}}\eta_{\text{BNC}\rightarrow\text{NMC}} \sim C_{2,\text{BNC}}\sqrt{\frac{4}{|\kappa|}\left(\frac{a_{\text{BNC}\rightarrow\text{NMC}}}{a_{\text{BNC}}} - 1\right)} \sim C_{2,\text{BNC}}\sqrt{\frac{4}{|\kappa|}\frac{a_{\text{BNC}\rightarrow\text{NMC}}}{a_{\text{BNC}}}},\tag{E.9}$$

where $a_{\text{BNC}\rightarrow\text{NMC}}$ indicates the value of the scale factor for which the evolution has reduced to the standard NMC domination. This enhancement occurs because, although the conformal Hubble parameter vanishes at the bounce, the relevant long-wavelength condition is $k\eta \ll 1$. Since the particle horizon

$$d_{\text{H}} := \int dt \frac{1}{a(t)} = \eta,\tag{E.10}$$

also vanishes at the bounce, all perturbation modes satisfy the long-wavelength condition and behave as effectively super-horizon modes in the sense of $kd_{\text{H}} \ll 1$, even though $k/\mathcal{H}$ diverges. The bounce therefore introduces only a small correction to the amplitude and does not modify the spectral dependence.

Appendix F

Solution of the Scalar Sector in the NMC-Epoch

While equations (8.15) are already much simpler than the full set of equations, they can acquire a very compact form by defining the variable

$$\tilde{\mathcal{R}} := \mathcal{R} - \frac{\mathcal{H}}{4} v_{\text{DM}}. \quad (\text{F.1})$$

With this choice of variable, the Equations (8.15) can be written as

$$\begin{aligned} \tilde{\mathcal{R}}'' - 2\mathcal{H} \frac{2k^2 - 3\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \tilde{\mathcal{R}}' + \frac{4}{3} \frac{2k^2 - 3\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \tilde{\mathcal{R}} &= 0, \\ \delta'_{\text{DM}} + 6\mathcal{H} v'_{\text{DM}} + (k^2 + 3\mathcal{H}^2) v_{\text{DM}} - \frac{36\mathcal{H}^2}{4k^2 - 3\mathcal{H}^2} \tilde{\mathcal{R}}' + \frac{24k^2\mathcal{H}}{4k^2 - 3\mathcal{H}^2} \tilde{\mathcal{R}} &= 0, \\ \frac{v'_{\text{DM}}}{k} + 6\mathcal{H} \frac{v_{\text{DM}}}{k} + \delta_{\text{DM}} + \frac{4}{\mathcal{H}} \frac{2k^2 - 9\mathcal{H}^2}{k^2 + 3\mathcal{H}^2} \tilde{\mathcal{R}}' + \frac{6(2k^2 + \mathcal{H}^2)}{4k^2 - 3\mathcal{H}^2} \tilde{\mathcal{R}} &= 0. \end{aligned} \quad (\text{F.2})$$

Since during the NMC epoch the comoving Hubble parameter scales as $\mathcal{H} = -2/\eta$, we can solve analytically the first of the Equations (F.2), yielding

$$\tilde{\mathcal{R}} = \tilde{C}_{1,\text{NMC}} \left(1 - i\sqrt{\frac{2}{3}}k\eta - \frac{2}{3}k^2\eta^2 \right) \frac{e^{i\sqrt{\frac{2}{3}}k\eta}}{\eta^3} + \tilde{C}_{2,\text{NMC}} \left(1 + i\sqrt{\frac{2}{3}}k\eta - \frac{2}{3}k^2\eta^2 \right) \frac{e^{-i\sqrt{\frac{2}{3}}k\eta}}{\eta^3}, \quad (\text{F.3})$$

which can be used in the other two Equations (F.2) to find the density perturbation and the velocity field

$$\begin{aligned}\delta_{\text{DM}} = & \tilde{C}_{1,\text{NMC}} \frac{12}{20} \left(20 - i20\sqrt{\frac{2}{3}}k\eta - 3\frac{2}{3}k^2\eta^2 - i\left(\frac{2}{3}\right)^{3/2}k^3\eta^3 \right) \frac{e^{i\sqrt{\frac{2}{3}}k\eta}}{\eta^3} + \\ & + \tilde{C}_{2,\text{NMC}} \left(20 + i20\sqrt{\frac{2}{3}}k\eta - 3\frac{2}{3}k^2\eta^2 + i\left(\frac{2}{3}\right)^{3/2}k^3\eta^3 \right) \frac{e^{-i\sqrt{\frac{2}{3}}k\eta}}{\eta^3} + \\ & + \tilde{C}_{3,\text{NMC}} (12 - k\eta) \frac{e^{k\eta}}{\eta} + \tilde{C}_{4,\text{NMC}} (12 + k\eta) \frac{e^{-k\eta}}{\eta},\end{aligned}$$

$$v_{\text{DM}} = \tilde{C}_{1,\text{NMC}} \frac{2}{5} \left(5 - i5\sqrt{\frac{2}{3}}k\eta - \frac{2}{3}k^2\eta^2 \right) \frac{e^{i\sqrt{\frac{2}{3}}k\eta}}{\eta^2} + \tilde{C}_{2,\text{NMC}} \frac{2}{5} \left(5 + i5\sqrt{\frac{2}{3}}k\eta - \frac{2}{3}k^2\eta^2 \right) \frac{e^{-i\sqrt{\frac{2}{3}}k\eta}}{\eta^2} + \quad (\text{F.4})$$

$$+ \tilde{C}_{3,\text{NMC}} e^{k\eta} + \tilde{C}_{4,\text{NMC}} e^{-k\eta}. \quad (\text{F.5})$$

where the $\tilde{C}_{i,\text{NMC}}$ are integration constants. The variable $\tilde{\mathcal{R}}$ and the velocity field v_{DM} combine to give the simple form for the comoving curvature perturbation in Equation (8.16), after the definition of free parameters $C_{1,\text{NMC}} := -\frac{8}{15}\tilde{C}_{1,\text{NMC}}$, $C_{2,\text{NMC}} := -\frac{8}{15}\tilde{C}_{2,\text{NMC}}$, $C_{3,\text{NMC}} := -\frac{1}{2}\tilde{C}_{3,\text{NMC}}$ and $C_{4,\text{NMC}} := -\frac{1}{2}\tilde{C}_{4,\text{NMC}}$.

Bibliography

- [1] Dario Bettoni, Stefano Liberati, and Lorenzo Sindoni. Extended Λ CDM: generalized non-minimal coupling for dark matter fluids. *JCAP*, 2011(11):007, November 2011.
- [2] Dario Bettoni, Valeria Pettorino, Stefano Liberati, and Carlo Baccigalupi. Non-minimally coupled dark matter: effective pressure and structure formation. *JCAP*, 2012(7):027, July 2012.
- [3] Giovanni Gandolfi, Andrea Lapi, and Stefano Liberati. Self-gravitating equilibria of non-minimally coupled dark matter halos. *The Astrophysical Journal*, 910(1):76, March 2021.
- [4] Giovanni Gandolfi, Andrea Lapi, and Stefano Liberati. Empirical Evidence of Nonminimally Coupled Dark Matter in the Dynamics of Local Spiral Galaxies? *ApJ*, 929(1):48, April 2022.
- [5] Gianfranco Bertone and Dan Hooper. History of dark matter. *Reviews of Modern Physics*, 90:045002, 2018.
- [6] F. Zwicky. Die rotverschiebung von extragalaktischen nebeln. *Helvetica Physica Acta*, 6:110–127, 1933.
- [7] Vera C. Rubin, Jr. Ford, W. Kent, and Norbert Thonnard. Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 ($R=4$ kpc) to UGC 2885 ($R=122$ kpc). *The Astrophysical Journal*, 238:471–487, 1980.
- [8] Vera C. Rubin, David Burstein, Jr. Ford, W. Kent, and Norbert Thonnard. Rotation velocities of 16 SA galaxies and a comparison of SA, SB, and SC rotation properties. *The Astrophysical Journal*, 289:81–104, 1985.
- [9] V. C. Rubin, W. K. Ford, Jr., and N. Thonnard. Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 ($R=4$ kpc) to UGC 2885 ($R=122$ kpc). , 238:471–487, June 1980.

- [10] Massimo Persic, Paolo Salucci, and Fulvio Stel. The universal rotation curve of spiral galaxies — I. The dark matter connection. , 281(1):27–47, July 1996.
- [11] Paolo Salucci. The distribution of dark matter in galaxies. , 27(1):2, February 2019.
- [12] Gary Steigman, David N. Schramm, and James E. Gunn. Cosmological limits to the number of massive leptons. *Physical Review Letters*, 43:30–33, 1979.
- [13] P. J. E. Peebles. Large-scale structure of the universe. *The Astrophysical Journal*, 263:L1–L5, 1982.
- [14] J. Richard Bond, Alexander S. Szalay, and Michael S. Turner. Formation of galaxies in a gravitationally dominated universe. *Physical Review Letters*, 48:1636–1639, 1982.
- [15] George R. Blumenthal, S. M. Faber, Joel R. Primack, and Martin J. Rees. Formation of galaxies and large-scale structure with cold dark matter. *Nature*, 311:517–525, 1984.
- [16] Simon D. M. White, Julio F. Navarro, August E. Evrard, and Carlos S. Frenk. The baryon content of galaxy clusters: a challenge to cosmological orthodoxy. , 366(6454):429–433, December 1993.
- [17] Steven W. Allen, August E. Evrard, and Adam B. Mantz. Cosmological Parameters from Observations of Galaxy Clusters. , 49(1):409–470, September 2011.
- [18] Christian Garrel, Marguerite Pierre, Patrick Valageas, Dominique Eckert, Federico Marulli, Alfonso Veropalumbo, Florian Pacaud, Nicolas Clerc, Mauro Sereno, Keiichi Umetsu, Lauro Moscardini, Sunayana Bhargava, Christophe Adami, Lucio Chiappetti, Fabio Gastaldello, Elias Koulouridis, Jean-Paul Le Fevre, and Manolis Plionis. The XXL survey. XLVI. Forward cosmological analysis of the C1 cluster sample. *A&A*, 663:A3, July 2022.
- [19] Adam B. Mantz, R. Glenn Morris, Steven W. Allen, Rebecca E. A. Canning, Lucie Baumont, Bradford Benson, Lindsey E. Bleem, Steven R. Ehlert, Benjamin Floyd, Ricardo Herbonnet, Patrick L. Kelly, Shuang Liang, Anja von der Linden, Michael McDonald, David A. Rapetti, Robert W. Schmidt, Norbert Werner, and Adam Wright. Cosmological constraints from gas mass fractions of massive, relaxed galaxy clusters. *Monthly Notices of the Royal Astronomical Society*, 510(1):131–145, February 2022.
- [20] Priyamvada Natarajan, Liliya L. Williams, Marusa Bradac, Claudio Grillo, Agniva Ghosh, Keren Sharon, and Jenny Wagner. Strong lensing by galaxy clusters, 2024.
- [21] Keiichi Umetsu, Mauro Sereno, Maggie Lieu, Hironao Miyatake, Elinor Medezinski, Atsushi J. Nishizawa, Paul Giles, Fabio Gastaldello, Ian G. McCarthy, Martin Kilbinger,

- Mark Birkinshaw, Stefano Ettori, Nobuhiro Okabe, I-Non Chiu, Jean Coupon, Dominique Eckert, Yutaka Fujita, Yuichi Higuchi, Elias Koulouridis, Ben Maughan, Satoshi Miyazaki, Masamune Oguri, Florian Pacaud, Marguerite Pierre, David Rapetti, and Graham P. Smith. Weak-lensing analysis of x-ray-selected xxl galaxy groups and clusters with subaru hsc data. *The Astrophysical Journal*, 890(2):148, February 2020.
- [22] M. Markevitch, A. H. Gonzalez, D. Clowe, A. Vikhlinin, W. Forman, C. Jones, S. Murray, and W. Tucker. Direct Constraints on the Dark Matter Self-Interaction Cross Section from the Merging Galaxy Cluster 1E 0657-56. , 606(2):819–824, May 2004.
- [23] Douglas Clowe, Maru sa Brada c, Anthony H. Gonzalez, Maxim Markevitch, Scott W. Randall, Christine Jones, and Dennis Zaritsky. A Direct Empirical Proof of the Existence of Dark Matter. , 648(2):L109–L113, September 2006.
- [24] D. Paraficz, J.-P. Kneib, J. Richard, A. Morandi, M. Limousin, E. Jullo, and J. Martinez. The bullet cluster at its best: weighing stars, gas, and dark matter. *A&A*, 594:A121, 2016.
- [25] G. Hinshaw et al. Nine-year wilkinson microwave anisotropy probe (WMAP) observations: Cosmological parameter results. *The Astrophysical Journal Supplement Series*, 208:19, 2013.
- [26] Planck Collaboration. Planck 2018 results. vi. cosmological parameters. *Astronomy & Astrophysics*, 641:A6, 2020.
- [27] Gerard Jungman, Marc Kamionkowski, and Kim Griest. Supersymmetric dark matter. *Physics Reports*, 267:195–373, 1995.
- [28] Jonathan L. Feng. Dark matter candidates from particle physics and methods of detection. *Annual Review of Astronomy and Astrophysics*, 48(Volume 48, 2010):495–545, 2010.
- [29] Marc Kamionkowski. Wimp and axion dark matter, 1997.
- [30] KIM GRIEST. The search for the dark matter: Wimps and machos. *Annals of the New York Academy of Sciences*, 688(1):390–407, 1993.
- [31] Gianfranco Bertone and Dan Hooper. History of dark matter. *Reviews of Modern Physics*, 90(4), October 2018.
- [32] Hsin-Chia Cheng, Jonathan L. Feng, and Konstantin T. Matchev. Kaluza-klein dark matter. *Phys. Rev. Lett.*, 89:211301, Oct 2002.
- [33] Julio F. Navarro, Carlos S. Frenk, and Simon D. M. White. A Universal Density Profile from Hierarchical Clustering. , 490(2):493–508, December 1997.

- [34] Ricardo A. Flores and Joel R. Primack. Observational and Theoretical Constraints on Singular Dark Matter Halos. , 427:L1, May 1994.
- [35] G. Gentile, P. Salucci, U. Klein, D. Vergani, and P. Kalberla. The cored distribution of dark matter in spiral galaxies. *Monthly Notices of the Royal Astronomical Society*, 351(3):903–922, 07 2004.
- [36] W. J. G. et al. de Blok. High-Resolution Rotation Curves and Galaxy Mass Models from THINGS. , 136(6):2648–2719, December 2008.
- [37] Se-Heon Oh, Deidre A. Hunter, Elias Brinks, Bruce G. Elmegreen, Andreas Schruba, Fabian Walter, Michael P. Rupen, Lisa M. Young, Caroline E. Simpson, Megan C. Johnson, Kimberly A. Herrmann, Dana Ficut-Vicas, Phil Cigan, Volker Heesen, Trisha Ashley, and Hong-Xin Zhang. High-resolution Mass Models of Dwarf Galaxies from LITTLE THINGS. , 149(6):180, June 2015.
- [38] Oliver M"uller, Marcel S. Pawlowski, Yves Revaz, Aku Venhola, Marina Rejkuba, Michael Hilker, and Katharina Lutz. A too-many-dwarf-galaxy-satellites problem in the m 83 group. *A&A*, 684:L6, 2024.
- [39] Michael Boylan-Kolchin, James S. Bullock, and Manoj Kaplinghat. Too big to fail? the puzzling darkness of massive milky way subhaloes. *Monthly Notices of the Royal Astronomical Society: Letters*, 415(1):L40–L44, July 2011.
- [40] Volker Springel, Jie Wang, Mark Vogelsberger, Aaron Ludlow, Adrian Jenkins, Amina Helmi, Julio F. Navarro, Carlos S. Frenk, and Simon D. M. White. The aquarius project: The subhaloes of galactic haloes. *Monthly Notices of the Royal Astronomical Society*, 391(4):1685–1711, December 2008.
- [41] Jürg Diemand, Michael Kuhlen, Piero Madau, Marcel Zemp, Ben Moore, Doug Potter, and Joachim Stadel. Clumps and streams in the local dark matter distribution. *Nature*, 454(7205):735–738, August 2008.
- [42] Stacy S. McGaugh, Federico Lelli, and James M. Schombert. Radial acceleration relation in rotationally supported galaxies. *Phys. Rev. Lett.*, 117:201101, Nov 2016.
- [43] Federico Lelli, Stacy S. McGaugh, James M. Schombert, and Marcel S. Pawlowski. One Law to Rule Them All: The Radial Acceleration Relation of Galaxies. , 836(2):152, February 2017.

- [44] Chiara Di Paolo, Paolo Salucci, and Adnan Erkurt. The universal rotation curve of low surface brightness galaxies - IV. The interrelation between dark and luminous matter. , 490(4):5451–5477, December 2019.
- [45] F. Donato, G. Gentile, P. Salucci, C. Frigerio Martins, M. I. Wilkinson, G. Gilmore, E. K. Grebel, A. Koch, and R. Wyse. A constant dark matter halo surface density in galaxies. *Monthly Notices of the Royal Astronomical Society*, 397(3):1169–1176, 08 2009.
- [46] Fiorenza Donato, Gianfranco Gentile, and Paolo Salucci. Cores of dark matter haloes correlate with stellar scalelengths. , 353(2):L17–L22, September 2004.
- [47] Paolo Salucci, Nicola Turini, and Chiara di Paolo. Paradigms and scenarios for the dark matter phenomenon. *Universe*, 6(8), 2020.
- [48] Amr El-Zant, Isaac Shlosman, and Yehuda Hoffman. Dark Halos: The Flattening of the Density Cusp by Dynamical Friction. , 560(2):636–643, October 2001.
- [49] C. Tonini, A. Lapi, and P. Salucci. Angular momentum transfer in dark matter halos: Erasing the cusp. *The Astrophysical Journal*, 649(2):591, oct 2006.
- [50] Andrew Pontzen and Fabio Governato. Cold dark matter heats up. , 506(7487):171–178, February 2014.
- [51] Jonathan Freundlich, Avishai Dekel, Fangzhou Jiang, Guy Ishai, Nicolas Cornuault, Sharon Lapiner, Aaron A Dutton, and Andrea V Macciò. A model for core formation in dark matter haloes and ultra-diffuse galaxies by outflow episodes. *Monthly Notices of the Royal Astronomical Society*, 491(3):4523–4542, November 2019.
- [52] Gianfranco Bertone, Dan Hooper, and Joseph Silk. Particle dark matter: evidence, candidates and constraints. , 405(5-6):279–390, January 2005.
- [53] R. Adhikari, M. Agostini, N. Anh Ky, T. Araki, M. Archidiacono, et al. A white paper on kev sterile neutrino dark matter. *Journal of Cosmology and Astroparticle Physics*, 2017(1):025, January 2017.
- [54] Steen Hannestad, Alessandro Mirizzi, and Georg G. Raffelt. New cosmological mass limit on thermal relic axions. *Journal of Cosmology and Astroparticle Physics*, 2005(07):002, 2005.
- [55] Steen Hannestad, Alessandro Mirizzi, Georg G. Raffelt, and Yvonne Y. Y. Wong. Cosmological constraints on neutrino plus axion hot dark matter. *Journal of Cosmology and Astroparticle Physics*, 2007(08):015, 2007.

- [56] Steen Hannestad, Alessandro Mirizzi, Georg G. Raffelt, and Yvonne Y. Y. Wong. Cosmological constraints on neutrino plus axion hot dark matter: update after wmap-5. *Journal of Cosmology and Astroparticle Physics*, 2008(04):019, 2008.
- [57] Steen Hannestad, Alessandro Mirizzi, Georg G. Raffelt, and Yvonne Y. Y. Wong. Neutrino and axion hot dark matter bounds after wmap-7. *Journal of Cosmology and Astroparticle Physics*, 2010(08):001, 2010.
- [58] Salah Eddine Ennadifi. Massive dark photons as hot dark matter. *Acta Physica Polonica B*, 53(11-A2), 2022.
- [59] Subinoy Das, Anshuman Maharana, Vivian Poulin, and Ravi Kumar Sharma. Nonthermal neutrino-like hot dark matter in light of the S_8 tension. *Physical Review D*, 105(10):103503, 2022.
- [60] Fabian Hervas Peters, Aurel Schneider, Jozef Bucko, Sambit K. Giri, and Gabriele Parimbelli. Constraining hot dark matter sub-species with weak lensing and the cosmic microwave background radiation. *Astronomy & Astrophysics*, 687:A??, 2024.
- [61] Mark R. Lovell, Carlos S. Frenk, Vincent R. Eke, Adrian Jenkins, Liang Gao, and Tom Theuns. The properties of warm dark matter haloes. *Monthly Notices of the Royal Astronomical Society*, 439(1):300–317, 03 2014.
- [62] Rachel Kennedy, Carlos Frenk, Shaun Cole, and Andrew Benson. Constraining the warm dark matter particle mass with milky way satellites. *Monthly Notices of the Royal Astronomical Society*, 442(3):2487–2495, 2014.
- [63] H. J. de Vega and N. G. Sanchez. Model-independent analysis of dark matter points to a particle mass at the kev scale. *Monthly Notices of the Royal Astronomical Society*, 404(2):885–898, 2010.
- [64] H. J. de Vega and N. G. Sanchez. Cosmological evolution of warm dark matter fluctuations. ii. solution from small to large scales and kev sterile neutrinos. *Physical Review D*, 85(4):043517, 2012.
- [65] Jihn E. Kim. Axino as cold dark matter candidate. *Physical Review Letters*, 89:101802, 2002.
- [66] Osamu Seto and Masahide Yamaguchi. Axino warm dark matter and $\omega_b - \omega_{DM}$ coincidence. *Physical Review D*, 75(12):123506, 2007.

- [67] Kyu Jung Bae, Ayuki Kamada, Seng Pei Liew, and Keisuke Yanagi. Light axinos from freeze-in: Production processes, phase space distributions, and $\text{ly-}\alpha$ forest constraints. *Journal of Cosmology and Astroparticle Physics*, 2018(01):054, 2018.
- [68] Cannon M. Vogel and Kevork N. Abazajian. Entering the era of measuring sub-galactic dark matter structure: Accurate transfer functions for axino, gravitino, and sterile neutrino thermal warm dark matter. *Physical Review D*, 108(4):043520, 2023.
- [69] Heinz R. Pagels and Joel R. Primack. Supersymmetry, cosmology, and new tev physics. *Physical Review Letters*, 48:223–226, 1982.
- [70] George R. Blumenthal, Heinz Pagels, and Joel R. Primack. Galaxy formation by dissipationless particles heavier than neutrinos. *Nature*, 299:37–38, 1982.
- [71] J. Richard Bond, Alexander S. Szalay, and Michael S. Turner. Formation of galaxies in a gravitino-dominated universe. *Physical Review Letters*, 48:1636–1639, 1982.
- [72] Kevork Abazajian, George M. Fuller, and Mitesh Patel. Sterile neutrino hot, warm, and cold dark matter. *Physical Review D*, 64(2):023501, 2001.
- [73] Takehiko Asaka, Mikhail Shaposhnikov, and Alexander Kusenko. Opening a new window for warm dark matter. *Physics Letters B*, 638(5–6):401–406, 2006.
- [74] Brian W. O’Shea and Michael L. Norman. Population iii star formation in a lambda wdm universe. *The Astrophysical Journal*, 648(1):31–46, 2006.
- [75] Dmitry Gorbunov, Alexey Khmelnitsky, and Valery Rubakov. Is gravitino still a warm dark matter candidate? *Journal of High Energy Physics*, 2008(12):055, 2008.
- [76] Cannon M. Vogel and Kevork N. Abazajian. Entering the era of measuring sub-galactic dark matter structure: Accurate transfer functions for axino, gravitino sterile neutrino thermal warm dark matter, 2023.
- [77] Pratika Dayal and Sambit K. Giri. Warm dark matter constraints from the jwst. *Monthly Notices of the Royal Astronomical Society*, 2023.
- [78] Mark R. Lovell, Vincent Eke, Carlos S. Frenk, Liang Gao, Adrian Jenkins, Tom Theuns, Jie Wang, Simon D. M. White, Alexey Boyarsky, and Oleg Ruchayskiy. The haloes of bright satellite galaxies in a warm dark matter universe. *Monthly Notices of the Royal Astronomical Society*, 420(3):2318–2324, 2012.

- [79] Donnino Anderhalden, Aurel Schneider, Andrea V. Macciò, Juerg Diemand, and Gianfranco Bertone. The galactic halo in mixed dark matter cosmologies. *Journal of Cosmology and Astroparticle Physics*, 2013(03):014, 2013.
- [80] Emil Polisensky and Massimo Ricotti. Constraints on the dark matter particle mass from the number of milky way satellites. *Physical Review D*, 83(4):043506, 2011.
- [81] Vid Iršič, Matteo Viel, Martin G. Haehnelt, James S. Bolton, Stefano Cristiani, George D. Becker, Valentina D’Odorico, Guido Cupani, Tae-Sun Kim, Trystyn A. M. Berg, Sebastian López, Sara Ellison, Lise Christensen, Kelly D. Denney, and Gábor Wörseck. New constraints on the free-streaming of warm dark matter from intermediate and small scale lyman- α forest data. *Physical Review D*, 96(2):023522, July 2017.
- [82] Ethan O. Nadler, Simon Birrer, Daniel Gilman, Risa H. Wechsler, Xiaolong Du, Andrew Benson, Anna M. Nierenberg, and Tommaso Treu. Dark matter constraints from a unified analysis of strong gravitational lenses and milky way satellite galaxies. *The Astrophysical Journal*, 917(1):7, 2021.
- [83] Ioana A. Zelko, Tommaso Treu, Kevork N. Abazajian, Daniel Gilman, Andrew J. Benson, Simon Birrer, Anna M. Nierenberg, and Alexander Kusenko. Constraints on sterile neutrino models from strong gravitational lensing, milky way satellites, and the lyman- α forest. *Physical Review Letters*, 129(19), November 2022.
- [84] Nilanjan Banik, Jo Bovy, Gianfranco Bertone, Denis Erkal, and T. J. L. de Boer. Novel constraints on the particle nature of dark matter from stellar streams. *Journal of Cosmology and Astroparticle Physics*, 2021(10):043, 2021.
- [85] David N. Spergel and Paul J. Steinhardt. Observational evidence for self-interacting cold dark matter. *Physical Review Letters*, 84(17):3760–3763, 2000.
- [86] Francis-Yan Cyr-Racine, Kris Sigurdson, Jesús Zavala, Torsten Bringmann, Mark Vogelsberger, and Christoph Pfrommer. Ethos – an effective theory of structure formation: From dark particle physics to the matter distribution of the universe. *Physical Review D*, 93(12):123527, 2016.
- [87] Tirtha Sankar Ray, Sambo Sarkar, and Abinash Kumar Shaw. Constraints on dark matter self-interaction from galactic core size. *Journal of Cosmology and Astroparticle Physics*, 2022(09):011, 2022.

- [88] Miguel Rocha, Annika H. G. Peter, James S. Bullock, Manoj Kaplinghat, Shea Garrison-Kimmel, Jorge Onorbe, and Leonidas A. Moustakas. Cosmological simulations with self-interacting dark matter i: Constant-density cores and substructure. *Monthly Notices of the Royal Astronomical Society*, 430(1):81–104, 2013.
- [89] Joshua D. Simon, Alberto D. Bolatto, Adam Leroy, Leo Blitz, and Elinor L. Gates. High-resolution measurements of the halos of four dark matter-dominated galaxies: Deviations from a universal density profile. *The Astrophysical Journal*, 621(2):757–776, 2005.
- [90] W. J. G. de Blok, Fabian Walter, Elias Brinks, Carsten Trachternach, Se-Heon Oh, and Jr. Kennicutt, Robert C. High-resolution rotation curves and galaxy mass models from things. *The Astronomical Journal*, 136(6):2648–2719, November 2008.
- [91] Joshua J. Adams, Joshua D. Simon, Maximilian H. Fabricius, Remco C. E. van den Bosch, John C. Barentine, Ralf Bender, Karl Gebhardt, Gary J. Hill, Jeremy D. Murphy, Rob C. Swaters, Jens Thomas, and Glenn van de Ven. Dwarf galaxy dark matter density profiles inferred from stellar and gas kinematics. *The Astrophysical Journal*, 789(1):63, 2014.
- [92] Mark Vogelsberger, Jesús Zavala, and Abraham Loeb. Subhaloes in self-interacting galactic dark matter haloes. *Monthly Notices of the Royal Astronomical Society*, 423(4):3740–3752, 2012.
- [93] Jesús Zavala, Mark Vogelsberger, and Matthew G. Walker. Constraining self-interacting dark matter with the milky way’s dwarf spheroidals. *Monthly Notices of the Royal Astronomical Society: Letters*, 431(1):L20–L24, 2013.
- [94] Susmita Adhikari, Arka Banerjee, Kimberly K. Boddy, Francis-Yan Cyr-Racine, Harry Desmond, Cora Dvorkin, Bhuvnesh Jain, Felix Kahlhoefer, Manoj Kaplinghat, Anna M. Nierenberg, Annika H. G. Peter, Andrew Robertson, Jeremy Sakstein, and Jesús Zavala. Astrophysical tests of dark matter self-interactions. *Journal of Cosmology and Astroparticle Physics*, 2022(11):024, 2022.
- [95] Matthew R. Buckley, Jesús Zavala, Francis-Yan Cyr-Racine, Kris Sigurdson, and Mark Vogelsberger. Scattering, damping, and acoustic oscillations: Simulating the structure of dark matter halos with relativistic force carriers. *Physical Review D*, 90(4):043524, 2014.
- [96] Ran Huo, Manoj Kaplinghat, Zhen Pan, and Hai-Bo Yu. Signatures of self-interacting dark matter in the matter power spectrum and the cmb. *Physics Letters B*, 783:76–81, 2018.

- [97] Wayne Hu, Rennan Barkana, and Andrei Gruzinov. Fuzzy cold dark matter: The wave properties of ultralight particles. *Phys. Rev. Lett.*, 85:1158–1161, Aug 2000.
- [98] Lam Hui, Jeremiah P. Ostriker, Scott Tremaine, and Edward Witten. Ultralight scalars as cosmological dark matter. *Phys. Rev. D*, 95:043541, Feb 2017.
- [99] Samuel D. McDermott and Samuel J. Witte. Cosmological evolution of light dark photon dark matter. , 101(6):063030, March 2020.
- [100] James S. Bolton, Andrea Caputo, Hongwan Liu, and Matteo Viel. Comparison of low-redshift lyman- α forest observations to hydrodynamical simulations with dark photon dark matter. *Phys. Rev. Lett.*, 129:211102, Nov 2022.
- [101] Masato Shirasaki, Tomoaki Ishiyama, and Shin’ichiro Ando. Virial halo mass function in the planck cosmology. *The Astrophysical Journal*, 922(1):89, nov 2021.
- [102] Andrea Lapi, Tommaso Ronconi, Lumen Boco, Francesco Shankar, Nicoletta Krachmalnicoff, Carlo Baccigalupi, and Luigi Danese. Astroparticle constraints from cosmic reionization and primordial galaxy formation. *Universe*, 8(9):476, September 2022.
- [103] E. Ritondale, S. Vegetti, G. Despali, M. W. Auger, L. V. E. Koopmans, and J. P. McKean. Low-mass halo perturbations in strong gravitational lenses at redshift $z \sim 0.5$ are consistent with cdm. *Monthly Notices of the Royal Astronomical Society*, 485(2):2179–2193, May 2019.
- [104] Isabella P. Carucci, Francisco Villaescusa-Navarro, Matteo Viel, and Andrea Lapi. Warm dark matter signatures on the 21cm power spectrum: Intensity mapping forecasts for ska. *Journal of Cosmology and Astroparticle Physics*, 2015(7):047, July 2015.
- [105] Robert J J Grand and Simon D M White. Dark matter annihilation and the galactic centre excess. *Monthly Notices of the Royal Astronomical Society: Letters*, 511(1):L55–L59, 03 2022.
- [106] Daniel R. Weisz and Michael Boylan-Kolchin. Local group ultra-faint dwarf galaxies in the reionization era. *Monthly Notices of the Royal Astronomical Society: Letters*, 469(1):L83–L88, 07 2017.
- [107] A. Burkert. Fuzzy Dark Matter and Dark Matter Halo Cores. , 904(2):161, December 2020.
- [108] Oliver et al. Newton. Constraints on the properties of warm dark matter using the satellite galaxies of the Milky Way. , 2021(8):062, August 2021.

- [109] Giovanni Gandolfi, Andrea Lapi, Tommaso Ronconi, and Luigi Danese. Astroparticle constraints from the cosmic star formation rate density at high redshift: Current status and forecasts for jwst. *Universe*, 8:589, 11 2022.
- [110] Vasiliki A. Mitsou. Shedding Light on Dark Matter at Colliders. *International Journal of Modern Physics A*, 28(31):1330052, December 2013.
- [111] Spyros Argyropoulos, Oleg Brandt, and Ulrich Haisch. Collider searches for dark matter through the higgs lens. *Symmetry*, 13(12):2406, December 2021.
- [112] XENON1T Collaboration. Dark matter search results from a one ton-year exposure of xenon1t. *Physical Review Letters*, 121(11):111302, September 2018.
- [113] R. Bernabei, P. Belli, V. Caracciolo, R. Cerulli, A. Di Marco, A. Leoncini, V. Merlo, F. Montecchia, F. Cappella, A. d’Angelo, A. Incicchitti, C. J. Dai, X. H. Ma, X. D. Sheng, and Z. P. Ye. Annual modulation results from dama/libra. *International Journal of Modern Physics: Conference Series*, 51:2361008, December 2023.
- [114] Fermi-LAT Collaboration. The fermi galactic center gev excess and implications for dark matter. *The Astrophysical Journal*, 840(1):43, May 2017.
- [115] Juan de Dios Zornoza. Review on indirect dark matter searches with neutrino telescopes. *Universe*, 7(11), 2021.
- [116] M. Milgrom. A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. , 270:365–370, July 1983.
- [117] Jacob D. Bekenstein. Relativistic gravitation theory for the modified newtonian dynamics paradigm. *Phys. Rev. D*, 70:083509, Oct 2004.
- [118] Jean-Philippe Bruneton and Gilles Esposito-Farese. Publisher’s Note: Field-theoretical formulations of MOND-like gravity [Phys. Rev. D 76, 124012 (2007)]. , 76(12):129902, December 2007.
- [119] Benoît Famaey and Stacy S. McGaugh. Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions. *Living Reviews in Relativity*, 15(1):10, December 2012.
- [120] W. J. G. de Blok and S. S. McGaugh. Testing Modified Newtonian Dynamics with Low Surface Brightness Galaxies: Rotation Curve FITS. , 508(1):132–140, November 1998.
- [121] Robert H. Sanders and Stacy S. McGaugh. Modified Newtonian Dynamics as an Alternative to Dark Matter. , 40:263–317, January 2002.

- [122] Pengfei Li, Federico Lelli, Stacy S. McGaugh, and James M. Schombert. Fitting the radial acceleration relation to individual sparc galaxies. *A&A*, 615:A3, 2018.
- [123] Oliver Müller, Federico Lelli, Benoit Famaey, Marcel S. Pawłowski, Katja Fahrion, Marina Rejkuba, Michael Hilker, and Helmut Jerjen. The cen a galaxy group: Dynamical mass and missing baryons. *A&A*, 662:A57, 2022.
- [124] Stacy S. McGaugh, Federico Lelli, James M. Schombert, Pengfei Li, Tiffany Visgaitis, Kaelee S. Parker, and Marcel S. Pawłowski. The baryonic tully–fisher relation in the local group and the equivalent circular velocity of pressure-supported dwarfs. *The Astronomical Journal*, 162(5):202, oct 2021.
- [125] Anthony Aguirre, Joop Schaye, and Eliot Quataert. Problems for modified newtonian dynamics in clusters and the $\text{Ly}\alpha$ forest? *The Astrophysical Journal*, 561(2):550–558, November 2001.
- [126] Douglas Scott, Martin White, Joanne D. Cohn, and Elena Pierpaoli. Cosmological difficulties with modified newtonian dynamics (or: La fin du monde?). *arXiv e-prints*, April 2001.
- [127] Garry W. Angus, Benoit Famaey, and Hongsheng Zhao. Can mond take a bullet? analytical comparisons of three versions of mond beyond spherical symmetry. *Monthly Notices of the Royal Astronomical Society*, 371(1):138–146, September 2006.
- [128] Stacy S. McGaugh. A tale of two paradigms: the mutual incommensurability of Λ CDM and MOND. *Canadian Journal of Physics*, 93(2):250–259, February 2015.
- [129] Theodorus Maria Nieuwenhuizen. How zwicky already ruled out modified gravity theories without dark matter. *Fortschritte der Physik*, 65(6-8):1600050, 2017.
- [130] Sibel Boran, Shantanu Desai, Emre O. Kahya, and R. P. Woodard. Gw170817 falsifies dark matter emulators. *Physical Review D*, 97(4):041501, February 2018.
- [131] Jacob D. Bekenstein. Relation between physical and gravitational geometry. *PRD*, 48(8):3641–3647, October 1993.
- [132] Jean-Philippe Bruneton, Stefano Liberati, Lorenzo Sindoni, and Benoit Famaey. Reconciling MOND and dark matter? *JCAP*, 2009(3):021, March 2009.
- [133] Dario Bettoni, Mattia Colombo, and Stefano Liberati. Dark matter as a Bose-Einstein Condensate: the relativistic non-minimally coupled case. *JCAP*, 2014(2):004, February 2014.

- [134] Dario Bettoni and Stefano Liberati. Dynamics of non-minimally coupled perfect fluids. *JCAP*, 2015(8):023–023, August 2015.
- [135] Giovanni Gandolfi, Andrea Lapi, and Stefano Liberati. Self-gravitating Equilibria of Non-minimally Coupled Dark Matter Halos. *ApJ*, 910(1):76, March 2021.
- [136] Giovanni Gandolfi, Balakrishna S. Haridasu, Stefano Liberati, and Andrea Lapi. Looking for Traces of Nonminimally Coupled Dark Matter in the X-COP Galaxy Clusters Sample. *ApJ*, 952(2):105, August 2023.
- [137] Andrea Giusti. Mond-like fractional laplacian theory. *Phys. Rev. D*, 101:124029, Jun 2020.
- [138] Michele Caputo. Linear models of dissipation whose q is almost frequency independent—ii. *Geophysical Journal International*, 13(5):529–539, 11 1967.
- [139] Mateusz Kwaśnicki. Ten equivalent definitions of the fractional laplace operator. *Fractional Calculus and Applied Analysis*, 20(1):7–51, February 2017.
- [140] Anna Lischke, Guofei Pang, Mamikon Gulian, Emin Song, Christian Glusa, Xiaoning Zheng, Zander Rojas-Palermo, Massoud Khan, Mees Al Alzahrani, George Em Karniadakis, and Panagiotis Stinis. What is the fractional laplacian? a comparative review with new results. *Journal of Computational Physics*, 404:109009, 2020.
- [141] N. S. Landkof. Foundations of modern potential theory. 1972.
- [142] A. M. Frassino and O. Panella. Quantization of nonlocal fractional field theories via the extension problem. *Phys. Rev. D*, 100:116008, Dec 2019.
- [143] A. V. Balakrishnan. Fractional powers of closed operators and the semigroups generated by them. *Pacific Journal of Mathematics*, 10:419–437, 1960.
- [144] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*, volume 30 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.
- [145] Gianluca Calcagni and Giuseppe Nardelli. Representations of the fractional d’alembertian and initial conditions in fractional dynamics. *Chaos, Solitons amp; Fractals*, 201:117401, December 2025.
- [146] Paolo Salucci, Irina A. Yegorova, and Niv Drory. The disc mass of spiral galaxies. *Monthly Notices of the Royal Astronomical Society*, 388(1):159–164, 07 2008.
- [147] R. Dehghani, P. Salucci, and H. Ghaffarnejad. Navarro-Frenk-White dark matter profile and the dark halos around disk systems. , 643:A161, November 2020.

- [148] E. V. Karukes and P. Salucci. The universal rotation curve of dwarf disc galaxies. *Monthly Notices of the Royal Astronomical Society*, 465(4):4703–4722, November 2016.
- [149] A. Lapi, P. Salucci, and L. Danese. Precision scaling relations for disk galaxies in the local universe. *The Astrophysical Journal*, 859(1):2, May 2018.
- [150] K. C. Freeman. On the Disks of Spiral and S0 Galaxies. , 160:811, June 1970.
- [151] Aaron A. Dutton and Andrea V. Macciò. Cold dark matter haloes in the Planck era: evolution of structural parameters for Einasto and NFW profiles. *MNRAS*, 441(4):3359–3374, July 2014.
- [152] Daniel Foreman-Mackey, David W. Hogg, Dustin Lang, and Jonathan Goodman. emcee: The MCMC Hammer. *PASP*, 125(925):306, March 2013.
- [153] Benjamin Nelson, Eric B. Ford, and Matthew J. Payne. Run dmc: An efficient, parallel code for analyzing radial velocity observations using n-body integrations and differential evolution markov chain monte carlo. *The Astrophysical Journal Supplement Series*, 210(1):11, dec 2013.
- [154] C.J.F. {ter Braak} and J.A. Vrugt. Differential evolution markov chain with snooker updater and fewer chains. *Statistics and Computing*, 18(4):435–446, 2008.
- [155] A. Burkert. Fuzzy Dark Matter and Dark Matter Halo Cores. , 904(2):161, December 2020.
- [156] Benjamin P. Moster, Thorsten Naab, and Simon D. M. White. Galactic star formation and accretion histories from matching galaxies to dark matter haloes. *Monthly Notices of the Royal Astronomical Society*, 428(4):3121–3138, February 2013.
- [157] Stacy S. McGaugh, Federico Lelli, and James M. Schombert. Radial acceleration relation in rotationally supported galaxies. *Physical Review Letters*, 117(20), November 2016.
- [158] Federico Lelli, Stacy S. McGaugh, and James M. Schombert. SPARC: Mass Models for 175 Disk Galaxies with Spitzer Photometry and Accurate Rotation Curves. , 152(6):157, December 2016.
- [159] Jonathan Freundlich, Benoit Famaey, Pierre-Antoine Oria, Michal Bílek, Oliver Müller, and Rodrigo Ibata. Probing the radial acceleration relation and the strong equivalence principle with the coma cluster ultra-diffuse galaxies. *Astronomy & Astrophysics*, 658:A26, 2022.

- [160] Giovanni Gandolfi, Andrea Lapi, and Stefano Liberati. Empirical evidence of nonminimally coupled dark matter in the dynamics of local spiral galaxies? *The Astrophysical Journal*, 929(1):48, April 2022.
- [161] P. Salucci and A. Burkert. Dark matter scaling relations. *The Astrophysical Journal*, 537(1):L9–L12, July 2000.
- [162] Gianfranco Gentile, Benoit Famaey, HongSheng Zhao, and Paolo Salucci. Universality of galactic surface densities within one dark halo scale-length. 461, 2009.
- [163] A. Burkert. The structure and dark halo core properties of dwarf spheroidal galaxies. *The Astrophysical Journal*, 808(2):158, jul 2015.
- [164] F. Donato, G. Gentile, P. Salucci, C. Frigerio Martins, M. I. Wilkinson, G. Gilmore, E. K. Grebel, A. Koch, and R. Wyse. A constant dark matter halo surface density in galaxies. *Monthly Notices of the Royal Astronomical Society*, 397(3):1169–1176, 07 2009.
- [165] Heling Deng, Mark P. Hertzberg, and Mohammad Hossein Namjoo. Can light dark matter solve the core-cusp problem? *Physical Review D*, 98(2):023513, July 2018.
- [166] L. Portinari, J. Sommer-Larsen, and R. Tantaló. On the mass-to-light ratio and the initial mass function in disc galaxies. *Monthly Notices of the Royal Astronomical Society*, 347(3):691–719, January 2004.
- [167] Paolo Salucci, Andrea Lapi, Chiara Tonini, Gianfranco Gentile, Irina Yegorova, and Ulrich Klein. The universal rotation curve of spiral galaxies – ii. the dark matter distribution out to the virial radius. *Monthly Notices of the Royal Astronomical Society*, 378(1):41–47, 06 2007.
- [168] Gabriele U. Varieschi. Newtonian Fractional-Dimension Gravity and Rotationally Supported Galaxies. *Mon. Not. Roy. Astron. Soc.*, 503(2):1915–1931, 2021.
- [169] Gabriele U. Varieschi. Relativistic Fractional-Dimension Gravity. *Universe*, 7(10):387, 2021.
- [170] Gabriele U. Varieschi. Newtonian fractional-dimension gravity and the external field effect. *Eur. Phys. J. Plus*, 137(11):1217, 2022.
- [171] Gabriele U. Varieschi. Newtonian Fractional-Dimension Gravity and Galaxies without Dark Matter. *Universe*, 9(6):246, 2023.
- [172] Gabriele U. Varieschi. Newtonian Fractional-Dimension Gravity and the Mass-Dimension Field Equation. *Universe*, 11(12):388, 2025.

- [173] Gabriele U. Varieschi. Fractional-Dimension Gravity and the Milky Way Galaxy. 7 2026.
- [174] Gianluca Calcagni. Fractal universe and quantum gravity. *Phys. Rev. Lett.*, 104:251301, 2010.
- [175] Gianluca Calcagni. Multi-scale gravity and cosmology. *JCAP*, 12:041, 2013.
- [176] Gianluca Calcagni. Classical and quantum gravity with fractional operators. *Classical and Quantum Gravity*, 38(16):165005, jul 2021.
- [177] Gianluca Calcagni and Gabriele U. Varieschi. Gravitational potential and galaxy rotation curves in multi-fractional spacetimes. *JHEP*, 08:024, 2022.
- [178] Iván Salvador-García and Gianluca Calcagni. Cosmology of fractional gravity. 4 2026.
- [179] D. Eckert, S. Ettori, E. Pointecouteau, S. Molendi, S. Paltani, C. Tchernin, and The X-COP Collaboration. The xmm cluster outskirts project (x-cop). *Astronomische Nachrichten*, 338(2–3):293–298, March 2017.
- [180] Stefano Ettori, V. Ghirardini, D. Eckert, E. Pointecouteau, F. Gastaldello, M. Sereno, M. Gaspari, S. Ghizzardi, M. Roncarelli, and M. Rossetti. Hydrostatic mass profiles in x-cop galaxy clusters. *Astronomy & Astrophysics*, 621:A39, January 2019.
- [181] V. Ghirardini, D. Eckert, S. Ettori, E. Pointecouteau, S. Molendi, M. Gaspari, M. Rossetti, S. De Grandi, M. Roncarelli, H. Bourdin, P. Mazzotta, E. Rasia, and F. Vazza. Universal thermodynamic properties of the intracluster medium over two decades in radius in the x-cop sample. *Astronomy & Astrophysics*, 621:A41, January 2019.
- [182] D. Eckert, S. Ettori, E. Pointecouteau, R. F. J. van der Burg, and S. I. Loubser. The gravitational field of x-cop galaxy clusters. *Astronomy & Astrophysics*, 662:A123, June 2022.
- [183] Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. , 641:A6, September 2020.
- [184] Craig L. Sarazin. *X-ray emission from clusters of galaxies*. 1988.
- [185] Alfonso Cavaliere and Andrea Lapi. The astrophysics of the intracluster plasma. , 533(3):69–94, December 2013.
- [186] R. A. Sunyaev and Ia. B. Zeldovich. Microwave background radiation as a probe of the contemporary structure and history of the universe. , 18:537–560, January 1980.

- [187] Yoel Rephaeli. Cosmic Microwave Background Comptonization by Hot Intracluster Gas. , 445:33, May 1995.
- [188] Ayumu Terukina, Lucas Lombriser, Kazuhiro Yamamoto, David Bacon, Kazuya Koyama, and Robert C. Nichol. Testing chameleon gravity with the coma cluster. *Journal of Cosmology and Astroparticle Physics*, 2014(4):013, April 2014.
- [189] Harry Wilcox, David Bacon, Robert C. Nichol, Philip J. Rooney, Ayumu Terukina, A. Kathy Romer, Kazuya Koyama, Gong-Bo Zhao, Ross Hood, Robert G. Mann, Matt Hilton, Maria Manolopoulou, Martin Sahlén, Chris A. Collins, Andrew R. Liddle, Julian A. Mayers, Nicola Mehrtens, Christopher J. Miller, John P. Stott, and Pedro T. P. Viana. The xmm cluster survey: Testing chameleon gravity using the profiles of clusters. *Monthly Notices of the Royal Astronomical Society*, 452(2):1171–1183, September 2015.
- [190] Jeremy Sakstein, Harry Wilcox, David Bacon, Kazuya Koyama, and Robert C. Nichol. Testing gravity using galaxy clusters: New constraints on beyond horndeski theories. *Journal of Cosmology and Astroparticle Physics*, 2016(07):019, July 2016.
- [191] Balakrishna S. Haridasu, Purnendu Karmakar, Marco De Petris, Vincenzo F. Cardone, and Roberto Maoli. Testing generalized scalar-tensor theories of gravity with clusters of galaxies. *Physical Review D*, 107(12):124059, June 2023.
- [192] Sreekanth Harikumar and Marek Biesiada. Moffat’s modified gravity tested on x-cop galaxy clusters. *The European Physical Journal C*, 82(3), March 2022.
- [193] Yacer Boumechta, Balakrishna S. Haridasu, Lorenzo Pizzuti, Minahil Adil Butt, Carlo Baccigalupi, and Andrea Lapi. Constraining chameleon screening using galaxy cluster dynamics. *Physical Review D*, 108(4):044007, August 2023.
- [194] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified gravity and cosmology. *Physics Reports*, 513(1–3):1–189, March 2012.
- [195] S. Nojiri, S.D. Odintsov, and V.K. Oikonomou. Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution. *Physics Reports*, 692:1–104, 2017.
- [196] *Modified Gravity and Cosmology: An Update by the CANTATA Network*. Springer International Publishing, 2021.
- [197] M. Milgrom. A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *ApJ*, 270:365–370, July 1983.

- [198] Benoît Famaey and Stacy S. McGaugh. Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions. *Living Reviews in Relativity*, 15(1):10, December 2012.
- [199] Erik P. Verlinde. Emergent gravity and the dark universe. *SciPost Physics*, 2(3), May 2017.
- [200] Youngsub Yoon, Sangmin Lee, and Jae-Weon Lee. Understanding galaxy rotation curves with verlinde’s emergent gravity. *Classical and Quantum Gravity*, 40(2):02LT01, January 2023.
- [201] Alexey Vikhlinin, Andrey Kravtsov, William Forman, Christine Jones, Maxim Markevitch, Stephen S. Murray, and Leon Van Speybroeck. Chandra sample of nearby relaxed galaxy clusters: Mass, gas fraction, and mass-temperature relation. *The Astrophysical Journal*, 640(2):691–709, April 2006.
- [202] Veronica Biffi, Stefano Borgani, Giuseppe Murante, Emanuela Rasia, Federico Sembolini, Susana Planelles, Nhat-Minh Truong, Andrea Ragagnin, Dusan Fabjan, Massimo Gaspari, Klaus Dolag, and Gian Luigi Granato. On the nature of hydrostatic equilibrium in galaxy clusters. *The Astrophysical Journal*, 827(2):112, August 2016.
- [203] S. Ansarifard, V. Biffi, S. Borgani, K. Dolag, A. Ragagnin, N. Truong, S. Planelles, F. Sembolini, E. Rasia, D. Fabjan, L. Gazzola, M. Gaspari, G. Murante, L. Tornatore, W. Cui, R. Mostoghiu, and G. L. Granato. The three hundred project: Correcting for the hydrostatic-equilibrium mass bias in x-ray and sz surveys. *Astronomy & Astrophysics*, 634:A113, February 2020.
- [204] L. Pizzuti, B. Sartoris, S. Borgani, and A. Biviano. Calibration of systematics in constraining modified gravity models with galaxy cluster mass profiles. *Journal of Cosmology and Astroparticle Physics*, 2020(04):024–024, April 2020.
- [205] Ricardo Herbonnet, Anja Buddendiek, Florian Pacaud, Henk Hoekstra, Adam B. Mantz, I-Non Chiu, M. W. Sommer, Christian Klein, Andrej Dvornik, C. Sifón, I. Bartalucci, Stefano Andreon, D. Eckert, Stefano Ettori, Fabio Gastaldello, Vittorio Ghirardini, Paul Giles, C. P. Haines, K. Kettula, Ian G. McCarthy, B. Maughan, G. W. Pratt, E. Rasia, K. Rines, Mauro Sereno, and R. F. J. van der Burg. Cccp and meneacs: (updated) weak-lensing masses for 100 galaxy clusters. *Monthly Notices of the Royal Astronomical Society*, 497(4):4684–4703, August 2020.
- [206] D. Eckert, S. Ettori, E. Pointecouteau, D. Nagai, S. Molendi, B. J. Maughan, P. Mazzotta, E. Rasia, M. Roncarelli, M. Rossetti, H. Bourdin, M. Gaspari, and G. W. Pratt. Non-

- thermal pressure support in x-cop galaxy clusters. *Astronomy & Astrophysics*, 621:A40, January 2019.
- [207] Aaron A. Dutton and Andrea V. Macciò. Cold dark matter haloes in the planck era: evolution of structural parameters for einasto and nfw profiles. *Monthly Notices of the Royal Astronomical Society*, 441(4):3359–3374, May 2014.
- [208] J. Binney. Dynamics of elliptical galaxies and other spheroidal components. , 20:399–429, January 1982.
- [209] Ewa L. Lokas and Gary A. Mamon. Properties of spherical galaxies and clusters with an nfw density profile. *Monthly Notices of the Royal Astronomical Society*, 321(1):155–166, February 2001.
- [210] Asher Wasserman, Pieter van Dokkum, Aaron J. Romanowsky, Jean Brodie, Shany Danieli, Duncan A. Forbes, Roberto Abraham, Christopher Martin, Matt Matuszewski, Alexa Villaume, John Tamanas, and Stefano Profumo. Spatially resolved stellar kinematics of the ultra-diffuse galaxy dragonfly 44. ii. constraints on fuzzy dark matter. *The Astrophysical Journal*, 885(2):155, November 2019.
- [211] H. C. Plummer. On the problem of distribution in globular star clusters: (plate 8.). *Monthly Notices of the Royal Astronomical Society*, 71:460–470, 1911.
- [212] Pengfei Li, Federico Lelli, Stacy McGaugh, and James Schombert. A comprehensive catalog of dark matter halo models for sparc galaxies. *The Astrophysical Journal Supplement Series*, 247(1):31, March 2020.
- [213] Matthew G. Walker, Mario Mateo, and Edward W. Olszewski. Stellar velocities in the carina, fornax, sculptor, and sextans dsph galaxies: Data from the magellan/mmfs survey. *The Astronomical Journal*, 137(2):3100–3108, February 2009.
- [214] Mario Mateo, Edward W. Olszewski, and Matthew G. Walker. The velocity dispersion profile of the remote dwarf spheroidal galaxy leo i: A tidal hit and run? *The Astrophysical Journal*, 675(1):201–233, March 2008.
- [215] Ivan de Martino, Antonaldo Diaferio, and Luisa Ostorero. Dynamics of dwarf galaxies in $f(r)$ gravity. *Monthly Notices of the Royal Astronomical Society*, 519(3):4424–4433, March 2023.
- [216] Lukas Leisman, Martha P. Haynes, Steven Janowiecki, Gregory Hallenbeck, Gyula Józsa, Riccardo Giovanelli, Elizabeth A. K. Adams, David Bernal Neira, John M. Cannon,

- William F. Janesh, Katherine L. Rhode, and John J. Salzer. (almost) dark galaxies in the alfalfa survey: Isolated h i-bearing ultra-diffuse galaxies. *The Astrophysical Journal*, 842(2):133, June 2017.
- [217] Pavel E. Mancera Piña, T. van der Hulst, Jarle Brinchmann, Federico Lelli, James Schombert, Elizabeth A. K. Adams, Lukas Leisman, Tom A. Oosterloo, Filippo Fraternali, Attila Popping, and Reynier F. Peletier. The evolution of ultra-diffuse galaxies in nearby galaxy clusters from the kapteyn iac weave int clusters survey. *Monthly Notices of the Royal Astronomical Society*, 485(1):1036–1052, January 2019.
- [218] Pavel E. Mancera Piña, Filippo Fraternali, Tom A. Oosterloo, Elizabeth A. K. Adams, Antonino Marasco, Emanuele M. Di Teodoro, Lorenzo Posti, Kyle A. Oman, Lukas Leisman, Yipeng Shi, T. van der Hulst, and Federico Lelli. Robust h i kinematics of gas-rich ultra-diffuse galaxies: Hints of a weak-feedback formation scenario. *Monthly Notices of the Royal Astronomical Society*, 495(4):3636–3655, July 2020.
- [219] Chandreyee Sengupta, A. Bouchard, S. Roychowdhury, A. Basu, S. Dutta, and B. B. Nath. Dark matter and h i in ultra-diffuse galaxy ugc 2162. *Monthly Notices of the Royal Astronomical Society*, 488(3):3222–3230, September 2019.
- [220] Pieter G. van Dokkum, Roberto Abraham, Allison Merritt, Jielai Zhang, Marla Geha, and Charlie Conroy. Forty-seven milky way-sized, extremely diffuse galaxies in the coma cluster. *The Astrophysical Journal Letters*, 798(2):L45, January 2015.
- [221] Mariana P. Júlio, José A. Vázquez-Mata, Benoit Famaey, M. Rejkuba, S. Mieske, K. Fahrion, E. Emsellem, M. Sarzi, J. Fensch, T.-O. Husser, and M. M. Roth. The muse-faint survey. iii. constraining scalar field dark matter with antlia b. *Astronomy & Astrophysics*, 678:A38, October 2023.
- [222] Eric F. Bell and Roelof S. de Jong. Stellar mass-to-light ratios and the tully-fisher relation. *The Astrophysical Journal*, 550(1):212–229, March 2001.
- [223] James Schombert, Stacy McGaugh, and Federico Lelli. The mass-to-light ratios and the star formation histories of disc galaxies. , 483(2):1496–1512, February 2019.
- [224] Johannes U. Lange, Frank C. van den Bosch, Andrew R. Zentner, Wenting Wang, Andrew P. Hearin, Surhud More, Alexie Leauthaud, Rachel Mandelbaum, Jenny E. Greene, and Guinevere Kauffmann. Updated results on the galaxy–halo connection from satellite kinematics in sdss. *Monthly Notices of the Royal Astronomical Society*, 487(3):3112–3129, June 2019.

- [225] Rachel Mandelbaum, W. Wang, Ying Zu, Martin White, Jeremy Tinker, Surhud More, Christopher Hirata, Erin Sheldon, David W. Hogg, Donald P. Schneider, Ricardo Reyes, and Guinevere Kauffmann. Strong bimodality in the host halo mass of central galaxies from galaxy-galaxy lensing. *Monthly Notices of the Royal Astronomical Society*, 457(3):3200–3218, April 2016.
- [226] Rob A. Swaters, T. S. van Albada, T. van der Hulst, and Renzo Sancisi. The westerbork h i survey of spiral and irregular galaxies. i. h i imaging of late-type dwarf galaxies. *Astronomy & Astrophysics*, 390:829–861, August 2002.
- [227] Teymoor Saifollahi, Michael A. Beasley, Ignacio Trujillo, Reynier F. Peletier, Arjen van der Wel, and Johan H. Knapen. The number of globular clusters around the iconic udg df44 is as expected for dwarf galaxies. *Monthly Notices of the Royal Astronomical Society*, 502(4):5921–5934, April 2021.
- [228] A. O. Barvinsky, Yu. V. Gusev, G. A. Vilkovisky, and V. V. Zhytnikov. Asymptotic behaviors of the heat kernel in covariant perturbation theory. *Journal of Mathematical Physics*, 35(7):3543–3559, 1994.
- [229] Edward Witten. Noncommutative Geometry and String Field Theory. *Nucl. Phys. B*, 268:253–294, 1986.
- [230] Arnau Bas i Beneito, Gianluca Calcagni, and Leslaw Rachwal. Classical and quantum nonlocal gravity, 2022.
- [231] Gianluca Calcagni, Breno L. Giacchini, Leonardo Modesto, Tibério de Paula Netto, and Leslaw Rachwal. Renormalizability of nonlocal quantum gravity coupled to matter, 2023.
- [232] Leonardo Modesto. Super-renormalizable quantum gravity. *Phys. Rev. D*, 86:044005, Aug 2012.
- [233] Tirthabir Biswas, Erik Gerwick, Tomi Koivisto, and Anupam Mazumdar. Towards singularity- and ghost-free theories of gravity. *Phys. Rev. Lett.*, 108:031101, Jan 2012.
- [234] Tirthabir Biswas, Tomi Koivisto, and Anupam Mazumdar. Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity. *Journal of Cosmology and Astroparticle Physics*, 2010(11):008, nov 2010.
- [235] Luca Buoninfante, Breno L. Giacchini, and Tibério de Paula Netto. Black holes in non-local gravity, 2022.

- [236] Stefano Foffa, Michele Maggiore, and Ermis Mitsou. Cosmological dynamics and dark energy from nonlocal infrared modifications of gravity. *International Journal of Modern Physics A*, 29(22):1450116, August 2014.
- [237] Salvatore Capozziello and Francesco Bajardi. Nonlocal gravity cosmology: An overview. *International Journal of Modern Physics D*, 31(06), December 2021.
- [238] Michele Maggiore and Michele Mancarella. Nonlocal gravity and dark energy. *Physical Review D*, 90(2), July 2014.
- [239] Enis Belgacem, Yves Dirian, Stefano Foffa, and Michele Maggiore. Nonlocal gravity. conceptual aspects and cosmological predictions. *Journal of Cosmology and Astroparticle Physics*, 2018(03):002, mar 2018.
- [240] Gianluca Calcagni. Classical and quantum gravity with fractional operators. *Class. Quant. Grav.*, 38(16):165005, 2021. [Erratum: *Class.Quant.Grav.* 38, 169601 (2021)].
- [241] Gianluca Calcagni, Daniele Oriti, and Johannes Thürigen. Spectral dimension of quantum geometries. *Classical and Quantum Gravity*, 31(13):135014, 2014.
- [242] Gianluca Calcagni, Daniele Oriti, and Johannes Thürigen. Dimensional flow in discrete quantum geometries. *Physical Review D*, 91(8), April 2015.
- [243] G. W. Gibbons and S. W. Hawking. Action integrals and partition functions in quantum gravity. *Phys. Rev. D*, 15:2752–2756, May 1977.
- [244] James W. York. Role of conformal three-geometry in the dynamics of gravitation. *Phys. Rev. Lett.*, 28:1082–1085, Apr 1972.
- [245] Charles W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman, San Francisco, 1973.
- [246] B. Bertotti, L. Iess, and P. Tortora. A test of general relativity using radio links with the Cassini spacecraft. *Nature*, 425, 2003.
- [247] James M. Bardeen. Gauge-invariant cosmological perturbations. *Phys. Rev. D*, 22:1882–1905, Oct 1980.
- [248] Enrico Barausse. *The physics of gravitational waves*, 2023.
- [249] Maud Jaccard, Michele Maggiore, and Ermis Mitsou. Bardeen variables and hidden gauge symmetries in linearized massive gravity. *Phys. Rev. D*, 87:044017, Feb 2013.

- [250] Gong, Yungui and Hou, Shaoqi. Gravitational wave polarizations in $f(r)$ gravity and scalar-tensor theory. *EPJ Web Conf.*, 168:01003, 2018.
- [251] Salvatore Capozziello and Maurizio Capriolo. Gravitational waves in non-local gravity. *Classical and Quantum Gravity*, 38(17):175008, August 2021.
- [252] Taishi Katsuragawa, Tomohiro Nakamura, Taishi Ikeda, and Salvatore Capozziello. Gravitational waves in $f(r)$ gravity: Scalar waves and the chameleon mechanism. *Phys. Rev. D*, 99:124050, Jun 2019.
- [253] Planck collaboration. Planck2018 results: Vi. cosmological parameters. *Astronomy & Astrophysics*, 641:A6, September 2020.
- [254] M. Bartelmann and P. Schneider. Weak gravitational lensing. *Phys. Rep.*, 340(4-5):291–472, January 2001.
- [255] Brouwer Margot M. et al. The weak lensing radial acceleration relation: Constraining modified gravity and cold dark matter theories with KiDS-1000. *A&A*, 650:A113, June 2021.
- [256] T. Mistele, S. McGaugh, F. Lelli, J. Schombert, and P. Li. Radial acceleration relation of galaxies with joint kinematic and weak-lensing data. *JCAP*, 2024(4):020, April 2024.
- [257] Keiichi Umetsu, Adi Zitrin, Daniel Gruen, Julian Merten, Megan Donahue, and Marc Postman. Clash: Joint analysis of strong-lensing, weak-lensing shear, and magnification data for 20 galaxy clusters*. *The Astrophysical Journal*, 821(2):116, April 2016.
- [258] Filippo Bouchè, Salvatore Capozziello, Vincenzo Salzano, and Keiichi Umetsu. Testing non-local gravity by clusters of galaxies. *European Physical Journal C*, 82(7):652, July 2022.
- [259] Saboura Zamani, Vincenzo Salzano, and Dario Bettoni. Gravitational lensing from clusters of galaxies to test disformal couplings theories, 2024.
- [260] J. Wang, S. Bose, C. S. Frenk, L. Gao, A. Jenkins, V. Springel, and S. D. M. White. Universal structure of dark matter haloes over a mass range of 20 orders of magnitude. , 585(7823):39–42, September 2020.
- [261] Olivier Minazzoli and Tiberiu Harko. New derivation of the lagrangian of a perfect fluid with a barotropic equation of state. *Physical Review D*, 86(8), October 2012.
- [262] Fabrizio Nesti and Paolo Salucci. The dark matter halo of the milky way, ad 2013. *Journal of Cosmology and Astroparticle Physics*, 2013(07):016–016, July 2013.

- [263] Charles Alcock, Edward Farhi, and Angela Olinto. Strange stars. *Astrophys. J.*, 310:261–272, 1986.
- [264] Julio Arrechea, Stefano Liberati, and Vania Vellucci. Whispers from the quantum core: the ringdown of semiclassical stars. *JCAP*, 12:004, 2024.
- [265] R. A. Konoplya and A. Zhidenko. Quasinormal modes of black holes: From astrophysics to string theory. *Rev. Mod. Phys.*, 83:793–836, 2011.
- [266] The Event Horizon Telescope Collaboration. First m87 event horizon telescope results. i. the shadow of the supermassive black hole. *The Astrophysical Journal Letters*, 875(1):L1, apr 2019.
- [267] Volker Perlick and Oleg Yu. Tsupko. Calculating black hole shadows: Review of analytical studies. *Physics Reports*, 947:1–39, February 2022.
- [268] Raúl Carballo-Rubio, Héloïse Delaporte, Astrid Eichhorn, and Aaron Held. Disentangling photon rings beyond General Relativity with future radio-telescope arrays. *JCAP*, 2024(5):103, May 2024.
- [269] Rebecca L. Larson, Steven L. Finkelstein, Dale D. Kocevski, Taylor A. Hutchison, Jonathan R. Trump, Pablo Arrabal Haro, Volker Bromm, Nikko J. Cleri, Mark Dickinson, Seiji Fujimoto, Jeyhan S. Kartaltepe, Anton M. Koekemoer, Casey Papovich, Nor Pirzkal, Sandro Tacchella, Jorge A. Zavala, Micaela Bagley, Peter Behroozi, Jaclyn B. Champagne, Justin W. Cole, Intae Jung, Alexa M. Morales, Guang Yang, Haowen Zhang, Adi Zitrin, Ricardo O. Amorín, Denis Burgarella, Caitlin M. Casey, Óscar A. Chávez Ortiz, Isabella G. Cox, Katherine Chworowsky, Adriano Fontana, Eric Gawiser, Andrea Grazian, Norman A. Grogin, Santosh Harish, Nimish P. Hathi, Michaela Hirschmann, Benne W. Holwerda, Stéphanie Juneau, Gene C. K. Leung, Ray A. Lucas, Elizabeth J. McGrath, Pablo G. Pérez-González, Jane R. Rigby, Lise-Marie Seillé, Raymond C. Simons, Alexander de la Vega, Benjamin J. Weiner, Stephen M. Wilkins, and L. Y. Aaron and Yung. A ceers discovery of an accreting supermassive black hole 570 myr after the big bang: Identifying a progenitor of massive z gt; 6 quasars. *The Astrophysical Journal Letters*, 953(2):L29, August 2023.
- [270] Philip Bull et al. Beyond Λ CDM: Problems, solutions, and the road ahead. *Phys. Dark Univ.*, 12:56–99, 2016.
- [271] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *Phys. Rept.*, 692:1–104, 2017.

- [272] Sean M. Carroll. The Cosmological constant. *Living Rev. Rel.*, 4:1, 2001.
- [273] S. Perlmutter et al. Measurements of Ω and Λ from 42 High Redshift Supernovae. *Astrophys. J.*, 517:565–586, 1999.
- [274] Gianfranco Bertone, Dan Hooper, and Joseph Silk. Particle dark matter: Evidence, candidates and constraints. *Phys. Rept.*, 405:279–390, 2005.
- [275] Andrea Lapi, Lumen Boco, Marcos M. Cueli, Balakrishna S. Haridasu, Tommaso Ronconi, Carlo Baccigalupi, and Luigi Danese. Little Ado about Everything: η CDM, a Cosmological Model with Fluctuation-driven Acceleration at Late Times. *Astrophys. J.*, 959(2):83, 2023.
- [276] Benoit Famaey and Stacy McGaugh. Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions. *Living Rev. Rel.*, 15:10, 2012.
- [277] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.*, 641:A6, 2020. [Erratum: *Astron. Astrophys.* 652, C4 (2021)].
- [278] Eleonora Di Valentino, Olga Mena, Supriya Pan, Luca Visinelli, Weiqiang Yang, Alessandro Melchiorri, David F. Mota, Adam G. Riess, and Joseph Silk. In the realm of the Hubble tension—a review of solutions. *Class. Quant. Grav.*, 38(15):153001, 2021.
- [279] A. G. Adame et al. DESI 2024 VI: cosmological constraints from the measurements of baryon acoustic oscillations. *JCAP*, 02:021, 2025.
- [280] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev. D*, 23:347–356, 1981.
- [281] Andreas Albrecht and Paul J. Steinhardt. Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking. *Phys. Rev. Lett.*, 48:1220–1223, 1982.
- [282] Alexei A. Starobinsky. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B*, 91:99–102, 1980.
- [283] Lev Kofman, Andrei D. Linde, and Alexei A. Starobinsky. Reheating after inflation. *Phys. Rev. Lett.*, 73:3195–3198, 1994.
- [284] Patrick Peter and Nelson Pinto-Neto. Has the universe always expanded? *Phys. Rev. D*, 65:023513, 2002.
- [285] Thomas P. Sotiriou. Covariant Effective Action for Loop Quantum Cosmology from Order Reduction. *Phys. Rev. D*, 79:044035, 2009.

- [286] Carlos Barragan, Gonzalo J. Olmo, and Helios Sanchis-Alepuz. Bouncing Cosmologies in Palatini $f(R)$ Gravity. *Phys. Rev. D*, 80:024016, 2009.
- [287] Fabio Finelli and Robert Brandenberger. On the generation of a scale invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase. *Phys. Rev. D*, 65:103522, 2002.
- [288] J. M. Stewart and M. Walker. Perturbations of space-times in general relativity. *Proceedings of the Royal Society of London Series A*, 341(1624):49–74, October 1974.
- [289] James M. Bardeen. Gauge-invariant cosmological perturbations. , 22(8):1882–1905, October 1980.
- [290] Hideo Kodama and Misao Sasaki. Cosmological Perturbation Theory. *Progress of Theoretical Physics Supplement*, 78:1, January 1984.
- [291] G. F. R. Ellis and M. Bruni. Covariant and gauge-invariant approach to cosmological density fluctuations. *Phys. Rev. D*, 40:1804–1818, Sep 1989.
- [292] Chung-Pei Ma and Edmund Bertschinger. Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges. , 455:7, December 1995.
- [293] Viatcheslav F. Mukhanov, H. A. Feldman, and Robert H. Brandenberger. Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions. *Phys. Rept.*, 215:203–333, 1992.
- [294] Viatcheslav F. Mukhanov. Quantum Theory of Gauge Invariant Cosmological Perturbations. *Sov. Phys. JETP*, 67:1297–1302, 1988.
- [295] T. S. Bunch and P. C. W. Davies. Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting. *Proc. Roy. Soc. Lond. A*, 360:117–134, 1978.
- [296] Robert H. Brandenberger. The Matter Bounce Alternative to Inflationary Cosmology. 6 2012.
- [297] Steven Weinberg. *Cosmology*. Oxford University Press, 2008.
- [298] Daniel Baumann. *Cosmology*. Cambridge University Press, 7 2022.
- [299] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

- [300] M. Novello and S. E. Perez Bergliaffa. Bouncing Cosmologies. *Phys. Rept.*, 463:127–213, 2008.

