SISSA
Scuola
Internazionale
Superiore di
Studi Avanzati

PhD course in Theoretical Particle Physics

Black holes, Heun functions and instanton counting

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Academic Year 2022/23

Abstract

In this thesis we explore and exploit the relationships between black hole physics, the mathematics of Heun's equation, Liouville conformal field theory and 4d supersymmetric gauge theories. We study the modular properties of a class of degenerate conformal blocks, including the recently introduced irregular or confluent conformal blocks, computing their connection matrices. In a certain limit, these connection matrices descend to the connection matrices for Heun functions, which are a generalization of the hypergeometric functions. We give explicit formulae, computable in terms of the Nekrasov partition functions of a class of 4d supersymmetric gauge theories. The above study is motivated by the frequent appearance of Heun's equation in physics, one interesting example being the perturbations of black holes. Using the connection formulae that we have obtained, we analytically solve the equations governing the perturbations of the 4d Kerr black hole and of the 5d AdS-Schwarzschild black hole.

Acknowledgements

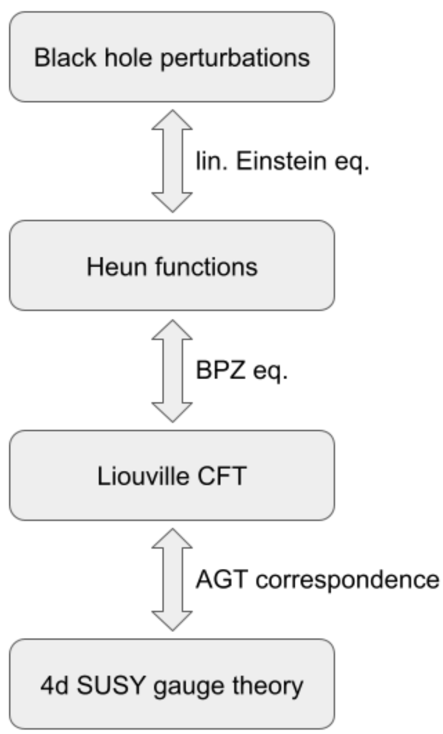
I want to thank Giulio Bonelli and Alessandro Tanzini for being the best supervisors I could have ever hoped for, both on an academic and on a personal level. I also want to thank my friend and colleague Cristoforo Iossa, without whom this thesis would not have been possible.

Contents

Introduction

Overview

This thesis deals with a chain of interconnected topics, graphically depicted as



The remaining pages will be an elaboration of the above diagram. Let us try to give an overview of its elements and its interconnections.

Black hole perturbations

The recent experimental verification of gravitational waves [1] renewed the interest in the theoretical studies of General Relativity and black hole physics. A particularly interesting aspect is the development of exact computational techniques to produce high precision tests of General Relativity. From this perspective, the study of exact solutions of differential equations rather than their approximate or numerical solutions is of paramount importance both to deepen our comprehension of physical phenomena and to reveal possible physical fine structure effects. Many interesting properties of black holes are already encoded in their *perturbations equations*, i.e. the Einstein equations linearized around a black hole background. In this thesis we study two particular examples, with different motivations: in chapter 1 we study the four-dimensional Kerr black hole, ubiquitous in astrophysics. Its perturbation equation gives rise to the so-called Teukolsky equation [2], which we solve exactly and from which we extract analytical formulae for physical observables such as its quasinormal modes, greybody factor and Love numbers. Then, in chapter 3 we study the 5d AdS-Schwarzschild black hole, which is interesting both as a black hole and from a dual holographic perspective. Studying its perturbation equation, we obtain an exact expression for the thermal two-point function of the dual 4d CFT which encodes a wealth of fascinating physics related to the richness of the black hole geometry. For example, two-point functions encode the transport properties of the system, see e.g. [3, 4], the approach to equilibrium [5], as well as chaotic dynamics via pole-skipping [6, 7]. Thermal four-point functions serve as an important diagnostic of quantum chaos [8, 9]. Thermal correlators have also been used to formulate a version of the information paradox [10], as well as to look for subtle signatures of the black hole singularity [11–14].

Heun functions

Heun's equation [15] is the most general second order linear differential equation with four regular singularities on the Riemann sphere. It is the next case in the Fuchsian series after the hypergeometric equation, which displays three regular singularities [16]. The Heun equation — along with its confluences — enters many problems in theoretical and mathematical physics, geometry and other branches of quantitative sciences[1] (see for example [17, 18]). For this reason, many studies appeared in the literature about it, see for example [19] for a general introduction and [20, 21] for studies on the connection problem. By its very definition, the Heun function also solves the classical Poincaré uniformisation problem of a Riemann sphere with four punctures [22, 23]. We also remind that Heun's equation arises from the linear system whose isomonodromic deformation problem is described by the Painlevé VI equation [24–26].
It turns out that the equations that one obtains from studying black hole perturbations

[1] For a huge bibliography take a look at https://theheunproject.org .

often fall into the class of Heun equations. This is true for many types of black holes in different backgrounds and in particular for the 4d Kerr and 5d Ads-Schwarzschild black holes mentioned above. Therefore, solving the physical black hole problem reduces mathematically to solving Heun's equation. In chapter 2 we study Heun's equation and its confluent variants, arriving at an exact solution.

Liouville CFT

Liouville conformal field theory is a two-dimensional conformal field theory related to many physical systems and integrable models. Recent developments, including its relation with supersymmetric gauge theories, equivariant localisation and duality in quantum field theory produced new tools which are very effective to study long-standing classical problems in the theory of differential equations. Indeed, it has been known for a long time that the study of two-dimensional Conformal Field Theories [27] and of the representations of its infinite-dimensional symmetry algebra provide exact solutions to partial differential equations in terms of conformal blocks and the appropriate fusion coefficients. The prototypical example is the null-state equation at level 2 for primary operators of Virasoro algebra which reduce, in the large central charge limit, to a Schrödinger-like equation with regular singularities, corresponding to a potential term with at most quadratic poles. In this way one can engineer solutions of second-order linear differential equations of Fuchsian type by making use of the appropriate two dimensional CFT[2]. While under the operator/state correspondence the vertex operators in the above construction correspond to primary (highest weight) states, one can insert more general irregular vertex operators corresponding to universal Whittaker states. The latter generate irregular singularities in the corresponding null-state equation and therefore allow engineering more general potentials with singularities of order higher than two. Schematically, given a multi-vertex operator $\mathcal{O}_V(z_1,\ldots,z_N)$ satisfying the OPE

$$T(z)\mathcal{O}_V(z_1,\ldots,z_N) \sim V(z;z_i)\mathcal{O}_V(z_1,\ldots,z_N) \quad \text{as} \quad z\sim z_i \tag{0.0.1}$$

one finds the corresponding level 2 null-state equation

$$[b^{-2}\partial_z^2 + \sum_i V(z;z_i)]\Psi(z) = 0 \quad \Psi(z) = \langle \Phi_{2,1}(z)\mathcal{O}_V(z_1,\ldots,z_N)\rangle \tag{0.0.2}$$

satisfied by the correlation function of the multi-vertex and the level 2 degenerate field $\Phi_{2,1}(z)$. If the multi-vertex contains primary operators only, the OPE (0.0.1) and the potential in (0.0.2) contain at most quadratic poles, while the insertions of irregular vertices generate higher order singularities in $\sum_i V(z;z_i)$. Actually, $V(z;z_i)$ is a function in z and in differential operators with respect to the z_i. The dependence on the latter is removed

[2] Our analysis is here limited - for the sake of presenting the general method - to second order linear differential equations, but all we say can be generalized to higher order equations by considering higher level degenerate field insertions, as already considered in [27].

by the semiclassical limit $b \to 0$ of Liouville CFT[3], corresponding to large Virasoro central charge $c \to \infty$. In this way, one finds a Schrödinger-like equation

$$\epsilon_1^2 \frac{d^2\Psi(z)}{dz^2} + V_{CFT}(z)\Psi(z) = 0\,, \tag{0.0.3}$$

where ϵ_1 is a parameter which stays finite in the large central charge limit and plays the rôle of the Planck constant. The advantage of this approach is that the explicit solution of the connection problem on the z-plane for equation (0.0.3) can be derived from the explicit computation of the full CFT_2 correlator (1.3.5) and from its expansions in different intermediate channels.

In chapter 2 we perform a detailed study of irregular correlators in Liouville Conformal Field Theory, of the related Virasoro conformal blocks with irregular singularities and of their connection formulae. Upon considering their semi-classical limit, we provide explicit expressions of the connection matrices for the Heun function and a class of its confluences. These result from the semi-classical limit of Virasoro conformal blocks for the five-point correlation function of four primaries and a degenerate field and a class of its coalescence limits to irregular conformal blocks. While the five-point correlator satisfies a linear PDE, namely the BPZ equation [27], its confluences satisfy a PDE obtained by an appropriate rescaling procedure. As we will discuss in detail, BPZ equations reduce in the semi-classical limit to ODEs. For the particular five-point correlation function mentioned above, this gets identified with Heun's equation upon a suitable dictionary. Following a class of coalescences of the singularities and/or specific parameter scalings, from the configuration of four regular points one naturally obtains a set of confluent irregular blocks satisfying the corresponding confluent BPZ equations. The Heun functions and its confluences are solutions of the resulting semiclassical reduced equations. Crucially, the language and the properties of Liouville CFT, in particular crossing symmetry allow us to determine the connection coefficients for Heun functions using the known three-point function of Liouville CFT and conformal blocks, thereby solving Heun's equation. Let us also mention that the method we use can be generalised to general Fuchsian equations and their confluences upon considering the relevant conformal blocks.

The mathematical interest of Liouville quantum field theory has been highlighted by A.M. Polyakov who proposed to interpret it as a quantum extension of the Poincaré uniformisation problem [28]. A consequence of the above interpretation is that one can make use of the classical limit of Liouville theory to obtain new exact solutions of classical uniformisation [29]. This inspired the work of several authors [30–33] and received a renewed interest after the discovery of AGT correspondence [34–40].

[3] This is not to be confused with the semiclassical approximation of the Schrödinger equation.

4d SUSY gauge theory

A crucial ingredient to accomplish this program outlined above, namely solving Heun's equation using techniques from 2d CFT, is a deep control on the analytic structure of regular and irregular Virasoro conformal blocks. This has been recently obtained after the seminal AGT paper [41], where conformal blocks of Virasoro algebra have been identified with concrete combinatorial formulae arising from equivariant instanton counting in the context of $\mathcal{N}=2$ four-dimensional supersymmetric gauge theories [42, 43]. The explicit solution of the instanton counting problem has been decoded in the CFT language in terms of overlap of universal Whittaker states in [44–47]. More precisely, the wave function $\Psi(z)$ in 0.0.2 corresponds to the insertion of a BPS surface observable in the gauge theory path integral [48]. The specific case studied in chapter 1 corresponds to a surface observable in the $SU(2)$ $\mathcal{N}=2$ gauge theory with $N_f=3$ fundamental hypermultiplets, while in chapter 2 we extend the analysis to all $N_f \leq 4$. The relevant gauge theory moduli space in these cases is the one of *ramified* instantons [49], with vortices localised on the surface defect, the z-variable providing the fugacity for the vortex counting. In the simplest cases the latter is indeed captured by hypergeometric functions [50].

An important consequence of the AGT correspondence between CFT correlation functions and exact BPS partition functions in $\mathcal{N}=2$ four dimensional gauge theories has been the discovery of the so called "Kiev formula" in the theory of Painlevé transcendents [51], which established the latter to be a further class of special functions with an explicit combinatorial expression in terms of equivariant volumes of instanton moduli spaces [42, 52]. This correspondence between Painlevé and gauge theory has been extended to the full Painlevé confluence diagram in [53], used in [54] to produce recurrence relations for instanton counting for general gauge groups and studied in terms of blow-up equations in [55–57]. These results are related via the AGT correspondence to the $c=1$ limit of Liouville conformal field theory. On the other hand, it is well-known that a direct relation exists between the linear system associated to Painlevé VI equation and the Heun equation [58]. Further studies on this subject appeared recently in [37, 59–61]. This perspective has been analyzed in the context of black hole physics in [62–65] where it was suggested that some physical properties of black holes, such as their greybody factor and quasinormal modes, can be studied in a particular regime in terms of Painlevé equations. Numerical checks appeared in [66, 67]. A decisive step forward about the quasinormal mode problem has been taken in [68], where a different approach making use of the Seiberg-Witten quantum curve of an appropriate supersymmetric gauge theory has been advocated to justify their sprectrum and whose evidence was also supported by comparison with numerical analysis of the gravitational equation (see also [69, 70] for further developments). This view point has been further analysed in [71], where the context is widely generalized to D-branes and other types of gravitational backgrounds in various dimensions. From the CFT_2 viewpoint, the gauge theoretical approach corresponds to the large Virasoro central charge limit recalled above.

According to the Alday-Gaiotto-Tachikawa (AGT) correspondence [41], a precise gauge theoretical counterpart of Liouville CFT is given by the BPS sector of four dimensional $\mathcal{N}=2$ $SU(2)$ gauge theory in the so-called Ω-background [43]. In particular the four-point conformal block of Liouville primary fields on the Riemann sphere gets identified with the Nekrasov partition function [42] of $SU(2)$ gauge theory with four fundamental hypermultiplets. In this context, the confluence procedure is interpreted as the decoupling of massive hypermultiplets [44] or the limit to strongly interacting Argyres-Douglas theories [47, 53] in the $SU(2)$ Seiberg-Witten theory. Degenerate field insertions in the CFT correlator correspond to surface operator insertions in the gauge theory [48]. The latter therefore satisfy BPZ equations and their confluent limits. The importance of the AGT correspondence is that it maps more complicated aspects of one side to easier ones of the other, basically it provides a proof of gauge theory dualities once reinterpreted as modular properties in CFT [27]. Moreover, it provides an explicit combinatorial expression for Virasoro conformal blocks in terms of Nekrasov partition function. We exploit this correspondence to provide concrete formulae for the connection matrices for the relevant conformal blocks and their confluences. The semi-classical limit of CFT coincides via AGT correspondence with an asymmetric limit in the Ω-background parameters known as the Nekrasov-Shatashvili (NS) limit [72]. This provides a quantization procedure of the classical integrable systems associated to the Seiberg-Witten theory [73]. From this viewpoint Heun equations can be interpreted as Schrödinger equations for these quantum systems. All in all, the connection problem for (confluent) Heun equations can be restated as a connection problem for semi-classical conformal blocks. The latter can be computed in very explicit terms via AGT correspondence by equivariant localisation in supersymmetric gauge theory in the NS limit. Let us here notice that the classifying group of the solutions of the Heun equation [74] is the D_4 Coxeter group, generated by the permutations of the four regular singular points and by the swaps of each couple of indices of the local solutions but a reference one. This concretely realises in the NS limit the action of the D_4 group on the vevs of surface operators in the $N_f = 4$ $SU(2)$ gauge theory.

Structure of the thesis and original contributions

This thesis is based on three of my publications. Chapter 1 is based on [75], where we study the 4d Kerr black hole. We solve its perturbation equation which reduces to the confluent Heun equation, for which we obtain the previously unknown connection coefficients in terms of the Liouville three-point function and the Nekrasov partition function. This result allows us also to give a new, exact expression for the greybody factor of the black hole, which in some approximation limits reduces to previously known formulae [76, 77]. We also provide a proof of the quantization conditions for the quasinormal modes and

spin-weighted spheroidal harmonics conjectured in [68]. Finally, we discuss the use of the precise asymptotics of our solution to determine the tidal deformation profile in the far away region of the Kerr black hole and compare it to recent results on the associated Love numbers in the static [78] and quasi-static [79, 80] regimes.
Chapter 2 is based on [81]. We extend the study of irregular Liouville correlation functions and Heun functions initiated in the previous chapter and obtain exact expressions for the Heun functions and its connection coefficients for all of its confluence diagram up to rank 1 singularities. This extends the range of validity of our formulae to a vast number of applications.
Finally, chapter 3 is based on [82] where we study the 5d AdS-Schwarzschild black hole and its holographically dual thermal CFT. We provide the first explicit result for the thermal two-point function in this dual CFT based on the results for Heun functions of the previous chapter. In the large spin limit this exact formula produces the solution to the heavy-light light-cone bootstrap [83, 84]. We also reproduce the available perturbative results from the literature [85–99] and make new predictions.

A final word of warning: between the three independent chapters there are some slight discrepancies in notation due to convenience and pre-existing conventions in the topics that are treated. Each chapter is of course consistent in itself, and the notation and conventions are explained in the beginning of each chapter and in its appendices.

Chapter 1

Kerr black hole perturbations and instanton counting

In this chapter, for the sake of concreteness and with a specific application to the Kerr black hole problem in mind, we study equation (0.0.2) for $N_f = 3$ in the case of two regular and one irregular singularity of rank 1. In Sect.1.1 we review the relativistic massless wave equation in the Kerr black hole background, giving rise to the Teukolsky equation, whose solution can be obtained by separation of variables. In Sect.1.2 we recall how both the radial and angular parts reduce, under an appropriate dictionary, to (0.0.2) with an irregular singularity of rank 1 at infinity and two regular singularities, which is the *confluent* Heun equation [19]. We provide the explicit exact solution of the connection coefficients in Sect.1.3. The efficiency of the instanton expansion in the exact solution against the numerical integration is demonstrated by a detailed quantitative analysis in Subsect.1.3.3. In Sect.1.4 we apply these results to Kerr black hole physics. We perform the study of the greybody factor of the Kerr black hole at finite frequency for which we give an exact formula. This reduces to the well-known result of Maldacena and Strominger [76] in the zero frequency limit and in the semiclassical regime reproduces the results computed via standard WKB approximation in [77]. By using the explicit solution of the connection problem, we also provide a proof of the exact quantization of Kerr black hole quasinormal modes as proposed in [68]. By solving the angular Teukolsky equation, we also prove the analogue dual quantization condition on the corresponding parameters of the spin-weighted spheroidal harmonics. Finally, we discuss the use of the precise asymptotics of our solution to determine the tidal deformation profile in the far away region of the Kerr black hole and compare it to recent results on the associated Love numbers in the static [78] and quasi-static [79, 80] regimes. We observe that our method naturally distinguishes the source and response terms in the solution without needing analytic continuation in the angular momentum [100, 101] and provides an alternative regularization procedure for the computation of static Love numbers.

1.1 Perturbations of Kerr black holes

The Kerr metric describes the spacetime outside of a stationary, rotating black hole in asymptotically flat space. In Boyer-Lindquist coordinates it reads:

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - \mathrm{a}^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2+\mathrm{a}^2)^2 - \Delta\mathrm{a}^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\phi^2 \\ &- \frac{2\mathrm{a}\sin^2\theta(r^2+\mathrm{a}^2-\Delta)}{\Sigma}dt\, d\phi\,, \end{aligned} \tag{1.1.1}$$

where

$$\Sigma = r^2 + \mathrm{a}^2\cos^2\theta\,, \quad \Delta = r^2 - 2Mr + \mathrm{a}^2\,. \tag{1.1.2}$$

The horizons are given by the roots of Δ:

$$r_\pm = M \pm \sqrt{M^2 - \mathrm{a}^2}\,. \tag{1.1.3}$$

Two other relevant quantities are the Hawking temperature and the angular velocity at the horizon:

$$T_H = \frac{r_+ - r_-}{8\pi M r_+}\,, \quad \Omega = \frac{\mathrm{a}}{2Mr_+}\,. \tag{1.1.4}$$

Perturbations of the Kerr metric by fields of spin $s = 0, -1, -2$ are described by the Teukolsky equation [2], who found that an Ansatz of the form

$$\Phi_s = e^{im\phi - i\omega t}S_{\lambda,s}(\theta, \mathrm{a}\omega)R_s(r)\,. \tag{1.1.5}$$

permits a separation of variables of the partial differential equation. One gets[1] the following equations for the radial and the angular part (see for example [102] eq.25):

$$\begin{aligned} &\Delta\frac{d^2R}{dr^2} + (s+1)\frac{d\Delta}{dr}\frac{dR}{dr} + \left(\frac{K^2 - 2is(r-M)K}{\Delta} - \Lambda_{\lambda,s} + 4is\omega r\right)R = 0\,, \\ &\partial_x(1-x^2)\partial_x S_\lambda + \left[(cx)^2 + \lambda + s - \frac{(m+sx)^2}{1-x^2} - 2csx\right]S_\lambda = 0\,. \end{aligned} \tag{1.1.6}$$

Here $x = \cos\theta$, $c = \mathrm{a}\omega$ and

$$K = (r^2 + \mathrm{a}^2)\omega - \mathrm{a}m, \quad \Lambda_{\lambda,s} = \lambda + \mathrm{a}^2\omega^2 - 2\mathrm{a}m\omega\,. \tag{1.1.7}$$

λ has to be determined as the eigenvalue of the angular equation with suitable boundary conditions imposing regularity at $\theta = 0, \pi$. In general no closed-form expression is known, but for small $\mathrm{a}\omega$ it is given by $\lambda = \ell(\ell+1) - s(s+1) + \mathcal{O}(\mathrm{a}\omega)$. We give a way to calculate it to arbitrary order in $\mathrm{a}\omega$ in subsection 1.4.3.

[1] Dropping the $_s$ subscript to ease the notation

For later purposes it is convenient to write both equations in the form of a Schrödinger equation. For the radial equation we define

$$z = \frac{r - r_-}{r_+ - r_-}, \quad \psi(z) = \Delta(r)^{\frac{s+1}{2}} R(r)\,. \tag{1.1.8}$$

With this change of variables the inner and outer horizons are at $z = 0$ and $z = 1$, respectively, and $r \to \infty$ corresponds to $z \to \infty$. We obtain the differential equation

$$\frac{d^2\psi(z)}{dz^2} + V_r(z)\psi(z) = 0 \tag{1.1.9}$$

with potential

$$V_r(z) = \frac{1}{z^2(z-1)^2} \sum_{i=0}^{4} \hat{A}_i^r z^i\,. \tag{1.1.10}$$

The coefficients $\hat{A}_i^r$ depend on the parameters of the black hole and the frequency, spin and angular momentum of the perturbation. Their explicit expression is given in Appendix 1.A. For the angular part instead we define

$$z = \frac{1+x}{2}, \quad y(z) = \sqrt{1-x^2}\frac{S_\lambda}{2}\,. \tag{1.1.11}$$

After this change of variables, $\theta = 0$ corresponds to $z = 1$, and $\theta = \pi$ to $z = 0$. The equation now reads

$$\frac{d^2y(z)}{dz^2} + V_{ang}(z)y(z) = 0\,, \tag{1.1.12}$$

with potential

$$V_{ang}(z) = \frac{1}{z^2(z-1)^2} \sum_{i=0}^{4} \hat{A}_i^\theta z^i\,. \tag{1.1.13}$$

Again, we give the explicit expressions of the coefficients $\hat{A}_i^\theta$ in Appendix 1.A. When written as Schrödinger equations, it is evident that the radial and angular equations share the same singularity structure. They both have two regular singular points at $z = 0, 1$ and an irregular singular point of Poincaré rank one at $z = \infty$. Such a differential equation is well-known in the mathematics literature as the confluent Heun equation [19].

1.2 The confluent Heun equation and conformal field theory

1.2.1 The confluent Heun equation in standard form

The confluent Heun equation (CHE) is a linear differential equation of second order with regular singularities at $z = 0$ and 1, and an irregular singularity of rank 1 at $z = \infty$. In its standard form it is written as

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\right)\frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)}w = 0\,. \tag{1.2.1}$$

By defining $w(z) = P(z)^{-1/2}\psi(z)$ with $P(z) = e^{\epsilon z} z^{\gamma}(z-1)^{\delta}$, we can bring the standard form of the CHE into the form of a Schrödinger equation:

$$\frac{d^2\psi(z)}{dz^2} + V_{Heun}(z)\psi(z) = 0 \tag{1.2.2}$$

where the potential is

$$V_{Heun}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4} A_i^H z^i\,, \tag{1.2.3}$$

with coefficients A_i given in terms of the parameters of the standard form of the CHE by

$$\begin{aligned}
A_0^H &= \frac{\gamma(2-\gamma)}{4}\,,\\
A_1^H &= q + \frac{\gamma}{2}(\gamma+\delta-\epsilon-2)\,,\\
A_2^H &= -q - \alpha - \frac{\gamma^2}{4} + \frac{\delta}{2} - \frac{(\delta-\epsilon)^2}{4} + \frac{\gamma}{2}(1-\delta+2\epsilon)\,,\\
A_3^H &= \alpha - \frac{\epsilon}{2}(\gamma+\delta-\epsilon)\,,\\
A_4^H &= -\frac{\epsilon^2}{4}\,.
\end{aligned} \tag{1.2.4}$$

1.2.2 The confluent Heun equation as a BPZ equation

In this section we work at the level of chiral conformal field theory/conformal blocks, which are fixed completely by the Virasoro algebra. Throughout this chapter we work with conformal momenta related to the conformal weight by $\Delta = \frac{Q^2}{4} - \alpha^2$. The representation theory of the Virasoro algebra contains degenerate Verma modules of weight $\Delta_{r,s} = \frac{Q^2}{4} - \alpha_{r,s}^2$ with $\alpha_{r,s} = -\frac{br}{2} - \frac{s}{2b}$, where $Q = b + \frac{1}{b}$ and b is related to the central charge as $c = 1 + 6Q^2$. At level 2, the degenerate field $\Phi_{2,1}$ has weight $\Delta_{2,1} = -\frac{1}{2} - \frac{3}{4}b^2$ and satisfies the null-state equation

$$(b^{-2}L_{-1}^2 + L_{-2})\cdot\Phi_{2,1}(z) = 0\,. \tag{1.2.5}$$

When this field is inserted in correlation functions, equation (1.2.5) translates into a differential equation for the correlator called BPZ equation [27]. Consider then the following conformal block with a degenerate field insertion, which by a slight abuse of notation we denote by

$$\Psi(z) := \langle\Delta, \Lambda_0, m_0|\Phi_{2,1}(z)V_2(1)|\Delta_1\rangle\,. \tag{1.2.6}$$

$\Phi_{2,1}$ is the degenerate field mentioned above, $V_2(1)$ is a primary operator of weight $\Delta_2 = \frac{Q^2}{4} - \alpha_2^2$ inserted at $z=1$ and $|\Delta_1\rangle$ is a primary state of weight $\Delta_1 = \frac{Q^2}{4} - \alpha_1^2$ corresponding via the state-operator correspondence to the insertion of $V_1(0)$. The state $\langle\Delta,\Lambda_0,m_0|$, called an irregular state of rank 1, is a more exotic kind of state, defined in [103] as:

$$\langle\Delta,\Lambda_0,m_0| = \sum_Y\sum_p \langle\Delta|L_Y m_0^{|Y|-2p}\Lambda_0^{|Y|}Q_\Delta^{-1}\big([2^p,1^{|Y|-2p}],Y\big)\,. \tag{1.2.7}$$

The first sum runs over Young tableaux Y, $|Y|$ denotes the total number of boxes in the tableau and Q is the Shapovalov form $Q_\Delta(Y,Y') = \langle\Delta|L_Y L_{-Y'}|\Delta'\rangle$. The notation $[2^p,1^{|Y|-2p}]$ refers to a Young tableau with p columns of two boxes and $|Y|-2p$ columns of single boxes. p then runs from 0 to $|Y|/2$. All in all this implies the following relations, derived in [103]:

$$\begin{aligned}
&\langle\Delta,\Lambda_0,m_0|L_0 = \left(\Delta+\Lambda_0\frac{\partial}{\partial\Lambda_0}\right)\langle\Delta,\Lambda_0,m_0|\,,\\
&\langle\Delta,\Lambda_0,m_0|L_{-1} = m_0\Lambda_0\langle\Delta,\Lambda_0,m_0|\,,\\
&\langle\Delta,\Lambda_0,m_0|L_{-2} = \Lambda_0^2\langle\Delta,\Lambda_0,m_0|\,,\\
&\langle\Delta,\Lambda_0,m_0|L_{-n} = 0 \quad \text{for } n\geq 3\,,
\end{aligned} \tag{1.2.8}$$

so it is a kind of coherent state for the Virasoro algebra. The investigation of these kind of states in CFT was motivated by the AGT conjecture [41] according to which they are related to asymptotically free gauge theories [44, 46, 47]. Indeed, this state can be obtained by colliding two primary operators mimicking the decoupling of a mass in the gauge theory [47, 103]. The result of the collision, understood as a scaling limit of an OPE, naturally has nonzero overlap with any Verma module. This gives a so-called Whittaker state [47] [104] [105], denoted by $\langle\Lambda_0,m_0|$ that makes no reference to any Verma module and is completely characterized by the following action of the Virasoro generators:

$$\begin{aligned}
&\langle\Lambda_0,m_0|L_0 = \Lambda_0\frac{\partial}{\partial\Lambda_0}\langle\Lambda_0,m_0|\,,\\
&\langle\Lambda_0,m_0|L_{-1} = m_0\Lambda_0\langle\Lambda_0,m_0|\,,\\
&\langle\Lambda_0,m_0|L_{-2} = \Lambda_0^2\langle\Lambda_0,m_0|\,,\\
&\langle\Lambda_0,m_0|L_{-n} = 0 \quad \text{for } n\geq 3\,,
\end{aligned} \tag{1.2.9}$$

The state introduced here is the projection of a Whittaker state onto a specific Verma module. Indeed, inserting explicitly the projector gives back the series (1.2.7):

$$\langle\Delta,\Lambda_0,m_0| := \Lambda_0^{-\Delta}\langle\Lambda_0,m_0|\sum_{Y,Y'}Q_\Delta^{-1}(Y,Y')L_{-Y'}|\Delta\rangle\langle\Delta|L_Y = \sum_Y\sum_p\langle\Delta|L_Y m_0^{|Y|-2p}\Lambda_0^{|Y|}Q_\Delta^{-1}\big([2^p,1^{|Y|-2p}],Y\big)\,, \tag{1.2.10}$$

where the overlap of the Whittaker state with a primary is defined as $\langle \Lambda_0, m_0|\Delta\rangle = \Lambda_0^\Delta$. This correlator satisfies the following BPZ equation (see Appendix 1.B for details):

$$\begin{aligned}0 =&\langle \Delta, \Lambda_0, m_0|\big(b^{-2}\partial_z^2 + L_{-2}\cdot\big)\Phi_{2,1}(z)V_2(1)|V_1\rangle = \\ =&\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z - \frac{z\partial_z - \Lambda_0\partial_{\Lambda_0} + \Delta_{2,1} + \Delta_2 + \Delta_1 - \Delta}{z(z-1)} + \frac{\Delta_2}{(z-1)^2} + \frac{\Delta_1}{z^2} + \frac{m_0\Lambda_0}{z} + \Lambda_0^2\right)\Psi(z)\,.\end{aligned} \tag{1.2.11}$$

We now take a double-scaling limit known as the Nekrasov-Shatashvili (NS) limit in the AGT dual gauge theory [106], which corresponds to the semiclassical limit of large Virasoro central charge in the CFT. This amounts to introducing a new parameter $\hbar$, and sending $\epsilon_2 = \hbar b \to 0$, while keeping fixed

$$\begin{aligned}&\epsilon_1 = \hbar/b,\\ &\hat{\Delta} = \hbar^2\Delta,\ \hat{\Delta}_1 = \hbar^2\Delta_1,\ \hat{\Delta}_2 = \hbar^2\Delta_2,\\ &\Lambda = 2i\hbar\Lambda_0,\ m_3 = \frac{i}{2}\hbar m_0\,.\end{aligned} \tag{1.2.12}$$

Furthermore, arguments from CFT [107] and the AGT conjecture tell us that in this limit the correlator exponentiates and the z-dependence appears only at subleading order:

$$\Psi(z) \propto \exp\frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F}^{\text{inst}}(\epsilon_1) + \epsilon_2\mathcal{W}(z;\epsilon_1) + \mathcal{O}(\epsilon_2^2)\right). \tag{1.2.13}$$

Introducing the normalized wavefunction $\psi(z) = \lim_{\epsilon_2\to 0}\Psi(z)/\langle\Delta, \Lambda_0, m_0|V_2(1)|\Delta_1\rangle$ and multiplying everything by $\hbar^2$, the BPZ equation in the NS limit becomes

$$0 = \left(\epsilon_1^2\partial_z^2 - \frac{1}{z}\frac{1}{z-1}\big(-\Lambda\partial_\Lambda\mathcal{F}^{\text{inst}} + \hat{\Delta}_2 + \hat{\Delta}_1 - \hat{\Delta}\big) + \frac{\hat{\Delta}_2}{(z-1)^2} + \frac{\hat{\Delta}_1}{z^2} - \frac{m_3\Lambda}{z} - \frac{\Lambda^2}{4}\right)\psi(z)\,. \tag{1.2.14}$$

All other terms vanish in the limit. It takes the form of a Schrödinger equation:

$$\epsilon_1^2\frac{d^2\psi(z)}{dz^2} + V_{CFT}(z)\psi(z) = 0 \tag{1.2.15}$$

with potential

$$V_{CFT}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}A_i z^i\,. \tag{1.2.16}$$

Written in this form it is clear that the BPZ equation for this correlation function takes the form of the confluent Heun equation. Using conformal momenta instead of dimensions we write $\hat{\Delta}_i = \frac{1}{4} - a_i^2$, where we have used $\hat{\Delta}_i = \hbar^2\Delta_i$, $\hbar Q = \epsilon_1 + \epsilon_2 = \epsilon_1$ and defined $a_i := \hbar\alpha_i$.

Defining furthermore $E := a^2 - \Lambda\partial_\Lambda \mathcal{F}^{\text{inst}}$, the coefficients of the potential are

$$\begin{aligned}
A_0 &= \frac{\epsilon_1^2}{4} - a_1^2\,,\\
A_1 &= -\frac{\epsilon_1^2}{4} + E + a_1^2 - a_2^2 - m_3\Lambda\,,\\
A_2 &= \frac{\epsilon_1^2}{4} - E + 2m_3\Lambda - \frac{\Lambda^2}{4}\,,\\
A_3 &= -m_3\Lambda + \frac{\Lambda^2}{2}\,,\\
A_4 &= -\frac{\Lambda^2}{4}\,.
\end{aligned} \tag{1.2.17}$$

Comparing with the coefficients A_i^H of the CHE in (1.2.4) and setting $\epsilon_1 = 1$ to match the coefficient of the second derivative, we can identify the parameters of the standard form with the parameters of the CFT as:

$$\boxed{\begin{aligned}
&\alpha = \theta''\Lambda(1 + \theta a_1 + \theta' a_2 + \theta'' m_3)\,,\\
&\gamma = 1 + 2\theta a_1\,,\\
&\delta = 1 + 2\theta' a_2\,,\\
&\epsilon = \theta''\Lambda\,,\\
&q = E - \frac{1}{4} - (\theta a_1 + \theta' a_2)^2 - (\theta a_1 + \theta' a_2) + \theta''\Lambda\left(\frac{1}{2} + \theta a_1 - \theta'' m_3\right)\,,
\end{aligned}} \tag{1.2.18}$$

for any choice of signs $\theta, \theta', \theta'' = \pm 1$. These $8 = 2^3$ dictionaries reflect the symmetries of the equation, which is invariant independently under $a_1 \to -a_1$, $a_2 \to -a_2$ and $(m_3, \Lambda) \to -(m_3, \Lambda)$.

1.2.3 The radial dictionary

We see that the BPZ equation takes the same form as the radial and angular equations of the black hole perturbation equation if we set $\epsilon_1 = 1$. We will do this from now on. This implies $b = \hbar$. Comparing with the coefficients $\hat{A}_i^r$ we find the following eight dictionaries between the parameters of the radial equation in the black hole problem and the CFT:

$$
\begin{aligned}
E &= \frac{1}{4} + \lambda + s(s+1) + \mathrm{a}^2\omega^2 - 8M^2\omega^2 - \left(2M\omega^2 + is\omega\right)(r_+ - r_-)\,, \\
a_1 &= \theta\left(-i\frac{\omega - m\Omega}{4\pi T_H} + 2\mathrm{i}M\omega + \frac{s}{2}\right)\,, \\
a_2 &= \theta'\left(-i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2}\right)\,, \\
m_3 &= \theta''\left(-2iM\omega + s\right)\,, \\
\Lambda &= -2i\theta''\omega(r_+ - r_-)\,,
\end{aligned}
\tag{1.2.19}
$$

where $\theta, \theta', \theta'' = \pm 1$. We will make the following choice for the dictionary from now on:

$$
\boxed{
\begin{aligned}
E &= \frac{1}{4} + \lambda + s(s+1) + \mathrm{a}^2\omega^2 - 8M^2\omega^2 - \left(2M\omega^2 + is\omega\right)(r_+ - r_-)\,, \\
a_1 &= -i\frac{\omega - m\Omega}{4\pi T_H} + 2\mathrm{i}M\omega + \frac{s}{2}\,, \\
a_2 &= -i\frac{\omega - m\Omega}{4\pi T_H} - \frac{s}{2}\,, \\
m_3 &= -2iM\omega + s\,, \\
\Lambda &= -2i\omega(r_+ - r_-)\,,
\end{aligned}
}
\tag{1.2.20}
$$

which corresponds to $\theta = \theta' = \theta'' = +1$. Using AGT this dictionary gives the following masses in the gauge theory (see Appendix 1.C for details):

$$
\begin{aligned}
m_1 &= a_1 + a_2 = -i\frac{\omega - m\Omega}{2\pi T_H} + 2iM\omega\,, \\
m_2 &= a_2 - a_1 = -2iM\omega - s\,, \\
m_3 &= -2iM\omega + s\,.
\end{aligned}
\tag{1.2.21}
$$

This is the same result as the one found in [68] except for a shift in E, which is due to a different definition of the $U(1)$-factor. For $s = 0$ the values are purely imaginary and correspond to physical Liouville momenta. For $s \neq 0$ the conformal block gets analytically continued. Note that in the supersymmetric gauge theory the masses are naturally complex parameters.

1.2.4 The angular dictionary

Comparing instead (1.2.17) with the $\hat{A}_i^\theta$ in (1.A.4) we find the following eight dictionaries between the parameters of the angular equation in the black hole problem and the CFT:

$$\begin{aligned} E &= \frac{1}{4} + c^2 + s(s+1) - 2cs + \lambda\,, \\ a_1 &= \theta\left(-\frac{m-s}{2}\right)\,, \\ a_2 &= \theta'\left(-\frac{m+s}{2}\right)\,, \\ m_3 &= -\theta'' s\,, \\ \Lambda &= \theta'' 4c\,, \end{aligned} \tag{1.2.22}$$

where again $\theta, \theta', \theta'' = \pm 1$ and our choice from here on will be $\theta = \theta' = \theta'' = +1$, i.e.:

$$\boxed{\begin{aligned} E &= \frac{1}{4} + c^2 + s(s+1) - 2cs + \lambda\,, \\ a_1 &= -\frac{m-s}{2}\,, \\ a_2 &= -\frac{m+s}{2}\,, \\ m_3 &= -s\,, \\ \Lambda &= 4c\,. \end{aligned}} \tag{1.2.23}$$

Using AGT this dictionary gives the following masses in the gauge theory (see Appendix 1.C for details):

$$\begin{aligned} m_1 &= a_1 + a_2 = -m\,, \\ m_2 &= a_2 - a_1 = -s\,, \\ m_3 &= -s\,. \end{aligned} \tag{1.2.24}$$

Again we note the discrepancy with [68] due to the different $U(1)$-factor.

1.3 The connection problem

Exploiting crossing symmetry of Liouville correlation functions we can connect different asymptotic expansions of the solutions of BPZ equations around different field insertion points. The DOZZ formula can be obtained exploiting the known connection formulae for hypergeometric functions [108, 109]. Here we do the reverse, namely knowing the DOZZ formula we reconstruct connection formulae for irregular degenerate conformal blocks. Asymptotic expansions are computed via OPEs with regular and irregular insertions. To this end, we recall that the OPE of the degenerate field of our interest and a primary field reads [27]:

$$\Phi_{2,1}(z,\bar{z})V_{\alpha_i}(w,\bar{w}) = \sum_{\pm} \mathcal{C}^{\alpha_{i\pm}}_{\alpha_{2,1},\alpha_i} |z-w|^{2k_\pm} \left(V_{\alpha_{i\pm}}(w,\bar{w}) + \mathcal{O}(|z-w|^2)\right) , \tag{1.3.1}$$

where $\alpha_{i\pm} := \alpha_i \pm \frac{-b}{2}$, and $k_\pm = \Delta_{\alpha_{i\pm}} - \Delta_{\alpha_i} - \Delta_{2,1}$ is fixed by the L_0 action. The OPE coefficient $\mathcal{C}^{\alpha_{i\pm}}_{\alpha_{2,1},\alpha_i}$ is computed in terms of DOZZ factors [110] [111] (see Appendix 1.B.2), namely

$$\mathcal{C}^{\alpha_{i\pm}}_{\alpha_{2,1},\alpha_i} = G^{-1}(\alpha_{i\pm})C(\alpha_{i\pm}, \frac{-b-Q}{2}, \alpha_i) \, . \tag{1.3.2}$$

The OPE with the irregular state is constrained by conformal symmetry, and the leading behavior is fixed by the action of L_0, L_1, L_2 instead of just L_0. The overall factors are again given in terms of DOZZ factors (see Appendix 1.B.3). One finds

$$\begin{aligned}
&\langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, m_0 | \Phi_{2,1}(z,\bar{z}) = \\
&= \mathcal{C}^{\alpha_+}_{\alpha,\alpha_{2,1}} \left| \sum_{\pm,k} \mathcal{A}_{\alpha_+,m_{0\pm}} (\pm\Lambda)^{-\frac{1}{2}\pm m_3 + b\alpha_+} z^{\frac{1}{2}(bQ-1\pm 2m_3)} e^{\pm\Lambda z/2} z^{-k} \langle \Delta_{\alpha_+}, \Lambda_0, m_{0\pm}; k| \right|^2 + \\
&+ \mathcal{C}^{\alpha_-}_{\alpha,\alpha_{2,1}} \left| \sum_{\pm,k} \mathcal{A}_{\alpha_-,m_{0\pm}} (\pm\Lambda)^{-\frac{1}{2}\pm m_3 - b\alpha_-} z^{\frac{1}{2}(bQ-1\pm 2m_3)} e^{\pm\Lambda z/2} z^{-k} \langle \Delta_{\alpha_-}, \Lambda_0, m_{0\pm}; k| \right|^2 .
\end{aligned} \tag{1.3.3}$$

Here the irregular state depending on $\Lambda_0, \bar{\Lambda}_0$ denotes the full (chiral⊗antichiral) state, and the modulus squared of the chiral states (depending only on Λ_0) also has to be understood as a tensor product. The coefficients $\mathcal{A}$ are given by

$$\begin{aligned}
\mathcal{A}_{\alpha_+,m_{0+}} &= \frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}+m_3-b\alpha_+)} \, , \; \mathcal{A}_{\alpha_+,m_{0-}} = \frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}-m_3-b\alpha_+)} \, , \\
\mathcal{A}_{\alpha_-,m_{0+}} &= \frac{\Gamma(1+2b\alpha_-)}{\Gamma(\frac{1}{2}+m_3+b\alpha_-)} \, , \; \mathcal{A}_{\alpha_-,m_{0-}} = \frac{\Gamma(1+2b\alpha_-)}{\Gamma(\frac{1}{2}-m_3+b\alpha_-)} \, .
\end{aligned} \tag{1.3.4}$$

Since the results presented in this section are formulated purely in a CFT context, they will be written for finite b unless otherwise specified.

1.3.1 Connection formulae for the irregular 4 point function

Let us consider the irregular correlator

$$\Psi(z,\bar{z}) = \langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, m_0 | \Phi_{2,1}(z,\bar{z}) V_{\alpha_2}(1,\bar{1}) | \Delta_{\alpha_1} \rangle \, . \tag{1.3.5}$$

The physical, crossing symmetric correlator has to be built using the Whittaker state (1.2.9) introduced before which makes no reference to Δ_α. Here instead we use the state projected onto the Verma module Δ_α which provides us with the explicit expression (1.2.7). In particular, the $\Lambda_0 \to 0$ limit is simple: it is just a primary state with the usual normalization. In any case, we still expect (1.3.5) to be crossing symmetric and we will exploit this in what follows. In the next chapter (2) we will show that the result presented here is consistent with crossing symmetry of the physical correlator. The asymptotics of Ψ for $z \sim 1, \infty$, respectively $t, u-$channels, are given by the OPEs. Due to crossing symmetry, the two expansions have to agree, therefore

$$\Psi(z,\bar{z}) = K^{(t)}_{\alpha_{2+},\alpha_{2+}}|f^{(t)}_{\alpha_{2+}}(z)|^2 + K^{(t)}_{\alpha_{2-},\alpha_{2-}}|f^{(t)}_{\alpha_{2-}}(z)|^2 = K^{(u)}_{\alpha_+,\alpha_+}|f^{(u)}_{\alpha_+}(z)|^2 + K^{(u)}_{\alpha_-,\alpha_-}|f^{(u)}_{\alpha_-}(z)|^2 . \tag{1.3.6}$$

where

$$\begin{aligned} K^{(t)}_{\alpha_{2+},\alpha_{2+}} &= \mathcal{C}^{\alpha_{2+}}_{\alpha_{2,1}\alpha_2} C(\alpha,\alpha_{2+},\alpha_1)\,,\; K^{(t)}_{\alpha_{2-},\alpha_{2-}} = \mathcal{C}^{\alpha_{2-}}_{\alpha_{2,1}\alpha_2} C(\alpha,\alpha_{2-},\alpha_1)\,,\\ K^{(u)}_{\alpha_+,\alpha_+} &= \mathcal{C}^{\alpha_+}_{\alpha_{2,1}\alpha} C(\alpha_+,\alpha_2,\alpha_1)\,,\; K^{(u)}_{\alpha_-,\alpha_-} = \mathcal{C}^{\alpha_-}_{\alpha_{2,1}\alpha} C(\alpha_-,\alpha_2,\alpha_1)\,, \end{aligned} \tag{1.3.7}$$

are the DOZZ factors for the two fusion channels in the t and u-channel OPEs and

$$\begin{aligned} f^{(t)}_{\alpha_{2+}}(z) &= \langle\Delta_\alpha,\Lambda_0,m_0|V_{\alpha_{2+}}(1)|\Delta_{\alpha_1}\rangle(z-1)^{\frac{bQ+2b\alpha_2}{2}}\left(1+\mathcal{O}(z-1)\right)\,,\\ f^{(t)}_{\alpha_{2-}}(z) &= \langle\Delta_\alpha,\Lambda_0,m_0|V_{\alpha_{2-}}(1)|\Delta_{\alpha_1}\rangle(z-1)^{\frac{bQ-2b\alpha_2}{2}}\left(1+\mathcal{O}(z-1)\right)\left(1+\mathcal{O}(z-1)\right)\,,\\ f^{(u)}_{\alpha_+}(z) &= \sum_\pm \langle\Delta_{\alpha_+},\Lambda_0,m_{0\pm}|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle\mathcal{A}_{\alpha_+,m_{0\pm}}e^{\pm\frac{\Lambda z}{2}}(\pm\Lambda)^{-\frac{1}{2}\pm m_3+b\alpha_+}z^{\frac{1}{2}(bQ-1\pm 2m_3)}\left(1+\mathcal{O}(z^{-1})\right)\,,\\ f^{(u)}_{\alpha_-}(z) &= \sum_\pm \langle\Delta_{\alpha_-},\Lambda_0,m_{0\pm}|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle\mathcal{A}_{\alpha_-,m_{0\pm}}e^{\pm\frac{\Lambda z}{2}}(\pm\Lambda)^{-\frac{1}{2}\pm m_3-b\alpha_-}z^{\frac{1}{2}(bQ-1\pm 2m_3)}\left(1+\mathcal{O}(z^{-1})\right)\,, \end{aligned} \tag{1.3.8}$$

give the expansions of the conformal blocks in the two fusion channels of the t and u-channels. Here and in the following the chiral correlators have to be understood as conformal blocks, we have extracted the DOZZ factors and they appear in (1.3.7). Note that in line with the definition (1.2.7), the irregular state contributes to the DOZZ factor the same as a regular state. Here, as noted in section 1.2.2, $f^{(t,u)}_\pm$ in the NS limit are (up to a rescaling by one of the correlators, to keep them finite) the two linearly independent confluent Heun functions expanded around 1 and ∞, respectively. We remark that due to the presence of the irregular singularity the $\alpha_\pm$ channels at infinity contribute with two different irregular states each, corresponding to $m_{0\pm}$. This is consistent with the fact that the irregular state comes from the collision of two primary operators [47]. The two expansions are related via a connection matrix M by

$$f^{(t)}_i(z) = M_{ij}f^{(u)}_j(z)\,,\; i=\alpha_{2\pm}\,,\; j=\alpha_\pm\,. \tag{1.3.9}$$

This equation, combined with the requirement of crossing symmetry (1.3.6) gives the constraints

$$K^{(t)}_{ij}M_{ik}M_{jl} = K^{(u)}_{kl}\,. \tag{1.3.10}$$

Equations (1.3.10) give 3 quadratic equations for the 4 entries M_{ij}. Other constraints come from noticing that the M_{ij} have to respect the symmetry under reflection of the momenta. The sign ambiguity inherent in the quadratic constraints (1.3.10) is resolved by imposing that for $\Lambda \to 0$ they reduce to the known hypergeometric connection matrix, since

$$\langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, m_0 | \Phi_{2,1}(z,\bar{z}) V_{\alpha_2}(1,\bar{1}) | \Delta_{\alpha_1} \rangle \to \langle \Delta_\alpha | \Phi_{2,1}(z,\bar{z}) V_{\alpha_2}(1,\bar{1}) | \Delta_{\alpha_1} \rangle \,, \text{ as } \Lambda \to 0 \,, \tag{1.3.11}$$

and conformal blocks of the regular degenerate 4 point functions are hypergeometric functions. This gives

$$\begin{aligned} M_{\alpha_{2+},\alpha_+} &= \frac{\Gamma(-2b\alpha)\Gamma(1+2b\alpha_2)}{\Gamma(\frac{1}{2}+b(\alpha_1+\alpha_2-\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1+\alpha_2-\alpha))} \,, \\ M_{\alpha_{2-},\alpha_-} &= \frac{\Gamma(2b\alpha)\Gamma(1-2b\alpha_2)}{\Gamma(\frac{1}{2}+b(\alpha_1-\alpha_2+\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1-\alpha_2+\alpha))} \,, \\ M_{\alpha_{2+},\alpha_-} &= \frac{\Gamma(2b\alpha)\Gamma(1+2b\alpha_2)}{\Gamma(\frac{1}{2}+b(\alpha_1+\alpha_2+\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1+\alpha_2+\alpha))} \,, \\ M_{\alpha_{2-},\alpha_+} &= \frac{\Gamma(-2b\alpha)\Gamma(1-2b\alpha_2)}{\Gamma(\frac{1}{2}+b(\alpha_1-\alpha_2-\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1-\alpha_2-\alpha))} \,. \end{aligned} \tag{1.3.12}$$

Note that M_{ij} is given by the hypergeometric connection matrix even for finite Λ, since all Λ corrections are encoded in the asymptotics of the functions (1.3.8). Proceeding in the same way we can find connection coefficients between $0,1$. Using crossing symmetry we have

$$\Psi(z,\bar{z}) = K^{(s)}_{\alpha_{1+},\alpha_{1+}} |f^{(s)}_{\alpha_{1+}}(z)|^2 + K^{(s)}_{\alpha_{1-},\alpha_{1-}} |f^{(s)}_{\alpha_{1-}}(z)|^2 = K^{(t)}_{\alpha_{2+},\alpha_{2+}} |f^{(t)}_{\alpha_{2+}}(z)|^2 + K^{(t)}_{\alpha_{2-},\alpha_{2-}} |f^{(t)}_{\alpha_{2-}}(z)|^2 \,, \tag{1.3.13}$$

where

$$\begin{aligned} f^{(s)}_{\alpha_{1+}}(z) &\simeq \langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_{1+}} \rangle z^{\frac{bQ+b\alpha_1}{2}} \,, \\ f^{(s)}_{\alpha_{1-}}(z) &\simeq \langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_{1-}} \rangle z^{\frac{bQ-b\alpha_1}{2}} \,. \end{aligned} \tag{1.3.14}$$

Imposing again

$$f^{(s)}_i(z) = N_{ij} f^{(t)}_j(z) \,, \tag{1.3.15}$$

substituting (1.3.15) in (1.3.13) and imposing that $f^{(s,t)}$ reduce to hypergeometric functions as $\Lambda \to 0$ we find (see Appendix 1.B.2)

$$\begin{aligned}
N_{\alpha_{1+},\alpha_{2+}} &= \frac{\Gamma(-2b\alpha_2)\Gamma(1+2b\alpha_1)}{\Gamma(\frac{1}{2}+b(\alpha_1-\alpha_2+\alpha))\Gamma(\frac{1}{2}+b(\alpha_1-\alpha_2-\alpha))}\,,\\
N_{\alpha_{1-},\alpha_{2-}} &= \frac{\Gamma(2b\alpha_2)\Gamma(1-2b\alpha_1)}{\Gamma(\frac{1}{2}+b(-\alpha_1+\alpha_2-\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1+\alpha_2+\alpha))}\,,\\
N_{\alpha_{1+},\alpha_{2-}} &= \frac{\Gamma(2b\alpha_2)\Gamma(1+2b\alpha_1)}{\Gamma(\frac{1}{2}+b(\alpha_1+\alpha_2-\alpha))\Gamma(\frac{1}{2}+b(\alpha_1+\alpha_2+\alpha))}\,,\\
N_{\alpha_{1-},\alpha_{2+}} &= \frac{\Gamma(-2b\alpha_2)\Gamma(1-2b\alpha_1)}{\Gamma(\frac{1}{2}+b(-\alpha_1-\alpha_2+\alpha))\Gamma(\frac{1}{2}+b(-\alpha_1-\alpha_2-\alpha))}\,.
\end{aligned} \tag{1.3.16}$$

1.3.2 AGT dual of irregular correlators and NS limit

The irregular correlators appearing in the asymptotics of the functions (1.3.8) can be efficiently computed as Nekrasov partition functions thanks to the AGT correspondence [41]. In particular, the irregular conformal block is identified with [103]

$$\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle = \mathcal{Z}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3)\,, \tag{1.3.17}$$

where $\mathcal{Z}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3)$ is the Nekrasov instanton partition function of $SU(2)$ $\mathcal{N}=2$ gauge theory in the Ω-background (see Appendix 1.C). While the analysis in the last section was completely general, in order to apply the obtained results to the Teukolsky equation, one needs to take the NS limit $\epsilon_2 \to 0$, $\epsilon_1 = 1$ as discussed in section 1.2.2. In this limit the correlators diverge, but rescaling the functions in (1.3.8) by one of the correlators, the resulting ratios are finite. In a slight abuse of notation, we write the connection coefficients in the NS limit as

$$\begin{aligned}
M_{a_{2+},a_+} &= \frac{\Gamma(-2a)\Gamma(1+2a_2)}{\Gamma(\frac{1}{2}+a_1+a_2-a)\Gamma(\frac{1}{2}-a_1+a_2-a)}\,,\\
M_{a_{2-},a_-} &= \frac{\Gamma(2a)\Gamma(1-2a_2)}{\Gamma(\frac{1}{2}+a_1-a_2+a)\Gamma(\frac{1}{2}-a_1-a_2+a)}\,,\\
M_{a_{2+},a_-} &= \frac{\Gamma(2a)\Gamma(1+2a_2)}{\Gamma(\frac{1}{2}+a_1+a_2+a)\Gamma(\frac{1}{2}-a_1+a_2+a)}\,,\\
M_{a_{2-},a_+} &= \frac{\Gamma(-2a)\Gamma(1-2a_2)}{\Gamma(\frac{1}{2}+a_1-a_2-a)\Gamma(\frac{1}{2}-a_1-a_2-a)}\,,
\end{aligned} \tag{1.3.18}$$

and similarly

$$\begin{aligned}
N_{a_{1+},a_{2+}} &= \frac{\Gamma(-2a_2)\Gamma(1+2a_1)}{\Gamma(\frac{1}{2}+a_1-a_2+a)\Gamma(\frac{1}{2}+a_1-a_2-a)}\,,\\
N_{a_{1-},a_{2-}} &= \frac{\Gamma(2a_2)\Gamma(1-2a_1)}{\Gamma(\frac{1}{2}-a_1+a_2-a)\Gamma(\frac{1}{2}-a_1+a_2+a)}\,,\\
N_{a_{1+},a_{2-}} &= \frac{\Gamma(2a_2)\Gamma(1+2a_1)}{\Gamma(\frac{1}{2}+a_1+a_2-a)\Gamma(\frac{1}{2}+a_1+a_2+a)}\,,\\
N_{a_{1-},a_{2+}} &= \frac{\Gamma(-2a_2)\Gamma(1-2a_1)}{\Gamma(\frac{1}{2}-a_1-a_2+a)\Gamma(\frac{1}{2}-a_1-a_2-a)}\,,
\end{aligned} \tag{1.3.19}$$

where $a_i = \hbar\alpha_i = b\alpha_i$ for $\epsilon_1 = \hbar/b = 1$.

1.3.3 Plots of the connection coefficients

In the following we illustrate the power of the connection coefficients obtained above by comparing our analytical solution to the numerical one. Furthermore this illustrates how to evaluate the connection coefficients. For simplicity we focus on the connection problem between $z = 0$ and 1. The confluent Heun function $w(z)$ solving the CHE in standard form (2.3.24) can be expanded as a power series near $z = 0$ as

$$w(z) = 1 - \frac{q}{\gamma}z + \frac{\alpha\gamma + q(q-\gamma-\delta+\epsilon)}{2\gamma(\gamma+1)}z^2 + \mathcal{O}(z^3)\,. \tag{1.3.20}$$

We are interested in analytically continuing this series toward the other singular point at $z = 1$. This problem is solved by our connection coefficients, we just need to identify the functions and parameters: in terms of the function $\psi(z)$ solving the CHE in Schrödinger form (1.2.2), we have around $z = 0$:

$$\psi(z) = e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z) = z^{\frac{1}{2}+\theta a_1}\left(1+\mathcal{O}(z)\right) = \hat{f}^{(s)}_{\alpha_{1\theta}}(z)\,, \tag{1.3.21}$$

where we have introduced the normalized s-channel function, related to the s-channel function defined before by $f^{(s)}_{\alpha_{1\theta}}(z) = \langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_{1\theta}}\rangle \hat{f}^{(s)}_{\alpha_{1\theta}}(z)$. Similarly, we define the normalized t-channel function, related to the one defined before by $f^{(t)}_{\alpha_{2\theta'}}(z) = \langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_{2\theta'}}(1)|\Delta_{\alpha_1}\rangle \hat{f}^{(t)}_{\alpha_{2\theta'}}(z)$. It is a solution to the CHE given as a power series around the singular point $z = 1$ which can be obtained by the Fröbenius method:

$$\hat{f}^{(t)}_{\alpha_{2\theta'}}(z) = (1-z)^{\frac{1}{2}+\theta' a_2}\left(1 - \frac{1/4 - a_1^2 - a_2^2 + E}{1+2\theta' a_2}(1-z) + \mathcal{O}((1-z)^2)\right)\,. \tag{1.3.22}$$

The s- and t-channel solutions are related by $f_i^{(s)} = N_{ij} f_j^{(t)}$, with the coefficients N_{ij} given before, which we now give more explicitly:

$$\boxed{\begin{aligned}\hat{f}^{(s)}_{\alpha_{1\theta}}(z) = &\frac{\Gamma(-2a_2)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 - a_2 + a))\Gamma(\frac{1}{2}+\theta a_1 - a_2 - a))} \frac{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1}\rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_{1\theta}}\rangle} \hat{f}^{(t)}_{\alpha_{2+}}(z) + \\ &+ \frac{\Gamma(2a_2)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 + a_2 + a))\Gamma(\frac{1}{2}+\theta a_1 + a_2 - a))} \frac{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2-}}(1) | \Delta_{\alpha_1}\rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_{1\theta}}\rangle} \hat{f}^{(t)}_{\alpha_{2-}}(z)\end{aligned}} \tag{1.3.23}$$

for $\theta = \pm$. A further complication arises from the fact that the parameter in the CHE is E, but in the connection formula the parameter a appears which is related to E in a nontrivial way and has to be obtained by inverting the Matone relation[2] [112, 113] (See Appendix 1.C for details):

$$E = a^2 - \Lambda \partial_\Lambda \mathcal{F}^{\text{inst}}. \tag{1.3.24}$$

Everything has to be computed for general ϵ_1, ϵ_2 using Nekrasov formulae and then specialized to the NS limit by setting $\epsilon_1 = 1$ and taking the limit $\epsilon_2 \to 0$ in the end. To work consistently at one instanton one also needs to expand the Gamma functions since they contain a which is given as an instanton expansion. We get

$$\boxed{\begin{aligned}&\hat{f}^{(s)}_{\alpha_{1\theta}}(z) = \frac{\Gamma(-2a_2)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 - a_2 + \sqrt{E}))\Gamma(\frac{1}{2}+\theta a_1 - a_2 - \sqrt{E}))} \hat{f}^{(t)}_{\alpha_{2+}}(z)\times \\ &\times \left[1 - \left(\frac{\theta a_1 + a_2}{\frac{1}{2} - 2E} + \frac{\frac{1}{4} - E + a_1^2 - a_2^2}{\sqrt{E}\,(1-4E)}\left[\psi^{(0)}\left(\frac{1}{2} - \sqrt{E} + \theta a_1 - a_2\right) - \psi^{(0)}\left(\frac{1}{2} + \sqrt{E} + \theta a_1 - a_2\right)\right]\right) m_3 \Lambda\right] + \\ &+ \frac{\Gamma(2a_2)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 + a_2 + \sqrt{E}))\Gamma(\frac{1}{2}+\theta a_1 + a_2 - \sqrt{E}))} \hat{f}^{(t)}_{\alpha_{2-}}(z)\times \\ &\times \left[1 - \left(\frac{\theta a_1 - a_2}{\frac{1}{2} - 2E} + \frac{\frac{1}{4} - E + a_1^2 - a_2^2}{\sqrt{E}\,(1-4E)}\left[\psi^{(0)}\left(\frac{1}{2} - \sqrt{E} + \theta a_1 + a_2\right) - \psi^{(0)}\left(\frac{1}{2} + \sqrt{E} + \theta a_1 + a_2\right)\right]\right) m_3 \Lambda\right] \\ &+ \mathcal{O}(\Lambda^2).\end{aligned}} \tag{1.3.25}$$

Here $\psi^{(0)}(z) = \frac{d}{dz}\log\Gamma(z)$ is the digamma function. The higher instanton corrections to the connection coefficients can be computed in an analogous way. We have identified $\hat{f}^{(s)}_{\alpha_{1\theta}}(z) = e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z)$ by using the power series expansion near $z=0$. We can then use the connection formula given above to obtain the power series expansion near $z=1$ in terms of $\hat{f}^{(t)}_{\alpha_{2\pm}}(z)$, and compare it to the numerical solution. In the following we illustrate the power of the connection formula by giving random values (in a suitable range) to the various parameters and plotting the confluent Heun function numerically versus the three-term power expansion at $z=1$, computed analytically by using the connection formula

[2] From the gauge theory viewpoint $a \in \mathbb{C}$ parametrizes the Cartan flat direction of the potential for the scalar field component φ of the $\mathcal{N}=2$ vector multiplet. The parameter E in the gauge theory is the gauge invariant co-ordinate $\langle \mathrm{Tr}\varphi^2 \rangle$ of the Coulomb branch.

from 0 to 1. Here we use the dictionary between the parameters of the CHE in standard form and the CFT parameters given in (2.3.25), with $\theta = +1, \theta' = -1, \theta'' = -1$.

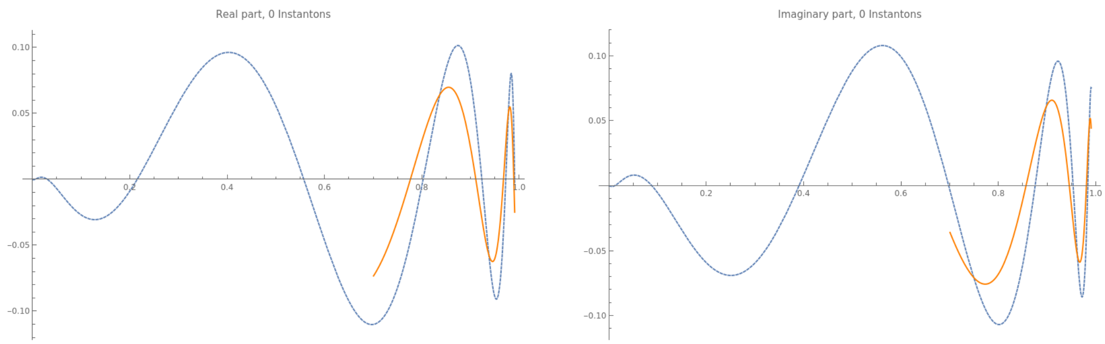


Figure 1.1: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2} (z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z = 1$ (solid, orange), obtained analytically using the connection coefficients computed at zero instantons. The validity of the series expansion around $z = 1$ (orange) is limited to a neighborhood of $z = 1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.970123 + 1.36981i$, $a_2 = -0.386424 - 2.99783i$, $E = 5.41627 + 6.40871i$, $m_3 = 1.68707 - 0.707722i$, $\Lambda = 1.96772 + 1.80414i$.

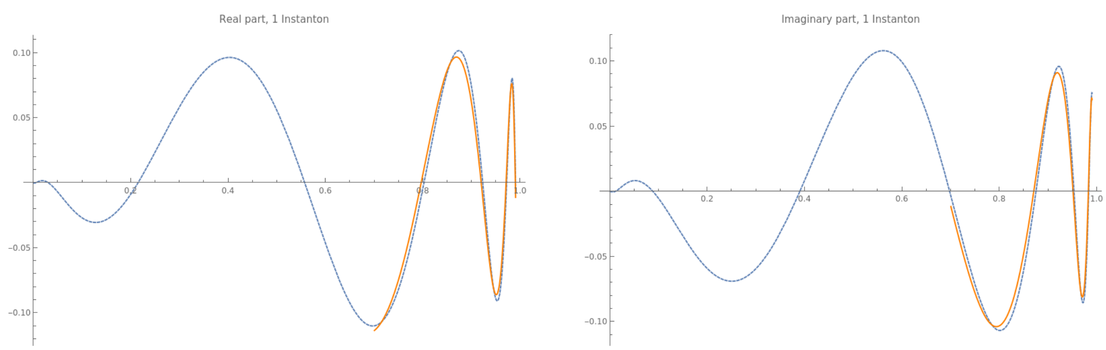


Figure 1.2: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2} (z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z = 1$ (solid, orange), obtained analytically using the connection coefficients computed at one instanton. The validity of the series expansion around $z = 1$ (orange) is limited to a neighborhood of $z = 1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.970123 + 1.36981i$, $a_2 = -0.386424 - 2.99783i$, $E = 5.41627 + 6.40871i$, $m_3 = 1.68707 - 0.707722i$, $\Lambda = 1.96772 + 1.80414i$.

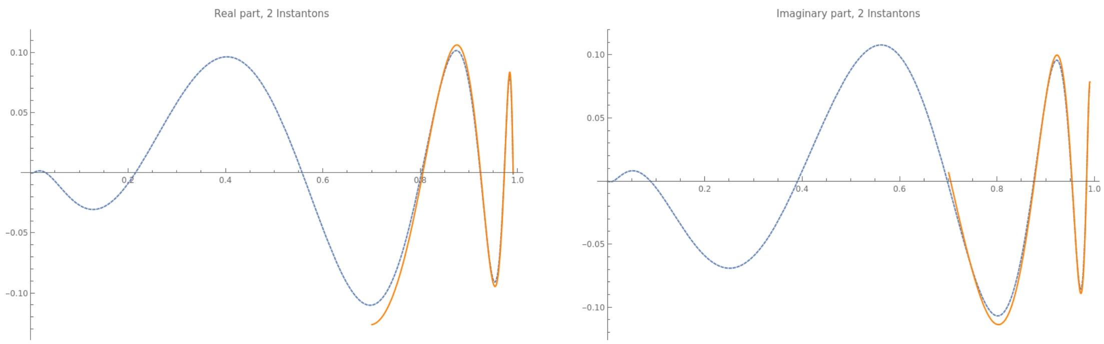


Figure 1.3: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z=1$ (solid, orange), obtained analytically using the connection coefficients computed at two instantons. The validity of the series expansion around $z=1$ (orange) is limited to a neighborhood of $z=1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.970123+1.36981i$, $a_2 = -0.386424-2.99783i$, $E = 5.41627+6.40871i$, $m_3 = 1.68707-0.707722i$, $\Lambda = 1.96772+1.80414i$.

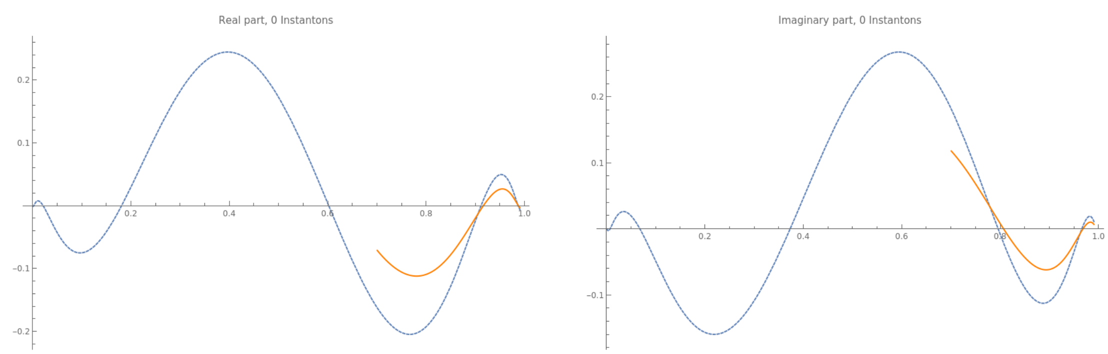


Figure 1.4: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z=1$ (solid, orange), obtained analytically using the connection coefficients computed at zero instantons. The validity of the series expansion around $z=1$ (orange) is limited to a neighborhood of $z=1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.5+1.24031i$, $a_2 = -0.5+1.55419i$, $E = 5.52396$, $m_3 = 0.92039+1.36765i$, $\Lambda = 1.60238+1.25941i$.

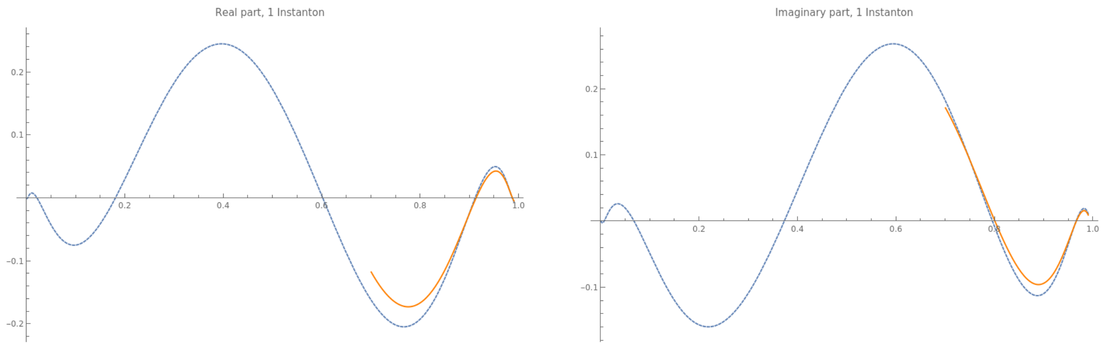


Figure 1.5: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z=1$ (solid, orange), obtained analytically using the connection coefficients computed at one instanton. The validity of the series expansion around $z=1$ (orange) is limited to a neighborhood of $z=1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.5 + 1.24031i$, $a_2 = -0.5 + 1.55419i$, $E = 5.52396$, $m_3 = 0.92039 + 1.36765i$, $\Lambda = 1.60238 + 1.25941i$.

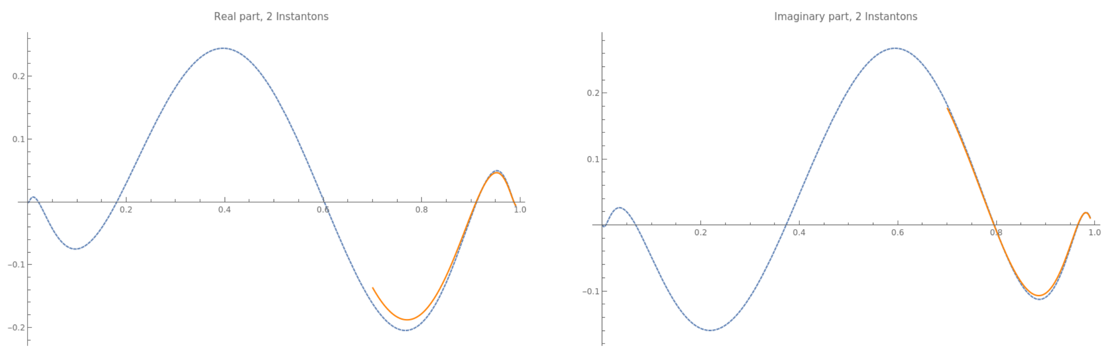


Figure 1.6: Real and imaginary parts of the rescaled confluent Heun function $e^{\epsilon z/2} z^{\gamma/2}(z-1)^{\delta/2} w(z)$ (blue, dashed), computed numerically, and of the three-term power expansion near $z=1$ (solid, orange), obtained analytically using the connection coefficients computed at two instantons. The validity of the series expansion around $z=1$ (orange) is limited to a neighborhood of $z=1$, but going to higher orders in the expansion to extend the validity is straightforward. The values of the parameters are: $a_1 = 0.5 + 1.24031i$, $a_2 = -0.5 + 1.55419i$, $E = 5.52396$, $m_3 = 0.92039 + 1.36765i$, $\Lambda = 1.60238 + 1.25941i$.

As a concluding remark, we notice that already the first instanton correction significantly improves the approximation.

1.4 Applications to the black hole problem

There are several interesting physical quantities in the black hole problem which are governed by the Teukolsky equation. Having the explicit expression for the connection coefficients allows us to compute them exactly. We turn to this now.

1.4.1 The greybody factor

While all our analysis has been for classical black holes, it is known that quantum black holes emit thermal radiation from their horizons [114]. However, the spacetime outside of the black hole acts as a potential barrier for the emitted particles, so that the emission spectrum as measured by an observer at infinity is no longer thermal, but is given by $\frac{\sigma(\omega)}{\exp\frac{\omega-m\Omega}{T_H}-1}$, where $\sigma(\omega)$ is the so-called greybody factor. Incidentally, it is the same as the absorption coefficient of the black hole, which tells us the ratio of a flux of particles incoming from infinity which penetrates the potential barrier and is absorbed by the black hole [114] [115]. More precisely, the radial equation with $s = 0$ has a conserved flux, given by the "probability flux" when written as a Schrödinger equation: $\phi = \mathrm{Im}\psi^\dagger(z)\partial_z\psi(z)$ for z on the real line. The absorption coefficient is then defined as the ratio between the flux ϕ_{abs} absorbed by the black hole (ingoing at the horizon) and the flux ϕ_{in} incoming from infinity. For non-zero spin, the potential (1.1.10) becomes complex, and the flux is no longer conserved. In that case the absorption coefficient can be computed using energy fluxes [116], but for simplicity we stick here to $s = 0$.

The exact result

On physical grounds we impose the boundary condition that there is only an ingoing wave at the horizon:

$$R(r \to r_+) \sim (r - r_+)^{-i\frac{\omega-m\Omega}{4\pi T_H}}\,, \tag{1.4.1}$$

so the wavefunction near the horizon is given by

$$\psi(z) = \hat{f}^{(t)}_{\alpha_{2+}}(z) = (z-1)^{\frac{1}{2}+a_2}\left(1 + \mathcal{O}(z-1)\right)\,, \tag{1.4.2}$$

with $a_2 = -i\frac{\omega-m\Omega}{4\pi T_H}$ and recall that the time-dependent part goes like $e^{-i\omega t}$. This boundary condition is independent of whether $\omega - m\Omega$ is positive or negative: an observer near the horizon always sees an ingoing flux into the horizon, but when $\omega - m\Omega < 0$ it is outgoing according to an observer at infinity. This phenomenon is known as superradiance [117]. In any case, this gives the flux

$$\phi_{abs} = \mathrm{Im}a_2 \tag{1.4.3}$$

ingoing at the horizon. Using our connection formula, we find that near infinity the wavefunction behaves as

$$\begin{aligned}\psi(z) &= \frac{M_{\alpha_{2+},\alpha_-} f^{(u)}_{\alpha_-}(z)}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + \frac{M_{\alpha_{2+},\alpha_+} f^{(u)}_{\alpha_+}(z)}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} = \\ &= M_{\alpha_{2+},\alpha_-} \Lambda^{-\frac{1}{2}-a} \sum_{\pm} \mathcal{A}_{\alpha_-, m_{3\pm}} e^{\pm \frac{\Lambda z}{2}} (\Lambda z)^{\pm m_3} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0\pm} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} \left(1 + \mathcal{O}(z^{-1})\right) + \\ &+ (\alpha \to -\alpha) \,. \end{aligned} \tag{1.4.4}$$

At infinity, the ingoing part of the wave is easy to identify: recalling that $\Lambda = -2i\omega(r_+ - r_-)$ it corresponds to the positive sign in the exponential. So the flux incoming from infinity is

$$\begin{aligned}\phi_{in} &= \mathrm{Im}\frac{\Lambda}{2} \left| M_{\alpha_{2+},\alpha_-} \mathcal{A}_{\alpha_-, m_{3+}} \Lambda^{-\frac{1}{2}-a+m_3} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + (\alpha \to -\alpha) \right|^2 = \\ &= -\frac{1}{2} \left| \frac{\Gamma(1+2a)\Gamma(2a)\Gamma(1+2a_2)\Lambda^{-a+m_3}}{\Gamma\left(\frac{1}{2}+m_3+a\right)\prod_\pm \Gamma\left(\frac{1}{2} \pm a_1 + a_2 + a\right)} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + (a \to -a) \right|^2 .\end{aligned} \tag{1.4.5}$$

The minus sign comes from the fact that we have simplified Λ and we have $\mathrm{Im}\Lambda = -|\Lambda|$. Note that also the flux at the horizon is negative (for non-superradiant modes). So the full absorption coefficient/greybody factor, defined as the flux going into the horizon normalized by the flux coming in from infinity is:

$$\sigma = \frac{\phi_{abs}}{\phi_{in}} = \frac{-\mathrm{Im} 2a_2}{\left| \frac{\Gamma(1+2a)\Gamma(2a)\Gamma(1+2a_2)\Lambda^{-a+m_3}}{\Gamma\left(\frac{1}{2}+m_3+a\right)\prod_\pm \Gamma\left(\frac{1}{2} \pm a_1 + a_2 + a\right)} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + (a \to -a) \right|^2} . \tag{1.4.6}$$

This is the exact result, given as a power series in Λ. The correlators have to be understood as computed in the NS limit with $\epsilon_1 = 1$. The ratio of correlators can be written in terms of the NS free energy (see Appendix 1.D), and substituting the dictionary (1.2.20) we get

$$\boxed{\begin{aligned}&\sigma = \frac{\phi_{abs}}{\phi_{in}} = \frac{\omega - m\Omega}{2\pi T_H} \times \\ &\times \left| \frac{\Gamma(1+2a)\Gamma(2a)\Gamma(1-i\frac{\omega-m\Omega}{2\pi T_H})(-2i\omega(r_+ - r_-))^{-a-2iM\omega} e^{-i\omega(r_+-r_-)} \exp\left(\frac{\partial \mathcal{F}^{\text{inst}}}{\partial a_1}\right)|_{a_1=a, a_2=-a}}{\Gamma\left(\frac{1}{2}-2iM\omega+a\right)\Gamma\left(\frac{1}{2}-i\frac{\omega-m\Omega}{2\pi T_H}+2iM\omega+a\right)\Gamma\left(\frac{1}{2}-2iM\omega+a\right)} + (a \to -a) \right|^{-2} .\end{aligned}} \tag{1.4.7}$$

Here $\mathcal{F}^{\text{inst}}(\Lambda, a_1, a_2, m_1, m_2, m_3)$ is the instanton part of the NS free energy as defined in Appendix 1.C computed for general $\vec{a} = (a_1, a_2)$ and after taking the derivative one substitutes the values $\vec{a} = (a, -a)$ appropriate for $SU(2)$. The same holds for the second summand but one substitutes $\vec{a} = (-a, a)$ in the end. To write this result fully in terms of the parameters of the black hole problem using the dictionary (1.2.20), one has to invert the relation $E = a^2 - \Lambda\partial_\Lambda \mathcal{F}^{\text{inst}}$ to obtain $a(E)$, which can be done order by order in Λ. In the literature, the absorption coefficient for Kerr black holes has been calculated using various approximations. As a consistency check, we show that our result reproduces the known results in the appropriate regimes.

Comparison with asymptotic matching

In [76], the absorption coefficient is calculated via an asymptotic matching procedure. They work in a regime in which $\mathrm{a}\omega \ll 1$ such that the angular eigenvalue $\lambda \approx \ell(\ell+1)$, and solve the Teukolsky equation for $s = 0$ asymptotically in the regions near and far from the outer horizon. Then one also takes $M\omega \ll 1$ such that there exists an overlap between the far and near regions and one can match the asymptotic solutions. For us these limits imply that also $|\Lambda| = 4\omega\sqrt{M^2 - \mathrm{a}^2} \ll 1$, so we expand our exact transmission coefficient to lowest order in $\mathrm{a}\omega$, $M\omega$ and Λ. Since from the dictionary (1.2.20) $E = a^2 + \mathcal{O}(\Lambda) = \frac{1}{4} + \ell(\ell+1) + \mathcal{O}(\mathrm{a}\omega, M\omega)$, in this limit we have $a = \ell + \frac{1}{2}$. Then the second term in the denominator of (1.4.6) which contains Λ^a vanishes for $\Lambda \to 0$ while the first one survives and passes to the numerator. The instanton part of the NS free energy also vanishes, $\mathcal{F}^{\text{inst}}(\Lambda \to 0) = 0$. (1.4.7) then becomes

$$\sigma \approx \frac{\omega - m\Omega}{2\pi T_H}(2\omega(r_+ - r_-))^{2\ell+1} \left| \frac{\Gamma(\ell+1)\,\Gamma\left(\ell+1-i\frac{\omega-m\Omega}{2\pi T_H}\right)\Gamma(\ell+1)}{\Gamma(2\ell+2)\Gamma(2\ell+1)\Gamma(1-i\frac{\omega-m\Omega}{2\pi T_H})} \right|^2 . \tag{1.4.8}$$

Using the relation $\frac{\Gamma(\ell+1)}{\Gamma(2\ell+2)} = \frac{\sqrt{\pi}}{2^{2\ell+1}\Gamma(\ell+\frac{3}{2})}$ (and sending $i \to -i$ inside the modulus squared) we reduce precisely to the result of [76] (eq. 2.29):

$$\boxed{\sigma \approx \frac{\omega - m\Omega}{2T_H}\frac{(r_+ - r_-)^{2\ell+1}\omega^{2\ell+1}}{2^{2\ell+1}} \left| \frac{\Gamma(\ell+1)\Gamma\left(\ell+1+i\frac{\omega-m\Omega}{2\pi T_H}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma(2\ell+1)\Gamma\left(1+i\frac{\omega-m\Omega}{2\pi T_H}\right)} \right|^2 ,} \tag{1.4.9}$$

which is valid for $M\omega, \mathrm{a}\omega \ll 1$.

Comparison with semiclassics

We now show that the exact absorption coefficient reduces to the semiclassical result obtained via a standard WKB analysis of the equation

$$\epsilon_1^2 \partial_z^2 \psi(z) + V(z)\psi(z) = 0 . \tag{1.4.10}$$

where we have reintroduced the small parameter ϵ_1 which plays the role of the Planck constant to keep track of the orders in the expansion. For the Teukolsky equation (which has $\epsilon_1 = 1$) the semiclassical regime is the regime in which $\ell \gg 1$. Following [77], we also take $M\omega \ll 1$ and $s = 0$ such that there are two zeroes of the potential between the outer horizon and infinity for real values of z which we denote by z_1 and z_2 with $z_2 > z_1$, between which there is a potential barrier for the particle ($V(z)$ becomes negative, notice the "wrong sign" in front of the second derivative). Without these extra conditions, the potential generically becomes complex, or does not form a barrier. The main difference with the regime used for the asymptotic matching procedure in the previous section is that there we worked to leading order in $M\omega, a\omega$. Now we still assume them to be small but keep all orders, while working to first subleading order in ϵ_1.

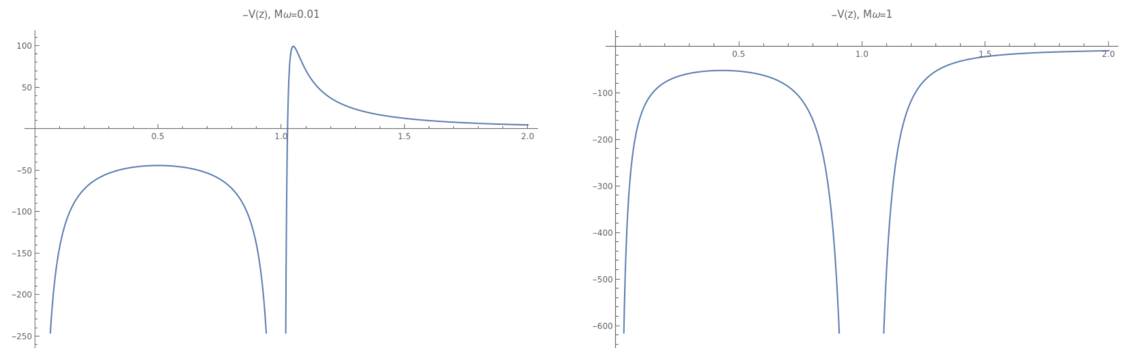


Figure 1.7: Forms of the potential $-V(z)$ for $M = 1$, $a = 0.5$, $\lambda = 10$, $m = 0$, $s = 0$, and $\omega = 0.01$ (left) and $\omega = 1$ (right). We see that for $M\omega$ not small enough, the potential does not form a barrier.

The standard WKB solutions are

$$\psi(z) \propto V(z)^{-\frac{1}{4}} \exp\left(\pm \frac{i}{\epsilon_1} \int_{z_*}^{z} \sqrt{V(z')}dz'\right) , \tag{1.4.11}$$

where z_* is some arbitrary reference point, usually taken to be a turning point of the potential, here corresponding to a zero. The absorption coefficient is given by the transmission coefficient from infinity to the horizon and captures the tunneling amplitude through this potential barrier. It is simply given by

$$\sigma \approx \exp\left(\frac{2i}{\epsilon_1} \int_{z_1}^{z_2} \sqrt{V(z')}dz'\right) = \exp\left(-\frac{2}{\epsilon_1} \int_{z_1}^{z_2} \sqrt{|V(z')|}dz'\right) . \tag{1.4.12}$$

On the other hand it is known that in the semiclassical limit the potential of the BPZ equation reduces to the Seiberg-Witten differential of the AGT dual gauge theory [41], which for us is $SU(2)$ gauge theory with $N_f = 3$: $V(z) \to -\phi_{SW}^2(z)$. The integral between the two zeroes then corresponds to half a B-cycle, so we identify

$$\boxed{\sigma \approx \exp\left(-\frac{2}{\epsilon_1} \int_{z_1}^{z_2} \phi_{SW}(z')dz'\right) = \exp\left(-\frac{1}{\epsilon_1} \oint_B \phi_{SW}(z')dz'\right) =: \exp\left(-\frac{a_D}{\epsilon_1}\right) ,} \tag{1.4.13}$$

where we have chosen an orientation of the B-cycle. Our exact absorption coefficient reduces to this expression in the semiclassical limit $\epsilon_1 \to 0$. The detailed calculation is deferred to Appendix 1.D.

1.4.2 Quantization of quasinormal modes

With the explicit expression of the connection matrix (1.3.12) in our hands we can extract the quantization condition for the quasinormal modes. The correct boundary conditions for quasinormal modes is only an ingoing wave at the horizon and only an outgoing one at infinity (see e.g. [102], eq. (80)), that is

$$\begin{aligned} &R_{\mathrm{QNM}}(r \to r_+) \sim (r - r_+)^{-i\frac{\omega - m\Omega}{4\pi T_H} - s} \\ &R_{\mathrm{QNM}}(r \to \infty) \sim r^{-1-2s+2iM\omega} e^{i\omega r} \,. \end{aligned} \tag{1.4.14}$$

In terms of the function $\psi(z)$ satisfying the Teukolsky equation in Schrödinger form:

$$\begin{aligned} &\psi_{\mathrm{QNM}}(z \to 1) \sim (z-1)^{\frac{1}{2}+a_2} \,, \\ &\psi_{\mathrm{QNM}}(z \to \infty) \sim e^{-\Lambda z/2} \left(\Lambda z\right)^{-m_3} \,. \end{aligned} \tag{1.4.15}$$

However, imposing the ingoing boundary condition at the horizon and using the connection formula, we get that near infinity

$$\begin{aligned} &\psi_{\mathrm{QNM}}(z \to \infty) \sim \\ \sim &\left(\Lambda^a M_{\alpha_{2+},\alpha_+} \mathcal{A}_{\alpha_+ m_{0-}} \frac{\langle \Delta_{\alpha+}, \Lambda_0, m_{0-} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + \Lambda^{-a} M_{\alpha_{2+},\alpha_-} \mathcal{A}_{\alpha_- m_{0-}} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0-} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} \right) \times \\ &\times e^{-\Lambda z/2} \left(\Lambda z\right)^{-m_3} + \\ + &\left(\Lambda^a M_{\alpha_{2+},\alpha_+} \mathcal{A}_{\alpha_+ m_{0+}} \frac{\langle \Delta_{\alpha+}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + \Lambda^{-a} M_{\alpha_{2+},\alpha_-} \mathcal{A}_{\alpha_- m_{0+}} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} \right) \times \\ &\times e^{\Lambda z/2} \left(\Lambda z\right)^{m_3} \,, \end{aligned} \tag{1.4.16}$$

which contains both an ingoing an an outgoing wave at infinity. In order to impose the correct boundary condition (1.4.15) we need to impose that the coefficient of the ingoing wave vanishes:

$$\begin{aligned} &\Lambda^a M_{\alpha_{2+},\alpha_+} \mathcal{A}_{\alpha_+ m_{0+}} \frac{\langle \Delta_{\alpha+}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} + \Lambda^{-a} M_{\alpha_{2+},\alpha_-} \mathcal{A}_{\alpha_- m_{0+}} \frac{\langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_{2+}}(1) | \Delta_{\alpha_1} \rangle} = 0 \\ &\Longrightarrow 1 + \Lambda^{-2a} \frac{M_{\alpha_{2+},\alpha_-} \mathcal{A}_{\alpha_- m_{0+}} \langle \Delta_{\alpha-}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle}{M_{\alpha_{2+},\alpha_+} \mathcal{A}_{\alpha_+ m_{0+}} \langle \Delta_{\alpha+}, \Lambda_0, m_{0+} | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle} = 0 \,. \end{aligned} \tag{1.4.17}$$

Identifying in the NS limit

$$\begin{aligned}&\frac{\langle\Delta_{\alpha-},\Lambda_0,m_{0+}|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle}{\langle\Delta_{\alpha+},\Lambda_0,m_{0+}|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle}=\frac{\mathcal{Z}(\Lambda,a+\frac{\epsilon_2}{2},m_1,m_2,m_3+\frac{\epsilon_2}{2})}{\mathcal{Z}(\Lambda,a-\frac{\epsilon_2}{2},m_1,m_2,m_3+\frac{\epsilon_2}{2})}=\\&=\exp\frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F}^{\text{inst}}(\Lambda,a+\frac{\epsilon_2}{2},m_1,m_2,m_3+\frac{\epsilon_2}{2})-\mathcal{F}^{\text{inst}}(\Lambda,a-\frac{\epsilon_2}{2},m_1,m_2,m_3+\frac{\epsilon_2}{2})\right)\to\\&\to\exp\frac{\partial_a\mathcal{F}^{\text{inst}}(\Lambda,a,m_1,m_2,m_3)}{\epsilon_1}\,.\end{aligned}\tag{1.4.18}$$

Moreover,

$$\begin{aligned}\frac{M_{\alpha_2+,\alpha_-}\mathcal{A}_{\alpha_-m_{0+}}}{M_{\alpha_2+,\alpha_+}\mathcal{A}_{\alpha_+m_{0+}}}&=\frac{\Gamma\left(\frac{2a}{\epsilon_1}\right)\Gamma\left(1+\frac{2a}{\epsilon_1}\right)}{\Gamma\left(-\frac{2a}{\epsilon_1}\right)\Gamma\left(1-\frac{2a}{\epsilon_1}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{a_2+a_1-a}{\epsilon_1}\right)\Gamma\left(\frac{1}{2}+\frac{a_2-a_1-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{a_2+a_1+a}{\epsilon_1}\right)\Gamma\left(\frac{1}{2}+\frac{a_2-a_1+a}{\epsilon_1}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{m_3-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_3+a}{\epsilon_1}\right)}=\\&=\frac{\Gamma\left(\frac{2a}{\epsilon_1}\right)\Gamma\left(1+\frac{2a}{\epsilon_1}\right)}{\Gamma\left(-\frac{2a}{\epsilon_1}\right)\Gamma\left(1-\frac{2a}{\epsilon_1}\right)}\prod_{i=1}^{3}\frac{\Gamma\left(\frac{1}{2}+\frac{m_i-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)}=e^{-i\pi}\left(\frac{\Gamma\left(1+\frac{2a}{\epsilon_1}\right)}{\Gamma\left(1-\frac{2a}{\epsilon_1}\right)}\right)^2\prod_{i=1}^{3}\frac{\Gamma\left(\frac{1}{2}+\frac{m_i-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)}=\\&=\exp\left[-i\pi+2\log\frac{\Gamma\left(1+\frac{2a}{\epsilon_1}\right)}{\Gamma\left(1-\frac{2a}{\epsilon_1}\right)}+\sum_{i=1}^{3}\log\frac{\Gamma\left(\frac{1}{2}+\frac{m_i-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)}\right].\end{aligned}\tag{1.4.19}$$

Including also the Λ factor (restoring the factor of ϵ_1), we identify the exponent with (see Appendix 1.C)

$$\frac{1}{\epsilon_1}\left[-i\pi\epsilon_1-2a\log\frac{\Lambda}{\epsilon_1}+2\epsilon_1\log\frac{\Gamma\left(1+\frac{2a}{\epsilon_1}\right)}{\Gamma\left(1-\frac{2a}{\epsilon_1}\right)}+\epsilon_1\sum_{i=1}^{3}\log\frac{\Gamma\left(\frac{1}{2}+\frac{m_i-a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)}\right]=-i\pi+\frac{1}{\epsilon_1}\partial_a\mathcal{F}^{1-\text{loop}}\,.\tag{1.4.20}$$

The instanton and one loop part combine to give the full NS free energy, and hence (1.4.17) can be conveniently rewritten for $\epsilon_1=1$ (as required by the dictionary), as

$$1-e^{\partial_a\mathcal{F}}=0\Rightarrow\partial_a\mathcal{F}=2\pi in\,,n\in\mathbb{Z}\,.\tag{1.4.21}$$

To solve for the quasinormal mode frequencies, we need to invert the relation $E=a^2-\Lambda\partial_\Lambda\mathcal{F}^{\text{inst}}$ to obtain $a(E)$. Then the quantization condition for the quasinormal mode frequencies that we have derived reads

$$\boxed{\partial_a\mathcal{F}\left(-2i\omega(r_+-r_-),a(E),-i\frac{\omega-m\Omega}{2\pi T_H}+2iM\omega,-2iM\omega-s,-2iM\omega+s,1\right)=2\pi in\,,n\in\mathbb{Z}\,,}\tag{1.4.22}$$

with $E=\frac{1}{4}+\lambda+s(s+1)+\mathrm{a}^2\omega^2-8M^2\omega^2-\left(2M\omega^2+is\omega\right)(r_+-r_-)$. This gives an equation that is solved for a discrete set of ω_n, in agreement with [68][3].

[3] In order to match with [68], it is important to notice that they use the variable $-ia$ instead of a, have a different $U(1)$ factor as previously noticed, and a sign difference in the definition of the free energy $\mathcal{F}$.

1.4.3 Angular quantization

Yet another application of the connection formulae is the computation of the angular eigenvalue λ. To this end, we impose regularity of the angular eigenfunctions at $z = 0, 1$. According to the angular dictionary (1.2.23),

$$\frac{1 \pm 2a_1}{2} = \frac{1}{2} \mp \frac{m-s}{2}\,,\ \frac{1 \pm 2a_2}{2} = \frac{1}{2} \mp \frac{m+s}{2}\,, \tag{1.4.23}$$

therefore, according to (1.1.11) the behavior of S_λ as $z \to 0$ is given by

$$S_\lambda(z \to 0) \propto z^{\mp\frac{m-s}{2}}\,. \tag{1.4.24}$$

Since $\lambda_{s,m} = \lambda^*_{s,-m}$, $\lambda_{-s,m} = \lambda_{s,m} + 2s$ [118], we can restrict without loss of generality to the case $m, -s \geq 0$. Regularity of S_λ as $z \to 0$ requires the boundary condition

$$y_{m>s}(z \to 0) = \hat{f}^{(s)}_{\alpha_{1-}}(z) \simeq z^{\frac{1}{2}+\frac{m-s}{2}}\,. \tag{1.4.25}$$

Therefore near $z \to 1$,

$$\begin{aligned}
&y_{m>s}(z \to 1) = \\
&= N_{a_{1-},a_{2-}} \frac{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_{2-}}(1)|\Delta_{\alpha_1}\rangle}{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_{1-}}\rangle}\hat{f}^{(t)}_{\alpha_{2-}}(z) + N_{a_{1-},a_{2+}} \frac{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_{2+}}(1)|\Delta_{\alpha_1}\rangle}{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_{1-}}\rangle}\hat{f}^{(t)}_{\alpha_{2+}}(z) \simeq \\
&\simeq \frac{\Gamma(-m-s)\Gamma(1+m-s)}{\Gamma(\frac{1}{2}-a-s)\Gamma(\frac{1}{2}+a-s)} \frac{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_{2-}}(1)|\Delta_{\alpha_1}\rangle}{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_{1-}}\rangle}(1-z)^{\frac{1}{2}+\frac{m+s}{2}} + \\
&+ \frac{\Gamma(m+s)\Gamma(1+m-s)}{\Gamma(\frac{1}{2}-a+m)\Gamma(\frac{1}{2}+a+m)} \frac{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_{2+}}(1)|\Delta_{\alpha_1}\rangle}{\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_{1-}}\rangle}(1-z)^{\frac{1}{2}-\frac{m+s}{2}}\,.
\end{aligned} \tag{1.4.26}$$

Let us start by assuming $m + s > 0$. Then the second term in (1.4.26) has a pole at $z = 1$ for generic values of a, and the first gamma function is divergent as it stands. However both divergences are cured by imposing that

$$a = \ell + \frac{1}{2}\,, \tag{1.4.27}$$

for some positive integer $\ell \geq m \geq -s$. Analogously if $m + s \leq 0$, regularity is ensured by imposing $a = \ell + \frac{1}{2}$ with $\ell \geq m \geq -s$. Therefore in general the quantization condition for the angular eigenvalue is

$$a(\Lambda, E, m_1, m_2, m_3) = \ell + \frac{1}{2}\,,\ \ell \geq \max(m\,, -s)\,. \tag{1.4.28}$$

As before, a is obtained by inverting the expression $E = a^2 - \Lambda\partial_\Lambda\mathcal{F}^{\text{inst}}$ order by order in Λ. Let us denote by

$$\lambda_0 = \lambda(\Lambda = 0) = \ell(\ell+1) - s(s+1)\,. \tag{1.4.29}$$

Moreover, their $\partial_a\mathcal{F}$ is shifted by a factor of $-i\pi$ with respect to ours.

Then the above quantization condition for the angular eigenvalue λ can be more conveniently written as

$$\lambda - \lambda_0 = 2cs - c^2 - \Lambda\partial_\Lambda \mathcal{F}^{\text{inst}}\left(\Lambda, \ell + \frac{1}{2}, -m, -s, -s\right)\Big|_{\Lambda=4c}, \tag{1.4.30}$$

which is the result already obtained in [68].

1.4.4 Love numbers

Applying an external gravitational field to a self-gravitating body generically causes it to deform, much in the same way as an external electric field polarizes a dielectric material. The response of the body to the external gravitational tidal field is captured by the so-called tidal response coefficients or Love numbers, named after A. E. H. Love who first studied them in the context of the Earth's response to the tides [119]. In general relativity, the tidal response coefficients are generally complex, and the real part captures the conservative response of the body, whereas the imaginary part captures dissipative effects. There is some naming ambiguity where sometimes only the real, conservative part is called the Love number, whereas sometimes the full complex response coefficient is called Love number. For us the Love number will be the full complex response coefficient. For four-dimensional Kerr black holes, the conservative (real part) of the response coefficient to static external perturbations has been found to vanish [78, 80]. Moreover, Love numbers are measurable quantities that can be probed with gravitational wave observations [120, 121]. Using our conformal field theory approach to the Teukolsky equation we compute the Love number of a slowly rotating Kerr black hole at linear order in the frequency of the perturbation. The extension of our computation to higher orders is straightforward.

Definition of Love number and the intermediate region

For the definition of Love numbers we follow [80] and [78], to which we refer for a more complete introduction. In the case of a static external perturbation ($\omega = 0$), one imposes the ingoing boundary condition on the radial part of the perturbing field at the horizon, which then behaves near infinity as

$$\begin{aligned} R(r \to \infty) &= Ar^{\ell-s}(1 + \mathcal{O}(r^{-1})) + Br^{-\ell-s-1}(1 + \mathcal{O}(r^{-1})) \\ &= Ar^{\ell-s}\left[(1 + \mathcal{O}(r^{-1})) + k_{\ell m}^{(s)}\left(\frac{r}{r_+ - r_-}\right)^{-2\ell-1}(1 + \mathcal{O}(r^{-1}))\right] \end{aligned} \tag{1.4.31}$$

for some constants A and B. The Love number $k_{\ell m}^{(s)}$ is then defined as the coefficient of $(r/(r_+ - r_-))^{-2\ell-1}$ (note that this differs from the definition in [80] where they define it as the coefficient of $(r/2M)^{-2\ell-1}$ instead). In the non-static case however, the definition of Love number is less clear, since the behaviour of the radial function at infinity is now qualitatively different from (1.4.31): it is oscillatory (cf. (1.4.4)) due to the term $\propto \omega^2$ in the potential (1.A.3). For small frequencies we can however define an intermediate regime $r \gg M$, $r\omega \ll 1$ in which the multipole expansion (1.4.31) is still valid and we can read off the Love numbers in the same way as in the static case. Recall the Teukolsky equation written as a Schrödinger equation:

$$\frac{d^2\psi(z)}{dz^2} + V_{CFT}(z)\psi(z) = 0 \tag{1.4.32}$$

with the potential (1.2.14)

$$V_{CFT}(z) = -\frac{1}{z}\frac{1}{z-1}\big(-\Lambda\partial_\Lambda \mathcal{F}^{\text{inst}} + \hat{\Delta}_2 + \hat{\Delta}_1 - \hat{\Delta}\big) + \frac{\hat{\Delta}_2}{(z-1)^2} + \frac{\hat{\Delta}_1}{z^2} - \frac{m_3\Lambda}{z} - \frac{\Lambda^2}{4}. \tag{1.4.33}$$

The intermediate regime corresponds to $z \gg 1$, $\Lambda z \ll 1$. Expanding in these variables the potential reads:

$$\frac{V_{CFT}(z)}{\Lambda^2} = \frac{\frac{1}{4} - E}{\Lambda^2 z^2}\left(1 + \mathcal{O}(z^{-1}, \Lambda z)\right). \tag{1.4.34}$$

We see that in this regime the leading term in the potential is the one $\propto 1/z^2$, and the multipole expansion holds. In a sense we are taking z to be big enough to be far from the horizon, but not so far as to reach the oscillatory region at infinity, as already mentioned in [79]. In the static case this intermediate region where the multipole expansion is valid extends all the way to infinity. On the CFT side, the conformal blocks in this regime are computed by expanding the irregular state as in (1.2.7) and doing the OPE of the degenerate state near infinity term by term. This gives an expansion in Λz and z^{-1}.

1.4.5 Slowly rotating Kerr Love numbers

Let us compute the Kerr Love numbers up to first order in $M\omega \sim M\Omega$. In order to do this we have to consider only the first instanton correction since $\Lambda \propto M\omega$. The wavefunction up to one instanton can be derived from the conformal blocks in the intermediate regime. Schematically,

$$\psi(z) \sim \frac{\langle \Delta, \Lambda_0, m_0|\phi(z)V_2(1)|\Delta_1\rangle}{\langle \Delta, \Lambda_0, m_0|V_2(1)|\Delta_1\rangle} \simeq \frac{\left(\langle\Delta| + \frac{m_0\Lambda_0}{2\Delta}\langle\Delta|L_1\right)\phi(z)V_2(1)|\Delta_1\rangle}{\left(\langle\Delta| + \frac{m_0\Lambda_0}{2\Delta}\langle\Delta|L_1\right)V_2(1)|\Delta_1\rangle}. \tag{1.4.35}$$

Imposing the ingoing boundary condition at the horizon, this gives the following wavefunction in the intermediate regime:

$$\psi(z) = \left[1 + \frac{m_3\Lambda}{\frac{1}{2} - 2a^2}\left(\left(1 - \frac{1}{z}\right)\partial_{1/z} + z - \frac{1}{2}\right)\right]\sum_{\theta=\pm} M_{a_2+a_\theta} z^{\frac{1}{2}-\theta a}\left(1 - \frac{1}{z}\right)^{\frac{1}{2}+a_2} \times$$
$$\times\, {}_2F_1\left(\frac{1}{2} + a_2 + \theta a - a_1, \frac{1}{2} + a_2 + \theta a + a_1; 1 + 2\theta a; \frac{1}{z}\right) + \mathcal{O}\left(\Lambda^2\right). \tag{1.4.36}$$

Note that the first instanton contributes at this order only if $s \neq 0$ since for zero spin $m_3\Lambda \sim \mathcal{O}(M^2\omega^2)$. For a slowly rotating black hole the connection coefficients start with $\mathcal{O}((M\omega)^0) = \mathcal{O}((M\Omega)^0)$ terms. Indeed substituting the dictionary we find

$$\begin{aligned} M_{a_2+a_+} &= \frac{\Gamma(-1-2\ell-2\Delta\ell)\Gamma(1-2i\frac{\omega-m\Omega}{4\pi T_H}-s)}{\Gamma(-\ell-\Delta\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(-\ell-\Delta\ell-2iM\omega-s)} = \\ &= \frac{\ell!(\ell+s)!}{(2\ell+1)!}(-1)^{s+1}\frac{(2iM\omega)(-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{2\Delta\ell} + \mathcal{O}(M\omega)\,, \\ M_{a_2+a_-} &= \frac{\Gamma(1+2\ell)\Gamma(1-s)}{\Gamma(\ell+1)\Gamma(\ell-s+1)} + \mathcal{O}(M\omega)\,, \end{aligned} \tag{1.4.37}$$

where $a = \ell + 1/2 + \Delta\ell$. It turns out that the first correction to a vanishes, so $\Delta\ell \sim \mathcal{O}(M^2\omega^2)$. Also note that all the Gamma functions are finite since $s \leq 0$. Plugging in the dictionary and expanding the hypergeometrics gives

$$\begin{aligned} & {}_2F_1\left(\frac{1}{2}+a_2+a-a_1, \frac{1}{2}+a_2+a+a_1; 1+2a; \frac{1}{z}\right) \simeq \\ & \simeq {}_2F_1\left(1+\ell-s-2iM\omega, 1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega; 2+2\ell; \frac{1}{z}\right)\,, \\ & {}_2F_1\left(\frac{1}{2}+a_2-a-a_1, \frac{1}{2}+a_2-a+a_1; 1-2a; \frac{1}{z}\right) \simeq \\ & \simeq \sum_{n=0}^{2\ell} \frac{(-\ell-s-2iM\omega)_{(n)}(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!} + \\ & + \frac{\Gamma(-2\ell-2\Delta\ell)\Gamma(1+\ell-s-2iM\omega)\Gamma(1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{\Gamma(-\ell-s-2iM\omega)\Gamma(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(2\ell+2)} z^{-2\ell-1} \times \\ & \times {}_2F_1\left(1+\ell-s-2iM\omega, 1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega; 2+2\ell; \frac{1}{z}\right)\,. \end{aligned} \tag{1.4.38}$$

Note that

$$\frac{\Gamma(-2\ell-2\Delta\ell)\Gamma(1+\ell-s-2iM\omega)\Gamma(1+\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)}{\Gamma(-\ell-s-2iM\omega)\Gamma(-\ell-2i\frac{\omega-m\Omega}{4\pi T_H}+2iM\omega)\Gamma(2\ell+2)} M_{a_2+a_-} = -M_{a_2+a_+} + \mathcal{O}\left((M\omega)^2\right)\,, \tag{1.4.39}$$

therefore at this order the hypergeometrics simplify one against the other up to a finite polynomial, hence

$$\psi(z) = \left[1 + \frac{m_3\Lambda}{\frac{1}{2} - 2a^2}\left(\left(1 - \frac{1}{z}\right)\partial_{1/z} + z - \frac{1}{2}\right)\right] \times$$
$$\times\, M_{a_2+a_-} z^{\frac{1}{2}+a}\left(1 - \frac{1}{z}\right)^{\frac{1}{2}+a_2} \sum_{n=0}^{2\ell} \frac{(-\ell - s - 2iM\omega)_{(n)}(-\ell - 2i\frac{\omega - m\Omega}{4\pi T_H} + 2iM\omega)_{(n)}}{(-2\ell)_{(n)}} \frac{z^{-n}}{n!} + \mathcal{O}(M^2\omega^2)\,. \tag{1.4.40}$$

The radial wavefunction is given by

$$R(r) = \Delta^{-\frac{s+1}{2}}(r)\psi(z)\,, \tag{1.4.41}$$

where

$$z = \frac{r}{2M} + \mathcal{O}\left(M^2\Omega^2\right)\,,\ \Delta(r)^{-\frac{s+1}{2}} = (r_+ - r_-)^{-s-1} z^{-s-1}\left(1 - \frac{1}{z}\right)^{-\frac{s+1}{2}}\,. \tag{1.4.42}$$

To find the Love numbers, we need the ratio between the coefficient of $r^{-\ell-s-1}$ (the response) and the coefficient of $r^{\ell-s}$ (the source). The term coming from the first instanton in (1.4.40) will not contribute at this order. Indeed this term gives

$$\psi(z) \supset \frac{-4iM^2\omega s}{\ell(\ell+1)}\left(\left(1 - \frac{1}{z}\right)\partial_{1/z} + z - \frac{1}{2}\right) M_{a_2+a_-} z^{\ell+1}\left(1 - \frac{1}{z}\right)^{\frac{1-s}{2}} \sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!} +$$
$$+ \mathcal{O}(M^2\omega^2) =$$
$$= \frac{-4iM^2\omega s}{\ell(\ell+1)} M_{a_2+a_-} z^{\ell+1}\left(1 - \frac{1}{z}\right)^{\frac{1-s}{2}}\left(-z\ell + \frac{2\ell+s}{2} + \left(1 - \frac{1}{z}\right)\partial_{1/z}\right)\sum_{n=0}^{\ell+s} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{z^{-n}}{n!} +$$
$$+ \mathcal{O}(M^2\omega^2)\,. \tag{1.4.43}$$

After taking into account the factor of Δ from (1.4.41), one sees that this contribution to $R(r)$ does not contain the power that we are interested in. Focusing on the zero instanton contribution, the $(1 - 1/z)$ prefactor has an $\mathcal{O}(M\omega)$ term in the exponent that has to be expanded, resulting in

$$R(r) \supset i\frac{\omega - m\Omega}{4\pi T_H}\frac{M_{a_2+a_-}}{(r_+ - r_-)^{s+1}}\frac{r^{\ell-s}}{((2M)^{\ell+1}}\left(1 + s\frac{2M}{r} + \frac{s(s+1)}{2}\left(\frac{2M}{r}\right)^2\right) \times$$
$$\times \sum_{k=1}^{\infty}\sum_{n=0}^{2\ell} \frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}}\frac{\left(\frac{r}{2M}\right)^{-n-k}}{n!k}\,. \tag{1.4.44}$$

This term contains the correct power, with coefficient

$$R(r) \supset \frac{M_{a_2+a_-}}{(r_+-r_-)^{s+1}}\frac{r^{\ell-s}}{((2M)^{\ell+1}}i\frac{\omega-m\Omega}{4\pi T_H}\left(\frac{r}{2M}\right)^{-2\ell-1}\left(\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell+1-n)}+\right.$$
$$\left.+\,s\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell-n)}+\frac{s(s+1)}{2}\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell-1-n)}\right). \tag{1.4.45}$$

A surprising identity reveals that

$$\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell+1-n)}+s\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell-n)}+\frac{s(s+1)}{2}\sum_{n=0}^{\ell+s}\frac{(-\ell-s)_{(n)}(-\ell)_{(n)}}{(-2\ell)_{(n)}n!(2\ell-1-n)}=$$
$$=\frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!}(-1)^s\,, \tag{1.4.46}$$

therefore

$$R(r) \supset \frac{M_{a_{2+}a_-}}{(r_+-r_-)^{s+1}}\frac{r^{\ell-s}}{(2M)^{\ell+1}}\left[1+i\frac{\omega-m\Omega}{4\pi T_H}\left(\frac{r}{2M}\right)^{-2\ell-1}\frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!}(-1)^s\right]. \tag{1.4.47}$$

Noticing that $1/4\pi T_H \simeq 2M$ finally gives the Love number

$$k^s_{a,m} = 2iM\,(\omega-m\Omega)\,(-1)^s\,\frac{(\ell+s)!\,(\ell-s)!\,(\ell!)^2}{(2\ell)!\,(2\ell+1)!}+\mathcal{O}(M^2\omega^2, M^2\Omega^2, M^2\omega\Omega)\,. \tag{1.4.48}$$

This result matches with formula (6.17) in [80]. Note that the Love number remains purely imaginary for a small frequency perturbation, and that it vanishes in the case of a static perturbation of a Schwarzschild black hole.

Appendix

1.A The radial and angular potentials

Both the radial and angular part of the Teukolsky equation can be written as a Schrödinger equation:

$$\frac{d^2\psi(z)}{dz^2} + V(z)\psi(z) = 0 \tag{1.A.1}$$

with potential

$$V(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\hat{A}_i z^i\,. \tag{1.A.2}$$

For the radial part, the coefficients are given by

$$\begin{aligned}
\hat{A}^r_0 &= \frac{\mathrm{a}^2(1-m^2) - M^2 + 4\mathrm{a}mM\omega(M-\sqrt{M^2-\mathrm{a}^2}) + 4M^2\omega^2(a^2-2M^2) + 8M^3\sqrt{M^2-\mathrm{a}^2}\omega^2}{4(\mathrm{a}^2-M^2)} + \\
&+ (is)\frac{am\sqrt{M^2-\mathrm{a}^2} - 2\mathrm{a}^2M\omega + 2M^2\omega(M-\sqrt{M^2-\mathrm{a}^2})}{2(\mathrm{a}^2-M^2)} - \frac{s^2}{4}\,,\\
\hat{A}^r_1 &= \frac{4\mathrm{a}^2\lambda - 4M^2\lambda + (8\mathrm{a}mM\omega + 16\mathrm{a}^2M\omega^2 - 32M^3\omega^2)\sqrt{M^2-\mathrm{a}^2} + 4\mathrm{a}^4\omega^2 - 36\mathrm{a}^2M^2\omega^2 + 32M^4\omega^2}{4(\mathrm{a}^2-M^2)} + \\
&+ (is)\left(-i + \frac{(2\mathrm{a}^2\omega - am)\sqrt{M^2-\mathrm{a}^2}}{\mathrm{a}^2-M^2}\right) + s^2\,,\\
\hat{A}^r_2 &= -\lambda - 5\mathrm{a}^2\omega^2 + 12M^2\omega^2 - 12M\omega^2\sqrt{M^2-\mathrm{a}^2} + (is)(i - 6\omega\sqrt{M^2-\mathrm{a}^2}) - s^2\,,\\
\hat{A}^r_3 &= 8\mathrm{a}^2\omega^2 - 8M^2\omega^2 + 8M\omega^2\sqrt{M^2-\mathrm{a}^2} + (is)4\omega\sqrt{M^2-\mathrm{a}^2}\,,\\
\hat{A}^r_4 &= 4(M^2-\mathrm{a}^2)\omega^2\,,
\end{aligned} \tag{1.A.3}$$

while for the angular part they are

$$\begin{aligned}
\hat{A}_0^\theta &= -\frac{1}{4}(-1+m-s)(1+m-s)\,,\\
\hat{A}_1^\theta &= c^2+s+2cs-ms+s^2+\lambda\,,\\
\hat{A}_2^\theta &= -s-(c+s)(5c+s)-\lambda\,,\\
\hat{A}_3^\theta &= 4c(2c+s)\,,\\
\hat{A}_4^\theta &= -4c^2\,.
\end{aligned} \tag{1.A.4}$$

1.B CFT calculations

1.B.1 The BPZ equation

To calculate the BPZ equation for the correlator (1.2.6) we first evaluate the correlator with an extra insertion of the energy-momentum tensor:

$$\begin{aligned}
&\langle\Delta,\Lambda_0,m_0|T(w)\Phi_{2,1}(z)V_2(y)|V_1\rangle =\\
&=\sum_{n\geq 0}\frac{1}{w^{n+2}}\langle\Delta,\Lambda_0,m_0|[L_n,\Phi_{2,1}(z)V_2(y)]|V_1\rangle+\left(\frac{\Delta_1}{w^2}+\frac{m_0\Lambda_0}{w}+\Lambda_0^2\right)\langle\Delta,\Lambda_0,m_0|\Phi_{2,1}(z)V_2(y)|V_1\rangle =\\
&=\left(\frac{z}{w}\frac{1}{w-z}\partial_z+\frac{\Delta_{2,1}}{(w-z)^2}+\frac{y}{w}\frac{1}{w-y}\partial_y+\frac{\Delta_2}{(w-y)^2}+\frac{\Delta_1}{w^2}++\frac{m_0\Lambda_0}{w}+\Lambda_0^2\right)\langle\Delta,\Lambda_0,m_0|\Phi_{2,1}(z)V_2(y)|V_1\rangle\,.
\end{aligned} \tag{1.B.1}$$

Now we can simply compute

$$\begin{aligned}
\langle\Delta,\Lambda_0,m_0|L_{-2}\cdot\Phi_{2,1}(z)V_2(y)|V_1\rangle &= \oint_{C_z}\frac{dw}{w-z}\langle\Delta,\Lambda_0,m_0|T(w)\Phi_{2,1}(z)V_2(y)|V_1\rangle =\\
=\left(-\frac{1}{z}\partial_z+\frac{y}{z}\frac{1}{z-y}\partial_y+\frac{\Delta_2}{(z-y)^2}+\frac{\Delta_1}{z^2}+\frac{m_0\Lambda_0}{z}+\Lambda_0^2\right)&\langle\Delta,\Lambda_0,m_0|\Phi_{2,1}(z)V_2(y)|V_1\rangle\,.
\end{aligned} \tag{1.B.2}$$

Using the Ward identity for L_0:

$$\big(z\partial_z+y\partial_y-\Lambda_0\partial_{\Lambda_0}+\Delta_{2,1}+\Delta_2+\Delta_1-\Delta\big)\langle\Delta,\Lambda_0,m_0|\Phi_{2,1}(z)V_2(y)|V_1\rangle=0 \tag{1.B.3}$$

we can eliminate ∂_y. Then setting $y=1$ we obtain

$$\begin{aligned}
&\langle\Delta,\Lambda_0,m_0|L_{-2}\cdot\Phi_{2,1}(z)V_2(1)|V_1\rangle =\\
&=\left(-\frac{1}{z}\partial_z-\frac{1}{z}\frac{1}{z-1}\big(z\partial_z-\Lambda_0\partial_{\Lambda_0}+\Delta_{2,1}+\Delta_2+\Delta_1-\Delta\big)+\frac{\Delta_2}{(z-1)^2}+\frac{\Delta_1}{z^2}+\frac{m_0\Lambda_0}{z}+\Lambda_0^2\right)\Psi(z)
\end{aligned} \tag{1.B.4}$$

which gives the BPZ equation

$$\begin{aligned}
0 =&\langle\Delta,\Lambda_0,m_0|\big(b^{-2}\partial_z^2+L_{-2}\cdot\big)\Phi_{2,1}(z)V_2(1)|V_1\rangle =\\
&=\left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z-\frac{1}{z}\frac{1}{z-1}\big(z\partial_z-\Lambda_0\partial_{\Lambda_0}+\Delta_{2,1}+\Delta_2+\Delta_1-\Delta\big)+\frac{\Delta_2}{(z-1)^2}+\frac{\Delta_1}{z^2}+\frac{m_0\Lambda_0}{z}+\Lambda_0^2\right)\Psi(z)\,.
\end{aligned} \tag{1.B.5}$$

1.B.2 DOZZ factors

We normalize vertex operators so that the DOZZ three-point function[110, 111] reads

$$C(\alpha_1,\alpha_2,\alpha_3)=\frac{1}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3+\frac{Q}{2})\Upsilon_b(\alpha_1+\alpha_2-\alpha_3+\frac{Q}{2})\Upsilon_b(\alpha_2+\alpha_3-\alpha_1+\frac{Q}{2})\Upsilon_b(\alpha_3+\alpha_1-\alpha_2+\frac{Q}{2})}\,, \tag{1.B.6}$$

where

$$\begin{aligned}\Upsilon_b(x)&=\frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}\,,\\ \Gamma_b(x)&=\frac{\Gamma_2(x|b,b^{-1})}{\Gamma_2(\frac{Q}{2}|b,b^{-1})}\,,\end{aligned} \tag{1.B.7}$$

and Γ_2 is the double gamma function. Υ_b satisfies the shift relation

$$\Upsilon_b(x+b)=\gamma(bx)b^{1-2bx}\Upsilon_b(x)\,. \tag{1.B.8}$$

Moreover $\gamma(x)=\Gamma(x)/\Gamma(1-x)$, and satisfies the following relations

$$\begin{aligned}\gamma(-x)\gamma(x)&=-\frac{1}{x^2}\,,\\ \gamma(x+1)&=-x^2\gamma(x)\,,\\ \gamma(x)&=\frac{1}{\gamma(1-x)}\,.\end{aligned} \tag{1.B.9}$$

The two-point function normalization is given in terms of the DOZZ factors, that is

$$\langle\Delta_\alpha|\Delta_\alpha\rangle=G(\alpha)=C(\alpha,-\frac{Q}{2},\alpha)=\frac{1}{\Upsilon_b(0)\Upsilon_b(0)\Upsilon_b(2\alpha)\Upsilon_b(2\alpha+Q)}\,. \tag{1.B.10}$$

The regular OPE coefficient appearing in section 1.3 can be explicitly computed in terms of DOZZ factors, that is

$$\mathcal{C}^{\alpha_{i\pm}}_{\alpha_{2,1},\alpha_i}=G^{-1}(\alpha_{i\pm})C(\alpha_{i\pm},\frac{-b-Q}{2},\alpha_i)=\gamma(-b^2)\gamma(\mp 2b\alpha_i)b^{2b(\pm2\alpha+Q)}\,. \tag{1.B.11}$$

Another relevant ratio is

$$\frac{C(\alpha_1,\alpha_2,\alpha_{3+})}{C(\alpha_1,\alpha_2,\alpha_{3-})}=b^{-8b\alpha_3}\prod_{\pm,\pm}\gamma(\frac{1}{2}+b(\pm\alpha_1\pm\alpha_2+\alpha_3))\,, \tag{1.B.12}$$

that is readily computed from the shift relation (1.B.8). With these relations at our disposal, we can evaluate ratios of the Ks appearing in equations (1.3.10). In particular,

$$\frac{K^{(t)}_{\alpha_{2-},\alpha_{2-}}}{K^{(t)}_{\alpha_{2+},\alpha_{2+}}}=\frac{G^{-1}(\alpha_{2-})C(\alpha_{2-},\frac{-b-Q}{2},\alpha_2)C(\alpha,\alpha_{2-},\alpha_1)}{G^{-1}(\alpha_{2+})C(\alpha_{2+},\frac{-b-Q}{2},\alpha_2)C(\alpha,\alpha_{2+},\alpha_1)}=\frac{\gamma(2b\alpha_2)}{\gamma(-2b\alpha_2)}\prod_{\pm,\pm}\gamma(\frac{1}{2}+b(\pm\alpha\pm\alpha_1-\alpha_2))\,, \tag{1.B.13}$$

and similarly

$$\frac{K^{(u)}_{\alpha_+,\alpha_+}}{K^{(u)}_{\alpha_-,\alpha_-}} = \frac{\gamma(-2b\alpha)}{\gamma(2b\alpha)} \prod_{\pm,\pm} \gamma(\frac{1}{2} + b(\alpha \pm \alpha_1 \pm \alpha_2)) \,. \tag{1.B.14}$$

1.B.3 Irregular OPE

Following [47] let us make the following Ansatz for the OPE with the irregular state

$$\langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, m_0 | \Phi_{2,1}(z,\bar{z}) = \sum_\beta \tilde{\mathcal{C}}^{\beta}_{\alpha,\alpha_{2,1}} \left| \sum_{\mu_0,k} \mathcal{A}_{\beta,\mu_0} z^\zeta \Lambda_0^\lambda e^{\gamma\Lambda_0 z} z^{-k} \langle \Delta_\beta, \Lambda_0, \mu_0; k | \right|^2 , \tag{1.B.15}$$

with all the parameters to be determined. Here $\langle \Delta_\beta, \Lambda_0, \mu_0, k|$ is the k-th irregular descendant, that schematically has the form

$$|\Delta_\beta, \Lambda_0, \mu_0; k\rangle \sim \sum L_{-J} \Lambda_0^{-k''} \partial^{k'}_{\Lambda_0} |\Delta_\beta, \Lambda_0, \mu_0\rangle \tag{1.B.16}$$

where the sum runs over all k', k'', J such that $k' + k'' + |J| = k$, with appropriate coefficients that can be determined from the Ward identities. Note that in principle the parameters ζ, λ, γ depend both on β and μ_0. The first constraint comes from comparing with the regular OPE, namely

$$\langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, m_0 | \Phi_{2,1}(z,\bar{z}) | \Delta_\beta \rangle \sim \langle \Delta_\alpha, \Lambda_0, \bar{\Lambda}_0, \mu_0 | \Delta_{\beta_\pm} \rangle = \langle \Delta_\alpha | \Delta_{\beta_\pm} \rangle \Rightarrow \beta_\pm = \alpha \,. \tag{1.B.17}$$

The other coefficients can be fixed by acting with the Virasoro generators on the left and right hand sides of the Ansatz (1.B.15). Focusing on the chiral correlator and comparing powers of Λ and z we have

$$\begin{aligned} \langle \Delta_\alpha, \Lambda_0, m_0 | \Phi_{2,1}(z) L_0 &= (\Delta_\alpha + \Lambda_0 \partial_{\Lambda_0} - \Delta_{2,1} - z\partial_z) \langle \Delta_\alpha, \Lambda_0, m_0 | \Phi_{2,1}(z) = \\ &= \sum_k z^{\zeta - k} \Lambda_0^\lambda e^{\gamma \Lambda_0 z} (\Delta_\alpha - \Delta_{2,1} - \zeta + k + \lambda + \Lambda_0 \partial_{\Lambda_0}) \langle \Delta_\beta, \Lambda_0, \mu_0; k | = \\ &= \sum_k z^{\zeta - k} \Lambda_0^\lambda e^{\gamma \Lambda_0 z} (\Delta_\beta + k + \Lambda_0 \partial_{\Lambda_0}) \langle \Delta_\beta, \Lambda_0, \mu_0; k | \,, \end{aligned} \tag{1.B.18}$$

that gives the constraint

$$\lambda - \zeta = \Delta_\beta - \Delta_\alpha + \Delta_{2,1} \,. \tag{1.B.19}$$

Now let us consider the action of L_{-1}. We have

$$\begin{aligned} \langle \Delta_\alpha, \Lambda_0, m_0 | \Phi_{2,1}(z) L_{-1} &= (m_0 \Lambda_0 - \partial_z) \langle \Delta_\alpha, \Lambda_0, m_0 | \Phi_{2,1}(z) = \\ &= \sum_k z^\zeta \Lambda_0^\lambda e^{\gamma \Lambda_0 z} ((m_0 - \gamma) \Lambda_0 z^{-k} + (k - \zeta) z^{-k-1}) \langle \Delta_\beta, \Lambda_0, \mu_0; k | = \\ &= z^\zeta \Lambda_0^\lambda e^{\gamma \Lambda_0 z} \left(\langle \Delta_\beta, \Lambda_0, \mu_0 | \mu_0 \Lambda_0 + z^{-1} \langle \Delta_\beta, \Lambda_0, \mu_0; 1 | L_{-1} + \dots \right) . \end{aligned} \tag{1.B.20}$$

Comparing powers,

$$\begin{aligned}&\mathcal{O}(z^{\zeta}) \Rightarrow m_0 - \gamma = \mu_0\,,\\ &\mathcal{O}(z^{\zeta-1}) \Rightarrow \mu_0\Lambda_0\langle\Delta_\beta,\Lambda_0,\mu_0;1| - \zeta\langle\Delta_\beta,\Lambda_0,\mu_0| = \langle\Delta_\beta,\Lambda_0,\mu_0;1|L_{-1}\,.\end{aligned} \tag{1.B.21}$$

The first irregular descendant is of the form[4]

$$\langle\Delta_\beta,\Lambda_0,\mu_0;1| = A\langle\Delta_\beta,\Lambda_0,\mu_0|L_1 + B\partial_{\Lambda_0}\langle\Delta_\beta,\Lambda_0,\mu_0|\,, \tag{1.B.22}$$

therefore equation (1.B.21) gives

$$\mu_0\Lambda_0\left(A\langle\Delta_\beta,\Lambda_0,\mu_0|L_1 + B\partial_{\Lambda_0}\langle\Delta_\beta,\Lambda_0,\mu_0|\right) - \zeta\langle\Delta_\beta,\Lambda_0,\mu_0| = A\langle\Delta_\beta,\Lambda_0,\mu_0|L_1L_{-1} + B\partial_{\Lambda_0}\langle\Delta_\beta,\Lambda_0,\mu_0|L_{-1}\,. \tag{1.B.23}$$

The RHS gives

$$\begin{aligned}&A(2\Delta_\beta + 2\Lambda_0\partial_{\Lambda_0})\langle\Delta_\beta,\Lambda_0,\mu_0| + A\mu_0\Lambda_0\langle\Delta_\beta,\Lambda_0,\mu_0|L_1 + B\mu_0\langle\Delta_\beta,\Lambda_0,\mu_0| + B\mu_0\Lambda_0\partial_{\Lambda_0}\langle\Delta_\beta,\Lambda_0,\mu_0| =\\ &= (2A\Delta_\beta + B\mu_0)\langle\Delta_\beta,\Lambda_0,\mu_0| + (2A\Lambda_0 + B\mu_0\Lambda_0)\partial_{\Lambda_0}\langle\Delta_\beta,\Lambda_0,\mu_0| + A\mu_0\Lambda_0\langle\Delta_\beta,\Lambda_0,\mu_0|L_1\,.\end{aligned} \tag{1.B.24}$$

Comparing term by term, we obtain equations for A, B

$$\begin{aligned}&2A\Delta_\beta + B\mu_0 = -\zeta\,,\\ &2A\Lambda_0 + B\mu_0\Lambda_0 = B\mu_0\Lambda_0\,,\\ &\Rightarrow A = 0\,,\; B = -\frac{\zeta}{\mu_0}\,.\end{aligned} \tag{1.B.25}$$

Another constraint comes from the action of L_2. We have

$$\begin{aligned}\langle\Delta_\alpha,\Lambda_0,m_0|\Phi_{2,1}(z)L_{-2} &= (\Lambda_0^2 - z^{-1}\partial_z + \Delta_{2,1}z^{-2})\langle\Delta_\alpha,\Lambda_0,m_0|\Phi_{2,1}(z) =\\ &= \sum_k z^{\zeta}\Lambda_0^{\lambda}e^{\gamma\Lambda_0 z}(\Lambda_0^2 z^{-k} - \gamma\Lambda_0 z^{-k-1} + (k-\zeta+\Delta_{2,1})z^{-k-2})\langle\Delta_\beta,\Lambda_0,\mu_0;k| =\\ &= z^{\zeta}\Lambda_0^{\lambda}e^{\gamma\Lambda_0 z}\left(\langle\Delta_\beta,\Lambda_0,\mu_0|\Lambda_0^2 + z^{-1}\langle\Delta_\beta,\Lambda_0,\mu_0;1|L_{-2} + \dots\right) =\\ &= z^{\zeta}\Lambda_0^{\lambda}e^{\gamma\Lambda_0 z}\left(\langle\Delta_\beta,\Lambda_0,\mu_0|\Lambda_0^2 - z^{-1}\frac{\zeta}{\mu_0}(2\Lambda_0 + \Lambda_0^2\partial_{\Lambda_0})\langle\Delta_\beta,\Lambda_0,\mu_0| + \dots\right)\end{aligned} \tag{1.B.26}$$

The previous equation is trivially satisfied at order $\mathcal{O}(z^{\zeta})$, and comparing at order $\mathcal{O}(z^{\zeta-1})$ gives

$$(-\Lambda_0^2\frac{\zeta}{\mu_0}\partial_{\Lambda_0} - \gamma\Lambda_0)\langle\Delta_\beta,\Lambda_0,\mu_0| = -\frac{\zeta}{\mu_0}(2\Lambda_0 + \Lambda_0^2\partial_{\Lambda_0})\langle\Delta_\beta,\Lambda_0,\mu_0|\,, \tag{1.B.27}$$

that finally gives

$$\gamma = 2\frac{\zeta}{\mu_0}\,. \tag{1.B.28}$$

[4] The term $\sim \Lambda^{-1}$ cannot be determined at this order. Luckily, it doesn't play any role in the following discussion.

The last constraint we need is most easily obtained by looking at the null-state equation satisfied by the irregular 3 point function (1.B.17). We have

$$\langle\Delta_\alpha,\Lambda_0,m_0|T(w)\Phi_{2,1}(z)|\Delta_{\alpha_\pm}\rangle = \langle\Delta_\alpha,\Lambda_0,m_0|\left(\tfrac{m_0\Lambda_0}{w}+\Lambda_0^2+\tfrac{\Delta_{\alpha_\pm}}{w^2}+\tfrac{\Delta_{2,1}}{(w-z)^2}+\tfrac{z/w}{w-z}\partial_z\right)\Phi_{2,1}(z)|\Delta_{\alpha_\pm}\rangle\,, \tag{1.B.29}$$

therefore

$$\left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z+\frac{\Delta_{\alpha\pm}}{z^2}+\frac{m_0\Lambda_0}{z}+\Lambda_0^2\right)\langle\Delta_\alpha,\Lambda_0,m_0|\Phi_{2,1}(z)|\Delta_{\alpha_\pm}\rangle=0\,. \tag{1.B.30}$$

Substituting the irregular OPE and looking at the leading term as $z\to\infty$ gives

$$\left(\frac{\gamma\Lambda_0}{b}\right)^2+\Lambda_0^2=0\Rightarrow\gamma=\pm ib\,. \tag{1.B.31}$$

Putting all the constraints together yields, for a fixed channel $\beta=\alpha_\theta$, $\theta=\pm$,

$$\begin{aligned}
&\gamma=\pm ib\,,\\
&\zeta=\frac{1}{2}\left(b^2\pm ibm_0\right)=\frac{1}{2}\left(bQ-1\pm 2m_3\right)\,,\\
&\lambda-\zeta=-\frac{1}{2}bQ+\theta b\alpha_\theta\,,\\
&\mu_0=m_{0\pm}=m_0\pm(-ib)\,.
\end{aligned} \tag{1.B.32}$$

Finally, the irregular OPE reads

$$\begin{aligned}
\langle\Delta_\alpha,\Lambda_0,\bar\Lambda_0,m_0|\Phi_{2,1}(z,\bar z)=\tilde{\mathcal{C}}^{\alpha_+}_{\alpha,\alpha_{2,1}}&\left|\sum_{\pm,k}\mathcal{A}_{\alpha_+,m_{0\pm}}\Lambda^{-\frac12 bQ+b\alpha_+}(\Lambda z)^{\frac12(bQ-1\pm2m_3)}e^{\pm\Lambda z/2}z^{-k}\langle\Delta_{\alpha_+},\Lambda_0,m_{0\pm};k|\right|^2+\\
+\tilde{\mathcal{C}}^{\alpha_-}_{\alpha,\alpha_{2,1}}&\left|\sum_{\pm,k}\mathcal{A}_{\alpha_-,m_{0\pm}}\Lambda^{-\frac12 bQ-b\alpha_-}(\Lambda z)^{\frac12(bQ-1\pm2m_3)}e^{\pm\Lambda z/2}z^{-k}\langle\Delta_{\alpha_-},\Lambda_0,m_{0\pm};k|\right|^2.
\end{aligned} \tag{1.B.33}$$

where we absorbed a $2ib$ factor in the OPE coefficients for later convenience, $\Lambda=2ib\Lambda_0$ and $m_3=\frac{i}{2}bm_0$. Here the irregular state depending on $\Lambda_0,\bar\Lambda_0$ denotes the full (chiral$\otimes$antichiral) state, and the modulus squared of the chiral states (depending only on Λ_0) has to be understood as a tensor product. Now we can fix the OPE coefficients $\tilde{\mathcal{C}}^{\alpha_\pm}_{\alpha,\alpha_{2,1}},\mathcal{A}_{\alpha_\pm,m_{0\pm}}$ making use of the NSE for the full irregular three point function. Namely,

$$\left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z+\frac{\Delta_{\alpha\pm}}{z^2}+\frac{m_0\Lambda_0}{z}+\Lambda_0^2\right)\langle\Delta_\alpha,\Lambda_0,\bar\Lambda_0,m_0|\Phi_{2,1}(z,\bar z)|\Delta_{\alpha_\pm}\rangle=0\,. \tag{1.B.34}$$

If we define $\langle\Delta_\alpha,\Lambda_0,\bar\Lambda_0,m_0|\Phi_{2,1}(z,\bar z)|\Delta_{\alpha_\pm}\rangle=\left|e^{-\frac{\Lambda z}{2}}(\Lambda z)^{\frac12(bQ+2b\alpha_\pm)}\right|^2G_\pm(z,\bar z)$, then $G(z,\bar z)$ will satisfy

$$\left(z\partial_z^2+(1+2b\alpha_\pm-\Lambda z)\partial_z-\frac{\Lambda}{2}(1+2m_3+2b\alpha_\pm)\right)G_\pm(z,\bar z)=0\,. \tag{1.B.35}$$

Note that we can rewrite the previous equation using the natural variable $w = \Lambda z$, and obtain

$$\left(w\partial_w^2 + (1+2b\alpha_\pm - w)\partial_w - \frac{1}{2}(1+2m_3+2b\alpha_\pm)\right) G_\pm(w,\bar{w}) = 0\,. \tag{1.B.36}$$

Equation (1.B.36) is the confluent hypergeometric equation, therefore[5]

$$G_\pm(w,\bar{w}) = K_\pm^{(1)}\left|{}_1F_1\left(\frac{1}{2}+m_3+b\alpha_\pm, 1+2b\alpha_\pm, w\right)\right|^2 + K_\pm^{(2)}\left|w^{-2b\alpha_\pm}{}_1F_1\left(\frac{1}{2}+m_3-b\alpha_\pm, 1-2b\alpha_\pm, w\right)\right|^2. \tag{1.B.37}$$

Expanding the correlator near zero and comparing the solution (1.B.37) with the regular OPE near zero,

$$K_\pm^{(1)}\left|(\Lambda z)^{\frac{1}{2}bQ+b\alpha_\pm}\right|^2 + K_\pm^{(2)}\left|(\Lambda z)^{\frac{1}{2}bQ-b\alpha_\pm}\right|^2 = G(\alpha)\mathcal{C}^{\alpha}_{\alpha_{2,1}\alpha_\pm}\left|z^{\frac{1}{2}bQ\mp b\alpha_\pm}\right|^2, \tag{1.B.38}$$

and hence

$$\begin{aligned} K_+^{(1)} &= 0\,,\; K_+^{(2)} = G(\alpha)\mathcal{C}^{\alpha}_{\alpha_{2,1}\alpha_+}\left|\Lambda^{-\frac{1}{2}bQ+b\alpha_+}\right|^2, \\ K_-^{(1)} &= G(\alpha)\mathcal{C}^{\alpha}_{\alpha_{2,1}\alpha_-}\left|\Lambda^{-\frac{1}{2}bQ-b\alpha_-}\right|^2,\; K_-^{(2)} = 0\,. \end{aligned} \tag{1.B.39}$$

Now expanding the confluent hypergeometric near infinity and matching with the OPE we can finally fix all the coefficients. Recall that as $w \to \infty$

$$\begin{aligned} {}_1F_1\left(\frac{1}{2}+m_3+b\alpha_\pm, 1+2b\alpha_\pm, w\right) &\simeq \frac{\Gamma(1+2b\alpha_\pm)}{\Gamma(\frac{1}{2}+m_3+b\alpha_\pm)}e^w w^{-\frac{1}{2}+m_3-b\alpha_\pm} + \\ &+ \frac{\Gamma(1+2b\alpha_\pm)}{\Gamma(\frac{1}{2}-m_3+b\alpha_\pm)}(-1)^{-\frac{1}{2}-m_3-b\alpha_\pm}(w)^{-\frac{1}{2}-m_3-b\alpha_\pm}, \\ w^{-2b\alpha_\pm}{}_1F_1\left(\frac{1}{2}+m_3-b\alpha_\pm, 1-2b\alpha_\pm, w\right) &\simeq \frac{\Gamma(1-2b\alpha_\pm)}{\Gamma(\frac{1}{2}+m_3-b\alpha_\pm)}e^w w^{-\frac{1}{2}+m_3-b\alpha_\pm} + \\ &+ \frac{\Gamma(1-2b\alpha_\pm)}{\Gamma(\frac{1}{2}-m_3-b\alpha_\pm)}(-1)^{-\frac{1}{2}-m_3+b\alpha_\pm}(w)^{-\frac{1}{2}-m_3-b\alpha_\pm}. \end{aligned} \tag{1.B.40}$$

[5] Note that in principle also mixed terms could appear. However, they cannot be there in order to correctly match the behavior near zero.

Let us concentrate on the α_+ channel. Expanding the full correlator and matching z powers gives

$$\begin{aligned}
&G(\alpha)\mathcal{C}^{\alpha}_{\alpha_{2,1}\alpha_+}\left|\Lambda^{-\frac{1}{2}bQ+b\alpha_+}\right|^2\times\\
&\times\left|\frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}+m_3-b\alpha_+)}e^{w/2}w^{\frac{bQ}{2}-\frac{1}{2}+m_3}+\frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}-m_3-b\alpha_+)}(-1)^{-\frac{1}{2}-m_3+b\alpha_+}e^{-w/2}(w)^{\frac{bQ}{2}-\frac{1}{2}-m_3}\right|^2=\\
&=G(\alpha_+)\tilde{\mathcal{C}}^{\alpha_+}_{\alpha,\alpha_{2,1}}\left|\sum_{\pm}\mathcal{A}_{\alpha_+,m_0\pm}\Lambda^{-\frac{1}{2}bQ+b\alpha_+}(\Lambda z)^{\frac{1}{2}(bQ-1\pm 2m_3)}e^{\pm\Lambda z/2}\right|^2 .
\end{aligned}\tag{1.B.41}$$

Finally from equation (1.B.41) we can read off the coefficients (the coefficients for the α_- channel are obtained simply sending $\alpha_+\to-\alpha_-$)

$$\begin{aligned}
\tilde{\mathcal{C}}^{\alpha_\pm}_{\alpha,\alpha_{2,1}}&=\mathcal{C}^{\alpha_\pm}_{\alpha_{2,1}\alpha}\,,\\
\mathcal{A}_{\alpha_+,m_0+}&=\frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}+m_3-b\alpha_+)}\,,\\
\mathcal{A}_{\alpha_+,m_0-}&=\frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}-m_3-b\alpha_+)}(-1)^{-\frac{1}{2}-m_3+b\alpha_+}\,,\\
\mathcal{A}_{\alpha_-,m_0+}&=\frac{\Gamma(1+2b\alpha_-)}{\Gamma(\frac{1}{2}+m_3+b\alpha_-)}\,,\\
\mathcal{A}_{\alpha_-,m_0-}&=\frac{\Gamma(1+2b\alpha_-)}{\Gamma(\frac{1}{2}-m_3+b\alpha_-)}(-1)^{-\frac{1}{2}-m_3-b\alpha_-}\,.
\end{aligned}\tag{1.B.42}$$

Two remarks about equations (1.B.42): first of all, the OPE is symmetric in $\alpha\to-\alpha$, as it should be. Moreover, we expect the full irregular 3 point correlator to be symmetric under the simultaneous transformation $\Lambda\to-\Lambda$, $m_3\to-m_3$. Under this transformation

$$\begin{aligned}
&\mathcal{A}_{\alpha_+,m_3+}\Lambda^{-\frac{1}{2}(bQ-2b\alpha_+)}e^{\frac{\Lambda z}{2}}(\Lambda z)^{\frac{1}{2}(bQ-1+2m_3)}\to\\
&\to(-1)^{-\frac{1}{2}-m_3+b\alpha_+}\frac{\Gamma(1-2b\alpha_+)}{\Gamma(\frac{1}{2}-m_3-b\alpha_+)}\Lambda^{-\frac{1}{2}(bQ-2b\alpha_+)}e^{-\frac{\Lambda z}{2}}(\Lambda z)^{\frac{1}{2}(bQ-1-2m_3)}=\\
&=\mathcal{A}_{\alpha_+,m_3-}\Lambda^{-\frac{1}{2}(bQ-2b\alpha_+)}e^{-\frac{\Lambda z}{2}}(\Lambda z)^{\frac{1}{2}(bQ-1-2m_3)}\,,
\end{aligned}\tag{1.B.43}$$

and the same happens for the other channel. This suggests that the $(-1)^{-\frac{1}{2}-m_3\pm b\alpha_\pm}$ factor naturally multiplies Λ in the irregular OPE. Therefore, after this minor change we obtain formulae (1.3.3), (1.3.4).

1.C Nekrasov formulae

1.C.1 The AGT dictionary

The irregular conformal blocks of the form $\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle$ can be efficiently computed as a gauge theory instanton partition function thanks to the AGT correspondence [41]. Concretely, we have

$$\langle\Delta_\alpha, \Lambda_0, m_0|V_{\alpha_2}(1)|\Delta_{\alpha_1}\rangle = \mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1, m_2, m_3) \tag{1.C.1}$$

where $\mathcal{Z}^{\text{inst}}$ is the Nekrasov instanton partition function of $\mathcal{N}=2$ $SU(2)$ gauge theory with three hypermultiplets. The Nekrasov partition function contains a fundamental mass scale $\hbar = \sqrt{\epsilon_1\epsilon_2}$ which sets the units in which everything is measured. The mapping of parameters between CFT and gauge theory is then

$$\begin{aligned} &\epsilon_1 = \frac{\hbar}{b}\,, \quad \epsilon_2 = \hbar b\,, \quad \epsilon = \epsilon_1+\epsilon_2 \quad \longrightarrow \quad Q = \frac{\epsilon}{\hbar}\,,\\ &\Delta_i = \frac{Q^2}{4} - \alpha_i^2 = \frac{\frac{\epsilon^2}{4} - a_i^2}{\hbar^2}\,, \quad a_i := \hbar\alpha_i\,,\\ &\Lambda = 2i\hbar\Lambda_0\,, \quad m_3 = \frac{i}{2}\hbar m_0\,,\\ &m_1 = a_1 + a_2\,, \quad m_2 = -a_1 + a_2\,. \end{aligned} \tag{1.C.2}$$

The factors of $i\hbar$ in Λ and m_3 do not appear in [103] where the irregular state is defined because they drop terms of the form $\sqrt{-\epsilon_1\epsilon_2}$.

1.C.2 The instanton partition function

The $SU(2)$ partition function is given by the $U(2)$ partition function divided by the $U(1)$-factor:

$$\mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1, m_2, m_3, \epsilon_1, \epsilon_2) = \mathcal{Z}^{-1}_{U(1)}(\Lambda, m_1, m_2, \epsilon_1, \epsilon_2)\mathcal{Z}^{\text{inst}}_{U(2)}(\Lambda, a, m_1, m_2, m_3, \epsilon_1, \epsilon_2) \tag{1.C.3}$$

where the $U(2)$ partition function is given by a combinatorial formula which we review now. We often suppress the dependence on ϵ_1, ϵ_2. We mostly follow the notation of [41].

Let $Y = (\lambda_1 \geq \lambda_2 \geq ...)$ be a Young tableau where λ_i is the height of the i-th column and we set $\lambda_i = 0$ when i is larger than the width of the tableau. Its transpose is denoted by $Y^T = (\lambda_1' \geq \lambda_2' \geq ...)$. For a box s at the coordinate (i,j) we define the arm-length $A_Y(s)$ and the leg-length $L_Y(s)$ with respect to the tableau Y as

$$A_Y(s) = \lambda_i - j\,, \quad L_Y(s) = \lambda_j' - i\,. \tag{1.C.4}$$

Note that they can be negative when s is outside the tableau. Define a function E by

$$E(a, Y_1, Y_2, s) = a - \epsilon_1 L_{Y_2}(s) + \epsilon_2(A_{Y_1}(s)+1)\,. \tag{1.C.5}$$

Using the notation $\vec{a} = (a_1, a_2)$ with $a_1 = -a_2 = a$ and $\vec{Y} = (Y_1, Y_2)$ the contribution of a vector multiplet is

$$z_{\text{vector}}^{\text{inst}}(\vec{a}, \vec{Y}) = \prod_{i,j=1}^{2} \prod_{s\in Y_i} \frac{1}{E(a_i - a_j, Y_i, Y_j, s)} \prod_{t\in Y_j} \frac{1}{\epsilon_1 + \epsilon_2 - E(a_j - a_i, Y_j, Y_i, t)} \tag{1.C.6}$$

and that of an (antifundamental) hypermultiplet

$$z_{\text{matter}}^{\text{inst}}(\vec{a}, \vec{Y}, m) = \prod_{i=1}^{2} \prod_{s\in Y_i} \left(a + m + \epsilon_1 \left(i - \frac{1}{2}\right) + \epsilon_2 \left(j - \frac{1}{2}\right)\right) . \tag{1.C.7}$$

This is different from the formula given in [41] because our masses are shifted with respect to theirs by $\epsilon/2$. Finally, the $U(2)$ partition function is given by

$$\mathcal{Z}_{U(2)}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3) = \sum_{\vec{Y}} \Lambda^{|\vec{Y}|} z_{\text{vector}}^{\text{inst}}(\vec{a}, \vec{Y}) \prod_{n=1}^{3} z_{\text{matter}}^{\text{inst}}(\vec{a}, \vec{Y}, m_n) , \tag{1.C.8}$$

where $|\vec{Y}|$ denotes the total number of boxes in Y_1 and Y_2.

The $U(1)$-factor on the other hand can be obtained by decoupling one mass from the $U(1)$-factor for $N_f = 4$. Before decoupling, the third and fourth masses are given by

$$m_3 = a_3 + a_4 , \quad m_4 = a_3 - a_4 \tag{1.C.9}$$

where a_3 and a_4 are related to the momenta of the two vertex operators that collide to form the irregular state. The $U(1)$-factor is

$$\mathcal{Z}_{U(1)}^{N_f=4} = (1-q)^{2(a_2+\epsilon/2)(a_3+\epsilon/2)/\epsilon_1\epsilon_2} . \tag{1.C.10}$$

The decoupling limit is given by $q \to 0$, $m_4 \to \infty$ with $qm_4 \equiv \Lambda$ finite. This gives the $N_f = 3$ $U(1)$-factor

$$\mathcal{Z}_{U(1)} = e^{-(m_1+m_2+\epsilon)\Lambda/2\epsilon_1\epsilon_2} . \tag{1.C.11}$$

For reference, we give the one-instanton partition functions:

$$\begin{aligned}
\mathcal{Z}_{U(2)}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3) &= 1 + \frac{\prod_{i=1}^{3}\left(-a + m_i + \frac{\epsilon}{2}\right)}{2a\epsilon_1\epsilon_2(-2a+\epsilon)}\Lambda - \frac{\prod_{i=1}^{3}\left(a + m_i + \frac{\epsilon}{2}\right)}{2a\epsilon_1\epsilon_2(2a+\epsilon)}\Lambda + \mathcal{O}(\Lambda^2) \\
\mathcal{Z}_{SU(2)}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3) &= 1 - \frac{\epsilon^2 - 4a^2 - 4m_1m_2}{2\epsilon_1\epsilon_2(\epsilon+2a)(\epsilon-2a)} m_3\Lambda + \mathcal{O}(\Lambda^2)
\end{aligned} \tag{1.C.12}$$

1.C.3 The Nekrasov-Shatashvili limit

While the above formulae are valid for arbitrary ϵ_1, ϵ_2, in the context of the black hole we work in the Nekrasov-Shatashvili (NS) limit which is defined by $\epsilon_2 \to 0$ while keeping ϵ_1 finite [106]. Furthermore we set $\epsilon_1 = 1$. The conformal blocks $\langle \Delta_\alpha, \Lambda_0, m_0 | V_{\alpha_2}(1) | \Delta_{\alpha_1} \rangle$ then need to be understood as being computed as a partition function in the NS limit. This is done by computing it for arbitrary ϵ_1, ϵ_2 and taking $\epsilon_1 = 1$ and $\epsilon_2 \to 0$ in the end, because the partition function itself diverges in this limit, while the ratios appearing e.g. in the connection formulas remain finite. Furthermore, we define the instanton part of the NS free energy as

$$\mathcal{F}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3, \epsilon_1) = \epsilon_1 \lim_{\epsilon_2 \to 0} \epsilon_2 \log \mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1, m_2, m_3, \epsilon_1, \epsilon_2) . \tag{1.C.13}$$

One also uses the Matone relation [112]

$$E = a^2 - \Lambda \partial_\Lambda \mathcal{F}^{\text{inst}} , \tag{1.C.14}$$

which can be inverted order by order in Λ to obtain $a(E)$. For reference, we give some relevant quantities computed up to one instanton, with $\epsilon_1 = 1$ and the leading power of ϵ_2.

$$\begin{aligned}
&\mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1, m_2, m_3) = 1 - \frac{\frac{1}{4} - a^2 - m_1 m_2}{\frac{1}{2} - 2a^2} \frac{m_3 \Lambda}{\epsilon_2} + \mathcal{O}(\Lambda^2) \\
&\mathcal{F}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3) = -\frac{\frac{1}{4} - a^2 - m_1 m_2}{\frac{1}{2} - 2a^2} m_3 \Lambda + \mathcal{O}(\Lambda^2) \\
&a(E) = \sqrt{E} - \frac{\frac{1}{4} - E + a_1^2 - a_2^2}{\sqrt{E}\,(1 - 4E)} m_3 \Lambda + \mathcal{O}(\Lambda^2) \\
&\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1 - \frac{\theta' \epsilon_2}{2}, m_2 - \frac{\theta' \epsilon_2}{2}, m_3)}{\mathcal{Z}^{\text{inst}}_{SU(2)}(\Lambda, a, m_1 - \frac{\theta \epsilon_2}{2}, m_2 + \frac{\theta \epsilon_2}{2}, m_3)} = 1 - \frac{\theta(m_1 - m_2) + \theta'(m_1 + m_2)}{1 - 4a^2} m_3 \Lambda + \mathcal{O}(\Lambda^2) .
\end{aligned} \tag{1.C.15}$$

Finally, we define the full NS free energy, including the classical and the one-loop part by

$$\begin{aligned}
\partial_a \mathcal{F}(\Lambda, a, m_1, m_2, m_3, \epsilon_1) = &- 2a \log \frac{\Lambda}{\epsilon_1} + 2\epsilon_1 \log \frac{\Gamma\left(1 + \frac{2a}{\epsilon_1}\right)}{\Gamma\left(1 - \frac{2a}{\epsilon_1}\right)} + \epsilon_1 \sum_{i=1}^{3} \log \frac{\Gamma\left(\frac{1}{2} + \frac{m_i - a}{\epsilon_1}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_i + a}{\epsilon_1}\right)} + \\
&+ \partial_a \mathcal{F}^{\text{inst}}(\Lambda, a, m_1, m_2, m_3, \epsilon_1) .
\end{aligned} \tag{1.C.16}$$

1.D The semiclassical absorption coefficient

We give the detailed reduction of the full absorption coefficient in the semiclassical regime to the final result $\sigma = \exp -a_D/\epsilon_1$, with

$$a_D := \oint_B \phi_{SW}(z)dz = \lim_{\epsilon_1 \to 0} \partial_a \mathcal{F} \tag{1.D.1}$$

where $\phi_{SW}(z)$ is the Seiberg-Witten differential of the $\mathcal{N} = 2$ $SU(2)$ gauge theory with three flavours and $\mathcal{F}$ is the full NS free energy introduced in the previous section. First we restore the powers of ϵ_1 which were previously set to one in the exact absorption coefficient and substitute the AGT dictionary (see Appendix 1.C):

$$\sigma = \frac{-\mathrm{Im}\dfrac{m_1+m_2}{\epsilon_1}}{\left| \frac{\Gamma\left(1+\frac{2a}{\epsilon_1}\right)\Gamma\left(\frac{2a}{\epsilon_1}\right)\Gamma\left(1+\frac{m_1+m_2}{\epsilon_1}\right)\left(\frac{\Lambda}{\epsilon_1}\right)^{\frac{-a+m_3}{\epsilon_1}}}{\prod_{i=1}^3 \Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)} \frac{\mathcal{Z}^{\mathrm{inst}}_{SU(2)}\left(\Lambda, a+\frac{\epsilon_2}{2}, m_1, m_2, m_3+\frac{\epsilon_2}{2}\right)}{\mathcal{Z}^{\mathrm{inst}}_{SU(2)}\left(\Lambda, a, m_1-\frac{\epsilon_2}{2}, m_2-\frac{\epsilon_2}{2}, m_3\right)} + (a \to -a) \right|^2} \tag{1.D.2}$$

In the regime where we have two real turning points and where we have obtained the semiclassical transmission coefficient we have $|\Lambda| \ll 1$. Then a can be obtained order by order in an expansion in Λ, starting from $a = \ell + \frac{1}{2} + \mathcal{O}(\Lambda)$ by using the relation $E = a^2 - \Lambda\partial_\Lambda \mathcal{F}^{\mathrm{inst}}$. Since $\Lambda\partial_\Lambda\mathcal{F}^{\mathrm{inst}}$ is real for real a, we see that all terms in the expansion and therefore a itself are real. We anticipate that in σ the term surviving in the semiclassical limit will be the first term in the denominator. This can be seen quickly by approximating $\Gamma(z) \approx e^{z\log z}$ for large z. In the semiclassical limit $a \gg m_i$ and the contribution of the five Gamma functions containing a in the first term goes like $e^{\frac{a}{\epsilon_1}\log\frac{a}{\epsilon_1}}$. Extracting the term $\epsilon_1^{-\frac{a}{\epsilon_1}}$ cancels the explicit power of ϵ_1 outside, and the rest of the exponential blows up. On the other hand the behaviour of the second term in the denominator of the transmission coefficient can be obtained by sending $a \to -a$, so we see that in this case the exponential vanishes, and indeed the dominant term is the first one. The Gamma functions give the correct semiclassical one-loop contributions using Stirling's formula, and the ratio of partition functions gives the correct instanton contribution to a_D.

More in detail, we can split the contributions to a_D as

$$a_D = a_D^{1-\mathrm{loop}} + a_D^{\mathrm{inst}} = a^{1-\mathrm{loop}}_{D,\mathrm{vector}} + \sum_{i=1}^3 a^{1-\mathrm{loop}}_{D,\mathrm{matter}} + a^{\mathrm{inst}}_{D,\mathrm{vector}} + \sum_{i=1}^3 a^{\mathrm{inst}}_{D,\mathrm{matter}}\,. \tag{1.D.3}$$

We take all matter multiplets to be in the antifundamental representation of $SU(2)$. The vector and matter multiplet one-loop contributions to a_D are

$$\begin{aligned} a^{1-\text{loop}}_{D,\text{vector}}(a) &= -8a + 4a\log\frac{2a}{\Lambda} + 4a\log\frac{-2a}{\Lambda} \\ a^{1-\text{loop}}_{D,\text{matter}}(a,m) &= (a-m)\left[1-\log\left(\frac{-a+m}{\Lambda}\right)\right] + (a+m)\left[1-\log\left(\frac{a+m}{\Lambda}\right)\right] . \end{aligned} \tag{1.D.4}$$

These are antisymmetric under $a \to -a$ as they should be. On the other hand, in the absorption coefficient we have several Gamma functions, which we can expand in the semi-classical limit using Stirling's approximation $\log\Gamma(z) = (z-1/2)\log z - z + \mathcal{O}(z^{-1})$. We neglect the constant factors of 2π since we have the same amount of Gamma functions in the numerator and denominator and they will cancel. We have for the vectormultiplet

$$\begin{aligned} \epsilon_1 \log\left|\Gamma\left(\frac{2a}{\epsilon_1}\right)\Gamma\left(1+\frac{2a}{\epsilon_1}\right)\right|^{-2} &\to 8a - 8a\log 2a + 8a\log\epsilon_1 \\ &= -a^{1-\text{loop}}_{D,\text{vector}}(a) - 4\pi i a - 8a\log\frac{\Lambda}{\epsilon_1} , \end{aligned} \tag{1.D.5}$$

for the matter multiplets

$$\begin{aligned} \epsilon_1\log\left|\Gamma\left(\frac{1}{2}+\frac{m+a}{\epsilon_1}\right)\right|^2 &\to (a-m)\log(a-m) + (a+m)\log(a+m) - 2a(1+\log\epsilon_1) \\ &= -a^{1-\text{loop}}_{D,\text{matter}}(a,m) + i\pi(a-m) + 2a\log\frac{\Lambda}{\epsilon_1} \end{aligned} \tag{1.D.6}$$

and there is one more Gamma function:

$$\left|\Gamma\left(1+\frac{m_1+m_2}{\epsilon_1}\right)\right|^2 \to i\frac{m_1+m_2}{\epsilon_1}e^{-i\pi(m_1+m_2)/\epsilon_1}. \tag{1.D.7}$$

The last contribution is

$$\left|\frac{\Lambda}{\epsilon_1}\right|^{2\frac{a-m_3}{\epsilon_1}} = \left(\frac{\Lambda}{\epsilon_1}\right)^{\frac{2a}{\epsilon_1}} e^{i\pi\frac{a+m_3}{\epsilon_1}}. \tag{1.D.8}$$

Putting it all together we have

$$-\mathrm{Im}\frac{m_1+m_2}{\epsilon_1}\left|\frac{\prod_{i=1}^3\Gamma\left(\frac{1}{2}+\frac{m_i+a}{\epsilon_1}\right)\left(\frac{\Lambda}{\epsilon_1}\right)^{\frac{a-m_3}{\epsilon_1}}}{\Gamma\left(1+\frac{2a}{\epsilon_1}\right)\Gamma\left(\frac{2a}{\epsilon_1}\right)\Gamma\left(1+\frac{m_1+m_2}{\epsilon_1}\right)}\right|^2 \to e^{-a_D^{1-\text{loop}}/\epsilon_1} . \tag{1.D.9}$$

Now let us look at the instanton partition functions:

$$\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a, m_1-\frac{\epsilon_2}{2}, m_2-\frac{\epsilon_2}{2}, m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a+\frac{\epsilon_2}{2}, m_1, m_2, m_3+\frac{\epsilon_2}{2}\right)} = \frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, \tilde{a}-\frac{\epsilon_2}{2}, m_1-\frac{\epsilon_2}{2}, m_2-\frac{\epsilon_2}{2}, \tilde{m}_3-\frac{\epsilon_2}{2}\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, \tilde{a}, m_1, m_2, \tilde{m}_3\right)} \tag{1.D.10}$$

where we have defined $\tilde{a} = a + \frac{\epsilon_2}{2}$ and $\tilde{m}_3 = m_3 + \frac{\epsilon_2}{2}$. Now, looking at the explicit expressions for the $U(2)$ Nekrasov partition functions, we see that the part corresponding to the gauge field depends only on $a_1 - a_2$, and the part corresponding the the hypermultiplets only on $a_1 + m_i$ and $a_2 + m_i$. So we see that

$$\begin{aligned}&\mathcal{Z}^{\text{inst}}_{U(2)}\left(\Lambda, a_1 = \tilde{a} - \frac{\epsilon_2}{2}, a_2 = -\tilde{a} + \frac{\epsilon_2}{2}, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}, \tilde{m}_3 - \frac{\epsilon_2}{2}\right) = \\ &= \mathcal{Z}^{\text{inst}}_{U(2)}\left(\Lambda, a_1 = \tilde{a} - \epsilon_2, a_2 = -\tilde{a}, m_1, m_2, \tilde{m}_3\right) .\end{aligned} \tag{1.D.11}$$

The $U(1)$ part behaves as

$$\mathcal{Z}^{-1}_{U(1)}(\Lambda, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}) = e^{(m_1+m_2-\epsilon_2)\Lambda/2\epsilon_1\epsilon_2} = e^{-\Lambda/2\epsilon_1}\mathcal{Z}^{-1}_{U(1)}(\Lambda, m_1, m_2)\,, \tag{1.D.12}$$

therefore

$$\begin{aligned}&\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}, m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a + \frac{\epsilon_2}{2}, m_1, m_2, m_3 + \frac{\epsilon_2}{2}\right)} = e^{-\Lambda/2\epsilon_1}\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, \tilde{a} - \epsilon_2, -\tilde{a}, m_1, m_2, \tilde{m}_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, \tilde{a}, -\tilde{a}, m_1, m_2, \tilde{m}_3\right)} = \\ &= e^{-\Lambda/2\epsilon_1}\exp\frac{1}{\epsilon_1\epsilon_2}\left\{\mathcal{F}^{\text{inst}}\left(\Lambda, \tilde{a} - \epsilon_2, -\tilde{a}, m_1, m_2, \tilde{m}_3\right) - \mathcal{F}^{\text{inst}}\left(\Lambda, \tilde{a}, -\tilde{a}, m_1, m_2, \tilde{m}_3\right)\right\} = \\ &= e^{-\Lambda/2\epsilon_1}\exp -\frac{1}{\epsilon_1}\frac{\partial}{\partial a_1}\mathcal{F}^{\text{inst}}\left(\Lambda, a_1, a_2, m_1, m_2, \tilde{m}_3\right)|_{a_1=\tilde{a},\, a_2=-\tilde{a}}\,.\end{aligned} \tag{1.D.13}$$

Now there are no more factors of $1/\epsilon_2$ so we can safely drop the tildes. On the other hand, by symmetry considerations which are most easily seen in the expression as a conformal block, and using the fact that Λ and the three masses are purely imaginary while a is real, we have

$$\begin{aligned}&\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}, m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a + \frac{\epsilon_2}{2}, m_1, m_2, m_3 + \frac{\epsilon_2}{2}\right)} = \frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(-\Lambda, a, -m_1 + \frac{\epsilon_2}{2}, -m_2 + \frac{\epsilon_2}{2}, -m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(-\Lambda, a + \frac{\epsilon_2}{2}, -m_1, -m_2, -m_3 - \frac{\epsilon_2}{2}\right)} = \\ &= \frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda^*, a^*, m_1^* + \frac{\epsilon_2}{2}, m_2^* + \frac{\epsilon_2}{2}, m_3^*\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda^*, a^* + \frac{\epsilon_2}{2}, m_1^*, m_2^*, m_3^* - \frac{\epsilon_2}{2}\right)}\end{aligned} \tag{1.D.14}$$

And therefore, repeating the same steps as above,

$$\left(\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}, m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a + \frac{\epsilon_2}{2}, m_1, m_2, m_3 + \frac{\epsilon_2}{2}\right)}\right)^* = e^{\Lambda/2\epsilon_1}\exp\frac{1}{\epsilon_1}\frac{\partial}{\partial a_2}\mathcal{F}^{\text{inst}}\left(\Lambda, a_1, a_2, m_1, m_2, m_3\right)|_{a_1=a,\, a_2=-a} \tag{1.D.15}$$

Now using $\partial_a\mathcal{F} = \partial_{a_1}\mathcal{F} - \partial_{a_2}\mathcal{F}$ we have:

$$\left|\frac{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a, m_1 - \frac{\epsilon_2}{2}, m_2 - \frac{\epsilon_2}{2}, m_3\right)}{\mathcal{Z}^{\text{inst}}_{SU(2)}\left(\Lambda, a + \frac{\epsilon_2}{2}, m_1, m_2, m_3 + \frac{\epsilon_2}{2}\right)}\right|^2 = e^{-a_D^{\text{inst}}/\epsilon_1} \tag{1.D.16}$$

which combined with the one-loop part finally gives

$$\boxed{\sigma \approx e^{-a_D/\epsilon_1}}. \tag{1.D.17}$$

This result is valid for $\ell \gg 1$ and $M\omega, \mathrm{a}\omega \ll 1$, while keeping all orders in $M\omega, \mathrm{a}\omega$.

Chapter 2

Irregular Liouville correlators and Heun functions

We perform a detailed study of a class of irregular correlators in Liouville Conformal Field Theory, of the related Virasoro conformal blocks with irregular singularities and of their connection formulae. Upon considering their semi-classical limit, we provide explicit expressions of the connection matrices for the Heun function and a class of its confluences. Their calculation is reduced to concrete combinatorial formulae from conformal block expansions. The technique that we implement here has already been developed for the confluent Heun function for linear perturbations of Kerr black holes in the previous chapter (1) and is further refined and generalised here.

This chapter is organised as follows. In section 2.1, as a warm-up, we recall the relation between four-point conformal blocks with the insertion of three primary fields and one level 2 degenerate field and hypergeometric functions and we study in detail the confluences to irregular conformal blocks and the related special functions. We obtain the connection formulae for the latter as solutions of the constraints imposed by crossing symmetry. In section 2.2 we systematically study the five point conformal blocks with the insertion of four primary fields and one level 2 degenerate field. We focus on the explicit computation of the connection formulae as solutions of the constraints imposed by crossing symmetry for the regular case and a class of its confluences. In each case, we also compute the semi-classical limit. In section 2.3 we provide a dictionary between semiclassical CFT data and Heun equations in the standard form, we apply the results of the previous section identifying the relevant semiclassical CFT blocks with Heun functions and provide the connection formulae. Few technical points are relegated to the Appendices. A list of symbols should help the reader in following our computations.

The accompanying table collects the dictionary between (irregular) conformal blocks, supersymmetric gauge theories and the corresponding Heun functions.

CFT - CB		$SU(2)$ Gauge Theory	Heun
$\mathfrak{F}$	Regular	$N_f = 4$	HeunG
${}_1\mathfrak{F}$	Confluent	$N_f = 3$	HeunC
${}_{\frac{1}{2}}\mathfrak{F}$	Reduced Confluent	$N_f = 2$ asymmetric	HeunRC
${}_1\mathfrak{D}_1$	Doubly Confluent	$N_f = 2$ symmetric	HeunDC
${}_1\mathfrak{E}_{\frac{1}{2}}$	Reduced Doubly Confluent	$N_f = 1$	HeunRDC
${}_{\frac{1}{2}}\mathfrak{E}_{\frac{1}{2}}$	Doubly Reduced Doubly Confluent	$N_f = 0$	HeunDRDC

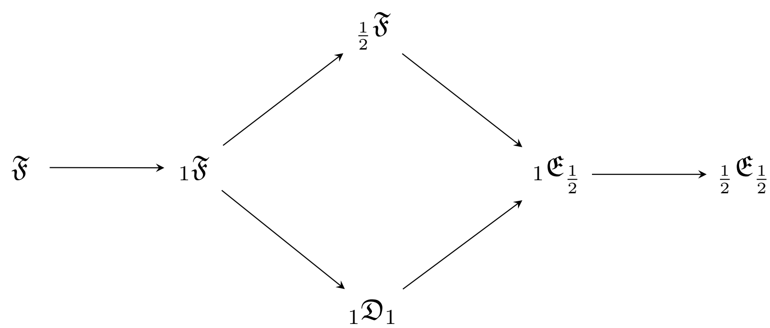


Figure 2.0.1: Confluence diagram of conformal blocks.

CFT symbols

b Liouville coupling constant

Q Liouville background charge $Q = b + b^{-1}$

α_∞, α_1, α_t, α_0 Liouville momenta of non-degenerate primary insertions

Δ_∞, Δ_1, Δ_t, Δ_0 Scaling dimensions of non-degenerate primary insertions, $\Delta_i = \frac{Q^2}{4} - \alpha_i^2$

α Intermediate momentum, with corresponding scaling dimension Δ

$\alpha_{2,1}$ Degenerate Liouville momentum $\alpha_{2,1} = -\frac{2b+b^{-1}}{2}$, with corresponding scaling dimension $\Delta_{2,1}$

$\alpha_{i\theta}$ Momentum shifted by $-\theta\frac{b}{2}$ with $\theta = \pm 1$, namely $\alpha_{i\theta} = \alpha_i - \theta\frac{b}{2}$

$V_{\alpha_i}(x)$ Primary operator of momentum α_i inserted at x

$|\Delta_i\rangle$ Primary state, corresponding to a primary operator of dimension Δ_i inserted at zero (at infinity if $\langle\Delta_\infty|$)

μ L_1-momentum of an irregular insertion of rank 1

μ' Intermediate L_1-momentum

$\mu_{i\theta}$ L_1-momentum shifted by $-\theta\frac{b}{2}$ with $\theta = \pm 1$

$|\mu, \Lambda\rangle$ Irregular state of rank 1 with eigenvalues $\mu\Lambda, -\frac{\Lambda^2}{4}$ inserted at zero (at infinity if $\langle\mu, \Lambda|$)

$|\Lambda^2\rangle$ Irregular state of rank $\frac{1}{2}$ with eigenvalues $-\frac{\Lambda^2}{4}$ inserted at zero (at infinity if $\langle\Lambda^2|$)

${}_x\mathfrak{F}_y$ Conformal block (CB) expanded around regular insertions, with an insertion of rank[1] x resp. y at ∞ resp. 0

[1]Here and in the following, if x or y are zero we drop the label for simplicity.

$_x\mathfrak{D}_y$ CB with at least one variable expanded around an irreg. singularity of rank 1, with an insertion of rank x resp. y at ∞ resp. 0

$_x\mathfrak{E}_y$ CB with at least one variable expanded around an irreg. singularity of rank $\frac{1}{2}$, with an insertion of rank x resp. y at ∞ resp. 0

$_x\widetilde{\mathfrak{F}}_y$, $_x\widetilde{\mathfrak{D}}_y$, $_x\widetilde{\mathfrak{E}}_y$ CB without classical part, i.e. normalized as $1 + \dots$

G_α Liouville two point function

$C_{\alpha_1\alpha_2\alpha_3}$ Liouville three point function

$C_{\mu\alpha}$ Pairing of a primary and a rank 1 irregular state

C_α Pairing of a primary and a rank $\frac{1}{2}$ irregular state

$C_{\alpha_1\alpha_2}^{\alpha_3}$ OPE coefficient involving three primaries

$B_{\mu\alpha}^{\mu'}$ OPE coefficient involving one primary and two irregular vertices of rank 1

$B_{\alpha_{2,1}}$ OPE coefficient involving a degenerate field and two irregular vertices of rank 1/2

CFT symbols - semiclassics

a_∞, a_1, a_t, a_0 Semiclassical Liouville momenta

a Semiclassical intermediate momentum

$a_{i\theta}$ Semiclassical momentum shifted by $-\theta\frac{b^2}{2}$ with $\theta = \pm 1$

m Semiclassical L_1-momentum

m' Semiclassical intermediate L_1-momentum

m_θ Semiclassical L_1-momentum shifted by $-\theta\frac{b^2}{2}$ with $\theta = \pm 1$

L Semiclassical highest eigenvalue of irregular states

$_x\mathcal{F}_y$ Semiclassical CB expanded around regular insertions, with an insertion of rank x resp. y at ∞ resp. 0

$_x\mathcal{D}_y$ Semiclassical CB with at least one variable expanded around an irreg. singularity of rank 1, with an insertion of rank x resp. y at ∞ resp. 0

$_x\mathcal{E}_y$ Semiclassical CB with at least one variable expanded around an irreg. singularity of rank $\frac{1}{2}$, with an insertion of rank x resp. y at ∞ resp. 0

F Logarithm of a classical conformal block

W Logarithm of a semiclassical conformal block

u Log-derivative of a classical CB, up to constants

${}_x\widetilde{\mathcal{F}}_y$, ${}_x\widetilde{\mathcal{D}}_y$, ${}_x\widetilde{\mathcal{E}}_y$ Semiclassical CB rescaled so that they start as $1 + \dots$

Heun symbols

α, β, γ, δ, ϵ Parameters of the Heun equations

q Accessory parameter of the Heun equations

$w(z)$ Solutions of the Heun equations in standard form

$P_i^{-1}(z)w(z)$ Solutions of the Heun equations in normal form

HeunG General Heun function

HeunC Confluent Heun function expanded near a regular singularity

HeunC_∞ Confluent Heun function expanded near the irregular singularity

HeunRC Reduced confluent Heun function expanded near a regular singularity

HeunRC_∞ Reduced confluent Heun function expanded near the irregular singularity

HeunDC Doubly confluent Heun function expanded near an irregular singularity

$\mathrm{HeunRDC}_0$ Reduced doubly confluent Heun function expanded near the irregular singularity at zero

$\mathrm{HeunRDC}_\infty$ Reduced doubly confluent Heun function expanded near the irregular singularity at infinity

HeunDRDC Doubly reduced doubly confluent Heun function expanded near an iregular singularity

2.1 Warm-up: 4-point degenerate conformal blocks and classical special functions

We start reviewing standard facts about four-point degenerate conformal blocks on the sphere and their confluence limits. In particular we review their relation to the hypergeometric function and its confluent limits, namely Whittaker and Bessel functions.

The hypergeometric function is the solution to the most general second-order linear ODE with three regular singularities. On the CFT side it arises as the four-point conformal block on the Riemann sphere when one of the insertions is a degenerate vertex operator.

2.1.1 Hypergeometric functions

Consider the four-point conformal block on the sphere with one degenerate field insertion $\Phi_{2,1}$ of momentum $\alpha_{2,1} = -\frac{2b+b^{-1}}{2}$ (corresponding to $\Delta_{2,1} = -\frac{1}{2} - \frac{3b^2}{4}$):

$$\langle \Delta_\infty | V_1(1) \Phi_{2,1}(z) | \Delta_0 \rangle \,. \tag{2.1.1}$$

In the following we will drop the subscript $2,1$ and just denote by $\Phi(z)$ this degenerate field. The corresponding BPZ equation takes the form

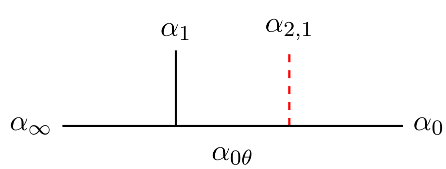
$$\left(b^{-2}\partial_z^2 - \left(\frac{1}{z-1} + \frac{1}{z} \right) \partial_z + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_0}{z^2} + \frac{\Delta_\infty - \Delta_1 - \Delta_{2,1} - \Delta_0}{z(z-1)} \right) \langle \Delta_\infty | V_1(1) \Phi(z) | \Delta_0 \rangle = 0 \,. \tag{2.1.2}$$

This equation has regular singularities at $0, 1, \infty$. As mentioned above, the corresponding conformal blocks should therefore be expressed in terms of hypergeometric functions. Indeed, the above differential equation by definition is solved by the conformal blocks corresponding to the correlator (2.1.1), which in turn are given in terms of hypergeometric functions. In particular, the conformal block corresponding to the expansion $z \sim 0$ is

$$\mathfrak{F}\left({}^{\alpha_1}_{\alpha_\infty} \alpha_{0\theta} {}^{\alpha_{2,1}}_{\alpha_0} ; z \right) = z^{\frac{bQ}{2} + \theta b \alpha_0} (1-z)^{\frac{bQ}{2} + b\alpha_1} {}_2F_1\left(\tfrac{1}{2} + b\left(\theta\alpha_0 + \alpha_1 - \alpha_\infty\right), \tfrac{1}{2} + b\left(\theta\alpha_0 + \alpha_1 + \alpha_\infty\right), 1 + 2b\theta\alpha_0, z \right) \,, \tag{2.1.3}$$

where $\theta = \pm$ and $\alpha_{0\pm} = \alpha_0 \pm \frac{-b}{2}$ are the two fusion channels allowed by the degenerate fusion rules. Similar formulae hold for the expansions around $z \sim 1$ and ∞. Conventionally, this conformal block is denoted diagrammatically by

$$\mathfrak{F}\left({}^{\alpha_1}_{\alpha_\infty} \alpha_{0\theta} {}^{\alpha_{2,1}}_{\alpha_0} ; z \right) = \tag{2.1.4}$$



We now want to expose the interplay between crossing symmetry, DOZZ factors and the connection formulae for the hypergeometric functions. To this end, let us expand the correlator once for $z \sim 0$ and once for $z \sim 1$:

$$\begin{aligned}\langle\Delta_\infty|V_1(1)\Phi_{2,1}(z)|\Delta_0\rangle &= \sum_{\theta=\pm} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C_{\alpha_\infty\alpha_1\alpha_{0\theta}} \left|\mathfrak{F}\begin{pmatrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & ; z \\ \alpha_\infty & & \alpha_0\end{pmatrix}\right|^2 = \\ &= \sum_{\theta'=\pm} C^{\alpha_{1\theta'}}_{\alpha_{2,1}\alpha_1} C_{\alpha_\infty\alpha_{1\theta'}\alpha_0} \left|\mathfrak{F}\begin{pmatrix}\alpha_0 & & \alpha_{2,1} \\ & \alpha_{1\theta'} & ; 1-z \\ \alpha_\infty & & \alpha_1\end{pmatrix}\right|^2 .\end{aligned} \tag{2.1.5}$$

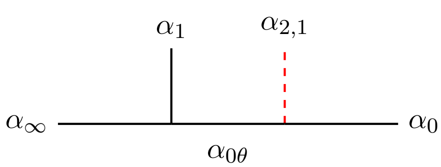
Here $C_{\alpha\beta\gamma}$ are the DOZZ three-point functions, and $C^{\alpha}_{\beta\gamma} = G^{-1}_{\alpha} C_{\alpha\beta\gamma}$ are the OPE coefficients (see Appendix 2.A.1). Equation (2.1.5) is just the statement of crossing symmetry, due to the associativity of the OPE. The two expansions are related by the connection matrix $\mathcal{M}_{\theta\theta'}$ as follows

$$\mathfrak{F}\begin{pmatrix}\alpha_1 & & \alpha_{2,1} \\ & \alpha_{0\theta} & ; z \\ \alpha_\infty & & \alpha_0\end{pmatrix} = \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_1; b\alpha_\infty) \mathfrak{F}\begin{pmatrix}\alpha_0 & & \alpha_{2,1} \\ & \alpha_{1\theta'} & ; 1-z \\ \alpha_\infty & & \alpha_1\end{pmatrix} . \tag{2.1.6}$$

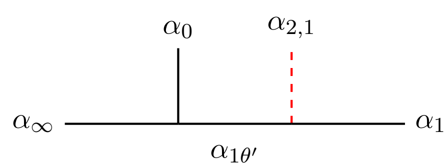
Plugging the latter into (2.1.5) determines $\mathcal{M}_{\theta\theta'}$ to be

$$\mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha_1; b\alpha_\infty) = \frac{\Gamma(-2\theta' b\alpha_1)\Gamma(1+2\theta b\alpha_0)}{\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 + b\alpha_\infty\right)\Gamma\left(\frac{1}{2}+\theta b\alpha_0 - \theta' b\alpha_1 - b\alpha_\infty\right)} , \tag{2.1.7}$$

which is indeed the connection matrix for hypergeometric functions. Diagrammatically, we can express the connection formula as



$$= \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}$$



. (2.1.8)

2.1.2 Whittaker functions

Colliding the singularities at 1 and ∞ of the hypergeometric functions we obtained the Whittaker functions, which are simply related to the confluent hypergeometric function. They have a regular singularity at 0 and an irregular singularity of rank 1 at ∞. To describe the confluence of two regular singularities in CFT we introduce the rank 1 irregular state, denoted by $\langle\mu, \Lambda|$. It lives in a Whittaker module and it is defined by the following properties

$$\begin{aligned}&\langle\mu,\Lambda|L_0 = \Lambda\partial_\Lambda\langle\mu,\Lambda| \\ &\langle\mu,\Lambda|L_{-1} = \mu\Lambda\langle\mu,\Lambda| \\ &\langle\mu,\Lambda|L_{-2} = -\frac{\Lambda^2}{4}\langle\mu,\Lambda| \\ &\langle\mu,\Lambda|L_{-n} = 0\,, \quad n>2\,.\end{aligned} \tag{2.1.9}$$

Note that the action of L_0 is not diagonal, and hence $\langle\mu,\Lambda|$ makes no reference to any Verma module. Equivalently, one can describe this state by a confluence limit of primary operators:

$$\langle\mu,\Lambda| \propto \lim_{\eta\to\infty} t^{\Delta_t-\Delta}\langle\Delta|V_t(t) \tag{2.1.10}$$

with[2]

$$\Delta = \frac{Q^2}{4} - \left(\frac{\mu+\eta}{2}\right)^2, \quad \Delta_t = \frac{Q^2}{4} - \left(\frac{\mu-\eta}{2}\right)^2, \quad t = \frac{\eta}{\Lambda}. \tag{2.1.11}$$

We fix the normalization of the irregular state by giving its overlap with a primary state, namely

$$\langle\mu,\Lambda|\Delta\rangle = |\Lambda|^{2\Delta}C_{\mu\alpha}\,, \tag{2.1.12}$$

with

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b\left(\frac{Q}{2}+\mu+\alpha\right)\Upsilon_b\left(\frac{Q}{2}+\mu-\alpha\right)}\,. \tag{2.1.13}$$

The Λ-dependence is fixed by the L_0-action, and $C_{\mu\alpha}$ is a normalization function that only depends on μ and α, and is calculated in Appendices 2.A.2, 2.B.1. The notation reflects the fact that C can be interpreted as a collided three-point function [47]. The correlator

$$\langle\mu,\Lambda|\Phi(z)|\Delta\rangle \tag{2.1.14}$$

satisfies the BPZ equation

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} + \frac{\mu\Lambda}{z} - \frac{\Lambda^2}{4}\right)\langle\mu,\Lambda|\Phi(z)|\Delta\rangle = 0\,, \tag{2.1.15}$$

that has a rank 1 irregular singularity at $z=\infty$ and a regular singularity at $z=0$. Correspondingly, we expect this correlator to be given in terms of confluent hypergeometric functions. Indeed, for $z\sim 0$ one finds by solving the differential equation that the corresponding *confluent* (or *irregular*) conformal block is given by a Whittaker function. In particular, the two solutions are $z^{\frac{b^2}{2}}M_{b\mu,\pm b\alpha}(b\Lambda z)$, where the Whittaker M-function has a simple expansion around $z\sim 0$:

$$M_{b\mu,b\alpha}(b\Lambda z) = (b\Lambda z)^{\frac{1}{2}+b\alpha}\left(1+\mathcal{O}(b\Lambda z)\right)\,. \tag{2.1.16}$$

We can compute the confluent conformal block as

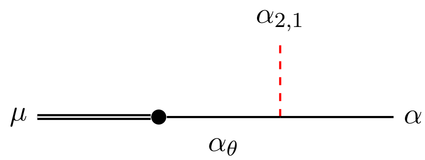
$${}_1\mathfrak{F}\left(\mu\,\alpha_\theta \begin{matrix}\alpha_{2,1}\\ \alpha\end{matrix};\Lambda z\right) = \Lambda^{\Delta_\theta}(b\Lambda)^{-\frac{1}{2}-\theta b\alpha}z^{\frac{b^2}{2}}M_{b\mu,\theta b\alpha}(b\Lambda z)\,. \tag{2.1.17}$$

by expanding the OPE between $\Phi(z)$ and $|\Delta\rangle$ and projecting on $\langle\mu,\Lambda|$. Comparing this with the expansion of M one obtains the prefactors written above. Here the subscript 1

[2] Note that this procedure mimics the decoupling of a mass in the AGT-dual gauge theory.

indicates the presence of a rank 1 irregular singularity at infinity. We represent this block diagramatically by

$$
{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,{}^{\alpha_{2,1}}_{\alpha};\Lambda z\right) =
$$



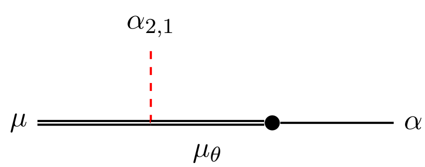
. (2.1.18)

The double line denotes the rank 1 irregular state, and the fat dot the projection onto a primary state. For $z \sim \infty$ we get an intrinsically different kind of confluent conformal block since we are now expanding z near an *irregular singularity* of rank 1, dubbed in [104] confluent conformal block *of 2nd kind*. We denote such a conformal block by the letter $\mathfrak{D}$ and find

$$
\begin{aligned}
{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_+\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}} e^{-i\pi b\mu} b^{b\mu} (\Lambda z)^{\frac{b^2}{2}} W_{-b\mu,b\alpha}(e^{-i\pi} b\Lambda z)\,, \\
{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_-\,\alpha;\frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}} b^{-b\mu} (\Lambda z)^{\frac{b^2}{2}} W_{b\mu,b\alpha}(b\Lambda z)\,,
\end{aligned}
\tag{2.1.19}
$$

where W is the Whittaker function with a simple asymptotic expansion around $z \sim \infty$. This block is obtained by doing the OPE between the irregular state and the degenerate field, which is derived in Appendix 2.B.1, and then projecting on $|\Delta\rangle$. Once again, the prefactors are fixed by comparing with the expansion of W. We represent this conformal block diagramatically by

$$
{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_\theta\,\alpha;\frac{1}{\Lambda z}\right) =
$$



. (2.1.20)

Crossing symmetry now implies

$$
\langle \mu,\Lambda|\Phi(z)|\Delta\rangle = \sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\mu\alpha_\theta} \left| {}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,{}^{\alpha_{2,1}}_{\alpha};\Lambda z\right)\right|^2 = \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\alpha_{2,1},\mu} C_{\mu_{\theta'}\alpha} \left| {}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right)\right|^2 .
\tag{2.1.21}
$$

Here B is the irregular OPE coefficient arising from the OPE between the irregular state and the degenerate field. We calculate it in Appendices 2.A.2, 2.B.1, and it is given by

$$
B^{\mu_\pm}_{\alpha_{2,1},\mu} = e^{i\pi\left(\frac{1}{2}\pm b\mu+\frac{b^2}{4}\right)} .
\tag{2.1.22}
$$

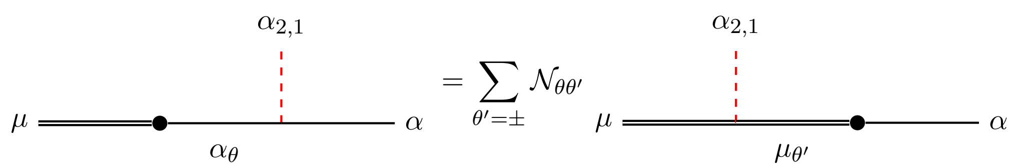
As for the hypergeometric function, we can make an Ansatz for the connection formula for these irregular conformal blocks of the form

$$
b^{\theta b\alpha}\,{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,{}^{\alpha_{2,1}}_{\alpha};\Lambda z\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu} \mathcal{N}_{\theta\theta'}(b\alpha,b\mu)\; {}_1\mathfrak{D}\left(\mu\,{}^{\alpha_{2,1}}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right) .
\tag{2.1.23}
$$

The constraints coming from crossing symmetry (2.1.21) are solved by the irregular connection coefficients

$$\mathcal{N}_{\theta\theta'}(b\alpha, b\mu) = \frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)} e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)} . \tag{2.1.24}$$

These are just the connection coefficients for Whittaker functions. In fact, in Appendix 2.B.1 we argue the other way around, namely we determine the normalization function $C_{\mu\alpha}$ and the irregular OPE coefficient $B^{\mu\pm}_{\alpha_{2,1},\mu}$ by using the known connection coefficients $\mathcal{N}_{\theta\theta'}$ for Whittaker functions. This shows the consistency of our approach. Let us emphasize for latter purposes that the functions $\mathcal{N}_{\theta\theta'}$ solve the constraint (2.1.21), which will appear later in a different context. We represent this connection formula diagrammatically by



(2.1.25)

2.1.3 Bessel functions

There is a natural limiting procedure which reduces a rank 1 irregular singularity to a rank 1/2 one. To describe the latter in CFT, let us introduce the rank 1/2 irregular state $\langle\Lambda^2|$ via defining properties

$$\begin{aligned} \langle\Lambda^2|L_0 &= \Lambda^2\partial_{\Lambda^2}\langle\Lambda^2| \\ \langle\Lambda^2|L_{-1} &= -\frac{\Lambda^2}{4}\langle\Lambda^2| \\ \langle\Lambda^2|L_{-n} &= 0\,, \quad n>1\,. \end{aligned} \tag{2.1.26}$$

It can be obtained from the rank 1 irregular state via the limit[3]

$$\langle\Lambda^2| = \lim_{\mu\to\infty}\langle\mu, -\frac{\Lambda^2}{4\mu}| \,. \tag{2.1.27}$$

We see that reducing a rank 1 to a rank 1/2 singularity corresponds to further decoupling a mass in the AGT dual gauge theory. We normalize the rank 1/2 state as

$$\langle\Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta}C_\alpha\,, \quad C_\alpha = 2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,. \tag{2.1.28}$$

This normalization function is calculated in Appendices 2.A.3, 2.B.2. Consider the following correlation function involving the rank 1/2 state:

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle\,. \tag{2.1.29}$$

[3]Note that this limit corresponds to the well known holomorphic decoupling limit of a massive hypermultiplet in the AGT dual gauge theory.

which correspondingly displays a rank 1/2 singularity at infinity. This is reflected in the BPZ equation

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} - \frac{\Lambda^2}{4z}\right)\langle\Lambda^2|\Phi(z)|\Delta\rangle = 0\,. \tag{2.1.30}$$

Solving this differential equation one finds that the corresponding *rank 1/2 irregular* conformal block is given by a modified Bessel function $I_\nu(x)$ as

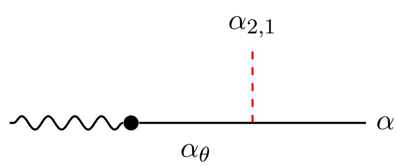
$$_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \Gamma(1+2\theta b\alpha)\Lambda^{2\Delta_\theta}\left(\frac{b\Lambda}{2}\right)^{-2\theta b\alpha} z^{\frac{bQ}{2}} I_{2\theta b\alpha}(b\Lambda\sqrt{z})\,. \tag{2.1.31}$$

Here the subscript $\frac{1}{2}$ indicates the presence of a rank 1/2 singularity at infinity. This conformal block is obtained by doing the OPE between Φ and $|\Delta\rangle$ and then projecting the result on $\langle\Lambda^2|$. The prefactors are fixed by comparing this with the following expansion of the Bessel function

$$I_{2\theta b\alpha}(b\Lambda\sqrt{z}) = \frac{(b\Lambda\sqrt{z}/2)^{2\theta b\alpha}}{\Gamma(1+2\theta b\alpha)}\left(1+\mathcal{O}(b\Lambda\sqrt{z})\right)\,. \tag{2.1.32}$$

We represent this conformal block diagramatically by

$$_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \tag{2.1.33}$$

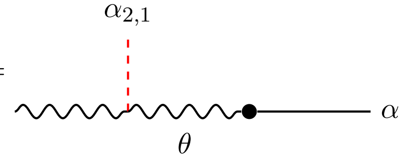


Here the wiggly line denotes the rank 1/2 irregular state, and the fat dot represents the pairing with a primary state. For $z \sim \infty$ we get a different kind of irregular conformal block, since we are now expanding for z near an irregular singularity of rank 1/2. We denote such a conformal block by the letter $\mathfrak{E}$

$$\begin{aligned}
_{\frac{1}{2}}\mathfrak{E}^{(+)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}e^{-\frac{i\pi}{2}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(e^{-i\pi}b\Lambda\sqrt{z})\,,\\
_{\frac{1}{2}}\mathfrak{E}^{(-)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(b\Lambda\sqrt{z})\,,
\end{aligned} \tag{2.1.34}$$

where K is the modified Bessel function of the second kind, which has a nice asymptotic expansion for $z \sim \infty$. This block is obtained from the OPE between the irregular rank 1/2 state and the degenerate field which we derived in Appendix 2.B.2, and then by taking the scalar product with $|\Delta\rangle$. We represent this block diagramatically by

$$_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) = \tag{2.1.35}$$



Crossing symmetry implies that

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle = \textstyle\sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\alpha_\theta} \left|{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right)\right|^2 = \textstyle\sum_{\theta'=\pm} B_{\alpha_{2,1}} C_\alpha \left|{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right)\right|^2 . \tag{2.1.36}$$

Here $B_{\alpha_{2,1}}$ is the irregular OPE coefficient arising from the OPE between the irregular rank 1/2 state and the degenerate field:

$$B_{\alpha_{2,1}} = 2^{b^2} e^{\frac{i\pi bQ}{2}} . \tag{2.1.37}$$

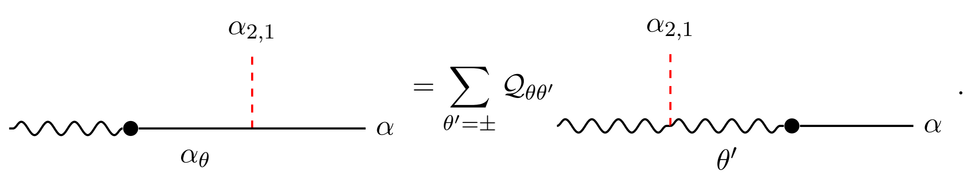
These functions are derived in Appendix 2.B.2. We can now make an Ansatz for the connection formula for these irregular conformal blocks:

$$b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \sum_{\theta'=\pm} b^{-1/2}\mathcal{Q}_{\theta\theta'}(b\alpha)\;{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) . \tag{2.1.38}$$

The crossing symmetry condition (2.1.36) gives constraints on the irregular connection coefficients, which are solved by

$$\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}}\Gamma(1+2\theta b\alpha) e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)} . \tag{2.1.39}$$

These are of course nothing else than the connection coefficients for Bessel functions, including the relevant prefactors. Similar constraints of the form (2.1.36) will reappear later. We represent the connection formula by



(2.1.40)

2.2 5-point degenerate conformal blocks, confluences and connection formulae

In this section we consider the relevant CFT correlators obeying the BPZ equations which reduce to Heun equations in the appropriate classical limit. Notice that for more than three vertix insertions BPZ equations on the sphere are richer than the corresponding ODE due to the presence of the corresponding moduli. This implies that a suitable classical limit (NS limit), engineered to decouple the moduli dynamics, is needed to recover the corresponding ODE.

We derive explicit connection formulae for the relevant conformal blocks by making use of crossing symmetry of the CFT correlators. In the classical limit, these generate explicit solutions of the connection problem for the Heun equations.

2.2.1 Regular conformal blocks

General case

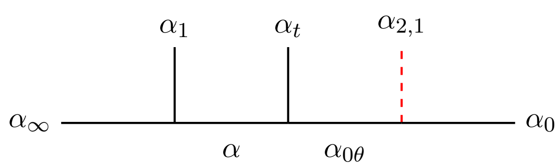
The five-point function with one degenerate insertion in Liouville CFT satisfies the BPZ equation

$$\left(b^{-2}\partial_z^2 + \frac{\Delta_1}{(z-1)^2} - \frac{\Delta_1 + t\partial_t + \Delta_t + z\partial_z + \Delta_{2,1} + \Delta_0 - \Delta_\infty}{z(z-1)} + \frac{\Delta_t}{(z-t)^2} + \frac{t}{z(z-t)}\partial_t - \frac{1}{z}\partial_z + \frac{\Delta_0}{z^2}\right)\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle = 0\,. \tag{2.2.1}$$

The five-point function can be expanded in the region $z \ll t \ll 1$ as follows

$$\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle = \sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t\alpha_{0\theta}} C_{\alpha_\infty\alpha_1\alpha}\mathfrak{F}\begin{pmatrix}\alpha_1 & \alpha & \alpha_t & \alpha_{0\theta} & \alpha_{2,1} \\ \alpha_\infty & & & & \alpha_0\end{pmatrix}; t, \frac{z}{t}\Big)\mathfrak{F}\Big(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_t}\ \alpha_{0\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; \bar{t}, \frac{\bar{z}}{\bar{t}}\Big)\,. \tag{2.2.2}$$

As usual the conformal blocks can be computed via OPEs. The result is naturally an expansion in the variables t and z/t. Conformal blocks are usually denoted diagrammatically as

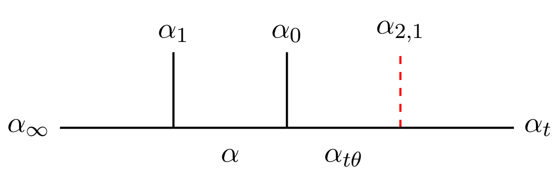


$$= \mathfrak{F}\Big(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_t}\ \alpha_{0\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\Big)\,. \tag{2.2.3}$$

An explicit combinatorial formula for this conformal block is given in Appendix 2.D. The same correlator can be expanded for $z \sim t$ and small t after the Möbius transformation $x \to \frac{x-t}{1-t}$, yielding

$$\begin{aligned}\langle\Delta_\infty|V_1(1)V_t(t)\Phi(z)|\Delta_0\rangle &= \left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\right|^2\langle\Delta_\infty|V_1(1)V_0\left(\frac{t}{t-1}\right)\Phi\left(\frac{z-t}{1-t}\right)|\Delta_t\rangle = \\ &= \sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t} C^{\alpha}_{\alpha_0\alpha_{t\theta}} C_{\alpha_\infty\alpha_1\alpha}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\Big(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_0}\ \alpha_{t\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix}; \frac{t}{t-1}, \frac{t-z}{t}\Big)\right|^2\,.\end{aligned} \tag{2.2.4}$$

Diagramatically, this conformal block is



$$= \mathfrak{F}\Big(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_0}\ \alpha_{t\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix}; \frac{t}{t-1}, \frac{t-z}{t}\Big)\,. \tag{2.2.5}$$

We notice that the diagrams just represent the order in which the OPEs are performed, neglecting factors such as Jacobians that arise from the Möbius transformations. By crossing symmetry the two expansions should agree, so that

$$\sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t\alpha_{0\theta}} C_{\alpha_\infty\alpha_1\alpha}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\frac{z}{t}\right)\right|^2 =$$
$$= \sum_{\theta=\pm}\int d\alpha\, C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t} C^{\alpha}_{\alpha_0\alpha_{t\theta}} C_{\alpha_\infty\alpha_1\alpha}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\right|^2 . \tag{2.2.6}$$

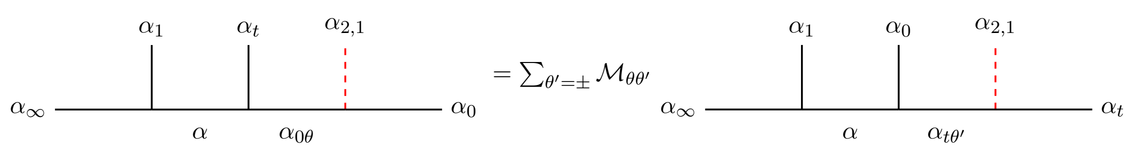
which can be conveniently recast as

$$\int d\alpha\, C_{\alpha_\infty\alpha_1\alpha}\sum_{\theta=\pm}\left(C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t\alpha_{0\theta}}\left|\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\frac{z}{t}\right)\right|^2 + \right.$$
$$\left. - C^{\alpha_{t\theta}}_{\alpha_{2,1}\alpha_t} C^{\alpha}_{\alpha_0\alpha_{t\theta}}\left|(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)\right|^2\right) = 0 . \tag{2.2.7}$$

By imposing the vanishing of the integrand we get a constraint analogous to (2.1.5), which analogously to (2.1.6) we solve as[4]

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_t}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\frac{z}{t}\right) =$$
$$= \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_t;b\alpha)e^{i\pi(\Delta-\Delta_0-\Delta_{2,1}-\Delta_t)}(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right), \tag{2.2.8}$$

where $\mathcal{M}_{\theta\theta'}$ are the hypergeometric connection coefficients defined in (2.1.7). Note indeed that in (2.2.8) the functional form of the connection coefficients depends on the local properties of the conformal block in the vicinity of the degenerate vertex insertion as can be seen form the factorized form of (2.2.7). Diagrammatically, the connection formula (2.2.8) reads



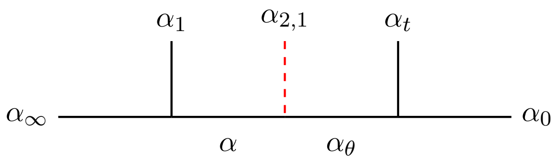
(2.2.9)

[4]The phase appearing in the RHS of equation (2.2.8) is fixed imposing that the overall leading powers of

$$(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,{}^{\alpha_0}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right) \sim e^{-i\pi(\Delta-\Delta_t-\Delta_{2,1}-\Delta_0)}t^{\Delta-\Delta_0-\Delta_{t\theta}}(t-z)^{\Delta_{t\theta}-\Delta_{2,1}-\Delta_{2,1}}(1+\dots)$$

agree with the leading powers of the OPEs of the full correlator, where no explicit phase appears.

Conformal blocks for small z can also be connected to the expansion for $z \sim 1, z \sim \infty$ passing through the region $t \ll z \ll 1$. The conformal block in that region is



$$= \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_\theta\ \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; z, \frac{t}{z}\right). \qquad (2.2.10)$$

Then, crossing symmetry relates this block to the expansion for $z \sim 0$ via

$$\langle \Delta_\infty | V_1(1) V_t(t) \Phi(z) | \Delta_0 \rangle = \langle \Delta_\infty | V_1(1) \Phi(z) V_t(t) | \Delta_0 \rangle \,, \qquad (2.2.11)$$

therefore, by comparing (2.2.11) with (2.2.3) we get

$$\begin{aligned}&\sum_{\theta=\pm} \int d\alpha\, C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} C^{\alpha}_{\alpha_t \alpha_{0\theta}} C_{\alpha_\infty \alpha_1 \alpha} \left| \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_t}\ \alpha_{0\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\right)\right|^2 = \\ &= \sum_{\theta=\pm} \int d\alpha\, C^{\alpha}_{\alpha_{2,1}\alpha_\theta} C^{\alpha_\theta}_{\alpha_t \alpha_0} C_{\alpha_\infty \alpha_1 \alpha} \left| \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_\theta\ \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; z, \frac{t}{z}\right)\right|^2 ,\end{aligned} \qquad (2.2.12)$$

and following the same argument as for the previous case we find

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_t}\ \alpha_{0\theta}\ \begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix}; t, \frac{z}{t}\right) = \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(b\alpha_0, b\alpha; b\alpha_t) \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; z, \frac{t}{z}\right). \qquad (2.2.13)$$

Now we can connect expansions in the intermediate region to expansions for $z \sim \infty$ again invoking crossing symmetry. Performing the transformation $x \to t/x$ on the LHS of (2.2.11) we get

$$\langle \Delta_\infty | V_1(1) \Phi(z) V_t(t) | \Delta_0 \rangle = \left| t^{\Delta_\infty + \Delta_1 + \Delta_{2,1} - \Delta_0 - \Delta_t} z^{-2\Delta_{2,1}} \right|^2 \langle \Delta_0 | V_t(1) V_1(t) \Phi\left(\frac{t}{z}\right) | \Delta_\infty \rangle \,, \qquad (2.2.14)$$

that implies

$$\begin{aligned}&\sum_{\theta=\pm} \int d\alpha\, C_{\alpha_t \alpha_0 \alpha} C^{\alpha}_{\alpha_{2,1}\alpha_\theta} C^{\alpha_\theta}_{\alpha_\infty \alpha_1} \left| \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; z, \frac{t}{z}\right)\right|^2 = \\ &= \sum_{\theta=\pm} \int d\alpha\, C_{\alpha_t \alpha_0 \alpha} C^{\alpha_{\infty\theta}}_{\alpha_{2,1}\alpha_\infty} C^{\alpha}_{\alpha_{\infty\theta}\alpha_1} \left| t^{\Delta_\infty + \Delta_1 + \Delta_{2,1} - \Delta_0 - \Delta_t} z^{-2\Delta_{2,1}} \mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix}\ \alpha\ {}^{\alpha_1}\ \alpha_{\infty\theta'}\ \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \frac{1}{z}\right)\right|^2 ,\end{aligned} \qquad (2.2.15)$$

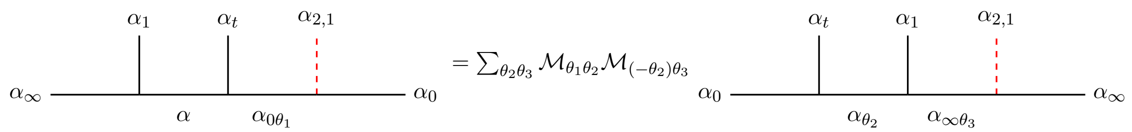
and finally

$$\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; z, \tfrac{t}{z}\right) = \textstyle\sum_{\theta'} \mathcal{M}_{\theta\theta'}(b\alpha, b\alpha_\infty; b\alpha_1) t^{\Delta_\infty + \Delta_1 + \Delta_{2,1} - \Delta_0 - \Delta_t} z^{-2\Delta_{2,1}} \mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix}\ \alpha\ {}^{\alpha_1}\ \alpha_{\infty\theta'}\ \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \tfrac{1}{z}\right). \qquad (2.2.16)$$

Combining equations (2.2.13) and (2.2.16) we can write

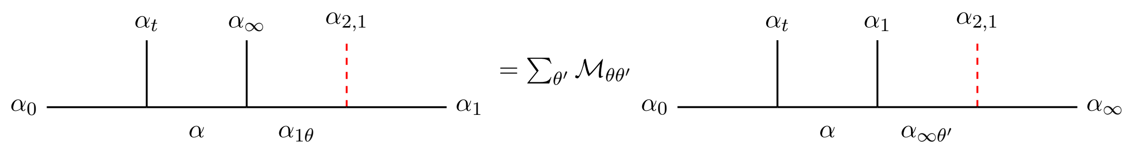
$$\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\\ \end{matrix}\,\alpha_{0\theta_1}\,\begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};t,\frac{z}{t}\right)=$$
$$=\sum_{\theta_2\theta_3}\mathcal{M}_{\theta_1\theta_2}(b\alpha_0,b\alpha;b\alpha_t)\mathcal{M}_{(-\theta_2)\theta_3}(b\alpha,b\alpha_\infty;b\alpha_1)t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha_{\theta_2}\,\begin{matrix}\alpha_1\\ \end{matrix}\,\alpha_{\infty\theta_3}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\frac{1}{z}\right). \tag{2.2.17}$$

Diagrammatically, this reads



(2.2.18)

The diagrams provide a straightforward way to generalize the connection formula to an arbitrary pair of points. Indeed, writing down the diagram it is immediate to guess the correct $\mathcal{M}_{\theta\theta'}$ factors and the conformal blocks that will enter the connection formula. As an example, the connection formula for the expansions for $z \sim 1$ and $z \sim \infty$ with $t \ll 1$ are given by

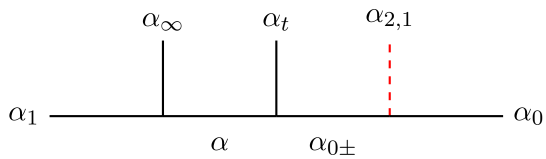


(2.2.19)

that is

$$t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_t-\Delta_0}(1-t)^{\Delta_\infty+\Delta_0+\Delta_{2,1}-\Delta_t-\Delta_1}(z-t)^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_0\\ \alpha_t\end{matrix}\,\alpha\,\begin{matrix}\alpha_\infty\\ \end{matrix}\,\alpha_{1\theta}\,\begin{matrix}\alpha_{2,1}\\ \alpha_1\end{matrix};t,\frac{z-1}{z-t}\right)=$$
$$=\sum_{\theta'}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha_\infty;b\alpha)t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t}z^{-2\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix}\alpha_t\\ \alpha_0\end{matrix}\,\alpha\,\begin{matrix}\alpha_1\\ \end{matrix}\,\alpha_{\infty\theta'}\,\begin{matrix}\alpha_{2,1}\\ \alpha_\infty\end{matrix};t,\frac{1}{z}\right). \tag{2.2.20}$$

Note that combining all the previous formulae we manage to analytically continue the expansion in $z \sim 0$ of the conformal block in all the complex plane for $t \ll 1$. It is straightforward to generalize the previous formulae for $t \sim 1, t \sim \infty$. All in all, for any value of t we can connect all the possible expansions in z. The analytic continuation in the $t-$plane is more involved and can be done via the fusion kernel. As a concluding remark, note that there is a Möbius tranformation in each region of expansions of the correlator, say $z \ll t \ll 1$ for reference, that only exchanges α_∞ and α_1 and that does not change the region of validity of the expansion. This transformation is usually called *braiding*. This gives, up to a Jacobian,



$$= \mathfrak{F}\begin{pmatrix} \alpha_\infty & \alpha & \alpha_t & \alpha_{0\theta} & \alpha_{2,1} & ; \frac{t}{t-1}, \frac{z}{t}\frac{t-1}{z-1} \\ \alpha_1 & & & & \alpha_0 & \end{pmatrix}. \tag{2.2.21}$$

Braiding changes the expansion variables in the conformal blocks according to the new positions of the insertions and as such can be used to generate other expansions and the related connection coefficients.

Semiclassical limit

Let us consider the semiclassical limit of Liouville theory, that is the double scaling limit

$$b \to 0,\ \alpha_i \to \infty,\ b\alpha_i = a_i \text{ finite.} \tag{2.2.22}$$

In this limit the conformal blocks and the corresponding BPZ equation greatly simplify. The divergence exponentiates and the z dependence becomes subleading, namely[5]

$$\mathfrak{F}\begin{pmatrix} \alpha_1 & \alpha & \alpha_t & \alpha_{0\theta} & \alpha_{2,1} & ; t, \frac{z}{t} \\ \alpha_\infty & & & & \alpha_0 & \end{pmatrix} = t^{\Delta-\Delta_t-\Delta_{0\theta}} z^{\frac{bQ}{2}+\theta b\alpha_0} \exp\left[\frac{1}{b^2}\left(F(t) + b^2 W(z/t,t) + \mathcal{O}(b^4)\right)\right]. \tag{2.2.23}$$

Here $F(t)$ is the classical conformal block, related to the conformal block without degenerate insertion via

$$\mathfrak{F}\begin{pmatrix} \alpha_1 & \alpha & \alpha_t & ; t \\ \alpha_\infty & & \alpha_0 & \end{pmatrix} = t^{\Delta-\Delta_t-\Delta_0} e^{b^{-2}\left(F(t)+\mathcal{O}(b^2)\right)}. \tag{2.2.24}$$

The divergences in the conformal blocks can be cured by dividing by the conformal block without the degenerate insertion. We denote the resulting finite, semiclassical conformal block by the letter $\mathcal{F}$:

$$\mathcal{F}\begin{pmatrix} a_1 & a & a_t & a_{0\theta} & a_{2,1} & ; t, \frac{z}{t} \\ a_\infty & & & & a_0 & \end{pmatrix} = \lim_{b\to 0} \frac{\mathfrak{F}\begin{pmatrix} \alpha_1 & \alpha & \alpha_t & \alpha_{0\theta} & \alpha_{2,1} & ; t, \frac{z}{t} \\ \alpha_\infty & & & & \alpha_0 & \end{pmatrix}}{\mathfrak{F}\begin{pmatrix} \alpha_1 & \alpha & \alpha_t & ; t \\ \alpha_\infty & & \alpha_0 & \end{pmatrix}} = t^{-\theta a_0} z^{\frac{1}{2}+\theta a_0} e^{-\frac{\theta}{2}\partial_{a_0} F(t)} \left(1 + \mathcal{O}(t, z/t)\right). \tag{2.2.25}$$

Note that the conformal block with the degenerate insertion and $z, t \sim 0$ contains a classical conformal block depending on $a_{0\theta} = a_0 - \theta\frac{b^2}{2}$. Dividing by the four-point function without the degenerate insertion, which depends on a_0, gives an incremental ratio that in

[5]Here and in the following we do not indicate the dependence of F and W on the rescaled momenta.

the limit (2.2.22) becomes the derivative $\partial_{a_0} F(t)$. The BPZ equation (2.2.1) simplifies in the semiclassical limit as well. The $t-$derivative acting on the conformal block gives

$$t\partial_t \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\end{matrix}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\tfrac{z}{t}\right) = b^{-2}\left(-\tfrac{1}{4}-a^2+a_t^2+a_0^2+t\partial_t F(a_i,a,t)+\mathcal{O}(b^2)\right)\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t\end{matrix}\,\alpha_{0\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_0\end{matrix};t,\tfrac{z}{t}\right), \tag{2.2.26}$$

therefore the $t-$derivative becomes a multiplication by a z-independent factor at leading order in b^2 and the BPZ equation becomes an ODE. Defining

$$u^{(0)} = \lim_{b\to 0} b^2 t\partial_t \log \mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t \\ \alpha_0\end{matrix};t\right), \tag{2.2.27}$$

where the superscript indicates that the block is expanded for $t\sim 0$, the BPZ equation (2.2.1) in the semiclassical limit reads

$$\left(\partial_z^2+\frac{\frac{1}{4}-a_1^2}{(z-1)^2}-\frac{\frac{1}{2}-a_1^2-a_t^2-a_0^2+a_\infty^2+u^{(0)}}{z(z-1)}+\frac{\frac{1}{4}-a_t^2}{(z-t)^2}+\frac{u^{(0)}}{z(z-t)}+\frac{\frac{1}{4}-a_0^2}{z^2}\right)\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}\,a\,\begin{matrix}a_t\end{matrix}\,a_{0\theta}\,\begin{matrix}a_{2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right)=0. \tag{2.2.28}$$

The solution of the previous ODE for $z\sim t$ is given by the semiclassical block

$$(t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}\,a\,\begin{matrix}a_0\end{matrix}\,a_{t\theta}\,\begin{matrix}a_{2,1} \\ a_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right) = \lim_{b\to 0}(t-1)^{-\Delta_{2,1}}\frac{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_0\end{matrix}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_0 \\ \alpha_t\end{matrix};\frac{t}{t-1}\right)} =$$

$$= \lim_{b\to 0}\frac{e^{i\pi(\Delta-\Delta_0-\Delta_{2,1}-\Delta_t)}(1-t)^{\Delta_\infty-\Delta_1-\Delta_t-\Delta_{2,1}-\Delta_0}\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_0\end{matrix}\,\alpha_{t\theta}\,\begin{matrix}\alpha_{2,1} \\ \alpha_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}\,\alpha\,\begin{matrix}\alpha_t \\ \alpha_0\end{matrix};t\right)}, \tag{2.2.29}$$

therefore the connection formula (2.2.8) descends to the semiclassical blocks to be

$$\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}\,a\,\begin{matrix}a_t\end{matrix}\,a_{0\theta}\,\begin{matrix}a_{,2,1} \\ a_0\end{matrix};t,\frac{z}{t}\right) = \sum_{\theta'}\mathcal{M}_{\theta\theta'}(a_0,a_t;a)(t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix}\,a\,\begin{matrix}a_0\end{matrix}\,a_{t\theta'}\,\begin{matrix}a_{2,1} \\ a_t\end{matrix};\frac{t}{t-1},\frac{t-z}{t}\right). \tag{2.2.30}$$

Note that the intermediate momentum a can be computed as a function of the parameters appearing in the semiclassical BPZ equation inverting the relation (2.2.27). Similarly, keeping $t\sim 0$ we can analytically continue the solution to the other singularities, that is for $z\sim 1$ and $z\sim\infty$. In particular, we can directly connect $z\sim 0$ and $z\sim\infty$ passing though the intermediate region. The semiclassical block for $z \sim \infty$ reads

$$t^{-\frac{1}{2}} z \mathcal{F}\begin{pmatrix} a_t & a & a_1 & a_{\infty\theta} & a_{2,1} \\ a_0 & & & & a_\infty \end{pmatrix}; t, \frac{1}{z}\Big) =$$

$$= \lim_{b\to 0} \frac{t^{\Delta_\infty+\Delta_1+\Delta_{2,1}-\Delta_0-\Delta_t} z^{-2\Delta_{2,1}} \mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha \begin{matrix}\alpha_1\end{matrix} \alpha_{\infty\theta'} \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \frac{1}{z}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix} \alpha \begin{matrix}\alpha_t \\ \alpha_0\end{matrix}; t\right)} = \lim_{b\to 0} \frac{t^{\Delta_{2,1}} z^{-2\Delta_{2,1}} \mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha \begin{matrix}\alpha_1\end{matrix} \alpha_{\infty\theta'} \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \frac{1}{z}\right)}{\mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha \begin{matrix}\alpha_1 \\ \alpha_\infty\end{matrix}; t\right)} . \tag{2.2.31}$$

The connection formula (2.2.17) from $z \sim 0$ to $z \sim \infty$ involves a conformal block with two shifted momenta, that is

$$\mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha_{\theta'} \begin{matrix}\alpha_1\end{matrix} \alpha_{\infty\theta} \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \tfrac{1}{z}\right) = t^{\Delta_{\theta'}-\Delta_1-\Delta_{\infty\theta}} \left(\tfrac{t}{z}\right)^{\frac{bQ}{2}+\theta b\alpha_\infty} \exp\left[\tfrac{1}{b^2} F\left(a - \theta' \tfrac{b^2}{2}, t\right) + W\left(a - \theta'\tfrac{b^2}{2}, t\right) + \mathcal{O}(b^2)\right] . \tag{2.2.32}$$

At first order in b^2

$$F\left(a - \theta' \frac{b^2}{2}, t\right) + b^2 W\left(a - \theta' \frac{b^2}{2}, t\right) = F(a,t) - \frac{\theta' b^2}{2} \partial_a F(a,t) + b^2 W(a,t) + \mathcal{O}(b^4) , \tag{2.2.33}$$

therefore in the semiclassical limit

$$\mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha_{\theta'} \begin{matrix}\alpha_1\end{matrix} \alpha_{\infty\theta} \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \frac{1}{z}\right) \sim t^{-\theta'\alpha} e^{-\frac{\theta'}{2}\partial_a F(t)} \mathfrak{F}\left(\begin{matrix}\alpha_t \\ \alpha_0\end{matrix} \alpha \begin{matrix}\alpha_1\end{matrix} \alpha_{\infty\theta} \begin{matrix}\alpha_{2,1} \\ \alpha_\infty\end{matrix}; t, \frac{1}{z}\right) , \quad \text{as } b \to 0 . \tag{2.2.34}$$

This is consistent with the fact that we expect only two linearly independent z behaviors. The connection formula (2.2.17) simplifies to

$$\mathcal{F}\left(\begin{matrix}a_1 \\ a_\infty\end{matrix} a \begin{matrix}a_t\end{matrix} a_{0\theta} \begin{matrix}a_{2,1} \\ a_0\end{matrix}; t, \frac{z}{t}\right) =$$

$$= \sum_{\theta'} \left(\sum_{\sigma} \mathcal{M}_{\theta\sigma}(a_0, a; a_t) \mathcal{M}_{(-\sigma)\theta'}(a, a_\infty; a_1) t^{-\sigma a} e^{-\frac{\sigma}{2}\partial_a F} \right) t^{-\frac{1}{2}} z \mathcal{F}\left(\begin{matrix}a_t \\ a_0\end{matrix} a \begin{matrix}a_1\end{matrix} a_{\infty\theta'} \begin{matrix}a_{2,1} \\ a_\infty\end{matrix}; t, \frac{1}{z}\right) . \tag{2.2.35}$$

Explicitly, the connection coefficients are

$$\sum_{\sigma=\pm} \mathcal{M}_{\theta\sigma}(a_0, a; a_t) \mathcal{M}_{(-\sigma)\theta'}(a, a_\infty; a_1) t^{-\sigma a} e^{-\frac{\sigma}{2}\partial_a F} =$$

$$= \sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_0)\Gamma(-2\theta' a_\infty) t^{-\sigma a} e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma\left(\frac{1}{2} + \theta a_0 - \sigma a + a_t\right)\Gamma\left(\frac{1}{2} + \theta a_0 - \sigma a - a_t\right)\Gamma\left(\frac{1}{2} - \sigma a - \theta' a_\infty + a_1\right)\Gamma\left(\frac{1}{2} - \sigma a - \theta' a_\infty - a_1\right)} . \tag{2.2.36}$$

For future reference, the semiclassical block for small t and $z \sim 1$ is given by

$$(t(1-t))^{-\frac{1}{2}}\,(t-z)\mathcal{F}\left(\begin{matrix} a_0 & & a_\infty & & a_{2,1} \\ & a & & a_{1\theta} & \\ a_t & & & & a_1 \end{matrix}; t, \tfrac{1-z}{t-z}\right) = \lim_{b\to 0}\,(t(1-t))^{\Delta_{2,1}}\,(t-z)^{-2\Delta_{2,1}}\frac{\mathfrak{F}\left(\begin{matrix} \alpha_0 & & \alpha_\infty & & \alpha_{2,1} \\ & \alpha & & \alpha_{1\theta} & \\ \alpha_t & & & & \alpha_1 \end{matrix}; t, \tfrac{1-z}{t-z}\right)}{\mathfrak{F}\left(\begin{matrix} \alpha_0 & & \alpha_\infty \\ & \alpha & \\ \alpha_t & & \alpha_1 \end{matrix}; t\right)}\,. \tag{2.2.37}$$

Similarly one can obtain the connection coefficients for the other $t-$expansions. As an example, let us schematically consider the case $t \gg 1$. The semiclassical block for $z \sim 0$ reads

$$t^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0 \end{matrix}; \frac{1}{t}, z\right) = \lim_{b\to 0}\frac{t^{-\Delta_{2,1}}\mathfrak{F}\left(\begin{matrix} \alpha_t & & \alpha_1 & & \alpha_{2,1} \\ & \alpha & & \alpha_{0\theta} & \\ \alpha_\infty & & & & \alpha_0 \end{matrix}; \frac{1}{t}, z\right)}{\mathfrak{F}\left(\begin{matrix} \alpha_t & & \alpha_1 \\ & \alpha & \\ \alpha_\infty & & \alpha_0 \end{matrix}; \frac{1}{t}\right)}\,. \tag{2.2.38}$$

Still the $t-$derivative decouples, leaving behind

$$u^{(\infty)} = \lim_{b\to 0} b^2 t\partial_t \log t^{\Delta-\Delta_t-\Delta_1-\Delta_0}\mathfrak{F}\left(\begin{matrix} \alpha_t & & \alpha_1 \\ & \alpha & \\ \alpha_\infty & & \alpha_0 \end{matrix}; \frac{1}{t}\right)\,. \tag{2.2.39}$$

Note that the semiclassical BPZ equation formally remains the same, with the substitution[6] of $u^{(0)}$ with $u^{(\infty)}$. Indeed, the intermediate momentum α is now determined in terms of $u^{(\infty)}$. The $z \sim 1$ expansion gives

$$(t-1)^{\frac{1}{2}}e^{i\theta\pi a}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0 \end{matrix}; \tfrac{1}{t-1}, 1-z\right) = \lim_{b\to 0}(t-1)^{-\Delta_{2,1}}e^{i\theta\pi b\alpha}\frac{\mathfrak{F}\left(\begin{matrix} \alpha_t & & \alpha_0 & & \alpha_{2,1} \\ & \alpha & & \alpha_{1\theta} & \\ \alpha_\infty & & & & \alpha_1 \end{matrix}; \tfrac{1}{1-t}, 1-z\right)}{\mathfrak{F}\left(\begin{matrix} \alpha_t & & \alpha_0 \\ & \alpha & \\ \alpha_\infty & & \alpha_1 \end{matrix}; \tfrac{1}{1-t}\right)}\,, \tag{2.2.40}$$

and the corresponding connection formula reads

$$t^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta} & \\ a_\infty & & & & a_0 \end{matrix}; \tfrac{1}{t}, z\right) = \sum_{\theta'=\pm 1}\mathcal{M}_{\theta\theta'}\,(a_0, a_1; a)\,(t-1)^{\frac{1}{2}}e^{i\theta\pi a}\mathcal{F}\left(\begin{matrix} a_t & & a_1 & & a_{2,1} \\ & a & & a_{0\theta'} & \\ a_\infty & & & & a_0 \end{matrix}; \tfrac{1}{t-1}, 1-z\right)\,. \tag{2.2.41}$$

All other connection formulae at $t \gg 1$ can be obtained similarly. The same can be done when $t \sim 1$. Note that again the semiclassical BPZ equation looks formally as (2.2.28) upon the substitution[7] of $u^{(0)}$ with

$$u^{(1)} = \lim_{b\to 0} b^2 t\partial_t \log \mathfrak{F}\left(\begin{matrix} \alpha_0 & & \alpha_t \\ & \alpha & \\ \alpha_\infty & & \alpha_1 \end{matrix}; 1-t\right)\,. \tag{2.2.42}$$

[6] From the gauge theory viewpoint this amounts to a change of frame from the electric to the monopole one.

[7] This is the dyon frame.

2.2.2 Confluent conformal blocks

General case

Consider the correlation function

$$\langle \mu, \Lambda | V_1(1) \Phi(z) | \Delta_0 \rangle \,. \tag{2.2.43}$$

It solves the BPZ equation

$$\left(b^{-2}\partial_z^2 - \left(\frac{1}{z} + \frac{1}{z-1}\right)\partial_z + \frac{\Lambda\partial_\Lambda - \Delta_{2,1} - \Delta_1 - \Delta_0}{z(z-1)} + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_0}{z^2} + \frac{\mu\Lambda}{z} - \frac{\Lambda^2}{4} \right) \langle \mu, \Lambda | \Phi(z) V_1(1) | \Delta_0 \rangle = 0 \,, \tag{2.2.44}$$

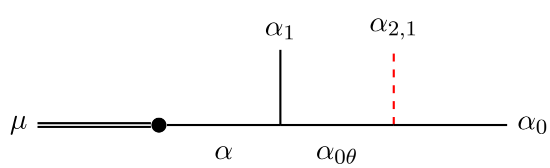
and can be decomposed into *confluent* conformal blocks in different ways. They are all given as collision limits of *regular* conformal blocks.

Small Λ blocks We focus first on the case where the conformal blocks are given as an expansion in Λ. The block for $z \sim 0$ is defined as[8]

$$ {}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right) = \Lambda^{\Delta} z^{\frac{bQ}{2}+\theta b\alpha_0}\,{}_1\widetilde{\mathfrak{F}}\left(\mu\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right) = \Lambda^{\Delta} z^{\frac{bQ}{2}+\theta b\alpha_0}\lim_{\eta\to\infty}\widetilde{\mathfrak{F}}\left({}^{\frac{\eta-\mu}{2}}_{\frac{\eta+\mu}{2}}\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\frac{\Lambda}{\eta},z\right). \tag{2.2.45}$$

This is nothing but the standard collision limit of $\langle \Delta_\infty |$ and $V_t(t)$ as defined in (2.1.11). The tilde on the conformal block means it has no classical part, i.e. is normalized such that the first term is 1. This conformal block can also be computed directly by doing the OPE of $\Phi(z)$ with $|\Delta_0\rangle$, then the OPE of $V_1(1)$ with the result which we specify to be in the Verma module Δ_α, and then contracting with $\langle \mu, \Lambda |$. In the diagrammatic notation introduced in section 2.1.2, we represent it by

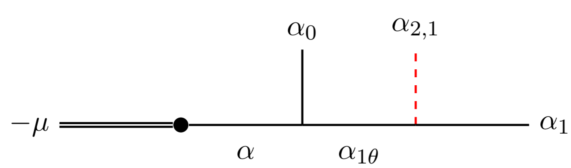
$${}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right) = $$



. (2.2.46)

The double line represents the rank 1 irregular state, and the dot the pairing with a primary state. For $z \sim 1$, the corresponding block can be expressed as

$$e^{\mu\Lambda}\,{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right) = $$

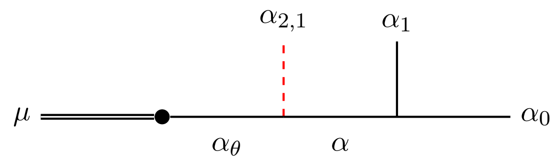


, (2.2.47)

[8]The argument $\frac{\eta+\mu}{2}$ should appear with a minus sign as in Appendix 2.A.2. Here and in the following we don't write it due to the symmetry of the conformal block. The reader wishing to compare with the Nekrasov partition function should take this sign into account as in Appendix 2.C.

where the exponential factor and the argument $-\mu$ arise from the corresponding Möbius transformation[9]. In the intermediate region, where $z \gg 1$ but $\Lambda z \ll 1$, the corresponding block is

$$z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right) =$$



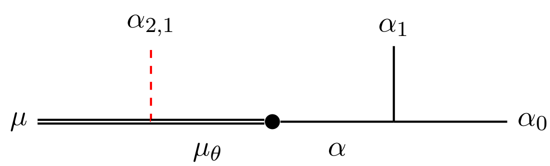
. (2.2.48)

In the deep irregular region where $z \gg 1$ and $\Lambda z \gg 1$, the conformal block is given by a different collision limit, proposed in [104]:

$$\begin{aligned}{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right) &= e^{\theta b\Lambda z/2}\Lambda^{\Delta_{2,1}+\Delta}\left(\Lambda z\right)^{-\theta b\mu+\frac{b^2}{2}}\times\\ \times\lim_{\eta\to\infty}\left(1-\frac{\eta}{\Lambda z}\right)^{-\frac{bQ}{2}-\theta\frac{b}{2}(\mu-\eta)}&\widetilde{\mathfrak{F}}\left({}^{\alpha_1}_{\alpha_0}\ \alpha\ {}^{\frac{\mu-\eta}{2}}\ {}_{\frac{\mu+\eta-\theta b}{2}}\ {}^{\alpha_{2,1}}_{\frac{\mu+\eta}{2}};\frac{\Lambda}{\eta},\frac{\eta}{\Lambda z}\right).\end{aligned} \tag{2.2.49}$$

Whenever z approaches an irregular singularity of rank 1, we denote the corresponding conformal block by $\mathfrak{D}$. This conformal block can also be computed directly by doing the OPE between $\langle\mu,\Lambda|$ and $\Phi(z)$, then the OPE of the result with $V_1(1)$ and contracting with $|\Delta_0\rangle$. Diagramatically, we write

$${}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right) =$$

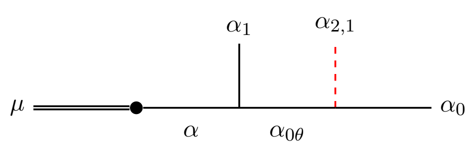


. (2.2.50)

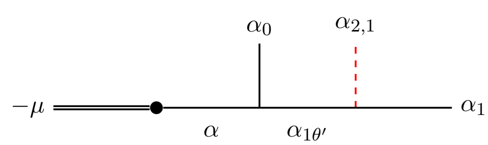
The connection problem between 0 and 1 is solved in the same way as for the regular conformal blocks, since we are never near the irregular singularity. The result is

$${}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_1}\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right) = \sum\nolimits_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_1;b\alpha)e^{\mu\Lambda}\,{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta'}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right). \tag{2.2.51}$$

Diagrammatically:



$= \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}$



. (2.2.52)

[9]Actually, doing the Möbius transformation one gets $-\Lambda$ but since the block depends only on $\mu\Lambda$ and Λ^2 except for the classical part, one can trade $-\Lambda$ for $-\mu$.

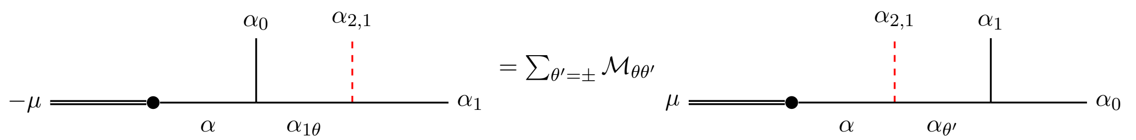
Instead, to solve the connection problem between 1 and ∞ one has to do two steps: from 1 to the intermediate region, and then to ∞. At each step we decompose the correlator into conformal blocks in the different regions and then use crossing symmetry to determine the connection coefficients. The relevant formulae for the irregular state are reviewed in Appendix 2.B.1. We have

$$\begin{aligned}\langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C_{\mu\alpha}\sum_{\theta=\pm} C^{\alpha_{1\theta}}_{\alpha_{2,1}\alpha_1}C^{\alpha}_{\alpha_{1\theta}\alpha_0}\left|e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right)\right|^2 = \\ &= \int d\alpha\, C_{\mu\alpha}\sum_{\theta'=\pm} C^{\alpha}_{\alpha_{2,1}\alpha_{\theta'}}C^{\alpha_{\theta'}}_{\alpha_1\alpha_0}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right)\right|^2 .\end{aligned} \tag{2.2.53}$$

We recognize this condition from the hypergeometric function (2.1.5). Therefore we can readily solve it in terms of the hypergeometric connection coefficients $\mathcal{M}$ and the connection formula between 0 and the intermediate region is then

$$e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta}\ {}^{\alpha_{2,1}}_{\alpha_1};\Lambda,1-z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha;b\alpha_0)z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\tfrac{1}{z}\right). \tag{2.2.54}$$

Diagrammatically:



(2.2.55)

If one decomposes the correlator into conformal blocks in the intermediate region and near ∞, one obtains the crossing symmetry condition

$$\begin{aligned}\langle\mu,\Lambda|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0}\sum_{\theta=\pm} C_{\mu\alpha_\theta}C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\frac{1}{z}\right)\right|^2 = \\ &= \int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0}\sum_{\theta'=\pm} C_{\mu_{\theta'}\alpha}B^{\mu_{\theta'}}_{\alpha_{2,1}\mu}\left|{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\frac{1}{\Lambda z}\right)\right|^2 .\end{aligned} \tag{2.2.56}$$

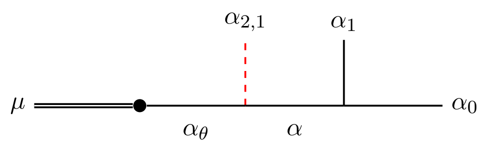
This condition is analogous to the one we found for the Whittaker functions (2.1.21) so that the connection formula between the intermediate region and ∞ reads

$$b^{\theta b\alpha}z^{-\Delta_{2,1}-\Delta_1-\Delta_0}{}_1\mathfrak{F}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda z,\tfrac{1}{z}\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu){}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\ {}^{\alpha_1}_{\alpha_0};\Lambda,\tfrac{1}{\Lambda z}\right) \tag{2.2.57}$$

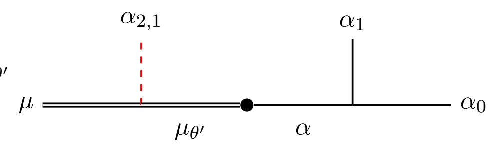
with irregular connection coefficients as in (2.B.18):

$$\mathcal{N}_{\theta\theta'}(b\alpha,b\mu) = \frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)} . \tag{2.2.58}$$

In diagrams:



$$= \sum_{\theta'=\pm} \mathcal{N}_{\theta\theta'}$$

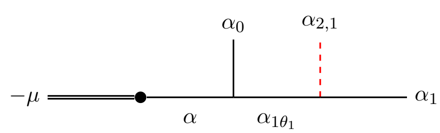


. (2.2.59)

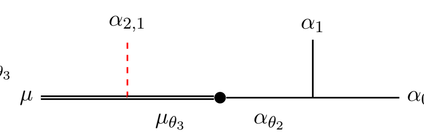
Let us write explicitly the more interesting connection formula between 1 and ∞, which is obtained by concatenating the two connection formulae above. Since the $\mathfrak{F}$ block in the intermediate region has different arguments in formula (2.2.54) and (2.2.57), we need to rename some of them. In the end we obtain the following connection formula from 1 directly to ∞:

$$e^{\mu\Lambda}{}_1\mathfrak{F}\left(-\mu\ \alpha\ {}^{\alpha_0}\ \alpha_{1\theta_1}\ {}^{\alpha_{2,1}}_{\alpha_1}; \Lambda, 1-z\right) =$$
$$= \sum_{\theta_2,\theta_3=\pm} b^{-\frac{1}{2}+\theta_2 b\alpha_{\theta_2}-\theta_3 b\mu} \mathcal{M}_{\theta_1\theta_2}(b\alpha_1, b\alpha; b\alpha_0)\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2}, b\mu)\,{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta_3}\ \alpha_{\theta_2}\ {}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right). \tag{2.2.60}$$

Again, in diagrams this is represented by:



$$= \sum_{\theta_2,\theta_3=\pm} \mathcal{M}_{\theta_1\theta_2}\mathcal{N}_{(-\theta_2)\theta_3}$$

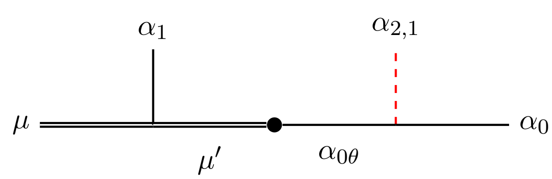


, (2.2.61)

where we have suppressed the arguments of the connection coefficients for brevity.

Large Λ blocks The conformal blocks considered up to now are expansions in Λ. One can however play the same game using expansions in $\frac{1}{\Lambda}$. For example, for large Λ and for $z \sim 0$, we have

$${}_1\mathfrak{D}\left(\mu\ {}^{\alpha_1}\ \mu'\ \alpha_{0\theta}\ {}^{\alpha_{2,1}}_{\alpha_0}; \frac{1}{\Lambda}, \Lambda z\right) =$$



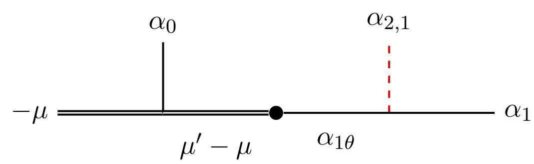
. (2.2.62)

One can compute it via OPE as in (2.B.1) or as a collision limit of a regular conformal block as proposed in [104]:

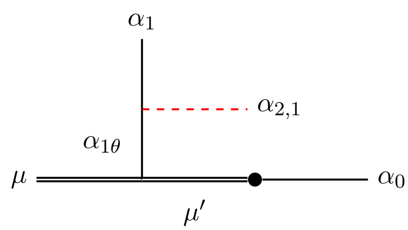
$$
\begin{aligned}
{}_1\mathfrak{D}\left(\mu \overset{\alpha_1}{\ } \mu' \ \alpha_{0\theta} \overset{\alpha_{2,1}}{\underset{\alpha_0}{\ }}; \frac{1}{\Lambda}, \Lambda z\right) =& e^{-(\mu'-\mu)\Lambda}\Lambda^{\Delta_{0\theta}+2\mu'(\mu'-\mu)} z^{\frac{bQ}{2}+\theta b\alpha_0} \times \\
&\times \lim_{\eta\to\infty}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} \widetilde{\mathfrak{F}}\left(\overset{\alpha_1}{\frac{\eta+\mu}{2}} \ \frac{\eta-\mu}{2}+\mu' \overset{\frac{\eta-\mu}{2}}{\ } \alpha_{0\theta} \overset{\alpha_{2,1}}{\underset{\alpha_0}{\ }}; \frac{\eta}{\Lambda}, \frac{\Lambda z}{\eta}\right) .
\end{aligned}
\tag{2.2.63}
$$

Similarly, we have a conformal block for large Λ and $z \sim 1$, which as usual we can write in the same form as the one for $z \sim 0$ by doing a Möbius transformation:

$$
e^{\mu\Lambda}{}_1\mathfrak{D}\left(-\mu \overset{\alpha_0}{\ } \mu'-\mu \ \alpha_{1\theta} \overset{\alpha_{2,1}}{\underset{\alpha_1}{\ }}; \frac{1}{\Lambda}, \Lambda(1-z)\right) =
$$



$=$



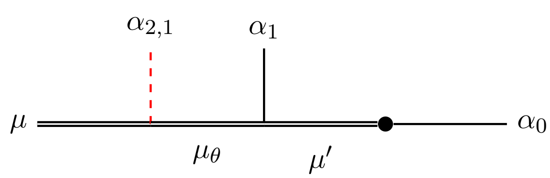
. (2.2.64)

The first line of (2.2.64) is the diagrammatic representation of the conformal block, while the second line is an equality of two a priori seemingly different conformal blocks, which can be checked by explicit computation. This is consistent with the fact that the corresponding DOZZ factors are equal:

$$
B^{\mu'-\mu}_{-\mu\alpha_0} C_{\mu'-\mu,\alpha_{1\theta}} = B^{\mu'}_{\mu\alpha_{1\theta}} C_{\mu',\alpha_0} \, , \tag{2.2.65}
$$

as can easily be proven by using their explicit expressions given in Appendix 2.A.2. The most exotic block is the one for large Λ and large z, which by a slight abuse of notation we still denote by $\mathfrak{D}$:

$$
{}_1\mathfrak{D}\left(\mu \overset{\alpha_{2,1}}{\ } \mu_\theta \overset{\alpha_1}{\ } \mu' \ \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right) =
$$



. (2.2.66)

This block is fully irregular in the sense that to calculate it, we have to perform two irregular OPEs as indicated by the diagram. It is more convenient to calculate it as a collision limit of a regular block:

$$ {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_\theta \;{}^{\alpha_1}\; \mu' \; \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right) = e^{\theta b \Lambda z/2} \Lambda^{\Delta_{2,1}} \left(\Lambda z\right)^{-\theta b\mu + \frac{b^2}{2}} e^{-(\mu'-\mu_\theta)\Lambda} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu_\theta)} \times $$
$$ \times \lim_{\eta\to\infty} \left(1-\frac{\eta}{\Lambda z}\right)^{\Delta_{2,1}-(\mu_\theta-\mu)(\eta-\mu_\theta)} \left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu_\theta)(\eta-\mu')-(\mu'-\mu_\theta)(\mu_\theta-\mu)} \widetilde{\mathfrak{F}}\left(\begin{matrix}\alpha_{2,1} \\ \frac{\eta+\mu}{2}\end{matrix} \;{}_{\frac{\eta-\mu}{2}+\mu_\theta}\; {}^{\alpha_1} \;{}_{\frac{\eta-\mu}{2}+\mu'}\; \begin{matrix}\frac{\eta-\mu}{2} \\ \alpha_0\end{matrix}; \frac{\eta}{\Lambda}, \frac{\Lambda z}{\eta}\right). \tag{2.2.67} $$

Having defined all the necessary conformal blocks we now derive their connection formulae. Let us start by connecting $z \sim 1$ with ∞. Expanding the correlator in these regions, we get the crossing symmetry condition

$$ \langle \mu, \Lambda | \Phi(z) V_1(1) | \Delta_0 \rangle = \int \mathrm{d}\mu' \sum_{\theta=\pm} B^{\mu'-\mu}_{-\mu\alpha_0} C_{\mu'-\mu,\alpha_{1\theta}} C^{\alpha_{1\theta}}_{\alpha_1 \alpha_{2,1}} \left| e^{\mu\Lambda}\, {}_1\mathfrak{D}\left(-\mu \;{}^{\alpha_0}\; \mu'-\mu \; \alpha_{1\theta} \; \begin{matrix}\alpha_{2,1} \\ \alpha_1\end{matrix}; \frac{1}{\Lambda}, \Lambda(1-z)\right)\right|^2 = $$
$$ = \int \mathrm{d}\mu' \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'}_{\mu_{\theta'}\alpha_1} C_{\mu',\alpha_0} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{\theta'} \;{}^{\alpha_1}\; \mu' \; \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 . \tag{2.2.68} $$

Using the following remarkable identity, which can easily be proven using the explicit expression of the structure functions given in Appendix 2.A.2,

$$ B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'}_{\mu_{\theta'}\alpha_1} C_{\mu'\alpha_0} = B^{\mu'-\mu}_{-\mu\alpha_0} B^{\mu'-\mu_{\theta'}}_{\mu'-\mu,\alpha_{2,1}} C_{\mu'-\mu_{\theta'},\alpha_1} \,, \tag{2.2.69} $$

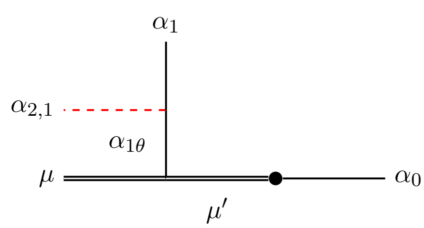
we find that the above crossing symmetry condition (after relabelling the dummy variable $\theta' \to -\theta'$) becomes:

$$ \langle \mu, \Lambda | \Phi(z) V_1(1) | \Delta_0 \rangle = \int \mathrm{d}\mu' B^{\mu'-\mu}_{-\mu\alpha_0} \sum_{\theta=\pm} C_{\mu'-\mu,\alpha_{1\theta}} C^{\alpha_{1\theta}}_{\alpha_1 \alpha_{2,1}} \left| e^{\mu\Lambda}\, {}_1\mathfrak{D}\left(-\mu \;{}^{\alpha_0}\; \mu'-\mu \; \alpha_{1\theta} \; \begin{matrix}\alpha_{2,1} \\ \alpha_1\end{matrix}; \frac{1}{\Lambda}, \Lambda(1-z)\right)\right|^2 = $$
$$ = \int \mathrm{d}\mu' B^{\mu'-\mu}_{-\mu\alpha_0} \sum_{\theta'=\pm} B^{\mu'_{\theta'}-\mu}_{\mu'-\mu,\alpha_{2,1}} C_{\mu'_{\theta'}-\mu,\alpha_1} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{-\theta'} \;{}^{\alpha_1}\; \mu' \; \alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 . \tag{2.2.70} $$

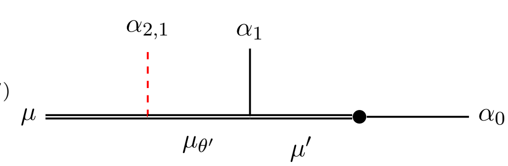
We recognize this constraint from the Whittaker functions (2.1.24), and can readily write the connection formula from 1 to ∞:

$$ b^{\theta b \alpha_1} e^{\mu\Lambda}{}_1\mathfrak{D}\left(-\mu \;{}^{\alpha_0}\; \mu'-\mu \; \alpha_{1\theta} \; \begin{matrix}\alpha_{2,1} \\ \alpha_1\end{matrix}; \tfrac{1}{\Lambda}, \Lambda(1-z)\right) = \textstyle\sum_{\theta'} b^{-\frac{1}{2}+\theta' b(\mu'-\mu)} \mathcal{N}_{\theta(-\theta')}(b\alpha_1, b\mu'-b\mu)\, {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{\theta'} \;{}^{\alpha_1}\; \mu' \; \alpha_0; \tfrac{1}{\Lambda}, \tfrac{1}{z}\right), \tag{2.2.71} $$

where $\mathcal{N}$ are the connection coefficients for the Whittaker functions (2.1.24). Diagrammatically this is clear:



$$ = \textstyle\sum_{\theta'=\pm} \mathcal{N}_{\theta(-\theta')} $$



. (2.2.72)

To connect 0 and ∞ we expand the correlator in the relevant regions. By crossing symmetry we have:

$$\langle \mu, \Lambda | V_1(1)\Phi(z)|\Delta_0\rangle = \int \mathrm{d}\mu' \sum_{\theta=\pm} B^{\mu'}_{\mu\alpha_1} C_{\mu'\alpha_{0\theta}} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_1}\; \mu' \;\alpha_{0\theta}\; {}^{\alpha_{2,1}}_{\alpha_0}; \frac{1}{\Lambda}, \Lambda z\right)\right|^2 =$$
$$= \int \mathrm{d}\mu' \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'_{\theta'}}_{\mu_{\theta'}\alpha_1} C_{\mu'_{\theta'},\alpha_0} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{\theta'} \;{}^{\alpha_1}\; \mu'_{\theta'} \;\alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 , \tag{2.2.73}$$

for later convenience we have labelled the intermediate channel in the second line by $\mu'_{\theta'}$ instead of μ'. By using an identity similar to (2.2.69):

$$B^{\mu_{\theta'}}_{\mu\alpha_{2,1}} B^{\mu'_{\theta'}}_{\mu_{\theta'}\alpha_1} C_{\mu'_{\theta'},\alpha_0} = B^{\mu'}_{\mu\alpha_1} B^{\mu'_{\theta'}}_{\mu'\alpha_{2,1}} C_{\mu'_{\theta'}\alpha_0} , \tag{2.2.74}$$

the above crossing symmetry equation then becomes:

$$\langle \mu, \Lambda | V_1(1)\Phi(z)|\Delta_0\rangle = \int \mathrm{d}\mu' B^{\mu'}_{\mu\alpha_1} \sum_{\theta=\pm} C_{\mu'\alpha_{0\theta}} C^{\alpha_{0\theta}}_{\alpha_{2,1}\alpha_0} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_1}\; \mu' \;\alpha_{0\theta}\; {}^{\alpha_{2,1}}_{\alpha_0}; \frac{1}{\Lambda}, \Lambda z\right)\right|^2 =$$
$$= \int \mathrm{d}\mu' B^{\mu'}_{\mu\alpha_1} \sum_{\theta'=\pm} B^{\mu'_{\theta'}}_{\mu'\alpha_{2,1}} C_{\mu'_{\theta'}\alpha_0} \left| {}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{\theta'} \;{}^{\alpha_1}\; \mu'_{\theta'} \;\alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\right)\right|^2 . \tag{2.2.75}$$

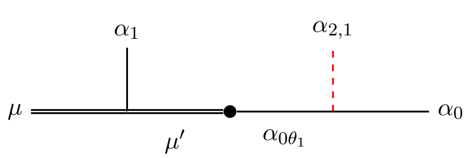
We recognize this constraint from the Whittaker functions (2.1.21) and can readily write the connection formula from 0 to ∞:

$$b^{\theta b\alpha_0}\,{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_1}\; \mu' \;\alpha_{0\theta}\; {}^{\alpha_{2,1}}_{\alpha_0}; \tfrac{1}{\Lambda}, \Lambda z\right) = \textstyle\sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu'} \mathcal{N}_{\theta\theta'}(b\alpha_0, b\mu')\,{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_{2,1}}\; \mu_{\theta'} \;{}^{\alpha_1}\; \mu'_{\theta'} \;\alpha_0; \tfrac{1}{\Lambda}, \tfrac{1}{z}\right). \tag{2.2.76}$$

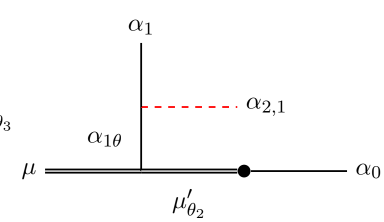
Combining (2.2.76) with the inverse of (2.2.71) we obtain the connection formula from 0 to 1:

$$b^{\theta_1 b\alpha_0}\,{}_1\mathfrak{D}\left(\mu \;{}^{\alpha_1}\; \mu' \;\alpha_{0\theta_1}\; {}^{\alpha_{2,1}}_{\alpha_0}; \frac{1}{\Lambda}, \Lambda z\right) =$$
$$= \sum_{\theta_2,\theta_3=\pm} b^{-\frac{1}{2}-\theta_2 b\mu'} \mathcal{N}_{\theta_1\theta_2}(b\alpha_0, b\mu') b^{\frac{1}{2}-\theta_2 b(\mu'_{\theta_2}-\mu)+\theta_3 b\alpha_1} \mathcal{N}^{-1}_{(-\theta_2)\theta_3}(b\mu'_{\theta_2}-b\mu, b\alpha_1) e^{\mu\Lambda}\,{}_1\mathfrak{D}\left(-\mu \;{}^{\alpha_0}\; \mu'_{\theta_2}-\mu \;\alpha_{1\theta_3}\; {}^{\alpha_{2,1}}_{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\right). \tag{2.2.77}$$

Diagrammatically:



$= \sum_{\theta_2,\theta_3=\pm} \mathcal{N}_{\theta_1\theta_2} \mathcal{N}^{-1}_{(-\theta_2)\theta_3}$



. (2.2.78)

One might expect the existence of conformal blocks expanded in an intermediate region, as was the case for small Λ. Indeed, in the case of large Λ one can define a block expanded in the intermediate region $\frac{1}{\Lambda} \ll z \ll 1$. However, by the identity (2.2.74), this block is actually the same as the block (2.2.66) corresponding to $z \sim \infty$, in the sense that the analytic continuation between the two is trivial. Similarly, one can define another intermediate block in the region $\frac{1}{\Lambda} \ll 1 - z \ll 1$ which is also the same as (2.2.66) by virtue of the identity (2.2.69).

Semiclassical limit

In the semiclassical limit $b \to 0$ and $\alpha_i, \mu, \Lambda \to \infty$ such that $a_i = b\alpha_i$, $m = b\mu$, $L = b\Lambda$ are finite. We denote the quantities which are finite in the semiclassical limit by latin letters instead of greek ones.

Small L blocks The conformal blocks in this limit are expected to exponentiate, and the z-dependence becomes subleading: schematically they take the form

$$\mathfrak{F}(\Lambda, z) \sim e^{\frac{1}{b^2}F(L)+W(L,z)+\mathcal{O}(b^2)}\,, \tag{2.2.79}$$

and they diverge in this limit. The classical conformal block $F(L)$ is related to the conformal block $\mathfrak{F}$ without the degenerate field insertion, i.e.

$${}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right) = \Lambda^{\Delta} e^{\frac{1}{b^2}\left(F(L)+\mathcal{O}(b^2)\right)}\,. \tag{2.2.80}$$

Normalizing by this block, we obtain finite semiclassical conformal blocks. Consider for concreteness the block corresponding to the expansion for $z \sim 0$. We define the corresponding (finite) semiclassical conformal block by

$${}_1\mathcal{F}\left(m\;a\;{}^{a_1}\;a_{0\theta}\;{}^{a_{2,1}}_{a_0};L,z\right) = \lim_{b\to 0} \frac{{}_1\mathfrak{F}\left(\mu\;\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;{}^{\alpha_{2,1}}_{\alpha_0};\Lambda,z\right)}{{}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right)} = e^{-\frac{\theta}{2}\partial_{a_0}F} z^{\frac{1}{2}+\theta a_0}\left(1+\mathcal{O}(L,z)\right). \tag{2.2.81}$$

The term $\exp -\frac{\theta}{2}\partial_{a_0}F$ on the RHS of the above equation comes from the fact that the leading behaviour of the numerator is $\exp b^{-2}F(a_{0\theta})$ while the denominator behaves as $\exp b^{-2}F(a_0)$. The fact that the z-dependence is subleading means that to leading order, the Λ-derivative in the BPZ equation (2.2.44) becomes z-independent, since we have $\Lambda\partial_\Lambda\mathfrak{F}(\Lambda,z) \sim b^{-2}\Lambda\partial_\Lambda F(\Lambda)\mathfrak{F}(\Lambda,z)$. Then the BPZ equation in the semiclassical limit reduces to an ODE. In particular, multiplying (2.2.44) by b^2, this semiclassical conformal block now satisfies the equation

$$\left(\partial_z^2 + \frac{u - \frac{1}{2} + a_0^2 + a_1^2}{z(z-1)} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{mL}{z} - \frac{L^2}{4}\right) {}_1\mathcal{F}\left(m\; a^{\;a_1}\; a_{0\theta}\; {}^{a_{2,1}}_{a_0}; L, z\right) = 0\,. \tag{2.2.82}$$

We have introduced

$$u = \lim_{b\to 0} b^2 \Lambda \partial_\Lambda \log {}_1\mathfrak{F}\left(\mu\, \alpha\, {}^{\alpha_1}_{\alpha_0}; \Lambda\right) = \frac{1}{4} - a^2 + \mathcal{O}(L) \tag{2.2.83}$$

Similarly, we define the semiclassical block for $z \sim 1$ to be

$$\begin{aligned} &{}_1\mathcal{F}\left(-m\; a^{\;a_0}\; a_{1\theta}\; {}^{a_{2,1}}_{a_1}; L, 1-z\right) = \lim_{b\to 0} \frac{e^{\mu\Lambda}\, {}_1\mathfrak{F}\left(-\mu\; \alpha^{\;\alpha_0}\; \alpha_{1\theta}\; {}^{\alpha_{2,1}}_{\alpha_1}; \Lambda, 1-z\right)}{{}_1\mathfrak{F}\left(\mu\, \alpha\, {}^{\alpha_1}_{\alpha_0}; \Lambda\right)} = \\ &= \lim_{b\to 0} \frac{{}_1\mathfrak{F}\left(-\mu\; \alpha^{\;\alpha_0}\; \alpha_{1\theta}\; {}^{\alpha_{2,1}}_{\alpha_1}; \Lambda, 1-z\right)}{{}_1\mathfrak{F}\left(-\mu\, \alpha\, {}^{\alpha_0}_{\alpha_1}; \Lambda\right)} = e^{-\frac{\theta}{2}\partial_{a_1} F} (1-z)^{\frac{1}{2}+\theta a_1} \left(1 + \mathcal{O}(L, 1-z)\right), \end{aligned} \tag{2.2.84}$$

and in the deep irregular region:

$$ {}_1\mathcal{D}\left(m^{\;a_{2,1}}\; m_\theta\; a\; {}^{a_1}_{a_0}; L, \tfrac{1}{Lz}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta m} \frac{{}_1\mathfrak{D}\left(\mu^{\;\alpha_{2,1}}\; \mu_\theta\; \alpha\; {}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right)}{{}_1\mathfrak{F}\left(\mu\, \alpha\, {}^{\alpha_1}_{\alpha_0}; \Lambda\right)} = e^{-\frac{\theta}{2}\partial_m F} e^{\theta L z/2} L^{-\frac{1}{2}-\theta m} z^{-\theta m} \left(1 + \mathcal{O}(L, 1/Lz)\right). \tag{2.2.85}$$

The explicit power of b is needed to combine with Λ to form L. All these blocks satisfy the same equation (2.2.82). Note that in the connection formula (2.2.60) we have four different conformal blocks on the right hand side. Since in the semiclassical limit the BPZ equation becomes a second-order ODE, these four different blocks have to reduce to the two linearly independent solutions near the irregular singular point. They are given by

$$ {}_1\mathfrak{D}\left(\mu^{\;\alpha_{2,1}}\; \mu_\theta\; \alpha\; {}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right) = e^{\theta b \Lambda z/2} \Lambda^{\Delta_{2,1}+\Delta} (\Lambda z)^{-\theta b \mu + \frac{b^2}{2}} e^{\frac{1}{b^2} F(a) + W(a) + \mathcal{O}(b^2)}\,, \tag{2.2.86}$$

where we have suppressed the dependence of F and W on the other parameters. Instead, in (2.2.60) we have

$$ {}_1\mathfrak{D}\left(\mu^{\;\alpha_{2,1}}\; \mu_\theta\; \alpha_{\theta'}\; {}^{\alpha_1}_{\alpha_0}; \Lambda, \frac{1}{\Lambda z}\right) = e^{\theta b \Lambda z/2} \Lambda^{\Delta_{2,1}+\Delta_{\theta'}} (\Lambda z)^{-\theta b \mu + \frac{b^2}{2}} e^{\frac{1}{b^2} F(a_{\theta'}) + W(a_{\theta'}) + \mathcal{O}(b^2)}\,. \tag{2.2.87}$$

Since we are taking the limit $b \to 0$, we can safely substitute $W(a_{\theta'}) \to W(a)$. This is not true for $F(a_{\theta'})$ however, since it multiplies a pole in b^2. Instead, in the semiclassical limit we have

$$ {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha_{\theta'}\ \begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda,\frac{1}{\Lambda z}\right) \sim \Lambda^{\theta' a} e^{-\frac{\theta'}{2}\partial_a F(a)}\, {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_\theta\ \alpha\ \begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda,\frac{1}{\Lambda z}\right), \quad \text{as}\, b \to 0\,, \tag{2.2.88} $$

as in (2.2.34). Therefore, we can simplify the connection formula from 1 to ∞ (2.2.60) in the semiclassical limit and state it as

$$ {}_1\mathcal{F}\left(-m\ a\ {}^{a_0}\ a_{1\theta}\ \begin{matrix}a_{2,1}\\ a_1\end{matrix};L,1-z\right) = \sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{N}_{(-\sigma)\theta'}(a,m)L^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right) {}_1\mathcal{D}\left(m\ {}^{a_{2,1}}\ m_{\theta'}\ a\ \begin{matrix}a_1\\ a_0\end{matrix};L,\tfrac{1}{Lz}\right), \tag{2.2.89} $$

with connection coefficients

$$ \sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{N}_{(-\sigma)\theta'}(a,m)L^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F} = \sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_1)e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m-\sigma a\right)}L^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}{\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a-a_0\right)\Gamma\left(\frac{1}{2}-\sigma a-\theta' m\right)}\,. \tag{2.2.90} $$

Note that all the powers of b appearing in (2.2.60) have been absorbed to give finite quantities.[10]

The connection formula from 0 to 1 trivially reduces to the semiclassical one:

$$ {}_1\mathcal{F}\left(m\ a\ {}^{a_1}\ a_{0\theta}\ \begin{matrix}a_{2,1}\\ a_0\end{matrix};L,z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(a_0,a_1;a)\,{}_1\mathcal{F}\left(-m\ a\ {}^{a_0}\ a_{1\theta'}\ \begin{matrix}a_{2,1}\\ a_1\end{matrix};L,1-z\right)\,. \tag{2.2.91} $$

Large L blocks For the conformal blocks valid for large Λ, the story is analogous. Taking the semiclassical limit, the conformal blocks are expected to exponentiate and the z-dependence becomes subleading. Schematically we have

$$ \mathfrak{D}(\Lambda^{-1},z) \sim e^{\frac{1}{b^2}F_D(L^{-1})+W_D(L^{-1},z)+\mathcal{O}(b^2)}\,. \tag{2.2.92} $$

Here F_D is the classical conformal block for large [11] Λ and is related to the conformal block without the degenerate field insertion, i.e.

$$ {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_1}\ \mu'\ \alpha_0;\frac{1}{\Lambda}\right) = e^{-(\mu'-\mu)\Lambda}\Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)}e^{\frac{1}{b^2}(F_D(L^{-1})+\mathcal{O}(b^2))}\,. \tag{2.2.93} $$

We use this block as a normalization for large Λ. For $z \sim 0$ we have

$$ {}_1\mathcal{D}\left(m\ {}^{a_1}\ m'\ a_{0\theta}\ \begin{matrix}a_{2,1}\\ a_0\end{matrix};\tfrac{1}{L},Lz\right) = \lim_{b\to 0} b^{\theta a_0}\frac{{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_1}\ \mu'\ \alpha_{0\theta}\ \begin{matrix}\alpha_{2,1}\\ \alpha_0\end{matrix};\frac{1}{\Lambda},\Lambda z\right)}{{}_1\mathfrak{D}\left(\mu\ {}^{\alpha_1}\ \mu'\ \alpha_0;\frac{1}{\Lambda}\right)} = L^{\theta a_0}e^{-\frac{\theta}{2}\partial_{a_0}F_D}z^{\frac{1}{2}+\theta a_0}(1+\mathcal{O}(L^{-1},Lz))\,. \tag{2.2.94} $$

[10] Note also that the Gamma functions in the denominator precisely correspond to the one-loop factors of the three hypermultiplets of the corresponding AGT dual gauge theory.

[11] As the notation suggests, it is nothing else but the dual prepotential of the gauge theory.

This block and all the other large-L blocks defined in the following satisfy the same equation (2.2.82) as the small-L blocks, with the substitution

$$u \to u_D = \lim_{b\to 0} b^2 \Lambda \partial_\Lambda \log {}_1\mathfrak{D}\left(\mu \,{}^{\alpha_1}\, \mu' \,\alpha_0; \frac{1}{\Lambda}\right) . \tag{2.2.95}$$

For $z \sim 1$ we have the block

$$\begin{aligned}{}_1\mathcal{D}\Big(-m^{a_0}\; m'-m\; a_{1\theta}\; {}^{a_{2,1}}_{a_1}; \frac{1}{L}, L(1-z)\Big) &= \lim_{b\to 0} b^{\theta a_1} \frac{e^{\mu\Lambda}{}_1\mathfrak{D}\Big(-\mu\;{}^{\alpha_0}\;\mu'-\mu\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\Big)}{{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_1}\,\mu'\,\alpha_0; \frac{1}{\Lambda}\right)} = \\ = \lim_{b\to 0} \frac{{}_1\mathfrak{D}\Big(-\mu\;{}^{\alpha_0}\;\mu'-\mu\;\alpha_{1\theta}\;{}^{\alpha_{2,1}}_{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\Big)}{{}_1\mathfrak{D}\left(-\mu^{\alpha_0}\,\mu'-\mu\,\alpha_1; \frac{1}{\Lambda}\right)} &= L^{\theta a_1} e^{-\frac{\theta}{2}\partial_{a_1} F_D} (1-z)^{\frac{1}{2}+\theta a_1}\left(1+\mathcal{O}(L^{-1}, L(1-z))\right),\end{aligned} \tag{2.2.96}$$

and for $z \sim \infty$:

$$\begin{aligned}{}_1\mathcal{D}\Big(m\;{}^{a_{2,1}}\; m_\theta\;{}^{a_1}\; m'\; a_0; \frac{1}{L}, \frac{1}{z}\Big) &= \lim_{b\to 0} b^{-\frac{1}{2}+\theta(m'-m)} \frac{{}_1\mathfrak{D}\Big(\mu\;{}^{\alpha_{2,1}}\;\mu_\theta\;{}^{\alpha_1}\;\mu'\;\alpha_0; \frac{1}{\Lambda}, \frac{1}{z}\Big)}{{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_1}\,\mu'\,\alpha_0; \frac{1}{\Lambda}\right)} = \\ &= e^{\theta L z/2} e^{-\theta L/2} e^{-\frac{\theta}{2}\partial_m F_D} L^{-\frac{1}{2}+\theta(m'-m)} z^{-\theta m}\left(1+\mathcal{O}(L^{-1}, z^{-1})\right).\end{aligned} \tag{2.2.97}$$

In the connection formula from 0 to 1 for large Λ (2.2.77), there appear four different conformal blocks on the right hand side. In the semiclassical limit these four reduce to two, by the same argument as for small Λ. Indeed we have

$$\begin{aligned}&e^{\mu\Lambda}{}_1\mathfrak{D}\Big(-\mu\;{}^{\alpha_0}\;\mu'_{\theta_2}-\mu\;\alpha_{1\theta_3}\;{}^{\alpha_{2,1}}_{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\Big) = e^{-(\mu'_{\theta_2}-\mu)\Lambda}\Lambda^{\Delta_{1\theta_3}+2\mu'_{\theta_2}(\mu'_{\theta_2}-\mu)}(1-z)^{\frac{bQ}{2}+\theta b\alpha_1} e^{\frac{1}{b^2}F_D(\mu'_{\theta_2})+W_D(\mu'_{\theta_2})} \\ &\sim e^{\theta_2 L/2}\Lambda^{-\theta_2(2m'-m)} e^{-\frac{\theta_2}{2}\partial_{m'}F_D(m')} e^{\mu\Lambda}{}_1\mathfrak{D}\Big(-\mu\;{}^{\alpha_0}\;\mu'-\mu\;\alpha_{1\theta_3}\;{}^{\alpha_{2,1}}_{\alpha_1}; \frac{1}{\Lambda}, \Lambda(1-z)\Big), \quad \text{as } b\to 0 .\end{aligned} \tag{2.2.98}$$

The connection formula (2.2.77) from 0 to 1 in the semiclassical limit then becomes

$$\begin{aligned}&{}_1\mathcal{D}\Big(m^{a_1}\; m'\; a_{0\theta}\;{}^{a_{2,1}}_{a_0}; \frac{1}{L}, Lz\Big) = \\ &= \sum_{\theta'=\pm}\left(\sum_{\sigma=\pm}\mathcal{N}_{\theta\sigma}(a_0, m')\mathcal{N}^{-1}_{(-\sigma)\theta'}(m'-m, a_1) e^{\frac{\sigma}{2}L} L^{-\sigma(2m'-m)} e^{-\frac{\sigma}{2}\partial_{m'}F_D(m')}\right){}_1\mathcal{D}\Big(-m^{a_0}\; m'-m\; a_{1\theta'}\;{}^{a_{2,1}}_{a_1}; \frac{1}{L}, L(1-z)\Big),\end{aligned} \tag{2.2.99}$$

where explicitly the connection coefficients read:

$$\sum_{\sigma=\pm}\mathcal{N}_{\theta\sigma}(a_0,m')\mathcal{N}^{-1}_{(-\sigma)\theta'}(m'-m,a_1)e^{\frac{\sigma}{2}L}L^{-\sigma(2m'-m)}e^{-\frac{\sigma}{2}\partial_{m'}F_D(m')}=$$

$$=\sum_{\sigma=\pm}\frac{\Gamma(1+2\theta a_0)\Gamma(-2\theta' a_1)e^{\frac{\sigma}{2}L}L^{-\sigma(2m'-m)}e^{-\frac{\sigma}{2}\partial_{m'}F_D(m')}e^{i\pi\left(\frac{1-\sigma}{2}\right)(\theta a_0-\theta' a_1-2m'+m)}}{\Gamma\left(\frac{1}{2}+\theta a_0-\sigma m'\right)\Gamma\left(\frac{1}{2}-\theta' a_1-\sigma(m'-m)\right)}\,. \tag{2.2.100}$$

Again, all the spurious powers of b and Λ have beautifully recombined to give the finite combination L.
The connection formula from 1 to ∞ (2.2.71) on the other hand becomes

$${}_1\mathcal{D}\Big(-m^{a_0}\; m'-m\; a_{1\theta}\; \overset{a_{2,1}}{a_1};\tfrac{1}{L},L(1-z)\Big)=\textstyle\sum_{\theta'=\pm}\mathcal{N}_{\theta(-\theta')}(a_1,m'-m)\,{}_1\mathcal{D}\Big(m\;{}^{a_{2,1}}\;m_{\theta'}\;{}^{a_1}\;m'\;a_0;\tfrac{1}{L},\tfrac{1}{z}\Big)\,, \tag{2.2.101}$$

where $\mathcal{N}$ is:

$$\mathcal{N}_{\theta(-\theta')}(a_1,m'-m,)=\frac{\Gamma(1+2\theta a_1)}{\Gamma\left(\frac{1}{2}+\theta a_1+\theta'(m'-m)\right)}e^{i\pi\left(\frac{1+\theta'}{2}\right)\left(\frac{1}{2}-(m'-m)+\theta a_1\right)}\,. \tag{2.2.102}$$

2.2.3 Reduced confluent conformal blocks

General case

Consider the correlation function

$$\langle\Lambda^2|V_1(1)\Phi(z)|\Delta_0\rangle\,, \tag{2.2.103}$$

which solves the BPZ equation

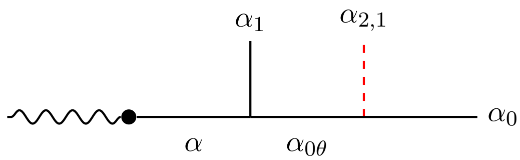
$$\left(b^{-2}\partial_z^2-\left(\frac{1}{z}+\frac{1}{z-1}\right)\partial_z+\frac{\Lambda^2\partial_{\Lambda^2}-\Delta_{2,1}-\Delta_1-\Delta_0}{z(z-1)}+\frac{\Delta_1}{(z-1)^2}+\frac{\Delta_0}{z^2}-\frac{\Lambda^2}{4z}\right)\langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle=0\,. \tag{2.2.104}$$

We can decompose it into irregular conformal blocks in different ways. The blocks corresponding to the expansion of z around a regular singular point can be given as a further decoupling limit of the confluent conformal blocks. For the blocks corresponding to the expansion of z around the irregular singular point of rank 1/2, no closed form expression presently known to us. The block for $z\sim 0$ can be defined as

$${}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;\overset{\alpha_{2,1}}{\alpha_0};\Lambda^2,z\right)=\lim_{\eta\to\infty}(4\eta)^{\Delta}\,{}_1\mathfrak{F}\left(-\eta\;\alpha\;{}^{\alpha_1}\;\alpha_{0\theta}\;\overset{\alpha_{2,1}}{\alpha_0};\frac{\Lambda^2}{4\eta},z\right)\,. \tag{2.2.105}$$

We multiply by the factor of $(4\eta)^{\Delta}$ to take care of the leading divergence in the limit. In the diagrammatic notation of section 2.1.3, we represent it by

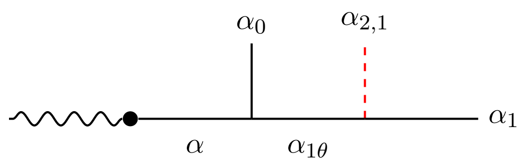
$$ {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \,{}^{\alpha_1}\, \alpha_{0\theta} \,{}^{\alpha_{2,1}}_{\alpha_0}; \Lambda^2, z\right) = $$



. (2.2.106)

As indicated by the diagram, all OPEs are regular in this case. The wiggly line represents the rank 1/2 irregular state, and the dot the pairing with a primary. The block for $z \sim 1$ is then simply

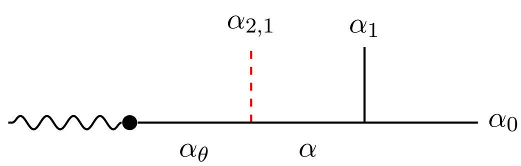
$$ e^{i\pi\Delta} e^{\frac{\Lambda^2}{4}} {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \,{}^{\alpha_0}\, \alpha_{1\theta} \,{}^{\alpha_{2,1}}_{\alpha_1}; e^{-i\pi}\Lambda^2, 1-z\right) = $$



. (2.2.107)

The overall phase compensates the sign in $e^{-i\pi}\Lambda^2$ such that the classical part is still $\Lambda^{2\Delta}$. In the intermediate region where $1 \ll z \ll \frac{1}{\Lambda^2}$ the corresponding block is

$$ z^{-\Delta_{2,1}-\Delta_1-\Delta_0} {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta \,{}^{\alpha_{2,1}}\, \alpha \,{}^{\alpha_1}_{\alpha_0}; \Lambda^2 z, \frac{1}{z}\right) = $$



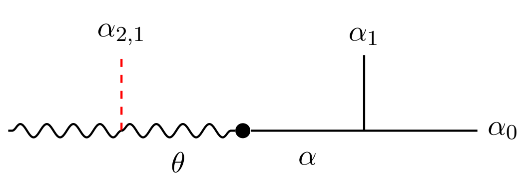
. (2.2.108)

Instead, in the deep irregular region, where $z \gg \frac{1}{\Lambda^2} \gg 1$, a decoupling limit of the form (2.1.27) does not work. Of course one can still calculate this block by solving the BPZ equation iteratively with a series Ansatz, or directly using the Ward identities determining the descendants of the OPE with the irregular state (see Appendix 2.B.1). In any case we will denote the conformal block in this region by

$$ {}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\ \alpha \,{}^{\alpha_1}_{\alpha_0}; \Lambda^2, \frac{1}{\Lambda\sqrt{z}}\right) \sim (\Lambda^2)^{\Delta_{2,1}+\Delta}(\Lambda\sqrt{z})^{\frac{1}{2}+b^2} e^{\theta b \Lambda\sqrt{z}} \left[1 + \mathcal{O}\left(\Lambda^2, \frac{1}{\Lambda\sqrt{z}}\right)\right]. \tag{2.2.109} $$

The $\sim$ refers to the fact that this expansion is asymptotic. In diagrams we represent this block by

$$ {}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\ \alpha \,{}^{\alpha_1}_{\alpha_0}; \Lambda^2, \frac{1}{\Lambda\sqrt{z}}\right) = $$



. (2.2.110)

The solution of the connection problems goes in the same way as for the (unreduced) confluent Heun equation (section 2.2.2). In particular the connection problem between 0 and 1 works in the same way as for the general Heun equation. We have

$$ {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_1}\,\alpha_{0\theta}\,{}^{\alpha_{2,1}}_{\alpha_0};\Lambda^2,z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_0,b\alpha_1;b\alpha)e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_0}\,\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right). \tag{2.2.111} $$

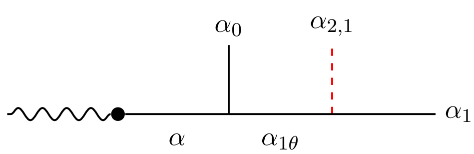
To solve the connection problem between 1 and ∞ one has to do two steps: from 1 to the intermediate region, and then to ∞. In each step we decompose the correlator into conformal blocks in the different regions and then use crossing symmetry to determine the connection coefficients. The relevant formulae for the rank 1/2 irregular state are reviewed in Appendix 2.B.2. We have

$$ \begin{aligned} \langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C_\alpha \sum_{\theta=\pm} C^{\alpha_{1\theta}}_{\alpha_{2,1}\alpha_1} C^{\alpha}_{\alpha_{1\theta}\alpha_0}\left|e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_0}\,\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)\right|^2 = \\ &= \int d\alpha\, C_\alpha \sum_{\theta'=\pm} C^{\alpha}_{\alpha_{2,1}\alpha_{\theta'}} C^{\alpha_{\theta'}}_{\alpha_1\alpha_0}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_{2,1}}\,\alpha_{\theta'}\,{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\frac{1}{z}\right)\right|^2. \end{aligned} \tag{2.2.112} $$

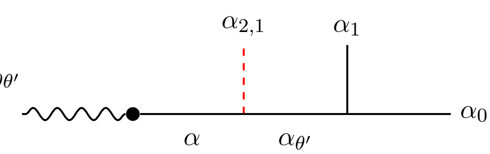
This is precisely the same condition as for the hypergeometric functions (2.1.5). The connection formula between 1 and the intermediate region is then

$$ e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_0}\,\alpha_{1\theta}\,{}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right) = \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(b\alpha_1,b\alpha;b\alpha_0)z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,{}^{\alpha_{2,1}}\,\alpha_{\theta'}\,{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\tfrac{1}{z}\right). \tag{2.2.113} $$

Diagrammatically:



$= \sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}$



. (2.2.114)

Now we decompose the correlator into conformal blocks in the intermediate region and near ∞, obtaining the crossing symmetry condition

$$ \begin{aligned} \langle\Lambda^2|\Phi(z)V_1(1)|\Delta_0\rangle &= \int d\alpha\, C^{\alpha}_{\alpha_1,\alpha_0} \sum_{\theta=\pm} C_{\alpha_\theta} C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left|z^{-\Delta_{2,1}-\Delta_1-\Delta_0}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,{}^{\alpha_{2,1}}\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda^2 z,\frac{1}{z}\right)\right|^2 = \\ &= \int d\alpha\, C^{\alpha}_{\alpha_1\alpha_0} \sum_{\theta'=\pm} C_\alpha B_{\alpha_{2,1}}\left|{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\ \alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)\right|^2. \end{aligned} \tag{2.2.115} $$

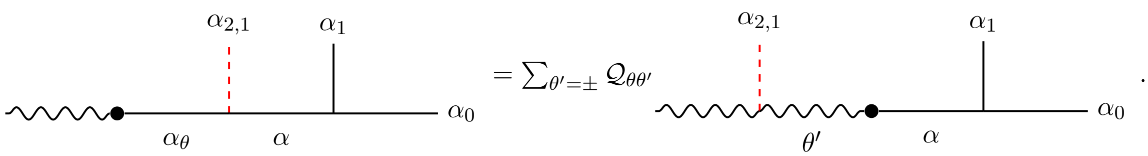
We recognize this condition from the Bessel functions (2.1.36). We then immediately find the connection formula between the intermediate region and ∞:

$$b^{2\theta b\alpha} z^{-\Delta_{2,1}-\Delta_1-\Delta_0} {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta \, {}^{\alpha_{2,1}} \, \alpha \, {}^{\alpha_1}_{\alpha_0}; \Lambda^2 z, \tfrac{1}{z}\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}} \mathcal{Q}_{\theta\theta'}(b\alpha) \, {}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1} \, \alpha \, {}^{\alpha_1}_{\alpha_0}; \Lambda^2, \tfrac{1}{\Lambda\sqrt{z}}\right) \tag{2.2.116}$$

with irregular connection coefficients as in (2.B.36):

$$\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}} \Gamma(1+2\theta b\alpha) e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)} . \tag{2.2.117}$$

In diagrams:

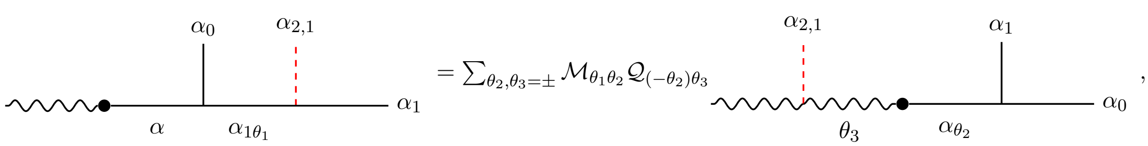


(2.2.118)

Let us write explicitly the more interesting connection formulae between 1 and ∞, which is obtained by concatenating the two connection formulae above. Since the $\mathfrak{F}$ block in the intermediate region has different arguments in formula (2.2.113) and (2.2.116), we need to rename some arguments. In the end we obtain the following connection formula from 1 directly to ∞:

$$\begin{aligned} &e^{i\pi\Delta} e^{\frac{\Lambda^2}{4}} {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_0} \, \alpha_{1\theta_1} \, {}^{\alpha_{2,1}}_{\alpha_1}; e^{-i\pi}\Lambda^2, 1-z\right) = \\ &= \sum_{\theta_2,\theta_3=\pm} \mathcal{M}_{\theta_1\theta_2}(b\alpha_1, b\alpha; b\alpha_0) \mathcal{Q}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2}) b^{-\frac{1}{2}+\theta_2 b\alpha_{\theta_2}} {}_{\frac{1}{2}}\mathfrak{E}^{(\theta_3)}\left(\alpha_{2,1} \, \alpha_{\theta_2} \, {}^{\alpha_1}_{\alpha_0}; \Lambda^2, \frac{1}{\Lambda\sqrt{z}}\right) . \end{aligned} \tag{2.2.119}$$

Diagrammatically we have



(2.2.120)

where we have suppressed the arguments of the connection coefficients for brevity.

Semiclassical limit

The story works the same way here as for the confluent case. In the semiclassical limit the BPZ equation becomes

$$\left(\partial_z^2 + \frac{u - \frac{1}{2} + a_1^2 + a_0^2}{z(z-1)} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_0^2}{z^2} - \frac{L^2}{4z}\right) {}_{\frac{1}{2}}\mathfrak{F}(z) = 0 \, , \tag{2.2.121}$$

for *any* semiclassical block. Here u is given by

$$u = \lim_{b\to 0} b^2\Lambda^2\partial_{\Lambda^2}\log {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L^2) \tag{2.2.122}$$

by the same argument as before. The finite semiclassical conformal blocks are defined by normalizing by the same block without the degenerate field insertion, i.e. the semiclassical block for $z \sim 0$ is

$${}_{\frac{1}{2}}\mathcal{F}\left(a \, {}^{a_1} \, a_{0\theta} \, {}^{a_{2,1}}_{a_0};L^2,z\right) = \lim_{b\to 0}\frac{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_1} \, \alpha_{0\theta} \, {}^{\alpha_{2,1}}_{\alpha_0};\Lambda^2,z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_0}_{\alpha_1};\Lambda^2\right)} = e^{-\frac{\theta}{2}\partial_{a_0}F}z^{\frac{1}{2}+\theta a_0}(1+\mathcal{O}(L^2,z))\,. \tag{2.2.123}$$

Here $F = \lim_{b\to 0} b^2 \log\left[\Lambda^{-2\Delta}{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2\right)\right]$.

$${}_{\frac{1}{2}}\mathcal{F}\left(a \, {}^{a_0} \, a_{1\theta} \, {}^{a_{2,1}}_{a_1};-L^2,1-z\right) = \lim_{b\to 0}\frac{e^{i\pi\Delta}e^{\frac{\Lambda^2}{4}}{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_0} \, \alpha_{1\theta} \, {}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2\right)} =$$

$$= \lim_{b\to 0}\frac{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_0} \, \alpha_{1\theta} \, {}^{\alpha_{2,1}}_{\alpha_1};e^{-i\pi}\Lambda^2,1-z\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha \, {}^{\alpha_0}_{\alpha_1};e^{-i\pi}\Lambda^2\right)} = e^{-\frac{\theta}{2}\partial_{a_1}F}(1-z)^{\frac{1}{2}+\theta a_1}(1+\mathcal{O}(L^2,1-z))\,. \tag{2.2.124}$$

In the deep irregular region we define the semiclassical block as

$${}_{\frac{1}{2}}\mathcal{E}^{(\theta)}\left(a_{2,1} \; a \, {}^{a_1}_{a_0};L^2,\tfrac{1}{L\sqrt{z}}\right) = \lim_{b\to 0} b^{-\frac{1}{2}}\frac{{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1} \; \alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda\sqrt{z}}\right)}{{}_{\frac{1}{2}}\mathfrak{F}\left(\mu \, \alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2\right)} = (L\sqrt{z})^{-\frac{1}{2}}e^{\theta L\sqrt{z}}(1+\mathcal{O}(L^2,\tfrac{1}{L\sqrt{z}}))\,. \tag{2.2.125}$$

All these blocks satisfy the same equation (2.2.121). As for the confluent case, in the connection formula between 1 and ∞ we have four different $\mathfrak{E}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1} \; \alpha_{\theta'} \, {}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda z}\right) \sim (\Lambda^2)^{\theta' a}e^{-\frac{\theta'}{2}\partial_a F}{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1} \; \alpha \, {}^{\alpha_1}_{\alpha_0};\Lambda^2,\frac{1}{\Lambda z}\right), \quad \text{as } b\to 0\,, \tag{2.2.126}$$

as in (2.2.34). Now that we have defined the semiclassical conformal blocks, we state the connection formulae. The connection formula from 0 to 1 (2.2.111) reduces trivially in the semiclassical limit to

$$ {}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_1}\,a_{0\theta}\,{}^{a_{2,1}}_{a_0};L^2,z\right)=\sum_{\theta'=\pm}\mathcal{M}_{\theta\theta'}(a_0,a_1;a)\,{}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_0}\,a_{1\theta}\,{}^{a_{2,1}}_{a_1};-L^2,1-z\right). \tag{2.2.127} $$

The connection formula from 1 to ∞ (2.2.119) becomes

$$ {}_{\frac{1}{2}}\mathcal{F}\left(a\,{}^{a_0}\,a_{1\theta}\,{}^{a_{2,1}}_{a_1};-L^2,1-z\right)=\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{Q}_{(-\sigma)\theta'}(a)L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right){}_{\frac{1}{2}}\mathcal{E}^{(\theta')}\left(a_{2,1}\ a\ {}^{a_1}_{a_0};L^2,\frac{1}{L\sqrt{z}}\right), \tag{2.2.128} $$

with connection coefficients[12]

$$ \begin{aligned} &\sum_{\sigma=\pm}\mathcal{M}_{\theta\sigma}(a_1,a;a_0)\mathcal{Q}_{(-\sigma)\theta'}(a)L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}=\\ &=\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\Gamma(1+2\theta a_1)2^{-2\sigma a}L^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-2\sigma a\right)}}{\sqrt{2\pi}\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a+a_0\right)\Gamma\left(\frac{1}{2}+\theta a_1-\sigma a-a_0\right)}. \end{aligned} \tag{2.2.129} $$

2.2.4 Doubly confluent conformal blocks

General case

Via a further collision limit we reach a correlator that solves the BPZ equation

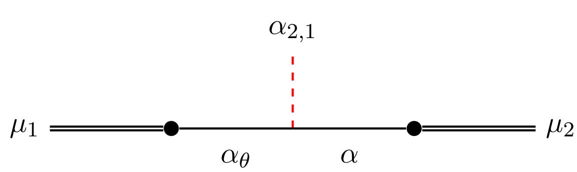
$$ \left(b^{-2}\partial_z^2-\frac{1}{z}\partial_z+\frac{\mu_1\Lambda_1}{z}-\frac{\Lambda_1^2}{4}+\frac{\Lambda_2\partial_{\Lambda_2}}{z^2}+\frac{\mu_2\Lambda_2}{z^3}-\frac{\Lambda_2^2}{4z^4}\right)\langle\mu_1,\Lambda_1|\Phi(z)|\mu_2,\Lambda_2\rangle=0\,. \tag{2.2.130} $$

This correlator can be expanded in the intermediate region $\Lambda_2\ll z\ll\Lambda_1^{-1}$ and near the two irregular singularities, that is either $z\gg\Lambda_1^{-1}\gg1$ or $z\ll\Lambda_2\ll1$. Note that in (2.2.130) one of the three parameters Λ_1,Λ_2,z is redundant. Indeed the conformal blocks will only depend on two ratios. The conformal blocks in these regions can easily be computed as a collision limit. Explicitly, in the intermediate region $\Lambda_2\ll z\ll\Lambda_1^{-1}$

$$ {}_1\mathfrak{F}_1\left(\mu_1\ \alpha_\theta\,{}^{\alpha_{2,1}}\alpha\ \mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)=\Lambda_1^{\Delta_\theta}\Lambda_2^{\Delta}z^{\frac{bQ}{2}+\theta b\alpha}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{F}}\left(\mu_1\ \alpha_\theta\,{}^{\alpha_{2,1}}\alpha\ {}^{\frac{\eta-\mu_2}{2}}_{\frac{\eta+\mu_2}{2}};\Lambda_1 z,\frac{\Lambda_2}{z\eta}\right). \tag{2.2.131} $$

This conformal block is the result of the projection of the Whittaker module $|\mu_2,\Lambda_2\rangle$ on a Verma module Δ and of $\langle\mu_1,\Lambda_1|$ on Δ_θ. We represent this block by the diagram

$$ {}_1\mathfrak{F}_1\left(\mu_1\ \alpha_\theta\,{}^{\alpha_{2,1}}\alpha\ \mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)= $$



. (2.2.132)

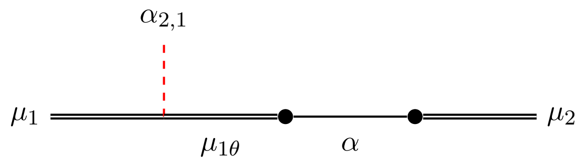
[12] Note that the Gamma functions in the denominator precisely correspond to the one-loop factors of the two hypermultiplets of the corresponding AGT dual gauge theory.

The expansion near the irregular singularity at infinity can be obtained by colliding in (2.2.49) the insertions far from the Whittaker state in the confluent conformal block. This gives

$$ {}_1\mathfrak{D}_1\left(\mu_1 \;{}^{\alpha_{2,1}}\; \mu_{1\theta}\;\alpha\;\mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1 z}\right) = e^{\theta b\Lambda_1 z/2}\Lambda_1^{\Delta+\Delta_{2,1}}\Lambda_2^{\Delta}\left(\Lambda_1 z\right)^{-\theta b\mu_1+\frac{b^2}{2}}\lim_{\eta\to\infty}{}_1\widetilde{\mathfrak{D}}\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta}\;\alpha\;{}^{\frac{\eta-\mu_2}{2}}_{\frac{\eta+\mu_2}{2}};\tfrac{\Lambda_1\Lambda_2}{\eta},\tfrac{1}{\Lambda_1 z}\right). \tag{2.2.133} $$

We represent this block diagrammatically by

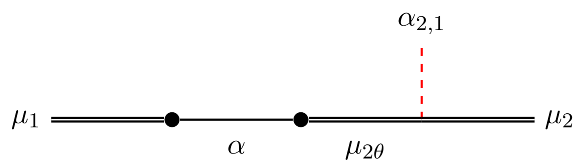
$$ {}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta}\;\alpha\;\mu_2;\Lambda_1\Lambda_2,\frac{1}{\Lambda_1 z}\right) = $$



. (2.2.134)

Finally, the expansion near the irregular singularity at zero is easily obtained from (2.2.133) by exchanging Λ_1 and Λ_2 and sending $z \to 1/z$, up to a Jacobian. The corresponding conformal block is

$$ z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\tfrac{z}{\Lambda_2}\right) = $$



. (2.2.135)

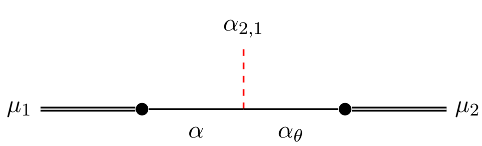
Expanding now the correlator first near 0 and then in the intermediate region, crossing symmetry implies

$$ \begin{aligned} \langle\mu_1,\Lambda_1|\Phi(z)|\mu_2,\Lambda_2\rangle &= \int d\alpha\, G_\alpha^{-1}C_{\mu_1\alpha}G_\alpha^{-1}\sum_{\theta=\pm}B^{\mu_{2\theta}}_{\alpha_{2,1},\mu_2}C_{\mu_{2\theta}\alpha}\left|z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right)\right|^2 = \\ &= \int d\alpha\, G_\alpha^{-1}C_{\mu_1\alpha}\sum_{\theta'=\pm}C^{\alpha_{\theta'}}_{\alpha_{2,1}\alpha}C_{\mu_2\alpha_{\theta'}}\left|{}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;{}^{\alpha_{2,1}}\alpha_{\theta'}\;\mu_2;\Lambda_1 z,\frac{\Lambda_2}{z}\right)\right|^2 . \end{aligned} \tag{2.2.136} $$

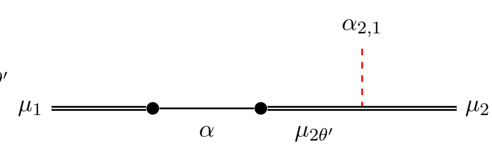
We recognize this condition from (2.1.21), and we can readily write down the solution to the connection problem:

$$ b^{\theta b\alpha}{}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;{}^{\alpha_{2,1}}\alpha_\theta\;\mu_2;\Lambda_1 z,\tfrac{\Lambda_2}{z}\right) = \textstyle\sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu_2}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu_2)z^{-2\Delta_{2,1}}{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta'}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\tfrac{z}{\Lambda_2}\right). \tag{2.2.137} $$

In diagrams:



$= \sum_{\theta'=\pm}\mathcal{N}_{\theta\theta'}$

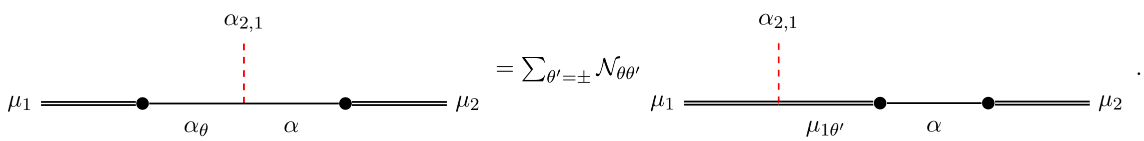


. (2.2.138)

A similar argument works for the connection between the intermediate region and infinity. We obtain

$$b^{\theta b\alpha}{}_1\mathfrak{F}_1\left(\mu_1\ \alpha_\theta\ {}^{\alpha_{2,1}}\alpha\ \mu_2;\Lambda_1 z,\tfrac{\Lambda_2}{z}\right) = \textstyle\sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu_1}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu_1)\,{}_1\mathfrak{D}_1\left(\mu_1\ {}^{\alpha_{2,1}}\ \mu_{1\theta'}\ \alpha\ \mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1 z}\right). \tag{2.2.139}$$

Or, diagrammatically:

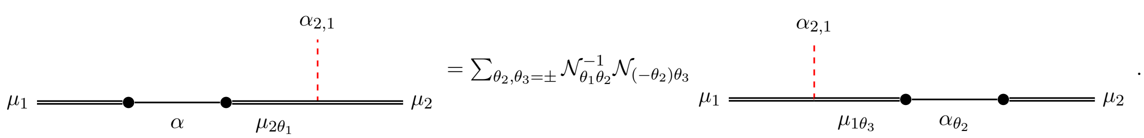


(2.2.140)

Concatenating the previous connection formulae we can connect 0 directly with ∞ as follows

$$b^{-\frac{1}{2}-\theta_1 b\mu_2} z^{-2\Delta_{2,1}}\,{}_1\mathfrak{D}_1\left(\mu_2\ {}^{\alpha_{2,1}}\ \mu_{2\theta_1}\ \alpha\ \mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right) =$$
$$= \sum_{\theta_2,\theta_3=\pm} b^{\theta_2 b\alpha}\mathcal{N}^{-1}_{\theta_1\theta_2}(b\mu_2,b\alpha) b^{-\frac{1}{2}+\theta_2 b\alpha-\theta' b\mu_1}\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2},b\mu_1)\,{}_1\mathfrak{D}_1\left(\mu_1\ {}^{\alpha_{2,1}}\ \mu_{1\theta_3}\ \alpha_{\theta_2}\ \mu_2;\Lambda_1\Lambda_2,\frac{1}{\Lambda_1 z}\right). \tag{2.2.141}$$

In diagrams:



(2.2.142)

Semiclassical limit

Let us now consider the semiclassical limit of the doubly confluent conformal blocks. Once again, the divergence as $b\to 0$ is expected to exponentiate, that is

$$z^{-2\Delta_{2,1}}\,{}_1\mathfrak{D}_1\left(\mu_2\ {}^{\alpha_{2,1}}\ \mu_{2\theta}\ \alpha\ \mu_1;\Lambda_1\Lambda_2,\tfrac{z}{\Lambda_2}\right) = z^{-2\Delta_{2,1}} e^{\frac{\theta b\Lambda_2}{2z}}\Lambda_2^{\Delta+\Delta_{2,1}}\Lambda_1^{\Delta}\left(\tfrac{\Lambda_2}{z}\right)^{-\theta b\mu_2+\frac{b^2}{2}}\exp\left(b^{-2}F\left(L_1L_2\right)+W\left(L_1L_2,zL_2^{-1}\right)\right), \tag{2.2.143}$$

where F is the classical conformal block defined by

$${}_1\mathfrak{F}_1\left(\mu_1\ \alpha\ \mu_2,\Lambda_1\Lambda_2\right) = \left(\Lambda_1\Lambda_2\right)^{\Delta}\exp\left(b^{-2}F+\mathcal{O}(b^0)\right), \tag{2.2.144}$$

and the ${}_1\mathfrak{F}_1$ block is given by

$$\langle\mu_1,\Lambda_1|\mu_2,\Lambda_2\rangle = \int d\alpha\, C_{\mu_1\alpha}C_{\mu_2\alpha}\left|{}_1\mathfrak{F}_1\left(\mu_1\ \alpha\ \mu_2,\Lambda_1\Lambda_2\right)\right|^2 . \tag{2.2.145}$$

We define the semiclassical block near zero to be

$$z\,{}_1\mathcal{D}_1\left(m_2\;{}^{a_{2,1}}\;m_{2\theta}\;a\;m_1;L_1L_2,\tfrac{z}{L_2}\right)=\lim_{b\to0}b^{-\frac{1}{2}-\theta b\mu_2+\frac{b^2}{2}}z^{-2\Delta_{2,1}}\frac{{}_1\mathfrak{D}_1\left(\mu_2\;{}^{\alpha_{2,1}}\;\mu_{2\theta}\;\alpha\;\mu_1;\Lambda_1\Lambda_2,\frac{z}{\Lambda_2}\right)}{{}_1\mathfrak{F}_1(\mu_1\;\alpha\;\mu_2,\Lambda_1\Lambda_2)}\,, \tag{2.2.146}$$

The semiclassical blocks satisfy the equation

$$\left(\partial_z^2+\frac{m_1L_1}{z}-\frac{L_1^2}{4}+\frac{u}{z^2}+\frac{m_2L_2}{z^3}-\frac{L_2^2}{4}\frac{1}{z^4}\right)z\,{}_1\mathcal{D}_1\left(m_2\;{}^{a_{2,1}}\;m_{2\theta}\;a\;m_1;L_1L_2,\frac{z}{L_2}\right)=0\,, \tag{2.2.147}$$

with the u parameter defined as usual to be the leftover of the Λ_2 derivative, that is

$$u=\frac{1}{4}-a^2+L_2\partial_{L_2}F\left(L_1L_2\right)\,. \tag{2.2.148}$$

Similarly, the semiclassical block near the irregular singularity at infinity is defined to be

$${}_1\mathcal{D}_1\left(m_1\;{}^{a_{2,1}}\;m_{1\theta}\;a\;m_2;L_1L_2,\frac{1}{L_1z}\right)=\lim_{b\to0}b^{-\frac{1}{2}-\theta b\mu_1+\frac{b^2}{2}}\frac{{}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta}\;\alpha\;\mu_2;\Lambda_1\Lambda_2,\frac{1}{\Lambda_1z}\right)}{{}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;\mu_2,\Lambda_1\Lambda_2\right)}\,, \tag{2.2.149}$$

and satisfies the same equation (2.2.147). In equation (2.2.141) 4 different blocks near infinity appear in the RHS. However they collapse to two of them in the semiclassical limit as in the previous cases. That is,

$${}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta}\;\alpha_{\theta'}\;\mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1z}\right)\sim\left(\Lambda_1\Lambda_2\right)^{\theta'a}e^{-\frac{\theta'}{2}\partial_aF}\,{}_1\mathfrak{D}_1\left(\mu_1\;{}^{\alpha_{2,1}}\;\mu_{1\theta}\;\alpha\;\mu_2;\Lambda_1\Lambda_2,\tfrac{1}{\Lambda_1z}\right),\quad\text{as } b\to0\,, \tag{2.2.150}$$

as in (2.2.34). Finally, the connection formula (2.2.141) in the semiclassical limit becomes

$$\begin{aligned}&z\,{}_1\mathcal{D}_1\left(m_2\;{}^{a_{2,1}}\;m_{2\theta}\;a\;m_1;L_1L_2,\frac{z}{L_2}\right)=\\&=\sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{N}^{-1}_{\theta\sigma}(m_2,a)\mathcal{N}_{(-\sigma)\theta'}(a,m_1)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}\right){}_1\mathcal{D}_1\left(m_1\;{}^{a_{2,1}}\;m_{1\theta'}\;a\;m_2;L_1L_2,\frac{1}{L_1z}\right),\end{aligned} \tag{2.2.151}$$

where explicitly the connection coefficients read

$$\begin{aligned}&\sum_{\sigma=\pm}\mathcal{N}^{-1}_{\theta\sigma}(m_2,a)\mathcal{N}_{(-\sigma)\theta'}(a,m_1)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}=\\&=\sum_{\sigma=\pm}\frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\left(L_1L_2\right)^{\sigma a}e^{-\frac{\sigma}{2}\partial_aF}}{\Gamma\left(\frac{1}{2}+\theta m_2-\sigma a\right)\Gamma\left(\frac{1}{2}-\theta'm_1-\sigma a\right)}e^{i\pi\left(\frac{1+\theta}{2}\right)\left(-\frac{1}{2}-m_2-\sigma a\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m_1-\sigma a\right)}\,,\end{aligned} \tag{2.2.152}$$

2.2.5 Reduced doubly confluent conformal blocks

General case

Consider the correlation function

$$\langle \mu, \Lambda_1 | \Phi(z) | \Lambda_2^2 \rangle \,, \tag{2.2.153}$$

which solves the BPZ equation

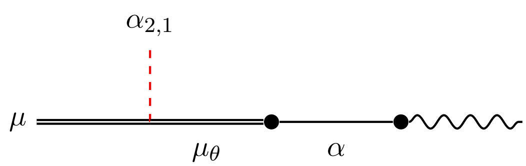
$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\mu\Lambda_1}{z} - \frac{\Lambda_1^2}{4} + \frac{\Lambda_2^2 \partial_{\Lambda_2^2}}{z^2} - \frac{\Lambda_2^2}{4z^3} \right) \langle \mu, \Lambda_1 | \Phi(z) | \Lambda_2^2 \rangle = 0 \,. \tag{2.2.154}$$

One of the parameters among Λ_1, Λ_2, z is redundant and can be set to an arbitrary value via a rescaling. We keep them all generic for convenience. We have three different conformal blocks, corresponding to the expansion of z near the two irregular singular points, and for z in the intermediate region. The block for $z \sim \infty$ is given by the decoupling limit of the corresponding doubly confluent block (2.2.133):

$${}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu \,{}^{\alpha_{2,1}}\, \mu_\theta \ \alpha \,; \Lambda_1 \Lambda_2^2, \tfrac{1}{\Lambda_1 z} \right) = e^{\theta b \Lambda_1 z/2} \Lambda_1^{\Delta + \Delta_{2,1}} (\Lambda_2^2)^{\Delta} \, (\Lambda_1 z)^{-\theta b \mu + \frac{b^2}{2}} \lim_{\eta\to\infty} {}_1\widetilde{\mathfrak{D}}_1\left(\mu \,{}^{\alpha_{2,1}}\, \mu_\theta \ \alpha \ \eta ; -\tfrac{\Lambda_1\Lambda_2^2}{4\eta}, \tfrac{1}{\Lambda_1 z} \right). \tag{2.2.155}$$

Equivalently, this block can be computed by doing the OPE $\langle \mu, \Lambda_1 | \Phi(z)$, projecting the result onto the Verma module Δ_α and contracting the result with $|\Lambda_2^2\rangle$. We denote it diagrammatically by

$${}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu \,{}^{\alpha_{2,1}}\, \mu_\theta \ \alpha \,; \Lambda_1 \Lambda_2^2, \frac{1}{\Lambda_1 z} \right) = \tag{2.2.156}$$

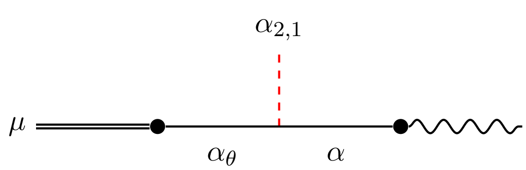


Also for the intermediate region $\Lambda_2^2 \ll z \ll \frac{1}{\Lambda_1}$ we have a closed form expression, given by

$${}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu \ \alpha_\theta \,{}^{\alpha_{2,1}}\, \alpha \ ; \Lambda_1 z, \frac{\Lambda_2^2}{z} \right) = \Lambda_1^{\Delta_\theta} (\Lambda_2^2)^{\Delta} z^{\frac{bQ}{2} + \theta b \alpha} \lim_{\eta\to\infty} {}_1\widetilde{\mathfrak{F}}_1\left(\mu \ \alpha_\theta \,{}^{\alpha_{2,1}}\, \alpha \ \eta ; \Lambda_1 z, -\frac{\Lambda_2^2}{4\eta z} \right). \tag{2.2.157}$$

This conformal block can also be computed directly by projecting $|\Lambda_2^2\rangle$ onto the Verma module Δ, then doing the OPE of $\Phi(z)$ term by term with the resulting expansion and then contracting with $\langle \mu, \Lambda_1|$. In diagrams

$${}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu \ \alpha_\theta \,{}^{\alpha_{2,1}}\, \alpha \ ; \Lambda_1 z, \frac{\Lambda_2^2}{z} \right) = \tag{2.2.158}$$

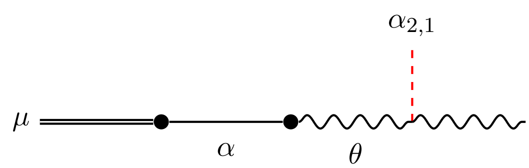


For the expansion around the irregular singular point of half rank no explicit, closed form expression is known to us. In any case one can calculate the expansion iteratively via other methods as for (2.2.109). We denote the corresponding conformal block in this region, where $z \ll \Lambda_2^2$ and $\Lambda_1\Lambda_2^2 \ll 1$ by

$$
{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\tfrac{\sqrt{z}}{\Lambda_2}\right) \sim e^{\theta b\Lambda_2/\sqrt{z}}\left(\tfrac{\sqrt{z}}{\Lambda_2}\right)^{-\frac{1}{2}-b^2} z^{-2\Delta_{2,1}}\Lambda_1^{\Delta}(\Lambda_2^2)^{\Delta_{2,1}+\Delta}\left[1+\mathcal{O}\left(\tfrac{\sqrt{z}}{\Lambda_2},\Lambda_1\Lambda_2^2\right)\right]. \tag{2.2.159}
$$

Diagrammatically,

$$
{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right) =
$$



. (2.2.160)

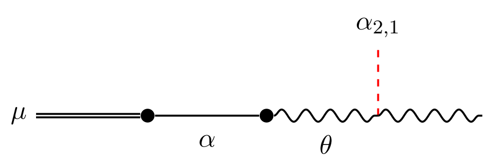
To connect 0 with the intermediate region we decompose

$$
\begin{aligned}
\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle &= \int d\alpha\, C_{\mu\alpha}G_\alpha^{-1}\sum_{\theta=\pm} C_\alpha B_{\alpha_{2,1}}\left|{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\right|^2 = \\
&= \int d\alpha\, C_{\mu\alpha}G_\alpha^{-1}\sum_{\theta'=\pm} C_{\alpha_{\theta'}}C^{\alpha_{\theta'}}_{\alpha_{2,1},\alpha}\left|{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right)\right|^2 .
\end{aligned} \tag{2.2.161}
$$

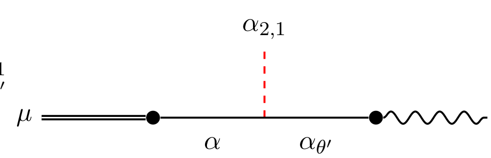
We recognize this constraint from (2.1.36). Its solution is

$$
b^{-\frac{1}{2}}{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right) = \sum_{\theta'=\pm} b^{2\theta' b\alpha}\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha)\,{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha\ {}^{\alpha_{2,1}}\ \alpha_{\theta'}\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right). \tag{2.2.162}
$$

In diagrams we write



$= \sum_{\theta'=\pm}\mathcal{Q}^{-1}_{\theta\theta'}$



. (2.2.163)

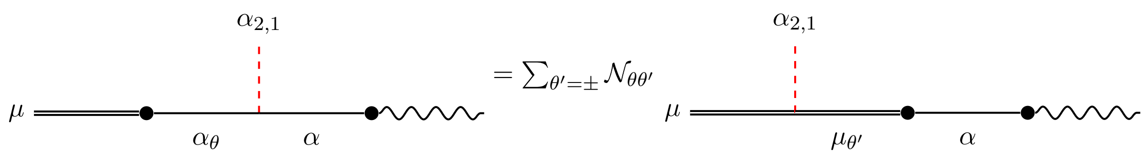
Instead, to connect from the intermediate region to ∞ we decompose

$$
\begin{aligned}
\langle\mu,\Lambda_1|\Phi(z)|\Lambda_2^2\rangle &= \int \mathrm{d}\alpha\, C_\alpha G_\alpha^{-1}\sum_{\theta=\pm} C_{\mu\alpha_\theta}C^{\alpha_\theta}_{\alpha_{2,1}\alpha}\left|{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ;\Lambda_1 z,\frac{\Lambda_2^2}{z}\right)\right|^2 = \\
&= \int \mathrm{d}\alpha\, C_\alpha G_\alpha^{-1}\sum_{\theta'=\pm} C_{\mu_{\theta'}\alpha}B^{\mu_{\theta'}}_{\alpha_{2,1}\mu}\left|{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\,;\Lambda_1\Lambda_2^2,\frac{1}{\Lambda_1 z}\right)\right|^2 .
\end{aligned} \tag{2.2.164}
$$

This is just the same constraint as for the Whittaker functions (2.1.21). The solution is

$$b^{\theta b\alpha}{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ;\Lambda_1 z,\tfrac{\Lambda_2^2}{z}\right)=\textstyle\sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu)_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta'}\ \alpha\ ;\Lambda_1\Lambda_2^2,\tfrac{1}{\Lambda_1 z}\right). \tag{2.2.165}$$

Diagrammatically

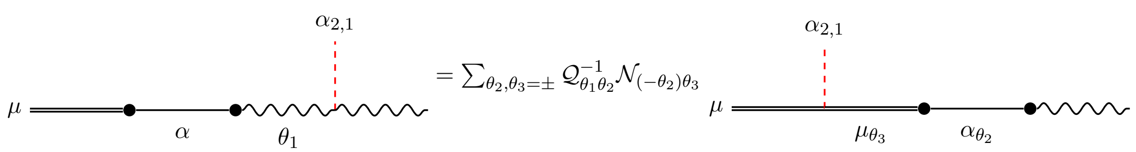


(2.2.166)

To connect from 0 to ∞ we just need to concatenate the two connection formulae above to obtain

$$b^{-\frac{1}{2}}{}_1\mathfrak{E}^{(\theta_1)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\tfrac{\sqrt{z}}{\Lambda_2}\right)=\textstyle\sum_{\theta_2,\theta_3=\pm} b^{2\theta_2 b\alpha}\mathcal{Q}^{-1}_{\theta_1\theta_2}(b\alpha)b^{-\frac{1}{2}+\theta_2 b\alpha-\theta_3 b\mu}\mathcal{N}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2},b\mu)_1\mathfrak{D}_{\frac{1}{2}}\left(\mu\ {}^{\alpha_{2,1}}\ \mu_{\theta_3}\ \alpha_{\theta_2}\,;\Lambda_1\Lambda_2^2,\tfrac{1}{\Lambda_1 z}\right). \tag{2.2.167}$$

In diagrams



(2.2.168)

Semiclassical limit

The BPZ equation in this limit becomes

$$\left(\partial_z^2-\frac{L_1^2}{4}+\frac{mL_1}{z}+\frac{u}{z^2}-\frac{L_2^2}{4z^3}\right){}_1\mathfrak{F}_{\frac{1}{2}}=0\,. \tag{2.2.169}$$

for *any* semiclassical block. Here u is given by

$$u=\lim_{b\to 0} b^2\Lambda_2^2\partial_{\Lambda_2^2}\log{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)=\frac{1}{4}-a^2+\mathcal{O}(L_1L_2^2)\,, \tag{2.2.170}$$

where ${}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)$ is the conformal block corresponding to $\langle\mu,\Lambda_1|\Lambda_2^2\rangle$ with intermediate momentum α. The finite semiclassical conformal blocks are defined as before by normalizing by the same block without the degenerate field insertion, i.e. for $z\sim 0$

$${}_1\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(m\,a\,a_{2,1}\,;L_1L_2^2,\tfrac{\sqrt{z}}{L_2}\right)=\lim_{b\to 0} b^{-\frac{1}{2}}\frac{{}_1\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\mu\,\alpha\,\alpha_{2,1}\,;\Lambda_1\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)}{{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right)}\sim e^{\theta L_2/\sqrt{z}}L_2^{-\frac{1}{2}}z^{\frac{3}{4}}(1+\mathcal{O}(L_1L_2^2,\sqrt{z}/L_2)) \tag{2.2.171}$$

For $z \sim \infty$ instead we have

$$
{}_1\mathcal{D}_{\frac{1}{2}}\left(m \, {}^{a_{2,1}} \, m_\theta \; a\,; L_1 L_2^2, \frac{1}{L_1 z}\right) = \lim_{b\to 0} b^{-\frac{1}{2}-\theta m} \frac{{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_\theta \; \alpha\,; \Lambda_1\Lambda_2^2, \frac{1}{\Lambda_1 z}\right)}{{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha; \Lambda_1\Lambda_2^2\right)} \sim
$$
$$
\sim e^{-\frac{\theta}{2}\partial_m F} e^{\theta L_1 z/2} L_1^{-\frac{1}{2}-\theta m} z^{-\theta m}\left(1 + \mathcal{O}(L_1 L_2^2, 1/L_1 z)\right). \tag{2.2.172}
$$

Here

$$
F = \lim_{b\to 0} b^2 \log\left[(\Lambda_1\Lambda_2^2)^{-\Delta}\,{}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha; \Lambda_1\Lambda_2^2\right)\right]. \tag{2.2.173}
$$

Both these blocks satisfy the same BPZ equation (2.2.169). Analogously to the previous confluences, in the connection formula between 0 and ∞ we have four different $\mathfrak{D}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$$
{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_\theta \; \alpha_{\theta'}\,; \Lambda_1\Lambda_2^2, \tfrac{1}{\Lambda_1 z}\right) \sim \left(\Lambda_1\Lambda_2^2\right)^{\theta' a} e^{-\frac{\theta'}{2}\partial_a F}\,{}_1\mathfrak{D}_{\frac{1}{2}}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_\theta \; \alpha\,; \Lambda_1\Lambda_2^2, \tfrac{1}{\Lambda_1 z}\right), \quad \text{as } b \to 0\,, \tag{2.2.174}
$$

as in (2.2.34). Now that we have defined the semiclassical conformal blocks, we state the connection formula. (2.2.167) in the semiclassical limit becomes

$$
{}_1\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(m\, a\, a_{2,1}\,; L_1 L_2^2, \tfrac{\sqrt{z}}{L_2}\right) = \sum_{\theta'}\left(\sum_{\sigma=\pm} \mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{N}_{(-\sigma)\theta'}(a,m)\left(L_1L_2^2\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}\right){}_1\mathcal{D}_{\frac{1}{2}}\left(m \, {}^{a_{2,1}} \, m_{\theta'} \; a\,; L_1 L_2^2, \tfrac{1}{L_1 z}\right). \tag{2.2.175}
$$

With connection coefficients[13]

$$
\sum_{\sigma=\pm} \mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{N}_{(-\sigma)\theta'}(a,m)\left(L_1L_2^2\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} =
$$
$$
= \frac{1}{\sqrt{2\pi}} \sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a)\Gamma(-2\sigma a)}{\Gamma\left(\frac{1}{2}-\theta' m - \sigma a\right)}\left(\frac{L_1L_2^2}{4}\right)^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F} e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\sigma a\right)} e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-m-\sigma a\right)}. \tag{2.2.176}
$$

Note that the factors of b appearing in (2.2.167) precisely combine with all the factors of Λ_1, Λ_2 to give the finite L_1, L_2.

2.2.6 Doubly reduced doubly confluent conformal blocks

General case

Decoupling the last mass we land on the last correlator of our interest, which solves the BPZ equation

$$
\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z - \frac{\Lambda_1^2}{4}\frac{1}{z} + \frac{\Lambda_2^2\partial_{\Lambda_2^2}}{z^2} - \frac{\Lambda_2^2}{4}\frac{1}{z^3}\right)\langle\Lambda_1^2|\Phi(z)|\Lambda_2^2\rangle = 0\,, \tag{2.2.177}
$$

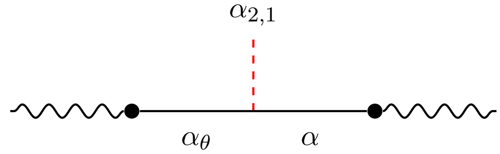
[13] Note that the Gamma functions in the denominator precisely correspond to the one-loop factor of the single hypermultiplet of the corresponding AGT dual gauge theory.

Again, one of the parameters among Λ_1, z, Λ_2 is redundant and can be set to an arbitrary value via a rescaling. We keep them generic for convenience. We can decompose the above correlator into conformal blocks in three different regions, that is for $z \ll \Lambda_2^2 \ll 1$, $z \gg \Lambda_1^{-2} \gg 1$, or for z in the intermediate region $\Lambda_2^2 \ll z \ll \Lambda_1^{-2}$. The conformal block in the intermediate region is again a block that can be expressed as a collision limit

$$ {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha\, ; \Lambda_1^2 z, \frac{\Lambda_2^2}{z}\right) = (\Lambda_1^2)^{\Delta_\theta} (\Lambda_2^2)^{\Delta} z^{\frac{bQ}{2}+\theta b\alpha} \lim_{\eta\to\infty} {}_1\widetilde{\mathfrak{F}}_{\frac{1}{2}}\left(\eta\ \alpha_\theta\ {}^{\alpha_{2,1}}\ \alpha\ ; \frac{-\Lambda_1^2}{4\eta} z, \frac{\Lambda_2^2}{z}\right) . \tag{2.2.178} $$

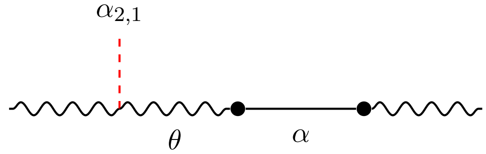
This conformal block can also be computed directly by projecting $|\Lambda_2^2\rangle$ onto the Verma module Δ, then doing the OPE of $\Phi(z)$ term by term with the resulting expansion and then contracting with $\langle \Lambda_1^2|$. In diagrams we represent it by

$$ {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha_\theta\, \alpha_{2,1}\, \alpha\, ; \Lambda_1^2 z, \frac{\Lambda_2^2}{z}\right) = \tag{2.2.179} $$



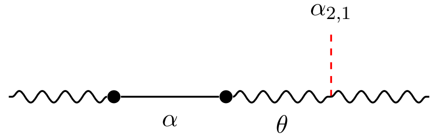
The block corresponding to the expansion for $z \gg \Lambda_1^{-2}$

$$ {}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\, \alpha\, ; \Lambda_1^2\Lambda_2^2, \frac{1}{\Lambda_1\sqrt{z}}\right) \sim (\Lambda_1^2)^{\Delta_{2,1}+\Delta} (\Lambda_2^2)^{\Delta} (\Lambda_1\sqrt{z})^{\frac{1}{2}+b^2} e^{\theta b \Lambda_1 \sqrt{z}} \left[1 + \mathcal{O}\left(\Lambda_1^2\Lambda_2^2, \frac{1}{\Lambda_1\sqrt{z}}\right)\right] = \tag{2.2.180} $$



and similarly for the expansion for $z \ll \Lambda_2^2$

$$ z^{-2\Delta_{2,1}}\, {}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha\, \alpha_{2,1}\, ; \Lambda_1^2\Lambda_2^2, \frac{\sqrt{z}}{\Lambda_2}\right) \sim (\Lambda_1^2)^{\Delta} (\Lambda_2^2)^{\Delta_{2,1}+\Delta} z^{-2\Delta_{2,1}} \left(\frac{\sqrt{z}}{\Lambda_2}\right)^{-\frac{1}{2}-b^2} e^{\theta b \Lambda_2/\sqrt{z}} \left[1 + \mathcal{O}\left(\Lambda_1\Lambda_2^2, \frac{\sqrt{z}}{\Lambda_2}\right)\right] = \tag{2.2.181} $$



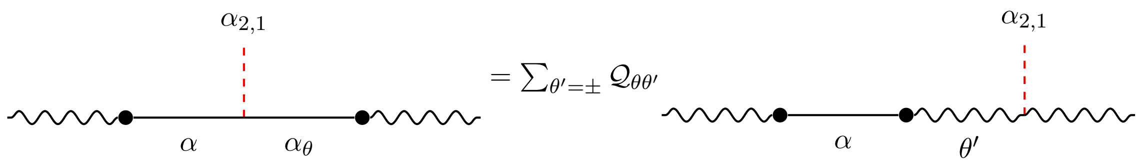
To connect the intermediate region with $z \sim 0$ we decompose the correlator as

$$ \begin{aligned} \langle \Lambda_1^2|\Phi(z)|\Lambda_2^2\rangle &= \int d\alpha\, C_\alpha G_\alpha^{-1} \sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\alpha_\theta} \left| {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha\, \alpha_{2,1}\, \alpha_\theta\, ; \Lambda_1^2 z, \frac{\Lambda_2^2}{z}\right)\right|^2 = \\ &= \int d\alpha\, C_\alpha G_\alpha^{-1} \sum_{\theta'=\pm} A_{-\frac{b}{2}} C_\alpha \left| z^{-2\Delta_{2,1}}\, {}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}_{\frac{1}{2}}\left(\alpha\, \alpha_{2,1}\, ; \Lambda_1^2\Lambda_2^2, \frac{\sqrt{z}}{\Lambda_2}\right)\right|^2 . \end{aligned} \tag{2.2.182} $$

This is the same constraint as in (2.1.36). Therefore the connection formula is

$$b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,\alpha_\theta\,;\Lambda_1^2 z,\frac{\Lambda_2^2}{z}\right)=\sum_{\theta'=\pm} b^{-\frac{1}{2}}\mathcal{Q}_{\theta\theta'}(b\alpha)z^{-2\Delta_{2,1}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)\,, \tag{2.2.183}$$

Diagrammatically

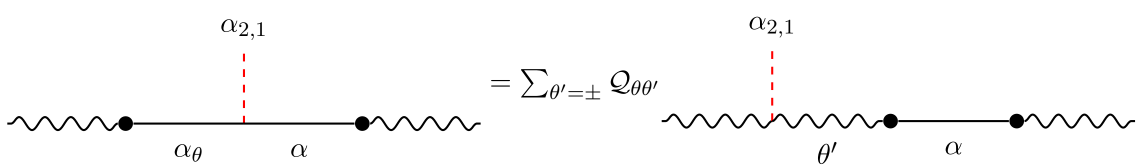


(2.2.184)

Similarly, the connection formula between the intermediate region and ∞ is

$$b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha\,;\Lambda_1^2 z,\frac{\Lambda_2^2}{z}\right)=\sum_{\theta'=\pm} b^{-\frac{1}{2}}\mathcal{Q}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\,. \tag{2.2.185}$$

In diagrams:

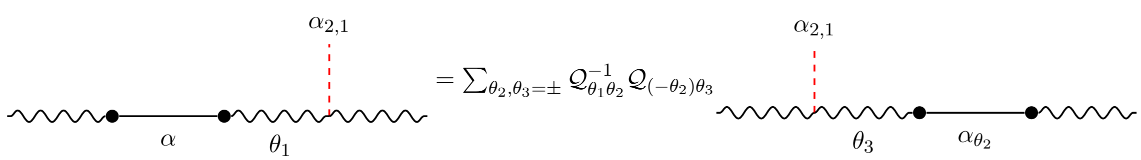


(2.2.186)

As in the previous cases, we can easily obtain a connection formula connecting the two irregular singularities, namely

$$b^{-\frac{1}{2}}z^{-2\Delta_{2,1}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta_1)}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\tfrac{\sqrt{z}}{\Lambda_2}\right)=\textstyle\sum_{\theta_2,\theta_3=\pm} b^{2\theta_2 b\alpha}\mathcal{Q}^{-1}_{\theta_1\theta_2}(b\alpha)b^{-\frac{1}{2}+2\theta_2 b\alpha}\mathcal{Q}_{(-\theta_2)\theta_3}(b\alpha_{\theta_2})\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta_3)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha_{\theta_2}\,;\Lambda_1^2\Lambda_2^2,\tfrac{1}{\Lambda_1\sqrt{z}}\right)\,. \tag{2.2.187}$$

Diagrammatically:



(2.2.188)

Semiclassical limit

The BPZ equation in this limit becomes

$$\left(\partial_z^2-\frac{L_1^2}{4z}+\frac{u}{z^2}-\frac{L_2^2}{4z^3}\right){}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}=0\,. \tag{2.2.189}$$

for *any* semiclassical block. Here u is given by

$$u = \lim_{b\to 0} b^2\Lambda_2^2\partial_{\Lambda_2^2}\log {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1^2L_2^2)\,, \tag{2.2.190}$$

where ${}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)$ is the conformal block corresponding to $\langle\Lambda_1^2|\Lambda_2^2\rangle$ with intermediate momentum α. The finite semiclassical conformal blocks are defined as before by normalizing by the same block without the degenerate field insertion, i.e. for $z\sim 0$

$$z\,{}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;L_1^2L_2^2,\tfrac{\sqrt{z}}{L_2}\right) = \lim_{b\to 0} b^{-1/2}\frac{z^{-2\Delta_{2,1}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha\,\alpha_{2,1}\,;\Lambda_1^2\Lambda_2^2,\frac{\sqrt{z}}{\Lambda_2}\right)}{{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)} \sim e^{\theta L_2/\sqrt{z}}L_2^{-1/2}z^{3/4}(1+\mathcal{O}(L_1^2L_2^2,\sqrt{z}/L_2))\,. \tag{2.2.191}$$

For $z\sim\infty$ instead we have

$${}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a_{2,1}\,a\,;L_1^2L_2^2,\tfrac{1}{L_1\sqrt{z}}\right) = \lim_{b\to 0} b^{-1/2}\frac{{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)}{{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)} \sim e^{\theta L_1\sqrt{z}}L_1^{-1/2}z^{1/4}(1+\mathcal{O}(L_1^2L_2^2,1/L_1\sqrt{z}))\,. \tag{2.2.192}$$

Here

$$F = \lim_{b\to 0} b^2\log\left[(\Lambda_1^2\Lambda_2^2)^{-\Delta}\,{}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha;\Lambda_1^2\Lambda_2^2\right)\right]\,. \tag{2.2.193}$$

Both these blocks satisfy the same BPZ equation (2.2.189). Analogously to the previous confluences, in the connection formula between 0 and ∞ we have four different $\mathfrak{E}$ blocks appearing, which should reduce to two in the semiclassical limit. Indeed, we have

$${}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha_{\theta'}\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right) \sim \left(\Lambda_1^2\Lambda_2^2\right)^{\theta' a}e^{-\frac{\theta'}{2}\partial_a F}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}_{\frac{1}{2}}\left(\alpha_{2,1}\,\alpha\,;\Lambda_1^2\Lambda_2^2,\frac{1}{\Lambda_1\sqrt{z}}\right)\,,\quad \text{as } b\to 0\,, \tag{2.2.194}$$

as in (2.2.34). Now that we have defined the semiclassical conformal blocks, we state the connection formula. (2.2.187) in the semiclassical limit becomes

$$z\,{}_{\frac{1}{2}}\mathcal{E}^{(\theta)}_{\frac{1}{2}}\left(a\,a_{2,1}\,;L_1^2L_2^2,\tfrac{\sqrt{z}}{L_2}\right) = \sum_{\theta'}\left(\sum_{\sigma=\pm}\mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{Q}_{(-\sigma)\theta'}(a)\,(L_1L_2)^{2\sigma a}\,e^{-\frac{\sigma}{2}\partial_a F}\right){}_{\frac{1}{2}}\mathcal{E}^{(\theta')}_{\frac{1}{2}}\left(a_{2,1}\,a\,;L_1^2L_2^2,\tfrac{1}{L_1\sqrt{z}}\right)\,. \tag{2.2.195}$$

With connection coefficients[14]

$$\begin{aligned}&\sum_{\sigma=\pm}\mathcal{Q}^{-1}_{\theta\sigma}(a)\mathcal{Q}_{(-\sigma)\theta'}(a)\,(L_1L_2)^{2\sigma a}\,e^{-\frac{\sigma}{2}\partial_a F} = \\ =&\frac{1}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a)\Gamma(-2\sigma a)\left(\frac{L_1L_2}{4}\right)^{2\sigma a}e^{-\frac{\sigma}{2}\partial_a F}e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\sigma a\right)}e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-2\sigma a\right)}\,.\end{aligned} \tag{2.2.196}$$

Note that the factors of b appearing in (2.2.187) precisely combine with all the factors of Λ_1,Λ_2 to give the finite L_1,L_2.

[14]Note also that there are no Gamma functions in the denominator corresponding to the fact that we have no hypermultiplets in the corresponding AGT dual gauge theory.

2.3 Heun equations, confluences and connection formulae

In this section we derive the explicit connection formulae for Heun functions and its confluences by identifying the semi-classical conformal blocks with the Heun functions and using the results so far obtained.

2.3.1 The Heun equation

In the following we identify the semiclassical BPZ equation (2.2.28) with Heun's equation via a dictionary between the relevant parameters. Moreover, we establish a precise relation between the Heun functions and the semiclassical *regular* conformal blocks. This is further used to obtain explicit formulae for the relevant connection coefficients. WLOG, we focus on the case $t \sim 0$. The connection formulae for $t \sim 1$, $t \sim \infty$ can be easily derived by matching the Heun equation and its local solutions with the corresponding semiclassical BPZ equations and the associated semiclassical conformal blocks.

The dictionary

Let us start giving the dictionary with CFT. The Heun equation reads

$$\begin{aligned}&\left(\frac{d^2}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{d}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}\right)w(z)=0\,,\\&\alpha+\beta+1=\gamma+\delta+\epsilon\,,\end{aligned} \tag{2.3.1}$$

where the condition $\alpha+\beta+1=\gamma+\delta+\epsilon$ ensures that the exponents of the local solutions at infinity are given by α, β. Here and in the following we restrict to generic values of the parameters. Define $w(z) = P_4(z)\psi(z)$ with

$$P_4(z) = z^{-\gamma/2}(1-z)^{-\delta/2}(t-z)^{-\epsilon/2}\,. \tag{2.3.2}$$

$\psi(z)$ then satisfies the Heun equation in normal form, which is easily compared with the semiclassical BPZ equation (2.2.28). We get $2^4 = 16$ dictionaries corresponding to the $(\mathbb{Z}_2)^4$ symmetry associated to flipping the signs of the momenta. We choose the following:

$$\begin{aligned}a_0 &= \frac{1-\gamma}{2}\,,\\a_1 &= \frac{1-\delta}{2}\,,\\a_t &= \frac{1-\epsilon}{2}\,,\\a_\infty &= \frac{\alpha-\beta}{2}\,,\\u^{(0)} &= \frac{-2q+2t\alpha\beta+\gamma\epsilon-t(\gamma+\delta)\epsilon}{2(t-1)}\,.\end{aligned} \tag{2.3.3}$$

The inverse dictionary is

$$\begin{aligned}
\alpha &= 1 - a_0 - a_1 - a_t + a_\infty \,, \\
\beta &= 1 - a_0 - a_1 - a_t - a_\infty \,, \\
\gamma &= 1 - 2a_0 \,, \\
\delta &= 1 - 2a_1 \,, \\
\epsilon &= 1 - 2a_t \,, \\
q &= \frac{1}{2} + t(a_0^2 + a_t^2 + a_1^2 - a_\infty^2) - a_t - a_1 t + a_0(2a_t - 1 + t(2a_1 - 1)) + (1-t)u^{(0)} \,.
\end{aligned} \tag{2.3.4}$$

The two linearly independent solutions for $z \sim 0$ of (2.2.28) are related by $a_0 \to -a_0$. This corresponds to the identification of the two linearly independent solutions of (2.3.1) for $z \sim 0$ as

$$\begin{aligned}
w_-^{(0)}(z) &= \mathrm{HeunG}\left(t, q, \alpha, \beta, \gamma, \delta, z\right) \,, \\
w_+^{(0)}(z) &= z^{1-\gamma}\mathrm{HeunG}\left(t, q - (\gamma - 1)(t\delta + \epsilon), \alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \delta, z\right) \,,
\end{aligned} \tag{2.3.5}$$

where by definition

$$\mathrm{HeunG}\left(t, q, \alpha, \beta, \gamma, \delta, z\right) = 1 + \frac{q}{t\gamma} z + \mathcal{O}(z^2) \,. \tag{2.3.6}$$

The Heun function can be identified with the semiclassical conformal blocks introduced before. In particular comparing with (2.2.25) we get the two solutions

$$\begin{aligned}
w_-^{(0)}(z) &= P_4(z)\, t^{\frac{1}{2} - a_t - a_0} e^{-\frac{1}{2}\partial_{a_0} F(t)} \mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix} a \begin{matrix} a_t \\ \, \end{matrix} a_{0-} \begin{matrix} a_{2,1} \\ a_0 \end{matrix} ; t, \frac{z}{t} \right) , \\
w_+^{(0)}(z) &= P_4(z)\, t^{\frac{1}{2} - a_t + a_0} e^{\frac{1}{2}\partial_{a_0} F(t)} \mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix} a \begin{matrix} a_t \\ \, \end{matrix} a_{0+} \begin{matrix} a_{2,1} \\ a_0 \end{matrix} ; t, \frac{z}{t} \right) .
\end{aligned} \tag{2.3.7}$$

Note that HeunG is an expansion in z, while the semiclassical conformal blocks are expanded both in z and t. To match the two expansions one has to express the accessory parameter q in terms of the Floquet exponent a as a series in t. This can be done substituting the dictionary as explained in Appendix 2.C.
The solutions for $z \sim t$ are given by

$$\begin{aligned}
w_-^{(t)}(z) &= \mathrm{HeunG}\left(\frac{t}{t-1}, \frac{q - t\alpha\beta}{1-t}, \alpha, \beta, \epsilon, \delta, \frac{z-t}{1-t} \right) , \\
w_+^{(t)}(z) &= (t-z)^{1-\epsilon}\mathrm{HeunG}\left(\frac{t}{t-1}, \frac{q - t\alpha\beta}{1-t} - (\epsilon - 1)\left(\frac{t}{t-1}\delta + \gamma \right), \alpha + 1 - \epsilon, \beta + 1 - \epsilon, 2 - \epsilon, \delta, \frac{z-t}{1-t} \right) .
\end{aligned} \tag{2.3.8}$$

Comparing with the semiclassical blocks (2.2.29) we get

$$w_-^{(t)}(z) = P_4(z)\, t^{\frac{1}{2}-a_0-a_t}(1-t)^{\frac{1}{2}-a_1} e^{-\frac{1}{2}\partial_{a_t}F(t)} \left((t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix} a^{\,a_0}\, a_{t-} \begin{matrix} a_{2,1} \\ a_t \end{matrix}; \frac{t}{t-1}, \frac{t-z}{t}\right)\right),$$
$$w_+^{(t)}(z) = P_4(z)\, t^{\frac{1}{2}-a_0+a_t}(1-t)^{\frac{1}{2}-a_1} e^{\frac{1}{2}\partial_{a_t}F(t)} \left((t-1)^{\frac{1}{2}}\mathcal{F}\left(\begin{matrix} a_1 \\ a_\infty \end{matrix} a^{\,a_0}\, a_{t+} \begin{matrix} a_{2,1} \\ a_t \end{matrix}; \frac{t}{t-1}, \frac{t-z}{t}\right)\right). \tag{2.3.9}$$

The two solutions for $z \sim 1$ read

$$w_-^{(1)}(z) = \left(\frac{z-t}{1-t}\right)^{-\alpha} \mathrm{HeunG}\left(t, q+\alpha(\delta-\beta), \alpha, \delta+\gamma-\beta, \delta, \gamma, t\frac{1-z}{t-z}\right),$$
$$w_+^{(1)}(z) = \left(\frac{z-t}{1-t}\right)^{-\alpha-1+\delta} (1-z)^{1-\delta}\mathrm{HeunG}\left(t, q-\alpha(\beta+\delta-2)+(\delta-1)(\alpha+\beta-1-t\gamma), \alpha+1-\delta, 1+\gamma-\beta, 2-\delta, \gamma, t\frac{1-z}{t-z}\right), \tag{2.3.10}$$

and matching with (2.2.37) gives

$$w_-^{(1)}(z) = P_4(z) e^{\pm i\pi(a_1+a_t)}(1-t)^{\frac{1}{2}-a_t} e^{-\frac{1}{2}\partial_{a_1}F(t)} \left((t(1-t))^{-\frac{1}{2}}(t-z)\mathcal{F}\left(\begin{matrix} a_0 \\ a_t \end{matrix} a^{\,a_\infty}\, a_{1-} \begin{matrix} a_{2,1} \\ a_1 \end{matrix}; t, \frac{1-z}{t-z}\right)\right)$$
$$w_+^{(1)}(z) = P_4(z) e^{\pm i\pi(-a_1+a_t)}(1-t)^{\frac{1}{2}-a_t} e^{\frac{1}{2}\partial_{a_1}F(t)} \left((t(1-t))^{-\frac{1}{2}}(t-z)\mathcal{F}\left(\begin{matrix} a_0 \\ a_t \end{matrix} a^{\,a_\infty}\, a_{1+} \begin{matrix} a_{2,1} \\ a_1 \end{matrix}; t, \frac{1-z}{t-z}\right)\right). \tag{2.3.11}$$

The $\pm$ ambuiguity in the overall phase depends on the choice of branch corresponding to

$$P_4(z)\mathcal{F}\left(\begin{matrix} a_0 \\ a_t \end{matrix} a^{\,a_\infty}\, a_{1\theta} \begin{matrix} a_{2,1} \\ a_1 \end{matrix}; t, \frac{1-z}{t-z}\right) \propto (t-1)^{\theta a_1+a_t} = e^{\pm i\pi(\theta a_1+a_t)}(1-t)^{\theta a_1+a_t}. \tag{2.3.12}$$

Finally, the two solutions near $z \sim \infty$ are given by

$$w_+^{(\infty)}(z) = z^{-\alpha}\mathrm{HeunG}\left(t, q-\alpha\beta(1+t)+\alpha(\delta+t\epsilon), \alpha, \alpha-\gamma+1, \alpha-\beta+1, \alpha+\beta+1-\gamma-\delta, \frac{t}{z}\right),$$
$$w_-^{(\infty)}(z) = z^{-\beta}\mathrm{HeunG}\left(t, q-\alpha\beta(1+t)+\beta(\delta+t\epsilon), \beta, \beta-\gamma+1, \beta-\alpha+1, \alpha+\beta+1-\gamma-\delta, \frac{t}{z}\right). \tag{2.3.13}$$

Comparing with (2.2.31) we get

$$\begin{aligned} w_+^{(\infty)}(z) &= P_4(z) e^{\pm i\pi(1-a_1-a_t)} e^{\frac{1}{2}\partial_{a_\infty}F(t)} \left(t^{-\frac{1}{2}} z\mathcal{F}\left(\begin{matrix} a_t \\ a_0 \end{matrix} a^{\,a_1}\, a_{\infty+} \begin{matrix} a_{2,1} \\ a_\infty \end{matrix}; t, \frac{1}{z}\right)\right), \\ w_-^{(\infty)}(z) &= P_4(z) e^{\pm i\pi(1-a_1-a_t)} e^{-\frac{1}{2}\partial_{a_\infty}F(t)} \left(t^{-\frac{1}{2}} z\mathcal{F}\left(\begin{matrix} a_t \\ a_0 \end{matrix} a^{\,a_1}\, a_{\infty-} \begin{matrix} a_{2,1} \\ a_\infty \end{matrix}; t, \frac{1}{z}\right)\right), \end{aligned} \tag{2.3.14}$$

where again the $\pm$ in the phase depends on the choice of branch corresponding to

$$P_4(z) = z^{-\frac{1}{2}+a_0}(1-z)^{-\frac{1}{2}+a_1}(t-z)^{-\frac{1}{2}+a_t} = e^{\mp i\pi(1-a_1-a_t)} z^{-\frac{1}{2}+a_0}(z-1)^{-\frac{1}{2}+a_1}(z-t)^{-\frac{1}{2}+a_t}. \tag{2.3.15}$$

Connection formulae

Finally we are in the position to give the connection formulae for the Heun function. Let us start with $z \sim 0$ and $z \sim t$. The corresponding connection formula can be read off from (2.2.30), which in the Heun notation reads

$$w_-^{(0)}(z) = \frac{\Gamma(1-\epsilon)\Gamma(\gamma)e^{\frac{1}{2}(\partial_{a_t}-\partial_{a_0})F}}{\Gamma\left(\frac{1+\gamma-\epsilon}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\epsilon}{2}-a(q)\right)}(1-t)^{-\frac{\delta}{2}}w_-^{(t)}(z) + \frac{\Gamma(\epsilon-1)\Gamma(\gamma)e^{\frac{1}{2}(-\partial_{a_t}-\partial_{a_0})F}}{\Gamma\left(\frac{-1+\gamma+\epsilon}{2}+a(q)\right)\Gamma\left(\frac{-1+\gamma+\epsilon}{2}-a(q)\right)}t^{\epsilon-1}(1-t)^{-\frac{\delta}{2}}w_+^{(t)}(z)\,, \tag{2.3.16}$$

for the other solution one finds

$$w_+^{(0)}(z) = \frac{\Gamma(1-\epsilon)\Gamma(2-\gamma)e^{\frac{1}{2}(\partial_{a_t}+\partial_{a_0})F}}{\Gamma\left(1+\frac{1-\gamma-\epsilon}{2}+a(q)\right)\Gamma\left(1+\frac{1-\gamma-\epsilon}{2}-a(q)\right)}t^{1-\gamma}(1-t)^{-\frac{\delta}{2}}w_-^{(t)}(z) + \frac{\Gamma(\epsilon-1)\Gamma(\gamma)e^{\frac{1}{2}(-\partial_{a_t}+\partial_{a_0})F}}{\Gamma\left(\frac{1-\gamma+\epsilon}{2}+a(q)\right)\Gamma\left(\frac{1-\gamma+\epsilon}{2}-a(q)\right)}t^{\epsilon-\gamma}(1-t)^{-\frac{\delta}{2}}w_+^{(t)}(z)\,. \tag{2.3.17}$$

Here $a(q)$ has to be computed inverting the relation (2.2.27) and substituting the dictionary as shown explicitly in Appendix 2.C, formula (2.C.13). The result to first order is

$$\begin{aligned} a(q) &= \frac{1}{16}\sqrt{3-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2)}\times \\ &\times\left(8-\frac{4(-1+2q-\epsilon(\gamma+\epsilon-2))(-3+4q+(\alpha-\beta)-\gamma^2-\delta(\delta-2)-2\gamma(\epsilon-1)-\epsilon(\epsilon-2))}{(3-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2))(2-4q+\gamma^2+2\gamma(\epsilon-1)+\epsilon(\epsilon-2))}t\right)+\mathcal{O}(t^2)\,. \end{aligned} \tag{2.3.18}$$

In Appendix 2.C we also explain how to compute the classical conformal block F and its derivatives (see formula 2.C.10). For example, to first order

$$\partial_{a_t}F(t) = \frac{\left(4a(q)^2-\alpha^2+2\alpha\beta-\beta^2-2\delta+\delta^2\right)(1-\epsilon)}{2-8a(q)^2}t+\mathcal{O}(t^2)\,. \tag{2.3.19}$$

The connection formula for $w_+^{(0)}(z)$ can be obtained from (2.3.16) by multiplying by $z^{1-\gamma}$, substituting

$$q \to q-(\gamma-1)(t\delta+\epsilon),\ \alpha \to \alpha+1-\gamma,\ \beta \to \beta+1-\gamma,\ \gamma \to 2-\gamma \tag{2.3.20}$$

as in (2.3.5), and noting that

$$\begin{aligned} &\mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t\alpha\beta}{1-t},\alpha,\beta,\epsilon,\delta,\frac{z-t}{1-t}\right) = \\ &= \left(\frac{z}{t}\right)^{1-\gamma}\mathrm{HeunG}\left(\frac{t}{t-1},\frac{q-t(\alpha+1-\gamma)(\beta+1-\gamma)-(\gamma-1)(t\delta+\epsilon)}{1-t},\alpha+1-\gamma,\beta+1-\gamma,\epsilon,\delta,\frac{z-t}{1-t}\right)\,. \end{aligned} \tag{2.3.21}$$

Similarly, the connection formula from $z \sim 0$ to $z \sim \infty$ can be read off from (2.2.35), and gives

$$
\begin{aligned}
w_-^{(0)}(z) = &\left(\sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\Gamma(\gamma)\Gamma(\beta-\alpha) t^{\frac{\gamma+\epsilon-1}{2}-\sigma a(q)} e^{-\frac{1}{2}\left(\partial_{a_0}-\partial_{a_\infty}+\sigma\partial_a\right)F} e^{i\pi\left(\frac{\delta+\gamma}{2}\right)}}{\Gamma\left(\frac{\gamma-\epsilon+1}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\epsilon-1}{2}-\sigma a(q)\right)\Gamma\left(1+\frac{\beta-\alpha-\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\beta-\alpha+\delta}{2}-\sigma a(q)\right)} \right) w_+^{(\infty)}(z) + \\
&+ \left(\sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\Gamma(\gamma)\Gamma(\alpha-\beta) t^{\frac{\gamma+\epsilon-1}{2}-\sigma a(q)} e^{-\frac{1}{2}\left(\partial_{a_0}-\partial_{a_\infty}+\sigma\partial_a\right)F} e^{i\pi\left(\frac{\delta+\gamma}{2}\right)}}{\Gamma\left(\frac{\gamma-\epsilon+1}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\epsilon-1}{2}-\sigma a(q)\right)\Gamma\left(1+\frac{\alpha-\beta-\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\alpha-\beta+\delta}{2}-\sigma a(q)\right)} \right) w_-^{(\infty)}(z) \,.
\end{aligned}
\tag{2.3.22}
$$

Let us conclude the section by giving the connection formulae from 1 to infinity. This can be derived from (2.2.20), and gives

$$
\begin{aligned}
w_-^{(1)}(z) = &- (1-t)^{\frac{1}{2}-a_t} \frac{\Gamma(\beta-\alpha)\Gamma(\delta) e^{-\frac{1}{2}\left(\partial_{a_1}+\partial_{a_\infty}\right)F(t)}}{\Gamma\left(\frac{\delta-\alpha+\beta}{2}+a(q)\right)\Gamma\left(\frac{\delta-\alpha+\beta}{2}-a(q)\right)} w_+^{(\infty)}(z) + \\
&- (1-t)^{\frac{1}{2}-a_t} \frac{\Gamma(\alpha-\beta)\Gamma(\delta) e^{-\frac{1}{2}\left(\partial_{a_1}-\partial_{a_\infty}\right)F(t)}}{\Gamma\left(\frac{\delta+\alpha-\beta}{2}+a(q)\right)\Gamma\left(\frac{\delta+\alpha-\beta}{2}-a(q)\right)} w_-^{(\infty)}(z) \,.
\end{aligned}
\tag{2.3.23}
$$

The connection formulae involving the other solutions can be read off from the previous ones, and the formulae involving different pairs of points can be similarly derived by considering the corresponding semiclassical conformal blocks. We conclude by stressing again that the connection formulae involving different regions in the $t-$plane are completely analogous to the previous ones, since all the singularities are regular. This will not be the case in the following.

2.3.2 The confluent Heun equation

The dictionary

Here we establish the dictionary between our results of section 2.2.2 on confluent conformal blocks and the confluent Heun equation (CHE) in standard notation, which reads

$$
\frac{d^2 w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon \right) \frac{dw}{dz} + \frac{\alpha z - q}{z(z-1)} w = 0 \,. \tag{2.3.24}
$$

By defining $w(z) = P_3(z)\psi(z)$ with $P_3(z) = e^{-\epsilon z/2} z^{-\gamma/2} (1-z)^{-\delta/2}$, we get rid of the first derivative and bring the equation to normal form, which can easily be compared with the semiclassical BPZ equation (2.2.82). We can read off the dictionary between the CFT parameters and the parameters of the CHE:

$$\begin{aligned} a_0 &= \frac{1-\gamma}{2}\,, \\ a_1 &= \frac{1-\delta}{2}\,, \\ m &= \frac{\alpha}{\epsilon} - \frac{\gamma+\delta}{2}\,, \\ L &= \epsilon\,, \\ u &= \frac{1}{4} - q + \alpha - \frac{(\gamma+\delta-1)^2}{4} - \frac{\delta\epsilon}{2}\,, \end{aligned} \tag{2.3.25}$$

where

$$u = \lim_{b\to 0} b^2 \Lambda \partial_\Lambda \log {}_1\mathfrak{F}\left(\mu\,\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda\right) = \frac{1}{4} - a^2 + \mathcal{O}(L) \tag{2.3.26}$$

as in (2.2.82). This relation can then be inverted to find a in terms of the parameters of the CHE: we denote this by $a(q)$. We write the solutions to the CHE in standard form in the notation of Mathematica, and their relation to the conformal blocks used before. We focus first on the blocks given as an expansion for small L. Then, near $z = 0$ we have the two linearly independent solutions

$$\begin{aligned} &\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)\,, \\ &z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta),\alpha+(1-\gamma)\epsilon,2-\gamma,\delta,\epsilon;z\right)\,, \end{aligned} \tag{2.3.27}$$

where the confluent Heun function has the following expansion around $z = 0$:

$$\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z) = 1 - \frac{q}{\gamma}z + \mathcal{O}(z^2)\,. \tag{2.3.28}$$

Comparing with the semiclassical conformal blocks in (2.2.2) we identify

$$\begin{aligned} &\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z) = P_3(z) e^{-\frac{1}{2}\partial_{a_0}F}\,{}_1\mathcal{F}\left(m\;a\,{}^{a_1}\,a_{0-}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};L,z\right)\,, \\ &z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta),\alpha+(1-\gamma)\epsilon,2-\gamma,\delta,\epsilon;z\right) = P_3(z) e^{\frac{1}{2}\partial_{a_0}F}\,{}_1\mathcal{F}\left(m\;a\,{}^{a_1}\,a_{0+}\,\begin{matrix}a_{2,1}\\ a_0\end{matrix};L,z\right)\,, \end{aligned} \tag{2.3.29}$$

where

$$F = \lim_{b\to 0} b^2 \log\left[\Lambda^{-\Delta}\,{}_1\mathfrak{F}\left(\mu\,\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda\right)\right]\,. \tag{2.3.30}$$

Doing a Möbius transformation $z \to 1 - z$ we obtain solutions around $z = 1$, which being a regular singularity can again be written in terms of HeunC. This amounts to sending $\gamma \to \delta$, $\delta \to \gamma$, $\epsilon \to -\epsilon$, $\alpha \to -\alpha$, $q \to q - \alpha$. The two solutions are therefore

$$\begin{aligned} &\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)\,, \\ &(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right)\,. \end{aligned} \tag{2.3.31}$$

Again, comparing with the semiclassical conformal blocks in (2.2.2), we identify

$$
\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z) = P_3(z)e^{-\frac{1}{2}\partial_{a_1}F}\,{}_1\mathcal{F}\left(-m\ a\ {}^{a_0}\ a_{1-}\ \begin{matrix}a_{2,1}\\ a_1\end{matrix};L,1-z\right),\\
&(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right) =\\
&\qquad = P_3(z)e^{\frac{1}{2}\partial_{a_1}F}\,{}_1\mathcal{F}\left(-m\ a\ {}^{a_0}\ a_{1+}\ \begin{matrix}a_{2,1}\\ a_1\end{matrix};L,1-z\right).
\end{aligned}
\tag{2.3.32}
$$

Around the irregular singular point $z=\infty$, we write the solutions in terms of a different function HeunC_∞:

$$
\begin{aligned}
&z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1})\\
&e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1}),
\end{aligned}
\tag{2.3.33}
$$

where the function HeunC_∞ has a simple asymptotic expansion around $z=\infty$:

$$
\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1}) \sim 1+\frac{\alpha^2-(\gamma+\delta-1)\alpha\epsilon+(\alpha-q)\epsilon^2}{\epsilon^3}z^{-1}+\mathcal{O}(z^{-2}). \tag{2.3.34}
$$

Comparing with the semiclassical conformal blocks we identify

$$
\begin{aligned}
&z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{\frac{1}{2}\partial_m F}L^{\frac{1}{2}+m}\,{}_1\mathcal{D}\left(m\,{}^{a_{2,1}}\ m_+\ a\ \begin{matrix}a_1\\ a_0\end{matrix};L,\frac{1}{z}\right)\\
&e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1}) = e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{-\frac{1}{2}\partial_m F}L^{\frac{1}{2}-m}\,{}_1\mathcal{D}\left(m\,{}^{a_{2,1}}\ m_-\ a\ \begin{matrix}a_1\\ a_0\end{matrix};L,\frac{1}{z}\right).
\end{aligned}
\tag{2.3.35}
$$

The phase $e^{\mp\frac{i\pi\delta}{2}}$ comes from the fact that near $z=\infty$

$$
P_3(z)\sim e^{-\epsilon z/2}z^{-\gamma/2}(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}e^{-\epsilon z/2}z^{-\gamma/2-\delta/2}. \tag{2.3.36}
$$

The second solution around $z=\infty$ can be found by using the manifest symmetry $(m,L)\to(-m,-L)$ of the semiclassical BPZ equation which according to the dictionary gives the symmetry $(q,\alpha,\epsilon)\to(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),-\epsilon)$ of the CHE in normal form.

For the large-L blocks the story is analogous. The dictionary (2.3.25) is the same, up to the substitution

$$
u\to u_D = \lim_{b\to 0} b^2\Lambda\partial_\Lambda \log {}_1\mathfrak{D}\left(\mu\ {}^{\alpha_1}\ \mu'\ \alpha_0;\tfrac{1}{\Lambda}\right) = -(m'-m)L+\tfrac{1}{4}-a_0^2+2m'(m'-m)+\mathcal{O}(L^{-1}). \tag{2.3.37}
$$

This relation can be inverted to find m' in terms of the parameters of the CHE. We will call this $m'(q)$. With this dictionary we can identify solutions of the CHE with conformal blocks as follows: near $z=0$ we have

$$
\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z) = P_3(z)e^{-\frac{1}{2}\partial_{a_0}F_D}\,{}_1\mathcal{D}\left(m\,{}^{a_1}\ m'\ a_{0-}\ \begin{matrix}a_{2,1}\\ a_0\end{matrix};\frac{1}{L},Lz\right),\\
&z^{1-\gamma}\mathrm{HeunC}\left(q+(1-\gamma)(\epsilon-\delta),\alpha+(1-\gamma)\epsilon,2-\gamma,\delta,\epsilon;z\right) = P_3(z)e^{\frac{1}{2}\partial_{a_0}F_D}\,{}_1\mathcal{D}\left(m\,{}^{a_1}\ m'\ a_{0+}\ \begin{matrix}a_{2,1}\\ a_0\end{matrix};\frac{1}{L},Lz\right),
\end{aligned}
\tag{2.3.38}
$$

with F_D given in (2.2.93). Near $z=1$ we have

$$
\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)=P_3(z)e^{-\frac{1}{2}\partial_{a_1}F_D}\,{}_1\mathcal{D}\left(-m^{a_0}\;m'-m\;a_{1-}\;\begin{matrix}a_{2,1}\\a_1\end{matrix};\frac{1}{L},L(1-z)\right),\\
&(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right)=\\
&\qquad=P_3(z)e^{\frac{1}{2}\partial_{a_1}F_D}\,{}_1\mathcal{D}\left(-m^{a_0}\;m'-m\;a_{1+}\;\begin{matrix}a_{2,1}\\a_1\end{matrix};\frac{1}{L},L(1-z)\right).
\end{aligned}
\tag{2.3.39}
$$

While near $z=\infty$ we have

$$
\begin{aligned}
&z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1})=e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{L/2}e^{\frac{1}{2}\partial_m F_D}L^{\frac{1}{2}-(m'-m)}\,{}_1\mathcal{D}\left(m\;{}^{a_{2,1}}\;m_+\;{}^{a_1}\;m'\;a_0;\frac{1}{L},\frac{1}{z}\right)\\
&e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1})=e^{\mp\frac{i\pi\delta}{2}}P_3(z)e^{-L/2}e^{-\frac{1}{2}\partial_m F_D}L^{\frac{1}{2}+(m'-m)}\,{}_1\mathcal{D}\left(m\;{}^{a_{2,1}}\;m_+\;{}^{a_1}\;m'\;a_0;\frac{1}{L},\frac{1}{z}\right).
\end{aligned}
\tag{2.3.40}
$$

As the careful reader should have noticed, we identify the small-L and large-L conformal blocks with the same confluent Heun functions. The only difference is in the expansion of the accessory parameter: in one case it is given in terms of the Floquet exponent a as an expansion in L, and in the other case in terms of the parameter m' as an expansion in L^{-1}.

Connection formulae

The connection formula between $z=0,1$ written in (2.2.91) for the semiclassical conformal blocks can now be restated as:

$$
\begin{aligned}
\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)=&\frac{\Gamma(1-\delta)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F+\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1+\gamma-\delta}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\delta}{2}-a(q)\right)}\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)+\\
+&\frac{\Gamma(\delta-1)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F-\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{\gamma+\delta-1}{2}+a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-a(q)\right)}(1-z)^{1-\delta}\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right).
\end{aligned}
\tag{2.3.41}
$$

The quantities $a(q)$ and F can be computed as explained in Appendix 2.C.
The connection formula between $z=1,\infty$ written in (2.2.89) reads in terms of confluent

Heun functions:

$$
\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)= \\
&=\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\epsilon^{-\frac{1}{2}-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F+\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F(a)}}{\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)\Gamma\left(\frac{1+\gamma+\delta}{2}-\frac{\alpha}{\epsilon}-\sigma a(q)\right)}\right)\times \\
&\quad\times z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z)+ \\
&+\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\epsilon^{-\frac{1}{2}+\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F+\frac{1}{2}\partial_m F-\frac{\sigma}{2}\partial_a F(a)}}{\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)\Gamma\left(\frac{1-\gamma-\delta}{2}+\frac{\alpha}{\epsilon}-\sigma a(q)\right)}\right)\times \\
&\quad\times e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z)\,.
\end{aligned}
\tag{2.3.42}
$$

Here the phase ambiguity comes from (2.3.35), i.e. corresponds to the choice $(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}z^{-\delta/2}$. A similar expression can be found connecting $z = 0$ and ∞. All connection coefficients given above are calculated in a series expansion in L. Therefore they are not valid for large L and in that case one has to use different connection formulae, which are derived in section 2.2.2 for the large-L semiclassical conformal blocks. Here we restate those results in the language of Heun functions. The connection formula from $z = 0$ to $z = 1$, valid for large L is given by

$$
\begin{aligned}
&\mathrm{HeunC}(q,\alpha,\gamma,\delta,\epsilon;z)= \\
&=\left(\sum_{\sigma=\pm}\frac{\Gamma(\gamma)\Gamma(1-\delta)e^{\frac{\sigma}{2}\epsilon}\epsilon^{-\sigma\left(2m'(q)-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}\right)}e^{-\frac{1}{2}\partial_{a_0}F_D+\frac{1}{2}\partial_{a_1}F_D-\frac{\sigma}{2}\partial_{m'}F_D}e^{i\pi\left(\frac{1-\sigma}{2}\right)\left(\frac{\alpha}{\epsilon}-\delta-2m'(q)\right)}}{\Gamma\left(\frac{\gamma}{2}-\sigma m'(q)\right)\Gamma\left(1-\frac{\delta}{2}-\sigma\left(m'(q)-\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}\right)\right)}\right)\times \\
&\quad\times\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)+ \\
&+\left(\sum_{\sigma=\pm}\frac{\Gamma(\gamma)\Gamma(\delta-1)e^{\frac{\sigma}{2}\epsilon}\epsilon^{-\sigma\left(2m'(q)-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta}{2}\right)}e^{-\frac{1}{2}\partial_{a_0}F_D-\frac{1}{2}\partial_{a_1}F_D-\frac{\sigma}{2}\partial_{m'}F_D}e^{i\pi\left(\frac{1-\sigma}{2}\right)\left(\frac{\alpha}{\epsilon}-2m'(q)-1\right)}}{\Gamma\left(\frac{\gamma}{2}-\sigma m'(q)\right)\Gamma\left(\frac{\delta}{2}-\sigma\left(m'(q)-\frac{\alpha}{\epsilon}-\frac{\gamma+\delta}{2}\right)\right)}\right)\times \\
&\quad\times\mathrm{HeunC}\left(q-\alpha-(1-\delta)(\epsilon+\gamma),-\alpha-(1-\delta)\epsilon,2-\delta,\gamma,-\epsilon;1-z\right)\,,
\end{aligned}
\tag{2.3.43}
$$

where the quantities $m'(q)$ and F_D are computed as explained in Appendix 2.C.
The connection formula from $z = 1$ to ∞ is simpler and reads

$$
\begin{aligned}
&\mathrm{HeunC}(q-\alpha,-\alpha,\delta,\gamma,-\epsilon;1-z)= \\
&=e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F_D-\frac{1}{2}\partial_m F_D}\epsilon^{-\frac{1}{2}-\frac{\alpha}{\epsilon}+\frac{\gamma+\delta-\epsilon}{2}+m'(q)}\frac{\Gamma(\delta)e^{i\pi\left(\frac{\alpha}{\epsilon}-\frac{\gamma}{2}-m'(q)\right)}}{\Gamma\left(-\frac{\alpha}{\epsilon}+\frac{\gamma}{2}+\delta+m'(q)\right)}z^{-\frac{\alpha}{\epsilon}}\mathrm{HeunC}_\infty(q,\alpha,\gamma,\delta,\epsilon;z^{-1})+ \\
&+e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F_D+\frac{1}{2}\partial_m F_D}\epsilon^{-\frac{1}{2}+\frac{\alpha}{\epsilon}-\frac{\gamma+\delta-\epsilon}{2}-m'(q)}\frac{\Gamma(\delta)}{\Gamma\left(\frac{\alpha}{\epsilon}-\frac{\gamma}{2}+m'(q)\right)}e^{-\epsilon z}z^{\frac{\alpha}{\epsilon}-\gamma-\delta}\mathrm{HeunC}_\infty(q-\gamma\epsilon,\alpha-\epsilon(\gamma+\delta),\gamma,\delta,-\epsilon;z^{-1})\,.
\end{aligned}
\tag{2.3.44}
$$

2.3.3 The reduced confluent Heun equation

The dictionary

Here we establish the dictionary between our results of section 2.2.3 on reduced confluent conformal blocks the reduced confluent Heun equation (RCHE) in standard notation, which reads

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dw}{dz} + \frac{\beta z - q}{z(z-1)}w = 0\,. \tag{2.3.45}$$

This is of course just the CHE specialized to[15] $\epsilon = 0$. The interesting difference with respect to the CHE is the behaviour for $z \to \infty$, which is no longer controlled by ϵ and the degree of the singularity gets lowered to $1/2$. By defining $w(z) = P_2(z)\psi(z)$ with $P_2(z) = z^{-\gamma/2}(1-z)^{-\delta/2}$, we pass to the normal form which is easily compared with the semiclassical BPZ equation (2.2.121). The dictionary between the CFT parameters and the parameters of the RCHE reads:

$$\begin{aligned}
a_0 &= \frac{1-\gamma}{2}\,,\\
a_1 &= \frac{1-\delta}{2}\,,\\
L &= 2i\sqrt{\beta}\,,\\
u &= \frac{1}{4} - q + \beta - \frac{(\gamma+\delta-1)^2}{4}\,,
\end{aligned} \tag{2.3.46}$$

where

$$u = \lim_{b\to 0} b^2\Lambda^2\partial_{\Lambda^2}\log {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L^2) \tag{2.3.47}$$

as in (2.2.121). This relation can then be inverted to find a in terms of the parameters of the RCHE: we denote this by $a(q)$. We therefore infer the relation between the solutions of the RCHE in standard form and the conformal blocks defined before. Near $z = 0$ we have the following two linearly independent solutions to the RCHE in standard form (2.3.45):

$$\begin{aligned}
&\text{HeunRC}(q,\beta,\gamma,\delta;z)\,,\\
&z^{1-\gamma}\text{HeunRC}\left(q-(1-\gamma)\delta,\beta,2-\gamma,\delta;z\right)\,,
\end{aligned} \tag{2.3.48}$$

where

$$F = \lim_{b\to 0} b^2\log\left[\Lambda^{-2\Delta}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda^2\right)\right]\,. \tag{2.3.49}$$

Since HeunRC is nothing else than HeunC with $\epsilon = 0$, it has the following expansion around $z = 0$:

$$\text{HeunRC}(q,\beta,\gamma,\delta;z) = 1 - \frac{q}{\gamma}z + \mathcal{O}(z^2)\,. \tag{2.3.50}$$

[15] This corresponds to the usual decoupling limit $m \to \infty, L \to 0$ such that mL remains finite.

Comparing with the conformal blocks in (2.2.3) we identify

$$\begin{aligned}
&\mathrm{HeunRC}(q,\beta,\gamma,\delta;z) = P_2(z) e^{-\frac{1}{2}\partial_{a_0}F} {}_{\frac{1}{2}}\mathcal{F}\left(a\ {}^{a_1}\ a_{0-}\ {}^{a_{2,1}}_{a_0}; L^2, z\right)\,,\\
&z^{1-\gamma}\mathrm{HeunRC}\left(q-(1-\gamma)\delta,\beta,2-\gamma,\delta;z\right) = P_2(z) e^{\frac{1}{2}\partial_{a_0}F} {}_{\frac{1}{2}}\mathcal{F}\left(a\ {}^{a_1}\ a_{0+}\ {}^{a_{2,1}}_{a_0}; L^2, z\right)\,,
\end{aligned} \tag{2.3.51}$$

Doing a Möbius transformation $z \to 1-z$ we obtain the solutions around $z=1$. Since this is a regular singularity the solution can again be written in terms of HeunRC. This amounts to sending $\gamma \to \delta$, $\delta \to \gamma$, $\beta \to -\beta$, $q \to q-\beta$. The two solutions are therefore

$$\begin{aligned}
&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)\,,\\
&(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right)\,.
\end{aligned} \tag{2.3.52}$$

Comparig with the conformal blocks we identify

$$\begin{aligned}
&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z) = P_2(z) e^{-\frac{1}{2}\partial_{a_1}F} {}_{\frac{1}{2}}\mathcal{F}\left(a\ {}^{a_0}\ a_{1-}\ {}^{a_{2,1}}_{a_1}; -L^2, 1-z\right)\,,\\
&(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right) =\\
&\quad = P_2(z) e^{\frac{1}{2}\partial_{a_1}F} {}_{\frac{1}{2}}\mathcal{F}\left(a\ {}^{a_0}\ a_{1+}\ {}^{a_{2,1}}_{a_1}; -L^2, 1-z\right)\,.
\end{aligned} \tag{2.3.53}$$

The new behaviour arises for $z \to \infty$, where we write the solutions in terms of another function HeunRC_∞:

$$\begin{aligned}
&e^{2i\sqrt{\beta z}} z^{\frac{1}{4}-\frac{\gamma+\delta}{2}} \mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})\\
&e^{-2i\sqrt{\beta z}} z^{\frac{1}{4}-\frac{\gamma+\delta}{2}} \mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}})\,.
\end{aligned} \tag{2.3.54}$$

The function HeunRC_∞ has a simple asymptotic expansion around $z=\infty$:

$$\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}}) \sim 1 - \frac{q-\beta+\left(\frac{\gamma+\delta}{2}-\frac{3}{4}\right)\left(\frac{\gamma+\delta}{2}-\frac{1}{4}\right)}{i\sqrt{\beta}} z^{-\frac{1}{2}} + \mathcal{O}(z^{-1})\,. \tag{2.3.55}$$

Comparing with the conformal blocks we identify

$$\begin{aligned}
&e^{2i\sqrt{\beta z}} z^{\frac{1}{4}-\frac{\gamma+\delta}{2}} \mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}}) = e^{\mp\frac{i\pi\delta}{2}} P_2(z) L^{\frac{1}{2}} {}_{\frac{1}{2}}\mathcal{E}^{(+)}\left(a_{2,1}\ a\ {}^{a_1}_{a_0}; L^2, \frac{1}{L\sqrt{z}}\right)\\
&e^{-2i\sqrt{\beta z}} z^{\frac{1}{4}-\frac{\gamma+\delta}{2}} \mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}}) = e^{\mp\frac{i\pi\delta}{2}} P_2(z) L^{\frac{1}{2}} {}_{\frac{1}{2}}\mathcal{E}^{(-)}\left(a_{2,1}\ a\ {}^{a_1}_{a_0}; L^2, \frac{1}{L\sqrt{z}}\right)\,.
\end{aligned} \tag{2.3.56}$$

Note that due to the nature of the rank 1/2 singularity at infinity, the expansion is in inverse powers of $\sqrt{z}$. The phase $e^{\mp\frac{i\pi\delta}{2}}$ comes from the fact that near $z=\infty$

$$P_2(z) \sim z^{-\gamma/2}(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}} z^{-\gamma/2-\delta/2} \,. \tag{2.3.57}$$

The second solution around $z=\infty$ can be found by using the manifest symmetry $L \to -L$ of the BPZ equation which according to the dictionary gives the symmetry $\beta \to e^{2\pi i}\beta$ of the RCHE in normal form.

Connection formulae

The connection formula between $z=0,1$ written in (2.2.127) for the semiclassical conformal blocks can now be restated as:

$$\begin{aligned}\mathrm{HeunRC}(q,\beta,\gamma,\delta;z) &= \frac{\Gamma(1-\delta)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F+\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{1+\gamma-\delta}{2}+a(q)\right)\Gamma\left(\frac{1+\gamma-\delta}{2}-a(q)\right)}\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)+ \\ &+\frac{\Gamma(\delta-1)\Gamma(\gamma)e^{-\frac{1}{2}\partial_{a_0}F-\frac{1}{2}\partial_{a_1}F}}{\Gamma\left(\frac{\gamma+\delta-1}{2}+a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-a(q)\right)}(1-z)^{1-\delta}\mathrm{HeunRC}\left(q-\beta-(1-\delta)\gamma,-\beta,2-\delta,\gamma;1-z\right)\,,\end{aligned} \tag{2.3.58}$$

where the quantities $a(q)$ and F are computed as explained in Appendix 2.C.
The connection formula between $z=1,\infty$ written in (2.2.128) reads

$$\begin{aligned}&\mathrm{HeunRC}(q-\beta,-\beta,\delta,\gamma;1-z)= \\ &=\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\left(e^{i\pi}\beta\right)^{-\frac{1}{4}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F-\frac{\sigma}{2}\partial_a F}}{2\sqrt{\pi}\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)}\right)e^{2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,\beta,\gamma,\delta;z^{-\frac{1}{2}})+ \\ &+\left(\sum_{\sigma=\pm}\frac{\Gamma(-2\sigma a(q))\Gamma(1-2\sigma a(q))\Gamma(\delta)\left(e^{-i\pi}\beta\right)^{-\frac{1}{4}+\sigma a(q)}e^{\pm\frac{i\pi\delta}{2}-\frac{1}{2}\partial_{a_1}F-\frac{\sigma}{2}\partial_a F}}{2\sqrt{\pi}\Gamma\left(\frac{1-\gamma+\delta}{2}-\sigma a(q)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}-\sigma a(q)\right)}\right)e^{-2i\sqrt{\beta z}}z^{\frac{1}{4}-\frac{\gamma+\delta}{2}}\mathrm{HeunRC}_\infty(q,e^{2\pi i}\beta,\gamma,\delta;z^{-\frac{1}{2}})\,.\end{aligned} \tag{2.3.59}$$

Here the phase ambiguity comes from (2.3.56), i.e. corresponds to the choice $(-z)^{-\delta/2} = e^{\pm\frac{i\pi\delta}{2}}z^{-\delta/2}$. A similar expression can be found connecting $z=0$ and ∞.

2.3.4 The doubly confluent Heun equation

The dictionary

The doubly confluent Heun equation (DCHE) reads

$$\left(\frac{d^2}{dz^2}+\frac{\delta+\gamma z+z^2}{z^2}\frac{d}{dz}+\frac{\alpha z-q}{z^2}\right)w(z)=0\,. \tag{2.3.60}$$

Again putting the DCHE in its normal form via the substitution $w(z)=\tilde{P}_2(z)\psi(z)$ with

$$\tilde{P}_2(z)=e^{\frac{1}{2}\left(\frac{\delta}{z}-z\right)}z^{-\frac{\gamma}{2}} \tag{2.3.61}$$

we find the $2^2 = 4$ different dictionaries with (2.2.147) corresponding to the $\mathbb{Z}_2^2$ symmetries $(m_i, L_i) \to (-m_i, -L_i)$ for $i = 1, 2$. For brevity we only write one of them, namely

$$\begin{aligned} L_1 &= 1\,, \\ L_2 &= \delta\,, \\ m_1 &= \frac{1}{2}\left(2\alpha - \gamma\right)\,, \\ m_2 &= 1 - \frac{\gamma}{2}\,, \\ u &= \frac{1}{4}\left(-4q + 2\gamma - \gamma^2 - 2\delta\right)\,. \end{aligned} \tag{2.3.62}$$

and the inverse dictionary is

$$\begin{aligned} \alpha &= 1 + m_1 - m_2\,, \\ \delta &= L_2\,, \\ \gamma &= 2(1 - m_2)\,, \\ q &= -\frac{1}{2}\left(L_2 + 2u + 2m_2(m_2 - 1)\right)\,, \\ L_1 &= 1\,. \end{aligned} \tag{2.3.63}$$

We denote the two solutions of the DCHE near the irregular singularity at zero as

$$\begin{aligned} &\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right)\,, \\ &e^{\frac{\delta}{z}} z^{2-\gamma} \text{HeunDC}\left(\delta + q + \gamma - 2, \alpha - \gamma + 2, \gamma, -\delta, z\right)\,, \end{aligned} \tag{2.3.64}$$

where HeunDC has the following asymptotic expansion around $z = 0$:

$$\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) \sim 1 + \frac{q}{\delta} z + \frac{q(q-\gamma) - \alpha\delta}{2\delta^2} z^2 + \mathcal{O}(z^3)\,. \tag{2.3.65}$$

Comparing with the semiclassical block (2.2.146) we get

$$\begin{aligned} &\text{HeunDC}\left(q, \alpha, \gamma, \delta, z\right) = \tilde{P}_2(z) L_2^{\frac{1}{2} - m_2} e^{-\frac{1}{2}\partial_{m_2} F} \left(z\, {}_1\mathcal{D}_1\left(m_2 \; {}^{a_{2,1}} \; m_{2-} \; a \; m_1; L_2, \frac{z}{L_2}\right)\right)\,, \\ &\text{HeunDC}\left(\delta + q + \gamma - 2, \alpha - \gamma + 2, \gamma, -\delta, z\right) = \tilde{P}_2(z) L_2^{\frac{1}{2} + m_2} e^{\frac{1}{2}\partial_{m_2} F} \left(z\, {}_1\mathcal{D}_1\left(m_2 \; {}^{a_{2,1}} \; m_{2+} \; a \; m_1; L_2, \frac{z}{L_2}\right)\right)\,. \end{aligned} \tag{2.3.66}$$

The solutions near the irregular singularity at infinity are given by

$$\begin{aligned} &z^{-\alpha} \text{HeunDC}\left(q - \alpha(\alpha + 1 - \gamma), \alpha, 2(\alpha + 1) - \gamma, \delta, -\frac{\delta}{z}\right)\,, \\ &e^{-z} z^{\alpha - \gamma} \text{HeunDC}\left(q + \delta + (\gamma - \alpha)(\alpha - 1), \gamma - \alpha, -2(\alpha - 1) + \gamma, -\delta, -\frac{\delta}{z}\right)\,. \end{aligned} \tag{2.3.67}$$

Comparing with the semiclassical block (2.2.149) we find

$$\begin{aligned}&\text{HeunDC}\left(q-\alpha(\alpha+1-\gamma),\alpha,2(\alpha+1)-\gamma,\delta,-\frac{\delta}{z}\right)=\tilde{P}_2(z)e^{\frac{1}{2}\partial_{m_1}F}{}_1\mathcal{D}_1\left(m_1\,{}^{a_{2,1}}\,m_{1+}\;a\;m_2;L_2,\frac{1}{z}\right),\\&\text{HeunDC}\left(q+\delta+(\gamma-\alpha)(\alpha-1),\gamma-\alpha,-2(\alpha-1)+\gamma,-\delta,-\frac{\delta}{z}\right)=\tilde{P}_2(z)e^{-\frac{1}{2}\partial_{m_1}F}{}_1\mathcal{D}_1\left(m_1\,{}^{a_{2,1}}\,m_{1-}\;a\;m_2;L_2,\frac{1}{z}\right).\end{aligned}\tag{2.3.68}$$

Connection formulae

In this case the only connection formula is the one between zero and infinity. This can be obtained from equation (2.2.151) and reads

$$\begin{aligned}\text{HeunDC}\,(q,\alpha,\gamma,\delta,z)=&\left(\sum_{\sigma=\pm}\frac{\Gamma\left(-2\sigma a\right)\Gamma\left(1-2\sigma a\right)\delta^{-\frac{1}{2}+\frac{\gamma}{2}+\sigma a}}{\Gamma\left(\frac{1}{2}-\left(1-\frac{\gamma}{2}\right)-\sigma a\right)\Gamma\left(\frac{1}{2}-\frac{2\alpha-\gamma}{2}-\sigma a\right)}\right)\times\\&\times e^{\frac{1}{2}\left(-\partial_{m_1}-\partial_{m_2}-\sigma\partial_a\right)F}z^{-\alpha}\text{HeunDC}\left(q-\alpha(\alpha+1-\gamma),\alpha,2(\alpha+1)-\gamma,\delta,-\frac{\delta}{z}\right)+\\+&\left(\sum_{\sigma=\pm}\frac{\Gamma\left(-2\sigma a\right)\Gamma\left(1-2\sigma a\right)\delta^{-\frac{1}{2}+\frac{\gamma}{2}+\sigma a}e^{i\pi\left(\frac{1+\gamma}{2}-\alpha-\sigma a\right)}}{\Gamma\left(\frac{1}{2}-\left(1-\frac{\gamma}{2}\right)-\sigma a\right)\Gamma\left(\frac{1}{2}+\frac{2\alpha-\gamma}{2}-\sigma a\right)}e^{\frac{1}{2}\left(\partial_{m_1}-\partial_{m_2}-\sigma\partial_a\right)F}\right)\times\\&\times e^{-z}z^{\alpha-\gamma}\text{HeunDC}\left(q+\delta+(\gamma-\alpha)(\alpha-1),\gamma-\alpha,-2(\alpha-1)+\gamma,-\delta,-\frac{\delta}{z}\right),\end{aligned}\tag{2.3.69}$$

2.3.5 The reduced doubly confluent Heun equation

The dictionary

Here we establish the dictionary between our results of section 2.2.5 on reduced doubly confluent conformal blocks and the reduced doubly confluent Heun equation (RDCHE) in the standard form, which reads

$$\frac{d^2w}{dz^2}-\frac{dw}{dz}+\frac{\beta z-q+\epsilon z^{-1}}{z^2}w=0\,.\tag{2.3.70}$$

By defining $w(z)=e^{z/2}\psi(z)$ we get rid of the first derivative and bring the equation to the normal form which is to be compared with the semiclassical BPZ equation (2.2.169). The resulting dictionary between the CFT parameters and the parameters of the RDCHE is

$$\begin{aligned}&L_1=1\,,\\&L_2=2i\sqrt{\epsilon}\,,\\&m=\beta\,,\\&u=-q\,.\end{aligned}\tag{2.3.71}$$

The fact that $L_1 = 1$ is of course consistent with the fact that it is a redundant parameter. Here

$$u = \lim_{b\to 0} b^2\Lambda_2^2\partial_{\Lambda_2^2}\log {}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\Lambda_2^2\right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1L_2^2) \tag{2.3.72}$$

as in (2.2.169). This relation can then be inverted to find a in terms of the parameters of the RDCHE: we denote this by $a(q)$. We can now write the solutions to the RDCHE in standard form and their relation to the conformal blocks by comparison. Near $z = 0$ we denote the two linearly independent solutions to the RDCHE in standard form (2.3.70) by:

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunRDC}_0(q,\beta,\epsilon;\sqrt{z})\,,\\ &e^{-2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunRDC}_0(q,\beta,e^{2\pi i}\epsilon;\sqrt{z})\,. \end{aligned} \tag{2.3.73}$$

The two solutions are related by the manifest symmetry $L_2 \to -L_2$ of the BPZ equation which according to the dictionary (2.3.71) gives the symmetry $\epsilon \to e^{2\pi i}\epsilon$ of the RDCHE in normal form. The function $\mathrm{HeunRDC}_0$ has the following asymptotic expansion around $z = 0$:

$$\mathrm{HeunRDC}_0(q,\beta,\epsilon;\sqrt{z}) \sim 1 - \frac{\frac{3}{16}+q}{i\sqrt{\epsilon}}\sqrt{z} + \mathcal{O}\left(z\right). \tag{2.3.74}$$

Note again that due to the presence of a rank 1/2 singularity, the expansion is in powers of $\sqrt{z}$. Comparing with the semiclassical conformal blocks in (2.2.5) we identify

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunRDC}_0(q,\beta,\epsilon;\sqrt{z}) = e^{z/2}L_2^{\frac{1}{2}}\,{}_1\mathcal{E}^{(+)}_{\frac{1}{2}}\left(m\,a\,a_{2,1}\,;L_2^2,\frac{\sqrt{z}}{L_2}\right),\\ &e^{-2i\sqrt{\epsilon/z}}z^{3/4}\mathrm{HeunRDC}_0(q,\beta,e^{2\pi i}\epsilon;\sqrt{z}) = e^{z/2}L_2^{\frac{1}{2}}\,{}_1\mathcal{E}^{(-)}_{\frac{1}{2}}\left(m\,a\,a_{2,1}\,;L_2^2,\frac{\sqrt{z}}{L_2}\right). \end{aligned} \tag{2.3.75}$$

For $z \sim \infty$ instead we have the two solutions

$$\begin{aligned} &z^{\beta}\mathrm{HeunRDC}_\infty(q,\beta,\epsilon;z^{-1})\,,\\ &e^{z}z^{-\beta}\mathrm{HeunRDC}_\infty(q,-\beta,-\epsilon;-z^{-1})\,. \end{aligned} \tag{2.3.76}$$

The function $\mathrm{HeunRDC}_\infty(q,\beta,\epsilon;z^{-1})$ has the following asymptotic expansion around $z = \infty$:

$$\mathrm{HeunRDC}_\infty(q,\beta,\epsilon;z^{-1}) \sim 1 + (q+\beta-\beta^2)z^{-1} + \mathcal{O}\left(z^{-2}\right). \tag{2.3.77}$$

Comparing with the semiclassical conformal blocks we identify

$$\begin{aligned} &z^{\beta}\mathrm{HeunRDC}_\infty(q,\beta,\epsilon;z^{-1}) = e^{z/2}e^{-\frac{1}{2}\partial_m F}\,{}_1\mathcal{D}_{\frac{1}{2}}\left(m\,{}^{a_{2,1}}\,m_-\;a\,;L_2^2,\frac{1}{z}\right),\\ &e^{z}z^{-\beta}\mathrm{HeunRDC}_\infty(q,-\beta,-\epsilon;-z^{-1}) = e^{z/2}e^{\frac{1}{2}\partial_m F}\,{}_1\mathcal{D}_{\frac{1}{2}}\left(m\,{}^{a_{2,1}}\,m_+\;a\,;L_2^2,\frac{1}{z}\right). \end{aligned} \tag{2.3.78}$$

These solutions are related by the symmetry $(m, L_1) \to (-m, -L_1)$ of the semiclassical BPZ equation. Notice that one can rescale the BPZ equation such that it only depends on the combination $L_1 z$ and the coefficient of the cubic pole is $-L_1 L_2^2/4$. By setting $L_1 = 1$ according to the dictionary with the RDCHE, the above symmetry descends to the symmetry $(\beta, \epsilon, z) \to (-\beta, -\epsilon, -z)$ of the RDCHE in normal form. Furthermore, in the equation above

$$F = \lim_{b\to 0} b^2 \log \left[(\Lambda_1 \Lambda_2^2)^{-\Delta}{}_1\mathfrak{F}_{\frac{1}{2}} \left(\mu\, \alpha; \Lambda_1 \Lambda_2^2 \right) \right] \tag{2.3.79}$$

as in (2.2.172).

Connection formulae

The connection formula between $z = 0$ and ∞ written in (2.2.175) for the semiclassical conformal blocks can now be restated as:

$$\begin{aligned} & e^{2i\sqrt{\epsilon/z}} z^{3/4} \mathrm{HeunRDC}_0(q, \beta, \epsilon; \sqrt{z}) = \\ & = \left(\sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\beta-\sigma a(q)\right)} \epsilon^{\frac{1}{4}+\sigma a(q)} e^{\frac{1}{2}\partial_m F - \frac{\sigma}{2}\partial_a F} e^{-i\pi\left(\frac{1}{4}+\sigma a(q)\right)} e^{i\pi\left(\frac{1}{2}-\beta-\sigma a(q)\right)} \right) z^{\beta} \mathrm{HeunRDC}_\infty(q, \beta, \epsilon; z^{-1}) + \\ & + \left(\sum_{\sigma=\pm} \frac{\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\beta-\sigma a(q)\right)} \epsilon^{\frac{1}{4}+\sigma a(q)} e^{\frac{1}{2}\partial_m F - \frac{\sigma}{2}\partial_a F} e^{-i\pi\left(\frac{1}{4}+\sigma a(q)\right)} \right) e^{z} z^{-\beta} \mathrm{HeunRDC}_\infty(q, -\beta, -\epsilon; -z^{-1}) , \end{aligned} \tag{2.3.80}$$

where the quantities $a(q)$ and F are computed as explained in Appendix 2.C.

2.3.6 The doubly reduced doubly confluent Heun equation

The dictionary

Here we establish the dictionary between our results of section 2.2.6 on doubly reduced doubly confluent conformal blocks and the corresponding Heun equation (DRDCHE) which reads

$$\frac{d^2 w}{dz^2} + \frac{z - q + \epsilon z^{-1}}{z^2} w = 0 . \tag{2.3.81}$$

This already takes the normal form of the semiclassical BPZ equation (2.2.189) and we immediately read off the dictionary:

$$\begin{aligned} L_1 &= 2i , \\ L_2 &= 2i\sqrt{\epsilon} , \\ u &= -q , \end{aligned} \tag{2.3.82}$$

where

$$u = \lim_{b\to 0} b^2 \Lambda_2^2 \partial_{\Lambda_2^2} \log {}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}} \left(\alpha; \Lambda_1^2 \Lambda_2^2 \right) = \frac{1}{4} - a^2 + \mathcal{O}(L_1^2 L_2^2) \tag{2.3.83}$$

as in (2.2.189). This relation can be inverted to find a in terms of the parameters of the DRDCHE: we denote this by $a(q)$. Near $z=0$ we denote the two linearly independent solutions to (2.3.81) by

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\text{HeunDRDC}(q,\epsilon;\sqrt{z})\,,\\ &e^{-2i\sqrt{\epsilon/z}}z^{3/4}\text{HeunDRDC}(q,e^{2\pi i}\epsilon;\sqrt{z})\,. \end{aligned} \tag{2.3.84}$$

The DRDC Heun function has a simple asymptotic expansion around $z=0$:

$$\text{HeunDRDC}(q,\epsilon;\sqrt{z})\sim 1-\frac{\frac{3}{16}+q}{i\sqrt{\epsilon}}\sqrt{z}+\mathcal{O}(z)\,. \tag{2.3.85}$$

Note that in the expansion, z appears with a square-root, and therefore mapping $z\to e^{2\pi i}z$ gives another solution. Comparing with the semiclassical conformal blocks in (2.2.6), we identify

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\text{HeunDRDC}(q,\epsilon;\sqrt{z}) = zL_2^{1/2}\,{}_{\frac{1}{2}}\mathcal{E}_{\frac{1}{2}}^{(+)}\left(a\,a_{2,1}\,;-4L_2^2,\frac{\sqrt{z}}{L_2}\right)\,,\\ &e^{-2i\sqrt{\epsilon/z}}z^{3/4}\text{HeunDRDC}(q,e^{2\pi i}\epsilon;\sqrt{z}) = zL_2^{1/2}\,{}_{\frac{1}{2}}\mathcal{E}_{\frac{1}{2}}^{(-)}\left(a\,a_{2,1}\,;-4L_2^2,\frac{\sqrt{z}}{L_2}\right)\,. \end{aligned} \tag{2.3.86}$$

Around $z=\infty$ we have the two linearly independent solutions

$$\begin{aligned} &e^{2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}})\,,\\ &e^{-2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}})\,, \end{aligned} \tag{2.3.87}$$

which we identify with the conformal blocks

$$\begin{aligned} &e^{2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}}) = \sqrt{2i}\,{}_{\frac{1}{2}}\mathcal{E}_{\frac{1}{2}}^{(+)}\left(a_{2,1}\,a\,;-4L_2^2,\frac{1}{2i\sqrt{z}}\right)\,,\\ &e^{-2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}}) = \sqrt{2i}\,{}_{\frac{1}{2}}\mathcal{E}_{\frac{1}{2}}^{(-)}\left(a_{2,1}\,a\,;-4L_2^2,\frac{1}{2i\sqrt{z}}\right)\,. \end{aligned} \tag{2.3.88}$$

Connection formulae

The connection formula (2.2.195) from 0 to ∞ in terms of the DRDC Heun functions is

$$\begin{aligned} &e^{2i\sqrt{\epsilon/z}}z^{3/4}\text{HeunDRDC}(q,\epsilon;\sqrt{z}) =\\ &=\left(\frac{-i}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\epsilon^{\frac{1}{4}+\sigma a(q)}e^{-\frac{\sigma}{2}\partial_a F}\right)e^{2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;(\epsilon z)^{-\frac{1}{2}})+\\ &+\left(\frac{1}{2\pi}\sum_{\sigma=\pm}\Gamma(1-2\sigma a(q))\Gamma(-2\sigma a(q))\epsilon^{\frac{1}{4}+\sigma a(q)}e^{-\frac{\sigma}{2}\partial_a F}e^{-2\pi i\sigma a(q)}\right)e^{-2i\sqrt{z}}z^{1/4}\text{HeunDRDC}(q,\epsilon;\left(e^{2\pi i}\epsilon z\right)^{-\frac{1}{2}})\,, \end{aligned} \tag{2.3.89}$$

where the quantities $a(q)$ and F are computed as explained in Appendix 2.C.

Appendix

2.A DOZZ factors and irregular generalizations

2.A.1 Regular case

We use conventions where $\Delta = \frac{Q^2}{4} - \alpha^2$, i.e. physical range of the momentum is $\alpha \in i\mathbb{R}^+$. The formula proposed by DOZZ for the Liouville three-point function is then [110, 111]

$$\begin{aligned}&\langle \Delta_1 | V_2(1) | \Delta_3 \rangle = C_{\alpha_1\alpha_2\alpha_3} = \\ &= \frac{\Upsilon_b'(0)\Upsilon_b(Q+2\alpha_1)\Upsilon_b(Q+2\alpha_2)\Upsilon_b(Q+2\alpha_3)}{\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\frac{Q}{2}+\alpha_1-\alpha_2+\alpha_3)\Upsilon_b(\frac{Q}{2}-\alpha_1+\alpha_2+\alpha_3)}\,.\end{aligned} \tag{2.A.1}$$

We neglect the dependence on the cosmological constant since its value is arbitrary and is not needed for the following discussion. We will not define the special function Υ_b and state all its remarkable properties, instead we refer to [122]. The most important property for us is the functional relation

$$\Upsilon_b(x+b) = \gamma(bx) b^{1-2bx}\Upsilon_b(x)\,, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{2.A.2}$$

The normalization of the states is obtained from the three-point function by taking the operator in the middle to be the identity operator, i.e. with $\Delta = 0$ which in our conventions means $\alpha = -\frac{Q}{2}$. One finds

$$\lim_{\epsilon\to 0} C_{\alpha_1, -\frac{Q}{2}+\epsilon, \alpha_2} = 2\pi\delta(\alpha_1-\alpha_2) G_{\alpha_1}\,, \tag{2.A.3}$$

with the two-point function G_α given by

$$G_\alpha = \frac{\Upsilon_b(2\alpha+Q)}{\Upsilon_b(2\alpha)}\,. \tag{2.A.4}$$

We use it to raise and lower indices: For example, OPE coefficients are given by

$$C^{\alpha_1}_{\alpha_2\alpha_3} = G^{-1}_{\alpha_1} C_{\alpha_1\alpha_2\alpha_3}\,. \tag{2.A.5}$$

We will be interested in the case where one of the fields is the degenerate field $\Phi_{2,1}$ with $\alpha_{2,1} = -\frac{2b+b^{-1}}{2}$, corresponding to $\Delta_{2,1} = -\frac{1}{2} - \frac{3b^2}{4}$. The fusion rules in this case impose that only two Verma modules appear in the OPE of this field with a primary:

$$\Phi_{2,1}(z)|\Delta\rangle = \sum_{\theta=\pm} z^{\frac{bQ}{2}+\theta b\alpha} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} |\Delta_\theta\rangle \left(1 + \mathcal{O}(z)\right) , \tag{2.A.6}$$

with

$$\alpha_\pm = \alpha \pm \left(-\frac{b}{2}\right) , \quad \Delta_\pm = \Delta_{\alpha_\pm} = \Delta \pm b\alpha - \frac{b^2}{4} . \tag{2.A.7}$$

Since the degenerate field is not in the physical spectrum, i.e. $\alpha_{2,1} \notin i\mathbb{R}^+$, the OPE coefficients $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$ have to be computed by analytic continuation of the DOZZ formula. This is tricky and is most easily performed by considering a four-point function, where the intermediate momentum is integrated over. During the analytic continuation one picks up residues of poles that cross the integration contour, and this in fact automatically imposes the fusion rules. In any case, the result is [123]:

$$C^{\alpha_+}_{\alpha_{2,1},\alpha} = 1 , \quad C^{\alpha_-}_{\alpha_{2,1},\alpha} = b^{2bQ} \frac{\gamma(2b\alpha)}{\gamma(bQ + 2b\alpha)} . \tag{2.A.8}$$

2.A.2 Rank 1

In section 2.1.2 we introduced the rank 1 irregular state, which can be given as a confluence limit of primary operators (here we consider only the chiral half):

$$\langle\mu, \Lambda| \propto \lim_{\eta\to\infty} t^{\Delta_t - \Delta} \langle\Delta|V_t(t) \tag{2.A.9}$$

with

$$\Delta = \frac{Q^2}{4} - \alpha^2 , \quad \alpha = -\frac{\eta+\mu}{2} , \quad \Delta_t = \frac{Q^2}{4} - \alpha_t^2 , \quad \alpha_t = \frac{\eta-\mu}{2} , \quad t = \frac{\eta}{\Lambda} . \tag{2.A.10}$$

This reproduces the desired Ward identities for the irregular state. To determine its normalization, we perform the collision limit on a (chiral+antichiral) three-point function, keeping track of the DOZZ factors. Although irrelevant for the Ward identities, the signs of α, α_t in (2.A.10) are crucial now. We find

$$\lim_{\eta\to\infty} (t\bar{t})^{\Delta_t - \Delta} \langle\Delta|V_t(t,\bar{t})|\Delta_0\rangle = (\Lambda\bar{\Lambda})^{\Delta_0} \lim_{\eta\to\infty} \eta^{-2\Delta_0} C_{-\frac{\eta+\mu}{2}, \frac{\eta-\mu}{2}, \alpha_0} . \tag{2.A.11}$$

Note that consistently with the main text, we consider the chiral and antichiral parts formally as independent and distinguish them by letting the "complex conjugation" formally act only on the coordinates t, Λ and not on the momenta α_0, μ, η. The asymptotic behaviour of the Υ_b function, valid for large imaginary x is:

$$\log \Upsilon_b\left(\frac{Q}{2} + x\right) = -\frac{1}{2}\Delta_x \log \Delta_x + \frac{1+Q^2}{12} \log \Delta_x + \frac{3}{2}\Delta_x + \mathcal{O}(x^0) . \tag{2.A.12}$$

We therefore find the following asymptotic behaviour of the DOZZ factor:

$$C_{-\frac{\eta+\mu}{2},\frac{\eta-\mu}{2},\alpha_0} \sim (-\eta^2)^{\Delta_0-\mu(Q-\mu)} \frac{\Upsilon_b(Q+2\alpha_0)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha_0)\Upsilon_b(\frac{Q}{2}+\mu-\alpha_0)} \,. \tag{2.A.13}$$

This suggests that we get a finite limit in (320) if we substract the factor of $(-\eta^2)^{-\mu(Q-\mu)}$ by hand. This can also be achieved by changing the power of t that we substract in the definition (2.A.9), but this would change the L_0-action on the irregular state, which we avoid. It is however precisely what is done in [47]. In any case, we find the following normalization of the irregular state:

$$\langle\mu,\Lambda|\Delta_0\rangle = \lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}|t|^{2\Delta_t-2\Delta}\langle\Delta|V_t(t,\bar{t})|\Delta_0\rangle = |\Lambda|^{2\Delta_0}C_{\mu\alpha_0}\,, \tag{2.A.14}$$

with normalization function

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha)\Upsilon_b(\frac{Q}{2}+\mu-\alpha)} \,. \tag{2.A.15}$$

The choice of the branch for the phase is consistent with the result found in 2.B.1.
In the text we also consider a different kind of collision limit, which reproduces the OPE between a primary operator and the irregular state. Performing this collision limit while keeping track of the DOZZ factors, we can extract the corresponding irregular OPE coefficient. In particular, consider the following correlation function, which we expand for large Λ:

$$\langle\mu,\Lambda|V_1(1)|\Delta_0\rangle = \int d\mu' B^{\mu'}_{\mu\alpha_1}C_{\mu'\alpha_0}\left|{}_1\mathfrak{D}\left(\mu \,{}^{\alpha_1}\, \mu' \,\alpha_0\,;\frac{1}{\Lambda}\right)\right|^2 \,. \tag{2.A.16}$$

Here $B^{\mu'}_{\mu\alpha_1}$ is the OPE coefficient corresponding to the OPE between the irregular state and V_1, $C_{\mu'\alpha_0}$ is the normalization function defined above and ${}_1\mathfrak{D}$ is just the corresponding conformal block. Following [104], we can express an irregular three-point function equivalently as a limit of a regular four-point function:

$$\begin{aligned}&\langle\mu,\Lambda|V_1(1)|\Delta_0\rangle = \lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}\int d\mu' C^{\alpha(\eta)}_{\alpha_\infty(\eta),\alpha_1}C_{\alpha(\eta),\alpha_t(\eta),\alpha_0}\times\\ &\times\left|e^{-(\mu'-\mu)\Lambda}\left(-\frac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(\frac{\Lambda}{\eta}\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\mathfrak{F}\left(\begin{matrix}\alpha_1\\ \alpha_\infty(\eta)\end{matrix}\,\alpha(\eta)\,\begin{matrix}\alpha_t(\eta)\\ \alpha_0\end{matrix};\frac{\eta}{\Lambda}\right)\right|^2,\end{aligned} \tag{2.A.17}$$

with

$$\alpha_\infty(\eta) = -\frac{\eta+\mu}{2}\,,\quad \alpha_t(\eta) = \frac{\eta-\mu}{2}\,,\quad \alpha(\eta) = -\frac{\eta-\mu}{2}-\mu'\,. \tag{2.A.18}$$

Several comments are in order: First, notice that in line with the definition of the irregular state we have multiplied by the same factors of $(-\eta^2)^{\mu(Q-\mu)}$ and $\left(\Lambda\bar{\Lambda}/\eta^2\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}$ as in

(2.A.14). Second, the remaining factors which we have put by hand are equal to 1 in the limit:

$$\lim_{\eta\to\infty} e^{-(\mu'-\mu)\Lambda}\left(-\tfrac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(1-\tfrac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} = \lim_{\eta\to\infty} e^{-(\mu'-\mu)\Lambda}\left(1-\tfrac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')} = 1\,. \tag{2.A.19}$$

Therefore all the factors that we put by hand are the same as if we had computed (2.A.17) by doing the OPE between V_1 and $|\Delta_0\rangle$ instead of between $\langle\mu,\Lambda|$ and V_1. This ensures crossing symmetry of the irregular three-point function. Furthermore, the factors inside the modulus square in the limit give the irregular conformal block up to an overall divergence, i.e.:

$$\begin{aligned} &e^{-(\mu'-\mu)\Lambda}\left(-\frac{\Lambda}{\eta}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\left(\frac{\Lambda}{\eta}\right)^{\Delta_\infty(\eta)-\Delta_t(\eta)}\left(1-\frac{\eta}{\Lambda}\right)^{\Delta_1-(\mu'-\mu)(\eta-\mu')}\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t(\eta)\\ \alpha_\infty(\eta) & \alpha(\eta) & \alpha_0\end{matrix};\frac{\eta}{\Lambda}\right)\longrightarrow \\ &\longrightarrow \eta^{-\Delta_0-\Delta_1-2\mu'(\mu'-\mu)}{}_1\mathfrak{D}\left(\mu\,{}^{\alpha_1}\,\mu'\,\alpha_0\,;\frac{1}{\Lambda}\right)\,,\quad \text{as}\,\eta\to\infty\,. \end{aligned} \tag{2.A.20}$$

This leaves us with

$$\begin{aligned} &\lim_{\eta\to\infty}(-\eta^2)^{\mu(Q-\mu)}(\eta^2)^{-\Delta_0-\Delta_1-2\mu'(\mu'-\mu)}C^{\alpha(\eta)}_{\alpha_\infty(\eta),\alpha_1}C_{\alpha(\eta),\alpha_t(\eta),\alpha_0} = \\ &= \frac{e^{-i\pi(\Delta_1+2\mu'(\mu'-\mu))}\Upsilon_b(Q+2\alpha_1)}{\Upsilon_b(\frac{Q}{2}+\mu'-\mu-\alpha_1)\Upsilon_b(\frac{Q}{2}+\mu'-\mu+\alpha_1)}\frac{e^{-i\pi\Delta_0}\Upsilon_b(Q+2\alpha_0)}{\Upsilon_b(\frac{Q}{2}+\mu'+\alpha_0)\Upsilon_b(\frac{Q}{2}+\mu'-\alpha_0)}\,, \end{aligned} \tag{2.A.21}$$

which remarkably has a finite limit. We recognize $C_{\mu'\alpha_0}$ and therefore we can identify

$$B^{\mu'}_{\mu\alpha_1} = \frac{e^{-i\pi(\Delta_1+2\mu'(\mu'-\mu))}\Upsilon_b(Q+2\alpha_1)}{\Upsilon_b(\frac{Q}{2}+\mu'-\mu-\alpha_1)\Upsilon_b(\frac{Q}{2}+\mu'-\mu+\alpha_1)}\,. \tag{2.A.22}$$

Specializing this formula to the case when V_1 is a degenerate field is again tricky and involves analytic continuation. It is simpler to perform the collision limit again. The fusion rules now imply that $\alpha(\eta) = \alpha_\infty(\eta) \pm (-b/2)$, i.e. $\mu' = \mu_\pm = \mu \pm (-b/2)$. Performing the collision limit using the degenerate OPE coefficients 2.A.8 one finds

$$B^{\mu_\theta}_{\mu\alpha_{2,1}} = e^{i\pi\left(\frac{1}{2}+\theta b\mu+\frac{b^2}{4}\right)}\,, \tag{2.A.23}$$

in agreement with the result (2.B.17).

2.A.3 Rank 1/2

Unfortunately, for the rank 1/2 state the situation is not as nice. It is clear that if we decouple another mass, the normalization function $C_{\mu\alpha}$ will diverge badly, since there are no Υ_b-functions in the numerator to compensate the divergence of the denominator. Indeed, it behaves as

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b(\frac{Q}{2}+\mu+\alpha)\Upsilon_b(\frac{Q}{2}+\mu-\alpha)} \to \text{const.} \times e^{3\mu^2}(-\mu^2)^{-\frac{1+Q^2}{6}-\mu^2+\Delta}e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)\,, \quad \text{as } \mu \to \infty\,. \tag{2.A.24}$$

The constant comes from the $\mathcal{O}(x^0)$ term in the expansion of the Υ_b-function (2.A.12). We neglect it in the following/consider it substracted by hand. This suggests we define

$$\langle\Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta}C_\alpha = \lim_{\mu\to\infty} e^{-3\mu^2}(-\mu^2)^{\frac{1+Q^2}{6}+\mu^2}\langle-\tfrac{\Lambda^2}{4\mu}|\Delta\rangle = |\Lambda^2|^{2\Delta}2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,, \tag{2.A.25}$$

where the factor of $-\frac{1}{4}$ is needed to reproduce the Ward identity $\langle\Lambda^2|L_1 = -\frac{\Lambda^2}{4}\langle\Lambda^2|$. This gives the normalization function for the rank 1/2 state as

$$C_\alpha = 2^{-4\Delta}e^{-2\pi i\Delta}\Upsilon_b(Q+2\alpha)\,, \tag{2.A.26}$$

in agreement with the result (2.B.34). Since no collision limit is known that reproduces the OPE between a primary and the rank 1/2 state, we cannot determine the corresponding OPE coefficient in the way we did in the previous section for the rank 1 state. For the case of a degenerate field however, we determine the OPE coefficient in Appendix 2.B.2.

2.B Irregular OPEs

2.B.1 Rank 1

The form of the (chiral) OPE of a general vertex operator with the irregular state introduced in section 2.1.2 is fixed by the Ward identities to be:

$$\langle\mu,\Lambda|V^{\Delta}_{\mu,\mu'}(z) = \sum_{k=0}^{\infty} z^{2\mu'(\mu'-\mu)-k}\Lambda^{\Delta+2\mu'(\mu'-\mu)}e^{-(\mu'-\mu)\Lambda z}\langle\mu',\Lambda;k|\,. \tag{2.B.1}$$

Here $V^{\Delta}_{\mu,\mu'}(z)$ is a vertex operator of weight Δ which maps from the Whittaker module specified by (μ,Λ), to the module specified by (μ',Λ). Furthermore $\langle\mu',\Lambda;k|$ are the ("generalized") descendants of the irregular state. They take the form

$$\langle\mu',\Lambda;k| = \sum c_{ijY}\Lambda^{-i}\partial^j_\Lambda\langle\mu',\Lambda|L_Y\,, \tag{2.B.2}$$

where c_{ijY} are coefficients fixed by the Ward identities and the sum runs over $i,j \geq 0$ and all Young tableaux Y such that $i+j+|Y| = k$. Furthermore we normalize $\langle\mu',\Lambda;0| \equiv \langle\mu',\Lambda|$. We then write the full (chiral+antichiral) OPE between the irregular state and a degenerate field as

$$\langle\mu,\Lambda|\Phi(z) = \sum_{\theta=\pm} B^{\mu_\theta}_{\mu,\alpha_{2,1}}\left|\sum_{k=0}^{\infty} e^{\theta b\Lambda z/2}\Lambda^{-\theta b\mu+\Delta_{2,1}+\frac{b^2}{2}}z^{-\theta b\mu+\frac{b^2}{2}-k}\right|^2\langle\mu_\theta,\Lambda;k,\bar{k}|\,, \tag{2.B.3}$$

where $B^{\mu_\theta}_{\mu,\alpha_{2,1}}$ are the corresponding irregular OPE coefficients. We have anticipated the fact that for the OPE with the degenerate field $\mu' = m_\pm = \mu \pm \frac{-b}{2}$ as will be shown later from the BPZ equation. Furthermore we now have both chiral and antichiral descendants which we label by k and $\bar{k}$, respectively.
We want to determine the irregular OPE coefficients B and the normalization function C introduced in (2.1.12). To this end consider the correlation function

$$\langle \mu, \Lambda | \Phi(z) | \Delta \rangle \,. \tag{2.B.4}$$

We can decompose it into irregular conformal blocks doing the OPE left or right as

$$\langle \mu, \Lambda | \Phi(z) | \Delta \rangle = \sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\mu\alpha_\theta} \left| {}_1\mathfrak{F}\left(\mu \, \alpha_\theta \, {\alpha_{2,1} \atop \alpha} ; \Lambda z \right) \right|^2 = \sum_{\theta'=\pm} B^{\mu_{\theta'}}_{\alpha_{2,1},\mu} C_{\mu_{\theta'}\alpha} \left| {}_1\mathfrak{D}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_{\theta'} \, \alpha ; \frac{1}{\Lambda z} \right) \right|^2 . \tag{2.B.5}$$

Here $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$ is just the usual (regular) OPE coefficient given in terms of the DOZZ formula, B is the irregular OPE coefficient to be determined, and $C_{\mu\alpha}$ is the normalization function of the irregular state, to be determined also. It is defined by

$$\langle \mu, \Lambda | \Delta \rangle = |\Lambda|^{2\Delta} C_{\mu\alpha} \,. \tag{2.B.6}$$

To determine B and C we use the BPZ equation

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} + \frac{\mu\Lambda}{z} - \frac{\Lambda^2}{4} \right) \langle \mu, \Lambda | \Phi(z) | \Delta \rangle = 0 \,. \tag{2.B.7}$$

This equation can be solved exactly and has the two solutions $z^{\frac{b^2}{2}} M_{b\mu,\pm b\alpha}(b\Lambda z)$, where M denotes the Whittaker function. It has a simple expansion around $z \sim 0$:

$$M_{b\mu,b\alpha}(b\Lambda z) = (b\Lambda z)^{\frac{1}{2}+b\alpha} \left(1 + \mathcal{O}(b\Lambda z)\right) \,. \tag{2.B.8}$$

Comparing this expansion with the leading term in the OPE between $\Phi(z)$ and $|\Delta\rangle$ we can identify

$${}_1\mathfrak{F}\left(\mu \, \alpha_\theta \, {\alpha_{2,1} \atop \alpha} ; \Lambda z \right) = \Lambda^{\Delta_\theta} z^{\frac{b^2}{2}} (b\Lambda)^{-\frac{1}{2}-\theta b\alpha} M_{b\mu,\theta b\alpha}(b\Lambda z) \,. \tag{2.B.9}$$

On the other hand, there exist two other solutions to the BPZ equation which have a simple expansion around $z \sim \infty$, namely the Whittaker W functions $W_{\pm b\mu,b\alpha}(\pm b\Lambda z)$. They have an asymptotic expansion at ∞ given by

$$W_{b\mu,b\alpha}(b\Lambda z) \sim e^{-b\Lambda z/2} (b\Lambda z)^{b\mu} \left(1 + \mathcal{O}((b\Lambda z)^{-1})\right) \,, \tag{2.B.10}$$

valid in the Stokes sector $|\arg(b\Lambda z)| < \frac{3\pi}{2}$. An important fact is that this function is invariant under $\alpha \to -\alpha$. We see that the expansion of the Whittaker W function (times the factor $z^{b^2/2}$) has exactly the form of the OPE between the irregular state and the degenerate field, with

$$\mu' = \mu_\pm = \mu \pm \left(-\frac{b}{2}\right) . \tag{2.B.11}$$

(Note that with this convention, $\mu_\pm$ corresponds to $W_{\mp b\mu, b\alpha}(\mp b\Lambda z)$. This may seem confusing but we like to keep the expression $\mu_\pm$ analogous to the fusion rules with a regular state which give $\alpha_\pm = \alpha \pm \frac{-b}{2}$).
Comparing the expansion of the W function with the irregular OPE (2.B.3), we can identify

$$\begin{aligned} {}_1\mathfrak{D}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_+ \, \alpha; \frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}} e^{-i\pi b\mu} b^{b\mu} (\Lambda z)^{\frac{b^2}{2}} W_{-b\mu, b\alpha}(e^{-i\pi} b\Lambda z) \, , \\ {}_1\mathfrak{D}\left(\mu \, {}^{\alpha_{2,1}} \, \mu_- \, \alpha; \frac{1}{\Lambda z}\right) &= \Lambda^{\Delta+\Delta_{2,1}} b^{-b\mu} (\Lambda z)^{\frac{b^2}{2}} W_{b\mu, b\alpha}(b\Lambda z) \, . \end{aligned} \tag{2.B.12}$$

For simplicity we focus on the branch specified by $-\Lambda = e^{-i\pi}\Lambda$ and use the asymptotic expansion (2.B.10) for both $b\Lambda z$ and $e^{-i\pi} b\Lambda z \to \infty$. This is valid for $-\frac{\pi}{2} < \arg(b\Lambda z) < \frac{3\pi}{2}$. The modulus squared has to be understood as acting by sending $\Lambda z \to \bar{\Lambda}\bar{z}$ and correspondingly $e^{-i\pi}\Lambda z \to e^{+i\pi}\bar{\Lambda}\bar{z}$. Since we have assumed $-\frac{\pi}{2} < \arg(b\Lambda z) < \frac{3\pi}{2}$, we also have $-\frac{\pi}{2} < \arg(e^{i\pi} b\bar{\Lambda}\bar{z}) < \frac{3\pi}{2}$, so all the asymptotic expansions are in their domain of validity. Similar expressions hold in the other Stokes sectors.
We can now restate the crossing symmetry condition (2.B.5) in terms of Whittaker functions and use the known connection formulae for them (see https://dlmf.nist.gov/13.14) to determine the normalization function C and the OPE coefficient B. We have

$$M_{\kappa,\mu}(z) = \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\kappa+\mu\right)} e^{i\pi\left(\frac{1}{2}-\kappa+\mu\right)} W_{\kappa,\mu}(z) + \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}-\kappa+\mu\right)} e^{-i\pi\kappa} W_{-\kappa,\mu}(e^{-i\pi} z) \, . \tag{2.B.13}$$

Plugging this into (2.B.5) using the identifications of the conformal blocks with the Whittaker functions we obtain the condition

$$\begin{aligned} \langle \mu, \Lambda | \Phi(z) | \Delta \rangle = & |\Lambda|^{2\Delta+2\Delta_{2,1}+b^2} \sum_{\theta=\pm} b^{-1-2\theta b\alpha} C^{\alpha_\theta}_{\alpha_{2,1},\alpha} C_{\mu\alpha_\theta} \Gamma(1+2\theta b\alpha)^2 \times \\ & \times \left| \frac{e^{i\pi\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}}{\Gamma\left(\frac{1}{2}+b\mu+\theta b\alpha\right)} z^{\frac{b^2}{2}} W_{b\mu,b\alpha}(b\Lambda z) + \frac{e^{-i\pi b\mu}}{\Gamma\left(\frac{1}{2}-b\mu+\theta b\alpha\right)} z^{\frac{b^2}{2}} W_{-b\mu,b\alpha}(e^{-i\pi} b\Lambda z) \right|^2 = \\ & = |\Lambda|^{2\Delta+2\Delta_{2,1}+b^2} B^{\mu_+}_{\alpha_{2,1},\mu} C_{\mu_+\alpha} \left| e^{-i\pi b\mu} b^{b\mu} z^{\frac{b^2}{2}} W_{-b\mu,b\alpha}(e^{-i\pi} b\Lambda z) \right|^2 + \\ & + |\Lambda|^{2\Delta+2\Delta_{2,1}+b^2} B^{\mu_-}_{\alpha_{2,1},\mu} C_{\mu_-,\alpha} \left| b^{-b\mu} z^{\frac{b^2}{2}} W_{b\mu,b\alpha}(b\Lambda z) \right|^2 , \end{aligned} \tag{2.B.14}$$

where we have used the fact that $W_{\kappa,-\mu}(z) = W_{\kappa,\mu}(z)$. Using the expression (2.A.8) for the coefficients $C^{\alpha_\theta}_{\alpha_{2,1},\alpha}$, the cancellation of the cross-terms in the modulus squared gives the following functional equation for $C_{\mu\alpha}$:

$$\frac{C_{\mu\alpha_+}}{C_{\mu\alpha_-}} = e^{-2\pi ib\alpha} b^{2bQ+4b\alpha} \frac{\gamma(-2b\alpha)\gamma\left(\frac{1}{2}+b\mu+b\alpha\right)}{\gamma(bQ+2b\alpha)\gamma\left(\frac{1}{2}+b\mu-b\alpha\right)}\,, \tag{2.B.15}$$

which is solved in terms of the usual Υ_b-function:

$$C_{\mu\alpha} = \frac{e^{-i\pi\Delta}\Upsilon_b(Q+2\alpha)}{\Upsilon_b\left(\frac{Q}{2}+\mu+\alpha\right)\Upsilon_b\left(\frac{Q}{2}+\mu-\alpha\right)}\,, \tag{2.B.16}$$

up to normalization and a periodic function of α with period b. We see however that the minimal choice is consistent with the result obtained by the collision limit in 2.A.2. Once we know the expression for $C_{\mu\alpha}$,we can compute the irregular OPE coefficients $B^{\mu\pm}_{\alpha_{2,1},\mu}$ from the diagonal terms in (2.B.14). The result is

$$B^{\mu\pm}_{\alpha_{2,1},\mu} = e^{i\pi\left(\frac{1}{2}\pm b\mu+\frac{b^2}{4}\right)}\,. \tag{2.B.17}$$

Again, we find that this is in agreement with the result found by the collision limit in 2.A.2. For completeness, let us write the connection formula for the conformal blocks $\mathfrak{F}$ and $\mathfrak{D}$, which solves the crossing symmetry constraint (2.B.5). Using the identification of the conformal blocks with the Whittaker functions with the correct prefactors we find

$$b^{\theta b\alpha}{}_1\mathfrak{F}\left(\mu\,\alpha_\theta\,\begin{matrix}\alpha_{2,1}\\ \alpha\end{matrix};\Lambda z\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}-\theta' b\mu}\mathcal{N}_{\theta\theta'}(b\alpha,b\mu)\,{}_1\mathfrak{D}\left(\mu\,\overset{\alpha_{2,1}}{}\,\mu_{\theta'}\,\alpha;\frac{1}{\Lambda z}\right)\,, \tag{2.B.18}$$

with irregular connection coefficients

$$\mathcal{N}_{\theta\theta'}(b\alpha,b\mu) = \frac{\Gamma(1+2\theta b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\alpha-\theta' b\mu\right)} e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}-b\mu+\theta b\alpha\right)}\,. \tag{2.B.19}$$

The inverse relation is

$$b^{-\frac{1}{2}-\theta b\mu}{}_1\mathfrak{D}\left(\mu\,\overset{\alpha_{2,1}}{}\,\mu_\theta\,\alpha;\frac{1}{\Lambda z}\right) = \sum_{\theta'=\pm} b^{\theta' b\alpha}\mathcal{N}^{-1}_{\theta\theta'}(b\mu,b\alpha)\,{}_1\mathfrak{F}\left(\mu\,\alpha_{\theta'}\,\begin{matrix}\alpha_{2,1}\\ \alpha\end{matrix};\Lambda z\right)\,, \tag{2.B.20}$$

with

$$\mathcal{N}^{-1}_{\theta\theta'}(b\mu,b\alpha) = \frac{\Gamma(-2\theta' b\alpha)}{\Gamma\left(\frac{1}{2}+\theta b\mu-\theta' b\alpha\right)} e^{i\pi\left(\frac{1+\theta}{2}\right)\left(-\frac{1}{2}-b\mu-\theta' b\alpha\right)}\,. \tag{2.B.21}$$

As a final remark, note that the Whittaker W-functions have a non-trivial monodromy around ∞. However, since for the correlator we considered, the monodromy around 0 and

∞ is the same, and by construction we have no monodromy around 0, the combination of W-functions appearing in the correlator expanded for large Λz is precisely such that the monodromy cancels. This can be checked also purely locally by carefully using the asymptotic expansions of the W-functions and its Stokes sectors. In particular, any other correlator involving this irregular state will have the same asymptotic behaviour and thus the normalization function $C_{\mu\alpha}$ ensures also the absence of monodromies for any other correlator.

2.B.2 Rank 1/2

Let us repeat the same arguments for the rank 1/2 irregular state introduced in section 2.1.3. The (chiral) OPE between the irregular state and the degenerate field is fixed by the Ward identities to be:

$$\langle\Lambda^2|\Phi_{\Lambda,\pm}(z) = \sum_{k=0}^{\infty}(\Lambda^2)^{-\frac{1}{4}-\frac{b^2}{4}} z^{\frac{1}{4}+\frac{b^2}{2}-\frac{k}{2}} e^{\pm b\Lambda\sqrt{z}}\langle\Lambda^2;\frac{k}{2}| \,. \tag{2.B.22}$$

Here $\langle\Lambda^2;\frac{k}{2}|$ are the ("generalized") descendants of the irregular state. They take the form

$$\langle\Lambda^2;\frac{k}{2}| = \sum c_{ijY}\Lambda^{-i}\partial_\Lambda^j\langle\Lambda^2|L_Y \,, \tag{2.B.23}$$

where c_{ijY} are coefficients fixed by the Ward identities and the sum runs over $i,j \geq 0$ and all Young tableaux Y such that $i+j+2|Y| = k$. In particular, note that only the integer descendants (i.e. $k \in 2\mathbb{Z}$) can contain Virasoro generators L_Y. Furthermore we normalize $\langle\Lambda^2;0| \equiv \langle\Lambda^2|$. Since both z-behaviours in (2.B.22) given by $\pm$ live in the same Bessel module specified by Λ, there is no canonical way of choosing a basis of solutions, in contrast to the rank 1 case. This ambiguity does not affect the physical correlator, since we have to sum over both solutions with the corresponding OPE coefficients. Changing the basis of conformal blocks changes the OPE coefficients in a way that the physical correlator is invariant. Consider the following correlation function involving the rank 1/2 state:

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle \,. \tag{2.B.24}$$

We can decompose it into conformal blocks by doing the OPE left and right:

$$\langle\Lambda^2|\Phi(z)|\Delta\rangle = \textstyle\sum_{\theta=\pm} C^{\alpha_\theta}_{\alpha_{2,1},\alpha}C_{\alpha_\theta}\left|{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right)\right|^2 = \textstyle\sum_{\theta'=\pm} B_{\alpha_{2,1}}C_\alpha\left|{}_{\frac{1}{2}}\mathfrak{C}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right)\right|^2 . \tag{2.B.25}$$

Here C_α is the normalization function of the irregular state, defined by

$$\langle\Lambda^2|\Delta\rangle = |\Lambda^2|^{2\Delta}C_\alpha \,, \tag{2.B.26}$$

which is to be determined. We also want to determine the irregular OPE coefficient $B_{\alpha_{2,1}}$. To do so, consider the BPZ equation that the correlator obeys:

$$\left(b^{-2}\partial_z^2 - \frac{1}{z}\partial_z + \frac{\Delta}{z^2} - \frac{\Lambda^2}{4z}\right)\langle\Lambda^2|\Phi(z)|\Delta\rangle = 0\,. \tag{2.B.27}$$

Solving this differential equation one identifies the conformal block corresponding to the expansion near 0 with a modified Bessel function:

$$_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \Gamma(1+2\theta b\alpha)\Lambda^{2\Delta_\theta}\left(\frac{b\Lambda}{2}\right)^{-2\theta b\alpha} z^{\frac{bQ}{2}} I_{2\theta b\alpha}(b\Lambda\sqrt{z})\,. \tag{2.B.28}$$

The prefactors are fixed by looking at the OPE between Φ and $|\Delta\rangle$ and using the expansion of the Bessel function:

$$I_{2\theta b\alpha}(b\Lambda\sqrt{z}) = \frac{(b\Lambda\sqrt{z}/2)^{2\theta b\alpha}}{\Gamma(1+2\theta b\alpha)}\left(1+\mathcal{O}(b\Lambda\sqrt{z})\right)\,. \tag{2.B.29}$$

On the other hand there are two other solutions to the BPZ equation given by the modified Bessel functions of the second kind $K_{2b\alpha}(\pm b\Lambda\sqrt{z})$. They have a nice behaviour at ∞, given by the asymptotic formula

$$K_{2b\alpha}(b\Lambda\sqrt{z}) \sim \sqrt{\frac{\pi}{2b\Lambda\sqrt{z}}}e^{-b\Lambda\sqrt{z}}(1+\mathcal{O}((b\Lambda\sqrt{z})^{-1}))\,. \tag{2.B.30}$$

Furthermore $K_{2b\alpha}(b\Lambda\sqrt{z}) = K_{-2b\alpha}(b\Lambda\sqrt{z})$. This expansion has precisely the form of the OPE between the irregular state and the degenerate field (2.B.22). We can therefore identify the necessary prefactors and defi1ne the irregular conformal blocks for $z\sim\infty$:

$$\begin{aligned} {}_{\frac{1}{2}}\mathfrak{E}^{(+)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}e^{-\frac{i\pi}{2}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(e^{-i\pi}b\Lambda\sqrt{z})\,,\\ {}_{\frac{1}{2}}\mathfrak{E}^{(-)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) &= \sqrt{\frac{2b}{\pi}}(\Lambda^2)^{\Delta-\frac{b^2}{4}}z^{\frac{bQ}{2}}K_{2b\alpha}(b\Lambda\sqrt{z})\,. \end{aligned} \tag{2.B.31}$$

We can now restate the crossing symmetry condition (2.B.25) in terms of Bessel functions and use the known connection formulae for them (see e.g. dlmf.nist.gov/10.27) to determine the normalization function C and the OPE coefficient $B_{\alpha_{2,1}}$. We have

$$I_\nu(z) = \frac{i}{\pi}e^{i\pi\nu}K_\nu(z) - \frac{i}{\pi}K_\nu(e^{-i\pi}z)\,. \tag{2.B.32}$$

Plugging this formula into (2.B.25) using the identifications between the conformal blocks and Bessel functions, one finds that the vanishing of the cross-terms gives the condition

$$\frac{C_{\alpha_+}}{C_{\alpha_-}} = 2^{-8b\alpha}b^{2bQ+8b\alpha}e^{-4\pi i b\alpha}\frac{\gamma(-2b\alpha)}{\gamma(bQ+2b\alpha)}\,. \tag{2.B.33}$$

We take the simplest solution, namely

$$C_\alpha = 2^{-4\Delta} e^{-2\pi i \Delta} \Upsilon_b(Q + 2\alpha)\,. \tag{2.B.34}$$

This is in agreement with the result found in 2.A.3. Once we have the expression for C_α, we can compute the irregular OPE coefficients from the diagonal terms of the crossing symmetry condition. The result is

$$B_{\alpha_{2,1}} = 2^{b^2} e^{\frac{i\pi b Q}{2}}\,. \tag{2.B.35}$$

We see that the OPE coefficients are independent of $\pm$, which is a reflection of the fact that we have a symmetry rotating the basis of conformal blocks into each other and leaving the physical correlator invariant.

For completeness, let us write also the connection formula for the irregular conformal blocks:

$$b^{2\theta b\alpha}\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_\theta\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right) = \sum_{\theta'=\pm} b^{-\frac{1}{2}}\,\mathcal{Q}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta')}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right)\,. \tag{2.B.36}$$

with irregular connection coefficients

$$\mathcal{Q}_{\theta\theta'}(b\alpha) = \frac{2^{2\theta b\alpha}}{\sqrt{2\pi}}\Gamma(1+2\theta b\alpha)e^{i\pi\left(\frac{1-\theta'}{2}\right)\left(\frac{1}{2}+2\theta b\alpha\right)}\,. \tag{2.B.37}$$

The inverse relation is

$$b^{-\frac{1}{2}}\,{}_{\frac{1}{2}}\mathfrak{E}^{(\theta)}\left(\alpha_{2,1}\,\alpha;\,\frac{1}{\Lambda\sqrt{z}}\right) = \sum_{\theta'=\pm} b^{2\theta' b\alpha}\,\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha)\,{}_{\frac{1}{2}}\mathfrak{F}\left(\alpha_{\theta'}\,\alpha_{2,1}\,\alpha;\,\Lambda\sqrt{z}\right)\,. \tag{2.B.38}$$

with irregular connection coefficients

$$\mathcal{Q}^{-1}_{\theta\theta'}(b\alpha) = \frac{2^{-2\theta' b\alpha}}{\sqrt{2\pi}}\Gamma(-2\theta' b\alpha)e^{-i\pi\left(\frac{1+\theta}{2}\right)\left(\frac{1}{2}+2\theta' b\alpha\right)}\,. \tag{2.B.39}$$

2.C Classical conformal blocks and accessory parameters

In this Appendix we give explicit combinatorial expressions for the classical conformal blocks used in the main text.

2.C.1 The regular case

Let us start with the case of regular conformal blocks. Via the AGT correspondence [41] the four-point regular conformal block is given by

$$\begin{aligned}\mathfrak{F}\left({\alpha_1 \atop \alpha_\infty}\alpha{\alpha_t \atop \alpha_0};t\right) &= t^{\Delta-\Delta_t-\Delta_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)}\times\\ &\times\sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_\infty\right)\,,\end{aligned} \tag{2.C.1}$$

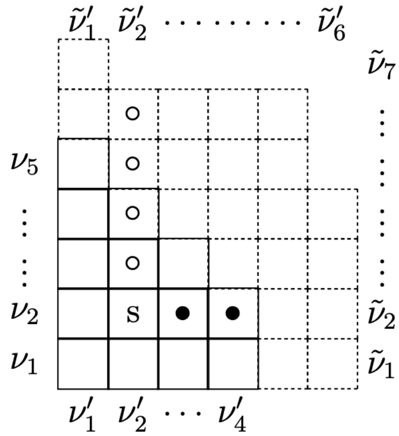


Figure 2.C.1: Arm length $A_{\tilde{Y}}(s) = 4$ (white circles) and leg length $L_Y(s) = 2$ (black dots) of a box at the site $s = (2, 2)$ for the pair of superimposed diagrams Y (solid lines) and $\tilde{Y}$ (dotted lines).

where the sum runs over all pairs of Young tableaux (Y_1, Y_2). We denote the size of the pair $|\vec{Y}| = |Y_1| + |Y_2|$, and [52, 124]

$$
\begin{aligned}
&z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \mu\right) = \prod_{k=1,2} \prod_{(i,j)\in Y_k} \left(\alpha_k + \mu + b^{-1}\left(i - \frac{1}{2}\right) + b\left(j - \frac{1}{2}\right)\right) ,\\
&z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) = \prod_{k,l=1,2} \prod_{(i,j)\in Y_k} E^{-1}\left(\alpha_k - \alpha_l, Y_k, Y_l, (i,j)\right) \prod_{(i',j')\in Y_l} \left(Q - E\left(\alpha_l - \alpha_k, Y_l, Y_k, (i',j')\right)\right)^{-1} ,\\
&E\left(\alpha, Y_1, Y_2, (i,j)\right) = \alpha - b^{-1} L_{Y_2}((i,j)) + b\left(A_{Y_1}((i,j)) + 1\right) .
\end{aligned}
\tag{2.C.2}
$$

Here $L_Y((i,j)), A_Y((i,j))$ denote respectively the leg-length and the arm-length of the box at the site (i,j) of the tableau Y. If we denote a Young tableau as $Y = (\nu'_1 \geq \nu'_2 \geq \dots)$ and its transpose as $Y^T = (\nu_1 \geq \nu_2 \geq \dots)$, then L_Y and A_Y read

$$
A_Y(i,j) = \nu'_i - j\,, \;\; L_Y(i,j) = \nu_j - i\,. \tag{2.C.3}
$$

Note that they can be negative if the box (i,j) are the coordinates of a box outside the tableau. Also, the previous formulae has to be evaluated at $\vec{\alpha} = (\alpha_1, \alpha_2) = (\alpha, -\alpha)$. Comparing (2.C.1) with (2.2.24) we find the explicit expression for the classical conformal block F:

$$
F(t) = \lim_{b\to 0} b^2 \log\left[(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)} \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \alpha_t + \theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \alpha_1 + \theta\alpha_\infty\right)\right] . \tag{2.C.4}
$$

This turns into a combinatorial expression of the u parameter defined as

$$u^{(0)} = \lim_{b\to 0} b^2 t\partial_t \log \mathfrak{F}\begin{pmatrix} \alpha_1 & & \alpha_t \\ & \alpha & ; t \\ \alpha_\infty & & \alpha_0 \end{pmatrix} = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F(t) \tag{2.C.5}$$

in terms of the intermediate momentum α. After substituting the dictionary with the Heun equation this gives a combinatorial expression of the accessory parameter q in terms of the Floquet exponent $a = b\alpha$. Inverting this relation order by order in t allows us to compute the connection coefficients in terms of the accessory parameter. Let us carry out explicitly a first order computation for the sake of clarity. At one instanton the relevant pairs of Young tableaux are $\vec{Y} = ((1),(0))$ and $\vec{Y} = ((0),(1))$. The various contributions give

$$\begin{aligned} z_{\text{hyp}}\left(\vec{\alpha},((1),(0)),\mu\right) &= \frac{Q}{2} + \alpha + \mu\,, \\ z_{\text{hyp}}\left(\vec{\alpha},((0),(1)),\mu\right) &= \frac{Q}{2} - \alpha + \mu\,, \end{aligned} \tag{2.C.6}$$

and since $A_{(0)}(i=1,j=1) = L_{(0)}(i=1,j=1) = -1$ and $A_{(1)}(i=1,j=1) = L_{(1)}(i=1,j=1) = 0$,

$$\begin{aligned} &E\left(0,(1),(1),(i=1,j=1)\right) = b\,, \\ &E\left(2\alpha,(1),(0),(i=1,j=1)\right) = Q + 2\alpha\,, \end{aligned} \tag{2.C.7}$$

therefore

$$\begin{aligned} z_{\text{vec}}\left(\vec{\alpha},((1),(0))\right) &= \prod_{l=1,2} E^{-1}\left(\alpha-\alpha_l,(1),Y_l,(i=1,j=1)\right) \prod_{k=1,2}\left(Q - E\left(\alpha-\alpha_k,(1),Y_k,(i'=1,j'=1)\right)\right)^{-1} \\ &= \frac{1}{-2\alpha\left(Q+2\alpha\right)}\,, \\ z_{\text{vec}}\left(\vec{\alpha},((0),(1))\right) &= \prod_{l=1,2} E^{-1}\left(-\alpha-\alpha_l,(1),Y_l,(i=1,j=1)\right) \prod_{k=1,2}\left(Q - E\left(-\alpha-\alpha_k,(1),Y_k,(i'=1,j'=1)\right)\right)^{-1} \\ &= \frac{1}{2\alpha\left(Q-2\alpha\right)}\,. \end{aligned} \tag{2.C.8}$$

Note that and that every time (i,j) have to run into an empty tableau, the corresponding term contributes with 1. Finally, substituting the previous results in (2.C.4) we get

$$F(t) = \frac{\left(\frac{1}{4} - a^2 - a_1^2 + a_\infty^2\right)\left(\frac{1}{4} - a^2 - a_t^2 + a_0^2\right)}{\frac{1}{2} - 2a^2} t + \mathcal{O}(t^2)\,. \tag{2.C.9}$$

In the main text we will need the derivatives of F expressed in terms of Heun parameters. For example,

$$\partial_{a_t} F(t) = \frac{\left(4a^2 - \alpha^2 + 2\alpha\beta - \beta^2 - 2\delta + \delta^2\right)(1-\epsilon)}{2 - 8a^2} t + \mathcal{O}(t^2)\,. \tag{2.C.10}$$

Moreover,

$$u^{(0)} = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + \frac{\left(\frac{1}{4} - a^2 - a_1^2 + a_\infty^2\right)\left(\frac{1}{4} - a^2 - a_t^2 + a_0^2\right)}{\frac{1}{2} - 2a^2} t + \mathcal{O}(t^2)\,. \tag{2.C.11}$$

Note that the relation between $u^{(0)}$ and a is quadratic at $t = 0$, therefore we will have two solutions for $a(u^{(0)})$:

$$a = \pm\sqrt{-\frac{1}{4} - u^{(0)} + a_t^2 + a_0^2}\left(1 - \frac{\left(-1 + 2a_0^2 + 2a_1^2 - 2a_\infty^2 + 2a_t^2 - 2u^{(0)}\right)\left(-1 + 4a_t^2 - 2u^{(0)}\right)}{2\left(-1 + 4a_0^2 + 4a_t^2 - 4u^{(0)}\right)\left(-1 + 2a_0^2 + 2a_t^2 - 2u^{(0)}\right)} t + \mathcal{O}(t^2)\right)\,. \tag{2.C.12}$$

Substituting the dictionary (2.3.3) we obtain

$$a = \pm\frac{1}{2}\sqrt{(\alpha+\beta-\delta)^2 - 4q} \mp \frac{t\left(\delta(q(\alpha+\beta+1) - \gamma(\alpha\beta+q)) + (q-\alpha\beta)(2q - \gamma(\alpha+\beta-1)) + \delta^2(-q)\right)}{\sqrt{(\alpha+\beta-\delta)^2-4q}(4q - (\alpha+\beta-\delta-1)(\alpha+\beta-\delta+1))} + O\left(t^2\right)\,. \tag{2.C.13}$$

Note that that all the connection formulae near the various singularity are all symmetric under $a \to -a$. The sign has to be carefully chosen only when connecting to the intermediate region. Finally, we are in the position to expand the connection coefficients. For example, one would have, choosing the lower sign in a,

$$\begin{aligned}\Gamma\left(\frac{1+\gamma-\epsilon}{2} + a\right) &\simeq \Gamma\left(\frac{1+\gamma-\epsilon-\sqrt{-4q+(\alpha+\beta-\delta)^2}}{2}\right)\times \\ &\times\left(1 + \frac{t\left(\delta(q(\alpha+\beta+1) - \gamma(\alpha\beta+q)) + (q-\alpha\beta)(2q-\gamma(\alpha+\beta-1)) + \delta^2(-q)\right)\psi_0\left(\frac{1+\gamma-\epsilon-\sqrt{-4q+(\alpha+\beta-\delta)^2}}{2}\right)}{\sqrt{(\alpha+\beta-\delta)^2-4q}(4q-(\alpha+\beta-\delta-1)(\alpha+\beta-\delta+1))}\right)\,,\end{aligned} \tag{2.C.14}$$

where ψ_0 is the Digamma function.

2.C.2 The confluent case

In order to discuss the confluent classical conformal block, let us write the four-point conformal block appearing in (2.2.39), that is

$$\begin{aligned}\mathfrak{F}\begin{pmatrix}\alpha_t & & \alpha_1 \\ & \alpha & & ; \frac{1}{t} \\ \alpha_\infty & & \alpha_0\end{pmatrix} &= t^{-\Delta+\Delta_1+\Delta_0}(1-t^{-1})^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)}\times \\ &\times\sum_{\vec{Y}} t^{-|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_t+\theta\alpha_\infty\right) z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right)\,.\end{aligned} \tag{2.C.15}$$

Note that in the decoupling limit (2.2.45), that is

$$\alpha_t + \alpha_\infty = -\mu\,,\ \alpha_t - \alpha_\infty = \eta\,,\ t = \frac{\Lambda}{\eta}\,, \tag{2.C.16}$$

where then $\eta \to \infty$,

$$\begin{aligned} &z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \alpha_t - \alpha_\infty\right) \sim (\alpha_t - \alpha_\infty)^{2|\vec{Y}|} \sim \left(\frac{\Lambda}{t}\right)^{2|\vec{Y}|}, \\ &z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \alpha_t + \alpha_\infty\right) = z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu\right), \\ &(1 - t^{-1})^{2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_t)} \sim e^{-(\frac{Q}{2}+\alpha_1)\Lambda}. \end{aligned} \tag{2.C.17}$$

Therefore the confluent 3-point function (2.2.80) has the following combinatorial expression

$$ {}_1\mathfrak{F}\left(\mu\,\alpha\,{}^{\alpha_1}_{\alpha_0};\Lambda\right) = \Lambda^{\Delta} e^{(\frac{Q}{2}+\alpha_1)\Lambda} \textstyle\sum_{\vec{Y}} \Lambda^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right) z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, -\mu\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha}, \vec{Y}, \alpha_1 + \theta\alpha_0\right). \tag{2.C.18}$$

As for the previous case, this turns into a combinatorial expression of the u parameter defined in equation 2.2.83 in terms of the intermediate momentum a, that after substituting the dictionary with the CHE gives a combinatorial expression for the accessory parameter in terms of the Floquet exponent. Again, inverting this relation is useful for computing the explicit connection coefficients. Similarly we can give an explicit expression of the classical conformal block for big Λ appearing in (2.2.93), that is

$$\begin{aligned} {}_1\mathfrak{D}\left(\mu\,{}^{\alpha_1}\,\mu'\,\alpha_0;\frac{1}{\Lambda}\right) &= \lim_{\eta\to\infty} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)} e^{-(\mu'-\mu)\Lambda} \left(1 - \frac{\eta}{\Lambda}\right)^{\Delta_1 - (\mu'-\mu)(\eta-\mu') - (\frac{Q}{2}+\alpha_1)(Q+\eta-\mu)} \times \\ &\times \sum_{\vec{Y}} \left(\frac{\eta}{\Lambda}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}(\eta), \vec{Y}\right) \prod_{\theta=\pm} z_{\text{hyp}}\left(\vec{\alpha}(\eta), \vec{Y}, \frac{\eta-\mu}{2} + \theta\alpha_0\right) z_{\text{hyp}}\left(\vec{\alpha}(\eta), \vec{Y}, \alpha_1 + \theta\frac{-\eta-\mu}{2}\right), \end{aligned} \tag{2.C.19}$$

where

$$\vec{\alpha}(\eta) = \left(-\frac{\eta-\mu}{2} - \mu', \frac{\eta-\mu}{2} + \mu'\right). \tag{2.C.20}$$

Again, this gives an explicit expression of the classical conformal block $F_D(L^{-1})$ recalling that

$$ {}_1\mathfrak{D}\left(\mu\,{}^{\alpha_1}\,\mu'\,\alpha_0;\frac{1}{\Lambda}\right) = e^{-(\mu'-\mu)\Lambda} \Lambda^{\Delta_0+\Delta_1+2\mu'(\mu'-\mu)} e^{\frac{1}{b^2}(F_D(L^{-1})+\mathcal{O}(b^2))}. \tag{2.C.21}$$

2.C.3 The reduced confluent case

To obtain the reduced confluent classical block we decouple the momentum μ starting from (2.C.15) as follows

$$\Lambda = -\frac{\Lambda_1\Lambda_2}{4\mu}, \text{ as } \mu \to \infty. \tag{2.C.22}$$

This gives

$$ {}_{\frac{1}{2}}\mathfrak{F}\left(\alpha\,\begin{matrix}\alpha_1\\ \alpha_0\end{matrix};\Lambda^2\right)=\Lambda^{2\Delta}\sum_{\vec{Y}}\left(\frac{\Lambda^2}{4}\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm}z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right)\,. \tag{2.C.23} $$

This gives for the classical conformal blocks

$$ F(L^2)=\lim_{b\to 0}b^2\log\sum_{\vec{Y}}\left(\frac{\Lambda^2}{4}\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)\prod_{\theta=\pm}z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\theta\alpha_0\right)\,. \tag{2.C.24} $$

2.C.4 The doubly confluent case

Let us consider the following decoupling limit of (2.C.15):

$$ \alpha_1+\alpha_0=-\mu_2\,,\;\;\alpha_1-\alpha_0=\eta\,,\;\;\Lambda\to\frac{\Lambda_1\Lambda_2}{\eta}\,,\;\text{as}\;\eta\to\infty\,. \tag{2.C.25} $$

This gives

$$ {}_1\mathfrak{F}_1\left(\mu_1\;\alpha\;\mu_2,\Lambda_1\Lambda_2\right)=\left(\Lambda_1\Lambda_2\right)^{\Delta}e^{\frac{\Lambda_1\Lambda_2}{2}}\sum\nolimits_{\vec{Y}}\left(\Lambda_1\Lambda_2\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu_1\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu_2\right)\,, \tag{2.C.26} $$

and

$$ F(L_1L_2)=\lim\nolimits_{b\to 0}b^2\log\left[e^{\frac{\Lambda_1\Lambda_2}{2}}\sum\nolimits_{\vec{Y}}\left(\Lambda_1\Lambda_2\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu_1\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu_2\right)\right]\,. \tag{2.C.27} $$

2.C.5 The reduced doubly confluent case

We now decouple μ_2 in (2.C.26) as follows

$$ \Lambda_2\to-\frac{\Lambda_2^2}{4\mu_2}\,,\;\text{as}\;\mu_2\to\infty\,. \tag{2.C.28} $$

Again,

$$ {}_1\mathfrak{F}_{\frac{1}{2}}\left(\mu\,\alpha;\Lambda_1\frac{\Lambda_2^2}{4}\right)=\left(\Lambda_1\Lambda_2^2\right)^{\Delta}\sum_{\vec{Y}}\left(\Lambda_1\Lambda_2^2\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu\right)\,. \tag{2.C.29} $$

Therefore the corresponding classical conformal block gives

$$ F(L_1L_2^2)=\lim_{b\to 0}b^2\log\sum_{\vec{Y}}\left(\Lambda_1\frac{\Lambda_2^2}{4}\right)^{|\vec{Y}|}z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right)z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},-\mu\right)\,. \tag{2.C.30} $$

2.C.6 The doubly reduced doubly confluent case

Decoupling the last momentum μ in (2.C.29) by setting

$$\Lambda_1 \to -\frac{\Lambda_1^2}{4\mu_1}\,, \text{ as } \mu \to \infty \tag{2.C.31}$$

gives

$${}_{\frac{1}{2}}\mathfrak{F}_{\frac{1}{2}}\left(\alpha; \Lambda_1^2\Lambda_2^2\right) = \left(\Lambda_1^2\Lambda_2^2\right)^{\Delta} \sum_{\vec{Y}} \left(\frac{\Lambda_1^2\Lambda_2^2}{16}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right)\,. \tag{2.C.32}$$

The corresponding classical conformal block gives

$$F(L_1^2L_2^2) = \lim_{b\to 0} b^2 \log\left[\sum_{\vec{Y}} \left(\frac{\Lambda_1^2\Lambda_2^2}{16}\right)^{|\vec{Y}|} z_{\text{vec}}\left(\vec{\alpha}, \vec{Y}\right)\right]\,. \tag{2.C.33}$$

2.D Combinatorial formula for the degenerate 5-point block

As for the four-point blocks in the previous Appendix, we give an explicit combinatorial expression for the degenerate 5-point conformal block introduced in 2.2.1 via the AGT correspondence. It can be computed as the partition function of $\mathcal{N}=2$ gauge theory with four flavours and a surface operator, or equivalently as a quiver gauge theory with specific masses fixed by the fusion rules of the degenerate field. Using the representation as a quiver gauge theory we find

$$\begin{aligned}
&\mathfrak{F}\left(\begin{matrix}\alpha_1 & & \alpha_t & & \alpha_{2,1} \\ & \alpha & & \alpha_{0\theta} & & ;t,\frac{z}{t} \\ \alpha_\infty & & & & \alpha_0\end{matrix}\right) = t^{\Delta-\Delta_t-\Delta_{0\theta}} z^{\frac{bQ}{2}+\theta b\alpha_0}(1-t)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}-\alpha_t)}\left(1-\frac{z}{t}\right)^{-2(\frac{Q}{2}+\alpha_t)(\frac{Q}{2}+\alpha_{2,1})}(1-z)^{-2(\frac{Q}{2}+\alpha_1)(\frac{Q}{2}+\alpha_{2,1})}\times \\
&\times \sum_{\vec{Y},\vec{W}} t^{|\vec{Y}|}\left(\frac{z}{t}\right)^{|\vec{W}|} z_{\text{vec}}\left(\vec{\alpha},\vec{Y}\right) z_{\text{vec}}\left(\vec{\alpha_{0\theta}},\vec{W}\right)\prod_{\sigma=\pm} z_{\text{hyp}}\left(\vec{\alpha},\vec{Y},\alpha_1+\sigma\alpha_\infty\right) z_{\text{hyp}}\left(\vec{\alpha_{0\theta}},\vec{W},\alpha_{2,1}+\sigma\alpha_0\right) z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\alpha_{0\theta}},\vec{W};\alpha_t\right)\,,
\end{aligned} \tag{2.D.1}$$

where the sum runs over two pairs of Young tableaux $\vec{Y}=(Y_1,Y_2)$ and $\vec{W}=(W_1,W_2)$. $\vec{\alpha_{0\theta}}$ has to be understood as $(\alpha_{0\theta},-\alpha_{0\theta})$ and we recall that $\alpha_{2,1}=-\frac{2b+b^{-1}}{2}$. Furthermore z_{vec} and z_{hyp} are defined as in (2.C.2). The new ingredient is the contribution of a bifundamental, defined as

$$\begin{aligned}
&z_{\text{bifund}}\left(\vec{\alpha},\vec{Y},\vec{\beta},\vec{W};\alpha_t\right) = \\
&= \prod_{k,l=1,2}\prod_{(i,j)\in Y_k}\left[E\left(\alpha_k-\beta_l,Y_k,W_l,(i,j)\right)-\left(\frac{Q}{2}+\alpha_t\right)\right]\prod_{(i',j')\in W_l}\left[Q-E\left(\beta_l-\alpha_k,W_l,Y_k,(i',j')\right)-\left(\frac{Q}{2}+\alpha_t\right)\right]\,,
\end{aligned} \tag{2.D.2}$$

with E as in (2.C.2).

Since all other conformal blocks are defined in terms of this degenerate 5-point block, the expression (2.D.1) can be used to compute any other block. In particular one can verify explicitly that the various confluence limits are finite.

Chapter 3

Holographic thermal correlators from supersymmetric instantons

In this chapter we study the thermal two-point function in a holographic four-dimensional CFT[1] [126–128] using techniques coming from four-dimensional supersymmetric gauge theories [42, 72, 73, 129, 130].

Finite temperature dynamics of CFTs is particularly rich in $d > 2$, where propagation of energy is not fixed by symmetries. On the gravity side, this is related to the presence of a propagating graviton in the spectrum of the theory, namely gravity waves.[2] On the field theory side, it is due to the fact that conformal symmetry is finite-dimensional in $d > 2$. This richness comes at a price that even for the simplest finite temperature observables no explicit solutions are available in $d > 2$.[3]

Here we provide the first example of such an explicit result. The thermal two-point function is computed by studying the wave equation on the black hole background [135–137]. This equation is of the Heun type [17–19], and the retarded two-point function is given in terms of its connection coefficients, which have been computed in the previous chapter (2). Starting with [72], a growing body of problems of this class have been solved using the connection to Seiberg-Witten theory and more precisely the Nekrasov-Shatashvili (NS) functions. These ideas have been applied to the study of black hole perturbation theory in [75, 138–144][4].

In particular this connection allows us to express the thermal two-point function in terms

[1]We consider a finite-temperature CFT on the sphere, $S^1_\beta \times S^3$, and on the plane, $S^1_\beta \times \mathbb{R}^3$. The former is related to the black hole geometry, and the latter to the black brane. The requirement of being holographic implies a large CFT central charge ($c_T \gg 1$), and a large gap in the spectrum of higher spin single trace operators ($\Delta_{\text{gap}} \gg 1$) [125].

[2]Another characteristic feature of black holes in $d > 2$ is the existence of stable orbits [99, 131, 132].

[3]Here we refer to the black hole phase. For the thermal AdS phase some explicit results exist [133]. They are also available in $d \leq 2$, see e.g. [134].

[4]See also [145–147] for a different approach based on Painlevé equations.

of the NS free energy [72] of an $SU(2)$ gauge theory with four fundamental hypermultiplets, and to study some of its basic properties both analytically and numerically. One particularly interesting regime is the large spin limit, where the exact formula produces the solution to the heavy-light light-cone bootstrap [83, 84]. We reproduce the available perturbative results from the literature [85–99] and make new predictions.

3.1 Holographic two-point function at finite temperature

3.1.1 Black hole

We consider a holographic conformal field theory at finite temperature. Above the Hawking-Page transition [148], this theory is dual to a black hole in AdS [149]. Here we will specialize to the case of AdS_5, where the black hole metric is

$$ds^2 = -f(r)\, dt^2 + f(r)^{-1}\, dr^2 + r^2 d\Omega_3^2. \tag{3.1.1}$$

Setting the AdS radius to 1, the redshift factor takes the form

$$f(r) = r^2 + 1 - \frac{\mu}{r^2} \equiv \left(1 - \frac{R_+^2}{r^2}\right)(r^2 + R_+^2 + 1), \tag{3.1.2}$$

where the Schwarzschild radius is given by

$$R_+ = \sqrt{\frac{\sqrt{1+4\mu} - 1}{2}}. \tag{3.1.3}$$

The dimensionless parameter μ is related to the black hole mass M by

$$\mu = \frac{8 G_N M}{3\pi} \ . \tag{3.1.4}$$

We are interested in the two-point function of a scalar operator $\mathcal{O}(x)$ with dimension Δ, dual to a massive scalar ϕ in the bulk with mass [150]

$$m = \sqrt{\Delta(\Delta - 4)}. \tag{3.1.5}$$

In order to compute this two-point function, we need to solve the wave equation on the black hole background,

$$(\Box - m^2)\phi = 0. \tag{3.1.6}$$

Expanding the solution into Fourier modes, we have

$$\phi(t, r, \Omega) = \int d\omega \sum_{\ell, \vec{m}} e^{-i\omega t} Y_{\ell\vec{m}}(\Omega) \psi_{\omega\ell}(r). \tag{3.1.7}$$

Our conventions for spherical harmonics $Y_{\ell\vec{m}}$ can be found in Appendix A of [12]. The wave equation then takes the form (see [102] and references there)

$$\left(\frac{1}{r^3}\partial_r(r^3 f(r)\partial_r) + \frac{\omega^2}{f(r)} - \frac{\ell(\ell+2)}{r^2} - \Delta(\Delta-4)\right)\psi_{\omega\ell} = 0. \tag{3.1.8}$$

We are interested in the retarded Green's function, and therefore we impose ingoing boundary conditions on the solution ϕ at the horizon,

$$\psi^{\text{in}}_{\omega\ell}(r) = (r - R_+)^{-\frac{i\omega}{2}\frac{R_+}{2R_+^2+1}} + \ldots \tag{3.1.9}$$

The solution ψ^{in} behaves near the AdS boundary $r \to \infty$ as

$$\psi^{\text{in}}_{\omega\ell}(r) = \mathcal{A}(\omega,\ell)(r^{\Delta-4} + \ldots) + \mathcal{B}(\omega,\ell)(r^{-\Delta} + \ldots). \tag{3.1.10}$$

The two-point function is then the ratio of the response $\mathcal{B}(\omega,\ell)$ to the source $\mathcal{A}(\omega,\ell)$ [135],

$$G_R(\omega,\ell) = \frac{\mathcal{B}(\omega,\ell)}{\mathcal{A}(\omega,\ell)}. \tag{3.1.11}$$

Our conventions for the thermal two-point function in the CFT dual are collected in Appendix 3.A.

The wave equation takes a particularly convenient form under the transformations

$$z = \frac{r^2}{r^2 + R_+^2 + 1}\ , \tag{3.1.12}$$

$$\psi_{\omega\ell}(r) = \left(r^3 f(r)\frac{dz}{dr}\right)^{-1/2}\chi_{\omega\ell}(z)\ . \tag{3.1.13}$$

We then obtain Heun's differential equation in normal form,

$$\left(\partial_z^2 + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} - \frac{\frac{1}{2} - a_0^2 - a_1^2 - a_t^2 + a_\infty^2 + u}{z(z-1)} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} + \frac{u}{z(z-t)} + \frac{\frac{1}{4} - a_0^2}{z^2}\right)\chi_{\omega\ell}(z) = 0. \tag{3.1.14}$$

Here the horizon is at $z = t$ and the AdS boundary is at $z = 1$.

In (3.1.14) we introduced a set of parameters that acquire a natural interpretation in the context of gauge theory that we discuss in the next section. They are defined in Table 3.1.1.

Finally, u is given by

$$u = -\frac{\ell(\ell+2) + 2(2R_+^2+1) + R_+^2\Delta(\Delta-4)}{4(R_+^2+1)} + \frac{R_+^2}{1+R_+^2}\frac{\omega^2}{4(2R_+^2+1)}. \tag{3.1.15}$$

Gauge theory	t	a_0	a_t	a_1	a_∞
Black hole	$\frac{R_+^2}{2R_+^2+1}$	0	$\frac{i\omega}{2}\frac{R_+}{2R_+^2+1}$	$\frac{\Delta-2}{2}$	$\frac{\omega}{2}\frac{\sqrt{R_+^2+1}}{2R_+^2+1}$

Table 3.1.1: Map from gauge theory to the black hole wave equation parameters.

The purely ingoing solution behaves near the black hole horizon as

$$\chi^{\rm in}_{\omega\ell}(z) = (t-z)^{\frac{1}{2}-a_t} + \dots \,. \tag{3.1.16}$$

Close to the AdS boundary it takes the form

$$\chi^{\rm in}_{\omega\ell}(z) \propto \mathcal{A}(\omega,\ell)\Big(\frac{1-z}{1+R_+^2}\Big)^{\frac{1}{2}-a_1} + \mathcal{B}(\omega,\ell)\Big(\frac{1-z}{1+R_+^2}\Big)^{\frac{1}{2}+a_1} + \dots \,.$$

The solutions to Heun's equation are known as Heun functions, see e.g. [19], and these can be written as an infinite series expanded around one of the singular points $z = 0, t, 1, \infty$. The problem of finding *the response function* (3.1.11) therefore reduces to finding the so-called *connection formulae* for the Heun function which express a given solution around one singular point (3.1.16) in terms of the basis of solutions around another singular point (3.1.17). The corresponding connection coefficients were computed explicitly in the previous chapter (2), [5] and we use these results in the present chapter.

3.1.2 Black brane

The black brane is dual to CFT on $S^1 \times \mathbb{R}^3$, and can be obtained by taking the high-temperature limit $T \to \infty$ of the black hole, while keeping $\frac{\omega}{T} \equiv \hat{\omega}$ and $\frac{\ell}{T} \equiv |\mathbf{k}|$ fixed. Here $\hat{\omega}$ and $\mathbf{k}$ are the dimensionless energy and three-momentum of the resulting theory on $S^1 \times \mathbb{R}^3$ in units of temperature. Recall that for the AdS-Schwarzschild black hole [149]

$$T = \frac{1}{\sqrt{2}\pi}\sqrt{\frac{1+4\mu}{\sqrt{1+4\mu}-1}}, \tag{3.1.17}$$

and the high-temperature limit corresponds to $\mu \to \infty$.

In this way we get the map between the gauge theory and gravity parameters for the black brane (to avoid clutter we switch from $\hat{\omega}$ to ω), see Table 3.1.2.
For u the relation takes the following form,

$$u = \frac{\omega^2 - 2\mathbf{k}^2}{8\pi^2} - \frac{1}{4}(\Delta-2)^2. \tag{3.1.18}$$

[5]See also [37, 151–155] for explicit relations between NS functions and the Heun equation.

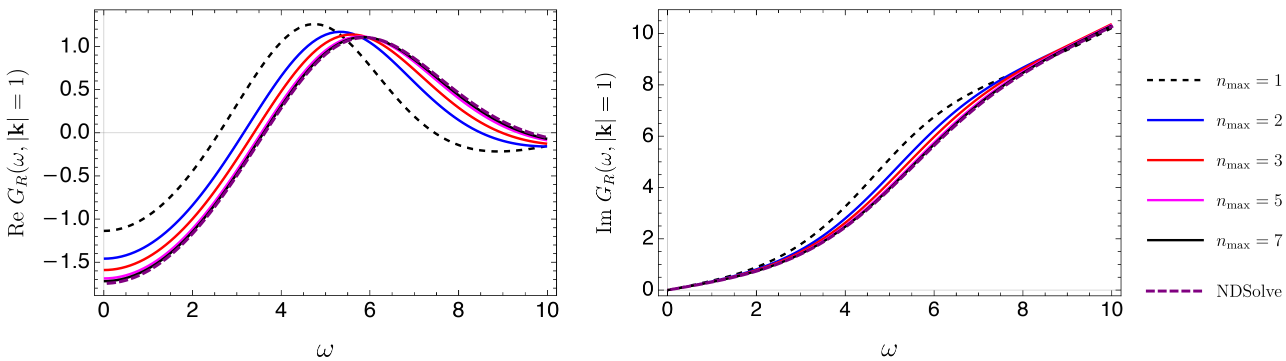
Gauge theory	t	a_0	a_t	a_1	a_∞
Black brane	$\frac{1}{2}$	0	$\frac{i\omega}{4\pi}$	$\frac{\Delta-2}{2}$	$\frac{\omega}{4\pi}$

Table 3.1.2: Map from gauge theory to the black brane wave equation parameters.

Finally, we define the two-point function as follows,

$$G_R^{\text{brane}}(\omega, |\mathbf{k}|) = \lim_{T\to\infty} \frac{G_R(\omega T, |\mathbf{k}|T)}{T^{4a_1}}, \tag{3.1.19}$$

see Appendix 3.B for the detailed derivation.



(a) (b)

Figure 3.1.1: We plot the retarded two-point function $G_R^{\text{brane}}(\omega, |\mathbf{k}|)$, given by (3.2.5) and (3.2.7), for $|\mathbf{k}| = 1$, $\Delta = 5/2$, as a function of ω and the maximal number of instantons $n_{\max}$ in the truncated sum (3.2.8). a) The real part of the retarded two-point function Re $G_R^{\text{brane}}(\omega, 1)$. b) The imaginary part of the retarded two-point function Im $G_R^{\text{brane}}(\omega, 1)$. We set $T = 1$. We also compare our results with the direct numerical solution of the differential equation (we used NDSolve in Mathematica), see e.g. [156], and find beautiful agreement between the two methods. An analogous plot can be generated for the $|\mathbf{k}|$-dependence as well, and again we observed perfect agreement between our formulas and the direct numerical solution of the differential equation.

3.2 Exact thermal two-point function

Heun's equation coincides with the quantum Seiberg-Witten curve describing the gauge theory with four flavors ($N_f = 4$) and therefore it can be solved exactly using the Nekrasov-Shatashvili (NS) functions [72]. Another way of understanding this connection is by using the AGT correspondence and the fact that Heun's equation corresponds to the semiclassical limit of the BPZ equation satisfied by the five-point function with one degenerate insertion, see for instance [37, 152–154]. For a review and a detailed list of references see [157]. Let us review the basic idea behind the exact solution of the connection problem. We consider a five-point function in the Liouville theory where one of the fields has been analytically continued to have degenerate quantum numbers. This five-point function satisfies the BPZ equation, which expresses the shortening of the Verma module of the degenerate field [27]. The BPZ equation reduces to the Heun equation in the semi-classical (large central charge) limit of the Liouville theory. The four singular points in the Heun equation correspond to insertions of the four operators (the fifth operator being the degenerate field). Crossing symmetry of the five-point function leads to crossing relations between the Virasoro blocks in different OPE channels. In the semi-classical limit these descend to the connection formulae for the solutions of the Heun equation. Thanks to the DOZZ formula [110, 111] the three-point functions that enter the crossing relations are explicitly known. Similarly, via the AGT correspondence [130] the relevant Virasoro blocks are expressed in terms of the partition function of the four-dimensional gauge theory which enters our final result. On the gauge theory side, the semi-classical limit corresponds to the so-called Nekrasov-Shatashvili limit [72]. The resulting expression for the connection coefficients can be found in [139].

From the gauge theory point of view, the parameters a_0, a_1, a_t, a_∞ are related to the masses of the hypermultiplets, $t \sim e^{-1/g^2_{\mathrm{YM}}}$ is the instanton counting parameter, and u parameterizes the moduli space of vacua. The latter is related to the VEV a of the scalar in the vector multiplet via the (quantum) Matone relation [158, 159]

$$u = -a^2 + a_t^2 - \frac{1}{4} + a_0^2 + t\partial_t F, \tag{3.2.1}$$

where F is the instanton part of the NS free energy defined in (3.C.2). The dictionary (3.2.1) requires a careful treatment close to the points $2a = \mathbb{Z}$, where the NS function exhibits non-analyticity, see e.g. [160, 161]. We leave a more detailed discussion of this region for future work.

In particular this hidden connection between Heun's equation and supersymmetric gauge theory makes it possible to compute the connection coefficients $\mathcal{A}$ and $\mathcal{B}$ in (3.1.10) using the NS free energy, as done in [139].

Let

$$\chi^{(t),\mathrm{in}}_{\omega\ell}(z) = (t-z)^{\frac{1}{2}-a_t} + \ldots \tag{3.2.2}$$

be the ingoing solution[6] of the wave equation (3.1.14) at the horizon ($z \sim t$) and let

$$\chi^{(1),\pm}_{\omega\ell}(z) = (1-z)^{\frac{1}{2}\pm a_1} + \ldots \tag{3.2.3}$$

[6] Here we have chosen the ingoing solution since we are interested in computing the retarded Green's function. Alternatively, the advanced Green's function can be computed by choosing the outgoing solution, resulting in a minor modification of (3.2.5).

be the two independent solutions at infinity ($z \sim 1$). The connection formula reads

$$\chi_{\omega\ell}^{(t),\mathrm{in}}(z) = \sum_{\theta'=\pm}\left(\sum_{\sigma=\pm}\mathcal{M}_{-\sigma}(a_t,a;a_0)\mathcal{M}_{(-\sigma)\theta'}(a,a_1;a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right)$$
$$t^{\frac{1}{2}-a_0-a_t}(1-t)^{a_t-a_1}e^{\frac{1}{2}(-\partial_{a_t}-\theta'\partial_{a_1})F}\chi_{\omega\ell}^{(1),\theta'}(z), \tag{3.2.4}$$

where

$$\mathcal{M}_{\theta\theta'}(\alpha_0,\alpha_1;\alpha_2) = \frac{\Gamma(-2\theta'\alpha_1)}{\Gamma\left(\frac{1}{2}+\theta\alpha_0-\theta'\alpha_1+\alpha_2\right)}\frac{\Gamma(1+2\theta\alpha_0)}{\Gamma\left(\frac{1}{2}+\theta\alpha_0-\theta'\alpha_1-\alpha_2\right)},$$

and F is the instanton part of the NS free energy defined in (3.C.2).

The exact formula for the retarded two-point function (3.1.11) then reads

$$\boxed{\begin{aligned}&G_R(\omega,\ell) = \left(1+R_+^2\right)^{2a_1}e^{-\partial_{a_1}F}\\&\frac{\sum_{\sigma'=\pm}\mathcal{M}_{-\sigma'}(a_t,a;a_0)\mathcal{M}_{(-\sigma')+}(a,a_1;a_\infty)t^{\sigma' a}e^{-\frac{\sigma'}{2}\partial_a F}}{\sum_{\sigma=\pm}\mathcal{M}_{-\sigma}(a_t,a;a_0)\mathcal{M}_{(-\sigma)-}(a,a_1;a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}}\end{aligned}} \tag{3.2.5}$$

where the parameters $t, a_0, a_t, a_1, a_\infty, u$ were defined in terms of ω, ℓ and the mass of the black hole μ in Table 3.1.1 and equation (3.1.15). The instanton part of the free energy F depends on all parameters, $F(t, a, a_0, a_t, a_1, a_\infty)$. Finally, we can eliminate a from the problem using the Matone relation (3.2.1). In this way the right hand side of (3.2.5) is fully fixed in terms of ω, ℓ and μ.

Based on general grounds, $G_R(\omega,\ell)$ should be analytic in the upper half-plane (causality), it satisfies $\mathrm{Im}\ G_R(\omega,\ell) = -\mathrm{Im}\ G_R(-\omega,\ell)$ (KMS), and finally $\mathrm{Im}\ G_R(\omega,\ell) \geq 0$ for $\omega > 0$ (unitarity), see e.g. appendix B in [162]. In fact from the standard dispersive representation of $G_R(\omega,\ell)$ it follows that

$$[G_R(-\omega,\ell)]^* = G_R(\omega,\ell), \quad \omega \in \mathbb{R}. \tag{3.2.6}$$

In the following we mostly limit our analysis to $\omega \in \mathbb{R}$ and it is easy to check that (3.2.5) indeed satisfies (3.2.6). The argument for this goes as follows. First, we notice that for real ω and ℓ, the relevant a is either purely imaginary or purely real. Second, we notice that (3.2.5) is invariant under the change $a \to \pm a$, $a_\infty \to \pm a_\infty$. Finally, the instanton partition function for real t is a real analytic function of its parameters, $F^*(a, a_0, a_t, a_1, a_\infty) = F(a^*, a_0^*, a_t^*, a_1^*, a_\infty^*)$. The property (3.2.6) then follows.

For the black brane, upon taking the limit (3.1.19) the result takes the form

$$G_R^{\mathrm{brane}}(\omega,|\mathbf{k}|) = \pi^{4a_1}\frac{G_R(\omega,\ell)}{\left(1+R_+^2\right)^{2a_1}}, \tag{3.2.7}$$

where $G_R(\omega,\ell)$ is taken from (3.2.5), but a_i, t, and u are now mapped to $(\omega,\mathbf{k})$ according to Table 3.1.2 and equation (3.1.18). In (3.2.7) the temperature for the theory on $S^1\times\mathbb{R}^3$ is set to 1.

The exact expressions presented above involve in a crucial way the NS free energy. As explained in Appendix 3.C, the NS free energy is computed as a (convergent) series expansion in the instanton counting parameter t,

$$F=\sum_{n\geq 1}^{\infty} c_n(a,a_0,a_t,a_1,a_\infty)t^n\ . \tag{3.2.8}$$

The coefficients $c_n(a,a_0,a_t,a_1,a_\infty)$ in this series have a precise combinatorial definition in terms of Young diagrams. Hence in principle we can determine all of them. Given (3.2.8) one can straightforwardly solve the Matone relation (3.2.1) as a series in t as well.

We can also write the above equation in a compact way by using the full NS free energy F^{NS} (3.C.6), which is the sum of the instanton part F, the one-loop part $F^{1-\text{loop}}$, and the classical term $F^{\text{p}}=-2a\log t$. The formula becomes

$$G_R(\omega,\ell)=(1+R_+^2)^{2a_1}\frac{\Gamma\left(-2a_1\right)}{\Gamma\left(2a_1\right)}\frac{\mathcal{G}(t,a,a_0,a_1,a_\infty,a_t)}{\mathcal{G}(t,a,a_0,-a_1,a_\infty,a_t)} \tag{3.2.9}$$

with

$$\mathcal{G}(t,a,a_0,a_1,a_\infty,a_t)=e^{-\frac{1}{2}\partial_{a_1}F^{\text{NS}}}\sinh\left(\frac{\partial_a F^{\text{NS}}}{2}\right)\ . \tag{3.2.10}$$

This is the typical form of the Fredholm determinant in this class of theories [163, eq. 8.12], [164, eq. 5.6], see also [165, 166]. Note that the result for the two-point function has the following simple property: under $\Delta\to 4-\Delta$ we have $G_R\to\frac{1}{G_R}$. This property is manifest in (3.2.9) after noticing that under this transformation $a_1\to -a_1$. It is also expected on general grounds because sending $\Delta\to 4-\Delta$ switches the boundary conditions [167], so that the source and response are interchanged.

One case where the exact Green's function (3.2.5) becomes analytically tractable is the limit where ℓ is the only large parameter. On the gauge theory side this means that the VEV of the scalar a is much larger than all other parameters. In this limit one can use Zamolodchikov's formula for the Virasoro conformal blocks [168] and the AGT correspondence [130] to show that [169]

$$\begin{aligned}F=&\,a^2\left(\log\frac{t}{16}+\pi\frac{K(1-t)}{K(t)}\right)\\&+\left(a_1^2+a_t^2-\frac{1}{4}\right)\log(1-t)\\&+2\left(a_0^2+a_t^2+a_1^2+a_\infty^2-\frac{1}{4}\right)\log\left(\frac{2}{\pi}K(t)\right)+\mathcal{O}(a^{-2})\,.\end{aligned} \tag{3.2.11}$$

Here $K(t)$ is the complete elliptic integral of the first kind. Solving the Matone relation (3.2.1) for a, we find

$$a = -\frac{(\ell+1)\sqrt{1-2t}K(t)}{\pi} + \mathcal{O}(\ell^{-1}). \tag{3.2.12}$$

In Appendix 3.F we use (3.2.12) to show that the imaginary part of G_R is exponentially small at large ℓ.

Let us conclude this section with a practical comment. When doing the actual computations we truncate the series in t at some maximal instanton number $n_{\max}$. Given $n_{\max}$ and the corresponding $F^{n_{\max}}$, we then solve (3.2.1) for a as a function of u perturbatively in t. This step requires solving a linear equation at every new order in t. Finally, we plug both $F^{n_{\max}}$ and $a^{n_{\max}}(u)$ in (3.2.9) and evaluate $G_R^{n_{\max}}(\omega, \ell)$. We present an example of this procedure for $n_{\max} \leq 7$ and the case of the black brane in Figure 3.1.1.[7] We find a beautiful agreement between our result and the direct numerical solution of the wave equation.

With the methods we used, going to higher $n_{\max}$ gets computationally costly rather quickly. For example, in the case of the $N_f = 4$ theory that we are interested in, going beyond 5-10 instantons appears challenging on a laptop. Hence to fully exploit the power of our method it would be important to identify the range of parameters for which $G_R(\omega, \ell)$ can be reliably computed with a few instantons. It would also be desirable to develop a more efficient way of computing the NS functions (either analytically or numerically).[8]

3.3 Relation to the heavy-light conformal bootstrap

The thermal two-point function computed in the previous section is directly related to the four-point correlation function of local operators $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ [171, 172]. Here $\mathcal{O}_L$ is the light or probe operator of dimension Δ_L from the previous section,[9] and $\mathcal{O}_H$ is a heavy operator with $\Delta_H \sim c_T$ that is dual to a black hole microstate, where c_T parameterizes the two-point function of canonically normalized stress tensors. For the precise relationship between $\mu \sim \frac{\Delta_H}{c_T}$, Δ_H and c_T see e.g. [85].

More precisely, we define the four-point function as follows

$$G(z, \bar{z}) \equiv \langle \mathcal{O}_H(0) \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(1, 1) \mathcal{O}_H(\infty) \rangle, \tag{3.3.1}$$

where all operators for simplicity are taken to be real scalars. The insertion at infinity is given by $\mathcal{O}_H(\infty) = \lim_{x_4 \to \infty} |x_4|^{2\Delta_H} \mathcal{O}_H(x_4)$. We also used conformal symmetry to put all four operators in a two-dimensional plane with coordinate $z = x^1 + ix^2$.

[7] Alternatively, we can use (3.2.5) to compute $G_R(\omega, a)$ and we can use (3.2.1) to evaluate the map $\ell(\omega, a)$ (or $\mathbf{k}(\omega, a)$). This is possible because the dependence on spin ℓ (or momentum $\mathbf{k}$) enters the problem only through the parameter u, which does not appear in the exact formula (3.2.5).

[8] For example using TBA-like techniques as in [170] and references there.

[9] In this section we switch from Δ to Δ_L to make the distinction between the light and heavy operators more obvious.

We choose the normalization of operators such that in the short distance limit $z, \bar{z} \to 1$ we have

$$G(z,\bar{z}) = \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}} + \dots \,. \tag{3.3.2}$$

This four-point function admits an OPE expansion in various channels, see e.g. [173]. We focus on the heavy-light channel, in which the expansion of the four-point function takes the form

$$G(z,\bar{z}) = \sum_{\mathcal{O}_{\Delta,\ell}} \lambda^2_{H,L,\mathcal{O}_{\Delta,\ell}} \frac{g^{\Delta_{H,L},-\Delta_{H,L}}_{\Delta,\ell}(z,\bar{z})}{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}}, \tag{3.3.3}$$

where $\Delta_{H,L} \equiv \Delta_H - \Delta_L$, and $\lambda_{H,L,\mathcal{O}_{\Delta,\ell}} \in \mathbb{R}$ are the three-point functions. Finally, the expressions for the conformal blocks $g^{\Delta_{H,L},-\Delta_{H,L}}_{\Delta,\ell}(z,\bar{z})$ can be found for example in [174, 175].

We next consider the $\Delta_H, c_T \to \infty$ limit of the expansion of $G(z,\bar{z})$ above with $\mu = \frac{160}{3}\frac{\Delta_H}{c_T}$ kept fixed. In this limit the spectrum of operators becomes effectively continuous and the contribution of descendants is suppressed [173].[10] Specializing to $d = 4$, we get the following expression for the OPE expansion,

$$G(z,\bar{z}) = \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} d\omega \, g_{\omega,\ell} (z\bar{z})^{\frac{\omega-\Delta-\ell}{2}} \frac{z^{\ell+1} - \bar{z}^{\ell+1}}{z-\bar{z}} \,, \tag{3.3.4}$$

where we introduced $\omega = \Delta'_H - \Delta_H$, and $g_{\omega,\ell}$ for the product of the three-point functions $\lambda^2_{H,L,\mathcal{O}_{\Delta_{H'},\ell}}$ and the density of primaries. Thanks to unitarity we have $g_{\omega,\ell} \geq 0$ and KMS symmetry implies that

$$g_{-\omega,\ell} = e^{-\beta\omega} g_{\omega,\ell} \,. \tag{3.3.5}$$

We can now state the precise relationship between the heavy-light four-point function and the thermal two-point function [99],

$$g_{\omega,\ell} = \frac{\ell+1}{2\pi(\Delta_L-1)(\Delta_L-2)} \frac{\text{Im}\, G_R(\omega,\ell)}{1-e^{-\beta\omega}} \,, \tag{3.3.6}$$

where β and Δ_H are related in the standard way, $\beta = \frac{\partial S(\Delta_H)}{\partial \Delta_H}$. In this formula $S(\Delta_H)$ is the effective density of primaries of dimension Δ_H. This relation is the combination of the eigenstate thermalization hypothesis [171, 172, 176, 177] and the standard relations between

[10] This requires an extra assumption on which operators dominate the OPE, see e.g. the discussion in [99].

various thermal two-point functions [162]. The factor $\ell + 1$ originates from summing over $\vec{m}$ of the spherical harmonics $Y_{\ell\vec{m}}$, see Appendix A of [12] for details.

There is a natural limit in which the general expression (3.3.6) simplifies: it is the large spin limit $\ell \to \infty$. As explained in detail in [99, 132], in this limit the relevant states are orbits which are stable perturbatively in $\frac{1}{\ell}$. These states manifest themselves in $G_R(\omega, \ell)$ as poles (also known as quasi-normal modes) with imaginary part which is non-perturbative in spin ℓ. Therefore, perturbatively in ℓ, Im $G_R(\omega, \ell)$ effectively becomes the sum of $\delta(|\omega| - \omega_{n\ell})$, where $\omega_{n\ell} = \Delta_L + \ell + 2n + \gamma_{n\ell}$ and $\gamma_{n\ell} \to 0$ at large spin. Notice that for $|\omega| \sim \ell$, $[(1 - e^{-\beta\omega})^{-1}]_{\text{pert}}$ becomes a step function $\theta(\omega)$, and in this way $g_{\omega,\ell}$ reduces at large spin to the expected sum over heavy-light double-twist operators $\mathcal{O}_H \Box^n \partial^\ell \mathcal{O}_L$.

We can summarize this as follows

$$
\begin{aligned}
g^{\text{pert}}_{\omega,\ell} &= \theta(\omega) \frac{\ell + 1}{2\pi(\Delta_L - 1)(\Delta_L - 2)} \text{Im } G^{\text{pert}}_R(\omega, \ell) \\
&= \sum_{n=0}^{\infty} c_{n\ell} \delta(\omega - \omega_{n\ell}) \ ,
\end{aligned}
\tag{3.3.7}
$$

where the relation holds for all the terms which contribute as powers at large spin ℓ, namely $\frac{1}{\ell^{\#}}$. We signified this by writing Im $G^{\text{pert}}_R(\omega, \ell)$ (see also Section 3.4 for a more precise definition). Here $c_{n\ell}$ is the square of the OPE coefficients of double-twist operators. In writing (3.3.7) we also used the fact that at fixed ω, Im $G_R(\omega, \ell)$ is nonperturbative in spin at large ℓ.[11] We establish this fact in Appendix 3.F.

The large spin expansion of the heavy-light four-point function was actively explored in the last few years [85–98]. One of the basic observations of these works is that in $d > 2$ the effective expansion parameter is $\frac{\mu}{\ell^{\frac{d-2}{2}}}$. We can therefore equivalently study the small μ expansion of the exact results. This is what we do in the next section.

3.4 Small μ expansion

In the previous section we explained how to compute the dimensions and OPE data of heavy-light double-twist operators using the exact two-point function (3.2.5). Now we would like to carry out this procedure perturbatively in $1/\ell$. Note that the expected perturbative parameter is $\frac{\mu}{\ell}$ [85–98], so that instead of taking the large spin limit, we can equivalently consider the limit of small black holes. This is a natural limit from the point of view of the Nekrasov-Shatashvili functions, which are defined as a perturbative expansion in $t \sim \mu$ for small μ.

[11] In principle, non-perturbative in spin effects are accessible to the light-cone bootstrap [178] thanks to the Lorentzian inversion formula [179–181]. However, such effects have not been yet explored in the context of the heavy-light bootstrap.

3.4.1 Exact quantization condition and residues

In the small μ and large spin expansion, the Green's function (3.2.5) simplifies considerably. To see this, note that at small μ the Matone relation (3.2.1) becomes

$$a = \pm\frac{\ell+1}{2} + \mathcal{O}(\mu), \tag{3.4.1}$$

where we plugged in the dictionary from Table 3.1.1. Since the Green's function is invariant under $a \to -a$, it does not matter what sign we pick in (3.4.1). Choosing the minus sign in (3.4.1), the ratio of the $\sigma = -1$ term to the $\sigma = 1$ term in both the numerator and the denominator of (3.2.5) scales as $\mu^{\ell+1}$, which is exponentially small in spin. Neglecting this nonperturbative correction, we find

$$\begin{aligned} G_R^{\text{pert}}(\omega,\ell) &= (1+R_+^2)^{2a_1} e^{-\partial_{a_1}F} \\ &\frac{\Gamma(-2a_1)\Gamma(1/2-a+a_1-a_\infty)\Gamma(1/2-a+a_1+a_\infty)}{\Gamma(2a_1)\Gamma(1/2-a-a_1-a_\infty)\Gamma(1/2-a-a_1+a_\infty)}. \end{aligned} \tag{3.4.2}$$

In a sense, this expression is a generalization of the semi-classical Virasoro vacuum block [182, 183] to $d = 4$. Indeed, via (3.3.7) it encodes the contribution of the identity and multi-stress tensor contributions in the light-light channel, schematically $\mathcal{O}_L \times \mathcal{O}_L \sim 1 + T + T^2 + \ldots$. The effects non-perturbative in spin (which are intimately related to the presence of the black hole horizon) are, on the other hand, encoded in the contribution of the double-twist operators $\mathcal{O}_L \times \mathcal{O}_L \sim \mathcal{O}_L \Box^n \partial^\ell \mathcal{O}_L$.

We can now explicitly read off the poles and residues of (3.4.2). There are poles in the function $\Gamma(1/2 - a + a_1 - a_\infty)$ at positive energies $\omega = \omega_{n\ell}$, which are nothing but the dimensions of the double-twist operators. The locations of these poles are determined by the following quantization condition,

$$\omega_{n\ell}: \quad n = a + a_\infty - a_1 - 1/2, \quad n \geq 0. \tag{3.4.3}$$

Geometrically this corresponds to the quantization of the quantum A-period associated to the Seiberg-Witten geometry. The relation (3.4.3) implicitly defines the scaling dimensions of the double-twist operators $\omega_{n,\ell}$ via the black hole to gauge theory dictionary in Table 3.1.1 and (3.1.15), along with the Matone relation (3.2.1). Computing the residues of the two-point function (3.4.2) and using (3.3.7) and Table 3.1.2 then gives

$$\begin{aligned} c_{n\ell} &= \frac{(\ell+1)\Gamma(\Delta+n-1)\Gamma(2a_\infty-n)}{\Gamma(\Delta)\Gamma(\Delta-1)\Gamma(n+1)\Gamma(2a_\infty-n-\Delta+2)} \\ &\times \frac{(1+R_+^2)^{\Delta-2}e^{-\partial_{a_1}F}}{2}\left(\frac{d(a+a_\infty)}{d\omega}\right)^{-1}\Bigg|_{\omega=\omega_{n\ell}}. \end{aligned} \tag{3.4.4}$$

Note that, since F is defined by a power series in μ whose coefficients are rational functions, it is straightforward to invert (3.4.3) to any desired order in μ by perturbing around the $\mu = 0$ result. In this sense, (3.4.3) and (3.4.4) represent an exact solution for the bootstrap data.

3.4.2 Anomalous dimensions and OPE data

To organize the perturbative series, let us define

$$\begin{aligned} \omega_{n\ell} &= \omega_{n\ell}^{(0)} + \sum_{i=1}^{\infty} \mu^i \gamma_{n\ell}^{(i)} \, , \\ c_{n\ell} &= c_{n\ell}^{(0)} \left(1 + \sum_{i=1}^{\infty} \mu^i c_{n\ell}^{(i)} \right) \, . \end{aligned} \tag{3.4.5}$$

We then plug these expansions into (3.4.3) and (3.4.2), using the dictionary in Table 3.1.1 and (3.1.15), the Matone relation (3.2.1), and the definitions in Appendix 3.C. At zeroth order in μ, we reproduce the OPE coefficients in generalized free field theory, see e.g. [86, 96],

$$\omega_{n\ell}^{(0)} = \Delta + \ell + 2n \, , \tag{3.4.6}$$

$$c_{n\ell}^{(0)} = \frac{(\ell+1)\Gamma(\Delta+n-1)\Gamma(\Delta+n+\ell)}{\Gamma(\Delta)\Gamma(\Delta-1)\Gamma(n+1)\Gamma(n+\ell+2)}, \tag{3.4.7}$$

namely we have the following identity

$$\sum_{n,\ell=0}^{\infty} c_{n\ell}^{(0)} (z\bar{z})^{\frac{\omega_{n\ell}^{(0)}-\Delta-\ell}{2}} \frac{z^{\ell+1} - \bar{z}^{\ell+1}}{z - \bar{z}} = \frac{1}{(1-z)^{\Delta}(1-\bar{z})^{\Delta}} \, . \tag{3.4.8}$$

Now let us go to first order in μ. We find

$$\gamma_{n\ell}^{(1)} = -\frac{\Delta^2 + \Delta(6n-1) + 6n(n-1)}{2(\ell+1)} \, , \tag{3.4.9}$$

$$\begin{aligned} c_{n\ell}^{(1)} = \frac{1}{2} \bigg(& 3(\Delta-2) - \frac{3(\Delta+2n-1)}{\ell+1} + \\ & (3(\ell+2n+\Delta) - 2\gamma_1)(\psi^{(0)}(2+\ell+n) - \psi^{(0)}(\Delta+\ell+n)) \bigg) \, , \end{aligned} \tag{3.4.10}$$

where $\psi^{(m)}(x) = d^{m+1} \log \Gamma(x)/dx^{m+1}$ is the polygamma function of order m. These results agree with the light-cone bootstrap computations [89, 96, 98].

At second order $\mathcal{O}(\mu^2)$ the answers become more complicated, and are displayed explicitly in Appendix 3.D. Already at this order only $\mathcal{O}(1/\ell^2)$ results are available in the literature, which is the leading term in the large spin expansion. We find complete agreement with the result of [96].

At k-th order $\mathcal{O}(\mu^k)$ we find the following structure

$$\gamma_{n\ell}^{(k)} = \sum_{j=0}^{2k+1} R_j^{(k)}(n,\ell)\Delta^j, \tag{3.4.11}$$

where $R_j^{(k)}(n,\ell)$ are polynomials of degree $k-j$ in n and are meromorphic functions of ℓ. The singularities occur at $\ell_{\text{sing}} \in \mathbb{Z}$ and $-k-1 \leq \ell_{\text{sing}} \leq k-1$. These singularities are however spurious and occur because for $\ell < k$ it is not justified to drop the $\sigma = -1$ term when going from (3.2.5) to (3.4.2).
For the three-point functions $c_{n\ell}^{(k)}$ the structure is very similar, the main difference being that the analogs of $R_j^{(k)}(n,\ell)$ can also depend on $\psi^{(m)}(\Delta+n+\ell) - \psi^{(m)}(2+n+\ell)$ with $m \leq k-1$.

3.4.3 The imaginary part of quasi-normal modes

Until now, in computing the position of the poles of $G_R(\omega,\ell)$, we have neglected the imaginary part, which is exponentially suppressed at large spin.[12] This exponential suppression of the imaginary part means that the large spin quasinormal modes thermalize very slowly, so they give the leading contribution to the late time Green's function to leading order in the $1/c_T$ expansion.
Let us now compute the leading behavior of the imaginary part, for which we must consider the exact Green's function (3.2.5). In the large spin expansion, the numerator of (3.2.5) is finite, so the poles arise when the denominator vanishes. Therefore we must solve

$$0 = \sum_{\sigma=\pm} \mathcal{M}_{-\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)-}(a, a_1, a_\infty) t^{\sigma a} e^{-\frac{\sigma}{2}\partial_a F}. \tag{3.4.12}$$

We make an ansatz

$$\text{Im } \omega_{n\ell} = i\sum_{k=1}^{\infty} f_{n\ell}^{(k)} \mu^{\ell+1/2+k}, \tag{3.4.13}$$

where $f_{n\ell}^{(k)}$ are real. Note that the imaginary part behaves as μ^ℓ at large ℓ, as expected from the tunneling calculation in [99]. The first contribution to the imaginary part is at order $\mu^{\ell+3/2}$, which is consistent with numerical evidence [184]. As shown in Appendix 3.E, the explicit form of the leading contribution to the imaginary part is

$$f_{n\ell}^{(1)} = -\frac{2^{-4\ell}\pi^2}{(\ell+1)^2}\omega_{n\ell}^{(0)} \frac{\Gamma(\Delta+n+\ell)}{\Gamma(\Delta+n-1)} \frac{\Gamma(n+\ell+2)}{\Gamma(n+1)\Gamma(\frac{\ell+1}{2})^4}. \tag{3.4.14}$$

It should be possible to check this expression using the techniques of [185]. Note that $\text{Im } \omega_{n\ell} < 0$ as expected from causality.

[12] Physically, this is related to the fact that classically stable orbits can decay quantum-mechanically due to tunneling, see e.g. [131].

Appendix

3.A Conventions

Here we collect our conventions for various thermal two-point functions. Let us start with the case of the black hole. This is dual to a holographic CFT on $S^1 \times S^3$, with the radius of S^1 being β and the radius of S^3 set to 1. We have for the retarded two-point function

$$i\theta(t)\langle[\mathcal{O}(t,\vec{n}),\mathcal{O}(0,\vec{n}')]\rangle_\beta = \frac{1}{4\pi(\Delta-1)(\Delta-2)}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\sum_{\ell=0}^{\infty}(\ell+1)G_R(\omega,\ell)\frac{\sin(\ell+1)\theta}{\sin\theta}, \tag{3.A.1}$$

where $\vec{n}\cdot\vec{n}' = \cos\theta$ and $\vec{n}^2 = \vec{n}'^2 = 1$, so that $\vec{n},\vec{n}' \in S^3$. $G_R(\omega,\ell)$ is given by (3.2.5). We also used for partial waves $C_\ell^{(1)}(\cos\theta) = \frac{\sin(\ell+1)\theta}{\sin\theta}$.

For the Euclidean two-point function we have

$$\langle\mathcal{O}(\tau,\vec{n})\mathcal{O}(0,\vec{n}')\rangle_\beta = \int_{-\infty}^{\infty} d\omega\, e^{-\omega\tau}\sum_{\ell=0}^{\infty} g_{\omega,\ell}\frac{\sin(\ell+1)\theta}{\sin\theta}, \quad 0<\tau<\beta, \tag{3.A.2}$$

where $g_{\omega,\ell}$ is given in (3.3.6) and τ is the Euclidean time. KMS symmetry or invariance under $\tau \to \beta-\tau$ holds thanks to (3.3.5). We normalize the operators such that the unit operator contributes as $\frac{e^{-\tau\Delta}}{([1-e^{-\tau+i\theta}][1-e^{-\tau-i\theta}])^\Delta}$. The Wightman function can be obtained through Wick rotation by taking $\tau \to \epsilon + it$ and then $\epsilon \to 0$.

For the black brane, or holographic CFT on $S^1 \times \mathbb{R}^{d-1}$ with the radius of S^1 set to 1, we have for the retarded two-point function

$$i\theta(t)\langle[\mathcal{O}(t,\mathbf{x}),\mathcal{O}(0,0)]\rangle_{\beta=1} = \frac{1}{(4\pi)^2(\Delta-1)(\Delta-2)}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\int_{-\infty}^{\infty} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}G_R^{\text{brane}}(\omega,\mathbf{k}). \tag{3.A.3}$$

$G_R^{\text{brane}}(\omega,\mathbf{k})$ is given by (3.2.7).

For the Euclidean two-point function we have

$$\langle\mathcal{O}(\tau,\mathbf{x})\mathcal{O}(0,0)\rangle_{\beta=1} = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\, e^{-\omega\tau}\int_{-\infty}^{\infty} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}g_{\omega,\mathbf{k}}, \quad 0<\tau<1, \tag{3.A.4}$$

where τ is the Euclidean time and $g_{\omega,\mathbf{k}}$ is given by (3.B.2). We normalize operators such that the unit operator contributes as $\frac{1}{(\tau^2+\mathbf{x}^2)^\Delta}$. KMS symmetry or invariance under $\tau \to 1-\tau$ holds thanks to (3.B.5). The Wightman function can be obtained through Wick rotation by taking $\tau \to \epsilon + it$ and then $\epsilon \to 0$.

3.B From black hole to black brane

Let us describe in a bit more detail the infinite temperature limit that takes us from the black hole to the black brane. This is one example of the so-called macroscopic limits considered in [173] and we simply apply the formulas of that paper to our case.

First of all, we introduce the limiting retarded two-point function as follows,

$$G_R^{\text{brane}}(\omega, |\mathbf{k}|) = \lim_{T\to\infty} \frac{G_R(\omega T, |\mathbf{k}|T)}{T^{4a_1}}, \tag{3.B.1}$$

where $G_R(\omega, |\mathbf{k}|)$ is the retarded thermal two-point function for a CFT on $S^1 \times \mathbb{R}^3$ with $(\omega, |\mathbf{k}|)$ measured in units of temperature on S^1. Let us also introduce

$$g_{\omega,\mathbf{k}}^{\text{brane}} = \frac{1}{2\pi(\Delta-1)(\Delta-2)} \frac{\text{Im } G_R^{\text{brane}}(\omega, |\mathbf{k}|)}{1-e^{-\omega}}. \tag{3.B.2}$$

At the level of the two-point function we consider the following limit

$$G^{\text{brane}}(w, \bar{w}) \equiv \lim_{T\to\infty} T^{-2\Delta} G\Big(z = 1 - \frac{w}{T}, \bar{z} = 1 - \frac{\bar{w}}{T}\Big). \tag{3.B.3}$$

Plugging this formula in the OPE expansion (3.3.4) we get

$$\begin{aligned} G^{\text{brane}}(w, \bar{w}) &= \lim_{T\to\infty} T^{-4} \int_0^\infty d|\mathbf{k}||\mathbf{k}| \times T^2 \int_{-\infty}^\infty d\omega \times T\, g_{\omega,\mathbf{k}} e^{-\frac{(w+\bar{w})}{2}(\omega-|\mathbf{k}|)} \frac{e^{-w|\mathbf{k}|} - e^{-\bar{w}|\mathbf{k}|}}{\bar{w}-w} \times T \\ &= \int_0^\infty d|\mathbf{k}||\mathbf{k}| \int_{-\infty}^\infty d\omega\, g_{\omega,\mathbf{k}} e^{-\frac{(w+\bar{w})}{2}(\omega-|\mathbf{k}|)} \frac{e^{-w|\mathbf{k}|} - e^{-\bar{w}|\mathbf{k}|}}{\bar{w}-w}, \end{aligned} \tag{3.B.4}$$

where we converted the sum to an integral, $\sum_\ell \to T \int d|\mathbf{k}|$.

The KMS symmetry becomes

$$g_{-\omega,\mathbf{k}} = e^{-\omega} g_{\omega,\mathbf{k}}. \tag{3.B.5}$$

We next consider the two-point function on $S^1 \times \mathbb{R}^{d-1}$,

$$\langle \mathcal{O}(\tau, \mathbf{x}) \mathcal{O}(0,0) \rangle_\beta = G^{\text{brane}}\Big(\tau + i|\mathbf{x}|, \tau - i|\mathbf{x}|\Big). \tag{3.B.6}$$

In terms of these variables we get

$$\begin{aligned}\langle \mathcal{O}(\tau,\mathbf{x})\mathcal{O}(0,0)\rangle_\beta &= \int_0^\infty d|\mathbf{k}|\,|\mathbf{k}|\int_{-\infty}^{\infty} d\omega\, g_{\omega,\mathbf{k}} e^{-\omega\tau}\frac{\sin|\mathbf{k}||\mathbf{x}|}{|\mathbf{x}|}\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} d^3\mathbf{k}\int_{-\infty}^{\infty} d\omega\, e^{i\mathbf{k}\cdot\mathbf{x}}e^{-\omega\tau}g_{\omega,\mathbf{k}}.\end{aligned} \tag{3.B.7}$$

The result is indeed invariant under KMS symmetry $\tau \to 1-\tau$ (recall that we have set $\beta = 1$). By analytically continuing to Lorentzian time we see that $g_{\omega,\mathbf{k}}$ is the Fourier transform of the Wightman two-point function.

Note that taking the limit (3.B.3) does not change the normalization of the scalar operator, since

$$\lim_{T\to\infty} T^{-2\Delta}\frac{1}{\left(1-(1-\frac{w}{T})\right)^{\Delta}\left(1-(1-\frac{\bar{w}}{T})\right)^{\Delta}} = \frac{1}{(w\bar{w})^\Delta} = \frac{1}{(\tau^2+\mathbf{x}^2)^\Delta}. \tag{3.B.8}$$

In other words if the operator was unit-normalized it will continue to be unit-normalized after taking the limit.

Let us finish with a few formulas for the vacuum correlators. In Fourier space, the vacuum Wightman two-point function $\langle \mathcal{O}(t,\mathbf{x})\mathcal{O}(0,0)\rangle_0 = \frac{1}{(-(t-i\epsilon)^2+\mathbf{x}^2)^\Delta}$ takes the form

$$\int_{-\infty}^{\infty} dt\, d^3\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\frac{1}{(-(t-i\epsilon)^2+\mathbf{x}^2)^\Delta} = \theta(\omega)\theta(\omega^2-\mathbf{k}^2)\frac{2\pi^3}{\Gamma(\Delta)\Gamma(\Delta-1)}\Big(\frac{\omega^2-\mathbf{k}^2}{4}\Big)^{\Delta-2}. \tag{3.B.9}$$

It is expected that (3.B.9) controls the large ω asymptotics of the thermal correlators [186, 187].

From (3.B.7) we get that

$$g_{\omega,\mathbf{k}} = \lim_{\epsilon\to 0}\frac{1}{4\pi^3}\int_{-\infty}^{\infty} d^3\mathbf{k}\int_{-\infty}^{\infty} d\omega\, e^{-i\mathbf{k}\cdot\mathbf{x}}e^{it\omega}\langle \mathcal{O}(\epsilon+it,\mathbf{x})\mathcal{O}(0,0)\rangle_\beta. \tag{3.B.10}$$

Formulas (3.B.9), (3.B.5) together with (3.B.2) imply that

$$\lim_{|\omega|\gg 1, |\omega|\gg|\mathbf{k}|} \mathrm{Im}\ G_R^{\mathrm{brane}}(\omega,|\mathbf{k}|) \simeq -\sin\pi\Delta\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\mathrm{sign}(\omega)\Big(\frac{|\omega|}{2}\Big)^{2(\Delta-2)}. \tag{3.B.11}$$

Via dispersion relations for $G_R^{\mathrm{brane}}(\omega,|\mathbf{k}|)$ this leads to the following asymptotic behavior for the real part,

$$\lim_{|\omega|\gg 1, |\omega|\gg|\mathbf{k}|} \mathrm{Re}\ G_R^{\mathrm{brane}}(\omega,|\mathbf{k}|) \simeq \cos\pi\Delta\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\Big(\frac{|\omega|}{2}\Big)^{2(\Delta-2)}, \tag{3.B.12}$$

where everywhere we tacitly assumed that Δ is not an integer. For the black hole case ($t < \frac{1}{2}$) we get in the same way

$$\lim_{|\omega|/T\gg 1,\ell} G_R(\omega,\ell) \simeq e^{-\pi i \Delta \mathrm{sign}(\omega)} \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)} \left(\frac{|\omega|}{2}\right)^{2(\Delta-2)}. \tag{3.B.13}$$

We can also derive the large ω and fixed ℓ behavior of the Green's function directly from our exact expression (3.2.5). Let us start with the black hole case. By solving the Matone relation (3.2.1) order by order in the instanton expansion, one finds in this limit $\partial_a F = ic_1(t)\omega + \mathcal{O}(\omega^0)$, $\partial_{a_1} F = c_3(t)(\Delta-2) + \mathcal{O}(\omega^{-1})$, and $a = ic_2(t)\omega + \mathcal{O}(\omega^0)$, with $c_i(t) \in \mathbb{R}$. Since the Green's function (3.2.5) is invariant under $a \to -a$, we can choose $c_2 > 0$ without loss of generality. With this specification, the $\sigma = 1$ term in (3.2.5) dominates over the $\sigma = -1$ term. Expanding the gamma functions at large ω and using the dictionary in Table 3.1.1, we find

$$\begin{aligned} G_R(\omega,\ell) &\approx (1+R_+^2)^{2a_1} e^{-\partial_{a_1} F} \frac{\Gamma(-2a_1)}{\Gamma(2a_1)} (a_\infty - a)^{2a_1} (-a-a_\infty)^{2a_1} \\ &\approx \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)} \left(\frac{|\omega|}{2}\right)^{2(\Delta-2)} e^{-\pi i \Delta \mathrm{sign}(\omega)} \Big(c(t)\Big)^{\Delta-2}, \end{aligned} \tag{3.B.14}$$

where

$$c(t) = \frac{e^{-c_3(t)}(1-t)(4c_2(t)^2 + 2t^2 - 3t + 1)}{1-2t}. \tag{3.B.15}$$

The OPE predicts that $c(t) = 1$.

We do not have complete analytic control over the constants $c_2(t)$ and $c_3(t)$, but we checked that (3.B.15) approaches 1 by computing the first few orders in the instanton expansion, see Figure 3.B.1. Hence we recover (3.B.13). The black brane results (3.B.11) and (3.B.12) correspond to $t \to \frac{1}{2}$ in Figure 3.B.1.

3.C The Nekrasov-Shatashvili function

We denote by $Y = (\nu_1, \nu_2, ...)$ a partition (or Young tableau) and by $Y^t = (\nu_1^t, \nu_2^t, ...)$ its transpose. We also use $\vec{Y} = (Y_1, Y_2)$ to denote a vector of Young tableaux. The leg-length and the arm-length are defined by $h_Y(s) = \nu_j - i$ and $v_Y(s) = \nu_i^t - j$, where $s = (i,j)$ is a box. We define

$$\begin{aligned} z_{\mathrm{h}}\left(\vec{a},\vec{Y},\mu\right) &= \prod_{I=1,2} \prod_{s\in Y_I} \left(a_I + \mu + \epsilon_1\left(i-\frac{1}{2}\right) + \epsilon_2\left(j-\frac{1}{2}\right)\right), \\ z_{\mathrm{v}}\left(\vec{a},\vec{Y}\right) &= \prod_{I,J=1}^{2} \prod_{s\in Y_I} \frac{1}{a_I - a_J - \epsilon_1 v_{Y_J}(s) + \epsilon_2(h_{Y_I}(s)+1)} \prod_{s\in Y_J} \frac{1}{a_I - a_J + \epsilon_1(v_{Y_I}(s)+1) - \epsilon_2 h_{Y_I}(s)}. \end{aligned} \tag{3.C.1}$$

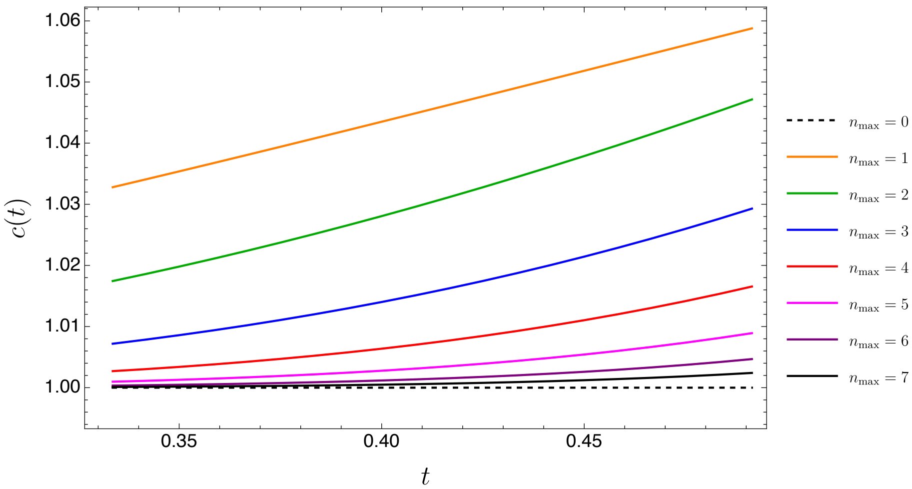


Figure 3.B.1: $c(t)$ defined in (3.B.15) as a function of the black hole mass (here parameterized by t), and the maximum instanton number $n_{\max}$. Based on the OPE we expect that $c(t)$ is independent of t and is equal to 1.

In this paper we always take $\vec{a} = (a, -a)$ and $\epsilon_1 = 1$. The instanton part of the NS function is defined as

$$F = \lim_{\epsilon_2 \to 0} \epsilon_2 \log \left((1-t)^{-2\epsilon_2^{-1}(\frac{1}{2}+a_1)(\frac{1}{2}+a_t)} \sum_{\vec{Y}} t^{|\vec{Y}|} z_{\text{v}} \left(\vec{a}, \vec{Y}\right) \prod_{\theta=\pm} z_{\text{h}} \left(\vec{a}, \vec{Y}, a_t + \theta a_0\right) z_{\text{h}} \left(\vec{a}, \vec{Y}, a_1 + \theta a_\infty\right) \right) . \tag{3.C.2}$$

Physically, a corresponds to the VEV of the scalar in the vector multiplet, ϵ_i are two Ω-background parameters regulating the infrared divergence in the localization computation, and a_0, a_∞, a_1, a_t are related to the masses m_i of the hypermultiplets via

$$m_1 = a_t + a_0, \quad m_2 = a_t - a_0, \quad m_3 = a_1 + a_\infty, \quad m_4 = a_1 - a_\infty \ . \tag{3.C.3}$$

This function takes the form of a convergent series expansion in t,

$$F = \sum_{n \geq 1}^{\infty} c_n(a, a_0, a_t, a_1, a_\infty) t^n, \tag{3.C.4}$$

where the c_n coefficients are rational functions defined via (3.C.2). For example we have

$$c_1(a, a_0, a_t, a_1, a_\infty) = \frac{\left(4a^2 - 4a_0^2 + 4a_t^2 - 1\right)\left(4a^2 + 4a_1^2 - 4a_\infty^2 - 1\right)}{8 - 32a^2}. \tag{3.C.5}$$

The full NS function $F^{\rm NS}$ includes, on top of the instanton part F, the classical and one-loop parts. We have

$$\begin{aligned}F^{\rm NS} =&F - a^2\log t - \psi^{(-2)}\left(-a-a_0-a_t+\frac{1}{2}\right) - \psi^{(-2)}\left(a-a_0-a_t+\frac{1}{2}\right) - \psi^{(-2)}\left(-a+a_0-a_t+\frac{1}{2}\right)\\ &- \psi^{(-2)}\left(a+a_0-a_t+\frac{1}{2}\right) - \psi^{(-2)}\left(-a-a_1-a_\infty+\frac{1}{2}\right) - \psi^{(-2)}\left(a-a_1-a_\infty+\frac{1}{2}\right)\\ &- \psi^{(-2)}\left(-a-a_1+a_\infty+\frac{1}{2}\right) - \psi^{(-2)}\left(a-a_1+a_\infty+\frac{1}{2}\right)\\ &+ \psi^{(-2)}(2a+1) + \psi^{(-2)}(1-2a)\ ,\end{aligned} \tag{3.C.6}$$

where $\psi^{(-2)}(x)$ is the polygamma function of negative order, $\psi^{(-n)}(x) = \frac{1}{(n-2)!}\int_0^z dt\,(z-t)^{n-2}\log\Gamma(t)$.

3.D $\mathcal{O}(\mu^2)$ OPE data of double-twist operators

Here we display the results for the OPE data at order μ^2. These expressions are in full agreement with [96] at order $1/\ell^2$, and provide new predictions at higher orders in $1/\ell$. We find

$$\begin{aligned}\gamma^{(2)}_{n\ell} =& -\frac{\left((\Delta-1)\Delta+6(\Delta-1)n+6n^2\right)^2}{8(\ell+1)^3} - \frac{n(\Delta+n-2)(\Delta+2n-2)^2}{2(\ell+2)} - \frac{(n+1)(\Delta+n-1)(\Delta+2n)^2}{2\ell}\\ &+ \frac{(\Delta-1)\Delta(8\Delta+1)+65n^4+130(\Delta-1)n^3+(3\Delta(27\Delta-43)+133)n^2+(\Delta-1)\left(16\Delta^2+\Delta+68\right)n}{16(\ell+1)}\\ &- \frac{(n-1)n(\Delta+n-3)(\Delta+n-2)}{32(\ell+3)} - \frac{(n+1)(n+2)(\Delta+n-1)(\Delta+n)}{32(\ell-1)},\\ c^{(2)}_{n\ell} =& \frac{1}{8}(\Delta-2)(9\Delta-44) - \frac{(2n+3)(\Delta+n-1)(\Delta+n)}{32(\ell-1)} - \frac{3(\Delta+2n-1)\left((\Delta-1)\Delta+6n^2+6(\Delta-1)n\right)}{4(\ell+1)^3}\\ &+ \frac{(\Delta+2n-1)\left(\Delta(16\Delta-71)+130n^2+130(\Delta-1)n+212\right)}{32(\ell+1)} - \frac{(\Delta+n-1)(\Delta+2n)(\Delta+4n+2)}{2\ell}\\ &- \frac{(n-1)n(2\Delta+2n-5)}{32(\ell+3)} - \frac{n(\Delta+2n-2)(3\Delta+4n-6)}{2(\ell+2)} + \frac{1}{4}(\psi^{(0)}(n+\ell+2)-\psi^{(0)}(n+\ell+\Delta))\\ &\times\left(9\Delta^2 - \frac{89\Delta}{2} + \frac{\left((\Delta-1)\Delta+6n^2+6(\Delta-1)n\right)^2}{2(\ell+1)^3} - \frac{3(\Delta+2n-1)\left((\Delta-1)\Delta+6n^2+6(\Delta-1)n\right)}{(\ell+1)^2}\right.\\ &+ \frac{(\Delta-1)(\Delta(4\Delta-73)+36)-65n^4-130(\Delta-1)n^3+(3(67-27\Delta)\Delta-493)n^2-(\Delta-1)(\Delta(16\Delta-71)+428)n}{4(\ell+1)}\\ &+ (18\Delta-89)n + \frac{2n(\Delta+n-2)(\Delta+2n-2)^2}{\ell+2} + \frac{2(n+1)(\Delta+n-1)(\Delta+2n)^2}{\ell} + \frac{(n-1)n(\Delta+n-3)(\Delta+n-2)}{8(\ell+3)}\\ &\left.+ \frac{(n+1)(n+2)(\Delta+n-1)(\Delta+n)}{8(\ell-1)} + \left(9\Delta-\frac{71}{2}\right)\ell+9\right) + \left(\Delta(\Delta+2)+6n^2+6n(\Delta+\ell)+3\ell^2+3(\Delta+1)\ell\right)^2\\ &\times\frac{(\psi^{(0)}(n+\ell+2)-\psi^{(0)}(n+\ell+\Delta))^2+\psi^{(1)}(n+\ell+\Delta)-\psi^{(1)}(n+\ell+2)}{8(\ell+1)^2}\ .\end{aligned}$$

Similar expressions up to order μ^5 can be found in the supplemental files.

3.E The imaginary part of quasi-normal modes

In this appendix we spell out some details for the computation of (3.4.14). The condition for a pole in $G_R(\omega,\ell)$ follows from (3.2.5) and reads

$$t^{-2a}e^{\partial_a F}\left(\frac{\Gamma(2a)\Gamma\left(-a-a_t+\frac{1}{2}\right)}{\Gamma(-2a)\Gamma\left(a-a_t+\frac{1}{2}\right)}\right)^2-\frac{\Gamma\left(a+a_1-a_\infty+\frac{1}{2}\right)\Gamma\left(a+a_1+a_\infty+\frac{1}{2}\right)}{\Gamma\left(-a+a_1-a_\infty+\frac{1}{2}\right)\Gamma\left(-a+a_1+a_\infty+\frac{1}{2}\right)}=0\,. \tag{3.E.1}$$

By using the ansatz (3.4.13) as well as the dictionary in Table 3.1.1 and the perturbative solution for the real part (3.4.5), we obtain

$$\begin{aligned}
&\mathrm{Im}\left(\frac{\Gamma\left(a+a_1-a_\infty+\frac{1}{2}\right)\Gamma\left(a+a_1+a_\infty+\frac{1}{2}\right)}{\Gamma\left(-a+a_1-a_\infty+\frac{1}{2}\right)\Gamma\left(-a+a_1+a_\infty+\frac{1}{2}\right)}\right)=\mu^{\ell+1/2}\left(\frac{\Gamma(n+1)\Gamma(n+\Delta-1)}{\Gamma(\ell+n+2)\Gamma(\ell+n+\Delta)}\frac{(-1)^\ell}{3\omega_{n\ell}^{(0)}-2\gamma_{n\ell}^{(1)}}f_{n\ell}^{(1)}+\mathcal{O}(\mu)\right)\\
&\mathrm{Im}\left(t^{-2a}e^{\partial_a F}\left(\frac{\Gamma(2a)\Gamma\left(-a-a_t+\frac{1}{2}\right)}{\Gamma(-2a)\Gamma\left(a-a_t+\frac{1}{2}\right)}\right)^2\right)=-\mu^{\ell+1/2}\left(\frac{\Gamma\left(\frac{\ell}{2}+1\right)^4}{\Gamma(\ell+1)^2\Gamma(\ell+2)^2}\frac{(-1)^\ell\omega_{n\ell}^{(0)}}{3\omega_{n\ell}^{(0)}-2\gamma_{n\ell}^{(1)}}+\mathcal{O}(\mu)\right)\,,
\end{aligned}$$

leading to (3.4.14).

3.F The large ℓ/large k, fixed ω limit

Using the asymptotic behavior (3.2.12), we can investigate the behavior of G_R at large ℓ. We start with the real part of G_R, for which the leading behavior comes from the $\sigma=1$ terms in (3.2.5). Expanding at large a, we find

$$\mathrm{Re}\ G_R(\omega,\ell)\approx(1+R_+^2)^{\Delta-2}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}e^{-\partial_{a_1}F}(-a)^{4a_1}\approx\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\left(\frac{\ell}{2}\right)^{2(\Delta-2)} \tag{3.F.1}$$

Note that this is independent of the temperature.

Now let us turn to the imaginary part. The leading contribution comes from expanding to first order in the $\sigma=-1$ term in both the numerator and denominator of (3.2.5). We find

$$\begin{aligned}
\mathrm{Im}\ G_R(\omega,\ell)\approx&-\frac{2(1+R_+^2)^{2a_1}e^{\partial_aF-\partial_{a_1}F}t^{-2a}\sin(2\pi a)\sin(2\pi a_1)}{\cos(2\pi(a-a_1))+\cos(2\pi a_\infty)}\frac{\Gamma(2a)^2\Gamma(-2a_1)\Gamma\left(\frac{1}{2}-a+a_1-a_\infty\right)\Gamma\left(\frac{1}{2}-a+a_1+a_\infty\right)}{\Gamma(-2a)^2\Gamma(2a_1)\Gamma\left(\frac{1}{2}+a-a_1-a_\infty\right)\Gamma\left(\frac{1}{2}+a-a_1+a_\infty\right)}\\
&\times\mathrm{Im}\left(\frac{\Gamma\left(\frac{1}{2}-a-a_t\right)^2}{\Gamma\left(\frac{1}{2}+a-a_t\right)^2}\right)\\
\approx&-\frac{\Gamma(-2a_1)}{\Gamma(2a_1)}(1+R_+^2)^{2a_1}e^{\partial_aF-\partial_{a_1}F}t^{-2a}2^{8a+1}(-a)^{4a_1}\sin(2\pi a_1)\sinh(2\pi|a_t|),
\end{aligned}$$

where in the second equality we took the large a limit. Plugging in the asymptotic behavior (3.2.12) and the dictionary given in Table 3.1.2 gives

$$\mathrm{Im}\ G_R(\omega,\ell)\approx\frac{2\pi\sinh(\pi\omega\sqrt{t(1-2t)})}{\Gamma(\Delta-1)\Gamma(\Delta-2)}\left(\frac{\ell}{2}\right)^{2(\Delta-2)}\exp\left(-2(\ell+1)\sqrt{1-2t}K(1-t)\right)\,. \tag{3.F.2}$$

We see that the imaginary part decays exponentially with spin.
To compute the large $|\mathbf{k}|$ behavior for the black brane, we can take the infinite temperature limit of (3.F.1) and (3.F.2). Using the definition (3.1.19) of the brane two-point function, we find

$$G_R^{\text{brane}}(\omega,|\mathbf{k}|) \approx \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\left(\frac{|\mathbf{k}|}{2}\right)^{2(\Delta-2)} + i\frac{2\pi\sinh\left(\frac{\omega}{2}\right)}{\Gamma(\Delta-1)\Gamma(\Delta-2)}\left(\frac{|\mathbf{k}|}{2}\right)^{2(\Delta-2)}\exp\left(-\sqrt{\frac{\pi}{2}}\frac{|\mathbf{k}|}{\Gamma\left(\frac{3}{4}\right)^2}\right). \tag{3.F.3}$$

The rate of exponential decay of the imaginary part matches the result from [135].

Conclusions

In this thesis we studied Heun's equation and a class of its confluences, deriving exact formulae for its connection coefficients. On the way, we studied novel objects in Liouville CFT and their connection to 4d gauge theory. Finally we applied the results to the study of black holes in the context of general relativity.

We started in chapter 1 by studying the confluent Heun equation with applications to the 4d Kerr black hole in mind. In this context, the confluent Heun equation is better known as the Teukolsky equation [2]. We provided exact formulae for its solutions, expanded around different singular points and for the connection matrices relating these solutions and demonstrated their validity by comparing with the numerical integration of the Teukolsky equation. This control of the solutions of the equation allowed us to study several physical quantities of interest. First, we derived an exact expression for the greybody factor of the black hole at finite frequency, which extends previously known results, obtained in different approximations. It matches the result of Maldacena and Strominger [76] in the zero frequency limit and the results obtained by studying the Teukolsky equation in the WKB approximation [77] upon taking the semiclassical limit. Second, we studied the quasinormal modes of the Kerr black hole which were conventionally calculated using purely numerical techniques. In [68] an analytic quantization condition for the quasinormal modes was given in the form of a well-motivated conjecture. We gave a physicist's proof of their formula by deriving it from first principles in Liouville CFT. An analogous quantization condition was proven for the angular part of the Teukolsky equation, giving the eigenvalues of the spin-weighted spheroidal harmonics. Finally, we studied the tidal deformations of the black hole, encoded in the so-called Love numbers. Our exact solution, valid at finite frequency allows to naturally distinguish the source and response term, a much discussed topic in the literature [100, 101]. We compared our expressions to the recent results on the Love numbers in the static [78] and quasi-static [79, 80] regimes.

In chapter 2 we extended the techniques introduced in chapter 1 to the general Heun equation and a class of its confluences, giving an extensive treatment of all the differential equations, their fundamental solutions in different regimes, and their connections matrices. On a more technical level, we studied in detail the role and the properties of the exotic rank 1 and rank 1/2 Whittaker states in Liouville CFT, in particular their physical normalization and interplay with Virasoro primary states, elaborating on results contained already (some-

what obscurely) in [47]. We also gave a detailed treatment of the large-modulus expansions of the corresponding confluent conformal blocks, extending and clarifying previous studies [104, 105].
Finally, in chapter 3 we applied our techniques to the study of the 5d AdS-Schwarzschild black hole and its holographically dual thermal CFT and managed to calculate the explicit expression for the thermal two-point function in this dual CFT. This is a completely new result and we checked its validity by comparing with previously obtained approximate results. In particular, exploiting the relationship between the thermal two-point function and the heavy-light four-point function we connect with the heavy-light lightcone bootstrap [83, 84]. In the large spin limit, we explained how to extract arbitrarily many terms contributing to the anomalous dimensions and OPE data of the double-twist operators from our exact expression. To zeroth order we reproduced results of generalized free field theory [86, 96], while to first order we found complete agreement with the lightcone bootstrap computations [89, 96, 98]. Already at second order we extended results which were previously only partially known [96], while giving the recipe to compute arbitrarily many higher orders. From our exact expression, we also extracted the imaginary part of the Green's function at large spin which is related to the thermalization properties of the quasinormal modes. We reproduced properties expected from tunneling arguments [99] and from numerical computations [184] and gave a recipe to extract arbitrarily many orders contributing to the imaginary part of the Green's function in the large spin regime from our exact formula.

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